Exit problems of Lévy processes with applications in finance

Eerste passage problemen van Lévy processen met toepassingen in financiering

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Universiteit Utrecht
op gezag van de Rector Magnificus, Prof. dr. W.H. Gispen,
ingevolge het besluit van het College voor Promoties
in het openbaar te verdedigen op

maandag 2 juni 2003 des namiddags te 4:15 uur

door

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geboren op 6 oktober 1976 te Dongen

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2000 Mathematics subject classification: Primary 60G40, 60G99, 60J30, 60J99, 90A09, 90A10, 91A09. Secondary 28D10, 60J65.

Keywords and key phrases. option pricing, perpetual option, call option, put option, Russian option, American option, Canadised option, integral option, optimal stopping, Laplace transform, Brownian motion, Bessel process, Lévy process, reflected Lévy process, Markov additive process, semimartingales, exit problem, first passage time, Wiener-Hopf factorisation, Wald martingale, scale function, h-transform, exponential decay, excursion theory, incomplete markets, utility maximisation, direct approach, variational problem, constrained trade.

Cover ontwerp: Ian Coenen

ISBN 90-393-3385-8

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Introduction

Risk plays a role in everyone's daily life. Some people are prepared to take more risk for example to achieve some goal, other people are quite 'risk-averse' and prefer to play safe. In financial markets similar patterns turn up with 'risk-loving' and 'risk-averse' agents. There risk can be *traded*: If an agent considers his financial position as too risky, he may limit the risk he is exposed to by buying an appropriate *contingent claim*. Such a claim is a contract between buyer and seller, where the latter promises to pay the former some payment or series of payments in the future. Contingent refers to the fact that at the time of agreement of the contract the actual size of the payment can be uncertain: The payment often depends on future developments (such as the price level of a certain stock one year from now). An example of a contingent claim is a *put (call) option*, which gives the right to sell (buy) a certain asset at a specified price until or at a future date.

History

The valuation of contingent claims is one of the main issues studied in modern finance: What is a fair price of a particular contingent claim? In other words, how much should the buyer of the claim pay to the seller such that both parties are satisfied (e.g. no of two parties can achieve a riskless profit)?

If the contract specifies that the holder has the right to exercise at a given future date, then this contingent claim is called a *European* option. In the literature, the pricing of European options goes back as early as Bachelier [17]. In 1900 he was the first to use Brownian motion with drift to model stock price fluctuations. In 1973 the papers of Black and Scholes [27] and Merton [97] appeared and would turn out to be milestones in the field: they established the important notions of *hedging* and *arbitrage free pricing*, which are currently common knowledge of traders worldwide. Harrison, Kreps and Pliska [66, 67] extended their ideas and put them on a firm mathematical basis using stochastic calculus. Almost 25 years later, in 1997, the Nobel Prize in Economics was awarded to Merton and Scholes for their path-breaking work (Black, who died in 1995, would undoubtedly have shared in the prize, had he still been alive.)

An option that turns up in practice more often, however, is the one where the holder has the right to exercise his contract at any time prior to the given future date. Claims of this type are called *American* options and their feature of intermediate exercise causes their valuation to be more complex and mathematically challenging. In this case, the question of the value of the option is intimately connected to that of the optimal exercise time of the holder. In 1965 McKean [99] was the first to give an analysis on pricing American options. He transformed the problem of pricing the American put into a *Stefan* or *free boundary* problem for the heat equation and solved this up to the free boundary. This free boundary corresponds to the *optimal exercise boundary*: It is optimal to exercise the put the first time that the stock price hits or falls below this space-time curve. Since then, a significant volume of literature has appeared on different aspects of pricing American options. See Myneni [103] for a review of the theory and methods of pricing American type options.

Modelling the stock price

The continuous time models we discussed until now all used the geometric Brownian motion as model for the evolution of the stock price. However, from extensive empirical research it appeared that this model is not ideal: It is not capable of replicating some of the features commonly seen in financial data, such as heavy tails and asymmetry. Recently, there has been a lot of interest in replacing the geometric Brownian motion by an exponential Lévy model which performs better empirically. A Lévy process is a stochastic process with stationary independent increments, whose paths are right-continuous and have left limits. The class of Lévy processes has a quite rich structure as is also demonstrated by the fact that it is in one-to-one correspondence with the class of infinitely divisible distributions. It is this flexibility that makes Lévy processes suitable for many modelling purposes. As most recent examples of stock price models driven by Lévy processes we mention the normal inverse Gaussian model proposed by Barndorff-Nielsen [20], the hyperbolic model of Eberlein [52], the variance-gamma model first explored by Madan and Seneta [92, 91] and the truncated stable family introduced by Koponen [32, 42, 81].

Replacing the standard geometric Brownian motion as model for the stock by an exponential Lévy process generally leads to several problems of different nature in answering the questions of valuation of contingent claims. In a market where the stock prices are driven by Brownian motion, the market is *complete*, that is, for every claim there exists a self-financing trading strategy such that the corresponding portfolio *replicates* the claim. By arbitrage arguments it then follows that the fair, arbitrage-free price of such a claim is equal to the initial value of its corresponding hedging strategy. Moreover, it turns out that the price can be evaluated as the expectation of the discounted claim under a *(local) martingale measure* equivalent to the 'real word' or 'objective' measure. This is a measure, also called a *risk neutral measure*, under which the discounted stock price becomes a local martingale. In a complete arbitrage free market an equivalent martingale measure exists and is unique.

Introducing jumps, however, generally leads to an *incomplete* market model. That is, in this market not all claims that can necessarily be replicated by a self-financing portfolio and if the market is free of arbitrage there exist infinitely many equivalent (local) martingale measures. It is therefore not clear what the fair price of these claims should be. Since a non-attainable claim can not be completely hedged against, for a particular agent the fair price of the claim will depend on his/her attitude towards risk. A possible approach is therefore to consider the pricing problem in the context of utility theory and link the pricing problem with utility optimisation problems of the agent.

A different approach of pricing in incomplete markets is based on selecting a particular local martingale measure as pricing measure. In analogy with the complete setting the price of the claim is then computed under this measure. In the literature different selection criteria have been developed, such as entropy minimisation and Esscher transformation, although it seems that the final word about this issue has not yet been spoken. For a review of the literature we refer to Chan [38] and references therein.

Whichever of the two approaches is taken, replacing the geometric Brownian motion model by an exponential Lévy process leads to many *mathematical* issues which need to be resolved to completely settle the problem of pricing options. The presence of jumps asks for adaptations of much of the previously mentioned theory connected to the classic geometric Brownian motion model. For example, in this model the value function of a European option with payoff only depending on the final value of the stock satisfies a partial differential equation. The possibility of jumps of the price process, however, introduces non-locality in the operator and, in the second approach mentioned above, we are led to the study of pseudo-differential equations. See e.g. [32] for recent work in this direction.

Organisation and outline of this thesis

The rest of this thesis consists of five self-contained chapters, each with its own summary and introduction, followed by a list of references. We give now an outline of the contents.

In the first chapter we study four options of American type in the context of the geometric Brownian motion model: the American put and call, the Russian option and the integral option. The value of the last two options was earlier computed in the papers [83, 116, 117]. We give an alternative derivation of their value exploiting properties of Brownian motion and Bessel processes. The four options we consider are all options of perpetual type, that is, they never expire. From a practical point of view perpetual options do not seem of much use, since in practice the time of expiration is always finite. However, following an appealing idea of Carr [36], one can build an approximating sequence of *perpetual-type* options that converges pointwise to the value of the corresponding finite time American option. This approximation procedure is also called *Canadization*. In Carr [36] numerical evidence was given for this convergence, here we give a mathematical proof. Next we compute for the three mentioned options the first approximation.

The second chapter proposes the *phase type* Lévy processes as a new model for the stock price. These are jump-diffusions whose positive and negative jumps form compound Poisson processes with jump distributions of *phase type*. Phase type distributions have a rational Laplace transform and are dense in all distributions. As a consequence, phase type Lévy processes form a class that is dense in all Lévy processes. Apart from this flexibility in modelling, the main reason for coming up with this new model is the analytical tractability of the pricing of many options under this model. We illustrate this by solving the problem of pricing the perpetual American put and Russian option under the phase type Lévy model. For the valuation we followed the second approach as sketched above, choosing as martingale measure the Esscher transform.

In the third chapter we study the same problems but now for the class of Lévy processes without negative jumps. We restrict ourselves to this class, since it contains already a lot of the rich structure of Lévy processes while still being analytically tractable due to many available results exploiting the fact that the jumps of the Lévy process have one sign. A recent study [37] offers empirical evidence supporting the case of a model where the risky asset is driven by a spectrally negative Lévy process. For this class of Lévy processes, we review theory on first exit times of finite and semi-infinite intervals. Subsequently, we determine the Laplace transform of the exit time and exit position from an interval containing the origin of the process reflected at its supremum. The proof relies on the application of Itô-excursion theory to the excursions of the reflected process away from zero. Combining the obtained results with martingale methods, we solve for the optimal stopping problem connected to the valuation of American perpetual put and Russian option and their Canadized versions, where we simply assumed the equivalent martingale measure already to have been chosen for us.

The fourth chapter complements the study on Lévy processes without negative jumps of the previous chapter. We find the Laplace transform of the first exit time of a finite interval containing the origin of the process reflected at its infimum. Then we turn our attention to these reflected processes killed upon leaving a finite interval containing zero and determine their resolvent measures. Invoking the R-theory of irreducible Markov chains developed by Tuomen and Tweedie [124], we are able to give a relatively complete description of the ergodic behaviour of their transition probabilities. The obtained results on Lévy processes in this and the previous chapter have also applications in the context of the theories of queueing, dams and insurance risk.

Finally, the fifth chapter considers the utility-optimisation problem of an agent that operates in a general semimartingale market and seeks to trade so as to maximise his utility from inter-temporal consumption and final wealth. In this setting existence is established following a direct variational approach, invoking a famous result of Komlós [80]. Also a characterisation for the optimal consumption and final wealth plan is given. The earlier mentioned problem of pricing contingent claims can be treated in this framework.

Publication details

The first four chapters presented in this thesis have been submitted to or accepted by refereed journals. The first chapter is a joint work with Andreas Kyprianou and has been accepted for publication in *the Annals of Applied Probability* as

> Kyprianou, A.E. and Pistorius, M.R. Perpetual options and Canadization through fluctuation theory.

Chapter 2 came out of a project with Søren Asmussen and Florin Avram and has been submitted to *Stochastic Processes and their Applications*.

> Asmussen, S., Avram, F. and Pistorius, M.R. American and Russian options under exponential phase type Lévy models.

The third chapter was written jointly with Florin Avram and Andreas Kyprianou and has been accepted in abbreviated form for publication in *the Annals of Applied Probability* as

> Avram, F., Kyprianou, A.E. and Pistorius, M.R. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options.

The fourth and fifth chapter have been submitted to *Journal of Theoretical Probability* and *Journal of Economic Theory* respectively.

> Pistorius, M.R. On exit and ergodicity of the reflected spectrally negative Lévy process reflected at its infimum.

> Pistorius, M.R. A direct approach to existence and characterisation of optimal consumption and investment in semimartingale markets.

Chapter I

Perpetual options and Canadisation

In this article it is shown that one is able to evaluate the price of perpetual calls, puts, Russian and integral options directly as the Laplace transform of a stopping time of an appropriate diffusion using standard fluctuation theory. This approach is offered in contrast to the approach of optimal stopping through free boundary problems [see volume 39,1 of Theory of Probability and its Applications]. Following ideas in [36], we discuss the Canadisation of these options as a method of approximation to their finite time counterparts. Fluctuation theory is again used in this case.

1 Introduction

We begin by introducing the standard stochastic model of a complete arbitrage free market. The market consists of a bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves in time deterministically such that

$$B_t = B_0 \mathrm{e}^{rt}, \quad B_0 > 0, \quad r \geq 0, \quad t \geq 0. \tag{1}$$

The value of the risky asset $S = \{S_t : t \geq 0\}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with the following components. Ω is the space of continuous functions $\omega = \{\omega_t\}_{t\geq 0}$, from $[0, \infty)$ to $\mathbb{R}$ with $\omega_0 = 0$. $\mathcal{F}$ is the smallest σ-algebra on Ω such that for every $t \geq 0$, the map $\omega \mapsto \omega_t$ of Ω to $\mathbb{R}$ is $\mathcal{F}/\mathcal{B}$-measurable, where $\mathcal{B}$ is the Borel-σ-algebra on $\mathbb{R}$. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is such that $W = W(\omega) = \{\omega_t : t \geq 0\} = \{W_t : t \geq 0\}$ is a Wiener process starting from the origin. Let $\mathcal{F}_t^0$ be the σ-algebra generated by W up to time t, then the filtration $\mathbf{F}$ is a flow of σ-algebras $\{\mathcal{F}_t : t \geq 0\}$, which are equal to the closure of $\cap_{s>t}\mathcal{F}_s^0$ by the $\mathbb{P}$-null sets of $\mathcal{F}$. The dynamics of the risky asset under $\mathbb{P}$ are given by an exponential of a Brownian motion with drift

$$S_t = s \exp\{\sigma W_t + \mu t\},$$

where $s > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$.

An option is a contract between the seller and the buyer, in which the buyer receives payments of the seller if certain events happen. Options may be divided two classes: *American* type options, which can be exercised at any time before the expiration date and *European* type options, which have exercise only at expiration. A *perpetual* option is an American type option with no expiration date. The buyer of a perpetual has the right to exercise it at any time t and receive then a payment π_t, which depends in some way on the underlying stock price S. Note that the zero time point is always taken to be the instant at which the contract commences. Examples of perpetual options are the call, the put, the Russian option [116, 117], and the integral option [83], with payments $\pi^c, \pi^p, \pi^r, \pi^i$ respectively:

$$\pi_t^p = \mathrm{e}^{-\lambda t}(K - S_t)^+, \qquad \pi_t^c = \mathrm{e}^{-\lambda t}(S_t - K)^+, \tag{2}$$

$$\pi_t^r = \mathrm{e}^{-\lambda t} \max\left\{\max_{u \le t} S_u, s\psi\right\}, \qquad \pi_t^i = \mathrm{e}^{-\lambda t}\left[\int_0^t S_u \mathrm{d}u + s\varphi\right], \tag{3}$$

where $\lambda, K, \psi, \varphi > 0$ are constants.

Remark The parameter K is called the *strike* price, s is usually taken as the value of the stock at time zero and we use y^+ to denote $\max\{y, 0\}$. The parameter λ can be considered as a continuous dividend rate. In order for the arbitrage free price of the Russian, call and integral perpetual option to be finite, λ has to be positive, whereas the price of put remains finite for $\lambda = 0$. See also [117, 50, 119]. Note $s\psi$ can be understood to be the supremum of the risky asset price process over some pre-contract period. Likewise, $s\varphi$ can be understood to be the integral of the stock price over some pre-contract period.

The payoffs of the perpetual call and put differ fundamentally from that of the Russian and integral option. The payoff of call and put only depend on the value of the underlying stock S at the exercise time, whereas the Russian and integral options are path dependent options. That is to say, that the payoff π_t depends on the whole path of the stock price S from some instant at or before the contract begins and up to time t.

Two fundamental questions that can be asked of American-type perpetual options are:

Q1. *What is the arbitrage free price of the option?* and

Q2. *What is an optimal time to exercise?*

Theorems 1 and 2 (see also for example [119] and [75]) give answers to these questions, but in a form that is not handy from an applied perspective. In order to state these theorems, we must first introduce a little more notation.

Throughout this article we shall use the letters s and x with the assumed relation

$$s = \exp\{\sigma x\}$$

to represent the relationship between the starting points of S and W. We introduce the measure $\mathbb{P}_x$ which is a translation of the measure $\mathbb{P}$ such that under $\mathbb{P}_x$, W is a Wiener process with initial position $W_0 = x$. Now introduce the measure $\mathbb{P}_x^\chi$ under which $W_t - \chi t$ is a Wiener process starting from x. The measures $\mathbb{P}_x^\chi$ and $\mathbb{P}_x$ are related through the Girsanov change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\chi}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\left\{\chi(W_t - x) - \frac{1}{2}\chi^2 t\right\}.$$

Henceforth it is understood that $\mathbb{E}_x^\chi$ refers to expectation with respect to $\mathbb{P}_x^\chi$. Note the value of the risky asset under $\mathbb{P}^{\mu/\sigma}$ satisfies $S_t = \exp\{\sigma W_t\}$.

Finally let $\mathcal{T}_{t,\infty}$ be the set of **F**-stopping times valued in $[t,\infty)$ and $\overline{\mathcal{T}}_{t,\infty}$ the set of **F**-stopping times valued in $[t,\infty]$ where $t \geq 0$.

Suppose now that $\pi = \{\pi_t : t \geq 0\}$ is an **F**-adapted sequence of non-negative payments. The following well established theorem addresses Q1 when the option holder has even the right never to exercise, corresponding to the case that their exercise time is infinite with possibly positive probability.

Theorem 1 *The arbitrage free price* $\Pi(t,s)$ *for an American type perpetual option at time* t *into the contract, with payments* π *and* S *starting at* s *satisfies*

$$\Pi(t,s) = \operatorname*{ess\,sup}_{\tau\in\overline{\mathcal{T}}_{t,\infty}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-r(\tau-t)}\pi_\tau \,\middle|\, \mathcal{F}_t\right].$$

In particular, the arbitrage free price of the option is given by

$$\sup_{\tau\in\overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-r\tau}\pi_\tau\right]. \tag{4}$$

If we formulate the problem insisting that the buyer must exercise within an almost surely finite time then exactly the same result holds except that $\overline{\mathcal{T}}_{t,\infty}$ should be replaced by $\mathcal{T}_{t,\infty}$.

The next Theorem, taken from [119], addresses Q2.

Theorem 2 *Suppose that the payments* π *are* $\mathcal{F}_t$*-measurable, càdlàg, without negative jumps and*

$$\{\mathrm{e}^{-r\tau}\pi_\tau : \tau \in \overline{\mathcal{T}}_{0,\infty}\}$$

is uniformly integrable with respect to $\mathbb{P}^{(r/\sigma-\sigma/2)}$. *Then*

$$\tau^* = \inf\{t \geq 0 : \Pi(t,s) \leq \pi_t\}$$

is an optimal exercise time for (4).

Again, when the problem of pricing is reformulated so that the buyer must exercise within an almost surely finite time, in the above Theorem we can replace $\overline{\mathcal{T}}_{0,\infty}$ by $\mathcal{T}_{0,\infty}$.

In reviewing the literature concerning perpetual options one finds two dominant methods that are used for their evaluation.

Free boundary problem approach. The first method has been nicely characterised in a series of papers [119, 117, 83] that appeared all together in volume 39,1 of Theory of Probability and its Applications. However its origin can be traced back as far as McKean's paper [99] in 1965. In these papers an approach based on free boundary problems, sometimes called Stephan problems, is applied to perpetual American call and put options, Russian options and integral options. Based on heuristic reasoning, the solution to an appropriate free boundary problem is taken as a candidate price for the option at hand. Then this solution is shown to be equal to the supremum (4) as a consequence of it being a solution to the free boundary problem.

Fluctuation theory approach. The second approach [119, 87], used for evaluating American call and put options, consists of proving that the optimal stopping time has the form of a hitting time of the stock price at some level, say a. Given that $(K - S_t)^+$ (or indeed $(S_t - K)^+$) is constant at such a hitting time, the price of the option is essentially proportional to the Laplace transform of the hitting time optimised over the level a. The computations for this procedure are very elementary once the optimal stopping time is realized as a hitting time.

In the case of the Russian perpetual option, it is also worth mentioning the paper [64]. In this paper the authors use two important properties to recover the price of the Russian perpetual. The first is that for continuous Markov processes Z, if τ_v is a hitting time of Z then, the expectation $E_z(\mathrm{e}^{-\lambda\tau_v} Z_{\tau_v})$ is a solution to a certain elliptic equation with boundary conditions. The second fact is the strong Markov property. These two essentially are enough to show that the optimal stopping time is that of a hitting time of an appropriate diffusion and also give the analytical form of the solution.

Below we give the conclusion of both the fluctuation theory and free boundary methods for perpetual calls and puts and the conclusion achieved by the first of these two methods for perpetual Russian and integral options. Recall that r and σ are parameters of the market (B, S) and λ is a parameter appearing in the claims outlined in (2) and (3).

Let $x_1 < 0 < x_2$ be the two roots of the quadratic equation

$$x^2 - \left(1 - \frac{2r}{\sigma^2}\right) x - \left(\frac{2\lambda + 2r}{\sigma^2}\right) = 0. \tag{5}$$

Theorem 3 *The arbitrage free price of a perpetual call and put at time t into the contract, $\Pi^{\text{call}}(t,s)$ and $\Pi^{\text{put}}(t,s)$, with payoff π^c and π^p respectively, are given by*

$$\Pi^{\text{call}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^C(S_t) \textit{ and } \Pi^{\text{put}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^P(S_t), \tag{6}$$

where

$$\Pi^C(s) = \begin{cases} (s_2 - K)\,(s/s_2)^{x_2} & \textit{if } s < s_2 \\ s - K & \textit{if } s \geq s_2 \end{cases}$$

and

$$\Pi^P(s) = \begin{cases} (K - s_1)\,(s/s_1)^{x_1} & \textit{if } s > s_1 \\ K - s & \textit{if } s \leq s_1. \end{cases}$$

Here

$$s_1 = K\frac{x_1}{x_1 - 1} < K\frac{x_2}{x_2 - 1} = s_2$$

are the optimal exercise boundaries. That is to say that the holder should exercise if the value of the asset exceeds or falls below s_2 and s_1 in the case of the call and put respectively.

Consider now the equation

$$y^2 - \left(1 + \frac{2r}{\sigma^2}\right) y - \left(\frac{2\lambda}{\sigma^2}\right) = 0 \tag{7}$$

with roots $y_1 < 0 < 1 < y_2$.

Theorem 4 *The arbitrage free price $\Pi^{\text{russ}}(t, s, \psi)$ of a perpetual Russian option at time t into the contract, with payoff π^r satisfies*

$$\Pi^{\text{russ}}(t, s, \psi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{R}}(\Psi_t),$$

where $\Psi_t := (\sup_{0 \le u \le t} S_u \vee s\psi)/S_t$ and

$$\Pi^{\mathrm{R}}(\psi) = \begin{cases} \tilde{\psi} \cdot \frac{y_2 \psi^{y_1} - y_1 \psi^{y_2}}{y_2 \tilde{\psi}^{y_1} - y_1 \tilde{\psi}^{y_2}}, & 1 \le \psi < \tilde{\psi}, \\ \psi, & \psi \ge \tilde{\psi}. \end{cases} \tag{8}$$

Here

$$\tilde{\psi} = \left| \frac{y_2}{y_1} \cdot \frac{y_1 - 1}{y_2 - 1} \right|^{\frac{1}{y_2 - y_1}}$$

is the optimal exercise boundary. That is to say that the holder should exercise if the process Ψ_t exceeds or equals $\tilde{\psi}$.

Theorem 5 *The arbitrage free price $\Pi^{\text{int}}(t, s, \varphi)$ of a perpetual integral option at time t into the contract with payoff π^i satisfies*

$$\Pi^{\text{int}}(t, s, \varphi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{I}}(\Phi_t),$$

where $\Phi_t := \left(\int_0^t S_u du + \varphi s\right)/S_t$ and

$$\Pi^{I}(\varphi) = \begin{cases} \varphi^* \frac{u(\varphi)}{u(\varphi^*)}, & 0 \le \varphi < \varphi^*, \\ \varphi, & \varphi \ge \varphi^*, \end{cases} \tag{9}$$

where

$$u(\varphi) = \int_0^\infty \mathrm{e}^{-2z/\sigma^2} z^{-y_2} (1 + \varphi z)^{y_1} \mathrm{d}z$$

and φ^* is the root of the equation $\varphi u'(\varphi) = u(\varphi)$. Here φ^* is the optimal exercise boundary, such that the holder should exercise once the process Φ_t exceeds or equals φ^*.

In this paper we shall show that the pricing of Russian and integral perpetual options can also be reduced to evaluating a Laplace transform of the hitting time of an appropriate diffusion, followed by a simple optimisation over the hitting level. These new proofs will rely heavily on fluctuation theory of Brownian motion and Bessel processes thus remaining loyal to ideas used in pricing perpetual calls and puts as explained in the second method above.

Several different proofs for pricing perpetual Russian options and one proof for the pricing of integral options already exist, [116, 117, 50, 83, 64]. One might therefore question the motivation behind providing alternative proofs. The first reason is that the methods used in this paper can and have been applied in markets where the underlying is assumed to be driven by a spectrally one sided Lévy process. The interested reader is referred to [16]. The free boundary problem approach in principle may also be applicable in this case. However knowledge of solutions to integro-differential equations is needed as opposed to fluctuation theory of Lévy processes. The existence of solutions to such integro-differential equations in general is less understood than the available tools for fluctuation theory. Secondly, the fluctuation techniques also give us an approach to deal with the issue of Canadisation.

The rest of this paper is organised as follows. In the next section, for the sake of completeness and later reflection, we review the derivation of the arbitrage free price of perpetual calls and puts in the context of fluctuation theory. Continuing in this vein, in Section 3 we show how the value of the Russian perpetual option can be established in a similar way. The strength of Section 3 centres around Theorem 6 which evaluates the Laplace transform of a hitting time of a Brownian motion reflected at its supremum. Section 4 deals with the integral option. In this case the optimal stopping time turns out to be that of a Bessel squared process with drift. This follows from the close relationship between exponential Brownian motion and Bessel squared processes (cf. [128, 129]). This connection also appears in the study of Asian options in [62].

Recently it has been proposed by Carr in [36] that finite expiry American type options can be approximated by a randomisation of the expiry date using an independent exponential distribution. This is what Carr refers to as Canadisation. The effect of randomisation is to make the optimal exercise boundary a constant, just like in the perpetual case. A better approximation to a fixed time expiry than this can be made by randomising using a sum of n independent exponential distributions (hence an Erlang distribution) whose total mean is the length of the contract. As n tends to infinity, it is possible to show convergence to the price of the finite expiry American option. These ideas work equally well for the Russian and integral option and we discuss them in Section 5.

On a final note we should say that the use of fluctuation theory, as indicated in the title of this paper, in effect constitutes only half of the pricing procedure. There is still a strength of optimal stopping theory found in Theorems 1 and 2 which give the foundation on which we build. For standard references in the context of these the reader is referred to [118], [105] and [87].

2 Perpetual call and put options

Combining Theorem 1 with the actual form of the system of payments for call and put (2), we find by a simple Markovian decomposition of the process S_t that the the price $\Pi^{\text{call}}, \Pi^{\text{put}}$ of a perpetual call and put satisfy (6), where

$$\Pi^{\text{call}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^{\mathrm{C}}(S_t) = \mathrm{e}^{-\lambda t} \sup_{\tau \in \overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma - \sigma/2)} \left[\mathrm{e}^{-(r+\lambda)\tau}(S_\tau - K)^+\right] \quad (10)$$

$$\Pi^{\text{put}}(t,s) = \mathrm{e}^{-\lambda t}\Pi^{\mathrm{P}}(S_t) = \mathrm{e}^{-\lambda t} \sup_{\tau \in \overline{\mathcal{T}}_{0,\infty}} \mathbb{E}_x^{(r/\sigma - \sigma/2)} \left[\mathrm{e}^{-(r+\lambda)\tau}(K - S_\tau)^+\right]. \quad (11)$$

Proposition 1 *The optimal stopping times in (10) and (11) are of the form*

$$\inf\{t \geq 0 : S_t \geq \mathrm{e}^{\sigma h}\} \text{ and } \inf\{t \geq 0 : S_t \leq \mathrm{e}^{\sigma l}\},$$

respectively, where h and l are some real constants.

Proof By choosing $\tau = 0$, we see that $\Pi^{\text{call}}(0,s) \geq (s-K)^+$, $\Pi^{\text{put}}(0,s) \geq (K-s)^+$, that is, perpetual calls and puts are always at least as valuable as the direct payoff. Noting that the function $x \mapsto (x-K)^+$ is increasing and convex, we see $\Pi^{\text{call}}(0,\cdot)$ is increasing and convex, since integration and taking the supremum preserve monotonicity and convexity. Furthermore, Π^{call} is bounded above by $\sup_\tau \mathbb{E}_x^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-(r+\lambda)\tau}S_\tau] < \infty$. Similarly, by the properties of $x \mapsto (K-x)^+$, $\Pi^{\text{put}}(0,\cdot)$ is bounded by K, decreasing and convex. Theorem 2 implies the optimal stopping times for the call and put are given by $\inf\{t \geq 0 : \Pi^{\text{call}}(0,S_t) = (S_t - K)^+\}$ and $\inf\{t \geq 0 : \Pi^{\text{put}}(0,S_t) = (K - S_t)^+\}$ respectively, which combined with above remarks completes the proof. □

Remark If we define for any Borel set B

$$\tau_B^W = \inf\{t \geq 0 : W_t \in B\},$$

then both the stopping times in the above proposition can be expressed respectively as $\tau^W_{[h,\infty)}$ and $\tau^W_{(-\infty,l]}$ under $\mathbb{P}_x^{(r/\sigma-\sigma/2)}$.

By Proposition 1, the supremum over all stopping times in $\overline{\mathcal{T}}_{0,\infty}$ in equations (10) and (11) is equal to the supremum over all hitting times $\{\tau^W_{[k,\infty)} : k \in \mathbb{R}\}$ and $\{\tau^W_{(-\infty,k]} : k \in \mathbb{R}\}$ respectively. Thanks to the continuity of Brownian motion, there is no overshoot at these stopping times. Thus the prices Π^{call}, Π^{put} are given by $\Pi^{\text{call}}(0,s) = \sup_{h\in\mathbb{R}} V_h^{(1)}(s)$ and $\Pi^{\text{put}}(0,s) = \sup_{l\in\mathbb{R}} V_l^{(2)}(s)$ where

$$V_h^{(1)}(s) = \begin{cases} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{[h,\infty)}}\right](\mathrm{e}^{\sigma h} - K)^+ & \log s < \sigma h, \\ (s-K)^+ & \log s \geq \sigma h, \end{cases} \quad (12)$$

and

$$V_l^{(2)}(s) = \begin{cases} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{(-\infty,l]}}\right](K - \mathrm{e}^{\sigma l})^+ & \log s > \sigma l, \\ (K-s)^+ & \log s \leq \sigma l. \end{cases} \quad (13)$$

Remark The functions $V_h^{(1)}$ and $V_l^{(2)}$ in equations (12) and (13) have a clear financial interpretation. $V_h^{(1)}$ is the value of an option that "knocks in" on exceedance of the level $\exp \sigma h$ with call rebate, that is, the option expires as soon as the stock exceeds the level $\exp \sigma h$ and pays out then the amount $(\exp \sigma h - K)^+$. By optimising over all possible values of h we find the value of the perpetual call. Similarly, $V_l^{(2)}$ is the value function of an option which expires if the stock value falls below the level $\exp \sigma l$ and then pays out the amount $(K - \exp \sigma l)^+$.

Thus, the computation of the prices $\Pi^{\mathrm{call}}, \Pi^{\mathrm{put}}$ boils down to the computation of the Laplace transform of a hitting time of Brownian motion at a certain (constant) level, followed by an optimisation over that level. This Laplace transform has a well known explicit formula to be found in any standard text on Brownian motion and can for example easily be derived using the Wald martingale. We thus quote without reference that

$$\begin{aligned} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{[h,\infty)}}\right] &= \mathrm{e}^{-\sigma x_2(h-x)} \\ \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-(r+\lambda)\tau^W_{(-\infty,l]}}\right] &= \mathrm{e}^{\sigma x_1(x-l)} \end{aligned}$$

when $h > x$ and $l < x$ respectively. Recall that x_1 and x_2 are the roots of the quadratic equation (5).

Proof of Theorem 3 This follows as a simple optimisation procedure in (12) and (13). □

Remark Notice the optimal stopping times for the optimal stopping problem are not necessarily finite, depending on the sign of $r - \sigma^2/2$. If, for example, $r < \sigma^2/2$ and the risky asset starts below the optimal exercise value s_2, the optimal stopping time for a call is infinite with positive $\mathbb{P}^{(r/\sigma-\sigma/2)}$-probability. Had we insisted that the holder should exercise in an almost surely finite time, there would have been no optimal exercise strategy in this case.

3 Perpetual Russian option

Following the lead of [119], the first step in solving this problem consists in recognising that under $\mathbb{P}_x^{(r/\sigma-\sigma/2)}$, $s^{-1}\mathrm{e}^{-rt}S_t$ acts as a Girsanov change of measure, which adds an extra drift σ to the Wiener process W. If we insist now that the claimants of the Russian option must exercise within an almost surely finite time we can use the above change of measure together with Theorem 1 to get

$$\Pi^{\mathrm{russ}}(t,s,\psi) = S_t \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,\infty}} \mathbb{E}_x^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\lambda\tau}\frac{\overline{S}_\tau \vee \psi s}{S_\tau}\,\middle|\,\mathcal{F}_t\right], \tag{14}$$

where $\overline{S}_t := \max_{0\le u\le t} S_u$. Introduce the new stochastic process $\Psi = \{\Psi_t, t \ge 0\}$ with $\Psi_t = (\overline{S}_t \vee \psi s)/S_t$. Note that it can be easily verified that Ψ is a Markov process (see [117]). Suppose now that the underlying Brownian motion has been running not since time zero, but since some time $-M < 0$ and further that, given $\mathcal{F}_0$, the exponential of the current distance of the Brownian motion from its previous maximum is ψ. In this instance Ψ can be understood to be the exponential of the excursions of a Brownian motion with drift away from its maximum given that at time zero its value is ψ. With this in mind, for each $\gamma \in \mathbb{R}$ let us introduce a new measure $\overline{\mathbb{P}}^{\chi}_{\psi}$ under which we assume that $\Psi_0 = \psi$ and that $W_t - \chi t$ is a Wiener process. We shall reserve the special notation $\overline{\mathbb{P}}^{\gamma} = \overline{\mathbb{P}}^{\gamma}_1$ In light of the fact that Ψ is a Markov process we can thus rewrite (14) as

$$\Pi^{\text{russ}}(t, s, \psi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{R}}(\Psi_t) \tag{15}$$

with

$$\Pi^{\mathrm{R}}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}[\mathrm{e}^{-\lambda\tau}\Psi_\tau],$$

where $\overline{\mathbb{E}}^{\chi}_{\psi}$ is expectation with respect to $\overline{\mathbb{P}}^{\chi}_{\psi}$ and, in effect, we may now take $\Psi_t := \overline{S}_t/S_t$ (which is not a function of s). Moreover, on account of Theorem 2, the optimal stopping time in (15) is given by

$$\inf\{s \geq 0 : \Pi^{\mathrm{R}}(\Psi_s) \leq \Psi_s\}. \tag{16}$$

Proposition 2 *The optimal stopping time in (15) is given by*

$$\tau^* = \inf\{t \geq 0 : \Psi_s \geq \tilde{\psi}\} \tag{17}$$

for some constant $\tilde{\psi} \geq 0$.

Proof By choosing the stopping time $\tau = 0$ we see that $\Pi^{\mathrm{R}}(\psi) \geq \psi$. Now note that we can write

$$\Pi^{\mathrm{R}}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{1}[\mathrm{e}^{-\lambda\tau}(\overline{S}_\tau \vee \psi s)/S_\tau]$$

where the dependency on s is superficial as it disappears through cancellation in the ratio. Since for every $\omega \in \Omega$ the function $\psi \mapsto (\overline{S}_\tau \vee \psi s)/S_\tau$ is a convex increasing function, $\Pi^{\mathrm{R}}(\cdot)$ inherits these properties, as integration over ω and taking the supremum over τ preserve monotonicity and convexity. Combining these facts with Theorem 2 completes the proof of optimality of τ^*. Finally, from the expression for the Laplace transform of a hitting time of Ψ of the form (17), stated in the forthcoming Theorem 6, we deduce that the optimal level $\tilde{\psi}$ is finite (since under $\overline{\mathbb{P}}^{(r/\sigma+\sigma/2)}$ the Laplace transform is $o(\tilde{\psi}^{-1})(\tilde{\psi} \to \infty)$) and also that the stopping time τ^* is almost surely finite. Thus $\tau^* \in \mathcal{T}_{0,\infty}$. □

It can now be seen that, just like the previous section, the valuation of the Russian option can be achieved by the evaluation of the Laplace transform of a crossing time. The following Theorem tells us what we need to know.

Theorem 6 *For Borel sets B let*

$$\tau^{\Psi}_B = \inf\{t \geq 0 : \log \Psi_t \in B\}.$$

Setting $\eta = \sqrt{2\lambda + \chi^2}/\sigma$, *we have for* $b \geq 0, \chi \in \mathbb{R}$ *and* $\log \psi \in [0, b]$

$$\overline{\mathbb{E}}^{\chi}_{\psi}[\mathrm{e}^{-\lambda \tau^{\Psi}_{[b,\infty)}}] = \left(\frac{\psi}{\mathrm{e}^b}\right)^{\frac{\chi}{\sigma}} \frac{\sigma\eta\cosh(\eta\log\psi) - \chi\sinh(\eta\log\psi)}{\sigma\eta\cosh(\eta b) - \chi\sinh(\eta b)}.$$

Proof of Theorem 4 From (15), Proposition 2, the continuity of Brownian motion and then Theorem 6, it follows as a matter of checking that that $\Pi^{\mathrm{R}}(\psi)$ is equal to the supremum over all $m \geq 1$ of $V^{(3)}_m(\psi)$ where

$$V^{(3)}_m(\psi) = \begin{cases} m \cdot \frac{y_2\psi^{y_1} - y_1\psi^{y_2}}{y_2 m^{y_1} - y_1 m^{y_2}} & 1 \leq \psi \leq m, \\ \psi & \psi > m. \end{cases}$$

Here y_1 and y_2 are the two solutions to the quadratic equation (7). By elementary optimisation we find, that Π^{R} is given by equation (8). □

We conclude this section by proving Theorem 6.

Proof of Theorem 6 First we prove the identity for $\sigma = 1$ and $\psi = 1$. Let $\overline{W} = \{\overline{W}_t, t \geq 0\}$ with $\overline{W}_t = \sup_{s \leq t} W_s$ denote the supremum of W. The process $\{\log\Psi_t = \overline{W}_t - W_t : t \geq 0\}$ can be written as the excursion process of W away from its supremum. Now let $L = \{L_t : t \geq 0\}$ be local time at zero of $\overline{W} - W$. It is well known that this process can be taken as simply the supremum; that is $L = \overline{W}$. Setting $\eta = \sqrt{2\lambda + \chi^2}$, we use Girsanov's theorem to find that

$$\begin{aligned} \overline{\mathbb{E}}^{\chi}\left[\mathrm{e}^{-\lambda\tau^{\Psi}_{[b,\infty)}}\right] &= \overline{\mathbb{E}}\left[\mathrm{e}^{-(\lambda+\chi^2/2)\tau^{\Psi}_{[b,\infty)} + \chi W(\tau^{\Psi}_{[b,\infty)})}\right] = \overline{\mathbb{E}}^{\eta}\left[\mathrm{e}^{(\chi-\eta)W(\tau^{\Psi}_{[b,\infty)})}\right] \\ &= \overline{\mathbb{E}}^{\eta}\left[\mathrm{e}^{(\chi-\eta)(W(\tau^{\Psi}_{[b,\infty)}) - \overline{W}(\tau^{\Psi}_{[b,\infty)})) + \overline{W}(\tau^{\Psi}_{[b,\infty)})}\right] \\ &= \mathrm{e}^{(\eta-\chi)b}\,\overline{\mathbb{E}}^{\eta}\left[\mathrm{e}^{(\chi-\eta)L(\tau^{\Psi}_{[b,\infty)})}\right], \end{aligned} \tag{18}$$

where we used that $L = \overline{W}$.

Now recall that, since $\log\Psi$ is recurrent under $\overline{\mathbb{P}}^{\eta}$, Itô theory of excursions tells us that under $\overline{\mathbb{P}}^{\eta}$ the suprema of excursions of $\log\Psi$ away from zero $h = \{h_t : t \geq 0\}$ form a Poisson point process indexed by the local time L. Since $L(\tau^{\Psi}_{[b,\infty)})$ is the time in this Poisson point process at which the first excursion with height greater or equal to b occurs, $L(\tau^{\Psi}_{[b,\infty)})$ is exponentially distributed with parameter $\nu[b,\infty)$ where ν is the characteristic measure of the Poisson point process h.

In order to proceed with the right hand side of (18) we need to supply an expression for $\nu[b,\infty)$. Under $\mathbb{P}^{\eta}$ and for $x > 0$, the set $\{\tau^W_{[y,\infty)} < \tau^W_{(-\infty,-x]}\}$ coincides with the set $\{h_t \leq t + x; 0 \leq t \leq y\}$ of excursions of W away from its supremum, up to local time y, which have height smaller than $x + t$ at local time t. Let $N_t(b)$ denote the number of excursions of maximal height greater or equal to b up to local time t. Then,

$$\begin{aligned} \mathbb{P}^{\eta}\left(h_t \leq t + x; 0 \leq t \leq y\right) &= \mathbb{P}^{\eta}\left(N_t(x+t) = 0, 0 \leq t \leq y\right) \\ &= \exp\left\{-\int_0^y \nu([x+t,\infty))\mathrm{d}t\right\}. \end{aligned} \tag{19}$$

On the other hand, we know from diffusion theory (e.g. [112]) that

$$\mathbb{P}^{\eta}\left(\tau^{W}_{[y,\infty)} < \tau^{W}_{(-\infty,-x]}\right) = \frac{s(0)-s(-x)}{s(y)-s(-x)}, \qquad x>0, \tag{20}$$

where s denotes the scale function of a Brownian motion with drift η $[s(x) = (1-\mathrm{e}^{-2\eta x})/2\eta]$. Comparing (19) and (20) we find for positive x that $\nu([x,\infty)) = s'(x)/s(x)$.

Now returning to the right hand side of (18), we have that $L(\tau^{\Psi}_{[b,\infty)})$ is exponentially distributed with parameter $s'(b)/s(b)$ and hence

$$\overline{\mathbb{E}}^{\eta}\left[\mathrm{e}^{-(\eta-\chi)L(\tau^{\Psi}_{[b,\infty)})}\right] = \frac{s'(b)}{(\eta-\chi)s(b)+s'(b)}.$$

After some algebra we then recover the result in Theorem 6 for $\psi = 1$.

Consider now the case that $\log\psi \in (0,b)$ and $\sigma = 1$. Note that, $\{\log\Psi_t, t \le \tau^{\Psi}_{(0,b)^c}\}$ has under $\overline{\mathbb{P}}^{\chi}_{\psi}$ the same law as $\{-W_t, t\le \tau^{W}_{(-b,0)^c}\}$ under $\mathbb{P}^{\chi}_{-\log\psi}$. Set $\mathcal{A} := \mathcal{A}(\log\psi - b, \log\psi)$ equal to the event that W exits the interval $\log\psi + (-b,0)$ below and let $\mathcal{A}^c$ denote the complement. The strong Markov property of Ψ now implies that

$$\begin{aligned}\overline{\mathbb{E}}^{\chi}_{\psi}\left[\mathrm{e}^{-\lambda\tau^{\Psi}_{[b,\infty)}}\right] &= \mathbb{E}^{\chi}\left[\mathrm{e}^{-\lambda\tau^{W}_{(-\infty,-(b-\log\psi)]}}\mathbf{1}_{\mathcal{A}}\right] \\ &\quad + \mathbb{E}^{\chi}\left[\mathrm{e}^{-\lambda\tau^{W}_{[\log\psi,\infty)}}\mathbf{1}_{\mathcal{A}^c}\right]\cdot\overline{\mathbb{E}}^{\chi}\left[\mathrm{e}^{-\lambda\tau^{\Psi}_{[b,\infty)}}\right].\end{aligned}$$

The first and second expectation follow from equation (20) applied to a Wiener process killed at an independent exponential time with parameter λ, which has as scale function proportional to $\mathrm{e}^{-\gamma x}\sinh(\eta x)$. The third expectation follows from the first part of the proof. A simple algebraic exercise leads to the stated result.

In order to remove the condition $\sigma = 1$, it suffices to consider the Laplace transform of the first time the process $\log\Psi^{1/\sigma}$ enters $[b/\sigma,\infty)$. □

4 Perpetual integral option

Analogously to what was done at the beginning of the last section and following the procedure in [119], we combine Theorem 1 with the Girsanov density $s^{-1}\exp\{-rt\}S_t$ under $\mathbb{P}^{(r/\sigma-\sigma/2)}_x$ and insist that the option holder must exercise in an almost surely finite time to achieve

$$\Pi^{\mathrm{int}}(t,s,\varphi) = S_t \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,\infty}} \mathbb{E}^{(r/\sigma+\sigma/2)}_x\left[\mathrm{e}^{-\lambda\tau}\frac{\int_t^\tau S_u du + (s\varphi + \int_0^t S_u du)}{S_\tau}\middle|\mathcal{F}_t\right].$$

We introduce the new stochastic process $\Phi = \{\Phi_t, t\ge 0\}$ with

$$\Phi_t := \frac{\int_0^t S_u du + s\varphi}{S_t},$$

which can easily be verified to be a Markov process. For convenience let us now assume that the Brownian motion driving the stock has been observed since some time $-M \leq 0$ and we shall interpret the constant φ to be the quantity $s^{-1}\int_{-M}^{0} S_u du$ (and assume that this is $\mathcal{F}_0$ measurable). Thus if $\tilde{\mathbb{P}}^{\chi}_{\varphi}$ is the probability measure under which W is a $\mathbb{P}^{\chi}_0$-Brownian motion but the process Φ has value at time zero equal to φ, then it follows that

$$\Pi^{\mathrm{int}}(t,s,\varphi) = \mathrm{e}^{-\lambda t} S_t \Pi^{\mathrm{I}}(\Phi_t)$$

with

$$\Pi^{\mathrm{I}}(\varphi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\varphi} \left[\mathrm{e}^{-\lambda\tau} \Phi_\tau \right], \tag{21}$$

where $\tilde{\mathbb{E}}^{\chi}_{\varphi}$ is expectation with respect to $\tilde{\mathbb{P}}^{\chi}_{\varphi}$ and, in effect, we may now take $\Phi_t := \int_{-M}^{t} S_u du / S_t$ (which is not a function of s).

As before we have the following result, which characterises the optimal stopping time in (21) as a hitting time of the process Φ.

Proposition 3 *The optimal stopping time in* (21) *is a hitting time of the form*

$$\inf\left\{t \geq 0 : \Phi_t \geq \tilde{\varphi}\right\}, \qquad \tilde{\varphi} \geq 0. \tag{22}$$

Analogously as in Proposition 2, we can prove the form of the optimal stopping time. The finiteness of the optimal stopping time follows from the forthcoming Theorem 7.

The problem of pricing the perpetual integral option, just as in the case of the perpetual Russian option, is reduced to the evaluation of a Laplace transform of a stopping time of a Markov process. The following Theorem essentially gives the analytical structure to the final price given in Theorem 5.

Theorem 7 *For Borel sets B let*

$$\tau^{\Phi}_{B} = \inf\{t \geq 0 : \Phi_t \in B\}.$$

For $\varphi \in [0,b)$, $\lambda \geq 0$ and $\chi \geq 0$ we have

$$\tilde{\mathbb{E}}^{\chi}_{\varphi}\left[\mathrm{e}^{-\lambda \tau^{\Phi}_{[b,\infty)}}\right] = \frac{u_\lambda(\varphi)}{u_\lambda(b)} \tag{23}$$

where the function u_λ is given by

$$u_\lambda(x) = \int_0^\infty \mathrm{e}^{-2y/\sigma^2} y^{-(z_1+1)} (1+yx)^{z_2} \mathrm{d}y.$$

with $z_1 < z_2$ the roots of $z^2 - (2\chi/\sigma)z - (2\lambda/\sigma^2) = 0$. In particular, $\tilde{\mathbb{P}}^{\chi}_{\varphi}(\tau^{\Phi}_{[b,\infty)} < \infty) = 1$.

We shall shortly prove this Theorem but let us proceed by showing that the price of the integral option can now be quickly obtained.

Proof of Theorem 5 The proof is given along the same lines as the proof of Theorem 4. We start by noting that Proposition 3 in conjunction with the relation (21), the continuity of Φ and Theorem 7 with $\chi = r/\sigma + \sigma/2$ implies that $\Pi^{\mathrm{I}}(\varphi)$ is equal to the supremum over all $m \geq 1$ of $V_m^{(4)}(\varphi)$ where

$$V_m^{(4)}(\varphi) = m \cdot u_\lambda(\varphi)/u_\lambda(m) \qquad 0 \leq \varphi \leq m.$$

The function $f(m) := m/u_\lambda(m)$ is positive and differentiable such that $f(0) = 0$ and $f(m)$ decreases to 0 as $m \to \infty$. Since u_λ is increasing and strictly convex it thus follows that there is a unique point in $[0, \infty)$ satisfying $f'(m) = 0$ or equivalently $u_\lambda(m) = m u'_\lambda(m)$. The theorem is proved. □

We now conclude this section by proving the main result, Theorem 7. We will set $\sigma = 1$. The case of general σ is reduced to the case $\sigma = 1$ by noting that, by the scaling property of Brownian motion, $\{\Phi_t, t \geq 0\}$ has the same law under $\tilde{\mathbb{P}}^{\chi}_{\phi}$ as $\{\sigma^{-2}\Phi_{\sigma^2 t}, t \geq 0\}$ under $\tilde{\mathbb{P}}^{\chi/\sigma}_{\varphi\sigma^2}$.

Let $R^{(\gamma)}$ denote a Bessel process with dimension $d = 2(\gamma + 1)$ (or *index* γ) and starting in $R^{(\gamma)}(0) = 1$. The main idea behind the proof is to take advantage of Lamperti's relation [88]

$$\exp(W_t + \chi t) = \left[R^{(2\chi)}\left(A_t^{(\chi)}/4\right)\right]^2, \tag{24}$$

where $\chi \geq 0$ and

$$A_t^{(\chi)} = \int_0^t \exp(W_s + \chi s)\mathrm{d}s.$$

See [112] for background on Bessel processes. Thus $\tau^{\Phi}_{[b,\infty)}$ may be considered to be of the form

$$\tau^{\Phi}_{[b,\infty)} = \inf\left\{t \geq 0 : R^{(2\chi)}\left(\frac{1}{4}A_t^{(\chi)}\right) \leq \sqrt{\frac{4}{b}\left(\frac{1}{4}A_t^{(\chi)}\right) + \frac{\varphi}{b}}\right\}.$$

One can now see that the necessary fluctuation theory we need concerns Bessel processes. Unlike the case of the Russian option the necessary fluctuation results we shall apply are quite deep and specific. We summarise them in the following two Lemmas whose proofs can be found in [128] and [127] respectively. The first Lemma is not too difficult to recover from the Girsanov Theorem, but the second needs considerably more work to prove.

Lemma 1 *Let $\hat{P}^{\chi}_x$ be the law of a Bessel process with index χ started from $x > 0$ and $\hat{E}^{\chi}_x$ expectation with respect to this measure. For any stopping time T, define $I_T = \int_0^T [R(s)]^{-2} ds$ where $\{R(t) : t \geq 0\}$ is a Bessel process. Suppose that T is $\hat{P}^{\chi}_x$-almost surely finite, then for $\lambda \geq 0$*

$$\hat{E}^{\chi}_x\left[\mathrm{e}^{-\lambda I_T}\right] = \hat{E}^{\nu}_x\left[\left(\frac{x}{R(T)}\right)^{(\nu-\chi)}\right],$$

where $\nu = \sqrt{2\lambda + \chi^2}$.

Lemma 2 *Define for Bessel processes* $\{R(t) : t \geq 0\}$ *stopping times of the form*

$$T(b) = \inf\{t \geq 0 : R(t) \leq b\sqrt{1+t}\}.$$

For any $\chi \geq 0, x > b, m \geq 0$ *we have*

$$\hat{E}^{\chi}_x \left[\left(\frac{1}{1+T(b)} \right)^m \right] = \frac{U(m, \chi+1, x^2/2)}{U(m, \chi+1, b^2/2)}, \tag{25}$$

where U *is the confluent hyper-geometric Kummer's function of the second kind. That is to say that for real valued* a, b, z,

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zt} t^{a-1} (1+t)^{b-a-1} \mathrm{d}t. \tag{26}$$

(See [89] for a description of this class of functions).

Proof of Theorem 7 We give the proof for $\sigma = 1$. The first thing to note is that the time change $A^{(\chi)}_t$ satisfies the inverse relation (see [112])

$$4 \int_0^{A^{(\chi)}_t/4} \left[R^{(2\chi)}(s) \right]^{-2} \mathrm{d}s = t \qquad t \geq 0. \tag{27}$$

Thus we can rewrite $\tau^{\Phi}_{[b,\infty)}$ in the form

$$\tau^{\Phi}_{[b,\infty)} = 4 \int_0^{\tilde{T}} [R(s)]^{-2} \, \mathrm{d}s \tag{28}$$

under $\hat{P}^{(2\chi)}_1$, where

$$\tilde{T} = \inf\left\{ t : R(t) \leq \sqrt{(4t + \varphi)/b} \right\}.$$

Bessel processes have a scaling property that can be considered to be inherited from Brownian motion. Namely that if R is a Bessel with index χ with $R(0) = 1$, then for any constant $c > 0$, $R' := \{c^{-1/2} R(ct), t \geq 0\}$ is also an Bessel process with index χ but starting from $R'(0) = c^{-1/2}$. It thus follows after a brief calculation that $\tilde{T}$ is equal in $\hat{P}^{(2\chi)}_1$-law to $(\varphi/4) \cdot T(\sqrt{4/b})$ under $\hat{P}^{(2\chi)}_z$ where $z = (4/\varphi)^{1/2}$. It is not hard to verify that $\lim_{a \downarrow 0} U(a,b,z) = 1$, where U is given in (26). From (25) we see that for $\chi \geq 0$ and $x > b$

$$\hat{P}^{\chi}_x (T(b) < \infty) = \lim_{m \downarrow 0} \hat{E}^{\chi}_x \left[\left(\frac{1}{1+T(b)} \right)^m I(T(b) < \infty) \right] = 1.$$

Hence, by the previous remark, also $\tilde{T}$ is finite $\hat{P}^{(2\chi)}$-almost surely and, since $R^{(\chi)}$ with $\chi \geq 0$ does not reach zero, we deduce that $\tilde{\mathbb{P}}^{\chi}_{\varphi}(\tau^{\Phi}_{[b,\infty)} < \infty) = 1$.

Combining this observation with Lemma 1, one can check that

$$\tilde{\mathbb{E}}^{\chi}_{\varphi} \left[\mathrm{e}^{-\lambda \tau^{\Phi}_{[b,\infty)}} \right] = \left(\frac{b}{\varphi} \right)^{-z_1} \hat{E}^{(z_2 - z_1)}_{\sqrt{4/\varphi}} \left[\left(\frac{1}{1 + T(\sqrt{4/b})} \right)^{-z_1} \right].$$

Applying Lemma 2 one finds, after some algebra, the stated expression. $\square$

5 Canadisation

From a financial point of view, perpetual options may be considered as rather theoretical objects, since in the real world options never have an infinite time of expiration. As we will show below, perpetual-type options can be linked to American type options of finite expiration.

Let us consider an American type option with finite expiration T and system of pay-off functions $\{\pi_t : 0 \leq t \leq T\}$, which are càdlàg and without negative jumps. The holder of the option has the right to exercise it at any time *before* T. If the holder does not exercise before this finite time then he receives a payment π_T at expiry. By considering Theorems 1 and 2 for the sequence of payments $\{\pi_{t\wedge T} : t \geq 0\}$ we have the arbitrage free price of this an American type

$$\Pi_T(x) = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-r\tau}\pi_\tau\right]$$

with optimal stopping time

$$\tau^* = \inf\{0 \leq t \leq T : \Pi_T(t) \leq \pi_t\},$$

where the hedging capital, as in section 1, is given by

$$\Pi_T(t,x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-r(\tau-t)}\pi_\tau\,|\mathcal{F}_t\right].$$

Unlike the perpetual case, the optimal stopping time is (in general) not a hitting time of a level. In many cases it will be the crossing time of a non-flat space time boundary. For this optimal exercise boundary no explicit formula's are known. For an account of the American put with finite time of expiration see for example [103]. Since no explicit solution is known for this problem, we consider instead a reasonable approximation. We follow the lead of [36]. The idea is to randomise T in a sensible way. That is, to replace T by an independent random variable.

Let $T_1, T_2, \ldots$ be a sequence of independent exponential variables with mean T, which are also independent of $\mathbf{F}$ and denote their probability measures and expectation respectively by P and E. An n-step approximation is understood to mean replacing the claim process $\pi_{t\wedge T}$ by $\pi_{t\wedge T^{(n)}}$ where $T^{(n)} = n^{-1}\sum_1^n T_i$, which has a Gamma$(n, n/T)$-distribution. Note by the strong law of large numbers $T^{(n)} \to T$ almost surely as n tends to infinity. The next result shows this approximation procedure makes sense.

Proposition 4 *Let the payments π be non-negative, $\{\mathcal{F}_t\}$ adapted, càdlàg and without negative jumps and suppose there are $\epsilon, C > 0$ such that the family $\{\mathrm{e}^{-r\tau}\pi_\tau : \tau \in \mathcal{T}_{0,T+\epsilon}\}$ is uniformly integrable with respect to $\mathbb{P}^{(r/\sigma-\sigma/2)}$ and*

$$sup_{\tau\in\mathcal{T}_{0,\infty},u>T+\epsilon}\mathbb{E}^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-r(\tau\wedge u)}\pi_{\tau\wedge u}] \leq C,$$

Then the sequence $\{\Pi^{(n)} : n \geq 1\}$ given by

$$\Pi^{(n)}(x) = \sup_{\tau\in\mathcal{T}_{0,\infty}} E \times \mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\mathrm{e}^{-r(\tau\wedge T^{(n)})}\pi_{\tau\wedge T^{(n)}}\right]$$

converges for each x to $\Pi_T(x)$ as n tends to infinity.

Proof For simplicity, write $g_t = e^{-rt}\pi_t$, $\chi = (r/\sigma - \sigma/2)$ and $P^{\chi}_x = P \times \mathbb{P}^{(r/\sigma-\sigma/2)}_x$. By an extension of Theorem 2 to the finite expiration case, we know there exists an optimal stopping time $\tau^* \in \mathcal{T}_{0,T}$ such that $\Pi_T = \mathbb{E}^{\chi}[g_{\tau^*}]$. Note that $\tau^* \in \mathcal{T}_{0,\infty}$ and hence $\Pi^{(n)} \geq E^{\chi}[g_{\tau^* \wedge T^{(n)}}]$. By Fatou's lemma and the fact that g has only non-negative jumps, we find that

$$\begin{aligned}\liminf_{n\to\infty} \Pi^{(n)}(x) &\geq \liminf_{n\to\infty} E^{\chi}_x[g_{\tau^*\wedge T^{(n)}}] \geq E^{\chi}_x[\liminf_{n\to\infty} g_{\tau^*\wedge T^{(n)}}] \\ &\geq \mathbb{E}^{\chi}_x[g_{\tau^*\wedge T}] = \Pi_T(0,x).\end{aligned}$$

To finish the proof we thus have to proof that

$$\limsup_{n\to\infty} \Pi^{(n)}(x) = \limsup_{n\to\infty} \sup_{\tau\in\mathcal{T}_{0,\infty}} E^{\chi}_x[g_{\tau\wedge T^{(n)}}] \leq \Pi_T(x).$$

Using the bound on $E^{\chi}_x[g_{\tau\wedge T^{(n)}}|T^{(n)}]$, we find that

$$\begin{aligned}\Pi^{(n)}(x) &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} E^{\chi}_x[g_{\tau\wedge T^{(n)}}\mathbf{1}_{\{T^{(n)}\leq T+\epsilon\}}] + \sup_{\tau\in\mathcal{T}_{0,\infty}} E^{\chi}_x[g_{\tau\wedge T^{(n)}}\mathbf{1}_{\{T^{(n)}> T+\epsilon\}}] \\ &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[g_{\tau\wedge(T+\epsilon)}] + C\cdot P(T^{(n)} > T+\epsilon),\end{aligned}$$

which after taking the limsup for $n \to \infty$ converges to $\Pi_{T+\epsilon}(x)$, by virtue of the fact that $T^{(n)}$ converges to T almost surely. The proof is completed by showing that $\Pi_{T+\epsilon}(x)$ tends to $\Pi_T(x)$ as ϵ tends to zero. To do so, note that

$$\begin{aligned}\left|\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[g_{\tau\wedge(T+\epsilon)}] - \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[g_{\tau\wedge T}]\right| &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[|g_{\tau\wedge(T+\epsilon)} - g_{\tau\wedge T}|] \\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[|g_{\tau} - g_T|\mathbf{1}_{\{T<\tau\leq T+\epsilon\}}] \\ &\leq \mathbb{E}^{\chi}_x[|g_{\tau_\epsilon} - g_T|\mathbf{1}_{\{T<\tau_\epsilon\leq T+\epsilon\}}] + \epsilon,\end{aligned}$$

where τ_ϵ is an ϵ-optimal stopping time, that is τ_ϵ is chosen such that

$$\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^{\chi}_x[|g_{\tau} - g_T|\mathbf{1}_{\{T<\tau\leq T+\epsilon\}}] - \epsilon \leq \mathbb{E}^{\chi}_x[|g_{\tau_\epsilon} - g_T|\mathbf{1}_{\{T<\tau_\epsilon\leq T+\epsilon\}}].$$

(The existence of this ϵ-optimal stopping time follows since there is always a sequence of stopping times approximating the supremum on the left-hand side.) The expectation on the right-hand side of the previous line converges to zero by uniform integrability. Hence it follows that $\Pi_{T+\epsilon}(x)$ can be made arbitrarily close to $\Pi_T(x)$ by making ϵ sufficiently small. □

Remark If the value function $T \mapsto \Pi_T$ considered as function of the expiration T is a concave function, we find from Jensen's inequality that

$$\Pi_T = \Pi_{E[\widetilde{T}]} \geq E[\Pi_{\widetilde{T}}] \geq \sup_{\tau\in\mathcal{T}_{0,\infty}} E\times\mathbb{E}^{(r/\sigma-\sigma/2)}_x\left[\mathrm{e}^{-r(\tau\wedge\widetilde{T})}\pi_{\tau\wedge\widetilde{T}}\right],$$

where $\widetilde{T}$ is a random variable independent of $\mathbf{F}$ with P-expectation T.

The Canadisation of an American-type option is the 1-step approximation as described above. That is to say the expiration date is randomised by an independent exponential distribution with parameter $\alpha = T^{-1}$. In all the cases we are interested in, American calls and puts, Russians and integrals, their Canadised price are of the form

$$\begin{aligned}\widehat{\Pi}(\chi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} E_\chi\left[\mathrm{e}^{-r(\tau\wedge T_1)}f(\Gamma_{\tau\wedge T_1})\right]\\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}} E_\chi\left[\mathrm{e}^{-(r+\alpha)\tau}f(\Gamma_\tau)+\alpha\int_0^\tau \mathrm{e}^{-(r+\alpha)s}f(\Gamma_s)\mathrm{d}s\right],\end{aligned}$$

where $\Gamma = \{\Gamma_t : t \geq 0\}$ is a continuous Markov process starting from χ under some measure whose expectation operator is E_χ and f is a non-negative, monotone increasing, convex function. It can be easily checked using Theorem 2 that the optimal stopping time is of the form

$$\tau^* = \inf\{t \geq 0 : \widehat{\Pi}(\Gamma_t) \leq f(\Gamma_t)\}.$$

Hence on account of the properties of f, we can reason as in the previous sections to conclude that τ^* is hitting time of the Markov process Γ.

In the following examples, note that it is no longer necessary that the parameter λ is positive in order to guarantee the existence of a solution. A finite albeit random expiry date removes this necessity.

5.1 1-Step American Put Approximation ($\lambda = 0$)

The first approximation $\Pi^{\text{put}}_{T_1}(s)$ to the price of a American put with expiration T

$$\Pi^{\text{put}}_T(s) = \sup\nolimits_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}_x^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-r\tau}(K-S_\tau)^+]$$

is equal to the supremum over all $l > 0$ of

$$\mathbb{E}_x^{(r/\sigma-\sigma/2)}[\mathrm{e}^{-(r+\alpha)\tau(l)}(K-S_{\tau(l)})^+] + \alpha\mathbb{E}_x^{(r/\sigma-\sigma/2)}\left[\int_0^{\tau(l)}\mathrm{e}^{-(r+\alpha)t}(K-S_t)^+\mathrm{d}t\right]$$

where $\tau(l) = \tau^W_{(-\infty,l]}$ and $\alpha = T^{-1}$. Using the resolvent of the Brownian motion killed upon entering the negative half-line (see e.g. [30])

$$\begin{aligned}&\alpha^{-1}\mathbb{P}^\chi_x(W_{e(\alpha)}\in\mathrm{d}y, e(\alpha)<\tau^W_{(-\infty,0]})/\mathrm{d}y\\ &\qquad = 2\Delta^{-1}\mathrm{e}^{-x_2y}\left(\mathrm{e}^{-\rho x}\sinh(\Delta x) - \mathbf{1}_{\{x\geq y\}}\mathrm{e}^{\rho(x-y)}\sinh(\Delta x)\right),\end{aligned}$$

where $2\Delta = x_2 - x_1$ and $2\rho = x_2 + x_1$ with $x_1 < x_2$ are the roots of $x^2 - (1-2r/\sigma^2)x - 2(\alpha+r)/\sigma^2 = 0$ and $e(\alpha)$ is an independent exponential random variable with parameter α, we find after some algebra $\Pi^{\text{put}}_{T_1}(s)$ to be equal to

$$\begin{cases} \left(\frac{s}{K}\right)^{x_1} K\left(\frac{x_2}{x_2-x_1}\frac{\alpha}{\alpha+r} - \frac{x_2-1}{x_2-x_1}\right) + \left(\frac{s}{l_*}\right)^{x_1} K\frac{x_2}{x_2-x_1}\frac{r}{r+\alpha} \\ K\frac{\alpha}{r+\alpha} - s + \left(\frac{s}{K}\right)^{x_2} K\left(\frac{1-x_1}{x_2-x_1} + \frac{x_1}{x_2-x_1}\frac{\alpha}{\alpha+r}\right) + \left(\frac{s}{l_*}\right)^{x_1} K\frac{x_2}{x_2-x_1}\frac{r}{r+\alpha} \\ K - s \end{cases}$$

if $s \geq K, s \in (l^*, K)$ and $s \leq l^*$ respectively, where the optimal exercise level is given by

$$l_* = K\left(\frac{-rx_1}{r+\alpha-rx_1}\right)^{\frac{1}{x_2}}.$$

5.2 1-Step Russian Option Approximation ($\lambda = 0$)

According to the preceding a first approximation to the price of a Russian option with expiry T

$$\Pi^{\text{russ}}_T(s,\psi) = s \cdot \Pi^{\text{R}}_T(\psi) = s \cdot \sup_{\tau \in \mathcal{T}_{0,T}} \overline{\mathbb{E}}^{(r/\sigma+\sigma/2)}_{\psi}[\Psi_\tau]$$

is equal to $\Pi^{\text{russ}}_{T_1}(s,\psi) = s \cdot \Pi^{\text{R}}_{T_1}(\psi)$ where $\Pi^{\text{R}}_{T_1}(\psi)$ is equal to the supremum over all $b > 0$ of

$$\mathrm{e}^{b}\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{[b,\infty)}}\right] + \alpha\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Psi_t \mathrm{d}t\right] \tag{29}$$

with $\alpha = T^{-1}$ and $\gamma = (r/\sigma + \sigma/2)$. By an application of Itô's lemma to the process $\exp(-\alpha t)\Psi_t$, we find that

$$\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Psi_t \mathrm{d}t\right] = -\frac{1}{r+\alpha}\Bigg(e^{b}\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{[b,\infty)}}\right] - \psi - \overline{\mathbb{E}}^{\gamma}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t} S_t^{-1}\mathrm{d}M_t\right]\Bigg).$$

where $M_t = \overline{S}_t$. Recalling that $\overline{W} = L$ and setting $A = \{\tau^{\Psi}_{\{0\}} < \tau^{\Psi}_{[b,\infty)}\}$ the second expectation on the right hand side can now be written

$$\sigma\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\int_0^{\tau^{\Psi}_{[b,\infty)}} \mathrm{e}^{-\alpha t}\Psi_t \mathrm{d}L_t\right] = \sigma\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{\{0\}}}\mathbf{1}_A\right] \times \times \int_0^{\infty} \mathrm{d}t\overline{\mathbb{E}}\left[\mathrm{e}^{-(\alpha+\gamma^2/2)L^{-1}(t)+\chi t}\mathbf{1}_{\left\{\sup_{0\leq s<L^{-1}(t)} \overline{W}_s - W_s < b/\sigma\right\}}\right].$$

An application of the Girsanov theorem together with the techniques used in the proof of Theorem 6 concerning the two sided exit problem yields

$$\overline{\mathbb{E}}^{\gamma}_{\psi}\left[\mathrm{e}^{-\alpha\tau^{\Psi}_{\{0\}}}\mathbf{1}_A\right] = \psi^{\gamma/\sigma}\frac{(\mathrm{e}^{b\eta/\sigma}\psi^{-\eta/\sigma} - \mathrm{e}^{-b\eta/\sigma}\psi^{\eta/\sigma})}{2\sinh(b\eta/\sigma)},$$

where $\eta = \sqrt{2\alpha + \gamma^2}$. The integral on the right hand side of the last but one display can also be written

$$\int_0^\infty \mathrm{d}t\overline{\mathbb{P}}^\eta(\sup_{0\le s<L^{-1}(t)}(\overline{W}_s - W_s) < b/\sigma)\mathrm{e}^{\gamma t-\eta t} = \frac{\sinh(b\eta/\sigma)}{\eta\cosh(b\eta/\sigma) - \gamma\sinh(b\eta/\sigma)},$$

where the equality follows by using the fact that the first excursion of height exceeding b/σ appears after a length of local time which is exponentially distributed with parameter $s'(b/\sigma)/s(b/\sigma)$ where the scale function $s(x)$ is taken proportional to $\mathrm{e}^{-\eta x}\sinh(\eta x)$.

Thus, after some algebra, we find that the first approximation is given by

$$\Pi^{\mathrm{russ}}_{T_1}(s,\psi) = s\cdot\left\{\frac{r}{r+\alpha}b_*\cdot\frac{y_2\psi^{y_1} - y_1\psi^{y_2}}{y_2 b_*^{y_1} - y_1 b_*^{y_2}} + \frac{\alpha}{r+\alpha}\left(\psi + \frac{b_*^{y_2}\psi^{y_1} - b_*^{y_1}\psi^{y_2}}{y_2 b_*^{y_1} - y_1 b_*^{y_2}}\right)\right\},$$

where $y_1 = (\gamma+\eta)/\sigma$ and $y_2 = (\gamma-\eta)/\sigma$ are the roots of $y^2 - (1+2r/\sigma^2)y - 2\alpha/\sigma^2 = 0$ and the optimal exercise level b_* is the unique solution of

$$r(y_2(1-y_1)b^{y_1} + y_1(y_2-1)b^{y_2}) + \alpha(y_2-y_1)b^{y_1+y_2-1} = 0. \tag{30}$$

Note that uniqueness follows since the function of b in (30) is concave and differentiable with a positive derivative at 1.

5.3 1-step Integral Option Approximation ($\lambda = 0$)

We now show how to find an approximation to the price of the integral option with expiry T, that is, we approximate

$$\Pi^{\mathrm{int}}_T(s,\varphi) = s\cdot\Pi^{\mathrm{I}}_T(\varphi) = s\cdot\sup_{\tau\in\mathcal{T}_{0,T}}\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_\varphi[\Phi_\tau].$$

The first approximation $\Pi^{\mathrm{I}}_{T_1}$ to the price Π^{I}_T is given by the supremum over all $b>0$ of

$$b\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_\varphi\left[\mathrm{e}^{-\alpha\tau^\Phi_{[b,\infty)}}\right] + \alpha\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_\varphi\left[\int_0^{\tau^\Phi_{[b,\infty)}}\mathrm{e}^{-\alpha t}\Phi_t\mathrm{d}t\right],$$

where $\alpha = T^{-1}$. An application of Itô's lemma to Φ_t shows that

$$\mathrm{d}\Phi_t = (1 - r\Phi_t)\mathrm{d}t - \sigma\Phi_t\mathrm{d}W^{(r/\sigma+\sigma/2)}_t.$$

where $W^{(r/\sigma+\sigma/2)} = \{W_t - (r/\sigma+\sigma/2)t, t\ge 0\}$ is a standard Wiener process under $\mathbb{P}^{(r/\sigma+\sigma/2)}$. Applying now partial integration to $\exp(-\alpha t)\Phi_t$ results in

$$\alpha\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_\varphi\left[\int_0^{\tau^\Phi_{[b,\infty)}}\mathrm{e}^{-\alpha t}\Phi_t\mathrm{d}t\right] = \left(\frac{1+\alpha\varphi}{\alpha+r} - \frac{(1+\alpha b)}{\alpha+r}\tilde{\mathbb{E}}^{(r/\sigma+\sigma/2)}_\varphi[\mathrm{e}^{-\alpha\tau^\Phi_{[b,\infty)}}]\right).$$

Recalling formula (23) we find,

$$\Pi^{\mathrm{int}}_{T_1}(s,\varphi) = s\cdot\left\{\frac{1+\alpha\varphi}{\alpha+r} + \frac{m_* r - 1}{\alpha+r}\cdot\frac{u_\alpha(\varphi)}{u_\alpha(m_*)}\right\}.$$

where, following the line of reasoning of the proof of Theorem 5, $m_* > 0$ is uniquely determined by $u'_\alpha(m_*)(rm_* - 1) = u_\alpha(m_*)r$.

For related work on analytical approximation to Asian or integral type options, see also [33].

5.4 n-Step Russian Option Approximation ($\lambda = 0$)

On a final note we consider how one would evaluate an n-step approximation by using a dynamic programming algorithm with the Russian option. Let $\alpha_n = \alpha/n$ and write $e_i = n^{-1}T_i$ for $i = 1, ..., n$. Define the subsequent stages $h_n, \ldots, h_1$ by

$$\begin{aligned} h_n(\psi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}\left[\Psi_{\tau\wedge e_n}\right] \\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\alpha_n\tau}\Psi_\tau + \alpha_n\int_0^\tau \mathrm{e}^{-\alpha_n t}\Psi_t \mathrm{d}t\right] \end{aligned}$$

and for $m = n-1, \ldots, 1$

$$\begin{aligned} h_m(\psi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}\left[\Psi_{\tau\wedge\sum_m^n e_i}\right] \\ &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \overline{\mathbb{E}}_\psi^{(r/\sigma+\sigma/2)}\left[\mathrm{e}^{-\alpha_n\tau}\Psi_\tau + \alpha_n\int_0^\tau \mathrm{e}^{-\alpha_n t}h_{m+1}(\Psi_t)\mathrm{d}t\right]. \end{aligned}$$

Using the Markov property it can be checked that the price $\Pi^{(n)}(x)$ of the n-approximation is equal to $h_1(\psi)$, the final outcome of the above dynamic programming algorithm, for all possible starting values ψ of the Markov process. Note each step in the dynamic programming algorithm requires solution of a problem of the form

$$\sup_{\tau\in\mathcal{T}_{0,\infty}} E_\chi\left[\mathrm{e}^{-\alpha_n\tau}f(\Gamma_\tau) + \alpha_n\int_0^\tau \mathrm{e}^{-\alpha_n s}g(\Gamma_s)\mathrm{d}s\right],$$

where g is another non-negative, convex, monotone increasing function. It can be reasoned similarly to previously using Theorem 2 that for each stage of the algorithm, the optimal stopping time is still a hitting time. Note that the optimal stopping time for the nth approximation $\Pi^{(n)}$ is a *randomised* $\mathbf{F}$-stopping time. In future work, we will investigate the convergence properties of these randomised stopping times. The American and integral options can be dealt with similarly.

Acknowledgements

The origin of this work was inspired by discussions with F. Avram to whom we are very grateful. Both authors would also like to thank K. Dajani and R. Doney for helpful comments and discussion.

Chapter II

American options under exponential phase–type Lévy models

Consider the American put and Russian option [116, 117, 50] with the stock price modelled as an exponential Lévy process. We find an explicit expression for the price in the dense class of Lévy processes with phase–type jumps in both directions. The solution rests on the reduction to the first passage time problem for (reflected) Lévy processes and on an explicit solution of the latter in the phase–type case via martingale stopping and Wiener-Hopf factorisation. The same type of approach is also applied to the more general class of regime switching Lévy processes with phase-type jumps.

1 Introduction

Consider a model of a financial market with two assets, a savings account with value $B = \{B_t\}_{t\geq 0}$ and an asset with price process $S = \{S_t\}_{t\geq 0}$. The evolution of B is deterministic, with

$$B_t = \exp(rt), \qquad r > 0, t \geq 0,$$

and the asset price is random and evolves according to the exponential model

$$S_t = S_0 \exp(X_t), \qquad S_0 = \exp(x), \qquad t \geq 0,$$

where $X = \{X_t\}_{t\geq 0}$ is some Lévy process. If X has no jumps, it can be represented by $X_t = \sigma W_t + \mu t$, with $x, \mu, \sigma \in \mathbf{R}$ and $W = \{W_t\}_{t\geq 0}$ a standard Wiener process; this is the classical Black-Scholes model. There has been considerable interest in replacing the classical Black-Scholes model by exponential

Lévy models allowing also for jumps. This development is motivated by superior fits to the data and hence improved pricing formulas and hedging strategies, as well as by theoretical considerations outlined in [61].

The search for a special Lévy model to outperform the Black-Scholes model was initiated by Mandelbrot [93, 94] and Fama [57, 58] followed by Merton, with the jump-diffusion with Gaussian jumps, and continues nowadays in the work of Carr, Chang, Madan, Geman and Yor who propose the variance-gamma model [91, 35], of Eberlein who proposes the hyperbolic model [52], of Barndorff-Nielsen with the normal inverse Gaussian model [20], of Kou who proposed a jump-diffusion with exponential jumps [82] and of Koponen who introduced the Koponen family [81], which was later extended (e.g. [32, 42]). There are still many statistical issues which will need to be resolved before an appropriate replacement of the Black Scholes model can emerge. Our paper addresses only the issue of the analytical tractability of pricing certain perpetual American type options. We propose a jump-diffusion model where the jumps form a compound Poisson process with jump distribution of *phase type* (e.g. [104, 9, 10], see further Section 2). On the one hand this *phase type* model is rich enough, since this class of processes is known to be dense in the class of all Lévy processes, and on the other hand for many options the model is analytically tractable.

We illustrate this in the case of the American put option and the Russian option. The last one was originally introduced by Shepp and Shiryaev in the context of the Black-Scholes model [50, 116, 117, 64, 85]. The pricing of the Russian option rests on a well known reduction to the *first passage time problem* for a Lévy process reflected at its supremum, making it somewhat more difficult than the analogous problem for the unconstrained Lévy process (which is used to solve the pricing problem for barrier and perpetual American options). We note that special solutions of this problem – see [16] and [102] – are currently available only under spectrally one sided Lévy models. The purpose of our note is to draw attention to the fact that under the phase–type assumption, easily implementable solutions for both the unconstrained and the reflected first passage time problems exist as well for *spectrally two sided* Lévy processes (and hence for the pricing of perpetual American put and Russian options). In fact, we show that the method employed – of obtaining barrier crossing probabilities via a martingale stopping approach – works equally for barrier problems under the much more general class of *regime switching* exponential Lévy models with phase–type jumps, or for the regime switching Brownian motion recommended for example by Guo [65]. Their analytical tractability suggests that this potentially very flexible class of models (which depart from the unrealistic assumption of independent increments of the Lévy models) deserves to be more fully investigated.

The rest of the paper is organised as follows. Section 2 presents the model, the problem and its reduction to the first passage time problems for (reflected) Lévy processes. The martingale stopping approach for (non-)reflected Lévy processes is reviewed in Section 3, including explicit formulae for the pricing of the perpetual American put option and the Russian option. Finally, the solution of the first passage problem for reflected regime switching phase–type

Lévy models via an embedding into a regime switching Brownian motion is presented in Section 4. Most proofs are relegated to Section 5.

2 Model and problem

We introduce now the model we consider.

2.1 Phase-type distributions

A distribution F on $(0,\infty)$ is *phase-type* if it is the distribution of the absorption time ζ in a finite state continuous time Markov process $J=\{J_t\}_{t\geq 0}$ with one state Δ absorbing and the remaining ones $1,\dots,m$ transient. That is, $F(t)=\mathbb{P}(\zeta\leq t)$ where $\zeta=\inf\{s>0:\ J_s=\Delta\}$. The parameters are m, the restriction $\boldsymbol{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\boldsymbol{\alpha}=(\alpha_1\ \dots\ \alpha_m)$ where $\alpha_i=\mathbb{P}(J_0=i)$. For any $i=1,\dots,m$, let t_i be the intensity of a transition $i\to\Delta$ and write $\boldsymbol{t}=(t_1\ \dots\ t_m)'$ for the (column) vector of such intensities. Note that $\boldsymbol{t}=-\boldsymbol{T}\boldsymbol{1}$, where $\boldsymbol{1}$ denotes a column vector of ones. It follows that the cumulative distribution F is given by:

$$1-F(x)=\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{1}, \tag{1}$$

the density is $f(x)=\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{t}$ and the Laplace transform is given by $\hat{F}[s]=\int_0^\infty \mathrm{e}^{-sx}F(\mathrm{d}x)=\boldsymbol{\alpha}(s\boldsymbol{I}-\boldsymbol{T})^{-1}\boldsymbol{t}$. Note that $\hat{F}[s]$ can be extended to the complex plane except at a finite number of poles (the eigenvalues of $\boldsymbol{T}$). A representation of the form (1) for the distribution function F is called *minimal* if there exists no number $k<m$, k-vector $\boldsymbol{b}$ and $k\times k$-matrix $\boldsymbol{G}$ such that $1-F(x)=\boldsymbol{b}\mathrm{e}^{\boldsymbol{G}x}\boldsymbol{1}$.

Phase-type distributions include and generalise exponential distributions in series and/or parallel and form a dense class in the set of all distributions on $(0,\infty)$. They have found numerous applications in applied probability, see for example [9, 10] for surveys. Much of the applicability of the class comes from the probabilistic interpretation, in particular the fact that that the overshoot distributions $F(x+y)/(1-F(x))$ belong to a finite vector space. More precisely, the overshoot distribution is again phase–type with the same m and $\boldsymbol{T}$ but α_i replaced by $\mathbb{P}(J_x=i|\zeta>x)$, which is reminiscent of the memoryless property of the exponential distribution ($m=1$) and explains the availability of many matrix formulas which generalise the scalar exponential case.

2.2 Lévy phase-type models

Let $X=\{X_t\}_{t\geq 0}$ be a Lévy process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$, a stochastic basis that satisfies the usual conditions. We consider X which can be represented as follows

$$X_t\ =\mu t+\sigma W_t+\sum_{k=1}^{N^{(+)}(t)}U_k^{(+)}-\sum_{\ell=1}^{N^{(-)}(t)}U_\ell^{(-)}, \tag{2}$$

where W is standard Brownian motion, $N^{(\pm)}$ are Poisson processes with rates of arrival $\lambda^{(\pm)}$ and $U^{(\pm)}$ are i.i.d. random variables with respective jump size distributions $F^{(\pm)}$ of phase–type with parameters $m^{(\pm)}, \boldsymbol{T}^{(\pm)}, , \boldsymbol{\alpha}^{(\pm)}$. All processes are assumed to be independent. Equivalently, for $s \in \mathrm{i}\mathbf{R}$, the Lévy exponent κ of X, defined by $\kappa(s) = \log \mathbb{E}[\exp(sX_1)]$, is

$$\kappa(s) \;=\; s\mu + s^2\frac{\sigma^2}{2} + \lambda^{(+)}\big(\hat{F}^{(+)}[-s] - 1\big) + \lambda^{(-)}\big(\hat{F}^{(-)}[s] - 1\big), \tag{3}$$

where $\hat{F}^{(\pm)}[s] = \boldsymbol{\alpha}^{(\pm)}(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})^{-1}\boldsymbol{t}^{(\pm)}$. As above, $\kappa(s)$ can be extended to the complex plane except a finite number of poles; this extension will also be denoted by κ. To avoid trivialities, in the sequel we will exclude the case that X has monotone paths.

Any Lévy process may be approximated arbitrarily closely by processes of the form (2):

Proposition 1 *For any Lévy process X, there exists a sequence $X(n)$ of Lévy processes of the form (2) such that $X(n) \to X$ in $D[0, \infty)$.*

Proof Let d be some metric on D. Choose first $X'(n)$ as an independent sum of a linear drift, a Brownian component and a compound Poisson process such that $d(X, X'(n)) \leq 1/n$. Use next the denseness of phase–type distributions to find $X(n)$ of the form (2) with $d(X(n), X'(n)) \leq 1/n$. QED

Remark 1 *The approximation in Proposition 1 is easy to carry out in practice: the compound approximation is obtained by just restricting the Lévy measure to $\{|x| > \epsilon\}$, and to get to phase–type jumps, the relevant methodology for fitting a phase–type distributions to a given distribution (or a set of data) is developed in [8] for traditional maximum likelihood and in [28] in a Bayesian setting.*

In complete markets (with a unique risk-neutral martingale measure $\mathbb{P}^*$ under which $\mathbb{E}^*[\exp(X_t)] = \exp(rt)$ where r is the risk-less discount rate), arbitrage free pricing is equivalent to computing expectations under this measure $\mathbb{P}^*$. Under the Lévy model (2) with non-zero jump component however, the market is incomplete, i.e. not all claims can be hedged against. In this case there are infinitely many equivalent martingale measures, and some choice must be made. We use here the so called Cramér-Esscher transform or exponential tilting proposed by Gerber and Shiu [63], which preserves the Lévy structure, and as shown in Chan [39], is indeed the solution to some of the most common criteria for selecting an equivalent martingale measure. Note that the Esscher transform preserves the phase-type structure of the log-price X (see Appendix A). From now on we assume that we are working *under the chosen equivalent martingale measure*. That is, we assume that the Lévy exponent κ satisfies under $\mathbb{P}$

$$\kappa(1) = \log \mathbb{E}[\exp(X_1)] = r, \tag{EMM}$$

Remark 2 *Many of the computations involving Lévy processes are based on finding the roots of the "Cramér-Lundberg equation" (see [10] for terminology)*

$$\kappa(s) = a \tag{4}$$

(for some a). From this perspective, working under the equivalent martingale measure means $s = 1$ is one of the roots of this equation when $a = r$.

Remark 3 *Using Appendix A, we can easily convert parameters of X under the real world measure into parameters under the Esscher transform and vice versa.*

2.3 American put option

The a-discounted perpetual American put option with strike K gives the holder the right to exercise at any $\{\mathcal{F}_t\}$-stopping time τ yielding the pay-out

$$\mathrm{e}^{-a\tau}(K - S_\tau)^+, \qquad a \geq 0, \tag{5}$$

where $c^+ = \max\{c, 0\}$. Recall that the process X satisfies (EMM). Then the arbitrage-free price corresponding to the chosen martingale measure is given by

$$U^*(x) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-(r+a)\tau}(K - S_\tau)^+] \tag{6}$$

where the supremum runs over all $\{\mathcal{F}_t\}$-stopping times τ, $\mathbb{E}_x$ denotes the expectation with respect to the measure $\mathbb{P}_x$ under which $\log S_0 = X_0 = x$. Let $I_\delta = \inf_{0 \leq t \leq \eta(\delta)} X_t$ denote the infimum of X up to $\eta(\delta)$, an independent exponential random variable with parameter $\delta = r + a$. Mordecki [100] has shown that, for a general Lévy process X,

$$U^*(x) = \mathbb{E}_x[\mathrm{e}^{-\delta T(k^*)}(K - \mathrm{e}^{X_{T(k^*)}})],$$

where the optimal stopping time $T(k^*)$ is given by the first passage time of the process X below the level k^*,

$$T = T(k^*) = \inf\{t \geq 0 : X_t \leq k^*\}, \tag{7}$$

where $\exp(k^*) = K\ \mathbb{E}[\mathrm{e}^{I_\delta}]$. (cf. Darling et al. [47] for the solution of a similar optimal stopping problem in discrete time).

2.4 The Russian option

The Russian option is an American type option which gives the holder the right to exercise at any almost surely finite $\{\mathcal{F}_t\}$-stopping time τ yielding pay-outs

$$\mathrm{e}^{-a\tau} \max\left\{M_0, \sup_{0 \leq u \leq \tau} S_u\right\}, \qquad M_0 \geq S_0, a > 0.$$

The constant M_0 can be viewed as representing the "starting maximum" of the stock price (say, the maximum over some previous period $(-t_0, 0]$). The positive discount factor a is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite.

Since X satisfies (EMM), the arbitrage-free price of the Russian option for this martingale measure is given by

$$V^*(x,m) = \sup_{\tau} \mathbb{E}_x \left[\mathrm{e}^{-(r+a)\tau} \max\left\{ \mathrm{e}^m, \sup_{0\le u\le \tau} S_u \right\} \right], \tag{8}$$

where the supremum is taken over the set $\mathcal{T}$ of all almost surely finite $\{\mathcal{F}_t\}$-measurable stopping times and $m = \log(M_0)$. Let $\overline{X}_t = \sup_{s\le t} X_s$ denote the supremum of the Lévy process and let $Y_t = \overline{X}_t \vee (m-x) - X_t$ denote the process reflected at its supremum level (started at $y = m - x$). The key simplification discovered by Shepp and Shiryaev (for the standard Black-Scholes model) is that the optimal stopping time must be of the form

$$\tau \ = \tau(k) = \ \inf\{t > 0 : Y_t \ge k\}, \tag{9}$$

i.e. τ must be the first time when the *reflected process* Y up-crosses a certain *constant* (positive) exercise level k^* (which may be found by solving a one dimensional optimisation problem). If X is a general Lévy process, Theorem 1 below states that the optimal stopping time is still of the form (9).

Theorem 1 *Let X be a general Lévy process which satisfies (EMM). Then the value function $V^*(x,m)$ of the* two dimensional stopping problem *(8) is given by:*

$$V^*(x,m) = \mathrm{e}^x v^*(m-x), \tag{10}$$

where $v^*(m-x)$ is the solution of the one dimensional stopping problem *of finding a function v^* and a $\tau^* \in \mathcal{T}$ such that*

$$v^*(y) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau+Y_\tau}] = \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau^*+Y_{\tau^*}}], \tag{11}$$

where $\mathbb{P}_y^{(1)}$ denotes the "tilted" probability measure given on $\mathcal{F}_t$ by $\mathrm{d}\mathbb{P}^{(1)}|_{\mathcal{F}_t} = \exp(X_t - x - \kappa(1)t)\mathrm{d}\mathbb{P}_x|_{\mathcal{F}_t}$, with $Y_0 = y$. The function v^* is convex and the optimal stopping time τ^* is the same in both problems, i.e. $\tau^* = \tau(k^*)$ with k^* given by

$$k^* = \operatorname*{Arg\,Max}_{k\ge 0}[\mathrm{e}^k v_k^{(1)}(0)]$$

where $v_k^{(1)}(y) = \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau(k)+Y_{\tau(k)}-k}]$. Moreover, $k^* = \operatorname*{Arg\,Max}_{k\ge y}[\mathrm{e}^k v_k^{(1)}(y)]$ for all $y > 0$.

In Section 5 we provide the proof. The proof draws on the experience of [116, 117, 85] and uses standard optimal stopping theory. In Section 3.3, an explicit expression is given for the optimal level k^* if X is of the form (2).

To explicitly solve our problem of pricing the American put and the Russian option driven by phase type Lévy processes X, the next goal will be the explicit evaluation of the first passage time functions of the process X at the stopping time (7) needed in (6) and of the process Y at the stopping time (9) required in

(11). The evaluation may be achieved in principle by solving the corresponding Feynman-Kac integro-differential equation, which is tractable for this phase-type Lévy model and worked out in Section 5.3. In the next section, however, we will follow a different approach, exploiting the probabilistic interpretation of phase-type distributions and the fact that distributions of phase type have a rational Laplace transform.

3 First passage time

In this section we first review the Wiener-Hopf decomposition and first passage time problem for the class of phase-type Lévy processes. Results on Wiener-Hopf factorisations have appeared before in the literature at different places. Here we aim to develop a self-contained presentation illustrating our methods. Next we solve the first passage time process of the phase-type Lévy process reflected at its supremum. For background on passage problems for Markov chains, we refer to [78].

3.1 Wiener-Hopf factorisation

We provide now a statement of the Wiener-Hopf factorisation for the class of phase-type Lévy processes (2). Let $M_a = \sup_{t\leq\eta(a)} X_t$ and $I_a = \inf_{t\leq\eta(a)} X_t$ be the supremum and infimum of X at an independent exponential random variable $\eta(a)$ with mean a^{-1}, respectively. Set for $\pm\Re(s) \geq 0$ and $a > 0$

$$\kappa_a^-(s) = \mathbb{E}[\exp(sI_a)], \qquad \kappa_a^+(s) = \mathbb{E}[\exp(sM_a)]. \tag{12}$$

The functions $s \mapsto \kappa_a^{\mp}(s)$ are analytic for s with $\pm\Re(s) > 0$, respectively. Note by a Tauberian theorem that $\kappa_a^-(\infty) = \mathbb{P}(I_a = 0)$ and $\kappa_a^+(-\infty) = \mathbb{P}(M_a = 0)$. For $a > 0$, the functions $s \mapsto \kappa^{\mp}(s)$ satisfy the Wiener-Hopf factorisation

$$a/(a - \kappa(s)) = \kappa_a^+(s)\kappa_a^-(s) \qquad \text{for all } s \text{ with } \Re(s) = 0,$$

see e.g. [25, Thm. 1]. Denote by $\mathcal{I}^{(\mp)} = \{i : \pm\Re(\rho_i) < 0\}$ the set of roots ρ_i with negative and positive real part respectively of the Cramèr-Lundberg equation

$$\kappa(\rho) = \kappa_X(\rho) = a, \tag{13}$$

where a root occurs as many times as its multiplicity. Since $|\kappa_a^+(s)|$ and $|\kappa_a^-(s)|$ take both values in $[0, 1]$ for s with $\Re(s) = 0$, there are no roots of (13) with zero real part when $a > 0$. Similarly, we write $\mathcal{J}^{(\mp)} = \{i : \pm\Re(\eta_i) < 0\}$ for the set of roots of $a/(a - \kappa(\eta)) = 0$ with positive and negative real part respectively, taking again multiplicity into account. Note that if $i \in \mathcal{J}^{(\mp)}$, η_i is an eigenvalue of $\pm\boldsymbol{T}^{(\mp)}$, although the converse need not necessarily be true. Since the (analytic continuation of the) Laplace transform $\hat{F}$ of a (non-defective) phase type distribution F is a ratio f_1/f_2 of two polynomials f_1, f_2 with degree(f_1)< degree(f_2), we note from (3) that under the model (2) the function κ is the ratio $\tilde{p}/\tilde{q}$ of two polynomials $\tilde{p}, \tilde{q}$ where degree($\tilde{q}$) $-$degree($\tilde{p}$)

$= 2, 1, 0$ according to whether $(\sigma \neq 0)$, $(\sigma = 0, \mu \neq 0)$ or $(\mu = \sigma = 0)$, respectively.

Lemma 1 *Let X be a Lévy process of the form (2) and suppose $a > 0$.*

1. *On the half-plane $\pm\Re(s) \geq 0$ the Wiener-Hopf factor $\kappa_a^{\mp}$ of X is given by $\kappa_a^{\mp} = \varphi_a^{\mp}$, where*

$$\varphi_a^{\mp}(s) = \frac{\prod_{j\in\mathcal{J}^{(\mp)}}(\pm s - \eta_j)}{\prod_{j\in\mathcal{J}^{(\mp)}}(-\eta_j)} \cdot \frac{\prod_{i\in\mathcal{I}^{(\mp)}}(-\rho_i)}{\prod_{i\in\mathcal{I}^{(\mp)}}(s-\rho_i)}, \tag{14}$$

   where the first factor is to be taken equal to 1 if X has no negative or positive jumps respectively.

2. *Moreover, $\#\mathcal{I}^{(-)} = \#\mathcal{J}^{(-)}$ or $\#\mathcal{J}^{(-)} + 1$ according to whether ($\sigma = 0$ and $\mu \geq 0$) or ($\sigma \neq 0$ or $\mu < 0$). If the representation of $F^{(-)}$ is minimal, $\#\mathcal{J}^{(-)} = m^{(-)}$.*

3. *If the roots of (13) with negative real part are distinct, then*

$$\mathbb{P}(-I_a \in \mathrm{d}x) = \sum_{j\in\mathcal{I}^{(-)}} A_j^-(-\rho_j)\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad x > 0 \tag{15}$$

   where $\boldsymbol{A}^- = (A_i^-, i \in \mathcal{I}^{(-)})$ are the partial fractions coefficients of the expansion:

$$\kappa_a^-(s) - \kappa_a^-(\infty) = \textstyle\sum_{i\in\mathcal{I}^{(-)}} A_i^- \rho_i(\rho_i - s)^{-1}. \tag{16}$$

Remark 4 *The assumption of distinct roots is only made for convenience; indeed, when the equation $\kappa(s) = a$ has multiple roots, let $n^{(-)}$ denote the number of* different *roots with positive/negative real part and $m^{(-,j)}$ the multiplicity of a root ρ_j with $j \in \mathcal{I}^{(-)}$. Then we find that for $k = 1, \ldots, m^{(-,j)}$ the coefficient $A_{j,k}^-$ of $(-\rho_j)^k/(s-\rho_j)^k$ in the partial fraction decomposition of $\kappa_a^-(s) - \kappa_a^-(\infty)$ is given by*

$$A_{j,k}^- = \frac{1}{(m-k)!}\frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}}\frac{\kappa_a^-(s)(s-\rho_j)^m}{(-\rho_j)^k}\bigg|_{s=\rho_j} \qquad \text{with } m = m^{(-,j)}.$$

By straightforward Laplace inversion, we conclude that

$$\mathbb{P}(-I_a \in \mathrm{d}x) = \sum_{j=1}^{n^{(-)}}\sum_{k=1}^{m^{(-,j)}} A_{j,k}^-(-\rho_j)\frac{(-\rho_j x)^{k-1}}{(k-1)!}\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad x > 0.$$

Example For a spectrally positive Lévy process, (14) yields $\kappa_a^-(s) = \frac{\rho_-}{\rho_- - s}$, where ρ_- is the unique negative root of (13). For Kou's jump-diffusion [82] with two-sided exponential jumps, (14) yields $\kappa_a^-(s) = \frac{\rho_1\rho_2}{(\rho_1 - s)(\rho_2 - s)}\frac{\mu_- + s}{\mu_-}$, where ρ_1, ρ_2 are the negative roots and μ_- is the rate of negative jumps. These explicit

expressions are at the root of various explicit computations and approximations in the literature on ruin probabilities and first-time passage barrier options.
Example (**Ruin probabilities**). For phase-type Levy processes X, equation (15) yields an explicit expression for the ruin probability

$$\mathbb{P}_x(\exists t \leq \eta(a) : X_t < 0) = \mathbb{P}(-I_a > x) = \sum_{j \in \mathcal{I}^{(-)}} A_j^- \mathrm{e}^{\rho_j x}, \qquad x \geq 0 \tag{17}$$

in case the roots $\rho_i, i \in \mathcal{I}^{(-)}$ are all distinct. For multiple roots with negative real part, similar expressions can be derived using Remark 4. This formula generalises those of [101] who considered X with negative mixed exponential jumps. Also, Erlang approximations of *finite time* ruin probabilities may be obtained, generalising those for the classical ruin model of Asmussen, Avram and Usabel [12]. See also the subsection on the American put below.

Remark 5 *From the proof of Lemma 1, we note that Lemma 1 holds for the slightly more general class of Lévy processes where the jumps form a compound Poisson process with a jump distribution which has rational Laplace transform.*

Proof 1. Following an analogous reasoning as in [6], one can show that the distributions of M_a and $-I_a$ are of phase type. Since the analytic continuation of the Laplace transform of a phase type distribution is a ratio of two polynomials and $\varphi_a^{\pm}(0) = 1$, we deduce from the Wiener-Hopf factorisation that $\varphi_a^{\pm} = \kappa_a^{\pm}$.
2. Since the jumps of X form a compound Poisson process, we see that $\mathbb{P}(M_a = 0)$ $[\mathbb{P}(I_a = 0)]$ is positive iff $\sigma = 0$ and $\mu \leq [\geq]0$. Combining this with (14), a Tauberian theorem, the form of κ and recalling that there are no roots with zero real part, the first statement follows. The second statement follows from the definition of minimality.
3. The third statement follows by Laplace-Stieltjes inversion of (14). QED

3.2 First passage time for X

The first passage time problem consists in computing the joint moment generating function

$$u_k(x) = u_k(x, a, b) = \mathbb{E}_x[\mathrm{e}^{-aT + b(X_T - k)}] \tag{18}$$

of the crossing time

$$T = T(k) = \inf\{t > 0 : X_t \leq k\}$$

and of the shortfall $X_T - k$, with k, $a > 0$ and b such that $u_k(x)$ is finite. The subscript x in $\mathbb{E}_x$ refers to $X_0 = x$.

At the crossing time $T(k)$, we must either have a downwards jump of X, or the component $\mu t + \sigma W_t$ must take the process X down to the barrier k. Denote by G_0 the event that the last alternative occurs, by G_i, $i = 1, \ldots, m^{(-)}$, the event that the first occurs and the up-crossing of k occurs in phase i, i.e. that $J(X_{T(k)-} - k) = i$ where J is the underlying phase process for the jump causing the up-crossing, and by $M^{(-)}$ the set of all phases during which down-crossing of a level may occur. Thus, calling the state where the Lévy process is moving continuously phase 0, $M^{(-)} = \{1, \ldots, m^{(-)}\}$ if the Brownian component is zero and if the drift points opposite to the barrier; otherwise, $M^{(-)} = \{0, \ldots, m^{(-)}\}$. Let $\pi_i = \mathbb{E}_x[\exp(-aT(k))\mathbf{1}_{G_i}]$ denote the discounted probability of up-crossing in phase i, where $X_0 = x$. Moreover, let $\mathbf{1}_i$ denote a vector of zeros with a 1 on the ith position, $\boldsymbol{\pi} = (\pi_i, i \in M^{(-)})$, and let $\hat{\boldsymbol{f}}^{(-)}[b]$ denote the vector (depending on the phase at the level crossing) of Laplace transforms at b of the absolute shortfall $|X_{T(k)} - k|$. This vector can be analytically continuated to the complex plane except a finite number of poles (the eigenvalues of $\boldsymbol{T}^{(-)}$). This analytic extension will also be denoted by $\hat{\boldsymbol{f}}^{(-)}$. Note that, if $0 \in M^{(-)}$, then the first component of $\hat{\boldsymbol{f}}^{(-)}[b]$ is 1, and the other components are given by $(b\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$ by the phase assumption and if $0 \notin M^{(-)}$, the first component is missing. The next result gives an explicit expression for the moment-generating function $u_k(x)$ in terms of the roots with negative real part.

Proposition 2 *Subject to (2) we have:*

1. *For any nonnegative function f and $x > k$:*

$$\mathbb{E}_x[\mathrm{e}^{-aT(k)} f(X_{T(k)} - k)] = \boldsymbol{\pi}\mathcal{G}f \tag{19}$$

where $\mathcal{G}f = (\int_0^\infty f(-z)F_i^{(-)}(\mathrm{d}z), i \in M^{(-)})$ with $F_0^{(-)}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$ and $1 - F_i^{(-)}(z) = \mathbf{1}_i \exp(\boldsymbol{T}^{(-)}z)\mathbf{1}$ for $i \neq 0$.

Moreover, assuming all the roots of the equation (4) with negative real part to be distinct the following hold true:

2. *For $x > k$ the vector $\boldsymbol{\pi}$ solves the system*

$$\boldsymbol{\pi}\hat{\boldsymbol{f}}^{(-)}[\rho_i] = \mathrm{e}^{\rho_i(x-k)}, \qquad \forall i \in \mathcal{I}^{(-)}. \tag{20}$$

If the representation for $F^{(-)}$ is minimal, $\boldsymbol{\pi}$ is the unique solution of (20).

3. *In particular, $u_k(x)$ defined in (18) is for $x > k$ given by*

$$\mathrm{e}^{bx}u_k(x) = u_{k-x}(0) = \kappa_a^-(b)^{-1} \sum_{j\in\mathcal{I}^{(-)}} A_j^- \rho_j \mathrm{e}^{\rho_j(x-k)}/(\rho_j - b). \tag{21}$$

where A_j^- is defined in (16).

4. *The resolvent of X killed upon entering $(-\infty, k]$ is for $k < 0$ and $y > k$ given by*

$$\mathbb{P}(X_{\eta(a)} \in \mathrm{d}y, \eta(a) < T(k))/\mathrm{d}y =$$
$$\sum_{i\in\mathcal{I}^{(+)}} \sum_{j\in\mathcal{I}^{(-)}} \frac{A_i^+ A_j^-(-\rho_j\rho_i)}{\rho_j - \rho_i} \mathrm{e}^{-\rho_i y}[\mathrm{e}^{-(\rho_j-\rho_i)k} - \mathrm{e}^{(\rho_j-\rho_i)(-y)^+}], \tag{22}$$

where $\eta(a)$ denotes,as before, an independent exponential random variable with parameter a, $c^+ = \max\{c, 0\}$ and $\boldsymbol{A}^+ = (A_i^+, i \in \mathcal{I}^{(+)})$ are the partial fraction coefficients of the expansion of $\kappa_a^+(s) - \kappa_a^+(-\infty)$ into $\rho_i/(\rho_i - s)$ for $i \in \mathcal{I}^{(+)}$.

Remark 6 *Taking Laplace transform of (21) in $x - k$, we recover a formula of [25] $\hat{u}_0(s) = (b-s)^{-1}\left(1 - \frac{\kappa_a^-(s)}{\kappa_a^-(b)}\right)$ for $\Re(s) \geq 0$.*

Proof of Proposition 2 1. Splitting the probability space in $G_0, \dots, G_{m^{(-)}}$ and using the fact that conditionally on the phase in which the up-crossing occurs, the time of overshoot $T(k)$ and the shortfall $X_{T(k)} - k$ are independent, yields the decomposition

$$\mathbb{E}_x[\mathrm{e}^{-aT} f(X_T - k)] = \mathbb{E}_x[\mathrm{e}^{-aT}\mathbf{1}_{G_0}] + \sum_{i=1}^{m^{(-)}} \mathbb{E}_x[\mathrm{e}^{-aT}\mathbf{1}_{G_i}]\mathbb{E}_i[f(X_T - k)],$$

where we wrote $T = T(k)$ and respectively used $\mathbb{E}_x, \mathbb{E}_i$ to denote the expectation under $\mathbb{P}$ conditioned on $\{X_0 = x\}$ and G_i. This yields (19).
2. The system (20) is derived by an optional stopping approach. By applying Itô s formula to the function $f(t, X_t) = \exp(-at + bX_t)$ for any a and b with $\Re(b) = 0$ (which ensures that $\kappa(b)$ is well defined), we find that

$$\begin{aligned} M_t &= f(t, X_t) - f(0, X_0) - \int_0^t Gf(s, X_s)\mathrm{d}s \qquad (23) \\ &= \exp(-at + bX_t) - \exp(bX_0) - (\kappa(b) - a)\int_0^t \exp(-as + bX_s)\mathrm{d}s, \end{aligned}$$

is a zero-mean martingale, where $G = \frac{\partial}{\partial t} + \Gamma$ with Γ the infinitesimal generator of $\{X_t, t \geq 0\}$ (note that $Gf(t, X_t) = (\kappa(b) - a)f(t, X_t))$. Applying for $a \geq 0$ Doob's optional stopping theorem with the stopping time $T(k) \wedge t$ and noting that $\sup_t |M_{T(k)\wedge t}|$ is bounded we find $\mathbb{E}_x[M_{T(k)}] = 0$. By a computation as above we can expand this for $x > k$ as

$$0 = \mathrm{e}^{bk}\boldsymbol{\pi}\hat{\boldsymbol{f}}^{(-)}[b] - \mathrm{e}^{bx} - (\kappa(b) - a)\mathbb{E}_x\left[\int_0^{T(k)} \exp(-as + bX_s)\mathrm{d}s\right]. \qquad (24)$$

By analytic continuation, the identity (24) can be extended to the half plane $\Re(b) < 0$ except finitely many poles (the eigenvalues of $\boldsymbol{T}^{(-)}$, recall that $\boldsymbol{T}^{(-)}$ has negative eigenvalues). By choosing b with $\Re(b) < 0$ to be a root of the equation $\kappa(b) = a$, we find (20). By Lemma 1 the number of equations is equal to the number of unknowns, if the representation of $F^{(-)}$ is minimal. The distinct roots assumption implies the linear independence of $\hat{\boldsymbol{f}}^{(-)}[\rho_i]$, as proved in Section 5. Hence the "Wald system" (20) is nonsingular, yielding $\boldsymbol{\pi}$.
3. Suppose first $b, a > 0$, and note that $\mathrm{e}^{bx}u_k(x) = u_{k-x}(0)$. Define $A = \{T(k-x) < \eta(a)\}$. The strong Markov property of X applied at $T(k-x)$

together with the memoryless property of the exponential distribution imply that

$$\begin{aligned}\mathbb{E}[\exp(bI_a)\mathbf{1}_A] &= \mathbb{E}[\exp(bX_{T(k-x)})\mathbf{1}_A]\mathbb{E}[\exp\{bI_a\}] \\ &= \mathbb{E}[\exp(-aT(k-x)+bX(T(k-x)))]\kappa_a^-(b),\end{aligned}$$

where $\mathbf{1}_A$ denotes the indicator of the event A. Noting that $A=\{I_a<k-x\}$ and using (15) one finds the formula as stated. By analytic extension, the identity holds for all b for which the right-hand side of (21) is well defined.
4. For $k<0$ the set $\{\eta(a)<T(k)\}$ is the same as $\{I_a>k\}$ and that (from time-reversal) M_a has the same distribution as $X_{\eta(a)}-I_a$, we find that

$$\mathbb{P}(X_{\eta(a)}\in \mathrm{d}y, \eta(a)<T(k)) = \int_0^{-k}\mathbb{P}(-I_a\in \mathrm{d}z)\mathbb{P}(M_a\in \mathrm{d}(z+y)).$$

Inserting the expressions from (15), we find the stated expression. QED

Remark 7 *Again, the case when the equation (13) has multiple roots with negative real part poses no problem; expressions similar to (20) – (21) may be obtained by approximation.*

American put and Erlang approximations

Under the phase-type Lévy model (2) and assuming that the roots $\rho_j\in\mathcal{I}^{(-)}$ are different, the value of the American put option for $\mathrm{e}^x>\mathrm{e}^{k^*}=K\kappa_\delta^-(1)$ can be checked to be given by

$$\begin{aligned}U^*(x) &= K\mathbb{E}_x[\mathrm{e}^{-\delta T(k^*)}]-\mathrm{e}^{k^*}\mathbb{E}_x[\mathrm{e}^{-\delta T(k^*)+X_{T(k^*)}-k^*}] \\ &= K\sum\nolimits_{j\in\mathcal{I}^{(-)}}\mathrm{e}^{\rho_j(x-k^*)}A_j^-/(1-\rho_j),\end{aligned}\tag{25}$$

where $\rho_j=\rho_j(\delta)$ denote the roots of $\kappa(\rho)=\delta$ for $\delta=r+a$ (just insert the expressions for k^* and the joint moment-generating function $u_k(x)$ of $T(k^*)$ and $X_{T(k^*)}-k^*$). The important application here is with the parameter $\delta=r+(T-t)^{-1}$, where t,T denote the current and expiration time of a finite expiration option, first proposed in [34]. Recalling that $\kappa(1)=r$ we see that the optimal exercise level $k^*=k^*(t,T)$ is given by

$$\exp(k^*)=K\frac{\delta}{\delta-\kappa(1)}\,\frac{1}{\kappa_\delta^+(1)}=K(r(T-t)+1)\,\frac{1}{\kappa_\delta^+(1)}.$$

As noticed in [15] this time dependent approximation for the optimal exercise boundary of an American put with finite expiration time T is asymptotically exact when $t\to-\infty$ and also when $t\to T$.

We can also obtain the value of an American put on a stock paying proportional dividends. Indeed, the value of an American put option with payoff (5) on a stock paying dividends at rate $q\geq 0$ can be found by choosing $\mathbb{P}$ such that

$\kappa(1) = r - q$ (instead of r) and by replacing everywhere in (25) the parameters (r, a) by $(r - q, a + q)$.

Further refinements under the "two sided phase-type" model (2) may be obtained by Erlangizing the expiration time, a method which goes back at least as far as S. Ross [115], and which was first implemented in mathematical finance by Carr [36] (see also [15, 85]). By this approach, one can obtain a sequence of analytic formulae that converges pointwise to the price of the American put with finite time of expiration T, extending thus the spectrally negative results in [15, 16]. We give an outline how to obtain the first approximation, named by Carr a "Canadised" American put option. Letting $\eta(T^{-1})$ denote an independent exponential random variable with mean T, by standard optimal stopping theory (see the argument of Theorem 1), one shows that the optimal stopping time for this option is again of the form $T(k)$ for some $k < \log K$. Computing the value function U_1^* thus boils down to evaluating

$$\mathbb{E}_x[\mathrm{e}^{-r(T(k)\wedge\eta(T^{-1}))}(K - \mathrm{e}^{X_{T(k)\wedge\eta(T^{-1})}})^+] =$$
$$\mathbb{E}_x[\mathrm{e}^{-qT(k)}(K - \mathrm{e}^{X_{T(k)}})^+] + \frac{1}{qT}\int_k^{\log K}(K - \mathrm{e}^z)\mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}z, \eta(q) < T(k)),$$

where $q = r + T^{-1}$, followed by a one-dimensional optimisation (or continuous/smooth fit) to find the optimal level k_1^*. The evaluation of the second term in the display uses the resolvent of X killed upon entering $(-\infty, k]$, from Lemma 1. If $\sigma \neq 0$ and the roots of (4) are distinct, $U_1^*(x)$ is given by

$$\begin{cases} \frac{K}{qT} - \mathrm{e}^x + c(x) + \frac{K}{qT}\sum_{i,j}\frac{A_j^- A_i^+(-\rho_j\rho_i)}{\rho_j - \rho_i}[d_i(x) - e_{ij}(x)] & x \in (k, \log K) \\ c(x) + \frac{K}{qT}\sum_{i,j}\frac{A_j^- A_i^+(-\rho_j\rho_i)}{\rho_j - \rho_i}[d_j(x) - e_{ij}(x)] & x \geq \log K, \end{cases}$$

where the sum is over $i \in \mathcal{I}^{(+)}$ and $j \in \mathcal{I}^{(-)}$, $c(x) = K\frac{r}{q}\sum_{j\in\mathcal{I}^{(-)}} A_j^- \mathrm{e}^{\rho_j(x-k)}$, $d_i(x) = \frac{\mathrm{e}^{\rho_i x}K^{-\rho_i}}{\rho_i(1-\rho_i)}$, $e_{ij}(x) = d_i(k)\mathrm{e}^{\rho_j(x-k)}$ and the optimal exercise level $k = k_1^*$ is

$$\exp k_1^* = K \sup\left\{x \leq \log K : rT = \sum_{i\in\mathcal{I}^{(+)}} A_i^+ x^{\rho_i}/(\rho_i - 1)\right\}. \tag{26}$$

Since it follows from the definition of the A_i^+ that $\sum A_i^+ \frac{\rho_i}{\rho_i - 1} = \kappa^+_{T^{-1}}(1) - 1$ which is larger than rT, we note that $\exp k_1^* < K$.

3.3 First passage time for Y

We now consider the first passage time problem for Y, which, analogously, consists in computing the joint moment generating function

$$v_k(y) = v_k(y, a, b) = \mathbb{E}_y[\mathrm{e}^{-a\tau + b(Y_\tau - k)}] \tag{27}$$

of the crossing time

$$\tau = \tau(k) = \inf\{t > 0 : Y_t \geq k\}$$

and of the overshoot $Y_\tau - k$, with k, $a \geq 0$, and where b is such that $v_k(y)$ is finite. Under the measure $\mathbb{P}_y$, the process Y starts in y.

Analogously to the previous section 3.2, we note that at the crossing time $\tau(k)$, we must either have a downward jump of X, or the component $\mu t + \sigma W_t$ must take the process Y to the barrier k. Similarly, Y is taken to 0 by an upward jump of X or by the component $\mu t + \sigma W_t$. Denote by $M^{(\pm)}$ the set of all phases during which down- or up-crossing may occur (again calling the non-jumping time phase 0). Let $\widetilde{\pi}_i = \mathbb{E}_y[\mathrm{e}^{-a\tau}\mathbf{1}_{H_i}]$ denote the (discounted) probability of up-crossing in phase i, i.e. that $J(k - Y_{\tau(k)-}) = i$, where J is the underlying phase process for the jump causing the up-crossing. As before we let $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_i, i \in M^{(-)})$, and let $\hat{\boldsymbol{f}}^{(-)}[b]$ denote the vector (depending on the initial starting state) of Laplace transforms at b of the overshoot $Y_{\tau(k)} - k$. Let $L_t = \sup_{0\leq s\leq t} X_s \vee y$ be the running supremum of X, with L_t^c the continuous part of L and $\Delta L_t = L_t - L_{t-}$ the jump of L at time t. Introduce the dummy-variables $\delta_0 = \mathbb{E}_y[\int_0^{\tau(k)} \exp(-as)\mathrm{d}L_s^c]$ and

$$\delta_i = \mathbb{E}_y\left[\sum_{0<s\leq\tau(k)} \exp(-as)\mathbf{1}_{\{\Delta L_s>0, H_j\}}\right],\ j = 1, \ldots, m^{(+)},$$

where H_j is the event of crossing the supremum in phase j. Denote by $\boldsymbol{\delta}$ the row vector $\boldsymbol{\delta} = (\delta_i, i \in M^{(+)})$ and write $\boldsymbol{g}[\rho] = (g[\rho]_i, i \in M^{(+)})$ with $g[\rho]_0 = \rho$ and $g[\rho]_i = \rho\mathbf{1}_i(-\rho\boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}\mathbf{1}$.

Proposition 3 *Subject to (2), the joint moment generating function $v_k(y)$ defined in (27) is for $y \in [0, k)$ given by*

$$v_k(y) = \widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-b].$$

where $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_i, i \in M^{(-)})$ and $\boldsymbol{\delta} = (\delta_i, i \in M^{(+)})$ solve the system

$$\mathrm{e}^{-\rho_i y} = \mathrm{e}^{-\rho_i k}\widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[\rho_i] - \boldsymbol{\delta}\boldsymbol{g}[\rho_i] \qquad i = 1, \ldots, p. \tag{28}$$

If the roots ρ_i of $\kappa(\rho) = a$ are distinct and the representations of $F^{(\pm)}$ are minimal then $\widetilde{\boldsymbol{\pi}}$ and $\boldsymbol{\delta}$ are the unique solution of this system.

Proof The proof of the first part is analogous to the proof of the second part of Proposition 2 and left to the reader. To compute the vector $\widetilde{\boldsymbol{\pi}}$, we apply the optional stopping approach to the reflected process Y, using the martingale introduced by Kella and Whitt [77]. Note that L^c and $\Delta L_t = L_t - L_{t-}$ have finite expected variation resp. finite number of jumps in each finite time interval. From Kella and Whitt [77] we find then that for $a > 0, \gamma \in \mathrm{i}\mathbf{R}$

$$\begin{aligned} N_t = (\kappa(-\gamma) - a)\int_0^t \exp(-as + \gamma Y_s)\mathrm{d}s + \exp(\gamma Y_0) - \exp(-at + \gamma Y_t) \\ + \gamma\int_0^t \exp(-as)\mathrm{d}L_s^c + \sum_{0<s\leq t} \exp(-as)[1 - \exp(-\gamma\Delta L_s)] \end{aligned}$$

is a zero mean martingale (where we used that if ΔL_s or $\mathrm{d}L_s$ is positive then $Y_s = 0$). Applying, as before, Doob's optional stopping theorem with the stopping time $\tau(k) \wedge t$ and straightforwardly checking that $|N_{\tau(k)\wedge t}|$ can be dominated by an integrable function, we find $\mathbb{E}_y[N_{\tau(k)}] = 0$. Then, expanding $\mathbb{E}_y[N_{\tau(k)}] = 0$ for $y < k$ leads to

$$0 = (\kappa(-\gamma) - a)\mathbb{E}_y\left[\int_0^{\tau(k)} \exp(-as + \gamma Y_s)\mathrm{d}s\right] + \mathrm{e}^{\gamma y} - \mathrm{e}^{\gamma k}\widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-\gamma]$$
$$+ \gamma\delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i(1 - \hat{\boldsymbol{f}}^{(+)}[\gamma]_i). \qquad (29)$$

By analytic continuation, the identity (29) can be extended to hold for γ in the complex plane except finitely many poles (the eigenvalues of $-\boldsymbol{T}^{(-)}, \boldsymbol{T}^{(+)}$) Letting ρ_j to be a root of $\kappa(\rho) = a$, we find the system (28). Note that $\widetilde{\pi}_0 = 0$ iff $\sigma = 0, \mu \geq 0$ (or equivalently $0 \notin M^{(-)}$) and $\delta_0 = 0$ iff $\sigma = 0, \mu \leq 0$ (or equivalently $0 \notin M^{(+)}$). Thus, since $M^{(\pm)} = \mathcal{I}^{(\pm)}$ for minimal representations, the number of unknowns is equal to the number of equations. If the roots are distinct the equations are linear independent (to be proved in Section 5) yielding the unique solution. QED

If $\sigma \neq 0$, the solution of the system (28) is in matrix notation form:

$$\boxed{(\widetilde{\boldsymbol{\pi}} \ - \boldsymbol{\delta}) = (\mathrm{e}^{-\rho_1 y} \ \ldots \ \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}}, \qquad (30)$$

where $\widetilde{\boldsymbol{S}} = \begin{pmatrix} \widetilde{\boldsymbol{S}}_1 \\ \widetilde{\boldsymbol{S}}_2 \end{pmatrix}$ is a $p \times p$ matrix whose first $m^{(-)}+1$ rows $\widetilde{\boldsymbol{S}}_1$ are column-wise given by

$$\widetilde{\boldsymbol{k}}_1^{(j)} = \mathrm{e}^{-\rho_j k} \begin{pmatrix} 1 \\ \left(\rho_j \boldsymbol{I} - \boldsymbol{T}^{(-)}\right)^{-1} \boldsymbol{t}^{(-)} \end{pmatrix}$$

and whose last $m^{(+)} + 1$ rows $\widetilde{\boldsymbol{S}}_2$ are column-wise given by

$$\widetilde{\boldsymbol{k}}_2^{(j)} = -\rho_j \begin{pmatrix} 1 \\ \left(-\rho_j \boldsymbol{I} - \boldsymbol{T}^{(+)}\right)^{-1} \boldsymbol{1} \end{pmatrix},$$

where $\boldsymbol{1}$ is a vector of ones. From Proposition 3 we conclude now that, if $\sigma \neq 0$,

$$v_k(y) = (\mathrm{e}^{-\rho_1 y} \ \ldots \ \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}\hat{\boldsymbol{f}}_o^{(-)}[-b], \qquad y \in [0, k),$$

where $\hat{\boldsymbol{f}}_o^{(-)}[-b]$ denotes the column vector of Laplace transforms of the overshoots over k prolonged by 0's. Therefore, $v_k(y) = \sum_{i=1}^{p} \mathrm{e}^{-\rho_i y} A_i$ is a linear combination of the exponentials, with the vector $\boldsymbol{A}$ satisfying the linear system

$$\widetilde{\boldsymbol{S}}\boldsymbol{A} = \hat{\boldsymbol{f}}_o^{(-)}[-b]. \qquad (31)$$

Replacing in above paragraph the vectors $\widetilde{\boldsymbol{k}}_1^{(j)}$ by $(\rho_j \boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$ if $\sigma = 0$ and $\mu \geq 0$ and $\widetilde{\boldsymbol{k}}_2^{(j)}$ by $(-\rho_j \boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}\boldsymbol{1}$ if $\sigma = 0$ and $\mu \leq 0$, the result (31) for the corresponding matrix $\widetilde{\boldsymbol{S}}$ remains valid.

To connect to other results in the literature, we reformulate now the system (31) for $\boldsymbol{A}$ in terms of the eigenvalues of the matrices $\boldsymbol{T}^{(\pm)}$, allowing at the same time for a general Jordan structure. Recall that $\mathcal{J}^{(\pm)}$ denotes the set of (indices of) roots of $a/(a-\kappa(s)) = 0$. Let $\{\eta_j, j = 1, \ldots, n^{(\pm)}\}$ be an enumeration of all the *distinct* roots of $a/(a-\kappa(s)) = 0$ and denote by $m^{(\pm,j)}$ the multiplicity of a root η_j. Note that $\sum_{j=1}^{n^{(\pm)}} m^{(\pm,j)}$ is equal to $\#\mathcal{J}^{(\pm)}$. Then the system (31) becomes:

Proposition 4 *Under (2) and assuming the roots ρ_i of $\kappa(\rho) = a$ to be distinct, we have*

$$v_k(y) = \sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i y}, \qquad y \in [0, k), \tag{32}$$

where $A_1, \ldots, A_p$ uniquely solve the p equations

$$\sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i k} \frac{1}{(\rho_i - \eta_j)^l} = \frac{1}{(-b - \eta_j)^l} \tag{33}$$

$$\sum_{i=1}^{p} A_i \rho_i \frac{1}{(-\rho_i - \eta_j)^l} = 0, \tag{34}$$

where in (33) and (34) $l = 1, \ldots, m^{(\mp,j)}$, $j = 1, \ldots, n^{(\mp)}$ respectively and in addition $l = 0$ in (33) if $\sigma \neq 0$ or $\mu < 0$ [$l = 0$ in (34) if $\sigma \neq 0$ or $\mu > 0$].

In Section 5.3 we provide an independent proof using a martingale method.

The Russian option

Now we turn to the explicit solution of the optimal stopping problem connected to the pricing of the Russian option. Recall $\hat{\boldsymbol{f}}_o{}^{(-)}[-b]$ denotes the column vector of Laplace transforms of the overshoots of $Y_{\tau(k)}$ over k prolonged by 0's. Combining Theorem 1 with the results of the foregoing section leads to the following statement:

Corollary 1 *Let X be of the form (2) and satisfy (EMM). Assume that the roots of (13) are distinct. Then the price of the Russian option is given by*

$$v^*(y) = \begin{cases} \mathrm{e}^{k^*} \widetilde{\boldsymbol{\pi}}(y, k^*) \hat{\boldsymbol{f}}^{(-)}[-1] = \mathrm{e}^{k^*} \sum_{i=1}^{p} A_i(k^*) \mathrm{e}^{-\rho_i y}, & y \in [0, k^*); \\ \mathrm{e}^{y}, & y \geq k^*, \end{cases}$$

where ρ_i are the roots of $a = \kappa_1(\rho) = \kappa(\rho+1) - \kappa(1)$ and the $A_i = A_i(k^*)$ are given in (32) and are, just as $\widetilde{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{f}}^{(-)}$, computed under the measure $\mathbb{P}^{(1)}$. The optimal level k^* satisfies the following:

(i) If $\mu \geq \sigma = 0$ *and* $\iota := \frac{\lambda^{(-)}}{a+\lambda^{(-)}} \boldsymbol{\alpha}^{(-)} \hat{\boldsymbol{f}}^{(-)}[-1] > 1$, k^* *is a positive root of*

$$(\mathrm{e}^{-\rho_1 k}, \ldots, \mathrm{e}^{-\rho_p k}) \widetilde{\boldsymbol{S}}^{-1} \hat{\boldsymbol{f}}_o^{(-)}[-1] = \sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i k} = 1; \tag{35}$$

(ii) If $\mu < -a$ *and* $\sigma = 0$, k^* *is a positive root of*

$$(\rho_1 \mathrm{e}^{-\rho_1 k}, \ldots, \rho_p \mathrm{e}^{-\rho_p k}) \widetilde{\boldsymbol{S}}^{-1} \hat{\boldsymbol{f}}_o^{(-)}[-1] = \sum_{i=1}^{p} \rho_i A_i \mathrm{e}^{-\rho_i k} = -1; \tag{36}$$

(iii) If $\sigma \neq 0$, k^* *is positive and uniquely determined by (36).*

In the literature equations (35) and (36) are called the conditions of *continuous fit* and *smooth fit* respectively. As in [108, 16] we observe that whenever the process Y has positive probability of creeping across positive levels, that is $\mathbb{P}(Y_{\tau(k)} = k) > 0$, the optimal level k^* satisfies the condition of smooth fit; otherwise it satisfies the condition of continuous fit.

Example Consider the case where X is a Brownian motion with drift. Denote by $\rho_1 < 0 < \rho_2$ the two roots of

$$\kappa_1(s) = \kappa(s+1) - r = \frac{\sigma^2}{2} s^2 + \left(r + \frac{\sigma^2}{2}\right) s - r = a.$$

Since $\hat{\boldsymbol{f}}_o^{(-)}[1] = (1,0)'$ and $\boldsymbol{S} = \begin{pmatrix} \mathrm{e}^{-\rho_1 k} & \mathrm{e}^{-\rho_2 k} \\ -\rho_1 & -\rho_2 \end{pmatrix}$, we find by adding (35) to (36) that the optimal level k^* is given by

$$\exp\left((\rho_2 - \rho_1) k^*\right) = \frac{\rho_1(\rho_2 + 1)}{\rho_2(\rho_1 + 1)},$$

which is the formula found by Shepp and Shiryaev [117].

In the proof of the Corollary we will use the following auxiliary result, which is proved in Section 5.

Lemma 2 *Under the assumptions of Corollary 1, the following hold true under* $\mathbb{P}^{(1)}$*:*

(i) If $\mu \geq \sigma = 0$, *then* $v_k(0) \to \iota$ *and if* $\mu < \sigma = 0$, *then* $v_k'(y)|_{y=k^-} \to -\frac{a}{\mu}$ *as* $k \downarrow 0$.

(ii) If $\sigma \neq 0$, *then the function* $k \mapsto v_k'(y)|_{y=k^-}$ *is continuous and increasing on* $(0, \infty)$ *with* $\lim v_k'(y)|_{y=k^-} = 0$ *or* > 1 *if* $k \downarrow 0$ *and* $k \to \infty$ *respectively.*

Proof of Corollary 1 The only statements left to prove are the ones on the optimal level k^*, the rest follows from Theorem 1 and Propositions 3 and 4. Let us first consider the situation that $\mu \leq 0 = \sigma$. Note that if $\iota > 1$, then $v^*(0) > 1$ and we have that $k^* > 0$. Since $y \mapsto v^*(y) = \mathrm{e}^{k^*} v_{k^*}(y)$ is convex and hence continuous on $(0,\infty)$, it follows that the optimal level k^* satisfies $v_{k^*}(k^*-) = 1$, which is (35).

Now let us consider the case $\mu < -a$ or $\sigma \neq 0$. Note first that if k^* is an optimal level, $k \mapsto v_k(y)$ is maximised in $k = k^*$ for all y. Thus $\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(0)\right)|_{k=0^+} \leq 0$ is a necessary condition for $k^* = 0$ to be optimal. and if $k^* > 0$, then it satisfies

$$\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(y)\right)\bigg|_{k=k^*} = 0 \qquad \text{for all } y < k^*$$

and in particular $\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(k^*-)\right)|_{k=k^*} = 0$. Secondly we note that in this case Y is regular for $(0,\infty)$, that is the first time Y enters $(0,\infty)$ is almost surely 0, which yields the identity

$$\mathrm{e}^k v_k(k^-) = \mathrm{e}^k v_k(k) = \mathrm{e}^k \qquad \text{for all } k > 0. \tag{37}$$

Differentiating (37) with respect to k we find

$$\boxed{\frac{\partial}{\partial z}\left(\mathrm{e}^z v_z(k^-)\right)\bigg|_{z=k} + \mathrm{e}^k \,\frac{\partial}{\partial y} v_k(y)\bigg|_{y=k^-} = \mathrm{e}^k.} \tag{38}$$

Since by Lemma 2 $v_k'(y)|_{y=k^-}$ converges to a number smaller than 1 if k tends to 0, we deduce that in this case $k^* > 0$ and that k^* is a positive root of $v_k'(y)|_{y=k^-} = 1$ which is the equation (36). If $\sigma \neq 0$, we see from Lemma 2(ii) that there is a unique $c > 0$ such that $v_c'(c^-) = 1$. QED

The "Canadised" Russian option is understood to be the Russian option with an independent exponential random variable $\eta(\lambda)$ as expiration, an analogue of the Canadised American put. It can be considered as a first approximation to the Russian option with finite expiration $1/\lambda$. See [85, 16]. The value of the Canadised Russian option is given by $V_c^*(x,m) = \mathrm{e}^x v_c^*(m-x)$, where v_c^* is the value function of the optimal stopping problem

$$v_c^*(y) = \sup \mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau\wedge\eta(\lambda)) + Y_{\tau\wedge\eta(\lambda)}}].$$

where the supremum runs over τ in $\mathcal{T}$. Mimicking the proof of Theorem 1, we check that again the optimal stopping time is of the form (9). The quantities $\widetilde{\boldsymbol{\pi}}$ and $\boldsymbol{\delta}$ are now understood to be taken under the measure $\mathbb{P}_y^{(1)}$. Note that $\kappa_1(-1) = \kappa(0) - \kappa(1) = -r$ (see Appendix A). Then we can read off from equation (29) that for $y < k$ and with $\gamma = \lambda/(a+\lambda+r)$, we have that

$$\begin{aligned}\mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau_k\wedge\eta(\lambda)) + Y_{\tau_k\wedge\eta(\lambda)}}] = \widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-1](1-\gamma) \times \mathrm{e}^k \\ + \gamma(\mathrm{e}^y + \delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i(1 - \hat{\boldsymbol{f}}^{(+)}[1]_i)).\end{aligned}$$

By optimisation of this expression over all levels $k \geq 0$ (or by smooth/continuous fit), we find $v_c^*(y)$.

Example Let X be given by a jump-diffusion where the jumps have a negative hyper-exponential distribution. In the general setting we choose $\sigma > 0$, $\lambda^{(+)} = 0$, $-\boldsymbol{T}^{(-)} = \mathrm{diag}(\beta_1, \ldots, \beta_n)$, β_i different, and $\boldsymbol{\alpha}^{(-)} = (\alpha_1, \ldots, \alpha_n)$. From Appendix A, we find that the parameters of X under $\mathbb{P}^{(1)}$ are given by

$$\widetilde{\mu} = \mu + \sigma^2, \qquad \widetilde{\lambda}^{(+)} = 0, \qquad -\widetilde{\boldsymbol{T}}^{(-)} = \mathrm{diag}(1+\beta_1, \ldots, 1+\beta_n)$$

$$\widetilde{\lambda}^{(-)} = \lambda^{(-)}\boldsymbol{\alpha}^{(-)}(\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)} = \lambda^{(-)}\sum_{i=1}^{n}\alpha_i\frac{\beta_i}{\beta_i+1}$$

$$\widetilde{\boldsymbol{\alpha}}^{(-)} = \boldsymbol{\alpha}^{(-)}\mathrm{diag}(k_1, \ldots, k_n)/\hat{F}^{(-)}[1] = \frac{1}{\sum_{i=1}^{n}\frac{\alpha_i\beta_i}{\beta_i+1}}\left(\frac{\alpha_1\beta_1}{\beta_1+1}, \ldots, \frac{\alpha_n\beta_n}{\beta_n+1}\right),$$

where $\boldsymbol{k} = (\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$. Let ρ_i be the roots of $\kappa_1(s) = a$ and note that they are all distinct. Then the price $V^*(x, m)$ of the Russian option is given by $V^*(x, m) = \mathrm{e}^x v^*(m - x)$ where $v^*(y) = \mathrm{e}^y$ for $y \geq k^*$ and

$$v^*(y) = \mathrm{e}^{k^*}\textstyle\sum_{i=0}^{n+1} A_i \mathrm{e}^{-\rho_i y} \qquad 0 \leq y < k^* \tag{39}$$

with the A_i and k^* are determined by

$$\sum_{i=0}^{n+1} A_i \mathrm{e}^{-\rho_i k^*} = 1 \qquad \sum_{i=0}^{n+1} A_i\rho_i = 0 \qquad \sum_{i=0}^{n+1} A_i\rho_i \mathrm{e}^{-\rho_i k^*} = -1$$

$$\sum_{i=0}^{n+1} A_i\rho_i \mathrm{e}^{-\rho_i k^*}\frac{1}{1+\beta_j+\rho_i} = \frac{1}{1+\beta_j-1} \qquad (j = 1, \ldots, n).$$

The first equation in the second line is smooth fit condition which determines k^*. Write now $C_i = A_i\mathrm{e}^{-\rho_i k}$ then we can rewrite the previous system as

$$1 = \sum_{i=0}^{n+1} C_i = -\sum_{i=0}^{n+1} C_i\rho_i = \sum_{i=0}^{n+1} C_i\frac{\beta_j}{1+\beta_j+\rho_i} \quad (j = 1, \ldots, n)$$

to find the B_i and then to find the k^* the equation $\sum_{i=0}^{n+1} C_i\rho_i\mathrm{e}^{\rho_i k^*} = 0$. By a partial fraction argument based on the rational function

$$\frac{\prod_{j=0}^{n+1}(\rho_j+1)}{\prod_{j=1}^{n}(-\beta_j)}\frac{\prod_{j=1}^{n}(s-\beta_j)}{\prod_{j=0}^{n+1}(s+\rho_j+1)},$$

we see that

$$A_i\mathrm{e}^{-\rho_i k} = C_i = \frac{\prod_{j=0}^{n+1}(\rho_j+1)}{\prod_{j=0, j\neq i}^{n+1}(\rho_j-\rho_i)}\frac{\prod_{j=1}^{n}(1+\rho_i+\beta_j)}{\prod_{j=1}^{n}\beta_j}.$$

The found formula for the value of the Russian option coincides with the results from [102].

4 Regime-switching Lévy processes

4.1 Introduction

In this section, we study a certain class of regime-switching Lévy processes following an approach based on embedding first the Lévy model into a *continuous regime switching Brownian motion*, as proposed in [7] (see also [9], [10]).

Definition. A regime switching phase–type Lévy process X is a semi-Markov process to which is associated an ergodic finite state space Markov process J such that, conditional on $J_t = j$, X_t is a Lévy model of the form (2) with parameters depending on j. In the case of no jumps the process is called a regime switching Brownian motion.

The trick of passing from a phase–type regime switching Lévy process to a regime switching Brownian motion is to level out the positive jumps to sample path segments with slope +1 and the negative jumps to sample path segments with slope −1, and add an extra phase, say 0, for the "regular time" when the process evolves continuously. This embeds a process with phase-type jumps X in a continuous Markov additive process (J, X'), or regime switching Brownian motion, where the Markov component J_t is in phase 0 at a regular time and gives the current phase of the jump otherwise.

For a general regime switching Lévy process Z, let us denote by $\boldsymbol{F}_t[s]$ the $p \times p$ matrix with ijth element $\mathbb{E}_i[e^{sZ_t}; J_t = j]$. Then ([10] p. 41) $\boldsymbol{F}_t[s] = e^{t\boldsymbol{K}[s]}$ where

$$\boldsymbol{K}[s] \ = \boldsymbol{Q} + \{\kappa^{(j)}(s)\}_{\text{diag}} \tag{40}$$

and $\kappa^{(j)}(s)$ is the Lévy exponent in phase j. Many of the computations involving regime switching Lévy processes reduce to finding the eigenstructure of the matrix $\boldsymbol{K}[s]$. For example, Asmussen & Kella [13] solved the first passage time problem for reflected regime switching Brownian motion by introducing the (row) vector martingale

$$\mathrm{e}^{bY_t - at}\mathbf{1}_{J_t} - \mathrm{e}^{by}\mathbf{1}_{J_0} - b\int_0^t \mathrm{e}^{-au}\mathbf{1}_{J_u} dL_u - \int_0^t \mathrm{e}^{bY_u - au}\mathbf{1}_{J_u} du \ \ \boldsymbol{K}[b]$$

where $\mathbf{1}_i$ denotes a (row) vector with a 1 in the ith coordinate and 0's everywhere else and L represents the local time at 0. To use the vector martingale, one forms first scalar martingales obtained by choosing $b = \rho_j$ such that $\boldsymbol{K}[b]$ is singular and by multiplying the vector martingale by the right eigenvectors $\boldsymbol{h}^{(j)}$ of $\boldsymbol{K}[\rho_j]$, with the effect that the last term falls down, yielding the family of scalar martingales

$$M_t^{(j)} \ = \mathrm{e}^{-at + \rho_j Y_t}\boldsymbol{h}_{J_t}^{(j)} - \ \mathrm{e}^{\rho_j y}\boldsymbol{h}_0^{(j)} - b\int_0^t \mathrm{e}^{-as}\boldsymbol{h}_{J_s}^{(j)}\,\mathrm{d}L_s,$$

to which one may apply the optional stopping theorem.

4.2 First passage for regime switching reflected Lévy processes

Let now X be a regime-switching Lévy process with two regimes, where the regimes of X switch from 1 to 2 and vice versa at rates η_1 and η_2 respectively. We denote by $J \in \{1,2\}$ the corresponding Markov-process indicating the current regime of X. If $J_t = i \in \{1,2\}$, $X = X^i$ is of the form (2) with parameters μ_i, σ_i, $\lambda_i^{(\pm)}$, $\boldsymbol{T}_i^{(\pm)}$ and $\boldsymbol{\alpha}_i^{(\pm)}$. We assume that this representation is minimal. We study the first passage problem for $Y = \overline{X} - X$, X reflected at its supremum. In analogogy with the foregoing section, $M^{j(-)}$ will denote all states of the underlying phase processes of the jumps of Y^j causing the up-crossing of levels. Then we are interested in the joint moment generating function

$$v_k^{(i,j)}(y) = v_k^{(i,j)}(y,a,b) = \mathbb{E}_{y,i}[\exp(-a\tau + b(Y_\tau - k)\mathbf{1}_{\{J_\tau = j\}}]$$

of the crossing time

$$\tau = \inf\{t \geq 0 : Y_t \geq k\}$$

and the overshoot $Y_\tau - k$. Here $i,j \in \{1,2\}$, $a \geq 0$ and b such that $v_k^{(i,j)}$ is finite. $\mathbb{E}_{i,y}$ denotes the measure under which $\{Y_0 = y, J_0 = i\}$. By the Markov property, we find as before that the moment-generating function $v_k^{(i,j)}$ is given by

$$v_k^{(i,j)}(y) = \boldsymbol{\pi}^{(i,j)}\hat{\boldsymbol{f}}^{j(-)}[-b],$$

where $\hat{\boldsymbol{f}}^{j(-)}[-b]$ is the Laplace-transform of overshoots $Y_\tau^j - k$, with Y^j denoting Y being in regime $j \in \{1,2\}$, and where

$$\boldsymbol{\pi}^{(i,j)} = (\mathbb{E}_{i,y}[\mathrm{e}^{-a\tau}\mathbf{1}_{G_{j,j'}}], \qquad j' \in M^{j(-)}), \tag{41}$$

with $G_{j,j'} = \{J_\tau = j, \text{level } k \text{ crossed in phase } j'\}$. We embed now the regime-switching Lévy process X into a fluid process X' by levelling out positive jumps of X to sample path segments of X' with slope $+1$, and negative jumps of X to sample path segments of X' with slope -1. More precisely, the phase process $J' = (J, \widetilde{J})$ is defined as follows. The first component $J(t) = i \in \{1,2\}$, indicates that the regime-switching Lévy process X is at time t in regime i. The second component $\widetilde{J}$ takes value $\widetilde{J}(t) = j \in \{1,\ldots,m_i^{(+)}\}$ if, at time t, X' is in one of the segments with slope $+1$ (such that the state of the underlying phase process corresponding to the upward jump of X^i is j), and value $j \in \{-1,\ldots,-m_i^{(-)}\}$ if, at time t, X' is in one of the segments with slope -1 (such that the phase of the corresponding downward jump of X^i is j); when at time t the X'–process operates according to the Lévy exponent $s\mu_i + s^2\sigma_i^2/2$, we let $\widetilde{J}(t) = 0$. The resulting process is a particular case of a regime switching Brownian motion.

Let $\boldsymbol{K}_a[s]$ be the moment generating matrix of X' killed at rate a while $\widetilde{J}(t) = 0$ (note that then the crossing probabilities coincide with those of the original model). Then, from [10] p. 41, we find that $\boldsymbol{K}_a[s]$ is, in obvious block–

partitioned notation, given by

$$\boldsymbol{K}_a[s] = \left(\begin{array}{c|c} \boldsymbol{K}_a^{(1)}[s] & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{K}_a^{(2)}[s] \end{array}\right) + \left(\begin{array}{c|c} \widetilde{\boldsymbol{Q}}_{11} & \widetilde{\boldsymbol{Q}}_{12} \\ \hline \widetilde{\boldsymbol{Q}}_{21} & \widetilde{\boldsymbol{Q}}_{22} \end{array}\right) \tag{42}$$

where

$$\boldsymbol{K}_a^{(i)}[s] \;=\; \begin{pmatrix} -\lambda_i - a + s\mu_i + s^2\sigma_i^2/2 & \lambda_i^{(-)}\boldsymbol{\alpha}_i^{(-)} & \lambda_i^{(+)}\boldsymbol{\alpha}_i^{(+)} \\ \boldsymbol{t}_i^{(-)} & \boldsymbol{T}_i^{(-)} - s\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{t}_i^{(+)} & \boldsymbol{0} & \boldsymbol{T}_i^{(+)} + s\boldsymbol{I} \end{pmatrix} \tag{43}$$

and $\widetilde{\boldsymbol{Q}}_{ii}$ is the matrix of the size of $\boldsymbol{K}_a^{(i)}$ with $-\eta_i$ on position $(1,1)$ and zeros for the rest. and $\widetilde{\boldsymbol{Q}}_{ij}, i \neq j$ has a everywhere zeros except on $(1,1)$ where it has η_i as entry.

We determine now the eigenstructure of $\boldsymbol{K}_a[s]$. As before we see from (3) that under the model (2), κ_i, the Lévy exponent of X^i, is the ratio between two polynomials of degrees $p_i - \epsilon_i$ and p_i resp. where $p_i = m^{(+)} + m^{(-)} + \epsilon_i$ and $\epsilon_i = 2, 1, 0$ if $\sigma_i \neq 0$, $(\sigma_i = 0, \mu_i \neq 0)$ and $(\mu_i = \sigma_i = 0)$, respectively. Hence the equation

$$\eta_1\eta_2 = (\kappa_1(s) - a - \eta_1)(\kappa_2(s) - a - \eta_2) \tag{44}$$

has $p_1 + p_2$ roots which we denote by $\varrho_1, \ldots \varrho_{p_1+p_2}$. For each $r = 1, \ldots, p_1 + p_2$ define

$$\boldsymbol{h}^{(r)} = \begin{pmatrix} \gamma_r \boldsymbol{k}_1^{(r)} \\ -\boldsymbol{k}_2^{(r)} \end{pmatrix} \text{ where } \boldsymbol{k}_i^{(r)} = \begin{pmatrix} 1 \\ \big(\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(-)}\big)^{-1}\boldsymbol{t}_i^{(-)} \\ \big(-\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(+)}\big)^{-1}\boldsymbol{t}_i^{(+)} \end{pmatrix} \tag{45}$$

and $\gamma_r = (\kappa_2(\varrho_r) - a - \eta_2)/\eta_2$. By straightforward algebra we can check:

Lemma 3 *For $j = 1, \ldots, p_1 + p_2$, $\boldsymbol{K}_a[\varrho_j]\boldsymbol{h}^{(j)} = \boldsymbol{0}$.*

We adapt now the semi-Markov generalisation of the Kella-Whitt martingale introduced by Asmussen and Kella [13]. First, we introduce some more notation. By Y' we will denote the process X' reflected in its supremum, that is, $Y' = \{Y'_t, t \geq 0\}$ with

$$Y'_t = \sup_{0 \leq s \leq t} X'_s \vee Y'_0 - X'_t.$$

By $L' = \{L'_t, t \geq 0\}$ we will denote the supremum of X', $L'_t = \sup_{s \leq t} X'_s \vee Y'_0$. Finally, we introduce the time spent by Y' in *phase 0* (which is the time of the original regime switching Lévy process) up to time t by

$$T'_0(t) = \int_0^t \boldsymbol{1}_{\{\widetilde{J}(s)=0\}} \mathrm{d}s.$$

Let $\mathbb{P}_{(i,l),y}$ refer to the case $J_0 = (i,l), Y'_0 = y$ and $\tau' = \tau'_k = \inf\{t > 0 : J_0 = j, Y'_t = k\}$. It is immediate by a sample path comparison that $\tau = T'_0(\tau')$ and

$\pi_{j'}^{(i,j)} = \mathbb{E}_{(i,0),y}[\mathrm{e}^{-aT_0'(\tau')}\mathbf{1}_{\{J'_{\tau'}=(j,j')\}}]$ for $i,j \in \{1,2\}$ and $j' \in M^{j(-)}$. Finally, let

$$\delta_\ell^{(i,j)} = \mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT_0'(t)}\mathbf{1}_{\{J'_t=(j,\ell)\}}\mathrm{d}L'_t\right], \qquad j \geq 0.$$

By $\mathbf{1}_{J'_t} = \mathbf{1}_{(r,s)}$, we denote a row-vector of the length of $\boldsymbol{K}_a$ with all zeros but a one on position $(r-1)(m_1^{(+)}+m_1^{(-)}+1)+s+1$, which corresponds with phase s in regime r.

The theorem below identifies a vector martingale (46), a set of $p+1$ scalar martingales (47) and an "optional stopping system" (48).

Theorem 2

1. *The process*

$$\mathrm{e}^{-aT_0'(t)+bY_t'}\mathbf{1}_{J'_t} - \mathrm{e}^{bY_0'}\mathbf{1}_{J'_0} + b\int_0^t \mathrm{e}^{-aT_0'(u)}\mathbf{1}_{J'_u}dL'_u - \int_0^t \mathrm{e}^{-aT_0'(u)+bY_u'}\mathbf{1}_{J'_u}du\ \ \boldsymbol{K}_a[-b] \quad (46)$$

   is a mean zero (vector) $\mathbb{P}$–martingale.

2. *Let ϱ_r denote any root of the equation (44). Then*

$$M_t = \mathrm{e}^{-aT_0'(t)-\varrho_r Y_t'}h_{J'_t}^{(j)} - \mathrm{e}^{-\varrho_r y}h_{J'_0}^{(r)} - \varrho_r\int_0^t \mathrm{e}^{-aT_0'(s)}h_{J'_s}^{(j)}\,\mathrm{d}L'_s \qquad (47)$$

   are mean zero (scalar) martingales for each $j = 1,\ldots,p_1+p_2$.

3. *Let $i \in \{1,2\}$ and $y \in [0,k)$. If the eigenvalues of $\boldsymbol{T}_i^{(\pm)}$ have single geometric multiplicity and the roots ϱ_r are distinct, then the numbers*

$$\pi_0^{(i,j)},\ldots,\pi_{m_j^{(-)}}^{(i,j)} \quad \text{and} \quad \delta_0^{(i,j)},\ldots,\delta_{m_j^{(+)}}^{(i,j)} \qquad (j=1,2)$$

   are the unique solution of the $p = p_1+p_2$ linear equations

$$\begin{aligned}
\mathrm{e}^{-\varrho_1 y}h_{(i,0)}^{(1)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_1 k}h_{j,\ell}^{(1)} - \varrho_1\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h_{j,\ell}^{(1)},\\
\mathrm{e}^{-\varrho_2 y}h_{(i,0)}^{(2)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_2 k}h_{j,\ell}^{(2)} - \varrho_2\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h_{j,\ell}^{(2)},\\
&\vdots \\
\mathrm{e}^{-\varrho_p y}h_{(i,0)}^{(p)} &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_p k}h_{j,\ell}^{(p)} - \varrho_p\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_\ell^{(i,j)}h_{j,\ell}^{(p)}.
\end{aligned} \qquad (48)$$

where $h^{(r)}_{j,\ell}$ is the coordinate of $\boldsymbol{h}^{(r)}$ corresponding to regime j and phase ℓ.

The proof is provided in Section 5.5.

5 Proofs

5.1 Proof of Theorem 1

We start with a lemma which explores properties of v^*:

Lemma 4 *The function $v^* : [0,\infty) \to [1,\infty)$ is convex. If $v^*(0) > 1$, then there exists a unique $k^* \in (0,\infty]$ such that $v^* > \exp$ if $k^* = \infty$ and*

$$\begin{cases} \exp(x) < v^*(x) \le \exp(k^*) & \text{if } k^* < \infty \text{ and } 0 \le x < k^*, \\ \exp(x) = v^*(x) & \text{if } k^* < \infty \text{ and } k^* \le x. \end{cases}$$

If $v^*(0) = 1$, then $v^* = \exp$.

Proof For τ arbitrary it holds that

$$\mathbb{E}^{(1)}_y[\mathrm{e}^{-a\tau+Y_\tau}] = \mathbb{E}[\mathrm{e}^{-q\tau+\overline{X}_\tau\vee y}] = \mathbb{E}[\mathrm{e}^{\overline{X}_{\eta(q)}\vee y}\mathbf{1}_{\{\tau<\eta(q)\}}], \tag{49}$$

where $q = a + r$ and $\eta(q)$ is an independent exponential random variable with parameter q. Since $\kappa(1) = r$ and $q/(q-\kappa(1)) = q/a$ is equal to $\kappa^+_q(1) \times \kappa^-_q(1)$, it follows that the expectation on the right-hand side of the previous display is finite uniformly in τ. The assertions follow from the following two observations:

(1) $v^*(x) \ge \mathrm{e}^x$, which follows by choosing $\tau = 0$ in (11);

(2) $x \mapsto v^*(x)$ and $x \mapsto \mathrm{e}^{-x}v^*(x)$ are both convex. Moreover, they are non-decreasing and non-increasing respectively.

Observation (2) is shown as follows. For each fixed $\tau \in \mathcal{T}$ and ω the functions $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega) - x)$ and $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega))$ are both convex. Also they are non-increasing and non-decreasing respectively. Integration over ω and taking the supremum over τ preserve these properties. QED

Proof of Theorem 1 Let $f_t = \exp(-at + \sup_{0\le s\le t} X_s \vee m)$ denote the system of pay-off functions belonging to the problem (8). Note that f_t has no negative jumps and, by (49), $\{\mathrm{e}^{-r\tau}f_\tau : \tau \in \mathcal{T}\}$ is uniformly integrable with respect to $\mathbb{P}$. Under these conditions, it is straightforward to check that Theorem 2 in Shiryaev et al. [119] continues to hold. (Theorem 2 in [119] is stated in the setting of the standard complete Black-Scholes market, but the completeness plays no role in the proof.) This now implies that the optimal stopping time in (8) is given by

$$\begin{aligned}\tau^* &= \inf\{t \geq 0 : \operatorname*{ess\,sup}_{\tau\in\mathcal{T},\tau\geq t} \mathbb{E}[\mathrm{e}^{-r(\tau-t)} f_\tau|\mathcal{F}_t] \leq f_t\}\\ &= \inf\{t \geq 0 : \sup_{\tau\in\mathcal{T}} \mathbb{E}_{X_t,\overline{X}_t\vee m}[\mathrm{e}^{-r\tau} f_\tau] \leq \mathrm{e}^{at} f_t\}\\ &= \inf\{t \geq 0 : V^*(X_t, \overline{X}_t \vee m) = \mathrm{e}^{X_t} v^*(\overline{X}_t \vee m - X_t) \leq \mathrm{e}^{\overline{X}_t \vee m}\}\end{aligned}$$

where in the second line, we used the Markov property of $(X_t, \overline{X}_t \vee m)$ and $\mathbb{P}_{x,z}$, $z \geq m$, denotes the probability measure under which the process $(X_t, \overline{X}_t \vee m)$ starts in (x, z). The final line follows by using the $\mathbb{P}$-martingale $M = \{M_t\}_{t\geq 0}$ with $M_t = \exp(X_t - X_0 - rt)$ as equivalent change of measure. The final line of the previous display combined with Lemma 4 implies that the optimal stopping time is a crossing time τ_{k^*} of Y, where the optimal level k^* can be found by optimisation. Since $\tau_k \to \infty$ if k tends to infinity, we deduce from (49) that k^* is finite. QED

5.2 Proof of linear independence

Here we show that the vectors $\hat{\boldsymbol{f}}^{(-)}[\rho_i], i \in \mathcal{I}^{(-)}$ are linearly independent. We assume that the roots ρ_i with $i \in \mathcal{I}^{(-)}$ are distinct and that $\boldsymbol{T}^{(-)}$ has no eigenvalues with multiple geometric multiplicity. We distinguish between the cases ($\sigma = 0$ and $\mu \geq 0$) and ($\sigma \neq 0$ or $\mu < 0$).

- $\sigma = 0$ and $\mu \geq 0$. Writing $\boldsymbol{C}$ for the matrix of (generalised) eigenvectors of $\boldsymbol{T}^{(-)}$ and $\boldsymbol{J} = \boldsymbol{C}^{-1}\boldsymbol{T}^{(-)}\boldsymbol{C}$ for its Jordan normal form, we have to show the linear independence of the vectors $(\rho_i\boldsymbol{I} - \boldsymbol{J})^{-1}\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$ for $i \in \mathcal{I}^{(-)}$. We claim that this linear independence is equivalent with invertibility of the matrix $\boldsymbol{M}$ with rows $\sum_{k=1}^{j-1} m^{(-,k)} + 1$ till $\sum_{k=1}^{j} m^{(-,k)}$ given by

$$\left(\tfrac{\rho_i}{(\rho_i-\eta^{(-,j)})^\ell}, i \in \mathcal{I}^{(-)}\right), \qquad \ell = 1, \ldots, m^{(-,j)} \tag{50}$$

where $\eta^{(-,j)}$ are the eigenvalues of $\boldsymbol{T}^{(-)}$ with multiplicities $m^{(-,j)}$. The claim follows by linear algebra. Indeed, denoting by $\boldsymbol{v}^{(j,m)}$ the column of $\boldsymbol{C}$ that lies in the kernel of $(\boldsymbol{J} - \eta^{(-,j)}\boldsymbol{I})^m$, but not in the kernel of $(\boldsymbol{J} - \eta^{(-,j)}\boldsymbol{I})^{m-1}$, we see that $\mathbf{1}$ is not in the span of $\{\boldsymbol{v}^{(j,m)}\}_{j,m<m^{(-,j)}}$ (for suppose this were the case, then applying $\prod_j(\boldsymbol{J} - \eta^{(-,j)}\boldsymbol{I})^{m_j}$, where $m_j = m^{(+,j)} - 1$, to the vector $\mathbf{1}$ would lead to a contradiction). This implies that the vector $\boldsymbol{C}^{-1}\mathbf{1}$ (and hence $\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$) is non-zero in all coordinates corresponding to the eigenvectors $\boldsymbol{v}^{(j,m^{(+,j)})}$. Recalling the form of the inverse $(\lambda\boldsymbol{I} - \boldsymbol{J})^{-1}$ for the Jordan form $\boldsymbol{J}$, it follows by writing out the equations that the vectors $(\rho_i\boldsymbol{I} - \boldsymbol{J})^{-1}\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$ are linearly independent if and only if $\boldsymbol{M}$ is one-to-one and the claim follows.

Next we show that $\boldsymbol{M}$ is invertible. Consider now the system $\boldsymbol{M}\boldsymbol{c} = -\boldsymbol{v}$, where $\boldsymbol{v}$ is the vector with $\kappa_a^-(\infty)$ in coordinates $1, m^{(-,1)} + 1, m^{(-,2)} + 1, \ldots$ and the rest zeros. Recall we restricted ourselves to the cases where the roots of $\kappa(s) = a$ with negative real part are distinct and not in the spectrum of

$\boldsymbol{T}^{(-)}$. Then we can check that any solution $\boldsymbol{c}$ of this system gives rise to a partial fraction decomposition of $\kappa_a^-(s) - \kappa_a^-(\infty)$. Indeed, recall that we can write $\kappa_a^-(s) = p(s)/q(s)$ for polynomials p, q of degree $m^{(-)}$. Taking $\boldsymbol{c}$ to be a solution of above system we have that

$$p(s) = \left(\sum_{i \in \mathcal{I}^{(-)}} c_i \rho_i / (\rho_i - s) + \kappa_a^-(\infty) \right) q(s) \tag{51}$$

since both sides of the equation are polynomials of the same degree, any root of the left-hand side is also a root of the right-hand side with the same multiplicity and $(p/q)(\infty) = \kappa_a^+(-\infty) > 0$. By unicity of this partial fraction decomposition, we deduce that the square matrix $\boldsymbol{M}$ is invertible.

• $\sigma \neq 0$ or $\mu > 0$. Similarly as above, one verifies that the linear independence of the $m^{(-)}+1$ vectors $\hat{\boldsymbol{f}}^{(-)}[\rho_i]$ of length $m^{(-)}+1$ for $i \in \mathcal{I}^{(-)}$ is equivalent to the invertibility of the matrix $\widetilde{\boldsymbol{M}}$ with the final $\mathcal{I}^{(-)} - 1$ rows given by (50) and the first row of ones. Any solution of $\widetilde{\boldsymbol{M}}\widetilde{c} = \mathbf{1}_1$, where $\mathbf{1}_1$ is a vector of zeros with as first coordinate a 1, gives rise to a partial fraction decomposition of $\kappa_a^-(s)$. To check this, we write as above $\kappa_a^-(s) = p(s)/q(s)$, where p, q are polynomials of degree $m^{(-)}$ and $m^{(-)}+1$. Since the left- and right-hand side of (51) are polynomials with the same roots and the same degree and $\kappa_a^-(0) = 1$, equation (51) holds true. Following the same line of reasoning as above, we conclude that $\widetilde{\boldsymbol{M}}$ is invertible.

Noting that $\kappa_a^+(0)\kappa_a^-(0) = 1$ and $\lim_{s\to\infty} sa/(a - \kappa(s)) = 0$ if $\sigma \neq 0$, one can adapt the previous scheme to partial fraction decomposition of $a/(a - \kappa(s))$ to prove that the vectors $\widetilde{\boldsymbol{k}}^{(j)} = (\widetilde{\boldsymbol{k}}_1{}^{(j)}, \widetilde{\boldsymbol{k}}_2{}^{(j)})$ given in the remark after Proposition 3 are linearly independent.

5.3 Spectral proof of Proposition 4

Proof of Proposition 4 By the linear independence proved in Section 5.2, the system (33)–(34) has a unique solution. Define the function $\widetilde{v}$ on $[0, \infty)$ by the right-hand side of (32) for $y \leq k$ and by $\exp(b(y - k))$ for $y > k$. From the explicit form of $\widetilde{v}$ we straightforwardly check that $\widetilde{v}'(0^+) = 0$ (if $\sigma \neq 0$ or $\mu > 0$) and $\Gamma' \widetilde{v}(x) = a\widetilde{v}(x)$ for $x \in (0, k)$ where Γ' acts on $f \in C^2(0, k)$ as

$$\begin{aligned} \Gamma' f(x) = \frac{\sigma^2}{2} f''(x) - \mu\, f'(x) + \lambda^{(-)} \int_0^\infty \big(f(x+z) - f(x)\big) F^{(-)}(\mathrm{d}z) \\ + \lambda^{(+)} \int_0^\infty \big(f((x-z)^+) - f(x)\big) F^{(+)}(\mathrm{d}z), \end{aligned} \tag{52}$$

for $x \in (0, k)$. Applying then Itô's lemma to $\exp(-at)\widetilde{v}(Y_t)$ on the set $\{t \leq \tau_k\}$ and using the two foregoing properties of $\widetilde{v}$, it follows that $\{\exp(-a(t \wedge \tau_k)\widetilde{v}(Y_{t\wedge\tau_k}), t \geq 0\}$ is a martingale. Thus, by bounded convergence combined with the fact that $\widetilde{v}(y) = \exp(b(y - k))$ for $y > k$ and $\widetilde{v}(k^-) = 1$ if $\sigma \neq 0$ or

$\mu < 0$, we deduce

$$\begin{aligned}\widetilde{v}(y) &= \lim_{t\to\infty} \mathbb{E}_y[\mathrm{e}^{-a(\tau_k\wedge t)}\widetilde{v}(Y_{\tau_k\wedge t})]\\ &= \mathbb{E}_y[\mathrm{e}^{-a\tau_k+b(Y_{\tau_k}-k)}] = v_k(y),\end{aligned}$$

which completes the proof. QED

5.4 Proof of Lemma 2

We work under $\mathbb{P}^{(1)}$, but omit the subscript (1) to lighten the notation. (i)-(ii) Recall v_k satisfies $\Gamma' v_k = av_k$, where Γ' is given in (52). In the case $\mu \geq \sigma = 0$, one finds that $v_k(0) = \frac{\lambda^{(-)}}{\lambda^{(-)}+a}\int_0^\infty v_k(x)F^{(-)}(\mathrm{d}x)$. Taking then the limit of $k\downarrow 0$ and using that $v_k(x) = \mathrm{e}^{x-k}$ for $x \geq k$, it follows that $\lim_{k\downarrow 0} v_k(0) = \iota$. In the case $\mu < \sigma = 0$, we take the limit of $x \uparrow k$ to find that $\Gamma' v_k(k^-) = av_k(k^-)$ which reads as $-\mu v_k'(k^-) + \lambda^{(+)}\int_0^i v_k((k-x)^+)F^{(+)}(\mathrm{d}x) = a + \lambda^{(+)}$. Letting then k tend to zero shows that $v_k'(k^-) \to -a/\mu$.
(iii) Denote by $\tau' = \tau'(\epsilon) = \inf\{t \geq 0 : Y_t < k - \epsilon\}$, write $\tau = \tau(k)$ and let ς be the first jump time of Y. Writing $\kappa(\epsilon) = k - \epsilon$, we find that

$$\begin{aligned}\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\varsigma\}}] - 1 &= v_k(k(\epsilon))\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}]\\ &\quad + \mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\tau'\wedge\varsigma\}}] - 1 \qquad (53)\\ &= (v_k(k(\epsilon)) - 1)\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}]\\ &\quad + \mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a(\tau\wedge\tau')}\mathbf{1}_{\{\tau\wedge\tau'<\varsigma\}}] - 1, \qquad (54)\end{aligned}$$

where we used the Markov property and that $Y_\tau = k$ on $\{\tau < \varsigma\}$. Since the jump component is independent of the rest of the process and has total jump rate λ, we find, invoking the theory developed on level-passage, that

$$w_k(x) := \mathbb{E}_x[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\varsigma\}}] = \tilde{s}(x)/\tilde{s}(k),$$

where $\tilde{s}(x) = \tilde{\rho}_2\mathrm{e}^{-\tilde{\rho}_1 x} - \tilde{\rho}_1\mathrm{e}^{-\tilde{\rho}_2 x}$ with $\tilde{\rho}_1 < 0 < \tilde{\rho}_2$ the roots of $\frac{\sigma^2}{2}s^2 + \mu s = a + \lambda$ (where the parameters are the ones under $\mathbb{P}^{(1)}$). Recalling that $\kappa_1(-1) = -r$ we find that $\frac{\sigma^2}{2} - m - \lambda \leq -r$ and thus $\tilde{\rho}_1 < -1$. It is then a matter of algebra to verify that the function $k \mapsto w_k'(k^-)$ has positive derivative and is thus increasing and converges to 0 and $-\tilde{\rho}_1 > 1$ as $k \downarrow 0$ and $k \to \infty$ respectively. Similarly, we find that

$$\begin{aligned}t_k(k - \tfrac{\epsilon}{2}) &:= \mathbb{E}_{k-\frac{\epsilon}{2}}[\mathrm{e}^{-a(\tau\wedge\tau')}\mathbf{1}_{\{\tau\wedge\tau'<\varsigma\}}]\\ &= \cosh(\tfrac{\tilde{\rho}_1}{2}\epsilon) + \cosh(\tfrac{\tilde{\rho}_2}{2}\epsilon))/(1 + \cosh(\tfrac{(\tilde{\rho}_2-\tilde{\rho}_1)}{2}\epsilon)).\end{aligned}$$

Note that $(t_k(k - \frac{\epsilon}{2}) - 1)/\epsilon \to 0$ as $\epsilon \downarrow 0$ and $\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}] \to 1/2$ as $\epsilon \downarrow 0$.

Dividing then the left- and right-hand side of (54) by $\epsilon/2$ and letting ϵ go to zero we find that $w_k'(k^-) = v_k(k^-)$, which establishes (iii). QED

5.5 Proof of Theorem 2

Let the process $Z = \{Z_t, t \geq 0\}$ be given by

$$Z_t = Y'_t - \frac{a}{b}T'_0(t) = -X'_t + L'_t - \frac{a}{b}T'_0(t).$$

Since Z has continuous sample paths, applying Theorem 2.1 d) of [13]), we find that – without restrictions on b, $M = \{M_t, t \geq 0\}$ with

$$\begin{aligned} M_t = & \int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_0[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}L'_s \\ & - a\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s = 0)\mathrm{d}s \\ = & \int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_a[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{-aT'_0(s)}\mathbf{1}_{J_s}\mathrm{d}L'_s, \end{aligned}$$

is a zero mean $\mathbb{P}_{0,y}$ (row) martingale. We used that L'_t can increase only if X'_t is equal to its current supremum or $Y'_t = 0$. Moreover $\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s = 0)\mathrm{d}s = \int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ a diagonal matrix with a 1 on positions 1 and $p_1 + 1$ and the rest zeros. Choosing $-b$ to be a root of $\kappa(s) = a$ and multiplying by the zero-eigenvectors of $K_a[-b]$ (using Lemma 3) completes the proof of 1 and 2.

Since $M_{t\wedge\tau'}$ is bounded for all t, for each j, can we apply optional stopping theorem to M at $\tau' = \tau'_k$, i.e. $\mathbb{E}_{(i,0),y}[M_{\tau'}] = \mathbb{E}_{(i,0),y}[M_0] = 0$. Since $\sup_{s\leq t} X'_s$ can increase only when $Y'_t = 0$ and $J_t \geq 0$, we find

$$\begin{aligned} & \mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(s)}h^{(r)}_{J_s}\mathrm{d}L'_s\right] \\ & \qquad = \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}} h^{(r)}_{j,\ell}\mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(s)}I(J_s = (j,\ell))\,dL'_s\right] \end{aligned}$$

which is equal to $\sum_{j=1}^2\sum_{\ell=0}^{m_j^{(+)}} \delta_k^{(i,j)}h^{(r)}_{j,\ell}$. Similarly, we must have $J_{\tau'} \leq 0$ so that

$$\begin{aligned} \mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r Z_{\tau'}}h^{(r)}_{J_{\tau'}}\right] & = \mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r k - aT'_0(\tau')}h^{(r)}_{J_{\tau'}}\right] \\ & = \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}} \pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_r k}h^{(r)}_{j,\ell}. \end{aligned}$$

Thus the rth equation is the same as $\mathbb{E}_{0,y}[M_{\tau'}] = 0$. If the roots ϱ_r are different, the equations are linearly independent, which can be proved as in Section 5.2. QED

Appendix: Exponential tilting of X

Consider the probability measure $\mathbb{P}^{(u)}$ given by $\mathbb{P}^{(u)}(A) = \mathbb{E}[e^{uX_t - t\kappa(u)}; A]$, $A \in \mathcal{F}_t$. It is standard (e.g. [10] p. 38) that X is again a Lévy process w.r.t. $\mathbb{P}_s$, with Lévy exponent given by $\kappa_u(s) = \kappa(u+s) - \kappa(u)$ corresponding to the following change of parameters:

$\mathbb{P}$	μ	σ^2	$\lambda^{(+)}$	$F^{(+)}$	$\lambda^{(-)}$	$F^{(-)}$
$\mathbb{P}^{(u)}$	$\mu + u\sigma^2$	σ^2	$\lambda^{(+)}\hat{F}^{(+)}[-u]$	$F_u^{(+)}$	$\lambda^{(-)}\hat{F}^{(-)}[u]$	$F_{-u}^{(-)}$

where $F_u^{(+)}(dx) = e^{ux}F^{(+)}(dx)/\hat{F}^{(+)}[-u]$, $F_{-u}^{(-)}(dx) = e^{-ux}F^{(-)}(dx)/\hat{F}^{(-)}[u]$. These distributions are again phase–type, as follows by the following result from [5]:

Lemma 5 *Let F be phase–type with parameters $(\boldsymbol{\alpha}, \boldsymbol{T})$ and let*

$$F_u(dx) = e^{ux}F(dx)/\hat{F}[-u].$$

Define $\boldsymbol{k} = (-u\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}$ and let $\boldsymbol{\Delta}$ be the diagonal matrix with the k_i on the diagonal. Then F_u is phase–type with parameters

$$\boldsymbol{\alpha}_u = \boldsymbol{\alpha}\boldsymbol{\Delta}/\hat{F}^{(+)}[-u], \quad \boldsymbol{T}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{T}\boldsymbol{\Delta} + u\boldsymbol{I}.$$

Further, $\boldsymbol{t}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{t}$.

Chapter III

Exit problems for spectrally negative Lévy processes

We consider spectrally negative Lévy process and determine the joint Laplace transform of the exit time and exit position from an interval containing the origin of the process reflected in its supremum. In the literature of fluid models, this stopping time can be identified as *the time to buffer-overflow.* The Laplace transform is determined in terms of the scale functions that appear in the two sided exit problem of the given Lévy process. The obtained results together with existing results on two sided exit problems are applied to solving optimal stopping problems associated with the pricing of American and Russian options and their Canadised versions.

1 Introduction

In this paper we consider the class of spectrally negative Lévy processes. These are real valued random processes with stationary independent increments which have no positive jumps. Amongst others Emery [54], Suprun [121], Bingham [25] and Bertoin [24] have all considered fluctuation theory for this class of processes. Such processes are often considered in the context of the theories of dams, queues, insurance risk and continuous branching processes; see for example [31, 25, 26, 110]. Following the exposition on two sided exit problems in Bertoin [24] we study first exit from an interval containing the origin for spectrally negative Lévy processes reflected in their supremum (equivalently spectrally positive Lévy processes reflected in their infimum). In particular we derive the joint Laplace transform of the time to first exit and the overshoot. The aforementioned stopping time can be identified in the literature of fluid models as *the time to buffer overflow* (see for example [4, 70]). Together with existing results on exit problems we apply our results to certain optimal stopping problems that are now classically associated with mathematical finance.

In sections 2 and 3 we introduce notation and discuss and develop existing results concerning exit problems of spectrally negative Lévy processes. In section 4 an expression is derived for the joint Laplace transform of the exit time and exit position of the reflected process from an interval containing the origin. This Laplace transform can be written in terms of scale functions that already appear in the solution to the two sided exit problem. In Section 5 we outline two classes of optimal stopping problem which are associated with the pricing of American and Russian options. Sections 6 and 7 are devoted to solving these optimal stopping problems in terms of scale functions that appear in the afore mentioned exit problems. In Section 8 we consider a modification of these optimal stopping problems known as Canadisation (corresponding to the case that the expiry dates of option contracts are randomized with an independent exponential distribution) and show that explicit solutions are also available in terms of scale functions. Finally we conclude the paper with some explicit examples of the optimal stopping problems under consideration.

2 Spectrally negative Lévy processes

Let $X = \{X_t, t \geq 0\}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. Restricting ourselves to spectrally negative Lévy processes, the process X may be represented as

$$X_t = \mu t + \sigma W_t + J_t^{(-)}, \tag{1}$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion and $J^{(-)} = \{J_t^{(-)}, t \geq 0\}$ is a non-Gaussian spectrally negative Lévy process. Both processes are independent. We exclude the case that X has monotone paths.

The jumps of $J^{(-)}$ are all nonpositive and hence the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$. A standard property of Lévy processes, following from the independence and stationarity of their increments, is that, when the moment generating function of the process at time t exists, it satisfies

$$\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\ \psi(\theta)} \tag{2}$$

for some function $\psi(\theta)$, the cumulant, which is well defined at least on the non-negative complex half plane and will be referred to as the Lévy exponent of X. It can be checked that this function is strictly convex and tends to infinity as θ tends to infinity, see Bertoin [23, p. 188].

We restrict ourselves to the Lévy processes which have unbounded variation or have bounded variation and a Lévy measure which is absolutely continuous with respect to the Lebesgue measure

$$\Lambda(\mathrm{d}x) \ll \mathrm{d}x. \tag{AC}$$

We conclude this section by introducing for any Lévy process having $X_0 = 0$ the family of martingales

$$\exp\left(cX_t - \psi(c)t\right),$$

defined for any c for which $\psi(c) = \log \mathbb{E}[\exp cX_1]$ is finite, and further the corresponding family of measures $\{\mathbb{P}^c\}$ with Radon-Nikodym derivatives:

$$\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(cX_t - \psi(c)t\right). \tag{3}$$

For all such c (including $c = 0$) the measure $\mathbb{P}^c_x$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$.

Remark 1 *Under the measure $\mathbb{P}^c$ the characteristics of the process X have changed. How they have changed can be found out by looking at the cumulant of X under $\mathbb{P}^c$:*

$$\begin{aligned} \psi_c(\theta) &:= \log\left(\mathbb{E}^c[\exp(\theta X_1)]\right) \\ &= \log\left(\mathbb{E}[\exp\left((\theta + c)X_1 - \psi(c)\right)]\right) \\ &= \psi(\theta + c) - \psi(c), \qquad \theta \geq \min(-c, 0). \end{aligned} \tag{4}$$

3 Exit problems for Lévy processes

3.1 Scale functions

Bertoin [24] studies two sided exit problems of spectrally negative Lévy processes in terms of a class of functions known as q-scale functions. Here we give a slightly modified definition of these objects (Definition 2).

Definition 1 *Let $q \geq 0$ and then define $\Phi_c(q)$ as the largest root of $\psi_c(\theta) = q$.*

Definition 2 *For $q \geq 0$, the q-scale function $W^{(q)} : (-\infty, \infty) \to [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform*

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta \geq \Phi(q)$$

and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_c^{(q)}(x)$ to mean the q-scale function as defined above for $(X, \mathbb{P}^c)$.

It is known that the q-scale function is increasing on $(0, \infty)$. Furthermore, if X has unbounded variation or if X has bounded variation and satisfies (AC), the restricted function $W_v^{(q)}|_{(0,\infty)}$ is continuously differentiable. See Lambert [86] and Bertoin [24]. For every $x \geq 0$, we can extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane by the identity

$$W_v^{(q)}(x) = \sum_{k \geq 0} q^k W_v^{\star(k+1)}(x) \tag{5}$$

where $W_v^{\star k}$ denotes the k-th convolution power of $W_v = W_v^{(0)}$. The convergence of this series is plain from the inequality

$$W_v^{\star k+1}(x) \leq x^k W_v(x)^{k+1}/k!, \qquad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of W_v.

Remark 2 *For each $q \geq 0$, a spectrally negative Lévy process X has an absolutely continuous potential measure $U^q(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t$. Its density, say u^q, is related to the q-scale function $W^{(q)}$. Indeed, in Bingham [25] it is shown that there exists a version of the potential density u^q such that*

$$W^{(q)}(x) = u^q(-x) + u^q(x) = u^q(-x) + \Phi'(q)\exp(-\Phi(q)x)$$

where $\Phi(q) = \Phi_0(q)$.

Remark 3 *By Corollary VII.1.5 in Bertoin [23] $\lim_{x\downarrow 0} W_v(x) = 0$ if and only if X has unbounded variation. By the expansion (5) it also follows that, under the same condition, $\lim_{x\downarrow 0} W_v^{(q)}(x) = 0$.*

Remark 4 *We have the following relationship between scale functions*

$$W^{(u)}(x) = \mathrm{e}^{vx} W_v^{(u-\psi(v))}(x)$$

for v such that $\psi(v) < \infty$ and $u \geq \psi(v)$. To see this, simply take Laplace transforms of both sides. By analytical entension, we see that the identity remains valid for all $u \in \mathbb{C}$.

Equally important as far as the following discussion is concerned is the function $Z^{(q)}$ which is defined as follows.

Definition 3 *For $q \geq 0$ we define $Z^{(q)} : \mathbb{R} \to [1, \infty)$ by*

$$Z^{(q)}(x) = 1 + q\int_{-\infty}^{x} W^{(q)}(z)\mathrm{d}z. \tag{6}$$

Keeping with our earlier convention, we shall use $Z_c^{(q)}(x)$ in the obvious way. Just like $W^{(q)}$,the function $Z^{(q)}$ may be characterised by its Laplace transform and continuity on $(0, \infty)$. Indeed, we can check that

$$\int_0^\infty \mathrm{e}^{-\theta x} Z^{(q)}(x)\mathrm{d}x = \psi(\theta)/\theta(\psi(\theta) - q), \qquad \theta \geq \Phi(q).$$

Note that when $q \geq 0$ this function inherits some properties from $W^{(q)}(x)$. Specifically it is strictly increasing, is equal to the constant 1 for $x \leq 0$ and $Z^{(q)}|_{(0,\infty)} \in C^2(0, \infty)$. When $q = 0$ then $Z^{(0)}(x) = Z(x) = 1$. Also, by working with the analytic extention of $q \mapsto W_v^{(q)}(x)$ we can define $q \mapsto Z_v^{(q)}(x)$ for all $q \in \mathbb{C}$.

We state the following result for the limit of $Z^{(q)}(x)/W^{(q)}(x)$ as x tends to infinity. For the formulation of this result and in the sequel, we shall understand $0/\Phi(0)$ to mean $\lim_{\theta\downarrow 0} \theta/\Phi(\theta)$.

Lemma 1 *For $q \geq 0$, $\lim_{x\to\infty} Z^{(q)}(x)/W^{(q)}(x) = q/\Phi(q)$.*

Proof First suppose $q \geq 0$. The fact that $\theta \mapsto \psi(\theta)$ is increasing for $\theta \geq \Phi(0)$ in conjunction with equation (4) implies that $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) \geq 0$. Recalling that $1/\psi_{\Phi(q)}$ is the Laplace transform of $W_{\Phi(q)}$, we now deduce from a Tauberian theorem (e.g. [23, p. 10]) that

$$0 < W_{\Phi(q)}(\infty) := \lim_{x\to\infty} W_{\Phi(q)}(x) = 1/\psi'_{\Phi(q)}(0) < \infty. \tag{7}$$

Recall from Remark 4 that $W^{(q)}(x)$ is equal to $\exp(\Phi(q)x)$ times $W_{\Phi(q)}(x)$. By partial integration, we then find for $x \geq 0$

$$Z^{(q)}(x) = 1 + q(W^{(q)}(x) - W^{(q)}(0^+))/\Phi(q) - q\int_0^x \mathrm{e}^{\Phi(q)y} W'_{\Phi(q)}(y)\mathrm{d}y/\Phi(q),$$

where $W^{(q)}(0^+) := \lim_{x\downarrow 0} W^{(q)}(x)$. Then equation (7) in conjunction with dominated convergence implies that the integral $\int_0^x \mathrm{e}^{\Phi(q)(y-x)} W'_{\Phi(q)}(y)\mathrm{d}y/W_{\Phi(q)}(x)$ converges to zero as x tends to ∞; hence $Z^{(q)}(x)/W^{(q)}(x)$ converges to $q/\Phi(q)$.

Consider now the case $q = 0$. We know from e.g. Bertoin [23] that $\Phi(0) \geq 0$ if and only if X drifts to $-\infty$. Recalling that by Remark 4 one has $W(x) = \exp(\Phi(0)x)W_{\Phi(0)}(x)$, we see that $1/W(x)$ converges to zero for x tending to infinity if X drifts to $-\infty$. If X does not drift to $-\infty$, we find by the same Tauberian theorem mentioned in the previous paragraph that $W(x)^{-1} \sim x\psi(x^{-1})$ as $x \to \infty$. We finish the proof by noting that $\psi'(0^+) = 1/\Phi'(0^+)$, since $\psi(\Phi(q)) = q$. □

3.2 Exit from a finite interval

The following Proposition gives a complete account of the two sided exit problem for the class of spectrally negative Lévy processes we are interested in. Before stating the result, we first introduce the following passage times.

Definition 4 *We denote the passage times above and below k for X by*

$$T_k^- = \inf\{t \geq 0 : X_t \leq k\} \text{ and } T_k^+ = \inf\{t \geq 0 : X_t \geq k\}. \tag{8}$$

Proposition 1 *Let $q \geq 0$. The Laplace transform of the two-sided exit time $T_a^- \wedge T_b^+$ on the part of the probability space where X, starting in $x \in (a,b)$, exits the interval (a,b) above and below are respectively given by*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_b^+} I_{(T_b^+ < T_a^-)}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}; \tag{9}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_a^-} I_{(T_b^+ \geq T_a^-)}\right] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}. \tag{10}$$

Proof This result can be extracted directly out of existing literature. See for example Bertoin [23, Thm. VII.8] for a proof of (9). Combining this with

Bertoin [24, Cor. 1], we find equation (10). Note, in Bertoin [24] there is a small typographic mistake so that in equation (10) the function $\int_0^{x-a} W^{(q)}(y)\mathrm{d}y$ is used instead of $Z^{(q)}(x-a)-1$. □

Remark 5 *The strong Markov property, in conjunction with equation (9), is enough to prove that*

$$\mathrm{e}^{-q(T_b^+ \wedge T_a^- \wedge t)} W^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) \tag{11}$$

is a martingale. To see this let $\tau = T_b^+ \wedge T_a^-$ *and note that* $W^{(q)}(X_\tau - a)/W^{(q)}(b-a)$ *is another way of writing the indicator of* $\{T_b^+ < T_a^-\}$*. Thus, by (9)*

$$\begin{aligned} &W^{(q)}(x-a) \\ &\quad = \mathbb{E}_x \left[\mathrm{e}^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &\quad = \mathbb{E}_x \left[\mathbb{E}_x \left[\mathrm{e}^{-q\tau} W^{(q)}(X_\tau - a) | \mathcal{F}_t \right] \right] \\ &\quad = \mathbb{E}_x \left[I_{(t \le \tau)} \mathrm{e}^{-qt} \mathbb{E}_{X_t} \left[\mathrm{e}^{-q\tau} W^{(q)}(X_\tau - a) \right] + I_{(t \ge \tau)} \mathrm{e}^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &\quad = \mathbb{E}_x \left[I_{(t \le \tau)} \mathrm{e}^{-qt} W^{(q)}(X_t - a) + I_{(t \ge \tau)} \mathrm{e}^{-q\tau} W^{(q)}(X_\tau - a) \right] \\ &\quad = \mathbb{E}_x \left[\mathrm{e}^{-q(\tau \wedge t)} W^{(q)}(X_{\tau \wedge t} - a) \right]. \end{aligned}$$

Now that this constant expectation has been established, the martingale property follows by a similar manipulaton of the expression (11). Similarly, this technique can also be employed to prove that

$$\mathrm{e}^{-q(T_b^+ \wedge T_a^- \wedge t)} \left(Z^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) - \frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)} W^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a) \right)$$

and hence (by linearity) $\mathrm{e}^{-q(T_b^+ \wedge T_a^- \wedge t)} Z^{(q)}(X_{T_b^+ \wedge T_a^- \wedge t} - a)$ *is a martingale.*

3.3 Exit from a positive half-line

The purpose of this subsection is to evaluate the joint moment-generating function of the time X exits $[k, \infty)$ and its position at that time. The result is not new and a proof of a variant of our proposition below is due to Emery [54].

Proposition 2 *For* $u \ge 0$ *and* v *with* $\psi(v) < \infty$ *the joint Laplace transform of* T_k^- *and* $X_{T_k^-}$ *is given by*

$$\begin{aligned} &\mathbb{E}_x[\exp\{-uT_k^- + vX_{T_k^-}\} I_{(T_k^- < \infty)}] \\ &\qquad = \mathrm{e}^{vx} \left(Z_v^{(p)}(x-k) - W_v^{(p)}(x-k) p/\Phi_v(p) \right), \end{aligned} \tag{12}$$

where $p = u - \psi(v)$ *and* $x, k \in \mathbb{R}$ *and* $0/\Phi_0(0)$ *is understood in the limiting sense.*

Emery's proof relies on Wiener-Hopf factorization in combination with complex-analytic arguments. Here we present a probabilistic proof, expressing the joint Laplace transform in the previously introduced p-scale function and its anti-derivative.

Remark 6 *Note the formula in Proposition 2 is stated in terms of the functions* $W_v^{(p)}, Z_v^{(p)}$*. However, we can reformulate the formula in Proposition 2 and the forthcoming Theorem 1 entirely in terms of the scale function* $W^{(u)}$ *and the exponential function* $\exp(vx)$ *by using Remark 4. Furthermore* $\Phi_v(u-\psi(v)) = \Phi(u) - v$ *for* $u \geq 0$ *and* v *such that* $\psi(v) < \infty$*. Indeed, Remark 1 implies that* $\psi_v(\Phi(u)-v) = u - \psi(v)$ *and hence* $\Phi_v(u-\psi(v))$ *is well defined. Further, for* $u \geq 0$*, using Remark 1 again we have*

$$u - \psi(v) = \psi_v(\Phi_v(u-\psi(v))) = \psi(\Phi_v(u-\psi(v)) + v) - \psi(v);$$

for $u = 0$ *the identity follows by continuity. Using these facts, it can be checked that the Laplace transform* $f_{u,v}(\theta)$ *of* $\mathbb{E}_x[\exp\{-uT_0^- + vX_{T_0^-}\}]$ *is given by*

$$f_{u,v}(\theta) = (\psi(\theta) - u)^{-1}\left(\frac{\psi(\theta)-\psi(v)}{\theta - v} - \frac{u - \psi(v)}{\Phi(u) - v}\right),$$

which agrees with Bingham [25, Thm. 6.5].

Proof of Proposition 2 We start with checking that for $u > 0$ and $\psi(v) < \infty$ the left-hand side of (12) is finite. Indeed, if $v < 0$ and $\psi(v) < \infty$, we find by the Compensation Formula applied to the Poisson point process of jumps $(\Delta X_t, t \geq 0)$ of X that $\mathbb{E}_x[\exp(-uT_k^- + vX_{T_k^-})I_{(T_k^- < \infty)}]$ is bounded above by

$$\begin{aligned} &\mathrm{e}^{v(k-M)} + \mathrm{e}^{vk}\mathbb{E}_x\left[\sum_{t \geq 0} \mathrm{e}^{-ut + v\Delta X_t} I_{(\Delta X_t < -M)}\right] \\ &\qquad \leq \mathrm{e}^{v(k-M)} + \mathrm{e}^{vk}\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-ut}\mathrm{d}t \int_{-\infty}^{-M} \mathrm{e}^{vz}\Lambda(\mathrm{d}z)\right], \quad (13) \end{aligned}$$

where Λ is the Lévy measure of X and $M > 1$ is some fixed constant. Since v is chosen such that $\psi(v) < \infty$, the expression in (13) is finite (by the Lévy-Khintchine formula for ψ). Fix any $u_0 > 0$. Then, for all u with $|u - u_0| < u_0$ Fubini's theorem implies that the left-hand side of (12) is equal to the series $\sum_n d_n(u_0 - u)^n$ with positive coefficients

$$d_n = \mathbb{E}_x[(T_k^-)^n \exp\{-u_0 T_k^- + vX_{T_k^-}\} I_{(T_k^- < \infty)}]/n! \geq 0.$$

Thus, since this series absolutely converges for $|u - u_0| < u_0$, the left-hand side of (12) is analyic in $u > 0$.

Let now u satisfy $u \geq \psi(v) \vee 0$. Then by a change of measure and the two-sided exit probability (10) we find that

$$\begin{aligned}\mathbb{E}_x[\exp\{-uT_k^- + vX_{T_k^-}\}I_{(T_{k+m}^+ > T_k^-)}] &= \mathrm{e}^{vx}\mathbb{E}_x^v\left[\mathrm{e}^{-pT_k^-}I_{(T_{k+m}^+ > T_k^-)}\right] \\ &= \mathrm{e}^{vx}\left(Z_v^{(p)}(x-k) - W_v^{(p)}(x-k)\frac{Z_v^{(p)}(m)}{W_v^{(p)}(m)}\right), \quad (14)\end{aligned}$$

where $p = u - \psi(v)$. By Remark 6 and the properties of the u-scale function, for each fixed v with $\psi(v) < \infty$ the expression on the right-hand side of (14) is analytic in $u > 0$. Since both sides of (14) are analytic in $u > 0$, the identity theorem implies that the identity (14) is valid for $u > 0$ and v with $\psi(v) < \infty$. By taking the limit of u to zero on both sides of (14) with the help of the monotone convergence theorem for the left-hand side, we see the identity (14) is valid for all $u \geq 0$ and v with $\psi(v) < \infty$.

Let now m tend to infinity in (14). Then by monotone convergence (for the left-hand side) and Lemma 1 (for the right-hand side), we end up with the identity (12) for the stated range of u and v. □

Remark 7 *Following an analogous reasoning as in Remark 5, we see that*

$$\exp\{-u(T_k^- \wedge t) + vX_{T_k^- \wedge t}\}\left(Z_v^{(p)}(X_{T_k^- \wedge t} - k) - W_v^{(p)}(X_{T_k^- \wedge t} - k)p/\Phi_v(p)\right)$$

for $t \geq 0$, *is a* $\mathbb{P}$*-martingale.*

4 Exit problems for reflected Lévy processes

Denote by $\overline{X} = \{\overline{X}_t, t \geq 0\}$, with

$$\overline{X}_t = \max\left\{s, \sup_{0\leq u\leq t} X_u\right\},$$

the non-decreasing process representing the current maximum of X given that, at time zero, the maximum from some arbitrary prior point of reference in time is s. Further, let us alter slightly our notation so that now $\mathbb{P}_{s,x}$ refers to the Lévy process X which at time zero is given to have a current maximum s and position x. The notation $\mathbb{P}_{s,x}^c$ is also used in the obvious way. Further in the sequel, we shall frequently exchange between $\mathbb{P}_{s,x}^c$, $\mathbb{P}_{(s-x),0}^c$ and $\mathbb{P}_{-(s-x)}^c$ as appropriate.

We can address similar questions to those of the previous section of the process $Y = \overline{X} - X$. In this case, problems of two sided exit from a finite interval $[a, b] \subset (0, \infty)$ for the process Y are the same as for the process X. In this section we study one sided exit problems centred around the stopping time

$$\tau_k := \inf\{t \geq 0 : Y_t \notin [0, k)\}$$

defined for $k \geq 0$.

Theorem 1 *For $u \geq 0$ and v such that $\psi(v) < \infty$, the joint Laplace transform of τ_k and Y_{τ_k} is given by*

$$\mathbb{E}_{s,x}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}] = \mathrm{e}^{-vz}\left(Z_v^{(p)}(k-z) - W_v^{(p)}(k-z)\frac{pW_v^{(p)}(k) + vZ_v^{(p)}(k)}{W_v^{(p)\prime}(k) + vW_v^{(p)}(k)}\right),$$

where $z = s - x \geq 0$ and $p = u - \psi(v)$.

Proof Suppose first that u, v are such that $u \geq \psi(v) \vee 0$ and let $z = s - x$. Denote by $\tau_{\{0\}}$ the first time that Y hits zero. An application of the strong Markov property of Y at $\tau_{\{0\}}$ yields that $\mathbb{E}_{s,x}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}]$ is equal to

$$\mathbb{E}_{s,x}\left[\mathrm{e}^{-u\tau_k - vY_{\tau_k}} I_{(\tau_k < \tau_{\{0\}})}\right] + C\mathbb{E}_{s,x}\left[\mathrm{e}^{-u\tau_{\{0\}}} I_{(\tau_k > \tau_{\{0\}})}\right], \tag{15}$$

where $C = \mathbb{E}_{s,s}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}] = \mathbb{E}_{0,0}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}]$. Since

$$\{Y_t, t \leq \tau_{\{0\}}, \mathbb{P}_{s,x}\} \stackrel{d}{=} \{-X_t, t \leq T_0^+, \mathbb{P}_{-z}\} \tag{16}$$

and $\exp(vX_{T_{-k}^- \wedge T_0^+} - \psi(v)(T_{-k}^- \wedge T_0^+) + vz)$ is an equivalent change of measure under $\mathbb{P}_{-z}$ (since $T_{-k}^- \wedge T_0^+$ is almost surely finite), we can rewrite the first expectation on the right hand side of (15) as $\exp(-vz)$ times

$$\mathbb{E}_{-z}^v\left[\mathrm{e}^{(\psi(v)-u)T_{-k}^-} I_{(T_{-k}^- < T_0^+)}\right] = Z_v^{(p)}(k-z) - W_v^{(p)}(k-z)\frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \tag{17}$$

from Propostition 1. By (16), Remark 4 and again Propostition 1 we find for the second expectation on the right-hand side of (15)

$$\mathbb{E}_{-z}\left[\mathrm{e}^{-uT_0^+} I_{(T_{-k}^- > T_0^+)}\right] = \frac{W^{(u)}(k-z)}{W^{(u)}(k)} = \mathrm{e}^{-vz}\frac{W_v^{(p)}(k-z)}{W_v^{(p)}(k)}. \tag{18}$$

We compute C by excursion theory. To be more precise, we are going to make use of the compensation formula of excursion theory. For this we shall use standard notation (see Bertoin [23, Ch. 4]). Specifically, we denote by $\mathcal{E}$ the set of excursions away from zero of finite length

$$\mathcal{E} = \{\epsilon \in D : \exists \zeta = \zeta(\epsilon) > 0 \text{ such that } \epsilon(\zeta) = 0 \text{ and } \epsilon(x) > 0 \text{ for } 0 < x < \zeta\},$$

where $D = D([0,\infty))$ denotes the space of all càdlàg functions on $[0,\infty)$. Analogously, $\mathcal{E}^{(\infty)}$ denotes the set of excursions ϵ away from zero with infinite length $\zeta = \infty$. We are interested in the excursion process $e = \{e_t, t \geq 0\}$ of Y, which takes values in the space of excursions $\mathcal{E} \cup \mathcal{E}^{(\infty)}$ and is given by

$$e_t = \{Y_s, L^{-1}(t^-) \leq s < L^{-1}(t)\} \qquad \text{if } L^{-1}(t^-) < L^{-1}(t),$$

where L^{-1} is the right inverse of a local time L of Y at 0. We take the running supremum of X to be this local time (c.f. Bertoin [23, Ch. VII]). The space

$\mathcal{E}$ is endowed with the Itô-excursion measure n. A famous theorem of Itô now states that, if Y is recurrent, $\{e_t, t \geq 0\}$ is a Poisson point process taking values in $\mathcal{E}$ with characteristic measure n; if Y is transient, $\{e_t, t \leq L(\infty)\}$ is a Poisson point process stopped at the first point in $\mathcal{E}^{(\infty)}$. This stopped Poisson point process has the same characteristic measure n and is independent of $L(\infty)$, an exponentially distributed random variable with parameter $\Phi(0)$. For an excursion $\epsilon \in \mathcal{E}$ with lifetime $\zeta = \zeta(\epsilon)$, we denote by $\overline{\epsilon}$ the supremum of ϵ, that is, $\overline{\epsilon} = \sup_{s \leq \zeta} \epsilon(s)$. The point process of maximum heights $h = \{h_t : t \leq L(\infty)\}$ of excursions appearing in the process e is a Poisson point process (respectively stopped Poisson point process) if Y is recurrent (respectively transient).

Following the line of reasoning in Bertoin [23] concerning Proposition 1 we can also deduce the characteristic measure of the process h. Suppose first Y is recurrent. The event that X starting in 0 exits the interval $(-x, y)$ at y is equal to the event $A = \{h_t \leq t + x \forall\, t \leq x + y\}$. Hence from Proposition 1 we find by differentiation that

$$W(x)/W(x+y) = \exp\left(-\int_0^y n(\overline{\epsilon} \geq t + x)\mathrm{d}t\right) \Rightarrow n(\overline{\epsilon} \geq k) = W'(k)/W(k).$$

If Y is transient, we replace the event A by $A' = \{h_t \leq t+x \;\forall\; t \leq x+y, x+y < L(\infty)\}$. Denoting by $n^{(\infty)}$ the characteristic function of e on $\mathcal{E} \cup \mathcal{E}^{(\infty)}$, we find that $n^{(\infty)}(\overline{\epsilon} \geq k) = \Phi(0) + n(\overline{\epsilon} \geq k)$. Since $\Phi(0) \geq 0$ precisely if Y is transient and the stopped Poisson point process has the same characteristic measure n, we see that above display remains valid if we replace everywhere n by $n^{(\infty)}$, irrespective of whether Y is transient or not. Hence in the sequel, we shall also write n for $n^{(\infty)}$ to lighten the notation.

Now let $\rho_k = \inf\{t \geq 0 : \epsilon(t) \geq k\}$ and denote by ϵ_g the excursion starting at real time g, that is, $\epsilon_g = \{Y_{g+t}, 0 \leq t < \zeta(\epsilon_g)\}$. The promised calculation involving the compensation formula is as follows.

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-u\tau_k - vY_{\tau_k}}\right) &= \mathbb{E}\left(\sum_g \left\{\mathrm{e}^{-ug} I_{\left(\sup_{h<g} \overline{\epsilon}_h < k\right)}\right\}\left\{I_{(\overline{\epsilon}_g \geq k)} \mathrm{e}^{-u(\tau_k - g) - vY_{\tau_k}}\right\}\right) \\
&= \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-us} I_{\left(\sup_{h<s} \overline{\epsilon}_h < k\right)} L(\mathrm{d}s)\right) \int_{\mathcal{E}} I_{(\overline{\epsilon} \geq k)} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(\mathrm{d}\epsilon\right) \\
&= \int_0^\infty \mathbb{E}\left(\mathrm{e}^{-uL_t^{-1}} I_{\sup_{l<L_t^{-1}} \overline{\epsilon}_l < k, t < L(\infty)}\right) \mathrm{d}t \\
&\times \int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(\mathrm{d}\epsilon | \overline{\epsilon} \geq k\right) n\left(\overline{\epsilon} \geq k\right).
\end{aligned}$$

The suprema and the sum are taken over left starting points g of excursions. The desired expectation is now identified as the product of the two items in the last equality, say I_1 and I_2 which can now be evaluated separately. For the first, note that L_t^{-1} is a stopping time and hence an argument involving a change of measure yields

$$\begin{aligned} I_1 &= \int_0^\infty \mathbb{E}\left(\mathrm{e}^{-uL_t^{-1}+\Phi(u)t} I_{\sup_{l<L_t^{-1}} \overline{e}_l<k, t<L(\infty)}\right)\mathrm{e}^{-\Phi(u)t}\mathrm{d}t \\ &= \int_0^\infty \mathbb{P}^{\Phi(u)}\left(\sup_{l<L_t^{-1}} \overline{e}_l < k, t< L(\infty)\right)\mathrm{e}^{-\Phi(u)t}\mathrm{d}t. \end{aligned}$$

Since $\psi'_{\Phi(u)}(0) = \psi(\Phi(u)) \geq 0$, the process X drifts to infinity under $\mathbb{P}^{\Phi(u)}$. Thus, under $\mathbb{P}^{\Phi(u)}$ the reflected process Y is recurrent and $L(\infty) = \infty$. Thus, the probability in the previous integral is the chance that, in the Poisson point process of excursions (indexed by local time), the first excursion of height greater or equal to k occurs after time s. The intensity of the Poisson process (again indexed by local time) counting the number of excursions with height not smaller than x associated with measure $\mathbb{P}^{\Phi(u)}$ is $W'_{\Phi(u)}(x)/W_{\Phi(u)}(x)$. We deduce that

$$\mathbb{P}^{\Phi(u)}\left(\sup_{l<L_t^{-1}} \overline{e}_l < k\right) = \exp\left\{-t\frac{W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)}\right\},$$

so that

$$\begin{aligned} I_1 &= \int_0^\infty \exp\left\{-t\frac{\Phi(u)W_{\Phi(u)}(k)+W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)}\right\}\mathrm{d}t \\ &= \frac{W_{\Phi(u)}(k)}{\Phi(u)W_{\Phi(u)}(k)+W'_{\Phi(u)}(k)} = \frac{W^{(u)}(k)}{W^{(u)\prime}(k)}, \end{aligned}$$

where the final identity follows from Remark 4. Note that $I_1 \geq 0$, since $W^{(u)}$ is an increasing non-negative on $(0,\infty)$. Now turning to I_2, we begin by noting from before that $n(\overline{\epsilon} \geq k) = W'(k)/W(k)$. Our aim is now to prove that

$$\int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon|\overline{\epsilon} \geq k\right) = \frac{Z_v^{(p)}(k)\, W_v^{(p)\prime}(k)/W_v^{(p)}(k) - pW_v^{(p)}(k)}{W'(k)/W(k)} \tag{19}$$

and hence that

$$I_2 = Z_v^{(p)}(k)\, W_v^{(p)\prime}(k)/W_v^{(p)}(k) - pW_v^{(p)}(k). \tag{20}$$

We start with setting the function f on $(0,\infty)$ equal to

$$f(z) := \frac{Z_v^{(p)}(k-z) - W_v^{(p)}(k-z)\, Z_v^{(p)}(k)/W_v^{(p)}(k)}{1 - W(k-z)/W(k)} \qquad \text{for } z \geq 0 \tag{21}$$

and $f(0) := \lim_{z\downarrow 0} f(z)$. By de l'Hôpital's rule, we find that

$$f(0) = \frac{Z_v^{(p)}(k)\, W_v^{(p)\prime}(k)/W_v^{(p)}(k) - pW_v^{(p)}(k)}{W'(k)/W(k)}. \tag{22}$$

To prove (19), we will show that, with $\rho_\theta = \inf\{t \geq 0 : \epsilon(t) \geq \theta\}$,

$$M_\theta = \mathrm{e}^{-u\rho_\theta - v\epsilon(\rho_\theta)} f(\epsilon(\rho_\theta)) \qquad \theta \in (0,k]$$

is a martingale under the measure $n(\cdot|\bar{\epsilon} \geq k)$ with respect to the filtration $\{\mathcal{G}_\theta : \theta \in [0,k]\}$, where $\mathcal{G}_\theta = \sigma(\epsilon(t) : t \leq \rho_\theta)$. Let $\eta(\cdot) = n(\cdot|\bar{\epsilon} \geq k)$. To show that the sequence $\{M_\theta : \theta \in (0,k]\}$ is a martingale consider first that

$$\eta\left(M_k|\,\mathcal{G}_\theta\right) \;=\; \frac{n\left(\mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)}\mathbf{1}_{(\rho_k<\infty)}\middle|\,\mathcal{G}_\theta\right)}{n\left(\rho_k < \infty|\,\mathcal{G}_\theta\right)}.$$

Using the strong Markov property for excursions, we have that given $\mathcal{G}_\theta$ the law of the continuing excursion is that of $-X$ killed on entering $(-\infty, 0)$ with entrance law being that of $\epsilon(\rho_\theta)$. Thus, we find that

$$\begin{aligned}
& n\left(\mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)}\mathbf{1}_{(\rho_k\leq\infty)}\middle|\,\mathcal{G}_\theta\right) \\
&= \mathrm{e}^{-u\rho_\theta}\mathbb{E}_{-\epsilon(\rho_\theta)}\left(\mathrm{e}^{-uT^-_{-k}+vX_{T^-_{-k}}}\mathbf{1}_{\left(T^-_{-k}\leq\infty\right)}\mathbf{1}_{\left(T^-_{-k}\leq T^+_0\right)}\right) \\
&= \mathrm{e}^{-u\rho_\theta}\mathbb{E}^v_{-\epsilon(\rho_\theta)}\left(\mathrm{e}^{-pT^-_{-k}}\mathbf{1}_{\left(T^-_{-k}\leq T^+_0\right)}\right)\mathrm{e}^{-v\epsilon(\rho_\theta)} \\
&= \mathrm{e}^{-u\rho_\theta - v\epsilon(\rho_\theta)}\left(Z^{(p)}_v\left(k-\epsilon(\rho_\theta)\right) - W^{(p)}_v\left(k-\epsilon(\rho_\theta)\right)\frac{Z^{(p)}_v\left(k\right)}{W^{(p)}_v\left(k\right)}\right) \qquad (23)
\end{aligned}$$

and choosing $u = v = 0$ in the above calculation

$$n\left(\rho_k \leq \infty|\,\mathcal{G}_\theta\right) = 1 - W\left(k - \epsilon(\rho_\theta)\right)/W\left(k\right).$$

The martingale status of $\{M_\theta : \theta \in (0,k]\}$ is proved. By this martingale property

$$\int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon|\bar{\epsilon} \geq k\right) = n(M_\theta|\bar{\epsilon} \geq k), \qquad \text{for all } \theta \in (0,k].$$

If X has unbounded variation, almost all excursion ϵ leave continuously from zero and by right-continuity of the paths $\epsilon(\rho_\theta) \to \epsilon(\rho_0) = 0$ $n(\cdot|\bar{\epsilon} \geq k)$-almost surely as θ tends to zero. Noting that the function f defined in (21) and (22) is continuous and bounded (since it takes the value 1 for $z \geq k$), we find by bounded convergence that

$$n(M_k|\bar{\epsilon} \geq k) = \lim_{\theta\downarrow 0} n(M_\theta|\bar{\epsilon} \geq k) = n(M_0|\bar{\epsilon} \geq k).$$

Putting the pieces together from I_1 and I_2 and noting Remark 4 implies

$$W^{(u)}(k)/W^{(u)\prime}(k) = W^{(p)}_v(k)/(W^{(p)\prime}_v(k) + vW^{(p)}_v(k)),$$

we find

$$C = -W^{(p)}_v(k)\frac{pW^{(p)}_v(k) + vZ^{(p)}_v(k)}{W^{(p)\prime}_v(k) + vW^{(p)}_v(k)} + Z^{(p)}_v(k) \qquad (24)$$

and by substitution of (17), (18) and (24) in (15) a weaker version (in view of the restrictions on u and v) of the theorem is proved for X having unbounded variation.

Suppose now we are still under the regime that $u \geq \psi(v) \vee 0$ and that X has bounded variation. Note that one may now deduce that $\mathrm{e}^{vx}W_v^{(p)}(x)$ and $\mathrm{e}^{vx}Z_v^{(p)}(x)$ are positive eigenfunctions of the infinitesimal generator of X restricted to domains of the form $(0,a)$ for any $a \geq 0$. To see this apply the change of variable formula (e.g. [111, Thm. II.31]) to the martingales mentioned in Remark 5. Next, use these facts when applying the change of variable formula again to the process

$$\mathrm{e}^{-u(t\wedge\tau_k)-vY_{t\wedge\tau_k}} \times \left(Z_v^{(p)}(k - Y_{t\wedge\tau_k}) - \frac{vZ_v^{(q)}(k) + pW_v^{(p)}(k)}{W_v^{(p)\prime}(k) + vW_v^{(p)}(k)} W_v^{(p)}(k - Y_{t\wedge\tau_k}) \right), \quad (25)$$

$t \geq 0$, to deduce that it is a martingale. The expectation of the terminal value of this martingale must be equal to its initial value. This is the statement of the theorem.

The result is now established for u, v such that $u \geq \psi(v) \vee 0$ both for X having bounded and fox X having unbounded variation. Mimicking the extension argument in the proof of Proposition 2, we find that that the stated result is valid for $u \geq 0$ and v with $\psi(v) < \infty$. □

Remark 8 *When X has unbounded variation, one cannot use the method in the proof used for the case of bounded variation on account of the fact that the function $W_v^{(p)}$ is not necessarily smooth enough to use in conjunction with Itô's formula. Having proved Theorem 1 however, following the comments in Remark 5 it is not difficult to show that for all u, v as in Theorem 1, (25) is again martingale, where, as before, $p = u - \psi(v)$.*

When X has bounded variation, the method of proof used for the case of unbounded variation is valid up to establishing the identity (20) for I_2. The method can be pushed through in a similar way to the case of unbounded variation but, as we shall now explain, the given technique in the proof is considerably quicker.

For the case of bounded variation it is known that (e.g. [114] and more recently [126, 106]) an excursion ϵ starts with a jump almost surely and $n(\epsilon(\rho_0) \in \mathrm{d}x) = \mathtt{d}^{-1}\Lambda(-\mathrm{d}x)$ where Λ and $\mathtt{d}$ are the Lévy measure and drift of X respectively. The law of an excursion ϵ under n is then that of $-X$ killed upon entering the negative half-line with entrance law $n(\epsilon(\rho_0) \in \mathrm{d}x)$. Then by the computation in (23),

$$I_2 = \int_{-\infty}^{0} \mathrm{e}^{vx} \left(Z_v^{(p)}(k+x) - W_v^{(p)}(k+x) \frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \right) \mathtt{d}^{-1}\Lambda(\mathrm{d}x). \quad (26)$$

By showing that the right hand side of (26) and the right hand side of (22) are continuous in k and their Laplace transform with respect to k conincide, *one checks that these expression are equal. But this boils down to the fact that* $e^{vx}W_v^p(x)$ *and* $e^{vx}Z_v^p(x)$ *are positive eigenfunctions of the infinitesimal generator of* X *on finite open intervals, which lead to the quicker martingale proof that was presented.*

5 Russian and American options

Consider a financial market consisting of a riskless bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 \exp(rt) \qquad B_0 \geq 0, \qquad r \geq 0, \qquad t \geq 0. \tag{27}$$

The price of the risky asset is modeled as the exponential spectrally negative Lévy process

$$S_t = S_0 \exp(X_t), \qquad S_0 > 0, t \geq 0. \tag{28}$$

If $X_t = \mu t + \sigma W_t$ where, as before, $W = \{W_t, t \geq 0\}$ is a standard Brownian motion, we get the standard Black-Scholes model for the price of the asset. Extensive empirical research has shown that this (Gaussian) model is not capable of capturing certain features (such as skewness, asymmetry and heavy tails) which are commonly encountered in financial data, for example returns of stocks. To accommodate for these problems, an idea, going back to Merton [96], is to replace the Brownian motion as model for the log-price by a general Lévy process X. In this paper, we will retrict ourselves to the model where X is given by the spectrally negative Lévy process given in (1). This restriction is mainly motivated by analytical tractability and the availability of many results (such as those given in the previous sections) which exploit the fact that X is spectrally negative. It is worth mentioning however that in a recent study, Carr and Wu [37] have offered empirical evidence (based on a study of implied volatitity) to support the case of a model in which the risky asset is driven by a spectrally negative Lévy process. Specifically, a spectrally negative stable process of index $\alpha \in (1, 2)$. See the examples in the final section for further discussion involving this class of Lévy process.

The model (27) – (28) for our market is free of arbitrage since there exists an equivalent martingale measure, that is, there exists a measure (equivalent to the implicit measure of the risky asset) under which the process $\{S_t/B_t : t \geq 0\}$ is a martingale. We can choose this measure so that X remains a spectrally negative Lévy process under this measure. If $\sigma \geq 0$ and $J^{(-)} \neq 0$ or $\sigma = 0$ and $J^{(-)}$ has more than one jump- size the model is incomplete and has infinitely many equivalent martingale measures. Which one to choose for pricing, is an important issue in which we do not indulge in in this article. We refer the interested reader to the paper of Chan [39] and references therein. We thus assume that some martingale measure has been chosen and let $\mathbb{P}$ take the role of this measure. Note that this necessarily implies that $\psi(1) = r$.

Russian options were originally introduced by Shepp and Shiryaev [116, 117] within the context of the Black-Scholes market (the case that the underlying

Lévy process is a Brownian motion with drift). In this paper we shall consider perpetual Russian options under the given model of spectrally negative Lévy processes. This option gives the holder the right to exercise at any almost surely finite **F**-stopping time τ yielding payouts

$$e^{-\alpha\tau} \max\left\{M_0, \sup_{0\le u\le\tau} S_u\right\}, \qquad M_0 \ge S_0, \alpha > 0.$$

The constant M_0 can be viewed as representing the "starting" maximum of the stock price (say, the maximum over some previous period $(-t_0, 0])$. The discount factor α is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite (cf. [116, 117]).

Somewhat more studied are American put options which give the holder again the right to exercise at any **F**-stopping time τ yielding a payout

$$e^{-\alpha\tau}(K - S_\tau)^+, \qquad \alpha \ge 0,$$

where α can be regarded as the dividend rate of the underlying stock. The American put option has been dealt with as early as McKean [99].

Standard theory of pricing American-type options in the original Black-Scholes market directs one to solving optimal stopping problems. For the Russian and American put, the analogy in this context involves evaluating

$$V_r(M_0, S_0) := B_0 \sup_\tau \mathbb{E}_{\log S_0}\left[B_\tau^{-1} \cdot e^{-\alpha\tau} \max\left\{M_0, \sup_{0\le u\le\tau} S_u\right\}\right], \tag{29}$$

$$V_a(S_0) := B_0 \sup_\tau \mathbb{E}_{\log S_0}\left[B_\tau^{-1} \cdot e^{-\alpha\tau}(K - S_\tau)^+\right], \tag{30}$$

where the supremum is taken over all almost surely finite respectively all **F**-stopping times. That is, to find a stopping time which optimizes the expected discounted claim under the chosen risk neutral measure. We refer to the optimal stopping problems (29) and (30) as the *Russian optimal stopping problem* and as the *American optimal stopping problem* respectively. In the Sections 6 and 7, we will solve (29) and (30) respectively, by combining well known optimal stopping theory with the results on exit problems from the previous sections 3 and 4. The real object of interest is of course the finite time version with the extra constraint $\tau \le T$, where T is a given expiration time (this is closely related to the lookback option). Note however that Carr [36] has shown that a close relative of the perpetual version lies at the basis of a very efficient approximation for the finite time expiration option, justifying therefore the interest in perpetuals. We shall address this matter in more detail in Section 8.

6 The Russian optimal stopping problem

When dealing with Russian options, our method leans on the experience of Shepp and Shiryaev [116, 117], Duffie and Harrison [50], Graversen and Peškir

[64] and Kyprianou and Pistorius [85]; all of which deal with the perpetual Russian option within the standard Black-Scholes market. The first thing to note is that the optimal stopping problem (29), depending on the two-dimensional Markov process $(X, \overline{X})$, can be reduced to an optimal stopping problem depending only on the one-dimensional Markov process $Y = \overline{X} - X$, the reflection of X at its supremum. Indeed, by Shepp and Shiryaev's technique of performing a change of measure using the $\mathbb{P}_x$-martingale $\exp\{-rt - x\}S_t$, we get for all $\mathbb{P}_x$-a.s. finite $\mathbf{F}$-stopping times τ

$$\begin{aligned} &B_0 \mathbb{E}_x \left[B_\tau^{-1} \cdot \mathrm{e}^{-\alpha\tau} \max\left\{ M_0, \sup_{0\le u\le\tau} S_u \right\} \right] \\ &= S_0 \mathbb{E}_x \left[\frac{B_0 S_\tau}{B_\tau S_0} \times \mathrm{e}^{-\alpha\tau} \max\left\{ \frac{M_0}{S_\tau}, \sup_{0\le u\le\tau} \frac{S_u}{S_\tau} \right\} \right] \\ &= \mathrm{e}^x \mathbb{E}_x^1 \left[\mathrm{e}^{-\alpha\tau + \max\{\overline{X}_\tau, s\} - X_\tau} \right] \\ &= \mathrm{e}^x \mathbb{E}_{x-s}^1 \left[\mathrm{e}^{-\alpha\tau + Y_\tau} \right], \end{aligned}$$

where $x = \log S_0$ and $s = \log M_0$. Note that under $\mathbb{P}_{x-s}^1$ the process Y starts in $Y_0 = s - x$. In this way we are lead to the problem of finding a function w^R and an almost surely finite stopping time τ^* such that

$$w^R(z) = \sup_\tau \mathbb{E}_{-z}^1 \left[\mathrm{e}^{-\alpha\tau + Y_\tau} \right] = \mathbb{E}_{-z}^1 \left[\mathrm{e}^{-\alpha\tau^* + Y_{\tau^*}} \right]. \tag{31}$$

The value function $V_r(M_0, S_0)$ of the optimal stopping problem (29) is related to w^R by $V_r(M_0, S_0) = S_0 \times w^R(\log(M_0/S_0))$.

In view of the fact that the payoff in the modified optimal stopping problem (31) is now Markovian, well known theory of optimal stopping suggests that we should now expect the optimal stopping time to be an upcrossing time of the reflected process Y at a certain constant (positive) level k. Appealing to standard techniques using martingale optimality and exploiting the fluctuation theory discussed in the previous sections we are able to prove that this is indeed the case.

Our study of exit problems for the reflected Lévy processes in Section 4 yields an expression for the value of stopping at τ_k.

Corollary 1 *Suppose X is as in Theorem 1 with $\psi(1) = r$. Then, for $k \geq 0, z \geq 0$, $\alpha \geq 0$ and $q = \alpha + r$*

$$\mathbb{E}_{-z}^1 \left(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}} \right) = \mathrm{e}^z \left(Z^{(q)}(k - z) + \frac{Z^{(q)}(k) - qW^{(q)}(k)}{W^{(q)\prime}(k) - W^{(q)}(k)} W^{(q)}(k - z) \right). \tag{32}$$

Proof Noting that $\psi_1(-1) = \psi(0) - \psi(1) = -r$, we may apply Theorem 1 with $u = \alpha, v = -1$ and $\mathbb{P}$ replaced by $\mathbb{P}^1$. The proof is finished once we note that $p = \alpha + r = q$ and

$$\mathrm{e}^x \mathrm{e}^x (W_1)_{-1}^{(\alpha+r)} \equiv \mathrm{e}^x W_1^{(\alpha)} \equiv W^{(q)}$$

each time by Remark 4. □

To complete the solution of the optimal stopping problem (31), we need to find the optimal level $k = \kappa^*$. It turns out that the optimal level is given by

$$\kappa^* = \inf\{x : Z^{(q)}(x) \leq qW^{(q)}(x)\}. \tag{33}$$

Write w_k for the function of z on $[0, \infty)$ given in (32). Since under (AC) $W^{(q)}$ and $Z^{(q)}$ are differentiable on $(0, \infty)$, the function w_k is so on $\mathbb{R}\backslash\{k\}$. In the case of *bounded* variation and $W^{(q)}(0^+) < q^{-1}$, $W^{(q)}(0^+) \in (0, q^{-1})$ and one notes that the level κ^* can be achieved by a principle of *continuous fit*:

$$\lim_{z\uparrow\kappa^*} w_{\kappa^*}(z) = \lim_{z\downarrow\kappa^*} w_{\kappa^*}(z).$$

This principle was earlier encountered by Peškir and Shiryaev [108] in their study of a sequential testing problem for Poisson processes. If X has bounded variation and $W^{(q)}(0^+) \geq q^{-1}$, we see that $\kappa^* = 0$ and it is optimal to stop immediately.

On the other hand, if X has *unbounded* variation, $W^{(q)}(0^+) = 0$ and we find that κ^* can be recovered by a principle of *smooth fit:*

$$\lim_{z\uparrow\kappa^*} \frac{1}{z - \kappa^*}(w_{\kappa^*}(z) - w_{\kappa^*}(\kappa^*)) = \lim_{z\downarrow\kappa^*} \frac{1}{z - \kappa^*}(w_{\kappa^*}(z) - w_{\kappa^*}(\kappa^*))$$

For an optimal stopping problem involving a Wiener process, the principle of smooth fit was first discovered in 1955 by Mikhalevich. We see that by this choice of κ^* the function w_{κ^*} is of class C^2 on $\mathbb{R}\backslash\{\kappa^*\}$ and differentiable and continuous in κ^* respectively according to whether X has unbounded or bounded variation. The next theorem summarises the solution of the optimal stopping problem (31).

Theorem 2 *Define $u : [0, \infty) \to [0, \infty)$ by $u(z) = \mathrm{e}^z Z^{(q)}(\kappa^* - z)$ with κ^* given in (33). Then the solution to (31) is given by $w^R = u$ where $\tau^* = \tau_{\kappa^*}$ is the optimal stopping time.*

Before we start the proof we collect some useful facts:

Lemma 2 *Define the function $f : [0, \infty) \to \mathbb{R}$ by $f(x) = Z^{(q)}(x) - qW^{(q)}(x)$ and let κ^* be as in Theorem 2. Then the following two assertions hold true:*

(i) For $q \geq r$, f decreases monotonically to $-\infty$.

(ii) If $W^{(q)}(0^+) \geq q^{-1}$, $\kappa^* = 0$; otherwise $\kappa^* \geq 0$ is the unique root of $f(x) = 0$.

Proof (i) By Remark 4, the function f has derivative in $x \geq 0$

$$f'(x) = qW^{(q)}(x) - qW^{(q)\prime}(x) = q\mathrm{e}^{\Phi(q)x}\left((1 - \Phi(q))W_{\Phi(q)}(x) - W'_{\Phi(q)}(x)\right).$$

For $x \geq 0$ and $q \geq r$, this derivative is seen to be negative, since $W_{\Phi(q)}$ is positive and increasing on $(0, \infty)$ and $\Phi(q) \geq \Phi(r) = 1$ for $q \geq r$. By Lemma

1, $f(x)/(qW^{(q)}(x))$ tends to $\Phi(q)^{-1}-1$ as $x\to\infty$. By Remark 4, $W^{(q)}(x) = \exp(\Phi(q)x)W_{\Phi(q)}(x)$ tends to infinity and the statement follows.

(ii) If $W^{(q)}(0^+)\geq q^{-1}$, (i) implies that $\kappa^*=0$, whereas if $W^{(q)}(0^+)<q^{-1}$, we have existence and uniqueness of a positive root of $Z^{(q)}(x)=qW^{(q)}(x)$. □

Proof of Theorem 2 Suppose now first $W^{(q)}(0^+) = 0$ (that is, X has unbounded variation). From the properties of $Z^{(q)}$, we see that u lives in $C^1(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{k_*\})$. Hence Itô's lemma implies that $\exp\{-\alpha t\}u(\overline{X}_t-X_t)$ can be written as the sum of stochastic and Stieltjes integrals. The non-martingale component of these integrals can be expressed as $\exp\{-\alpha t\}$ times

$$(\widehat{\Gamma}_1-\alpha)u(\overline{X}_t-X_t)\mathrm{d}t+u'(\overline{X}_t-X_t)\mathrm{d}\overline{X}_t=(\widehat{\Gamma}_1-\alpha)u(Y_t)\mathrm{d}t, \tag{34}$$

where $\widehat{\Gamma}_1$ is the infinitesimal generator corresponding to the process $-X$ under $\mathbb{P}^1$ and the equality follows from the fact that the process $\overline{X}_t$ only increments when $Y_t=0$ (since Y reaches zero always by creeping in the absence of positive jumps of X) and $u'(0)=0$. From Remark 8 we know that $\exp\{-\alpha(t\wedge\tau_{\kappa^*})\}u(Y_{t\wedge\tau_{\kappa^*}})$ is a martingale, which implies that on $\{t\leq\tau_{\kappa^*}\}$, and hence on $\{X_t\leq\kappa^*\}$,

$$(\widehat{\Gamma}_1-\alpha)u(z)=0 \text{ for } z\in[0,\kappa_*).$$

Now recall that under the measure $\mathbb{P}^1_{s,x}$ the process $\exp\{-X_t+rt\}$ is a martingale. By a similar reasoning to the above, we can deduce that $(\widehat{\Gamma}_1+r)(\exp\{z\})=0$. Specifically, this implies for $z\geq\kappa_*$ that

$$(\widehat{\Gamma}_1-\alpha)u(z)=\left(\widehat{\Gamma}_1+r-(r+\alpha)\right)(\exp\{z\})\leq 0.$$

By the expression (34) for the non-martingale part of $\mathrm{d}\left(\exp\{-\alpha t\}u(Y_t)\right)$, we deduce that

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha t+Y_t}Z^{(q)}(\kappa^*-Y_t)\right]\leq\mathrm{e}^{(s-x)}Z^{(q)}(\kappa^*-s+x).$$

A argument similar to the one presented in Remark 5, now shows that the process $\{\exp\{-\alpha t\}u(Y_t), t\geq 0\}$ is a $\mathbb{P}^1_{s,x}$-supermartingale. Doob's optional stopping theorem for supermartingales together with the fact that $\exp\{z\}\leq u(z)$ implies that for all almost surely finite stopping times τ,

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha\tau+Y_\tau}\right]\leq\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha\tau}u(Y_\tau)\right]\leq u(s-x).$$

Since the inequalities above can be made equalities by choosing $\tau=\tau_{\kappa^*}$, the proof is complete for the case of unbounded variation.

If $W^{(q)}(0^+)\in(0,q^{-1})$ (X has bounded variation) we see that u lives in $C^0(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{\kappa^*\})$. Itô's lemma for this case is nothing more than the change of variable formula for Stieltjes integrals (cf. Protter [111]) and the rest of the proof follows exactly the same line of reasoning as above.

Finally the case $W^{(q)}(0^+)\geq q^{-1}$ (again X has bounded variation). Recall from Lemma 2 that, if $W^{(q)}(0^+)\geq q^{-1}$, then for $x>0$

$$Z^{(q)}(x)-qW^{(q)}(x)<0 \qquad\text{and}\qquad W^{(q)}(x)-W^{(q)\prime}(x)<0.$$

Hence, recalling that $Z^{(q)}(x) = 1$ for $x \leq 0$, we see from Corollary 1 that for any $k \leq 0$

$$\begin{aligned}\mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}}) &= \mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}} Z(k - Y_{\tau_k})) \\ &\leq \mathrm{e}^{(s-x)} Z^{(q)}(k - s + x).\end{aligned}$$

As before we conclude that, for any $k \geq 0$, $\{\mathrm{e}^{-\alpha(\tau_k \wedge t)} u(Y_{\tau_k \wedge t})\}_{t \geq 0}$ is a supermartingale and hence using similar reasoning to the previous case it is still the case that $\{\mathrm{e}^{-\alpha t} u(Y_t)\}_{t \geq 0}$ is a supermartingale. It follows that for all almost surely finite **F**-stopping times τ,

$$\mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau + Y_\tau}) \leq \mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau + Y_\tau} Z(k - Y_\tau)) \leq \mathrm{e}^{s-x} Z^{(q)}(k - s + x).$$

Taking $k = 0$, we see that for any a.s. finite stopping time τ,

$$\mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau + Y_\tau}) \leq \mathrm{e}^{s-x}$$

with equality for $\tau = 0$, which completes the proof. □

Remark 9 *Given that the optimal stopping time in (31) is of the form τ_k, here is another way of finding the optimal level κ^* if $W^{(q)}$ is twice differentiable. Let $\eta(q)$ be an independent exponential random variable. Since $\overline{X}_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ which is larger than 1 for $q \geq r$,*

$$\mathbb{E}[\mathrm{e}^{-q\tau_k + \overline{X}_{\tau_k}}] = \mathbb{E}[\mathrm{e}^{\overline{X}_{\tau_k}} \mathbf{1}_{\tau_k \leq \eta(q)}] \leq \mathbb{E}[\mathrm{e}^{\overline{X}_{\eta(q)}}] \leq \infty,$$

where $q \geq r$. Thus, there exists a finite κ^* such that for all $z \geq 0$ the right-hand side of (32) has its maximum at κ^*. By elementary optimization using the assumed differentiability combined with Lemma 2, one then deduces that κ^* is given by (33).

7 The American optimal stopping problem

For the American put option, the associated optimal stopping problem is to find a function w^A and a stopping time τ^* such that

$$w^A(x) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-q\tau}(K - S_\tau)^+] = \mathbb{E}_x[\mathrm{e}^{-q\tau^*}(K - S_{\tau^*})^+] \tag{35}$$

where the supremum is taken over all **F**-measurable stopping times and $q = \alpha + r$. Employing the methods from the previous section, we wil show that solution to the American optimal stopping problem lies with the downcrossing of an optimal constant level. Just as in the Russian optimal stopping problem, smooth or continuous fit (according to whether X has unbounded or bounded variation respectively) suggest that the optimal level k^* is given by

$$k^* = \log(K) + \log(q/\Phi(q)) + \log(\Phi_1(\alpha)/\alpha).$$

As in Remark 9, we may also use an optimization argument to find this level k^*. Recently, Avram et al. [15] studied the same problem for spectrally negative Lévy process where the jump process has bounded variation. See also references therein for further studies of yet simpler cases of spectrally negative Lévy processes. Here, we will employ a different method in the proof, which will enables us to treat the case of a jump component with unbounded variation as well.

Recall that we understand $0/\Phi_1(0)$ to mean $\lim_{\theta\downarrow 0}\theta/\Phi_1(\theta)$.

Theorem 3 *Let $\alpha \geq 0$ and define the function $w : \mathbb{R} \to \mathbb{R}$ by*

$$w(x) = KZ^{(q)}(x - k^*) - \mathrm{e}^x Z_1^{(\alpha)}(x - k^*). \tag{36}$$

The solution to the American put optimal stopping problem (35) is given by $w_A = w$ where $\tau^* = T_{k^*}^-$ is the optimal stopping time.

From Section 3 we find the following expression for the value of hte optimal stopping problem at $\tau = T_k^-$.

Corollary 2 *For $\alpha \geq 0, s \geq 0, k \leq \log K, q = \alpha + \psi(1)$ we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_k^-}\left(K - \exp\{X_{T_k^-}\}\right)^+\right] = K\left(Z^{(q)}(x-k) - W^{(q)}(x-k)\frac{q}{\Phi(q)}\right) - \mathrm{e}^x\left(Z_1^{(\alpha)}(x-k) - W_1^{(\alpha)}(x-k)\frac{\alpha}{\Phi_1(\alpha)}\right).$$

Proof Note that for $k \leq \log K$

$$\mathbb{E}_x\left[\mathrm{e}^{-(\alpha+r)T_k^-}(K - S_{T_k^-})^+\right] = K\mathbb{E}_x\left[\mathrm{e}^{-qT_k^-}\right] - \mathrm{e}^x\mathbb{E}_x\left[\mathrm{e}^{-qT_k^- + X_{T_k^-}}\right],$$

which gives the stated formula after invoking Proposition 2. □

Before we go to the proof of the Theorem, we summarise some usefull results which we will need further on.

Lemma 3 *Consider the function w and the level k^* as defined in Theorem 3. Then $k^* < \log K$. Moreover, it holds that*

(i) $w(x) \geq (K - \mathrm{e}^x)^+$ *and* $w(x) = \mathbb{E}_x[\mathrm{e}^{-qT_{k^*}^-}(K - S_{T_{k^*}^-})^+]$.

(ii) $\left\{\mathrm{e}^{-q(T_{k^*}^- \wedge t)}w(X_{T_{k^*}^- \wedge t}); t \geq 0\right\}$ *is a $\mathbb{P}$-martingale.*

We postpone the proof of the lemma to the end of this subsection and first prove the theorem.

Proof of Theorem 3 Let τ be any **F**-stopping time. If $W^{(q)}(0^+) = 0$ (X has unbounded variation), we see that, by the properties of $Z^{(q)}, Z_1^{(\alpha)}$, the function w is C^2 everywhere except in k^* where it is continuously differentiable. Hence, by applying Itô's lemma to $\mathrm{e}^{-q(T_{k^*}^- \wedge t)}w(X_{T_{k^*}^- \wedge t})$, and using Lemma 3(ii), it follows as before that $(\Gamma - q)w(x) = 0$ for all $x \geq k^*$, where Γ is the infinitesimal generator of X. Moreover, for $x \leq \log K$, we find

$$(\Gamma - q)(K - \mathrm{e}^x) = q(\mathrm{e}^x - K) - \psi(1)\mathrm{e}^x \leq -r\mathrm{e}^x < 0.$$

Hence, $(\Gamma - q)w(x) \leq 0$ for all x. Combining with (34), we deduce that $\{\exp(-qt)w(X_t) : t \geq 0\}$ is a $\mathbb{P}$-supermartingale. Doob's optimal stopping theorem and Lemma 3 (i) imply that for all $\mathbf{F}$-stopping times τ,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau}(K - S_\tau)^+\right] \leq \mathbb{E}_x\left[\mathrm{e}^{-q\tau}w(X_\tau)\right] \leq w(x). \tag{37}$$

Choosing $\tau = T^-_{k^*}$ forces the inequalities (37) to be equalities (Lemma 3(i)) and the proof for this case is complete.

If $W^{(q)}(0^+) > 0$ (X has bounded variation), we note that X is a positive drift minus a jump process of bounded variation. The same argument as in the proof of Theorem 2 justifies the use of the Change of Variable Formula. Then we use a similar reasoning as above to finish the proof. □

Proof of Lemma 3 By strict convexity of ψ combined with $\psi(0) = 0$, we deduce that for any $\alpha > -r$

$$r = \psi(1) < \frac{r}{\alpha + r}\psi\left(\frac{\alpha + r}{r}\right) + \frac{\alpha}{\alpha + r}\psi(0) \iff \alpha + r < \psi\left(\frac{\alpha + r}{r}\right).$$

Recalling that $\Phi(\psi(v)) = v$ for $v \geq 0$ and Φ is increasing, we see that the last inequality is equivalent to

$$\frac{1}{\Phi(\alpha + r)} > \frac{r}{\alpha + r} \iff \frac{(\Phi(\alpha + r) - 1)}{\Phi(\alpha + r)} < \frac{\alpha}{\alpha + r}.$$

Hence, using that for $\alpha \geq 0$

$$\alpha = \psi_1(\Phi_1(\alpha)) = \psi(\Phi_1(\alpha) + 1) - r \iff \Phi(\alpha + r) = \Phi_1(\alpha) + 1,$$

we deduce that $k^* < \log K$.

The first part of (i) follows by substituting the formula for k^* in Proposition 2 and recalling that, by Remark 4, $W^{(q)}(x) = \mathrm{e}^x W_1^{(\alpha)}(x)$. Since $w(x) = (K - \mathrm{e}^x)$ for $x \leq k^*$, we then find

$$\mathbb{E}_x\left[\mathrm{e}^{-qT^-_{k^*}}w(X_{T^-_{k^*}})\right] = w(x).$$

Combining this with the strong Markov property, we can prove, along the same line of reasoning as in Remark 5, that

$$\left\{\mathrm{e}^{-q(T^-_{k^*}\wedge t)}w(X_{T^-_{k^*}\wedge t}) : t \geq 0\right\}$$

is a $\mathbb{P}$-martingale. Finally we show that $w(x) \geq (K - \exp(x))^+$. For $x \leq k^*$ and $x \geq \log K$, we have $w(x) = (K - \mathrm{e}^x)^+$ and $w(x) \geq 0 = (K - \mathrm{e}^x)^+$, respectively.

For $\log K > x > k^*$, we find that

$$\begin{aligned} w(x) &= KZ^{(q)}(x-k^*) - \mathrm{e}^x Z_1^{(\alpha)}(x-k^*) \\ &= K - \mathrm{e}^x + \int_0^{x-k^*} [KqW^{(q)}(y) - \alpha \mathrm{e}^x W_1^{(\alpha)}(y)]\mathrm{d}y \\ &= (K-\mathrm{e}^x)^+ + \int_0^{x-k^*} [Kq - \alpha \mathrm{e}^{(x-y)}]W^{(q)}(y)\mathrm{d}y \geq (K-\mathrm{e}^x)^+ \end{aligned}$$

where we used again $\mathrm{e}^x W^{(\alpha)}(x) = W^{(q)}(x)$ and the definition of $Z^{(q)}$. Note that the integrand is positive, since for $y \in (0, x-k^*)$ we have that $W^{(q)}(y)$ as well as $Kq - \alpha \mathrm{e}^{x-y}$ are positive. □

Remark 10 *Since Φ_1 is the right-inverse of ψ_1, we note that $\lim_{\alpha\downarrow 0} \alpha/\Phi_1(\alpha) = \psi_1'(0)$, which is equal to $\psi'(1)$. Hence, for $\alpha = 0$, the expression for k^* coincides with the one found by Chan [38].*

Remark 11 *Denote by I_t, I the infimum of X up to t and up to an independent exponential time $\eta(q)$ with parameter $q \geq 0$ respectively,*

$$I_t = \inf\{X_s : 0 \leq s \leq t\} \qquad\qquad I = \inf\{X_s : 0 \leq s \leq \eta(q)\},$$

where as before we write $q = \alpha + r$. The solution to the optimal stopping problem (35) in terms of I was already implicit in the work of Darling et al. [47]. They showed that the optimal stopping problem

$$\sup_\tau \mathbb{E}[\mathrm{e}^{-q\tau}(\exp(R_\tau + x) - 1)^+] \tag{38}$$

where R is a random walk, has a solution in terms of $M = \sup_{0\leq s\leq \eta(q)} R_s$. Recently, Mordecki [100] has studied the optimal stopping problem one gets by replacing $(\exp(R_\tau + x) - 1)^+$ by $(1 - \exp(R_\tau + x))^+$ in (38). Moreover, Mordecki [100] showed that the structure of these optimal stopping problems is preserved in continuous time. In our notation and setting, the results read as follows.
The solution to (35) is given by

$$w^A(x) = \mathbb{E}[K\mathbb{E}(\mathrm{e}^I) - \mathrm{e}^{x+I}]^+ / \mathbb{E}(\mathrm{e}^I) \tag{39}$$

where the optimal stopping time is given by $\tau^* = T^-_{l^*}$ with $l^* = \log K\mathbb{E}(\mathrm{e}^I)$.
In Theorem 3, we end up with the same optimal stopping time. Indeed, for spectrally negative Lévy processes the Laplace transform of I is well known to be (e.g. [23])

$$\mathbb{E}(\mathrm{e}^I) = \frac{q}{\Phi(q)} \cdot \frac{\Phi(q)-1}{q-\psi(1)} = \frac{q}{\Phi(q)} \cdot \frac{\Phi_1(\alpha)}{\alpha},$$

where we used Remark 6 for the second equality. This implies $k^* = l^*$.

8 Canadised options

Suppose now we consider a claim structure in which the holder again receives a payout like that of the Russian or American put option. However, we also impose the restriction that the holder must claim before some time $\eta(\lambda)$, where $\eta(\lambda)$ is an **F**-independent exponential random variable with parameter λ. If the holder has not exercised by time $\eta(\lambda)$, then he/she is forced a rebate equal to the claim evaluated at time $\eta(\lambda)$. This is what is known in the literature as Canadisation (c.f. Carr [36]). In the next two subsections we will treat respectively the Canadised Russian and American put.

8.1 Canadised Russian options

We are thus interested in a solution to the optimal stopping problem

$$w^{CR}(z) = \sup \mathbb{E}^1_{-z}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right], \tag{40}$$

where the supremum is taken over almost surely finite stopping times τ. Using the fact that $\eta(\lambda)$ is independent of the Lévy process, we can rewrite this problem in the following form,

$$w^{CR}(z) = \sup_{\tau} \mathbb{E}^1_{-z}\left[\mathrm{e}^{-(\alpha+\lambda)\tau+Y_\tau} + \lambda\int_0^\tau \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right].$$

Given the calculations in Kyprianou and Pistorius [85], one should again expect to see that the optimal stopping time is of the form τ_k for some $k \geq 0$.

From now we write $p = \alpha + \lambda + r$.

Lemma 4 *For each $k \geq 0$,*

$$\mathbb{E}^1_{-z}\left[\mathrm{e}^{-\alpha(\tau_k\wedge\eta(\lambda))+Y_{(\tau_k\wedge\eta(\lambda))}}\right] = \left(\frac{p-\lambda}{p}\right)\mathrm{e}^z Z^{(p)}(k-z) + \frac{\lambda}{p}\mathrm{e}^z + \mathrm{e}^z\frac{(p-\lambda)\left(Z^{(p)}(k) - pW^{(p)}(k)\right)+\lambda}{p\left(W^{(p)\prime}(k) - W^{(p)}(k)\right)}W^{(p)}(k-z). \tag{41}$$

Proof Consider the Itô Lemma applied to the process $\exp\{-(\alpha+\lambda)t + Y_t\}$ on the event $\{t \leq \tau_k\}$. Denote by Γ_1 the infinitesimal generator of X under $\mathbb{P}^1$. Standard calculations making use of the fact that $(\Gamma_1 + r)(\exp\{-x\}) = 0$ yield

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\right) &= -(\alpha+\lambda)\,\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t - r\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t \\ &\quad +\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}\overline{X}_t + \mathrm{d}M_t,\end{aligned}$$

where $\mathrm{d}M_t$ is a martingale term. Taking expectations of the stochastic integral given by the above equalities we have

$$\begin{aligned}p\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k}\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right] &= \mathrm{e}^{(s-x)} - \mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+\lambda)\tau_k+Y_{\tau_k}}\right] \\ &\quad +\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k}\mathrm{e}^{-(\alpha+\lambda)t+Y_t}d\overline{X}_t\right].\end{aligned} \tag{42}$$

The last term in the previous expression can be dealt with by taking account of the fact that $\overline{X} = L$, the local time at the supremum of the process X. Recall that $\tau_{\{0\}}$ is the first time that Y reaches 0 and note that $\mathrm{d}\overline{X}_t = 0$ on the set where $\{\tau_k \leq \tau_{\{0\}}\}$. Letting $A \in \mathcal{F}_t$ be the set

$$A = \left\{ \sup\nolimits_{0 \leq u \leq L_t^{-1}} Y_u < k, t < L(\infty) \right\},$$

we have by the strong Markov property of $(X, \overline{X})$ and Propostion 1

$$\begin{aligned}
&\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t} d\overline{X}_t I_{\left(\tau_k > \tau_{\{0\}}\right)}\right] \\
&\quad = \mathbb{E}^1_{-(s-x)}\left[\mathrm{e}^{-(\alpha+\lambda)\tau_{\{0\}}} I_{\left(\tau_k > \tau_{\{0\}}\right)}\right] \mathbb{E}^1\left[\int_0^\infty I_{(t<\tau_k)} \mathrm{e}^{-(\alpha+\lambda)t+Y_t} dL_t\right] \\
&\quad = \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)} \mathbb{E}^1\left[\int_0^\infty I_A \mathrm{e}^{-(\alpha+\lambda)L_t^{-1}} \mathrm{d}t\right] \\
&\quad = \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)} \int_0^\infty \mathrm{e}^{-\Phi_1(\alpha+\lambda)t} \mathbb{P}^{1+\Phi_1(\alpha+\lambda)}(A)\,\mathrm{d}t, \qquad (43)
\end{aligned}$$

where in the last equality we have applied a change of measue with respect to $\mathbb{P}^1$ using the exponential density $\exp\{\Phi_1(\alpha+\lambda) X_t - (\alpha+\lambda) t\}$.

We can apply now techniques from excursion theory, similarly as in the proof of Theorem 1. The number of heights of the excursions of Y away from zero that exceed height k forms a Poisson process with intensity given by $W'_{1+\Phi_1(\alpha+\lambda)}(k)/W_{1+\Phi_1(\alpha+\lambda)}(k)$. The probability in the last line of (43) can now be re-written as

$$\mathbb{P}^{1+\Phi_1(\alpha+\lambda)}\left(\sup_{0\leq u \leq L_t^{-1}} Y_u < k, t < L(\infty)\right) = \exp\left\{-t\frac{W'_{1+\Phi_1(\alpha+\lambda)}(k)}{W_{1+\Phi_1(\alpha+\lambda)}(k)}\right\}.$$

Completing the integral in (43) much in the same way the integral I_1 was computed in Theorem 1 we end up with

$$\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t} d\overline{X}_t\right] = \mathrm{e}^{(s-x)} \frac{W^{(p)}(k-s+x)}{W^{(p)\prime}(k) - W^{(p)}(k)}.$$

Substituting this term back in (42) and combining with Corollary 1, we end up with the expression stated. □

Using continuous and smooth fit suggests that at the level

$$\kappa_* = \inf\{x \geq 0 : Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/(p-\lambda)\}$$

it is optimal to exercise the Canadised Russian. Next result shows this is indeed the case:

Theorem 4 *Define $h : [0,\infty) \to [0,\infty)$ by*

$$h(z) = (p-\lambda)\mathrm{e}^z Z^{(p)}(\kappa_* - z)/p + \lambda \mathrm{e}^z/p.$$

Then the solution to the optimal stopping problem (40) is $w^{CR} = h$ *where* $\tau^* = \tau_{\kappa_*}$ *is the optimal stopping time.*

The proof of the theorem uses the following observation:

Lemma 5 *Let* h *and* κ_* *be as in Theorem 4. If* $W^{(p)}(0^+) \geq p^{-1}$ *then* $\kappa_* = 0$. *If* $W^{(p)}(0^+) \leq p^{-1}$, $\kappa_* \geq 0$ *is the unique root of* $Z^{(p)}(x) - pW^{(p)}(x) = -\lambda/p$ *and for* $t \geq 0$

$$\mathrm{e}^{-(\alpha+\lambda)(\tau_{\kappa_*}\wedge t)} h(Y_{\tau_{\kappa_*}\wedge t}) + \lambda \int_0^{\tau_{\kappa_*}\wedge t} \mathrm{e}^{-(\alpha+\lambda)s+Y_s}\mathrm{d}s$$

is a $\mathbb{P}^1_{s,x}$*-martingale.*

Proof The statements involving κ_* follow from Lemma 2. Note that $h(s-x) = \exp\{s-x\}$ when $s-x \geq \kappa_*$. Let for $t \geq 0$

$$U_t = \mathrm{e}^{-(\alpha+\lambda)t} h(Y_t) + \lambda \int_0^t \mathrm{e}^{-(\alpha+\lambda)s+Y_s}\mathrm{d}s.$$

It is a matter of checking that the special choice of κ_* together with Lemma 4 imply that $h(s-x) = \mathbb{E}^1_{s,x}[U_{\tau_{\kappa_*}}]$ for all $s-x \geq 0$.

Starting from this fact and making use of the strong Markov property, we can prove that $h(s-x)$ is equal to $\mathbb{E}^1_{s,x}\left[U_{\tau_{\kappa_*}\wedge t}\right]$, in the same vein as Remark 5. The martingale property of $U_{t\wedge\tau_{\kappa_*}}$ will follow in a fashion similar to the proof of this fact. □

Proof of Theorem 4 First suppose $W^{(p)}(0^+) = 0$. We know that $U_{\tau_{\kappa_*}\wedge t}$ is a $\mathbb{P}^1_{s,x}$-martingale from the previous lemma. As earlier seen, $Z^{(p)}$ is twice differentiable everywhere with continuous derivatives except in κ_* where it is just continuously differentiable. The Itô formula applied to $U_{\tau_{\kappa_*}\wedge t}$ implies now that necessarily on $\{t \leq \tau_{\kappa_*}\}$, and hence on $\{Y_t \leq \kappa_*\}$,

$$\left(\widehat{\Gamma}_1 - (\alpha+\lambda)\right) h(Y_t)\mathrm{d}t + \lambda\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t + h'(Y_{t-})d\overline{X}_t = 0$$

$\mathbb{P}^1_{s,x}$-almost surely, where as before $\widehat{\Gamma}_1$ denotes the infinitesimal generator of $-X$. It can be easily checked that $h'(0) = 0$ by simple differentiation and use of the definition of κ_*. Since, as before, $\overline{X}_t$ only increments when $\overline{X}_{t-} = X_{t-}$ (and this when the process creeps) it follows that the integral with respect to $\mathrm{d}\overline{X}_t$ above is zero.

Recall that $\left(\widehat{\Gamma}_1 + r\right)(\exp\{y\}) = 0$. Since in the regime $z \geq \kappa_*$ $h(z)$ is equal to $\exp\{z\}$, we have on $\{Y_t \geq \kappa_*\}$

$$\left(\widehat{\Gamma}_1 - (\alpha+\lambda)\right) h(Y_{t-})\mathrm{d}t + \lambda\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t \quad = \quad \mathrm{e}^{Y_t}(\lambda\mathrm{e}^{-(\alpha+\lambda)t} - p)\mathrm{d}t,$$

which is non-positive. From these inequalities we now have, as before, that $\mathbb{E}^1_{s,x}(U_t) \leq h(s-x)$ for all $t \geq 0$ and $s-x \geq 0$. Computations along the lines in the previous Lemma show that this is sufficient to conclude that U_t is a $\mathbb{P}^1_{s,x}$-supermartingale.

We finish the proof of optimal stopping as in the previous optimal stopping problem. Note that

$$h(z) = \mathrm{e}^z + (p-\lambda)\mathrm{e}^z \int_0^{\kappa_*-z} W^{(p)}(y)\,\mathrm{d}y \geq \mathrm{e}^z.$$

By the supermartingale property and Doob's optional stopping theorem, for all almost surely finite stopping times τ, it follows that

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+\lambda)\tau+Y_\tau} + \lambda\int_0^\tau \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right] \leq \mathbb{E}^1_{s,x}(U_\tau) \leq h(s-x).$$

Since we can make these inequalities equalities by choosing $\tau = \tau_{\kappa_*}$, we are done.

If $W^{(p)}(0^+) \in (0, (p-\lambda)^{-1})$, the use of the Change of Variable Formula is justified by the same arguments as used in the proofs of Theorems 2, 3. The proof then goes the same as above.

Finally, if $W^{(p)}(0^+) \geq (p-\lambda)^{-1}$, we see from Lemma 5 that $Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/p$ for all x positive and the proof runs analogously as the one of Theorem 2. Indeed, one should find for all $k \geq 0$ and almost surely finite $\mathbf{F}$-stopping times τ,

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right] \leq \frac{p-\lambda}{p}\mathrm{e}^{s-x}Z^{(p)}(k-s+x) + \frac{\lambda}{p}\mathrm{e}^{s-x}.$$

Taking $k=0$ in the previous display, we conclude that for all almost surely finite stopping times τ

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right] \leq \mathrm{e}^{s-x}$$

with equality for $\tau = 0$. □

8.2 Canadised American Put

As before, as an extension of the perpetual option and as a first approximation to the finite time counter part, we now consider the problem of finding a rational value for the American put option with time of expiration given by the independent exponential random variable $\eta(\lambda)$. We solve the corresponding optimal stopping problem by taking $\alpha = 0$ and finding a function w^{AC} and a stopping time τ_* such that

$$w^{AC}(x) = \sup \mathbb{E}_x[\mathrm{e}^{-r(\tau\wedge\eta(\lambda))}(K - S_{\tau\wedge\eta(\lambda)})^+] = \mathbb{E}_x[\mathrm{e}^{-r(\tau^*\wedge\eta(\lambda))}(K - S_{\tau^*\wedge\eta(\lambda)})^+],$$

where the supremum is taken over all $\mathbf{F}$-stopping times τ. We expect that as in the American optimal stopping problem the optimal stopping time will be of the form T_k^-.

Proposition 3 *Let $k \le \log K$. Then,*

$$\mathbb{E}_x[\mathrm{e}^{-r(T_k^- \wedge \eta(\lambda))}(K - S_{T_k^- \wedge \eta(\lambda)})^+] = \frac{Kr}{q} Z^{(q)}(x-k) + \frac{K\lambda}{q} Z^{(q)}(x - \log K) \\ - \mathrm{e}^x Z_1^{(\lambda)}(x - \log K) - CW^{(q)}(x-k) + \frac{\lambda}{\Phi_1(\lambda)} \mathrm{e}^x W_1^{(\lambda)}(x-k) \quad (44)$$

where $q = r + \lambda$ and

$$C = \frac{K}{\Phi(q)} \left(r + \lambda \mathrm{e}^{\Phi(q)(k - \log K)} \right) + \frac{\lambda}{\Phi_1(\lambda)} \left(\mathrm{e}^k - K \mathrm{e}^{\Phi(q)(k - \log K} \right).$$

Proof First note that since $\eta(\lambda)$ is independent of X we can write

$$\mathbb{E}_x[\mathrm{e}^{-r(T_k^- \wedge \eta(\lambda))}(K - S_{T_k^- \wedge \eta(\lambda)})^+] = \mathbb{E}_x[\mathrm{e}^{-qT_k^-}(K - S_{T_k^-})^+] \\ + \lambda \mathbb{E}_x \left[\int_0^{T_k^-} \mathrm{e}^{-qt}(K - \mathrm{e}^{X_t})^+ \mathrm{d}t \right]. \quad (45)$$

Rewriting the second expectation on the right hand side as

$$\mathbb{E}_x \left[\int_0^{T_k^-} \mathrm{e}^{-qt}(K - \mathrm{e}^{X_t})^+ \mathrm{d}t \right] = q^{-1} \int_k^{\log K} (K - \mathrm{e}^y) \mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) \le T_k^-),$$

we see it can be evaluated using the expression for the resolvent density of X killed upon entering $(-\infty, k]$, which can be extracted directly from [24, Lemma 1]. After some calculations and combined with Corollary 2 we find the formula as stated. □

Continuous and smooth fit or optimization leads to

$$k_* = \log K + \frac{1}{\Phi(r+\lambda)} \log \left(r \frac{\Phi_1(\lambda)}{\lambda} \right)$$

as a candidate for the optimal exercise level.

Theorem 5 *Let $q = \lambda + r$ and define the function $v : \mathbb{R} \to \mathbb{R}$ by*

$$v(x) = KrZ^{(q)}(x - k_*)/q + K\lambda Z^{(q)}(x - \log K)/q - \mathrm{e}^x Z_1^{(\lambda)}(x - \log K).$$

Then the solution of the Canadised American Put optimal stopping problem is given by $w^{AC} = v$ where $\tau_* = T_{k_*}^-$ is the optimal stopping time.

The proof of this Theorem is left to the reader since it is in principle similar in nature to the case of the Canadised Russian optimal stopping problem. The following Lemma, whose proof is also left to the reader for the same reasons, serves as an interim step.

Lemma 6 *Consider the function v and the level k_* as defined in Theorem 5. Then $v(x) \ge (K - \mathrm{e}^x)^+$ and*

$$\left\{ \mathrm{e}^{-qt} v(X_t) + \lambda \int_0^t \mathrm{e}^{-qs}(K - \mathrm{e}^{X_s})^+ \mathrm{d}s : t \ge 0 \right\}$$

is a $\mathbb{P}$-supermartingale and a $\mathbb{P}$-martingale when stopped at $T_{k_*}^-$.

9 Examples

In this section we provide some explicit examples of the foregoing theory and check these against known results in the literature.

9.1 Exponential Brownian motion

In the case of the classical Black-Scholes geometric Brownian motion model the functions $W^{(q)}$ and $Z^{(q)}$ are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2 \epsilon} \mathrm{e}^{\gamma x} \sinh(\epsilon x), \; Z^{(q)}(x) = \mathrm{e}^{\gamma x} \cosh(\epsilon x) - \frac{\gamma}{\epsilon} \mathrm{e}^{\gamma x} \sinh(\epsilon x)$$

on $x \geq 0$ where $\epsilon = \epsilon(q) = \sqrt{(\frac{r}{\sigma^2} - \frac{1}{2})^2 + \frac{2q}{\sigma^2}}$ and $\gamma = \frac{1}{2} - \frac{r}{\sigma^2}$. Note $\gamma \pm \epsilon$ are the roots of $\frac{\sigma^2}{2}\theta^2 + (r - \frac{\sigma^2}{2})\theta - q = 0$. Let κ^* be given by

$$\exp(\kappa^*) = \left(\frac{\epsilon - \gamma + 1}{\epsilon + \gamma - 1} \cdot \frac{\epsilon + \gamma}{\epsilon - \gamma} \right)^{1/2\epsilon}, \tag{46}$$

then after some algebra we find the value function for the Russian optimal stopping problem is

$$w^R(x,s) = \mathrm{e}^s \left[\frac{\epsilon + \gamma}{2\epsilon} \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}} \right)^{\epsilon - \gamma} + \frac{\epsilon - \gamma}{2\epsilon} \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}} \right)^{-\epsilon - \gamma} \right]$$

for $s - x \in [0, \kappa^*)$ and e^s otherwise. This expression is the same as Shepp and Shiryaev [116, 117] found. In the same vein we find an expression for the Canadised Russian case. Indeed, let κ_* be the unique positive root of

$$(\epsilon - \gamma + 1)(\epsilon + \gamma)\mathrm{e}^{-\epsilon x} - (\epsilon + \gamma - 1)(\epsilon - \gamma)\mathrm{e}^{\epsilon x} - 2q^{-1}\epsilon\lambda \mathrm{e}^{-\gamma x} = 0,$$

where $\epsilon = \epsilon(p)$ for $p = r + \alpha + \lambda$. Then we find the value function to be given by

$$w^{CR}(s,x) = \mathrm{e}^s \left[\frac{q}{q+\lambda} \left(\frac{\epsilon + \gamma}{2\epsilon} \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa_*}} \right)^{\epsilon - \gamma} + \frac{\epsilon - \gamma}{2\epsilon} \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa_*}} \right)^{-\epsilon - \gamma} \right) + \frac{\lambda}{q+\lambda} \right]$$

for $s - x \in [0, \kappa_*)$ and e^s otherwise. Now we turn our attention to the perpetual American Put option. Note $Z_1^{(\lambda)}$ is equal to the expression for $Z^{(q)}$ but now with γ replaced by $\gamma - 1$. Plugging in the formulas for the scale functions $Z^{(q)}$ and $Z_1^{(\lambda)}$ in Theorem 3 and reordering the terms the value function is seen to be equal to

$$w^A(x) = \frac{K}{\epsilon - \gamma + 1} \exp\left((\gamma - \epsilon)(x - k^*)\right)$$

for $x \geq k^*$ and $K - \mathrm{e}^x$ otherwise. Here the optimal crossing level is $k^* = \log K \frac{q}{\lambda} \frac{\gamma + \epsilon - 1}{\gamma + \epsilon}$. Finally, we consider the case of the Canadised American put.

Define the functions $\mathbb{R} \to \mathbb{R}$

$$b(x) = K\frac{r}{q} \cdot \frac{\epsilon+\gamma}{2\epsilon}\mathrm{e}^{(\gamma-\epsilon)x} \qquad c(x) = K\frac{\lambda}{q} \cdot \frac{\epsilon-\gamma}{2\epsilon} \cdot \frac{1}{\epsilon+\gamma-1}\mathrm{e}^{(\gamma+\epsilon)x}$$

$$p(x) = K\mathrm{e}^{(\gamma-\epsilon)x}\left[\frac{\lambda}{q}\frac{\gamma+\epsilon}{2\epsilon} - \frac{\gamma+\epsilon-1}{2\epsilon}\right].$$

See Carr [36] for the special interpretation of b, c, p. For

$$k_* = \log K + \frac{1}{\epsilon+\gamma}\log(r(\epsilon+\gamma-1)/\lambda),$$

we find after some algebra that the value function for the Canadised American put problem can be represented by

$$w^{CA}(x) = \begin{cases} p(x-\log K) + b(x-k_*) & \text{if } x \geq \log K, \\ K\frac{\lambda}{q} - \mathrm{e}^x + b(x-k_*) + c(x-\log K) & \text{if } x \in (k_*, \log K), \\ K - \mathrm{e}^x & \text{if } x \leq k_*. \end{cases}$$

Recalling that $q = \lambda + r$ and taking λ to be T^{-1} this formula agrees with formula (8) in [36].

9.2 Jump-diffusion with hyperexponential jumps

Let $X = \{X_t, t \geq 0\}$ be a jump-diffusion given by

$$X_t = (a - \sigma^2/2)t + \sigma W_t - \sum_{i=1}^{N_t} Y_i,$$

where $\sigma > 0$, N is a Poisson process and $\{Y_i\}$ is a sequence of i.i.d. random variables with hyper-exponantial distribution

$$F(y) = 1 - \sum_{i=1}^{n} A_i \mathrm{e}^{-\alpha_i y}, \qquad y \geq 0,$$

where $A_i \geq 0$; $\sum_i A_i = 1$; and $0 \leq \alpha_1 \leq \ldots \leq \alpha_n$. The processes W, N, Y are independent. We claim that for $x \geq 0$ the function $Z^{(q)}$ of X is given by

$$Z^{(q)}(x) = \sum_{i=0}^{n+1} D_i \mathrm{e}^{\theta_i x}$$

where $\theta_i = \theta_i(q)$ are the roots of $\psi(\theta) = q$, where $\theta_{n+1} \geq 0$ and the rest of the roots are negative, and where

$$D_i = \prod_{k=1}^{n}(\theta_i/\alpha_k + 1) \Bigg/ \prod_{k=0, k\neq i}^{n+1} (\theta_i/\theta_k - 1).$$

Indeed, recall that $\psi(\lambda)/\lambda(\psi(\lambda)-q))$ is the Laplace transform of $Z^{(q)}$ and note that

$$\begin{aligned} D_i &= \frac{1}{\theta_i}\frac{\prod_{k=0}^{n+1}(-\theta_k)}{\prod_{k=0,k\neq i}^{n+1}(\theta_k-\theta_i)}\frac{\prod_{k=1}^{n}(\theta_i+\alpha_k)}{\prod_{k=1}^{n}\alpha_k} \\ &= \frac{q}{\theta_i}\frac{\prod_{k=1}^{n}(\theta_i+\alpha_k)}{\prod_{k=0,k\neq i}^{n+1}(\theta_k-\theta_i)} = \frac{\psi(\theta_i)}{\theta_i}\frac{1}{\psi'(\theta_i)} \end{aligned}$$

are the coefficients in the partial fraction expansion of $\psi(\lambda)/(\lambda(\psi(\lambda)-q))$. Hence we find for the value function of the Russian option $w^R = w^R_{\alpha,\kappa^*}$

$$w^R_{q,\kappa^*}(s,x) = \mathrm{e}^x \begin{cases} \sum_{i=0}^{n+1} D_i \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}}\right)^{1-\theta_i} & s-x\in[0,\kappa^*), \\ \mathrm{e}^{s-x} & s-x\geq\kappa^*; \end{cases}$$

where $\kappa^*\geq 0$ is the root of $r(x,\alpha) := \sum_{i=0}^{n+1}(\theta_i-1)D_i\mathrm{e}^{x\theta_i} = 0$. The Canadised Russian has then value function $w^{CR}(s,x) = uw^R_{\alpha+\lambda,\kappa_*}(s,x) + (1-u)\exp(s)$, where $u = (\alpha+\lambda)/(\alpha+\lambda+r)$ and κ_* is the root of $ur(x,\alpha+\lambda)+(1-u)\exp(x) = -\lambda/(\alpha+r)$.

In the case of the American put option, we find after some algebra that

$$w^A(x) = \begin{cases} K-\mathrm{e}^x & x\leq k^*, \\ K\sum_{i=0}^{n}\frac{D_i}{\theta_i-1}\frac{\theta_i-\theta_{n+1}}{\theta_{n+1}}\mathrm{e}^{\theta_i(x-k^*)} & x>k^*, \end{cases}$$

where $k^* = \log\left(K\frac{q}{\lambda}\frac{\theta_{n+1}-1}{\theta_{n+1}}\right)$. These two formulas can be checked to agree with the results in the literature [102, 100]. Finally, we consider the Canadised American put option. The value function w^{CA} can then be checked to be given by

$$w^{CA}(x) = \begin{cases} K-\mathrm{e}^x \\ K\frac{\lambda}{q}-\mathrm{e}^x+K\frac{r}{q}\sum_{i=0}^{n+1}D_i\mathrm{e}^{\theta_i(x-k_*)} \\ \frac{K}{q}\sum_{i=0}^{n}D_i\left[r\left(\frac{r}{\lambda}(\theta_{n+1}-1)\right)^{-\theta_i/\theta_{n+1}}-\lambda(\theta_i-1)^{-1}\right]\mathrm{e}^{\theta_i(x-\log K)} \end{cases}$$

for $x\leq k_*$, $x\in(k_8,\log K)$ and $x\geq\log K$ respectively, where $k_* = \log K + \log\left(\frac{r}{\lambda}(\theta_{n+1}-1)\right)/\theta_{n+1}$.

9.3 Stable jumps

We model X as

$$X_t = \sigma Z_t,$$

where Z is a standard stable process of index $\gamma\in(1,2]$. Its cumulant is given by $\psi(\theta) = (\sigma\theta)^\gamma$. Note the martingale restriction amounts to $1 = \sigma^\gamma$. By inverting the Laplace transform $(\psi(\theta)-q)^{-1}$, Bertoin [22] found that the q-scale function is given by

$$W^{(q)}(x)=\gamma\frac{x^{\gamma-1}}{\sigma^{\gamma}}E'_{\gamma}\left(q\frac{x^{\gamma}}{\sigma^{\gamma}}\right),\qquad x\geq 0$$

and hence $Z^{(q)}(x)=E_{\gamma}(q(x/\sigma)^{\gamma})$ for $x\geq 0$, where E_{γ} is the Mittag-Leffler function of index γ

$$E_{\gamma}(y)=\sum_{n=0}^{\infty}\frac{y^{n}}{\Gamma(1+\gamma n)},\qquad y\in\mathbb{R}.$$

From Theorem 2, 4 , 3 and 5 we can find closed formulas for the (Canadised) Russian and American put option. In particular, we note that for the American put and its Canadised version the optimal excercise levels k^{*},k_{*} are respectively given by

$$k^{*}=\log K\frac{q}{\alpha}\frac{q^{1/\gamma}-\sigma}{q^{1/\gamma}}\qquad k_{*}=\log K+\frac{\sigma}{q^{1/\gamma}}\log\frac{r(q^{1/\gamma}-\sigma)}{\sigma\lambda}.$$

Chapter IV

On exit and ergodicity of reflected Lévy processes

Consider a spectrally one-sided Lévy process X and reflect it at its past infimum I. Call this process Y. For spectrally positive X, Avram et al. [16] found an explicit expression for the law of the first time that $Y = X - I$ crosses a finite positive level a. Here we determine the Laplace transform of this crossing time for Y, if X is spectrally negative. Subsequently, we find an expression for the resolvent measure for Y killed upon leaving $[0, a]$. We determine the exponential decay parameter ϱ for the transition probabilities of Y killed upon leaving $[0, a]$, prove that this killed process is ϱ-positive and specify the ϱ-invariant function and measure. Restricting ourselves to the case where X has absolutely continuous transition probabilities, we also find the quasi-stationary distribution of this killed process. We construct then the process Y confined in $[0, a]$ and prove some properties of this process.

1 Introduction

A spectrally one-sided Lévy process is a real-valued stochastic process with stationary and independent increments which has jumps of one sign. In this paper we will study such a Lévy process reflected at its past infimum, that is, the Lévy process minus its past infimum. In applied probability, these reflected processes frequently occur, for example in the study of the water level in a dam, the work load in a queue or the stock level (See e.g. [3, 31, 110] and references therein.) Moreover, the reflected Lévy process occurs in relation with a problem associated with mathematical finance. See [85] and references therein.

The paper consists of three parts. In the first part, we study the level-crossing probabilities of the reflected Lévy process. For spectrally positive Lévy processes X, Avram et al. [16] found an explicit expression for the Laplace transform of the first exit-time of the reflected process from $[0, a]$. In Section

4.2 we complement this study by obtaining the Laplace transform of the exit time for the reflected process of the dual, a spectrally negative Lévy process. Subsequently, in Section 5, we solve for the resolvent measure of the transition probabilities of the reflected Lévy process killed upon leaving $[0, a]$.

From the study of [16] and Section 4.2, it appears that the Laplace transforms of the exit times of X and Y from finite and semi-finite intervals can be expressed in terms of $W^{q)}$ and $Z^{(q)}$, where $Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y$ and $W^{(q)}$ is the scale function of a spectrally negative Lévy process X that is killed at an independent exponential time with parameter q. See forthcoming Section 2 for a precise definition. The function $Z^{(q)}$ first occurred, although implicitly, in [22, 24]. By analogy of the theory of diffusions, the function $W^{(q)}$ is called a q-scale function since $\{\exp(-q(\widehat{T} \wedge t))W^{(q)}(X_{\widehat{T}\wedge t}), t \geq 0\}$ is a martingale where $\widehat{T}$ the first exit time of the positive real line. In Section 4.3 we show that the function $Z^{(q)}$ has an analogous property for the reflected process Y: $\{\exp(-qt)Z^{(q)}(Y_t), t \geq 0\}$ is a martingale. Therefore we call $Z^{(q)}$ the *adjoint* q-scale function for X.

Bertoin [24] investigated the exponential decay and ergodicity for completely asymmetric Lévy processes killed upon leaving a finite interval. The purpose of the second part is to extend Bertoin's study to reflected Lévy processes killed upon up-crossing a finite level. We determine the exponential decay parameter ϱ of the semi-group, prove that the process is ϱ-positive in the classification of Tuominen and Tweedie [124] and specify the ϱ-invariant function and measure. Restricting ourselves to Lévy processes whose one-dimensional distributions are absolutely continuous with respect to the Lebesgue measure, we also find the quasi-stationary distribution. Section 8 contains the main results in that direction. Section 7 contains a study of the transition probabilities of the reflected Lévy process with preparatory results.

Important elements in the proof of the ergodic properties and the exponential decay are the special form of the earlier computed resolvent measure together with special properties of fluctuation theory of completely asymmetric Lévy processes, elementary properties of analytic functions and the R-theory developed by Tuominen and Tweedie [124] for a general irreducible Markov process.

The third part, to be found in section 9, starts with the construction by h-transform of the reflected process conditioned to stay below the level a. We study then this process: we show that it is a positively recurrent Markov process and determine its stationary measure. If the one-dimensional distributions of the Lévy process are absolutely continuous, we observe that, as a direct consequence of the results of the second part mentioned above, the conditional probabilities of the reflected Lévy process conditioned on the fact that it exits $[0, a]$ after t, converge as t tends to infinity. The process constructed this way coincides with the earlier mentioned h-transform. Finally, we use excursion theory to determine the rate of convergence of the supremum of the reflected process to a. The obtained results reveal a similar pattern as those achieved by Lambert [86] in his study of a completely asymmetric Lévy process confined in a finite interval.

2 Preliminaries

This section reviews standard results on spectrally negative Lévy processes. For more background we refer to [25] or [23], Chapter VII.

Let $X = \{X_t, t \geq 0\}$ be a Lévy process without positive jumps defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. For all x the measure $\mathbb{P}_x$ will denote the translation of $\mathbb{P}$ under which $X_0 = x$. To avoid trivialities, we exclude the case where X has monotone paths. Since X has no positive jumps, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and is given by

$$\mathbb{E}[\exp(\theta X_t)] = \exp(t\ \psi(\theta))$$

for some function $\psi(\theta)$ which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0), \infty)$ the function ψ is strictly increasing and we denote its right-inverse function by $\Phi : [0, \infty) \to [\Phi(0), \infty)$. It is well known, that the asymptotic behaviour of X can be determined from the sign of $\psi'_+(0)$, the right-derivative of ψ at zero. Indeed, X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\psi'_+(0)$ is negative, zero or positive.

We use the notations $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$. Denote by I and S the past infimum and supremum of X respectively, that is,

$$I_t = \inf_{0\leq s\leq t}(X_t \wedge 0), \qquad S_t = \sup_{0\leq s\leq t}(X_t \vee 0)$$

and introduce the notations $Y = X - I$ and $\widehat{Y} = \widehat{X} - \widehat{I} = S - X$ for the Lévy process X reflected at its past infimum I and its dual, the process X reflected at its supremum. Denote by $\eta(q)$ an exponential random variable with parameter $q > 0$ which is independent of X. The Wiener-Hopf factorisation of X implies that $Y_{\eta(q)}$ and $I_{\eta(q)}$ are independent, where $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ and

$$\mathbb{E}[\exp(\theta I_{\eta(q)})] = \frac{q}{q - \psi(\theta)} \cdot \frac{\Phi(q) - \theta}{\Phi(q)}. \tag{1}$$

By time reversal one can show that the pairs $(Y_{\eta(q)}, -I_{\eta(q)})$ and $(S_{\eta(q)}, \widehat{Y}_{\eta(q)})$ have the same distribution.

3 Scale functions

As in e.g. [24, 16], a crucial role will be played by the function $W^{(q)}$, which is closely connected to the two-sided exit problem. To be precise we give a definition for $W^{(q)}$ and review some of its properties.

Definition 1 *For $q \geq 0$ the q-scale function $W^{(q)} : (-\infty, \infty) \to [0, \infty)$ is the unique function whose restriction to $[0, \infty)$ is continuous and has Laplace* *transform*

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q),$$

and is defined to be identically zero for $x < 0$.

By taking $q = 0$ we get the 0-scale function which is usually called just "the scale function" in the literature [23]. For $q > 0$, $W^{(q)}$ can be regarded as "the scale function" of the original process X killed at an independent exponential time with parameter q. It is known that $W = W^{(0)}$ is increasing, when restricted to $(0, \infty)$. Moreover, the value of W at 0 and infinity is connected to certain global properties of X. Indeed, $W(0)$ is zero precisely if X has unbounded variation. Secondly, $W(\infty) = \lim_{x\to\infty} W(x)$ is finite, precisely if X drifts to ∞, which follows from a Tauberian theorem in conjunction with the earlier mentioned fact that $\psi'_+(0) > 0$ if and only if X drifts to ∞.

Inverting now the Laplace transform (1), we find that

$$\mathbb{P}(\widehat{Y}_{\eta(q)} \in \mathrm{d}y) = \frac{q}{\Phi(q)} W^{(q)}(\mathrm{d}y) - qW^{(q)}(y)\mathrm{d}y, \qquad y \geq 0, \tag{2}$$

where $W^{(q)}(\mathrm{d}y)$ denotes the Stieltjes measure associated with $W^{(q)}$ with mass $W^{(q)}(0)$ at zero.

For every fixed x, we can extend the mapping $q \mapsto W^{(q)}(x)$ to the complex plane by the identity

$$W^{(q)}(x) = \sum_{k\geq 0} q^k W^{\star k+1}(x), \tag{3}$$

where $W^{\star k}$ denoted the k-th convolution power of $W = W^{(0)}$. The convergence of this series is plain from the inequality

$$W^{\star k+1}(x) \leq x^k W(x)^{k+1}/k! \qquad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of W. From the expansion (3) and the properties of W, we see that the q-scale function is continuous except possibly at zero and that it is increasing on $(0, \infty)$ for each $q \geq 0$.

Closely related to $W^{(q)}$ is the function $Z^{(q)}$. We recall the definition given in [16].

Definition 2 *The adjoint q-scale function $Z^{(q)}$ is defined by*

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(z)\mathrm{d}z. \tag{4}$$

Note that this function inherits some properties from $W^{(q)}(x)$. Specifically it is increasing, differentiable and strictly convex on $(0, \infty)$ and is equal to the constant 1 for $x \leq 0$. Moreover, if X has unbounded variation, $Z^{(q)}$ is C^2 on $(0, \infty)$.

Example 1 *A stable Lévy process X with index $\alpha \in (1,2]$ has as cumulant $\psi(\theta) = \theta^\alpha$; its scale function and adjoint are respectively computed in [22] as*

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha) \qquad Z^{(q)}(x) = E_\alpha(qx^\alpha),$$

where E_α is the Mittag-Leffler function with parameter α

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \qquad y \in \mathbb{R}. \tag{5}$$

In the case $\alpha = 2$, the process $X/\sqrt{2}$ is a Brownian motion and $W^{(q)}, Z^{(q)}$ reduce to

$$W^{(q)}(x) = q^{-\frac{1}{2}} \sinh(x\sqrt{q}) \qquad Z^{(q)}(x) = \cosh(x\sqrt{q}). \tag{6}$$

Hence for a standard Brownian motion $W^{(q)}, Z^{(q)}$ are found by replacing (x,q) by $(2x, q/2)$ in (6).

For later reference, we give four lemmas with some properties of $W^{(q)}$ and $Z^{(q)}$ which we will need later on.

Lemma 1 *The function $x \mapsto W^{(q)}(x)$ is right- and left-differentiable on $(0,\infty)$. Moreover, if X has unbounded variation or its Lévy measure has no atoms, $W^{(q)}$ is continuously differentiable on $(0,\infty)$.*

By $W^{(q)\prime}_\pm(x)$, we will denote the right and left-derivative of $W^{(q)}$ in x, respectively.

Proof In the proof of theorem VII.8 in [23] Bertoin shows that W satisfies for some constant K, $W(x) = K\exp(-\int_x^\infty \widehat{n}(h > t)\mathrm{d}t)$, where $\widehat{n}$ is the Itô excursion measure of $\widehat{Y} = S - X$ and h are excursion heights of excursions of $\widehat{Y}$ away from zero. From this representation, we deduce that

$$W'_+(x) = W(x)\widehat{n}(h > x) \qquad W'_-(x) = W(x)\widehat{n}(h \geq x). \tag{7}$$

It can be shown that the distribution of h under excursion measure $\widehat{n}$ has no atoms if X has unbounded variation (see [86]) or if X has bounded variation but its Lévy measure Λ has no atoms (one way to see this is to invoke equation (20) to show that $\widehat{n}(h = x) = \mathtt{d}^{-1}\Lambda(\{-x\})W(0)/W(x)$). Hence, under these conditions, W restricted to $(0,\infty)$ is continuously differentiable. Using the expansion (3) and the monotonicity of W, it is not hard to prove that the properties of W carry over to $W^{(q)}$ (see [86, Prop. 5.1]). □

The second lemma is immediate from (3) and the definitions of $Z^{(q)}$ and $W^{(q)\prime}_\pm$.

Lemma 2 *The mapping $(x,q) \mapsto Z^{(q)}(x)$ is continuous on $[0,\infty) \times \mathbb{R}$ and, for every $x \geq 0$, $q \mapsto Z^{(q)}(x)$ and $q \mapsto W^{(q)\prime}_\pm(x)$ are analytic functions.*

Using the expansion (3), one can check the following convolution identities to be true:

Lemma 3 *For $q, r \in \mathbb{C}$ and $a > 0$ we have*

$$W^{(q)} \star W^{(r)}(a) = \frac{1}{r-q}(W^{(r)}(a) - W^{(q)}(a)),$$
$$W^{(q)} \star \mathrm{d}W^{(r)}(a) = \frac{1}{r-q}(W^{(r)\prime}(a) - W^{(q)\prime}(a))$$

where for $q = r$ the expression is to be understood in the limiting sense and where $W^{(q)} \star \mathrm{d}W^{(r)}(a) = \int_0^a W^{(q)}(a-x)W^{(r)}(\mathrm{d}x)$.

The following result concerns the asymptotic behaviour of $W^{(q)}$ and $Z^{(q)}$. We write $f \sim g$ if $\lim(f/g) = 1$.

Lemma 4 **(i)** *For $q > 0$, we have as $x \to \infty$*

$$W^{(q)}(x) \sim \mathrm{e}^{\Phi(q)x}/\psi'(\Phi(q)), \qquad Z^{(q)}(x) \sim q\mathrm{e}^{\Phi(q)x}/\left(\Phi(q)\psi'(\Phi(q))\right).$$

(ii) *As $x \downarrow 0$ the ratio $(W(x) - W(0)/x$ converges to a positive constant or to $+\infty$.*

Proof (i) We can straightforwardly check that the Laplace-Stieltjes transforms of the functions $U(x) :=\mathrm{e}^{-\Phi(q)x}W^{(q)}(x)$ and $\widetilde{U}(x) := \mathrm{e}^{-\Phi(q)x}(Z^{(q)}(x) - 1)$ are given by

$$\int_0^\infty \mathrm{e}^{-\lambda x}U(\mathrm{d}x) = \frac{\lambda}{\psi(\lambda + \Phi(q)) - q} = \frac{\lambda}{\psi(\lambda + \Phi(q)) - \psi(\Phi(q))};$$
$$\int_0^\infty \mathrm{e}^{-\lambda x}\widetilde{U}(\mathrm{d}x) = \frac{q\lambda}{(\Phi(q) + \lambda)(\psi(\lambda + \Phi(q)) - q)},$$

where $\mathrm{d}U, \mathrm{d}\widetilde{U}$ denote the Stieltjes measure associated to $U, \widetilde{U}$ respectively, which respectively assign masses $W^{(q)}(0)$ and 0 to zero. Since $\psi'(\Phi(q)) > 0$, the statements follow using a Tauberian theorem (e.g. [23, p.10]).
(ii) Recall that the Laplace-Stieltjes transform of W is given by $\lambda/\psi(\lambda)$. If the Brownian coefficient $s := \lim_{\lambda\to\infty}\psi(\lambda)/\lambda^2$ is positive, the same Tauberian theorem implies that $W(x) \sim x/s$ for x tending to infinity. Set $\mathtt{d} := \lim_{\lambda\to\infty}\lambda/\psi(\lambda)$. If $s = \mathtt{d}^{-1} = 0$, that is X has no Brownian component and unbounded variation, we find, again using the Tauberian theorem, that $W(x)/x$ tends to infinity for $x \downarrow 0$. Similarly, if the Lévy measure of X has finite mass m, we can check $W(x) - W(0) \sim mx/\mathtt{d}$, whereas for X with bounded variation but infinite mass of the Lévy measure we can verify that $(W(x) - W(0))/x$ tends to infinity as $x \downarrow 0$. □

4 Exit problems

4.1 Two-sided exit

We now turn our attention to the two-sided exit problem and review the main results. Denote the passage times $\widehat{T}_a, T_a$ for X and $-X$ above and below the level a by

$$T_a = \inf\{t \geq 0 : X_t > a\} \qquad \widehat{T}_a = \inf\{t \geq 0 : -X_t > a\}.$$

The following result, the origins of which go back to Takács [122], expresses the (discounted) probabilities of exiting the interval $[0, a]$ above and below in terms of $W^{(q)}$ and $Z^{(q)}$.

Proposition 1 *For $q \geq 0$, the Laplace transforms of the two-sided exit time $\widehat{T}_0 \wedge T_a$ on the part of the probability space where X starts at $x \in [0, a]$ and exits the interval $[0, a]$ above and below are respectively given by*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_a} I_{(\widehat{T}_0 > T_a)}\right] = W^{(q)}(x)/W^{(q)}(a); \tag{8}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\widehat{T}_0} I_{(\widehat{T}_0 < T_a)}\right] = Z^{(q)}(x) - W^{(q)}(x)Z^{(q)}(a)/W^{(q)}(a). \tag{9}$$

Proof For $x \in (0, a)$, this result can be extracted directly out of existing literature. See for example [23, Thm. VII.8] for a proof of (8) using excursion theory. Combining this with [24, Cor 1], we find equation (9). Note by a small typographic mistake in [24] $\int_0^x W^{(q)}(x)\mathrm{d}x$ is used instead of $q\int_0^x W^{(q)}(y)\mathrm{d}y = Z^{(q)}(x) - 1$. Since 0 is regular for $(0, \infty)$ for X, the identities hold for $x = a$. Similarly, they hold for $x = 0$ if X has unbounded variation. If X has bounded variation, 0 is irregular for $(-\infty, 0)$ and hence $\widehat{T}_0 > 0$ almost surely. Since $T_\epsilon \downarrow 0$ almost surely if $\epsilon \downarrow 0$, the strong Markov property implies that

$$\begin{aligned}
\mathbb{E}_0\left[\mathrm{e}^{-qT_a} I_{(\widehat{T}_0 > T_a)}\right] &= \lim_{\epsilon \downarrow 0} \mathbb{E}_0\left[\mathrm{e}^{-qT_\epsilon} I_{(\widehat{T}_0 > T_\epsilon)} W^{(q)}(\epsilon)/W^{(q)}(a)\right] \\
&= W^{(q)}(0)/W^{(q)}(a),
\end{aligned}$$

whence (8) is valid for $x = 0$ as well. Analogously (9) is shown to hold for $x = 0$. □

Remark. Let n be the Itô-excursion measure associated to the excursions of Y away from zero and let h, ζ denote the height and lifetime of the generic excursion respectively. In [23, Prop VII.15] Bertoin related n and the scale function W as follows

$$n(h > a) = W(a)^{-1}.$$

The expansion (3) implies that $W^{(q)}(0) = W(0)$ and that $W^{(q)\prime}(0^+) = W'(0^+)$ if X has unbounded variation. Using Propositions 14 and 15 in [23] combined with Proposition 1, we find then the following links between n, $W^{(q)}$ and $Z^{(q)}$:

$$\begin{aligned}
n(\mathrm{e}^{-qT_a}, h > a) &= \lim_{x \downarrow 0} \frac{\mathbb{E}_x(\mathrm{e}^{-qT_a} I_{(\widehat{T}_0 > T_a)})}{W(x)} = W^{(q)}(a)^{-1}, \\
n(\mathrm{e}^{-q\zeta}, h > a) &= \frac{\mathbb{E}_a[\mathrm{e}^{-q\widehat{T}_0}]}{W^{(q)}(a)} = \frac{Z^{(q)}(a)}{W^{(q)}(a)} - \frac{q}{\Phi(q)},
\end{aligned}$$

where we used the strong Markov property and upward creeping of Y. Letting $q \to 0$ in the last equation yields that $n(h > a, \zeta < \infty) = n(h > a) + 1/W(\infty)$, which agrees with the fact that in case Y is not recurrent the excursions of Y away from zero form a Poisson point process stopped at rate $1/W(\infty)$.

4.2 Mixed exit

As a next step, we study exit problems of $[0, a]$ for the reflected Lévy processes Y and $\widehat{Y}$. The first passage time of a positive level $a > 0$ will be denoted by

$$\tau_a = \inf\{t \geq 0 : Y_t > a\} \qquad \text{and} \qquad \widehat{\tau}_a = \inf\{t \geq 0 : \widehat{Y}_t > a\},$$

where we will use τ_0 and $\widehat{\tau}_0$, respectively, to denote the first time that Y and $\widehat{Y}$ hit zero. The following result expresses the Laplace transforms of the exit times τ_a and $\widehat{\tau}_a$ in terms of the scale functions $W^{(q)}$ and $Z^{(q)}$. Note that $X_0 = x$ and hence $Y_0 = x$ under $\mathbb{P}_x$. Similarly, we see that $\widehat{Y}$ starts from x under the measure $\mathbb{P}_{-x}$.

Proposition 2 *Let $x \in [0, a]$ and $q \geq 0$. Then we have*

$$\textbf{(i)} \qquad \mathbb{E}_x[\mathrm{e}^{-q\tau_a}] = Z^{(q)}(x)/Z^{(q)}(a).$$

$$\textbf{(ii)} \qquad \mathbb{E}_{-x}[\mathrm{e}^{-q\widehat{\tau}_a}] = Z^{(q)}(a-x) - qW^{(q)}(a-x)W^{(q)}(a)/W_+^{(q)\prime}(a).$$

By analyticity in q (Lemma 2) and monotone convergence, we find from Proposition 2 the following expressions for the expectations of the stopping times τ_a and $\widehat{\tau}_a$ for $x \in [0, a]$:

$$\mathbb{E}_x[\tau_a] = \overline{W}(a) - \overline{W}(x), \qquad \mathbb{E}_{-x}[\widehat{\tau}_a] = W(a-x)\frac{W(a)}{W_+'(a)} - \overline{W}(a-x), \qquad (10)$$

where $\overline{W}(x) = \int_0^x W(y)\mathrm{d}y$. If X is a standard Brownian motion, we recall the form of the q-scale function given in the example in Section 3 and we find back the following well known identities:

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a}] = \cosh(x\sqrt{2q})/\cosh(a\sqrt{2q}), \qquad \mathbb{E}_x[\tau_a] = (a^2 - x^2)/2,$$

for $q \geq 0$ and $x \in [0, a]$.

Proof (ii) Denote by $\mathcal{D}$ the subset of $a \in (0, \infty)$ where $W_+^{(q)\prime}(a) > 0$. Since $x \mapsto W^{(q)}(x)$ is increasing, the complement of $\mathcal{D}$ in $(0, \infty)$ is a closed set with empty interior. For any $a \in \mathcal{D}$ the Laplace transform in (ii) can be directly inferred from Theorem 1 in [16]. However, for any positive $a \notin \mathcal{D}$, we would immediately reach a contradiction in view of the form of the Laplace transform of $\widehat{\tau}_a$ and the fact that $\widehat{\tau}_a$ is increasing in a. Hence (ii) holds for all $a > 0$ and $\mathcal{D} = (0, \infty)$.

(i) To prove the form of the first Laplace transform, we use ideas developed in [16]. From the two-sided exit probability (8) we can extract that

$$M_t = \exp(-q(t \wedge \widehat{T}_0 \wedge T_a))W^{(q)}(X(t \wedge \widehat{T}_0 \wedge T_a)) \qquad t \geq 0,$$

is a martingale. Indeed, combining (8) and the fact that $W^{(q)}(X_{\widehat{T}_0 \wedge T_a})/W^{(q)}(a)$ is almost surely equal to the indicator of $\{\widehat{T}_0 > T_a\}$, we find for $x \in \mathbb{R}$

$$\mathbb{E}_x[\mathrm{e}^{-q(\widehat{T}_0 \wedge T_a)}W^{(q)}(X_{\widehat{T}_0 \wedge T_a})] = W^{(q)}(x).$$

Combined with the Markov property of X we then see that

$$\begin{aligned}\mathbb{E}_x(\mathrm{e}^{-q(\widehat{T}_0\wedge T_a)}W^{(q)}(X_{\widehat{T}_0\wedge T_a})|\mathcal{F}_t) &= \mathrm{e}^{-qt}W^{(q)}(X_t)\mathbf{1}_{\{t<\widehat{T}_0\wedge T_a\}} \\ &\quad + \mathrm{e}^{-q(\widehat{T}_0\wedge T_a)}W^{(q)}(X_{\widehat{T}_0\wedge T_a})\mathbf{1}_{\{t\geq\widehat{T}_0\wedge T_a\}} \\ &= \mathrm{e}^{-q(t\wedge\widehat{T}_0\wedge T_a)}W^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a}),\end{aligned}$$

so that we have constance of expectation. Similarly, the martingale property follows. Exactly in the same vein, now using the exit probability in equation (9), we conclude that

$$\mathrm{e}^{-q(t\wedge\widehat{T}_0\wedge T_a)}\left(Z^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a}) - W^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a})\tfrac{Z^{(q)}(a)}{W^{(q)}(a)}\right), \qquad t\geq 0$$

is a martingale. By taking a linear combination, we conclude that

$$\mathrm{e}^{-q(t\wedge\widehat{T}_0\wedge T_a)}Z^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a}), \qquad t\geq 0$$

is a martingale as well. Recall that $Z^{(q)}(\cdot)$ is once (twice) continuously differentiable on $(0,\infty)$ if X has (un)bounded variation, respectively. Applying Itô's lemma to $\mathrm{e}^{-qt}Z^{(q)}(X_t)$ (Theorems II.31(32) in [111] in the case where X has (un)bounded variation) on the set $\{t\leq\widehat{T}_0\}$, we find that

$$\mathrm{e}^{-q(t\wedge\widehat{T}_0)}Z^{(q)}(X_{t\wedge\widehat{T}_0}) - \int_0^{t\wedge\widehat{T}_0}\mathrm{e}^{-qs}(\Gamma-q)Z^{(q)}(X_{s-})\mathrm{d}s$$

is a (local) martingale, where Γ is the infinitesimal generator of X.

The martingale property of $\mathrm{e}^{-q(t\wedge\widehat{T}_0\wedge T_a)}Z^{(q)}(X_{t\wedge\widehat{T}_0\wedge T_a})$ implies now that

$$(\Gamma-q)Z^{(q)}(x) = 0, \qquad x\in(0,a). \tag{11}$$

Let I^c be the continuous part of I. By applying (the appropriate version of) Itô's lemma to $N_t = \exp(-qt)Z^{(q)}(Y_t)$ and using $Z^{(q)}(x) = 1$ for $x\leq 0$, one can verify that

$$N_t - \int_0^t \mathrm{e}^{-qt}(\Gamma-q)Z^{(q)}(Y_{s-})\mathrm{d}s + q\int_0^t W^{(q)}(Y_{s-})\mathrm{d}I_s^c$$

is a local martingale. Note that the last term in the previous display is identically zero. Indeed, if X has bounded variation $I^c\equiv 0$; otherwise we see that $\mathrm{d}I_s^c$ is negative if and only if $Y_{s-}=0$ and $W^{(q)}(Y_{s-})=0$ in this case. Noting that $N_{t\wedge\tau_a}$ is bounded by $Z^{(q)}(a)$ we deduce from equation (11) that $N_{t\wedge\tau_a}$ is a uniformly integrable martingale. Hence, as $t\to\infty$,

$$Z^{(q)}(x) = \mathbb{E}_x[N_{t\wedge\tau_a}] \to \mathbb{E}_x[N_{\tau_a}] = Z^{(q)}(a)\mathbb{E}_x[\mathrm{e}^{-q\tau_a}] \qquad x\in[0,a],$$

where we used that $\mathbb{P}_x$-almost surely $\tau_a<\infty$ and $Y_{\tau_a}=a$. □

4.3 Martingales

Another consequence of Proposition 2 is the following martingale property, which justifies the name *adjoint q-scale function* for $Z^{(q)}$.

Proposition 3 *For $q \geq 0$,*

$$(\mathrm{e}^{-q(t\wedge \widehat{T}_0)} W^{(q)}(X_{t\wedge \widehat{T}_0}), t \geq 0) \qquad \textit{and} \qquad (\mathrm{e}^{-qt} Z^{(q)}(Y_t), t \geq 0)$$

are martingales.

Proof The first assertion follows by applying Lemma VII.11 in [23] to a spectrally negative Lévy process that is killed at an independent exponential time $\eta(q)$.

As before we write $N_t = \exp(-qt)Z^{(q)}(Y_t)$. From the proof of Proposition 2, we know that $(N_{t\wedge\tau_a}, t \geq 0)$ is a martingale. We now claim that $N_{t\wedge\tau_a}$ converges in L^1 to N_t as a tends to infinity. Since for $s \leq t$

$$\mathbb{E}|\mathbb{E}(N_{t\wedge\tau_a}|\mathcal{F}_s) - \mathbb{E}(N_t|\mathcal{F}_s)| \leq \mathbb{E}|N_{t\wedge\tau_a} - N_t|,$$

the claim implies that N_t is a martingale. Let us now prove the claim. Write

$$N_{t\wedge\tau_a} = N_t \mathbf{1}_{\{t<\tau_a\}} + N_{\tau_a}\mathbf{1}_{\{t\geq\tau_a\}}.$$

Since $\tau_a < \infty$ a.s., monotone convergence implies that the first term on the right-hand side increases to N_t in L^1 if a tends to infinity. For the second term we note that by the Cauchy-Schwarz-inequality

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a}\mathbf{1}_{\{t\geq\tau_a\}}] \leq (\mathbb{E}_x[\mathrm{e}^{-2q\tau_a}])^{1/2}\mathbb{P}_x(t \geq \tau_a)^{1/2}. \tag{12}$$

Recall that $\eta(r)$ denotes an independent exponential random variable with parameter r. Since $\tau_a \to \infty$ if $a \to \infty$ and we can check that, for $r > 0$,

$$\begin{aligned}\mathbb{P}_x(\tau_a \leq \eta(r)) = \mathbb{P}_x(\tau_a \leq t) + \mathbb{P}_x(\tau_a \in (t, \eta(r)], t < \eta(r)) \\ - \mathbb{P}_x(\tau_a \in (\eta(r), t], \eta(r) < t),\end{aligned}$$

there exists an a_r large enough such that $\mathbb{P}_x(\tau_a \leq t)$ is bounded by $\mathbb{P}_x(\tau_a \leq \eta(r))$ for all $a \geq a_r$. Combining this property with equations (12) and Proposition 2(i), we find that for a large enough

$$\mathbb{E}_x[N_{\tau_a}\mathbf{1}_{\{\tau_a\leq t\}}] \leq Z^{(2q)}(x)Z^{(q)}(a)/Z^{(2q)}(a).$$

By Lemma 4 (recalling that Φ is increasing), we conclude that the expectation in the previous display converges to zero, which finishes the proof. □

5 Resolvent measure

The Lévy process killed when it exits from $[0,a]$ has the strong Markov property; denote its transition probabilities by $(P^t, t \geq 0)$, that is, for a Borel set $A \subseteq [0,a]$ we have

$$P^t(x,A) = \mathbb{P}_x(X_t \in A, t < T_a \wedge \widehat{T}_0) \qquad \text{for } x \in [0,a].$$

and its q-resolvent kernel by

$$U^q(x,A) = \int_0^\infty P^t(x,A)\mathrm{e}^{-qt}\mathrm{d}t = \mathbb{E}_x\left(\int_0^{T_a \wedge \widehat{T}_0} \mathrm{e}^{-qt}\mathbf{1}_{\{X_t \in A\}}\mathrm{d}t\right), \qquad q \geq 0.$$

Since the Lévy process has an absolute continuous resolvent kernel, it follows from the Radon-Nikodym theorem that $U^q(x,\cdot)$ has a density with respect to the Lebesgue measure, which will be denoted by $u^q(x,\cdot)$. Suprun [121] showed that, for $x,y \in [0,a]$,

$$u^q(x,y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \tag{13}$$

is a version of this density. Now we consider the Lévy processes Y and $\widehat{Y}$ killed upon leaving $[0,a]$. These killed processes still have the strong Markov property and we write $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$ respectively to denote their transition probabilities. To be more precise, for Borel-sets $A \subseteq [0,a]$, we denote the transition probabilities of Y and $\widehat{Y}$ by

$$Q^t(x,A) = \mathbb{P}_x(Y_t \in A, t < \tau_a), \qquad \widehat{Q}^t(x,A) = \mathbb{P}_{-x}(\widehat{Y}_t \in A, t < \widehat{\tau}_a).$$

and the corresponding q-resolvent kernels by $R^q(x,A)$ and $\widehat{R}^q(x,A)$, respectively. We state the following result:

Theorem 1 **(i)** *The measure $R^q(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure and a version of its density is given by*

$$r^q(x,y) = \frac{Z^{(q)}(x)}{Z^{(q)}(a)}W^{(q)}(a-y) - W^{(q)}(x-y), \qquad x,y \in [0,a).$$

(ii) *Let $\widehat{r}^q(x,0) = W^{(q)}(a-x)W^{(q)}(0)/W_+^{(q)\prime}(a)$ for $x \geq 0$ and set*

$$\widehat{r}^q(x,y) = W^{(q)}(a-x)\frac{W_+^{(q)\prime}(y)}{W_+^{(q)\prime}(a)} - W^{(q)}(y-x) \qquad x,y \in [0,a], y \neq 0.$$

Then $\widehat{r}^q(x,0)\delta_0(\mathrm{d}y) + \widehat{r}^q(x,y)\mathrm{d}y$ is a version of the measure $\widehat{R}^q(x,\mathrm{d}y)$.

Example 2 *If X is a standard Brownian motion, a famous result of Lévy states that $|X| = Y$, where the equality is in law. Let τ' be the first exit time of $|X|$ from $[0,a]$ and as before $\eta(q)$ is an independent exponential random variable* *with parameter $q > 0$. Recalling from the example in Section 3 the form of the functions $W^{(q)}, Z^{(q)}$ for a Brownian motion and substituting in Theorem 1, we find, after some algebra,*

$$\mathbb{P}_x(|X|_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau') = \frac{\sqrt{q}}{\sqrt{2}} \cdot \frac{\sinh((a - |y - x|)\sqrt{2q}) + \sinh((a - x - y)\sqrt{2q})}{\cosh(a\sqrt{2q})} \mathrm{d}y$$

for $0 \leq x, y \leq a$. This formula is well known in the literature (e.g. [30, 3.1.1.6]).

Proof of part (i) Pick $x, y \in [0, a]$ arbitrary and let $q > 0$. By applying the strong Markov property of Y at the stopping time τ_x and using the lack of memory property of the exponential distribution, we find

$$\begin{aligned}\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) = \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_x) \\ + \mathbb{E}_0[\mathrm{e}^{-q\tau_x}]\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a)\end{aligned} \tag{14}$$

Analogously, the probability in (14) admits as second decomposition

$$\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) = \mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y) - \mathbb{E}_0[\mathrm{e}^{-q\tau_a}]\mathbb{P}_a(Y_{\eta(q)} \in \mathrm{d}y). \tag{15}$$

Combining the two decompositions (14) and (15) we find

$$\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) = \mathbb{P}_a(Y_{\eta(q)} \in \mathrm{d}y) - \frac{\mathbb{E}_0[\mathrm{e}^{-q\tau_a}]}{\mathbb{E}_0[\mathrm{e}^{-q\tau_x}]}\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y). \tag{16}$$

Our next step is to evaluate the probability $\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y)$. Applying as before the strong Markov property at the stopping time τ_0, we find the decomposition

$$\begin{aligned}\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y) &= \mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_0) + \mathbb{E}_x[\mathrm{e}^{-q\tau_0}]\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y) \\ &= \mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_0) + \mathbb{E}_x[\mathrm{e}^{-q\widehat{T}_0}]\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y)\end{aligned} \tag{17}$$

where in the second line we used that $(Y_t, t \leq \tau_0)$ has the same law as $(X_t, t \leq \widehat{T}_0)$. Suprun [121] showed that a version of the resolvent density of the process X killed upon entering the negative half-line is given by

$$q^{-1}\mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_0)/\mathrm{d}y = \mathrm{e}^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x - y). \tag{18}$$

By integrating this resolvent density over y (or letting $a \to \infty$ in equation (9)), we find the Laplace transform of $\widehat{T}_0$ to be equal to

$$\mathbb{E}_x[\mathrm{e}^{-q\widehat{T}_0}] = Z^{(q)}(x) - q\Phi(q)^{-1}W^{(q)}(x). \tag{19}$$

Substituting (19) and (18) into (17) and recalling that $Y_{\eta(q)}$ has an exponential distribution with parameter $\Phi(q)$ we end up with

$$\mathbb{P}_x(Y_{\eta(q)} \in \mathrm{d}y)/\mathrm{d}y = Z^{(q)}(x)\Phi(q)\mathrm{e}^{-\Phi(q)y} - qW^{(q)}(x - y).$$

Substituting this into equation (16) and recalling from Proposition 2 that the Laplace transform $\mathbb{E}_0[\mathrm{e}^{-q\tau_x}]$ is given by $Z^{(q)}(x)^{-1}$, we get the formula as stated in the Theorem for $q > 0$. For $q = 0$, the result follows by letting $q \downarrow 0$. $\square$

Proof of part (ii) Let $x, y \in [0, a]$ and let $q > 0$. Since $(\widehat{Y}_t; t < \widehat{\tau}_0)$ has the same law as $(-X_t; t < T_0)$, the strong Markov property of $\widehat{Y}$ enables us to write

$$\begin{aligned}\mathbb{P}_{-x}(\widehat{Y}_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)) = \mathbb{P}_{-x}(-X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_0 \wedge \widehat{T}_a) \\ + \frac{W^{(q)}(a-x)}{W^{(q)}(a)} \mathbb{P}_0(\widehat{Y}_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)),\end{aligned}$$

where we used the two-sided exit probability (8). The first quantity on the right-hand side is seen to be equal to $qu^q(a-x, a-y)\mathrm{d}y$, where u^q is given in (13). To evaluate the probability in the second term on the right-hand side we are going to make use of the Master formula of excursion theory (e.g. [23, Cor. IV.11]). We shall use standard notation (see Bertoin [23, Ch. IV]). To this end, we introduce the excursion process $\widehat{e} = (\widehat{e}_t, t \geq 0)$ of $\widehat{Y}$, which takes values in the space of excursions

$$\mathcal{E} = \{f \in D[0, \infty) : f \geq 0,\ \exists\, \zeta = \zeta(f) \text{ such that} f(\zeta) = 0\}.$$

of càdlàg functions f with a generic life time $\zeta = \zeta(f)$ and is given by

$$\widehat{e}_t = (\widehat{Y}_s, L^{-1}(t^-) \leq s < L^{-1}(t)) \qquad \text{if } L^{-1}(t^-) < L^{-1}(t)$$

where L^{-1} is the right-inverse of a local time L of $\widehat{Y}$ at zero; else $\widehat{e}_t = \partial$, some isolated point. We take the running supremum S to be this local time L (cf. [23, Ch. VII]). The space $\mathcal{E}$ is endowed with the Itô excursion measure $\widehat{n}$. A famous theorem of Itô states that $\widehat{e}$ is a Poisson point process with characteristic measure $\widehat{n}$, if $\widehat{Y}$ is recurrent; otherwise $(\widehat{e}_t, t \leq L(\infty))$ is a Poisson point process stopped at the first excursion of infinite lifetime. For an excursion $\epsilon \in \mathcal{E}$ its supremum is denoted by $\overline{\epsilon}$. By $\epsilon_g = (\widehat{Y}_{g+t}, t \leq \zeta_g)$ we denote the excursion of $\widehat{Y}$ with left-end point g, where ζ_g and $\overline{\epsilon}_g$ denote its lifetime and supremum respectively.

Letting $T_a(\epsilon) = \inf\{t \geq 0 : \epsilon(t) \geq a\}$ an application of the compensation formula yields for $y > 0$

$$\begin{aligned}&\mathbb{P}_0(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{\tau}(a)) \\ &\quad = \mathbb{E}\left[\sum_g I\left(\epsilon_g(\eta(q)) \in \mathrm{d}y, g < \eta(q) < g + \zeta_g, \eta(q) < \widehat{\tau}_a, \sup_{h<g} \overline{\epsilon}_h \leq a\right)\right] \\ &\quad = \mathbb{E}\left[\int \mathrm{e}^{-qs} I\left(\sup_{h<s} \overline{\epsilon}_h \leq a\right) \mathrm{d}S_s\right] \widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a(\epsilon)).\end{aligned}$$

The first factor can be inferred from [16] to be equal to $W^{(q)}(a)/W_+^{(q)\prime}(a)$. For the second factor, we distinguish between the case that X has bounded or unbounded variation.

If X has bounded variation, it is well known (e.g. [114] or [126] for a more recent reference) that an excursion starts with a jump almost surely. Denote by $\mathtt{d}$ and $\Lambda(\mathrm{d}x)$ the drift and Lévy measure of X, respectively. Note that in this case the time up to time t that the process $\widehat{Y}$ has spent in zero is equal to local time S_t divided by the drift $\mathtt{d}$. By the Markov property, under $\widehat{n}$, the excursion of $\widehat{Y}$, once in $(0,\infty)$, evolves as $-X$ killed at time T_0. Furthermore, the entrance law of an excursion of $\widehat{Y}$ under $\widehat{n}$ is given by $\Lambda/\mathtt{d}$. Indeed, letting $F:\mathcal{E}\to[0,\infty)$ be any bounded measurable functional on the space of excursions, we find that

$$\begin{aligned}
&\int F(\epsilon)\widehat{n}(\mathrm{d}\epsilon)\\
&=\mathbb{E}\left[\sum_{0\le s\le 1}F(e_s)\right]\\
&=\mathbb{E}\left[\sum_{0\le t<\infty}I(S_t\le 1, X_{t^-}=S_t, \Delta X_t<0)F(\{-X_{s+t}+X_{t^-}, s\le\widehat{\tau}_0\})\right]\\
&=\mathbb{E}\left[\int_0^\infty I(S_t\le 1, X_{t^-}=S_t)\mathrm{d}t\int_{-\infty}^0 F(\{-X_s-x, s\le T_{-x}\})\Lambda(\mathrm{d}x)\right]\\
&=\frac{1}{\mathtt{d}}\int_{-\infty}^0\mathbb{E}_x\left[F(\{-X_s, s\le T_0\})\right]\Lambda(\mathrm{d}x),
\end{aligned}\tag{20}$$

where on the first line we used as before the Master formula of excursion theory followed in the third line by an application of the compensation formula applied to the Poisson point process $(\Delta X_t, t\ge 0)$ with characteristic measure $\Lambda(\mathrm{d}x)$ combined with the independent increments property of X. Applying this identity to $F(\epsilon)=I(\epsilon(t)\in\mathrm{d}y, t<\zeta\wedge T_a(\epsilon))$, taking the Laplace transform in t and using (13), we find that

$$\begin{aligned}
&\widehat{n}(\epsilon(\eta(q))\in\mathrm{d}y, \eta(q)<\zeta\wedge T_a)\\
&\qquad=\frac{q}{\mathtt{d}}\int_{-\infty}^0\left(\frac{W^{(q)}(a+x)W^{(q)}(y)-W^{(q)}(a)W^{(q)}(y+x)}{W^{(q)}(a)}\right)\Lambda(\mathrm{d}x)\mathrm{d}y.
\end{aligned}$$

We claim that the following identity holds true for all $a>0$:

$$\mathtt{d}W_+^{(q)\prime}(a)=\int_{-\infty}^0\left(W^{(q)}(a)-W^{(q)}(a+x)\right)\Lambda(\mathrm{d}x)+qW^{(q)}(a).\tag{21}$$

To see this, first note that the right and left-hand sides of (21) have the same Laplace transform in a. Moreover, equation (7) and the decomposition (3) imply that $W_+^{(q)\prime}$ is bounded on any compact interval in $(0,\infty)$. It follows that the right-hand side of (21) is right-continuous in $a>0$, as is certainly the left-hand side of (21). This continuity combined with the almost sure unicity of the Laplace transform shows that the claim is true for $a>0$.

After some algebra, we find that

$$\widehat{n}(\epsilon(\eta(q) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a)/\mathrm{d}y = q\left(W_+^{(q)\prime}(y) - \frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(y)\right). \quad (22)$$

Substituting back the expression (22), we find the stated form of the density for $y > 0$. Noting that

$$1 - \mathbb{E}_{-x}[\mathrm{e}^{-q\widehat{\tau}_a}] = \int_{0+}^{a} q\widehat{r}^q(x, y)\mathrm{d}y + \mathbb{P}_{-x}(\widehat{Y}_{\eta(q)} = 0, \eta(q) < \widehat{\tau}_a),$$

we can verify, by combining Proposition 2 with the just found density, that $\widehat{R}^q(x, 0) = W^{(q)}(a-x)W^{(q)}(0)/W_+^{(q)\prime}(a)$ which finishes the proof in the bounded variation case.

Suppose now X has unbounded variation. Let $g(\widehat{\tau}_a)$ and $d(\widehat{\tau}_a)$ be the last time before and first time after $\widehat{\tau}_a$ that $\widehat{Y}$ visits zero. Consider now the excursion straddling $\widehat{\tau}_a$, $\{\widehat{Y}_t, t \in [g(\widehat{\tau}_a), d(\widehat{\tau}_a)\}$, and denote its law by $Q^{(a)}$ and the completed right-continuous filtration generated by this process by $\{\mathcal{G}_t, t \geq 0\}$. Since we are in the case of unbounded variation, 0 is regular for $\widehat{Y}$ for itself under $\mathbb{P}$. Then, in canonical notation, we have that X leaves continuously from zero $Q^{(a)}$-almost surely and $T(x) = \inf\{t \geq 0 : X_t \geq x\}$ decreases to zero almost surely under $Q^{(a)}$ as $x \downarrow 0$. By right-continuity of the paths, the sequence of measures $(Q_x^{t,(a)}, x > 0)$ with

$$Q_x^{t,(a)}(A) := Q^{(a)}(X \circ \theta_{T(x)}(t) \in A, t < T(a)) \qquad A \in \mathcal{F}_t$$

converges in finite distributions and weakly as a measure (by tightness) to $Q^{(a)}(\cdot, t < T(a))$. The strong Markov property implies that under $Q^{(a)}$ the shifted process $X \circ \theta_{T(x)}$ has the same law as $\widehat{X} = -X$ under $\mathbb{P}$ starting at x and conditioned to exit $[0, a]$ at a. Using (13) and (8), we find for $A \in \mathcal{F}_t$

$$\begin{aligned} Q_x^{t,(a)}(A) &= \int \mathbb{E}_{-y}[-X_t \in A|\widehat{T}_a < T_0]Q^{(a)}(X_{T(x)} \in \mathrm{d}y) \\ &= \int\int_A \frac{W(a) - W(a-z)}{W(a) - W(a-y)}\mathbb{P}_{-y}(-X_t \in \mathrm{d}z, t < \widehat{T}_a \wedge T_0)Q^{(a)}(X_{T(x)} \in \mathrm{d}y). \end{aligned}$$

From (13) it follows that

$$\lim_{x \downarrow 0} \mathbb{P}_{-x}(-X_{\eta(q)} \in \mathrm{d}y, \eta(q) < \widehat{T}_a|\widehat{T}_a < T_0) = \frac{W(a) - W(a-y)}{W'(a)}f(y, a)\mathrm{d}y$$

where the limit is in the sense of weak convergence and $f(y, a)$ is equal to the right-hand side of (22). Since $X_{T(x)}$ converges to zero $Q^{(a)}$-a.e., we deduce by bounded convergence that

$$Q^{(a)}(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T(a)) = \frac{W(a) - W(a-y)}{W'(a)}f(y, a)\mathrm{d}y. \quad (23)$$

By a computation based on the compensation formula for excursion theory (cf. proof of Theorem 4 in [40]), one can verify that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty I(S_t \le x) q\mathrm{e}^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in \mathrm{d}y, t-g_t < \widehat{\tau}_a)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\sum_{0\le g<\infty} I(S_g \le x) \int_g^{g+\zeta_g} q\mathrm{e}^{-q(t-g)} I(\widehat{Y}_{t-g} \in \mathrm{d}y, t-g < \widehat{\tau}_a)\mathrm{d}t\right] \\
&= x\widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a(\epsilon)),
\end{aligned}$$

where $g_t = \sup\{s \le t : \widehat{Y}_s = 0\}$. Thus, we find

$$\begin{aligned}
&\widehat{n}(\epsilon(\eta(q)) \in \mathrm{d}y, \eta(q) < \zeta \wedge T_a(\epsilon)) \\
&= \lim_{x\downarrow 0} \frac{1}{x}\mathbb{E}_{-x}\left[\int_0^\infty q\mathrm{e}^{-q(t-g_t)} I(\widehat{Y}_{t-g_t} \in \mathrm{d}y, t-g_t < \widehat{\tau}_a \wedge \widehat{\tau}_0)\right] \\
&= \lim_{x\downarrow 0} \frac{W(a)-W(a-x)}{x} \times \\
&\times \lim_{x\downarrow 0} \frac{1}{W(a)-W(a-x)} \int_0^\infty q\mathrm{e}^{-qt}\mathbb{E}_{-x}[I(X_{t-g_t} \in \mathrm{d}y, t-g_t < \widehat{T}_a \wedge T_0)\mathrm{d}t \\
&= W'(a)Q^{(a)}\left[\frac{1}{W(a)-W(a-X_{\eta(q)})} I(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T(a))\right]. \qquad (24)
\end{aligned}$$

Combining (24) and (23) we deduce that (22) is also valid in this case. □

Remark. If X drifts to $-\infty$, we can relate the conditionings in the proof of the theorem to those in the literature on spectrally negative Lévy processes conditioned to stay in a half line. Recall that, since X drifts to $-\infty$, we have $\Phi(0) > 0$ and $\psi'(\Phi(0)) > 0$. We write $W(x) = \mathrm{e}^{\Phi(0)x}W^{\#}(x)$ where $W^{\#}$ is the scale function of X under the measure $\mathbb{P}^{\#}$ which is for $A \in \mathcal{F}_t$ given by $\mathbb{P}^{\#}(A) = \mathbb{E}[\exp(\Phi(0)X_t)I_A]$. Since $\psi^{\#\prime}(0) = \psi'(\Phi(0)) > 0$, X drifts to $+\infty$ under $\mathbb{P}^{\#}$ and $W^{\#}$ is bounded. Then it follows from Proposition 1 that the probability $\mathbb{P}_{-x}(\widehat{T}_a < T_0)$ converges to $1-\exp(-\Phi(0)x)$ as $a \to \infty$. By bounded convergence we then find for $A_t \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{P}_{-x}(A_t|\widehat{T}_a < T_0) &= (W(a) - W(a-x))^{-1}\mathbb{P}_{-x}((W(a) - W(a+X_t))A_t) \\
&\to \mathbb{P}^{\downarrow}_{-x}(A_t) := \mathbb{E}_{-x}\left(\tfrac{1-\mathrm{e}^{\Phi(0)X_t}}{1-\mathrm{e}^{-\Phi(0)x}} I(A_t)\right) \qquad \text{as } a \to \infty.
\end{aligned}$$

Note that $\mathbb{P}^{\downarrow}_{-x}(A_t)$ is also equal to $\mathbb{P}_{-x}(A_t|S_\infty < 0)$. Hence the notation $\mathbb{P}^{\downarrow}_{-x}$ is justified since under this measure the process always stay below zero with probability one. As $x \downarrow 0$ the measures $\mathbb{P}^{\downarrow}_{-x}$ converge weakly (in the Skorohod topology) to a measure $\mathbb{P}^{\downarrow}$. For an analysis of this case, see [21].

6 Analytic continuation

In this subsection, we show that we can extend the resolvent measures $R^q(x,\cdot)$ and $\widehat{R}^q(x,\cdot)$ to some negative values of q. Let us define ϱ and $\widehat{\varrho}$ by

$$\varrho = \inf\{q \geq 0 : Z^{(-q)}(a) = 0\} \qquad \widehat{\varrho} = \inf\{q \geq 0 : W_+^{(-q)\prime}(a) = 0\}. \tag{25}$$

Continuity of $q \mapsto Z^{(q)}(a)$ (Lemma 2) combined with the fact $Z^{(0)}(a) \equiv 1$ implies that ϱ is positive. Similarly, combining the continuity of $q \mapsto W_+^{(q)\prime}(a)$ with the fact (from the proof of Proposition 2(ii)) that $W'_+(a)$ is positive for all $a > 0$, we see that $\widehat{\rho}$ is positive as well.

Proposition 4 *Let $x \in [0,a]$ and A a Borel subset of $[0,a]$. We have for $q < \varrho$*

$$\int_0^\infty \mathrm{e}^{qt} Q^t(x,A)\mathrm{d}t = \int_A \left\{ \frac{Z^{(-q)}(x)}{Z^{(-q)}(a)} W^{(-q)}(a-y) - W^{(-q)}(x-y) \right\} \mathrm{d}y$$

and for $q < \widehat{\varrho}$

$$\int_0^\infty \mathrm{e}^{qt} \widehat{Q}^t(x,A)\mathrm{d}t = \int_A \frac{W^{(-q)}(a-x)}{W_+^{(-q)\prime}(a)} W^{(-q)}(\mathrm{d}y) - \int_A W^{(-q)}(y-x)\mathrm{d}y.$$

Proof For $q \leq 0$, the statement (i) rephrases Theorem 1. By Lemma 2 and the properties of $q \mapsto W^{(q)}(x)$ as listed in [24, Lemma 4], we can extend the right-hand side for $q < \varrho$. The coefficient c_n of q^n in the corresponding expansion as a power series at zero is given in terms of the left-derivative of the left-hand side,

$$c_n = \textstyle\int_0^\infty t^n Q^t(x,A)\mathrm{d}t/n!.$$

We know that the series $\sum_n c_n q^n$ converges for $|q| < \varrho$. The statement follows. The proof of (ii) is similar and left to the reader. □

7 Irreducibility and continuity

Let μ denote any σ-finite measure on $([0,a), \mathcal{B}_{[0,a)})$, the interval $[0,a)$ endowed with the Borel σ-algebra $\mathcal{B}_{[0,a)}$. Examples are the Lebesgue measure λ and the Dirac measure δ_x at $x \in [0,a)$. One says that transition probabilities $(P^t, t \geq 0)$ are μ-irreducible if, for every $A \in \mathcal{B}_{[0,a)}$ with $\mu(A) > 0$, their potential $U(x,A)$ of A is positive for every $x \in [0,a)$. Before we formulate the result, we set the condition (R) by

$$\left\{ \begin{array}{c} X \text{ has jumps of absolute size smaller than } a \\ \text{or the Brownian coefficient } s = \lim_{\lambda\to\infty} \lambda^{-2}\psi(\lambda) \text{ is positive} \end{array} \right\}. \tag{R}$$

Proposition 5 *Q^t is λ-irreducible and under condition (R) $\widehat{Q}^t$ is $(\lambda + W(0)\delta_0)$-irreducible.*

Proof The first statement follows since, by Theorem 1,

$$r^0(x,y) = W(a-y) - W(x-y) > 0 \qquad \text{for all } x, y \in [0,a),$$

as W is increasing. For the second statement, we note that $\widehat{Q}^t(x, \mathrm{d}y) \geq P^t(a - x, \mathrm{d}(a-y))$. Thus $\widehat{r}^0(x,y) \geq u^0(a-x, a-y)$, where under condition (R) $u^0(a-x, a-y) > 0$ for all $x, y \in (0,a)$ by Corollary 3 in [24]. If $x = 0$, we see from (7) that for $y < a$

$$\widehat{r}^0(0,y)/\left(W(a)W(y)\right) = \frac{W_+'(y)}{W(y)} - \frac{W_+'(a)}{W(a)} = \widehat{n}(h > y) - \widehat{n}(h > a) > 0.$$

Finally, note that for $x \in [0,a)$ the measure $\widehat{R}^q(x, \mathrm{d}y)$ has an atom at zero if and only if X has bounded variation and thus if and only if $W(0) > 0$. □
Proposition 5 implies the following property of $Z^{(q)}$.

Corollary 1 *(i) For every $q < \varrho$ and $x \in [0,a)$, we have $Z^{(-q)}(x) > 0$.*
(ii) Similarly, for every $q < \widehat{\varrho}$ and $x \in (0,a)$, $W_+^{(-q)\prime} > 0$.

Proof (i) We know from Lemma 2 that $Z^{(-q)}(x) > 0$ if x is sufficiently small. Let x_0 be the smallest zero of $Z^{(-q)}(x) = 0$. If we had $x_0 < a$, then we would have $\int_0^\infty \mathrm{e}^{qt} Q^t(x_0, (x_0, a))\mathrm{d}t = 0$ by Proposition 4, which conflicts with the fact that Q^t is Lebesgue irreducible.

(ii) Suppose first there would exist an $x_0 \in (0,a)$ such that $W_+^{(-q)\prime}(x_0) < 0$. Then, by right-continuity and Proposition 4, we would find the contradictory statement $\int_0^\infty \mathrm{e}^{qt} Q^t(x_0 + \epsilon, (x_0, x_0 + \epsilon))\mathrm{d}t < 0$ for $\epsilon > 0$ small enough. Thus $W_+^{(-q)\prime} \geq 0$ on $(0,a)$ for all $q \leq \widehat{\varrho}$. Next suppose that there exists an $x_1 \in (0,a)$ with $W_+^{(-q)\prime}(x_1) = 0$. This would imply that $\frac{\partial}{\partial q} W_+^{(q)\prime}(x_1) = 0$, which conflicts with the second identity of Lemma 3 for $q = r$ combined with Corollary 4 in [24] (which also holds without (AC) as follows from part (i)). □
In order to be able to prove continuity in space and time of the transition probabilities $(Q^t, t \geq 0)$ and $(\widehat{Q}^t, t \geq 0)$, we restrict ourselves to Lévy processes X whose one-dimensional distributions are absolutely continuous with respect to the Lebesgue measure, that is,

$$\mathbb{P}_0(X_t \in \mathrm{d}y) \ll \mathrm{d}y \qquad \text{for all } t > 0. \tag{AC}$$

It is known that (AC) holds whenever the Brownian coefficient is positive or when the mass of the absolutely continuous part of the Lévy measure is infinite (see Tucker [123]). We use the standard notation $Q^t f(x) = \int_{[0,a)} f(y) Q^t(x, \mathrm{d}y)$. Recall that the family $(Q^t, t \geq 0)$ has the strong Feller property if for every bounded Borel function f, $Q^t f(\cdot)$ is a continuous function on $[0,a]$ for all $t > 0$. If a family of probability measures has the Feller as well as the strong Feller property it is called doubly Feller.

Proposition 6 *Assume (AC) is satisfied. Then the following hold true:*

(i) For every $x \in [0,a]$ and Borel set $A \subseteq [0,a]$ the mappings $t \mapsto Q^t(x,A)$ and $t \mapsto \widehat{Q}^t(x,A)$ are continuous on $(0,\infty)$.

(ii) For every $t > 0$, Q^t and $\widehat{Q}^t$ have the strong Feller property.

To prove Proposition 6 we need the following auxiliary results.

Lemma 5 *Assume (AC) holds.*

(i) The one-dimensional distributions of the reflected Lévy process Y are absolutely continuous, that is,

$$\mathbb{P}_x(Y_t \in \mathrm{d}y) \ll \mathrm{d}y \qquad \textit{for every } t > 0, x \geq 0.$$

(ii) For any $t > 0, x \geq 0$, the measure $\mathbb{P}_x(\widehat{Y}_t \in \mathrm{d}y)$ is absolutely continuous on $(0,\infty)$. If X has (un)bounded variation, $\mathbb{P}_x(\widehat{Y}_t = 0) > (=)0$.

Proof (ii) Let $N \subset (0,\infty)$ be an arbitrary Borel set of measure zero and fix $t > 0$. The form of the law of $\widehat{Y}_{\eta(q)}$ given in (2) combined with the absolute continuity of of $W^{(q)}(\mathrm{d}x)$ for $x > 0$ implies that

$$\mathbb{P}_0(\widehat{Y}_t \in N) = 0 \qquad \text{for Lebesgue-almost all } t > 0. \tag{26}$$

Next we note that $\mathbb{P}_x(\widehat{\tau}_0 \in \mathrm{d}t)$ has no atoms for $x > 0$. Indeed, since the sample paths of a Lévy process are continuous at each fixed time a.s. we see that under (AC)

$$\mathbb{P}_x(\widehat{\tau}_0 = t) = \mathbb{P}_{-x}(T_0 = t) \leq \mathbb{P}_{-x}(X_t = 0) = 0 \qquad x > 0. \tag{27}$$

Applying the Markov property at $\widehat{\tau}_0$ yields that

$$\mathbb{P}_x(\widehat{Y}_t \in N) = \mathbb{P}_x(\widehat{Y}_t \in N, t < \widehat{\tau}_0) + \int_0^t \mathbb{P}_0(\widehat{Y}_{t-s} \in N)\mathbb{P}_x(\widehat{\tau}_0 \in \mathrm{d}s). \tag{28}$$

Noting that $\mathbb{P}_x(\widehat{Y}_t \in N, t < \widehat{\tau}_0)$ is dominated by $\mathbb{P}_{-x}(-X_t \in N)$ and invoking (26) and (27), we deduce from (28) that $\mathbb{P}_x(\widehat{Y}_t \in N)$ is zero under (AC) for all $t, x > 0$. By an application of the Markov property at time $s \notin N$, we can now remove the "almost" in (26) and the first assertion follows. Recalling that $\widehat{Y}_t$ has the same law as $-I_t$ and using (2), we see that $\mathbb{P}_0(\widehat{Y}_t = 0)$ is zero for all $t > 0$ if and only if X has unbounded variation. The proof of (ii) is complete. The proof of (i) is similar to (ii) and is left to the reader. □

Lemma 6 *For $a > 0$, the distribution of $\widehat{\tau}_a$ has no atom, that is,*

$$\mathbb{P}_x(\widehat{\tau}_a = t) = 0 \qquad \textit{for every } x \in [0,a) \textit{ and } t \geq 0.$$

Under (AC), the same holds for the distribution of τ_a.

Proof Since a Lévy process (and also a reflected Lévy process) is almost surely continuous at time t, we have

$$\mathbb{P}_x(\tau_y = t) \leq \mathbb{P}_x(Y_t = y) \text{ and } \mathbb{P}_x(\widehat{\tau}_y = t) \leq \mathbb{P}_x(\widehat{Y}_t = y)$$

which are both zero under (AC) by the first and second part of the Lemma 5 respectively. Suppose now (AC) is not satisfied; X is then a drift minus pure jump process of bounded variation. Hence $\widehat{Y}$ can cross the level $a > 0$ only by a jump. However, the probability is zero that the Poisson point process $(\Delta X_t, t \geq 0)$ jumps at time t. □

Lemma 7 *Assume (AC) holds and let $A \subseteq \mathbb{R}$ be an arbitrary Borel set.*

(i) For every $t > 0$, $\mathbb{P}_x(Y_t \in \cdot)$ and $\mathbb{P}_x(\widehat{Y}_t \in \cdot)$ have the strong Feller property.

(ii) For every $x \geq 0$, $t \mapsto \mathbb{P}_x(Y_t \in A)$ and $t \mapsto \mathbb{P}_x(\widehat{Y}_t \in A)$ are continuous on $(0, \infty)$.

Proof (i) Let f be any bounded Borel function. Since Y_t under $\mathbb{P}_x$ has the same law as $X_t - (I_t \wedge (-x))$ under $\mathbb{P}_0$, we have

$$\mathbb{E}_x[f(Y_t)] = \mathbb{E}[f(X_t + x)\mathbf{1}_{\{I_t \geq -x\}}] + \mathbb{E}[f(X_t - I_t)\mathbf{1}_{\{I_t < -x\}}].$$

Recall that $P(X_t \in \mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure. Since $C_c(\mathbb{R})$, the continuous functions with compact support, are dense in L^1, it follows by dominated convergence that $x \mapsto \mathbb{E}_x[f(Y_t)]$ is continuous. The proof of the second statement is similar.

(ii) From the proof of Theorem 2.2 in [68] and Lemma 5, we can deduce, following an analogous line of reasoning, that there exists a version $(t, x, y) \mapsto q(t, x, y)$ of the density of the one-dimensional distributions of Y, such that for all Borel bounded f and for all $x \geq 0$

$$\mathbb{E}_x[f(Y_t)] = \int f(y) q_t(x, y) \mathrm{d}y$$

and $\int q_t(\cdot, z) q_s(z, \cdot) \mathrm{d}z = q_{t+s}(\cdot, \cdot)$ for all $s, t > 0$. Moreover, by part (i) $x \mapsto \mathbb{E}_x[f(Y_t)]$ is continuous. By the weak convergence of $q_\epsilon(x, z)\mathrm{d}z$ to the Dirac point measure at x as $\epsilon \downarrow 0$ and almost sure sample path continuity of Y at time ϵ, left- and right-continuity follow (as in the proof of [24, Lemma 2]).

To prove the second statement of (ii), we repeat the proof of part (i) where everywhere the Lebesgue measure $\mathrm{d}y$ is replaced by the measure $\mathrm{d}y + \delta_0(y)$, the Lebesgue measure $\mathrm{d}y$ with an atom of size one at zero. □

Proof of Proposition 6 We only prove the statements for Y, the proofs for $\widehat{Y}$ are similar.

(i) By the strong Markov property of Y applied at τ_a, we find that

$$\mathbb{P}_x(Y_t \in A) = Q^t(x, A) + \int_0^t \mathbb{P}_x(\tau_a \in \mathrm{d}s)\mathbb{P}_a(Y_{t-s} \in A).$$

The left-hand side is continuous in t on $(0,\infty)$ by Lemma 7. The same holds for the integral on the right-hand side, as the distribution of τ_a has no atom. Hence $t \mapsto Q^t(x,A)$ is continuous.
(ii) Proposition VI.1 in [23] states that Y has the Feller property. Combining this with Lemma 7, we see that Y is doubly Feller. From Chung [41], we know that a doubly Feller process killed upon hitting an open set remains doubly Feller. □

8 Ergodicity and exponential decay

Under the assumption (AC) Bertoin [24] identifies the decay parameter of the transition probabilities $(P^t, t \geq 0)$ of X killed upon leaving $[0,a]$ as $\rho = \rho(a)$ where

$$\rho(a) = \inf\{q \geq 0 : W^{(-q)}(a) = 0\}.$$

Recall from (25) that we defined $\varrho = \varrho(a)$ and $\widehat{\varrho} = \widehat{\varrho}(a)$ as

$$\varrho(a) = \inf\{q \geq 0 : Z^{(-q)}(a) = 0\} \qquad \widehat{\varrho}(a) = \inf\{q \geq 0 : W_+^{(-q)\prime}(a) = 0\}.$$

The result below concerns the ergodic properties of the transition probabilities Q^t and $\widehat{Q}^t$ and identifies their decay parameters as ϱ and $\widehat{\varrho}$ respectively. The proof uses the R-theory of irreducible Markov processes developed by Tuominen and Tweedie [124]. For the terminology of R-theory used in this section, we refer to [124].

Theorem 2 **(A)** *We have that $\varrho \in (0,\infty)$ and ϱ is a simple root of $q \mapsto Z^{(-q)}(a)$ and the following hold true:*

(i) Q^t is ϱ-recurrent and, more precisely, ϱ-positive.

(ii) $x \mapsto Z^{(-\varrho)}(x)$ is positive on $[0,a)$ and ϱ-invariant for Q^t; that is,

$$Q^t Z^{(-\varrho)}(x) = \mathrm{e}^{-\varrho t} Z^{(-\varrho)}(x) \qquad \text{for all } x \in [0,a). \tag{29}$$

(iii) $x \mapsto W^{(-\varrho)}(a-x)$ is positive almost everywhere on $(0,a)$ and the measure $\Pi(\mathrm{d}x) = W^{(-\varrho)}(a-x)\mathrm{d}x$ on $[0,a)$ is ϱ-invariant for Q^t, that is,

$$\Pi Q^t = \mathrm{e}^{-\varrho t}\Pi. \tag{30}$$

(iv) Assume (AC) is satisfied. Then for every $x \in [0,a]$ we have

$$\lim_{t\to\infty} \mathrm{e}^{\varrho t} Q^t(x,\cdot) = c^{-1} Z^{(-\varrho)}(x)\Pi(\cdot) \tag{31}$$

in the sense of weak convergence where $c = \frac{\mathrm{d}}{\mathrm{d}q} Z^{(q)}(a)|_{q=-\varrho} > 0$.

(B) *Suppose X satisfies (R). Then $\widehat{\varrho} \in (0,\infty)$ and $\widehat{\varrho}$ is a simple root of $q \mapsto W_+^{(-q)\prime}(a)$ and the following hold:*

(i) $\widehat{Q}^t$ *is* $\widehat{\varrho}$*-recurrent and, more precisely,* $\widehat{\varrho}$*-positive;*

(ii) $x \mapsto W^{(-\widehat{\varrho})}(a-x)$ *is positive on* $(0,a)$ *and* $\widehat{\varrho}$*-invariant for* $\widehat{Q}^t$*;*

(iii) $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$ *is almost everywhere positive on* $(0,a)$ *and the measure* $\widehat{\Pi}(\mathrm{d}x) = W^{(-\widehat{\varrho})}(\mathrm{d}x)$ *on* $[0,a)$ *is* $\widehat{\varrho}$*-invariant for* Q^t*;*

(iv) Assume (AC) is satisfied. Then for every $x \in [0,a]$ *we have*

$$\lim_{t\to\infty} \mathrm{e}^{\widehat{\varrho}t}\widehat{Q}^t(x,\cdot) = \widehat{c}^{-1}W^{(-\widehat{\varrho})}(a-x)\widehat{\Pi}(\cdot)$$

in the sense of weak convergence where $\widehat{c} = \frac{\mathrm{d}}{\mathrm{d}q}W_+^{(q)\prime}(a)|_{q=-\widehat{\varrho}} > 0$.

Remarks.

(i) Specialising Theorem 2(A,iv) and (B,iv) we get the following asymptotic identities for $t \to \infty$ and $x, y \in [0,a]$

$$\begin{aligned} \mathbb{P}_x(\tau_a > t) &\sim c' Z^{(-\varrho)}(x)\mathrm{e}^{-\varrho t}, && \text{for a constant } c' > 0; \\ \mathbb{P}_x(\widehat{\tau}_a > t) &\sim \tilde{c} W^{(-\widehat{\varrho})}(a-x)\mathrm{e}^{-\widehat{\varrho}t}, && \text{for a constant } \tilde{c} > 0; \\ \mathbb{P}_x(Y_t \in A | \tau_a > t) &\sim \Pi(A)/\Pi([0,a)), && \text{for Borel sets } A \subseteq [0,a). \end{aligned}$$

(ii) Take $\alpha \in (1,2]$. In the case X is stable process of index α we recall from the example in Section 3 that $Z^{(q)}(x) = E_\alpha(qx^\alpha)$. The roots introduced in (25) are hence respectively given by $\varrho = a^{-\alpha}r(\alpha)$ where $-r(\alpha)$ is the first negative root of E_α and $\widehat{\varrho} = a^{-\alpha}\tilde{r}(\alpha)$ where $-\tilde{r}(\alpha)$ is the first negative root of

$$\sum_{n=1}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}.$$

In the special case $\alpha = 2$, $X/\sqrt{2}$ is a standard Brownian motion and $E_2(-x) = \cos\sqrt{x}$ for $x > 0$. In particular, $r(2) = \pi^2/4$ and

$$\varrho = \pi^2/(4a^2) \qquad\qquad Z^{(-\varrho)}(x) = \cos\left(\frac{\pi}{2a}x\right).$$

Since in this case $W^{(-q)\prime}(x) = Z^{(-q)}(x) = \cos(x\sqrt{q})$, we see that $\widehat{\varrho} = \varrho$, as it should be.

(iii) By the decomposition (3) and Lemma 3 we find that

$$\begin{aligned} \frac{\partial}{\partial q} Z^{(q)}(a)|_{q=-\varrho} &= W^{(-\varrho)} \star Z^{(-\varrho)}(a) \\ \frac{\partial}{\partial q} W_+^{(q)\prime}(a)|_{q=-\widehat{\varrho}} &= \int_0^a W^{(-\widehat{\varrho})}(a-x)W^{(-\widehat{\varrho})}(\mathrm{d}x). \end{aligned}$$

Hence the constants $c, \widehat{c}$ in the Theorem make $\mu(\mathrm{d}x) = c^{-1}Z^{(-\varrho)}(x)\Pi(\mathrm{d}x)$ and $\widehat{\mu}(\mathrm{d}x) = \widehat{c}^{-1}W^{(-\widehat{\varrho})}(a-x)\widehat{\Pi}(\mathrm{d}x)$ into probability measures.

(iv) We now consider equation (25) to define mappings $a \mapsto \varrho(a)$ and $a \mapsto \widehat{\varrho}(a)$ from $(0,\infty)$ to $(0,\infty)$. Note that, by Corollary 1(i) and Theorem 2(A,ii), $Z^{(-q)}(x) > 0$ for all $x \in [0,a)$ and $q \leq \varrho$. Much in the same vein as [86], it then follows from Theorem 2 and Lemma 2 that the mapping ϱ is decreasing and continuously differentiable on $(0,\infty)$ with derivative

$$\varrho'(a) = -\varrho(a) W^{(-\varrho(a))}(a) / \tfrac{\partial}{\partial q} Z^{(q)}(a)|_{q=-\varrho(a)}. \tag{32}$$

Similarly, if we assume that W (and hence $W^{(-\varrho)}$) is twice continuously differentiable, we can show that $a \mapsto \widehat{\varrho}(a)$ is decreasing and continuously differentiable with derivative

$$\widehat{\varrho}'(a) = \tfrac{\partial^2}{\partial x^2} W^{(-\widehat{\varrho})}(x)|_{x=a} / \tfrac{\partial}{\partial q} W^{(q)\prime}(a)|_{q=-\widehat{\varrho}(a)}. \tag{33}$$

Introduce the set $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) < 0\}$. Note that it is open and its complement has an empty interior, as ϱ is decreasing. We have now the following relation between ρ on the one hand and $\widehat{\varrho}$ and ϱ on the other hand.

Corollary 2 *Suppose X satisfies (R). Then the following hold.*

(i) $\widehat{\varrho}(a) < \rho(a)$;

(ii) $\varrho(a) < [=]\rho(a)$ *if and only if* $a \in [\notin]\mathcal{D}_\varrho$. *Moreover,* $W^{(-\varrho(a))}(x) > 0$ *for* $x \in (0,a)$.

Proof of Corollary 2 Following the same line of reasoning as in the proof of Theorem 2, one can show that Theorem 2 (i)-(iv) and Corollary 4 in [24] continue to hold if assumption (AC) is replaced by the assumption (R).

(i) Since $\widehat{Q}^t > P^t$, we see that $\widehat{\varrho}$ is bounded above by ρ. Since $W^{(-\widehat{\varrho})}(0) = W(0)$ is nonnegative and $W^{(-\varrho)}(\mathrm{d}x)$ has no atoms in $(0,\infty)$, it follows from Theorem 2(B,iii) that $W^{(\widehat{\varrho}(a))}(a) > 0$. By definition of ρ and continuity of $q \mapsto W^{(q)}(a)$, this implies that $\widehat{\varrho} \neq \rho$ and hence $\widehat{\varrho} < \rho$.

(ii) Since $Q^t > P^t$ we see that ϱ is bounded above by ρ. The second statement follows directly from Corollary 4 and Theorem 2(iii) in [24]. Equation (32) combined with the positivity of $\frac{\partial}{\partial q} Z^{(q)}(a)|_{q=-\varrho(a)}$ implies that $a \in \mathcal{D}_\varrho$ if and only if $W^{(-\varrho(a))}(a) > 0$. As in (i) the first statement follows. □

Proof of Theorem 2 By a close reading of the proofs of Theorems 2 and 3 in [124] one notes that these remain valid under the requirement of irreducibility (instead of simultaneous irreducibility). By Proposition 5, we can thus use Theorems 2 and 3 from [124]. Recall from Section 6 that ϱ is positive. Moreover, Proposition 4 implies that $\varrho < \infty$, since otherwise $\int_0^\infty \mathrm{e}^{qt} Q^t(x,A)\mathrm{d}t$ would be finite for all $x \in [0,a)$ and $q > 0$, which would not agree with Theorem 2 in [124]. We identify ϱ as the decay parameter and show Q^t is ϱ recurrent. From Lemma 2 combined with Lemma 5 in [24] we know we can find a $\delta \in (0, a/2)$ such that $Z^{(-\varrho)}(x) > 1/2$ and $W^{(-\varrho)}(x) > 0$ if $x \in (0,\delta)$; Since $q \mapsto Z^{(q)}(x)$ and $q \mapsto W^{(q)}(x)$ are continuous as stated in Lemma 2 and [24, Lemma 4(i)] respectively, we find that, for every $x < \delta, y \in (a-\delta, a)$,

$$\lim_{q \uparrow \varrho} \frac{Z^{(-q)}(x) W^{(-q)}(a-y)}{Z^{(-q)}(a)} = \infty. \tag{34}$$

Let $A \subseteq (a-\delta, a)$ be any Borel set with positive Lebesgue measure. By Proposition 4, monotone convergence and Fatou's Lemma we deduce that

$$\int_0^\infty \mathrm{e}^{\varrho t} Q^t(x, A)\mathrm{d}t = \infty \qquad \text{for every } x \in (0, \delta). \tag{35}$$

Theorem 2 in [124] now implies that ϱ coincides with the decay parameter and Q^t is ϱ-recurrent. In particular, it implies that (35) holds for all $x \in [0, a)$ and non-null Borel-sets $A \subseteq [0, a)$. Hence, we deduce that $Z^{(-\varrho)}(x)$ and $W^{(-\varrho)}(x)$ are positive for all respectively Lebesgue almost all $x \in (0, a)$. By remark (iii) after the Theorem, we now see that $\frac{\partial}{\partial q} Z^{(-\varrho)}(a) > 0$ thus ϱ is a simple root.

Using the identity of Lemma 3 combined with the observation $Z^{(q)}(x) - 1 = q(1 \star W^{(q)}(x))$ and the form of the resolvent given in Theorem 1, one finds, after some algebra, that $\int \mathrm{e}^{-qt} Q^t Z^{(-\varrho)}(x)\mathrm{d}t = Z^{(-\varrho)}(x)/(q+\varrho)$. By unicity of the Laplace transform, we find that there exists a null set N such that (29) holds for $t \notin N$. Since N is a Lebesgue null-set, for any $t \in N$, there exists an $s \notin N$ such that $(t-s) \notin N$. Applying the Markov property at s, we see

$$Q^t Z^{(-\varrho)}(x) = Q^{(t-s)}(Q^s Z^{(-\varrho)}(x)) = \mathrm{e}^{-\varrho s} Q^{(t-s)} Z^{(-\varrho)}(x) = \mathrm{e}^{-\varrho t} Z^{(-\varrho)}(x),$$

from which we see that (29) holds for all $t > 0$. Hence $Z^{(-\varrho)}$ is the ϱ-invariant function for Q^t (unicity from Theorem 3 in [124]). Analogously, one can prove that $W^{(-\varrho)}(a-x)\mathrm{d}x$ is the ϱ-invariant measure for Q^t.

(iv) By Proposition 6 and Theorem 1 from [124], we are allowed to apply Theorem 5 and 7 in [124]. Noting that the ϱ-invariant measure Π has a finite mass and $Q^t 1(x)$ converges to zero for all x as t tends to ∞, we find from Theorem 7 in [124] that for Π-almost every (and hence Lebesgue-almost every) $x \in [0, a)$

$$Q^t(x, A)/Q^t(x, [0, a)) \to \Pi(A)/\Pi([0, a)) \qquad \text{as } t \to \infty.$$

Combining with Theorem 5(i) in [124] this proves (31) for almost every $x \in [0, a)$. The Markov property combined with the absolute continuity of Q^t under (AC) then implies that the last statement is valid for all $x \in [0, a]$. This completes the proof of part (A).

Part (B) follows along the same lines as part (A). By Proposition 5, we can again use Theorems 2 and 3 of [124], where the role of the measure m is now played by the Lebesgue measure with an atom of size one at zero. We invoke Lemma 5 in [24] to find a $\delta \in (0, a/2)$ such that $W^{(-\widehat{\varrho})}(y) > 0$ for $y \in (0, \delta)$. By the expansion (3) we see that $W_+^{(-\widehat{\varrho})\prime}(0) = W_+'(0) - \widehat{\varrho} W(0)^2$. Combined with Lemma 4 [and monotonicity of $\widehat{\varrho}(\cdot)$ in the compound Poisson case] this implies $W_+^{(-\widehat{\varrho})\prime}(0)$ is positive or infinite. By right-continuity (Lemma 2) of $x \mapsto W_+^{(-\widehat{\varrho})\prime}(x)$ we can then find a δ' such that $W^{(-\widehat{\varrho})\prime}(y) > 0$ for $y \in (0, \delta')$. Analogously as for part (A), we can then prove the $\widehat{\varrho}$-recurrence of $\widehat{Q}^t$ and the stated properties of $W_+^{(-\widehat{\varrho})\prime}(x), W^{(-\widehat{\varrho})}(x)$. To identify the $\widehat{\varrho}$-invariant function and measure we follow an analogous line of reasoning using the second identity in Lemma 3. The proof of (iv) goes along the same lines as above. □

9 The processes Y and $\widehat{Y}$ conditioned to stay below a

We study the processes Y and $\widehat{Y}$ conditioned to stay below a fixed level $a > 0$. We introduce the measures $\mathbb{P}^\diamond$ and $\widehat{\mathbb{P}}^\diamond$ by

$$\mathrm{d}\mathbb{P}^\diamond_{x|\mathcal{F}_t} = H_t \mathrm{d}\mathbb{P}_{x|\mathcal{F}_t} \qquad \text{and} \qquad \mathrm{d}\widehat{\mathbb{P}}^\diamond_{x|\mathcal{F}_t} = \widehat{H}_t \mathrm{d}\mathbb{P}_{-x|\mathcal{F}_t}$$

where

$$H_t = \mathrm{e}^{\varrho t}\mathbf{1}_{\{t<\tau_a\}}\frac{Z^{(-\varrho)}(Y_t)}{Z^{(-\varrho)}(x)} \qquad \text{and} \qquad \widehat{H}_t = \mathrm{e}^{\widehat{\varrho} t}\mathbf{1}_{\{t<\widehat{\tau}_a\}}\frac{W^{(-\widehat{\varrho})}(a-\widehat{Y}_t)}{W^{(-\widehat{\varrho})}(a-x)}.$$

Theorem 2 implies that $\mathbb{P}^\diamond_x$ and $\widehat{\mathbb{P}}^\diamond_x$ are h-transforms of $\mathbb{P}_x$ and $\mathbb{P}_{-x}$ respectively. Indeed, by the Markov property of Y under the probability measure $\mathbb{P}$:

$$\begin{aligned}\mathbb{E}_x(H_{t+s}|\mathcal{F}_t) &= \frac{\mathrm{e}^{\varrho(t+s)}}{Z^{(-\varrho)}(x)}\mathbb{E}_x(\mathbf{1}_{\{t+s<\tau_a\}}Z^{(-\varrho)}(Y_{t+s})|\mathcal{F}_t) \\ &= \frac{\mathrm{e}^{\varrho(t+s)}}{Z^{(-\varrho)}(x)}\mathbf{1}_{\{t<\tau_a\}}\mathbb{E}_{Y_t}(\mathbf{1}_{\{s<\tau_a\}}Z^{(-\varrho)}(Y_s))\end{aligned}$$

and the martingale property of H follows from Theorem 2(A,iii). Similarly, using Theorem 2(B,iii) we can verify that $\widehat{H}$ is a martingale under $\mathbb{P}_{-x}$. The next result proves properties of the constructed processes and shows that, if (AC) holds, the h-transforms are equal to the limit as t tends to infinity of the conditional probabilities of Y (resp. $\widehat{Y}$) exiting $[0,a]$ after t. Recall the measures μ and $\widehat{\mu}$ given in note (iii) after Theorem 2.

Theorem 3 *Let $x \in [0,a)$. The following are true:*

(i) Under $\mathbb{P}^\diamond$, Y has the strong Markov property and is positively recurrent with stationary probability measure μ. Moreover, we have in the sense of weak convergence

$$\lim_{t\to\infty} \mathbb{P}^\diamond_x(Y_t \in \cdot) = \mu. \tag{36}$$

(ii) If X satisfies (R), (i) continues to hold if we replace the triple $(Y, \mathbb{P}^\diamond, \mu)$ by $(\widehat{Y}, \widehat{\mathbb{P}}^\diamond, \widehat{\mu})$.

(iii) Suppose (AC) holds. Then the convergence in (i) and (ii) holds in total variation norm. Moreover, for any $s \geq 0$ and $A \in \mathcal{F}_s$, the conditional laws converge as $t \to \infty$

$$\mathbb{P}_x(A|\tau_a > t) \to \mathbb{P}^\diamond_x(A) \qquad \textit{and} \qquad \mathbb{P}_{-x}(A|\widehat{\tau}_a > t) \to \widehat{\mathbb{P}}^\diamond_x(A).$$

Example 3 *Let X be a standard Brownian motion and define the space-time function h by*

$$h(t,x,y) = \mathrm{e}^{\frac{\pi^2}{4a}t}\cos(\tfrac{\pi}{2a}y)/\cos(\tfrac{\pi}{2a}x), \qquad x,y \in [0,a).$$

Theorem 3 implies that the process Y conditioned to stay below a has infinitesimal generator L which acts on f in its domain $D = \{f \in C^2(0,a) : f'_+(0) = 0\}$ as

$$\begin{aligned} Lf &= \left(\frac{\partial}{\partial t} + \Delta\right)(fh) \\ &= \frac{1}{2}f'' - \frac{\pi}{2a}\tan\left(\frac{\pi x}{2a}\right) f'. \end{aligned} \tag{37}$$

By a famous theorem of Lévy, the process Y is in law equal to the process $|X|$. Hence, by symmetry and $\tan(x) = -\tan(-x)$, we find, as in [79], that the generator of Brownian motion conditioned to stay in $(-a,a)$, is given on $(-a,a)$ by (37) for all functions f in $C^2(-a,a)$. This conditioned Brownian motion is called the Brownian Taboo process with taboo states $\{-a,a\}$ in the nomenclature of [79].

Proof We only prove the part of the theorem involving Y, leaving the rest to the reader.
(i) It is well known that under $\mathbb{P}^\diamond_x$ and $\widehat{\mathbb{P}}^\diamond_x$ the strong Markov property is preserved [49, Thm. XVI.28 p. 329] and the process has as semi-group $P^\diamond_t(x, \mathrm{d}y) = Q^t(x, \mathrm{d}y)\mathrm{e}^{\varrho t}\frac{Z^{(-\varrho)}(y)}{Z^{(-\varrho)}(x)}$. The positive recurrence and invariance of μ for $P^\diamond_t$ are immediate from Theorem 2(A;i,iii) combined with the form of the resolvents of the process under $\mathbb{P}^\diamond_x$, which follows now directly from Theorem 1. The form of the constant follows from note (iii) after Theorem 2. To prove the convergence, we will use the regenerative property of Y under $\mathbb{P}^\diamond$. To be more precise, under $\mathbb{P}^\diamond$, Y is a Markov process and hence a delayed regenerative process, where the delay is the time to reach zero and a cycle starts at zero and ends again at the first return to zero after a crossing of the level $a/2$. Denoting by T^* the cycle length, we see from forthcoming Proposition 7 that T^* has a finite mean. Note that T^* has the same distribution as $\widehat{\tau}_0 \circ \theta_{\tau_{\frac{a}{2}}}$ under $\mathbb{P}^\diamond_0$. From Lemma 6 we see that the distribution of $\widehat{\tau}_0 \circ \theta_{\tau_{\frac{a}{2}}}$ under $\mathbb{P}^\diamond_0$ has no atoms. In particular, T^* is not concentrated on a lattice. Theorem V.1.2 from Asmussen [3] now implies the weak convergence (36).

(iii) Suppose now (AC) holds. As in the proof of Theorem 2(A; iv), we invoke Theorem 5(i) of [124] to find that (36) holds in total variation norm for Π-a.e. $x \in [0,a)$. Combining the Markov property with the absolute continuity of the transition probabilities $P^\diamond_t$ of Y under $\mathbb{P}^\diamond$ under (AC), we find

$$\|P^\diamond_t(x,\cdot) - \mu\| \le P^\diamond_s(x, \|P^\diamond_{t-s}(Y_s,\cdot) - \mu\|)$$

where $\|\cdot\|$ denotes the total variation norm. By bounded convergence the right-hand side converges to zero as t tends to infinity.

To prove the convergence of the conditional laws, pick $s, t > 0$. From the notes after Theorem 2, we see that the random variables

$$H_{t,s} = \frac{\mathbb{P}_x(\tau_a > t + s|\mathcal{F}_t)}{\mathbb{P}_x(\tau_a > t + s)} = \mathbf{1}_{\{\tau_a > t\}} \frac{\mathbb{P}_{Y_t}(\tau_a > s)}{\mathbb{P}_x(\tau_a > t + s)}$$

converge to H_t a.s. as $s \to \infty$. Since $\mathbb{E}_x(H_{t,s}) = 1 = \mathbb{E}_x(H_t)$, it follows from Scheffe's lemma that the preceding convergence holds in L^1. We deduce that $\mathbb{E}_x(AH_{t,s})$ converges to $\mathbb{E}_x(AH_t)$ for every $A \in L^\infty(\mathcal{F}_t)$. By the Markov property this means:

$$\lim_{s\to\infty} \mathbb{E}_x(A|\tau_a > t + s) = \mathbb{E}_x(AH_t) = \mathbb{E}^\diamond_x(A). \qquad \square$$

In the sequel, we will frequently use the fact (from the optional stopping theorem) that for every finite stopping time S and $A \in L_+(\mathcal{F}_S)$

$$\mathbb{P}^\diamond_x(A) = \mathbb{E}_x(AH_S), \qquad \widehat{\mathbb{P}}^\diamond_x(A) = \mathbb{E}_{-x}(A\widehat{H}_S).$$

We now collect some Laplace transforms of hitting times under $\mathbb{P}^\diamond$.

Proposition 7 *For any $0 < b < x < c < a, q \geq 0$ the following hold:*

(i) Two sided exit problem under $\mathbb{P}^\diamond$: if $T' = \inf\{t \geq 0 : Y_t \notin (b, c)\}$,

$$\mathbb{E}^\diamond_x(\mathrm{e}^{-qT'}\mathbf{1}_{\{Y_{T'}=c\}}) = \frac{Z^{(-\varrho)}(c)}{Z^{(-\varrho)}(x)} \frac{W^{(q-\varrho)}(x-b)}{W^{(q-\varrho)}(c-b)}.$$

(ii) Passage at an upper level:

$$\mathbb{E}^\diamond_x(\exp(-q\tau_c)) = \frac{Z^{(-\varrho)}(c)}{Z^{(-\varrho)}(x)} \frac{Z^{(q-\varrho)}(x)}{Z^{(q-\varrho)}(c)}$$

(iii) Passage time below a lower level: if $T'' = \inf\{t \geq 0 : Y_t =\notin (b, a]\}$,

$$\mathbb{E}^\diamond_x(\exp(-qT'')\mathbf{1}_{\{Y_{T''-}\in \mathrm{d}y\}}\mathbf{1}_{\{\Delta Y_{T''}\in \mathrm{d}z\}}) = \frac{Z^{(-\varrho)}(y+z)}{Z^{(-\varrho)}(x)} \times$$
$$\times \left(\frac{W^{(q-\varrho)}(x-b)W^{(q-\varrho)}(a-y)}{W^{(q-\varrho)}(a-b)} - W^{(q-\varrho)}(x-y) \right) \mathrm{d}y\Lambda(\mathrm{d}z).$$

Similarly, we state some Laplace transforms of hitting times under $\widehat{\mathbb{P}}^\diamond$.

Proposition 8 *For any $0 < b < x < c < a$ and $q \geq 0$ the following hold:*

(i) Two sided exit problem under $\widehat{\mathbb{P}}^\diamond$: if $T' = \inf\{t \geq 0 : \widehat{Y}_t \notin (b, c)\}$,

$$\widehat{\mathbb{E}}^\diamond_x(\mathrm{e}^{-qT'}\mathbf{1}_{\{X_{T'}=b\}}) = \frac{W^{(-\widehat{\varrho})}(a-b)}{W^{(-\widehat{\varrho})}(a-x)} \frac{W^{(q-\widehat{\varrho})}(c-x)}{W^{(q-\widehat{\varrho})}(c-b)}.$$

(ii) *Passage at an upper level: if* $\widehat{\tau}_{\{c\}} = \inf\{t \geq 0 : \widehat{Y}_t = c\}$ *and* $\widehat{Y}$ *has unbounded variation,*

$$\widehat{\mathbb{E}}_x^{\diamond}(\exp(-q\widehat{\tau}_{\{c\}})) = \frac{W^{(-\widehat{\varrho})}(a-c)}{W^{(-\widehat{\varrho})}(a-x)} \times \times \left[\frac{W^{(q-\widehat{\varrho})}(a-x)}{W^{(q-\widehat{\varrho})}(a-c)} - \frac{W^{(q-\widehat{\varrho})\prime}(a)}{W^{(q-\widehat{\varrho})\prime}(c)} \frac{W^{(q-\widehat{\varrho})}(c-x)}{W^{(q-\widehat{\varrho})}(a-c)}\right]$$

(iii) *Passage time above an upper level: if* $T'' = \inf\{t \geq 0 : \widehat{Y}_t \notin [0, c)\}$,

$$\widehat{\mathbb{E}}_x^{\diamond}(\exp(-qT'')\mathbf{1}_{\{X_{T''-}\in \mathrm{d}y\}}\mathbf{1}_{\{\Delta X_{T''}\in \mathrm{d}z\}}) = \frac{W^{(-\widehat{\varrho})}(a-y-z)}{W^{(-\widehat{\varrho})}(a-x)} \times \times \left(\frac{W^{(q-\widehat{\varrho})}(c-x)W^{(q-\widehat{\varrho})}(\mathrm{d}y)}{W_+^{(q-\widehat{\varrho})\prime}(c)} - W^{(q-\widehat{\varrho})}(y-x)\mathrm{d}y\right)\Lambda(\mathrm{d}z).$$

Proof of Propositions 7 and 8 Proposition 7 and statements (i) and (iii) of Proposition 8 can be proved adapting the line of reasoning followed in the proof of [86, Prop. 3.2].

The second statement of Proposition 8 follows from potential theory of Markov processes. Recall that under $\widehat{\mathbb{P}}^{\diamond}$ the process $\widehat{Y}$ is a Markov process with semi-group $\widehat{P}_t^{\diamond}(x, \mathrm{d}y) = \widehat{Q}^t(x, \mathrm{d}y)\mathrm{e}^{\widehat{\varrho}t}W^{(-\widehat{\varrho})}(a-y)/W^{(-\widehat{\varrho})}(a-x)$. If $\widehat{Y}$ has unbounded variation, Theorem 1 implies that the process $\widehat{Y}$ under $\widehat{\mathbb{P}}^{\diamond}$, has an absolutely continuous q-resolvent measure $\widehat{U}_q^{\diamond}$ with a version of its density given by

$$\widehat{u}_q^{\diamond}(x, y) = \widehat{r}^{q-\widehat{\varrho}}(x, y)W^{(-\widehat{\varrho})}(a-y)/W^{(-\widehat{\varrho})}(a-x).$$

Then one has the identity

$$\mathbb{E}_x^{\diamond}[\mathrm{e}^{-q\widehat{\tau}_{\{c\}}}] = \frac{\widehat{u}_q^{\diamond}(x, c)}{\widehat{u}_q^{\diamond}(c, c)},$$

which follows by approximating the potential density as in Theorem II.19(ii) in [23]. □

9.1 Excursion measure away from a point

Recall that a point $x \in [0, a)$ is said to be regular (for itself) under $\mathbb{P}^{\diamond}$ if

$$\mathbb{P}_x^{\diamond}(\inf\{s > 0 : Y_s = x\}) = 1.$$

Obviously, $x > 0$ is regular under $\mathbb{P}^{\diamond}$ if and only if $x > 0$ is regular under $\mathbb{P}$, hence if and only if X has unbounded variation under $\mathbb{P}$. We assume this throughout from now on. The local time at level at x, denoted by L^x is defined as the occupation density

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|Y_s - x| < \epsilon\}} \mathrm{d}s.$$

Let σ_s be its right-continuous inverse:

$$\sigma_s = \inf\{t > 0 : L_t^x > s\}, \qquad s \geq 0.$$

Analogously to what we did in the proof of Theorem 1, we now consider the excursion process $e = (e_s, s \geq 0)$ of Y away from $\{x\}$ where

$$e_s = (Y_u, \sigma_{s^-} \leq u < \sigma_s) \qquad \text{if } \sigma_{s^-} < \sigma_s$$

and else e_s takes the value ∂ where ∂ is an additional isolated point. A famous theorem of Itô states that e is a Poisson point process valued in the space $\mathcal{E}'$

$$\mathcal{E}' = \{f \in D[0,\infty) : \exists \zeta = \zeta(f) \text{ such that } f(0) = f(\zeta) = 0\}.$$

Its characteristic measure is denoted by n_x under $\mathbb{P}$ (and $n_x^\diamond$ under $\mathbb{P}^\diamond$) and is called the excursion measure away from $\{x\}$. In this section we present some useful formulas involving the local time L^x and the excursion measure $n_x^\diamond$. For every excursion of Y away from $\{x\}$, we denote its height by $m = m(\epsilon)$:

$$m(\epsilon) = \sup_{u \leq \zeta(\epsilon)} (\epsilon_u - \epsilon_0) = \sup_{u \leq \zeta(\epsilon)} \epsilon_u - x.$$

Recall that σ stands for the inverse of the local time L^x. As well known, σ is a subordinator. Define Laplace exponent $\Phi_x^\diamond$ by

$$\mathbb{E}_x^\diamond(\mathrm{e}^{-\lambda\sigma_t}) = \exp(-t\Phi_x^\diamond(\lambda)), \qquad \lambda \geq 0.$$

Analogously, replacing everywhere in the above definitions the process $(Y, \mathbb{P}^\diamond)$ by $(\widehat{Y}, \widehat{\mathbb{P}}^\diamond)$ we define the local time $\widehat{L}$, its inverse $\widehat{\sigma}$ with exponent $\widehat{\Phi}_x$ and the excursion process $\widehat{e}$ with its heights $\widehat{m}$ and characteristic measure $\widehat{n}_x^\diamond$.

Proposition 9 *(i) For any nonnegative λ and any $\eta \in [0, a-x]$,*

$$n_x^\diamond(1 - \mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta}) = \frac{Z^{(\lambda-\varrho)}(x+\eta)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(\eta)}.$$

In particular, for any nonnegative λ and for any $\eta \in [0, a-x]$,

$$\Phi_x^\diamond(\lambda) = \frac{Z^{(\lambda-\varrho)}(a)}{Z^{(\lambda-\varrho)}(x)W^{(\lambda-\varrho)}(a-x)} \qquad n_x^\diamond(m > \eta) = \frac{Z^{(-\varrho)}(x+\eta)}{Z^{(-\varrho)}(x)W^{(-\varrho)}(\eta)}.$$

(ii) For any nonnegative λ and any $\eta \in [0, a-x]$,

$$\widehat{n}_x^\diamond(1 - \mathbf{1}_{\{\widehat{m}<\eta\}}\mathrm{e}^{-\lambda\widehat{\zeta}}) = \frac{W^{(\lambda-\widehat{\varrho})\prime}(x+\eta)}{W^{(\lambda-\widehat{\varrho})\prime}(x)W^{(\lambda-\widehat{\varrho})}(\eta)}.$$

In particular, for any nonnegative λ and for any $\eta \in [0, a-x]$,

$$\widehat{\Phi}_x^\diamond(\lambda) = \frac{W^{(\lambda-\widehat{\varrho})\prime}(a)}{W^{(\lambda-\widehat{\varrho})\prime}(x)W^{(\lambda-\widehat{\varrho})}(a-x)} \qquad \widehat{n}_x^\diamond(m > \eta) = \frac{W^{(-\widehat{\varrho})\prime}(x+\eta)}{W^{(-\widehat{\varrho})\prime}(x)W^{(-\widehat{\varrho})}(\eta)}.$$

Propositions 9 can be proved much in the same vein as Proposition 4.2 in [86] and therefore we only sketch their proofs. Note that in both propositions the last two assertions follow easily from the first (by taking $\eta = a - x$ and $\lambda = 0$ respectively). Let $\tau_{\{c\}}$ and $\widehat{\tau}_{\{c\}}$ denote the first hitting time of $\{c\}$ by Y and $\widehat{Y}$ respectively:

$$\tau_{\{c\}} = \inf\{t \geq 0 : Y_t = c\} \qquad \widehat{\tau}_{\{c\}} = \inf\{t \geq 0 : \widehat{Y}_t = c\}.$$

The proof of the Proposition 9(i) starts from the identity

$$n_x^{\diamond}(1 - \mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta}) = \left[\mathbb{E}_x^{\diamond}(\int_0^{\tau_{\{x+\eta\}}} \mathrm{e}^{-\lambda t}\mathrm{d}L_t^x)\right]^{-1},$$

which follows by an application of the exponential formula to the Poisson point processes $m(e)$. Next step consists in establishing

$$\mathbb{E}_y^{\diamond}(\int_0^{\infty} \mathrm{e}^{-\lambda t}\mathrm{d}L_t^x) = u_\lambda^{\diamond}(y, x)$$

following a line of reasoning similar to the first lines of Proposition V.2 in [23]. An application of the optional sampling theorem yields that

$$(n_x^{\diamond}(1 - \mathbf{1}_{\{m<\eta\}}\mathrm{e}^{-\lambda\zeta}))^{-1} = u_\lambda^{\diamond}(x, x) - u_\lambda^{\diamond}(x + \eta, x)\mathbb{E}_x^{\diamond}[\mathrm{e}^{-\lambda\tau_{\{x+\eta\}}}]$$

and after substituting the expressions for $u_\lambda^{\diamond}$ and for the Laplace transform (from Proposition 7(ii)) we end up with the stated expression. The first assertion of Proposition 9(ii) follows analogously replacing everywhere Y by $\widehat{Y}$.

The previous propositions enable us to specify the asymptotic behaviour of the local time. Recall from the Theorem 3 that the stationary measure μ and $\widehat{\mu}$ of the conditioned processes are absolutely continuous with respective densities p and $\widehat{p}$, say.

Corollary 3 *(i) If $x \in (0, a)$ or $x = 0$ and $a \in \mathcal{D}_\varrho$, we have*

$$\lim_{t\to\infty} \frac{L_t^x}{t} = \frac{\mu(\mathrm{d}x)}{\mathrm{d}x} = p(x) \qquad a.s. \tag{38}$$

(ii) If $x \in (0, a)$ or $x = 0$ and $a \in \mathcal{D}_{\widehat{\varrho}}$, equation (38) if we replace the triple (L^x, μ, p) by $(\widehat{L}^x, \widehat{\mu}, \widehat{p})$.

Proof We deduce from Proposition 9 that $\Phi_x^{\diamond}$ has a right-derivative at 0 equal to

$$\frac{C(a)}{Z^{(-\varrho)}(x)W^{(-\varrho)}(a - x)} = p(x)^{-1}.$$

Hence using $\mathbb{E}_x^{\diamond}(\sigma_t) = t/p(x)$ and that $\{\sigma_t : t \geq 0\}$ is a Lévy process, the strong law of large numbers entails that almost surely

$$\lim_{t\to\infty} \frac{L_t^x}{t} = \lim_{t\to\infty} \frac{t}{\sigma_t} = p(x).$$

The proof of the second statement is similar and omitted. □

9.2 Convergence of the supremum

The fact that the conditioned processes is recurrent implies that under the measures $\mathbb{P}^\diamond$ and $\widehat{\mathbb{P}}^\diamond$ the suprema of Y and $\widehat{Y}$, $M_t = \sup\{Y_s; s \in [0,t]\}$ and $\widehat{M_t} = \sup\{\widehat{Y}_s; s \in [0,t]\}$ respectively, converge to a as t tends to infinity. Our purpose is to investigate the rate of convergence. We still assume that X has unbounded variation.

Let $f : [0,\infty) \to (0,\infty)$ be a decreasing function and write

$$l_f = \liminf_{t\to\infty} \frac{a - M_t}{f(t)}, \qquad L_f = \limsup_{t\to\infty} \frac{a - M_t}{f(t)}.$$

and $\widehat{l}_f, \widehat{L}_f$ for the corresponding quantities involving $\widehat{M}$. Recall that a real-valued function is said to be slowly varying at infinity if for any $\lambda > 0$,

$$\lim_{t\to\infty} \frac{g(\lambda t)}{g(t)} = 1.$$

Finally, we set the notations $\mathcal{D}_\varrho = \{a > 0 : \varrho'(a) < 0\}$ and $\mathcal{D}_{\widehat{\varrho}} = \{a > 0 : \widehat{\varrho}'(a) < 0\}$.

Theorem 4 *Assume $a \in \mathcal{D}_\varrho$ for the statements involving M_t. Assume that $W^{(-\widehat{\varrho})}(\cdot)$ is twice continuously differentiable and let $a \in \mathcal{D}_{\widehat{\varrho}}$ for the statements involving $\widehat{M_t}$. Then following three assertions hold.*

(i) The random variables $t(a - M_t)$ and $t(a - \widehat{M_t})$ converge in distribution as $t \to \infty$ to exponential random variables with parameters $|\varrho'(a)|$ and $|\widehat{\varrho}'(a)|$ respectively.

(ii) The random variables l_f and $\widehat{l}_f$ are 0 or ∞ almost surely according to whether $\int_1^\infty f(s)\mathrm{d}s$ converges or diverges.

(iii) Assume further that $t \mapsto tf(t)$ is increasing and slowly varying at infinity and let

$$\gamma_f = \inf\left\{\gamma > 0 : \int_1^\infty f(t)\mathrm{e}^{-\gamma t f(t)}\mathrm{d}t < \infty\right\},$$

with the convention $\inf\emptyset = +\infty$. Then $L_f = |\varrho'(a)|^{-1}\gamma_f$ and $\widehat{L}_f = |\widehat{\varrho}'(a)|^{-1}\gamma_f$ almost surely.

Remarks

(i) Let $I_f(\gamma) = \int^\infty \mathrm{d}t f(t)\mathrm{e}^{-\gamma t f(t)}$. One easily sees that if $\gamma_f < \infty$, I_f is finite (and decreasing) on (γ_f, ∞) and that if $\gamma_f > 0, I_f = \infty$ on $[0, \gamma_f)$.

(ii) If $\log_k$ denotes the k-th iterate of the logarithm, then for

$$\begin{aligned}
f(t) &= t^{-1}\log t, & L_f &= 0, & l_f &= 0;\\
f(t) &= t^{-1}\log_2 t, & L_f &= |\varrho'(a)|^{-1}, & l_f &= 0;\\
f(t) &= t^{-1}\log_3 t, & L_f &= \infty, & l_f &= 0.
\end{aligned}$$

(iii) Note that (32) implies that $Z^{(-\varrho(a))\prime}(a) < 0$ for $a \in \mathcal{D}_\varrho$. Similarly, if W is twice continuously differentiable and $a \in \mathcal{D}_{\widehat{\varrho}}$, it follows from (33) that $W^{(-\widehat{\varrho}(a))\prime\prime}(a) < 0$. These facts are crucially used in the proof of the theorem.

(iv) Recall that for $\alpha \in (1,2]$, $-r(\alpha)$ denotes the first negative root of E_α, where E_α is the Mittag-Leffler function of parameter α. In the case X is a stable process of index $\alpha \in (1,2]$,

$$\limsup_{t\to\infty} \frac{t(a-M_t)}{\log_2 t} = \frac{a^{\alpha+1}}{\alpha r(\alpha)} \qquad \text{a.s.,}$$

which yields in the case X is a Brownian motion

$$\limsup_{t\to\infty} \frac{t(a-M_t)}{\log_2 t} = \frac{2a^3}{\pi^2} \qquad \text{a.s..}$$

Following the lead of Lambert [86], the idea is to exploit the Poisson point character of the excursions away from $\{x\}$ under $\mathbb{P}^\diamond$ and $\widehat{\mathbb{P}}^\diamond$. First note that Corollary 3 allows one to translate statements involving asymptotics of M_{σ_t} and $\widehat{M}_{\widehat{\sigma}_t}$ for large t into statements on asymptotics of M_t and $\widehat{M}_t$ for large t respectively. An elementary observation is that M_{σ_t} and $\widehat{M}_{\widehat{\sigma}_t}$ are the maximum of the excursion heights $(m(\epsilon_s), s \le t)$ and $(\widehat{m}(\widehat{\epsilon}_s), s \le t)$ respectively. The study of the asymptotics of M_t and $\widehat{M}_t$ is thus reduced to that of the maxima of some Poisson point process. In Proposition 9 the distribution functions of the measures $n_x^\diamond$ and $\widehat{n}_x^\diamond$ were computed explicitly and this allows one to perform explicit computations on the excursion heights. The key step in the proof of the third assertion is to appeal to results of Robbins and Siegmund [113] on laws of the iterated logarithm for maxima and minima of uniform random variables. Since the arguments are quite close to those of [86], we leave the details to the reader.

Acknowledgement The author thanks an anonymous referee for his remarks which improved the presentation of this chapter.

Chapter V

Optimal consumption in semimartingale markets

Consider an agent with finite horizon whose trading is constrained and who operates in a market where the prices of the assets are modelled as semimartingales. In this setting, we establish an existence result for maximising the expected utility of the agent over attainable inter-temporal consumption and final wealth. To obtain existence we do not resort to convex duality methods. We also give a characterisation result of the optimal solution.

1 Introduction

A general issue studied in economic sciences is the behaviour of agents in financial markets. In both classical and modern theory, one uses utility functions to model the different levels of "satisfaction" of the agents corresponding to different distributions of wealth over intermediate consumption and trading in risky assets. In two seminal papers [96, 98], Merton was the first to study this problem in a continuous time framework. He assumed that the market was driven by Markov state price processes. In this setting he could use dynamic programming and the Bellman equation to derive a partial differential equation for the value function, which he solved explicitly in the case of constant coefficients.

More recently, Cox and Huang, Pliska and Karatzas and co-authors developed in several papers, e.g. [43, 73, 109], a martingale approach to the problem of utility maximisation. In a *complete* market setting— that is, all the admissible stochastic claims can be replicated by the traded assets— the problem is decomposed into two steps. Firstly, a variational problem is solved. See for a close study of this type of optimisation problems [19]. Secondly, a portfolio financing this consumption-final wealth plan is found by using a martingale representation theorem. The *incomplete* market setting is considerably more complicated. In Karatzas et al. [74] and He and Pearson [69] the incomplete market was studied in a continuous time diffusion setting. In these papers the central idea is to solve a dual variational problem and to find the solution of the original problem by convex duality methods. Then, in the paper [84], Kramkov and Schachermayer also employed a duality approach to solve the problem of maximisation of the utility of final wealth in a general semimartingale market. Recently, Karatzas and Žitković [76] have taken up the line of [84] and extended it to the setting where also cone restrictions are put on the agent's trading strategies, where the agent has random endowment and where the agent not only gets utility from final wealth but also from intermediate consumption.

In this paper, we study the problem of optimal consumption in the same semimartingale setting with constrained trade. However, following the line set out by Cuoco [45] in the continuous time diffusion setting, we use a direct primal approach to obtain existence, without resorting to duality methods. Although we could have followed Cuoco in using Levin [90]'s technique of *relaxation-projection* to obtain existence, our existence proof relies on a famous result of Komlós [80]. Compared to Cuoco [45] and Karatzas and Žitković [76], our utility function does not need to be differentiable or increasing in the consumption.

The rest of the paper is organised as follows. In Section 2 we formulate the model. In Section 3 we show the control problem can be equivalently reformulated as a static variational problem and in 4 existence is obtained. Section 5 studies the characterisation of optimal policies and in Section 6, we consider two specific examples: in Section 6.1 we consider the case of a jump-diffusion market and in Section 6.2, we specialise to the the case of p-integrable consumption-final wealth plans in a complete markets with no restrictions on the trading.

2 Formulation of the model

2.1 Setting

In this paper we consider the continuous time model with a finite time horizon $T > 0$. Let $S = (S_1, \dots, S_n)$ be an n-dimensional semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t : 0 \le t \le T\}, P)$ which satisfies the usual conditions (i.e. $\mathbf{F}$ is complete and right-continuous). The filtration $\mathbf{F}$ can be thought of as a model for the information structure of the market. Except for the integrands all processes which occur are assumed to be adapted to the filtration $\mathbf{F}$ and to have càdlàg paths (right-continuous paths with left limits).

Consider now a securities market where a bond and n risky assets are traded. The price process of the ith risky asset is given by $S_i = \{S_i(t), 0 \le t \le T\}$. We model the price of the bond $S_0 = \{S_0(t), 0 \le t \le T\}$ by $S_0(t) = \exp(\int_0^t r(s)\mathrm{d}s)$, where $r = \{r(t), 0 \le t \le T\}$ is some non-negative $\mathbf{F}$-adapted process. The process r represents the (risk-less) interest rate. We assume that for some constant $R > 0$

$$\int_0^T r(s)\mathrm{d}s < R \qquad P\text{-a.e.} \tag{1}$$

In the sequel we write $\gamma^0 = S_0^{-1}$.

Denote by $\mathcal{P}^0$ the set of all probability measures that are equivalent to P on $\mathcal{F}_T$ and under which the discounted price processes $\gamma^0 S_i$ are local martingales for $i = 1, \ldots, n$. We make the following assumption on our process $(S_0, S) = (S_0, S_1, \ldots, S_n)$ to ensure the absence of arbitrage (see Proposition 1 below).

Assumption 1 $\mathcal{P}^0 \neq \emptyset$.

For any interval $I \subset [0,T]$, we denote by $\mathcal{L}^0_+(I \times \Omega)$ the set of all processes $b = \{b(t), t \in I\}$ that are $\mathbf{F}$-adapted and nonnegative. In above setting of the security market operates an economic agent who seeks to maximise utility through consumption and investment. The agent is endowed with an initial capital $w_0 > 0$ and receives a stochastic income flow $y \in \mathcal{L}^0_+([0,T] \times \Omega)$, for example as labour income. We assume that the process y satisfies for some constant $K > 0$

$$\int_0^T \gamma^0(t) y(t) \mathrm{d}t \leq K. \qquad P\text{-a.e.} \tag{2}$$

At time t the agent can choose to buy and consume an amount $c_i(t)$ of the commodity i $(i = 1, \ldots, m)$ or to invest his/her money in the security market to generate a final wealth at time T or to be able to consume at some later time. A final wealth plan and w consumption plan c are elements of $\mathcal{L}^0_+(\{T\} \times \Omega)$ and $(\mathcal{L}^0_+([0,T] \times \Omega))^m$ respectively.

The agent's preferences for consumption-final wealth plans (c, w) are represented by a functional U:

$$U(c, w) = E\left[\int_0^T u(t, c(t))\mathrm{d}t + v(w)\right], \tag{3}$$

where preferences of intermediate consumption c are expressed through a time-additive function $u : [0,T] \times \mathbf{R}^m_+ \times \Omega \to [-\infty, \infty)$ and those of final wealth w through a function $v : \mathbf{R}_+ \times \Omega \to [-\infty, \infty)$. For each $c \in \mathbf{R}^m_+$, the function $(t, \omega) \mapsto u(t, c, \omega)$ is $\mathbf{F}$-adapted; for each $\omega \in \Omega$, the function $(t, c) \mapsto u(t, c, \omega)$ is measurable with respect to $\mathcal{B}([0,T] \times \mathbf{R}^m_+)$, the Borel σ-algebra of $[0,T] \times \mathbf{R}^m_+$. Similarly, the function v is measurable with respect to $\mathcal{B}(\mathbf{R}_+) \times \mathcal{F}_T$. The functions u, v may be random, reflecting the agent's changes in preferences. The agent who is just interested in inter-temporal consumption or just in final wealth is included in the model by setting v respectively u equal to zero. Later on we will impose conditions on the utility functions u and v to ensure that the functional U is well defined.

2.2 Constraints on trading

We begin with recalling some notations and facts from the theory of stochastic integration. For a comprehensive treatment of the subject of stochastic integration we refer to Dellacherie and Meyer [48], Protter [111] and Jacod and Shiryaev [71]. Let $X = (X_1, \ldots, X_n)$ be an n-dimensional semimartingale and let $\mathcal{L}(X)$ denote the space of all n-dimensional predictable processes integrable with respect to X. This space contains among others the locally bounded predictable processes. The stochastic integral of $H \in \mathcal{L}(X)$ with respect to X will be denoted as $\int H \mathrm{d}X = \int H_1 \mathrm{d}X_1 + \ldots + \int H_n \mathrm{d}X_n$, where $\int H_i \mathrm{d}X_i$ denotes the stochastic integral of H_i with respect to the real valued semimartingale X_i.

For $H \in \mathcal{L}(X)$ the stochastic integral $\int H \mathrm{d}X$ is again a semimartingale. Identifying two processes H, $\widetilde{H}$ if the integrals $\int H \mathrm{d}X$ and $\int \widetilde{H} \mathrm{d}X$ are indistinguishable, that is

$$P\left(\left\{\omega : \int_0^t H(s,\omega)\mathrm{d}X(s,w) = \int_0^t \widetilde{H}(s,\omega)\mathrm{d}X(s,\omega) \text{ for all } t \in [0,T]\right\}\right) = 1,$$

we write $\mathbf{L}(X)$ for the resulting space of equivalence classes. In the sequel we will slightly abuse notation by writing H if we mean the equivalence class of H. An integrand $H \in \mathbf{L}(X)$ is called *(locally) admissible* if $\int H \mathrm{d}X$ is (locally) bounded from below. The class of all admissible (resp. locally admissible) processes is denoted by $\mathbf{L}^a(X)$ (resp. $\mathbf{L}^a_{loc}(X)$). The *Emery distance* D between two real valued semimartingales Y and Z is defined as

$$D(Y,Z) = \sup_{U \in \mathcal{U}} E\left[\min\left\{1, \sup_{t \le T}\left|\int U \mathrm{d}Y - \int U \mathrm{d}Z\right|\right\}\right]$$

with the supremum taken over the set $\mathcal{U}$ of predictable real valued processes bounded in absolute value by one. For this metric the space of real valued semimartingales (up to distinguishability) is complete, see Emery [55]. Mémin [95] has shown that the space $\mathbf{L}(X)$ is complete with respect to the metric

$$d_X(H,G) = D\left(\int H \mathrm{d}X, \int G \mathrm{d}X\right). \tag{4}$$

From now on X denotes the n-dimensional semimartingale $X = \gamma^0 S$. We assume that trading takes place continuously in time and that there are no transaction costs. Let $H_0(t)$ and $H(t) = (H_i(t))_{i=1}^n$ denote the number of bonds and shares of different types (1 till n) the agent has in his/her portfolio at time t. Assume that $H_0 = \{H_0(t), 0 \le t \le T\}$ is progressively measurable with respect to the filtration $\mathbf{F}$. The trading of the agent is subject to certain restrictions (e.g. prohibition of short selling). To model these constraints, we follow [59] and impose the condition that H lies in a certain subset of $\mathbf{L}^a_{loc}(X)$. Let $\mathcal{H} \subseteq \mathbf{L}^a_{loc}(X)$ be a family of locally admissible integrands for X. We assume that $\mathcal{H}$ contains $H \equiv 0$, is closed in $\mathbf{L}^a_{loc}(X)$ with respect to the metric d_X given in (4) and is convex in the following sense: for any H and G in $\mathcal{H}$ and any predictable process $0 \le h \le 1$ the process $hG + (1-h)H$ belongs to $\mathcal{H}$. A strategy is called $\mathcal{H}$-constrained if $H \in \mathcal{H}$.

Example 1 *Using $\mathcal{H}$ various constraints on the choice of trading strategies can be modelled. For instance, in the following cases the set $\mathcal{H}$ satisfies our assumptions:*

(i) No constraints: $\mathcal{H} = \mathbf{L}^a_{loc}(X)$;

(ii) Short-selling of assets 1 to k is not allowed:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H_i \geq 0, 1 \leq i \leq k\};$$

(iii) Assets 1 *to k are non-tradeable:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H_i = 0, 1 \leq i \leq k\};$$

(iv) The strategy H has to lie in a closed convex set $A \subset \mathbf{R}^n$ containing 0*:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H \in A\};$$

(v) Minimum capital requirements:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : \sum_{i=1}^{n} H_i X_i \geq L\},$$

for some $L \leq 0$.

(vi) Constraints on the amounts of money $H_i X_i$ invested:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : (H_1 X_1, \ldots, H_n X_n) \in A\},$$

where $A \subset \mathbf{R}^n$ is a convex closed set containing 0*.*

(vii) Upper and lower bounds on the number of shares:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : L_i \leq H_i \leq U_i, 1 \leq i \leq n\},$$

where $L_i \leq 0 \leq U_i$ and L, U belong to $\mathbf{L}^a_{loc}(X)$.

(viii) Upper and lower bounds on the amounts of money $H_i X_i$ invested:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : L_i \leq H_i X_i \leq U_i, 1 \leq i \leq n\},$$

where L, U are as in (vii).

Set $\mathcal{Y}$ equal to the family of semimartingales $\mathcal{Y} = \{\int H \mathrm{d}X : H \in \mathcal{H}\}$. For any $Y \in \mathcal{Y}$ and any probability measure $P^* \sim P$ we denote by $A^Y_{P^*}$ the compensator of Y under P^*. We restrict ourselves to the family $\mathcal{P}$ of measures $P^* \sim P$ under which any $Y \in \mathcal{Y}$ is a special semimartingale* and under which there exists an increasing predictable process $A^{\mathcal{Y}}_{P^*}$ with

$$A^{\mathcal{Y}}_{P^*}(t) := \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} A^Y_{P^*}(t) < \infty \tag{5}$$

a.e. for all $t \in [0, T]$. Then, for any $P^* \in \mathcal{P}$, the process $A^{\mathcal{Y}}_{P^*}$ is the minimal upper bound for all predictable processes arising in the Doob-Meyer decomposition of the special semimartingales $Y \in \mathcal{Y}$ (called the *upper variation process* of Y, see [59]). Note that the set $\mathcal{P}^0$ of equivalent local martingale measures is contained in $\mathcal{P}$. Indeed, since under $P^0 \in \mathcal{P}^0$ any element in $\mathcal{Y}$ is a local martingale (cf. [2]), we have that $A^{\mathcal{Y}}_{P^0}$ is identically zero.

*A semimartingale X is a special semimartingale if it admits the decomposition $X_t = X_0 + M_t + A_t$, where M is a local martingale with $M_0 = 0$ and A is a predictable process which is locally of integrable variation with $A_0 = 0$

Example 2 *Let $P^* \sim P$. If the set $\mathcal{H}$ is a cone, the corresponding upper variation process $A^{\mathcal{Y}}_{P^*}$ is identically zero (since $A^{\lambda Y}_{P^*} = \lambda A^{Y}_{P^*}$ for $\lambda \geq 0$) and thus $\mathcal{P}$ is the set of measures $P^* \sim P$ under which any $Y \in \mathcal{Y}$ is a local supermartingale. Similarly, if $\mathcal{H}$ is a linear family, we find that the corresponding process $A^{\mathcal{Y}}_{P^*}$ is identically zero and $\mathcal{P}$ is the set of measures $P^* \sim P$ under which all $Y \in \mathcal{Y}$ are local martingales. As third example we consider $\mathcal{H}$ as in Example 1(vii). Let $P^* \sim P$ be a measure under which X is a special semimartingale and write $X = M + A$ for the canonical decomposition of X into a local martingale M under P^* and a predictable process A of bounded variation. For this class $\mathcal{H}$, one can then show, following [59], that*

$$A^{\mathcal{Y}}_{P^*}(t) = \sum_{i=1}^{n} \int_0^t U_i(s)\mathrm{d}A_i^+(s) - \sum_{i=1}^{n} \int_0^t L_i(s)\mathrm{d}A_i^-(s), \qquad t \geq 0,$$

(where $A = A^+ - A^-$ with $A^\pm$ predictable increasing processes) and that the set $\mathcal{P}$ contains the set of all $P^* \sim P$ under which X is a special semimartingale.

In the sequel we put the following restriction on the constraint family $\mathcal{H}$:

Assumption 2 *The family $\mathcal{H}$ is such that* $\sup_{P^* \in \mathcal{P}} E^*[A^{\mathcal{Y}}_{P^*}(T)] < \infty$.

We make this assumption for the ease and clarity of the presentation and without loss of generality. Indeed, since $A^{\mathcal{Y}}_{P^*}$ is locally bounded, Assumption 2 can be dispensed with using *localisation* arguments (see [111] for explanation and compare the proof of Proposition 4.2 in [59]). We omit the details.

2.3 The consumption problem

If the agent has followed trading strategy $(H_0, H) = (H_0, H_1, \ldots, H_n)$ up to time t, his/her wealth at time t is given by $\Pi(t) = H_0(t)S_0(t) + H(t) \cdot S(t)$ (where $\cdot$ denotes the inner product). For an agent with income y and initial wealth w_0, a pair (c, w) of a consumption plan $c \in \mathcal{L}^0_+([0,T] \times \Omega)$ and a final wealth $w \in \mathcal{L}^0_+(\{T\} \times \Omega)$ is called *$\mathcal{H}$-feasible* if there exists an $H \in \mathcal{H}$ and a non decreasing process $D \in \mathcal{L}^0_+([0,T] \times \Omega)$ such that Π is bounded from below, reaches the wealth w at time T, $\Pi(T) \geq w$ a.e., and satisfies for all $t \in [0,T]$

$$\Pi(t) = w_0 + \int_0^t H_0(s)\mathrm{d}S_0(s) + \int_0^t H(s)\mathrm{d}S(s) - C(t) \tag{6}$$

where

$$C(t) = \int_0^t \left(\sum_{i=1}^{m} c_i(s) - y(s) \right) \mathrm{d}s + D(t).$$

The set of all $\mathcal{H}$-feasible consumption-final wealth plans (c, w) is denoted by $\mathcal{A}^0(\mathcal{H})$. The process D in (6) covers the possibility of free disposal of wealth, that is the agent is allowed to reinvest not his entire wealth. The amount of wealth wasted up to time t is then given by $D(t)$. Equation (6) is the usual dynamic budget constraint. It states at time t the wealth is equal to the initial wealth plus trading gains minus net withdrawals. The problem facing the agent can now be stated as the following control problem:

$$\sup_{(c,w)\in\mathcal{A}^0(\mathcal{H})} U(c,w) = \sup_{(c,w)\in\mathcal{A}^0(\mathcal{H})} E\left[\int_0^T u(t,c(t))\mathrm{d}t + v(w)\right]. \qquad (\mathcal{V})$$

For the agent's optimisation problem $(\mathcal{V})$ to be well posed it is necessary that there are no *arbitrage possibilities* in the market attainable for the agent, that is, the agent can not make a risk-less profit using some $\mathcal{H}$-feasible trading strategy. To be more precise, an arbitrage possibility is a nonzero consumption-final wealth plan (c,w) that is $\mathcal{H}$-feasible for zero initial wealth w_0 and zero income y. To rule out strategies such as the doubling strategy of Harrison and Kreps [66], we imposed the condition that the wealth process Π is bounded from below.

Proposition 1 *Under Assumption 1 there are no arbitrage possibilities.*

Proof Use partial integration and recall that $\Pi = H_0S_0 + H\cdot S$ to find that (6) implies

$$\begin{aligned}\gamma^0(t)\Pi(t) + \int_0^t \gamma^0(s)\mathrm{d}C(s) &= -\int_0^t [H_0(s)S_0(s) + H(s)\cdot S(s)]r(s)\gamma^0(s)\mathrm{d}s \\ &\quad + \int_0^t \gamma^0(s)H(s)\mathrm{d}S(s) \\ &= \int_0^t H(s)\mathrm{d}(\gamma^0(s)S(s)) =: J_t \qquad (7)\end{aligned}$$

Since H is locally admissible, $J = \{J_t, t\in[0,T]\}$ is a local martingale under all measures $P^0 \in \mathcal{P}^0$. Since the left-hand side is bounded below, Fatou's lemma then implies that J is a supermartingale and we find that

$$E^0\left[\gamma^0(T)\Pi(T) + \int_0^T \gamma^0(s)\left(\sum_{i=1}^m c_i(s)\right)\mathrm{d}s\right] \le 0 \qquad \forall P^0\in\mathcal{P}^0.$$

Since $\gamma^0(t) \ge \exp(-R)$ for all $t\in[0,T]$, it follows that $(c,w) = (0,0)$ a.e. □

3 Reformulation

As in Cox and Huang [43] and Cuoco [45], the first step in establishing existence for $(\mathcal{V})$ is to reformulate the control problem $(\mathcal{V})$ as a static variational problem. We will write E^* for the expectation under the measure P^*. Recall the definition of the process $A^{\mathcal{Y}}_{P^*}$ given in (5).

Theorem 1 *Let* $(c,w) \in (L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$ *be a consumption-final wealth plan. Then the following two assertions are equivalent:*

(i) (c, w) *is* $\mathcal{H}$*-feasible;*

(ii) (c, w) *satisfies for all* $P^* \in \mathcal{P}$*:*

$$E^*\left[\int_0^T \gamma^0(t)\Big(\sum_{i=1}^m c_i(t) - y(t)\Big)\mathrm{d}t + \gamma^0(T)w\right] \leq w_0 + E^*\left[A^{\mathcal{Y}}_{P^*}(T)\right]. \qquad (8)$$

Letting (c, w) a consumption-final wealth plan in $(L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$, we get from this result that there exists an $\mathcal{H}$-constrained strategy that the agent can follow to attain (c, w) if, and only if, (c, w) satisfies (8) for all $P^* \in \mathcal{P}$. Thus (c, w) is optimal for the control problem $(\mathcal{V})$ if, and only if, (c, w) is optimal for the variational problem

$$\sup_{(c,w)\in F} U(c, w), \qquad (\mathcal{V}')$$

where F is the set of $(c, w) \in (L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$ that satisfy the inequality (8) for all $P^* \in \mathcal{P}$.

Proof $(i) \Rightarrow (ii)$ Let $\widetilde{H} \in \mathcal{H}$ be a strategy that implements (c, w), let P^* be any element of $\mathcal{P}$ and denote by $\widetilde{Y}$ the process $\widetilde{Y} = \int \widetilde{H}\mathrm{d}(\gamma^0 S)$. Since $A^{\widetilde{Y}}_{P^*}$ denotes the compensator of $\widetilde{Y}$, the process $\widetilde{Y} - A^{\widetilde{Y}}_{P^*}$ is a local martingale under P^*. Let $\mathcal{T}$ denote the set of $\mathbf{F}$-stopping times and write $(\tau_m)_m$ with $\tau_m \in \mathcal{T}$ for a fundamental sequence of the stopping times belonging to this local martingale. That is, the τ_m form an increasing sequence $\tau_m \uparrow T$ a.e. such that the stopped processes $\{(\widetilde{Y} - A^{\widetilde{Y}}_{P^*})(t\wedge\tau_m), t \geq 0\}$ are uniformly integrable martingales. Adapting the partial integration argument (7) for general initial wealth w_0 and income y and taking expectations we find that

$$E^*\left[\gamma^0(T\wedge\tau_m)\Pi(T\wedge\tau_m) + \int_0^{T\wedge\tau_m}\gamma^0(s)\mathrm{d}C(s)\right] = w_0 + E^*[A^{\widetilde{Y}}_{P^*}(T\wedge\tau_m)],$$

where we used that $E^*[(\widetilde{Y} - A^{\widetilde{Y}}_{P^*})(T\wedge\tau_m)] = \widetilde{Y}(0) - A^{\widetilde{Y}}_{P^*}(0) = 0$. Since the integrand on the left-hand side of the previous display is uniformly bounded from below, Fatou's lemma yields

$$E^*\left[\gamma^0(T)\Pi(T) + \int_0^T \gamma^0(s)\mathrm{d}C(s)\right] \leq w_0 + \liminf_{m\to\infty} E^*[A^{\widetilde{Y}}_{P^*}(T\wedge\tau_m)]. \qquad (9)$$

By monotonicity and the definition of $A^{\mathcal{Y}}_{P^*}$ it follows that $A^{\widetilde{Y}}_{P^*}(t) \leq A^{\mathcal{Y}}_{P^*}(T)$ which combined with (9) yields (8). $(ii) \Rightarrow (i)$ Define the process $W = \{W(t), 0 \leq t \leq T\}$ by

$$W(t) = \operatorname*{ess\,sup}_{P^*\in\mathcal{P}}\left\{E^*\left[L(T) + \gamma^0(T)w - A^{\mathcal{Y}}_{P^*}(T)|\mathcal{F}_t\right] + A^{\mathcal{Y}}_{P^*}(t)\right\},$$

where $L = \{L(t), t \in [0,T]\}$ is the process given by $L(t) = \int_0^t \gamma^0(u)[\sum_{i=1}^m c_i(u) - y(u)]\mathrm{d}u$. From (8) and Assumption 2 combined with (2) we see that $E^*[L(T) +$

$\gamma^0(T)w - A^{\mathcal{Y}}_{P^*}(T)]$ is uniformly bounded above and below for all $P^* \in \mathcal{P}$, respectively. By Theorem 2.1.1 in [53], the process W is a supermartingale under each $P^* \in \mathcal{P}$ with a càdlàg modification. Assume W denotes this modification. Since $A^{\mathcal{Y}}_{P^*}$ is increasing, then also the process $W - A^{\mathcal{Y}}_{P^*}$ is a supermartingale under any $P^* \in \mathcal{P}$. Recall that $A^{\mathcal{Y}}_{P^*} \equiv 0$ if P^* is an equivalent local martingale measure for $\gamma^0 S$. Combining with Assumption 1, (1) and (2), we deduce that W is uniformly bounded from below. The Constrained Optional Decomposition Theorem of Föllmer and Kramkov ([59, Theorem 4.1]) then implies that there exists a process $H \in \mathcal{H}$ and an increasing process G such that $W = W(0) + \int H \mathrm{d}(\gamma^0 S) - G$. Consider now the value process $\Pi = S_0(W - L)$. Note that Π is uniformly bounded from below, has càdlàg paths and has final value $\Pi(T) = w$. Moreover, we find by partial integration that

$$\mathrm{d}\Pi = H \mathrm{d}S + \gamma^0(\Pi - H \cdot S)\mathrm{d}S_0 - S_0 \mathrm{d}G - \mathrm{d}\widetilde{L}$$

where $\widetilde{L} = \int (c(s) - y(s))\mathrm{d}s$. Thus, Π satisfies (6) for the strategy $(H_0, H) = (\gamma^0(\Pi - H \cdot S), H)$ and $D \equiv \int S_0 \mathrm{d}G$. Since $H \in \mathcal{H}$ and H_0 is adapted with càdlàg paths (and hence is progressively measurable), we conclude that (c, w) is $\mathcal{H}$-feasible. □

4 Existence

To prove existence for $(\mathcal{V})$, we can restrict ourselves, by the equivalence result from the previous section, to the study of the variational problem $(\mathcal{V}')$. In order for the functional U in (3) to be well defined and to guarantee existence for $(\mathcal{V}')$, u and v have to satisfy a certain asymptotic growth condition. A similar condition, which goes back to [14], appeared in [19] in a different context. Fix some $P^{0\star} \in \mathcal{P}^0$ (which is non-empty by Assumption 1) and denote by $\xi^{0\star}$ the Radon-Nikodym derivative of $P^{0\star}$ with respect to P on $\mathcal{F}_T$ and write $\xi^{0\star}(t) = E[\xi^{0\star}|\mathcal{F}_t]$. Let λ be the Lebesgue measure on $[0,T]$.

$$\left\{\begin{array}{c} \text{For every } \epsilon > 0 \text{ there exist } \psi^\epsilon \in \mathcal{L}^1(P^{0\star}), \phi^\epsilon \in \mathcal{L}^1(\lambda \times P^{0\star}) \\ \text{such that for } P\text{-a.e. } \omega \in \Omega \text{ and } \lambda\text{-a.e. } t \in [0,T] \\ v(w,\omega) \leq \epsilon \xi^{0\star}(T,\omega)|w| + \xi^{0\star}(T,\omega)\psi^\epsilon(\omega) \text{ for all } w \in \mathbf{R}_+ \\ u(t,c,\omega) \leq \epsilon \xi^{0\star}(t,\omega)|c| + \xi^{0\star}(t,\omega)\phi^\epsilon(t,\omega) \text{ for all } c \in \mathbf{R}^m_+ \end{array}\right\}. \qquad (\gamma_1)$$

Now we can state the main result of this paper:

Theorem 2 *Let u, v satisfy the asymptotic growth property (γ_1) and suppose $u(t,\cdot,\omega), v(\cdot,\omega)$ are concave. Then the problems $(\mathcal{V})$ and $(\mathcal{V}')$ have an optimal solution.*

Note that the problem $(\mathcal{V})$ is feasible. Indeed, first note that since $H \equiv 0 \in \mathcal{H}$ (it is allowed for the agent not to trade), by (5) the process $A^{\mathcal{Y}}_{P^*}$ is non-negative for any $P^* \in \mathcal{P}$. Assumption 1 and Theorem 1 imply then that

$(c,w)= (c_1, c_2, \ldots, w)= (y, 0, \ldots, 0)$ is $\mathcal{H}$-feasible for $(\mathcal{V})$. Moreover, if in addition u or v is strictly concave, the problem $(\mathcal{V})$ has a unique solution (where we identify two solutions that differ only on a null-set).†

As the following example shows, a considerable class of utility functions satisfy the conditions of Theorem 2.

Example 3 *Let $b \in (0,1)$, $k_1 \geq 0$ and $k_2 > 0$ and suppose $1/\xi^{0\star} \in \mathcal{L}^{1/b}(\lambda \times P^{0\star})$. Consider utility functions for intermediate consumption u which are non-decreasing and concave in c and satisfy for $\lambda \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$ the growth condition*

$$u(t,c,\omega) \leq k_1 + k_2|c|^{1-b} \quad c \in \mathbf{R}^m_+. \tag{10}$$

Note this condition also appears in Cox and Huang [43] and Cuoco [45] for one-dimensional consumption. Then $u(z)$ is $o(|z|)$ as $|z| \to \infty$; that is, the function u is asymptotically dominated by $|z|$. The functions u defined by (10) satisfy growth property (γ_1), as follows from Example 4.2 and Proposition 4.3 in [19]. As concrete examples of utility functions satisfying the bound (10) we find the HARA *utility functions for m-dimensional consumption*

$$u(t,c) = e^{-\rho t}\frac{b}{1-b}\left(\frac{\alpha \cdot c}{b} + \beta\right)^{1-b} \qquad c \in \mathbf{R}^m_+$$

with $b \in (0,\infty)\backslash\{1\}, \alpha \in \mathbf{R}^m_+, \beta, \rho \geq 0$. If $b = 1$, $u(c,t) = e^{-\rho t}\log(\alpha \cdot c + \beta)$ and if $\beta = 1, b = \infty$, $u(c,t) = \exp(-\rho t - \alpha \cdot c)$.

Similarly, ignoring the time dependence and setting $m = 1$, one constructs examples of utility functions for final wealth. ◇

Remark. In our set-up also some optimisation problems connected to hedging can be incorporated. Consider an agent who has sold a contingent claim with pay-off V, a $\mathcal{F}_T$-measurable nonnegative random variable, at time T and now seeks to hedge against this claim. The trading of the agent is subject to certain restrictions which are incorporated in the set $\mathcal{H}$. Suppose that for this agent a super-hedge, a final wealth w that dominates V, is too expensive: there is no strategy $H \in \mathcal{H}$ such that the corresponding portfolio Π is bounded below and satisfies $\Pi(T) \geq V$ almost everywhere. The agent's attitude towards the risk of a shortfall $(V - w)^+$ is represented by the function

$$v(w,\omega) = -p^{-1}((V(\omega) - w)^+)^p, \qquad p \geq 1, w \geq 0.$$

The optimal final wealth for this agent is then given by the optimal solution of the corresponding control problem $(\mathcal{V})$. Since v only takes non-positive values and v is concave, Theorem 2 yields that there exists an optimal w_* for $(\mathcal{V})$ and an $\mathcal{H}$-constrained strategy which satisfies $\Pi(T) \geq w_*$ almost everywhere. See [60] for more on minimising shortfall in hedging problems.

†Similarly, in the setting of [76], only strict concavity in $t = T$ of the utility function is needed to get uniqueness

4.1 Proof of theorem 2

Firstly, we define the measure $\hat{\lambda} = \lambda + \delta_T$ on $[0,T]$ as a unit atom δ_T in $t = T$ added to the Lebesgue measure λ on $[0,T]$. Furthermore, we construct $\tilde{u}, \tilde{c}, \tilde{y}$ from u, v, c, y by setting $\tilde{u}, \tilde{c}, \tilde{y}$ on $[0,T)$ equal to u, c, y respectively and choosing $\tilde{u}(T,c)$, $\tilde{c}(T)$ and $\tilde{y}(T)$ to take the values $v(c_1)$, $w\mathbf{e}_1$ (where $\mathbf{e}_1$ is the first unit vector in $\mathbf{R}^m$) and 0 respectively. Then the static variational problem $(\mathcal{V}')$ becomes in the new notation:

$$\sup_{\tilde{c}\in\tilde{F}} E\left[\int_0^T \tilde{u}(t,\tilde{c}(t))\hat{\lambda}(\mathrm{d}t)\right] \qquad (\mathcal{V}')$$

where the set $\tilde{F}$ is given by the set of all $\tilde{c} \in (\mathcal{L}^0_+([0,T]\times\Omega))^m$ such that

$$\sup_{P^*\in\mathcal{P}} E^*\left[\int_0^T \gamma^0(t)\left(\tilde{s}(t) - \tilde{y}(t)\right)\hat{\lambda}(\mathrm{d}t) - A^{\mathcal{Y}}_{P^*}(T)\right] \le w_0,$$

where $\tilde{s} = \sum_{i=1}^m \tilde{c}_i$. Note that for $\tilde{c} \in (\mathcal{L}^1(\hat{\lambda}\times P^{0\star}))^m$ the expectation in $(\mathcal{V}')$ is not $+\infty$, because of the asymptotic growth property (γ_1).

Before we start with the proof of the theorem, we first take a closer look at the set $\tilde{F}$:

Lemma 1 *The set $\tilde{F}$ is convex, subset of some ball in $(\mathcal{L}^1(\hat{\lambda}\times P^{0\star}))^m$ and closed with respect to convergence almost everywhere.*

Proof Let $\tilde{c} \in \tilde{F}$, then we deduce – recalling (2) and that $A^{\mathcal{Y}}_{P^{0\star}} \equiv 0$ since $P^{0\star} \in \mathcal{P}^0$–

$$E^{0\star}\left[\int_0^T \gamma^0(t)\tilde{s}(t)\hat{\lambda}(\mathrm{d}t)\right] \le w_0 + E^{0\star}\left[\int_0^T \gamma^0(t)\tilde{y}(t)\hat{\lambda}(\mathrm{d}t)\right] \le w_0 + K, \quad (11)$$

where $\tilde{s} = \sum_{i=1}^m \tilde{c}_i$. Thus, since $\int_0^T r(s)\mathrm{d}s < R$ by (1), $\tilde{F}$ is subset of some ball in $(\mathcal{L}^1(\hat{\lambda}\times P^{0\star}))^m$ The convexity of $\tilde{F}$ easily follows from the linearity of the inequalities. Finally, let $(\tilde{c}_n)_n$ be a sequence in F converging almost everywhere to, say, $\hat{c}$. By Fatou's lemma

$$\begin{aligned} E^*\left[\int_0^T \gamma^0(t)\sum_{i=1}^m(\hat{c}_n)_i(t)\hat{\lambda}(\mathrm{d}t)\right] &\le \liminf_{n\to\infty} E^*\left[\int_0^T \gamma^0(t)\sum_{i=1}^m(\tilde{c}_n)_i(t)\hat{\lambda}(\mathrm{d}t)\right] \\ &\le w_0 + E^*\left[\int_0^T \gamma^0(t)\tilde{y}(t)\hat{\lambda}(\mathrm{d}t) + A^{\mathcal{Y}}_{P^*}(T)\right], \end{aligned}$$

for all $P^* \in \mathcal{P}$. Thus $\hat{c}$ is in $\tilde{F}$. □

By the previous Lemma we see that the program $(\mathcal{V}')$ is an optimisation problem over the closed and norm-bounded subset $\tilde{F}$ of $(\mathcal{L}^1(\hat{\lambda}\times P^{0\star}))^m$. Since $\mathcal{L}^1$ is not reflexive, the set $\tilde{F}$ lacks weak compactness. To obtain existence, we will need the concept of *K-convergence* [18]. Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, where Ω is a set and $\mathcal{S}$ and μ are a σ-algebra and a measure on Ω, respectively. Suppose $(f_n)_n$ is a sequence of integrable functions on this measure space. This sequence is said to K-converge to a function f if for every subsequence $(n') \subset (n)$, there exists a μ-null set $\tilde{N}$ such that, as $N \to \infty$,

$$\frac{1}{N}\sum_{n'=1}^{N} f_{n'}(\omega) \to f(\omega) \text{ for all } \omega \in \Omega \backslash \tilde{N}. \tag{12}$$

To obtain existence we use a deep result of Komlós [80]:

Theorem 3 (Komlós) *Let* $(f_n)_n$ *be functions satisfying* $\sup_n \int_\Omega |f_n| d\mu < \infty$. *Then there exists a* μ*-integrable* f_* *and a subsequence* $(n') \subset (n)$ *such that* $f_{n'}$ K*-converges to* f_* *as* $n' \to \infty$.

Proof of theorem 2 Define $s \equiv \sup(\mathcal{V}')$. Note that $s < \infty$ because of the growth property (γ_1). If $s = -\infty$ there is nothing to prove, so without loss of generality we can suppose $s \in \mathbf{R}$. Let $(\tilde{c}_n)_n \subset \tilde{F}$ be a maximising sequence for $(\mathcal{V}')$; that is, $E[\int_0^T \tilde{u}(t, \tilde{c}_n(t))\hat{\lambda}(\mathrm{d}t)] \uparrow s$ as $n \to \infty$. Since $\tilde{F}$ is subset of some ball in $(\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$, an application of Theorem 3 shows that there exists a subsequence $(n') \subset (n)$ for which $(\tilde{c}_{n'})_{n'}$ K-converges to a function $\tilde{c}_* : \Omega \times [0, T] \to \mathbf{R}^m$. By Lemma 1, $\tilde{F}$ is closed and convex; thus $\tilde{c}_* \in \tilde{F}$. By concavity of $\tilde{u}$, we find

$$\frac{1}{N}\sum_{n'=1}^{N} E\left[\int_0^T \tilde{u}(t, \tilde{c}_{n'}(t))\hat{\lambda}(\mathrm{d}t)\right] \le E\left[\int_0^T \tilde{u}\left(t, \frac{1}{N}\sum_{n'=1}^{N}\tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right]. \tag{13}$$

As N approaches infinity, the left-hand side of (13) converges to s, where we used the fact that convergence of a sequence of real numbers implies convergence of the averages of the first l terms as l goes to infinity. To obtain convergence on the right-hand side of (13), we first decompose the expectation into two parts, using the fact that $\tilde{u} = \tilde{u}^+ - \tilde{u}^-$, where $\tilde{u}^+ = \max\{0, \tilde{u}\}$ and $\tilde{u}^- = -\max\{0, -\tilde{u}\}$. The expectation can be decomposed since $E\int \tilde{u}^+ < \infty$ by growth property (γ_1). Note that limsup $E\int(u^+ - u^-) \le$ limsup $E\int u^+ -$ liminf $E\int u^-$. For the second part, Fatou's lemma combined with the continuity of u implies

$$\liminf_{N\to\infty} E\left[\int_0^T \tilde{u}^-\left(t, \frac{1}{N}\sum_{n'=1}^{N}\tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right] \ge E\left[\int_0^T \tilde{u}^-\left(t, \tilde{c}_*(t)\right)\hat{\lambda}(\mathrm{d}t)\right].$$

To deal with the first part we first note that the growth property (γ_1) implies that $\{\tilde{u}^+(t, s_r(t,\omega), \omega), r \in \mathbf{N}\}$, where $s_r = r^{-1}\sum_{n'=1}^{r} c_{n'}$, is uniformly integrable with respect to $\hat{\lambda} \times P$. Indeed, let $\epsilon > 0$, then by the growth property

(γ_1) there exists a $\tilde{\phi}^\epsilon \in \mathcal{L}^1(\hat{\lambda} \times P^{0\star})$ such that for each $A \in \mathcal{B}([0,T]) \times \mathcal{F}$

$$\begin{aligned}\int_A \tilde{u}^+(t, \tilde{s}_r(t,\omega), \omega)\mathrm{d}(\hat{\lambda} \times P) &\le \epsilon \int_A |\tilde{s}_r(t,\omega)|\mathrm{d}(\hat{\lambda} \times P^{0\star}) \\ &\quad + \int_A \tilde{\phi}^\epsilon(t,\omega)\mathrm{d}(\hat{\lambda} \times P^{0\star}) \\ &\le \epsilon \mathrm{e}^{-R}(w_0 + K) + \int_A \tilde{\phi}^\epsilon(t,\omega)\mathrm{d}(\hat{\lambda} \times P^{0\star}).\end{aligned}$$

By first choosing ϵ small enough and then choosing A, this quantity can be made arbitrarily small, uniformly in r. Since almost everywhere-convergence together with uniform integrability implies convergence in $\mathcal{L}^1$ and $u(t,\cdot,\omega)$ is continuous, we find that

$$\limsup_{N\to\infty} E\left[\int_0^T \tilde{u}^+\left(t, \frac{1}{N}\sum_{n'=1}^N \tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right] = E\left[\int_0^T \tilde{u}^+(t, \tilde{c}_*(t))\hat{\lambda}(\mathrm{d}t)\right].$$

Thus, $s \le E[\int_0^T \tilde{u}(t, \tilde{c}_*(t,\omega), \omega)\hat{\lambda}(\mathrm{d}t)]$ and $\tilde{c}_*$ is an optimal solution for $(\mathcal{V}')$. Combining with Theorem 1, we immediately find an existence result for the original problem $(\mathcal{V})$. □

5 Characterisation of optimal policies

In the previous section, Theorem 2 gives conditions under which the existence of an optimal consumption-final wealth plan (c_*, w_*) is guaranteed, but it tells nothing about the actual form of (c_*, w_*). In this section we assume that the utility functions u, v satisfy the conditions of Theorem 2. Denote by $\mathbf{e}$ the column vector of m ones and let $(c_*(t,\omega))_{\mathrm{diag}}$ be the diagonal matrix with $c_{*i}(t,\omega)$, $i = 1, \dots, m$ on its diagonal. Furthermore, by $B_\epsilon(c_*, w_*)$ we denote the set of all $(c, w) \in \mathcal{A}^0(\mathcal{H})$ that satisfy

$$\max\{|c_1 - c_{*1}|, \dots, |c_m - c_{*m}|, |w - w_*|\} \le \epsilon \max\{|c_{*1}|, \dots, |c_{*m}|, |w_*|\}$$

for almost every (t,ω). The column vector of coordinate-wise absolute values of the gradient of u, $(|\frac{\partial u}{\partial c_1}|, \dots, |\frac{\partial u}{\partial c_m}|)^\top$, we will denote for short by $|\nabla_c|u$. Then we have the following result, which relates the marginal utility at the optimal consumption-final wealth plan (c_*, w_*) with Radon-Nikodym derivatives of $P^* \in \mathcal{P}$ with respect to P. A similar result in the context of one-dimensional intertemporal consumption and where stock prices are modelled as Itô-processes can be found in Cuoco [45].

Proposition 2 *Suppose that $\mathcal{H}$ is such that the set $\mathcal{P}$ is convex, that $(c_*, w_*) \neq 0$ and that there exists a $\epsilon_0 \in (0,1)$ such that (u, v) is differentiable for $(c, w) \in B_{\epsilon_0}(c_*, w_*)$ and*

$$E\left[\int_0^T c_*(t)^\top \left|\nabla_c\right| u(t, c(t))\mathrm{d}t + w_* \left|\frac{\partial v}{\partial w}(w)\right|\right] < \infty. \tag{14}$$

for all $(c, w) \in B_{\epsilon_0}(c_*, w_*)$. *Let* (c_*, w_*) *be an optimal solution of* $(\mathcal{V})$ *Then there exists a sequence* $(\phi_n \xi_n)_n$ *with* $\phi_n > 0$ *and* ξ_n *the Radon-Nikodym derivative of* $P_n^* \in \mathcal{P}$ *with respect to* P *such that*

$$\begin{array}{c} (c_*(t,\omega))_{\mathrm{diag}}(\nabla_c u(t, c_*(t,\omega), \omega) - \gamma^0(t,\omega)\phi_n \xi_n(t,\omega)\mathbf{e}) \to 0 \\ w_*(\omega)(\frac{\partial v}{\partial w}(w_*(\omega), \omega) - \gamma^0(T,\omega)\phi_n \xi_n(T,\omega)) \to 0 \end{array} \tag{15}$$

in $\left(L^1(\lambda \times P)\right)^m$ *and* $L^1(P)$ *respectively as* $n \to \infty$. *If in addition*

$$\inf_{P^* \in \mathcal{P}} E^* \left[\int_0^T \gamma^0(t) c_*(t)^\top \mathbf{e} \mathrm{d}t + \gamma^0(T) w_* \right] > 0, \tag{16}$$

then (15) holds with $\phi_n = \phi > 0$ *for all* n.

Recall from Example 2 that the set $\mathcal{P}$ is convex for example if $\mathcal{H}$ is a linear family or a convex cone. If the optimal consumption c_* and the final wealth w_* are uniformly bounded away from zero, one has as an immediate consequence from above proposition that the marginal utility of inter-temporal consumption and that of final wealth at the optimal consumption-final wealth plan (c_*, w_*) are, up to a constant multiplicative factor, equal to a pointwise limit of 'state price densities' $\gamma_0 \xi_n$:

Corollary 1 *Assume that (14) and (16) hold and* $\mathcal{H}$ *is such that the set* $\mathcal{P}$ *is convex. Then there exist a* $\phi > 0$ *and a sequence* $(\xi_n)_n$ *with* $\xi_n(t) = E[\frac{\mathrm{d}P_n^*}{\mathrm{d}P} | \mathcal{F}_t]$ *for* $P_n^* \in \mathcal{P}$ *such that if* $c_{*i} > 0$ *a.e.*

$$(\nabla_c u(t, c_*(t,\omega), \omega))_i = \lim_{n\to\infty} \gamma^0(t,\omega)\phi\xi_n(t,\omega) \qquad \lambda \times P\text{-a.e.} \tag{17}$$

and if $w_* > 0$ *a.e.*

$$\frac{\partial v}{\partial w}(w_*(\omega), \omega) = \lim_{n\to\infty} \gamma^0(T,\omega)\phi\xi_n(\omega) \qquad P\text{-a.e.} \tag{18}$$

Proof of Proposition 2 Recall the notations $\tilde{u}$ and $\tilde{c}$ from Section 4.1 and write $\nabla_c \tilde{u}(s)$ for $\nabla_c \tilde{u}(s,c)|_{c=\tilde{c}_*(s)}$, the gradient of $\tilde{u}(s,c)$ in $c = \tilde{c}_*(s)$. Inspired by [45] we define the sets C_1 and C_2, which are subset of $(L^1(\hat{\lambda} \times P))^m$ by (8), 2) and Assumption 2), by

$$C_1 = \{\phi\gamma_0\xi_{P^*} \times \tilde{c}_* : \phi > 0, P^* \in \mathcal{P}\}$$

where $\xi_{P^*}(t) = E[\frac{\mathrm{d}P^*}{\mathrm{d}P} | \mathcal{F}_t]$ and

$$C_2 = \{(\tilde{c}_*)_{\mathrm{diag}} \nabla_c \tilde{u} - x : x \in \mathrm{cl}(C_1)\}.$$

where $\mathrm{cl}(C_1)$ denotes the closure of C_1 in $(L^1(\hat{\lambda} \times P))^m$. Since $\mathcal{P}$ is assumed to be convex, C_2 is convex as well. We argue by contradiction and suppose that there is no sequence $(\phi_n \xi_n)_n$ such that $\phi_n \gamma^0 \xi_n \tilde{c}_* \to \tilde{u}(\tilde{c}_*)_{\mathrm{diag}} \nabla_c$. Then

$C_2 \cap \{0\} = \emptyset$ and it follows therefore from the separating hyperplane theorem (e.g. [51, Thm. V.2.10]) that there exists an $f \in (L^\infty(\hat{\lambda} \times P))^m$ such that

$$E\left[\nabla_c \tilde{u}(s)^\top (\tilde{c}_*(s))_{\text{diag}} f(s) \hat{\lambda}(\mathrm{d}s)\right] - \phi E^*\left[\int_0^T \gamma^0(s) \mathbf{e}^\top (\tilde{c}_*(s))_{\text{diag}} f(s) \hat{\lambda}(\mathrm{d}s)\right] > 0$$

for all $P^* \in \mathcal{P}$ and $\phi > 0$. Writing $\hat{f} = (\tilde{c}_*)_{\text{diag}} f / \|f\|_{L^\infty}$, the above implies

$$E\left[\int_0^T \nabla_c \tilde{u}(s)^\top \hat{f}(s) \hat{\lambda}(\mathrm{d}s)\right] > 0 \geq E^*\left[\int_0^T \gamma^0(s) \mathbf{e}^\top \hat{f}(s) \hat{\lambda}(\mathrm{d}s)\right], \tag{19}$$

for all $P^* \in \mathcal{P}$. Note that for $\epsilon \in (0,1)$ the consumption plan $\tilde{c}_\epsilon = \tilde{c}_* + \epsilon \hat{f}$ satisfies $\tilde{c}_\epsilon \geq (1-\epsilon)\tilde{c}_* \geq 0$. But then by Theorem 1 the consumption-final wealth plan $(\tilde{c}_\epsilon, \tilde{c}_\epsilon(T))$ is $\mathcal{H}$-feasible for each $\epsilon \in (0,1)$. It follows from the optimality of $\tilde{c}_*$ that

$$0 \geq \lim_{\epsilon \downarrow 0} \frac{U(c_\epsilon, w_\epsilon) - U(c_*, w_*)}{\epsilon} = E\left[\int_0^T \nabla_c \tilde{u}(s, \tilde{c}_*(s)) \hat{f}(s) \hat{\lambda}(\mathrm{d}s)\right],$$

where the last equality follows from the dominated convergence theorem using the fact that by concavity of u for $\epsilon > 0$ small enough

$$\begin{aligned} \frac{\tilde{u}(t, \tilde{c}_\epsilon(t)) - \tilde{u}(t, \tilde{c}_*(t))}{\epsilon} &\leq \max\{|\nabla_c \tilde{u}(t, \tilde{c}_{\epsilon_0}(t))^\top \hat{f}(t)|, |\nabla_c \tilde{u}(t, \tilde{c}_*(t))^\top \hat{f}(t)|\} \\ &\leq (|\nabla_c| \tilde{u}(t, \tilde{c}_{\epsilon_0}(t)) + |\nabla_c| \tilde{u}(t, \tilde{c}_*(t)))^\top \tilde{c}_*(t) \end{aligned}$$

and that the last expression is integrable by (14). This contradicts (19) and thus proves (15).

Now suppose that in addition condition (16) holds and let $(\varphi_n \xi_n)_n$ be a sequence as in (15). Since $\varphi_n \|\gamma^0 \xi_n \mathbf{e}^\top \tilde{c}_*\|_{L^1}$ converges to $\|(\tilde{c}_*)^\top \nabla_c \tilde{u}\|_{L^1}$ (where $\|\cdot\|_{L^1}$ denotes the $L^1(\hat{\lambda} \times P)^m$-norm) and $\|\gamma^0 \xi_n \mathbf{e}^\top (\tilde{c}_*)\|_{L^1}$ is bounded away from zero, $(\varphi_n)_n$ is bounded. Hence we can find a subsequence, again denoted by $(\varphi_n)_n$, such that $\varphi_n \to \varphi > 0$. Then by the triangle inequality and (11), $\|\varphi \gamma_0 \xi_n \mathbf{e}^\top \tilde{c}_* - (\tilde{c}_*)^\top \nabla_c \tilde{u}\|_{L^1}$ is bounded by

$$\begin{aligned} &\|\gamma_0 \xi_n \mathbf{e}^\top \tilde{c}_*\|_{L^1} |\varphi - \varphi_n| + \|\varphi_n \gamma_0 \xi_n \mathbf{e}^\top \tilde{c}_* - (\tilde{c}_*)^\top \nabla_c \tilde{u}\|_{L^1} \\ &\qquad \leq (w_0 + K)|\varphi - \varphi_n| - \|\varphi_n \gamma_0 \xi_n \mathbf{e}^\top \tilde{c}_* - (\tilde{c}_*)^\top \nabla_c \tilde{u} \cdot \|_{L^1}, \end{aligned}$$

which tends to zero as $n \to \infty$. The proof is done. □

Remark. Suppose we are in the setting of Corollary 1 and suppose that $v(w, \omega)$ is differentiable and strictly concave in w with $\partial_w v(w, \omega)$ tending to zero and ∞ if $w \to \infty$ and $w \downarrow 0$ respectively. Then $\frac{\partial v}{\partial w}(\cdot, \omega) : [0, \infty] \to [0, \infty]$ is strictly decreasing and we denote its inverse function by $I(\cdot, \omega)$. Writing then ξ_* for the pointwise limit of the ξ_n from (18) we find for the optimal final wealth w_*

$$w_*(\omega) = I(\phi \gamma_0(T, \omega) \xi_*(\omega), \omega). \tag{20}$$

Note that under appropriate conditions on u, a similar result holds for c_{*i}. If ξ_* is a Radon-Nikodym derivative of some probability measure with respect to P, the formula (20) is reminiscent of the results found by Kramkov and Schachermayer [75]. The exact relation between the direct approach and the dual one as developed by [75, 76] is subject of ongoing research.

6 Examples

6.1 A Jump-diffusion model

In this section we consider as a specific model for the price of the risky assets a jump-diffusion driven by a Wiener process and an independent Poisson process.

Assume that on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ there exists a d-dimensional Wiener process $W = (W^{(1)}, \dots, W^{(d)})$ with independent components $(d \geq n)$ and an s-dimensional Poisson processes $N = (N^{(1)}, \dots, N^{(s)})$ with $\mathbf{F}$-adapted intensity ν and independent components. The price process of the ith risky assets $S_i = \{S_i(t), 0 \leq t \leq T\}$ is modelled by the stochastic differential equation

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i(t)\mathrm{d}t + \sigma_i(t)^\top \mathrm{d}W(t) + \rho_i(t)^\top \mathrm{d}N(t)\right].$$

Here μ and σ are $\mathbf{F}$-adapted and ρ is locally bounded from below and $\mathbf{F}$-predictable and they satisfy

$$\int_0^T |\mu(s)|\mathrm{d}s + \int_0^T |\sigma(s)|^2 \mathrm{d}s + \int_0^T |\rho(s)\cdot\nu(s)|\mathrm{d}s + \int_0^T |\nu(s)|\mathrm{d}s < \infty.$$

Moreover, we assume that the $n \times d$ matrix σ with rows σ_i has full rank n. Note that this market is incomplete if $d > n$ or the intensity ν is nonzero. The market is free of arbitrage if there exists a process $\kappa = (\kappa_i)_{i=1}^s$ which solves

$$\sigma(t)\kappa(t) = \mu(t) + \rho(t)\nu(t) - r(t)\mathbf{e}, \qquad t \in [0, T],$$

where $\mathbf{e} \in \mathbf{R}^n$ is the column-vector of ones and satisfies the so called *Novikov condition* condition

$$E\left[\exp\left\{2^{-1}\int_0^T |\kappa(s)|^2 \mathrm{d}s\right\}\right] < \infty. \tag{21}$$

Indeed, applying Itô's lemma to the process $Z = \{Z_t : t \in [0, T]\}$ with

$$Z(t) = \exp\left\{\int_0^t \kappa(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t |\kappa(s)|^2 \mathrm{d}s\right\}$$

shows that Z satisfies the stochastic differential equation $\mathrm{d}Z = Z\kappa\mathrm{d}W$ and is thus a local martingale. It is well known that under (21) (e.g. [71]) Z becomes a martingale. Moreover, by Girsanov's theorem the discounted price process $\gamma^0 S$ is a local martingale under the measure P^0 given by $\mathrm{d}P^0 = Z\mathrm{d}P$.

6.2 Complete markets

In this section we consider the case that the set of equivalent local martingale measures is a singleton, $\mathcal{P}^0 = \{P^0\}$ and no constraints are put on the trade, that is, the portfolios may take values in $\mathcal{H} = \mathbf{L}^a_{loc}(X)$. In this setting the market is arbitrage free (cf. Proposition 1) and complete.

Now we are also interested in the integrability of the consumption final-wealth plans (c, w). For $p \geq 1$ we define $\mathcal{A}^p$ as the subset of $\mathcal{A}^0 = \mathcal{A}^0(\mathbf{L}^a_{loc}(X))$ consisting of the p-integrable consumption-final wealth plans

$$\mathcal{A}^p \equiv \{(c,w) \in \mathcal{A}^0 : (c,w) \in (\mathcal{L}^p_+(\lambda \times P))^m \times \mathcal{L}^p_+(P)\}. \tag{22}$$

where $\mathcal{L}^p_+(\cdot)$ denotes the set of non-negative p-integrable functions with respect to the measure $\cdot$ between the brackets. In this setting, the agent faces the following problem for $p = 0$ or $p \geq 1$

$$\sup_{(c,w)\in\mathcal{A}^p} E\left[\int_0^T u(t, c(t,\omega), \omega)\mathrm{d}t + v(w(\omega), \omega)\right] \tag{$\mathcal{V}_p$}$$

By Theorem 1 we now can reformulate the dynamic control problem $(\mathcal{V}_p)$ as the following static variational problem. As before, we denote by ξ the Radon-Nikodym derivative of P^0 with respect to P, $\xi(t) = \xi^0(t) = E\left[\mathrm{d}P^0/\mathrm{d}P|\mathcal{F}_t\right]$.

$$\sup_{(c,w)} E\left[\int_0^T u\left(t, c(t)\right)\mathrm{d}t + v(w)\right] \text{ s.t. } E\left[\int_0^T \pi(t)z(t)\mathrm{d}t + \pi(T)w\right] \leq w_0. \tag{$\mathcal{V}'_p$}$$

where $\pi(t) = \gamma^0(t)\xi(t)$ and $z = \sum_{i=1}^m c_i - y$ and the supremum is taken over the set of (c, w) in $(\mathcal{L}^p_+(\lambda \times P))^m \times \mathcal{L}^p_+(P)$.

Existence

Note $(\mathcal{V}'_p)$ is an optimisation problem of the form studied by [19, 44]. As the following examples show, $(\mathcal{V}'_p)$ may have no solution.

Example 4 *Let the consumption plans be one-dimensional ($m = 1$) and assume that $\xi^{-1} \in \mathcal{L}^p_+(\lambda \times P)$. Set $u(t,c)$ equal to $c \times t$ whereas $v \equiv 0$. Consider then the sequence $(c_n, w_n)_n$ of consumption-final wealth plans given by $w_n = 0$ and*

$$c_n(t) = nk\mathbf{1}_{[T-\frac{1}{n},T]}(t) \cdot S^0(t)\xi^{-1}(t) \qquad n = 1, 2, \ldots$$

where $k = w_0 + E[\int_0^T \pi(t)y(t)dt]$. Since $\xi^{-p} \in \mathcal{L}^1_+(\lambda \times P)$, we see that $c_n \in \mathcal{L}^p_+(\lambda \times P)$ and that (c_n, w_n) satisfies the constraint in $(\mathcal{V}'_p)$ as equality. Hence the (c_n, w_n) are feasible for $(\mathcal{V}'_p)$. However, $E[\int_0^T u(t, c_n(t))dt] \to \infty$ as $n \to \infty$. In this case $(\mathcal{V}'_p)$ has no solution, since the agent would like to concentrate his/her consumption closer and closer to time $t = T$. ◇

Example 5 *We show that the supremum in* $(\mathcal{V}'_p)$ *may not be attained, although it is finite. We assume that* $(\lambda \times P)(\pi^0_t \in (1, 1+\epsilon)) > 0$ *for some* $\epsilon > 0$. *Consider* $(\mathcal{V}'_p)$ *in the same setting as in the previous example, but now with* $u(c,t) = c \cdot \mathbf{1}_A(t,\omega)$, *where* $A \equiv \{(t,\omega) : \pi^0(t,\omega) > 1\}$. *Note*

$$E\left[\int_0^T \mathbf{1}_A(t)c(t)\,dt\right] < E\left[\int_0^T \xi(t)\gamma^0(t)c(t)\,dt\right] \leq k \tag{23}$$

where k *is as in the previous example. From (23) we see that the supremum in* $(\mathcal{V}'_p)$ *is at most* k. *The sequence given by*

$$c_n(t,\omega) = \frac{k}{T}\frac{\mathbf{1}_{G_n}(t,\omega)}{E\left[\int_0^T \gamma^0(t)\xi(t)\mathbf{1}_{G_n}(t)\,dt\right]},$$

where $G_n = \{(t,\omega) : 1 < \gamma^0(t,\omega)\xi(t,\omega) < 1 + \frac{1}{n}\}$ *shows that in fact the supremum is* k. *However, since the first inequality in (23) is a strict one, a feasible consumption* c *plan with expected utility* $E\left[\int_0^T u(t,c(t))\,dt\right]$ *equal to* k *does not exist.* ◇

From the above examples, it appears, certain conditions on the asymptotic growth rate of u and v are needed to ensure existence in $(\mathcal{V}'_p)$. Therefore, following [19], we introduce the following asymptotic growth condition $(\widetilde{\gamma}_p)$ on u and v (closely connected to the condition (γ_1)):

$$\left\{\begin{array}{c} \text{For every } \epsilon > 0 \text{ there exist a } \psi^\epsilon \in \mathcal{L}^p(P) \text{ and a } \phi^\epsilon \in \mathcal{L}^p(\lambda \times P), \\ \text{such that for all } c \in \mathbf{R}^m_+ \text{ and } w \in \mathbf{R}_+ \\ v(w,\omega) \leq \epsilon\xi(T,\omega)w + \xi(T,\omega)\psi^\epsilon(\omega) \text{ for } P\text{-a.s. } \omega \in \Omega \\ u(t,c,\omega) \leq \epsilon\xi(t,\omega)|c| + \xi(t,\omega)\phi^\epsilon(t,\omega) \text{ for } \lambda \times P\text{-a.s. } (t,\omega) \in [0,T] \times \Omega \end{array}\right\} \tag{$\widetilde{\gamma}_p$}$$

In addition, it appears, that, in order to guarantee existence in $(\mathcal{V}'_p)$, u and v have to satisfy the condition of *essential non-satiation* (ς); i.e. there exist sets I, J with $P(I) > 0$ and $(\lambda \times P)(J) > 0$ such that

$$\left\{\begin{array}{ll} \arg\max_{w \in \mathbf{R}_+} v(w,\omega) = \emptyset & \text{for all } \omega \in I \\ \arg\max_{c \in \mathbf{R}^m_+} u(t,c,\omega) = \emptyset & \text{for all } (t,\omega) \in J \end{array}\right\} \tag{ς}$$

Now we can state the existence result for $(\mathcal{V}_p)$, $p \geq 1$, which follows immediately from combining [19, Theorem 2.8] and [19, §5.2] with Theorem 1:

Theorem 4 *Suppose that* $u(t,z,\omega)$ *and* $v(z,\omega)$ *are upper semi-continuous in* z *for a.e.* (t,ω) *in* $[0,T] \times \Omega$ *and a.e.* ω *in* Ω *respectively, that* $v(z,\omega)$ *is concave in* z *for a.e.* ω *in the purely atomic part* Ω^{pa} *of* $(\Omega, \mathbf{F}, P)$ *and that* u *and* v *are essentially nonsatiated* (ς). *Suppose also that* u, v *satisfy growth condition* $(\widetilde{\gamma}_p)$ *and that there exists some* $(\tilde{c}, \tilde{w}) \in \mathcal{L}^p_+(\lambda^m \times P) \times \mathcal{L}^p(P)$ *for which* $(t,\omega) \mapsto u(t,\tilde{c}(t,\omega),\omega)/\xi(t,\omega)$ *belongs to* $\mathcal{L}^p_+(\lambda \times P)$ *and* $\omega \mapsto v(\tilde{w}(\omega),\omega)/\xi(T,\omega)$ *belongs to* $\mathcal{L}^p(P)$. *Then problem* $(\mathcal{V}_p)$ *has an optimal solution.*

For concrete examples of utility functions satisfying the requirements of Theorem 4 we refer to Example 3, where for utility functions of intermediate consumption one replaces the requirement of concavity by upper semicontinuity. For example the utility function

$$u(t,c,\omega) = (1 - \exp(-|c|^2/2\sigma^2))\mathbf{1}_{F_t}(\omega) \qquad c \in \mathbf{R}^m_+, \sigma \neq 0,$$

where $F_t \in \mathcal{F}_t$ with $P(F_t) > 0$, satisfies the requirements of Theorem 4 but fails to satisfy those of Theorem 2, since it is not concave.

Characterisation of optimal consumption-final wealth plan

In this subsection, we look at characterisation of the optimal solutions of $(\mathcal{V}'_p)$ for $p = 0$ or $p \geq 1$. If $w_0 + E[\int_0^T \pi(t)y(t)\mathrm{d}t] = 0$, by Proposition 1, we only have $(c, w) = 0$ a.e. as admissible consumption-final wealth plan for $(\mathcal{V}'_p)$. So, let us assume $w_0 + E[\int_0^T \pi(t)y(t)\mathrm{d}t] > 0$. Moreover, we suppose $v(\cdot,\omega)$ is concave for P-a.e. ω in the purely atomic part Ω^{pa} of Ω. Then, from e.g. [1], we find that, (c_*, w_*) is optimal for $(\mathcal{V}'_p)$ if and only if (c_*, w_*) is a feasible consumption-final wealth plan for $(\mathcal{V}'_p)$ and there exists a $\zeta \geq 0$ such that the following two conditions hold:

$$\zeta\left(E\left[\int_0^T \pi(t)z(t)\mathrm{d}t + \pi(T)w\right] - w_0\right) = 0 \tag{CS}$$

$$\begin{cases} c_*(t,\omega) \in \mathrm{argmax}_{x\in\mathbf{R}^m_+} u(t,x,\omega) - \zeta x^\top \mathbf{e}\pi(t,\omega) & \lambda \times P\text{-a.e.} \\ w_*(\omega) \in \mathrm{argmax}_{x\in\mathbf{R}_+} v(x,\omega) - \zeta x\pi(T,\omega) & P\text{-a.e.,} \end{cases} \tag{PMP}$$

where, as before, we wrote $z = \sum_i c_i - y$ and $\pi = \gamma^0\xi$. The foregoing two equations are also known as complementary slackness (CS) and the pointwise maximum principle (PMP) respectively. Note no concavity of u and v is demanded, except for v on Ω^{pa}. If, in addition, u, v satisfy the condition of essential non-satiation (ς), $\zeta > 0$ in the above characterisation. Then the condition of complementary slackness is equivalent to

$$E\left[\int_0^T \left(\pi(t)\sum_{i=1}^m c_{*i}(t)\right)\mathrm{d}t + \pi(T)w_*\right] = w_0 + E\left[\int_0^T \pi(t)y(t)\mathrm{d}t\right] \tag{24}$$

Example 6 *Consider the problem $(\mathcal{V}'_p)$ for $p = 0$ or $p \geq 1$, where the agent faces a consumption-investment problem, where just one commodity is available and the utility of final wealth equal to zero. Suppose that for almost every $(\omega, t) \in \Omega \times [0,T]$ the utility function $u(t,\cdot,\omega)$ is differentiable, increasing and strictly concave on $\mathbf{R}_+$, with $u'(t,0,\omega) \equiv \lim_{z\downarrow 0}(\partial_z u)(t,z,\omega) = +\infty$ and $u'(t,\infty,\omega) \equiv \lim_{z\uparrow\infty}(\partial_z u)(t,z,\omega) = 0$. Note that in (24) the optimal w_* is zero a.e., by monotonicity of $u(t,\cdot,\omega)$. The above then implies that c_* is optimal in $(\mathcal{V}'_p)$ if and only if there exists a $\zeta > 0$ such that (24) and the pointwise maximum* *principle (PMP) are satisfied for a.e.* (t,ω) *in* $[0,T]\times\Omega$. *By differentiability and concavity of* $u(t,\cdot,\omega)$ *and since* $u'(t,0,\omega)=+\infty$, *(PMP) is equivalent to*

$$\partial_z u(t,z,\omega)\big|_{z=c_*(t,\omega)} = \zeta\pi(t,\omega) \text{ for } P\times\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T]. \tag{25}$$

For all (t,ω) *for which* $u(t,\cdot,\omega)$ *is strictly concave, it has a (strictly) decreasing derivative and* $\partial_z u(t,\cdot,\omega)$ *has an inverse* $\mathcal{I}(t,\cdot,\omega)\equiv(\partial_z u)^{-1}(t,\cdot,\omega):[0,\infty]\to[0,\infty]$. *Thus we can equivalently rewrite (25) as*

$$c_*(t,\omega)=\mathcal{I}(t,\pi(t,\omega),\omega) \text{ for } P\times\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T].$$

Now we introduce the function $\mathcal{J}$ *on* $\mathbf{R}_+$ *by*

$$\mathcal{J}(y)=E\left[\int_0^T \pi(t,\omega)\mathcal{I}(y\pi(t,\omega),t,\omega)\,dt\right]$$

and assume that $\mathcal{J}(y)<\infty$ *for all* $y\in(0,\infty)$. *By the monotone convergence theorem and the dominated convergence theorem* $\mathcal{J}$ *is (strictly) decreasing and continuous. Moreover, we find* $\lim_{y\downarrow 0}\mathcal{J}(y)=+\infty$ *and* $\mathcal{J}(\infty)=0$. *Thus* $\mathcal{H}$ *has an inverse* $\mathcal{K}\equiv\mathcal{J}^{-1}$ *and there is an unique* $\zeta=\mathcal{K}(w_0+E[\int_0^T\pi(t)y(t)\,dt])$ *satisfying (24). Hence, under the assumption that* $\mathcal{J}<\infty$ *on* $\mathbf{R}_+$, *the optimal consumption plan in* $(\mathcal{V}_0')$ *is given by*

$$c_*(t,\omega)=\mathcal{I}\left(t,\mathcal{K}\left(w_0+E\left[\int_0^T\pi(t)y(t)\,dt\right],\omega\right)\pi(t)\right).$$

If, for $p\geq 1$, c_* *is in addition* p*-integrable,* c_* *is the optimal consumption in* $(\mathcal{V}_p')$ *as well.* ◇

Acknowledgement. The author would like to thank Erik Balder for drawing the paper of Cuoco [45] to his attention and suggesting to generalise it.

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Samenvatting

Het leven van alledag is niet zonder risico. Sommige mensen zijn bereid meer risico te nemen om meer te kunnen bereiken, anderen spelen liever 'op safe' en proberen zo min mogelijk risico te lopen. In financiële markten zien we vergelijkbare patronen bij agenten die een hoog respectievelijk laag risico-profiel hebben. Op deze markt kan het risico verhandeld worden: een agent die vindt dat hij/zij te veel risico loopt kan zich indekken door een passende *optie* te kopen. Een optie is een contract tussen koper en verkoper waarin de laatste belooft de eerste een bepaalde betaling te doen in de toekomst. Op het moment van afsluiten is de hoogte van de betaling vaak onzeker en kan afhangen van toekomstige ontwikkelingen (zoals de koers van een bepaald aandeel volgend jaar). In grote lijnen kunnen opties ingedeeld worden in *Europese* en *Amerikaanse*. We noemen een optie Europees wanneer deze een bepaalde betaling uitkeert op een *vast* tijdstip in de toekomst. Bij een Amerikaanse optie kan de koper op *ieder moment* tot een vast eind-tijdstip het contract uitoefenen.

De waardering van opties is een van de belangrijkste vragen die bestudeerd wordt in de financiering. Wat is een eerlijke prijs voor een optie? Oftewel, hoe veel moet een koper van de optie aan de verkoper betalen zodat beiden tevreden zijn (en bijvoorbeeld geen van beiden winst kan behalen zonder risico te lopen)? In het geval van Amerikaanse opties komt nog een andere natuurlijke vraag op, namelijk wat is voor een koper het optimale moment om zijn Amerikaanse optie uit te oefenen?

In dit proefschrift bestuderen we deze vragen voor een aantal opties van Amerikaans type onder verschillende modellen voor de prijs van een aandeel. In het eerste hoofdstuk beschouwen we het klassieke model voor de prijs van een aandeel

$$S_t = S_0 \exp(X_t), \qquad S_0 = \exp(x), \qquad t \geq 0, \tag{1}$$

waar $X_t = \sigma W_t + (\mu - \frac{\sigma^2}{2})t$ een Brownse beweging W met drift $\mu - \frac{\sigma^2}{2}$ is. Onder dit model met de rente constant genomen, bekijken we een viertal opties van Amerikaans type, de call, put, de Russische optie en de integraal optie. Voor dit viertal leiden we de waarde af en bepalen het optimale moment om ze uit te oefenen. De opties waarvan we de oplossing hebben gegeven, zijn alle van perpetuele aard, dat wil zeggen ze verlopen nooit. Vanuit praktisch oogpunt lijkt dat niet erg bruikbaar, omdat opties in praktijk altijd eindige looptijd hebben. Echter, dankzij een schitterend idee van Peter Carr, is het mogelijk opties met eindige looptijd te benaderen met een bepaalde rij opties van perpetueel type, een procedure die ook wel *Canadizering* wordt genoemd. We geven een wiskundig bewijs voor deze convergentie en berekenen voor de vier eerder genoemde opties de eerste benadering.

Uitgebreid empirisch onderzoek heeft naar voren gebracht dat het klassieke geometrische Brownse beweging model niet ideaal is voor de modellering van de aandelen prijs. Het model is niet in staat om bepaalde eigenschappen die typerend zijn voor financiële data, zoals zware staarten en asymmetrie van de verdeling, te reproduceren. Vandaar de zoektocht naar een model dat empirisch gezien beter presteert. Het idee is om de prijs van het aandeel nu te modelleren door middel van (1) waar X nu een Lévy proces is. Een Lévy proces is een stochastisch proces dat onafhankelijke, gelijk verdeelde incrementen heeft en waarvan de paden rechts-continu zijn en linker limieten hebben. De klasse van Lévy processen heeft een zeer rijke structuur, wat onder andere blijkt uit het feit dat de klasse in één–op–één verhouding staat met de klasse van oneindig deelbare verdelingen. In Hoofdstuk 2 introduceren we een nieuw model voor de prijs van het aandeel, het *phase-type Lévy proces*. Dit model is een sprong-diffusie waar de sprongen een samengesteld Poisson proces vormen en een verdeling hebben die van *phase-type* is. Dit model is enerzijds rijk genoeg omdat het ieder Lévy proces willekeurig dicht kan benaderen en anderzijds kunnen onder dit model vele opties analytisch geprijsd worden. We illustreren dit door het bepalen van de van de prijs van de perpetuele Amerikaanse put en de Russische optie.

In het derde en vierde hoofstuk bestuderen we Lévy processen met sprongen in één richting. Voor deze klasse van Lévy processen bestuderen we *eerste passage* problemen: Wat is de kansverdeling van het eerste tijdstip dat het Lévy proces een bepaald niveau overschrijdt of een interval verlaat en zijn positie op dat moment? Dezelfde vraag beantwoorden we voor bepaalde *gespiegelde* Lévy processen. Ook geven we een vrij complete beschrijving van de *ergodische* eigenschappen van de overgangskansen van zulke gespiegelde Lévy processen. Daarbij bepalen we bijvoorbeeld hoe groot bij benadering de kans is dat zo'n gespiegeld Lévy proces pas na zeer lange tijd een eindig interval $[0, a)$ verlaat. De gevonden resultaten gecombineerd met martingaal technieken stellen ons in staat het optimale stop probleem op te lossen dat verbonden is met de waardering van de perpetuele Amerikaanse put en Russische optie. De gevonden resultaten in het tweede, derde en vierde hoofdstuk vinden daarnaast ook toepassing in de context van modellen voor verzekeringsrisico, wachtrijen en modellen voor het waterniveau in een dam.

Het laatste hoofdstuk, tenslotte, handelt over nutsmaximalisatie van een agent die actief is op een financiële markt. We beschouwen een algemeen model waarin prijzen van financiële producten worden gemodelleerd door semimartingalen en de agent voldoening verkrijgt uit zowel tussentijdse consumptie als zijn/haar vermogen op het eindtijdstip. In dit kader bewijzen we dat er voor de agent een optimale, nutsmaximaliserende manier bestaat om te consumeren, handelen en sparen en geven we een karakterisatie van zo'n optimaal plan.

Dankwoord

Hier wil ik een aantal mensen bedanken die bijgedragen hebben aan het totstandkomen van dit proefschrift.

- ⋄ Mijn co-promotor, *Andreas Kyprianou*, voor de dagelijkse begeleiding en het uitdagende gezamenlijke onderzoek dat geleid heeft tot Hoofdstuk 1 en 3 in dit proefschrift.
- ⋄ Mijn promotor, *Erik Balder*, voor zijn ideeën, adviezen, overzicht en geduld, en het geven van vrijheid om mijn eigen pad te volgen.
- ⋄ *Florin Avram* for inspiration and ideas, which were at the root of several chapters of this thesis, and for the wonderful hiking-mathematics sessions in France.
- ⋄ Mijn mede-promovendi en andere collegae op het Mathematisch Instituut, in het bijzonder *Igor, Manuel, Pieter, Alexa*, voor de goede sfeer en gezelligheid!
- ⋄ Mijn huisgenoot *Ian* voor het schitterende ontwerp van de cover!
- ⋄ Mijn ouders, *Jannie en Martijn*, en broer en zus, *Wouter en Eefke*, voor hun steun en belangstelling.

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Martijn Pistorius was born on 6th October 1976 in Dongen, Netherlands, where he also grew up. From 1989 till 1995 he was a pupil at the Sint Oelbert gymnasium in Oosterhout. In 1995 he went on to study mathematics at Utrecht University and passed his propaedeuse in 1996 (cum laude). In the years to come he got more and more fascinated by probability theory and its interplay with applications. He followed a student seminar and a condensed course at Eurandom on financial mathmematics. In august 1999 he finished his studies (cum laude) under supervision of Prof. dr. ir. E.J. Balder with a thesis 'Optimal consumption in semi-martingale markets'. During his study he was student member of the Educational Committee and gave tutorials as student-assistant. Shortly after he started working as a PhD-student supervised by dr. A.E. Kyprianou and Prof. dr. ir. E.J. Balder on topics in financial mathematics and probability theory. During his PhD-studies he attended seminars, courses, conferences, winter- and summer-schools in Aarhus, Besançon, Eindhoven (Eurandom) and Oegstgeest. Also he taught a variety of undergraduate mathematics tutorials. The research resulted in this thesis.