Sequents and Link Graphs

Contraction criteria for refinements of multiplicative linear logic

Sequenten en Linkgrafen
Contractiecriteria voor verfijningen van multiplicatieve lineaire logica
(met een samenvatting in het Nederlands)

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Preface

The results of my first preprint [**Puite 98**] are included in this thesis in Chapter 2 and Section 3.2. Lemma 3.2.4 explains the connection between the two-sided proof structures as defined in the preprint on the one hand, and the new definition as a special kind of link graphs on the other hand. Although the preprint concentrates on **MLL** and the switching criterion. In Chapter 4, however, we will extensively study non-commutative **MLL** (i.e. **NCLL**) and the *contraction* criterion, which we think aligns better with the next chapters. Chapter 4 is central in the sense that most proof theoretical notions and facts are defined in this chapter only; in this sense, **NCLL** will serve as the starting-point for most of the theory. Chapter 5 will be completely in the same line as Chapter 4. Hence, if possible, we will leave out the details of proofs in this chapter.

Chapter 6 is the result of my joint preprint with Richard Moot. This independent chapter will also appear as [**MP 00**]. The reader should observe that some notions in this final chapter are either new or differ from the corresponding ones in previous chapters; see Section 5.8 for the connections between the distinct descriptions. (In particular, there is a discrepancy between the numbering of the premisses/conclusions of a link: in the first chapters this numbering is anti-clockwise, starting with 0 at the main formula (page 49); in the last chapter (page 193) both the premisses and the conclusions are numbered from left to right. In many cases, however, these indices will be left out anyway, in which case the given geometrical order is meant. Also the notion of proof net alters; in the first chapters a proof net is a sequentializable proof structure; in the last chapter it is a proof structure which is correct in the sense of the correctness criterion. Anyway, as the main theorem of each chapter states that the sequentializable proof structures are precisely the correct ones, the original difference in definition does not matter.)

Chronologically, Chapter 6 should precede Chapter 5: the theory of $\mathbf{NL}\Diamond_{\mathcal{R}}$ led us to attempts to define a classical extension, and raised the demand for the corresponding contraction criterion. The new system **CNL** of [**dGL 00**] turned out to be exactly the calculus we were looking for, and our contraction criterion turned out to generalize to this classical conservative extension of **NL**. This generalization is naturally described within the framework of link graphs (Chapter 3), which are introduced precisely to this purpose.

We have listed the calculi that are central to this thesis in Appendix A.

For the notation we refer to page 245.

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CHAPTER 1

Introduction

1.1. Proof theory

Proof theory originated in Hilbert's programme for the foundations of mathematics. This programme was a reaction to the debate usually referred to as 'Grundlagenstreit', in which Brouwer openly doubted certain principles of reasoning. Hilbert on the contrary, tried to show the correctness of parts of mathematics, by means of broadly accepted finitary formalistic concepts. His ambitious programme was the beginning of an attempt to reduce mathematics to the art of mechanical manipulation of symbols according to certain rules; to completely formalize mathematics, including the logical steps in mathematical arguments. As an example, instead of considering the *construction* of the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, he based himself on their *axiomatization*: the axioms of Peano arithmetic (PA), consisting in the defining properties of addition and multiplication, and the principle of mathematical induction. However, Gödel's first incompleteness theorem for PA showed that there is a sentence A (in the appropriate formal first-order language) such that it has no proof (it cannot be deduced from the axioms of PA), but also its negation $\neg A$ has no proof. Yet one of the two holds in the standard model $(\mathbb{N}; +, \cdot, S, 0)$, so there is a sentence, holding for $\mathbb{N}$, which is nevertheless unprovable. (Below we will see that this sentence must hence be false in another model of PA.) All this said, Hilbert's programme failed at least to the extent that it had turned out impossible to fully capture the properties of the natural numbers by an axiomatic approach.

Gödel's *completeness* theorem clarifies the close relation between *truth* and *derivability*. Let us first illustrate the dichotomy in logic between semantics on the one hand and syntax on the other hand, by the example of group theory (but we can equally well consider PA, or any other theory). A group can be described as a structure $(G; \cdot, e, {}^{-1})$ consisting of an underlying set G, a binary multiplication $- \cdot -$ on G, a constant e in G and a unary map $(-)^{-1}$ on G, by definition satisfying the first-order sentences of associativity, the unitary law and the inverse law:

$$\begin{aligned}\Gamma_{\mathrm{gt}} := \{\ & \forall_{x,y,z}[x \cdot (y \cdot z) = (x \cdot y) \cdot z]\ , \\ & \forall_x[[x \cdot e = x] \wedge [e \cdot x = x]]\ , \\ & \forall_x[[x \cdot x^{-1} = e] \wedge [x^{-1} \cdot x = e]]\ \}\end{aligned}$$

Otherwise said: among all structures $\mathcal{G} = (G; \cdot, e, {}^{-1})$ groups are exactly those for which the axioms Γ_{gt} hold: $\mathcal{G} \models \Gamma_{\mathrm{gt}}$. We say a theorem A (in the first-order language of group theory) is *true* (or *holds*) (w.r.t. group theory) whenever it holds *for all* groups, in which case we say A is a *semantical* consequence of Γ_{gt}; on the other hand A is *derivable* (w.r.t. group theory) if there *exists* a formal proof from the axioms Γ_{gt}, in which case A is called a *syntactical* consequence of Γ_{gt}. Gödel's completeness theorem (for classical predicate logic) now states that these notions coincide:

$$(A \text{ holds}) \quad \forall_{\mathcal{G}}[\,[\mathcal{G}\models\Gamma_{\text{gt}}]\Rightarrow[\mathcal{G}\models A]\,] \quad\iff\quad \exists_{\mathcal{P}}[\mathcal{P} \text{ formally proves } \Gamma_{\text{gt}}\vdash A] \quad (A \text{ is derivable})$$

This was a deep result at the time, as the two notions are totally different: the first one is set theoretical, while the latter is rather of a finitary combinatorial nature.

In contrast to theories like group theory, in which one tries to capture a large class of non-isomorphic structures, Peano introduced the language of arithmetic with the intention to describe $(\mathbb{N};+,\cdot,S,0)$ up to isomorphism. However, as we saw in the discussion of Gödel's incompleteness theorem, there is a sentence A, holding for $\mathbb{N}$ but unprovable from PA. By the *completeness* theorem, A must be false in some model of PA. We conclude that Peano's axioms do not characterize a unique structure.

There are several formalizations of the notion of a 'proof'. The three principle types of formalism are Hilbert system, Natural deduction system and Sequent calculus system (see [**TS 96**]).

1.2. Hilbert systems

A Hilbert system is based on axiom schemes and only two rules, viz. *modus ponens* (→-elimination) and the rule of *generalization* (∀-introduction):

$$\frac{A \qquad A\to B}{B}\ \text{modus ponens} \qquad\qquad \frac{A}{\forall_y A[y/x]}\ \text{generalization (under certain conditions)}$$

A *derivation* is a finite tree labeled with formulas, such that the immediate successors of a node C are the premisses of a rule having C as a conclusion. The root of the tree is the conclusion of the whole derivation. The maximal elements (i.e. the leaves of the tree) are required to be axioms. On the one hand there are purely logical axioms which define the logical connectives, e.g.

$$A\to(B\to(A\wedge B)) \quad ; \quad A\to(B\to A) \quad ; \quad [A\to(B\to C)]\to[(A\to B)\to(A\to C)]$$

In addition, there will be axioms for the particular theory (for example, the axioms of group theory).

The following is an example of a derivation of $A\to A$ in a Hilbert system. Axioms are overlined, as they should be considered as conclusions of 0-ary rules.

$$\dfrac{\dfrac{}{A\to(A\to A)}\,\text{k} \qquad \dfrac{\dfrac{}{A\to((A\to A)\to A)}\,\text{k} \qquad \dfrac{}{[A\to((A\to A)\to A)]\to[(A\to(A\to A))\to(A\to A)]}\,\text{s}}{(A\to(A\to A))\to(A\to A)}\,\text{mp}}{A\to A}\,\text{mp}$$

1.3. Natural deduction systems

Another formalism is the system of Natural deduction, first introduced by Gentzen in [**Gentzen 35**], and later developed by [**Prawitz 65**]. While the logical connectives in a Hilbert style derivation are characterized by the logical *axioms*, in a Natural deduction system the *deduction rules* determine them: in addition to modus ponens and the rule of generalization, there are introduction and elimination rules for each of the logical connectives. Instead of merely deriving a formula A as in the Hilbert system formalism from logical and additional axioms at the top of our tree, we now deduce A from a set of so-called open assumptions, which we shall make more precise shortly.

A *deduction* in the system of Natural deduction again is a tree of formulas with a unique conclusion C. However, there is an additional 'linking structure'. Some of the leaves at the top of the tree are 'linked to' exactly one rule in the tree. These leaves are called 'closed' assumptions; the others are called 'open'. This is characteristic for Natural deduction: for some of the rules (viz. →-introduction, ∨-elimination, ∃-elimination and reductio ad absurdum), application of them allows certain open assumptions to become closed (by fixing the 'link'). Only the remaining open assumptions are the 'real assumptions' on the basis of which one is allowed to conclude C.

The Hilbert system and the Natural deduction system for a theory are deductively equivalent, in the sense that there is a (Hilbert style) derivation with conclusion C if and only if there is a deduction (in the Natural deduction formalism) with conclusion C and with (open) assumptions belonging to the axioms for the particular theory.

As an example, let us consider the ∧-introduction rule of Natural deduction. It states that, given a deduction with conclusion A and one with conclusion B

$$\begin{array}{c}\text{assumptions}\\ \cdots \\ \hline A\end{array} \qquad\qquad \begin{array}{c}\text{assumptions}\\ \cdots \\ \hline B\end{array} \quad ,$$

we can build a new deduction with conclusion $A \wedge B$:

$$\frac{\begin{array}{c}\text{assumptions}\\ \cdots \\ \hline A\end{array} \qquad \begin{array}{c}\text{assumptions}\\ \cdots \\ \hline B\end{array}}{A \wedge B}\,\wedge\mathrm{I}$$

Another rule, the →-introduction rule, states that, given a deduction with conclusion C, we can build a new deduction with conclusion $D \to C$, which allows (but does not oblige) for the open assumptions D being 'linked' (and hence becoming closed). With these rules we can form the following deduction, where the linking structure is indicated by the indices 1 and 2.

$$\frac{A \qquad B}{A \wedge B}\,\wedge\mathrm{I} \qquad\qquad \frac{\dfrac{A \qquad [B]^1}{A \wedge B}\,\wedge\mathrm{I}}{B \to (A \wedge B)}\,{\to}\mathrm{I},1 \qquad\qquad \frac{\dfrac{\dfrac{[A]^2 \qquad [B]^1}{A \wedge B}\,\wedge\mathrm{I}}{B \to (A \wedge B)}\,{\to}\mathrm{I},1}{A \to (B \to (A \wedge B))}\,{\to}\mathrm{I},2$$

We thus deduced the logical axiom $A \to (B \to (A \wedge B))$ of a Hilbert system from no assumptions. The other way around, this axiom gives rise to the deduction rule ∧I, as the following compound Hilbert style rule illustrates:

$$\left.\frac{B \qquad \dfrac{A \qquad \overline{A \to (B \to (A \wedge B))}}{B \to (A \wedge B)}\,\mathrm{mp}}{A \wedge B}\,\mathrm{mp}\;\right\}\wedge\,\mathrm{I}$$

1.4. λ-Calculus

The natural deduction rule ‘reductio ad absurdum’ (RAA) states that whenever $\neg A$ leads to a contradiction, we may conclude A. Logics *without* this rule are intuitionistic logics (see [**TvD 88**]). They admit the Brouwer-Heyting-Kolmogorov (BHK) interpretation, explaining by constructive methods what it means to prove a compound statement in terms of what it means to prove its components: a construction $\mathcal{D}$ proves $A \wedge B$ if it is a pair $(\mathcal{D}_0, \mathcal{D}_1)$ consisting of a proof $\mathcal{D}_0$ of A and a proof $\mathcal{D}_1$ of B; $\mathcal{D}$ proves $A \vee B$ if $\mathcal{D}$ is either of the form $(0, \mathcal{D}_0)$, and $\mathcal{D}_0$ proves A, or of the form $(1, \mathcal{D}_1)$, and $\mathcal{D}_1$ proves B; $\mathcal{D}$ proves $A \to B$ if $\mathcal{D}$ is a construction transforming any proof $\mathcal{D}_0$ of A into a proof $\mathcal{D}(\mathcal{D}_0)$ of B; $\perp$ is a proposition without proof. A formalized version of the BHK-interpretation is given by the so-called λ-calculus, consisting of ‘terms’ of certain ‘types’ (the latter for the moment being considered as logical formulas). For each type A there is a countably infinite supply of variables $x_A, y_A, \ldots$, and each of them is by definition a term of that type. Moreover, if t is a term of type B and x_A a variable (possibly occurring several times in B), then $\lambda_{x_A} t$ is a term of type $A \to B$, which closely corresponds to function abstraction, but also to $\to$-introduction; if t is a term of type $A \to B$ and s is a term of type A, then ts is a term of type B, which closely corresponds to function application, but also to $\to$-elimination. As a final example, if s is a term of type A and b is a term of type B, there is a term $\langle s, t\rangle$ of type $A \wedge B$. Let us now start with appropriate variables x_A and y_B, then $\langle x_A, y_B\rangle$ is a term of type $A \wedge B$, whence $\lambda_{y_B} \langle x_A, y_B\rangle$ is a term of type $B \to (A \wedge B)$, yielding that $\lambda_{x_A} \lambda_{y_B} \langle x_A, y_B\rangle$ is a term of type $A \to (B \to (A \wedge B))$. This illustrates the so-called Curry-Howard isomorphism, or the formulas-as-types paradigm: a term carries all information of a deduction, while the other way around a deduction defines a term. The ‘links’ between closed assumptions and rules are taken care of by the scope of λ_x, bound variables corresponding to closed assumptions, and free variables to open assumptions. This suggests that we may regard a type C as the collection of all its terms, and we can reformulate deducibility questions as follows: (the formula) C is deducible iff (the type) C contains a closed term. More general, C is deducible from (open) assumptions in Γ iff C contains a term with free variables of types among the types in Γ. As an example we give the construction tree of the above constructed λ-term, which coincides with the deduction tree mentioned earlier:

$$\dfrac{\dfrac{\dfrac{[x_A : A]^2 \qquad [y_B : B]^1}{\langle x_A, y_B\rangle : A \wedge B}\,\wedge\mathrm{I}}{\lambda_{y_B} \langle x_A, y_B\rangle : B \to (A \wedge B)}\,\to\mathrm{I},1}{\lambda_{x_A} \lambda_{y_B} \langle x_A, y_B\rangle : A \to (B \to (A \wedge B))}\,\to\mathrm{I},2$$

As a consequence of the Curry-Howard isomorphism, phenomena occurring in λ-calculus have a direct counterpart in natural deduction. For example, the so-called β-reduction

$$(\lambda_{x_A} t)s \text{ reduces to } t\,[s/x_A]$$

corresponds to a certain operation on deduction trees. This is actually the explanation for the word *isomorphism* instead of *bijection*.

1.5. Sequent calculus systems

The last type of formalism, also due to Gentzen, is the Sequent calculus system (or Gentzen system), which operates with sequents $\Gamma \vdash A$, where Γ is a finite set of formulas (in fact, a multiset). The intended meaning of "$\Gamma \vdash A$ is valid" is "A is implied by the conjunction of the formulas in Γ". A *(sequent) derivation* is a tree labeled with sequents, which propagate from the identity axioms

$$\frac{}{A \vdash A}$$

according to the inference rules towards the unique conclusion sequent. The dependence on (open) assumptions of a formula in a natural deduction, is now captured in the antecedent part of the corresponding sequent in the sequent calculus derivation. So no assumption management (as by means of the 'links' in natural deduction) is needed. The immediate counterparts of the →-elimination and →-introduction rule of natural deduction can be given by

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}\,{\to}\mathrm{E} \qquad\qquad \frac{\Gamma, A, \ldots, A \vdash B}{\Gamma \vdash A \to B}\,{\to}\mathrm{I} \tag{1}$$

In fact, the following formulation of the rules is more standard

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\,\mathrm{L}{\to} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\mathrm{R}{\to} \tag{2}$$

In order to show the pairs of rules (1) and (2) are equivalent (in the sense that we can derive the same sequents by their application) one needs the following so-called *structural rules* of *weakening* and *contraction*

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathrm{LW} \qquad\qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathrm{LC} \tag{3}$$

as well as the *cut rule*, defined by

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\textsc{Cut} \tag{4}$$

which relates to composition (or substitution) of deductions:

$$\left.\begin{array}{c} \Gamma \\ \cdots \\ \hline A \\ \\ \Delta \qquad \mathrm{A} \\ \cdots\cdots \\ \hline B \end{array}\right\} \mapsto \qquad \begin{array}{c} \begin{array}{cc} & \Gamma \\ & \cdots \\ \cline{2-2} \Delta & A \end{array} \\ \cdots\cdots \\ \hline B \end{array}$$

Intuitively we can read $\Gamma \vdash A$ as "there is a deduction in the system of Natural deduction of the formula A from the (open) assumptions Γ". Indeed, the latter holds precisely if $\Gamma \vdash A$ is derivable in the sequent calculus system.

1.6. Classical logic

The systems indicated above are formalizations of *Intuitionistic Logic* (**IL**). In a natural deduction system, the step to *Classical Logic* (**CL**) is made by adding the rule 'reductio ad absurdum' (or equivalently, the axiom scheme $(\neg A \to \bot) \to A$, i.e. $\neg\neg A \to A$). The corresponding increase in logical expressibility in sequent calculus is achieved by admitting sequents with multiple succedents, like $\Gamma \vdash \Delta$. This system reflects the symmetries of classical logic; every formula can either play a role as an assumption (occurring in the antecedent part Γ), or as a conclusion (occurring negated in the succedent part Δ). As

$\Gamma \vdash \Delta$ is equivalent to $\vdash \neg\Gamma, \Delta$, we could in fact restrict to *one-sided* sequents: sequents with an empty antecedent part.

In general, a sequent calculus system is called a *one-sided* or *two-sided* system depending on what the sequents are. An *intuitionistic* system is a two-sided system in which the sequents are restricted to a single succedent formula.

1.7. Cut elimination

In a Hilbert style or a natural deduction formalism, during a bottom-up search for a proof of a formula B, we encounter the problem that certain formulas must be guessed at. E.g. A in

$$\frac{A \qquad A \to B}{B}\,\mathrm{mp}$$

need not have any relation with B. The same problem occurs in a sequent calculus formalism when using the cut rule. However, Gentzen proved that the cut rule is actually superfluous. This central result in proof theory, with many consequences, is sometimes, as in Gentzen's original paper, referred to as the Hauptsatz:

Theorem 1.7.1. ***(Hauptsatz)*** *For every sequent calculus derivation in* **CL** *(or* **IL***) there is a* Cut*-free derivation of the same sequent.* ♢

For example, it implies the so-called Subformula Property: whenever a formula C is provable, there exists a proof involving only subformulas of C.

By means of cut elimination, we can also simplify natural deductions. Indeed, given a deduction tree, translate it into a sequent calculus derivation, eliminate the cut, and translate back. The resulting deduction is in some sense free of detours.

In the sequent calculus system, cut elimination can be viewed as a way to explicate the contents of a proof. Even better, Gentzen's constructive proof of the Hauptsatz provides us with an inductive procedure to calculate such a cut free representative. Some natural questions now arise, e.g. whether these cut free representatives are unique in some sense.

1.8. Denotational semantics

Denotational semantics is the semantics of proofs. The fundamental idea consists in interpreting cut elimination (a 'dynamic notion') by equality (a 'static notion'). Let us consider the congruence relation on all derivations for a fixed sequent such that a derivation is related to another one if it is obtained from the latter by a cut elimination step. Each derivation is now given an interpretation, called its *denotation*, such that congruent derivations have the same interpretation.

For **CL** this does not yield an interesting distinction between derivations for a fixed sequent: due to the structural rules and the resulting non-determinism of Gentzen's procedure of cut elimination, we can show that they must all have the same denotation (see [**GLT 90**]). The situation is different for **IL**, where the application of the structural rules is restricted to the antecedent part of sequents.

Coherence spaces are mathematical structures which were originally introduced by [**Girard 87**] as a denotational semantics for **IL**. If the denotations for derivations of a formula A are found in the coherence space $[\![A]\!]$, and those of B in $[\![B]\!]$, then those of $A \wedge B$ are found in $[\![A]\!] \wedge [\![B]\!]$, where the second $\wedge$ is a suitable operation on coherence spaces.

In this way, the logical connectives correspond to operations on coherence spaces. Some of these operations turned out to admit a decomposition in more primitive operations. For example, the function-space operation $\rightarrow$ (corresponding to the connective $\rightarrow$) can be defined in terms of two new operations, called *linear implication* ($\multimap$) and *of course* (or *storage*) (!), as follows:

$$X \rightarrow Y = (!X) \multimap Y$$

Another example is the sum type (corresponding to $\vee$), which has the following decomposition:

$$X \vee Y = (!X) \oplus (!Y)$$

See the survey [**Schellinx 00**] for a complete overview.

1.9. Linear logic

The remarkable observation that this semantical decomposition of the intuitionistic connectives has a syntactical counterpart, is the origin of Girard's linear logic: the semantical operations like $\multimap$, ! and $\oplus$ actually correspond to logical operations in their own right ([**Girard 87**]; an all-around introduction is given in [**Troelstra 92**]). This is achieved by a severe restriction, or better, by control of the use of the structural rules of weakening and contraction. They are given a logical status by the new logical unary connectives like !. E.g. the *structural* rules (3) are replaced by, among other things, the *logical* rules

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,\mathrm{W!} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,\mathrm{C!} \tag{5}$$

Because of the absence of structural rules, linear logic is *resource sensitive*: 'A' stands for "exactly one copy of A". The bridge with ordinary logic, where assumptions can be used ad libitum because of the weakening and contraction rules, is made by '!': '$!A$' stands for "any number of copies of the formula A".

So, historically, linear logic originated as the calculus extracted out of a semantics for **IL**. This refinement of **IL** survives in what is known as *Intuitionistic Linear Logic* (**ILL**). As for **CL**, in order to obtain *Classical Linear Logic* (**CLL**), we have to allow for sequents with multiple succedents ($\Gamma \vdash \Delta$), and generalize the rules accordingly.

As a refinement of both **IL** and **CL**, **CLL** can be considered as a tool to investigate behaviour and properties of intuitionistic and classical sequent calculus derivations. In [**Schellinx 94**] it is shown that every classical sequent derivation is a 'simplified' linear logic derivation. As an important consequence, cut elimination for **CL** (or **IL**) now follows from cut elimination for **CLL**: given a classical derivation, find a corresponding linear derivation, eliminate the CUT's and simplify the result. As cut elimination for **CLL** is essentially deterministic, the non-determinism of cut elimination for **CL** now can be captured in the choice among possible corresponding linear derivations. Indeed, this choice essentially fixes the cut elimination procedure in **CLL**, whence, by projection, that in **CL** as well.

1.10. Multiplicative linear logic

Derivations rules for the connectives in **CL** (or **IL**) can be given by several equivalent formulations. Because of the absence of weakening and contraction, however, such distinct formulations may become essentially different in **CLL** (or **ILL**). It turns out that the connectives split into two variants: context-sharing (or context-sensitive) variants (which are called *additive*), respectively context-free (or context-insensitive) variants (which are called *multiplicative*). For conjunction, the additive variant is defined by the pair of rules (6), and the multiplicative variant by (7).

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge_{\mathrm{a}} A_2 \vdash C}\,\mathrm{L}\wedge_{\mathrm{a},i} \qquad\qquad \frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \wedge_{\mathrm{a}} A_2}\,\mathrm{R}\wedge_{\mathrm{a}} \tag{6}$$

$$\frac{\Gamma, A_1, A_2 \vdash C}{\Gamma, A_1 \wedge_{\mathrm{m}} A_2 \vdash C}\,\mathrm{L}\wedge_{\mathrm{m}} \qquad\qquad \frac{\Gamma \vdash A_1 \qquad \Delta \vdash A_2}{\Gamma, \Delta \vdash A_1 \wedge_{\mathrm{m}} A_2}\,\mathrm{R}\wedge_{\mathrm{m}} \tag{7}$$

We can understand $\wedge_{\mathrm{a}}$ and $\wedge_{\mathrm{m}}$ as natural, but essentially distinct, operational aspects of classical conjunction: if we consider the formulas as data types, a datum of type $A \wedge_{\mathrm{a}} B$ is a datum which can be used exactly once to extract a datum of type A or a datum of type B. On the contrary, a datum of type $A \wedge_{\mathrm{m}} B$ is a pair of data. $\star$-Autonomous categories are algebraic structures defined by an axiomatization corresponding to linear logic. (Actually, they already existed before linear logic was invented; see [**Barr 79**]). In these categories, the cartesian product corresponds to $\wedge_{\mathrm{a}}$, while the tensor product corresponds to $\wedge_{\mathrm{m}}$.

The fragment of linear logic consisting precisely in these multiplicative connectives is *Multiplicative Linear Logic* (**MLL**). This fragment constitutes the core of linear sequent calculus: for a sequent $\Gamma \vdash \Delta$, a comma in Γ naturally corresponds to multiplicative conjunction, the entailment sign $\vdash$ to linear implication, and a comma in Δ to multiplicative disjunction.

1.11. Proof nets and proof structures

In [**Girard 87**] Girard introduces the system of proof nets for linear logic. Proof nets are graph-like structures that abstract from inessential distinctions due to the intrinsic order of rules in sequent derivations. The system of proof nets can be considered as a natural deduction system of the sequent calculus for linear logic, but with two notable differences. First of all, there is no need for parcels of assumptions; only one assumption at a time may be closed by a rule. And secondly, in the original definition of proof nets, elimination rules do not occur. Instead, by linear negation and Cut they are represented by introduction rules.

The fact that proof nets abstract from inessential order, can be seen as follows: whenever in an ordinary derivation there is a moment of choice, the proof net pursues both possibilities at the same time, but in a *parallel* fashion. But there is a cost: while derivations are recognized in linear time, it requires more effort to check whether a candidate for a proof net is really a proof net.

To every sequent S we can assign several potential proof nets. These so-called *proof structures* only depend on the formulas of S, and certain connections between their atomic subformulas. Indeed, every (cut-free) proof net is such a proof structure. A criterion determining whether a proof structure also is a proof net is called a *correctness criterion*. The original criterion for the system of **MLL** proof nets given in [**Girard 87**] is the *long trip condition*. It closely relates to the *tree condition*, which more or less states that all graphs, associated to a proof structure, must be trees (acyclic and connected) in order for the proof structure to be a proof net. Another criterion is the *contraction criterion* given by Danos in [**Danos 90**]: proof nets are those proof structures that can be contracted into one point, under a suitable contraction relation. Our contraction criteria in this thesis will be variations on this theme. The last criterion we mention here is Métayer's homological criterion ([**Métayer 94**]): by generalizing the ordinary definition of homology for graphs, proof nets turn out to be characterized among proof structures by their homology. The elegance of this criterion is the fact that it enables us to give completely algebraical proofs of proof theoretical phenomena of **MLL**. (See also [**Puite 96**] and [**PS 97**].)

1.12. Structural refinements

We have seen that linear logic emerges as a refinement of **IL**, in the sense that the intuitionistic connectives decompose into more primitive ones. Moreover, we saw that this system of linear logic is characterized by a restriction on the use of the structural rules of weakening and contraction. It is hence also referred to as a *substructural logic*. This restriction implies that the sequent $\vdash \Gamma, A, A$ is to be considered different from $\vdash \Gamma, A$.

Still structural rules abound, be it implicitly. They differ from weakening and contraction in the fact that they are *linear*. In a linear structural rule

$$\frac{\vdash \Gamma}{\vdash \widetilde{\Gamma}}$$

Γ and $\widetilde{\Gamma}$ consist in exactly the same formulas. Only the 'relative positions' of these formulas may change. The careful analysis of these connections and interactions of the formulas in a sequent leads to further refinements of **MLL**.

Do $\vdash A, B, \Gamma$ and $\vdash B, A, \Gamma$ stand for the same sequent? Under the structural rule of *exchange* these expressions indeed are equivalent. But often there are good reasons to abandon this structural rule too. Then the *order* of the formulas has to be taken into account. This gives rise to non-commutative "refinements" of **MLL**. The intuitionistic fragment of non-commutative **MLL** had already been known before the birth of linear logic, viz. as "Lambek calculus" (**L**) ([**Lambek 58**], see also [**Roorda 91, Lambek 95**]). There are several classical non-commutative calculi: cyclic linear logic (see [**Yetter 90**]) with formula cycles as sequents; the system of [**Abrusci 95**] with formula lists as sequents, which leads to the coexistence of two negations: a linear post-negation and a linear retro-negation, that are not involutive, but cancel each other. Both these systems conservatively extend **L**. Further we mention systems with coexisting commutative and non-commutative connectives, where the sequents are given extra structure (e.g. a series-parallel partial order), e.g. the system of [**Retoré 93**] (with the new non-commutative connective "before") and the system of [**AR 98**]. The specific system Non-commutative Cyclic Linear Logic (**NCLL**) corresponding to [**Yetter 90**] will be the subject of Chapter 4.

Let us go one step further and ask whether the two expressions $\vdash \ldots, (A, B), C, \ldots$ and $\vdash \ldots, A, (B, C), \ldots$ represent the same sequent? Here, the brackets denote the construction order of the sequent. There are many mathematical operations for which this so-called *bracketing* does matter, e.g. the cross product in Euclidian space (not in general $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$), or the Lie brackets of Lie algebra's (mostly $[\,[xy]z] \neq [x[yz]\,]$). As notion of sequent for non-associative and non-commutative refinements of

MLL, bracketings as they stand are inappropriate. It would force us to an undesirable further splitting of the connectives. For example, the one-sided counterpart of (7), given by

$$\frac{\vdash \Gamma \, , \, A \qquad \vdash B \, , \, \Delta}{\vdash \Gamma \, , \, A \wedge_{\mathrm{m}} B \, , \, \Delta}\,\mathrm{R}\wedge_{\mathrm{m}}$$

would require a bracketing, hence either

$$\frac{\vdash (\Gamma \, , \, A) \qquad \vdash (B \, , \, \Delta)}{\vdash \Big(\big(\Gamma \, , \, A \wedge^{\mathrm{l}}_{\mathrm{m}} B\big) \, , \, \Delta\Big)}\,\mathrm{R}\wedge^{\mathrm{l}}_{\mathrm{m}} \qquad \text{or} \qquad \frac{\vdash (\Gamma \, , \, A) \qquad \vdash (B \, , \, \Delta)}{\vdash \Big(\Gamma \, , \, \big(A \wedge^{\mathrm{r}}_{\mathrm{m}} B \, , \, \Delta\big)\Big)}\,\mathrm{R}\wedge^{\mathrm{r}}_{\mathrm{m}}$$

Apparently, we would obtain two distinct multiplicative conjunctions. This dichotomy is overcome in [**dGL 00**] by a minimal identification on bracketings. In Chapter 5 we consider the two-sided formulation of the system Classical Non-associative Lambek calculus (**CNL**) of [**dGL 00**].

The controlled reintroduction of some linear structural rules in the intuitionistic fragment of this system (cf. [**Morrill 96, Moortgat 97**]) will be the subject of Chapter 6. Likewise '!' controls the use of weakening and contraction in plain linear logic, new unary connectives ('$\Diamond$' and '$\Box$') are introduced in order to control the use of the additional linear structural rules. The next linear structural rule is an example of a weak form of (one direction of) associativity:

$$\frac{\Delta[\, A_1, (A_2, \Diamond A_3) \,] \ \vdash \ C}{\Delta[\, (A_1, A_2), \Diamond A_3 \,] \ \vdash \ C} \tag{8}$$

We thus obtain a wide variety of substructural logics. Depending on our purposes, we can vary the set of admitted linear structural rules. Adding plain associativity takes us back to **NCLL**, and also adding commutativity takes us back to **MLL**.

1.13. Categorial grammars

In linguistics, substructural logics can be used as parsing tools. In fact, Lambek calculus (**L**) was motivated by this application ([**Lambek 58**]); and only after the introduction of linear logic ([**Girard 87**]), **L** turned out to be the intuitionistic non-commutative fragment of linear logic.

We consider an example explaining the resource-sensitivity of natural language. "John · talks" is a well-formed sentence. Let us assign "John" the formula N and "talks" the formula $N \to S$, indicating that together with an N-word (noun phrase, to be precise) it yields a sentence (S). Then "John · talks" has the formula $N \wedge (N \to S)$, from which S is derivable. On the contrary "John · talks · Jim" would have a surplus of N (where "Jim" also is assigned the formula N).

Similarly, "John · sees · Jim" becomes an S, when we assign "sees" the formula $N \to (N \to S)$. But "John · sees" lacks an N in order to become an S.

Hence the logic underlying language should at least be a resource conscious one. But we also want "talks · John" to be underivable (although, as words they can constitute a sentence after permutation). There are hence good reasons to take **L** or even non-associative Lambek calculus (**NL**) as the underlying logic. In such a non-commutative setting "TALKS" will rather be given the formula $N \multimap S$, expressing that it combines only on the left with an N-word. Similarly, the word "SEES" will get the formula $(N \multimap S) \circ\!\!- N$, meaning that its combination with "JIM" on the right (i.e. "SEES · JIM") yields the formula $N \multimap S$ (indeed, the same as "TALKS"!), which in turn means that "JOHN · (SEES · JIM)" becomes an S.

A categorial grammar consists of a derivation system and a lexicon. The lexicon assigns formulas to words of a natural language. The derivation system describes the grammatical rules of the language. Taking our underlying logic as simple as possible (say, **NL**), grammatical rules can be described by a controlled reintroduction of the linear structural rules. We can plug-in such packages of structural rules, depending on the actual language.

The task of parsing a sentence now consists in first choosing a formula for every word in the sentence (out of the assigned formulas), and secondly searching for a deduction of the formula S, witnessing that the words, in the given order, constitute a grammatical correct sentence.

1.14. New contributions

The main theme of this thesis is the general two-sided theory of proof nets for the substructural logics **MLL**, **NCLL**, **CNL** and **NL**$\Diamond_{\mathcal{R}}$. For each of the calculi we will prove a correctness criterion, according to which proof nets are exactly those proof structures that appropriately contract. The intended contraction relation is defined on the space of *link graphs*, a new notion which turns out to be sufficiently general to capture both proof structures and sequents.

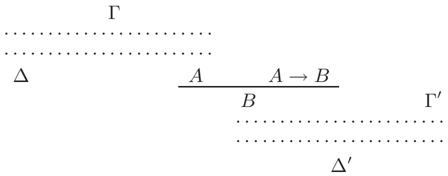
For **MLL** our contraction criterion is a combination of Danos' well-known contraction criterion for one-sided **MLL** (see [**Danos 90**]) and Lafonts criterion for parsing boxes (see [**Lafont 95**]). For **NCLL** our proof nets closely relate to those of [**Roorda 91**]. For **CNL** our contraction criterion extends the results of [**dGL 00**]. A notable difference in our approach is the fact that — although a sequent is defined in terms of a certain (tree) structure on the formulas — our notion of proof structure is free of any explicit structure on the leaves. Instead of asking whether a certain sequent (structure of formulas) is derivable, we will ask whether the *set* of formulas is somehow derivable; if yes, our criterion *yields* the required structure! The gap between 'derivable' and 'well-structured', though having decreased, has evidently not vanished. This approach implies that our notion of proof structure is independent of the particular two-sided calculus; of course, sequents, derivation rules, contraction steps and proof nets do depend on the calculus. For the intuitionistic restriction of **CNL**, extended by multiple modalities and a fixed set of structural rules (**NL**$\Diamond_{\mathcal{R}}$), our contraction criterion proves and generalizes a correctness criterion for the labeled sequent calculus of [**Gabbay 96**].

For the sequent calculus system **NCLL** we will prove strong normalization of cut elimination by means of a generalization of the cut rule (Subsection 4.2.2). For each of the calculi we will prove correctness of cut elimination w.r.t. our contraction criterion (Section 4.5, Section 5.5, Section 6.6). This requires quite deep investigations on the dependency between the contraction steps in a given conversion sequence; it gives rise to particular substructures called *block* and *component*. We think our proof might serve as a key towards further results on the parallelism of the conversion steps.

Let us briefly describe the two-sided system and some of its advantages in more detail.

On two-sided proof net systems. In the two-sided sequent calculus for **MLL**, sequents are of the form $\Gamma \vdash \Delta$. We will define a corresponding notion of proof net, which differs from the original notion of proof net in two respects: proof nets are allowed to have assumptions in addition to conclusions, and, secondly, rules are allowed to have more than one conclusion, while the main formula (the formula that has the remaining formulas as subformulas) need not be a conclusion. The latter is in contrast to the one-sided system, in which a logical rule has exactly one conclusion which moreover also is the main formula: it has as subformulas the premisses of the rule. In the new setting, rules for 'new' connectives are naturally obtained. An example is provided by the following novel rule, which closely relates to the modus ponens rule (→E) of Natural deduction. It states that, given a proof net with a conclusion A and one with an assumption B

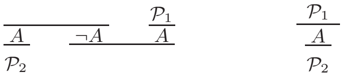
we can build a new proof net with an assumption $A \to B$:



The assumptions of proof nets in the new system can be interpreted as if they were negated conclusions. It is also possible to interpret the new rules as compound one-sided rules. It may thus be clear that our generalizations do not change logical derivability *from the one-sided point of view*: our system derives $\Gamma \vdash \Delta$ precisely if the one-sided system derives $\vdash \neg\Gamma, \Delta$ (Lemma 4.7.1). On the other hand, however, our two-sided theory turns out to be more appropriate in many respects, which we will briefly discuss now.

In the one-sided system some of the basic logical laws are expressed by operations in the formula language. By one-sidedness, they simply *cannot* be taken care of by the logical inference system. As an example, consider the *defined* operations of negation ($\neg(A \wedge B) := (\neg B) \vee (\neg A)$, etc.) and implication ($A \to B := (\neg A) \vee B$). A first gain of our new system consists in the fact that we can take such operations (like ¬ and →) as new primitive connectives. This means that $A \to B$ and $(\neg A) \vee B$, as they stand, are different formulas. By means of the two-sided theory we will see, however, that there is a close logical relation between both formulas. This is achieved by the particularly nice notion of *dualizability* (see Example 3.2.8). Roughly said, turning a two-sided proof structure up-side-down yields its dualization, and a proof net is dualizable if this dualization is a proof net as well. We will show that in the category of formulas and proof nets, these dual proof nets are exactly the isomorphisms (Theorem 4.8.3). Secondly, we prove that isomorphic formulas are exactly those that are equal in the one-sided setting (Theorem 4.6.3, Theorem 5.6.1). This gives a justification for the original one-sided formula identifications like $A \to B = (\neg A) \vee B$; indeed, two-sidedly these different formulas are apparently indistinguishable at the derivational level.

A second advantage is obtained with respect to subnets. The notion of a subnet in the original theory is, in our opinion, rather artificial. The following two proof nets should behave similarly, as they only differ by one cut elimination step. However, in the left hand side proof net, $\mathcal{P}_2$ with the negated assumption is a subnet, while in the right hand side proof net it is impossible to regard $\mathcal{P}_2$ as a subnet.



The solution to this artificial distinction is found in the two-sided theory, where also in the right hand side proof net $\mathcal{P}_2$ is a subnet, with A as an assumption. (In the one-sided theory, we sometimes encounter a temporary allowance for assumptions; cf. [**Danos 90**].) Hence the two-sided theory is a better illustration of the modular point of view: proof nets are plug-ins of several subnets. This, in turn, allows for proving our lemmas and theorems by induction on the size of the proof net. Without going into details, we will give one 'classical' example of this statement for the reader acquainted with **MLL**. Danos' splitting par lemma states that a correct (in some sense) proof structure without terminal par links (but with at least one non-terminal par link) has a *splitting* par link. This splitting par lemma yields the induction step when proving sequentialization: the two substructures, obtained by removing the splitting par, are still correct, whence sequentializable, and hence also the original correct proof structure is. Observe that removing this par link in a one-sided setting introduces an undesirable assumption; in the two-sided setting, however, this does not harm. By the way, another proof of sequentialization uses the splitting tensor lemma, where a splitting tensor is a terminal tensor which is a 'bridge' between two disjoint substructures. Now observe that, in our two-sided setting, we can introduce the overall notion of a *splitting formula*, which generalizes both the notion of splitting par and that of splitting tensor.

A very practical consequence of the two-sided system is the step towards the intuitionistic fragment (Section 4.9). The intuitionistic fragment of **MLL** is simply obtained as 'the theory of proof nets with one conclusion'. And this, in turn, is nothing else than the linear version of Natural deduction for **IL**. As a corollary, we find a nice alternative for the $\vee$-elimination rule, which in the non-linear system is given by:

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array}\qquad\begin{array}{c}[A]\\ \vdots\\ C\end{array}\qquad\begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\;\vee\mathrm{E}$$

As said, one of our aims consists in formulating and proving contraction criteria for the substructural calculi mentioned above. As a last advantage of the two-sided setting, observe that the absence of axiom links provides us with an easy description of the contraction steps (Section 4.4). Alternatively one can argue that, conversely, those contraction steps tell us that axioms should rather be regarded as single formulas and not as links.

Sequents and link graphs. In order to describe a substructural calculus, we will define a notion of sequent which is subtle enough to admit the appropriate structural fine treatment.

Sequents for **NCLL** (Chapter 4) will be defined as cyclic lists of formulas, each of which can play the role of an assumption or a conclusion. If we would require a separation of the assumptions and the conclusions (cf. the ordinary sequent notation $\Gamma \ \vdash \ \Delta$), our sequents would essentially (i.e. one-sidedly) be lists ($\vdash \neg\Gamma, \Delta$) instead of formula cycles. We would arrive at the two-sided counterpart of the non-commutative system of [**Abrusci 95**].

We will define a sequent for **CNL** (Chapter 5) by the geometrical notion of a *cyclic tree* (Section 5.1). This is equivalent to the definition as given in [**dGL 00**].

Each of the mentioned substructural calculi has its own elegant, though somewhat *ad hoc* definition(s). But since we are also interested in the relations between the different calculi, we will start from a very general definition of proof structure as a particular kind of the even more general so-called *link graphs*, to be defined in Section 3.1. Link graphs as such do not have any obvious logical meaning: we should consider them as our universe of discourse. In later chapters, for each calculus they will be used in order to define the basic objects (viz. the sequents and the derivable sequents), as well as the corresponding notions of proof structure and proof net. As we intend to prove that a sequent is derivable if and only if the corresponding proof structure converts to a certain form (in fact, as we shall see, itself a sequent), it turns out to be highly useful to have this overall notion of link graph in which we can formulate both proof structures and sequents, as well as the process of conversion.

Controlled linear structural rules. In the labeled sequent calculus of [**Gabbay 96**] derivability splits into logical derivability as well as structural derivability. In the corresponding theory of labeled proof nets ([**Moortgat 97**]), one uniquely decorates the formulas of an intuitionistic proof net by means of formal terms. All structural modification is now taken care of by *the term* of this proof net, by which we mean the term assigned to the unique conclusion. Our criterion in Chapter 6 is based on the Curry-Howard-like observation that the term of a proof net is in fact nothing else than the proof net itself. Hence, instead of defining the structural operations on *terms*, we can directly define them on *proof nets*. Moreover, we can extend this term assignment to *any* proof structure (even without the nice properties of an intuitionistic proof *net*): the term of a proof structure is given by the proof structure itself! The latter is quite a generalization, as the original way in which labels in a proof net propagate towards the unique conclusion, in no way generalizes to arbitrary proof structures. These considerations have been elaborated in joint work with Richard Moot and will appear as [**MP 00**]; see Chapter 6.

As **NL** is the intuitionistic (hence asymmetric) fragment of **CNL**, the former admits a finer description of the linear structural rules. E.g. to **NL** we can independently add either direction of associativity; in **CNL**, the two directions coincide. (The same phenomenon occurs with respect to the weakening and contraction rules for **IL**; e.g. the difference between left contraction (allowed) and right contraction (impossible) evaporates in **CL**.)

Side results. For **MLL**, it is impossible to uniquely assign a proof net to a sequent derivation. As the sequents are multisets, due to multiple occurrences an active or main formula of a derivation rule may be ambiguous. We will show that this problem is overcome for **NCLL** (and hence for further refinements). The non-periodicity of derivable sequents (Lemma 4.1.13) turns out to solve the original ambiguity (Lemma 4.2.5).

We will define a natural section of the projection from the two-side theory to the one-sided theory (Subsection 2.1.5). The fact that this map is a section means that a one-sided conclusion can *one-sidedly* be seen as either a negation-free (two-sided) assumption or a negation-free (two-sided) conclusion (Lemma 2.2.5). E.g., the one-sided conclusion $(\neg A) \vee B$ is a conclusion $A \rightarrow B$; the one-sided conclusion $(\neg B) \vee (\neg A)$ is an *assumption* $A \wedge B$. This provides a well-behaved embedding of one-sided systems into their two-sided counterparts.

In Subsection 2.3.2 we will study the intuitionistic restriction of associativity. Given two intuitionistic formulas A and C, a chain of equivalences $A \simeq B \simeq C$ (where $\simeq$ is generated by associativity) may pass a non-intuitionistic formula B. We will show that the well-known associative laws for the intuitionistic language are really enough to capture the original equivalence (Subsection 2.3.2).

A small but nice result is the easy calculation of the n-th Catalan number: the number of binary trees with $n+1$ leaves (Example 5.1.5). We will obtain it from the combinatorics of cyclic trees.

Refinements vs. refinements. Let us finish with the question in which sense the systems described in this thesis are *refinements* of **MLL**. Not every **MLL**-provable sequent can be ordered (or bracketed) such that it becomes provable in the more restrictive system (a counter example being provided by $A \wedge B \vdash B \wedge A$); this means that the restrictive systems derive less formulas, whence they are ‘refinements’ in the weak sense of ‘subsystems’. However, addition of plain associativity and commutativity to the restrictive systems gives rise to a system with the same expressionable strength as **MLL**. Indeed, a proof in **MLL** transforms into a proof in **NCLL**+commutativity (**CNL**+associativity+commutativity) by explicating the originally hidden use of commutativity (and associativity). These systems hence are ‘refinements’ in the stronger sense that every ‘coarse’ derivation in **MLL** decomposes in smaller steps; the same formulas are derivable, by more detailed proofs. (See also the work of [**Fleury 96**].)

CHAPTER 2

Preliminaries

Sequent calculi for fragments of classical linear logic are often given in a one-sided way: sequents only consist of conclusions, and hypotheses do not appear explicitly. This is because classical formulas by definition satisfy De Morgan identities like $(X \parr Y)^\perp = Y^\perp \otimes X^\perp$, whence it is possible to consider hypotheses X as conclusions $[X]^\perp$. This set of formulas will be called $\mathfrak{L}_1$, and introduced in Subsection 2.1.1.

Lambek Calculus treats intuitionistic sequents, having several hypotheses and exactly one conclusion. These may be considered as a particular kind of *two-sided* sequents, in which we allow both hypotheses and conclusions. It no longer is necessary to quotient by the De Morgan equivalence; even better: if we do not, we are able to derive it. This two-sided language is called $\mathfrak{L}_2$.

Every two-sided sequent or derivation has a one-sided counterpart, obtained by turning all hypotheses into conclusions by means of negation, and then taking the De Morgan quotient. The other way around, it is far less evident how to canonically assign a two-sided structure to a given one-sided one. We will present a solution to this problem in Subsection 2.1.5: every one-sided formula can be expressed uniquely (modulo outermost $[-]^\perp$) in atoms by means of $\otimes$ and $\parr$ and the defined operations $\multimap$ and $\multimapinv$; we take this expression as the corresponding $\mathfrak{L}_2$-formula, and add the sign $+$ $(-)$ depending on the absence (presence) of the outermost negation. So our solution at this stage uses a form of *polarization* of formulas, which in fact is not a severe generalization; the structures that we define in later sections, will anyhow need polarized formulas, and then the outermost negation will correspond to sign alternation.

In Section 2.2 we will show that the De Morgan equivalence is exactly the identification made when mapping $\mathfrak{L}_2$-formulas onto $\mathfrak{L}_1$-formulas. The canonical representatives of each equivalence class obtained by the already defined map in the opposite direction contain the intuitionistic $\mathfrak{L}_2$-formulas. The corresponding elements of $\mathfrak{L}_1$ are then called the intuitionistic $\mathfrak{L}_1$-formulas.

It is well-known what generates the equivalence relation of associativity in $\mathfrak{L}_2$. However, for e.g. intuitionistic $\mathfrak{L}_2$-formulas a chain of such equivalences may lead us out of the intuitionistic part. So in Section 2.3 we will give an explicit definition of associativity completely within the intuitionistic language.

2.1. Formulas

A *language* is a set of formulas. Three languages will play a central role in this thesis, viz. the one-sided classical language $\mathfrak{L}_1$; the two-sided classical language $\mathfrak{L}_2$; and the intuitionistic language $\mathfrak{L}_{2,\mathrm{i}}$. Formulas of an unspecified language are denoted by $F, G, H, \ldots$.

2.1.1. One-sided language. Starting from an infinite denumerable set of *atoms* $\mathcal{A} := \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and the set of their formal negations $\mathcal{A}^\perp := \{\alpha_1^\perp, \alpha_2^\perp, \alpha_3^\perp, \ldots\}$, which together constitute the set of *atomic formulas*, the *formulas* of the one-sided classical calculi are built up with the binary connectives $\otimes$ (*"times"*) and $⅋$ (*"par"*), representing multiplicative conjunction and disjunction respectively. The resulting set of formulas will be denoted by $\mathfrak{L}_1$, and we will refer to $\mathfrak{L}_1$-formulas as $X, Y, Z, W, \ldots$.

Linear negation $[-]^\perp : \mathfrak{L}_1 \to \mathfrak{L}_1$ (*"perp"*) of a formula is inductively defined by the 'De Morgan laws':

$$\begin{aligned} [\alpha_i]^\perp &:= \alpha_i^\perp \\ [\alpha_i^\perp]^\perp &:= \alpha_i \\ [X \otimes Y]^\perp &:= [Y]^\perp ⅋ [X]^\perp \\ [X ⅋ Y]^\perp &:= [Y]^\perp \otimes [X]^\perp \end{aligned}$$

which is easily shown to be an involution: $[[X]^\perp]^\perp = X$. Moreover, we define two operations for linear implication, viz. *"implies" (or "under")* and *"if" (or "over")*, by

$$\begin{aligned} (X \setminus Y :=)\quad X \multimap Y &:= [X]^\perp ⅋ Y \\ (X / Y :=)\quad X \multimapinv Y &:= X ⅋ [Y]^\perp. \end{aligned}$$

2.1.2. Two-sided language. Starting from an infinite denumerable set of *atoms* $\mathcal{A} := \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, the *formulas* of the two-sided classical calculi are built up with the unary connective $(-)^\perp$ and the binary connectives $\otimes$, $⅋$, $\multimap$ and $\multimapinv$. The resulting set of formulas will be denoted by $\mathfrak{L}_2$, and we will refer to $\mathfrak{L}_2$-formulas as $A, B, C, D, \ldots$.

Within $\mathfrak{L}_2$ we distinguish three particular subsets, viz. the *De Morgan normal forms* (see page 32 for the definition of the De Morgan normal form *of a formula*)

$$\mathfrak{L}_{2,\mathrm{nf}} := \mathcal{F} ::= \mathcal{A} \mid (\mathcal{A})^\perp \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} ⅋ \mathcal{F}$$

which are $\multimap$- and $\multimapinv$-free and moreover contain negations only of atoms; secondly the $\perp$-*free formulas* defined by

$$\mathfrak{L}_{2,\perp\text{-free}} := \mathcal{F} ::= \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} ⅋ \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid \mathcal{F} \multimapinv \mathcal{F}$$

and finally the *intuitionistic formulas*

$$\mathfrak{L}_{2,\mathrm{i}} := \mathcal{F} ::= \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid \mathcal{F} \multimapinv \mathcal{F}$$

which are the $\perp$- and $⅋$-free formulas.

The inclusion $\mathfrak{L}_{2,\mathrm{i}} \hookrightarrow \mathfrak{L}_2$ is called κ.

2.1.3. Data types. The name 'data type' will be used for a collection of similarly structured sets (so-called 'settings') of (polarized) formulas, e.g. lists or trees.

Given some formulas $F_0, \ldots, F_{m-1}$ (each $F_i \in \mathfrak{L}$), the *set* $S := \{F_0, \ldots, F_{m-1}\}$ contains at most m formulas. It is completely determined by its characteristic function $\mathfrak{L} \to \{0, 1\}$ which is precisely 1 on $F \in S$. In case we want to distinguish multiple occurrences of one and the same formula, we have to generalize to the notion of *multiset*, determined by a characteristic function $\mathfrak{L} \to \mathbb{N}$. A multiset of formulas is sometimes regarded as a set of formula occurrences.

The free abelian group generated by $\mathfrak{L}$-formulas $\mathbb{Z}^{\mathfrak{L}}$ consists of formal finite sums $\sum_{F \in \mathfrak{L}} s_F F$, each one characterized by the function $F \mapsto s_F : \mathfrak{L} \to \mathbb{Z}$. A formal sum can be regarded as a multiset in which negative multiplicities are allowed.

The collection $\mathfrak{Sets}$ ($\mathfrak{MSets}$) ($\mathbb{Z}^{\mathfrak{L}}$) of sets (multisets) (formal sums) is an example of a data type. The operation assigning the set (multiset) (formal sum) $S \cup T$ ($S \uplus T$) ($S{+}T$) to an ordered pair (S, T) of sets (multisets) (formal sums) corresponds to truncated addition (ordinary addition) (ordinary addition) of the associated characteristic functions $s, t : \mathfrak{L} \to \{0, 1\}$ ($\mathbb{N}$) ($\mathbb{Z}$). This is an associative and commutative binary operation on $\mathfrak{Sets}$ ($\mathfrak{MSets}$) ($\mathbb{Z}^{\mathfrak{L}}$).

The map $\{0, \ldots, m-1\} \to \mathfrak{L} : i \mapsto F_i$ itself determines the *list* $(F_0, \ldots, F_{m-1})$ of length m. The concatenation (Γ, Δ) of $\Gamma = (F_0, \ldots, F_{m-1})$ and $\Delta = (G_0, \ldots, G_{n-1})$ is defined to be the list determined by

$$\{0, \ldots, m+n-1\} \to \mathfrak{L} : i \mapsto \begin{cases} F_i & \text{if } i \in \{0, \ldots, m-1\} \\ G_{i-m} & \text{if } i \in \{m, \ldots, m+n-1\} \end{cases}$$

List concatenation clearly is an associative operation on the data type $\mathfrak{Lists}$ of lists

$$((\Gamma, \Delta), \Pi) = (\Gamma, (\Delta, \Pi))$$

allowing us to write (Γ, Δ, Π), or, more generally, $(\Gamma_0, \ldots, \Gamma_k)$. On the other hand it is not commutative: not in general $(\Gamma, \Delta) = (\Delta, \Gamma)$.

Let $\leftrightarrow$ be the smallest equivalence relation on $\mathfrak{Lists}$ satisfying *cyclic permutation* (or *rotation*), which is nothing else than commutativity (on the outermost level):

$$(\Gamma, \Delta) \leftrightarrow (\Delta, \Gamma)$$

The equivalence classes will be called *cyclic lists*, and denoted by $(\!\![F_0, \ldots, F_{m-1}]\!\!)$. List concatenation does not translate into an operation on this data type $\mathfrak{CLists}$, as

$$\Gamma \leftrightarrow \Gamma' \quad \& \quad \Delta \leftrightarrow \Delta' \quad \Longrightarrow \quad (\Gamma, \Delta) \leftrightarrow (\Gamma', \Delta')$$

does not generally hold.

The smallest *congruence* relation satisfying commutativity, or equivalently, the smallest equivalence relation satisfying the expansions of commutativity

$$(\Pi, \Gamma, \Delta, \Sigma) \leftrightarrow (\Pi, \Delta, \Gamma, \Sigma),$$

clearly has multisets as equivalence classes. The corresponding canonical projection $\theta : (\mathfrak{Lists}, (-,-)) \to (\mathfrak{MSets}, - \uplus -)$ obviously is a homomorphism:

$$\theta(\Gamma, \Delta) = \theta\Gamma \uplus \theta\Delta.$$

All data types mentioned so far can be regarded as associative quotients on the data type $\mathfrak{Trees}$ of (rooted binary) *trees*

$$\mathfrak{Trees} ::= \mathfrak{L} \mid \mathfrak{Trees} \odot \mathfrak{Trees}.$$

The data type $\mathfrak{CTrees}$ of *cyclic trees* is an example of a not merely associative quotient, and will be described in Section 5.1.

The collections $\mathfrak{Lists}$ and $\mathfrak{Trees}$ are *strictly non-commutative* data types, by which we mean that every list (tree) has a uniquely determined order on the formulas it contains.

We use the word *setting* for an element of a general data type, and one data type consists of *similar* settings.

In the next diagram we have indicated the different data types as equivalence relations on $\mathfrak{Trees}$. Every arrow represents the canonical projection from the collection of representatives to the collection of equivalence classes. Regarding all collections as quotients on $\mathfrak{Trees}$, every arrow represents taking a quotient corresponding to the smallest equivalence relation satisfying the indicated clause in addition to the original clauses the domain already satisfies. The actual clauses are given by the following instances and expansions of instances of associativity and commutativity on $\mathfrak{Trees}$:

$$(\Gamma \odot \Delta) \odot \Pi \leftrightarrow \Gamma \odot (\Delta \odot \Pi) \tag{\textsc{Ass}}$$

$$\Xi[(\Gamma \odot \Delta) \odot \Pi] \leftrightarrow \Xi[\Gamma \odot (\Delta \odot \Pi)] \tag{\underline{\textsc{Ass}}}$$

$$\Gamma \odot \Delta \leftrightarrow \Delta \odot \Gamma \tag{\textsc{Comm}}$$

$$\Xi[\Gamma \odot \Delta] \leftrightarrow \Xi[\Delta \odot \Gamma] \tag{\underline{\textsc{Comm}}}$$

$$\begin{tikzcd}
 & & \overset{\neg\text{ass} \,\&\, \neg\text{comm} \ -\odot-}{\boxed{\mathfrak{Trees}}} \arrow[dll, Rightarrow, "\underline{\textsc{Ass}}"'] \arrow[d, Rrightarrow, "\textsc{Ass} \,\&\, \textsc{Comm}"] \\
\overset{\text{ass} \,\&\, \neg\text{comm} \ (-,-)}{\boxed{\mathfrak{Lists}}} \arrow[dr, Rrightarrow, "\textsc{Comm}"] \arrow[ddrr, Rightarrow, bend right, "\underline{\textsc{Comm}}", "\theta"'] & & \boxed{\mathfrak{CTrees}} \arrow[dl, Rightarrow, "\underline{\textsc{Ass}}"', "\theta"] \\
 & \boxed{\mathfrak{CLists}} \arrow[dr, Rightarrow, "\underline{\textsc{Comm}}", "\theta"'] & \\
 & & \overset{\text{ass} \,\&\, \text{comm} \ -\uplus-}{\boxed{\mathfrak{MSets}}}
\end{tikzcd}$$

This diagram will be completed in Section 5.1, to be precise in Example 5.1.5, where we will concentrate on the various inverse images of the subcollection of $\mathfrak{MSets}$ consisting of one single multiset $\{e_0, \ldots, e_{m-1}\}$ which is actually a set (i.e. all e_i distinct).

2.1.4. Polarized formulas. Given a language $\mathfrak{L}$ and a formula $F \in \mathfrak{L}$, to F we formally associate two polarized formulas[1], viz. F^+ and F^- (as they stand). We define $\mathfrak{L}^+ := \{F^+ \mid F \in \mathfrak{L}\}$ and $\mathfrak{L}^- := \{F^- \mid F \in \mathfrak{L}\}$. Given a finite setting (e.g. a (multi)set, (cyclic) list, (cyclic) tree) of formulas Γ, by Γ^+ we mean Γ, formula-wise provided with a positive sign. By Γ^- we mean Γ, formula-wise provided with a negative sign and *in reversed order*. I.e. for trees we formally define:

$$\langle\!| F |\!\rangle^+ := \left\langle\!\left| F^+ \right|\!\right\rangle \qquad \langle\!| F |\!\rangle^- := \left\langle\!\left| F^- \right|\!\right\rangle$$

$$\left(\Gamma \diamond \Delta\right)^+ := \Gamma^+ \diamond \Delta^+ \qquad \left(\Gamma \diamond \Delta\right)^- := \Delta^- \diamond \Gamma^-$$

where $\langle\!| F |\!\rangle$ denotes the singleton tree and $\diamond$ denotes tree construction; for the other settings, the maps are those which commute with the canonical projection: $(\theta\Gamma)^+ := \theta(\Gamma^+)$ and $(\theta\Gamma)^- := \theta(\Gamma^-)$, which are well-defined. For (multi)sets (lists, trees) Γ and

---

[1] One should not confuse this with extant notions of polarity of formulas, as e.g. in Girard's 'ludics' (see [**Girard 98**]). Moreover, observe that the sign does not, in any way, propagate through a formula: $(A \otimes B)^+$ is not the same as $A^+ \otimes B^+$; the latter expression as it stands is not even a polarized formula.

Δ, the expression $\Gamma \vdash \Delta$ is an alternative denotation for the setting[2] $\Delta^+ \diamond \Gamma^-$, where $\diamond$ denotes setting construction (defined for the non-cyclic data types).

For a language $\mathfrak{L}$ we define $\mathfrak{L}^\pm := \mathfrak{L}^+ \cup \mathfrak{L}^-$. We consider $\mathfrak{L}$ as a subset of $\mathfrak{L}^\pm$ by means of $\iota : \mathfrak{L} \hookrightarrow \mathfrak{L}^\pm : F \mapsto F^+$. The map which changes the polarity on $\mathfrak{L}^\pm$ is called *sign alternation* $\tau : \mathfrak{L}^\pm \to \mathfrak{L}^\pm : \begin{cases} F^+ \mapsto F^- \\ F^- \mapsto F^+ \end{cases}$.

Let $\mathfrak{L}$ and $\mathfrak{K}$ be two languages. Given a map $\xi : \mathfrak{L} \to \mathfrak{K}$, it induces sign-preserving maps $\mathfrak{L}^+ \to \mathfrak{K}^+$, $\mathfrak{L}^- \to \mathfrak{K}^-$ and $\mathfrak{L}^\pm \to \mathfrak{K}^\pm$, which we will also denote by ξ.

Given a map $\xi : \mathfrak{L} \to \mathfrak{K}^\pm : F \mapsto (F^\bullet)^{\overline{F}}$ assigning to F the formula $F^\bullet \in \mathfrak{K}$ polarized by $\overline{F} \in \{+,-\}$, we extend it to a map $\xi^\pm : \mathfrak{L}^\pm \to \mathfrak{K}^\pm$ by[3] $F^+ \mapsto (F^\bullet)^{+\overline{F}} = \xi F$ and $F^- \mapsto (F^\bullet)^{-\overline{F}} = \tau\xi F$. In this way negatively polarizing an original argument $F \in \mathfrak{L}$ corresponds to sign alternation of the result ξF. Observe that $\xi^\pm$ and τ commute.

Given a map $\xi : \mathfrak{L}^\pm \to \mathfrak{K} : F^\rho \mapsto \xi(F^\rho)$, we define two maps $\xi^+, \xi^- : \mathfrak{L}^\pm \to \mathfrak{K}^\pm$ by

$$\begin{aligned} \xi^+ &: \mathfrak{L}^\pm \to \mathfrak{K}^\pm : F^\rho \mapsto \big(\xi(F^\rho)\big)^+ \\ \xi^- &: \mathfrak{L}^\pm \to \mathfrak{K}^\pm : F^\rho \mapsto \big(\xi(F^{-\rho})\big)^- \end{aligned}$$

For any map $\xi : \mathfrak{L}^\star \to \mathfrak{K}^*$ there is a corresponding structure-preserving map ξ from a particular data type of $\mathfrak{L}^\star$-settings to the same data type of $\mathfrak{K}^*$-settings, where $\star, * = \text{\textvisiblespace}$, $+$, $-$ or $\pm$.

2.1.5. π, ν and ψ. Every formula A of $\mathfrak{L}_2$ may be seen as a generalized operation on $\mathfrak{L}_1$-formulas, together with its arguments. Indeed, all primitive connectives of $\mathfrak{L}_2$ are either connectives or operations of $\mathfrak{L}_1$. Let us define π as the evaluation of this expression, to be more precise:

$$\begin{aligned} \pi(\alpha_i) &:= \alpha_i \\ \pi((A)^\perp) &:= [\pi(A)]^\perp \\ \pi(A \,\Box\, B) &:= \pi(A) \,\Box\, \pi(B) \qquad (\Box = \otimes, \parr, \multimap \text{ or } \multimapinv) \end{aligned}$$

This map actually computes the 'De Morgan quotient' on $\mathfrak{L}_2$, in a sense to be made precise in the next subsection (see Proposition 2.2.3).

The other way around, every formula X of $\mathfrak{L}_1$ is just a formula of $\mathfrak{L}_2$ when we replace the formal negations of atoms $\alpha_i^\perp$ by actual negations of atoms $(\alpha_i)^\perp$. The resulting formula $\nu(X)$ is $\multimap$- and $\multimapinv$-free and contains no negations but negations of atoms (i.e. belongs to $\mathfrak{L}_{2,\mathrm{nf}}$), and may formally be defined by:

$$\begin{aligned} \nu(\alpha_i) &:= \alpha_i \\ \nu(\alpha_i^\perp) &:= (\alpha_i)^\perp \\ \nu(X \otimes Y) &:= \nu(X) \otimes \nu(Y) \\ \nu(X \parr Y) &:= \nu(X) \parr \nu(Y) \end{aligned}$$

Now the map

$$\mathfrak{L}_1 \xrightarrow{\ \nu\ } \mathfrak{L}_2 \xrightarrow{\ \pi\ } \mathfrak{L}_1$$

---

[2] This formula is to be understood such that $U \diamond \emptyset = U = \emptyset \diamond U$.

[3] Here we use the convention that $-- = ++ = +$ and $+- = -+ = +$, i.e. $(\{+,-\},\cdot) \cong (\mathbb{Z}_2, +_2)$.

is the identity on $\mathfrak{L}_1$, showing π is surjective and ν is injective. Restricting the domain of π and the codomain of ν to $\mathfrak{L}_{2,\mathrm{nf}}$, also the next composite

$$\begin{tikzcd}\mathfrak{L}_{2,\mathrm{nf}} \arrow[r, tail, two heads, "\pi'"] & \mathfrak{L}_1 \arrow[r, tail, two heads, "\nu'"] & \mathfrak{L}_{2,\mathrm{nf}}\end{tikzcd}$$

is easily shown to be the identity, whence $\mathfrak{L}_1 \cong \mathfrak{L}_{2,\mathrm{nf}}$.

The map ν, however, turns out to be inappropriate for our purposes. The main problem is that the negation operation cannot be mimicked in a satisfying way. We solve this by defining another translation $\psi : \mathfrak{L}_1 \to \mathfrak{L}_2^{\pm}$, where the $[-]^{\perp}$ operation of $\mathfrak{L}_1$ will correspond to the sign alternation τ in $\mathfrak{L}_2^{\pm}$.

Definition 2.1.1. *Let X be an $\mathfrak{L}_1$-formula. We define $\psi(X) := (X^{\bullet})^{\overline{X}} \in \mathfrak{L}_2^{\pm}$ as follows:*

$$\begin{aligned}
\alpha_i &\mapsto \alpha_i^{+} \\
\alpha_i^{\perp} &\mapsto \alpha_i^{-} \\
X \otimes Y &\mapsto \begin{cases} (X^{\bullet} \otimes Y^{\bullet})^{+} & \textit{if } \overline{X} = + \textit{ and } \overline{Y} = +, \\ (Y^{\bullet} \multimapinv X^{\bullet})^{-} & \textit{if } \overline{X} = + \textit{ and } \overline{Y} = -, \\ (Y^{\bullet} \multimap X^{\bullet})^{-} & \textit{if } \overline{X} = - \textit{ and } \overline{Y} = +, \\ (Y^{\bullet} \parr X^{\bullet})^{-} & \textit{if } \overline{X} = - \textit{ and } \overline{Y} = - \end{cases} \\
Y \parr X &\mapsto \begin{cases} (Y^{\bullet} \parr X^{\bullet})^{+} & \textit{if } \overline{X} = + \textit{ and } \overline{Y} = +, \\ (Y^{\bullet} \multimap X^{\bullet})^{+} & \textit{if } \overline{X} = + \textit{ and } \overline{Y} = -, \\ (Y^{\bullet} \multimapinv X^{\bullet})^{+} & \textit{if } \overline{X} = - \textit{ and } \overline{Y} = +, \\ (X^{\bullet} \otimes Y^{\bullet})^{-} & \textit{if } \overline{X} = - \textit{ and } \overline{Y} = - \end{cases}
\end{aligned}$$

◊

For convenience we will sometimes write

$$\begin{aligned} \psi(X \otimes Y) &= \psi(X) \otimes \psi(Y) \qquad \text{and} \\ \psi(Y \parr X) &= \psi(Y) \parr \psi(X) \end{aligned}$$

where $\otimes$ and $\parr$ in the right hand side are defined as maps[4] $\mathfrak{L}_2^{\pm} \times \mathfrak{L}_2^{\pm} \to \mathfrak{L}_2^{\pm}$ by

$$A^{\rho} \otimes B^{\sigma} := \begin{cases} (A \otimes B)^{+} & \text{if } \rho = + \text{ and } \sigma = + \\ (B \multimapinv A)^{-} & \text{if } \rho = + \text{ and } \sigma = - \\ (B \multimap A)^{-} & \text{if } \rho = - \text{ and } \sigma = + \\ (B \parr A)^{-} & \text{if } \rho = - \text{ and } \sigma = - \end{cases}$$

$$B^{\sigma} \parr A^{\rho} := \begin{cases} (B \parr A)^{+} & \text{if } \rho = + \text{ and } \sigma = + \\ (B \multimap A)^{+} & \text{if } \rho = + \text{ and } \sigma = - \\ (B \multimapinv A)^{+} & \text{if } \rho = - \text{ and } \sigma = + \\ (A \otimes B)^{-} & \text{if } \rho = - \text{ and } \sigma = - \end{cases}$$

From the definition it is immediately clear that $X^{\bullet}$ is $\perp$-free, and that the following boolean characterization may be used in order to compute the sign $\overline{X}$:

---

[4] The expression $\alpha^{+} \otimes \beta^{-}$ as it stands is not an element of $\mathfrak{L}_2^{\pm}$; by definition of the operation $\otimes : \mathfrak{L}_2^{\pm} \times \mathfrak{L}_2^{\pm} \to \mathfrak{L}_2^{\pm}$ it equals $(\beta \multimapinv \alpha)^{-}$, which expression is an element of $\mathfrak{L}_2^{\pm}$.

$\mathfrak{L}_1$	
connectives	
0-ary	α_i, $\alpha_i^\perp$
1-ary	
2-ary	$\otimes$, $\parr$
operations	
0-ary	
1-ary	$[-]^\perp$
2-ary	$\multimap$, $\multimapinv$

$$\xrightarrow{\psi}$$
$$X \quad \mapsto \quad (X^\bullet)^{\overline{X}}$$
$$\xleftarrow{\pi}$$
$$\begin{array}{rcl} \pi(A) & \leftarrowtail & A^+ \\ [\pi(A)]^\perp & \leftarrowtail & A^- \end{array}$$

$\mathfrak{L}_2^\pm$	
connectives	
0-ary	α_i
1-ary	$(-)^\perp$
2-ary	$\otimes$, $\parr$, $\multimap$, $\multimapinv$
operations	
0-ary	
1-ary	
2-ary	

FIGURE 2.1. Definitions of formulas for both $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

LEMMA 2.1.2.

$$\begin{aligned} \overline{X \otimes Y} = + \quad & \textit{iff} \quad \overline{X} = + \quad \textit{and} \quad \overline{Y} = +; \\ \overline{Y \parr X} = + \quad & \textit{iff} \quad \overline{X} = + \quad \textit{or} \quad \overline{Y} = +. \end{aligned}$$

$\Diamond$

For a map in the reverse direction, we extend the domain of $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$ to $\mathfrak{L}_2^\pm$ by composing it with

$$\mu : \mathfrak{L}_2^\pm \to \mathfrak{L}_2 : \begin{cases} A^+ \mapsto & A \\ A^- \mapsto & (A)^\perp \end{cases}$$

Let us denote this extended map also by $\pi : \mathfrak{L}_2^\pm \to \mathfrak{L}_1$, so that

$$\begin{aligned} &\pi(A^+) = \pi(A) \\ &\pi(A^-) = \pi((A)^\perp) = [\pi A]^\perp \end{aligned}$$

and let us call χ the other composite $\mu\psi$.

$$\begin{tikzcd} \mathfrak{L}_1 \arrow[d, tail, "\psi"'] \arrow[dr, tail, dashed, "\chi"] & \\ \mathfrak{L}_2^\pm \arrow[r, "\mu"'] \arrow[dr, dashed, "\pi"'] & \mathfrak{L}_2 \arrow[d, "\pi"] \\ & \mathfrak{L}_1 \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{L}_{2,\perp\text{-free}}^\pm \arrow[r, tail, two heads, "\pi'"] & \mathfrak{L}_1 \arrow[r, tail, two heads, "\psi'"] & \mathfrak{L}_{2,\perp\text{-free}}^\pm \end{tikzcd}$$

In the next two lemmas we will see that both of these diagrams are identities, where in the second diagram the domain of π and the codomain of ψ have been restricted to $\mathfrak{L}_{2,\perp\text{-free}}^\pm$. This yields $\mathfrak{L}_1 \cong \mathfrak{L}_{2,\perp\text{-free}}^\pm$.

LEMMA 2.1.3. *For every $\mathfrak{L}_1$-formula X the following holds:*

$$\pi(X^\bullet) = \pi((X^\bullet)^+) = [\pi((X^\bullet)^-)]^\perp \tag{1}$$

$$\pi(X^\bullet) = \begin{cases} \pi\psi X & \text{if } \overline{X} = +, \\ [\pi\psi X]^\perp & \text{if } \overline{X} = - \end{cases} \tag{2}$$

$$\pi\psi X = X \tag{3}$$

$$\pi\chi X = X \tag{3'}$$

$$\psi([X]^\perp) = (X^\bullet)^{-\overline{X}} \tag{4}$$

$$([X]^\perp)^\bullet = X^\bullet \tag{4a}$$

$$\overline{[X]^\perp} = -\overline{X} \tag{4b}$$

◊

PROOF: For statement (1), observe that $\pi((X^\bullet)^+)$ equals by definition $\pi(X^\bullet)$. Also by definition $\pi((X^\bullet)^-) = [\pi(X^\bullet)]^\perp$, whence $\pi(X^\bullet) = [[\pi(X^\bullet)]^\perp]^\perp = [\pi((X^\bullet)^-)]^\perp$

Proof of (2): Directly from (1), by writing $\psi(X) = (X^\bullet)^{\overline{X}}$.

Proof of (3): By induction on $Z \in \mathfrak{L}_1$. For (3'), observe that (3) yields $X = \pi\psi X = (\pi\mu)\psi X = \pi(\mu\psi)X = \pi\chi X$, for the map χ defined on page 23.

As $\psi([X]^\perp) = (([X]^\perp)^\bullet)^{\overline{[X]^\perp}} = (X^\bullet)^{-\overline{X}}$ iff $([X]^\perp)^\bullet = X^\bullet$ and $\overline{[X]^\perp} = -\overline{X}$, it is clear that (4) is equivalent to "(4a) and (4b)". The proof of (4) is also by an easy induction on Z. ////

LEMMA 2.1.4. *For every $\mathfrak{L}_{2,\perp\text{-}free}$-formula A and $\rho = +, -$ the following holds:*

$$\psi\pi A = A^+ \tag{1}$$

$$(\pi A)^\bullet = A \tag{1a}$$

$$\overline{\pi A} = + \tag{1b}$$

$$\chi\pi A = A \tag{1'}$$

$$\psi\pi(A^\rho) = A^\rho \tag{2}$$

$$(\pi(A^\rho))^\bullet = A \tag{2a}$$

$$\overline{\pi(A^\rho)} = \rho \tag{2b}$$

$$\chi\pi(A^\rho) = \mu A^\rho \tag{2'}$$

◊

PROOF: As $\psi(\pi A) = ((\pi A)^\bullet)^{\overline{\pi A}}$, (1) is equivalent to "(1a) and (1b)". The proof of (1) is by induction on $C \in \mathfrak{L}_{2,\perp\text{-free}}$.

From (1) we obtain $\chi\pi A = \mu\psi\pi A = \mu(A^+) = A$, proving (1'). Moreover, (1) proves the "$\rho = +$"-case of (2) as well. For $\rho = -$:

$$\psi(\pi(A^-)) = \psi([\pi A]^\perp) \overset{\text{Lemma 2.1.3(4)}}{=} ((\pi A)^\bullet)^{-\overline{\pi A}} \overset{(1a)\&(1b)}{=} A^-.$$

Now (2) is equivalent to "(2a) and (2b)", while (2') follows after composition with μ. ////

The importance of this map ψ is the fact that it yields a partition of the $\mathfrak{L}_1$-formulas into two parts: the *even* X with $\overline{X} = +$ and the *odd* X with $\overline{X} = -$. As X and $[X]^\perp$ are of opposite *parity*, the involutive operation $[-]^\perp$ on $\mathfrak{L}_1$ yields a bijection $\mathfrak{L}_{1,\text{even}} \cong \mathfrak{L}_{1,\text{odd}}$, and the next diagram – where all arrows are bijections — commutes:

$$\begin{tikzcd}
\mathfrak{L}_{1,\text{even}} \arrow[r, shift left, tail, two heads, "\psi"] \arrow[d, tail, "{[-]^\perp}"'] & \mathfrak{L}^+_{2,\perp\text{-free}} \arrow[l, shift left, tail, two heads, "\pi"] \arrow[d, tail, "\tau"] \\
\mathfrak{L}_{1,\text{odd}} \arrow[r, shift left, tail, two heads, "\psi"] & \mathfrak{L}^-_{2,\perp\text{-free}} \arrow[l, shift left, tail, two heads, "\pi"]
\end{tikzcd}$$

In the next sections we will use ψ to embed one-sided classical link graphs, proof structures, sequents and derivations into their two-sided counterparts. Let us confine ourselves at this point to remarking what happens to the axiom rule of $\mathbf{MLL}_1$. The formulas X and $[X]^\perp$ in an $\mathbf{MLL}_1$-axiom

$$\frac{}{\vdash X, X^\perp}\,\mathrm{Ax}$$

play a completely symmetric role; indeed, one can also consider $X = [[X]^\perp]^\perp$ as the negated formula. This would leave us a choice for the $\mathfrak{L}_2$-formula in the corresponding $\mathbf{MLL}_2$-axiom

$$\frac{}{A \vdash A}\,\mathrm{Ax}$$

in case we would use ν, viz. either $A = \nu(X)$ or $A = \nu([X]^\perp)$. However, $(-)^\bullet$ maps both X and $[X]^\perp$ to the same $\mathfrak{L}_2$-formula A, which is hence preferable.

Example 2.1.5. Take $X = \alpha^\perp \otimes \beta$ in the above situation, then

$$\frac{}{\vdash \alpha^\perp \otimes \beta, \beta^\perp \parr \alpha}\,\mathrm{Ax}$$

translates to

$$\frac{}{\beta \multimap \alpha \vdash \beta \multimap \alpha}\,\mathrm{Ax}$$

instead of

$$\frac{}{(\alpha)^\perp \otimes \beta \vdash (\alpha)^\perp \otimes \beta}\,\mathrm{Ax} \qquad \text{or} \qquad \frac{}{(\beta)^\perp \parr \alpha \vdash (\beta)^\perp \parr \alpha}\,\mathrm{Ax}$$

◊

Example 2.1.6. An atom α_i is even, while its formal negation $\alpha_i^\perp$ is odd. Hence by Lemma 2.1.2 the $\mathfrak{L}_1$-formulas in the upper part of this box are even, and the others are odd.

$\alpha \otimes \beta$	$\beta \parr \alpha$
$\alpha \otimes \beta^\perp$	$\beta^\perp \parr \alpha$
$\alpha^\perp \otimes \beta$	$\beta \parr \alpha^\perp$
$\alpha^\perp \otimes \beta^\perp$	$\beta^\perp \parr \alpha^\perp$

These formulas may also be expressed — modulo outermost $[-]^\perp$ — in atoms by means of $\otimes$ and $\parr$ and the defined operations $\multimap$ and $\multimapinv$, which gives respectively

$\alpha \otimes \beta$	$\beta \parr \alpha$
$[\beta \multimapinv \alpha]^\perp$	$\beta \multimap \alpha$
$[\beta \multimap \alpha]^\perp$	$\beta \multimapinv \alpha$
$[\beta \parr \alpha]^\perp$	$[\alpha \otimes \beta]^\perp$

which explains the words 'even' and 'odd'. In Lemma 2.1.3 we saw that we can always express an $\mathfrak{L}_1$-formula X in this way:

$$X = \pi\psi X = \begin{cases} \pi(X^\bullet) & \text{if } \overline{X} = +; \\ [\pi(X^\bullet)]^\perp & \text{if } \overline{X} = -. \end{cases}$$

In Lemma 2.1.4 we saw that this way of expressing $\mathfrak{L}_1$-formulas is even unique: suppose X is expressible as πA or $[\pi A]^\perp$ where $A \in \mathfrak{L}_{2,\perp\text{-free}}$, then $X = \pi(A^\rho)$, whence $\psi X = \psi\pi(A^\rho) \overset{\text{Lemma 2.1.4}}{=} A^\rho$, which means that $A = X^\bullet$ and $\rho = \overline{X}$ are uniquely determined. ◊

2.1.6. Counting connectives and atoms. Let $\#_\otimes(-)$ and $\#_⅋(-)$ be the functions on $\mathfrak{L}_1$, assigning to a formula X the number of $\otimes$-symbols (⅋-symbols respectively) occurring in X. So formally:

$$\begin{aligned} \#_\otimes(\alpha_i) &:= 0 \\ \#_\otimes(\alpha_i^\perp) &:= 0 \\ \#_\otimes(X \otimes Y) &:= \#_\otimes(X) + 1 + \#_\otimes(Y) \\ \#_\otimes(X ⅋ Y) &:= \#_\otimes(X) + \#_\otimes(Y); \end{aligned}$$

for $\#_⅋(-)$ an analogue definition applies. Instantaneously we see that for all $X, Y \in \mathfrak{L}_1$ we have the equalities:

$$\begin{aligned} \#_\otimes([X]^\perp) &= \#_⅋(X); \\ \#_⅋([X]^\perp) &= \#_\otimes(X); \\ \#_\otimes(X \multimap Y) = \#_\otimes(Y \circ\!\!- X) &= \#_⅋(X) + \#_\otimes(Y); \\ \#_⅋(X \multimap Y) = \#_⅋(Y \circ\!\!- X) &= \#_\otimes(X) + 1 + \#_⅋(Y). \end{aligned}$$

We extend these functions to polarized formulas by

$$\begin{aligned} \#_\otimes(X^+) &:= \#_\otimes(X) & \qquad \#_\otimes(X^-) &:= \#_⅋(X) \\ \#_⅋(X^+) &:= \#_⅋(X) & \qquad \#_⅋(X^-) &:= \#_\otimes(X), \end{aligned}$$

such that $\#_\otimes(X^\rho) = \#_⅋(X^{-\rho})$. Let us finally extend these functions additively to settings of polarized $\mathfrak{L}_1$-formulas Γ, as in:

$$\#_\Box(\Gamma) := \sum_{X^\rho \in \Gamma} \#_\Box(X^\rho) \qquad (\Box = \otimes, ⅋).$$

Composing these maps $\#_\Box(-) : \mathfrak{L}_1 \to \mathbb{Z}$ with $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$, we get similar maps on $\mathfrak{L}_2$. The same holds for the extensions: composing $\#_\Box(-)$ with the sign preserving induced map $\pi : \mathfrak{L}_2^\pm \to \mathfrak{L}_1^\pm$ (the sign- and structure-preserving induced map π from the $\mathfrak{L}_2^\pm$-settings to the $\mathfrak{L}_1^\pm$-settings), yields similar maps $\#_\Box(-)$ on $\mathfrak{L}_2^\pm$ (the settings of polarized $\mathfrak{L}_2$-formulas).

Lemma 2.1.7. *For $\mathfrak{L}_2$-formulas A and B the following holds, where $\otimes$ and ⅋ are the maps $\mathfrak{L}_2^\pm \times \mathfrak{L}_2^\pm \to \mathfrak{L}_2^\pm$ as defined in Subsection 2.1.5:*

$$\begin{aligned} \#_\otimes(A^\rho \otimes B^\sigma) &:= \#_\otimes(A^\rho) + 1 + \#_\otimes(B^\sigma) \\ \#_\otimes(B^\sigma ⅋ A^\rho) &:= \#_\otimes(B^\sigma) + \#_\otimes(A^\rho); \end{aligned}$$

for $\#_⅋(-)$ an analogue result holds. ◊

Another useful map is the algebraic interpretation of a formula. Let $\langle \mathcal{A} \rangle$ be the free group generated by the atoms, the unit of which we will denote by Λ. We define $\llbracket - \rrbracket$ as the function from $\mathfrak{L}_1$ to $\langle \mathcal{A} \rangle$ by:

$$\begin{aligned} \llbracket \alpha_i \rrbracket &:= \alpha_i \\ \left\llbracket \alpha_i^{\perp} \right\rrbracket &:= \alpha_i^{-1} \\ \llbracket X \mathbin{\Box} Y \rrbracket &:= \llbracket X \rrbracket \cdot \llbracket Y \rrbracket \qquad (\Box = \otimes, \parr). \end{aligned}$$

Instantaneously we see that for all $X, Y \in \mathfrak{L}_1$ we have the equalities:

$$\begin{aligned} \left\llbracket [X]^{\perp} \right\rrbracket &= \llbracket X \rrbracket^{-1} ; \\ \llbracket X \multimap Y \rrbracket &= \llbracket X \rrbracket^{-1} \cdot \llbracket Y \rrbracket ; \\ \llbracket Y \multimapinv X \rrbracket &= \llbracket Y \rrbracket \cdot \llbracket X \rrbracket^{-1} . \end{aligned}$$

Extend $\llbracket - \rrbracket$ to a map on $\mathfrak{L}_1^{\pm}$ by

$$\llbracket X^{\rho} \rrbracket := \llbracket X \rrbracket^{\rho} \qquad (\rho = +1, -1)$$

and finally to *strictly non-commutative* settings (i.e. lists or trees) of polarized $\mathfrak{L}_1$-formulas Γ, as in:

$$\left\llbracket X_0^{\rho_0}, \ldots, X_{m-1}^{\rho_{m-1}} \right\rrbracket := \prod_{j=0}^{m-1} \left\llbracket X_j^{\rho_j} \right\rrbracket .$$

For $\mathfrak{L}_2$ similar maps may be defined, by composition with the appropriate extensions of $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$. We then find that

$$\llbracket A^{\rho} \otimes B^{\sigma} \rrbracket = \llbracket A^{\rho} \parr B^{\sigma} \rrbracket = \llbracket A \rrbracket^{\rho} \cdot \llbracket B \rrbracket^{\sigma} .$$

Only counting the positive occurrences of atoms (and neglecting the order in which they occur), yields a map $\llangle - \rrangle$ from $\mathfrak{L}_1$ to the free *abelian* group generated by the atoms $\mathbb{Z}^{\mathcal{A}}$, obtained as the composition $\langle \mathcal{A} \rangle \to \mathbb{Z}^{\mathcal{A}}$ after $\llbracket - \rrbracket$. A direct definition is given by:

$$\begin{aligned} \llangle \alpha_i \rrangle &:= \alpha_i \\ \left\llangle \alpha_i^{\perp} \right\rrangle &:= -\alpha_i \\ \llangle X \mathbin{\Box} Y \rrangle &:= \llangle X \rrangle + \llangle Y \rrangle \qquad (\Box = \otimes, \parr). \end{aligned}$$

Instantaneously we see that for all $X, Y \in \mathfrak{L}_1$ we have the equalities:

$$\begin{aligned} \left\llangle [X]^{\perp} \right\rrangle &= - \llangle X \rrangle ; \\ \llangle X \multimap Y \rrangle = \llangle Y \multimapinv X \rrangle &= \llangle Y \rrangle - \llangle X \rrangle . \end{aligned}$$

Extend $\llangle - \rrangle$ to a map on $\mathfrak{L}_1^{\pm}$ by

$$\llangle X^{\rho} \rrangle := \rho \llangle X \rrangle ,$$

and finally additively to settings of polarized $\mathfrak{L}_1$-formulas Γ, as in:

$$\llangle \Gamma \rrangle := \sum_{X^{\rho} \in \Gamma} \llangle X^{\rho} \rrangle .$$

The last expression is well-defined for multisets and cyclic lists (trees), since the order does not matter. For $\mathfrak{L}_2$ similar maps may be defined, by composition with the appropriate extensions of $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$. We then find that

$$\llangle A^{\rho} \otimes B^{\sigma} \rrangle = \llangle B^{\sigma} \parr A^{\rho} \rrangle = \rho \llangle A \rrangle + \sigma \llangle B \rrangle .$$

Let us finish this subsection by defining the so-called *count of negations* (cf. [**Pentus 93**]) of an $\mathfrak{L}_1$-formula X by

$$\natural(X) := \frac{1 + \#_{\otimes}(X) - \#_{\parr}(X) - \epsilon(\llparenthesis X \rrparenthesis)}{2},$$

where ϵ is the augmentation map $\mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}$ which maps $\sum c_i \alpha_i$ to $\sum c_i$. By the observation that

$$\begin{aligned} \natural(\alpha_i) &= 0 \\ \natural(\alpha_i^{\perp}) &= 1 \\ \natural(X \otimes Y) &= \natural(X) + \natural(Y) \\ \natural(X \parr Y) &= \natural(X) + \natural(Y) - 1 \end{aligned}$$

we see immediately that $\natural(X)$ is an integer. Furthermore,

$$\begin{aligned} \natural([X]^{\perp}) &= 1 - \natural(X); \\ \natural(X \multimap Y) = \natural(Y \multimapinv X) &= \natural(Y) - \natural(X). \end{aligned}$$

Knowing the extensions of $\#_{\Box}(-)$ and $\llparenthesis - \rrparenthesis$ to polarized $\mathfrak{L}_1$-formulas and settings of them, we can extend the defining equation $\natural(X) := \frac{1+\#_{\otimes}(X)-\#_{\parr}(X)-\epsilon(\llparenthesis X \rrparenthesis)}{2}$ accordingly, yielding:

$$\begin{aligned} \natural(X^{+}) &= \natural(X) \\ \natural(X^{-}) &= 1 - \natural(X) \\ \natural(X^{\rho}) &= \frac{1}{2} + \rho(\natural(X) - \frac{1}{2}) = \rho\natural(X) + \frac{1-\rho}{2} \qquad (\rho = +1, -1) \\ \natural(\Gamma) &= \frac{1 - |\Gamma|}{2} + \sum_{X^{\rho} \in \Gamma} \natural(X^{\rho}) \\ \natural(\Gamma \diamond \Delta) &= \natural(\Gamma) + \natural(\Delta) - \frac{1}{2}. \end{aligned}$$

Observe that $\natural(-)$ is not additive for settings, and moreover is no longer integer if $|\Gamma|$ is even.

Knowing that $\natural(X^{\rho})$ is an integer yields

$$1 + \#_{\otimes}(X^{\rho}) \equiv \#_{\parr}(X^{\rho}) + \epsilon(\llparenthesis X^{\rho} \rrparenthesis) \pmod{2}$$

and taking the sum over Γ yields

$$|\Gamma| + \#_{\otimes}(\Gamma) \equiv \#_{\parr}(\Gamma) + \epsilon(\llparenthesis \Gamma \rrparenthesis) \pmod{2}$$

Lemma 2.1.8. *For any setting Γ, the following holds:*

$$|\Gamma| + \#_{\otimes}(\Gamma) \equiv \#_{\parr}(\Gamma) + \epsilon(\llparenthesis \Gamma \rrparenthesis) \pmod{2}$$

$\Diamond$

Example 2.1.9. Suppose $\Gamma = ((C \multimapinv B)^+, A^+, (C \parr (B \multimap A))^-)$. Then

$$\begin{aligned}
\#_\otimes(\Gamma) &= \#_\otimes(C \multimapinv B) && + \#_\otimes(A) + \#_\parr(C \parr (B \multimap A)) \\
&= \#_\otimes(C) + \#_\parr(B) && + \#_\otimes(A) + \#_\parr(C) + 1 + \#_\otimes(B) + 1 + \#_\parr(A) = j + 2 \\
\#_\parr(\Gamma) &= \#_\parr(C \multimapinv B) && + \#_\parr(A) + \#_\otimes(C \parr (B \multimap A)) \\
&= \#_\parr(C) + 1 + \#_\otimes(B) + \#_\parr(A) + \#_\otimes(C) + \#_\parr(B) + \#_\otimes(A) && = j + 1
\end{aligned}$$

where $j = \#_\otimes(A, B, C) + \#_\parr(A, B, C)$, while

$$\begin{aligned}
[\![\Gamma]\!] &= [\![C \multimapinv B]\!] \quad \cdot [\![A]\!] \cdot [\![C \parr (B \multimap A)]\!]^{-1} \\
&= [\![C \multimapinv B]\!] \quad \cdot [\![A]\!] \cdot ([\![C]\!] \cdot [\![B]\!]^{-1} \cdot [\![A]\!])^{-1} \\
&= [\![C]\!] \cdot [\![B]\!]^{-1} \cdot [\![A]\!] \cdot [\![A]\!]^{-1} \cdot [\![B]\!] \cdot [\![C]\!]^{-1} \quad = \Lambda
\end{aligned}$$

◊

For later use, we also define at this place the *length* $\mathsf{l}(A)$ of an $\mathfrak{L}_2$-formula $A \in \mathfrak{L}_2$ by

$$\begin{aligned}
\mathsf{l}(\alpha_i) &:= 1 \\
\mathsf{l}((A)^\perp) &:= \mathsf{l}(A) + 1 \\
\mathsf{l}(A \mathbin{\square} B) &:= \mathsf{l}(A) + 1 + \mathsf{l}(B) \qquad (\square = \otimes, \parr, \multimap \text{ or } \multimapinv)
\end{aligned}$$

We extend it to polarized formulas by $\mathsf{l}(A^\rho) = \mathsf{l}(A)$, and finally additively to settings of polarized $\mathfrak{L}_2$-formulas Γ, as in:

$$\mathsf{l}(\Gamma) := \sum_{A^\rho \in \Gamma} \mathsf{l}(A^\rho).$$

2.2. De Morgan equivalence

2.2.1. Two-sided language.

Definition 2.2.1. *Let* $\equiv$ *(* “De Morgan equivalence” *) be the smallest equivalence relation on* $\mathfrak{L}_2$*-formulas satisfying:*

$$\begin{aligned}
A \equiv A', \quad B \equiv B' \quad &\Longrightarrow \quad A \mathbin{\square} B \equiv A' \mathbin{\square} B' \qquad (\square = \otimes, \multimapinv, \multimap \textit{ or } \parr) && (0\square) \\
A \equiv A' \quad &\Longrightarrow \quad (A)^\perp \equiv (A')^\perp && (0\perp) \\
&(A \otimes B)^\perp \equiv (B)^\perp \parr (A)^\perp && (1) \\
&(A \parr B)^\perp \equiv (B)^\perp \otimes (A)^\perp && (2) \\
&A \multimap B \equiv (A)^\perp \parr B && (3a) \\
&B \multimapinv A \equiv B \parr (A)^\perp && (3b) \\
&((A)^\perp)^\perp \equiv A && (4)
\end{aligned}$$

◊

We will often use the notion of a derivation tree in order to apply induction on $\equiv$. In this format, equivalences are derived by the following inference rules:

equivalence rules[5]

$$\frac{}{A \equiv A}\,\textsc{Refl} \qquad \frac{A \equiv B}{B \equiv A}\,\textsc{Symm} \qquad \frac{A \equiv B \qquad B \equiv C}{A \equiv C}\,\textsc{Trans}$$

congruence rules

$$\frac{A \equiv A' \qquad B \equiv B'}{A \mathbin{\Box} B \equiv A' \mathbin{\Box} B'}\,(0\Box) \qquad \frac{A \equiv A'}{(A)^\perp \equiv (A')^\perp}\,(0\perp)$$

De Morgan axioms

$$\frac{}{(A \otimes B)^\perp \equiv (B)^\perp \parr (A)^\perp}\,(1) \qquad \frac{}{(A \parr B)^\perp \equiv (B)^\perp \otimes (A)^\perp}\,(2)$$

$$\frac{}{A \multimap B \equiv (A)^\perp \parr B}\,(3a) \qquad \frac{}{B \multimapinv A \equiv B \parr (A)^\perp}\,(3b)$$

$$\frac{}{((A)^\perp)^\perp \equiv A}\,(4)$$

This set of rules is actually not the most efficient one. For example, let us show $(0\parr)$ is redundant, as it is derivable from others. First of all

$$\frac{\dfrac{\dfrac{}{((A \parr B)^\perp)^\perp \equiv A \parr B}\,(4)}{A \parr B \equiv ((A \parr B)^\perp)^\perp}\,\textsc{Symm} \qquad \dfrac{\dfrac{}{(A \parr B)^\perp \equiv (B)^\perp \otimes (A)^\perp}\,(2)}{((A \parr B)^\perp)^\perp \equiv ((B)^\perp \otimes (A)^\perp)^\perp}\,(0\perp)}{A \parr B \equiv ((B)^\perp \otimes (A)^\perp)^\perp}\,\textsc{Trans}$$

whence

$$\frac{\dfrac{\overline{\overline{A \parr B \equiv ((B)^\perp \otimes (A)^\perp)^\perp}} \qquad \dfrac{\dfrac{\dfrac{A \equiv A'}{(A)^\perp \equiv (A')^\perp}\,(0\perp) \qquad \dfrac{B \equiv B'}{(B)^\perp \equiv (B')^\perp}\,(0\perp)}{(B)^\perp \otimes (A)^\perp \equiv (B')^\perp \otimes (A')^\perp}\,(0\otimes)}{((B)^\perp \otimes (A)^\perp)^\perp \equiv ((B')^\perp \otimes (A')^\perp)^\perp}\,(0\perp)}{A \parr B \equiv ((B')^\perp \otimes (A')^\perp)^\perp}\,\textsc{Trans} \qquad \dfrac{\overline{\overline{A' \parr B' \equiv ((B')^\perp \otimes (A')^\perp)^\perp}}}{((B')^\perp \otimes (A')^\perp)^\perp \equiv A' \parr B'}\,\textsc{Symm}}{A \parr B \equiv A' \parr B'}\,\textsc{Trans}$$

Similarly, we can show $(0\multimap)$ and $(0\multimapinv)$ to be superfluous, given the other rules.

PROPOSITION 2.2.2. *For any two $\mathfrak{L}_2$-formulas A and B, if $A \equiv B$, then we can derive it by the rules* REFL, SYMM, TRANS, *$(0\otimes)$, $(0\perp)$ and the axioms (1), (2), (3a), (3b) and (4).* ◊

---

[5] Instead of SYMM and TRANS we can use the following equivalent rule $\dfrac{B \equiv A \qquad B \equiv C}{A \equiv C}$.

2.2.2. De Morgan quotient on two-sided language.

The following proposition characterizes the De Morgan equivalence of $\mathfrak{L}_2$-formulas.

Proposition 2.2.3. *The relation $\equiv$ on $\mathfrak{L}_2$-formulas is the kernel of the map π, i.e. for all $\mathfrak{L}_2$-formulas A and B the following holds:*

$$A \equiv B \qquad \textit{if and only if} \qquad \pi A = \pi B$$

$$\Diamond$$

Proof: $\boxed{\Longrightarrow}$ Consider $\sim \,:= \{(A, B) \mid \pi A = \pi B\}$. This is an equivalence relation satisfying:

- (0) (since π commutes with $\Box$ and $(-)^\perp$);
- (1) and (2) (by definition of $[-]^\perp$);
- (3a) and (3b) (by definition of $\multimap$ and $\multimapinv$);
- (4) (by the fact that $[-]^\perp$ is an involution).

As $\equiv$ is the smallest such equivalence relation, we must have that $\equiv\; \subseteq\; \sim$, i.e. if $A \equiv B$ then $\pi A = \pi B$.

$\boxed{\Longleftarrow}$ We prove this by induction on the total length of the formulas A and B, distinguishing the 11 cases $A = \alpha_i$; $A = A_1 \Box A_2$; $A = (\alpha_i)^\perp$; $A = (A_2 \Box A_1)^\perp$ and $A = ((A_1)^\perp)^\perp$. Suppose $\pi A = \pi B$.

A		πA		
		atomic	$X_1 \otimes X_2$	$X_1 \parr X_2$
α_i		α_i		
$A_1 \otimes A_2$			$\pi A_1 \otimes \pi A_2$	
$A_1 \parr A_2$				$\pi A_1 \parr \pi A_2$
$A_1 \multimap A_2$				$[\pi A_1]^\perp \parr \pi A_2$
$A_1 \multimapinv A_2$				$\pi A_1 \parr [\pi A_2]^\perp$
$(A')^\perp$:	$(\alpha_i)^\perp$	$\alpha_i^\perp$		
	$(A_2 \otimes A_1)^\perp$			$[\pi A_1]^\perp \parr [\pi A_2]^\perp$
	$(A_2 \parr A_1)^\perp$		$[\pi A_1]^\perp \otimes [\pi A_2]^\perp$	
	$(A_2 \multimap A_1)^\perp$		$[\pi A_1]^\perp \otimes \pi A_2$	
	$(A_2 \multimapinv A_1)^\perp$		$\pi A_1 \otimes [\pi A_2]^\perp$	
	$((A_1)^\perp)^\perp$	πA_1		

If $A = ((A_1)^\perp)^\perp$, then

$$\begin{aligned} \pi A_1 &= [[\pi A_1]^\perp]^\perp \qquad \text{(since } [-]^\perp \text{ is an involution)} \\ &= \pi((A_1)^\perp)^\perp \\ &= \pi A = \pi B \end{aligned}$$

whence

$$\begin{aligned} A = ((A_1)^\perp)^\perp &\equiv A_1 && \text{(by (4))} \\ &\equiv B && \text{(by induction hypothesis)} \end{aligned}$$

The case $B = ((B_1)^\perp)^\perp$ is treated similarly.

Now suppose A and B are of the other ten forms.

- If $\pi A = \alpha_i$, then $A = \alpha_i = B$ whence $A \equiv B$.
- If $\pi A = \alpha_i^\perp$, then $A = (\alpha_i)^\perp = B$ whence $A \equiv B$.
- If $\pi A = X_1 \otimes X_2$ for some X_1 and X_2, then $A = A_1 \otimes A_2$; $A = (A_2 \parr A_1)^\perp$; $A = (A_2 \multimap A_1)^\perp$ or $A = (A_2 \multimapinv A_1)^\perp$, while B is of one of the similar forms, yielding 16 subcases.
  E.g. in the subcase that $A = A_1 \otimes A_2$ and $B = (B_2 \multimapinv B_1)^\perp$, then from $\pi A = \pi B$ it follows that $\pi A_1 \otimes \pi A_2 = \pi((B_2 \multimapinv B_1)^\perp) = \pi B_1 \otimes [\pi B_2]^\perp$, whence $\pi A_1 = \pi B_1$ and $\pi A_2 = [\pi B_2]^\perp = \pi(B_2)^\perp$. This yields by induction hypothesis $A_1 \equiv B_1$ and $A_2 \equiv (B_2)^\perp$, whence

$$\begin{aligned} A = A_1 \otimes A_2 &\equiv B_1 \otimes (B_2)^\perp && \text{(by (0}\otimes\text{))} \\ &\equiv ((B_1)^\perp)^\perp \otimes (B_2)^\perp && \text{(by (4) and (0}\otimes\text{))} \\ &\equiv (B_2 \parr (B_1)^\perp)^\perp && \text{(by (2))} \\ &\equiv (B_2 \multimapinv B_1)^\perp = B && \text{(by (3b) and (0}\perp\text{))} \end{aligned}$$

  The other subcases are proved analogously.
- If $\pi A = X_1 \parr X_2$ for some X_1 and X_2, then $A = A_1 \parr A_2$; $A = A_1 \multimap A_2$; $A = A_1 \multimapinv A_2$, or $A = (A_2 \otimes A_1)^\perp$, while B is of one of the similar forms, yielding again 16 subcases that are also proved in a straightforward way.

////

From

$$\pi\nu = \mathrm{id}_{\mathfrak{L}_1} = \pi\chi$$

we see that $\pi A = \pi B$ is equivalent to $\nu\pi A = \nu\pi B$ and also to $\chi\pi A = \chi\pi B$ (which are conditions of $\mathfrak{L}_2$-formulas), whence we can also formulate Proposition 2.2.3 as

$$\begin{aligned} A \equiv B \quad & \text{if and only if} \quad \nu\pi A = \nu\pi B \\ & \text{if and only if} \quad \chi\pi A = \chi\pi B \end{aligned}$$

We call $\nu\pi A$ the *De Morgan normal form* of A and $\chi\pi A$ the *$\perp$-free normal form* of A (although $\chi\pi A$ may have an outermost $(-)^\perp$). As $\pi(\nu\pi A) = (\pi\nu)\pi A = \pi A$, we see that $\nu\pi A \equiv A$. Similar, $\chi\pi A \equiv A$.

COROLLARY 2.2.4. $\mathfrak{L}_1 \cong \mathfrak{L}_2/\!\equiv$ ◊

PROOF: We only use the result of Proposition 2.2.3 and the fact that π is epi. Let us denote the congruence class of $A \in \mathfrak{L}_2$ by $[A] \in \mathfrak{L}_2/\!\equiv$. As π is epi, for every $X \in \mathfrak{L}_1$ there is a $\xi X \in \mathfrak{L}_2$ such that $\pi\xi X = X$. (E.g. one can take $\xi = \nu$ or $\xi = \chi$.) Let us

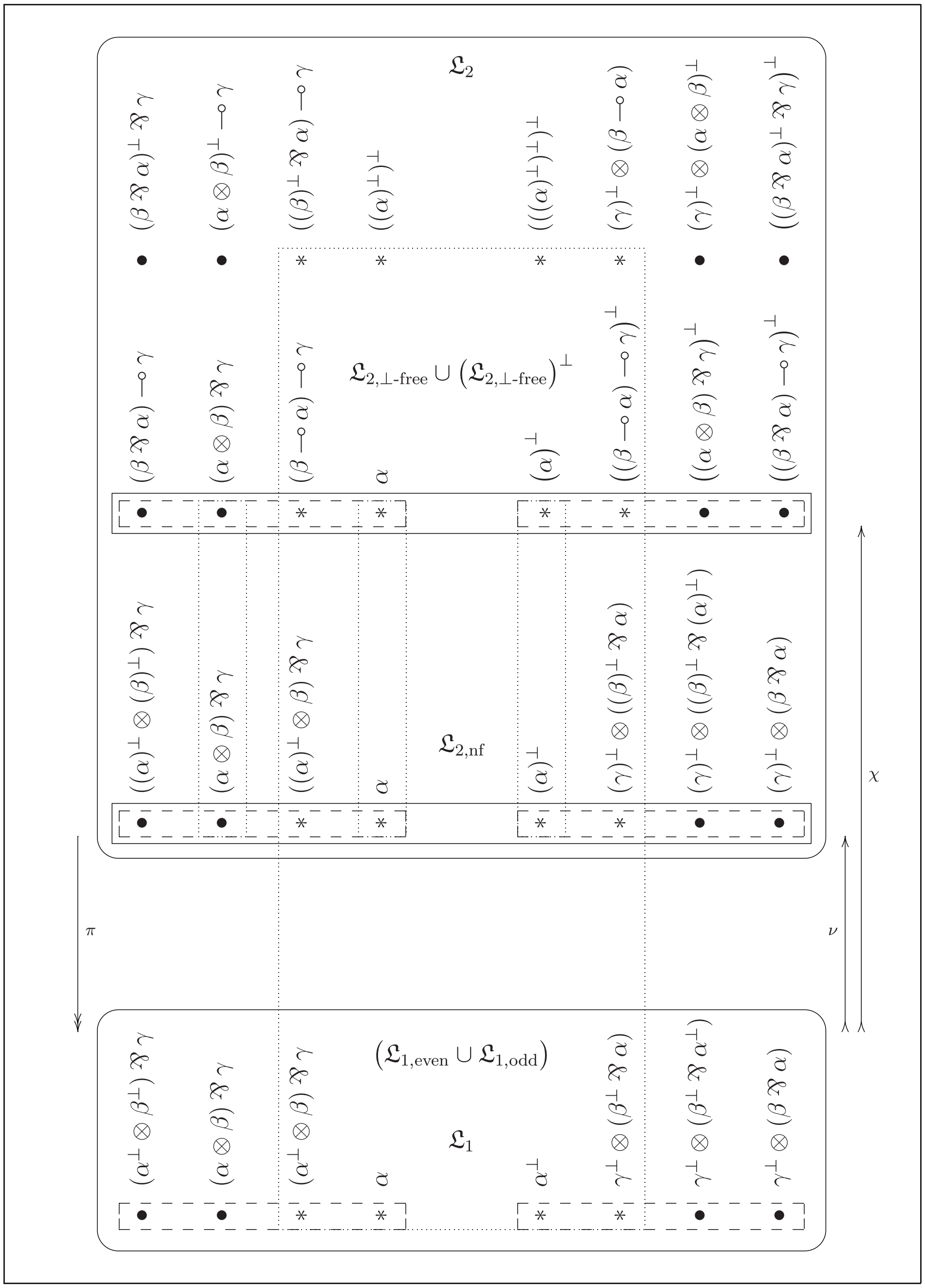


FIGURE 2.2. The projection $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$, and the canonical representatives νX and χX of $\pi^{-1}X$ for some $X \in \mathfrak{L}_1$.

consider the following maps:

$$\begin{aligned} \mathfrak{L}_1 &\leftrightarrows \mathfrak{L}_2/{\equiv} \\ X &\mapsto [\xi X] \\ \pi A &\hookleftarrow [A] \end{aligned}$$

The last map is well-defined, because $[A] = [B]$ implies $\pi A = \pi B$. The one composite reads

$$X \mapsto [\xi X] \mapsto \pi\xi X = X$$

while the other one reads

$$[A] \mapsto \pi A \mapsto [\xi\pi A] = [A]$$

(since $\pi(\xi\pi A) = (\pi\xi)\pi A = \pi A$, implying $\xi\pi A \equiv A$). This proves the bijective correspondence between the formulas of $\mathfrak{L}_1$ and the objects of $\mathfrak{L}_2/{\equiv}$. ////

At first sight dividing $\mathfrak{L}_2$ by $\equiv$ destroys the inductive construction of the objects; the connectives have become operations:

$$\begin{aligned} \widetilde{\alpha_i} &:= [\alpha_i] \\ \big(\,[A]\,\big)^{\perp} &:= [(A)^{\perp}] \\ [A] \,\Box\, [B] &:= [A \,\Box\, B] \qquad (\Box = \otimes, \parr, \multimap \text{ or } \multimapinv) \end{aligned}$$

which are well-defined by the requirements $(0\perp)$ and $(0\Box)$. The importance of this corollary is the fact that this quotient still has an inductive construction of the objects, viz. the same as $\mathfrak{L}_1$.

Combining the isomorphisms $\mathfrak{L}_1 \cong \mathfrak{L}_2/{\equiv}$ and $\mathfrak{L}_1 \cong \mathfrak{L}^{\pm}_{2,\perp\text{-free}}$ yields canonical representatives for each equivalence class: Given $A \in \mathfrak{L}_2$, we know πA is uniquely expressible as either πB or $\pi(B)^{\perp}$, where $B \in \mathfrak{L}_{2,\perp\text{-free}}$ (cf. Example 2.1.6). So A is equivalent to a unique member of $\mathfrak{L}_{2,\perp\text{-free}} \cup \big(\mathfrak{L}_{2,\perp\text{-free}}\big)^{\perp}$.

Lemma 2.2.5. *Let $A \in \mathfrak{L}_2$ and $B \in \mathfrak{L}_{2,\perp\text{-free}} \cup \big(\mathfrak{L}_{2,\perp\text{-free}}\big)^{\perp}$ such that $A \equiv B$. Then $B = \chi\pi A$.* $\Diamond$

Proof: We repeat the argument of Example 2.1.6. Write $B = \mu(C^{\rho})$ where $C \in \mathfrak{L}_{2,\perp\text{-free}}$. From $A \equiv B$ we get $\chi\pi A = \chi\pi B = \chi\pi\mu(C^{\rho}) = \chi\pi(C^{\rho}) = \mu\psi\pi(C^{\rho}) \overset{\text{Lemma 2.1.4}}{=} \mu(C^{\rho}) = B$. ////

Of course, we can also decide to take the formulas of $\mathfrak{L}_{2,\mathrm{nf}}$ as canonical representatives. We have sketched the situation in Figure 2.2.

2.2.3. Intuitionistic language. As we know $\equiv$ for $\mathfrak{L}_2$, we also know the restriction to $\mathfrak{L}_{2,\mathrm{i}} := \mathcal{F} ::= \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid \mathcal{F} \multimapinv \mathcal{F}$, the $\perp$- and $\parr$-free formulas. This is again an equivalence relation, but we cannot define it as the smallest equivalence relation satisfying the clauses of Definition 2.2.1, as these requirements are outside the language $\mathfrak{L}_{2,\mathrm{i}}$.

First we will show that $\equiv$ trivializes when restricted to $\mathfrak{L}_{2,\mathrm{i}}$.

Proposition 2.2.6. *For $\mathfrak{L}_{2,i}$-formulas A and B the following holds:*

$$A \equiv B \qquad \textit{if and only if} \qquad A = B$$

$\Diamond$

Proof: By Lemma 2.2.5 every $\mathfrak{L}_2$ formula C is $\equiv$-equivalent to exactly one formula $D \in \mathfrak{L}_{2,\perp\text{-free}} \cup \left(\mathfrak{L}_{2,\perp\text{-free}}\right)^{\perp}$ (viz. $\chi\pi C$). So every $\mathfrak{L}_2$ formula C is $\equiv$-equivalent to at most one intuitionistic formula $D \in \mathfrak{L}_{2,\mathrm{i}}$. In particular, an intuitionistic A is only $\equiv$-equivalent to itself.

Alternative proof: Suppose the $\mathfrak{L}_{2,\mathrm{i}}$-formulas A and B satisfy $A \equiv B$, then

$$A \overset{\text{Lemma 2.1.4(1')}}{=} \chi\pi A = \chi\pi B \overset{\text{Lemma 2.1.4(1')}}{=} B.$$

////

Let us extend the notion of "intuitionistic" objects to $\mathfrak{L}_2/{\equiv}$ by calling an equivalence class $[A]$ intuitionistic whenever $A \equiv B$ for some $B \in \mathfrak{L}_{2,\mathrm{i}}$. By the previous proposition, this gives a bijective correspondence between the intuitionistic $[A] \in \mathfrak{L}_2/{\equiv}$ and the intuitionistic $B \in \mathfrak{L}_{2,\mathrm{i}}$, viz.

$$\begin{aligned}[A] &\mapsto \chi\pi A\\ [B] &\mapsfrom B\end{aligned}$$

By the isomorphism $\left(\mathfrak{L}_2/{\equiv}\right) \cong \mathfrak{L}_1$ (defined by $[A] \mapsto \pi A$; see Corollary 2.2.4) the set of intuitionistic $[A] \in \mathfrak{L}_2/{\equiv}$ corresponds to a subset of $\mathfrak{L}_1$, which we will hence call *intuitionistic* $\mathfrak{L}_1$-formulas. These are just the π-images of $\mathfrak{L}_{2,\mathrm{i}}$-formulas. Obviously, since $\mathfrak{L}_{2,\mathrm{i}} \subseteq \mathfrak{L}_{2,\perp\text{-free}}$, the bijections in

$$\begin{tikzcd}
\mathfrak{L}_{1,\mathrm{even}} \arrow[r, shift left, tail, two heads, "\psi"] \arrow[d, tail, two heads, "{[-]^{\perp}}"'] & \mathfrak{L}^{+}_{2,\perp\text{-free}} \arrow[l, shift left, tail, two heads, "\pi"] \arrow[d, tail, two heads, "\tau"] \\
\mathfrak{L}_{1,\mathrm{odd}} \arrow[r, shift left, tail, two heads, "\psi"] & \mathfrak{L}^{-}_{2,\perp\text{-free}} \arrow[l, shift left, tail, two heads, "\pi"]
\end{tikzcd}$$

show that $\pi(\mathfrak{L}_{2,\mathrm{i}}) \subseteq \mathfrak{L}_{1,\mathrm{even}}$.

Let us define the following subsets of $\mathfrak{L}_1$ by simultaneous induction:

$$\begin{aligned}\mathfrak{L}_{1,\mathrm{even,i}} &:= \mathcal{F} ::= \ \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{G} \parr \mathcal{F} \mid \mathcal{F} \parr \mathcal{G}\\ \mathfrak{L}_{1,\mathrm{odd,i}} &:= \mathcal{G} ::= \mathcal{A}^{\perp} \mid \mathcal{G} \parr \mathcal{G} \ \mid \mathcal{G} \otimes \mathcal{F} \mid \mathcal{F} \otimes \mathcal{G}\end{aligned}$$

Lemma 2.2.7. *Every* $X \in \mathfrak{L}_{1,even,i}$ *satisfies* $\psi X \in \mathfrak{L}^{+}_{2,i}$. *Every* $X \in \mathfrak{L}_{1,odd,i}$ *satisfies* $\psi X \in \mathfrak{L}^{-}_{2,i}$. $\Diamond$

Proof: By simultaneous induction, using Definition 2.1.1. Let us show one case: Suppose we are given $F \in \mathcal{F} = \mathfrak{L}_{1,\mathrm{even,i}}$ and $G \in \mathcal{G} = \mathfrak{L}_{1,\mathrm{odd,i}}$, satisfying by induction hypothesis that $F^{\bullet}, G^{\bullet} \in \mathfrak{L}_{2,\mathrm{i}}$ and moreover that $\overline{F} = +$ and $\overline{G} = -$. Then $\psi(F \otimes G) = (G^{\bullet} \multimapinv F^{\bullet})^{-} \in \mathfrak{L}^{-}_{2,\mathrm{i}}$, as desired. ////

This lemma shows in particular that $\mathfrak{L}_{1,\mathrm{even,i}} \subseteq \mathfrak{L}_{1,\mathrm{even}}$ and $\mathfrak{L}_{1,\mathrm{odd,i}} \subseteq \mathfrak{L}_{1,\mathrm{odd}}$.

Lemma 2.2.8. *For all* $D \in \mathfrak{L}_{2,i}$ *the following holds:* $\pi(D^{+}) \in \mathfrak{L}_{1,even,i}$ *and* $\pi(D^{-}) \in \mathfrak{L}_{1,odd,i}$. $\Diamond$

Proof: By induction on D. Let us show one case. Given $D = A_1 \multimapinv A_2$, we have $\pi(D^{+}) = \pi D = \pi A_1 \multimapinv \pi A_2 = \pi A_1 \parr [\pi A_2]^{\perp} = \pi(A_1^{+}) \parr \pi(A_2^{-}) \in \mathcal{F} \parr \mathcal{G} \subseteq \mathcal{F}$ while $\pi(D^{-}) = [\pi D]^{\perp} = [\pi A_1 \multimapinv \pi A_2]^{\perp} = [\pi A_1 \parr [\pi A_2]^{\perp}]^{\perp} = \pi A_2 \otimes [\pi A_1]^{\perp} = \pi(A_2^{+}) \otimes \pi(A_1^{-}) \in \mathcal{F} \otimes \mathcal{G} \subseteq \mathcal{G}$. For the remaining cases, the following table may be useful.

D	$\pi(D^+)$	$\pi(D^-)$
α	α	$\alpha^\perp$
$A_1 \otimes A_2$	$\pi A_1 \otimes \pi A_2$	$[\pi A_2]^\perp \parr [\pi A_1]^\perp$
$A_1 \multimap A_2$	$[\pi A_1]^\perp \parr \pi A_2$	$[\pi A_2]^\perp \otimes \pi A_1$
$A_1 \multimapinv A_2$	$\pi A_1 \parr [\pi A_2]^\perp$	$\pi A_2 \otimes [\pi A_1]^\perp$

////

Using the fact that π and ψ are bijections, these two lemma's immediately yield the following proposition and corollary.

Proposition 2.2.9. *$\pi\big(\mathfrak{L}^+_{2,i}\big) = \mathfrak{L}_{1,even,i}$ and $\pi\big(\mathfrak{L}^-_{2,i}\big) = \mathfrak{L}_{1,odd,i}$* ◊

Corollary 2.2.10. *For $D \in \mathfrak{L}_2$ the following holds:*

$$\begin{array}{lcl} \psi\pi D \in \mathfrak{L}^+_{2,i} & \textit{if and only if} & \pi D \in \mathfrak{L}_{1,even,i} \\ \psi\pi D \in \mathfrak{L}^-_{2,i} & \textit{if and only if} & \pi D \in \mathfrak{L}_{1,odd,i} \end{array}$$

◊

$$\begin{tikzcd}
\mathfrak{L}_{1,\mathrm{even,i}} \arrow[r, shift left, tail, two heads, "\psi"] \arrow[d, tail, two heads, "{[-]^\perp}"'] & \mathfrak{L}^+_{2,\mathrm{i}} \arrow[l, shift left, tail, two heads, "\pi"] \arrow[d, tail, two heads, "\tau"] \\
\mathfrak{L}_{1,\mathrm{odd,i}} \arrow[r, shift left, tail, two heads, "\psi"] & \mathfrak{L}^-_{2,\mathrm{i}} \arrow[l, shift left, tail, two heads, "\pi"]
\end{tikzcd}$$

Observe that the negation of a formula of the form $\mathcal{F} \parr \mathcal{G}$ is of the form $\mathcal{F} \otimes \mathcal{G}$ (and not $\mathcal{G} \otimes \mathcal{F}$). Indeed, negation not only changes the order of the atomic subformulas, but also their parity (even/odd):

$$[\pi(A_1^+) \parr \pi(A_2^-)]^\perp = [\pi(A_2^-)]^\perp \otimes [\pi(A_1^+)]^\perp = \pi(A_2^+) \otimes \pi(A_1^-)$$

In Figure 2.2 we have indicated the intuitionistic $\mathfrak{L}_1$-formulas and the members of intuitionistic $[A] \in \mathfrak{L}_2/\equiv$ by $*$ instead of $\bullet$.

By an easy simultaneous induction we deduce that

Lemma 2.2.11. *If $X \in \mathfrak{L}_{1,even,i}$ then $\natural X = 0$. If $X \in \mathfrak{L}_{1,odd,i}$ then $\natural X = 1$.* ◊

The converse, however, does not hold. E.g. $X := (\alpha^\perp \otimes \beta^\perp) \parr (\alpha \parr \beta)$ has $\natural(X) = \frac{1+1-2-0}{2} = 0$, but $X \notin \mathfrak{L}_{1,\mathrm{even,i}}$.

2.3. Adding associativity

2.3.1. Two-sided language. In the previous section we have proved that the quotient of $\mathfrak{L}_2$ under De Morgan equivalence equals $\mathfrak{L}_1$. In this thesis we will not consider *languages* obtained by dividing out bigger equivalence relations. Observe, though, that many authors do, e.g. writing $A \otimes B \otimes C$. However, in **MLL** associativity (and commutativity) are derivable (e.g. we have proofs of $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$ and $(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)$, which is associativity of $\otimes$). In this section we study the equivalence relation $\simeq$ generated by associativity (in addition to the clauses of Definition 2.2.1).

Definition 2.3.1. *Let $\simeq$ be the smallest equivalence relation on $\mathfrak{L}_2$-formulas satisfying:*

$$
\begin{array}{rcll}
A \simeq A', \quad B \simeq B' & \Longrightarrow & A \mathbin{\Box} B \simeq A' \mathbin{\Box} B' \qquad (\Box = \otimes, \multimapinv, \multimap \text{ or } \parr) & (0\Box)\\
A \simeq A' & \Longrightarrow & (A)^\perp \simeq (A')^\perp & (0\perp)\\
 & & (A \otimes B)^\perp \simeq (B)^\perp \parr (A)^\perp & (1)\\
 & & (A \parr B)^\perp \simeq (B)^\perp \otimes (A)^\perp & (2)\\
 & & A \multimap B \simeq (A)^\perp \parr B & (3a)\\
 & & B \multimapinv A \simeq B \parr (A)^\perp & (3b)\\
 & & ((A)^\perp)^\perp \simeq A & (4)\\
 & & A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C & (5\otimes)\\
 & & A \parr (B \parr C) \simeq (A \parr B) \parr C & (5\parr)
\end{array}
$$

◊

Again, we can show some rules to be superfluous, given the other rules.

Proposition 2.3.2. *For any two $\mathfrak{L}_2$-formulas, if $A \simeq B$, then we can derive it by the rules* Refl, Symm, Trans, *(0⊗), (0⊥) and the axioms (1), (2), (3a), (3b), (4) and (5⊗).* ◊

The decision problem whether two $\mathfrak{L}_2$-formulas are $\simeq$-equivalent will be answered in Section 4.6, where we will establish a geometrical equivalent for $\simeq$. At this point we only mention the well-known law of general associativity.

Lemma 2.3.3. *Let A and B be two $\otimes$-only ($\parr$-only) $\mathfrak{L}_2$-formulas. If A and B have the same sequence of atoms, then $A \simeq B$.*

◊

Proof: The proof is by induction on the number n of atoms. ////

The converse of this lemma also holds, and is a consequence of:

Lemma 2.3.4. *For $\mathfrak{L}_2$-formulas A and B the following holds: if $A \simeq B$, then $[\![A]\!] = [\![B]\!]$ and $\natural(A) = \natural(B)$.*

◊

2.3.2. Intuitionistic language. We have seen that $\equiv$ trivializes when restricted to $\mathfrak{L}_{2,\mathrm{i}}$. The following example illustrates that this does not hold for $\simeq$.

Example 2.3.5. By Definition 2.3.1 we have the following chain of $\simeq$-equivalent $\mathfrak{L}_2$-formulas:

$$
\begin{aligned}
A \multimap (B \multimap C) &\simeq (A)^\perp \parr (B \multimap C)\\
&\simeq (A)^\perp \parr ((B)^\perp \parr C)\\
&\simeq ((A)^\perp \parr (B)^\perp) \parr C\\
&\simeq (B \otimes A)^\perp \parr C\\
&\simeq (B \otimes A) \multimap C
\end{aligned}
$$

whence the two intuitionistic formulas $A \multimap (B \multimap C)$ and $(B \otimes A) \multimap C$ are $\simeq$-equivalent. ◊

Definition 2.3.6. *Let $\simeq_{\mathrm{i}}$ be the smallest equivalence relation on $\mathfrak{L}_{2,i}$-formulas satisfying:*

$$A \simeq_{\mathrm{i}} A', \quad B \simeq_{\mathrm{i}} B' \quad \Longrightarrow \quad A \mathbin{\square} B \simeq_{\mathrm{i}} A' \mathbin{\square} B' \qquad (\square = \otimes, \multimapinv \text{ or } \multimap) \tag{0$\square$}$$

$$A \otimes (B \otimes C) \simeq_{\mathrm{i}} (A \otimes B) \otimes C \tag{5$\otimes$}$$

$$A \multimap (B \multimap C) \simeq_{\mathrm{i}} (B \otimes A) \multimap C \tag{5$\multimap$}$$

$$(A \multimapinv B) \multimapinv C \simeq_{\mathrm{i}} A \multimapinv (C \otimes B) \tag{5$\multimapinv$}$$

$$A \multimap (B \multimapinv C) \simeq_{\mathrm{i}} (A \multimap B) \multimapinv C \tag{5$\multimap\multimapinv$}$$

◊

$$\begin{tikzcd}
\textcircled{$=$} \arrow[r, equal] \arrow[d, hook] & \textcircled{$=$} \arrow[r, hook] \arrow[d, hook] & \textcircled{$\simeq_{\mathrm{i}}$} \arrow[r, hook] \arrow[d, hook] & \mathfrak{L}_{2,\mathrm{i}} \times \mathfrak{L}_{2,\mathrm{i}} \arrow[d, hook] \\
\textcircled{$=$} \arrow[r, hook] & \textcircled{$\equiv$} \arrow[r, hook] & \textcircled{$\simeq$} \arrow[r, hook] & \mathfrak{L}_2 \times \mathfrak{L}_2
\end{tikzcd}$$

We want to prove that $\simeq_{\mathrm{i}}$ is the restriction of $\simeq$ to $\mathfrak{L}_{2,\mathrm{i}}$. Below we will give an elementary proof of this fact (Proposition 2.3.8), fow which we will need the next lemma. An alternative proof, using the theory of dualizable proof nets, is given in Theorem 4.9.6.

Lemma 2.3.7. *Let $D, E \in \mathfrak{L}_2$. Suppose $D \simeq E$, and $\psi\pi D \in \mathfrak{L}^{\rho}_{2,i}$ or $\psi\pi E \in \mathfrak{L}^{\rho}_{2,i}$. Then both $\psi\pi D \in \mathfrak{L}^{\rho}_{2,i}$ and $\psi\pi E \in \mathfrak{L}^{\rho}_{2,i}$, and $(\pi D)^{\bullet} \simeq_{\mathrm{i}} (\pi E)^{\bullet}$.* ◊

Proof: The proof is by induction on the derivation tree of $D \simeq E$, which we may assume to consist of Refl, Symm, Trans, (0$\otimes$), (0$\perp$) and the axioms (1), (2), (3a), (3b), (4) and (5$\otimes$) only (see Proposition 2.3.2).

Let $D \simeq E$ be given such that $\psi\pi F \in \mathfrak{L}^{+(-)}_{2,\mathrm{i}}$ for $F = D$ or E, then — by Corollary 2.2.10 — $\pi F \in \mathfrak{L}_{1,\mathrm{even,i}}$ ($\mathfrak{L}_{1,\mathrm{odd,i}}$). Let us write $\mathcal{F}$ ($\mathcal{G}$) for $\mathfrak{L}_{1,\mathrm{even,i}}$ ($\mathfrak{L}_{1,\mathrm{odd,i}}$) in the sequel.

We only prove the case that the last inference is

$$\frac{}{A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C}\;(5\otimes)$$

Let us assume we know $\pi D = \pi A \otimes (\pi B \otimes \pi C) \in \mathcal{F}$ ($\mathcal{G}$). According to the definition of $\mathcal{F}$ and $\mathcal{G}$ we can distinguish four subcases:

- $\pi D = \pi A \otimes (\pi B \otimes \pi C) \in \mathcal{F} \otimes (\mathcal{F} \otimes \mathcal{F}) \subseteq \mathcal{F}$:
  Then also $\pi E = (\pi A \otimes \pi B) \otimes \pi C \in (\mathcal{F} \otimes \mathcal{F}) \otimes \mathcal{F} \subseteq \mathcal{F}$ and
  $$(\pi D)^{\bullet} = (\pi A)^{\bullet} \otimes \big((\pi B)^{\bullet} \otimes (\pi C)^{\bullet}\big) \simeq_{\mathrm{i}} \big((\pi A)^{\bullet} \otimes (\pi B)^{\bullet}\big) \otimes (\pi C)^{\bullet} = (\pi E)^{\bullet}$$
  by axiom (5$\otimes$) for $\simeq_{\mathrm{i}}$.
- $\pi D = \pi A \otimes (\pi B \otimes \pi C) \in \mathcal{F} \otimes (\mathcal{F} \otimes \mathcal{G}) \subseteq \mathcal{G}$:
  Then also $\pi E = (\pi A \otimes \pi B) \otimes \pi C \in (\mathcal{F} \otimes \mathcal{F}) \otimes \mathcal{G} \subseteq \mathcal{G}$ and
  $$(\pi D)^{\bullet} = \big((\pi C)^{\bullet} \multimapinv (\pi B)^{\bullet}\big) \multimapinv (\pi A)^{\bullet} \simeq_{\mathrm{i}} (\pi C)^{\bullet} \multimapinv \big((\pi A)^{\bullet} \otimes (\pi B)^{\bullet}\big) = (\pi E)^{\bullet}$$
  by axiom (5$\multimapinv$) for $\simeq_{\mathrm{i}}$.
- $\pi D = \pi A \otimes (\pi B \otimes \pi C) \in \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{F}) \subseteq \mathcal{G}$:
  Then also $\pi E = (\pi A \otimes \pi B) \otimes \pi C \in (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{F} \subseteq \mathcal{G}$ and
  $$(\pi D)^{\bullet} = \big((\pi C)^{\bullet} \multimap (\pi B)^{\bullet}\big) \multimapinv (\pi A)^{\bullet} \simeq_{\mathrm{i}} (\pi C)^{\bullet} \multimap \big((\pi B)^{\bullet} \multimapinv (\pi A)^{\bullet}\big) = (\pi E)^{\bullet}$$

by axiom $(5{\multimap}{\multimapinv})$ (and Symm) for $\simeq_{\mathrm{i}}$.

- $\pi D = \pi A \otimes (\pi B \otimes \pi C) \in \mathcal{G} \otimes (\mathcal{F} \otimes \mathcal{F}) \subseteq \mathcal{G}$:
  Then also $\pi E = (\pi A \otimes \pi B) \otimes \pi C \in (\mathcal{G} \otimes \mathcal{F}) \otimes \mathcal{F} \subseteq \mathcal{G}$ and

$$(\pi D)^{\bullet} = \big((\pi B)^{\bullet} \otimes (\pi C)^{\bullet}\big) \multimap (\pi A)^{\bullet} \simeq_{\mathrm{i}} (\pi C)^{\bullet} \multimap \big((\pi B)^{\bullet} \multimap (\pi A)^{\bullet}\big) = (\pi E)^{\bullet}$$

  by axiom $(5\multimap)$ (and Symm) for $\simeq_{\mathrm{i}}$.

////

Proposition 2.3.8. *Let $D, E \in \mathfrak{L}_{2,i}$. Then*

$$D \simeq E \qquad \text{if and only if} \qquad D \simeq_{\mathrm{i}} E$$

$\Diamond$

Proof: The axioms of a $\simeq_{\mathrm{i}}$-derivation are derivable in $\simeq$ (cf. Example 2.3.5 for the $(5\multimap)$-clause). The same holds for the rules $(0\Box)$, which are even the same for $\simeq$. Hence $D \simeq_{\mathrm{i}} E$ implies $D \simeq E$.

The other way around, suppose $D \simeq E$. As $D, E \in \mathfrak{L}_{2,\perp\text{-free}}$, Lemma 2.1.4 yields $D = (\pi D)^{\bullet}$, $E = (\pi E)^{\bullet}$ and $\psi\pi D = D^{+} \in \mathfrak{L}^{+}_{2,\mathrm{i}}$. Hence by Lemma 2.3.7

$$D = (\pi D)^{\bullet} \simeq_{\mathrm{i}} (\pi E)^{\bullet} = E$$

////

As said, an alternative proof, using the theory of dualizable proof nets, is given in Theorem 4.9.6.

We have not used our lemma at full strength here, so we can ask whether we could have formulated it in a more efficient way. However, if — in Lemma 2.3.7 — we would have required that "both $\psi\pi D \in \mathfrak{L}^{\rho}_{2,\mathrm{i}}$ and $\psi\pi E \in \mathfrak{L}^{\rho}_{2,\mathrm{i}}$", the Trans-step would have failed. If we would have required that only $\psi\pi D \in \mathfrak{L}^{\rho}_{2,\mathrm{i}}$, the Symm-step would have failed. Finally, the conclusion that "both $\psi\pi D \in \mathfrak{L}^{\rho}_{2,\mathrm{i}}$ and $\psi\pi E \in \mathfrak{L}^{\rho}_{2,\mathrm{i}}$" was needed in order to be able to make use of the induction hypothesis, in particular in the Trans-step and in the $(0\otimes)$-step, where we obtained that $(\pi D)^{\bullet}$ and $(\pi E)^{\bullet}$ have coinciding main connective $\otimes$, $\multimap$ or $\multimapinv$, allowing us to apply $(0\Box)$ for $\simeq_{\mathrm{i}}$.

Let us, for completeness, mention a somewhat more general consequence of Lemma 2.3.7. Recall that

$$\mu : \mathfrak{L}_2^{\pm} \to \mathfrak{L}_2 : \begin{cases} A^{+} \mapsto & A \\ A^{-} \mapsto & (A)^{\perp} \end{cases}$$

Proposition 2.3.9. *Let $D \in \mathfrak{L}_{2,i}$ and $E \in \mathfrak{L}_{2,\perp\text{-free}}$. Let $\rho, \sigma \in \{+,-\}$. Then*

$$\mu(D^{\rho}) \simeq \mu(E^{\sigma}) \qquad \text{if and only if} \qquad E \in \mathfrak{L}_{2,i} \text{ and } \rho = \sigma \text{ and } D \simeq_{\mathrm{i}} E$$

$\Diamond$

Proof: $D \simeq_{\mathrm{i}} E$ implies $D \simeq E$, whence also $\mu(D^{\rho}) \simeq \mu(E^{\sigma})$ by a possible application of $(0\perp)$ for $\simeq$ in case $\rho = \sigma = -$.

The other way around, first of all we know by Lemma 2.1.4 that

$$\begin{aligned} D &= (\pi D)^{\bullet} = (\pi\mu(D^{\rho}))^{\bullet}, \\ E &= (\pi E)^{\bullet} = (\pi\mu(E^{\sigma}))^{\bullet}, \\ \psi\pi\mu(D^{\rho}) &= \psi\pi(D^{\rho}) = D^{\rho} \in \mathfrak{L}^{\rho}_{2,\mathrm{i}} \text{ and} \\ \psi\pi\mu(E^{\sigma}) &= \psi\pi(E^{\sigma}) = E^{\sigma}. \end{aligned}$$

Now suppose $\mu(D^\rho) \simeq \mu(E^\sigma)$. Then applying Lemma 2.3.7 on $\mu(D^\rho), \mu(E^\sigma) \in \mathfrak{L}_2$ gives

$$E^\sigma = \psi\pi\mu(E^\sigma) \in \mathfrak{L}^{\rho}_{2,\mathrm{i}} \text{ and}$$
$$D = (\pi\mu(D^\rho))^\bullet \simeq_{\mathrm{i}} (\pi\mu(E^\sigma))^\bullet = E$$

from which we deduce $E \in \mathfrak{L}_{2,\mathrm{i}}$ and $\rho = \sigma$ and $D \simeq_{\mathrm{i}} E$. ////

The next lemma will be the intuitionistic counterpart of Lemma 2.3.3. For $\mathfrak{L}_2$-formulas A and B we define

$$A \overset{+}{\to} B := A \multimap B$$
$$A \overset{-}{\to} B := B \multimapinv A$$

Lemma 2.3.10. *Let*

$$A = A_n \overset{\rho_n}{\to} (A_{n-1} \overset{\rho_{n-1}}{\to} (\dots (A_1 \overset{\rho_1}{\to} \alpha)\dots)) \textit{ and}$$
$$B = B_m \overset{\sigma_m}{\to} (B_{m-1} \overset{\sigma_{m-1}}{\to} (\dots (B_1 \overset{\sigma_1}{\to} \alpha)\dots))$$

be two $\mathfrak{L}_{2,i}$-formulas where all A_i and B_j are $\multimap$- and $\multimapinv$-free (i.e. they are $\otimes$-only). Let $A_{k_1}, \dots, A_{k_{n'}}$ be the subsequence of $A_1, \dots, A_{n-1}, A_n$ consisting of the A_i for which ρ_i is positive, and let $A_{k'_1}, \dots, A_{k'_{n''}}$ be the subsequence of $A_1, \dots, A_{n-1}, A_n$ consisting of the A_i for which ρ_i is negative. Let $B_{l_1}, \dots, B_{l_{m'}}$ be the subsequence of $B_1, \dots, B_{m-1}, B_m$ consisting of the B_j for which σ_j is positive, and let $B_{l'_1}, \dots, B_{l'_{m''}}$ be the subsequence of $B_1, \dots, B_{m-1}, B_m$ consisting of the B_j for which σ_j is negative. Suppose the sequence of atoms of the n' formulas $A_{k_1}, \dots, A_{k_{n'}}$ together equals the sequence of atoms of $B_{l_1}, \dots, B_{l_{m'}}$ together. Moreover, suppose the sequence of atoms of the n'' formulas $A_{k'_{n''}}, \dots, A_{k'_1}$ together equals the sequence of atoms of $B_{l'_{m''}}, \dots, B_{l'_1}$ together. Then $A \simeq_{\mathrm{i}} B$. $\Diamond$

Proof: By induction on n we can first show that

$$\begin{aligned} &A_n \overset{\rho_n}{\to} (\dots (A_1 \overset{\rho_1}{\to} \alpha)\dots) \\ \simeq_{\mathrm{i}} &\big(((\dots (A_{k_1} \otimes A_{k_2})\dots) \otimes A_{k_{n'}}) \multimap \alpha\big) \multimapinv (A_{k'_{n''}} \otimes (\dots (A_{k'_2} \otimes A_{k'_1})\dots)) \end{aligned}$$

where the right hand side is to be understood as

$$\big(\alpha\big) \multimapinv (A_{k'_{n''}} \otimes (\dots (A_{k'_2} \otimes A_{k'_1})\dots))$$

if $n' = 0$ and $n'' \neq 0$; as

$$\big(((\dots (A_{k_1} \otimes A_{k_2})\dots) \otimes A_{k_{n'}}) \multimap \alpha\big)$$

if $n' \neq 0$ and $n'' = 0$; as α if $n = 0$.

Now let A and B be given as described in the lemma, then

$$A \simeq_{\mathrm{i}} \big(((\dots (A_{k_1} \otimes A_{k_2})\dots) \otimes A_{k_{n'}}) \multimap \alpha\big) \multimapinv (A_{k'_{n''}} \otimes (\dots (A_{k'_2} \otimes A_{k'_1})\dots))$$
$$B \simeq_{\mathrm{i}} \big(((\dots (B_{l_1} \otimes B_{l_2})\dots) \otimes B_{l_{m'}}) \multimap \alpha\big) \multimapinv (B_{l'_{m''}} \otimes (\dots (B_{l'_2} \otimes B_{l'_1})\dots))$$

and $A \simeq_{\mathrm{i}} B$ easily follows. ////

CHAPTER 3

Link graphs and proof structures

In this chapter we will define proof structures corresponding to different calculi. Each particular calculus has its own elegant, although somewhat *ad hoc*, definition(s), but since we are also interested in the relations between the different calculi, we will start from a very general definition of proof structure as a particular kind of the even more general so-called *link graphs*, to be defined in Section 3.1. Link graphs as such do not have any obvious logical meaning: we should consider them as our universe of discourse. In later chapters for each calculus they will be used in order to define the basic objects (viz. the sequents and the derivable sequents), as well as the corresponding notions of proof structure and proof net. As we intend to prove that a sequent is derivable if and only if the corresponding proof structure converts to a certain form (in fact, as we shall see, itself a sequent), it turns out to be highly useful to have this overall notion of link graph in which we can formulate both proof structures and sequents, as well as the process of conversion.

In Section 3.2 we will define proof structures for both the one-sided and the two-sided language. Contrary to sequents, this definition is independent of the particular one-sided (two-sided) calculus we work in, so e.g. proof structures for $\mathbf{MLL}_1$, $\mathbf{NCLL}_1$ and $\mathbf{CNL}_1$ all are the same. Identity proof structures, cut elimination and dualization will be defined in Subsection 3.2.3 for two-sided proof structures, and under the appropriate maps (to be defined in Subsection 3.2.5) these notions immediately translate to one-sided structures.

3.1. Link graphs

In order to define proof structures and sequents, we will introduce a concept which captures both notions, and which moreover is general enough to be adapted to the different calculi. Within each particular calculus, however, structures may be described by more simple means.

A *link graph* should be thought of as a graph with edges and vertices, where some vertices of valence 1 are called open ends, and the rest of the vertices are called links. In each link the order of the attached edges may be prescribed by means of a cyclic list, and one of these attached edges may play the role of *principal* edge. Moreover, a labeling assigns zero or one polarized formulas to every edge extremity. We formalize these notions in the following definition.

Definition 3.1.1. *Let $\mathfrak{L}$ be a language. Let $\mathcal{E}$ be a set of (formal) edges. We call $\widetilde{\mathcal{E}} := \cup_{\eta\in\mathcal{E}}\{\hat{\eta},\check{\eta}\}$ the set of edge extremities, or* ends.

We define three kinds of links:

- *A* rooted link *in $\mathcal{E}$ is a list of ends $(e_0,\ldots,e_{m-1})_\theta$ ($m \geq 1$) labeled by a certain type θ. The end e_0 will be called the* principal end*, and will also be referred to by e_m.*

- *A* cyclic link *in $\mathcal{E}$ is a cyclic list of ends $\llparenthesis e_0, \dots, e_{m-1} \rrparenthesis_\theta$ ($m \geq 0$) labeled by a certain type θ.*
- *A* set link *in $\mathcal{E}$ is a set of ends $\{e_0, \dots, e_{m-1}\}_\theta$ ($m \geq 0$) labeled by a certain type θ.*

An $\mathfrak{L}$-labeling *of $\mathcal{E}$ is a partial map $\lambda : \widetilde{\mathcal{E}} \rightarrowtail \mathfrak{L}^{\pm}$ to the set of polarized $\mathfrak{L}$-formulas.*

An $\mathfrak{L}$-link graph *$\mathcal{P} := (\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$ consists of a finite set $\mathcal{E}$ of edges, a finite set $\mathcal{L}$ of links in $\mathcal{E}$, called the* context links*, a finite set $\mathcal{L}'$ of links in $\mathcal{E}$, called the* connector links *and an $\mathfrak{L}$-labeling λ of $\mathcal{E}$. These data are required to satisfy the following properties:*

- *The context links are either cyclic links or set links;*
- *The connector links are rooted links;*
- *Each end e occurs at most once in the multiset $\bigcup \mathcal{L} \cup \bigcup \mathcal{L}'$. If e does not occur in $\bigcup \mathcal{L} \cup \bigcup \mathcal{L}'$ it is called an* open end*; if e occurs in $\bigcup \mathcal{L}$ ($\bigcup \mathcal{L}'$) it is called a* context (connector) end*;*
- *The domain of λ is exactly the set of open ends together with the connector ends.*

An open end e is called an hypothesis *(*conclusion*) of $\mathcal{P}$ if $\lambda(e)$ is F^- (F^+).* ♢

Actually, in Chapter 4 and Chapter 5 there will by only one type $\theta = \odot$ for the context links. For a link graph in Chapter 4, all the context links are cyclic links of arbitrary valence, except for Section 4.10, in which all the context links of a link graph are set links of arbitrary valence. For a link graph in Chapter 5, all the context links are cyclic links with fixed valence 3. In Chapter 6, a link graph may have cyclic context links of several types, with valence 2 or valence 3.

We will abuse language, by referring to an open end or a connector end e by means of the assigned label occurrence $\lambda(e)$. If $G_0^+, G_1^+, \dots, G_{n-1}^+, F_0^-, F_1^-, \dots, F_{m-1}^-$ are the open ends of a link graph $\mathcal{P}$, we will say $\mathcal{P}$ is a link graph of $F_0, \dots, F_{m-1} \vdash G_0, \dots, G_{n-1}$, where the two expressions separated by the ' $\vdash$ '-sign are to be understood as multisets, because different ends may be labeled by one and the same polarized formula.

We suppose the two ends $\hat{\eta}$ and $\check{\eta}$ of one edge η play a completely symmetric role; wherever we mention $\hat{\eta}$ respectively $\check{\eta}$, we also mean $\check{\eta}$ respectively $\hat{\eta}$. By e^+ (e^-) we mean "e, labeled by a positively (negatively) polarized formula $\lambda(e) = F^+$ (F^-)".

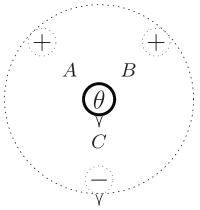
Definition 3.1.2. *Let $\mathcal{P}$ be a link graph, and let $l = (e_0, \dots, e_{m-1})_\theta$ be a connector link of $\mathcal{P}$. The positively (negatively) polarized ends of l are called the* premisses *(*conclusions*) of l. A $\pm$-alteration is an index j ($0 \leq j < m$) such that e_j and e_{j+1} are of opposite sign. If there is an edge η such that $\hat{\eta}^- \in l$ and $\check{\eta}^+ \in l'$ (where l' is also a connector link of $\mathcal{P}$), l is* above *l'.*

The principal end $e_0 = e_m$ of this connector link l is also called the main end *of l, while the other ends are called the* active ends *of l. If the main end of l is a conclusion (premiss) of l, then l is called a* right *(*left*) link.* ♢

We graphically represent a context link l of type θ by

$$\textcircled{θ},$$

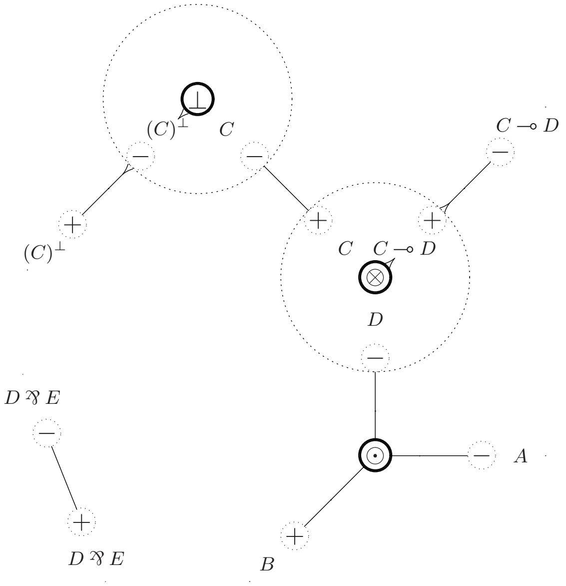
to which we attach the cyclic list (set) of ends counterclockwisely (in some order), in case it is a cyclic (set) link. A connector link will be represented similarly, with an emphasized principal end; now also the labels of the connector ends have to be given. The rooted link $(C^-, B^+, A^+)_\theta$ is depicted as



Observe that there is a priori no relation between $\lambda(\hat{\eta})$ and $\lambda(\check{\eta})$.

The *underlying graph* of a link graph is obtained by replacing links by vertices and adding a vertex for each open end, connecting them by the edges appropriately.

EXAMPLE 3.1.3.



Let $\mathcal{P}$ be the depicted link graph, i.e. there are seven edges; there is one cyclic context link $\llparenthesis e_0, e_1, e_2 \rrparenthesis_{\odot}$; and there are two connector links: one $\perp$-link which is above a $\otimes$-link. The link of type $\perp$ is a right link $\big(\big((C)^\perp\big)^-, C^-\big)_\perp$ with two conclusions: an active end and a main end $\big((C)^\perp\big)^-$. The link of type $\otimes$ is a left link $\big(\big(C \multimap D\big)^+, C^+, D^-\big)_\otimes$ with two premisses and one conclusion, one premiss $\big(C \multimap D\big)^+$ being the main end. Moreover, the six ends which do not occur in any link are the open ends, and they split up in three hypotheses $(D \parr E)^-, (C \multimap D)^-$ and A^- and three conclusions $\big((C)^\perp\big)^+, (D \parr E)^+$ and B^+. Hence $\mathcal{P}$ is an $\mathfrak{L}_2$-link graph of $D \parr E, C \multimap D, A \vdash (C)^\perp, D \parr E, B$. $\Diamond$

Let $\mathfrak{L}$ be $\mathfrak{L}_1$, $\mathfrak{L}_2$ or $\mathfrak{L}_{2,\mathrm{i}}$. An $\mathfrak{L}$-link graph $\mathcal{P}_1 := (\mathcal{E}_1, \mathcal{L}_1, \mathcal{L}'_1, \lambda_1)$ is a *sub $\mathfrak{L}$-link graph* of the $\mathfrak{L}$-link graph $\mathcal{P} := (\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$, notation $\mathcal{P}_1 \subseteq_{\mathrm{lg}} \mathcal{P}$, whenever $\mathcal{E}_1 \subseteq \mathcal{E}$; $\mathcal{L}_1 \subseteq \mathcal{L}$; $\mathcal{L}'_1 \subseteq \mathcal{L}'$ and $\operatorname{graph} \lambda_1 \subseteq \operatorname{graph} \lambda$. Observe that

$$(\text{open ends of } \mathcal{P}_1) \subseteq (\text{open ends of } \mathcal{P})$$

need not hold; some open ends of $\mathcal{P}_1$ may be connector ends of $\mathcal{P}$.

3.2. Proof structures

Proof structures for multiplicative linear logic are usually defined as the smallest set containing axiom-links $\overline{X^{\perp}\qquad X}$ and closed under disjoint union and under the lower attachment of the links

$$\frac{X\qquad Y}{X\otimes Y}\ ,\quad \frac{X\qquad Y}{X\parr Y}\quad\text{and}\quad \underline{X^{\perp}\qquad X}$$

(cf. [**Girard 87**]). This definition closely approximates our definition of $\mathfrak{L}_1$-proof structure (see Subsection 3.2.1), the only difference being the fact that we do not consider axiom-links nor cut-links, but only axiomatic edges and cut edges, which cannot be composed as in

$$\overline{X^{\perp}\qquad \underline{X\qquad}}\,\underline{\overline{\qquad X^{\perp}}}\,\overline{\qquad X}\qquad\text{or}\qquad \underline{\overline{X^{\perp}\qquad X}}$$

We generalize the usual notion of proof structure in two directions, which finally yields the $\mathfrak{L}_2$-proof structures of Subsection 3.2.2.

First, we allow a proof structure to have open hypotheses (cf. [**Danos 90**]); roughly said, we consider the smallest set containing sole formulas, and closed under disjoint union and under the (lower and upper) attachment of links like

$$\overline{A^{\perp}\qquad A}\ ,\quad \frac{A\qquad B}{A\otimes B}\ ,\ \text{et cetera.}$$

Furthermore, this generalization enables us to introduce links corresponding to left rules (e.g. $\dfrac{A\otimes B}{A\qquad B}$) as well as links corresponding to 'new' connectives, e.g. to $-\multimap-$ (definable as $(-)^{\perp}\parr-$), with link $\dfrac{A}{B\qquad B\multimap A}$.

In this section we will more precisely define proof structures for each of the languages $\mathfrak{L}=\mathfrak{L}_1,\mathfrak{L}_2$ and $\mathfrak{L}_{2,\mathrm{i}}$. As usual, the proof structures will correspond to pseudo-derivations, while a particular subset of them, the proof nets, will correspond to real derivations. We will define proof structures as $\mathfrak{L}$-link graphs $\mathcal{P}:=(\mathcal{E},\mathcal{L},\mathcal{L}',\lambda)$ without context links ($\mathcal{L}=\emptyset$), and with (hence total) labeling $\lambda:\widetilde{\mathcal{E}}\to\mathfrak{L}^{\pm}$ satisfying two additional conditions, viz. a *link condition* and an *edge condition*. We will consider at most three types of (connector) links, viz.

- *tensor links* $(e_0,e_1,e_2)_{\otimes}$ (indicated by $\otimes$),
- *par links* $(e_0,e_1,e_2)_{\parr}$ (indicated by $\parr$) and
- *negation links* $(e_0,e_1)_{\perp}$ (indicated by $\perp$).

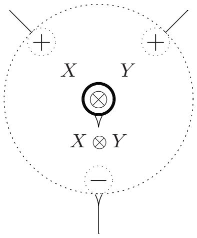
We will not require any graph theoretical properties, like acyclicity or connectedness of the underlying graph. Moreover, we do not require the underlying graph to have a planar representation; the edges are allowed to intersect. The only information that counts is to which link the edges are connected, and in what order.

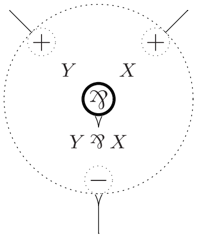
For every rule L□ (R□) in the corresponding sequent calculus (see Section 4.1), there will be a link subtype, i.e. an allowed way of labeling the ends of a link by corresponding main and active formulas. However, there are no links corresponding to the identity rules Ax and Cut. Instead, we will have axiomatic and cut edges. An edge η is *axiomatic* if each of $\hat{\eta}$ and $\check{\eta}$ is not the main end of any link, whereas η is a *cut edge* if both $\hat{\eta}$ and $\check{\eta}$ are the main ends of two links. Let us call the number of main ends η possesses the *role* of η, so that η is an axiomatic edge (a cut edge) iff it has role 0 (2).

3.2.1. One-sided proof structures.

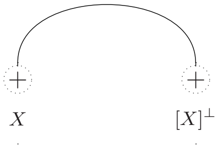
Definition 3.2.1. *An* $\mathfrak{L}_1$-proof structure $\mathcal{P} := (\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$ *is an* $\mathfrak{L}_1$*-link graph with* $\mathcal{L} = \emptyset$ *and with connector links of type* $\otimes$ *and* $\parr$*, whose ends are labeled — depending on the link type — in one of the following ways:*

$$\Big(\big(X \otimes Y\big)^{-}, Y^{+}, X^{+}\Big)_{\otimes} \qquad \Big(\big(Y \parr X\big)^{-}, X^{+}, Y^{+}\Big)_{\parr}$$

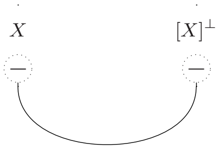




Moreover, for all edges $\eta \in \mathcal{E}$ *the two ends* $\hat{\eta}$ *and* $\check{\eta}$ *are labeled as follows:*



(axiomatic edge)



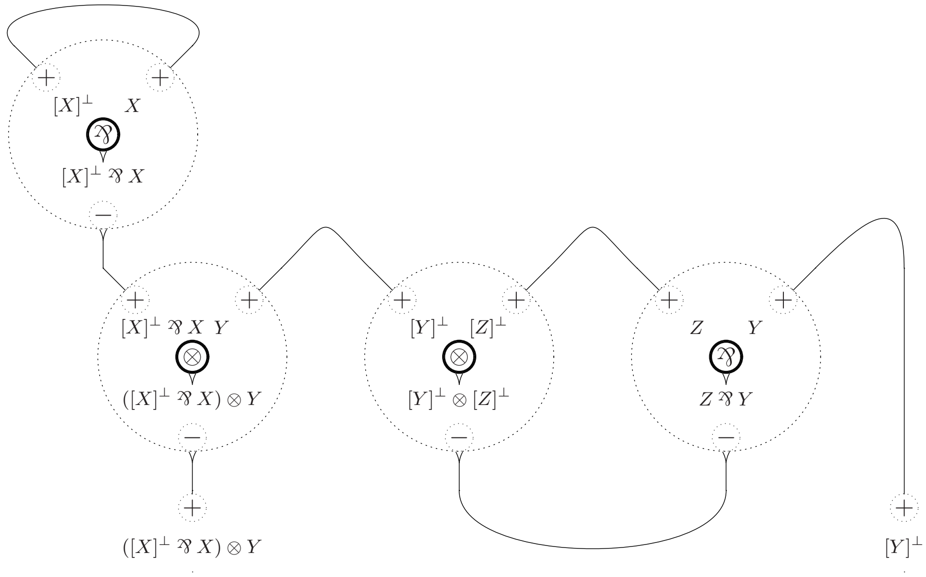
(cut edge)

Finally, all open ends are conclusions of $\mathcal{P}$*.* ◊

Note that the connector links of an $\mathfrak{L}_1$-proof structure are right links: main ends are conclusions of the links. Even stronger: for a connector link l, an end is the main end of l if and only if it is a conclusion of l. As negatively polarized ends are always connector ends (indeed, each open end is a conclusion of $\mathcal{P}$, whence positively polarized), we conclude that an end is the main end of some connector link if and only if it is negatively polarized.

This implies that axiomatic edges are exactly the edges of the first type in the previous definition, and that cut edges are exactly those of the third type in the previous definition.

Example 3.2.2.



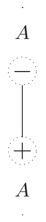
This $\mathfrak{L}_1$-proof structure $\mathcal{P}$ contains seven edges, among which four axiomatic edges and one cut edge. There are two right $\parr$-links and two right $\otimes$-links. The two open ends are the conclusions, viz. $\big(([X]^\perp \parr X) \otimes Y\big)^+$ and $\big([Y]^\perp\big)^+$, so $\mathcal{P}$ is a proof structure of $\vdash ([X]^\perp \parr X) \otimes Y, [Y]^\perp$. $\Diamond$

3.2.2. Two-sided proof structures.

Definition 3.2.3. *An* $\mathfrak{L}_2$-proof structure $\mathcal{P} := (\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$ *is an* $\mathfrak{L}_2$*-link graph with* $\mathcal{L} = \emptyset$ *and with connector links of type* $\otimes$, $\parr$ *and* $\perp$*, whose ends are labeled — depending on the link type — in one of the following ways (see Figure 3.1):*

$$\begin{array}{llll}
\Big(\big((A)^\perp\big)^+, A^+\Big)_\perp & [L\perp] & \Big(\big((A)^\perp\big)^-, A^-\Big)_\perp & [R\perp] \\
\Big(\big(A \otimes B\big)^+, A^-, B^-\Big)_\parr & [L\otimes] & \Big(\big(A \otimes B\big)^-, B^+, A^+\Big)_\otimes & [R\otimes] \\
\Big(\big(B \multimapinv A\big)^+, B^-, A^+\Big)_\otimes & [L\multimapinv] & \Big(\big(B \multimapinv A\big)^-, A^-, B^+\Big)_\parr & [R\multimapinv] \\
\Big(\big(B \multimap A\big)^+, B^+, A^-\Big)_\otimes & [L\multimap] & \Big(\big(B \multimap A\big)^-, A^+, B^-\Big)_\parr & [R\multimap] \\
\Big(\big(B \parr A\big)^+, B^-, A^-\Big)_\otimes & [L\parr] & \Big(\big(B \parr A\big)^-, A^+, B^+\Big)_\parr & [R\parr]
\end{array}$$

Moreover, for all edges $\eta \in \mathcal{E}$ the two ends $\hat{\eta}$ and $\check{\eta}$ are labeled by one and the same formula A, polarized by opposite signs.



◊

Observe that the links of type $\otimes$ together may be described by

$$\big(\tau(A^\rho \otimes B^\sigma), B^\sigma, A^\rho\big)_\otimes$$

while those of type $\parr$ are of the form

$$\big(\tau(B^\sigma \parr A^\rho), A^\rho, B^\sigma\big)_\parr$$

Recall that $\otimes$ and $\parr$ are defined as maps $\mathfrak{L}_2^\pm \times \mathfrak{L}_2^\pm \to \mathfrak{L}_2^\pm$ by

$$A^\rho \otimes B^\sigma := \begin{cases} (A \otimes B)^+ & \text{if } \rho = + \text{ and } \sigma = + \\ (B \multimapinv A)^- & \text{if } \rho = + \text{ and } \sigma = - \\ (B \multimap A)^- & \text{if } \rho = - \text{ and } \sigma = + \\ (B \parr A)^- & \text{if } \rho = - \text{ and } \sigma = - \end{cases}$$

$$B^\sigma \parr A^\rho := \begin{cases} (B \parr A)^+ & \text{if } \rho = + \text{ and } \sigma = + \\ (B \multimap A)^+ & \text{if } \rho = + \text{ and } \sigma = - \\ (B \multimapinv A)^+ & \text{if } \rho = - \text{ and } \sigma = + \\ (A \otimes B)^- & \text{if } \rho = - \text{ and } \sigma = - \end{cases}$$

So, even more general, all ternary links are of the form

$$\big(\tau(A^\rho \Box B^\sigma), B^\sigma, A^\rho\big)_\Box \qquad (\Box = \otimes, \parr)$$

Depending on the labeling of its main end $(A\Box(B))^\nu$ (where $\Box$ is one of the connectives $\otimes, \multimapinv, \multimap, \parr$ or $(-)^\perp$) we will refer to a link by means of its subtype L$\Box$ (for left links; i.e. in case $\nu = +$) and R$\Box$ (for right links; i.e. in case $\nu = -$). This nomenclature may be confusing: an L$\otimes$-link is not a tensor link, but a par link; an L$\parr$-link is not a par link, but a tensor link.

The edge condition for $\mathfrak{L}_2$-proof structures implies that $\mathcal{E}$ may be regarded as a set of formula occurrences. Let us say that an edge (i.e. a formula occurrence) $A \in \mathcal{E}$ is a conclusion (premiss; main formula; active formula) of a link l if $A^- \in l$ ($A^+ \in l$; A^σ is the main end of l; A^σ is an active end of l). This terminology implies that A can be at most once the conclusion of a link and also at most once the premiss of link. But both can happen simultaneously. And moreover, A may be main (active) formula of up to two links. The number of links A is main formula of, is still called the *role* of A. In accordance with the definition of axiomatic (cut) edge, we call A an *axiomatic (cut) formula* if A has role 0 (2).

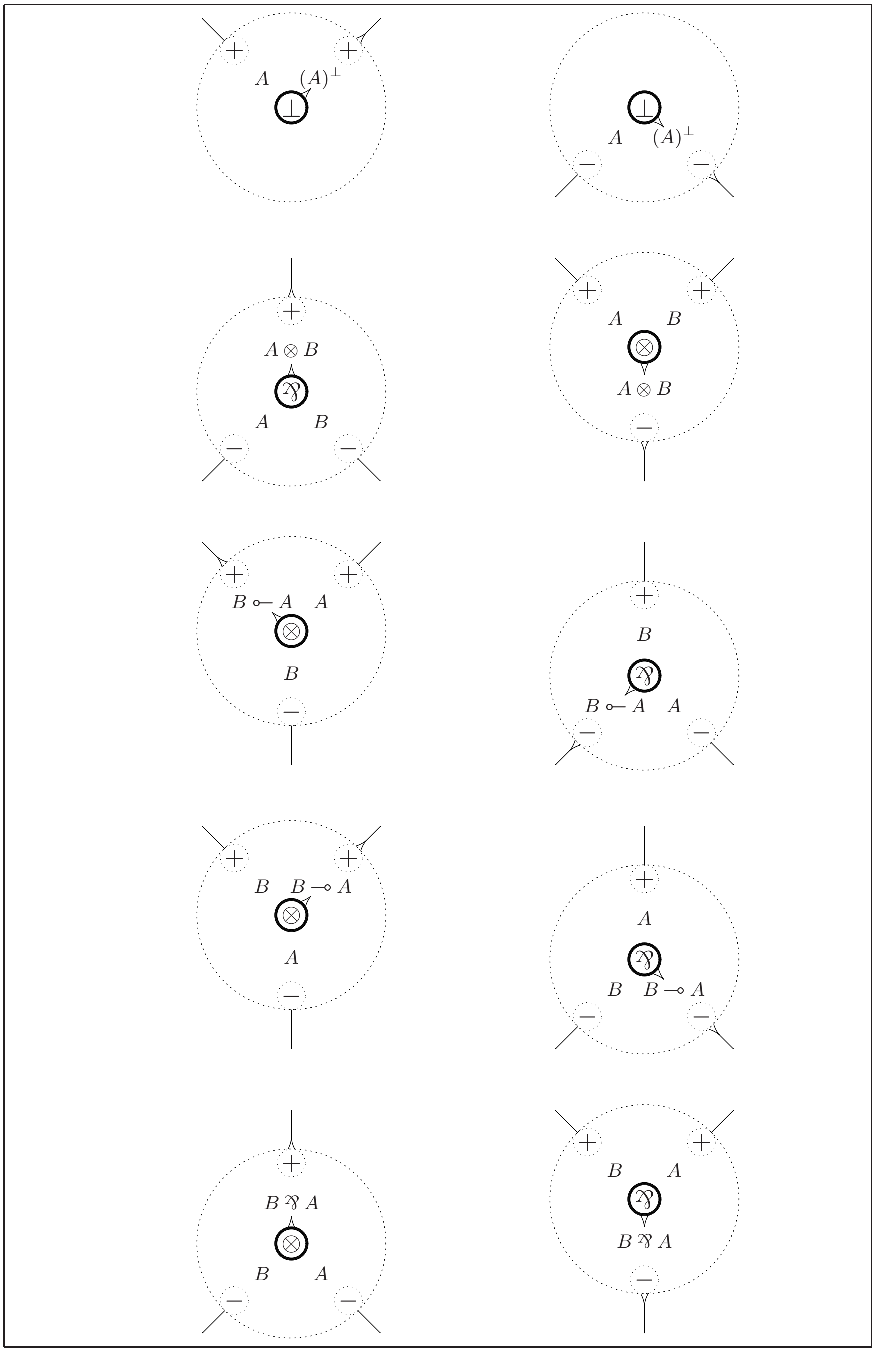
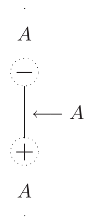


Figure 3.1. The links of $\mathfrak{L}_2$.

If A is not the conclusion (premiss) of any link, then $\mathcal{P}$ has A^- (A^+) among its open ends, whence this end is an hypothesis (conclusion) of $\mathcal{P}$. In this case we will also say that A is an hypothesis (conclusion) of $\mathcal{P}$. Again, A may simultaneously be hypothesis and conclusion of $\mathcal{P}$, the most simple example being the one edge proof structure



which we will soon denote by

$$A$$

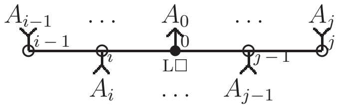
These remarks inspire an alternative definition of an $\mathfrak{L}_2$-proof structure, which we will formulate in the following lemma.

LEMMA 3.2.4. *Every $\mathfrak{L}_2$-proof structure one-to-one corresponds to a pair $\langle \mathcal{E}, \mathcal{L}' \rangle$ consisting of a multiset $\mathcal{E}$ of $\mathfrak{L}_2$-formulas and a set $\mathcal{L}'$ of 'links in $\mathcal{E}$' (i.e. lists of polarized formulas of the form as in Definition 3.2.3 where the formulas belong to $\mathcal{E}$), satisfying the requirements that every formula of $\mathcal{E}$ is at most once a conclusion of a link, and at most once a premiss of a link.* ♢

Exploiting this idea, we obtain an alternative graphical representation of $\mathfrak{L}_2$-proof structures if we contract each edge into the corresponding single formula A, and draw a connection to a link l 'above' ('below') A if A is a conclusion (premiss) of l. We will draw these connections as arrows pointing towards the link, unless A^- (A^+) is the main end of l, in which case the arrow points away from the link. For typographic reasons, links of type $\otimes$ (resp. $⅋$, $\perp$) will then be represented by solid (resp. dashed, solid) horizontal bars. We should keep in mind that the order still does matter. Note that this representation only works due to the fact that our links at most have two ±-alterations. The representation of a general left link

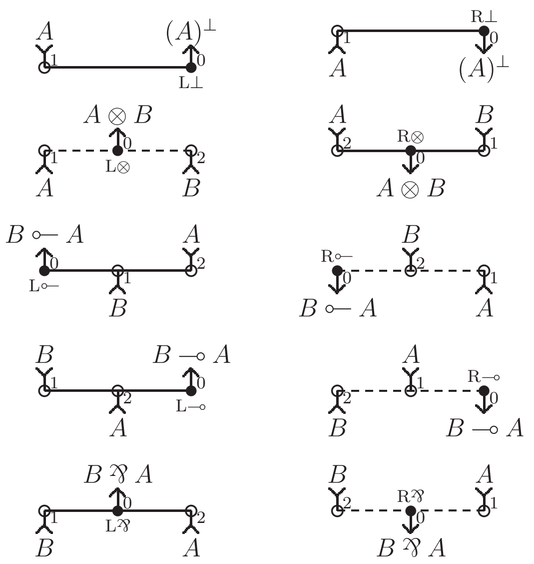
$$\left(A_0^+, A_1^+, \ldots, A_{i-1}^+, A_i^-, \ldots, A_{j-1}^-, A_j^+, \ldots, A_{m-1}^+\right)_\otimes$$

is given by

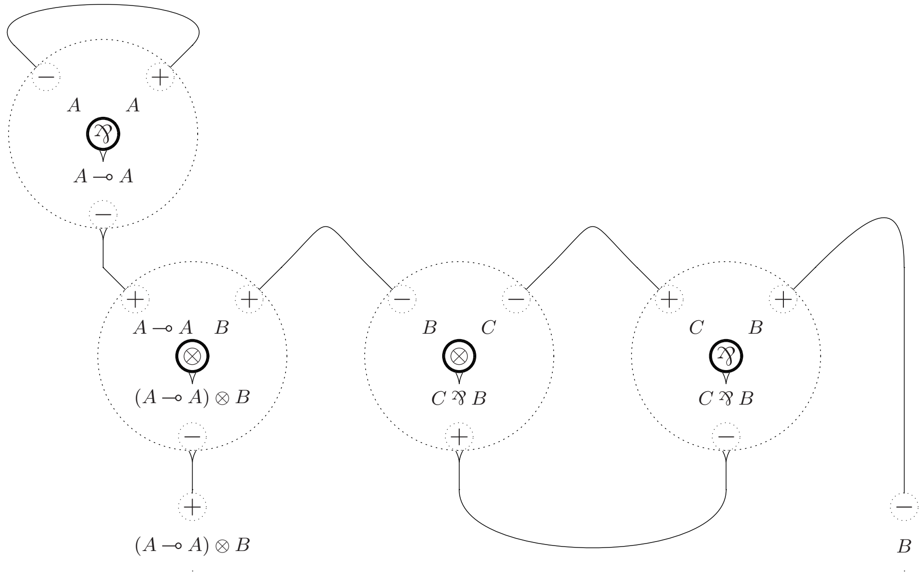


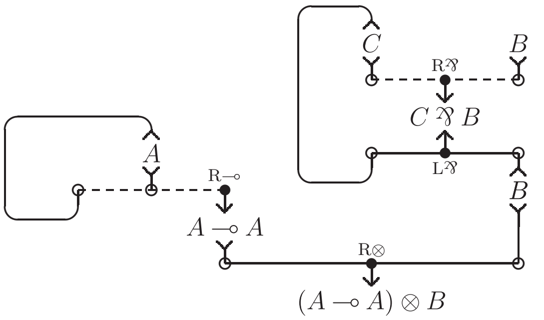
and similar for a right link.

Let us give the representations of the possible links occurring in $\mathfrak{L}_2$-proof structures:



Example 3.2.5. The following are graphical representations of one and the same $\mathfrak{L}_2$-proof structure $\mathcal{P}$.





$\mathcal{P}$ contains seven formulas (edges), among which four axiomatic formulas and one cut formula. There are four links: a [R$\multimap$]-link and a [R$\parr$]-link of type $\parr$, and a [R$\otimes$]-link and a [L$\parr$]-link of type $\otimes$. Moreover, $\mathcal{P}$ has one hypothesis, viz. $\big(B\big)^-$, and one conclusion, viz. $\big((A \multimap A) \otimes B\big)^+$, so $\mathcal{P}$ is a proof structure of $B \vdash (A \multimap A) \otimes B$.

Observe that in the second representation, every depicted formula D plays the role of an edge with labels

$$\begin{array}{c} D^- \\ | \\ D^+ \end{array}$$

The bend connections are inevitable for the [R$\multimap$]-link (being above itself) and the [L/R$\parr$]-links (being above each other).

In Subsection 3.2.5 we will see that this $\mathfrak{L}_2$-proof structure is essentially the same as the one in Example 3.2.2. ♢

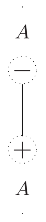
The $\mathfrak{L}_{2,\mathrm{i}}$-proof structures are defined as the obvious particular subclass of $\mathfrak{L}_2$-proof structures.

Definition 3.2.6. *An* $\mathfrak{L}_{2,\mathrm{i}}$-proof structure $\mathcal{P} := (\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$ *is an* $\mathfrak{L}_{2,i}$*-link graph which, considered as an* $\mathfrak{L}_2$*-link graph is an* $\mathfrak{L}_2$*-proof structure.* ♢

This definition implies that the connector links are of type $\otimes$ and $\parr$ (no $\perp$), their ends being labeled in one of the following ways by polarized $\mathfrak{L}_{2,\mathrm{i}}$-formulas:

$$\begin{array}{llll}
\Big(\big(A \otimes B\big)^+, A^-, B^-\Big)_{\parr} & [\mathrm{L}\otimes] & \Big(\big(A \otimes B\big)^-, B^+, A^+\Big)_{\otimes} & [\mathrm{R}\otimes] \\
\Big(\big(B \multimapinv A\big)^+, B^-, A^+\Big)_{\otimes} & [\mathrm{L}\multimapinv] & \Big(\big(B \multimapinv A\big)^-, A^-, B^+\Big)_{\parr} & [\mathrm{R}\multimapinv] \\
\Big(\big(B \multimap A\big)^+, B^+, A^-\Big)_{\otimes} & [\mathrm{L}\multimap] & \Big(\big(B \multimap A\big)^-, A^+, B^-\Big)_{\parr} & [\mathrm{R}\multimap]
\end{array}$$

Moreover, for all edges $\eta \in \mathcal{E}$ the two ends $\hat{\eta}$ and $\check{\eta}$ are labeled by one and the same $\mathfrak{L}_{2,\mathrm{i}}$-formula A, polarized by opposite signs.



Definition 3.2.7. *Let* $\mathfrak{L}$ *be* $\mathfrak{L}_1$*,* $\mathfrak{L}_2$ *or* $\mathfrak{L}_{2,i}$*, and let* $\mathcal{P}$ *be an* $\mathfrak{L}$*-proof structure. An* $\mathfrak{L}$*-link graph* $\mathcal{P}_1$ *is a* sub $\mathfrak{L}$-proof structure *of* $\mathcal{P}$*, notation* $\mathcal{P}_1 \subseteq_{ps} \mathcal{P}$*, whenever* $\mathcal{P}_1 \subseteq_{lg} \mathcal{P}$ *(see page 43) and* $\mathcal{P}_1$ *is an* $\mathfrak{L}$*-proof structure itself.* ♢

3.2.3. Basic operations. In this subsection we will define some general operations on $\mathfrak{L}_2$-proof structures. First we will define for an $\mathfrak{L}_2$-formula A its identity proof structure $\mathcal{I}(A)$. Secondly, for an $\mathfrak{L}_2$-proof structure $\mathcal{P}$ we will define its cut-free proof structure $\mathcal{P}'$, which has the same hypotheses and conclusions as $\mathcal{P}$, but no cut formula anymore. Finally, we will introduce the notion of a dualization $\mathcal{P}^*$ of a proof structure $\mathcal{P}$.

For fixed connective $\Box \neq \perp$, let us consider a left link

$$\Big(\big(B \,\Box\, A\big)^+, B^\sigma, A^\rho\Big)_\theta \quad [\mathrm{L}\Box]$$

which is of type $\theta = \otimes$ or $\parr$. Then there is a corresponding right link

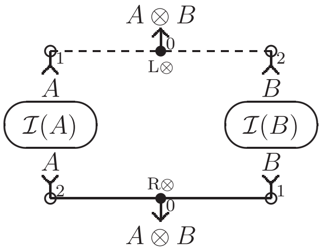
$$\Big(\big(B \,\Box\, A\big)^-, A^{-\rho}, B^{-\sigma}\Big)_{\overline{\theta}} \quad [\mathrm{R}\Box]$$

which we have obtained by reversing the order of the original link, altering the polarities of the ends, and changing the type $\theta = \otimes$ ($\parr$) to $\overline{\theta} = \parr$ ($\otimes$). We call these two links *dual* with respect to each other. Also, the links

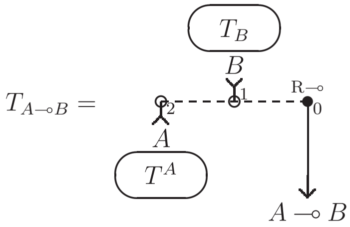
$$\Big(\big((A)^\perp\big)^+, A^+\Big)_\perp \quad [\mathrm{L}\perp] \quad \text{and} \quad \Big(\big((A)^\perp\big)^-, A^-\Big)_\perp \quad [\mathrm{R}\perp]$$

are called dual.

Because of the existence of dual links, we are able to define the *identity* proof structure $\mathcal{I}(A)$ of $A \vdash A$ (for every $\mathfrak{L}_2$-formula A), using only *atomic* axiomatic formulas (called an η-*expanded* proof structure): for an atom α_i we can take the proof structure consisting of the sole formula α_i and no links, while for a complex formula $A \,\Box\, (B)$ we can paste the $[\mathrm{L}\Box]$-link and its dual $[\mathrm{R}\Box]$-link to the inductively obtained identity proof structure(s) for A (and B), as in:



To each $\mathfrak{L}_2$-formula A we can inductively assign two proof structures, called the *upper* and *lower construction tree* of A. Denoting the set of positive *atomic* subformulas[1] of A by $P(A)$ and the set of negative ones by $N(A)$, the upper construction tree T_A is a proof structure of $P(A) \vdash N(A), A$, while the lower construction tree T^A is a proof structure of $N(A), A \vdash P(A)$.



In these trees there are no cut formulas, and the axiomatic formulas are exactly the atomic subformulas. The two trees T_A and T^A of a formula A may be pasted into one proof structure by connecting the corresponding atomic open ends[2]. This is another way to obtain the identity proof structure $\mathcal{I}(A)$ of $A \vdash A$.

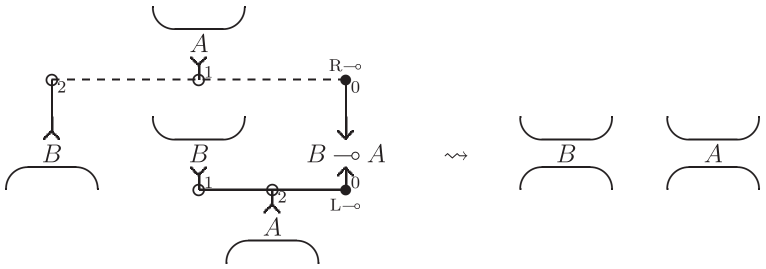
Next, we will introduce the *cut elimination* procedure for proof structures. Note that a cut formula is the main formula of two *dual* links $l = (C^-, (B')^\sigma, \big((A')^\rho\big))_\theta$ and $l^* = (C^+, \big((A'')^{-\rho}\big), (B'')^{-\sigma})_{\overline{\theta}}$, where B' and B'' are occurrence of one and the same formula B; similar for A' and A'' in case they are present. Now a reduction step is defined in the following way. Delete these links and the cut formula, and let the active formulas pairwise *collaps*: first, if the occurrences B' and B'' are different, identify them; otherwise delete them. Second (in case applicable), if the occurrences A' and A'' are still different (after the possible identification of B and B''), identify them; otherwise delete them. It is clear that the number of links decreases by 2 (and that the number of formulas decreases by $m = |l| = |l^*|$), implying that this reduction is noetherian (i.e. strongly normalizing). Moreover this reduction is confluent, whence normal forms are unique.

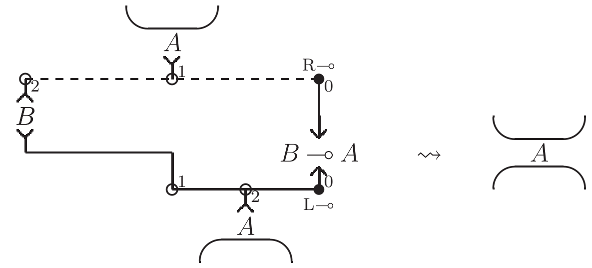
We give three examples:

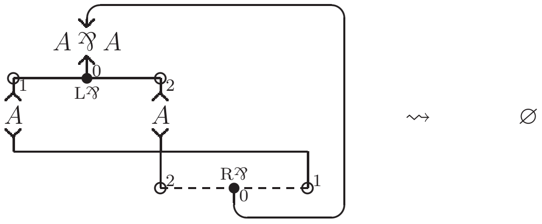
---

[1] The *positive (negative) atomic subformulas* of A are those corresponding with occurrences α ($\alpha^\perp$) in $\pi A \in \mathfrak{L}_1$.

[2] In Subsection 3.2.4 we demonstrate that every η-expanded cut-free proof structure is obtained as the union of some trees, followed by *an* identification of the atomic formulas.

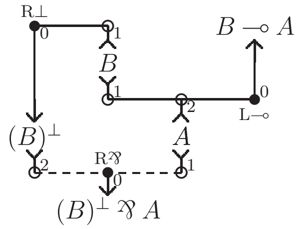


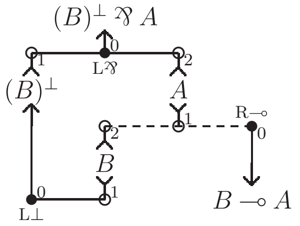




Given a proof structure $\mathcal{P}$, we define its *dualization* $\mathcal{P}^*$ to be the link graph obtained by replacing every link by its dual, simultaneously reversing the labeling of every edge. The result is easily shown to be a proof structure again. If an edge A is an hypothesis of $\mathcal{P}$ (i.e. the end A^- is an open end of $\mathcal{P}$) and is a premiss of a link l (i.e. $A^+ \in l$), then in its dualization $\mathcal{P}^*$ this edge A is a conclusion of $\mathcal{P}^*$ (A^+ is an open end now), and moreover is a conclusion of the link l^* (i.e. $A^- \in l^*$). Hence, if $\mathcal{P}$ is a proof structure of $\Gamma \vdash \Delta$, then $\mathcal{P}^*$ is a proof structure of $\Delta \vdash \Gamma$. Note that the operation $\mathcal{P} \mapsto \mathcal{P}^*$ is an involution.

Example 3.2.8. The following proof structures are the dualizations of each other. In Section 4.6 these will prove the relation between $\multimap$ and ⅋.





◊

3.2.4. η-Expanded cut-free proof structures and axiom linkings. Let $\mathcal{P}$ be an η-expanded cut-free $\mathfrak{L}_2$-proof structure, i.e. all axiomatic formulas are atomic and there are no cut formulas. This is equivalent to the statement that each compound formula has role 1 (i.e. is the main formula of exactly one link). Suppose a compound formula $A \otimes B$ of $\mathcal{P}$ is the main formula of, say, a R$\otimes$ link l. Then the active formula A (B) of l is either atomic, or compound and hence the main formula of another link. Carrying on we will recognize the upper construction tree $T_{A \otimes B}$ in $\mathcal{P}$. In general, every compound formula C of $\mathcal{P}$ will thus determine a tree T_C or T^C. For leaves we even better can say: every hypothesis (conclusion) of $\mathcal{P}$ will thus determine a tree T^C (T_C). This is the intuition behind the following proposition.

Proposition 3.2.9. *An η-expanded cut-free proof structure $\mathcal{P}$ of $A_0, \ldots, A_{m-1} \vdash B_0, \ldots, B_{n-1}$ can be constructed by first taking the union of the construction trees T^{A_i} of $N(A_i), A_i \vdash P(A_i)$ and T_{B_j} of $P(B_j) \vdash N(B_j), B_j$, and then identifying each $\alpha_k \in \bigcup_i P(A_i) \cup \bigcup_j N(B_j)$ with an $\alpha_k \in \bigcup_i N(A_i) \cup \bigcup_j P(B_j)$ in one way or the other.* ◊

Proof: By induction on the size of $\mathcal{P}$. If $\mathcal{E} = \emptyset$ the result is clear. Otherwise, choose a formula $C \in \mathcal{E}$ with maximal length $\mathsf{l}(C)$ (defined in Subsection 2.1.6). If $\mathsf{l}(C) = 1$ then all formulas are atomic, whence there are no links, and the result clearly holds. Otherwise, C is compound and hence the main formula of exactly one link l. It cannot be the active formula of another link, since then there would be a formula with strictly greater length. So C is a leaf. Removing C and l yields a strictly smaller proof structure $\mathcal{P}'$ for which the result holds by induction hypothesis. But then the result also holds for $\mathcal{P}$ (with the same identification of the atomic subformulas as used for $\mathcal{P}'$). ////

Observe that the other way around construction trees are proof structures which are η-expanded and cut-free, properties which are preserved under disjoint union and identification of the atomic subformulas. So η-expanded and cut-free proof structures are precisely those "identified construction forests".

From this proposition we can infer that there is a certain restriction in the labeling of the open ends of any proof structure: proof structures have *balanced* leaves:

Corollary 3.2.10. *If $\mathcal{P}$ is a proof structure of $A_0, \ldots, A_{m-1} \vdash B_0, \ldots, B_{n-1}$, then the multisets $\bigcup_i P(A_i) \cup \bigcup_j N(B_j)$ and $\bigcup_i N(A_i) \cup \bigcup_j P(B_j)$ coincide; i.e.*

$$\sum_i \langle\!\langle A_i \rangle\!\rangle = \sum_j \langle\!\langle B_j \rangle\!\rangle .$$

◊

Proof: Given $\mathcal{P}$ with the mentioned leaves, first eliminate the cut formulas, yielding a cut-free proof structure $\mathcal{P}'$ with the same leaves. Now expand the non-atomic axiomatic formulas C by replacing them by $\mathcal{I}(C)$, yielding an η-expanded cut-free proof structure $\mathcal{P}''$, still with the same open ends. Then the above proposition applies, and there is an identification of the atomic subformulas of the trees. As a consequence, the two multisets $\bigcup_i P(A_i) \cup \bigcup_j N(B_j)$ and $\bigcup_i N(A_i) \cup \bigcup_j P(B_j)$ are equal. Considering both multisets as elements of the free abelian group generated by the atoms $\mathbb{Z}^{\mathcal{A}}$ (with non-negative coefficients), we have

$$\sum_i P(A_i) + \sum_j N(B_j) = \sum_i N(A_i) + \sum_j P(B_j)$$

whence

$$\sum_i P(A_i) - \sum_i N(A_i) = \sum_j P(B_j) - \sum_j N(B_j)$$

i.e.

$$\sum_i \langle\!\langle A_i \rangle\!\rangle = \sum_j \langle\!\langle B_j \rangle\!\rangle$$

////

Proposition 3.2.9 shows that, given the multisets of hypotheses $A_0, \ldots, A_{m-1}$ and conclusions $B_0, \ldots, B_{n-1}$, an η-expanded cut-free proof structure $\mathcal{P}$ is completely determined by a bijection $\bigcup_i P(A_i) \cup \bigcup_j N(B_j) \to \bigcup_i N(A_i) \cup \bigcup_j P(B_j)$, pairing occurrences of one and the same atom $\alpha_k \in \mathcal{A}$. Stated differently, $\mathcal{P}$ is determined by an *axiom linking*: a (fixed point free) involution p on the multiset $\bigcup_i P(A_i) \cup \bigcup_j N(B_j) \cup \bigcup_i N(A_i) \cup \bigcup_j P(B_j)$, such that x is an occurrence of α_k in $\bigcup_i P(A_i) \cup \bigcup_j N(B_j)$ if and only if $p(x)$ is an occurrence of α_k too in $\bigcup_i N(A_i) \cup \bigcup_j P(B_j)$.

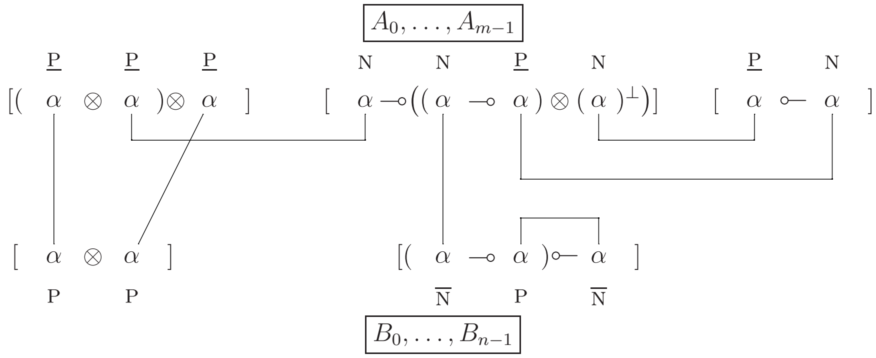
Example 3.2.11. Let $\alpha \in \mathcal{A}$. Let hypotheses be given by

$$\begin{aligned} A_0 &:= (\alpha \otimes \alpha) \otimes \alpha \\ A_1 &:= \alpha \multimap \big((\alpha \multimap \alpha) \otimes (\alpha)^{\perp}\big) \\ A_2 &:= \alpha \multimapinv \alpha \end{aligned}$$

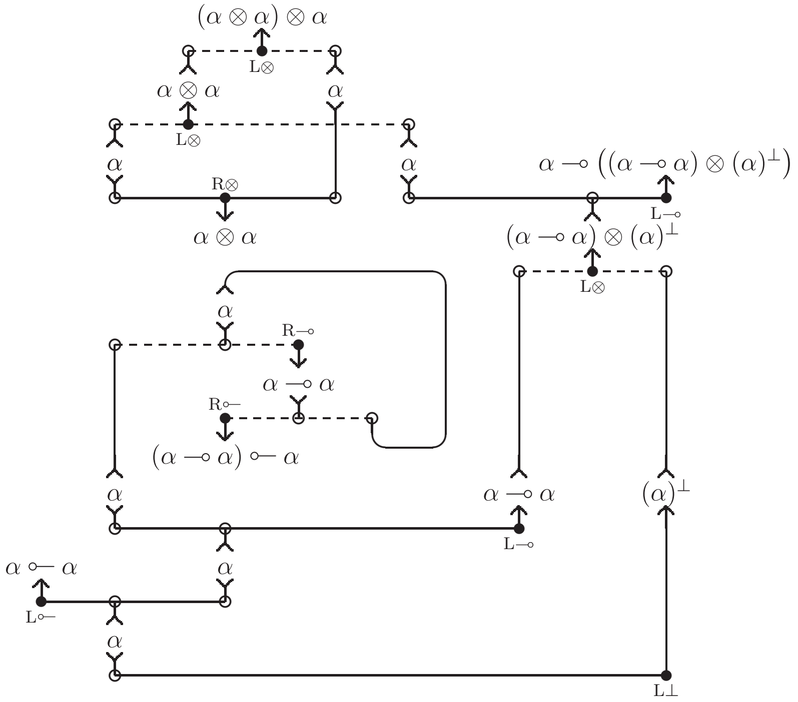
and conclusions by

$$\begin{aligned} B_0 &:= \alpha \otimes \alpha \\ B_1 &:= (\alpha \multimap \alpha) \multimapinv \alpha \end{aligned}$$

One easily verifies that these leaves are balanced. Let p be the involution determined by the following diagram.



This involution is indeed an axiom linking: x is an occurrence of α in $\bigcup_i P(A_i)$ (indicated by $\underline{\text{P}}$) or $\bigcup_j N(B_j)$ (indicated by $\overline{\text{N}}$) precisely when $p(x)$ is an occurrence of α in $\bigcup_i N(A_i)$ (indicated by N) or $\bigcup_j P(B_j)$ (indicated by P). It determines the following η-expanded cut-free proof structure:



◇

Let $\mathcal{P}$ be an η-expanded cut-free proof structure of $A_0, \ldots, A_{m-1} \vdash B_0, \ldots, B_{n-1}$, determined by an axiom linking p. Suppose p' is also an axiom linking, determining a proof structure $\mathcal{P}'$ of $A'_0, \ldots, A'_{m'-1} \vdash B'_0, \ldots, B'_{n'-1}$. If one of the B_k equals an A'_l, we consider the proof structure $\mathcal{P}_0$ obtained by taking the disjoint union of $\mathcal{P}$ and $\mathcal{P}'$, and unifying the formula B_k of $\mathcal{P}$ with A'_l of $\mathcal{P}'$. Let $\mathcal{P}_1$ be the (unique) cut-free proof structure obtained after cut elimination of $\mathcal{P}_0$. (If $C := B_k = A'_l$ is atomic, $\mathcal{P}_0$ is cut-fee already. If C is compound however, it it a cut-formula of $\mathcal{P}_0$.) Observe that $\mathcal{P}_1$ is still η-expanded. Now we define the *composite axiom linking* $p' \circ_{B_k = A'_l} p$ to be the (unique) axiom linking determined by $\mathcal{P}_1$.

The next example shows that composites can be calculated by 'connecting wires', disregarding possibly occurring 'cycles'. A formal definition can be given by means of Girard's '(result of the) execution formula' for the 'geometry of interaction' ([**Girard 87a, Girard 89**]):

$$p' \circ_{B_k = A'_l} p = (1 - j_C^2) \frac{q}{1 - j_C q} (1 - j_C^2) = (1 - j_C^2) \left(\sum_{N=0}^{\infty} q (j_C q)^N \right) (1 - j_C^2)$$

where $q := p + p'$ on the coproduct multiset $\bigcup_i P(A_i) \cup \bigcup_j N(B_j) \cup \bigcup_i N(A_i) \cup \bigcup_j P(B_j) + \bigcup_i P(A'_i) \cup \bigcup_j N(B'_j) \cup \bigcup_i N(A'_i) \cup \bigcup_j P(B'_j)$ while j_C is the partial bijection on the same multiset that relates the subformulas of B_k with those of A'_l.

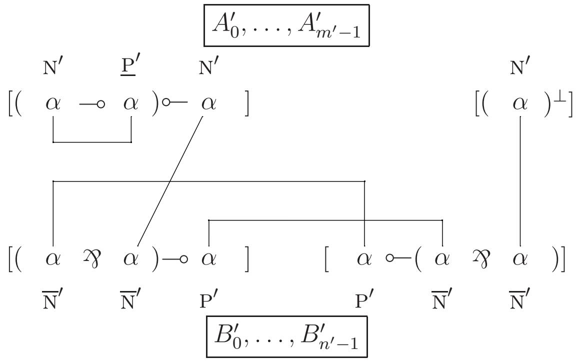
Example 3.2.12. Let α, p and $\mathcal{P}$ be as in Example 3.2.11. Let hypotheses be given by

$$\begin{aligned} A'_0 &:= (\alpha \multimap \alpha) \multimapinv \alpha = B_1 \\ A'_1 &:= (\alpha)^{\perp} \end{aligned}$$

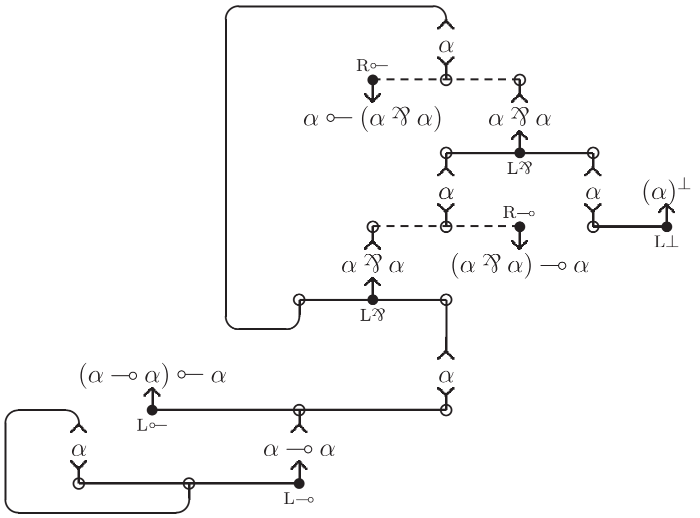
and conclusions by

$$\begin{aligned} B'_0 &:= (\alpha \parr \alpha) \multimap \alpha \\ B'_1 &:= \alpha \multimapinv (\alpha \parr \alpha) \end{aligned}$$

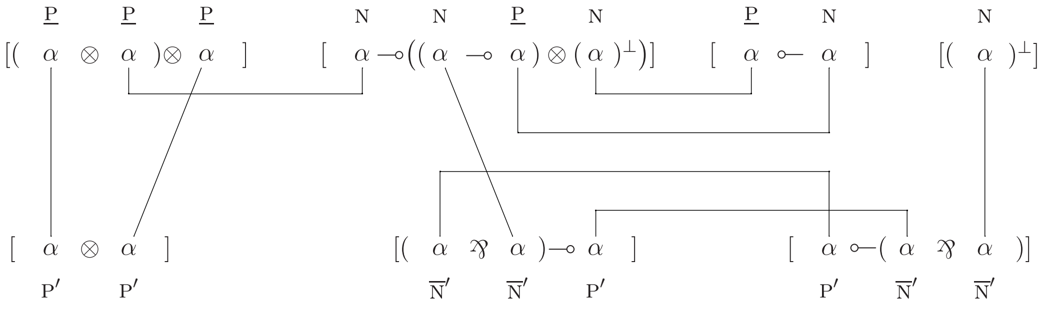
Let p' be the involution determined by the following diagram:



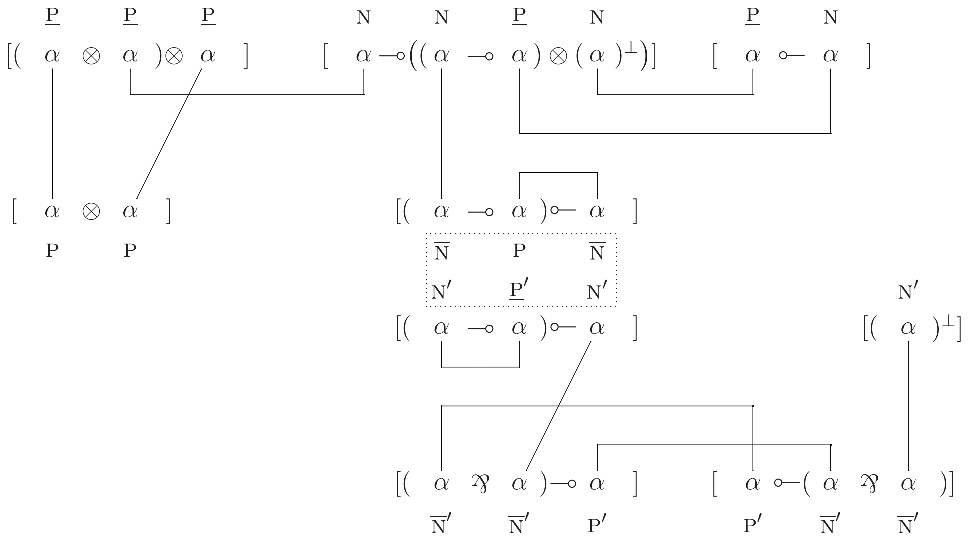
It corresponds to the following η-expanded cut-free proof structure $\mathcal{P}'$:



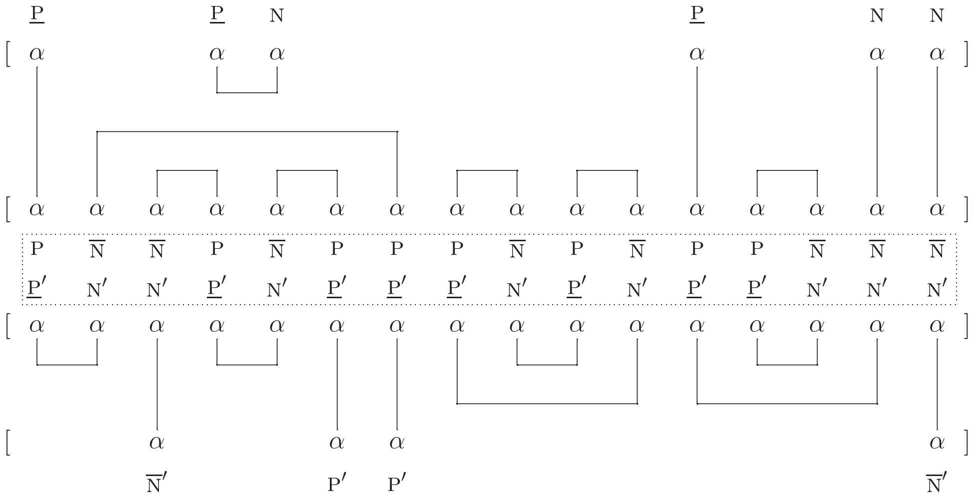
Connecting the conclusion $B_1 = (\alpha \multimap \alpha) \multimapinv \alpha$ of $\mathcal{P}$ to the hypothesis $A'_0 = (\alpha \multimap \alpha) \multimapinv \alpha$ of $\mathcal{P}'$ yields a proof structure $\mathcal{P}_0$. Eliminating the thus obtained cut formula $(\alpha \multimap \alpha) \multimapinv \alpha$ results in an η-expanded cut-free proof structure $\mathcal{P}_1$ which corresponds to the following axiom linking:



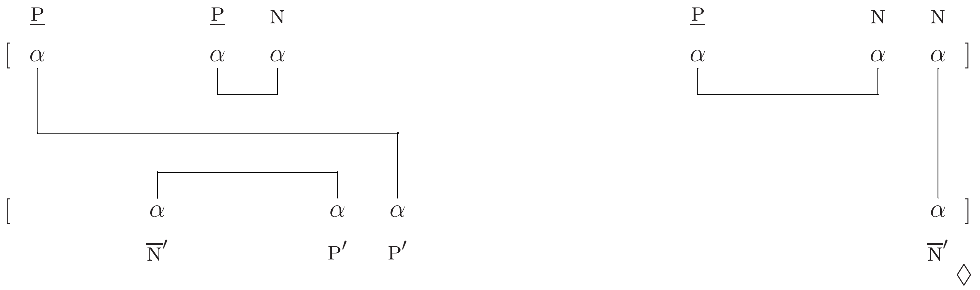
Hence, by definition, this is the value of the composite $p' \circ_{B_1=A'_0} p$



Another example is given by the next composite of an axiom linking p of $A \vdash B$ and p' of $B \vdash C$. We will not specify A, B and C, but only indicate the positive and negative atomic subformulas of each of them. The composite



turns out to be



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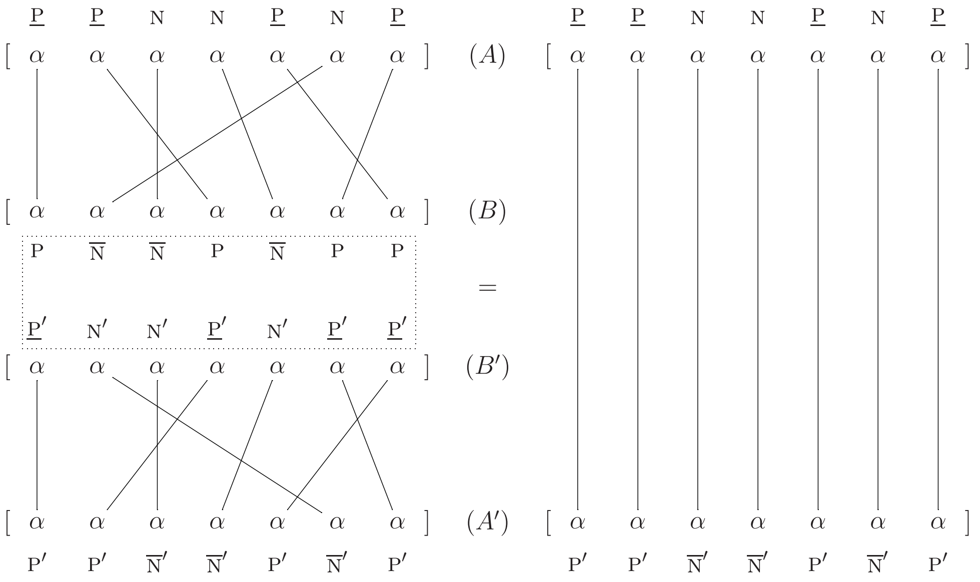
Given an $\mathfrak{L}_2$-formula A, we have defined the identity proof structure $\mathcal{I}(A)$ of $A \vdash A$. As this proof structure is η-expanded and cut-free, it makes sense to define the *identity axiom linking* i_A on A as the corresponding axiom linking. Let us write A (A') for the hypothesis (conclusion) occurrence of the formula A in $\mathcal{I}(A)$. Observe that i_A is an involution[3] on the multiset $P(A) \cup N(A') \cup N(A) \cup P(A')$, such that x is an occurrence of α_k in $P(A) \cup N(A')$ precisely when $i_A(x)$ is an occurrence of α_k too in $N(A) \cup P(A')$. Even stronger: x is an occurrence of α_k in $P(A)$ $(N(A'))$ precisely when $i_A(x)$ is an occurrence of α_k too in $P(A')$ $(N(A))$: there are no 'wires' of the form $\underline{\text{P}}\text{N}$ or $\overline{\text{N}}\text{P}$.

We will now show that whenever an arbitrary proof structure $\mathcal{P}$ of $A \vdash B$ has such an axiom linking without 'wires' of the form $\underline{\text{P}}\text{N}$ or $\overline{\text{N}}\text{P}$, then the composite of its axiom linking with the axiom linking of P^* yields the identity axiom linking. Let us first give an example which gives a good indication of this fact.



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[3] Note that i_A is far from the identity map $p(x) = x$ on the multiset $P(A) \cup N(A') \cup N(A) \cup P(A')$; indeed, as an axiom linking it is fixed point free. Though, if we denote by x' the subformula of A' corresponding to the subformula x of A, we can describe i_A by $i_A(x) = x'$.

PROPOSITION 3.2.13. *Let $\mathcal{P}$ be an η-expanded cut-free proof structure of $A \vdash B$, and $\mathcal{P}^*$ its dualization. Let p and p^* be the corresponding axiom linkings. Suppose that $x \in P(A) \cup N(A)$ is equivalent to $p(x) \in P(B) \cup N(B)$. Then $p \circ p^*$ and $p^* \circ p$ are identity axiom linkings.* $\diamondsuit$

PROOF: First of all p is an involution on the multiset $P(A) \cup N(B) \cup N(A) \cup P(B)$, such that x is an occurrence of α_k in $P(A) \cup N(B)$ iff $p(x)$ is an occurrence of α_k too in $N(A) \cup P(B)$. Similarly, p^* is an involution on the multiset $P(B') \cup N(A') \cup N(B') \cup P(A')$, such that y is an occurrence of α_k in $P(B') \cup N(A')$ iff $p^*(y)$ is an occurrence of α_k too in $N(B') \cup P(A')$, where A' and B' are the occurrences of A and B in $\mathcal{P}^*$. Let j_A (j_B) be the partial bijection on $P(A) \cup N(B) \cup N(A) \cup P(B) + P(B') \cup N(A') \cup N(B') \cup P(A')$ relating the corresponding subformula occurrences of A and A' (B and B'); let j denote their union, which is total. Now the fact that $\mathcal{P}^*$ and $\mathcal{P}$ are dualizations of each other leads to

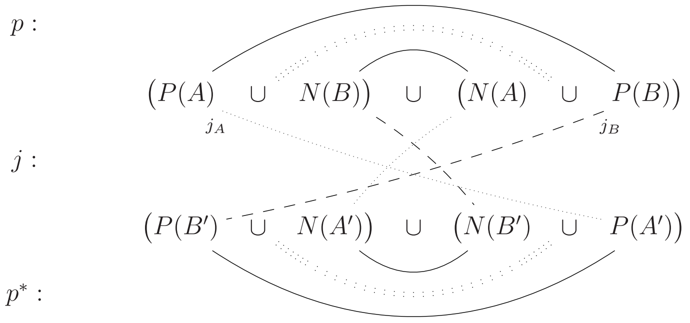
$$j(p + p^*) = (p + p^*)j$$

Hence for an occurrence x of α_k in $P(A)$:

$$\big(j_B(p+p^*)\big)^N(x) = \begin{cases} x \in P(A) & \text{if } N = 0 \\ j_B p(x) \in P(B') & \text{if } N = 1 \\ 0 & \text{if } N > 1 \end{cases}$$

whence $(1 - j_B^2)\left(\sum_{N=0}^{\infty}(p+p^*)(j_B(p+p^*))^N\right)(1 - j_B^2)(x) = (1 - j_B^2)(p+p^*)(x + j_B p x) = (1 - j_B^2)(px + j_A x) = j_A x = i_A x$. Here we used the assumption that there are no 'wires' of the form $\underline{\text{P}}\text{N}$ or $\overline{\text{N}}\text{P}$, whence $p(x) \in P(B)$ and $p^* j_B p(x) \in P(A')$. A similar computation can be done for $y \in N(A')$. This shows that $p^* \circ p = i_A$, as desired.

The other composite follows by interchanging the roles of $\mathcal{P}$ and $\mathcal{P}^*$.



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3.2.5. Translations. Given an $\mathfrak{L}_1$-proof structure $\mathcal{P}$ with labeling $\lambda : \widetilde{\mathcal{E}} \to \mathfrak{L}_1^{\pm}$, composing this labeling with $\psi^{\pm} : \mathfrak{L}_1^{\pm} \to \mathfrak{L}_2^{\pm}$ (see Subsection 2.1.5 for the definition of ψ and Subsection 2.1.4 for the meaning of $\psi^{\pm}$) yields an $\mathfrak{L}_2$-link graph $\psi\mathcal{P}$ which is easily shown to be an $\mathfrak{L}_2$-proof structure. On links this operation has the following effect

$$\begin{array}{ccc} \Big(\big(X \otimes Y\big)^-, Y^+, X^+\Big)_{\otimes} & \mapsto & \Big(\tau\psi(X \otimes Y), \psi Y, \psi X\Big)_{\otimes} \\ \Big(\big(Y \parr X\big)^-, X^+, Y^+\Big)_{\parr} & \mapsto & \Big(\tau\psi(Y \parr X), \psi X, \psi Y\Big)_{\parr} \end{array}$$

and by comparison of Definition 2.1.1 and Definition 3.2.3 we see that these links are indeed of the required form. Moreover, the edge labeling changes according to:

$$\begin{array}{ccc} X^+ \text{———} ([X]^\perp)^+ & \mapsto & A^\rho \text{———} A^{-\rho} \\ X^- \text{———} X^+ & \mapsto & A^{-\rho} \text{———} A^\rho \\ X^- \text{———} ([X]^\perp)^- & \mapsto & A^{-\rho} \text{———} A^\rho \end{array}$$

(where $A^\rho := \psi X$), whence it does satisfy the edge condition for $\mathfrak{L}_2$-proof structures as well.

Observe that main and active ends are preserved, whence also axiomatic and cut edges are preserved. However, conclusions and premisses of a link are not preserved, as the sign of an end label changes if the one-sided formula Z of this label is in $\mathfrak{L}_{1,\text{odd}}$. This also means that the conclusions Δ^+ of $\mathcal{P}$ split up into open ends $\psi^\pm\Delta^+ = \psi\Delta = (\Delta^\bullet_{\text{even}})^+, (\Delta^\bullet_{\text{odd}})^-$ of $\psi\mathcal{P}$, the latter hence being an $\mathfrak{L}_2$-proof structure of $\Delta^\bullet_{\text{odd}} \vdash \Delta^\bullet_{\text{even}}$.

Example 3.2.14. Let $\mathcal{P}$ be the $\mathfrak{L}_1$-proof structure of Example 3.2.2. Then $\psi\mathcal{P}$ is given by the proof structure of Example 3.2.5 in case $X, Y, Z \in \mathfrak{L}_{1,\text{even}}$, where $A = X^\bullet$, $B = Y^\bullet$ and $C = Z^\bullet$. ♢

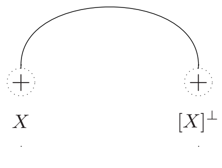
Remark 3.2.15. Composing the labeling $\lambda : \widetilde{\mathscr{E}} \to \mathfrak{L}_1^\pm$ of an $\mathfrak{L}_1$-proof structure $\mathcal{P}$ with the sign-preserving map $\nu : \mathfrak{L}_1^\pm \to \mathfrak{L}_2^\pm$ yields an $\mathfrak{L}_2$-link graph with only links of subtype [R⊗] and [R⅋]. However, this is not in general an $\mathfrak{L}_2$-proof structure as the edge condition may not be fulfilled. We might try to solve this by replacing the axiomatic (cut) edges by [R⊥] ([L⊥]) links

$$\Big(\big((\nu X)^\perp\big)^{-(+)}, \big(\nu X\big)^{-(+)}\Big)_\perp \qquad [\mathrm{R(L)}\perp]$$

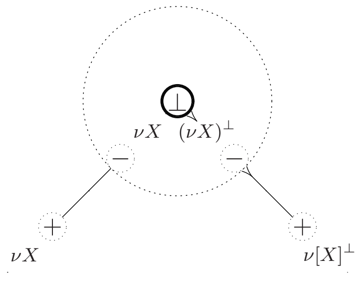
or, depending on your strategy concerning the symmetry between $[X]^\perp$ and $X = [[X]^\perp]^\perp$,

$$\Big(\big((\nu[X]^\perp)^\perp\big)^{-(+)}, \big(\nu[X]^\perp\big)^{-(+)}\Big)_\perp \qquad [\mathrm{R(L)}\perp]$$

but even then the obtained edges (with — at least — ends of opposite polarity) are not in general well typed, since $\nu[X]^\perp$ does not always equal $(\nu X)^\perp$ (or, in the second case, since $(\nu[X]^\perp)^\perp$ does not always equal νX).

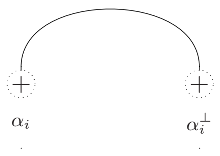


(axiomatic edge)

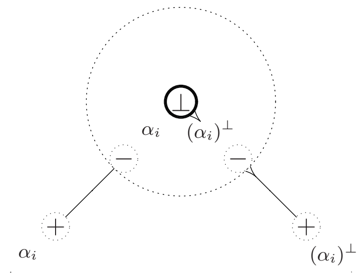


(its replacement)

However, for cut-free and η-expanded proof structures this construction does yield a well-defined $\mathfrak{L}_2$-proof structure $\nu\mathcal{P}$:



(axiomatic edge)



(its replacement)

$\Diamond$

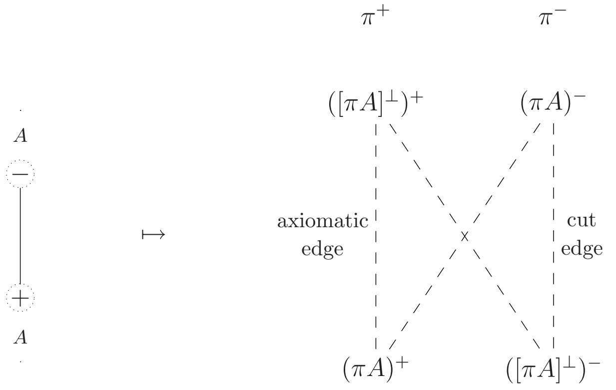
The other way around, given an $\mathfrak{L}_2$-proof structure $\mathcal{P}$, we will use $\pi : \mathfrak{L}_2^\pm \to \mathfrak{L}_1$ in order to define its one sided counterpart $\pi\mathcal{P}$. Recall from Subsection 2.1.4 that $\pi^+, \pi^- : \mathfrak{L}_2^\pm \to \mathfrak{L}_1^\pm$ are defined as follows:

$$\begin{aligned}
\pi^+(A^+) &= (\pi(A^+))^+ = (\pi A)^+ \\
\pi^+(A^-) &= (\pi(A^-))^+ = ([\pi A]^\perp)^+ \\
\pi^-(A^+) &= (\pi(A^-))^- = ([\pi A]^\perp)^- \\
\pi^-(A^-) &= (\pi(A^+))^- = (\pi A)^-
\end{aligned}$$

Given the labeling $\lambda : \widetilde{\mathcal{E}} \to \mathfrak{L}_2^\pm$ of $\mathcal{P}$, we define $\pi(\lambda) : \widetilde{\mathcal{E}} \to \mathfrak{L}_1^\pm$ as follows:

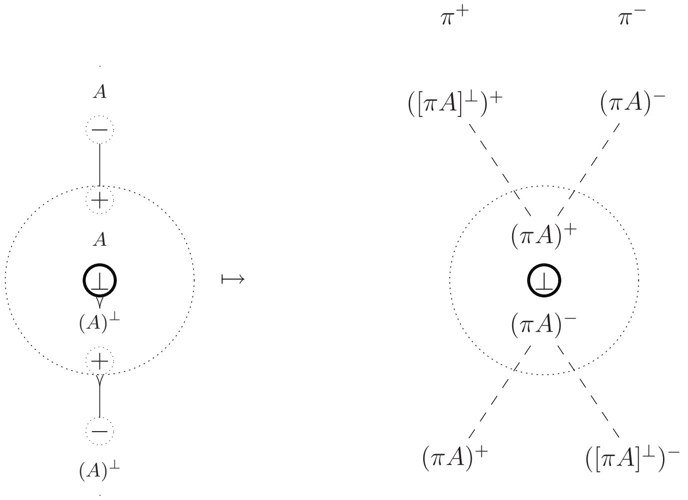
$$\pi(\lambda) : e \mapsto \begin{cases} \pi^+\lambda e & \text{if } e \text{ is an open end or an active connector end} \\ \pi^-\lambda e & \text{if } e \text{ is a main connector end} \end{cases}$$

Then, depending on the role of the ends, the labeling of an edge η will be turned into one of the following four possibilities:



whence the edge condition of $\mathfrak{L}_1$-proof structures is satisfied.

However, there are still links of type $\perp$ present, violating the link condition. Let us consider the labeling of a [L$\perp$] link in the resulting $\mathfrak{L}_1$-link graph:



We can solve this problem by successively deleting the $\perp$-links, letting collaps the involved edges: we identify them if they are (still) different and delete them if they are equal. Obviously, the result is independent of the order in which we proceed.

The result $\pi\mathcal{P} = (\mathcal{E}', \emptyset, \mathcal{L}', \pi(\lambda))$ is a one-sided proof structure. If $\mathcal{P}$ is a proof structure of $\Gamma \vdash \Delta$, then $\pi\mathcal{P}$ has open ends $\pi^+(\Delta^+, \Gamma^-) = \big(\pi\Delta, [\pi\Gamma]^\perp\big)^+$.

LEMMA 3.2.16. *(a) For every $\mathfrak{L}_1$-proof structure $\mathcal{P}$ it holds that $\pi\psi\mathcal{P} = \mathcal{P}$.*

(b) For every $\mathfrak{L}_2$-proof structure $\mathcal{P}$ that also is an $\mathfrak{L}_{2,\perp\text{-free}}$-link graph, it holds that $\psi\pi\mathcal{P} = \mathcal{P}$.

$\Diamond$

By means of this lemma, the three basic operations of the previous subsection immediately translate into corresponding operations on $\mathfrak{L}_1$-proof structures:

The identity proof structure $\mathcal{I}(X)$ of an $\mathfrak{L}_1$-formula X is $\pi\mathcal{I}(X^\bullet)$; for an $\mathfrak{L}_1$-proof structure $\mathcal{P}$ its cut-free proof structure $\mathcal{P}'$ can be defined by $\pi((\psi\mathcal{P})')$; and finally, $\mathcal{P}^* := \pi((\psi\mathcal{P})^*)$.

$$\begin{tikzcd}
\mathfrak{L}_1 \arrow[r, tail, "(-)^\bullet"] \arrow[d, dashed, "\mathcal{I}"'] & \mathfrak{L}_2 \arrow[d, "\mathcal{I}"] \\
\mathfrak{PS}_1 & \mathfrak{PS}_2 \arrow[l, two heads, "\pi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{PS}_1 \arrow[r, tail, "\psi"] \arrow[d, dashed, "(-)'"'] & \mathfrak{PS}_2 \arrow[d, "(-)'"] \\
\mathfrak{PS}_1 & \mathfrak{PS}_2 \arrow[l, two heads, "\pi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{PS}_1 \arrow[r, tail, "\psi"] \arrow[d, dashed, "(-)^*"'] & \mathfrak{PS}_2 \arrow[d, "(-)^*"] \\
\mathfrak{PS}_1 & \mathfrak{PS}_2 \arrow[l, two heads, "\pi"]
\end{tikzcd}$$

CHAPTER 4

Two-sided proof nets for Cyclic Linear Logic

In this chapter we will consider the theory of two-sided proof nets for Non-commutative Cyclic Linear Logic (**NCLL$_2$**). Sequents are cyclic lists of formulas, each of which can play the role of an hypothesis or a conclusion. If we would require a separation of the hypotheses and the conclusions, our sequents would essentially be lists, which would lead to the coexistence of two negations: a linear post-negation and a linear retro-negation (cf. the non-commutative system of [**Abrusci 95**]).

In Section 4.1 we will define the sequent calculus for **NCLL$_2$**. This section also introduces proof theoretical concepts for general calculi.

In Section 4.2 cut elimination is proved. We will prove strong normalization by means of a generalization of the cut rule in Subsection 4.2.2.

Proof nets will be defined in Section 4.3 as the proof structures of derivations. For **MLL**, it is impossible to uniquely assign a proof net to a sequent derivation. As the sequents are multisets, due to multiple occurrences an active or main formula of a derivation rule may be ambiguous. We will show that this problem is overcome for **NCLL** (and hence for further refinements). The non-periodicity of derivable sequents (Lemma 4.1.13) turns out to solve the original ambiguity (Lemma 4.2.5).

Notice that, contrary to the usual one-sided system, the translation of a derivation into a proof net is no longer from up (the axiom-links) to down, but from middle (the axiomatic *formulas*) to the leaves (the hypotheses and the conclusions). The right rules translate in links going down, as usual, but the left rules are mapped to links going up.

The correspondence between axiomatic (cut) formulas and the corresponding identity rules of sequent calculus will be settled (Proposition 4.3.5). We will also give a proof of the fact that proof nets abstract from inessential distinctions due to the intrinsic order of rules in sequent derivations.

A contraction criterion will be formulated and proved in Section 4.4. This criterion is a combination of Danos' contraction criterion for one-sided **MLL** (see [**Danos 90**]) and Lafonts criterion for parsing boxes (see [**Lafont 95**]). The contraction relation is terminating, though not confluent. However, we achieve confluence on a restricted domain, leading us to the main contraction theorem, Theorem 4.4.12. Our contraction criterion has the special property that a priori there is no order on the leaves of the proof structure; if the proof structure is *correct* (in the sense that it contracts properly), our criterion a posteriori provides the unique order of the leaves (Subsection 4.4.3).

We will prove correctness of cut elimination w.r.t. our contraction criterion in Section 4.5. This requires quite deep investigations on the dependency between the contraction steps in a given conversion sequence; it gives rise to particular substructures called *block* and *component*. We think our proof might serve as a key towards further results on the parallelism of the conversion steps.

In Section 4.6 we will particularly exploit the two-sidedness of our proof nets. By the notion of *dualizability* (see Example 3.2.8) provable equivalences will be distinguished.

Roughly said, turning a two-sided proof structure up-side-down yields its dualization, and a proof net is dualizable if this dualization is a proof net as well. For provably equivalent formulas A and B there are two proof nets $\mathcal{P}_1$ of $A \vdash B$ and $\mathcal{P}_2$ of $B \vdash A$. Now A and B are called dualizable-provably equivalent if moreover $\mathcal{P}_1$ and $\mathcal{P}_2$ are each others dualization (see Subsection 3.2.3). Formulas will be shown (Theorem 4.6.3) to be dualizable-provably equivalent precisely if they are equal modulo De Morgan equivalence and associativity, i.e. if $A \simeq B$ (see Definition 2.3.1).

By the maps in Subsection 3.2.5, we obtain the corresponding contraction criterion for one-sided proof structures in Section 4.7. We notice that one-sided proof structures are not a particular kind of two-sided proof structures; the 'identity map' ν from the one-sided language to the two-sided language does not in general extend to proof structures (see Remark 3.2.15). On the contrary, the translation ψ provides the appropriate embedding. This translation avoids having to map an axiomatic (cut) edge into a right (left) $\perp$-link.

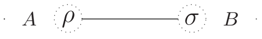
In Section 4.8 we will introduce the category of formulas and proof nets, where once again we observe the elegance of the two-sided theory. The isomorphisms in this category are exactly the dualizable proof nets of Section 4.6 (Theorem 4.8.3).

A very practical general consequence of the two-sided system is the step towards the intuitionistic fragment (Section 4.9), which is simply obtained as 'the theory of proof nets with one conclusion'. Starting from **NCLL**, we obtain a theory of proof nets for the Lambek calculus (**L**). This theory can be viewed as the linear and non-commutative version of Natural deduction for **IL**.

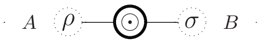
Finally, in Section 4.10 the connection with commutative **MLL** is made. **MLL** is obtained from **NCLL** by adding the rule of EXCHANGE.

4.1. Sequent calculus

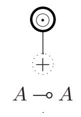
A sequent of $\mathbf{NCLL}_2$ is an $\mathfrak{L}_2$-link graph $\mathcal{P}$ containing exactly one cyclic link $l = \llparenthesis e_0, \ldots, e_{m-1} \rrparenthesis_{\odot}$ as context link, no connector links, and whose underlying graph is a tree, i.e. acyclic and connected. Because of the last requirement, every edge η has exactly one extremity $\hat{\eta}$ occurring in l, whence $\mathcal{P}$ may be represented by the cyclic list $\llparenthesis \lambda\check{\eta}_0, \ldots, \lambda\check{\eta}_{m-1} \rrparenthesis$ of open ends. Observe that a one-edge link graph

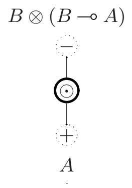


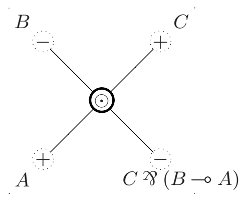
is not a sequent; there must be one context link, like in



Example 4.1.1. The following are examples of $\mathbf{NCLL}_2$-sequents with 0, 1, 2 respectively 4 open ends:





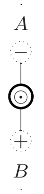


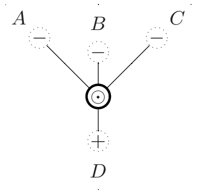
$(\![\;]\!)$ $\quad$ $(\![(A \multimap A)^+]\!)$ $\quad$ $(\![(B \otimes (B \multimap A))^-, A^+]\!)$ $\quad$ $(\![B^-, A^+, (C \parr (B \multimap A))^-, C^+]\!)$

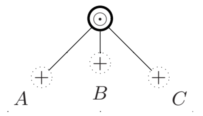
$\Diamond$

By $C_{m-1}, \ldots, C_i \vdash C_0, \ldots, C_{i-1}$ (where both sides are lists) we will denote the sequent $\big(\!\big[C_0^+, \ldots, C_{i-1}^+, C_i^-, \ldots, C_{m-1}^-\big]\!\big)$. A sequent which can be represented in this way is called a *separable* sequent. The last example shows that for $m \geq 4$ there exist non-separable m-sequents; for $m \leq 3$ every sequent is separable.

Example 4.1.2. The following are examples of separable $\mathbf{NCLL}_2$-sequents:







$A \vdash B$ $\qquad$ $A, B, C \vdash D$ $\qquad$ $\vdash A, B, C$ (or $\vdash B, C, A$ or $\vdash C, A, B$)

$\Diamond$

The calculus $\mathbf{NCLL}_2$ is defined by the following *(elementary) rules*:

NCLL$_2$

$$\frac{}{\llparenthesis A^+, A^- \rrparenthesis}\ \textsc{Ax}$$

$$\frac{\llparenthesis \Gamma, A^+ \rrparenthesis \qquad \llparenthesis \Delta, A^- \rrparenthesis}{\llparenthesis \Gamma, \Delta \rrparenthesis}\ \textsc{Cut}$$

$$\frac{\llparenthesis \Gamma, A^+ \rrparenthesis}{\big\llparenthesis \Gamma, ((A)^\perp)^- \big\rrparenthesis}\ \text{L}\perp \qquad \frac{\llparenthesis \Gamma, A^- \rrparenthesis}{\big\llparenthesis \Gamma, ((A)^\perp)^+ \big\rrparenthesis}\ \text{R}\perp$$

$$\begin{array}{c} \llparenthesis \Gamma, B^-, A^- \rrparenthesis \\ \hdashline \llparenthesis \Gamma, (A \otimes B)^- \rrparenthesis \end{array}\ \text{L}\otimes \qquad \frac{\llparenthesis \Gamma, A^+ \rrparenthesis \qquad \llparenthesis \Delta, B^+ \rrparenthesis}{\llparenthesis \Gamma, (A \otimes B)^+, \Delta \rrparenthesis}\ \text{R}\otimes$$

$$\frac{\llparenthesis \Gamma, A^+ \rrparenthesis \qquad \llparenthesis \Delta, B^- \rrparenthesis}{\llparenthesis \Gamma, (B \multimapinv A)^-, \Delta \rrparenthesis}\ \text{L}\multimapinv \qquad \begin{array}{c} \llparenthesis \Gamma, B^+, A^- \rrparenthesis \\ \hdashline \llparenthesis \Gamma, (B \multimapinv A)^+ \rrparenthesis \end{array}\ \text{R}\multimapinv$$

$$\frac{\llparenthesis \Gamma, A^- \rrparenthesis \qquad \llparenthesis \Delta, B^+ \rrparenthesis}{\llparenthesis \Gamma, (B \multimap A)^-, \Delta \rrparenthesis}\ \text{L}\multimap \qquad \begin{array}{c} \llparenthesis \Gamma, B^-, A^+ \rrparenthesis \\ \hdashline \llparenthesis \Gamma, (B \multimap A)^+ \rrparenthesis \end{array}\ \text{R}\multimap$$

$$\frac{\llparenthesis \Gamma, A^- \rrparenthesis \qquad \llparenthesis \Delta, B^- \rrparenthesis}{\llparenthesis \Gamma, (B \parr A)^-, \Delta \rrparenthesis}\ \text{L}\parr \qquad \begin{array}{c} \llparenthesis \Gamma, B^+, A^+ \rrparenthesis \\ \hdashline \llparenthesis \Gamma, (B \parr A)^+ \rrparenthesis \end{array}\ \text{R}\parr$$

which may be condensed to

NCLL$_2$

$$\frac{}{\llparenthesis A^+, A^- \rrparenthesis}\ \textsc{Ax}$$

$$\frac{\llparenthesis \Gamma, A^+ \rrparenthesis \qquad \llparenthesis \Delta, A^- \rrparenthesis}{\llparenthesis \Gamma, \Delta \rrparenthesis}\ \textsc{Cut}$$

$$\frac{\llparenthesis \Gamma, A^\rho \rrparenthesis}{\big\llparenthesis \Gamma, ((A)^\perp)^{-\rho} \big\rrparenthesis}\ \perp$$

$$\frac{\llparenthesis \Gamma, A^\rho \rrparenthesis \qquad \llparenthesis \Delta, B^\sigma \rrparenthesis}{\llparenthesis \Gamma, A^\rho \otimes B^\sigma, \Delta \rrparenthesis}\ \otimes \qquad \begin{array}{c} \llparenthesis \Gamma, B^\sigma, A^\rho \rrparenthesis \\ \hdashline \llparenthesis \Gamma, B^\sigma \parr A^\rho \rrparenthesis \end{array}\ \parr$$

Recall that the latter $\otimes$ and $\parr$ are the maps $\mathfrak{L}_2^\pm \times \mathfrak{L}_2^\pm \to \mathfrak{L}_2^\pm$ defined on page 22.

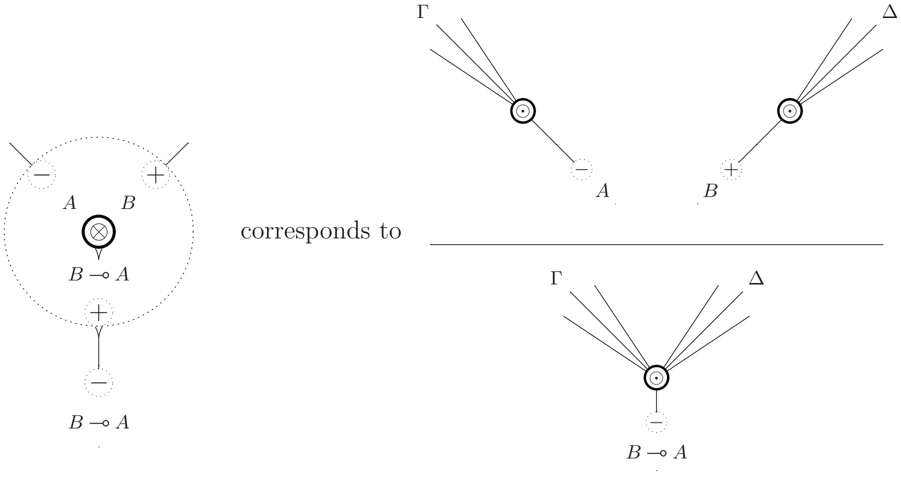
Every rule consists of a (possibly empty) list of premiss sequents, and one conclusion sequent. The distinguished (polarized) formulas in the former are called the *active formulas*, while those in the latter are called the *main formulas*. All other formulas (occurring in Γ and Δ) constitute the *context*. We make the following observation, which is characteristic for a multiplicative calculus:

REMARK 4.1.3. **(Multiplicativity)** The context of the conclusion sequent equals the concatenation of the context(s) of the premiss sequent(s) (in a specified order). $\lozenge$

The rules AX and CUT are the *identity rules*. The rule indicated by $\perp$ actually stands for two rules, which we call L$\perp$ (if the sign $-\rho$ of the main formula is negative) and R$\perp$ (if $-\rho = +$). Together these two rules constitute the *negation rules*. Furthermore, the rule indicated by $\otimes$ stands for the four so-called *tensor rules* R$\otimes$, L$\multimapinv$, L$\multimap$, L$\parr$, which we call L$\square$ or R$\square$ depending on the sign ($-$ respectively $+$) and the outermost connective $\square$ of its main formula $A^\rho \otimes B^\sigma$. Finally, the rule indicated by $\parr$ stands for the four so-called *par rules* L$\otimes$, R$\multimapinv$, R$\multimap$, R$\parr$, referring to the main formula $B^\sigma \parr A^\rho$. The negation rules, the tensor rules and the par rules together constitute the *logical rules*.

REMARK 4.1.4. Given signs ρ and σ and $\mathfrak{L}_2$-formulas A and B, the polarized formulas $A^\rho \otimes B^\sigma$ and $B^{-\sigma} \parr A^{-\rho}$ are equal up to their signs, which are opposite. Hence there is a unique connective $\square$ such that $A^\rho \otimes B^\sigma$ is the main formula of the tensor rule L$\square$ (R$\square$), while $B^{-\sigma} \parr A^{-\rho}$ is the main formula of the par rule R$\square$ (L$\square$). $\lozenge$

As a consequence of the above observation that contexts behave additively, a rule is completely determined by its active and main formulas, together with the way contexts should be concatenated. Now this information is precisely contained in the $\mathcal{L}_2$-links (see Definition 3.2.3 and Figure 3.1), when we interpret a tensor link $\otimes$ as to work on two premiss sequents, and a par link $\parr$ and a negation link $\perp$ as to work on one premiss sequent. E.g., the L$\multimap$-rule is completely determined by the L$\multimap$-link:



This correspondence will be exploited in the theory of proof nets in Section 4.3, and serves as our guide when defining the contraction relation in Section 4.4.

A composite of instances of elementary rules is called a *semi-derivation* (or *generalized rule*). Each sequent occurring in a semi-derivation is of the form

$$\left(\!\left[\widetilde{\Gamma}_0, A_0^{\rho_0}, \ldots, \widetilde{\Gamma}_{m-1}, A_{m-1}^{\rho_{m-1}}\right]\!\right), \tag{1}$$

where each $\widetilde{\Gamma}_i$ is either vacuous, or a so-called *context variable* Γ_j for some j. The $A_i^{\rho_i}$ in the premiss sequents are called the active formulas of the semi-derivation, while the $A_i^{\rho_i}$ in the conclusion sequent are called the main formulas. One can easily show that every context variable Γ_j of a premiss sequent occurs exactly once as a context variable of the conclusion sequent. There are no restrictions w.r.t. multiple occurrences of (polarized) formulas or context variables, as Example 4.1.5 illustrates.

The most simple example of a semi-derivation is a single sequent consisting of a single context variable $(\![\Gamma]\!)$, which is the composite of zero rules, having coinciding premiss sequent and conclusion sequent $(\![\Gamma]\!)$. A semi-derivation with no premiss sequents is a *derivation*. Of course, no sequent in a derivation can contain a context variable (since no axiom does). For a derivation $\mathcal{D}$, let $\llcorner\mathcal{D}\lrcorner$ denote its final sequent. A sequent is called *derivable* iff it occurs as the final sequent of some derivation.

Example 4.1.5. This semi-derivation has three main formulas, and seven active formulas.

$$\begin{array}{c}
\dfrac{\dfrac{\left(\!\left[\Gamma_1, \boxed{C^+}, \Gamma_2, \boxed{B^+}, \boxed{A^-}, \boxed{B^+}, \Gamma_2, \boxed{A^-}\right]\!\right) \qquad \left(\!\left[\Gamma_4, \boxed{B^+}\right]\!\right)}{(\![\Gamma_1, C^+, \Gamma_2, B^+, A^-, B^+, \Gamma_2, (B \multimap A)^-, \Gamma_4]\!)}\,{\scriptstyle \mathrm{L}\multimap} \qquad \left(\!\left[\Gamma_1, \boxed{C^-}\right]\!\right)}{(\![\Gamma_2, B^+, A^-, B^+, \Gamma_2, (B \multimap A)^-, \Gamma_4, \Gamma_1, \Gamma_1]\!)}\,{\scriptstyle \textsc{Cut}} \\
\hdashline
\left(\!\left[\Gamma_2, \boxed{(B \multimapinv A)^+}, \boxed{B^+}, \Gamma_2, \boxed{(B \multimap A)^-}, \Gamma_4, \Gamma_1, \Gamma_1\right]\!\right)
\end{array}\ {\scriptstyle \mathrm{R}\multimapinv}$$

The next semi-derivation (which is just an instance of R⊗) has three occurrences of the context variable Γ in the conclusion sequent. Each one corresponds with an occurrence of Γ in one of the premiss sequents (multiplicativity), but there is no *canonical* bijection between both sets of occurrences.

$$\dfrac{(\![\Gamma, (A \otimes A)^+, \Gamma, A^+]\!) \qquad (\![A^+, \Gamma, (A \otimes A)^+]\!)}{(\![\Gamma, (A \otimes A)^+, \Gamma, (A \otimes A)^+, \Gamma, (A \otimes A)^+]\!)}\,{\scriptstyle \mathrm{R}\otimes}$$

The absence of a canonical bijection also holds for the occurrences of the formula $(A \otimes A)^+$; it is impossible to distinguish between the occurrence which has just been introduced, the occurrence originating from the left premiss sequent, and the one originating from the right premiss sequent. In Lemma 4.2.5 we will show that nevertheless there will always be a canonical bijection between the non-main formula occurrences of the conclusion sequent of an elementary rule on the one hand, and the non-active formula occurrences of the premiss sequents on the other hand, *when these premiss sequents are derivable*. As a consequence, we will never encounter an instance of this semi-derivation in a derivation.

The following is an example of a semi-derivation without premiss sequents, i.e. an $\mathbf{NCLL}_2$-derivation $\mathcal{D}$. It shows that $\llcorner\mathcal{D}\lrcorner = (\![(C \multimapinv B)^+, A^+, (C \parr (B \multimap A))^-]\!)$ is derivable.

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{}{(\![A^+,A^-]\!)}\,\mathrm{Ax} \qquad \dfrac{}{(\![B^+,B^-]\!)}\,\mathrm{Ax}}{(\![A^+,(B\multimap A)^-,B^-]\!)}\,\mathrm{L}\!\multimap \qquad \dfrac{}{(\![C^-,C^+]\!)}\,\mathrm{Ax}}{(\![B^-,A^+,(C\parr(B\multimap A))^-,C^+]\!)}\,\mathrm{L}\!\parr \\ \hdashline (\![(C\multimapinv B)^+,A^+,(C\parr(B\multimap A))^-]\!) \end{array}\;\mathrm{R}\!\multimapinv$$

$\Diamond$

An expression

$$\begin{array}{c} (\![\Gamma_0]\!) \qquad \dots \qquad (\![\Gamma_{n-1}]\!) \\ \hline\hline (\![\Gamma]\!) \end{array} \tag{2}$$

given by a list of premiss sequents $(\![\Gamma_i]\!)$ and a conclusion sequent $(\![\Gamma]\!)$, all of the form (1), is an *induced rule* (or *derived rule*) if there is a semi-derivation with premiss sequents among the n sequents $(\![\Gamma_0]\!)$ up to $(\![\Gamma_{n-1}]\!)$ (used ad libitum) and conclusion sequent $(\![\Gamma]\!)$. More precisely, an induced rule is given by an expression (2) together with a semi-derivation $\mathcal{D}$ (with m premiss sequents $(\![\Delta_k]\!)$ and conclusion sequent $(\![\Gamma]\!)$) and a function $f : \{0,\dots,m-1\} \to \{0,\dots,n-1\}$, such that $(\![\Delta_k]\!) = \big(\!\big[\Gamma_{f(k)}\big]\!\big)$.

Given only the expression (2) of an induced rule, in many cases we can reconstruct information about the semi-derivation $\mathcal{D}$ and the function f because of the multiplicativity of our calculus. Indeed, comparing the context variables of the sequent $(\![\Gamma]\!)$ to the context variables of the $(\![\Gamma_i]\!)$, we can get an impression how many times $|f^{-1}(i)|$ a certain premiss sequent $(\![\Gamma_i]\!)$ actually occurs in $\mathcal{D}$. This method is useful in case the $(\![\Gamma_i]\!)$ have pair-wise disjoint context variables.

We will allow application of an induced rule in a (semi-)derivation, in which case it should be seen as an abbreviation of the corresponding semi-derivation. Formally, we define

$$\begin{array}{c} \mathcal{D}_0 \qquad\qquad\quad \mathcal{D}_{n-1} \\ (\![\Gamma_0]\!) \qquad \dots \qquad (\![\Gamma_{n-1}]\!) \\ \hline\hline (\![\Gamma]\!) \end{array}\;(\mathcal{D},f)$$

to denote[1]

$$\begin{array}{c} \mathcal{D}_{f(0)} \qquad\qquad\quad \mathcal{D}_{f(m-1)} \\ (\![\Delta_0]\!) \qquad \dots \qquad (\![\Delta_{m-1}]\!) \\ \ddots \quad \mathcal{D} \quad .\!\cdot\!\cdot \\ (\![\Gamma]\!) \end{array}$$

where some subderivations $\mathcal{D}_i$ may have disappeared, while others may have been copied, depending on $|f^{-1}(i)|$.

Example 4.1.6. Consider the semi-derivation

$$\mathcal{D} := \quad \dfrac{(\![\Gamma,A^\rho]\!) \qquad (\![\Delta,B^\sigma]\!)}{(\![\Gamma,A^\rho\otimes B^\sigma,\Delta]\!)}\,\otimes$$

---

[1] Observe that by definition (semi-)derivations only consist of elementary rules, so

$$\begin{array}{c} \mathcal{D}_0 \qquad\qquad\quad \mathcal{D}_{n-1} \\ (\![\Gamma_0]\!) \qquad \dots \qquad (\![\Gamma_{n-1}]\!) \\ \hline\hline (\![\Gamma]\!) \end{array}\;(\mathcal{D},f)$$

is not a derivation in the proper sense of the word; it only stands for a derivation. Cf. the fact that $1+1$ is not a natural number; it only stands for a natural number.

It gives rise to the induced rules

$$\begin{array}{c} \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Delta, B^{\sigma}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes B^{\sigma}, \Delta\rrparenthesis \end{array} \otimes := (\mathcal{D}, \{0,1\} \xrightarrow{\substack{0 \mapsto 0 \\ 1 \mapsto 1}} \{0,1\}) \qquad \begin{array}{c} \llparenthesis\Delta, B^{\sigma}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes B^{\sigma}, \Delta\rrparenthesis \end{array} \otimes' := (\mathcal{D}, \{0,1\} \xrightarrow{\substack{0 \mapsto 1 \\ 1 \mapsto 0}} \{0,1\})$$

Moreover, the next instance of $\mathcal{D}$

$$\mathcal{D}' := \quad \frac{\llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis}{\llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis} \otimes$$

is a witness for

$$\begin{array}{c} \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes'' := (\mathcal{D}', \{0,1\} \xrightarrow{\substack{0 \mapsto 0 \\ 1 \mapsto 0}} \{0\}) \qquad \begin{array}{c} \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Delta, B^{\sigma}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes''' := (\mathcal{D}', \{0,1\} \xrightarrow{\substack{0 \mapsto 0 \\ 1 \mapsto 0}} \{0,1\})$$

Now let us apply the induced rules $\otimes$, $\otimes'$ and $\otimes'''$ to a pair of different derivations with coinciding final sequent $\llparenthesis\Gamma, A^{\rho}\rrparenthesis$. Then

$$\begin{array}{c} \mathcal{D}_0 \qquad\qquad \mathcal{D}_1 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes \quad = \quad \begin{array}{c} \mathcal{D}_0 \qquad\qquad \mathcal{D}_1 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes$$

$$\begin{array}{c} \mathcal{D}_0 \qquad\qquad \mathcal{D}_1 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes' \quad = \quad \begin{array}{c} \mathcal{D}_1 \qquad\qquad \mathcal{D}_0 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes$$

$$\begin{array}{c} \mathcal{D}_0 \qquad\qquad \mathcal{D}_1 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes''' \quad = \quad \begin{array}{c} \mathcal{D}_0 \qquad\qquad \mathcal{D}_0 \\ \llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad \llparenthesis\Gamma, A^{\rho}\rrparenthesis \\ \hline \llparenthesis\Gamma, A^{\rho} \otimes A^{\rho}, \Gamma\rrparenthesis \end{array} \otimes$$

yielding three different derivations. ◊

When f is not surjective, the induced rule $\left(\begin{array}{c} \llparenthesis\Gamma_0\rrparenthesis \quad \ldots \quad \llparenthesis\Gamma_{n-1}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma\rrparenthesis \end{array}, \mathcal{D}, f \right)$ is said to have *dummy* premiss sequents. If it does not have dummy sequents, the expression (2) is superfluous, as it is completely determined by $\mathcal{D}$ and f. When f is not injective, $\left(\begin{array}{c} \llparenthesis\Gamma_0\rrparenthesis \quad \ldots \quad \llparenthesis\Gamma_{n-1}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma\rrparenthesis \end{array}, \mathcal{D}, f \right)$ is said to have *doublings*. An induced rule without dummy premiss sequents and without doublings (i.e. with f a bijection) is clearly multiplicative, since every semi-derivation is. Each semi-derivation $\mathcal{D}$ gives rise to a particular induced rule $(\mathcal{D}, \mathrm{id})$. In this case the list of premiss sequents of $(\mathcal{D}, \mathrm{id})$ can be given a natural tree structure (viz. corresponding to the tree structure of $\mathcal{D}$)[2].

The next trivial lemma shows that — as far as derivability is concerned — we may consider the premiss sequents of an induced rule to constitute a set.

LEMMA 4.1.7. *Suppose* $\left(\begin{array}{c} \llparenthesis\Gamma_0\rrparenthesis \quad \ldots \quad \llparenthesis\Gamma_{n-1}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma\rrparenthesis \end{array}, \mathcal{D}, f \right)$ *is an induced rule. Then*

$$\begin{array}{c} \llparenthesis\Gamma'_0\rrparenthesis \quad \ldots \quad \llparenthesis\Gamma'_{n'-1}\rrparenthesis \\ \hline\hline \llparenthesis\Gamma\rrparenthesis \end{array}$$

is an induced rule, whenever the lists of premiss sequents are equal as sets. ◊

---

[2] An example of such an induced rule would be

$$\frac{\llparenthesis\Gamma, A^{\rho}\rrparenthesis \qquad (\quad \llparenthesis\Delta, B^{\sigma}\rrparenthesis \qquad \llparenthesis\Pi, C^{\nu}\rrparenthesis \quad)}{\llparenthesis\Gamma, A^{\rho} \otimes (B^{\sigma} \otimes C^{\nu}), \Pi, \Delta\rrparenthesis}$$

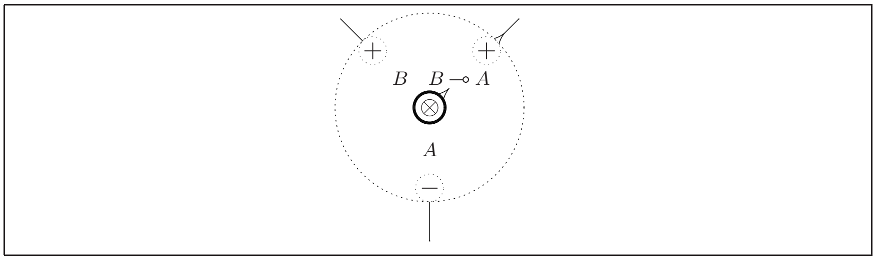


FIGURE 4.1. The L⊸-link of $\mathfrak{L}_2$: $\left(\left(B \multimap A\right)^{+}, B^{+}, A^{-}\right)_{\otimes}$.

PROOF: We know $\llparenthesis\Gamma_i\rrparenthesis = \left(\left\llbracket\Gamma'_{g(i)}\right\rrbracket\right)$ for some $g : \{0, \ldots, n-1\} \rightarrow \{0, \ldots, n'-1\}$. Now compose f with g. ////

An expression (2) is an *admissible rule* if from derivability of the premiss sequents we can conclude derivability of the conclusion sequent (but not necessarily by composition with a fixed semi-derivation). An expression (2) is *reversible* if the n *reversals*

$$\begin{array}{c}\llparenthesis\Gamma\rrparenthesis\\ \hline\hline \llparenthesis\Gamma_j\rrparenthesis\end{array}$$

are induced rules. It is *invertible* if these n reversals are admissible rules.

Every semi-derivation gives rise to an induced rule, by taking $f = \mathrm{id}$. Every induced rule is an admissible rule. The converse does not generally hold. In Section 4.2 we will show that in the CUT-free $\mathbf{NCLL}_2$ the CUT-rule is admissible, but not induced. Of course, adding admissible rules to our calculus does not harm, as the set of derivable sequents does not increase. The other way around, if a rule (2) can be added to our calculus harmlessly, it is an admissible rule.

Observe that a par rule may only be applied to a sequent in which the two active formulas B^{σ} and A^{ρ} are next to each other, in that order. In the (semi-)derivations we will indicate par rules by dashed horizontal lines. The reversal of a par rule will turn out to be an induced rule. So the par rules are reversible. In the CUT-free calculus this reversal is not an induced rule anymore (since the length[3] of the premiss sequent is strictly greater than the length of the conclusion sequent), but still admissible (simply since CUT is admissible). We conclude the par rules are only invertible now.

Given a calculus $\mathbf{C}$, two expressions RULE_1 and RULE_2 of the form (2) are *strongly equivalent* rules if RULE_1 is an induced rule in the calculus $\mathbf{C} + \mathrm{RULE}_2$, and visa versa. They are *equivalent* if RULE_1 is an admissible rule in $\mathbf{C} + \mathrm{RULE}_2$, and visa versa.

Under the presence of the identity rules, there are many alternative formulations of $\mathbf{NCLL}_2$. E.g. the following rules are all strongly equivalent in the calculus without L⊸, and hence each one may serve as an alternative to the L⊸ rule:

$$\frac{}{\llparenthesis(B \multimap A)^{-}, B^{-}, A^{+}\rrparenthesis}\multimap\mathrm{Ax}$$

---

[3] See Section 4.2 for a precise definition of the length $\mathsf{l}(\Gamma)$ of a list Γ.

$$\frac{([\Pi, (B \multimap A)^+])}{([\Pi, B^-, A^+])}\,\text{R}\multimap^{-1} \qquad \frac{([\Delta, B^+])}{([(B \multimap A)^-, \Delta, A^+])} \qquad \frac{([\Gamma, A^-])}{([(B \multimap A)^-, B^-, \Gamma])}$$

$$\frac{([\Delta, B^+]) \qquad ([\Gamma, A^-])}{([(B \multimap A)^-, \Delta, \Gamma])}\,\text{L}\multimap' \quad \frac{([\Pi, (B \multimap A)^+]) \qquad ([\Gamma, A^-])}{([\Pi, B^-, \Gamma])} \quad \frac{([\Pi, (B \multimap A)^+]) \qquad ([\Delta, B^+])}{([\Pi, \Delta, A^+])}$$

$$\frac{([\Pi, (B \multimap A)^+]) \qquad ([\Delta, B^+]) \qquad ([\Gamma, A^-])}{([\Pi, \Delta, \Gamma])}$$

In the same way as the L⊸-rule is completely determined by the L⊸-link, actually each of these rules is completely determined by the L⊸-link (see Figure 4.1), if we extend some of its ends D^ρ to open ends labeled by $D^{-\rho}$, and interpret it as to work on a number of premiss sequents, each one having as its unique active formula one of the remaining link ends E^σ.

Strong equivalence of the mentioned rules will be an immediate consequence of the theory of proof nets in Section 4.3. Also, observe that every rule is a particular instance of

$$\frac{([\Pi, (B \multimap A)^+])^c \qquad ([\Delta, B^+])^b \qquad ([\Gamma, A^-])^a}{\Big(\Big[\big(\Pi\big)^c, \big((B \multimap A)^-\big)^{1-c}, \big(\Delta\big)^b, \big(B^-\big)^{1-b}, \big(\Gamma\big)^a, \big(A^+\big)^{1-a}\Big]\Big)}\,\multimap(a, b, c)$$

where $a, b, c \in \{0, 1\}$, and $\big(X\big)^x$ stands for x copies of the formula (sequent, derivation, rule) X. Now we can show each of them is strongly equivalent to ⊸Ax (i.e. the instance $(a, b, c) = (0, 0, 0)$), which proves our claim.

As a consequence, we observe that the R⊸ rule is reversible, its reversal $\text{R}\multimap^{-1}$ being strongly equivalent to L⊸.

The same applies to the negation rules L⊥ and R⊥, and the other tensor rules, yielding the next lemma. (We only give three of the eight (four) strongly equivalent formulations, viz. $(a, b, c) = (0, 0, 0)$, $(0, 0, 1)$, $(1, 1, 0)$. Here b is dummy for the negation rules, having only one premiss sequent).

Lemma 4.1.8. *In each row every pair of rules is strongly equivalent.*

$$\frac{}{\big(\big[((A)^\perp)^{-\rho}, A^{-\rho}\big]\big)} \qquad \frac{\big(\big[\Pi, ((A)^\perp)^\rho\big]\big)}{([\Pi, A^{-\rho}])}\,\perp^{-1} \qquad \frac{([\Gamma, A^\rho])}{\big(\big[((A)^\perp)^{-\rho}, \Gamma\big]\big)}\,\perp$$

$$\frac{}{([A^\rho \otimes B^\sigma, B^{-\sigma}, A^{-\rho}])} \qquad \frac{([\Pi, B^{-\sigma} \parr A^{-\rho}])}{([\Pi, B^{-\sigma}, A^{-\rho}])}\,\parr^{-1} \qquad \frac{([\Delta, B^\sigma]) \qquad ([\Gamma, A^\rho])}{([A^\rho \otimes B^\sigma, \Delta, \Gamma])}\,\otimes$$

Let us write out the particular instances for the different signs ρ and σ.

$$\frac{}{(\![((A)^{\bot})^{-},A^{-}]\!)} \qquad \frac{(\![\Pi,((A)^{\bot})^{+}]\!)}{(\![\Pi,A^{-}]\!)}\,\mathrm{R}\bot^{-1} \qquad \frac{(\![\Gamma,A^{+}]\!)}{(\![((A)^{\bot})^{-},\Gamma]\!)}\,\mathrm{L}\bot$$

$$\frac{}{(\![((A)^{\bot})^{+},A^{+}]\!)} \qquad \frac{(\![\Pi,((A)^{\bot})^{-}]\!)}{(\![\Pi,A^{+}]\!)}\,\mathrm{L}\bot^{-1} \qquad \frac{(\![\Gamma,A^{-}]\!)}{(\![((A)^{\bot})^{+},\Gamma]\!)}\,\mathrm{R}\bot$$

$$\frac{}{(\![(A\otimes B)^{+},B^{-},A^{-}]\!)} \qquad \frac{(\![\Pi,(A\otimes B)^{-}]\!)}{(\![\Pi,B^{-},A^{-}]\!)}\,\mathrm{L}\otimes^{-1} \qquad \frac{(\![\Delta,B^{+}]\!) \qquad (\![\Gamma,A^{+}]\!)}{(\![(A\otimes B)^{+},\Delta,\Gamma]\!)}\,\mathrm{R}\otimes$$

$$\frac{}{(\![(B\multimapinv A)^{-},B^{+},A^{-}]\!)} \qquad \frac{(\![\Pi,(B\multimapinv A)^{+}]\!)}{(\![\Pi,B^{+},A^{-}]\!)}\,\mathrm{R}\!\multimapinv^{-1} \qquad \frac{(\![\Delta,B^{-}]\!) \qquad (\![\Gamma,A^{+}]\!)}{(\![(B\multimapinv A)^{-},\Delta,\Gamma]\!)}\,\mathrm{L}\!\multimapinv$$

$$\frac{}{(\![(B\multimap A)^{-},B^{-},A^{+}]\!)} \qquad \frac{(\![\Pi,(B\multimap A)^{+}]\!)}{(\![\Pi,B^{-},A^{+}]\!)}\,\mathrm{R}\!\multimap^{-1} \qquad \frac{(\![\Delta,B^{+}]\!) \qquad (\![\Gamma,A^{-}]\!)}{(\![(B\multimap A)^{-},\Delta,\Gamma]\!)}\,\mathrm{L}\!\multimap$$

$$\frac{}{(\![(B\parr A)^{-},B^{+},A^{+}]\!)} \qquad \frac{(\![\Pi,(B\parr A)^{+}]\!)}{(\![\Pi,B^{+},A^{+}]\!)}\,\mathrm{R}\!\parr^{-1} \qquad \frac{(\![\Delta,B^{-}]\!) \qquad (\![\Gamma,A^{-}]\!)}{(\![(B\parr A)^{-},\Delta,\Gamma]\!)}\,\mathrm{L}\!\parr$$

◊

COROLLARY 4.1.9. *The negation rules and the par rules are reversible.* ◊

PROOF: See the second column of the previous lemma. A direct proof uses their respective counterparts and the identity rules AX and CUT. ////

As an immediate consequence of this corollary we have

LEMMA 4.1.10. ***NCLL***$_2$ *satisfies the following* adjunctions*:*

$$\begin{array}{lcll} A\otimes(-) & \dashv & A\multimap(-) & \textit{(for all formulas A)}\\ (-)\otimes A & \dashv & (-)\multimapinv A & \textit{(for all formulas A)} \end{array}$$

i.e. the following expressions and their reversals are induced rules

$$\frac{A\otimes B\ \vdash\ C}{\overline{B\ \vdash\ A\multimap C}}\Updownarrow \qquad\qquad \frac{B\otimes A\ \vdash\ C}{\overline{B\ \vdash\ C\multimapinv A}}\Updownarrow$$

Recall that we use $D\ \vdash\ E$ to denote the sequent . ◊

The lemma even holds if we interpret the word 'adjunction' in the categorical sense, in the appropriate category; see Section 4.8.

Let us mention the unit and co-unit of the first adjunction. Taking $A\otimes B$ for C we find the unit is given by the derivable sequent

$$B\ \vdash\ A\multimap(A\otimes B),$$

and taking $A\multimap C$ for B yields the co-unit

$$A\otimes(A\multimap C)\ \vdash\ C.$$

The next Soundness Lemma is easily proved by induction on the derivation. For the definitions of the counting maps $\#_{\Box}(-)$, $\llbracket - \rrbracket$ and $\natural(-)$ we refer to Subsection 2.1.6.

Lemma 4.1.11. *Let* $([\Gamma])$ *be derivable in* $\boldsymbol{NCLL}_2$*. Then the following holds:*

$$|\Gamma| + \#_{\parr}(\Gamma) = \#_{\otimes}(\Gamma) + 2 \qquad \textit{and} \qquad \llbracket \Gamma \rrbracket = \Lambda$$

In particular, if $([B^+])$ *is derivable, then* $\#_{\parr}(B) - \#_{\otimes}(B) = 1$ *and* $\llbracket B \rrbracket = \Lambda$*.*

Also, if $A \vdash B$ *(i.e.* $([A^-, B^+])$*) is derivable, then*

$$\begin{aligned} \#_{\otimes}(A) - \#_{\parr}(A) &= \#_{\otimes}(B) - \#_{\parr}(B) \\ \llbracket A \rrbracket &= \llbracket B \rrbracket \\ \natural(A) &= \natural(B) \end{aligned}$$

so

$$\begin{aligned} &A \mapsto \#_{\otimes}(A) - \#_{\parr}(A) \\ &A \mapsto \llbracket A \rrbracket \qquad \textit{and} \\ &A \mapsto \natural(A) \end{aligned}$$

are so-called $\boldsymbol{NCLL}_2$*-derivability invariants.* ◊

As an example, observe that the derivable sequent

$$\Gamma = \big([(C \multimapinv B)^+, A^+, (C \parr (B \multimap A))^-]\big)$$

of Example 4.1.5 obeys this lemma, as shown in Example 2.1.9.

From this lemma it is clear that the empty sequent $([\quad])$ is not derivable.

The converse of the final statements in our lemma does not hold: if

$$\begin{aligned} \#_{\otimes}(A) - \#_{\parr}(A) &= \#_{\otimes}(B) - \#_{\parr}(B) \\ \llbracket A \rrbracket &= \llbracket B \rrbracket \end{aligned}$$

(and as a consequence also $\natural(A) - \natural(B)$) it need not be the case that $A \vdash B$ is derivable. However, in [**Pentus 93**] it it is shown that in this case A and B have a *join* (i.e. a formula C such that both $A \vdash C$ and $B \vdash C$ are derivable) as well as a *meet* (i.e. a formula D such that both $D \vdash A$ and $D \vdash B$ are derivable).

In general there are m lists which represent a sequent $\Delta := \big([A_0^{\rho_0}, \ldots, A_{m-1}^{\rho_{m-1}}]\big)$, viz. $\Gamma_i := A_i^{\rho_i}, \ldots, A_{i+m-1}^{\rho_{i+m-1}}$ $(0 \leq i < m)$, where the indices should be read modulo m. The sequent Δ is *periodic* whenever two of the m representing lists Γ_i coincide. If Δ is non-periodic, the occurrence of a particular (polarized) formula is completely determined by its context, in a way to be explained in the next example.

Example 4.1.12. The sequent

$$\Delta := \left(\left[\underset{(0)}{A^+}, \underset{(1)}{A^+}, \underset{(2)}{A^+}, \underset{(3)}{A^+}, \underset{(4)}{A^+}, \underset{(5)}{A^-}\right]\right)$$

is non-periodic; writing Δ as $([A^+, \Delta'])$ where Δ' is the list A^+, A^+, A^-, A^+, A^+ we know this particular A^+ is occurrence $\underset{(2)}{A^+}$.

On the contrary, the sequent

$$\left(\left[\underset{(0)}{A^+}, \underset{(1)}{A^+}, \underset{(2)}{A^-}, \underset{(3)}{A^+}, \underset{(4)}{A^+}, \underset{(5)}{A^-}, \right]\right)$$

is periodic; writing Δ as $\llparenthesis A^+, \Delta' \rrparenthesis$ where Δ' is the list A^+, A^-, A^+, A^+, A^- this particular A^+ is either occurrence $\underset{(0)}{A^+}$ or $\underset{(3)}{A^+}$. $\Diamond$

Lemma 4.1.13. *Derivable sequents are non-periodic.* $\Diamond$

Proof: Suppose a derivable Δ is periodic, then there is a list Γ such that

$$\Delta = \llparenthesis\ \overbrace{\Gamma, \dots, \Gamma}^{k \text{ times}}\ \rrparenthesis$$

where $k > 1$. By Lemma 4.1.11 we find

$$k\,|\Gamma| + k\#_{\parr}(\Gamma) = k\#_{\otimes}(\Gamma) + 2$$

whence 2 is a k-fold, implying $k = 2$. Hence

$$|\Gamma| + \#_{\parr}(\Gamma) = \#_{\otimes}(\Gamma) + 1$$

Secondly, $\llbracket \Gamma, \Gamma \rrbracket = \Lambda$, whence $2\epsilon(\langle\!\langle \Gamma \rangle\!\rangle) = 0$ and hence also $\epsilon(\langle\!\langle \Gamma \rangle\!\rangle) = 0$. Now Lemma 2.1.8 states

$$|\Gamma| + \#_{\otimes}(\Gamma) \equiv \#_{\parr}(\Gamma) + \epsilon(\langle\!\langle \Gamma \rangle\!\rangle) \pmod 2$$

which yields a contradiction. ////

4.2. Cut elimination

In this section we will show that the cut elimination theorem for **NCLL**$_2$ holds, i.e. in the Cut-free **NCLL**$_2$ the Cut-rule is admissible (although it is not an induced rule). Because of this property, shared by all decent variations of **MLL**, for mere derivability we can restrict our attention to the Cut-free fragments, which posses some very important special properties of logical derivations.

An important consequence of the cut elimination theorem is the *subformula property* (see e.g. [**Ono 98, TS 96**]).

Theorem 4.2.1. *If* $\llparenthesis \Gamma \rrparenthesis$ *is derivable in* ***NCLL***$_2$*, then there is a derivation* $\mathcal{D}$ *of* $\llparenthesis \Gamma \rrparenthesis$ *such that any formula appearing in* $\mathcal{D}$ *is a subformula of some formula in* Γ. $\Diamond$

Proof: It is easy to see that a Cut-free derivation $\mathcal{D}$ meets the requirements, since in every inference rule except the Cut, every formula appearing in the premiss sequent(s) is a subformula of some formula in the conclusion sequent. ////

From this subformula property we can infer the following result on *conservative extensions.* Let $J \subseteq \{(-)^{\perp}, \otimes, \parr, \multimap, \multimapinv\}$ and define $\mathfrak{L}_{2,J}$ as the corresponding language.

$$\mathfrak{L}_{2,J} := \mathcal{F} ::= \mathcal{A} \mid \Big((\mathcal{F})^{\perp} \Big)_{\text{if } (-)^{\perp} \in J} \mid \Big(\mathcal{F} \,\Box\, \mathcal{F} \Big)_{\text{if } \Box \in J}$$

By the J-fragment of **NCLL**$_2$ we mean the sequent calculus whose rules are the same as those of **NCLL**$_2$, except that we will take only the identity rules Ax$_A$ and Cut$_A$ where $A \in \mathfrak{L}_{2,J}$, and the logical rules L$\Box$ and R$\Box$ for connectives $\Box \in J$.

Theorem 4.2.2. *For any sequent* $\llparenthesis \Gamma \rrparenthesis$ *in* $\mathfrak{L}_{2,J}$*,* $\llparenthesis \Gamma \rrparenthesis$ *is derivable in* ***NCLL***$_2$ *if and only if it is derivable in the* J*-fragment of* ***NCLL***$_2$*.* $\Diamond$

Proof: The if-part is immediate. The other way around, suppose that a sequent $\llparenthesis \Gamma \rrparenthesis$ in $\mathfrak{L}_{2,J}$ is derivable in **NCLL**$_2$, then by the subformula property there is a derivation $\mathcal{D}$ of $\llparenthesis \Gamma \rrparenthesis$ such that any formula appearing in $\mathcal{D}$ is a subformula of some formula in Γ. But then this is actually a derivation in the J-fragment of **NCLL**$_2$.

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This theorem immediately applies to the intuitionistic fragment of $\mathbf{NCLL}_2$ (a.k.a. the *Lambek Calculus* **L**), obtained by taking $J = \{\otimes, \multimap, \multimapinv\}$ such that $\mathfrak{L}_{2,J} = \mathfrak{L}_{2,\mathrm{i}}$. Another well-known application is to the so-called *implicational* fragment of $\mathbf{NCLL}_2$, taking $J = \{\multimap, \multimapinv\}$.

The cut elimination theorem also implies decidability of $\mathbf{NCLL}_2$, i.e. there is an algorithm that can decide whether a given formula A is derivable in $\mathbf{NCLL}_2$ or not. Indeed, given a sequent $\llparenthesis \Gamma \rrparenthesis$, it is derivable if and only if it has a Cut-free derivation. As one easily shows by induction on the number of symbols in $\llparenthesis \Gamma \rrparenthesis$, there are only finitely many possible Cut-free derivations.

4.2.1. Weak normalization. Having seen some consequences of the cut elimination theorem, let us turn now to its proof. We define the *rank* of a $\textsc{Cut}_A$-rule by the length[4] $\mathsf{l}(A)$ of the cut formula A.

Deleting all Cut-rules does not yield a calculus in which $\textsc{Cut}_A$ is an induced rule. This follows from the fact that every semi-derivation $\mathcal{D}$ with premiss sequents $\llparenthesis \Gamma_0 \rrparenthesis$ up to $\llparenthesis \Gamma_{n-1} \rrparenthesis$ and conclusion sequent $\llparenthesis \Gamma \rrparenthesis$ in the Cut-free calculus satisfies[5]

$$\sum_{i=0}^{n-1} \mathsf{l}(\Gamma_i) \leq \mathsf{l}(\Gamma).$$

Indeed, if $\textsc{Cut}_A$ would be an induced rule, there would be a semi-derivation with several premiss sequents, which however all equal $\llparenthesis \Gamma, A^+ \rrparenthesis$ or $\llparenthesis \Delta, A^- \rrparenthesis$, and with conclusion sequent $\llparenthesis \Gamma, \Delta \rrparenthesis$. As Γ occurs only once in the conclusion sequent, there is precisely one premiss sequent $\llparenthesis \Gamma, A^+ \rrparenthesis$ in this semi-derivation, and similarly precisely one premiss sequent $\llparenthesis \Delta, A^- \rrparenthesis$. But this is impossible, because $\mathsf{l}(\llparenthesis \Gamma, A^+ \rrparenthesis) + \mathsf{l}(\llparenthesis \Delta, A^- \rrparenthesis) > \mathsf{l}(\llparenthesis \Gamma, \Delta \rrparenthesis)$.

Surprisingly the Cut-free calculus is still as strong as the original, by which we mean that a sequent $\llparenthesis \Gamma \rrparenthesis$, derivable in $\mathbf{NCLL}_2$, is still derivable in the Cut-free calculus, which we will prove here.

Let the *level* $\ell(\mathcal{D})$ of a derivation $\mathcal{D}$ be defined as the number of sequent occurrences that it contains. So inductively:

$$\ell\left(\frac{\begin{array}{ccc}\mathcal{D}_0 & & \mathcal{D}_{n-1}\\ \llparenthesis \Gamma_0 \rrparenthesis & \dots & \llparenthesis \Gamma_{n-1} \rrparenthesis\end{array}}{\llparenthesis \Gamma \rrparenthesis}\right) = 1 + \sum_{i=0}^{n-1} \ell(\mathcal{D}_i) \qquad (n = 0, 1, 2).$$

When $\mathcal{D} = \dfrac{\begin{array}{ccc}\mathcal{D}_0 & & \mathcal{D}_{n-1}\\ \llparenthesis \Gamma_0 \rrparenthesis & \dots & \llparenthesis \Gamma_{n-1} \rrparenthesis\end{array}}{\llparenthesis \Gamma \rrparenthesis}\,\textsc{Rule}$ we also call $\ell(\mathcal{D})$ the level of the rule occurrence Rule or the level of the sequent occurrence $\llparenthesis \Gamma \rrparenthesis$.

---

[4] Usually the rank of a $\textsc{Cut}_A$-rule is defined by the complexity $\mathsf{c}(A)$ of the cut formula A. This *complexity* $\mathsf{c}(A)$ of a formula $A \in \mathfrak{L}_2$ is defined by the number of connectives contained in it, so

$$\begin{aligned}\mathsf{c}(\alpha_i) &:= 0\\ \mathsf{c}((A)^{\perp}) &:= \mathsf{c}(A) + 1\\ \mathsf{c}(A \,\square\, B) &:= \mathsf{c}(A) + 1 + \mathsf{c}(B) \qquad (\square = \otimes, \parr, \multimap \text{ or } \multimapinv)\end{aligned}$$

However, the definition of the rank as the length also works for our purposes.

[5] Equality holds for the semi-derivation consisting of a single sequent $\llparenthesis \Gamma \rrparenthesis$, which is the composite of zero rules, having coinciding premiss sequent and conclusion sequent $\llparenthesis \Gamma \rrparenthesis$.

Lemma 4.2.3. *Let $\mathcal{D}$ be a derivation in $\boldsymbol{NCLL}_2$ of the form*

$$\dfrac{\overset{\mathcal{D}_1}{(\![\Gamma, C^+]\!)} \qquad \overset{\mathcal{D}_2}{(\![\Delta, C^-]\!)}}{(\![\Gamma, \Delta]\!)}\,\textsc{Cut}_C \quad ,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are Cut*-free derivations. Then there exists a* Cut*-free $\mathcal{D}'$ with the same conclusion.* $\Diamond$

Proof: Induction on rank $\mathsf{l}(C)$.

$$\text{Suppose the statement holds for all } \widetilde{\mathcal{D}} \text{ with } \mathsf{l}(\tilde{C}) < n_1. \tag{3}$$

Now we have to prove that the statement holds for all $\mathcal{D}$ with $\mathsf{l}(C) = n_1$.
Induction on level $\ell(\mathcal{D})$.

$$\text{Suppose the statement holds for all } \widetilde{\mathcal{D}} \text{ with } \mathsf{l}(\tilde{C}) = n_1 \text{ and } \ell(\widetilde{\mathcal{D}}) < n_2. \tag{4}$$

We have to show that it also holds for all $\mathcal{D}$ with $\mathsf{l}(C) = n_1$ and $\ell(\mathcal{D}) = n_2$. In case not both C^+ and C^- are the main formulas in $\mathcal{D}_1$ and $\mathcal{D}_2$, this is proved by an easy permutation of the rules. E.g.,

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{(\![\Gamma_1, A^\rho]\!)} \qquad \overset{\mathcal{D}_2'}{(\![B^\sigma, \Gamma_2, C^+, \Gamma_3]\!)}}{(\![\Gamma_1, A^\rho \otimes B^\sigma, \Gamma_2, C^+, \Gamma_3]\!)}\,\otimes \qquad \overset{\mathcal{D}_2}{(\![\Delta, C^-]\!)}}{(\![\Gamma_1, A^\rho \otimes B^\sigma, \Gamma_2, \Delta, \Gamma_3]\!)}\,\textsc{Cut}_C$$

of level $2 + \ell(\mathcal{D}_1') + \ell(\mathcal{D}_2') + \ell(\mathcal{D}_2)$ reduces to

$$\dfrac{\overset{\mathcal{D}_1'}{(\![\Gamma_1, A^\rho]\!)} \qquad \dfrac{\overset{\mathcal{D}_2'}{(\![B^\sigma, \Gamma_2, C^+, \Gamma_3]\!)} \qquad \overset{\mathcal{D}_2}{(\![\Delta, C^-]\!)}}{(\![B^\sigma, \Gamma_2, \Delta, \Gamma_3]\!)}\,\textsc{Cut}_C}{(\![\Gamma_1, A^\rho \otimes B^\sigma, \Gamma_2, \Delta, \Gamma_3]\!)}\,\otimes$$

in which the appearing Cut is of level $1 + \ell(\mathcal{D}_2') + \ell(\mathcal{D}_2)$ and of the same rank, so by (4) we can eliminate this Cut.

Then we concentrate on the case that both C^+ and C^- are main. When one of them is an axiom, $\textsc{Cut}_C$ is in fact nothing else than the identity induced rule. And otherwise C^+ and C^- must be $((A)^\perp)^+$ and $((A)^\perp)^-$, or $A^\rho \otimes B^\sigma$ and $\tau A^\rho \otimes B^\sigma$, i.e. $B^{-\sigma} \parr A^{-\rho}$ (in some order), in which case we replace $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{(\![\Gamma_1, A^\rho]\!)} \qquad \overset{\mathcal{D}_2'}{(\![\Gamma_2, B^\sigma]\!)}}{(\![\Gamma_1, A^\rho \otimes B^\sigma, \Gamma_2]\!)}\,\otimes \qquad \dfrac{\overset{\mathcal{D}_1''}{(\![\Delta, B^{-\sigma}, A^{-\rho}]\!)}}{(\![\Delta, B^{-\sigma} \parr A^{-\rho}]\!)}\,\parr}{(\![\Gamma_1, \Delta, \Gamma_2]\!)}\,\textsc{Cut}_{A^\rho \otimes B^\sigma}$$

first by

$$\dfrac{\overset{\mathcal{D}_1'}{(\![\Gamma_1, A^\rho]\!)} \qquad \dfrac{\overset{\mathcal{D}_2'}{(\![\Gamma_2, B^\sigma]\!)} \qquad \overset{\mathcal{D}_1''}{(\![\Delta, B^{-\sigma}, A^{-\rho}]\!)}}{(\![\Delta, \Gamma_2, A^{-\rho}]\!)}\,\textsc{Cut}_B}{(\![\Gamma_1, \Delta, \Gamma_2]\!)}\,\textsc{Cut}_A \quad .$$

Now Cut$_B$ may be eliminated by (3), after which the same holds for Cut$_A$. Note that elimination of Cut$_B$ might increase the level of Cut$_A$ — which is originally less than the level of Cut$_{A^\rho \otimes B^\sigma}$ — so the outer-induction on Cut rank is really needed. ////

Theorem 4.2.4. ***(Cut elimination)*** *Each derivable sequent* $(\![\Gamma]\!)$ *of* $\boldsymbol{NCLL}_2$ *has a* Cut*-free derivation.* $\Diamond$

Proof: Induction on the number of occurring Cut-rules in the original derivation $\mathcal{D}$. We can decrease this number by taking a Cut$_C$ of minimal level in $\mathcal{D}$, hence with no Cut-rules above it. By the previous lemma we can eliminate it. ////

We have proved the so-called Cut elimination theorem or weak normalization theorem for **NCLL**$_2$ here. In the next subsection we will construct two reduction processes which may be applied to arbitrary Cut's, instead of only Cut's of minimal level.

4.2.2. Strong normalization. Let us consider the process where the reduction step is defined as above, extended with the following replacement for the case that $\mathcal{D}_1$ (or $\mathcal{D}_2$) ends with a Cut, in which case we shall replace $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ (\![\Gamma_1, C^+, \Gamma_2, B^+]\!)\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ (\![\Gamma_3, B^-]\!)\end{matrix}}{(\![\Gamma_1, C^+, \Gamma_2, \Gamma_3]\!)}\,\text{Cut}_B \qquad \begin{matrix}\mathcal{D}_2\\ (\![\Delta, C^-]\!)\end{matrix}}{(\![\Gamma_1, \Delta, \Gamma_2, \Gamma_3]\!)}\,\text{Cut}_C \tag{5}$$

of level $2 + \ell(\mathcal{D}_1') + \ell(\mathcal{D}_2') + \ell(\mathcal{D}_2)$ by

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ (\![\Gamma_1, C^+, \Gamma_2, B^+]\!)\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ (\![\Delta, C^-]\!)\end{matrix}}{(\![\Gamma_1, \Delta, \Gamma_2, B^+]\!)}\,\text{Cut}_C \qquad \begin{matrix}\mathcal{D}_2'\\ (\![\Gamma_3, B^-]\!)\end{matrix}}{(\![\Gamma_1, \Delta, \Gamma_2, \Gamma_3]\!)}\,\text{Cut}_B \tag{6}$$

in which Cut$_C$ is of lower level and of the same rank.

We may however arrive at a loop, since reducing the Cut$_B$ in (6) will yield (5) again. But when we remain reducing the occurrence Cut$_C$ (instead of turning to another Cut$_{\tilde{C}}$ such as Cut$_B$), rank or — if not — level of this Cut will decrease. So we redefine one reduction step as the composition[6] of the elementary reduction steps as defined above, restricting our attention to a particular Cut$_C$; things may become exact by an inductive definition. This yields a reduction process in which exactly one Cut is eliminated at every step.

Another solution for the twist (5)$\leftrightsquigarrow$(6) is considering them to be one and the same derivation

$$\dfrac{\begin{matrix}\mathcal{D}_1'\\ (\![\Gamma_1, C^+, \Gamma_2, B^+]\!)\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ (\![\Gamma_3, B^-]\!)\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ (\![\Delta, C^-]\!)\end{matrix}}{(\![\Gamma_1, \Delta, \Gamma_2, \Gamma_3]\!)}\,\text{Cut}' \quad . \tag{7}$$

So we first need to generalize the notion of a Cut.

We define a *cut tree* $\mathcal{T}$ to be a link graph which can be constructed as a number $n \geq 1$ of sequents connected to each other by means of links belonging to a new type of

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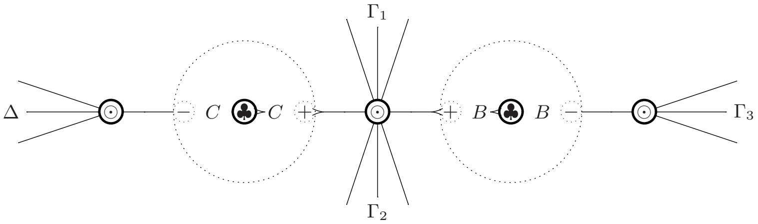
[6] Of course we mean tree-wise composition; at Cut$_{\tilde{C}}$ occurrences where both $\tilde{C}^+$ and $\tilde{C}^-$ are main, our operation may split up.

connector link[7] (called *cut link*) $(A^+, A^-)_{\textsc{Cut}}$ (indicated by ♣), and which underlying graph is a tree (implying there are $n-1$ such cut links). To $\mathcal{T}$ we assign a *generalized* Cut *rule*

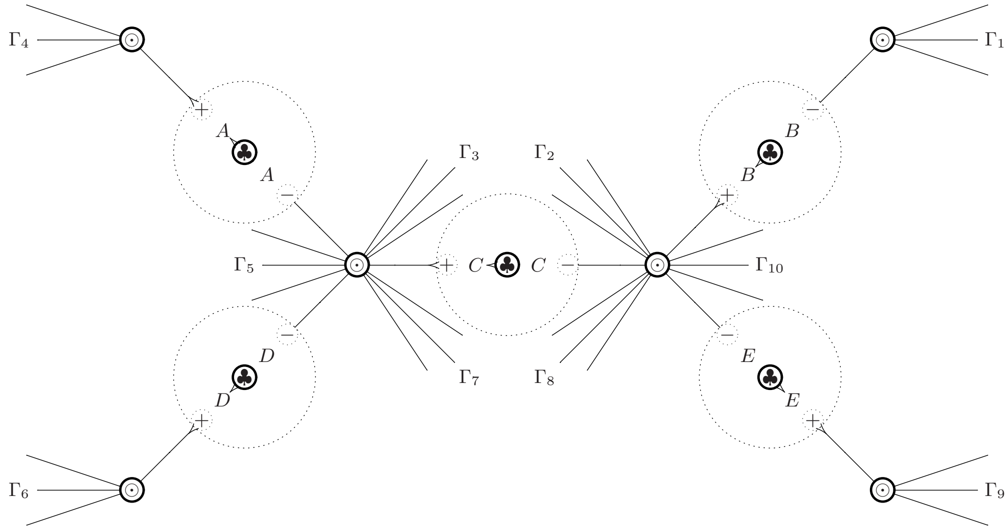
$$\frac{(\![\Gamma_0]\!) \qquad \ldots \qquad (\![\Gamma_{n-1}]\!)}{(\![\Gamma]\!)}\ \textsc{Cut}_{\mathcal{T}}$$

with premiss sequents the n sequents $(\![\Gamma_i]\!)$ constituting $\mathcal{T}$ (in some order), and with conclusion sequent the cyclic list $(\![\Gamma]\!)$ of polarized formulas obtained by enumerating the open ends of $\mathcal{T}$, when walking counterclockwisely around it.

E.g. (7) is $\textsc{Cut}_{\mathcal{T}}$ where $\mathcal{T}$ is given by



while the following cut tree



has generalized Cut rule

$$\frac{(\![\Gamma_6, D^+]\!) \quad (\![\Gamma_9, E^+]\!) \quad (\![\Gamma_4, A^+]\!) \quad (\![\Gamma_5, D^-, \Gamma_7, C^+, \Gamma_3, A^-]\!) \quad (\![\Gamma_8, E^-, \Gamma_{10}, B^+, \Gamma_2, C^-]\!) \quad (\![\Gamma_1, B^-]\!)}{(\![\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}]\!)}\ \textsc{Cut}_{\mathcal{T}}$$

Evidently for each cut tree $\mathcal{T}$ the associated generalized Cut rule $\textsc{Cut}_{\mathcal{T}}$ is an induced rule of $\mathbf{NCLL}_2$; there is a semi-derivation with the same premiss sequents, and containing one elementary Cut rule for each cut link. In particular, a one-sequent $(n = 1)$ cut tree

---

[7] The only reason we choose for a *connector* link, is the fact that labels are preserved. The choice for main formula (A^+) and active formula (A^-) is rather ad hoc.

$\mathcal{T}$ stands for the identity induced rule $\dfrac{\llparenthesis \Gamma \rrparenthesis}{\llparenthesis \Gamma \rrparenthesis}\,\textsc{Cut}_{\mathcal{T}}$. Let us now define $\mathbf{NCLL}_2'$ as $\mathbf{NCLL}_2$ without elementary Cut's, but with these generalized $\textsc{Cut}_{\mathcal{T}}$.

Next we will define the reduction steps for cut elimination by an operation on $\mathcal{T}$. Defining the length of a $\textsc{Cut}_{\mathcal{T}}$-rule as the sum $\sum_{i=1}^{n} \mathsf{l}(\Gamma_i)$ of the lengths of its premiss sequents, each step reduces the sum of all $\textsc{Cut}_{\mathcal{T}}$ lengths (the "total Cut length", from now on called the tcl), as desired for strong normalization. This is a consequence of deleting either a (non-empty) premiss sequent of the $\textsc{Cut}_{\mathcal{T}}$, or at least one connective in the concatenation of all premiss sequents.

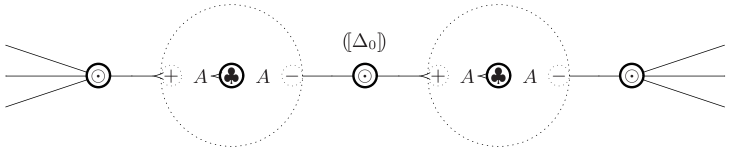
Consider a $\textsc{Cut}_{\mathcal{T}}$ of the form

$$\dfrac{\overset{\mathcal{D}_0}{\llparenthesis \Gamma_0, C_0^{\rho_0}, \Gamma_1, C_1^{\rho_1}, \ldots, \Gamma_{m-1}, C_{m-1}^{\rho_{m-1}} \rrparenthesis} \qquad \overset{\mathcal{D}_1}{\llparenthesis \Delta_1 \rrparenthesis} \quad \ldots \quad \overset{\mathcal{D}_{n-1}}{\llparenthesis \Delta_{n-1} \rrparenthesis}}{\llparenthesis \Delta \rrparenthesis}\,\textsc{Cut}_{\mathcal{T}} \quad .$$

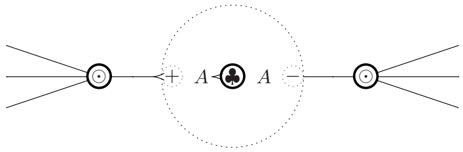
For $n = 1$, replace $\dfrac{\overset{\mathcal{D}_0}{\llparenthesis \Gamma \rrparenthesis}}{\llparenthesis \Gamma \rrparenthesis}\,\textsc{Cut}_{\mathcal{T}}$ by $\overset{\mathcal{D}_0}{\llparenthesis \Gamma \rrparenthesis}$, reducing the tcl, because Γ is not empty (the empty sequent $\llparenthesis \quad \rrparenthesis$ being underivable).

When $n > 1$, each premiss sequent contains at least one cut formula $C_k^{\rho_k}$, for a tree is connected.

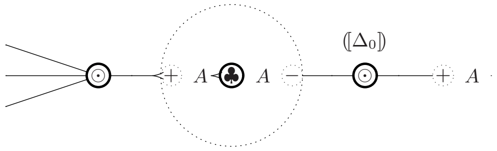
- For $\mathcal{D}_0$ an Ax, contract the sequent and the corresponding cut link of $\mathcal{T}$, resulting in a deletion of the first premisse.



$\rightsquigarrow$



or



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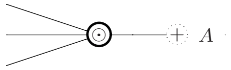
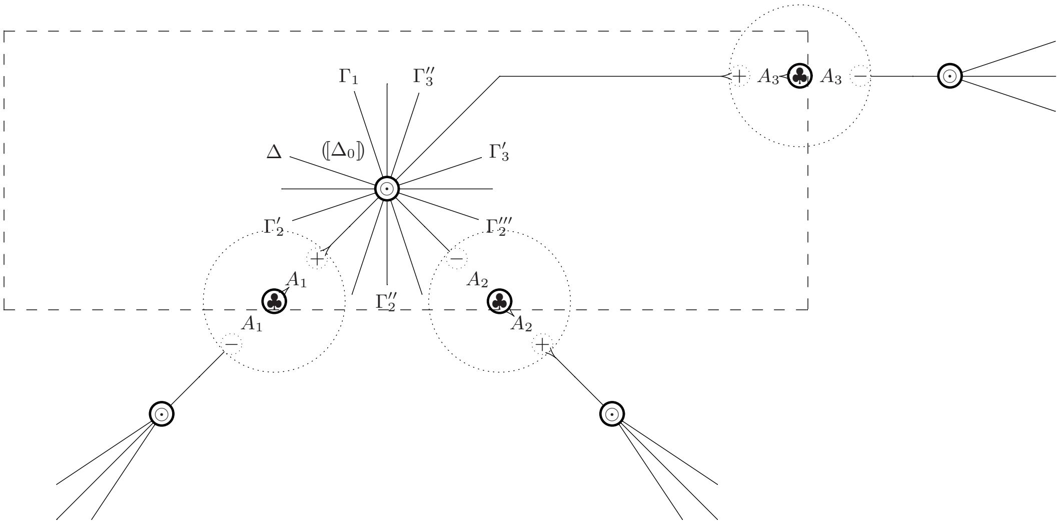
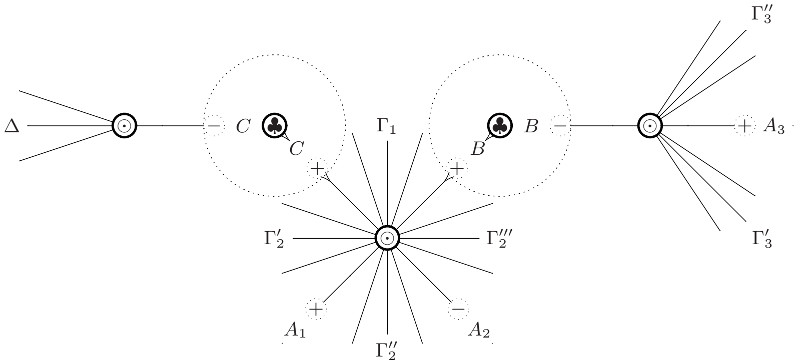


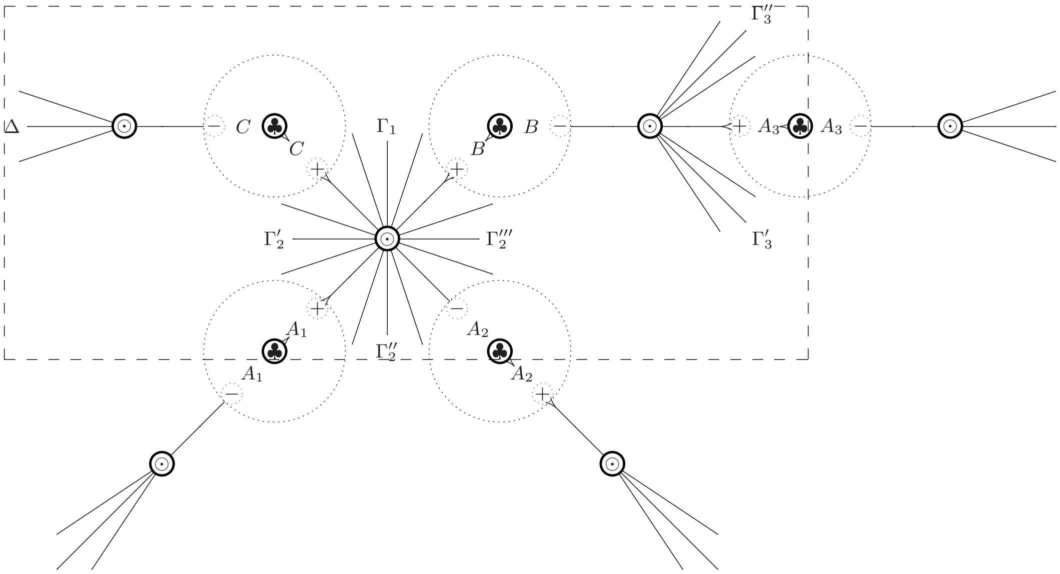
- For $\mathcal{D}_0$ ending with another $\textsc{Cut}_{\mathcal{T}'}$, extract from $\mathcal{T}$ the m subtrees connected to $\llparenthesis \Delta_0 \rrparenthesis$, and glue them to the appropriate open ends of $\mathcal{T}'$, yielding, say, $\mathcal{T}''$. (Alternatively said, replace the sequent $\llparenthesis \Delta_0 \rrparenthesis$ of $\mathcal{T}$ by $\mathcal{T}'$.) Now replace $\textsc{Cut}_{\mathcal{T}}$ and $\textsc{Cut}_{\mathcal{T}'}$ by this new $\textsc{Cut}_{\mathcal{T}''}$, reducing the tcl by $\mathsf{l}(\Delta_0)$. E.g., if $\textsc{Cut}_{\mathcal{T}}$ is given by $\mathcal{T}$ which equals



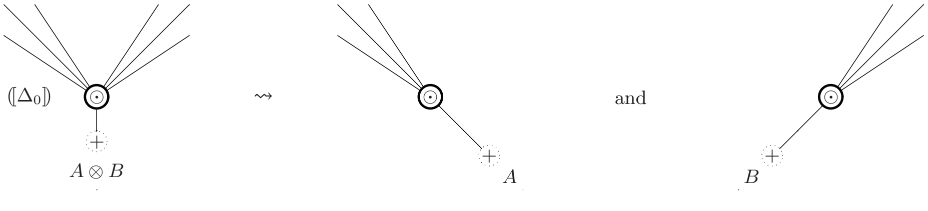
($(\!\![\Delta_0]\!\!)$ having $m = 3$ cut formulas) and $(\!\![\Delta_0]\!\!)$ is the conclusion sequent of $\textsc{Cut}_{\mathcal{T}'}$ where $\mathcal{T}'$ equals (cf. (7))



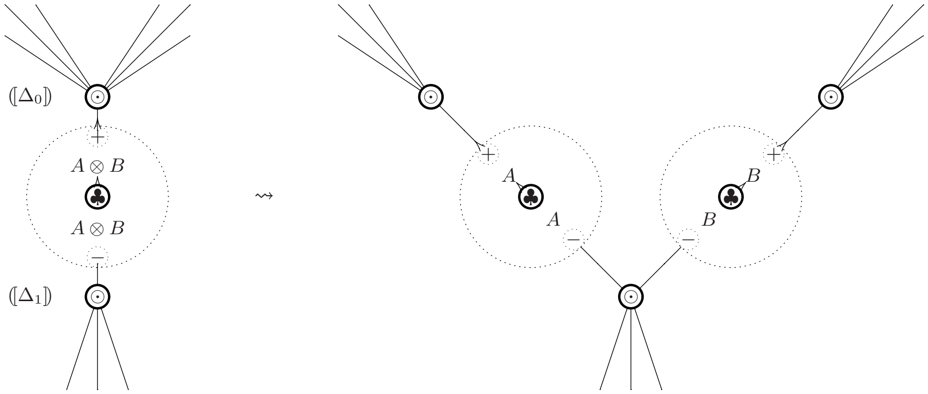
(implying that the cyclic list of open ends of $\mathcal{T}'$ coincides with $(\![\Delta_0]\!)$), then their composition can be replaced by $\textsc{Cut}_{\mathcal{T}''}$ where $\mathcal{T}''$ equals



- For a $\otimes$ inference with the main formula not *a* cut formula, we can split $\mathcal{T}$ into two parts, and apply the $\otimes$ after the two new $\textsc{Cut}$'s. A $⅋$ inference or a $\bot$ inference is even more simple.



- When we are not in these cases — for none of the $\mathcal{D}_i$ — every $\mathcal{D}_i$ ends with a logical rule with its main formula a cut formula. So each of the n premiss sequents determines one unique cut link amidst the $n-1$ cut links. This implies that at least one cut link is determined by two premiss sequents, say by $(\![\Delta_0]\!)$ (final sequent of $\mathcal{D}_0$) and $(\![\Delta_1]\!)$ (final sequent of $\mathcal{D}_1$). We can replace our original $\textsc{Cut}_{\mathcal{T}}$ by a $\textsc{Cut}_{\mathcal{T}'}$ with possibly one premiss sequent more (premiss sequent $j = 0$ or 1 splits if $\mathcal{D}_j$ is a $\otimes$ inference), but two connectives less, represented by a tree $\mathcal{T}'$ obtained from $\mathcal{T}$ by possibly cutting it in link j.



We have defined a (non-deterministic) reduction process, where each reduction step is very elementary (and not a disguised composition of several steps, as was the case in the first process we discussed). The price paid for this, is the enormous increase in complexity of the new "elementary" Cut rules.

4.2.3. Logical cuts and substitution. Given a derivation $\mathcal{D}$ with final sequent

$$\llparenthesis\big[A_0^{\rho_0},\ldots,A_{m-1}^{\rho_{m-1}}\big]\rrparenthesis\ ,$$

we say $A_k^{\rho_k}$ originates from a rule Rule of $\mathcal{D}$ if the former is the main formula of the latter:

$$\dfrac{\begin{array}{ccc}\dfrac{\overset{\mathcal{D}_2}{\cdots}}{\cdots A_k^{\rho_k}\cdots}\,\textsc{Rule} & \mathcal{D}_1 & \\ \ddots & \vdots & \iddots\end{array}}{\llparenthesis\big[A_0^{\rho_0},\ldots,A_{m-1}^{\rho_{m-1}}\big]\rrparenthesis}$$

Every formula of the final sequent originates unambiguously from exactly one rule, which is either an Ax or a logical rule. Here we presuppose a correspondence between the (polarized) formulas in the consecutive sequents of $\mathcal{D}$, which is allowed by the non-periodicity of derivable sequents[8]:

Lemma 4.2.5. *Given a derivation with final rule*

$$\frac{\llparenthesis\Gamma_0\rrparenthesis \qquad \ldots \qquad \llparenthesis\Gamma_{n-1}\rrparenthesis}{\llparenthesis\Gamma\rrparenthesis} \qquad\qquad (n=0,1,2)$$

there is a unique canonical bijective correspondence between all non-active formulas in the premiss sequents and the non-main formulas in the conclusion sequent. $\Diamond$

Proof: For a list $\Pi := (e_0,\ldots,e_{m-1})$, let us define $\Pi^k := (e_k,\ldots,e_{k+m-1})$ $(0 \le k < m)$, where the indices should be read modulo m.

If the final rule is Ax, there is nothing to prove, as there is clearly a unique bijection between two empty sets.

---

[8] This correspondence may seem trivial, but when our sequents would have been multisets instead of cyclic list, it would not be well-defined anymore (cf. **MLL**).

If the final rule is $\otimes$

$$\dfrac{\llparenthesis \Delta_0, A^{\rho} \rrparenthesis \qquad \llparenthesis \Delta_1, B^{\sigma} \rrparenthesis}{\llparenthesis \Delta_0, A^{\rho} \otimes B^{\sigma}, \Delta_1 \rrparenthesis}\otimes$$

we know by the non-periodicity (Lemma 4.1.13) of derivable sequents that there are unique k_0, k_1, k such that the following equalities of lists hold:

$$\begin{aligned} \Gamma_0^{k_0} &= \Delta_0, A^{\rho} \\ \Gamma_1^{k_1} &= \Delta_1, B^{\sigma} \\ \Gamma^{k} &= \Delta_0, A^{\rho} \otimes B^{\sigma}, \Delta_1 \end{aligned}$$

which establishes the unicity of the correspondence[9].

Finally, if the final rule is Cut or another logical rule, a similar argument applies. ////

Definition 4.2.6. *Let $\mathcal{D}$ be a derivation in $\boldsymbol{NCLL}_2$ of the form*

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \llparenthesis \Gamma, C^{+} \rrparenthesis\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \llparenthesis C^{-}, \Delta \rrparenthesis\end{matrix}}{\llparenthesis \Gamma, \Delta \rrparenthesis}\,\textsc{Cut}_C$$

We say this $\textsc{Cut}_C$ *is* logical *if both C^{+} and C^{-} originate from a logical rule.* ◊

If a $\textsc{Cut}_C$ is not logical, at least one of C^{+} and C^{-}, say C^{+}, originates from an Ax. Now replacing this Ax by $\mathcal{D}_2$ and the consecutive occurrences of C^{+} in $\mathcal{D}_1$ by Δ yields again a derivation $\mathcal{D}_1[\mathcal{D}_2]$ of $\llparenthesis \Gamma, \Delta \rrparenthesis$, called the *substitution* of $\mathcal{D}_2$ into $\mathcal{D}_1$. A derivation satisfying the (global) requirement that all appearing Cut's be logical, is called a *sober* derivation. Even without the cut elimination theorem one easily deduces that a derivable

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[9] Observe that we also use the fact that the $\otimes$ rule has distinguishable left and right premiss sequents. Otherwise,

$$\begin{aligned} \Gamma_0^{k_0} &= \Delta_0, A^{\rho} & & & \Gamma_1^{k_1'} &= \Delta_0, A^{\rho} \\ \Gamma_1^{k_1} &= \Delta_1, B^{\sigma} & &\textit{or} & \Gamma_0^{k_0'} &= \Delta_1, B^{\sigma} \\ \Gamma^{k} &= \Delta_0, A^{\rho} \otimes B^{\sigma}, \Delta_1 & & & \Gamma^{k} &= \Delta_0, A^{\rho} \otimes B^{\sigma}, \Delta_1 \end{aligned}$$

would have been the case, destroying the unicity of the correspondence iff $\llparenthesis \Gamma_0 \rrparenthesis = \llparenthesis \Gamma_1 \rrparenthesis$. This is the reason why we make a distinction between the left and the right premiss sequent of a rule. E.g. in the following derivation

$$\dfrac{\dfrac{\dfrac{\overline{\llparenthesis A^{-\rho}, A^{\rho} \rrparenthesis} \qquad \overline{\llparenthesis B^{-\sigma}, B^{\sigma} \rrparenthesis}}{\left(\left[A^{-\rho}, \boxed{A^{\rho} \otimes B^{\sigma}}, B^{-\sigma}\right]\right)}\otimes}{\left(\left[\boxed{A^{\rho} \otimes B^{\sigma}}, B^{-\sigma} \parr A^{-\rho}\right]\right)}\parr \qquad \overline{\llparenthesis A^{\rho} \otimes B^{\sigma}, B^{-\sigma} \parr A^{-\rho} \rrparenthesis}}{\left(\left[\boxed{A^{\rho} \otimes B^{\sigma}}, (B^{-\sigma} \parr A^{-\rho}) \otimes (B^{-\sigma} \parr A^{-\rho}), A^{\rho} \otimes B^{\sigma}\right]\right)}\otimes$$

the boxed $\boxed{A^{\rho} \otimes B^{\sigma}}$ correspond to each other. A *different* derivation is given by

$$\dfrac{\overline{\left(\left[\boxed{A^{\rho} \otimes B^{\sigma}}, B^{-\sigma} \parr A^{-\rho}\right]\right)} \qquad \dfrac{\dfrac{\overline{\llparenthesis A^{-\rho}, A^{\rho} \rrparenthesis} \qquad \overline{\llparenthesis B^{-\sigma}, B^{\sigma} \rrparenthesis}}{\llparenthesis A^{-\rho}, A^{\rho} \otimes B^{\sigma}, B^{-\sigma} \rrparenthesis}\otimes}{\llparenthesis A^{\rho} \otimes B^{\sigma}, B^{-\sigma} \parr A^{-\rho} \rrparenthesis}\parr}{\left(\left[\boxed{A^{\rho} \otimes B^{\sigma}}, (B^{-\sigma} \parr A^{-\rho}) \otimes (B^{-\sigma} \parr A^{-\rho}), A^{\rho} \otimes B^{\sigma}\right]\right)}\otimes$$

sequent always has a sober derivation; just eliminate the non-logical Cut's by means of these substitutions. Each such elimination reduces the total number of Cut's by one, while logical Cut's remain logical; hence the total number of non-logical Cut's decreases by at least one[10].

$$\dfrac{\dfrac{\begin{matrix}\overline{(\![C^-, C^+]\!)} & \mathcal{D}_1 & \\ \ddots & \vdots & \iddots\end{matrix}}{(\![\Gamma, C^+]\!)} \qquad \begin{matrix}\mathcal{D}_2\\ (\![C^-, \Delta]\!)\end{matrix}}{(\![\Gamma, \Delta]\!)}\,\textsc{Cut}_C \qquad \longrightarrow \qquad \dfrac{\begin{matrix}\mathcal{D}_2 & \\ \dfrac{(\![C^-, \Delta]\!)}{} & \mathcal{D}_1 \\ \ddots & \vdots \quad \iddots\end{matrix}}{(\![\Gamma, \Delta]\!)}$$

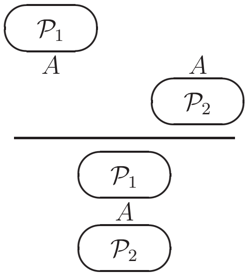
4.3. Proof nets

To a derivation $\mathcal{D}$ we inductively assign an $\mathfrak{L}_2$-proof structure $\mathcal{P}(\mathcal{D})$ with open ends in one-to-one correspondence to the open ends of the final sequent $\llcorner\mathcal{D}\lrcorner$ of $\mathcal{D}$, such that corresponding open ends are labeled by the same polarized formula. We use the representation of $\mathfrak{L}_2$-proof structure as formulated in Lemma 3.2.4.

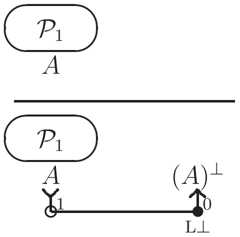
- **(Case** Ax**)** Take for $\mathcal{P}$ the one edge proof structure with end labels A^- and A^+:

$$A$$

- **(Case** Cut**)** Given $\mathcal{P}_1$ with open end $\hat{\eta}_1 = A^+$ and $\mathcal{P}_2$ with open end $\check{\eta}_2 = A^-$, unite them by identifying the edges η_1 and η_2:



- **(Case** $\perp$**)** Given $\mathcal{P}_1$ with open end A^ρ, attach the appropriate negation link. E.g., in case $\rho = +$:



- **(Case** $\otimes$**)** Given $\mathcal{P}_1$ with open end A^ρ and $\mathcal{P}_2$ with open end B^σ, make them the active ends of a new tensor link $\big(\tau(A^\rho \otimes B^\sigma), B^\sigma, A^\rho\big)_\otimes$, which yields a new open end $A^\rho \otimes B^\sigma$. E.g., in case $\rho = -$ and $\sigma = +$:

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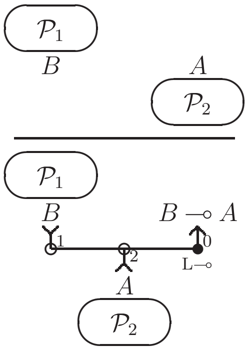
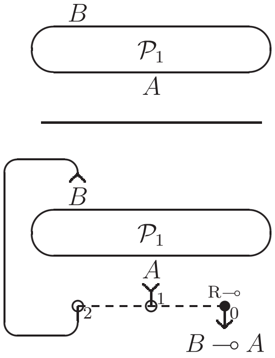
[10] In fact, one other non-logical Cut may become logical, on account of the disappearance of the Ax.



- **(Case $\parr$)** Given $\mathcal{P}_1$ with open ends A^ρ and B^σ, make them the active ends of a new par link $\big(\tau(B^\sigma \parr A^\rho), A^\rho, B^\sigma\big)_\parr$, which yields a new open end $B^\sigma \parr A^\rho$. E.g., in case $\rho = +$ and $\sigma = -$:



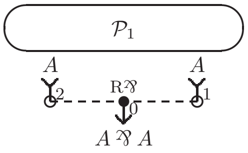
Remark 4.3.1. The operation $\mathcal{D} \mapsto \mathcal{P}(\mathcal{D})$ is well-defined on account of the correspondence between (polarized) formulas in the consecutive sequents of a derivation[11], settled in Subsection 4.2.3, and also because of the inductively obtained correspondence between the open ends of $\llcorner \mathcal{D}_i \lrcorner$ and $\mathcal{P}(\mathcal{D}_i)$. ◊

Definition 4.3.2. *An **NCLL**$_2$-*proof net *is an $\mathfrak{L}_2$-proof structure that can be obtained as the proof structure of an **NCLL**$_2$-derivation.* ◊

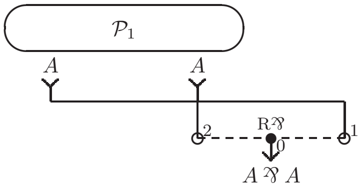
Example 4.3.3. Some examples of proof nets are given by

---

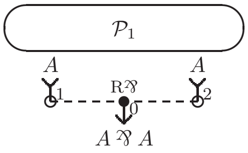
[11] Without the correspondence mentioned in Subsection 4.2.3, the operation would be ambiguous. E.g., if a derivation $\mathcal{D}_1$ with proof structure $\mathcal{P}(\mathcal{D}_1)$ would have final sequent $\llcorner \mathcal{D}_1 \lrcorner = \llbracket A^+, A^+ \rrbracket$ (which — however — is impossible), there actually would be two ways to infer $\llbracket (A \parr A)^+ \rrbracket$ from it. The corresponding proof structure would be either



or



(i.e.



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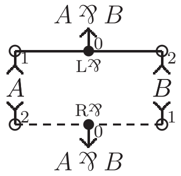

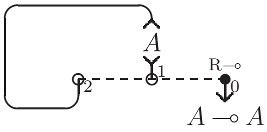

◇

By definition, there is a correspondence between the open ends of $\mathcal{P}(\mathcal{D})$ and those of $\llcorner\mathcal{D}\lrcorner$. Let us now consider an arbitrary sequent occurring in $\mathcal{D}$, say $\llcorner\mathcal{D}'\lrcorner$, where $\mathcal{D}'$ is a subderivation of $\mathcal{D}$. It is clear from the construction of $\mathcal{P}(\mathcal{D})$ that the latter contains $\mathcal{P}(\mathcal{D}')$, but the open ends of $\mathcal{P}(\mathcal{D}')$ need not be open ends of $\mathcal{P}(\mathcal{D})$ anymore; however, they are still present as ends (possibly connector ends) of $\mathcal{P}(\mathcal{D})$. In our derivation $\mathcal{D}$, let us call the set of polarized formulas corresponding to one and the same end A^ρ of $\mathcal{P}(\mathcal{D})$, the *A^ρ-clique*[12] of $\mathcal{D}$. Observe that — in the situation of Lemma 4.2.5 — the corresponding formulas are in the same clique. For CUT-free derivations, the elements of a non-empty clique constitute a chain $F_0 < F_1 < \cdots < F_{k-1}$ ($k \geq 1$; all F_i being occurrences in $\mathcal{D}$ of one and the same polarized formula): they can be linearly ordered by means of their height in $\mathcal{D}$; the first element F_0 is then the occurrence of A^ρ in the rule RULE$_1$ it originates from (i.e. the occurrence as main formula of a particular inference (possibly AX)), and the last element F_{k-1} is either the occurrence as active formula of a *logical* inference RULE$_2$, or the occurrence in the final sequent $\llcorner\mathcal{D}\lrcorner$. Let us denote both occasions as

$$\text{RULE}_1\ F_0 < F_1 < \cdots < F_{k-1}\ \text{RULE}_2$$

and

$$\text{RULE}_1\ F_0 < F_1 < \cdots < F_{k-1}\ \emptyset$$

respectively. In derivations with CUT, the last element F_{k-1} may also be the active formula of a CUT. Observe that — as a result of identifying the edges η_1 and η_2 when applying a CUT — the A^+-cliques of $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and similar for the A^--cliques. Hence, in general, an A^ρ-clique consists of several A^ρ-chains. In the sequel, let

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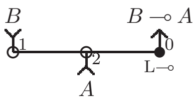
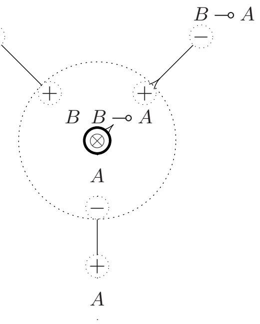
[12] Some cliques are empty; e.g. if $\mathcal{D}$ equals $\dfrac{\dfrac{}{\llparenthesis A^+, A^- \rrparenthesis}\,\text{Ax} \qquad \dfrac{}{\llparenthesis B^+, B^- \rrparenthesis}\,\text{Ax}}{\llparenthesis A^+, (B \multimap A)^-, B^- \rrparenthesis}\,\text{L}\multimap$ then $\mathcal{P}(\mathcal{D})$ equals



i.e.



and the main end of our connector link $(B \multimap A)^+$ does not correspond to any polarized formula of $\mathcal{D}$.

us denote an A^ρ-chain as

$$\begin{array}{r}
\textsc{Ax}\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\\
\textsc{Rule}\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\\
\textsc{Ax}\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Rule}^*\\
\textsc{Rule}\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Rule}^*
\end{array}$$

depending on the distinct cases for $\textsc{Rule}_1$ and $\textsc{Rule}_2/\emptyset$. Here, $\textsc{Rule}$ is a *logical* rule, and $\textsc{Rule}^*$ stands for either a *logical* rule $\textsc{Rule}$, or no rule ($\emptyset$) at all (in case the last element of our chain is the occurrence in the final sequent).

Recall that an edge η is *axiomatic* if each of $\hat{\eta}$ and $\check{\eta}$ is not the main end of any link, whereas η is a *cut edge* if both $\hat{\eta}$ and $\check{\eta}$ are the main ends of two links. The number of main ends η possesses is called the *role* of η, so that η is an axiomatic edge (a cut edge) iff it has role 0 (2). In terms of the formulation in Lemma 3.2.4, a formula A may be main (active) formula of up to two links. The number of links A is main formula of, is still called the *role* of A, and we call A an *axiomatic (cut) formula* if A has role 0 (2).

In the next proposition we will prove that for every formula A of $\mathcal{P}(\mathcal{D})$, the A-*clique* (defined as the union of the A^+-clique and the A^--clique, where A^+ and A^- are the ends of the edge A) is of one of the following forms[13]:

$$\textsc{Rule}^*\ \overleftarrow{A^{-\rho}\text{-chain}}\ \textsc{Ax}\left[\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\ \overleftarrow{A^{-\rho}\text{-chain}}\ \textsc{Ax}\right]\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Rule}^*$$

(called the *axiomatic form*)

$$\textsc{Rule}\left[\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\ \overleftarrow{A^{-\rho}\text{-chain}}\ \textsc{Ax}\right]\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Rule}^*$$

(called the *flow form*)

$$\textsc{Rule}\left[\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\ \overleftarrow{A^{-\rho}\text{-chain}}\ \textsc{Ax}\right]\ \overrightarrow{A^\rho\text{-chain}}\ \textsc{Cut}\ \overleftarrow{A^{-\rho}\text{-chain}}\ \textsc{Rule}$$

(called the *cut form*)

where by the expression between square brackets its $0, 1, 2, 3, \ldots$ times repetition is meant.

$\textsc{Example}$ 4.3.4. Suppose $\mathcal{D}$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\llparenthesis A^-, A^+\rrparenthesis}\,\textsc{Ax}_1 \qquad \dfrac{\dfrac{\dfrac{}{\llparenthesis A^+, A^-\rrparenthesis}\,\textsc{Ax}_2}{\bigl(\![A^+, (A^\perp)^+]\!\bigr)}\,\text{R}\perp \qquad \dfrac{\dfrac{}{\llparenthesis A^-, A^+\rrparenthesis}\,\textsc{Ax}_3}{\bigl(\![A^-, (A^\perp)^-]\!\bigr)}\,\text{L}\perp}{\llparenthesis A^+, A^-\rrparenthesis}\,\textsc{Cut}_{A^\perp}}{\llparenthesis A^-, A^+\rrparenthesis}\,\textsc{Cut}_1 \qquad \dfrac{\dfrac{}{\llparenthesis B^+, B^-\rrparenthesis}\,\textsc{Ax}_B \qquad \dfrac{\dfrac{}{\llparenthesis A^+, A^-\rrparenthesis}\,\textsc{Ax}_4}{\bigl(\![(A^\perp)^-, A^-]\!\bigr)}\,\widetilde{\text{L}\perp}}{\bigl(\![B^+, (A^\perp \parr B)^-, A^-]\!\bigr)}\,\text{L}\parr}{\bigl(\![A^-, B^+, (A^\perp \parr B)^-]\!\bigr)}\,\textsc{Cut}_2$$

Then $\mathcal{D}$ has proof net

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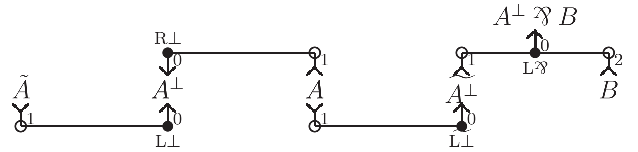
[13] By symmetry, we may suppose $\rho = +$ for the axiomatic form and for the cut form. The cases $\rho = +, -$ for the flow form, however, are really different.



There are 14 chains, viz. the columns in

		A^+	A^-		$\tilde{A}^-$	$\tilde{A}^+$							
		A^+		$(A^\perp)^+$	$\tilde{A}^-$		$(A^\perp)^-$					A^+	A^-
$\tilde{A}^-$	$\tilde{A}^+$	A^+			$\tilde{A}^-$			B^+	B^-		$(\widetilde{A^\perp})^-$		A^-
$\tilde{A}^-$		A^+						B^+		$(A^\perp \parr B)^-$			A^-
$\tilde{A}^-$								B^+		$(A^\perp \parr B)^-$			

Let us determine for every formula occurrence C of $\mathcal{P}(\mathcal{D})$ the corresponding C-clique. We have distinguished different occurrences of one and the same formula by a tilde.

The $A^\perp$-clique consists of the chains $\mathrm{R}\bot\ (A^\perp)^+\ \textsc{Cut}_{A^\perp}$ and $\mathrm{L}\bot\ (A^\perp)^-\ \textsc{Cut}_{A^\perp}$; it turns out to be in cut form.

The $\widetilde{A^\perp}$-clique consists of only one chain $\widetilde{\mathrm{L}\bot}\ (\widetilde{A^\perp})^-\ \mathrm{L}\parr$, and is in flow form.

The same holds for the $(A^\perp \parr B)$-clique, consisting of the chain $\mathrm{L}\parr\ (A^\perp \parr B)^- < (A^\perp \parr B)^-\ \emptyset$.

The B-clique consists of the chains $\textsc{Ax}_B\ B^+ < B^+ < B^+\ \emptyset$ and $\textsc{Ax}_B\ B^-\ \mathrm{L}\parr$, which meet in $\textsc{Ax}_B$. So this clique is in axiomatic form.

The remaining four A^+-chains and four A^--chains distribute over the A-clique and the $\tilde{A}$-clique; schematicly we can describe their constitution by

$$\begin{tikzcd}
& \textsc{Ax}_2 \arrow[dl, "A^-"] \arrow[dr, "A^+"] & & \textsc{Ax}_4 \arrow[dl, "A^-"] \arrow[dr, "A^+"] & \\
\mathrm{R}\bot & & \textsc{Cut}_2 & & \widetilde{\mathrm{L}\bot}
\end{tikzcd}$$

$$\begin{tikzcd}
& \textsc{Ax}_1 \arrow[dl, "\tilde{A}^-"] \arrow[dr, "\tilde{A}^+"] & & \textsc{Ax}_3 \arrow[dl, "\tilde{A}^-"] \arrow[dr, "\tilde{A}^+"] & \\
\emptyset & & \textsc{Cut}_1 & & \mathrm{L}\bot
\end{tikzcd}$$

whence both cliques are in axiomatic form.

We conclude that each of the axiomatic formulas of $\mathcal{P}(\mathcal{D})$ (viz. B, A and $\tilde{A}$) has clique in axiomatic form; that the cut formula $(A^\perp)$ has clique in cut form; and finally that each of the remaining formulas ($\widetilde{A^\perp}$ and $A^\perp \parr B$) has clique in flow form. Observe that $\textsc{Cut}_{A^\perp}$ is a logical cut (see Definition 4.2.6), unlike $\textsc{Cut}_1$ and $\textsc{Cut}_2$. $\Diamond$

Proposition 4.3.5. *Let $\mathcal{P} = \mathcal{P}(\mathcal{D})$ be the proof net of a derivation $\mathcal{D}$. Every link l of $\mathcal{P}$ of subtype $X\Box$ corresponds to an $X\Box$-rule of $\mathcal{D}$. For every formula A of $\mathcal{P}$, the A-clique (in $\mathcal{D}$) is in*

- *axiomatic form if A is an axiomatic formula,*
- *flow form if A is neither axiomatic nor cut,*
- *cut form if A is a cut formula.*

As a consequence, every formula A of $\mathcal{P}$ corresponds to n Ax_A rules and m Cut_A rules of $\mathcal{D}$, and the following holds:

$$n - m = \begin{cases} 1 & \textit{if } A \textit{ is an axiomatic formula,} \\ 0 & \textit{if } A \textit{ is neither axiomatic nor cut,} \\ -1 & \textit{if } A \textit{ is a cut formula.} \end{cases}$$

In particular, if $\mathcal{D}$ is sober, there is a bijective correspondence between the axiomatic (cut) formulas of $\mathcal{P}$ and the Ax *(*Cut*) rules of $\mathcal{D}$.* ◊

Proof: The proof is by induction on the definition of the derivation.

The proof net of $\overline{\llparenthesis A^+, A^- \rrparenthesis}^{\,\text{Ax}}$ is the sole formula A which is an axiomatic formula. It has clique $\emptyset \ \overleftarrow{A^+} \ \text{Ax} \ \overrightarrow{A^-} \ \emptyset$, which is in axiomatic form, while there are no other links as well as no other rules, which proves this case.

If $\mathcal{D} = \otimes(\mathcal{D}_1, \mathcal{D}_2)$, by induction hypothesis the X☐ links of $\mathcal{P}_i$ correspond to the X☐ rules of $\mathcal{D}_i$. Hence the X☐ links of $\mathcal{P}$ correspond to the X☐ rules of $\mathcal{D}$. Moreover, all old formulas of $\mathcal{P}$ correspond to similar cliques as before (only the chains with last element $C^\nu \ \emptyset$ change into either $C^\nu < C^\nu \ \emptyset$ or $C^\nu \ X\square$), while $A^\rho \otimes B^\sigma$ has clique $X\square \ A^\rho \otimes B^\sigma \ \emptyset$ which is in flow form. This proves the present case, since $A^\rho \otimes B^\sigma$ is neither axiomatic nor cut, while all other formulas remain of the same role.

The cases of another logical rule are proved similarly.

In case of an application of the Cut rule (say on the formula A), the first part of the proposition is clear, and also the second part w.r.t. all formulas different from A. When in $\mathcal{D}_i$ the A-clique is in a form with n_i Ax rules and m_i Cut rules, then in $\mathcal{D}$ the A-clique has $n := n_1 + n_2$ Ax rules and $m := m_1 + m_2 \mathbf{+1}$ Cut rules, where the two original cliques are connected by means of Cut, yielding one of the three described forms again. E.g., if $A \in \mathcal{P}_1$ has clique in flow form

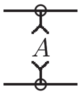
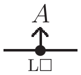
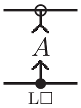
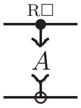
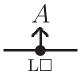
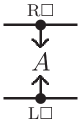
$$\text{Rule}\left[\ldots\right] \overrightarrow{A^+\text{-chain}} \ \emptyset$$

and $A \in \mathcal{P}_2$ has clique in axiomatic form

$$\emptyset \ \overleftarrow{A^-\text{-chain}} \ \text{Ax}\left[\ldots\right] \overrightarrow{A^+\text{-chain}} \ \text{Rule}^*$$

then $A \in \mathcal{P}$ has clique in flow form

$$\text{Rule}\left[\ldots\right] \overrightarrow{A^+\text{-chain}} \ \text{Cut} \ \overleftarrow{A^-\text{-chain}} \ \text{Ax}\left[\ldots\right] \overrightarrow{A^+\text{-chain}} \ \text{Rule}^*$$

$\mathcal{P}_1$ $n_1 - m_1$		$\mathcal{P}_2$ $n_2 - m_2$		$\mathcal{P}$ $n - m$	
	1		1		1
	1		0		0
	0		1		0
	0		0		-1

Depending on the role (0 or 1) of A in $\mathcal{D}_1$ respectively $\mathcal{D}_2$, we know by the induction hypothesis the value of $n_i - m_i$, from which we can compute $n - m = (n_1 - m_1) + (n_2 - m_2) - 1$, and this number turns out to correspond in the desired way with the role of the new formula A in $\mathcal{P}$.

For a sober derivation, observe that every formula A of its proof net corresponds to 0 Ax-rules on A or 0 Cut-rules on A; otherwise there would be a connecting chain between at least on Ax and one Cut, contradicting soberness. Hence we know that an axiomatic formula of the proof net of a sober derivation corresponds to exactly one Ax-rule; a cut formula corresponds to exactly one Cut-rule, while a formula which is neither axiomatic nor cut does not correspond to any Ax- or Cut-rule. ////

Different derivations may have the same proof net. The next theorem will characterize all derivations that have the same proof net.

Theorem 4.3.6. *Let $\mathcal{D}$ and $\mathcal{D}'$ be derivations of the same sequent $(\![\Gamma]\!)$. Then their respective proof nets $\mathcal{P}$ and $\mathcal{P}'$ are equal if and only if there exists a sequence of derivations $\mathcal{D} = \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{n-1} = \mathcal{D}'$ such that $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ differ only for a permutation of two consecutive inferences, or $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i+1}$ (or the other way around) by a substitution (i.e. elimination of a non-logical* Cut*).* $\Diamond$

Proof: The if-part is clear: attaching links in a different order does not give a different proof net, and neither does a substitution.

The other way around, suppose $\mathcal{P}$ and $\mathcal{P}'$ are equal. In both derivations $\mathcal{D}$ and $\mathcal{D}'$ we can substitute for all non-logical Cut-inferences, so that we may assume that both $\mathcal{D}$ and $\mathcal{D}'$ are sober derivations (i.e. with only logical Cut-rules). This implies that $\mathcal{D}$ and $\mathcal{D}'$ have the same logical rules (one for each link of $\mathcal{P} = \mathcal{P}'$), the same (logical) Cut-rules (one for each cut formula of $\mathcal{P}$) and the same Ax-rules (one for each axiomatic formula). Now we proceed in a way similar to that of the proof of the corresponding theorem for **MLL**, e.g. as in [**BvdW 95**], with this interesting difference that we do not rely on the notion of empire. (In fact the only thing one really needs — in *both* cases — is connectedness.)

Let I be the last inference of $\mathcal{D}$. Via a link l (respectively an axiomatic formula respectively a cut formula) of $\mathcal{P} = \mathcal{P}'$ we know this inference corresponds to a *similar* inference I' of $\mathcal{D}'$ (i.e. an inference of the same type (R$\otimes$, L$\otimes$, Cut et cetera) and with the same active and/or main formulas, but with possibly different context formulas). By induction on the height $\hbar$ of I' in the deduction tree $\mathcal{D}'$ we first show that after some permutations of two consecutive inferences $\mathcal{D}'$ may be turned into a derivation $\mathcal{D}''$ with last inference I'' similar to I and I'.

If $\hbar = 0$, I' is already the last inference of $\mathcal{D}'$.

If $\hbar > 0$, denote by I_1' the inference below I' in $\mathcal{D}'$. This inference has active formulas distinct from the main formula(s) of I', and moreover it corresponds to a similar inference I_1 of $\mathcal{D}$. We distinguish four cases.

In case I' is an Ax-rule, an active formula of I_1' would be a main formula of I'; contradiction.

In case I' is a negation rule or par rule, we can permute it with I_1' (yielding two similar inferences in the other order).

In case I' is a tensor rule or a Cut-rule, we can permute it in the subcases that I_1' is a negation rule, a tensor rule or a Cut-rule. As I_1' cannot be an Ax-rule, we only have to consider the subcase that I_1' is a par rule. In order to permute the two rules, we must show that the active formulas B_1 and B_2 of I_1' originate from the same subderivation $\mathcal{D}_j'$ of $\mathcal{D}'$ above I'.

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \ddots\ \ \vdots\ \ \iddots\\ \dfrac{(\![\ldots B_1^{\sigma_1}, B_2^{\sigma_2}]\!)}{(\![\ldots B_1^{\sigma_1} \parr B_2^{\sigma_2}]\!)}\,I_1\\ \ddots\ \ \vdots\ \ \iddots\\ (\![\ldots A_1^{\rho_1} \ldots B_1^{\sigma_1} \parr B_2^{\sigma_2}]\!)\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ (\![\ldots A_2^{\rho_2}]\!)\end{array}}{(\![\ldots A_1^{\rho_1} \otimes A_2^{\rho_2} \ldots B_1^{\sigma_1} \parr B_2^{\sigma_2}]\!)}\,I \qquad\qquad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \ddots\ \ \vdots\ \ \iddots\\ (\![\ldots A_1^{\rho_1}]\!)\end{array} \qquad \begin{array}{c}\mathcal{D}_2'\\ \ddots\ \ \vdots\ \ \iddots\\ (\![\ldots A_2^{\rho_2}]\!)\end{array}}{(\![\ldots A_1^{\rho_1} \otimes A_2^{\rho_2} \ldots B_1^{\sigma_1}, B_2^{\sigma_2}]\!)}\,I'}{(\![\ldots A_1^{\rho_1} \otimes A_2^{\rho_2} \ldots B_1^{\sigma_1} \parr B_2^{\sigma_2}]\!)}\,I_1' \\ \ddots\ \ \vdots\ \ \iddots \end{array}$$

$A_j^{\rho_j}$ is a leaf of the subnet $\mathcal{P}(\mathcal{D}_j')$ of $\mathcal{P}$, which equals the union of $\mathcal{P}(\mathcal{D}_1)$ and $\mathcal{P}(\mathcal{D}_2)$ together with a $\otimes$ link. Hence by connectedness $\mathcal{P}(\mathcal{D}_j') \subseteq \mathcal{P}(\mathcal{D}_j)$. If w.l.o.g. I_1 belongs to $\mathcal{D}_1$, then the corresponding link l_1 of $\mathcal{P}$ occurs in $\mathcal{P}(\mathcal{D}_1)$. But then B_1 and B_2 cannot belong to $\mathcal{D}_2'$, and hence must belong to $\mathcal{D}_1'$. This means that we can permute I' and I_1', as desired.

Now after this permutation, $\hbar$ has decreased by 1, so that we know by induction hypothesis that after some (more) permutations of two consecutive inferences $\mathcal{D}'$ is turned into a derivation $\mathcal{D}''$ with last inference I''.

Moreover, this last inference I'' of $\mathcal{D}''$ actually coincides with I. This is clear for the sequent below the bar which consist of the leaves of $\mathcal{P}$. Hence we are ready in the case of an AX-rule, while the result is immediately clear for a negation rule or a par rule. For a tensor rule or a CUT-rule the result follows again by inspection of the subnets of the subderivations $\mathcal{D}_j''$ of $\mathcal{D}''$ above I'', compared to the subnets $\mathcal{P}(\mathcal{D}_j)$.

Finally we show by induction on the coinciding subtree of $\mathcal{D}$ and $\mathcal{D}'$ that after some permutations of two consecutive inferences $\mathcal{D}'$ may be turned into $\mathcal{D}$.

If the coinciding subtree is $\mathcal{D}$ we are ready.

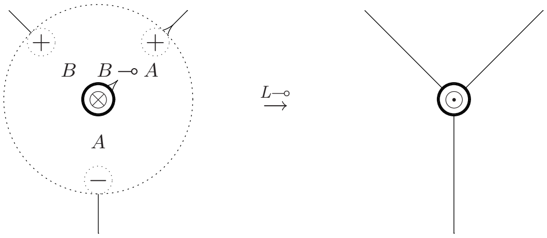
Otherwise there is a branch of $\mathcal{D}$ with an inference I of minimal level such that $\mathcal{D}$ and $\mathcal{D}'$ coincide below I. Let us call the subderivation above this coinciding subbranch $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}'}$ respectively. Then applying the previous result we know that after some permutation of consecutive inferences $\widehat{\mathcal{D}'}$ may be turned into a derivation with last inference exactly the same as I. Hence the coinciding subtree has increased and we may apply the induction hypothesis. ////

4.4. Contraction criterion

Let $\mathfrak{L}\mathfrak{G}_2$ denote the collection of $\mathfrak{L}_2$-link graphs with well-labeled (see Figure 3.1) connector links, viz. tensor links $(e_0, e_1, e_2)_{\otimes}$ (indicated by ⊗), par links $(e_0, e_1, e_2)_{⅋}$ (indicated by ⅋), and negation links $(e_0, e_1)_{\perp}$ (indicated by ⊥), and with context links $([e_0, \ldots, e_{m-1}])_{\odot}$ (indicated by ⊙). Observe that both proof structures and sequents belong to $\mathfrak{L}\mathfrak{G}_2$. On $\mathfrak{L}\mathfrak{G}_2$ we will define the following conversion relation. One easily checks that these conversion steps are well defined (i.e. they do yield an element of $\mathfrak{L}\mathfrak{G}_2$) and preserve the open ends.

DEFINITION 4.4.1.

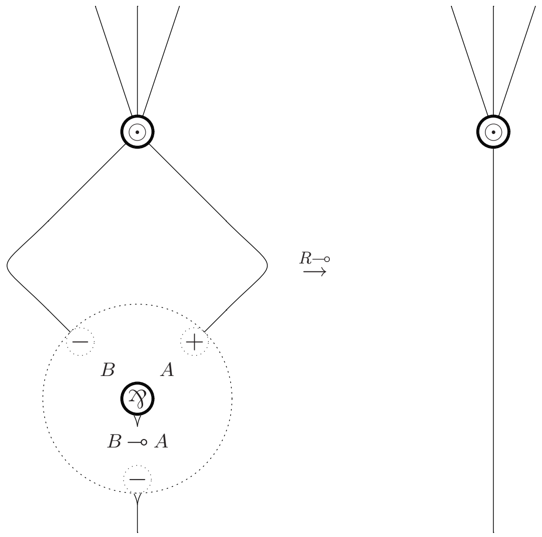
- ***[tens/neg]***(l) *Every tensor link or negation link l converts into a context link:*



and similar for R⊗, L∘−, L⅋, L⊥ and R⊥.

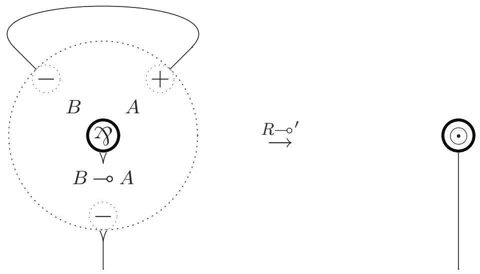
- ***[par]***(l_1, l_2) *Given a par link l_1, the active ends of which are connected to two consecutive ends of a single context link l_2 in the right order, then l_1 and l_2*

together convert into one context link as follows; the two edges disappear:



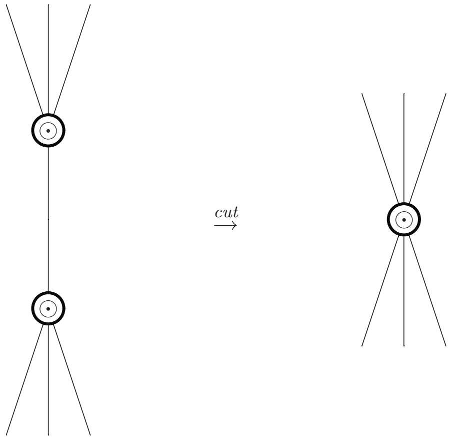
and similar for $L\otimes$, $R⟜$ *and* $R⅋$.

- ***[par′]***(l, η) *Given a par link* l*, the active ends of which belong to the same edge* η*, then* l *converts into a valence 1 context link, and* η *disappears:*



and similar for $L\otimes$, $R⟜$ *and* $R⅋$.

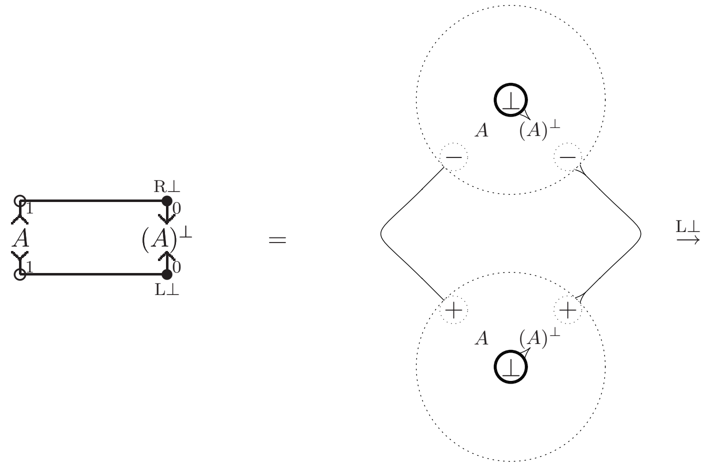
- ***[cut]***(l_1, l_2, η) *Given context links* l_1 *and* $l_2 \neq l_1$ *connected by an edge* η, *then* l_1 *and* l_2 *together convert into one context link and* η *disappears:*

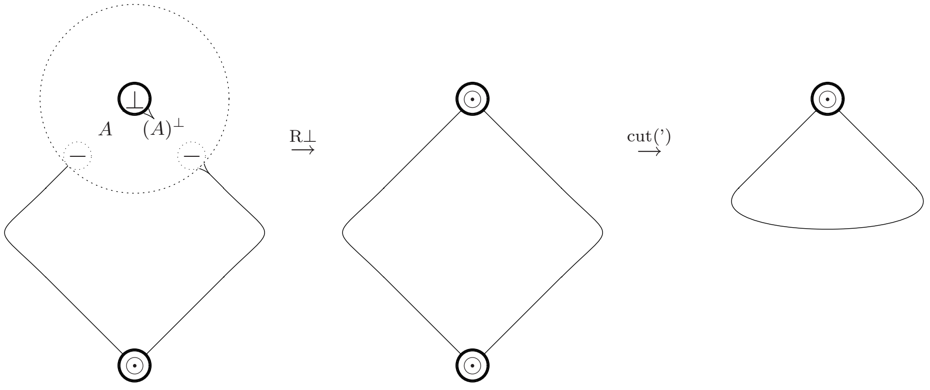


- ***[cut']***(l, η_1, η_2) *Given a context link* l *of valence 2 connected to two edges* η_1 *and* $\eta_2 \neq \eta_1$, *then* l *disappears and* η_1 *and* η_2 *are identified:*

◊

Example 4.4.2. Let us consider a conversion sequence of the following proof structure:





◊

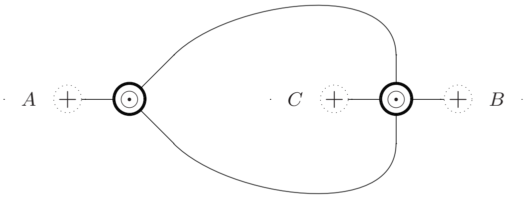
It is easy to see that this reduction relation is terminating; in each conversion step $\mathcal{P}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{L}'_1, \lambda_1) \to \mathcal{P}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{L}'_2, \lambda_2)$ the non-negative integer $\phi(\mathcal{P}) := |\mathcal{E}| + |\mathcal{L}| + 2\,|\mathcal{L}'|$ decreases by at least one (recall that $\mathcal{L}$ consists of the context links, while $\mathcal{L}'$ contains the connector links):

Lemma 4.4.3. *The conversion steps increase ϕ by $\Delta\phi$, given by:*

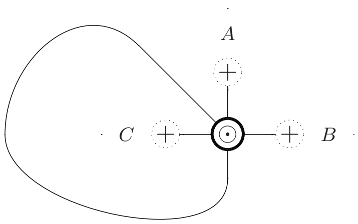
- ***[tens/neg]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (0, +1, -1)$*, so* $\Delta\phi = -1$.
- ***[par]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-2, 0, -1)$*, so* $\Delta\phi = -4$.
- ***[par′]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-1, +1, -1)$*, so* $\Delta\phi = -2$.
- ***[cut]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-1, -1, 0)$*, so* $\Delta\phi = -2$.
- ***[cut′]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-1, -1, 0)$*, so* $\Delta\phi = -2$.

◊

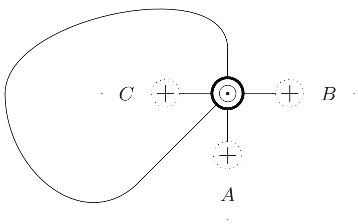
However, this conversion is not confluent. A counterexample is given by the following link graph



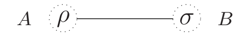
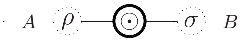
which converts — depending on the edge we apply [**cut**] on — to either



or



Observe that even the order in which the open ends are attached to the unique context link differs.

In the next subsections we will investigate this conversion applied on $\mathfrak{L}_2$-proof structures. First, it is shown that for any derivation $\mathcal{D}$ the corresponding proof net $\mathcal{P}(\mathcal{D})$ converts to the sequent $\llcorner\mathcal{D}\lrcorner$, ór to the one-edge link graph  with the same open ends as $\llcorner\mathcal{D}\lrcorner$ (the latter hence only possibly occurring in case $\llcorner\mathcal{D}\lrcorner$ equals ).

The other way around, we will prove that a proof structure $\mathcal{P}$ that converts to a sequent Γ or to a one-edge link graph Γ' (which corresponds to a unique sequent Γ), is actually a proof net, viz. the proof structure of a derivation $\mathcal{D}$, for which moreover $\llcorner\mathcal{D}\lrcorner$ equals Γ.

4.4.1. Completeness.

THEOREM 4.4.4. *(a) Let $\mathcal{D}$ be an $\boldsymbol{NCLL}_2$-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$ (or $\mathcal{P}(\mathcal{D}) \twoheadrightarrow (\llcorner\mathcal{D}\lrcorner)'$).*

(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a sequent or a corresponding one-edge link graph. Then there is a derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.

$\lozenge$

PROOF: **(a)** By induction on the derivation $\mathcal{D}$ we will prove that $\mathcal{P}(\mathcal{D})$ converts to the sequent $\llcorner\mathcal{D}\lrcorner$ or to the one-edge link graph $(\llcorner\mathcal{D}\lrcorner)'$. Actually, [**cut**$'$] turns out to be superfluous, but is added for practical use.

If $\mathcal{D}$ is AX, the corresponding proof net is the one-edge link graph corresponding to $\Gamma = ([A^+, A^-])$, so we are done in zero steps.

If $\mathcal{D}$ ends by CUT, the two subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$ have proof nets converting to a one-edge link graph or a sequent. Connecting the two proof nets in A then yields the proof net of $\mathcal{D}$, which converts to a one-edge link graph, a single sequent, or two sequents connected in A, further on converting by [**cut**].

If $\mathcal{D}$ ends by $\mathrm{R}\otimes$, $\mathrm{L}\circ\!-$, $\mathrm{L}\!-\!\circ$, $\mathrm{L}\parr$, $\mathrm{L}\bot$ or $\mathrm{R}\bot$, connecting the corresponding tensor or negation link to the inductively obtained proof net(s) yields the proof net of $\mathcal{D}$, that converts by induction hypothesis first to a single tensor or negation link, perhaps connected to one or two sequents, converting on by [**tens/neg**] and zero, one or two times [**cut**].

If $\mathcal{D}$ ends by $\mathrm{L}\otimes$, $\mathrm{R}\circ\!-$, $\mathrm{R}\!-\!\circ$ or $\mathrm{R}\parr$, connecting the corresponding par link to the inductively obtained proof net yields the proof net of $\mathcal{D}$, that converts to a single par link or to a par link connected to a sequent, converting on by [**par′**] respectively [**par**].

(b) By induction on the length of the conversion sequence we will prove that if $\mathcal{P} \twoheadrightarrow \Gamma$ (Γ'), then there is a derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.

If the length is zero, $\mathcal{P}$ is a one-edge proof structure (corresponding to the derivation Ax) or a sequent, which is not a proof structure.

Suppose the last conversion is a [**tens/neg**]-step $\mathcal{P}' \to \Gamma$. Since Γ has exactly one context node, $\mathcal{P}'$ only contains a single tensor link or negation link

$$\big(\tau(A^\rho \otimes B^\sigma), B^\sigma, A^\rho\big)_\otimes \qquad \text{or} \qquad \big((A^\bot)^\rho, A^\rho\big)_\bot.$$

So $\mathcal{P}$ consists of three (two) proof nets $\mathcal{P}_0$, $\mathcal{P}_1$ (and $\mathcal{P}_2$) attached to one another in a tensor link (negation link), each of which converts to one edge. We find three (two) derivations which may be combined by application of the corresponding tensor (negation) rule and a Cut (i.e. by the appropriate semi-derivation corresponding to the induced rule with $(a, b, c) = (1, 1, 1)$; see page 76). The proof net of this derivation indeed equals $\mathcal{P}$.

If the last conversion is [**par′**], the same reasoning yields $\mathcal{P}$ consists of two proof nets $\mathcal{P}_0$, $\mathcal{P}_1$ attached to one another in a par link in the given way, each of which converts to one edge. We find two derivations which may be combined by application of the corresponding par rule and a Cut. The proof net of this derivation indeed equals $\mathcal{P}$.

The case [**par**] is similar, although now $\mathcal{P}_1$ converts to a sequent Γ_1.

If the last conversion is [**cut**], $\mathcal{P}$ consists of two proof nets $\mathcal{P}_1$ and $\mathcal{P}_2$ attached in A, both converting to a sequent. Apply a Cut rule to the inductively obtained derivations.

If the last conversion is [**cut′**] $\mathcal{P}' \to \Gamma'$, the result holds since $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ is already a conversion sequence to a sequent. If the last conversion is [**cut′**] $\mathcal{P}' \to \Gamma$, $\mathcal{P}$ consists of two proof nets $\mathcal{P}_1$ and $\mathcal{P}_2$ attached in A, both converting to a sequent. Apply a Cut rule to the inductively obtained derivations. ////

Corollary 4.4.5. *Let* Γ *be an* $\boldsymbol{NCLL}_2$*-sequent. Then the following are equivalent:*

(i) Γ *is derivable in* $\boldsymbol{NCLL}_2$*;*

(ii) There is a proof structure $\mathcal{P}$ *and a conversion sequence* $\mathcal{P} \twoheadrightarrow \Gamma$ *or to its corresponding one-edge link graph* Γ'*.*

◊

Given $\mathcal{P}$ and a conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma$ (Γ'), we find a unique derivation $\mathcal{D}$ of Γ by the second part of this proof. It is clear that there is a correspondence between the conversion steps in $\mathcal{P} \twoheadrightarrow \Gamma$ (Γ') on the one hand, and the logical and Cut rules of $\mathcal{D}$ on the other hand. Moreover, knowing that $\mathcal{P} = \mathcal{P}(\mathcal{D})$, Proposition 4.3.5 gives us a correspondence between the links of $\mathcal{P}$ and the logical rules of $\mathcal{D}$. Observe, however, that a Cut formula of $\mathcal{P}$ may correspond to several Cut rules of $\mathcal{D}$ (and hence to several conversion steps [**cut**]), and that an Ax formula of $\mathcal{P}$ may correspond to several Ax rules of $\mathcal{D}$.

We claim that it is possible to adapt part **(b)** of the proof of Theorem 4.4.4 in such a way that the derivation $\mathcal{D}$ of Γ we find is sober, yielding, by Proposition 4.3.5, a bijective correspondence between the axiomatic (cut) formulas of $\mathcal{P}$ and the Ax (Cut) rules of $\mathcal{D}$ (in addition to the correspondence between the links of $\mathcal{P}$ and the *logical* rules of $\mathcal{D}$).

To this purpose, we strengthen Theorem 4.4.4 by adding the statement that there is a bijective correspondence between the axiomatic formulas C (of $\mathcal{P}$) and the Ax_C rules (of $\mathcal{D}$), and moreover that every axiomatic conclusion or hypothesis C^ρ (of $\mathcal{P}$) corresponds to a conclusion or hypothesis C^ρ of (the final sequent of) $\mathcal{D}$ that originates from an Ax:

$$\begin{array}{c} \overline{\llparenthesis C^{-\rho}, C^{\rho} \rrparenthesis} \quad \mathcal{D} \\ \ddots \qquad \vdots \qquad \iddots \\ \hline \llparenthesis \Delta, C^{\rho} \rrparenthesis \end{array}$$

We adapt the proof in case the length of the conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma$ (Γ') is non-zero: we know that $\mathcal{P}$ consists of some proof nets attached in a link with main formula A, or connected by a formula A, and each of these proof nets converts to a sequent or an edge. Now, if A is a cut formula of $\mathcal{P}$ we proceed as described earlier: apply a Cut rule (after possibly the logical rule) to the inductively obtained derivations. However, if A is not a cut formula, then A is an axiomatic leaf of at least one of the involved proof nets, whence we can apply a substitution instead of a Cut (after possibly the logical rule):

$$\text{not} \quad \dfrac{\begin{array}{c} \overline{\llparenthesis A^{-}, A^{+} \rrparenthesis} \quad \mathcal{D}_1 \\ \ddots \qquad \vdots \qquad \iddots \\ \hline \llparenthesis \Delta_1, A^{+} \rrparenthesis \end{array} \qquad \begin{array}{c} \mathcal{D}_2 \\ \llparenthesis A^{-}, \Delta_2 \rrparenthesis \end{array}}{\llparenthesis \Delta_1, \Delta_2 \rrparenthesis} \,\text{Cut}_A \qquad \text{but} \quad \begin{array}{c} \mathcal{D}_2 \\ \overline{\llparenthesis A^{-}, \Delta_2 \rrparenthesis} \quad \mathcal{D}_1 \\ \ddots \qquad \vdots \qquad \iddots \\ \hline \llparenthesis \Delta_1, \Delta_2 \rrparenthesis \end{array}$$

Observe that $\mathcal{D}$ is not unique anymore now: if A is an axiomatic formula of $\mathcal{P}$, then it is an axiomatic leaf of both proof nets $\mathcal{P}_1$ and $\mathcal{P}_2$, whence we can either substitute $\mathcal{D}_1$ in $\mathcal{D}_2$ or $\mathcal{D}_2$ in $\mathcal{D}_1$.

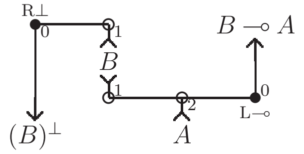
Theorem 4.4.6. *(a) Let $\mathcal{D}$ be an $\boldsymbol{NCLL_2}$-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$ (or $\mathcal{P}(\mathcal{D}) \twoheadrightarrow (\llcorner\mathcal{D}\lrcorner)'$).*

(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a sequent or a corresponding one-edge link graph. Then there is a sober *derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.*

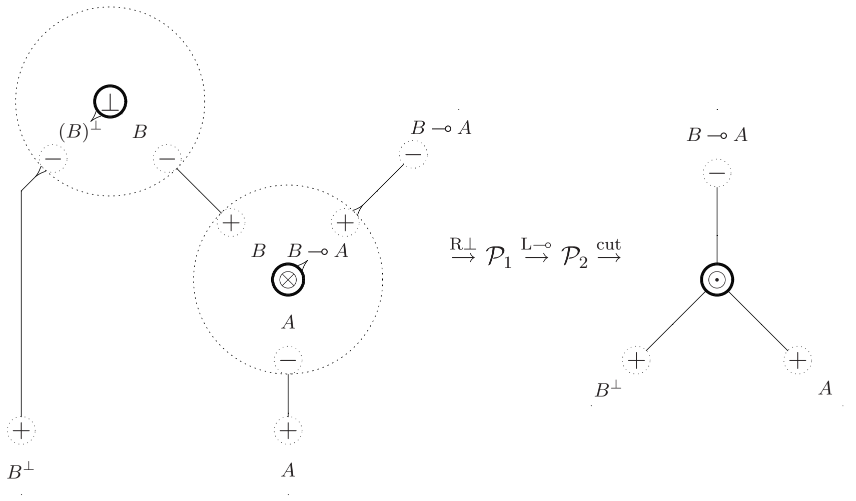
◊

Proof: **(b)** An outline of the elegant proof is given above. Alternatively we can proceed as follows. By Theorem 4.4.4 there is a derivation $\mathcal{D}$ of Γ with $\mathcal{P}(\mathcal{D}) = \mathcal{P}$. Eliminating the non-logical Cut's of $\mathcal{D}$ by means of substitutions yields a sober derivation $\mathcal{D}'$ (see Subsection 4.2.3), and by Theorem 4.3.6 $\mathcal{P}(\mathcal{D}') = \mathcal{P}(\mathcal{D})$. Hence $\mathcal{D}'$ meets the requirements. ////

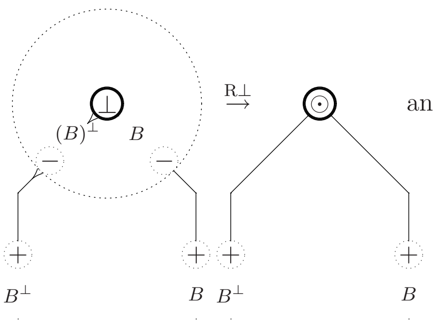
Example 4.4.7. Consider the proof structure



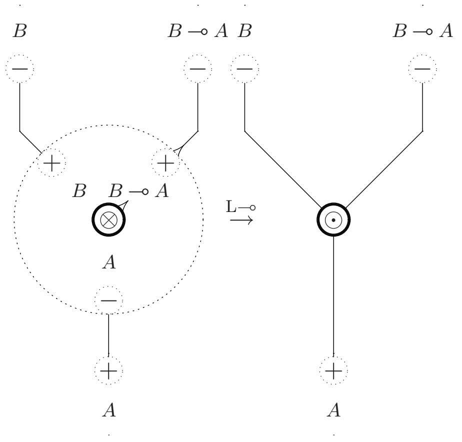
having an axiomatic formula B. It may be converted as indicated:



Analyzing the last step, we have to combine two inductively obtained derivations, corresponding to the sub conversions



and



say

$$\mathcal{D}_1 = \dfrac{\dfrac{}{(\![B^+, B^-]\!)}\,\text{Ax}}{\left(\![B^+, (B^\perp)^+]\!\right)}\,\text{R}\perp \qquad \text{and} \qquad \mathcal{D}_2 = \dfrac{\dfrac{}{(\![A^+, A^-]\!)}\,\text{Ax} \qquad \dfrac{}{(\![B^+, B^-]\!)}\,\text{Ax}}{(\![A^+, (B \multimap A)^-, B^-]\!)}\,\text{L}\multimap$$

The naive solution is a Cut on B; however, when we prefer sober derivations, a solution is given by

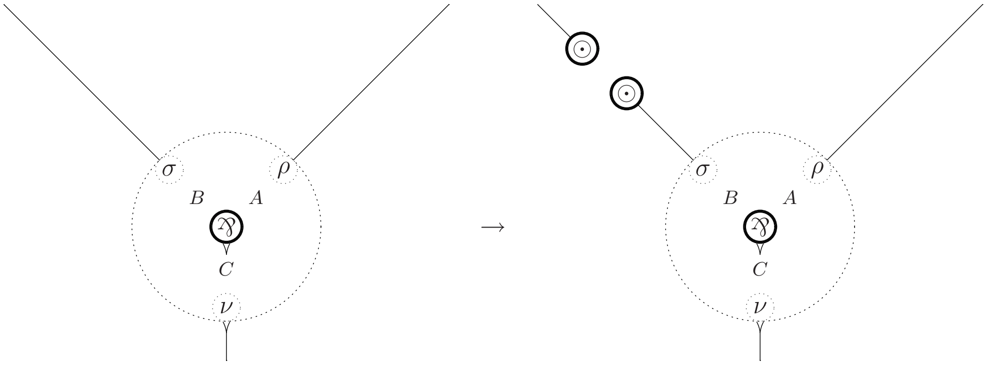
$$\mathcal{D}_1[\mathcal{D}_2] = \dfrac{\dfrac{}{(\![A^+, (B \multimap A)^-, B^-]\!)}\,\mathcal{D}_2}{\left(\![A^+, (B \multimap A)^-, (B^\perp)^+]\!\right)}\,\text{R}\perp \quad = \quad \dfrac{\dfrac{\dfrac{}{(\![A^+, A^-]\!)}\,\text{Ax} \qquad \dfrac{}{(\![B^+, B^-]\!)}\,\text{Ax}}{(\![A^+, (B \multimap A)^-, B^-]\!)}\,\text{L}\multimap}{\left(\![A^+, (B \multimap A)^-, (B^\perp)^+]\!\right)}\,\text{R}\perp$$

while another solution is given by

$$\mathcal{D}_2[\mathcal{D}_1] = \dfrac{\dfrac{}{\llparenthesis A^+, A^- \rrparenthesis}\,\text{Ax} \qquad \dfrac{}{\big(\big[B^+, (B^\perp)^+\big]\big)}\,\mathcal{D}_1}{\big(\big[A^+, (B \multimap A)^-, (B^\perp)^+\big]\big)}\,\text{L}\multimap \quad = \quad \dfrac{\dfrac{}{\llparenthesis A^+, A^- \rrparenthesis}\,\text{Ax} \qquad \dfrac{\dfrac{}{\llparenthesis B^+, B^- \rrparenthesis}\,\text{Ax}}{\big(\big[B^+, (B^\perp)^+\big]\big)}\,\text{R}\perp}{\big(\big[A^+, (B \multimap A)^-, (B^\perp)^+\big]\big)}\,\text{L}\multimap$$

◊

4.4.2. Confluence on $\mathfrak{LG}_2'$. Given an element $\mathcal{P}$ of $\mathfrak{LG}_2$ with n par links $(e_0, e_1, e_2)_{\invamp}$, we define a switching ω for $\mathcal{P}$ to be a choice, for each par link l, of one of the active ends of l. The *correction link graph* $\omega\mathcal{P}$ of $\mathcal{P}$ under the switching ω is obtained by replacing each par link as follows, where the non-chosen end is disconnected. (So for the following link l, $\omega(l) = A^\rho$.)



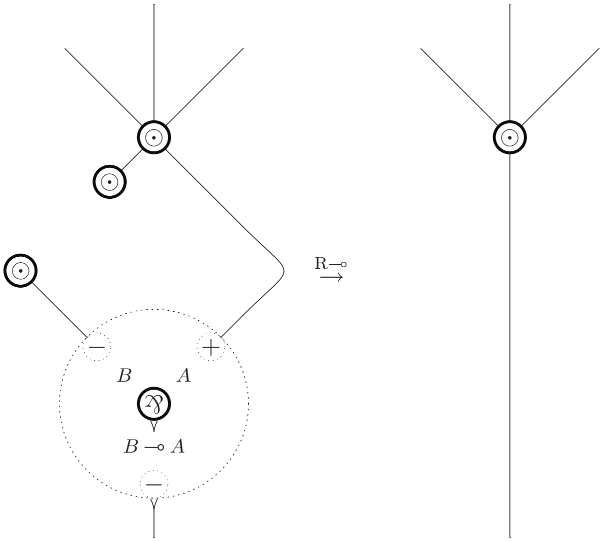
Let $\mathfrak{LG}_2'$ denote the collection of those elements $\mathcal{P}$ of $\mathfrak{LG}_2$ for which all the 2^n correction link graphs $\omega\mathcal{P}$ have a tree as underlying graph.

Lemma 4.4.8. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{LG}_2$ and suppose $\mathcal{P}_1 \rightarrow \mathcal{P}_2$. Then $\mathcal{P}_1 \in \mathfrak{LG}_2'$ if and only if $\mathcal{P}_2 \in \mathfrak{LG}_2'$.* ◊

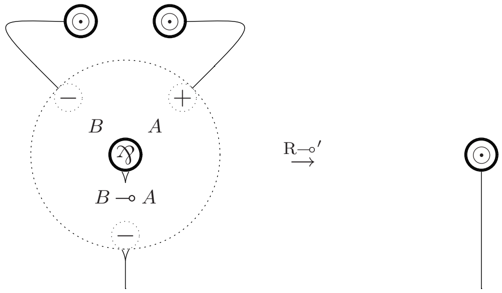
Proof: If the conversion step is $[\textbf{tens/neg}](l)$, $[\textbf{cut}](l_1, l_2, \eta)$ or $[\textbf{cut}'](l, \eta_1, \eta_2)$, the result is immediate, because a correction link graph of $\mathcal{P}_1$ is a tree if and only if the corresponding correction link graph of $\mathcal{P}_2$ is a tree.

If the conversion step is $[\textbf{par}](l_1, l_2)$, writing a switching for $\mathcal{P}_1$ as $(\omega; (l_1 \mapsto e_i))$, observe that the correction link graph $(\omega; (l_1 \mapsto e_1))\mathcal{P}_1$ is a tree if and only if $(\omega; (l_1 \mapsto$

$e_2))\mathcal{P}_1$ is a tree if and only if the corresponding correction link graph $\omega\mathcal{P}_2$ is a tree.



If the conversion step is $[\mathbf{par'}](l,\eta)$, writing a switching for $\mathcal{P}_1$ as $(\omega;(l\mapsto e_i))$, observe that the correction link graph $(\omega;(l\mapsto e_1))\mathcal{P}_1=(\omega;(l\mapsto e_2))\mathcal{P}_1$ is a tree if and only if the corresponding correction link graph $\omega\mathcal{P}_2$ is a tree.



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In particular, the conversion steps are well defined on $\mathfrak{LG}'_2$ (i.e. they do yield an element of $\mathfrak{LG}'_2$ when applied on an element of $\mathfrak{LG}'_2$).

Since sequents (one-edge link graphs) belong to $\mathfrak{LG}'_2$, we immediately obtain the next result.

Corollary 4.4.9. *If a proof structure $\mathcal{P}$ converts to a sequent Γ (one-edge link graph Γ'), then $\mathcal{P}\in\mathfrak{LG}'_2$.* ♢

So proof nets will only be found[14] in $\mathfrak{LG}'_2$. Now we can prove the confluence of this conversion relation on $\mathfrak{LG}'_2$.

---

[14] The other way around, a proof structure in $\mathfrak{LG}'_2$ is not necessarily a proof net. However, if we add

$$\frac{(\![\Gamma, B^{\sigma}, A^{\rho}]\!)}{(\![\Gamma, A^{\rho}, B^{\sigma}]\!)}\,\mathrm{Ex}$$

Lemma 4.4.10. *If $\mathcal{P} \in \mathfrak{LG}'_2$ converts in one step to $\mathcal{P}_1$ and $\mathcal{P}_2$, then both $\mathcal{P}_1$ and $\mathcal{P}_2$ convert in at most one step to a common $\mathcal{P}_3 \in \mathfrak{LG}'_2$.* $\Diamond$

$$\begin{tikzcd}
 & \mathcal{P}_1 \arrow[dr, dashed] & \\
\mathcal{P} \arrow[ur, "{[\mathbf{conv}_1]}"] \arrow[dr, "{[\mathbf{conv}_2]}"'] & & \mathcal{P}_3 \\
 & \mathcal{P}_2 \arrow[ur, dashed] &
\end{tikzcd}$$

Proof: If $[\mathbf{conv}_1]$ exactly equals $[\mathbf{conv}_2]$, the result is clear by

$$\begin{tikzcd}
 & \mathcal{P}_1 \arrow[dr, equal] & \\
\mathcal{P} \arrow[ur, "{[\mathbf{conv}_1]}"] \arrow[dr, "{[\mathbf{conv}_2]}"'] & & \mathcal{P}_3 . \\
 & \mathcal{P}_2 \arrow[ur, equal] &
\end{tikzcd}$$

So assume $[\mathbf{conv}_1] \neq [\mathbf{conv}_2]$.

If $[\mathbf{conv}_1]$ is $[\mathbf{tens/neg}](l)$, $[\mathbf{conv}_2]$ does not contain l. In all the subcases $[\mathbf{conv}_1]$ and $[\mathbf{conv}_2]$ turn out to commute:

- $[\mathbf{conv}_2]$ is $[\mathbf{tens/neg}](l')$ ($l' \neq l$ a tensor link or negation link): o.k.;
- $[\mathbf{conv}_2]$ is $[\mathbf{par}](l_1, l_2)$ (l_1 a par link and l_2 a context link): o.k.;
- $[\mathbf{conv}_2]$ is $[\mathbf{par'}](l_1, \eta)$ (l_1 a par link): o.k.;
- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_1, l_2, \eta)$ ($l_1 \neq l_2$ context links): o.k., since $[\mathbf{tens/neg}](l)$ does not identify l_1 and l_2;
- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l_1, \eta_1, \eta_2)$ (l_1 a context link and $\eta_1 \neq \eta_2$ edges): o.k., since $[\mathbf{tens/neg}](l)$ does not require edges to be different, and neither identifies η_1 and η_2.

The case $[\mathbf{conv}_1] = [\mathbf{par'}](l, \eta)$ is treated similarly.

If $[\mathbf{conv}_1]$ is $[\mathbf{par}](l_1, l_2)$, the following subcases remain (where $l'_1 \neq l_1$ is a par link and $l'_2 \neq l_2$ is a context link):

- $[\mathbf{conv}_2]$ is $[\mathbf{par}](l'_1, l_2)$: the diagram in Figure 4.2 of the form

$$\begin{tikzcd}
 & \mathcal{P}_1 \arrow[dr, "\mathrm{L}\otimes"] & \\
\mathcal{P} \arrow[ur, "\mathrm{R}\multimap"] \arrow[dr, "\mathrm{L}\otimes"'] & & \mathcal{P}_3 \\
 & \mathcal{P}_2 \arrow[ur, "\mathrm{R}\multimap"'] &
\end{tikzcd}$$

  shows that the statement holds;
- $[\mathbf{conv}_2]$ is $[\mathbf{par}](l_1, l'_2)$: impossible;
- $[\mathbf{conv}_2]$ is $[\mathbf{par}](l'_1, l'_2)$: o.k., disjoint redexes;

(called Exchange) to the rules of $\mathbf{NCLL}_2$ we obtain $\mathbf{MLL}_2$, and the proof nets (where a proof net is still defined as a proof structure that can be obtained as the proof structure of a derivation) of this calculus are exactly the proof structures in $\mathfrak{LG}'_2$. See Section 4.10.

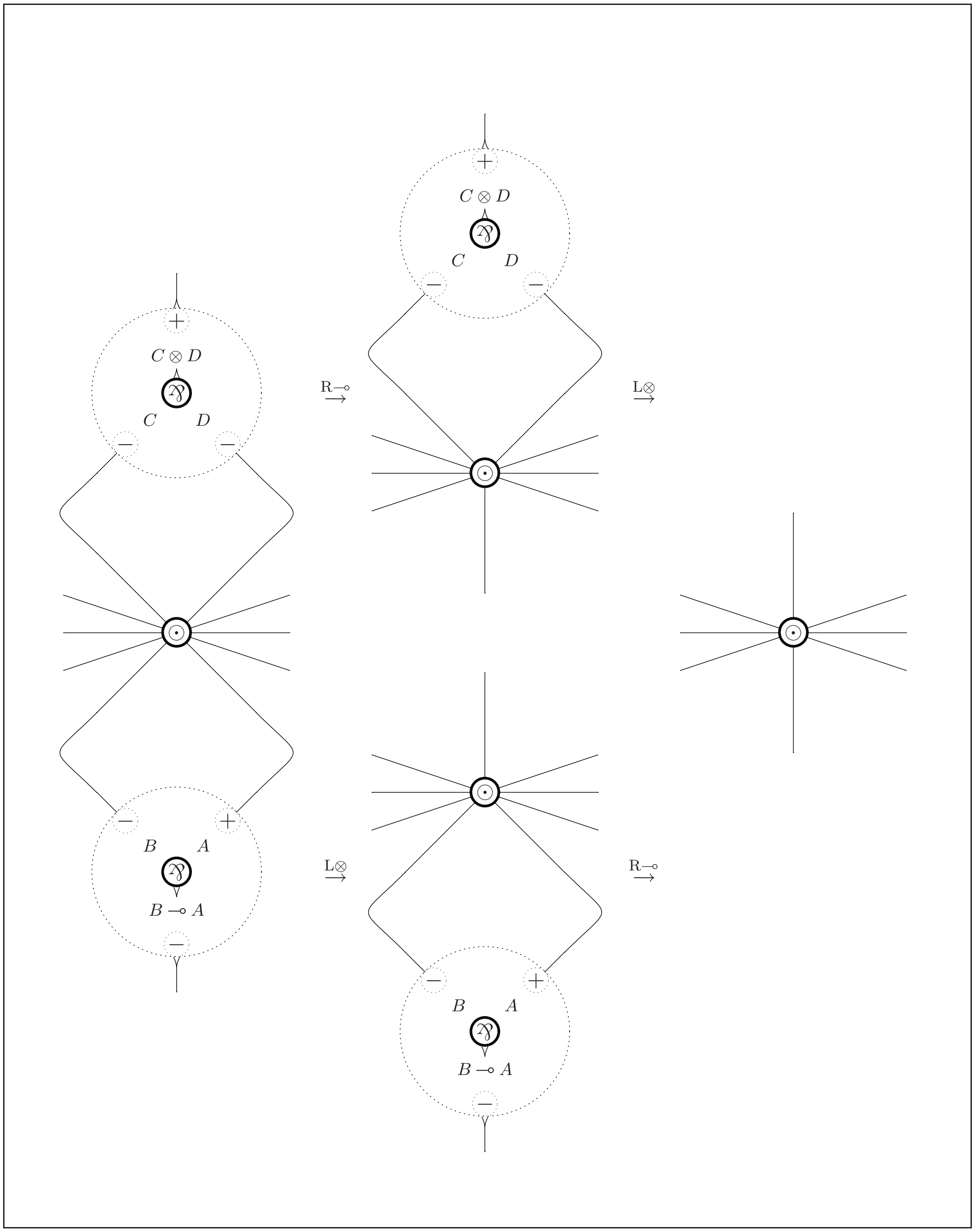
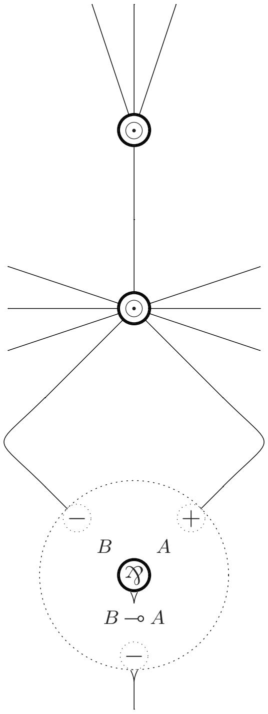


Figure 4.2.

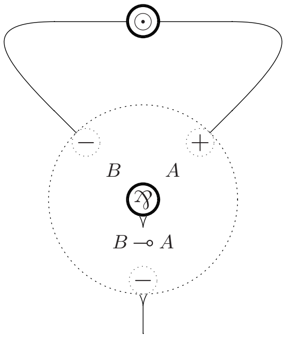
- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_2, l_2', \eta)$:



in this case the statement holds as well by
$$\begin{tikzcd}
& \mathcal{P}_1 \arrow[dr, "\text{cut}"] & \\
\mathcal{P} \arrow[ur, "\text{R}\multimap"] \arrow[dr, "\text{cut}"'] & & \mathcal{P}_3 \\
& \mathcal{P}_2 \arrow[ur, "\text{R}\multimap"'] &
\end{tikzcd}$$
;

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_2', l_2'', \eta)$ (l_2'' a context link different from l_2 and l_2'): o.k., disjoint redexes;

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}'](l_2, \eta_1, \eta_2)$:



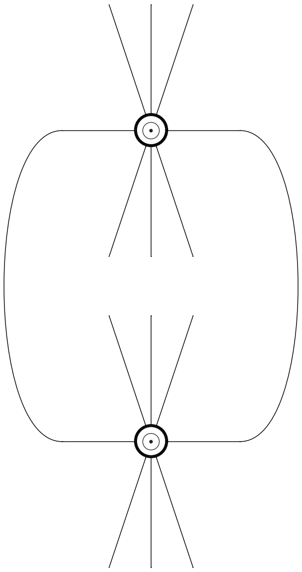
in this case the statement holds as well by

$$\begin{tikzcd}
 & \mathcal{P}_1 \arrow[dr, equal] & \\
\mathcal{P} \arrow[ur, "\text{R}\multimap"] \arrow[dr, "\text{cut}'"'] & & \mathcal{P}_3 \;; \\
 & \mathcal{P}_2 \arrow[ur, "\text{R}\multimap'"'] &
\end{tikzcd}$$

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}'](l_2', \eta_1, \eta_2)$: o.k., since $[\mathbf{par}](l_1, l_2)$ does not require edges to be different, and neither identifies η_1 and η_2.

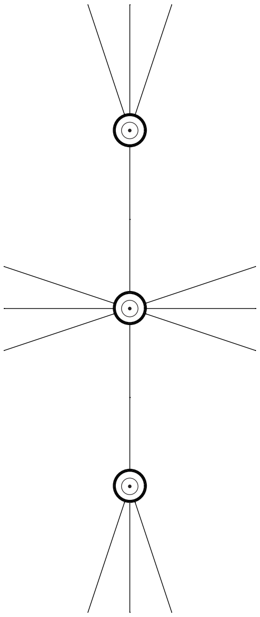
If $[\mathbf{conv}_1]$ is $[\mathbf{cut}](l_1, l_2, \eta)$, the following subcases remain (where l_1, l_2, l_3, l_4 are distinct context links and $\zeta, \zeta' \neq \eta$ distinct edges):

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_1, l_2, \zeta)$:



this case is impossible because of the requirement $\mathcal{P} \in \mathfrak{LG}_2'$;

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_2, l_3, \zeta)$:



in this case the statement holds as well by

$$\begin{tikzcd}
& \mathcal{P}_1 \arrow[dr, "\text{cut}"] & \\
\mathcal{P} \arrow[ur, "\text{cut}"] \arrow[dr, "\text{cut}"'] & & \mathcal{P}_3 \\
& \mathcal{P}_2 \arrow[ur, "\text{cut}"'] &
\end{tikzcd}$$

;

- $[\mathbf{conv}_2]$ is $[\mathbf{cut}](l_3, l_4, \zeta)$: o.k., disjoint redexes;
- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l_2, \eta, \zeta)$: o.k., $\mathcal{P}_1 = \mathcal{P}_2$;
- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l_3, \zeta, \zeta')$: o.k., since $[\mathbf{cut}](l_1, l_2, \eta)$ does not require edges to be different, and neither identifies ζ and ζ'.

Finally, if $[\mathbf{conv}_1]$ is $[\mathbf{cut'}](l, \eta_1, \eta_2)$, the following subcases remain (where $\eta_1, \eta_2, \eta_3, \eta_4$ are distinct edges and $l' \neq l$ a context link):

- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l', \eta_1, \eta_2)$: o.k., $\mathcal{P}_1 = \mathcal{P}_2$ (actually, this case is impossible because of the requirement $\mathcal{P} \in \mathfrak{LG}'_2$);
- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l', \eta_2, \eta_3)$: o.k., $\mathcal{P}_1 = \mathcal{P}_2$;
- $[\mathbf{conv}_2]$ is $[\mathbf{cut'}](l', \eta_3, \eta_4)$: o.k., disjoint redexes.

////

Observe how we avoid the case causing non-confluence (see page 101) by imposing $\mathcal{P} \in \mathfrak{LG}'_2$.

By means of Lemma 4.4.10 and Lemma 4.4.3 we can sharpen Corollary 4.4.5 into:

Theorem 4.4.11. *Let Γ be an $\boldsymbol{NCLL}_2$-sequent. Then the following are equivalent:*

(i) *Γ is derivable in $\boldsymbol{NCLL}_2$;*

(ii) *There is a proof structure $\mathcal{P}$ such that either all conversion sequences $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ (where $\mathcal{P}'$ is normal) satisfy $\mathcal{P}' = \Gamma$, or they all satisfy that $\mathcal{P}'$ equals Γ's corresponding one-edge link graph Γ'.*

$\diamondsuit$

Theorem 4.4.12. *Let $\mathcal{P}$ be a proof structure and $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ be an arbitrary conversion sequence to a normal form. Then $\mathcal{P}$ is a proof net if and only if $\mathcal{P}'$ is a sequent or a one-edge link graph.* $\diamondsuit$

4.4.3. Structurality. Suppose we are given a proof structure $\mathcal{P}$ which is a proof net, i.e. the proof structure of some (for the moment unknown) derivation $\mathcal{D}$ with final sequent Γ. As $\mathcal{P} \twoheadrightarrow \Gamma$ (or $\mathcal{P} \twoheadrightarrow \Gamma'$, where Γ' is Γ's corresponding one-edge link graph) we know $\mathcal{P}$ must have the same open ends as Γ. Hence the information that $\mathcal{P}$ is a proof net with $m > 0$ open ends $A_0^{\rho_0}, \ldots, A_{m-1}^{\rho_{m-1}}$, gives rise to $(m-1)!$ candidate sequents $\Big(\Big[A_0^{\rho_0}, A_{i_1}^{\rho_{i_1}}, \ldots, A_{i_{m-1}}^{\rho_{i_{m-1}}}\Big]\Big)$ (where $i = (k \mapsto i_k)$ ranges over the permutations of $\{1, \ldots m-1\}$), among which at least one is derivable. By confluence, we know that Γ is uniquely determined, and we can find Γ and *a* derivation $\mathcal{D}$ by arbitrarily performing conversions until we arrive at a normal form; this normal form then equals Γ (or Γ'), and the sequence of conversions can be translated into a derivation (see Subsection 4.4.1). This means that, although $\mathcal{P}$ a priori[15] does not give any information about the structure (i.e. order) of its open ends, this information is still contained in $\mathcal{P}$. In this case we will say that $\mathcal{P}$ *proves* Γ.

Remark 4.4.13. **(Implicit structurality)** A proof net implicitly determines a structure on its open ends (i.e. a sequent). $\diamondsuit$

The particular case $0 < m \leq 2$ is of a different kind. Now $(m-1)! = 1$, which means that a proof net $\mathcal{P}$ explicitly determines a derivable sequent $(\![A_0^{\rho_0}]\!)$ or $(\![A_0^{\rho_0}, A_1^{\rho_1}]\!) = (\![A_1^{\rho_1}, A_0^{\rho_0}]\!)$.

Proof structures with more than two open ends, however, can also be given explicit structurality by appropriately 'closing off' the open ends by means of $m-1$ (or just $m-2$) par links:

Example 4.4.14. Suppose proof net $\mathcal{P}$ has open ends A^-, B^+ and C^-. Then it proves either $(\![A^-, B^+, C^-]\!)$ or $(\![C^-, B^+, A^-]\!)$, but we will only discover the actual corresponding provable sequent after applying conversions (implicit structurality). However, if we already *know* $\mathcal{P}$ actually proves $(\![C^-, B^+, A^-]\!)$, then also $(\![C^-, B^+ \parr A^-]\!) = (\![C^-, (B \circ\!\!- A)^+]\!)$ (and hence also $(\![(C \multimap (B \circ\!\!- A))^+]\!)$) is provable, and attaching the corresponding par link(s) to $\mathcal{P}$ removes the original ambiguity. $\diamondsuit$

Recall that we say $\mathcal{P}$ is a proof net (respectively proof structure or link graph) of $\Gamma \vdash \Delta$ whenever it has Γ as multiset of hypotheses, and Δ as multiset of conclusions. Now, when both multisets are singletons, by the above discussion, a proof net of $A \vdash B$ really proves the *sequent* $\big(A \vdash B\big) = (\![A^-, B^+]\!)$.

---

[15] A *planar* graphical representation of the link graph $\mathcal{P}$ would give this information, but recall that our link graphs are defined by a tuple $(\mathcal{E}, \mathcal{L}, \mathcal{L}', \lambda)$ and not by a geometric picture.

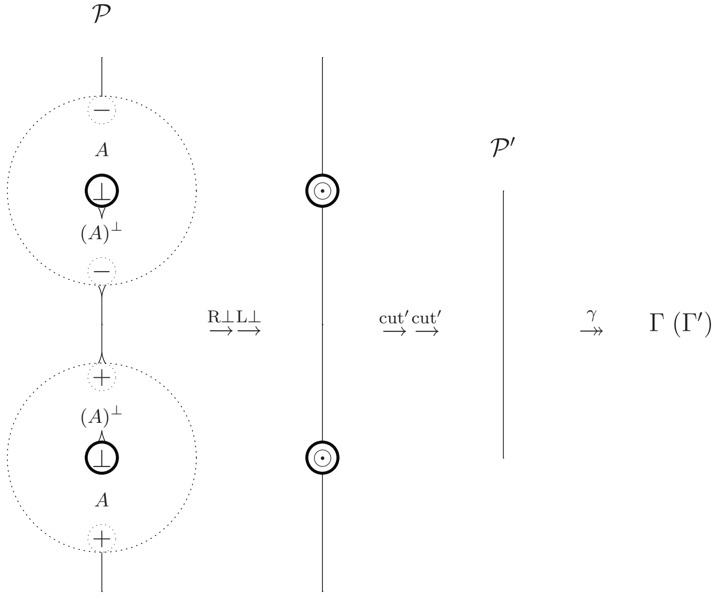
4.5. Cut elimination by means of proof nets

Recall that a cut formula is a formula which is the main formula of two dual links, and that a cut reduction step is defined by deleting these links and the cut formula, while pairwise letting *collaps* the active formulas: successively identifying them in case they are different (as occurrence of the same formula), or deleting them if they are identical (see Subsection 3.2.3).

THEOREM 4.5.1. *If $\mathcal{P}$ is a proof net proving Γ, and $\mathcal{P} \rightsquigarrow \mathcal{P}'$ by a cut reduction step, then $\mathcal{P}'$ is a proof net proving Γ as well.* ♢

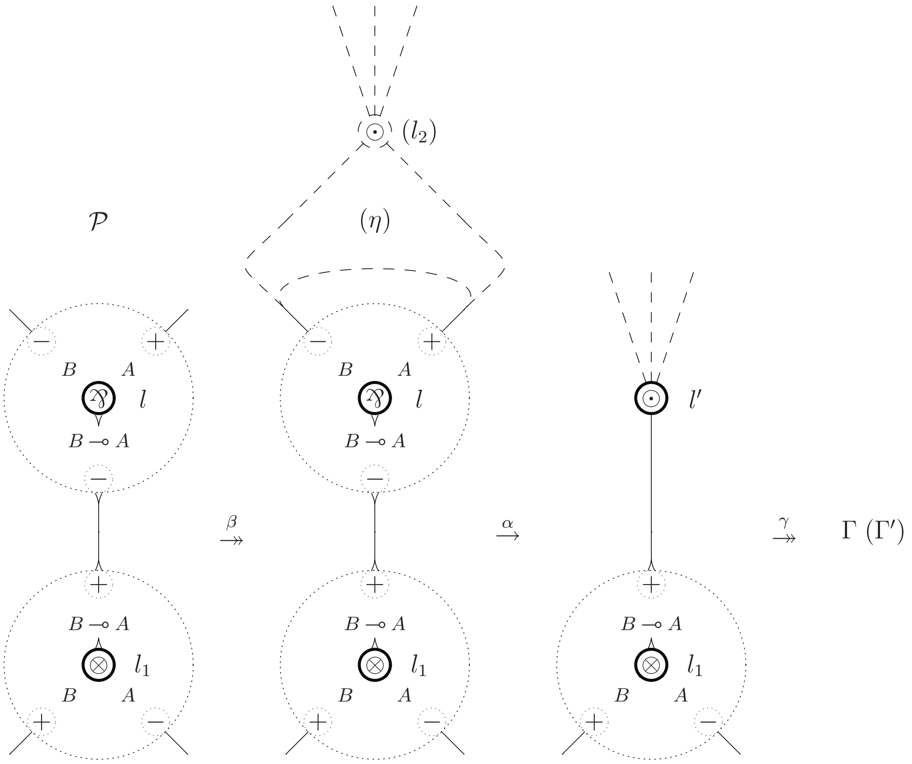
PROOF: We are given that there is a conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma\ (\Gamma')$. By Lemma 4.4.3 (Termination) and Lemma 4.4.10 (Confluence) we know all maximal conversion sequences $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\mathrm{nf}}$ satisfy $\mathcal{P}_{\mathrm{nf}} = \Gamma\ (\Gamma')$. Now suppose $\mathcal{P} \rightsquigarrow \mathcal{P}'$ is the cut reduction step eliminating a cut formula C of $\mathcal{P}$. We will show that also $\mathcal{P}' \twoheadrightarrow \Gamma\ (\Gamma')$.

First suppose $C = (A)^{\perp}$. Let us consider a conversion sequence of $\mathcal{P}$ with the following initial conversions, which is justified by confluence.

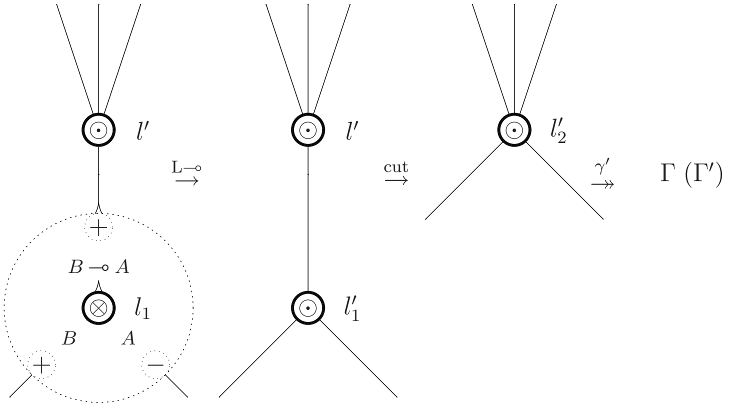


Here the two applications of $[\mathbf{cut'}]$ are allowed since the edges are different as a result of Corollary 4.4.9. Now the queue γ is a conversion sequence for $\mathcal{P}'$, which proves this case.

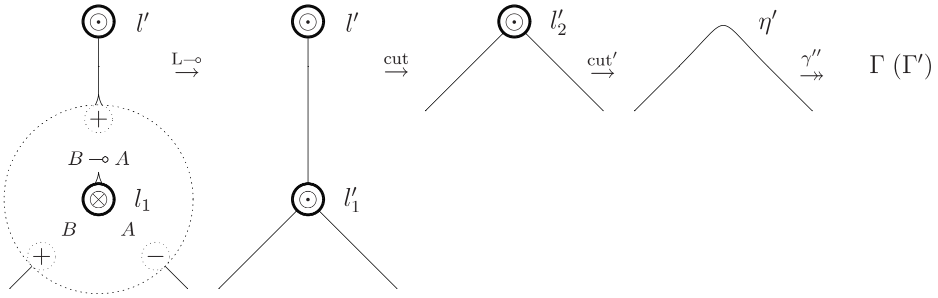
Now suppose $C = B \multimap A$ (the cases $A \otimes B$, $B \circ\!\!- A$ and $B \parr A$ are similar). Let l and l_1 be the corresponding par link respectively tensor link. Fix a conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma\ (\Gamma')$. At some point the par link l (appearing in $\mathcal{P}$ but absent in $\Gamma\ (\Gamma')$) must disappear, say at a conversion $\alpha := [\mathbf{par}](l, l_2)\ ([\mathbf{par'}](l, \eta))$, yielding a context link l' in either case. Of course, l remains untouched until α. *Now let us suppose that l_1 remains* *untouched until* α *as well.* Then our conversion sequence has the following form:



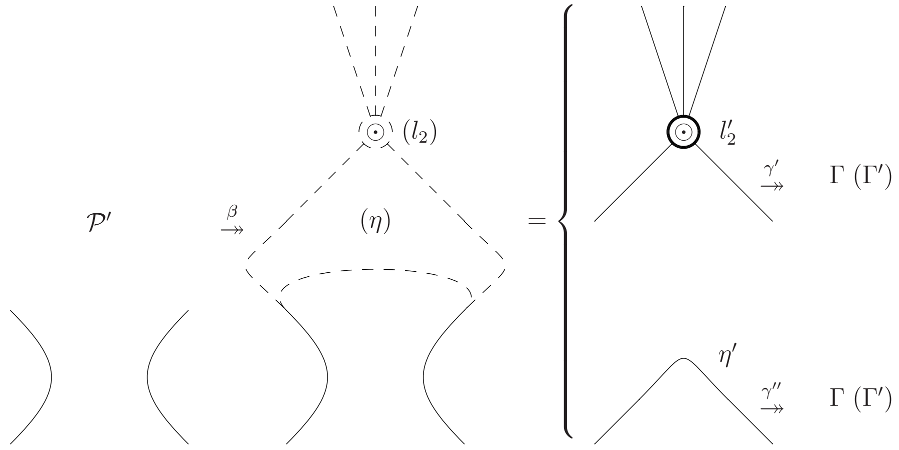
By confluence, we can alter the queue γ into



respectively



Executing the cut reduction step yields the proof structure $\mathcal{P}'$ to which we can apply β followed by γ' (γ''):



Observe that l_2 (η) plays the role of l'_2 (η') in γ' (γ'').

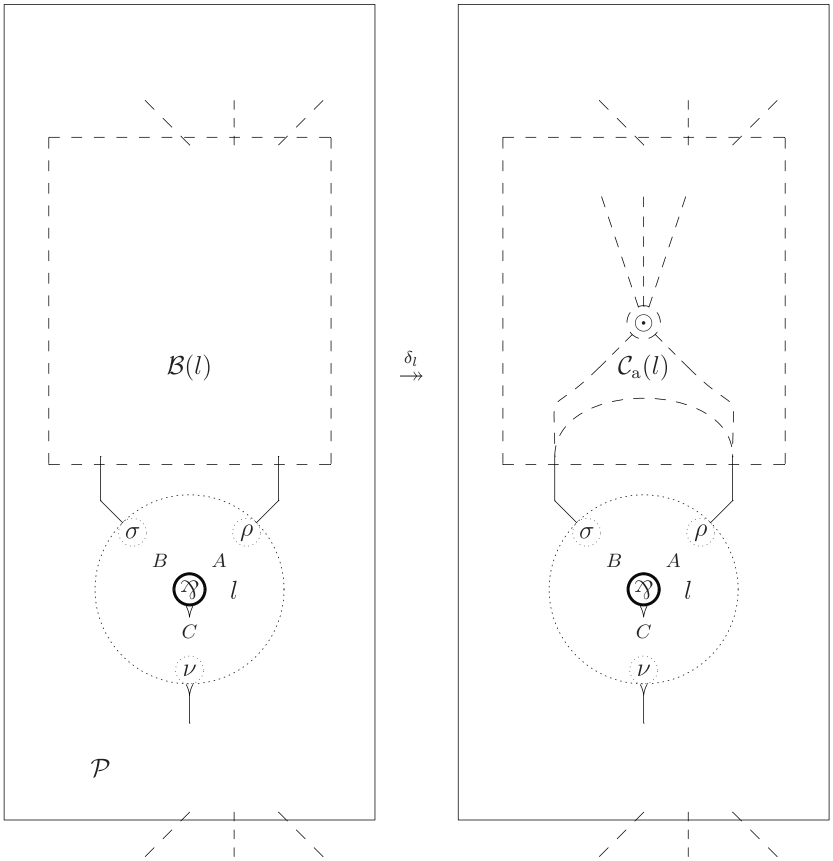
It remains to prove that it is possible to choose our conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma$ (Γ') such that l_1 remains untouched until $\alpha = [\mathbf{par}](l, l_2)$ ($[\mathbf{par'}](l, \eta)$).

Let $\mathcal{P} \overset{\delta}{\twoheadrightarrow} \Gamma$ (Γ') be an arbitrary conversion sequence:

$$\mathcal{P} = \mathcal{P}_m \xrightarrow{\delta_m} \mathcal{P}_{m-1} \xrightarrow{\delta_{m-1}} \ldots \xrightarrow{\delta_2} \mathcal{P}_1 \xrightarrow{\delta_1} \mathcal{P}_0 = \Gamma\ (\Gamma').$$

For each par link l of $\mathcal{P}$ with main end C^ν and active ends A^ρ and B^σ we will define a sub-proof structure $\mathcal{B}(l)$ of $\mathcal{P}$ (called the *block* of l in $\mathcal{P}$ w.r.t. δ) and a subsequence δ_l

satisfying the following properties:



- $\mathcal{B}(l)$ does not contain the edge C. Consequently it does not contain l either;
- $\mathcal{B}(l)$ has A^ρ and B^σ among its open ends.
- The conversion sequence δ_l acts completely within $\mathcal{B}(l)$, turning $\mathcal{B}(l)$ into a link graph $\mathcal{C}_{\mathrm{a}}(l)$ which together with l contains a redex for the conversion $\alpha_l = [\mathbf{par}](l, l_2)$ $([\mathbf{par'}](l, \eta))$.
- Our original conversion sequence may be replaced by

$$\mathcal{P} \overset{\delta_l}{\twoheadrightarrow} \cdot \overset{\alpha_l}{\rightarrow} \cdot \overset{\delta'_l}{\twoheadrightarrow} \Gamma\ (\Gamma')$$

We will only sketch the idea; the formal definition and proof may be given simultaneously by induction on the length of δ.

First of all, deleting all n par links $l^1, \ldots, l^n$ of $\mathcal{P} = \mathcal{P}_m$ (which turns some connector ends into open ends) yields $n+1$ sub-proof structures, called the *components* of $\mathcal{P}$. This even holds in a more general sense for all intermediate link graphs $\mathcal{P}_i$ between $\mathcal{P}$ and Γ (Γ'): deleting all n' present par links yields $n'+1$ sub-link graphs, called the *components of* $\mathcal{P} \overset{\delta}{\twoheadrightarrow} \Gamma$ (Γ') *at stage* i. Indeed, reasoning backwards from Γ (Γ'), we start with one component ($n' = 0$). After a number of conversions of the form $[\mathbf{tens/neg}]$, [**cut**] or [**cut′**], a contraction $\mathcal{P}_i \overset{\alpha_{l^1}}{\to} \mathcal{P}_{i-1}$ splits this sole component $\mathcal{P}_{i-1}$ into two parts and replaces one context link by a redex. The par link l^1 of this redex now serves as a boundary between the two new components $\mathcal{C}_{\mathrm{a}}(l^1)$ (containing A^ρ and B^σ among its open ends) and $\mathcal{C}_{\mathrm{m}}(l^1)$ (containing C^ν among its open ends), while (at this stage) $\mathcal{C}_{\mathrm{a}}(l^1)$ together with l contains a redex. All next conversions of the form [**tens/neg**], [**cut**] or [**cut′**] take place completely within one of the two components, and the next contraction $\mathcal{P}_j \overset{\alpha_{l^2}}{\to} \mathcal{P}_{j-1}$ ($j > i$) takes place in exactly one of the two components as well. In this way every contraction $\mathcal{P}_i \overset{\alpha_l}{\to} \mathcal{P}_{i-1}$ replaces *exactly one* of the $n' + 1$ present components of δ at stage $i - 1$ by two new components of δ at stage i (resulting in n' 'old' components and 2 new components $\mathcal{C}_{\mathrm{a}}(l)$ and $\mathcal{C}_{\mathrm{m}}(l)$). This yields $n' + 1$ components at a general stage i (where we suppose that the intermediate link graph $\mathcal{P}_i$ has n' par links), and $2n' + 1$ distinct components in the whole conversion sequence up to this stage (read from right to left), taking into account that we should not distinguish between *corresponding* components of $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$.

Let l be a par link which (dis)appears in $\mathcal{P}_i \overset{\alpha_l}{\to} \mathcal{P}_{i-1}$. We define the *block of* l *at stage* i *w.r.t.* δ to be the component $\mathcal{C}_{\mathrm{a}}(l)$ of δ at stage i, and further on (at stages $j > i$) it grows with the remaining conversions $\delta_{i+1}, \delta_{i+2}, \ldots$ in δ; it is clear that every conversion is completely inside the block, or completely outside the block, which proves our properties for $\mathcal{B}(l)$: the block of l at stage m w.r.t. δ.

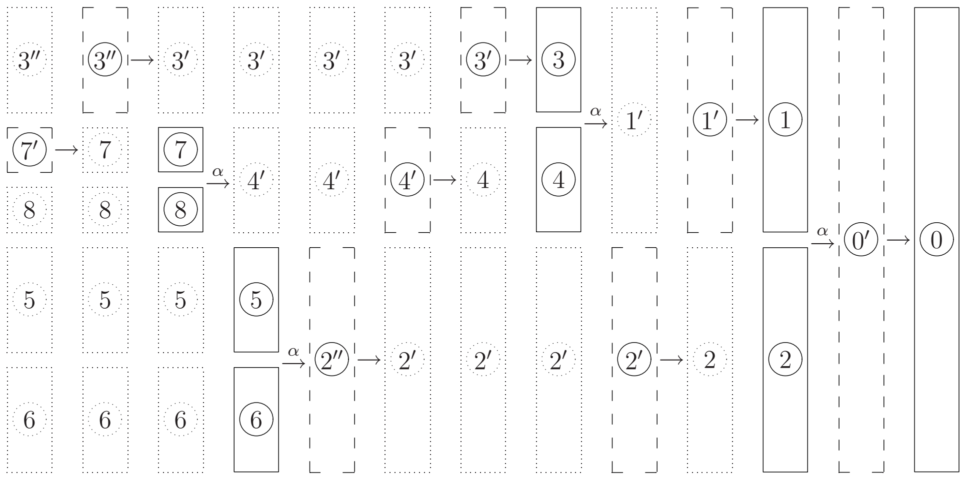
This shows that we can reorder our original conversion sequence δ by

$$\mathcal{P} \overset{\delta_l}{\twoheadrightarrow} \cdot \overset{\alpha_l}{\to} \cdot \overset{\delta_l'}{\twoheadrightarrow} \Gamma\ (\Gamma')$$

Observe that the edge C and hence also l_1 is outside $\mathcal{B}(l)$, showing that l_1 remains untouched until α_l. ////

Example 4.5.2. Suppose a proof net $\mathcal{P}$ contains 4 par links, then any conversion sequence $\mathcal{P} \overset{\delta}{\twoheadrightarrow} \Gamma\ (\Gamma')$ contains 4 conversions α of type [**par**] or [**par′**]. Let a particular δ be given. Reasoning backwards from $\Gamma\ (\Gamma')$ we find the components at each stage, e.g.

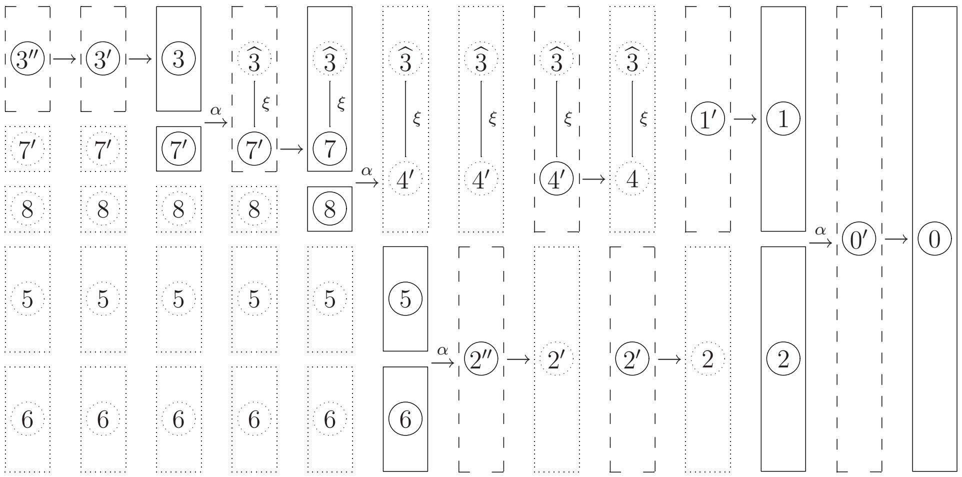
$$\mathcal{P}_{12} \overset{\delta_{12}}{\to} \mathcal{P}_{11} \overset{\delta_{11}}{\to} \mathcal{P}_{10} \overset{\delta_{10}}{\to} \mathcal{P}_9 \overset{\delta_9}{\to} \mathcal{P}_8 \overset{\delta_8}{\to} \mathcal{P}_7 \overset{\delta_7}{\to} \mathcal{P}_6 \overset{\delta_6}{\to} \mathcal{P}_5 \overset{\delta_5}{\to} \mathcal{P}_4 \overset{\delta_4}{\to} \mathcal{P}_3 \overset{\delta_3}{\to} \mathcal{P}_2 \overset{\delta_2}{\to} \mathcal{P}_1 \overset{\delta_1}{\to} \mathcal{P}_0$$



The proof net $\mathcal{P} = \mathcal{P}_{12}$ has 5 components, viz. ③″, ⑦′, ⑧, ⑤ and ⑥, which result is independent of δ. Observe that $\mathcal{P}_5$ contains $n' = 2$ par links. Hence at stage 5 there are $n' + 1 = 3$ components (③, ④ and ②′), and there have been $2n' + 1 = 5$ so far yet (⓪, ①, ②, ③ and ④). Here we consider ⓪′ the same component as ⓪, although as a sub link graph they differ for a conversion of the form [**tens/neg**], [**cut**] or [**cut′**]; similar for ①′ and ②′.

Let us concentrate on the par link l corresponding to the conversion $\alpha = \delta_5$. Suppose that $\mathcal{C}_{\mathrm{a}}(l)$ is ③. Then, reasoning backwards, we see $\mathcal{B}(l)$ equals ③″. Observe that ①′ consist of ④, attached via one edge ξ to a modification of ③, viz. α applied to the sub-link graph consisting of the par link l attached to ③, notation $\widehat{③}$. This edge ξ originally belongs either to ⑦′ or to ⑧; let us suppose the former. Now we can alter δ into

$$\mathcal{P}_{12} \xrightarrow{\delta_{11}} \cdot \xrightarrow{\delta_6} \cdot \xrightarrow{\delta_5} \cdot \xrightarrow{\delta_{12}} \cdot \xrightarrow{\delta_{10}} \cdot \xrightarrow{\delta_9} \cdot \xrightarrow{\delta_8} \cdot \xrightarrow{\delta_7} \mathcal{P}_4 \xrightarrow{\delta_4} \mathcal{P}_3 \xrightarrow{\delta_3} \mathcal{P}_2 \xrightarrow{\delta_2} \mathcal{P}_1 \xrightarrow{\delta_1} \mathcal{P}_0$$



If $\mathcal{C}_{\mathrm{a}}(l)$ would have been ④ instead, $\mathcal{B}(l)$ would equal the union of ⑦′, a par link and ⑧. The new conversion sequence would now start with δ_{12}, δ_{10}, δ_7 and δ_5. ◇

This result re-establishes Theorem 4.2.4 (Cut elimination) for derivations by means of Theorem 4.4.6: given a derivable sequent Γ, there is a proof net proving Γ (Corollary 4.4.5). By Theorem 4.5.1 also the normal form $\widehat{\mathcal{P}}$ of $\mathcal{P}$ under cut elimination proves Γ, whence by Theorem 4.4.6 there is a sober derivation $\mathcal{D}$ of Γ with $\mathcal{P}(\mathcal{D}) = \widehat{\mathcal{P}}$. As $\widehat{\mathcal{P}}$ is CUT-free (has no cut formulas), by soberness also $\mathcal{D}$ is CUT-free (contains no CUT rules; Proposition 4.3.5), alternatively proving Theorem 4.2.4 (Cut elimination).

LEMMA 4.5.3. *(a) An identity proof structures $\mathcal{I}(C)$ (see page 52) proves $C \vdash C$ and is hence a proof net.*

(b) If $\mathcal{P}$ is a proof net proving Γ*, and $\mathcal{P}'$ is obtained from $\mathcal{P}$ by the replacement of a non-atomic axiomatic formula C by the identity proof structure $\mathcal{I}(C)$, then also $\mathcal{P}'$ is a proof net proving* Γ*.*

◊

Proof: **(a)** By induction on C. For atomic C the result is immediate. For $C = A \,\Box\, (B)$ first convert $\mathcal{I}(A)$ (as well as $\mathcal{I}(B)$) to a single edge, and finally the remaining dual [L□]- and [R□]-link by means of [**tens**], [**par**] and [**cut′**] ([**neg**], [**neg**], [**cut′**] and [**cut′**]).

(b) First convert $\mathcal{I}(C)$ into a singe edge according to part (a), yielding $\mathcal{P}$, which by assumption converts to Γ (Γ'). ////

We will henceforth talk about the identity proof *net* $\mathcal{I}(C)$.

Suppose Γ is derivable, then there is a proof net $\mathcal{P}$ (proving Γ) which we can hence suppose to be cut-free (Theorem 4.5.1) as wel as η-expanded (Lemma 4.5.3). We have characterized such proof structures in Subsection 3.2.4, whence the search space for our proof net $\mathcal{P}$ is limited to the (finitely many) proof structures described in Proposition 3.2.9. This observation proves the following main theorem concerning proof search by means of proof nets.

Theorem 4.5.4. *(a) Let a sequent Γ be given. Then Γ is derivable if and only if the set*

$$\{\mathcal{P} \,|\, \mathcal{P} \textit{ is an } \eta\textit{-expanded cut-free proof structure with the same open ends as } \Gamma\}$$

contains a proof net proving Γ*.*

(b) Let a set of open ends be given, i.e. a multiset of hypotheses $A_0^-, \ldots, A_{m-1}^-$ and a multiset of conclusions $B_0^+, \ldots, B_{n-1}^+$. Then some cyclic list consisting of these open ends is derivable if and only if the set

$$\big\{\mathcal{P} \,|\, \mathcal{P} \textit{ is an } \eta\textit{-expanded cut-free proof structure of } A_0^-, \ldots, A_{m-1}^- \vdash B_0^+, \ldots, B_{n-1}^+\big\}$$

contains a proof net: a proof structure converting to a sequent or a single edge.

◊

In part (b) we have abstracted part (a) over Γ. For many applications one is interested in the existence of some derivable cyclic list Γ with given open ends. Instead of applying part (a) for each of the candidate cyclic lists Γ — there are in general $(n + m - 1)!$ such — we can check the RHS of part (b); in case of succes, this procedure automatically gives the witnessing sequent, showing the computational strength of our contraction criterion.

4.6. Dualizable proof nets

In Subsection 3.2.3 we introduced the dualization operation $\mathcal{P} \mapsto \mathcal{P}^*$ on proof structures, defined by replacing each link by its dual and reversing the labeling of every edge. In particular, if $\mathcal{P}$ has open ends given by the multiset Π, then $\mathcal{P}^*$ has open ends given by the multiset $\Pi^* := \tau\Pi$ (where τ is the sign alternation map). So a proof structure $\mathcal{P}$ of $\Gamma \vdash \Delta$ is turned into a proof structure $\mathcal{P}^*$ of $\Delta \vdash \Gamma$.

Observe that actually, in Example 3.2.8, both $\mathcal{P}$ and $\mathcal{P}^*$ are $\mathbf{NCLL}_2$-proof *nets*. However, it does not generally hold that a proof net yields another proof net under this transformation. Let us call a proof net $\mathcal{P}$ *dualizable* if its dualization $\mathcal{P}^*$ is a proof net as well.

In this section we will show that the collection of dualizable proof nets is — roughly speaking — equal to the collection of identity proof nets, modulo the 'De Morgan laws' and associativity.

Suppose we have a proof net $\mathcal{P}_1$ with open ends Π, and another proof net $\mathcal{P}_2$ with open ends Π^*. Then some sequent Γ consisting of the elements of Π is derivable, whence by Lemma 4.1.11 $|\Gamma| + \#_{\parr}(\Gamma) = \#_{\otimes}(\Gamma) + 2$. Similarly, some sequent Δ consisting of the elements of Π^* is derivable, whence $|\Delta| + \#_{\parr}(\Delta) = \#_{\otimes}(\Delta) + 2$. Using the fact that $\Pi^* = \tau\Pi$, the last line rewrites to $|\Gamma| + \#_{\otimes}(\Gamma) = \#_{\parr}(\Gamma) + 2$. We conclude that $\#_{\parr}(\Gamma) = \#_{\otimes}(\Gamma)$ and that $|\Gamma| = 2$.

In particular, if $\mathcal{P}_2 = \mathcal{P}_1^*$, the result holds, so every dualizable proof net has two open ends, say A^{ρ} and B^{σ}, thus proving the sequents $(\![A^{\rho}, B^{\sigma}]\!)$ as well as $(\![A^{-\rho}, B^{-\sigma}]\!)$. Moreover, there is a dualizable proof net proving $(\![A^{+}, B^{+}]\!)$ (and $(\![A^{-}, B^{-}]\!)$) if and only if there is a dualizable proof net proving $\big(\![A^{+}, (B^{\perp})^{-}]\!\big)$ (and $\big(\![A^{-}, (B^{\perp})^{+}]\!\big)$). Hence it is no restriction to study only dualizable proof nets with exactly one hypothesis and one conclusion.

Let us define the following relations $\dashv\vdash$ and $\dashv_{\mathrm{d}}\vdash$ on $\mathfrak{L}_2$:

$$\begin{array}{lll}
A \dashv\vdash B & :\Longleftrightarrow & A \vdash B \text{ is derivable} \quad \text{and} \quad B \vdash A \text{ is derivable} \\
 & \Longleftrightarrow & \text{there is a proof net } \mathcal{P}_1 \text{ of } A \vdash B \\
 & & \quad \text{and a proof net } \mathcal{P}_2 \text{ of } B \vdash A \\
 & \Longleftrightarrow & \text{there is a cut-free proof net } \mathcal{P}_1 \text{ of } A \vdash B \\
 & & \quad \text{and a cut-free proof net } \mathcal{P}_2 \text{ of } B \vdash A \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded proof net } \mathcal{P}_1 \text{ of } A \vdash B \\
 & & \quad \text{and a cut-free and } \eta\text{-expanded proof net } \mathcal{P}_2 \text{ of } B \vdash A \\
A \dashv_{\mathrm{d}}\vdash B & :\Longleftrightarrow & \text{there is a proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is a proof net of } B \vdash A \\
 & \overset{(*)}{\Longleftrightarrow} & \text{there is a cut-free proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is a cut-free proof net of } B \vdash A \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is a cut-free and } \eta\text{-expanded proof net of } B \vdash A
\end{array}$$

Recall that we mean by an *η-expanded* proof structure a proof structure with only *atomic* axiomatic formulas. The last equivalence in both definitions above is a consequence of the fact that we can replace a non-atomic axiomatic formula C by the identity proof net $\mathcal{I}(C)$ having only atomic axiomatic formulas and being invariant under dualization. After this operation the proof structure remains a proof net proving the same sequent as before (the dualization of which is also obtained by a replacement of C by $\mathcal{I}(C)$ and hence is also a proof net).

The equivalence marked by $(*)$ is a consequence of the fact that dualizing a proof structure commutes with a cut elimination step: suppose $\mathcal{P}$ is a dualizable proof net of $A \vdash B$ (i.e. $\mathcal{P}$ and $\mathcal{P}^*$ are proof nets). Then its reduct $\mathcal{P}'$ is a proof net (by the soundness of cut-elimination), the dualization $(\mathcal{P}')^*$ of which is nothing else but the reduct $(\mathcal{P}^*)'$ of $\mathcal{P}^*$, and hence a proof net itself. This implies that $\mathcal{P}'$ is a dualizable proof net. By induction we may conclude that the normal form (under cut elimination) $\widehat{\mathcal{P}}$ is a dualizable cut-free proof net of $A \vdash B$, the dualization of which is automatically cut-free as well.

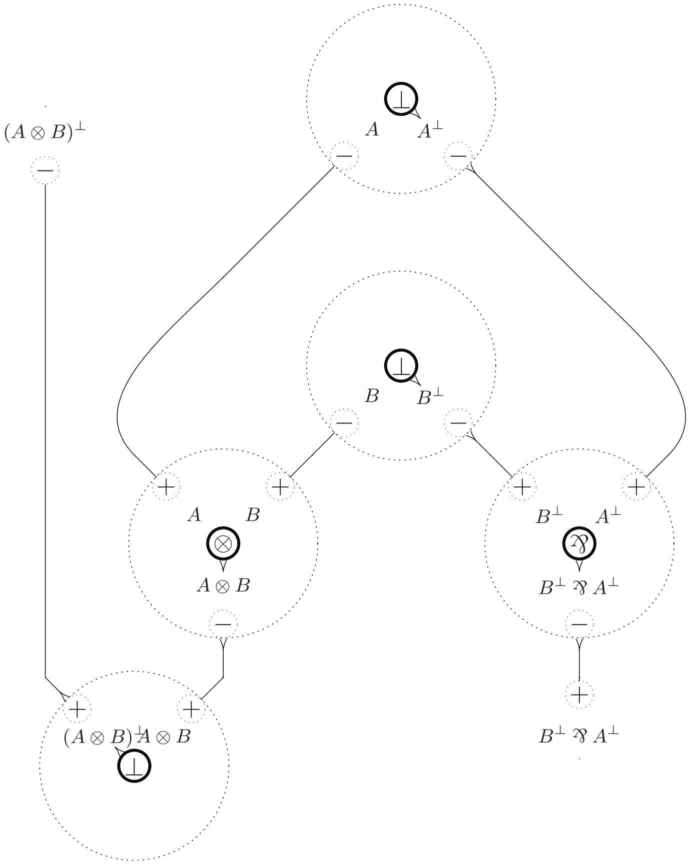
We say A is *(dualizable)-provably equivalent* to B iff $A \dashv\vdash B$ ($A \dashv_{\mathrm{d}}\vdash B$), and it is easy to check that both relations are congruence relations on $\mathfrak{L}_2$-formulas. In Example 4.6.4.3 we will see that $\dashv_{\mathrm{d}}\vdash$ is strictly contained in $\dashv\vdash$.

In order to completely characterize the congruence relation $\dashv_{\mathrm{d}}\vdash$, we will make use of $\mathfrak{L}_1$, which can be considered as the De Morgan quotient of $\mathfrak{L}_2$ (see Section 2.2). With the help of the next lemmas we will establish the fact that $\simeq$ (Section 2.3) and $\dashv_{\mathrm{d}}\vdash$ coincide.

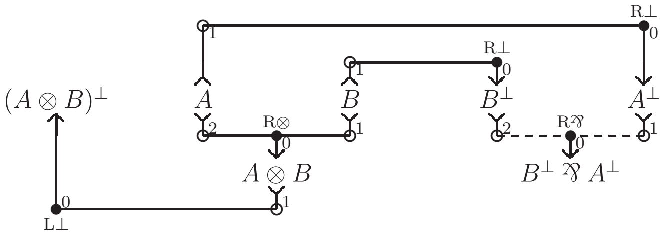
LEMMA 4.6.1. *Let A and B be two syntactic **MLL**-formulas.*
1. If $A \simeq B$, then A and B are dualizable-provably equivalent.
2. A is dualizable-provably equivalent with its normal form $\nu\pi(A)$. ◊

PROOF: Part 1. The relation $\dashv_{\mathrm{d}}\vdash := \{ (A, B) \mid A \dashv_{\mathrm{d}}\vdash B\}$ is an equivalence relation satisfying (where the numbers refer to Definition 2.3.1):

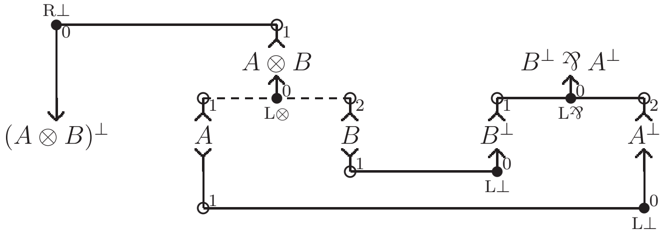
- (0□), by pasting the dual links L□ and R□ to the given dualizable proof net(s) ($\square = \otimes, \multimapinv, \multimap, \parr$ or $\perp$);
- (1), by the following dualizable proof net



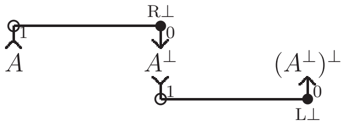
alternatively denoted by



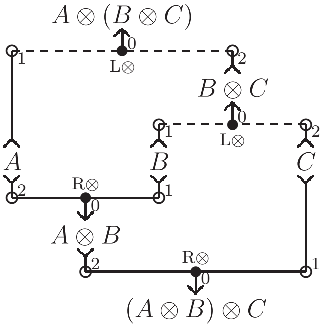
whose dualization is given by



- (2); similarly;
- (3a); by the dual proof nets of Example 3.2.8;
- (3b); similarly;
- (4), by the dualizable proof net



- (5⊗), by the dualizable proof net



- (5⅋); similarly.

As $\simeq$ is the smallest such equivalence relation, we must have that $\simeq\ \subseteq\ \dashv_{\mathrm{d}}\vdash$, i.e. if $A \simeq B$ then $A \dashv_{\mathrm{d}}\vdash B$.

Part 2. We know that $\nu\pi A \equiv A$, implying $\nu\pi A \simeq A$, which yields $\nu\pi A \dashv_{\mathrm{d}}\vdash A$ by part 1. ////

LEMMA 4.6.2. *Suppose $A \dashv_{\mathrm{d}}\vdash B$ and let $\mathcal{P}$ be a cut-free and η-expanded dualizable proof net of $A \vdash B$. Then each (atomic) axiomatic formula is a subformula of both A and B.*

As a consequence, A and B have the same multiset of atomic subformulas:

$$P(A) \cup N(A) = P(B) \cup N(B).$$

$\Diamond$

Proof: By Proposition 3.2.9 we know that $\mathcal{P}$ is the union of T^A and T_B, followed by an identification of the atomic formulas[16]. Now suppose that an atomic subformula α of A is identified with another atomic subformula α of A. (I.e. $\alpha \in N(A)$ is identified with $\alpha \in P(A)$.) Let A' be the smallest subformula of A containing both occurrences of α. If A' is the main formula of a tensor link l, then there is a switching ω' of T^A (extendible to a switching ω of $\mathcal{P}$) yielding a path from the one occurrence of α to A' as well as a path from the other occurrence of α to A'. But this yields (after identification of the two occurrences of α) a cycle in $\omega\mathcal{P}$ since l is a tensor link, in contradiction to Corollary 4.4.9. If l is a par link, then l^* is a tensor link, so the same argument applies and yields a cycle in $\omega\mathcal{P}^*$. Hence every atomic subformula α of A is identified with an atomic subformula α of B, even better: every positive (negative) atomic subformula α of A is identified with a positive (negative) atomic subformula α of B ////

Theorem 4.6.3. *For all $\mathfrak{L}_2$-formulas A and B the following holds:*

$$A \simeq B \qquad \textit{if and only if} \qquad A \dashv_{\mathrm{d}}\vdash B$$

$\Diamond$

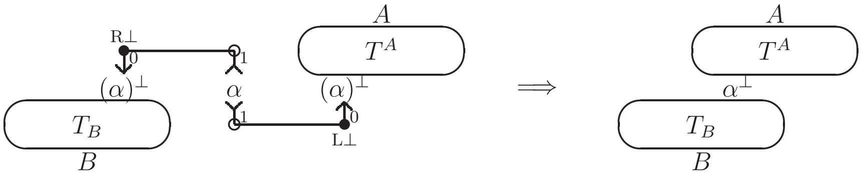
Proof: $\boxed{\Longrightarrow}$ This is Lemma 4.6.1.1.

$\boxed{\Longleftarrow}$ We first prove this direction for $\mathfrak{L}_{2,\mathrm{nf}}$-formulas (the De Morgan normal forms):

$$\mathfrak{L}_{2,\mathrm{nf}} := \mathcal{F} ::= \mathcal{A} \mid (\mathcal{A})^{\perp} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \parr \mathcal{F}$$

This will be done by induction on the size of a cut-free and η-expanded dualizable proof net of $A \vdash B$.

Suppose $A \dashv_{\mathrm{d}}\vdash B$ where A and B are $\mathfrak{L}_{2,\mathrm{nf}}$-formulas. Let $\mathcal{P}$ be a cut-free and η-expanded dualizable proof net of $A \vdash B$. Then we know by Proposition 3.2.9 that $\mathcal{P}$ is the union of T^A and T_B containing only L$\otimes$-, R$\otimes$-, L$\parr$- and R$\parr$-links, or $\perp$-links applied to atoms, followed by an identification of the atomic formulas, which is pairwise by Lemma 4.6.2. If $(\alpha)^{\perp}$ is a subformula of A, then α is an hypothesis of T^A. Hence it is a conclusion of T_B, yielding that $(\alpha)^{\perp}$ is a subformula of B. Contracting the two $\perp$-links and replacing $(\alpha)^{\perp}$ by the new atom $\alpha^{\perp}$ yields a proof net which moreover is $\perp$-free.



Hence let $\mathcal{P}$ be a cut-free and η-expanded dualizable proof net of $A \vdash B$, where $\mathcal{P}$ is the union of T^A and T_B containing only L$\otimes$-, R$\otimes$-, L$\parr$- and R$\parr$-links, followed by a pairwise

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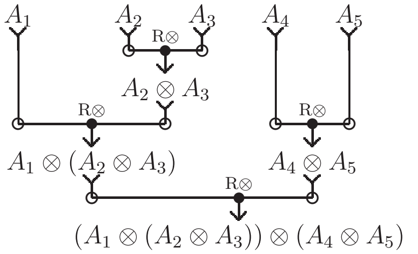
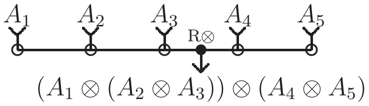
[16] Moreover, by Corollary 3.2.10 we know that the multisets $P(A) \cup N(B)$ (all conclusions of $T^A \cup T_B$ except B) and $P(B) \cup N(A)$ (all hypotheses of $T^A \cup T_B$ except A) coincide. However, this does not yet yield $P(A) = P(B)$ and $N(B) = N(A)$, what we actually want to prove now as it implies $P(A) \cup N(A) = P(B) \cup N(B)$.

identification of the (new) atomic formulas. Performing the contractions $\mathcal{P} \twoheadrightarrow \Gamma$ in the opposite direction provides us with a *planar* graphical representation of $\mathcal{P}$.

We will call a maximal connected component of $\mathcal{P}$ consisting entirely of a positive number of tensor (par) links a *tensor (par) cluster*. By the absence of $\perp$-links, every link belongs to exactly one cluster. Every internal formula of a cluster is neither axiomatic nor cut: for if it was an axiomatic formula, then for some switching of the par version of this cluster (in $\mathcal{P}$ or $\mathcal{P}^*$) this formula would be disconnected; and it can neither be a cut formula, since we assumed $\mathcal{P}$ to be cut-free[17]. We call a leaf of a cluster an active (main) formula, if it is an active (main) formula of some link of the cluster. Each cluster contains exactly one main formula, while by the absence of [L/R$\multimap$]- and [L/R$\multimapinv$]-links (besides the absence of [L/R$\perp$]-links) we know that all active formulas are positive active formulas. Let us be even more precise and show that a cluster $\mathcal{C}$ is a tree of links, all of the same subtype L$\otimes$, R$\otimes$, L$⅋$ or R$⅋$.





We show this by induction on the number of links in $\mathcal{C}$. Let $\mathcal{C}$ be a cluster, hence with at least one link. Choose a formula C belonging to $\mathcal{C}$ with maximal length $\mathsf{l}(C)$. If C would be an internal formula of $\mathcal{C}$ it would have role 1, whence it would be the main formula of one link, and an active formula of some other link l'. But then l' has as main formula a formula of bigger length; contradiction. So C is a leaf of $\mathcal{C}$, occurring in precisely one link l of $\mathcal{C}$ by connectedness. It can still not be an active formula of l, whence C is the main formula of l. Then the active formula A (B) of l is either a leaf of $\mathcal{C}$, or main formula of a link l_A (l_B) of $\mathcal{C}$. Removing l yields two components $\mathcal{C}_A$ (or possibly only A), and $\mathcal{C}_B$ (or possibly only B), which are disconnected because otherwise there would be a cycle in a correction link graph of $\mathcal{P}$ or $\mathcal{P}^*$. Now by induction hypothesis $\mathcal{C}_A$ (if present) is a tree of links, all of the same subtype, whence with a unique main formula, which must be A, being the main formula of l_A. Also, $\mathcal{C}_B$ (if present) is a tree of similar links with main formula B. But then $\mathcal{C}$ is a tree of similar links (with unique main formula C).

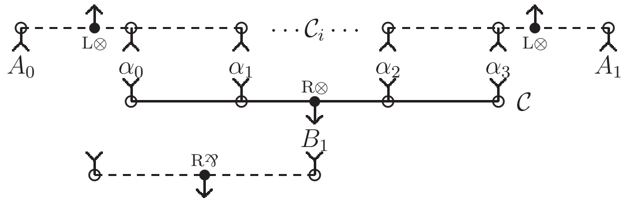
Knowing what the clusters look like, we turn back to our main proof. If there are no clusters, we get $A = \alpha = B$, whence $A \simeq B$. Now suppose there is at least one cluster. Then there is a cluster with only *atomic* active formulas. For if not, then for every cluster $\mathcal{C}_i$ we could choose a non-atomic active formula, which cannot be an axiomatic formula in our η-expanded dualizable proof net, and hence is the main formula of another cluster $\mathcal{C}_{i+1}$, yielding an infinite descending chain of subformulas; contradiction.

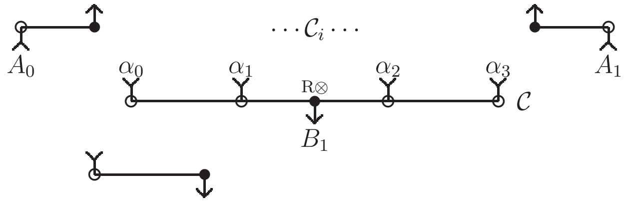
We may assume there is a tensor cluster $\mathcal{C}$ with only atomic active formulas (because there is a cluster with only atomic active formulas in $\mathcal{P}$, which is a tensor or par cluster, and hence a par or tensor cluster in $\mathcal{P}^*$). The active formulas are — on the other hand

---

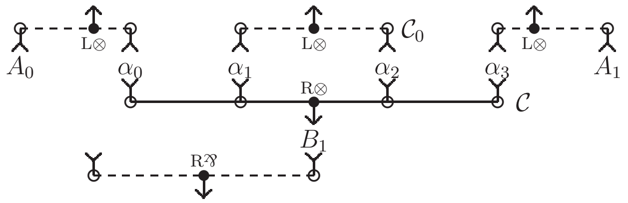
[17] Even if $\mathcal{P}$ would not be cut-free, an internal formula of a cluster is not a cut formula, since a cut formula is always between two dual links, whence (in the $\perp$-free case) between a tensor link and a par link.

— also active formulas of par clusters $\mathcal{C}_i$ (according to the definition of a cluster the $\mathcal{C}_i$ cannot be tensor clusters), and we want to show that at least one of these par clusters $\mathcal{C}_0$ has all its active formulas among those of $\mathcal{C}$. Well, if this would not be the case, then for each par cluster choosing one of its active formulas not among those of $\mathcal{C}$ can be extended to a switching where $\mathcal{C}$ is disconnected (after disconnecting its main formula as well in case B_1 is a *strict* subformula of B; see picture below); contradiction.





So there is a par cluster $\mathcal{C}_0$ having all its active formulas among those of $\mathcal{C}$:



Now repeating the same story in $\mathcal{P}^*$ yields that the active formulas of $\mathcal{C}$ are among those of $\mathcal{C}_0$. Hence $\mathcal{C}$ and $\mathcal{C}_0$ face each other, so — by the planarity of our representation — their main formulas C and C_0 are ⊗-only (⅋-only) $\mathfrak{L}_2$-formulas with the same sequence of atoms. This gives $C \simeq C_0$ by Lemma 2.3.3.

Replacing $\mathcal{C}$ and $\mathcal{C}_0$ by a unique new atom α_∞ results in a strictly smaller dualizable proof net $\mathcal{P}'$, yielding $A[\alpha_\infty/C_0] \simeq B[\alpha_\infty/C]$ by induction hypothesis. Backsubstituting C and C_0 (for which $C \simeq C_0$) we get $A[\alpha_\infty/C_0][C_0/\alpha_\infty] \simeq B[\alpha_\infty/C][C/\alpha_\infty]$, i.e. $A \simeq B$.

Now let arbitrary A and B be given for which $A \dashv_{\mathrm{d}}\vdash B$. Then by Lemma 4.6.1.2 $\nu\pi A \dashv_{\mathrm{d}}\vdash A \dashv_{\mathrm{d}}\vdash B \dashv_{\mathrm{d}}\vdash \nu\pi B$, hence $\nu\pi A \dashv_{\mathrm{d}}\vdash \nu\pi B$. By the result established above we obtain $\nu\pi A \simeq \nu\pi B$, whence also $A \simeq \nu\pi A \simeq \nu\pi B \simeq B$, i.e. $A \simeq B$.

////

We summarize the results of this section by:

$$\begin{tikzcd}
A = B \arrow[r, Rightarrow] & A \equiv B \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\text{Proposition 2.2.3}"'] & A \simeq B \arrow[d, Leftrightarrow, "\text{Theorem 4.6.3}"] & & \\
 & \pi A = \pi B & A \dashv_{\mathrm{d}}\vdash B \arrow[r, Rightarrow] & A \dashv\vdash B \arrow[r, Rightarrow, "\text{Lemma 4.1.11}"] & \begin{array}{c}\#_{\otimes}(A) - \#_{\parr}(A) \\ = \\ \#_{\otimes}(B) - \#_{\parr}(B)\end{array}
\end{tikzcd}$$

Observe that all implications above are strict:

Example 4.6.4. 1. $(\alpha \otimes \beta)^{\perp} \equiv (\beta)^{\perp} \parr (\alpha)^{\perp}$, but $(\alpha \otimes \beta)^{\perp} \neq (\beta)^{\perp} \parr (\alpha)^{\perp}$.
2. $\alpha \otimes (\beta \otimes \gamma) \simeq (\alpha \otimes \beta) \otimes \gamma$ but $\alpha \otimes (\beta \otimes \gamma) \not\equiv (\alpha \otimes \beta) \otimes \gamma$ since $\pi(\alpha \otimes (\beta \otimes \gamma)) = \alpha \otimes (\beta \otimes \gamma) \neq (\alpha \otimes \beta) \otimes \gamma = \pi((\alpha \otimes \beta) \otimes \gamma)$.
3. $\alpha \otimes (\alpha \multimap \alpha) \dashv\vdash \alpha$ but not by a dualizable proof net, since the formulas do not have the same multiset of atoms (Lemma 4.6.2). Actually, this provable equivalence is due to the following two derivable sequents:

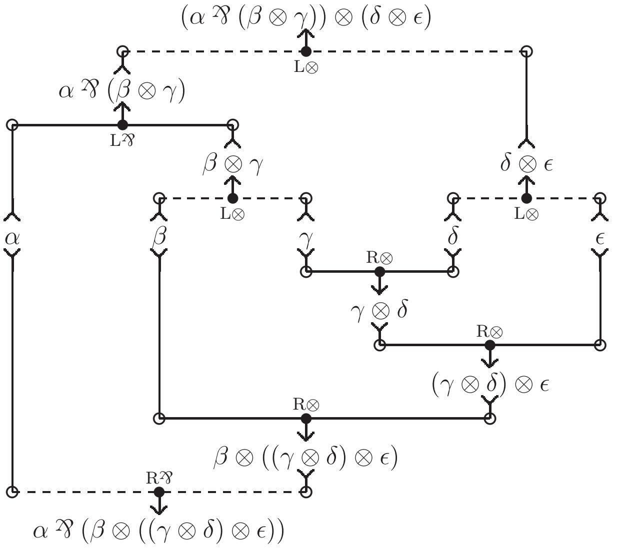
$$\begin{aligned} \beta \otimes (\beta \multimap \alpha) \;&\vdash\; \alpha \\ \alpha \;&\vdash\; \alpha \otimes (\beta \multimap \beta) \end{aligned}$$

which become witnesses for $\alpha \otimes (\alpha \multimap \alpha) \dashv\vdash \alpha$ when taking $\beta := \alpha$.
4. The first proof net in Example 4.6.6 (say, of $A \vdash B$) satisfies $\#_{\otimes}(A) - \#_{\parr}(A) = \#_{\otimes}(B) - \#_{\parr}(B)$, but $B \vdash A$ is not provable. ◊

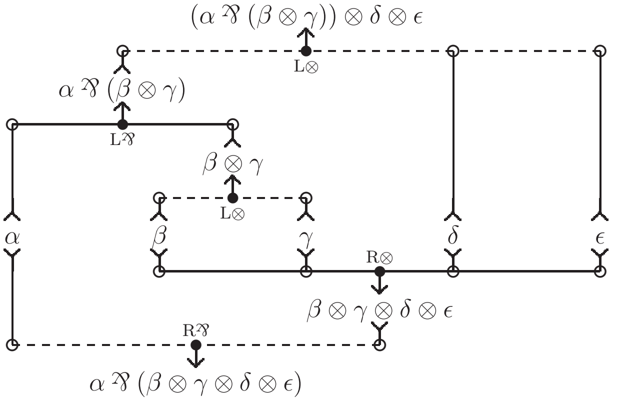
Lemma 4.6.5. *Suppose A and B are $\mathfrak{L}_2$-formulas with coinciding multisets of atomic subformulas (i.e. $P(A) \cup N(A) = P(B) \cup N(B)$), in which moreover each occurring atom has multiplicity one. Then $A \dashv\vdash B$ implies $A \dashv_{\mathrm{d}}\vdash B$.* ◊

Proof: Given $A \dashv\vdash B$, there is a cut-free and η-expanded proof net $\mathcal{P}_1$ of $A \vdash B$ and a cut-free and η-expanded proof net $\mathcal{P}_2$ of $B \vdash A$. We know that $\mathcal{P}_1$ is the union of T^A and T_B followed by an identification of the atoms, while $\mathcal{P}_2$ has a similar description. From the requirement in the lemma it follows that the mentioned identifications are unique in both cases, and hence they are the same. But then $\mathcal{P}_2 = \mathcal{P}_1^*$, which means that $A \dashv_{\mathrm{d}}\vdash B$. ////

EXAMPLE 4.6.6. Consider the proof net $\mathcal{P}$:=



The clusters of $\mathcal{P}$ may be represented as in the following diagram:



This proof net cannot be dualizable, since the tensor cluster on β, γ, δ and ϵ does not face exactly one par cluster, while we know from the proof of Theorem 4.6.3 that this is a necessary condition for $\otimes/\parr$-only dualizable proof nets. Moreover, $\mathcal{P}$ is the unique η-expanded cut-free proof net with these leaves. Hence

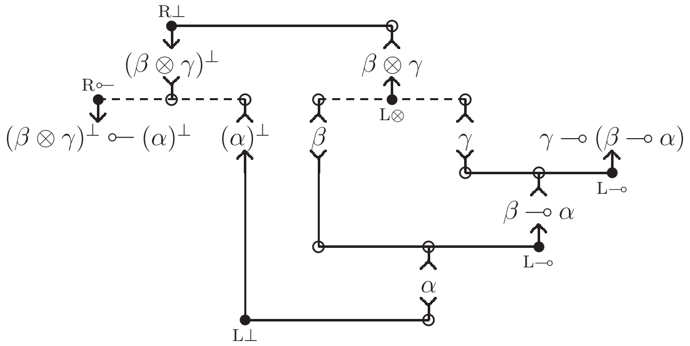
$$(\alpha \parr (\beta \otimes \gamma)) \otimes (\delta \otimes \epsilon) \vdash \alpha \parr (\beta \otimes ((\gamma \otimes \delta) \otimes \epsilon))$$

is provable, but from Lemma 4.6.5 we now can deduce that

$$\alpha \parr (\beta \otimes ((\gamma \otimes \delta) \otimes \epsilon)) \vdash (\alpha \parr (\beta \otimes \gamma)) \otimes (\delta \otimes \epsilon)$$

is not provable.

One easily checks that the next proof net is dualizable:



By Theorem 4.6.3 this implies that $(\beta\otimes\gamma)^\perp \multimapinv (\alpha)^\perp$ and $\gamma \multimap (\beta \multimap \alpha)$ must be $\simeq$-equivalent. We will show this by computing their normal forms.

The normal form of $(\beta\otimes\gamma)^\perp \multimapinv (\alpha)^\perp$ is

$$\begin{aligned}
\nu\pi((\beta\otimes\gamma)^\perp \multimapinv (\alpha)^\perp) &= \nu([\pi(\beta)\otimes\pi(\gamma)]^\perp \multimapinv [\pi(\alpha)]^\perp)\\
&= \nu([\beta\otimes\gamma]^\perp \multimapinv [\alpha]^\perp)\\
&= \nu((\gamma^\perp \parr \beta^\perp) \multimapinv \alpha^\perp)\\
&= \nu((\gamma^\perp \parr \beta^\perp) \parr [\alpha^\perp]^\perp)\\
&= \nu((\gamma^\perp \parr \beta^\perp) \parr \alpha)\\
&= ((\gamma)^\perp \parr (\beta)^\perp) \parr \alpha
\end{aligned}$$

while the normal form of $\gamma \multimap (\beta \multimap \alpha)$ is

$$\begin{aligned}
\nu\pi(\gamma \multimap (\beta \multimap \alpha)) &= \nu(\pi(\gamma) \multimap (\pi(\beta) \multimap \pi(\alpha)))\\
&= \nu(\gamma^\perp \parr (\beta^\perp \parr \alpha))\\
&= (\gamma)^\perp \parr ((\beta)^\perp \parr \alpha)
\end{aligned}$$

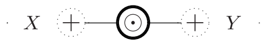
and these two normal forms are $\simeq$-equivalent by (5⅋). As we also know that formulas are $\simeq$-equivalent to their normal forms (actually by (0) up to (4)), we indeed find that $(\beta\otimes\gamma)^\perp \multimapinv (\alpha)^\perp$ and $\gamma \multimap (\beta \multimap \alpha)$ are $\simeq$-equivalent. ◊

4.7. One-sided nets

In this section we will use the theory of the previous sections to prove a contraction criterion for one-sided **NCLL**.

A sequent of $\mathbf{NCLL}_1$ is an $\mathfrak{L}_1$-link graph $\mathcal{P}$ containing exactly one cyclic link $l = \llbracket e_0, \ldots, e_{m-1} \rrbracket_{\odot}$ as context link, no connector links, and whose underlying graph is a tree, i.e. acyclic and connected. Moreover, all labels are positively polarized. As every edge η has exactly one extremity $\hat{\eta}$ occurring in l, $\mathcal{P}$ may be represented by the cyclic list $\llbracket \lambda\check{\eta}_0, \ldots, \lambda\check{\eta}_{m-1} \rrbracket$ of open ends. Observe that a one-edge link graph

is not a sequent; there must be one context link, like in



By $\vdash Z_0, \ldots, Z_{m-1}$ (where $Z_0, \ldots, Z_{m-1}$ is a list) we will denote the sequent $\big(\big[Z_0^+, \ldots, Z_{m-1}^+\big]\big)$.

The calculus $\mathbf{NCLL}_1$ is defined by the following *(elementary) rules*:

$$\mathbf{NCLL}_1$$

$$\frac{}{\big(\big[X^+, ([X]^\perp)^+\big]\big)}\ \textsc{Ax}$$

$$\frac{\llparenthesis \Pi, X^+ \rrparenthesis \qquad \big(\big[\Sigma, ([X]^\perp)^+\big]\big)}{\llparenthesis \Pi, \Sigma \rrparenthesis}\ \textsc{Cut}$$

$$\frac{\llparenthesis \Pi, X^+ \rrparenthesis \qquad \llparenthesis \Sigma, Y^+ \rrparenthesis}{\llparenthesis \Pi, (X \otimes Y)^+, \Sigma \rrparenthesis}\ \otimes \qquad \frac{\llparenthesis \Pi, Y^+, X^+ \rrparenthesis}{\llparenthesis \Pi, (Y \parr X)^+ \rrparenthesis}\ \parr$$

The map $\pi^+ : \mathfrak{L}_2^\pm \to \mathfrak{L}_1^\pm$ defined according to Subsection 2.1.4 and Subsection 2.1.5 by

$$\begin{aligned} \pi^+(A^+) &= (\pi A)^+ \\ \pi^+(A^-) &= \big(\pi((A)^\perp)\big)^+ = ([\pi A]^\perp)^+ \end{aligned}$$

extended to $\mathbf{NCLL}_2$-sequents yields $\mathbf{NCLL}_1$-sequents. One easily verifies that whenever

$$\frac{\llparenthesis \Gamma_0 \rrparenthesis \qquad \ldots \qquad \llparenthesis \Gamma_{n-1} \rrparenthesis}{\llparenthesis \Gamma \rrparenthesis} \tag{8}$$

is an instance of an elementary rule of $\mathbf{NCLL}_2$ different from a negation rule, then so is

$$\frac{\pi^+ \llparenthesis \Gamma_0 \rrparenthesis \qquad \ldots \qquad \pi^+ \llparenthesis \Gamma_{n-1} \rrparenthesis}{\pi^+ \llparenthesis \Gamma \rrparenthesis} \tag{9}$$

w.r.t. $\mathbf{NCLL}_1$, while an instance of a negation rule translates into an instance of the identity induced rule $\frac{\llparenthesis \Pi \rrparenthesis}{\llparenthesis \Pi \rrparenthesis}$. Hence a semi-derivation $\mathcal{D}$ (uniformly) translates into a composition of elementary $\mathbf{NCLL}_1$-rules and identity semi-derivations, which is hence an $\mathbf{NCLL}_1$-semi-derivation. As a consequence, if (8) is an induced rule of $\mathbf{NCLL}_2$, then the same holds for (9) w.r.t. $\mathbf{NCLL}_1$. In particular, derivability of a particular $\mathbf{NCLL}_2$-sequent Γ implies derivability of the $\mathbf{NCLL}_1$-sequent $\pi^+\Gamma$.

The other way around, the function $\psi^\pm : \mathfrak{L}_1^\pm \to \mathfrak{L}_2^\pm$ mapping X^+ to $\psi X = (X^\bullet)^{\overline{X}} \in \mathfrak{L}_2^\pm$ (and X^- to $\tau\psi X = (X^\bullet)^{-\overline{X}} \in \mathfrak{L}_2^\pm$; see Subsection 2.1.4 and Subsection 2.1.5) extended to $\mathbf{NCLL}_1$-sequents yields $\mathbf{NCLL}_2$-sequents, and whenever

$$\frac{\llparenthesis \Pi_0 \rrparenthesis \qquad \ldots \qquad \llparenthesis \Pi_{n-1} \rrparenthesis}{\llparenthesis \Pi \rrparenthesis} \tag{10}$$

is an instance of an elementary rule of $\mathbf{NCLL}_1$, then so is

$$\frac{\psi^\pm \llparenthesis \Pi_0 \rrparenthesis \qquad \ldots \qquad \psi^\pm \llparenthesis \Pi_{n-1} \rrparenthesis}{\psi^\pm \llparenthesis \Pi \rrparenthesis} \tag{11}$$

w.r.t. $\mathbf{NCLL}_2$ (possibly after interchanging the premiss sequents in case of a Cut rule[18]) which follows from the fact that

$$\begin{aligned} \psi(X \otimes Y) &= \psi(X) \otimes \psi(Y) \qquad \text{and} \\ \psi(Y \parr X) &= \psi(Y) \parr \psi(X) \end{aligned}$$

Observe that an instance of a tensor rule may translate into an instance of any of the four tensor rules R$\otimes$, L$\multimapinv$, L$\multimap$ or L$\parr$ of $\mathbf{NCLL}_2$ depending on the parity (see page 25) of X and Y, which shows that a semi-derivation (e.g. the tensor rule) does not in general translate into a semi-derivation. Nevertheless, concrete[19] derivations do translate into derivations, whence derivability of a particular $\mathbf{NCLL}_1$-sequent Π implies derivability of the $\mathbf{NCLL}_2$-sequent $\psi^{\pm}\Pi$.

Lemma 4.7.1. *(a) For all $\boldsymbol{NCLL}_2$-sequents Γ the following holds:*
Γ is $\boldsymbol{NCLL}_2$-derivable if and only if $\pi^+\Gamma$ is $\boldsymbol{NCLL}_1$-derivable.
(b) For all $\boldsymbol{NCLL}_1$-sequents Π the following holds:
Π is $\boldsymbol{NCLL}_1$-derivable if and only if $\psi^{\pm}\Pi$ is $\boldsymbol{NCLL}_2$-derivable.

$\Diamond$

Proof: **(a)** The 'only if'-part is the remark above. For the 'if'-part: suppose $\pi^+\Gamma$ is $\mathbf{NCLL}_1$-derivable. Then by the remark above, $\psi^{\pm}\pi^+\Gamma = \psi\pi\Gamma$ is $\mathbf{NCLL}_2$-derivable. Let $A^\rho \in \Gamma$. Define $B^\sigma := \psi^{\pm}\pi^+ A^\rho = \psi\pi A^\rho$. We claim that $\mu A^\rho \equiv \mu B^\sigma$, whence $\mu A^\rho \simeq \mu B^\sigma$, whence $\mu A^\rho \dashv_{\mathrm{d}}\vdash \mu B^\sigma$ by Lemma 4.6.1. This entails $\mu A^\rho \dashv\vdash \mu B^\sigma$, i.e. both $\llparenthesis(\mu B^\sigma)^+, (\mu A^\rho)^-\rrparenthesis$ and $\llparenthesis(\mu A^\rho)^+, (\mu B^\sigma)^-\rrparenthesis$ are derivable. Using reversibility of the negation rules (in case ρ or σ is negative) we obtain derivability of $\llparenthesis B^\sigma, A^{-\rho}\rrparenthesis$ and $\llparenthesis A^\rho, B^{-\sigma}\rrparenthesis$. Applying a Cut on the derivable sequent $\psi\pi\Gamma$ (containing B^σ) and $\llparenthesis A^\rho, B^{-\sigma}\rrparenthesis$, shows we can substitute A^ρ for $B^\sigma \in \psi\pi\Gamma$, the resulting sequent still being derivable. Continuing we obtain derivability of Γ. The claim $\mu A^\rho \equiv \mu B^\sigma$ follows from the fact that the images of both sides under $\pi : \mathfrak{L}_2 \to \mathfrak{L}_1$ coincide (Proposition 2.2.3): $\pi(\mu B^\sigma) = \pi\mu(\psi\pi A^\rho) = (\pi\mu\psi)\pi A^\rho = \pi A^\rho = \pi(\mu A^\rho)$. Recall that

$$\begin{tikzcd}
 & \mathfrak{L}_1 \arrow[dl, tail, "\psi"'] \arrow[d, tail, dashed, "\chi"] \\
\mathfrak{L}_2^{\pm} \arrow[r, "\mu"'] \arrow[dr, dashed, two heads, "\pi"'] & \mathfrak{L}_2 \arrow[d, two heads, "\pi"] \\
 & \mathfrak{L}_1
\end{tikzcd}$$

is actually the identity $\mathfrak{L}_1 \to \mathfrak{L}_1$ (see page 23).

(b) The 'only if'-part is the remark above. For the 'if'-part: suppose $\psi^{\pm}\Pi$ is $\mathbf{NCLL}_2$-derivable. Then by the remark above, $\pi^+\psi^{\pm}\Pi$ is $\mathbf{NCLL}_1$-derivable. Given $X^+ \in \Pi$, then $\pi^+\psi^{\pm}X^+ = \pi^+\psi X = (\pi\psi X)^+ = X^+$, so in fact $\pi^+\psi^{\pm}\Pi = \Pi$ is $\mathbf{NCLL}_1$-derivable, proving this part. Observe that it is not true that $\pi^+\psi^{\pm}$ is the identity on all of $\mathfrak{L}_1^{\pm}$; it maps X^-

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[18] In case $\dfrac{\llparenthesis \Pi, X^+ \rrparenthesis \qquad \llparenthesis \Sigma, ([X]^\perp)^+ \rrparenthesis}{\llparenthesis \Pi, \Sigma \rrparenthesis}\,\textsc{Cut}$ translates into $\dfrac{\llparenthesis \Gamma, A^- \rrparenthesis \qquad \llparenthesis \Delta, A^+ \rrparenthesis}{\llparenthesis \Gamma, \Delta \rrparenthesis}\,\textsc{Cut}$, formally the result is not an elementary rule of $\mathbf{NCLL}_2$. However, still it is an induced rule of $\mathbf{NCLL}_2$.

[19] In a concrete derivation, for every axiom $\dfrac{}{\llparenthesis X^+, ([X]^\perp)^+ \rrparenthesis}\,\textsc{Ax}$ the formulas X and $[X]^\perp$ are explicitly given as elements of $\mathfrak{L}_1$ instead of being kept variable.

to $\pi^+\psi^\pm X^- = \pi^+\tau\psi X = (\pi\tau\psi X)^+ \overset{\text{Lemma 2.1.3(4)}}{=} (\pi\psi[X]^\perp)^+ = ([X]^\perp)^+$, or, without using Lemma 2.1.3(4), $\pi^+\psi^\pm X^- = \pi^+\tau\psi X = (\pi\tau\psi X)^+ = ([\pi\psi X]^\perp)^+ = ([X]^\perp)^+$. However, Π only contains positively polarized $\mathfrak{L}_1$-formulas.

////

Lemma 4.7.1(b) completely answers the derivability question for $\mathbf{NCLL}_1$ in terms of $\mathbf{NCLL}_2$-derivability, for which we have established a contraction criterion in Section 4.4. We will now sketch a completely analogue contraction criterion for $\mathbf{NCLL}_1$.

First, an $\mathbf{NCLL}_1$-*proof net* is an $\mathfrak{L}_1$-proof structure (see Subsection 3.2.1) that can be obtained as the (one-sided) proof structure $\mathcal{P}_1(\mathcal{D})$ of an $\mathbf{NCLL}_1$-derivation $\mathcal{D}$ (cf. Definition 4.3.2), which is defined in the obvious way: Ax translates to an axiomatic edge; for the tensor (par) rule, make X^+ and Y^+ the active ends of a new tensor (par) link, which yields a new open end $(X \otimes Y)^+$ ($(Y \parr X)^+$); for Cut, identify the two edges (not necessarily yielding a cut edge). Observe that $\mathcal{P}_1(\mathcal{D})$ equals the π-image (see Subsection 3.2.5) of the two-sided proof net $\mathcal{P}(\psi^\pm\mathcal{D})$ of a corresponding two-sided derivation $\psi^\pm\mathcal{D}$:

$$\pi\mathcal{P}(\psi^\pm\mathcal{D}) = \mathcal{P}_1(\mathcal{D}).$$

More general, given a two-sided derivation $\mathcal{D}_2$, then

$$\pi\mathcal{P}(\mathcal{D}_2) = \mathcal{P}_1(\pi^+\mathcal{D}_2),$$

whence

$$\pi\mathcal{P}(\psi^\pm\mathcal{D}) = \mathcal{P}_1(\pi^+\psi^\pm\mathcal{D}) = \mathcal{P}_1(\mathcal{D})$$

where the last equality is a result of the fact that $\pi^+\psi^\pm\mathcal{D}$ and $\mathcal{D}$ are equal up to the order of the premiss sequents of some Cut rules.

We define a conversion relation on the collection $\mathfrak{LG}_1$ of $\mathfrak{L}_1$-link graphs with well-labeled (see Definition 3.2.1) connector links, viz. tensor links $(e_0, e_1, e_2)_\otimes$ (indicated by $\otimes$) and par links $(e_0, e_1, e_2)_\parr$ (indicated by $\parr$), and with context links $[\![e_0, \ldots, e_{m-1}]\!]_\odot$ (indicated by $\odot$), whose open ends are positively polarized. Up to the labeling, we take the conversion steps exactly the same as in Section 4.4. The translations $\psi : \mathfrak{PS}_1 \to \mathfrak{PS}_2 : \mathcal{P} \mapsto \psi\mathcal{P}$ and $\pi : \mathfrak{PS}_2 \to \mathfrak{PS}_1 : \mathcal{P} \mapsto \pi\mathcal{P}$ of Subsection 3.2.5 generalize to maps $\mathfrak{LG}_1 \to \mathfrak{LG}_2$ respectively $\mathfrak{LG}_2 \to \mathfrak{LG}_1$ in a straightforward way, which also extend $\psi^\pm$ and π^+ on the respective collections of sequents.

$$\begin{tikzcd}
\mathfrak{PS}_1 \arrow[r, tail, "\psi"] \arrow[d, hook] & \mathfrak{PS}_2 \arrow[r, two heads, "\pi"] \arrow[d, hook] & \mathfrak{PS}_1 \arrow[d, hook] \\
\mathfrak{LG}_1 \arrow[r, tail, "\psi"] & \mathfrak{LG}_2 \arrow[r, two heads, "\pi"] & \mathfrak{LG}_1 \\
\mathfrak{CLists}_1 \arrow[r, tail, "\psi^\pm"] \arrow[u, hook] & \mathfrak{CLists}_2 \arrow[r, two heads, "\pi^+"] \arrow[u, hook] & \mathfrak{CLists}_1 \arrow[u, hook]
\end{tikzcd}$$

We establish the one-sided counterpart of Theorem 4.4.4.

Theorem 4.7.2. *(a) Let $\mathcal{D}$ be an $\mathbf{NCLL}_1$-derivation. Then $\mathcal{P}_1(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$ (or $\mathcal{P}(\mathcal{D}) \twoheadrightarrow (\llcorner\mathcal{D}\lrcorner)'$).*

(b) Let $\mathcal{P} \twoheadrightarrow \Pi$ ($\mathcal{P} \twoheadrightarrow \Pi'$) be a conversion sequence from an $\mathfrak{L}_1$-proof structure to a sequent or a corresponding one-edge link graph. Then there is a derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}_1(\mathcal{D})$ and $\Pi = \llcorner \mathcal{D} \lrcorner$.

◊

Proof: **(a)** Directly, by induction on the derivation $\mathcal{D}$.

(b) Given a conversion sequence

$$\mathcal{P} = \mathcal{P}_m \overset{\delta_m}{\to} \mathcal{P}_{m-1} \overset{\delta_{m-1}}{\to} \ldots \overset{\delta_2}{\to} \mathcal{P}_1 \overset{\delta_1}{\to} \mathcal{P}_0 = \Pi \ (\Pi')$$

on $\mathfrak{LG}_1$, we embed it into $\mathfrak{LG}_2$: for every step $\mathcal{P}_i \to \mathcal{P}_{i-1}$ on $\mathfrak{LG}_1$ there is a corresponding step $\psi\mathcal{P}_i \to \psi\mathcal{P}_{i-1}$ on $\mathfrak{LG}_2$. This yields a conversion sequence $\psi\mathcal{P} \twoheadrightarrow \psi\Pi = \psi^{\pm}\Pi$ ($\psi\mathcal{P} \twoheadrightarrow \psi\Pi'$) whence, by Theorem 4.4.4, there is a two-sided derivation $\mathcal{D}_2$ with $\psi\mathcal{P} = \mathcal{P}(\mathcal{D}_2)$ and $\psi^{\pm}\Pi = \llcorner \mathcal{D}_2 \lrcorner$. As a consequence, $\mathcal{P} = \pi\psi\mathcal{P} = \pi\mathcal{P}(\mathcal{D}_2) = \mathcal{P}_1(\pi^+\mathcal{D}_2)$ and $\Pi = \pi^+\psi^{\pm}\Pi = \pi^+\llcorner \mathcal{D}_2 \lrcorner = \llcorner \pi^+\mathcal{D}_2 \lrcorner$. So $\pi^+\mathcal{D}_2$ is a one-sided derivation possessing the desired properties. ////

A direct proof without using two-sided link graphs would have forced us anyhow to generalize in one way or the other to one-sided proof structures *with hypotheses*. For example, if the last conversion is a [**tens**]-step $\mathcal{P}' \to \Pi$, then $\mathcal{P}'$ only contains a single tensor link

$$\Big(\big(X \otimes Y\big)^-, Y^+, X^+\Big)_{\otimes}$$

whence $\mathcal{P}$ consists of three 'proof nets' $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ attached to one another in a tensor link, each of which converts to one edge. For the proof structure $\mathcal{P}_0$ to be well-defined we have to alter the open end $\big(X \otimes Y\big)^-$ into $\big([X \otimes Y]^\perp\big)^+ = \big([Y]^\perp \parr [X]^\perp\big)^+$ and apply the induction hypothesis to this proof net, which formally is not a sub net of $\mathcal{P}$ anymore.

4.8. The category of proof nets

Proof structures for multiplicative linear logic are usually defined as the smallest set containing axiom-links[20] $\overline{X^\perp \qquad X}$ and closed under disjoint union and under the lower attachment of the links

$$\frac{X \qquad Y}{X \otimes Y} \ , \ \frac{X \qquad Y}{X \parr Y} \ \text{ and } \ \underline{X^\perp \qquad X}$$

(cf. [**Girard 87**]). The proof *nets* then correspond to one-sided sequent calculus derivations. In order to make a category out of $\mathfrak{L}_1$-formulas and proof nets, one takes as morphisms $X \to Y$ precisely the proof nets proving $\big([X]^\perp, Y\big)$. The identity morphism on Y is given by $\overline{Y^\perp \qquad Y}$, while composition of a proof net proving $\big([X]^\perp, Y\big)$

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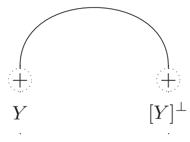
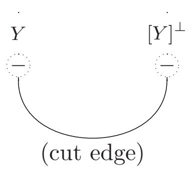
[20] In our definition of $\mathfrak{L}_1$-proof structure (see Subsection 3.2.1), we do not consider axiom-links nor cut-links, but only axiomatic edges and cut edges:

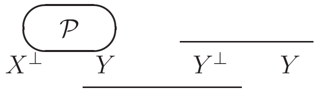


(axiomatic edge)



(cut edge)

and a proof net proving $\left([[Y]^\perp, Z]\right)$ into a proof net proving $\left([[X]^\perp, Z]\right)$ is given by the attachment of the link $\underline{\;Y^\perp \quad\quad Y\;}$ to their union. However, composing a proof net $\mathcal{P} : X \to Y$ with the identity morphism $Y \to Y$ yields



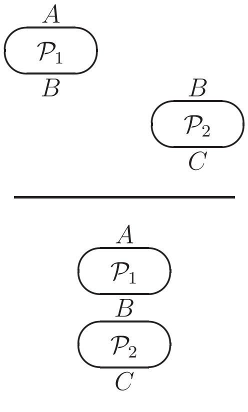
which differs from $\mathcal{P}$. So some identification has to be made in order to achieve a category. Moreover, observe that the hom-set $\mathrm{Hom}(X, Y)$ is exactly the same as $\mathrm{Hom}([Y]^\perp, [X]^\perp)$. This means that we should somehow indicate whether a proof net $\mathcal{P}$ with conclusions $[X]^\perp$ and $Y = [[Y]^\perp]^\perp$ is a morphism $X \to Y$ or $[Y]^\perp \to [X]^\perp$.

We will now show how our two-sided proof nets provide us with an elegant way to make a category out of formulas and proof nets. We take as collection of objects the $\mathfrak{L}_2$-formulas, and as morphisms $A \to B$ the $\mathbf{NCLL}_2$-proof nets of $A \vdash B$. The identity morphism on B is just the single formula B (i.e. the one-edge link graph), while composition is given by identifying the appropriate leaves:

- **(identity)** The identity arrow on B is defined to be the one-edge proof structure with end labels B^- and B^+:

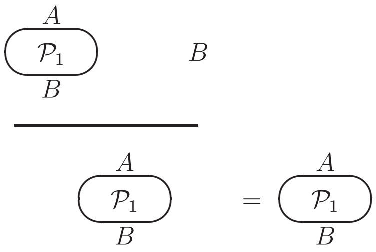
$$B$$

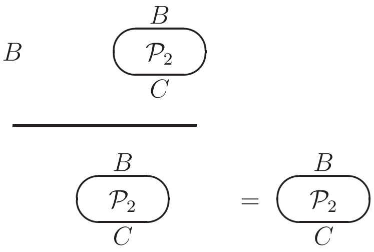
- **(composition)** Given $\mathcal{P}_1$ with open ends A^- and $\hat{\eta}_1 = B^+$ and $\mathcal{P}_2$ with open ends $\check{\eta}_2 = B^-$ and C^+, the composition $\mathcal{P}_2 \circ \mathcal{P}_1$ is defined to be the union of both, in which the edges η_1 and η_2 are identified:



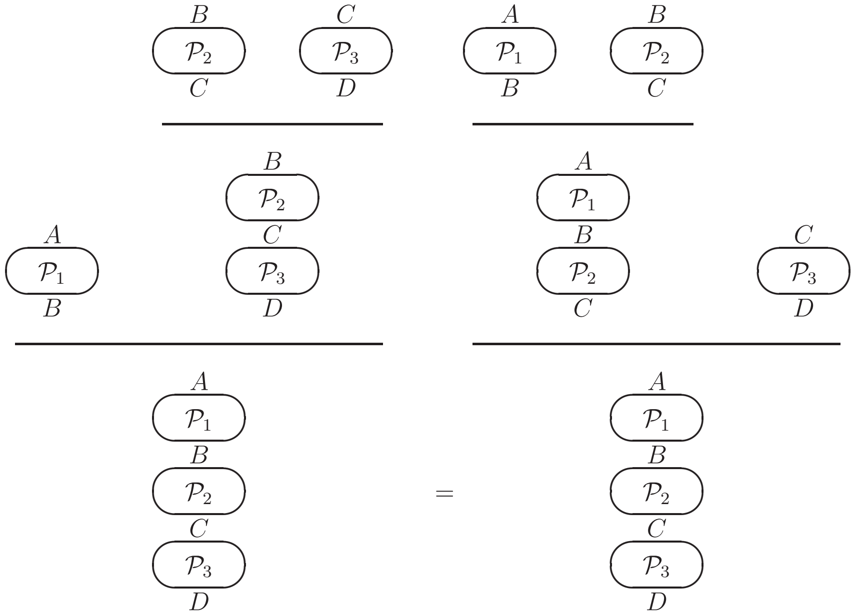
Let us verify the category axioms. Suppose we are given $A \xrightarrow{\mathcal{P}_1} B \xrightarrow{\mathcal{P}_2} C \xrightarrow{\mathcal{P}_3} D$.

- **(unit axiom)** $\mathrm{id}_B \circ \mathcal{P}_1 = \mathcal{P}_1$ and $\mathcal{P}_2 \circ \mathrm{id}_B = \mathcal{P}_2$.

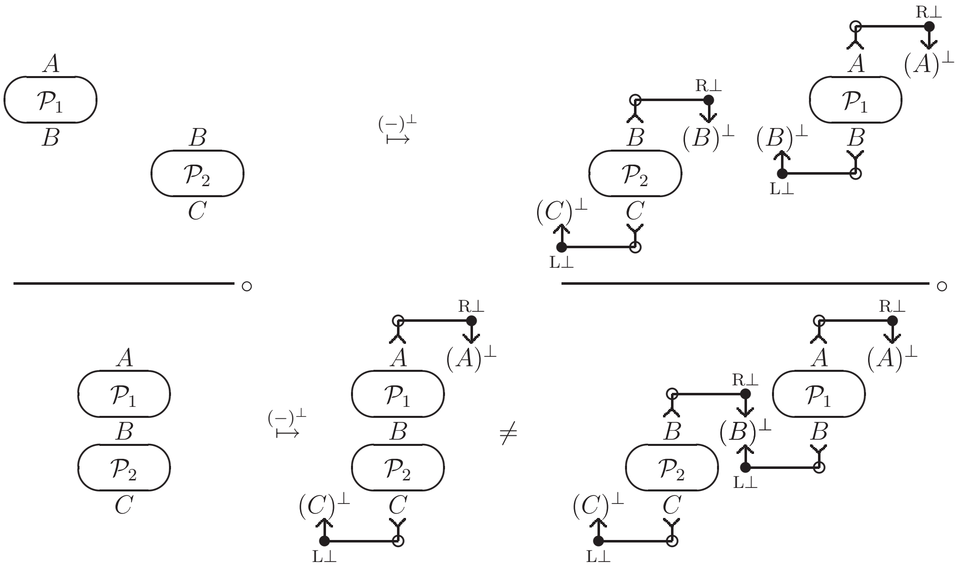




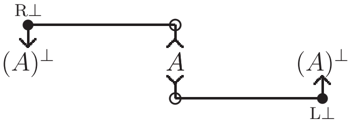
- **(associativity axiom)** $(\mathcal{P}_3 \circ \mathcal{P}_2) \circ \mathcal{P}_1 = \mathcal{P}_3 \circ (\mathcal{P}_2 \circ \mathcal{P}_1)$.



The function $(-)^{\perp} : \mathfrak{L}_2 \to \mathfrak{L}_2$ can be extended to a contravariant map on morphisms, mapping an arrow $A \xrightarrow{\mathcal{P}} B$ into an arrow $(B)^{\perp} \xrightarrow{(\mathcal{P})^{\perp}} (A)^{\perp}$ (by means of attaching a L⊥- and R⊥-link to $\mathcal{P}$). However, this map is not functorial, as $\mathcal{P}_2 \circ \mathcal{P}_1$ does not translate into $(\mathcal{P}_1)^{\perp} \circ (\mathcal{P}_2)^{\perp}$.

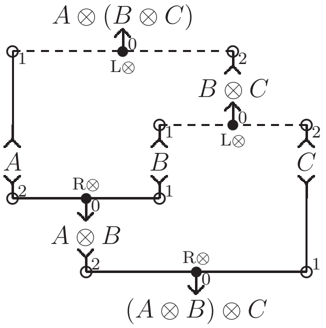


Also, the identity on A does not translate into the identity on $(A)^{\perp}$, but into



Similar remarks can be made for the binary functions $\otimes$, $\parr$, $\multimap$ and $\multimapinv : \mathfrak{L}_2 \times \mathfrak{L}_2 \to \mathfrak{L}_2$.

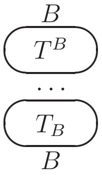
Observe that the only isomorphisms[21] are the identity arrows; indeed the number of links in $\mathcal{P}_2 \circ \mathcal{P}_1$ is the sum of the number of links in $\mathcal{P}_2$ and $\mathcal{P}_1$, which can only be zero if both $\mathcal{P}_1$ and $\mathcal{P}_2$ are identity arrows. So the following proof net does not give us an associativity *isomorphism*, which would be needed in order to provide our category with a monoidal structure.



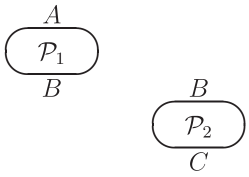
All these problems can be solved at once by going over to η-expanded cut-free proof nets. We define the category $\mathbb{NCLL}$ as follows. The objects are the $\mathfrak{L}_2$-formulas, and the morphisms $A \to B$ are the η-expanded cut-free $\mathbf{NCLL}_2$-proof nets of $A \vdash B$. The identity morphism on B is the identity proof net $\mathcal{I}(B)$ (see Subsection 3.2.3), while composition is given by identifying the appropriate leaves and applying cut elimination:

- **(identity)** The identity arrow on B is defined to be the identity proof net with end labels B^- and B^+:

$\mathcal{I}(B) =$



- **(composition)** Given $\mathcal{P}_1$ with open ends A^- and $\hat{\eta}_1 = B^+$ and $\mathcal{P}_2$ with open ends $\check{\eta}_2 = B^-$ and C^+, the composition $\mathcal{P}_2 \circ \mathcal{P}_1$ is defined to be the (unique) normal form of the union of both, in which the edges η_1 and η_2 are identified:



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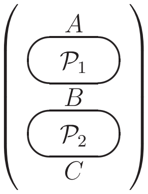
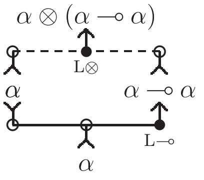
[21] An isomorphism is an arrow $A \xrightarrow{\mathcal{P}} B$ for which there is a left and right inverse $B \xrightarrow{\mathcal{P}^{-1}} A$.

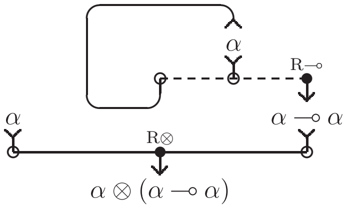
normal form of

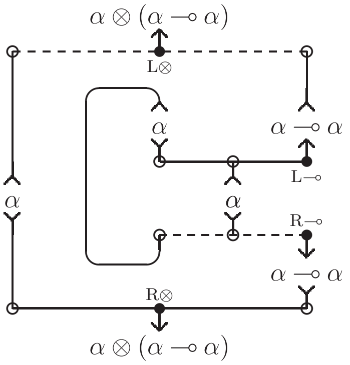


By the theory of Subsection 3.2.4 (for arbitrary η-expanded cut-free proof *structures*), we observe that the thus defined composition coincides with the composition on the corresponding axiom linkings.

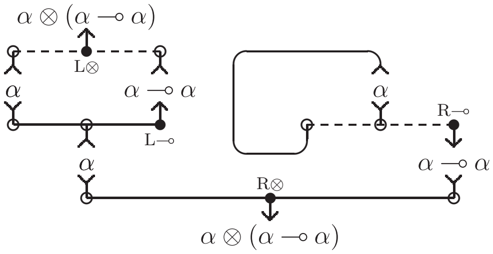
Example 4.8.1. Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{I} = \mathcal{I}(\alpha \otimes (\alpha \multimap \alpha))$ be the following proof nets.







Evidently $\mathcal{P}_1 \circ \mathcal{I} = \mathcal{P}_1$ and $\mathcal{I} \circ \mathcal{P}_2 = \mathcal{P}_2$. Observe that $\mathcal{P}_1 \circ \mathcal{P}_2$ must be $\mathcal{I}(\alpha) = \alpha$; the other composition is



♢

Lemma 4.8.2. *With these definitions of identity and composition,* $\mathbb{NCLL}$ *constitutes a category.* ♢

Proof: This follows from the confluence of cut elimination. Alternatively, observe that 'making connections between wires' (in the setting of axiom linkings) satisfies the unit axiom as well as associativity. ////

Theorem 4.8.3. *A proof net* $\mathcal{P}$ *is dualizable if and only if* $\mathcal{P}$ *is invertible in the category* $\mathbb{NCLL}$. ♢

Proof: $\boxed{\Longrightarrow}$ Suppose $\mathcal{P}$ is dualizable. By Lemma 4.6.2 we know that $P(A) = P(B)$ and $N(A) = N(B)$. Hence Proposition 3.2.13 applies, showing that $\mathcal{P} \circ \mathcal{P}^*$ and $\mathcal{P}^* \circ \mathcal{P}$ are the identities.

$\boxed{\Longleftarrow}$ Assume $\mathcal{P}$ of $A \vdash B$ has an inverse $\mathcal{P}'$, then the corresponding axiom linking cannot have 'wires' of the form $\underline{\mathrm{P}}\mathrm{N}$ (since $\mathcal{P}' \circ \mathcal{P} = i_A$ has not) or $\overline{\mathrm{N}}\mathrm{P}$ (since $\mathcal{P} \circ \mathcal{P}' = i_B$ has not). Hence all 'wires' of $\mathcal{P}$ and $\mathcal{P}'$ are of the form $\underline{\mathrm{P}}\mathrm{P}$ and $\overline{\mathrm{N}}\mathrm{N}$. Now a compound 'wire' like $\underline{\mathrm{P}}\mathrm{P}\underline{\mathrm{P}}'\mathrm{P}'$ is the identity, finishing the proof that $\mathcal{P}$ and $\mathcal{P}^*$ are dual to each other. ////

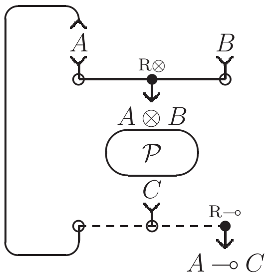
The 'adjunctions' in Lemma 4.1.10

$$\begin{array}{lcll} A \otimes (-) & \dashv & A \multimap (-) & \text{(for all formulas } A\text{)} \\ (-) \otimes A & \dashv & (-) \multimapinv A & \text{(for all formulas } A\text{)} \end{array}$$

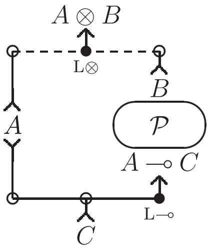
are easily shown to be real adjunctions in the categorical sense.

$$\begin{array}{c} A \otimes B' \vdash A \otimes B \vdash C \vdash C' \\ \hline\hline B' \vdash B \vdash A \multimap C \vdash A \multimap C' \end{array}$$

The bijective map is defined as follows: given a proof net $\mathcal{P}$ proving $A \otimes B \vdash C$, we define its image $\Psi\mathcal{P}$ as the normal form of



The other way around, given a proof net $\mathcal{P}$ proving $B \vdash A \multimap C$, we define its image $\Phi\mathcal{P}$ as the normal form of



These maps Ψ and Φ are indeed inverses of each other, and moreover

$$\Psi\big(\mathcal{P}_2 \circ \mathcal{P} \circ (A \otimes \mathcal{P}_1)\big) = (A \multimap \mathcal{P}_2) \circ \Psi(\mathcal{P}) \circ \mathcal{P}_1$$

4.9. Intuitionistic fragment

In this section we will study the intuitionistic fragment of $\mathbf{NCLL}_2$, which by definition is the sequent calculus whose rules are the same as those of $\mathbf{NCLL}_2$, except that we will take only the identity rules AX_A and CUT_A where $A \in \mathfrak{L}_{2,\mathrm{i}}$, and the logical rules $\mathrm{L}\square$ and $\mathrm{R}\square$ for connectives $\square \in \{\otimes, \multimap, \multimapinv\}$ (see Section 4.2). Derivations turn out to be of a special form, which shows that this fragment is the same as Lambek calculus **L** ([**Lambek 58**]). After having seen the theory of proof nets for **L** (cf. [**Roorda 91**]), we will establish the analogue of Theorem 4.6.3 for this calculus.

We also refer to Remark 6.3.12, where **L** is approached 'from below': as a variant of multimodal Lambek Calculus.

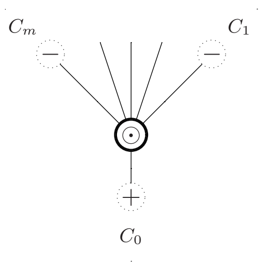
4.9.1. Lambek calculus. By Theorem 4.2.2 we know that for any sequent $(\![\Gamma]\!)$ in $\mathfrak{L}_{2,\mathrm{i}}$, $(\![\Gamma]\!)$ is derivable in $\mathbf{NCLL_2}$ if and only if it is derivable in the intuitionistic fragment of $\mathbf{NCLL_2}$, where the only-if part is a consequence of the subformula property for $\mathbf{NCLL_2}$. Now the resulting derivable sequents of this fragment are easily shown to satisfy the additional property of having only one conclusion: indeed, if each of the 0,1,2 premiss sequents of a rule (different from L$\perp$, R$\perp$, L$\parr$, R$\parr$) has exactly one conclusion, so has the conclusion sequent. Derivations hence only contain such one-conclusion sequents, which observation leads to the so-called Lambek calculus **L**.

Definition 4.9.1. *An **L**-sequent $\mathcal{P}$ is an $\boldsymbol{NCLL_2}$-sequent satisfying:*

- *$\mathcal{P}$ is actually an $\mathfrak{L}_{2,i}$-link graph;*
- *$\mathcal{P}$ has exactly one conclusion C^+.*

◊

As **L**-sequents are separable, we can denote a sequent $(\![C_0^+, C_1^-, \ldots, C_m^-]\!)$ by $C_m, \ldots, C_1 \vdash C_0$.



As rules for **L** we take those instances of the inference rules of $\mathbf{NCLL_2}$ in which the premiss sequents and the conclusion sequent are **L**-sequents. We have just seen that $\mathbf{NCLL_2}$ is a conservative extension of **L**: if an **L**-sequent $(\![\Gamma]\!)$ (considered as an $\mathbf{NCLL_2}$-sequent) is derivable in $\mathbf{NCLL_2}$, then it is derivable in the intuitionistic fragment of $\mathbf{NCLL_2}$, whence in **L** already.

L

$$\dfrac{}{(\![A^+, A^-]\!)}\ \text{Ax}$$

$$\dfrac{(\![\Gamma^-, A^+]\!) \qquad \big(\big[\Delta_1^-, C^+, \Delta_2^-, A^-\big]\big)}{\big(\big[\Gamma^-, \Delta_1^-, C^+, \Delta_2^-\big]\big)}\ \textsc{Cut}$$

$$\dfrac{\big(\big[\Gamma_1^-, C^+, \Gamma_2^-, B^-, A^-\big]\big)}{\big(\big[\Gamma_1^-, C^+, \Gamma_2^-, (A \otimes B)^-\big]\big)}\ \text{L}\otimes \qquad \dfrac{(\![\Gamma^-, A^+]\!) \qquad (\![\Delta^-, B^+]\!)}{(\![\Gamma^-, (A \otimes B)^+, \Delta^-]\!)}\ \text{R}\otimes$$

$$\dfrac{(\![\Gamma^-, A^+]\!) \qquad \big(\big[\Delta_1^-, C^+, \Delta_2^-, B^-\big]\big)}{\big(\big[\Gamma^-, (B \mathrel{\circ\!\!-} A)^-, \Delta_1^-, C^+, \Delta_2^-\big]\big)}\ \text{L}{\circ\!\!-} \qquad \dfrac{(\![\Gamma^-, B^+, A^-]\!)}{(\![\Gamma^-, (B \mathrel{\circ\!\!-} A)^+]\!)}\ \text{R}{\circ\!\!-}$$

$$\dfrac{\big(\big[\Gamma_1^-, C^+, \Gamma_2^-, A^-\big]\big) \qquad (\![\Delta^-, B^+]\!)}{\big(\big[\Gamma_1^-, C^+, \Gamma_2^-, (B \multimap A)^-, \Delta^-\big]\big)}\ \text{L}{\multimap} \qquad \dfrac{(\![\Gamma^-, B^-, A^+]\!)}{(\![\Gamma^-, (B \multimap A)^+]\!)}\ \text{R}{\multimap}$$

In the alternative notation, denoting the sequent $\big(\big[C_0^+, C_1^-, \dots, C_m^-\big]\big)$ by $C_m, \dots, C_1 \vdash C_0$, **L** is defined by the following rules.

L

$$\dfrac{}{A \vdash A}\ \text{Ax}$$

$$\dfrac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash C}{\Delta_1, \Gamma, \Delta_2 \vdash C}\ \textsc{Cut}$$

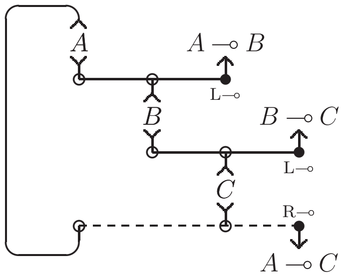
$$\dfrac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\ \text{L}\otimes \qquad \dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \text{R}\otimes$$

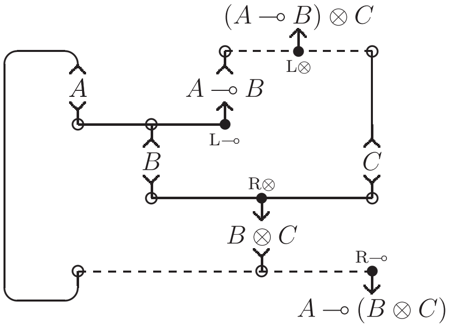
$$\dfrac{\Gamma \vdash A \qquad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, B \mathrel{\circ\!\!-} A, \Gamma, \Delta_2 \vdash C}\ \text{L}{\circ\!\!-} \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash B \mathrel{\circ\!\!-} A}\ \text{R}{\circ\!\!-}$$

$$\dfrac{\Gamma_1, A, \Gamma_2 \vdash C \qquad \Delta \vdash B}{\Gamma_1, \Delta, B \multimap A, \Gamma_2 \vdash C}\ \text{L}{\multimap} \qquad \dfrac{B, \Gamma \vdash A}{\Gamma \vdash B \multimap A}\ \text{R}{\multimap}$$

4.9.2. Proof nets and contraction criterion. Analogue to Definition 4.3.2, we define an **L**-*proof net* to be an $\mathfrak{L}_2$-proof structure that can be obtained as the (two-sided) proof structure $\mathcal{P}(\mathcal{D})$ of an **L**-derivation $\mathcal{D}$ (the latter considered as an $\mathbf{NCLL}_2$-derivation). Obviously, an **L**-proof net is actually an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure (see Definition 3.2.6).

Example 4.9.2. Some examples of **L**-proof nets are given by





◊

The conversion steps of Section 4.4 are well-defined on the restriction to the intuitionistic labeled elements of $\mathfrak{L}\mathfrak{G}_2$. The next two lemmas are a direct consequence of Theorem 4.4.4(b).

Lemma 4.9.3. *Let $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) be a conversion sequence from an $\mathfrak{L}_{2,i}$-proof structure to an $\boldsymbol{NCLL}_2$-sequent or a corresponding one-edge link graph. Then there is an $\boldsymbol{L}$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$, while Γ is an $\boldsymbol{L}$-sequent.* ◊

Proof: If $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) is a conversion sequence from an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure to an $\mathbf{NCLL}_2$-sequent or a corresponding one-edge link graph, then by Theorem 4.4.4(b) there is an $\mathbf{NCLL}_2$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$. Now all formulas occurring in $\mathcal{D}$ are contained in $\mathcal{P}$ which is an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure, whence $\mathcal{D}$ belongs to the intuitionistic fragment of $\mathbf{NCLL}_2$, and hence to **L**. Of course, $\Gamma = \llcorner\mathcal{D}\lrcorner$ is actually an **L**-sequent. ////

Lemma 4.9.4. *Let $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) be a conversion sequence from an $\mathfrak{L}_2$-proof structure to an $\boldsymbol{L}$-sequent or a corresponding one-edge link graph. Then there is an $\boldsymbol{L}$-derivation $\mathcal{D}$ with $\Gamma = \llcorner\mathcal{D}\lrcorner$.* ◊

Proof: If $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) is a conversion sequence from an $\mathfrak{L}_2$-proof structure to an **L**-sequent or a corresponding one-edge link graph, then by Theorem 4.4.4(b) there is an $\mathbf{NCLL}_2$-derivation $\mathcal{D}'$ with $\mathcal{P} = \mathcal{P}(\mathcal{D}')$ and $\Gamma = \llcorner\mathcal{D}'\lrcorner$. Now, by conservativity, Γ is also derivable in $\mathbf{L}$, say by $\mathcal{D}$. (Observe it need not hold that $\mathcal{P} = \mathcal{P}(\mathcal{D})$.) ////

As a corollary we find the intuitionistic counterpart of Theorem 4.4.4:

THEOREM 4.9.5. *(a) Let $\mathcal{D}$ be an $\boldsymbol{L}$-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$ (or $\mathcal{P}(\mathcal{D}) \twoheadrightarrow (\llcorner\mathcal{D}\lrcorner)'$).*

(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ ($\mathcal{P} \twoheadrightarrow \Gamma'$) be a conversion sequence from an $\mathfrak{L}_{2,i}$-proof structure to an $\boldsymbol{L}$-sequent or a corresponding one-edge link graph. Then there is an $\boldsymbol{L}$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.

◊

4.9.3. Dualizable L-proof nets. Let us define the following relations $\dashv\vdash_{\mathrm{i}}$ and $\dashv_{\mathrm{d}}\vdash_{\mathrm{i}}$ on $\mathfrak{L}_{2,\mathrm{i}}$:

$$\begin{aligned}
A \dashv\vdash_{\mathrm{i}} B \quad &:\Longleftrightarrow \quad A \vdash B \text{ is } \mathbf{L}\text{-derivable} \quad \text{and} \quad B \vdash A \text{ is } \mathbf{L}\text{-derivable}\\
&\Longleftrightarrow \quad \text{there is a cut-free and } \eta\text{-expanded } \mathbf{L}\text{-proof net } \mathcal{P}_1 \text{ of } A \vdash B\\
&\qquad\quad \text{and a cut-free and } \eta\text{-expanded } \mathbf{L}\text{-proof net } \mathcal{P}_2 \text{ of } B \vdash A\\
A \dashv_{\mathrm{d}}\vdash_{\mathrm{i}} B \quad &:\Longleftrightarrow \quad \text{there is an } \mathbf{L}\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that}\\
&\qquad\quad \text{its dualization } \mathcal{P}^* \text{ is an } \mathbf{L}\text{-proof net of } B \vdash A\\
&\Longleftrightarrow \quad \text{there is a cut-free and } \eta\text{-expanded } \mathbf{L}\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that}\\
&\qquad\quad \text{its dualization } \mathcal{P}^* \text{ is an } \mathbf{L}\text{-proof net of } B \vdash A
\end{aligned}$$

As cut-free and η-expanded $\mathbf{NCLL}_2$-proof nets with $\mathfrak{L}_{2,\mathrm{i}}$-labeled open ends are automatically $\mathbf{L}$-proof nets, we see that $\dashv\vdash_{\mathrm{i}}$ and $\dashv_{\mathrm{d}}\vdash_{\mathrm{i}}$ are just the restrictions of $\dashv\vdash$ and $\dashv_{\mathrm{d}}\vdash$ (defined in Section 4.6). Also, by Proposition 2.3.8, $\simeq_{\mathrm{i}}$ is just the restriction of $\simeq$ to $\mathfrak{L}_{2,\mathrm{i}}$. Hence Theorem 4.6.3 instantaneously leads to the following theorem, of which we will also give a direct proof using Lemma 2.3.10. This direct proof alternatively proves the complicated Proposition 2.3.8.

THEOREM 4.9.6. *For all $\mathfrak{L}_{2,i}$-formulas A and B the following holds:*

$$A \simeq_{\mathrm{i}} B \qquad \textit{if and only if} \qquad A \dashv_{\mathrm{d}}\vdash_{\mathrm{i}} B$$

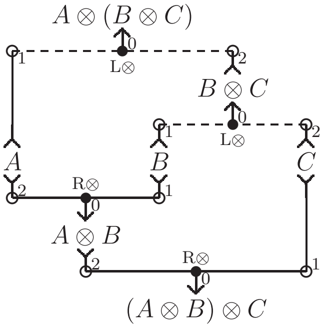
◊

$$\begin{tikzcd}
A \simeq_{\mathrm{i}} B \arrow[rr, Leftrightarrow, dashed, "\text{Theorem 4.9.6}"] \arrow[d, Leftrightarrow, dashed, "\text{Proposition 2.3.8}"'] & & A \dashv_{\mathrm{d}}\vdash_{\mathrm{i}} B \arrow[d, Leftrightarrow] \\
A \simeq B \arrow[rr, Leftrightarrow, "\text{special case of}", "\text{Theorem 4.6.3}"'] & & A \dashv_{\mathrm{d}}\vdash B
\end{tikzcd}$$

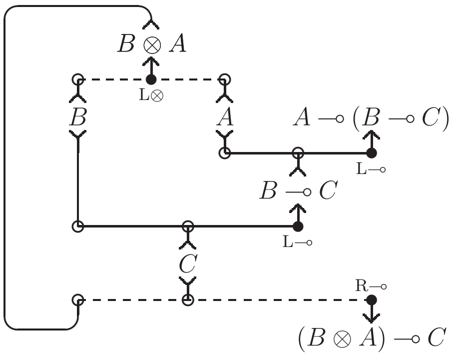
PROOF: The proof will be independent of Proposition 2.3.8.

$\boxed{\Longrightarrow}$ The relation $\dashv_{\mathrm{d}}\vdash_{\mathrm{i}} := \{\, (A,B) \in \mathfrak{L}_{2,\mathrm{i}} \times \mathfrak{L}_{2,\mathrm{i}} \mid A \dashv_{\mathrm{d}}\vdash_{\mathrm{i}} B\}$ is an equivalence relation satisfying (where the numbers refer to Definition 2.3.6):

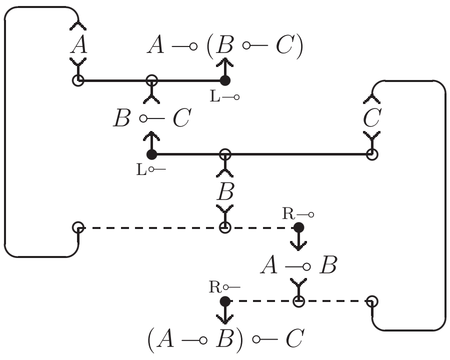
- (0$\Box$), by pasting the dual links L$\Box$ and R$\Box$ to the given dualizable proof nets ($\Box = \otimes, \multimapinv$ or $\multimap$);
- (5$\otimes$), by the dualizable proof net



- (5$\multimap$), by the dualizable proof net

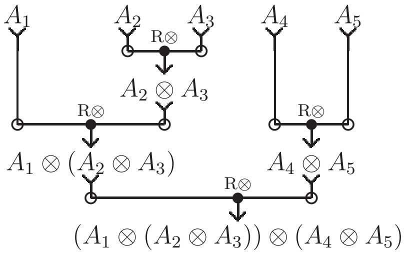


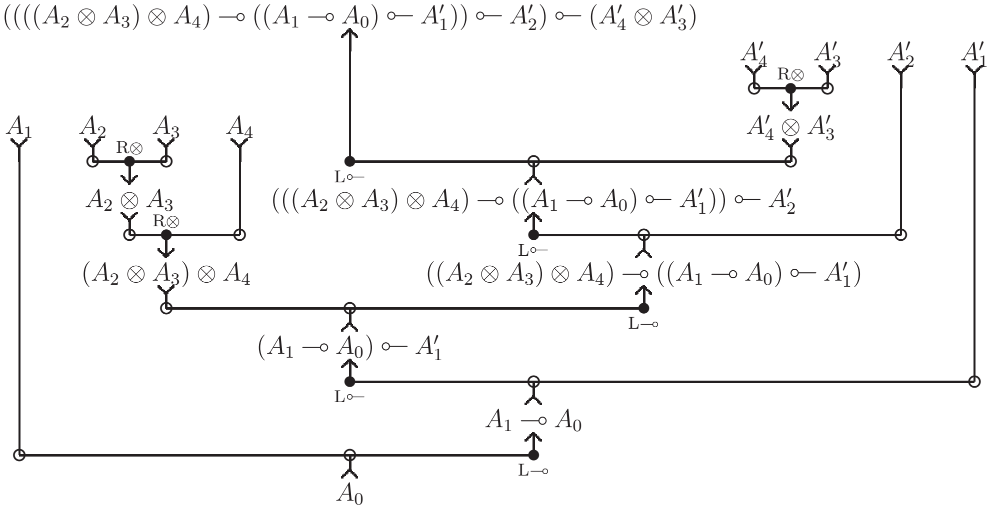
- (5$\multimapinv$); similarly;
- (5$\multimap\multimapinv$), by the dualizable proof net



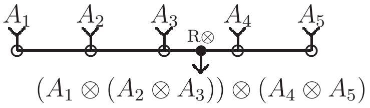
As $\simeq_{\mathrm{i}}$ is the smallest such equivalence relation, we must have that $\simeq_{\mathrm{i}} \ \subseteq \ \dashv_{\mathrm{d}}\vdash_{\mathrm{i}}$, i.e. if $A \simeq_{\mathrm{i}} B$ then $A \dashv_{\mathrm{d}}\vdash_{\mathrm{i}} B$.

$\boxed{\Longleftarrow}$ This proof is similar to that of Theorem 4.6.3, though somewhat more complicated as we do not restrict to $\mathfrak{L}_{2,\mathrm{nf}}$. Let $\mathcal{P}$ be a cut-free and η-expanded dualizable proof net of $A \ \vdash \ B$. Then we know (Proposition 3.2.9) that $\mathcal{P}$ is the union of T^A and T_B containing only $\otimes$-, $\multimapinv$- and $\multimap$-links, followed by an identification of the atomic formulas, which is pairwise by Lemma 4.6.2. The tensor clusters are now of the form:

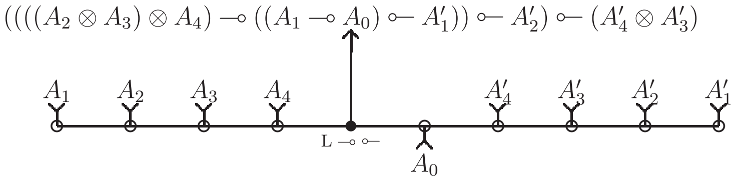




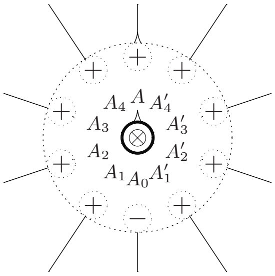
which we can abbreviate by



respectively



or



The par clusters are given by their dualizations.

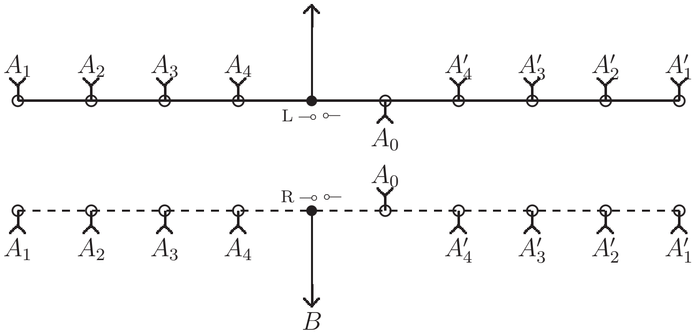
If there are no clusters, we get $A = \alpha = B$. Otherwise, there is a cluster with only atomic active formulas, which moreover we may suppose to be a tensor cluster. This cluster hence is a generalized R$\otimes$-link or a generalized L$\multimap\!\multimapinv$-link. It faces exactly one par cluster (which is hence a generalized L$\otimes$-link respectively a generalized R$\multimap\!\multimapinv$-link), and the result follows by induction by means of Lemma 2.3.3 (the proof of which only refers to (0$\otimes$) and (5$\otimes$) for the $\otimes$-case) respectively Lemma 2.3.10. E.g., let A be given by

$$\begin{aligned}&((((A_2 \otimes A_3) \otimes A_4) \multimap ((A_1 \multimap A_0) \multimapinv A'_1)) \multimapinv A'_2) \multimapinv (A'_4 \otimes A'_3)\\ &= (A'_4 \otimes A'_3) \xrightarrow{-} A'_2 \xrightarrow{-} ((A_2 \otimes A_3) \otimes A_4) \xrightarrow{+} A'_1 \xrightarrow{-} A_1 \xrightarrow{+} A_0\end{aligned}$$

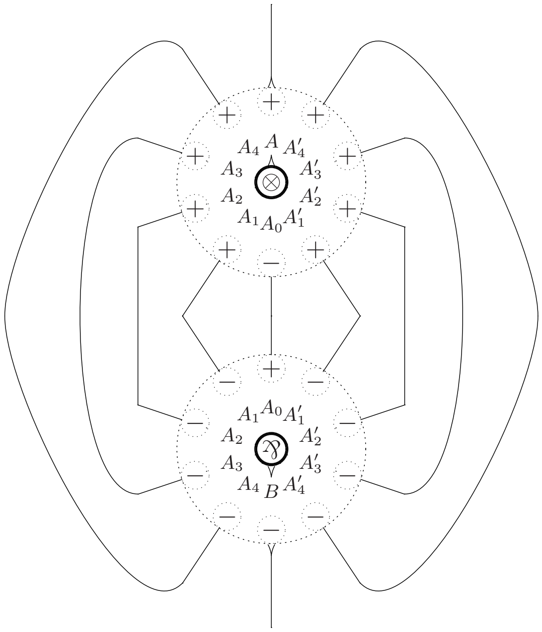
(where the second expression is supposed to be rightmost bracketed) corresponding to the generalized L$\multimap\!\multimapinv$-link above, where all the A_i and A'_i are atoms now. For Lemma 2.3.10 we have to consider the sequence of atoms of the two sequences

$$A_1, \quad ((A_2 \otimes A_3) \otimes A_4) \qquad \text{and} \qquad (A'_4 \otimes A'_3), \quad A'_2, \quad A'_1$$

which are exactly the geometrically obtained orders of the premisses in the generalized link before and after the main formula. By the planarity of our representation, a generalized R$\multimap\!\multimapinv$-link facing our tensor link has as main formula a formula B having the same respective sequences of atoms, whence Lemma 2.3.10 indeed applies, giving $A \simeq_{\mathrm{i}} B$:



are connected as follows:



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4.10. Adding Exchange

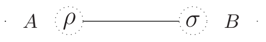
If we add Exchange

$$\frac{(\![\Gamma, B^\sigma, A^\rho]\!)}{(\![\Gamma, A^\rho, B^\sigma]\!)}\,\text{Ex}$$

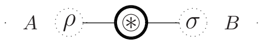
to the rules of $\mathbf{NCLL}_2$ we obtain $\mathbf{MLL}_2$. The proof nets (where a proof net is still defined as a proof structure that can be obtained as the proof structure of a derivation) of this calculus exactly are the proof structures in $\mathfrak{LG}'_2$ (see Subsection 4.4.2): proof structures for which all correction link graphs $\omega\mathcal{P}$ (ω a switching for $\mathcal{P}$) have a tree as underlying graph. This is the switching criterion of [**DR 89**]. Another criterion is the *contraction criterion* given by Danos in [**Danos 90**]: proof nets are those proof structures that can be contracted into one point, under a suitable contraction relation. Our contraction criterion for $\mathbf{MLL}_2$ is essentially the same. Yet an alternative criterion can be obtained by using tools from algebraic topology: by generalizing the ordinary definition of homology for graphs, proof nets turn out to be characterized among proof structures by their homology (Métayer's homological criterion, [**Métayer 94**]). The elegance of this criterion is the fact that it enables us to give completely algebraical proofs of proof theoretical phenomena of **MLL**: several characteristics of our proof structures (e.g. initial pars, splitting pars, etc.), can now be characterized by algebraic means. (See also [**Puite 96**] and [**PS 97**].)

Usually the rule of Exchange will be added as an implicit rule. Let us introduce the corresponding well-known coarser notion of sequent. We will still use link graphs for this purpose.

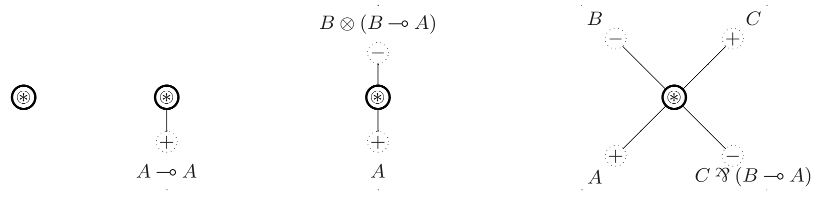
A sequent of $\mathbf{MLL}_2$ is an $\mathfrak{L}_2$-link graph $\mathcal{P}$ containing exactly one *set* link $l = \{e_0, \ldots, e_{m-1}\}_{\circledast}$ as context link, no connector links, and whose underlying graph is a tree, i.e. acyclic and connected. Because of the last requirement every edge η has exactly one extremity $\hat{\eta}$ occurring in l, whence $\mathcal{P}$ may be represented by the multiset $\{\lambda\check{\eta}_0, \ldots, \lambda\check{\eta}_{m-1}\}$ of open ends. Observe that a one-edge link graph



is not a sequent; there must be one context link, like in



Example 4.10.1. The following are examples of $\mathbf{MLL}_2$-sequents with 0, 1, 2 respectively 4 open ends:



$\{\ \}$ $\{(A \multimap A)^+\}$ $\{(B \otimes (B \multimap A))^-, A^+\}$ $\{B^-, A^+, (C \parr (B \multimap A))^-, C^+\}$

◊

By $C_{m-1}, \ldots, C_i \ \vdash \ C_0, \ldots, C_{i-1}$ (where both sides are multisets) we will denote the sequent $\{C_0^+, \ldots, C_{i-1}^+, C_i^-, \ldots, C_{m-1}^-\}$. In contrast to the sequents for $\mathbf{NCLL}_2$, all sequents can be represented in this way.

The calculus $\mathbf{MLL}_2$ is defined by the following (elementary) rules:

MLL₂

$$\frac{}{\{A^+, A^-\}}\ \textsc{Ax}$$

$$\frac{\{\Gamma, A^+\} \qquad \{\Delta, A^-\}}{\{\Gamma, \Delta\}}\ \textsc{Cut}$$

$$\frac{\{\Gamma, A^\rho\}}{\{\Gamma, ((A)^\perp)^{-\rho}\}}\ \perp$$

$$\frac{\{\Gamma, A^\rho\} \qquad \{\Delta, B^\sigma\}}{\{\Gamma, A^\rho \otimes B^\sigma, \Delta\}}\ \otimes \qquad \frac{\{\Gamma, B^\sigma, A^\rho\}}{\{\Gamma, B^\sigma \parr A^\rho\}}\ \parr$$

Recall that the latter $\otimes$ and $\parr$ are the maps $\mathfrak{L}_2^\pm \times \mathfrak{L}_2^\pm \to \mathfrak{L}_2^\pm$ defined on page 22.

In order to define proof nets, we postulate Lemma 4.2.5 for $\mathbf{MLL}_2$. This means that the inference rules have to be extended by the information as to which formulas correspond to each other.

The contraction criterion for $\mathbf{MLL}_2$ is now easily obtained from the theory in Section 4.4.

We think link graphs (with context links all of the type 'set link') admit a definition of homology similar to the original definition of Métayer, although some difficulties have to be overcome. (E.g., one single edge may belong to two 'pairs'.) It is, however, an open question whether we can reasonably refine this definition to link graphs with context links of the type 'cyclic link'. If yes, this could lead to a homological criterion for the non-commutative calculus $\mathbf{NCLL}_2$.

CHAPTER 5

A contraction criterion for CNL

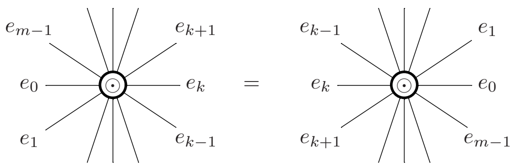
In this chapter we will investigate Classical Non-associative Lambek calculus (**CNL**), the calculus which remains after removing commutativity as well as associativity from **MLL**. Our notion of sequent must be subtle enough to admit this structural fine treatment. Moreover, we will define it in such a way that the calculus **NL** of the next chapter arises as the intuitionistic fragment. The one-sided non-associative linear logic of [**dGL 00**] will be obtained by projection on $\mathfrak{L}_1$.

5.1. Cyclic trees

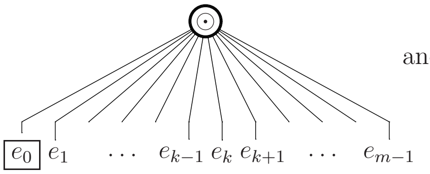
Given $m \geq 1$ distinct elements $e_0, e_1, \ldots, e_{m-1}$, a cyclic list $(\![e_0, \ldots, e_{m-1}]\!)$ can be represented by m lists $(e_k, e_{k+1}, \ldots, e_{k-1})$ (where $0 \leq k < m$). Each representative $(e_k, e_{k+1}, \ldots, e_{k-1})$ is determined by its first element e_k, and distinct representatives can be obtained from each other by *cyclic permutation* (or *rotation*), which is nothing else than commutativity on the outermost level:

$$(\Gamma, \Delta) \leftrightarrow (\Delta, \Gamma)$$

EXAMPLE 5.1.1. The cyclic list $(\![e_0, \ldots, e_{m-1}]\!)$

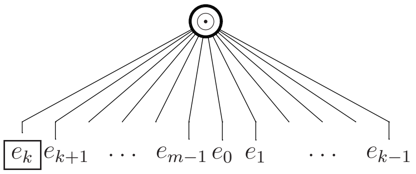


has m representatives, among which



$(e_0, \ldots, e_{m-1})$

and



$(e_k, e_{k+1} \ldots, e_{k-1})$

◊

We will introduce cyclic trees similarly as a quotient of the collection of rooted binary trees[1] instead of lists. Let $e_0, e_1, \ldots, e_{m-1}$ be $m \geq 2$ distinct elements. On the collection of

---

[1] A rooted binary tree can be denoted by a binary parenthesization, e.g.

$$A \odot ((B \odot D) \odot C) \quad \text{or} \quad A((BD)C).$$

This will be called a rooted binary tree *on the list* (A, B, D, C), or a rooted binary tree *with leaves* A, B, C, D (in some order).

all rooted binary trees with leaves $e_0, e_1, \ldots, e_{m-1}$ (in some order), let $\leftrightarrow$ be the smallest *equivalence* relation satisfying

$$(\Gamma \odot \Delta) \odot \Pi \leftrightarrow \Gamma \odot (\Delta \odot \Pi) \tag{Ass}$$

$$\Gamma \odot \Delta \leftrightarrow \Delta \odot \Gamma \tag{Comm}$$

Now we define[2] *cyclic trees on* $e_0, e_1, \ldots, e_{m-1}$ to be the equivalence classes of $\leftrightarrow$. We stress the fact that $\leftrightarrow$ is not defined as the smallest *congruence* relation w.r.t. $\odot$ satisfying (Ass) and (Comm), which would have trivialized it.

We define a $\mathbf{CNL}_2$-sequent to be an $\mathfrak{L}_2$-link graph $\mathcal{P} = (\mathcal{E}, \mathcal{L}, \emptyset, \lambda)$ containing only cyclic links $(\![e_0, e_1, e_2]\!)_\odot$ of valence 3 as context links, no connector links, and whose underlying graph is acyclic and connected. The underlying graph of a sequent has $m+|\mathcal{L}|$ vertices, while there are $|\mathcal{E}|$ edges. By acyclicity and connectedness

$$m + |\mathcal{L}| = |\mathcal{E}| + 1$$

As every link has valence 3, counting the total number of ends (edge extremities) yields

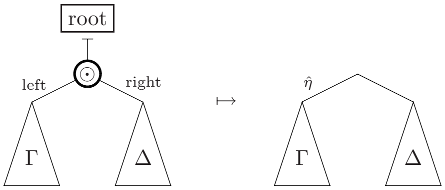
$$2\,|\mathcal{E}| = m + 3\,|\mathcal{L}|$$

We conclude that

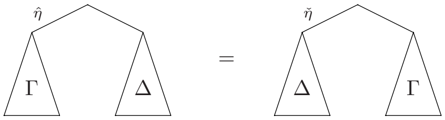
$$|\mathcal{L}| = m - 2$$
$$|\mathcal{E}| = 2m - 3$$

Lemma 5.1.2. *A* $\boldsymbol{CNL}_2$*-sequent one-to-one corresponds to a cyclic tree of polarized* $\mathfrak{L}_2$*-formulas.* ◊

Proof: Given a cyclic tree on $m \geq 2$ polarized formulas, let $\Gamma \odot \Delta$ be a representing rooted binary tree (where $m \geq 2$ guarantees there is an outermost $\odot$). Replacing the outermost $\odot$ and the corresponding root by one single edge η actually yields a sequent. Observe that the end $\hat{\eta}$ completely determines the original rooted binary tree $\Gamma \odot \Delta$.



The resulting link graph is invariant under $\leftrightarrow$-equivalence:



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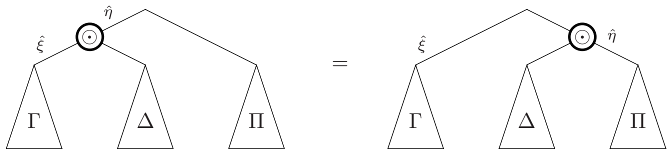
[2] The representatives of one cyclic tree hence are exactly those rooted binary trees which can be obtained from a particular representative by a form of associativity and commutativity. This seems paradoxical, since Lemma 5.1.2 shows that a cyclic tree is just a sequent for **CNL**: the calculus which remains after removing commutativity as well as associativity from **MLL**. However, the paradox evaporates when one realizes that only a very restrictive form of associativity and commutativity is divided out, viz. *on the outermost level* only.

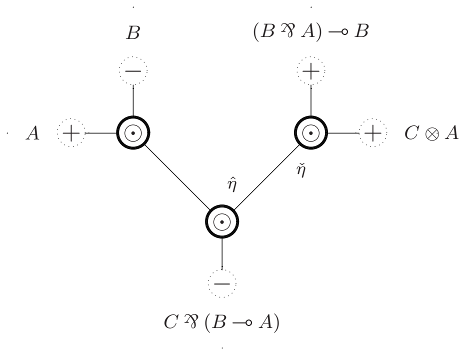


Hence we have established a well-defined map from cyclic trees to sequents, which is clearly surjective. It remains to prove that only $\leftrightarrow$-equivalent rooted binary trees yield the same sequent. Given two rooted binary trees Γ_1 and Γ_2 with the same image $\mathcal{P}$ under the above map, then each of them determines an end e_i of $\mathcal{P}$. Walking[3] around $\mathcal{P}$ in anticlockwise direction from e_1 to e_2 corresponds to a chain of equivalences (Ass) (in case the present end $\hat{\eta}$ is a context end) and (Comm) (in case $\hat{\eta}$ is an open end) connecting Γ_1 and Γ_2. (In this way, (Comm) is only applied with Γ a trivial rooted binary tree $\hat{\eta} = A^{\rho}$. As a matter of fact, we can possibly cut short our thus obtained walk by instances of (Comm) with complex Γ.) ////

The most simple sequent contains only one edge, and consequently has two open ends. Depending on the polarization of the labels, it will be denoted by $\vdash A, B$ or $A \vdash B$ or $A, B \vdash$.

Observe that sequents themselves do not have an outermost $\odot$; indeed, by the definition of a link graph, every open end is labeled by a polarized formula, whence a formal root is absent. Instead, every directed edge (or equivalently, every end) can be considered as the root of a representing rooted binary tree.

Example 5.1.3. Let $\mathcal{P}$ be the following sequent:



Then the two representatives corresponding to the ends $\hat{\eta}$ and $\check{\eta}$ are given by

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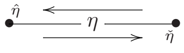
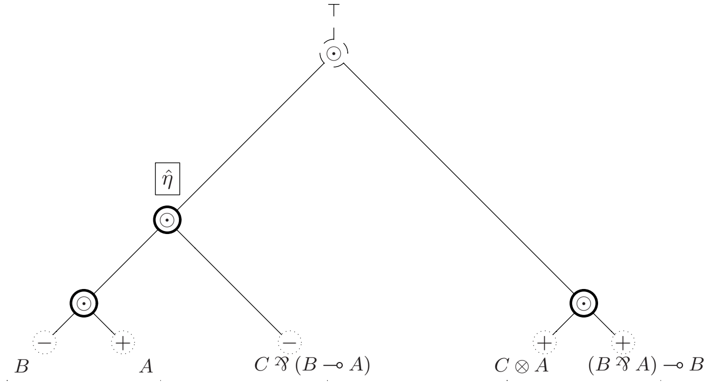
[3] Let us suppose that ends 'keep right':

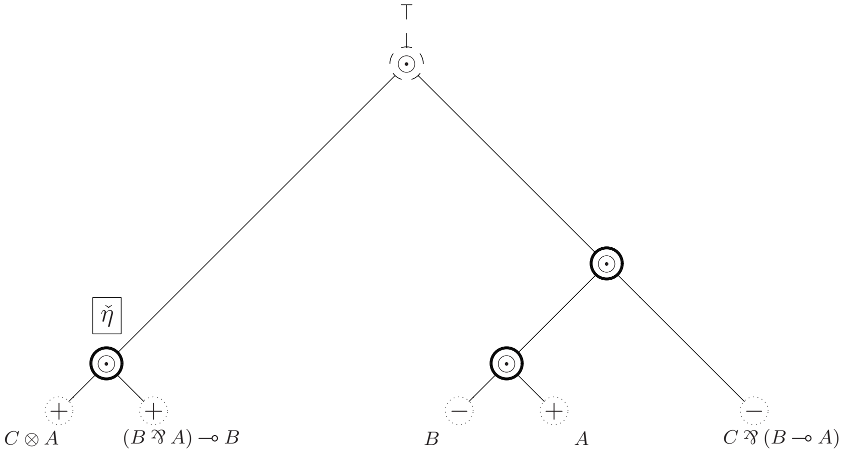


In this way ends correspond to *directed* edges.



$$\big((B^- \odot A^+) \odot (C \parr (B \multimap A))^-\big) \odot \big((C \otimes A)^+ \odot ((B \parr A) \multimap B)^+\big)$$

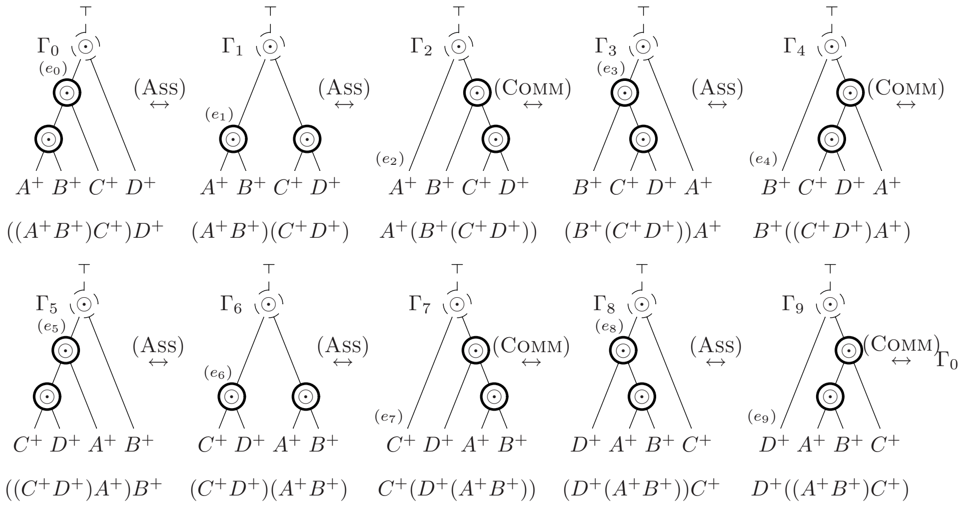
and



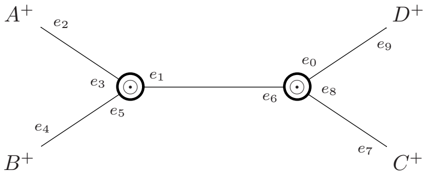
$$\big((C \otimes A)^+ \odot ((B \parr A) \multimap B)^+\big) \odot \big((B^- \odot A^+) \odot (C \parr (B \multimap A))^-\big)$$

As a representative is already determined by indicating the corresponding end e of $\mathcal{P}$, a distinguished end $\boxed{e}$ makes it superfluous to depict the outermost $\odot$ of a representative. $\Diamond$

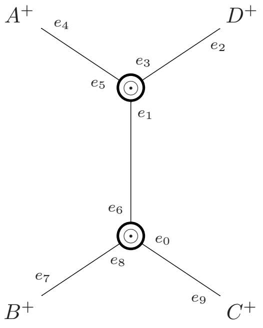
Example 5.1.4. Let us enumerate all those rooted binary trees Γ_i which are $\leftrightarrow$-equivalent with a given one Γ_0. We do this by 'counterclockwisely walking around' the corresponding sequent $\mathcal{P}$.



Together these ten rooted binary trees constitute the equivalence class (i.e. cyclic tree) corresponding to the sequent $\mathcal{P}$, given by



We know there are 5 rooted binary trees on a fixed list (A^+, B^+, C^+, D^+); three of them are enumerated above (Γ_0, Γ_1 and Γ_2); the remaining two (viz. $(A^+(B^+C^+))D^+$ and $A^+((B^+C^+)D^+)$) will be found in the equivalence class of



There are no other sequents with the same cyclic order $[\![A^+, B^+, C^+, D^+]\!]$ of open ends. ◊

Example 5.1.5. In this example we will investigate the combinatorics behind cyclic trees.

Let C_n be the number of binary parenthesizations of a fixed list of $n+1$ elements, e.g. for $n=3$:

$$((x_0x_1)x_2)x_3 \qquad (x_0(x_1x_2))x_3 \qquad x_0((x_1x_2)x_3) \qquad x_0(x_1(x_2x_3)) \qquad (x_0x_1)(x_2x_3)$$

Equivalently, the C_n count the number of abstract rooted binary trees with $n+1$ leaves. Now $C_0 = 1$, while every rooted binary tree with $n+1 \geq 2$ leaves is of the form $\Gamma \odot \Delta$ where Γ and Δ have length $k+1$ from 1 up to n respectively length $n+1-(k+1)$, whence

$$C_n = \sum_{k=0}^{n-1} C_k \, C_{n-k-1} \qquad (n \geq 1)$$

We will alternatively prove the well-known fact that C_n equals the nth *Catalan number*:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Observe that the right-hand-side satisfies the recursive definition

$$\begin{aligned} &\frac{1}{n+1}\binom{2n}{n}\bigg|_{n=0} = 1 \\ &\frac{\frac{1}{n+1}\binom{2n}{n}\big|_{n=m+1}}{\frac{1}{n+1}\binom{2n}{n}\big|_{n=m}} = \frac{(m+1)}{(m+2)}\,\frac{(2m+2)(2m+1)}{(m+1)(m+1)} = \frac{4m+2}{m+2} \qquad (m \geq 0) \end{aligned}$$

A sequent (cyclic tree of polarized $\mathfrak{L}_2$-formulas) $\mathcal{P}$ on $m \geq 2$ *distinct*[4] elements has $2m-3$ edges and hence can be represented by $4m-6$ distinct rooted binary trees, each one corresponding with one of the $4m-6$ ends (edge extremities) of $\mathcal{P}$. Now there are $m!$ ways to put the formulas in a list, and for every list there are C_{m-1} rooted binary trees on it. This yields a total number of $m!C_{m-1}$ rooted binary trees, and each contingent of $4m-6$ elements represents one and the same cyclic tree, yielding

$$\frac{m! \, C_{m-1}}{4m-6}$$

different cyclic trees.

The $m \geq 2$ open ends of a sequent $\mathcal{P}$ can be given a canonical cyclic order $\theta\mathcal{P} := (\!\![e_0, \ldots, e_{m-1}]\!\!)$ by walking counterclockwisely around the tree. The other way around, given a cyclic list $\Gamma := (\!\![e_0, \ldots, e_{m-1}]\!\!)$ of distinct elements, there are C_{m-2} cyclic trees $\mathcal{P}$ satisfying $\theta\mathcal{P} = \Gamma$. Indeed, declaring one open end as the root, we have to choose a rooted binary tree structure on the list of remaining $m-1$ open ends. Now there are $\frac{m!}{m} = (m-1)!$ cyclic lists, each of which corresponds to C_{m-2} cyclic trees, giving

$$\frac{m!}{m} C_{m-2}$$

cyclic trees. We conclude that

$$C_{(m-2)+1} = C_{m-1} = \frac{4m-6}{m} C_{m-2} = \frac{4(m-2)+2}{(m-2)+2} C_{m-2} \qquad (m-2 \geq 0)$$

---

[4] If some elements are equal, we have to correct for multiple occurrences. E.g. taking $C = A$ and $D = B$ in Example 5.1.4 causes a rotational order 2 symmetry on the sequent, and the $4m-6$ representatives pairwisely coincide.

which — together with $C_0 = 1$ — is in accordance with the above recursive definition of $\frac{1}{n+1}\binom{2n}{n}$, proving $C_n = \frac{1}{n+1}\binom{2n}{n}$.

In the next diagram we have indicated the different equivalence relations on the set of $m!C_{m-1}$ rooted binary trees. Every arrow represents the canonical projection from the set of representatives to the set of equivalence classes, which have the indicated size. Regarding all collections as quotients on the set of $m!C_{m-1}$ trees, every arrow represents taking a quotient corresponding to the smallest equivalence relation satisfying the indicated clause in addition to the original clauses the domain already satisfies. The actual clauses are given by the following instances and expansions of instances of associativity and commutativity:

$$(\Gamma \odot \Delta) \odot \Pi \leftrightarrow \Gamma \odot (\Delta \odot \Pi) \tag{\textsc{Ass}}$$

$$\Xi[(\Gamma \odot \Delta) \odot \Pi] \leftrightarrow \Xi[\Gamma \odot (\Delta \odot \Pi)] \tag{\underline{\textsc{Ass}}}$$

$$\Gamma \odot \Delta \leftrightarrow \Delta \odot \Gamma \tag{\textsc{Comm}}$$

$$\Xi[\Gamma \odot \Delta] \leftrightarrow \Xi[\Delta \odot \Gamma] \tag{\underline{\textsc{Comm}}}$$

$$\begin{tikzcd}
 & & \overset{m!\,C_{m-1}}{\boxed{\mathfrak{Trees}}} \arrow[dl, "\textsc{Ass}"', "{2,\ldots,m-1}"] \arrow[dr, "\textsc{Comm}", "2"'] \arrow[dd, Rrightarrow, "4m-6"', "\textsc{Ass}\ \&\ \textsc{Comm}"] \arrow[ddll, Rightarrow, bend right=40, "\underline{\textsc{Ass}}"', "C_{m-1}"] & & \\
 & \overset{m!\,C_{m-2}}{\boxed{\ldots}} \arrow[dl, "\underline{\textsc{Ass}}"', "C_{m-2}"] \arrow[dr, "\textsc{Comm}", "m"'] & & \overset{\frac{1}{2}m!\,C_{m-1}}{\boxed{\ldots}} \arrow[dl, "\textsc{Ass}"', "2m-3"] \arrow[dr, "\underline{\textsc{Comm}}", "2^{m-2}"'] & \\
\overset{m!}{\boxed{\mathfrak{Lists}}} \arrow[dr, Rrightarrow, "\textsc{Comm}", "m"'] \arrow[ddrr, Rightarrow, bend right=40, "\underline{\textsc{Comm}}", "m!"'] & & \overset{(m-1)!\,C_{m-2}}{\boxed{\mathfrak{CTrees}}} \arrow[dl, Rightarrow, "\underline{\textsc{Ass}}"', "\theta", "C_{m-2}"] \arrow[dr, "\underline{\textsc{Comm}}", "2^{m-2}"'] & & \overset{\frac{1}{2^{m-1}}m!\,C_{m-1}}{\boxed{\mathfrak{Mobiles}}} \arrow[dl, "\textsc{Ass}"', "2m-3"] \\
 & \overset{(m-1)!}{\boxed{\mathfrak{CLists}}} \arrow[dr, Rightarrow, "\theta", "\underline{\textsc{Comm}}", "(m-1)!"'] & & \overset{\frac{1}{2^{m-2}}(m-1)!\,C_{m-2}}{\boxed{\mathfrak{Molecules}}} \arrow[dl, "\underline{\textsc{Ass}}"', "(2m-5)(2m-7)\cdots 1"] & \\
 & & \overset{1}{\boxed{\mathfrak{MSets}}} & &
\end{tikzcd}$$

The quotient $\mathfrak{CTrees}$ of $\mathfrak{Trees}$ and the quotient $\mathfrak{CLists}$ of $\mathfrak{Lists}$ are indicated by an $\Rrightarrow$-arrow. Observe that the outermost square differs from the down most square in only one value of m.

$$\begin{tikzcd}[column sep=small]
 & \overset{m!\,C_{m-1}}{\boxed{\mathfrak{Trees}}} \arrow[dl, Rightarrow, "\underline{\textsc{Ass}}"', "C_{m-1}"] \arrow[dr, "\underline{\textsc{Comm}}", "2^{m-1}"'] & \\
\overset{m!}{\boxed{\mathfrak{Lists}}} \arrow[dr, Rrightarrow, "\underline{\textsc{Comm}}", "m!"'] & & \overset{\frac{1}{2^{m-1}}m!\,C_{m-1}}{\boxed{\mathfrak{Mobiles}}} \arrow[dl, "\underline{\textsc{Ass}}"', "(2m-3)(2m-5)\cdots1"] \\
 & \overset{1}{\boxed{\mathfrak{MSets}}} &
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
 & \overset{(m-1)!\,C_{m-2}}{\boxed{\mathfrak{CTrees}}} \arrow[dl, Rightarrow, "\underline{\textsc{Ass}}"', "\theta\, C_{m-2}"] \arrow[dr, "\underline{\textsc{Comm}}", "2^{m-2}"'] & \\
\overset{(m-1)!}{\boxed{\mathfrak{CLists}}} \arrow[dr, Rightarrow, "\theta\,\underline{\textsc{Comm}}", "(m-1)!"'] & & \overset{\frac{1}{2^{m-2}}(m-1)!\,C_{m-2}}{\boxed{\mathfrak{Molecules}}} \arrow[dl, "\underline{\textsc{Ass}}"', "(2m-5)(2m-7)\cdots1"] \\
 & \overset{1}{\boxed{\mathfrak{MSets}}} &
\end{tikzcd}$$

A *molecule* of polarized $\mathfrak{L}_2$-formulas one-to-one corresponds to the commutative version of a $\mathbf{CNL}_2$-sequent: an $\mathfrak{L}_2$-link graph $\mathcal{P} = (\mathcal{E}, \mathcal{L}, \emptyset, \lambda)$ containing only *set* links $\{e_0, e_1, e_2\}_{\circledast}$ of valence 3 as context links, no connector links, and whose underlying graph is acyclic and connected. 'Suspending' it by fixing one of its $2m-3$ edges yields a representing so-called *mobile*[5]. The number of mobiles is

$$\frac{m!\,C_{m-1}}{2^{m-1}} = \tfrac{1}{2^{m-1}}m!\tfrac{1}{m}\tfrac{(2m-2)!}{(m-1)!(m-1)!} = \frac{(2m-2)!}{2^{m-1}(m-1)!} = (2m-3)(2m-5)\cdots5\cdot3\cdot1$$

The operation $\odot$ defined on a collection of equivalence classes of trees by

$$[\Gamma] \odot [\Delta] := [\Gamma \odot \Delta]$$

is only well-defined when this is the collection of trees, of lists, of mobiles, or of multisets. Indeed, $\leftrightarrow$ has to be a congruence w.r.t. $\odot$ now:

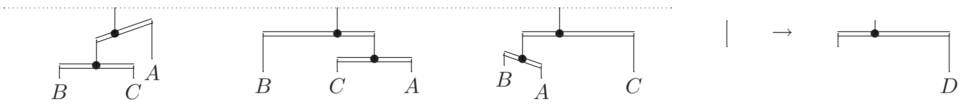
$$\Gamma \leftrightarrow \Gamma' \quad \& \quad \Delta \leftrightarrow \Delta' \quad \Longrightarrow \quad \Gamma \odot \Delta \leftrightarrow \Gamma' \odot \Delta'$$

For lists this operation is associative (but not commutative) and denoted by (Γ, Δ), while for mobiles this operation is commutative (but not associative).

We finish this example with some concrete numbers.

---

[5] A mobile, as found pending at the ceiling of many nurseries, is usually made of thread and straws. Every straw is free to turn around in the horizontal plane. These are the 3 distinct mobiles with $m = 3$ leaves, each one drawn in a certain position:



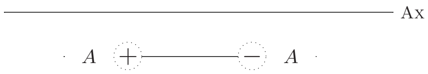
We can generate the mobiles with 4 leaves by inserting a next straw. In every depicted 3-mobile, any of the 5 vertical threads may be replaced by this new straw, yielding 15 4-mobiles. This explains the formula $(2m-3)(2m-5)\cdots5\cdot3\cdot1$.

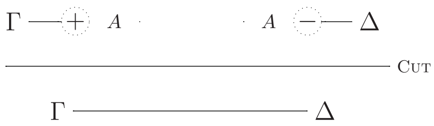
	m	0	1	2	3	4	5	6	7	8	9
	C_m	1	1	2	5	14	42	132	429	1430	4862
# trees	$m!\,C_{m-1}$	-	1	2	12	120	1680	30240	665280	17297280	518918400
# cyclic trees	$(m-1)!\,C_{m-2}$	-	-	1	2	12	120	1680	30240	665280	17297280
# mobiles	$\frac{m!\,C_{m-1}}{2^{m-1}}$	-	1	1	3	15	105	945	10395	135135	2027025
# lists	$m!$	1	1	2	6	24	120	720	5040	40320	362880
# cyclic lists	$(m-1)!$	-	1	1	2	6	24	120	720	5040	40320

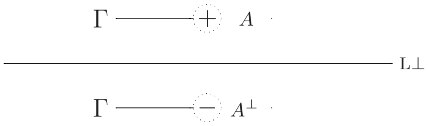
◊

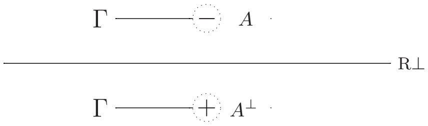
5.2. Sequent calculus

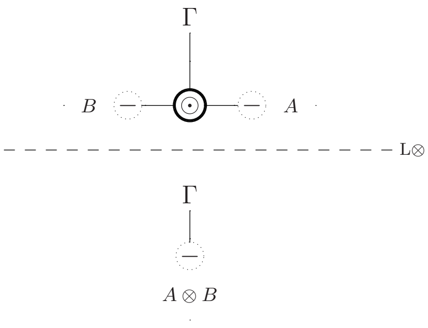
Let us define the sequent calculus to consist of the following rules.

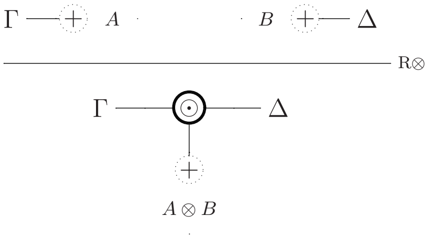


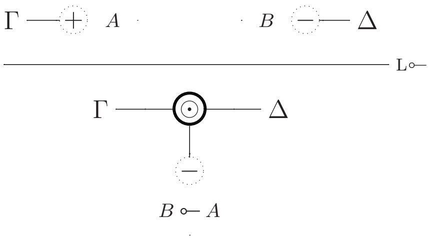


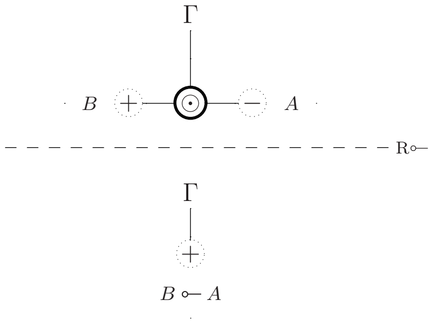


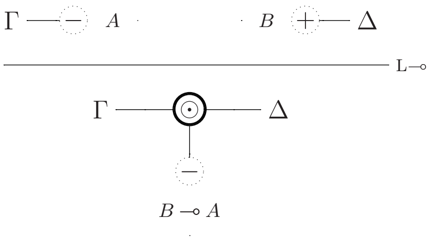


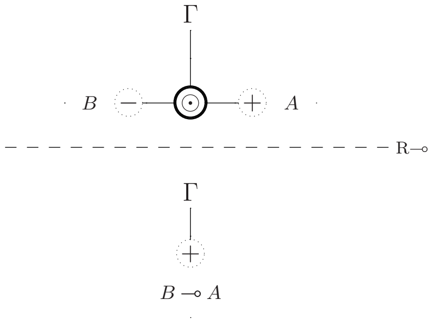


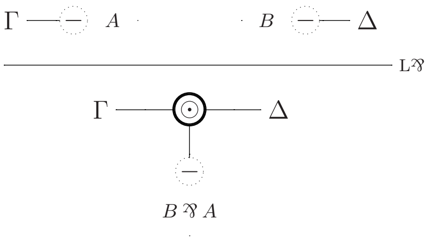


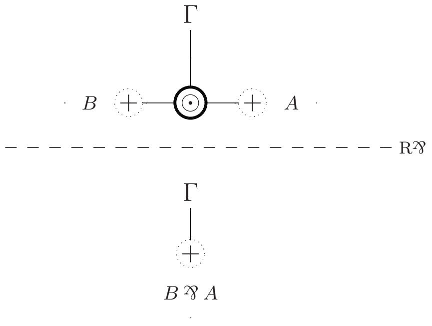




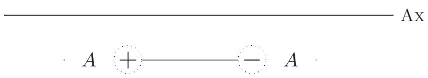


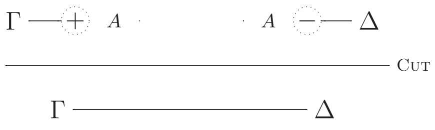


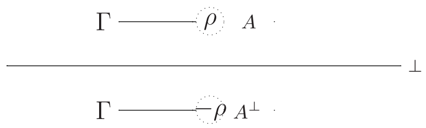


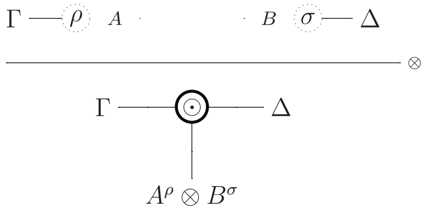


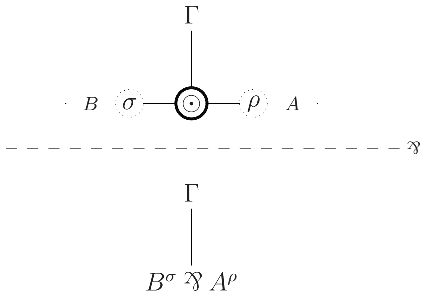
They may be compressed by









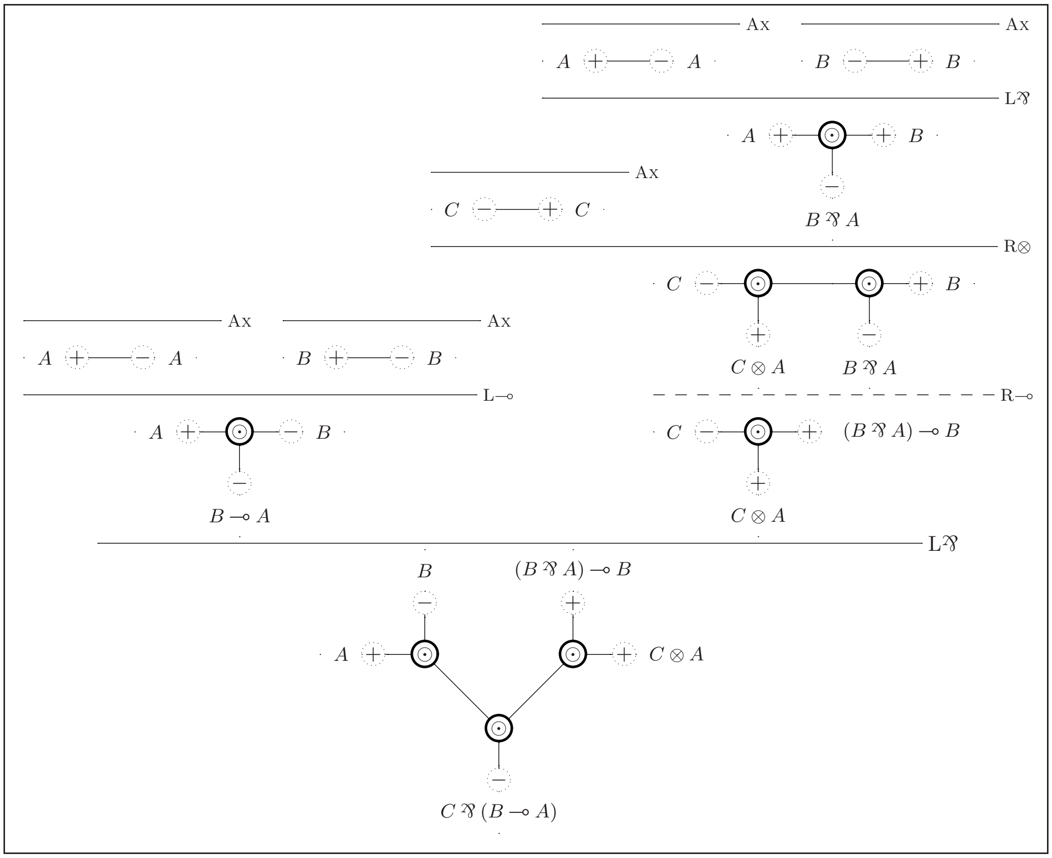


We divide the rules in four parts:

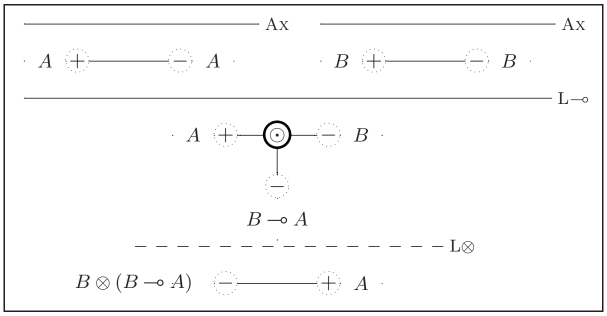
- the *identity rules* Ax and Cut;
- the *negation rules* L⊥ and R⊥;
- the *tensor rules* R⊗, L○—, L—○, L⅋;
- the *par rules* L⊗, R○—, R—○, R⅋.

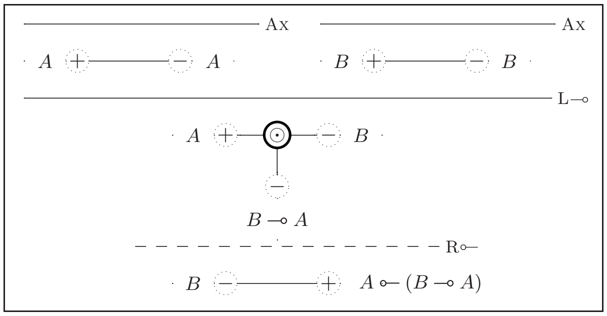
Observe that a par rule may only be applied to a so called outermost context link, i.e. a link with at least two ends connected to open ends. In the derivations we will indicate par rules by dashed horizontal lines.

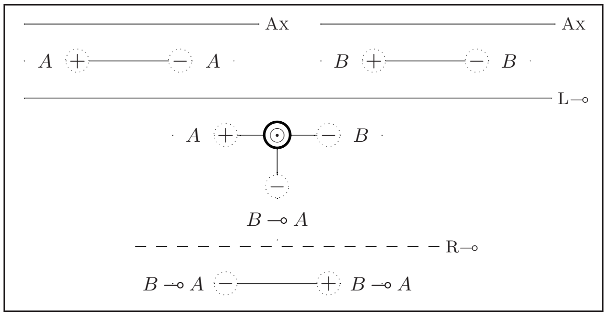
Example 5.2.1. One easily checks that the sequent in Example 5.1.3 is derivable in this calculus.



The following derivations also are examples for this calculus:



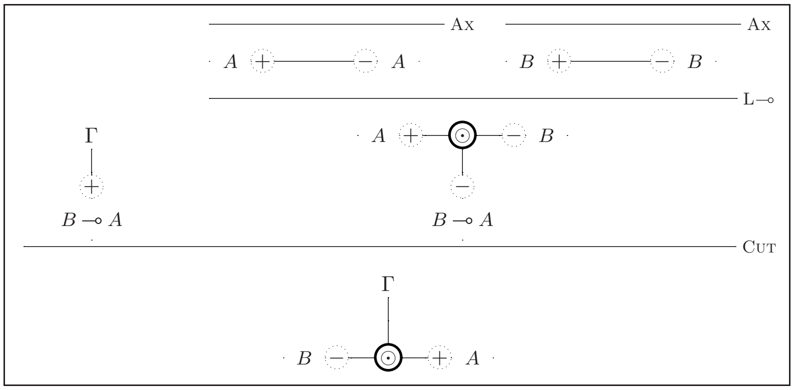




◊

Lemma 5.2.2. *The negation rules and the par rules are reversible.* ◊

Proof: The proof is by means of their respective counterparts and Cut. Let us prove the R⊸ case.



////

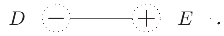
As an immediate consequence of the previous lemma we have

Lemma 5.2.3. *This calculus satisfies the following adjunctions:*

$$\begin{array}{lcll} A \otimes (-) & \dashv & A \multimap (-) & \textit{(for all formulas } A\textit{)} \\ (-) \otimes A & \dashv & (-) \multimapinv A & \textit{(for all formulas } A\textit{)} \end{array}$$

i.e.

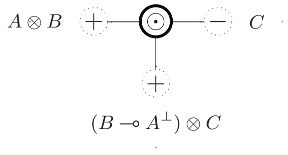
$$\begin{array}{c} A \otimes B \ \vdash \ C \\ \hline\hline B \ \vdash \ A \multimap C \end{array} \Updownarrow \qquad\qquad \begin{array}{c} B \otimes A \ \vdash \ C \\ \hline\hline B \ \vdash \ C \multimapinv A \end{array} \Updownarrow$$

where we used $D \vdash E$ *to denote the sequent*  . ◊

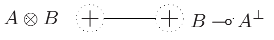
Let us mention the unit and co-unit of the first adjunction. Taking $A \otimes B$ for C we find the unit is given by $B \vdash A \multimap (A \otimes B)$, and taking $A \multimap C$ for B yields the co-unit $A \otimes (A \multimap C) \vdash C$.

In contrast to associative calculi, in a certain sense the tensor rules are reversible as well: suppose we know that a derivation ends with a tensor rule having main formula C^ν, then the two premiss sequents are uniquely determined. Compare this phenomenon to associative calculi, in which it is not a priori clear how to distribute the context formulas over the premiss sequents when executing a proof search algorithm. (In case the context contains n formulas, for **NCLL** there are $n+1$ choices, and for **MLL** there are in general 2^n choices.) In spite of this sort of reversibility, the tensor rules are not reversible in the usual sense, as is shown in the next example.

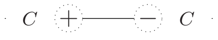
Example 5.2.4. Consider the derivable sequent Π, given by



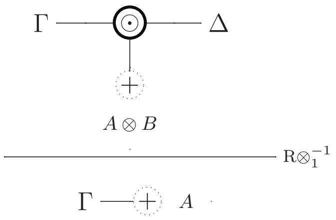
If we are given that $\big((B \multimap A^\perp) \otimes C\big)^+$ is most recently introduced, the premiss sequents must be



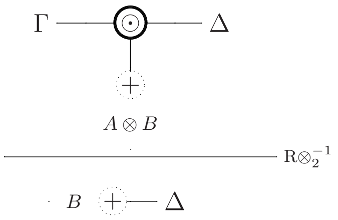
and



The two reversals of R⊗



and



are not in general admissible rules. As a counter example, derivability of our sequent Π would then imply e.g. derivability of

and

quod non. We conclude that R⊗ is not invertible, and as a consequence not reversible. ◊

5.3. Contraction criterion

This section will mainly follow Section 4.3 and Section 4.4. We will leave out the proofs, which are completely similar.

5.3.1. Proof nets. To a $\mathbf{CNL}_2$-derivation $\mathcal{D}$ we assign an $\mathfrak{L}_2$-proof structure $\mathcal{P}(\mathcal{D})$ in the obvious way. It has open ends in one-to-one correspondence to the open ends of the final sequent $\llcorner\mathcal{D}\lrcorner$ of $\mathcal{D}$, and corresponding open ends are labeled by the same polarized formula.

Definition 5.3.1. *A* $\boldsymbol{CNL}_2$-proof net *is an* $\mathfrak{L}_2$*-proof structure that can be obtained as the proof structure of a* $\boldsymbol{CNL}_2$*-derivation.* ◊

Example 5.3.2. Consider the first derivation of Example 5.2.1. Its proof net is given by Figure 5.1. ◊

This notion of proof net just determines a subset of the already encountered proof nets for $\mathbf{NCLL}_2$.

Lemma 5.3.3. *Let* $\theta : \mathfrak{CTrees} \to \mathfrak{CLists}$ *be the forgetful map as introduced in Example 5.1.5.*

(a) If $\mathcal{D}$ *is a* $\boldsymbol{CNL}_2$*-derivation, then* $\theta\mathcal{D}$ *is an* $\boldsymbol{NCLL}_2$*-derivation.*
(b) If Γ *is a derivable* $\boldsymbol{CNL}_2$*-sequent, then* $\theta\Gamma$ *is a derivable* $\boldsymbol{NCLL}_2$*-sequent.*
(c) Every $\boldsymbol{CNL}_2$*-proof net is an* $\boldsymbol{NCLL}_2$*-proof net.*

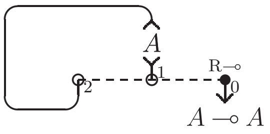
◊

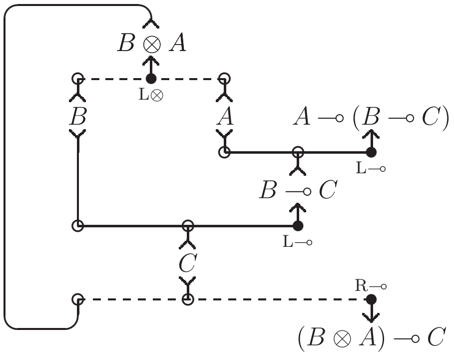
Proof: **(a)** Applying θ to the 0, 1 or 2 premiss sequent(s) and the conclusion sequent of a $\mathbf{CNL}_2$-inference yields an $\mathbf{NCLL}_2$-inference, whence the result follows by induction on $\mathcal{D}$.

(b) Directly, by part (a).

(c) Suppose $\mathcal{P}$ is an $\mathbf{CNL}_2$-proof net, say $\mathcal{P} = \mathcal{P}(\mathcal{D})$ where $\mathcal{D}$ is a $\mathbf{CNL}_2$-derivation. Now $\theta\mathcal{D}$ is an $\mathbf{NCLL}_2$-derivation by part (a), while $\mathcal{P}(\theta\mathcal{D})$ still equals $\mathcal{P}$, the latter hence being an $\mathbf{NCLL}_2$-proof net. ////

Example 5.3.4. Examples of an $\mathbf{NCLL}_2$-proof net which will turn out not to be a $\mathbf{CNL}_2$-proof net are given by





◊

We define the notions *logical cut*, *substitution of* $\mathcal{D}_2$ *into* $\mathcal{D}_1$, *sober derivation*, A^ρ*-clique*, *A-clique*, and *A-clique in axiomatic, flow, or cut form* similar as in Subsection 4.2.3 and Section 4.3.

Proposition 5.3.5. *Let* $\mathcal{P} = \mathcal{P}(\mathcal{D})$ *be the proof net of a derivation* $\mathcal{D}$*. Every link* l *of* $\mathcal{P}$ *of subtype* $X\Box$ *corresponds to an* $X\Box$*-rule of* $\mathcal{D}$*. For every formula* A *of* $\mathcal{P}$*, the* A*-clique (in* $\mathcal{D}$*) is in*

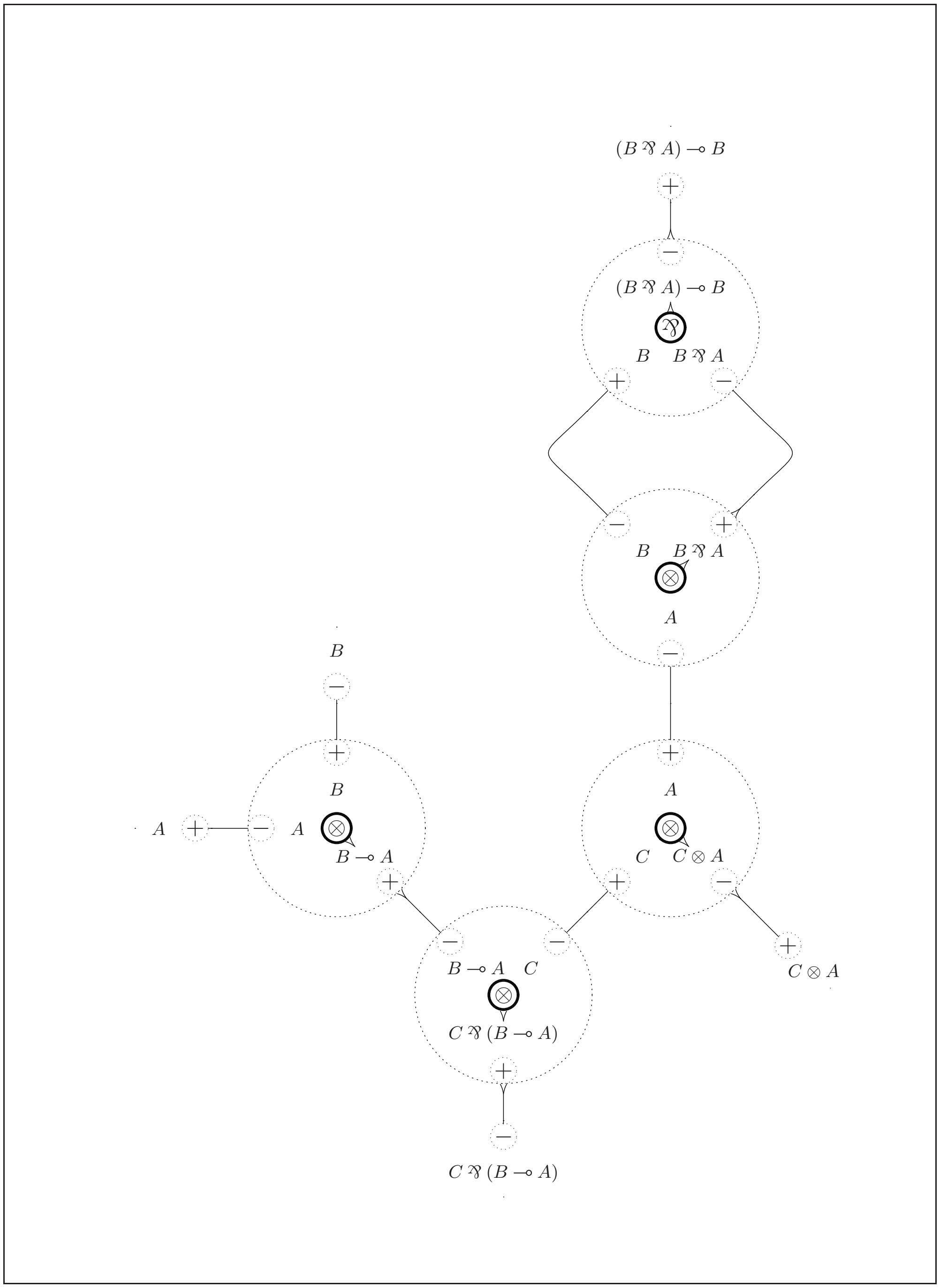

$(B \parr A) \multimap B$
+
−
$(B \parr A) \multimap B$
$\parr$
B
$B \parr A$
+
−
−
+
B
$B \parr A$
$\otimes$
A
−
B
−
+
B
A
+
−
A
$\otimes$
$B \multimap A$
+
+
A
$\otimes$
C
$C \otimes A$
+
−
−
−
$B \multimap A$
C
$\otimes$
$C \parr (B \multimap A)$
+
+
$C \otimes A$
−
$C \parr (B \multimap A)$

FIGURE 5.1.

- *axiomatic form if A is an axiomatic formula,*
- *flow form if A is neither axiomatic nor cut,*
- *cut form if A is a cut formula.*

As a consequence, every formula A of $\mathcal{P}$ corresponds to n $\textsc{Ax}_A$ rules and m $\textsc{Cut}_A$ rules of $\mathcal{D}$, and the following holds:

$$n - m = \begin{cases} 1 & \text{if } A \text{ is an axiomatic formula,} \\ 0 & \text{if } A \text{ is neither axiomatic nor cut,} \\ -1 & \text{if } A \text{ is a cut formula.} \end{cases}$$

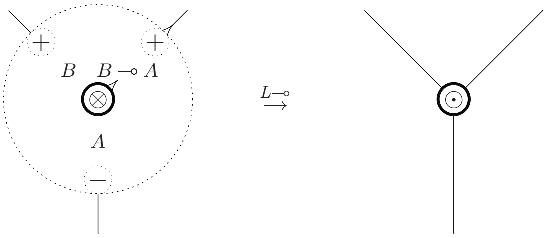
In particular, if $\mathcal{D}$ is sober, there is a bijective correspondence between the axiomatic (cut) formulas of $\mathcal{P}$ and the Ax *(*Cut*) rules of $\mathcal{D}$.* ◊

Theorem 5.3.6. *Let $\mathcal{D}$ and $\mathcal{D}'$ be derivations of the same sequent $(\![\Gamma]\!)$. Then their respective proof nets $\mathcal{P}$ and $\mathcal{P}'$ are equal if and only if there exists a sequence of derivations $\mathcal{D} = \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{n-1} = \mathcal{D}'$ such that $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ differ only for a permutation of two consecutive inferences, or $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i+1}$ (or the other way around) by a substitution (i.e. elimination of a non-logical* Cut*).* ◊

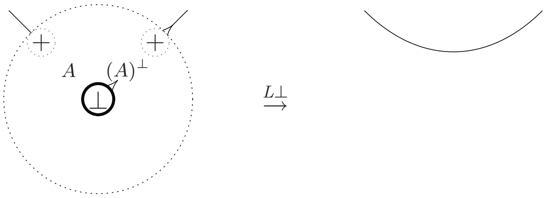
Let $\mathfrak{LG}_2^3$ denote the collection of $\mathfrak{L}_2$-link graphs with well-labeled (see Figure 3.1) connector links, viz. tensor links $(e_0, e_1, e_2)_\otimes$ (indicated by $\otimes$), par links $(e_0, e_1, e_2)_\parr$ (indicated by $\parr$), and negation links $(e_0, e_1)_\perp$ (indicated by $\perp$), and with context links $(\![e_0, e_1, e_2]\!)_\odot$ (indicated by $\odot$) *of valence 3*. Observe that both $\mathfrak{L}_2$-proof structures and $\mathbf{CNL}_2$-sequents belong to $\mathfrak{LG}_2^3$. On $\mathfrak{LG}_2^3$ we will define the following conversion relation. One easily checks that these conversion steps are well defined (i.e. they do yield an element of $\mathfrak{LG}_2^3$) and preserve the open ends.

Definition 5.3.7.

- ***[tens]***(l) *Every tensor link l converts into a context link:*

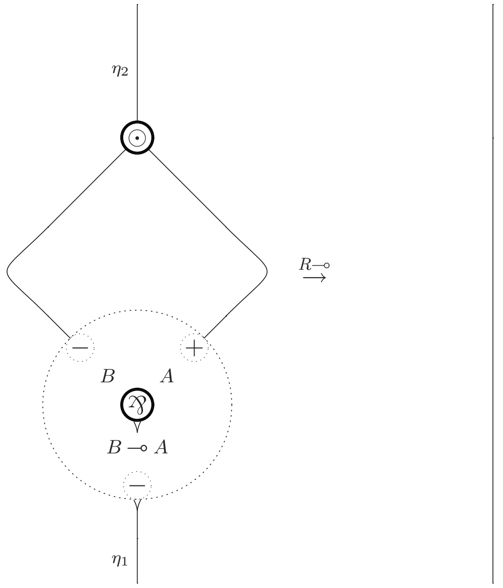


  and similar for $R\otimes$, $L\multimapinv$ and $L\parr$.
- ***[neg]***(l, η_1, η_2) *Every negation link l connected to two edges η_1 and $\eta_2 \neq \eta_1$ converts into a single edge:*



  and similar for $R\perp$.

- ***[par]***$(l_1, l_2, \eta_1, \eta_2)$ *Given a par link* l_1*, the active ends of which are connected to two consecutive ends of a single context link* l_2 *in the right order, then* l_1 *and* l_2 *together convert into a single edge if* $\eta_1 \neq \eta_2$*:*



and similar for $L\otimes$*,* $R\circ\!\!-$ *and* $R\parr$*.*

$\Diamond$

EXAMPLE 5.3.8. The proof net of Example 5.3.2 converts to the sequent of Example 5.1.3 in four **[tens]**-steps and one R$\multimap$-step.

The proof structures of Example 5.3.4 do not convert to a sequent. $\Diamond$

It is easy to see that this reduction relation is terminating; in each conversion step $\mathcal{P}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{L}'_1, \lambda_1) \rightarrow \mathcal{P}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{L}'_2, \lambda_2)$ the non-negative integer $\phi(\mathcal{P}) := |\mathcal{E}| + |\mathcal{L}| + 2\,|\mathcal{L}'|$ decreases by at least one (recall that $\mathcal{L}$ consists of the context links, while $\mathcal{L}'$ contains the connector links):

LEMMA 5.3.9. *The conversion steps increase* ϕ *by* $\Delta\phi$*, given by:*

- ***[tens]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (0, +1, -1)$*, so* $\Delta\phi = -1$*.*
- ***[neg]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-1, 0, -1)$*, so* $\Delta\phi = -3$*.*
- ***[par]*** $(\Delta\,|\mathcal{E}|\,, \Delta\,|\mathcal{L}|\,, \Delta\,|\mathcal{L}'|) = (-3, -1, -1)$*, so* $\Delta\phi = -6$*.*

$\Diamond$

Let $(\mathfrak{L}\mathfrak{G}_2^3)'$ denote the collection of those elements $\mathcal{P}$ of $\mathfrak{L}\mathfrak{G}_2^3$ for which all the 2^n correction link graphs $\omega\mathcal{P}$ have a tree as underlying graph (see Subsection 4.4.2), i.e. $(\mathfrak{L}\mathfrak{G}_2^3)' = \mathfrak{L}\mathfrak{G}_2^3 \cap \mathfrak{L}\mathfrak{G}'_2$.

LEMMA 5.3.10. *Let* $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{L}\mathfrak{G}_2^3$ *and suppose* $\mathcal{P}_1 \rightarrow \mathcal{P}_2$*. Then* $\mathcal{P}_1 \in (\mathfrak{L}\mathfrak{G}_2^3)'$ *if and only if* $\mathcal{P}_2 \in (\mathfrak{L}\mathfrak{G}_2^3)'$*.* $\Diamond$

In particular, the conversion steps are well defined on $(\mathfrak{L}\mathfrak{G}_2^3)'$ (i.e. they do yield an element of $(\mathfrak{L}\mathfrak{G}_2^3)'$ when applied on an element of $(\mathfrak{L}\mathfrak{G}_2^3)'$).

Since $\mathbf{CNL}_2$-sequents belong to $(\mathfrak{L}\mathfrak{G}_2^3)'$, we immediately obtain the next result.

Corollary 5.3.11. *If an $\mathfrak{L}_2$-proof structure $\mathcal{P}$ converts to a $\boldsymbol{CNL}_2$-sequent Γ, then $\mathcal{P} \in (\mathfrak{LG}_2^3)'$.* $\Diamond$

The conversion relation is confluent on $(\mathfrak{LG}_2^3)'$.

Lemma 5.3.12. *If $\mathcal{P} \in (\mathfrak{LG}_2^3)'$ converts in one step to $\mathcal{P}_1$ and $\mathcal{P}_2$, then both $\mathcal{P}_1$ and $\mathcal{P}_2$ convert in at most one step to a common $\mathcal{P}_3 \in (\mathfrak{LG}_2^3)'$.* $\Diamond$

$$\begin{tikzcd}
 & \mathcal{P}_1 \arrow[dr, dashed] & \\
\mathcal{P} \arrow[ur, "{[\mathbf{conv}_1]}"] \arrow[dr, "{[\mathbf{conv}_2]}"'] & & \mathcal{P}_3 \\
 & \mathcal{P}_2 \arrow[ur, dashed] &
\end{tikzcd}$$

5.3.2. Completeness.

Theorem 5.3.13. *(a) Let $\mathcal{D}$ be a $\boldsymbol{CNL}_2$-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$.*
(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a $\boldsymbol{CNL}_2$-sequent. Then there is a $\boldsymbol{CNL}_2$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.

$\Diamond$

Observe that by part (a) **CNL**$_2$-proof nets convert to **CNL**$_2$-sequents. Hence by Corollary 5.3.11 they will only be found in $(\mathfrak{LG}_2^3)'$.

Corollary 5.3.14. *Let Γ be a $\boldsymbol{CNL}_2$-sequent. Then the following are equivalent:*

(i) Γ is derivable in $\boldsymbol{CNL}_2$;
(ii) There is a proof structure $\mathcal{P}$ and a conversion sequence $\mathcal{P} \twoheadrightarrow \Gamma$.

$\Diamond$

Theorem 5.3.15. *(a) Let $\mathcal{D}$ be a $\boldsymbol{CNL}_2$-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$.*
(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a $\boldsymbol{CNL}_2$-sequent. Then there is a sober *$\boldsymbol{CNL}_2$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.*

$\Diamond$

By means of Lemma 5.3.12 and Lemma 5.3.9 we can sharpen Corollary 5.3.14 into:

Theorem 5.3.16. *Let Γ be a $\boldsymbol{CNL}_2$-sequent. Then the following are equivalent:*

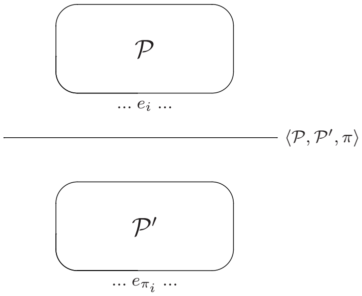
(i) Γ is derivable in $\boldsymbol{CNL}_2$;
(ii) There is a proof structure $\mathcal{P}$ such that all conversion sequences $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ (where $\mathcal{P}'$ is normal) satisfy $\mathcal{P}' = \Gamma$.

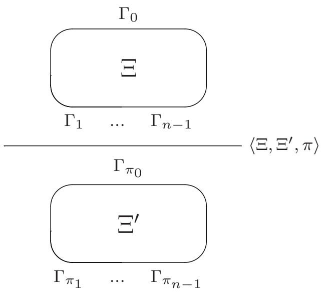
$\Diamond$

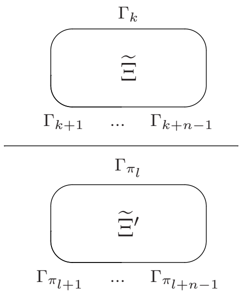
Theorem 5.3.17. *Let $\mathcal{P}$ be a proof structure and $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ be an arbitrary conversion sequence to a normal form. Then $\mathcal{P}$ is a $\boldsymbol{CNL}_2$-proof net if and only if $\mathcal{P}'$ is a $\boldsymbol{CNL}_2$-sequent.* $\Diamond$

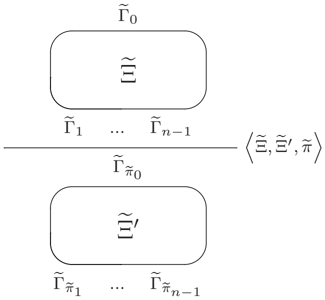
5.4. Adding structural rules

An *n-ary structural rule* is defined by a pair of cyclic n-trees $\mathcal{P}, \mathcal{P}'$ with open ends in bijective correspondence $\pi : e_i \mapsto e_{\pi_i}$. Declaring one open end of $\mathcal{P}$ ($\mathcal{P}'$) as the root enables us to enumerate the sequence of open ends of $\mathcal{P}$ ($\mathcal{P}'$), starting with the root. Hence the structural rule may be represented by means of a pair of rooted trees Ξ, Ξ' on $n-1$ elements, and a rearrangement of the coinciding n open ends $\pi \in \mathfrak{S}_n$, the symmetric group of degree n, where 'e_0' is the root of Ξ and 'e_{π_0}' is the root of Ξ'. Different defining tuples $\langle \Xi, \Xi', \pi \rangle$ and $\left\langle \widetilde{\Xi}, \widetilde{\Xi}', \widetilde{\pi} \right\rangle$ represent the same structural rule precisely if Ξ and $\widetilde{\Xi}$ (Ξ' and $\widetilde{\Xi}'$) differ only for the choice of the root, while $\widetilde{\pi}$ compensates π for this difference: $\widetilde{\pi}(i) = \pi(i +_n l) -_n k$.

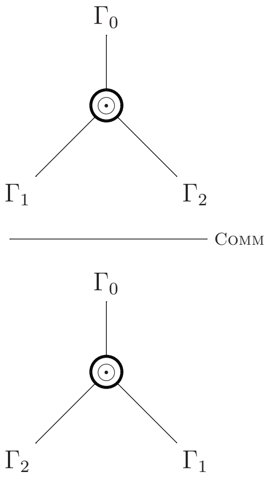


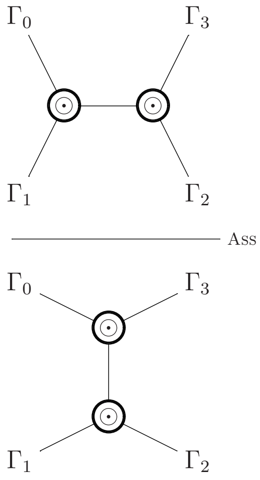


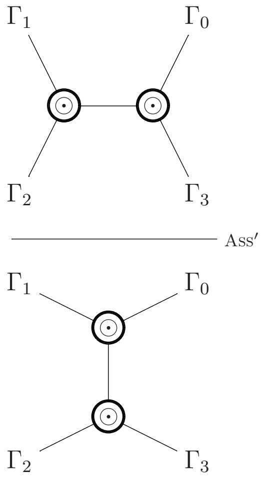




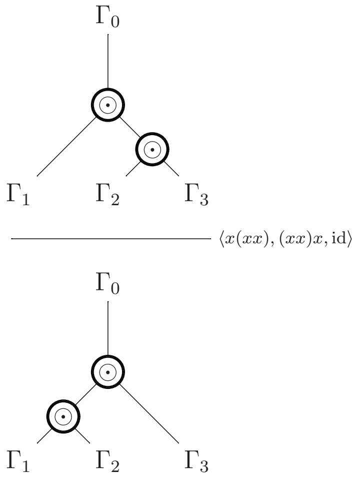
Example 5.4.1. The following are examples of structural rules.

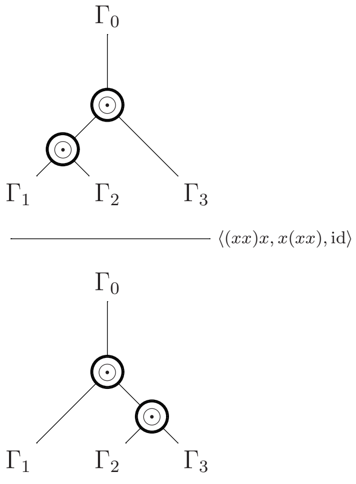




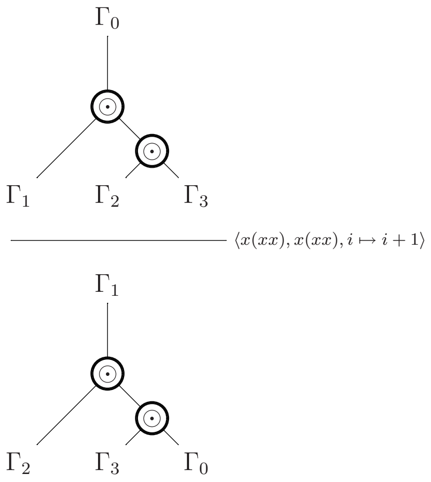


Observe that Ass and Ass′ coincide. Representatives of Ass $=$ Ass′ with $\pi_0 = 0$ are given by





depending[6] on the choice of the root of $\mathcal{P}$, while another representative (obtained by taking Γ_0 respectively Γ_1 as roots of $\mathcal{P}$ and $\mathcal{P}'$) is given by



$\Diamond$

Let $\mathcal{R}$ be a fixed set of structural rules. We call $\mathbf{CNL}_{2,\mathcal{R}}$ the calculus obtained by adding these structural rules to $\mathbf{CNL}_2$, in which sequents still are $\mathbf{CNL}_2$-sequents. To a $\mathbf{CNL}_{2,\mathcal{R}}$-derivation $\mathcal{D}$ we assign an $\mathfrak{L}_2$-proof structure $\mathcal{P}_{\mathcal{R}}(\mathcal{D})$ (called a $\mathbf{CNL}_{2,\mathcal{R}}$-proof net) by neglecting the structural rules occurring in $\mathcal{D}$.

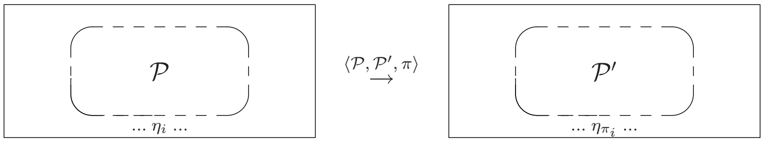
Proof nets in the calculus without structural rules ($\mathbf{CNL}_2$-proof nets or, more general, $\mathbf{NCLL}_2$-proof nets) satisfy the following property: they may be graphically represented by a planar graph, which induces an order on the open ends that coincides with the cyclic list of open ends of the sequent the proof net proves. This is easily shown by induction on $\mathcal{D}$, or from the contraction criterion Theorem 5.3.13 (Theorem 4.4.4) when we realize that the conversion steps, read from right to left, preserve planarity and the cyclic order of the open ends of the link graphs.

For $\mathbf{CNL}_{2,\mathcal{R}}$-proof nets, on the contrary, a final structural rule occurring in

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_1\end{matrix}}{\Gamma}\,\langle \Xi, \Xi', \pi\rangle$$

has no effect on $\mathcal{P}_{\mathcal{R}}(\mathcal{D})$ (i.e. the latter equals $\mathcal{P}_{\mathcal{R}}(\mathcal{D}_1)$), although the cyclic list of open ends of Γ may differ from that of Γ_1. Nevertheless, we can formulate a contraction criterion for $\mathbf{CNL}_{2,\mathcal{R}}$. We extend the conversion relation on $\mathfrak{LG}_2^3$ (Definition 5.3.7) by

- ***[struct]***($\mathcal{P}$) *If $\langle \mathcal{P}, \mathcal{P}', \pi\rangle$ is a structural rule in $\mathcal{R}$, a sub link graph $\mathcal{P}$ converts into $\mathcal{P}'$, while the open ends of $\mathcal{P}$ are correctly permuted:*



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[6] Indeed, the choice of root (Γ_0) for $\mathcal{P}$ fixes the root (Γ_{π_0}) of $\mathcal{P}'$ when $\pi_0 = 0$.

and denote the steps of this new relation by $\rightarrow_{\mathcal{R}}$. This relation $\rightarrow_{\mathcal{R}}$ is not in general terminating anymore, neither is it confluent on $(\mathfrak{LG}_2^3)'$. However, Lemma 5.3.10, Corollary 5.3.11, Theorem 5.3.13, Corollary 5.3.14, Theorem 5.3.15 still hold for this extended conversion relation:

Lemma 5.4.2. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{LG}_2^3$ and suppose $\mathcal{P}_1 \rightarrow_{\mathcal{R}} \mathcal{P}_2$. Then $\mathcal{P}_1 \in (\mathfrak{LG}_2^3)'$ if and only if $\mathcal{P}_2 \in (\mathfrak{LG}_2^3)'$.* $\Diamond$

In particular, the $\rightarrow_{\mathcal{R}}$-conversion steps are well defined on $(\mathfrak{LG}_2^3)'$ (i.e. they do yield an element of $(\mathfrak{LG}_2^3)'$ when applied on an element of $(\mathfrak{LG}_2^3)'$).

Since $\mathbf{CNL}_2$-sequents belong to $(\mathfrak{LG}_2^3)'$, we immediately obtain the next result.

Corollary 5.4.3. *If an $\mathfrak{L}_2$-proof structure $\mathcal{P}$ $\rightarrow_{\mathcal{R}}$-converts to a* ***CNL**$_2$*-sequent Γ, then $\mathcal{P} \in (\mathfrak{LG}_2^3)'$.* $\Diamond$

Theorem 5.4.4. *(a) Let $\mathcal{D}$ be a* ***CNL**$_{2,\mathcal{R}}$*-derivation. Then $\mathcal{P}_{\mathcal{R}}(\mathcal{D}) \twoheadrightarrow_{\mathcal{R}} \llcorner\mathcal{D}\lrcorner$.*

(b) Let $\mathcal{P} \twoheadrightarrow_{\mathcal{R}} \Gamma$ be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a ***CNL**$_2$*-sequent. Then there is a* ***CNL**$_{2,\mathcal{R}}$*-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}_{\mathcal{R}}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.*

$\Diamond$

Proof: **(a)** If the last rule is a structural rule

$$\mathcal{D} = \begin{array}{c} \mathcal{D}_1 \\ \Gamma_1 \\ \hline \Gamma \end{array} \langle \Xi, \Xi', \pi \rangle$$

we know by induction hypothesis that $\mathcal{P}_{\mathcal{R}}(\mathcal{D}_1) \twoheadrightarrow_{\mathcal{R}} \Gamma_1$, whence

$$\mathcal{P}_{\mathcal{R}}(\mathcal{D}) = \mathcal{P}_{\mathcal{R}}(\mathcal{D}_1) \twoheadrightarrow_{\mathcal{R}} \Gamma_1 \overset{\text{struct}}{\rightarrow}_{\mathcal{R}} \Gamma$$

The other cases are the same as those of Theorem 5.3.13, which are proved similar to Theorem 4.4.4.

(b) If the last conversion is a [**struct**]-step $\Gamma_1 \rightarrow_{\mathcal{R}} \Gamma$, also Γ_1 is a sequent, whence by induction hypothesis there is a $\mathbf{CNL}_{2,\mathcal{R}}$-derivation $\mathcal{D}_1$ of Γ_1 with $\mathcal{P} = \mathcal{P}_{\mathcal{R}}(\mathcal{D}_1)$. Now extend $\mathcal{D}_1$ with the appropriate structural rule. The other cases are the same as those of Theorem 5.3.13, which are proved similar to Theorem 4.4.4. ////

Observe that by part (a), $\mathbf{CNL}_{2,\mathcal{R}}$-proof nets $\rightarrow_{\mathcal{R}}$-convert to $\mathbf{CNL}_2$-sequents. Hence by Corollary 5.4.3 they will only be found in $(\mathfrak{LG}_2^3)'$.

Corollary 5.4.5. *Let Γ be a* ***CNL**$_2$*-sequent. Then the following are equivalent:*

(i) Γ is derivable in ***CNL**$_{2,\mathcal{R}}$*;*
(ii) There is a proof structure $\mathcal{P}$ and a conversion sequence $\mathcal{P} \twoheadrightarrow_{\mathcal{R}} \Gamma$.

$\Diamond$

Theorem 5.4.6. *(a) Let $\mathcal{D}$ be a* ***CNL**$_{2,\mathcal{R}}$*-derivation. Then $\mathcal{P}_{\mathcal{R}}(\mathcal{D}) \twoheadrightarrow_{\mathcal{R}} \llcorner\mathcal{D}\lrcorner$.*

(b) Let $\mathcal{P} \twoheadrightarrow_{\mathcal{R}} \Gamma$ be a conversion sequence from an $\mathfrak{L}_2$-proof structure to a ***CNL**$_2$*-sequent. Then there is a* sober ***CNL**$_{2,\mathcal{R}}$*-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}_{\mathcal{R}}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.*

$\Diamond$

If we take $\mathcal{R}$ to consist of Ass only, we actually obtain the variant $\mathbf{NCLL}_{2,2}$ of $\mathbf{NCLL}_2$, in which the sequents are required to contain at least two formulas.

Theorem 5.4.7. *Let $\mathcal{R} = \{$*Ass*$\}$ and let Γ be a* ***CNL**$_2$*-sequent. Then Γ is* ***CNL**$_{2,\mathcal{R}}$*-derivable if and only if $\theta\Gamma$ is* ***NCLL**$_{2,2}$*-derivable.* $\Diamond$

Proof: The ‘only if’-part is easy: a $\mathbf{CNL}_{2,\mathcal{R}}$-derivation becomes an $\mathbf{NCLL}_{2,2}$-derivation under θ; here Ass translates into the identity induced rule.

For the ‘if’-part, observe that every $\mathbf{NCLL}_{2,2}$-sequent Δ can be written as $\theta\Gamma_1$ for some $\mathbf{CNL}_2$-sequent Γ_1, while $\theta\Gamma_1 = \theta\Gamma_2$ implies that Γ_2 can be obtained from Γ_1 by several applications of Ass. The proof of this fact is similar to the proof of Lemma 2.3.3, applied to representing trees of Γ_1 and Γ_2. Now consider the final rule of a derivation $\mathcal{D}$ of $\theta\Gamma$ in $\mathbf{NCLL}_{2,2}$.

$$\mathcal{D} = \quad \dfrac{\begin{matrix}\mathcal{D}_0 \\ \theta\Gamma_0\end{matrix} \qquad \ldots \qquad \begin{matrix}\mathcal{D}_{n-1} \\ \theta\Gamma_{n-1}\end{matrix}}{\theta\Gamma}$$

By induction hypothesis there are $\mathbf{CNL}_{2,\mathcal{R}}$-derivations of Γ_0 up to Γ_{n-1}, and — possibly after some Ass-modifications in case of a par rule — we can apply the corresponding rule of $\mathbf{CNL}_2$. Modifying the result Γ' finally yields Γ.

$$\widetilde{\mathcal{D}} = \quad \begin{array}{c} \dfrac{\begin{matrix}\widetilde{\mathcal{D}}_0 \\ \begin{array}{c}\Gamma_0 \\ \hline\hline \Gamma_0'\end{array}\textsc{Ass}\end{matrix} \qquad \ldots \qquad \begin{matrix}\widetilde{\mathcal{D}}_{n-1} \\ \begin{array}{c}\Gamma_{n-1} \\ \hline\hline \Gamma_{n-1}'\end{array}\textsc{Ass}\end{matrix}}{\begin{array}{c}\Gamma' \\ \hline\hline \Gamma\end{array}\textsc{Ass}} \end{array}$$

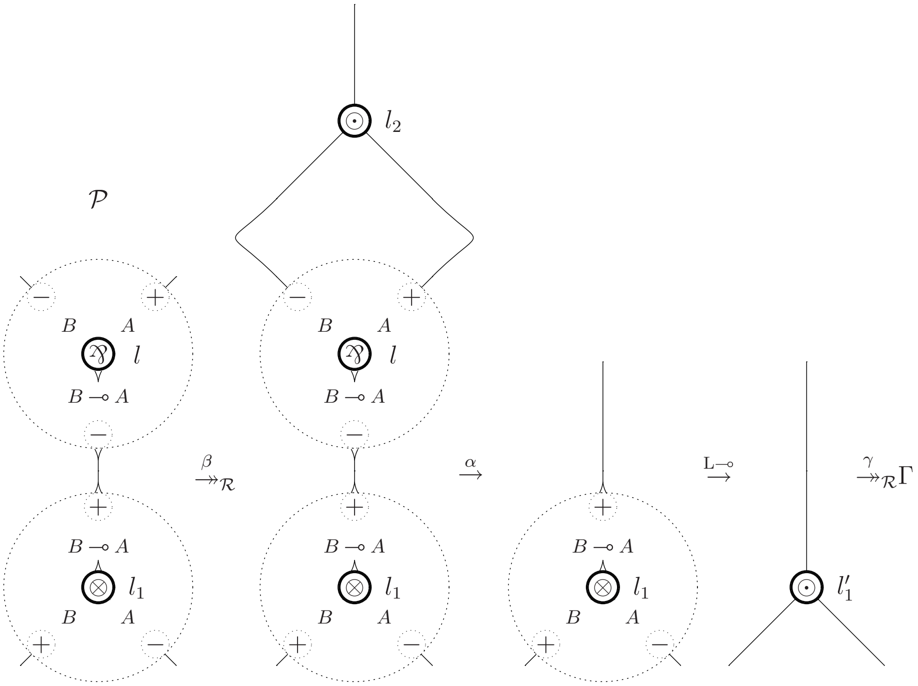
////

5.5. Cut elimination

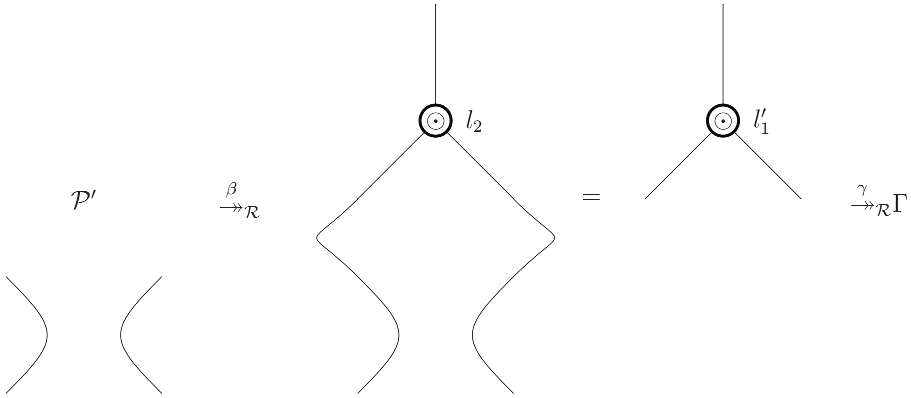
We will formulate the non-associative counterparts of the results in Section 4.5 for $\mathbf{CNL}_{2,\mathcal{R}}$ at once; the results for $\mathbf{CNL}_2$ follow by taking $\mathcal{R} = \emptyset$.

Theorem 5.5.1. *If $\mathcal{P}$ is a $\boldsymbol{CNL}_{2,\mathcal{R}}$-proof net proving the $\boldsymbol{CNL}_2$-sequent Γ, and $\mathcal{P} \rightsquigarrow \mathcal{P}'$ by a cut reduction step, then $\mathcal{P}'$ is a proof net proving Γ as well.* $\Diamond$

Proof: The proof will resemble that of Theorem 4.5.1. In the first part ($C = (A)^{\perp}$), observe that it is no restriction to assume that our conversion sequence $\mathcal{P} \twoheadrightarrow_{\mathcal{R}} \Gamma$ starts with all occurring **[neg]**-steps. In the last part ($C = B \multimap A$ etc.), let l and l_1 be the corresponding par link respectively tensor link. Fix a conversion sequence $\mathcal{P} \twoheadrightarrow_{\mathcal{R}} \Gamma$. The par link l disappears at a conversion $\alpha := [\mathbf{par}](l, l_2)$, yielding an edge. We claim that it is no restriction to assume that l_1 remains untouched until α. Indeed, reasoning backwards from Γ, after a number of conversions of the form **[tens]**, **[neg]** or **[struct]**, a contraction (**[par]**-step) splits the sole component into two parts, replacing one edge by a redex. The par link of this redex now serves as a boundary between the two new components; all next conversions of the form **[tens]**, **[neg]** or **[struct]** take place completely within one of the two components, and the same holds for the next contraction. This shows that we can reorder our original conversion sequence as desired, in which l_1 (and l) remains untouched until α. So our conversion sequence has the following form:



We have used the fact that it is no restriction either to assume that the next conversion step after α is L⊸. Executing the cut reduction step yields the proof structure $\mathcal{P}'$ to which we can apply β followed by γ:



Observe that l_2 plays the role of l'_1 in γ. ////

This in turn enables us to prove the next theorem.

Theorem 5.5.2. *(a) Let a* $\boldsymbol{CNL}_2$*-sequent* Γ *be given. Then* Γ *is* $\boldsymbol{CNL}_{2,\mathcal{R}}$*-derivable if and only if the set*

$$\{\mathcal{P} \mid \mathcal{P} \textit{ is an } \eta\textit{-expanded cut-free proof structure with the same open ends as } \Gamma\}$$

contains a proof structure $\rightarrow_{\mathcal{R}}$*-converting to* Γ.

(b) Let a cyclic list Δ *of open ends be given (i.e. an* $\boldsymbol{NCLL}_2$*-sequent). Then some cyclic tree with the same cyclic list of open ends as* Δ *is* $\boldsymbol{CNL}_{2,\mathcal{R}}$*-derivable if and only if the set*

$$\{\mathcal{P} \mid \mathcal{P} \textit{ is an } \eta\textit{-expanded cut-free proof structure with the same open ends as } \Delta\}$$

contains a proof structure $\rightarrow_{\mathcal{R}}$*-converting to a sequent with the same cyclic list of open ends as* Δ.

(c) Let a set of open ends be given, i.e. a multiset of hypotheses $A_0^-, \ldots, A_{m-1}^-$ *and a multiset of conclusions* $B_0^+, \ldots, B_{n-1}^+$*. Then some cyclic tree consisting of these open ends is* $\boldsymbol{CNL}_{2,\mathcal{R}}$*-derivable if and only if the set*

$$\big\{\mathcal{P} \mid \mathcal{P} \textit{ is an } \eta\textit{-expanded cut-free proof structure of } A_0^-, \ldots, A_{m-1}^- \vdash B_0^+, \ldots, B_{n-1}^+\big\}$$

contains a $\boldsymbol{CNL}_{2,\mathcal{R}}$*-proof net: a proof structure* $\rightarrow_{\mathcal{R}}$*-converting to a sequent.*

◊

5.6. Dualizable proof nets

Let us define the following relations $\overset{\textbf{CNL}}{\dashv\vdash}$ and $\overset{\textbf{CNL}}{\dashv_{\mathrm{d}}\vdash}$ on $\mathfrak{L}_{2,}$:

$$\begin{array}{lll}
A \overset{\textbf{CNL}}{\dashv\vdash} B & :\Longleftrightarrow & A \vdash B \text{ is } \mathbf{CNL}_2\text{-derivable} \quad \text{and} \quad B \vdash A \text{ is } \mathbf{CNL}_2\text{-derivable} \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded } \mathbf{CNL}_2\text{-proof net } \mathcal{P}_1 \text{ of } A \vdash B \\
 & & \quad \text{and a cut-free and } \eta\text{-expanded } \mathbf{CNL}_2\text{-proof net } \mathcal{P}_2 \text{ of } B \vdash A \\
A \overset{\textbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B & :\Longleftrightarrow & \text{there is a } \mathbf{CNL}_2\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is a } \mathbf{CNL}_2\text{-proof net of } B \vdash A \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded } \mathbf{CNL}_2\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is a } \mathbf{CNL}_2\text{-proof net of } B \vdash A
\end{array}$$

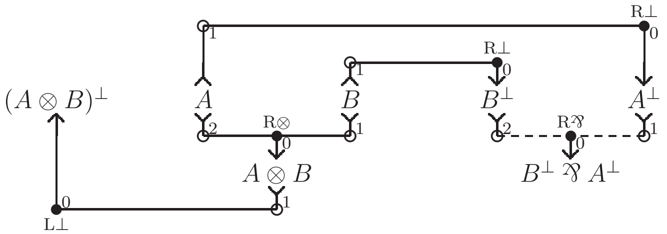
Theorem 5.6.1. *For all* $\mathfrak{L}_2$*-formulas* A *and* B *the following holds:*

$$A \equiv B \qquad \textit{if and only if} \qquad A \overset{\boldsymbol{CNL}}{\dashv_{\mathrm{d}}\vdash} B$$

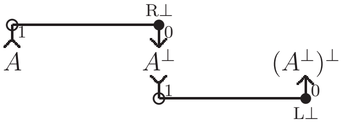
◊

Proof: $\boxed{\Longrightarrow}$ Similar to Lemma 4.6.1.1. The relation $\overset{\textbf{CNL}}{\dashv_{\mathrm{d}}\vdash} := \left\{ (A, B) \mid A \overset{\textbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B \right\}$ is an equivalence relation satisfying (where the numbers refer to Definition 2.2.1):

- (0□), by pasting the dual links L□ and R□ to the given dualizable proof net(s) ($\Box = \otimes, \multimapdotinv, \multimap, \invamp$ or $\bot$);
- (1), by the following dualizable proof net



- (2); similarly;
- (3a); by the dual proof nets of Example 3.2.8;
- (3b); similarly;
- (4), by the dualizable proof net



As $\equiv$ is the smallest such equivalence relation, we must have that $\equiv\ \subseteq\ \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash}$, i.e. if $A \equiv B$ then $A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B$.

$\boxed{\Longleftarrow}$ The proof resembles that of Theorem 4.6.3. We first show this direction for $\mathfrak{L}_{2,\mathrm{nf}}$-formulas (the De Morgan normal forms):

$$\mathfrak{L}_{2,\mathrm{nf}} := \mathcal{F} ::= \mathcal{A} \mid (\mathcal{A})^{\perp} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \parr \mathcal{F}$$

Actually, we will prove something stronger: for all $A, B \in \mathfrak{L}_{2,\mathrm{nf}}$, if $A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B$, then $A = B$. This will be done by induction on the size of a cut-free and η-expanded dualizable $\mathbf{CNL}_2$-proof net of $A \vdash B$.

Suppose $A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B$, in which A and B are $\mathfrak{L}_{2,\mathrm{nf}}$-formulas. By Lemma 5.3.3 we know also $A \dashv_{\mathrm{d}}\vdash B$. Let $\mathcal{P}$ be a cut-free and η-expanded dualizable proof net of $A \vdash B$. Then we know by Proposition 3.2.9 that $\mathcal{P}$ is the union of T^A and T_B containing only L⊗-, R⊗-, L⅋- and R⅋-links, or ⊥-links applied to atoms, followed by an identification of the atomic formulas, which is pairwise by Lemma 4.6.2 (since $A \dashv_{\mathrm{d}}\vdash B$). If $(\alpha)^{\perp}$ is a subformula of A, then α is an hypothesis of T^A. Hence it is a conclusion of T_B, yielding that $(\alpha)^{\perp}$ is a subformula of B. Contracting the two ⊥-links and replacing $(\alpha)^{\perp}$ by the new atom $\alpha^{\perp}$ yields a proof net which moreover is ⊥-free. Hence, let $\mathcal{P}$ be a cut-free and η-expanded **CNL**-dualizable proof net of $A \vdash B$, where $\mathcal{P}$ is the union of T^A and T_B containing only L⊗-, R⊗-, L⅋- and R⅋-links, followed by a pairwise identification of the (new) atomic formulas. Performing the contractions $\mathcal{P} \twoheadrightarrow \Gamma$ in the opposite direction provides us with a *planar* graphical representation of $\mathcal{P}$.

The clusters are exactly the same as in the proof of Theorem 4.6.3. If there are no clusters, we get $A = \alpha = B$. Otherwise there is a cluster $\mathcal{C}$ with only atomic active formulas, which we moreover may suppose to be a tensor cluster. It faces exactly one par cluster $\mathcal{C}_0$, and (which is characteristic for $\mathbf{CNL}_2$) $\mathcal{P}$ can only convert to a sequent when $\mathcal{C}$ and $\mathcal{C}_0$ are dual to each other. So their main formulas C and C_0 are coinciding ⊗-only (⅋-only) $\mathfrak{L}_2$-formulas: $C = C_0$.

Replacing $\mathcal{C}$ and $\mathcal{C}_0$ by a unique new atom α_∞ results in a strictly smaller dualizable proof net $\mathcal{P}'$, yielding $A[\alpha_\infty/C_0] = B[\alpha_\infty/C]$ by induction hypothesis. Backsubstituting C and C_0 (for which $C = C_0$) we get $A[\alpha_\infty/C_0][C_0/\alpha_\infty] = B[\alpha_\infty/C][C/\alpha_\infty]$, i.e. $A = B$.

Now let arbitrary A and B be given for which $A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B$. Then by the $\boxed{\Longrightarrow}$-part (knowing $\nu\pi A \equiv A$) $\nu\pi A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} \nu\pi B$, hence $\nu\pi A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} \nu\pi B$. By the result established above we obtain $\nu\pi A = \nu\pi B$, whence also $A \equiv \nu\pi A = \nu\pi B \equiv B$, i.e. $A \equiv B$.

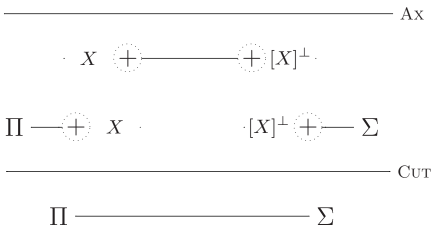
////

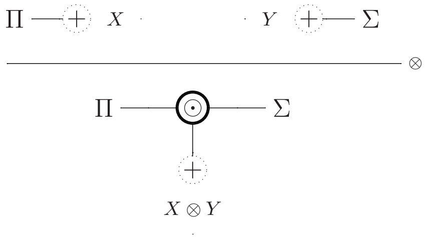
5.7. One-sided nets

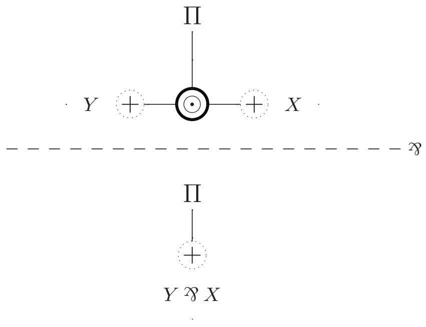
In this section we will use the theory of previous sections to prove a contraction criterion for one-sided **CNL**.

A sequent of $\mathbf{CNL}_1$ is an $\mathfrak{L}_1$-link graph $\mathcal{P}$ containing only cyclic links $(\![e_0, e_1, e_2]\!)_\odot$ of valence 3 as context links, no connector links, and whose underlying graph is acyclic and connected. Moreover, all labels are positively polarized.

The calculus $\mathbf{CNL}_1$ is defined by the following *(elementary) rules*:







The map $\pi^+ : \mathfrak{L}_2^\pm \to \mathfrak{L}_1^\pm$, defined according to Subsection 2.1.4 and Subsection 2.1.5 by

$$\pi^+(A^+) = (\pi A)^+$$
$$\pi^+(A^-) = \big(\pi((A)^\perp)\big)^+ = ([\pi A]^\perp)^+,$$

extended to $\mathbf{CNL}_2$-sequents, yields $\mathbf{CNL}_1$-sequents.

The other way around, the function $\psi^{\pm} : \mathfrak{L}_1^{\pm} \to \mathfrak{L}_2^{\pm}$ mapping X^+ to $\psi X = (X^{\bullet})^{\overline{X}} \in \mathfrak{L}_2^{\pm}$ (and X^- to $\tau\psi X = (X^{\bullet})^{-\overline{X}} \in \mathfrak{L}_2^{\pm}$; see Subsection 2.1.4 and Subsection 2.1.5), extended to $\mathbf{CNL}_1$-sequents, yields $\mathbf{CNL}_2$-sequents.

Lemma 5.7.1. *(a) For all $\boldsymbol{CNL}_2$-sequents Γ the following holds:*
Γ is $\boldsymbol{CNL}_2$-derivable if and only if $\pi^+\Gamma$ is $\boldsymbol{CNL}_1$-derivable.
(b) For all $\boldsymbol{CNL}_1$-sequents Π the following holds:
Π is $\boldsymbol{CNL}_1$-derivable if and only if $\psi^{\pm}\Pi$ is $\boldsymbol{CNL}_2$-derivable.

$\Diamond$

Proof: Similar to Lemma 4.7.1(b).

(a) For the 'if'-part, observe that for $A^{\rho} \in \Gamma$ and $B^{\sigma} := \psi^{\pm}\pi^{+}A^{\rho} = \psi\pi A^{\rho}$ it holds that $\mu A^{\rho} \equiv \mu B^{\sigma}$, from which it follows that $\mu A^{\rho} \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} \mu B^{\sigma}$ by Theorem 5.6.1, entailing $\mu A^{\rho} \overset{\mathbf{CNL}}{\dashv\vdash} \mu B^{\sigma}$. Hence both

and

are derivable.

(b) Similar to that of Lemma 4.7.1. ////

Lemma 5.7.1(b) completely answers the derivability question for $\mathbf{CNL}_1$ in terms of $\mathbf{CNL}_2$-derivability, for which we have established a contraction criterion in Section 5.3. We will now sketch a completely analogue contraction criterion for $\mathbf{CNL}_1$.

First, a $\mathbf{CNL}_1$-*proof net* is an $\mathfrak{L}_1$-proof structure (see Subsection 3.2.1) that can be obtained as the (one-sided) proof structure $\mathcal{P}_1(\mathcal{D})$ of a $\mathbf{CNL}_1$-derivation $\mathcal{D}$ (cf. Definition 4.3.2). Observe that $\mathcal{P}_1(\mathcal{D})$ equals the π-image (see Subsection 3.2.5) of the two-sided proof net $\mathcal{P}(\psi^{\pm}\mathcal{D})$ of a corresponding two-sided derivation $\psi^{\pm}\mathcal{D}$:

$$\pi\mathcal{P}(\psi^{\pm}\mathcal{D}) = \mathcal{P}_1(\mathcal{D}).$$

More general, given a two-sided derivation $\mathcal{D}_2$, then

$$\pi\mathcal{P}(\mathcal{D}_2) = \mathcal{P}_1(\pi^{+}\mathcal{D}_2),$$

whence

$$\pi\mathcal{P}(\psi^{\pm}\mathcal{D}) = \mathcal{P}_1(\pi^{+}\psi^{\pm}\mathcal{D}) = \mathcal{P}_1(\mathcal{D})$$

where the last equality is a result of the fact that $\pi^{+}\psi^{\pm}\mathcal{D}$ and $\mathcal{D}$ are equal up to the order of the premiss sequents of some Cut rules.

We define a conversion relation on the collection $\mathfrak{LG}_1^3$ of $\mathfrak{L}_1$-link graphs with well-labeled (see Definition 3.2.1) connector links, viz. tensor links $(e_0, e_1, e_2)_{\otimes}$ (indicated by $\otimes$) and par links $(e_0, e_1, e_2)_{\parr}$ (indicated by $\parr$), and with context links $[\![e_0, e_1, e_2]\!]_{\odot}$ (indicated by $\odot$) of valence 3, whose open ends are positively polarized. Up to the labeling, we define the conversion steps in exactly the same way as in Section 5.3. The translations $\psi : \mathfrak{PS}_1 \to \mathfrak{PS}_2 : \mathcal{P} \mapsto \psi\mathcal{P}$ and $\pi : \mathfrak{PS}_2 \to \mathfrak{PS}_1 : \mathcal{P} \mapsto \pi\mathcal{P}$ of Subsection 3.2.5 generalize to maps $\mathfrak{LG}_1^3 \to \mathfrak{LG}_2^3$ respectively $\mathfrak{LG}_2^3 \to \mathfrak{LG}_1^3$ in a straightforward way, which also extend $\psi^{\pm}$ and π^{+} on the respective collections of **CNL**-sequents.

$$\begin{tikzcd}
\mathfrak{PS}_1 \arrow[r, tail, "\psi"] \arrow[d, hook] & \mathfrak{PS}_2 \arrow[r, two heads, "\pi"] \arrow[d, hook] & \mathfrak{PS}_1 \arrow[d, hook] \\
\mathfrak{LG}_1^3 \arrow[r, tail, "\psi"] & \mathfrak{LG}_2^3 \arrow[r, two heads, "\pi"] & \mathfrak{LG}_1^3 \\
\mathfrak{CTrees}_1 \arrow[r, tail, "\psi^{\pm}"] \arrow[u, hook] & \mathfrak{CTrees}_2 \arrow[r, two heads, "\pi^{+}"] \arrow[u, hook] & \mathfrak{CTrees}_1 \arrow[u, hook]
\end{tikzcd}$$

We establish the one-sided counterpart of Theorem 5.3.13.

THEOREM 5.7.2. *(a) Let $\mathcal{D}$ be a $\boldsymbol{CNL}_1$-derivation. Then $\mathcal{P}_1(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$.*
(b) Let $\mathcal{P} \twoheadrightarrow \Pi$ be a conversion sequence from an $\mathfrak{L}_1$-proof structure to a $\boldsymbol{CNL}_1$-sequent. Then there is a $\boldsymbol{CNL}_1$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}_1(\mathcal{D})$ and $\Pi = \llcorner\mathcal{D}\lrcorner$.

◇

PROOF: **(a)** Directly, by induction on the derivation $\mathcal{D}$.
(b) Similar to that of Theorem 4.7.2(b): given a conversion sequence

$$\mathcal{P} = \mathcal{P}_m \overset{\delta_m}{\to} \mathcal{P}_{m-1} \overset{\delta_{m-1}}{\to} \dots \overset{\delta_2}{\to} \mathcal{P}_1 \overset{\delta_1}{\to} \mathcal{P}_0 = \Pi$$

on $\mathfrak{LG}_1^3$, we embed it into $\mathfrak{LG}_2^3$. ////

5.8. Intuitionistic fragment

In this section we will study the intuitionistic fragment of $\mathbf{CNL}_2$, which by definition is the sequent calculus whose rules are the same as those of $\mathbf{CNL}_2$, except that we will take only the identity rules AX_A and CUT_A where $A \in \mathfrak{L}_{2,\mathrm{i}}$, and the logical rules L$\square$ and R$\square$ for connectives $\square \in \{\otimes, \multimap, \multimapinv\}$ (see Section 4.2). Derivations turn out to be of a special form, and we call this fragment Non-associative Lambek calculus **NL**. After having seen the theory of proof nets for **NL**, we will establish the analogue of Theorem 5.6.1 for this calculus. Chapter 6 is the continuation of this section, where we generalize in three directions:

- we extend the connectives with unary tensor and par connectives $\Diamond$ and $\Box$;
- we allow connectives of different modes: $\Diamond_j$ and $\Box_j$; $\otimes_i$, $\multimap_i$ and $\multimapinv_i$, where j and i vary over given fixed finite sets of *modes* J respectively I;
- we allow structural rules.

This calculus, because of its special properties, admits its own description of link graphs, conversion steps, etcetera. In the sequel we will establish the connections between the respective notions.

5.8.1. Non-associative Lambek calculus. For any $\mathbf{CNL}_2$-sequent Γ with labeling in $\mathfrak{L}_{2,\mathrm{i}}$, Γ is derivable in $\mathbf{CNL}_2$ if and only if it is derivable in the intuitionistic fragment of $\mathbf{CNL}_2$, where the 'only if'-part is a consequence of the subformula property for $\mathbf{CNL}_2$. Now the resulting derivable sequents of this fragment are easily shown to satisfy the additional property of having only one conclusion: indeed, if each of the 0, 1 or 2 premiss sequents of a rule (different from L$\bot$, R$\bot$, L$⅋$, R$⅋$) has exactly one conclusion, so has the conclusion sequent. Derivations hence only contain such one-conclusion sequents, which observation leads to the so-called Non-associative Lambek calculus **L**.

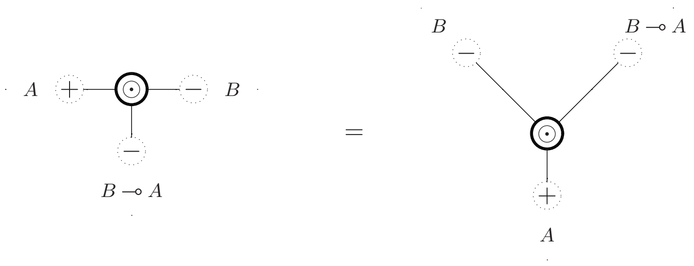
Definition 5.8.1. *An* ***NL****-sequent* $\mathcal{P}$ *is a* $\mathbf{CNL}_2$*-sequent satisfying:*

- $\mathcal{P}$ *is actually an* $\mathfrak{L}_{2,i}$*-link graph;*
- $\mathcal{P}$ *has exactly one conclusion* C^+*.*

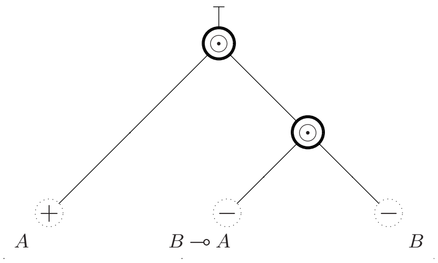
◊

Given an **NL**-sequent, let $\check{\eta}$ be the unique open end C^+. It determines a representative[7] $C^+ \odot \Gamma^-$ where Γ is a rooted binary tree of $\mathfrak{L}_{2,\mathrm{i}}$-formulas, which we also denote by $\Gamma \vdash C$.

Example 5.8.2. Let $\mathcal{P}$ be the following **NL**-sequent ($A, B \in \mathfrak{L}_{2,\mathrm{i}}$):

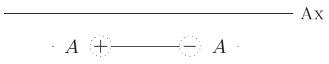


The representative determined by the open end A^+ is



i.e. $A^+ \odot \big((B \multimap A)^- \odot B^-\big)$, i.e. $B \odot (B \multimap A) \vdash A$. ◊

As rules for **NL** we take those instances of the inference rules of $\mathbf{CNL}_2$ where the premiss sequents and the conclusion sequent are **NL**-sequents. We have just seen that $\mathbf{CNL}_2$ is a conservative extension of **NL**: if an **NL**-sequent Γ (considered as a $\mathbf{CNL}_2$-sequent) is derivable in $\mathbf{CNL}_2$, then it is derivable in the intuitionistic fragment of $\mathbf{CNL}_2$, whence in **NL** already.



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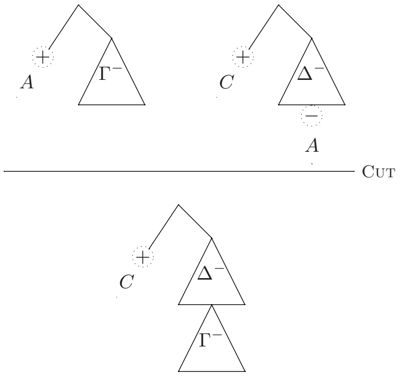
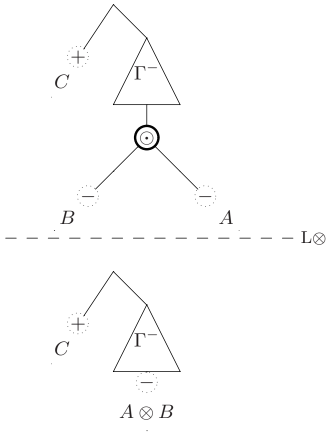
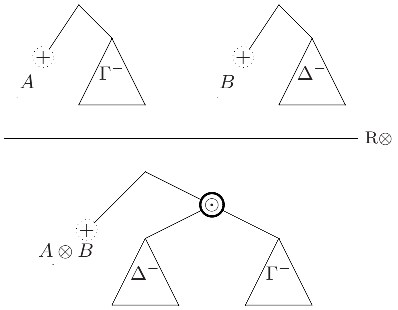
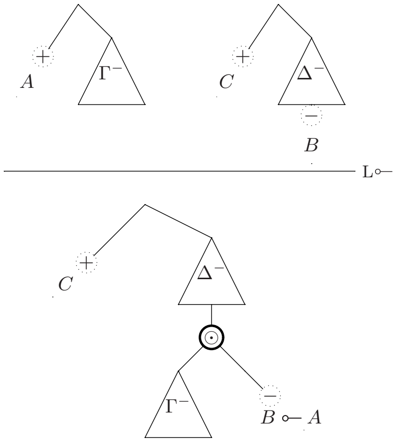
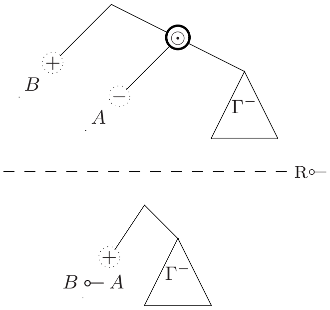
[7] Recall that Γ^- equals Γ, formula-wise provided with a negative sign and *in reversed order* (see Subsection 2.1.4).

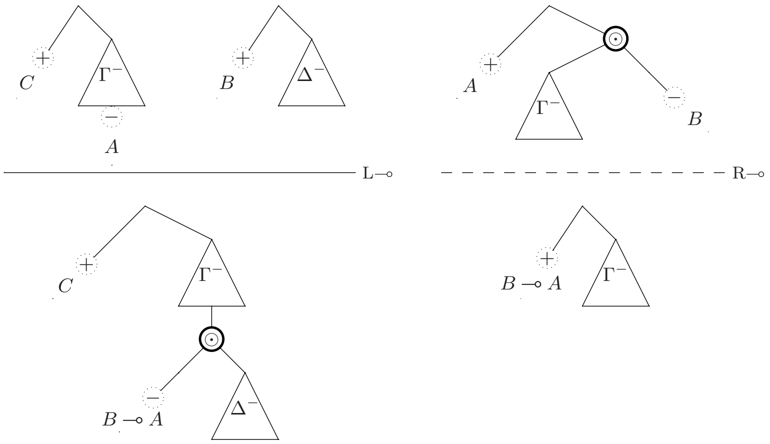

+
A
Γ⁻
+
C
Δ⁻
−
A
Cut
+
C
Δ⁻
Γ⁻


+
C
Γ⁻
⊙
−
B
−
A
L⊗
+
C
Γ⁻
−
A ⊗ B


+
A
Γ⁻
+
B
Δ⁻
R⊗
⊙
+
A ⊗ B
Δ⁻
Γ⁻


+
A
Γ⁻
+
C
Δ⁻
−
B
L⊸
+
C
Δ⁻
⊙
Γ⁻
−
B ⊸ A


⊙
+
B
−
A
Γ⁻
R⊸
+
B ⊸ A
Γ⁻



We also state the rules formulated in the alternative notation.

NL

$$\frac{}{A \vdash A}\;\text{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Delta[A] \vdash C}{\Delta[\Gamma] \vdash C}\;\text{Cut}$$

$$\begin{array}{c}\Gamma[A \odot B] \vdash C\\ \hdashline \Gamma[A \otimes B] \vdash C\end{array}\;\text{L}\otimes \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma \odot \Delta \vdash A \otimes B}\;\text{R}\otimes$$

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[(B \mathrel{\circ\!\!-} A) \odot \Gamma] \vdash C}\;\text{L}{\circ\!\!-} \qquad \begin{array}{c}\Gamma \odot A \vdash B\\ \hdashline \Gamma \vdash B \mathrel{\circ\!\!-} A\end{array}\;\text{R}{\circ\!\!-}$$

$$\frac{\Gamma[A] \vdash C \qquad \Delta \vdash B}{\Gamma[\Delta \odot (B \multimap A)] \vdash C}\;\text{L}{\multimap} \qquad \begin{array}{c}B \odot \Gamma \vdash A\\ \hdashline \Gamma \vdash B \multimap A\end{array}\;\text{R}{\multimap}$$

5.8.2. Proof nets and contraction criterion. We define an **NL**-*proof net* to be an $\mathfrak{L}_2$-proof structure that can be obtained as the (two-sided) proof structure $\mathcal{P}(\mathcal{D})$ of an **NL**-derivation $\mathcal{D}$ (the latter considered as a $\mathbf{CNL}_2$-derivation). It is obvious that an **NL**-proof net actually is an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure (see Definition 3.2.6).

The conversion steps of Section 5.3 are well-defined on the restriction to the intuitionistic labeled elements of $\mathfrak{LG}_2^3$. The next two lemmas are a direct consequence of Theorem 5.3.13(b).

Lemma 5.8.3. *Let $\mathcal{P} \twoheadrightarrow \Gamma$ be a conversion sequence from an $\mathfrak{L}_{2,i}$-proof structure to a $\boldsymbol{CNL}_2$-sequent. Then there is an **NL**-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$, while Γ is an **NL**-sequent.* ◊

Proof: If $\mathcal{P} \twoheadrightarrow \Gamma$ is a conversion sequence from an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure to a $\mathbf{CNL}_2$-sequent, then by Theorem 5.3.13(b) there is a $\mathbf{CNL}_2$-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$. Now all formulas occurring in $\mathcal{D}$ are contained in $\mathcal{P}$ which is an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure, whence $\mathcal{D}$ belongs to the intuitionistic fragment of $\mathbf{CNL}_2$, and hence to $\mathbf{NL}$. Of course, $\Gamma = \llcorner\mathcal{D}\lrcorner$ is actually an $\mathbf{NL}$-sequent. ////

Lemma 5.8.4. *Let $\mathcal{P} \twoheadrightarrow \Gamma$ be a conversion sequence from an $\mathfrak{L}_2$-proof structure to an **NL**-sequent. Then there is an **NL**-derivation $\mathcal{D}$ with $\Gamma = \llcorner\mathcal{D}\lrcorner$.* $\Diamond$

Proof: If $\mathcal{P} \twoheadrightarrow \Gamma$ is a conversion sequence from an $\mathfrak{L}_2$-proof structure to an $\mathbf{NL}$-sequent, then by Theorem 5.3.13(b) there is a $\mathbf{CNL}_2$-derivation $\mathcal{D}'$ with $\mathcal{P} = \mathcal{P}(\mathcal{D}')$ and $\Gamma = \llcorner\mathcal{D}'\lrcorner$. Now, by conservativity, Γ is also derivable in $\mathbf{NL}$, say by $\mathcal{D}$. (Observe it need not hold that $\mathcal{P} = \mathcal{P}(\mathcal{D})$.) ////

As a corollary, we find the intuitionistic counterpart of Theorem 5.3.13:

Theorem 5.8.5. *(a) Let $\mathcal{D}$ be an **NL**-derivation. Then $\mathcal{P}(\mathcal{D}) \twoheadrightarrow \llcorner\mathcal{D}\lrcorner$.*
*(b) Let $\mathcal{P} \twoheadrightarrow \Gamma$ be a conversion sequence from an $\mathfrak{L}_{2,i}$-proof structure to an **NL**-sequent. Then there is an **NL**-derivation $\mathcal{D}$ with $\mathcal{P} = \mathcal{P}(\mathcal{D})$ and $\Gamma = \llcorner\mathcal{D}\lrcorner$.*

$\Diamond$

5.8.3. Dualizable NL-proof nets. Let us define the following relations $\overset{\mathbf{NL}}{\dashv\vdash}$ and $\overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash}$ on $\mathfrak{L}_{2,\mathrm{i}}$:

$$\begin{array}{lll}
A \overset{\mathbf{NL}}{\dashv\vdash} B & :\Longleftrightarrow & A \vdash B \text{ is } \mathbf{NL}\text{-derivable} \quad \text{and} \quad B \vdash A \text{ is } \mathbf{NL}\text{-derivable} \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded } \mathbf{NL}\text{-proof net } \mathcal{P}_1 \text{ of } A \vdash B \\
 & & \quad \text{and a cut-free and } \eta\text{-expanded } \mathbf{NL}\text{-proof net } \mathcal{P}_2 \text{ of } B \vdash A \\
A \overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash} B & :\Longleftrightarrow & \text{there is an } \mathbf{NL}\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is an } \mathbf{NL}\text{-proof net of } B \vdash A \\
 & \Longleftrightarrow & \text{there is a cut-free and } \eta\text{-expanded } \mathbf{NL}\text{-proof net } \mathcal{P} \text{ of } A \vdash B \text{ such that} \\
 & & \quad \text{its dualization } \mathcal{P}^* \text{ is an } \mathbf{NL}\text{-proof net of } B \vdash A
\end{array}$$

As cut-free and η-expanded $\mathbf{CNL}_2$-proof nets with $\mathfrak{L}_{2,\mathrm{i}}$-labeled open ends are automatically $\mathbf{NL}$-proof nets, we see that $\overset{\mathbf{NL}}{\dashv\vdash}$ and $\overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash}$ are just the restrictions of $\overset{\mathbf{CNL}}{\dashv\vdash}$ and $\overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash}$ (defined in Section 5.6). Also, by Proposition 2.2.6, the restriction of $\equiv$ to $\mathfrak{L}_{2,\mathrm{i}}$ is equality ($=$). Hence Theorem 5.6.1 instantaneously leads to the following theorem, of which we will also give a direct proof, the latter alternatively proving Proposition 2.2.6.

Theorem 5.8.6. *For all $\mathfrak{L}_{2,i}$-formulas A and B the following holds:*

$$A = B \qquad \textit{if and only if} \qquad A \overset{\boldsymbol{NL}}{\dashv_{\mathrm{d}}\vdash} B$$

$\Diamond$

$$\begin{tikzcd}
A = B \arrow[r, Leftrightarrow, dashed, "\text{Theorem 5.8.6}"] \arrow[d, Leftrightarrow, dashed, "\text{Proposition 2.2.6}"'] & A \overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash} B \arrow[d, Leftrightarrow] \\
A \equiv B \arrow[r, Leftrightarrow, "\text{special case of} \\ \text{Theorem 5.6.1}"] & A \overset{\mathbf{CNL}}{\dashv_{\mathrm{d}}\vdash} B
\end{tikzcd}$$

Proof: The proof will be independent of Proposition 2.2.6.

$\boxed{\Longrightarrow}$ The relation $\overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash} := \left\{ (A, B) \in \mathfrak{L}_{2,\mathrm{i}} \times \mathfrak{L}_{2,\mathrm{i}} \mid A \overset{\mathbf{NL}}{\dashv_{\mathrm{d}}\vdash} B \right\}$ is an equivalence relation, and therefore reflexive.

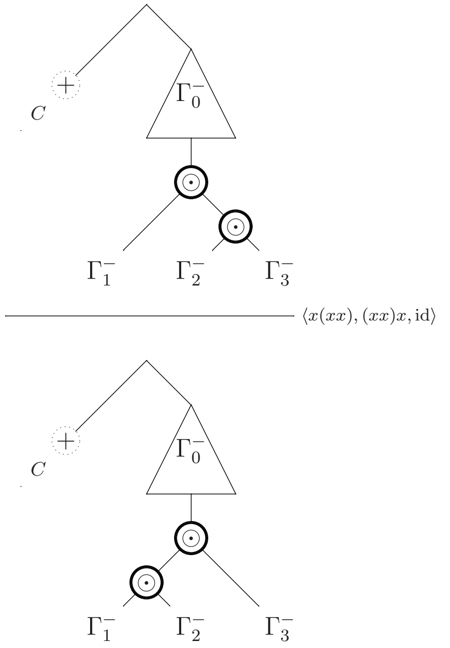
$\boxed{\Longleftarrow}$ Similar to the proof of Theorem 4.9.6: let $\mathcal{P}$ be a cut-free and η-expanded dualizable **NL**-proof net of $A \vdash B$. Then we know by Proposition 3.2.9 that $\mathcal{P}$ is the union of T^A and T_B containing only $\otimes$-, $\multimapinv$- and $\multimap$-links, followed by an identification of the atomic formulas, which is pairwise by Lemma 4.6.2 (since $A \dashv_{\mathrm{d}}\vdash B$). Performing the contraction $\mathcal{P} \twoheadrightarrow \Gamma$ in the opposite direction provides us with a **planar** graphical representation of $\mathcal{P}$.

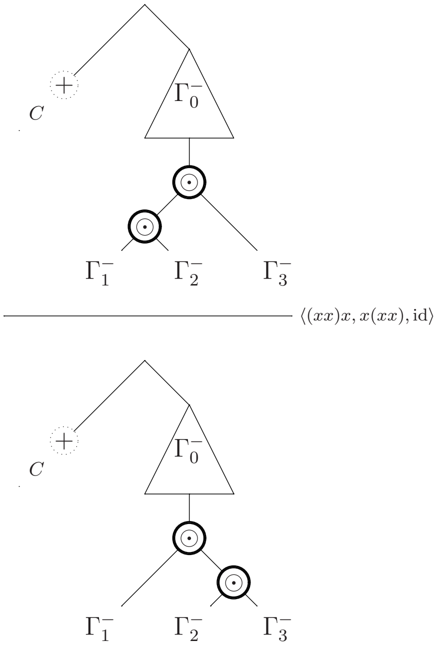
The clusters are exactly the same as in the proof of Theorem 4.9.6. If there are no clusters, we get $A = \alpha = B$. Otherwise there is a cluster $\mathcal{C}$ with only atomic active formulas, which moreover we may suppose to be a tensor cluster. Hence this cluster is a generalized R$\otimes$-link or a generalized L$\multimap\!\multimapinv$-link. It faces exactly one par cluster $\mathcal{C}_0$ (which hence is a generalized L$\otimes$-link respectively a generalized R$\multimap\!\multimapinv$-link), and $\mathcal{C}$ and $\mathcal{C}_0$ must be dual to each other. So their main formulas C and C_0 coincide. The result follows by induction. ////

5.8.4. Adding structural rules. Since **NL**-sequents have a distinguished conclusion, a fixed n-ary structural rule in general has n distinguishable appearances, depending on the open end which points towards the conclusion. I.e. every application of the structural rule in an **NL**-derivation uniquely determines a representative of this structural rule: if, say, Γ_0 of the upper sequent contains the unique conclusion, so does Γ_0 of the lower sequent, and we can declare each of the open ends of $\mathcal{P}$ and $\mathcal{P}'$ corresponding to Γ_0 as the root ($\pi_0 = 0$).

We can refine the calculus by adding these representatives separately and independently to the calculus. In this way, we can add left associativity without adding right associativity.

Example 5.8.7. In Example 5.4.1 we have given two representatives of Ass with $\pi_0 = 0$. They yield the following distinct structural rules with which we can extend **NL**.



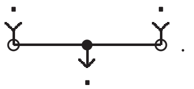


In our alternative notation they read

$$\frac{\Gamma_0[(\Gamma_3 \odot \Gamma_2) \odot \Gamma_1] \ \vdash\ C}{\Gamma_0[\Gamma_3 \odot (\Gamma_2 \odot \Gamma_1)] \ \vdash\ C}\,\langle x(xx), (xx)x, \mathrm{id}\rangle \qquad \frac{\Gamma_0[\Gamma_3 \odot (\Gamma_2 \odot \Gamma_1)] \ \vdash\ C}{\Gamma_0[(\Gamma_3 \odot \Gamma_2) \odot \Gamma_1] \ \vdash\ C}\,\langle (xx)x, x(xx), \mathrm{id}\rangle$$

which we will call RAss and LAss respectively. ◊

It is no restriction to perform the [**tens**]-steps before the [**par**]- and [**struct**]-steps. This is actually what happens in Chapter 6, where in Theorem 6.3.7 the condition states that the *underlying hypothesis structure* $\widehat{\mathcal{P}}$ of a proof structure $\mathcal{P}$ converts to a so-called *hypothesis tree*, i.e. a sequent. As all tensor links R⊗, L∘— and L—∘ have two premisses and one conclusion, it does no harm to depict the context links of intermediate link graphs in a conversion $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ ($\mathcal{P}$ an $\mathfrak{L}_{2,\mathrm{i}}$-proof structure) by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CHAPTER 6

Proof nets for the Multimodal Lambek Calculus

Since the introduction of proof nets as an elegant proof theory for the multiplicative fragment of linear logic in [**Girard 87**], a number of attempts have been made to adapt this proof theory to a variety of Lambek Calculi, as shown by work from e.g. [**Roorda 91**], [**Morrill 96**] and [**Moortgat 97**].

In this chapter we will present a new way to look at proof nets for the multimodal Lambek Calculus. We will show how we can uniformly handle both the unary and the binary connectives and how we have a natural correctness criterion for the *base logic* $\mathbf{NL}\diamond$ together with a set $\mathcal{R}$ of structural rules subject to a linearity condition.

First, we introduce proof structures in a way similar to Chapter 3. Then we will look at slightly more abstract graphs, which we will call hypothesis structures, and on which we will formulate a correctness criterion in the form of graph conversions. Proof nets will be those proof structures of which the hypothesis structure converts to a tree.

As our main result we will prove our proof net calculus is sound and complete with respect to the sequent calculus. In the following sections we will sketch a proof of cut elimination and show applications of our calculus to automated deduction.

The formalism we present here is related to a number of other proposals, notably to Danos' graph contractions [**Danos 90**], of which our contractions are a special case. As a result acyclicity and connectedness are a consequence of our correctness criterion.

Our approach is also related to the labeled proof nets of [**Moortgat 97**]. Our hypothesis structures correspond closely to the labels Moortgat assigns to proof nets. Advantages of our formalism are that we have a very direct correspondence between proof structures and their hypothesis structures, and that we can handle cyclic or disconnected proof structures unproblematically, whereas to acyclic and connected proof structures only can be assigned a meaningful label.

An open question concerns some natural notion of equality on conversion sequences. As defined now, a number of uninteresting permutations are possible in conversion sequences, which is somewhat against the spirit of proof nets as 'sequent proofs modulo permutations of inferences'.

This chapter consists of joint work with Richard Moot.

6.1. Structure Trees

Starting from a set of atoms $\{p_1, p_2, \ldots\}$, the *formulas* of the multimodal Lambek Calculus with $\diamond$ ($\mathbf{NL}\diamond$) are built up with the unary connectives $\diamond_j$ and $\Box_j$ and with the binary connectives[1] $\otimes_i$, $\multimap_i$ and $\multimapinv_i$, where j and i vary over given fixed finite sets of *modes* J respectively I.

Structure trees are built up from formulas with unary constructors $\langle - \rangle^j$ and binary constructors $- \odot_i -$, where again j and i vary over the modes. Derivable objects are

---

[1] The linguistic notation reads $\bullet_i$, $\backslash_i$ and $/_i$ for the respective binary connectives.

sequents $\Gamma \vdash C$ in which the antecedent part is a structure tree and in which the succedent part is a formula.

Given sets J and I of unary respectively binary modes, we define the set of *structure trees with holes over a set* S as follows:

$$\begin{aligned} \mathsf{tree}_\mathsf{S} ::= {} & \mathsf{S} \cup \{\, [\,] \,\} \\ \mid\ & \langle\, \mathsf{tree}_\mathsf{S} \,\rangle^J \\ \mid\ & \mathsf{tree}_\mathsf{S} \odot_I \mathsf{tree}_\mathsf{S} \end{aligned}$$

The *length* $\lambda(\Xi)$ and the *number of holes* $\kappa(\Xi)$ of such a tree Ξ is defined by

$$\begin{aligned} \lambda(A) &:= 1 \qquad (A \in \mathsf{S}) & \kappa(A) &:= 0 \qquad (A \in \mathsf{S}) \\ \lambda([\,]) &:= 1 & \kappa([\,]) &:= 1 \\ \lambda(\langle\, \Xi_1 \,\rangle^j) &:= \lambda(\Xi_1) & \kappa(\langle\, \Xi_1 \,\rangle^j) &:= \kappa(\Xi_1) \\ \lambda(\Xi_1 \odot_i \Xi_2) &:= \lambda(\Xi_1) + \lambda(\Xi_2) & \kappa(\Xi_1 \odot_i \Xi_2) &:= \kappa(\Xi_1) + \kappa(\Xi_2) \end{aligned}$$

Let $\mathsf{tree}_\mathsf{S}^{\lambda,\kappa}$ be the subset of tree_S consisting of trees with length λ and number of holes κ. Observe that $\mathsf{S} \subset \mathsf{tree}_\mathsf{S}^{1,0}$.

For every $\Xi \in \mathsf{tree}_\mathsf{S}^{\lambda,\kappa}$ there is a multiset $\|\Xi\|$ of $\lambda - \kappa$ elements of S which equals Ξ modulo structural information and holes. Moreover, we define $\langle\!\langle \Xi \rangle\!\rangle$ to be the order in which the elements of $\|\Xi\|$ occur in Ξ.

$$\begin{aligned} \|A\| &:= \{A\} \qquad (A \in \mathsf{S}) & \langle\!\langle A \rangle\!\rangle &:= A \qquad (A \in \mathsf{S}) \\ \|[\,]\| &:= \emptyset & \langle\!\langle [\,] \rangle\!\rangle &:= \emptyset \\ \big\|\langle\, \Xi_1 \,\rangle^j\big\| &:= \|\Xi_1\| & \big\langle\!\big\langle \langle\, \Xi_1 \,\rangle^j \big\rangle\!\big\rangle &:= \langle\!\langle \Xi_1 \rangle\!\rangle \\ \|\Xi_1 \odot_i \Xi_2\| &:= \|\Xi_1\| \cup \|\Xi_2\| & \langle\!\langle \Xi_1 \odot_i \Xi_2 \rangle\!\rangle &:= \langle\!\langle \Xi_1 \rangle\!\rangle\,, \langle\!\langle \Xi_2 \rangle\!\rangle \end{aligned}$$

Example 6.1.1.

$$\begin{gathered} \langle\, A_1 \odot_1 \langle\, A_2 \,\rangle^1 \,\rangle^1 \odot_2 (A_1 \odot_1 [\,]) \in \mathsf{tree}_\mathsf{S}^{4,1} \\ \big\|\langle\, A_1 \odot_1 \langle\, A_2 \,\rangle^1 \,\rangle^1 \odot_2 (A_1 \odot_1 [\,])\big\| = \{A_1, A_1, A_2\} \\ \langle\, [\,] \,\rangle^1 \odot_1 \big(\langle\, [\,] \,\rangle^1 \odot_1 (\langle\, [\,] \,\rangle^1 \odot_2 [\,])\big) \in \mathsf{tree}_\mathsf{S}^{4,4} \\ \big|\big\{ \Xi \in \mathsf{tree}_\mathsf{S}^{7,7} \,\big|\, \Xi \text{ is } \langle\ \rangle^j\text{-free} \big\}\big| = 132\ |I|^6 \end{gathered}$$

◊

The *dual* Ξ^* of a tree Ξ is defined by

$$\begin{aligned} (A)^* &:= A \qquad (A \in \mathsf{S}) \\ ([\,])^* &:= [\,] \\ (\langle\, \Xi_1 \,\rangle^j)^* &:= \langle\, \Xi_1^* \,\rangle^j \\ (\Xi_1 \odot_i \Xi_2)^* &:= \Xi_2^* \odot_i \Xi_1^* \end{aligned}$$

which is actually Ξ, in reversed order.

There is a substitution operation

$$\begin{gathered} \mathsf{tree}_\mathsf{S}^{\lambda,\kappa} \times \mathsf{tree}_\mathsf{S}^{l_1,k_1} \times \ldots \times \mathsf{tree}_\mathsf{S}^{l_\kappa,k_\kappa} \longrightarrow \mathsf{tree}_\mathsf{S}^{\lambda-\kappa+\sum_{j=1}^{\kappa} l_j\ ,\ \sum_{j=1}^{\kappa} k_j} \\ \big(\Xi, \Theta_1, \ldots, \Theta_\kappa\big) \mapsto \Xi[\ \Theta_1, \ldots, \Theta_\kappa\ ] \end{gathered}$$

which can be defined by induction on Ξ:

- if $\Xi \in \mathsf{S}$, then $\lambda = 1; \kappa = 0$, and we define the image of Ξ to be Ξ itself;
- if $\Xi = [\,]$, then $\lambda = 1; \kappa = 1$, and we define the image of (Ξ, Θ_1) to be $\Xi[\ \Theta_1\ ] := \Theta_1$;
- if $\Xi = \langle\, \Xi_1 \,\rangle^j$, then $\lambda = \lambda_1$ and $\kappa = \kappa_1$. By induction hypothesis we know that $\Xi_1[\ \Theta_1, \ldots, \Theta_{\kappa_1}\ ]$ is defined and belongs to $\mathsf{tree}_{\mathsf{S}}^{\lambda_1 - \kappa_1 + \sum_{j=1}^{\kappa_1} l_j \,,\, \sum_{j=1}^{\kappa_1} k_j}$. Now we define the image of $\big(\Xi, \Theta_1, \ldots, \Theta_\kappa\big)$ to be
$$\Xi[\ \Theta_1, \ldots, \Theta_\kappa\ ] := \langle\, \Xi_1[\ \Theta_1, \ldots, \Theta_{\kappa_1}\ ] \,\rangle^j$$
which belongs to
$$\mathsf{tree}_{\mathsf{S}}^{\lambda - \kappa + \sum_{j=1}^{\kappa} l_j \,,\, \sum_{j=1}^{\kappa} k_j},$$
as desired.
- if $\Xi = \Xi_1 \odot_i \Xi_2$, then $\lambda = \lambda_1 + \lambda_2$ and $\kappa = \kappa_1 + \kappa_2$. By induction hypothesis we know that $\Xi_1[\ \Theta_1, \ldots, \Theta_{\kappa_1}\ ]$ is defined and belongs to $\mathsf{tree}_{\mathsf{S}}^{\lambda_1 - \kappa_1 + \sum_{j=1}^{\kappa_1} l_j \,,\, \sum_{j=1}^{\kappa_1} k_j}$ while $\Xi_2[\ \Theta_{\kappa_1+1}, \ldots, \Theta_{\kappa_1+\kappa_2}\ ]$ is defined and belongs to $\mathsf{tree}_{\mathsf{S}}^{\lambda_2 - \kappa_2 + \sum_{j=\kappa_1+1}^{\kappa_1+\kappa_2} l_j \,,\, \sum_{j=\kappa_1+1}^{\kappa_1+\kappa_2} k_j}$. Now we define the image of $\big(\Xi, \Theta_1, \ldots, \Theta_\kappa\big)$ to be
$$\Xi[\ \Theta_1, \ldots, \Theta_\kappa\ ] := \Xi_1[\ \Theta_1, \ldots, \Theta_{\kappa_1}\ ] \odot_i \Xi_2[\ \Theta_{\kappa_1+1}, \ldots, \Theta_{\kappa_1+\kappa_2}\ ]$$
which belongs to
$$\mathsf{tree}_{\mathsf{S}}^{\lambda_1 - \kappa_1 + \sum_{j=1}^{\kappa_1} l_j + \lambda_2 - \kappa_2 + \sum_{j=\kappa_1+1}^{\kappa_1+\kappa_2} l_j \,,\, \sum_{j=1}^{\kappa_1} k_j + \sum_{j=\kappa_1+1}^{\kappa_1+\kappa_2} k_j} = \mathsf{tree}_{\mathsf{S}}^{\lambda - \kappa + \sum_{j=1}^{\kappa} l_j \,,\, \sum_{j=1}^{\kappa} k_j},$$
as desired.

EXAMPLE 6.1.2.

- Restricting our attention to the case where $\lambda = \kappa$ *(formal trees)*, this map is
$$\mathsf{tree}^{\lambda,\lambda} \times \mathsf{tree}^{l_1,l_1} \times \ldots \times \mathsf{tree}^{l_\lambda,l_\lambda} \to \mathsf{tree}^{\sum_{j=1}^{\lambda} l_j \,,\, \sum_{j=1}^{\lambda} l_j}$$
where we have deleted the subscript S, since these sets do not depend on it.
- Restricting our attention to substitution of elements of S, this map is
$$\mathsf{tree}_{\mathsf{S}}^{\lambda,\kappa} \times \mathsf{S}^\kappa \to \mathsf{tree}_{\mathsf{S}}^{\lambda,0}$$
- Restricting our attention to the case where $\kappa = 0$ *(S-trees)*, the substitution map is the identity $\mathsf{tree}_{\mathsf{S}}^{\lambda,0} \to \mathsf{tree}_{\mathsf{S}}^{\lambda,0}$ (proof by induction); there is nothing to substitute.

◊

LEMMA 6.1.3. *Given* $\big(\Xi, \Theta_1, \ldots, \Theta_\kappa\big) \in \mathsf{tree}_{\mathsf{S}}^{\lambda,\kappa} \times \mathsf{tree}_{\mathsf{S}}^{l_1,k_1} \times \ldots \times \mathsf{tree}_{\mathsf{S}}^{l_\kappa,k_\kappa}$*, the following holds:*
$$\|\Xi[\ \Theta_1, \ldots, \Theta_\kappa\ ]\| = \|\Xi\| \cup \bigcup_{j=1}^{\kappa} \|\Theta_j\|$$
◊

PROOF: By induction on Ξ. ////

We study sequents in which the antecedent part is a structure tree of formulas rather than a sequence or a multiset of formulas.
$$\mathsf{seq} := \big\{\, \Gamma \,\vdash\, C \mid n \geq 1; \Gamma \in \mathsf{tree}_{\mathsf{form}}^{n,0}; C \in \mathsf{form} \,\big\}$$

Atoms will be denoted by $p, q, \ldots$; modes by $i, j, \ldots$; formulas by $A, B, C, \ldots$; and form-trees by $\Gamma, \Delta, \ldots$. Observe that all sequents have non-empty antecedent part, since there are no empty trees.

Convention. Writing down a sequent like $\Delta[\,\Gamma_1, \Gamma_2\,] \vdash C$ implies that $\Delta \in \mathsf{tree}^{n,2}_{\mathsf{form}}$; $\Gamma_1 \in \mathsf{tree}^{n_1,0}_{\mathsf{form}}$ and $\Gamma_2 \in \mathsf{tree}^{n_2,0}_{\mathsf{form}}$, yielding $\Delta[\,\Gamma_1, \Gamma_2\,] \in \mathsf{tree}^{n-2+n_1+n_2,0}_{\mathsf{form}}$

There is a map $\mathsf{tree}^{n,0}_{\mathsf{form}} \to \mathsf{form}$, which replaces all $\langle\ \rangle^j$- and $\odot_i$-occurrences by $\Diamond_j$- and $\otimes_i$-occurrences. The image of Γ under this map will be denoted by $\Gamma^{\otimes}$.

$$\begin{aligned} \big(A\big)^{\otimes} &:= A \qquad (A \in \mathsf{form}) \\ \big(\langle\,\Xi_1\,\rangle^j\big)^{\otimes} &:= \Diamond_j \Xi_1^{\otimes} \\ \big(\Xi_1 \odot_i \Xi_2\big)^{\otimes} &:= \Xi_1^{\otimes} \otimes_i \Xi_2^{\otimes} \end{aligned}$$

6.2. The calculus

An *n-ary structural rule*

$$\frac{\Delta[\,\Xi^*[\,\Gamma_1, \ldots, \Gamma_n\,]\,] \ \vdash\ C}{\Delta[\,(\Xi')^*[\,\Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n}\,]\,] \ \vdash\ C}\ \langle \Xi, \Xi', \pi \rangle$$

is defined by a pair of formal trees[2] Ξ, Ξ' of length n and a rearrangement of the variables $\pi \in \mathfrak{S}_n$, the symmetric group of degree n. Observe that every subtree Γ_k occurs exactly once in both the upper and lower sequent of the inference, whence any non-linear structural rule like

$$\frac{\Delta[\,\Gamma_1 \odot_i \Gamma_1\,] \ \vdash\ C}{\Delta[\,\Gamma_1\,] \ \vdash\ C}\ \textsc{Contraction}_i \qquad \frac{\Delta[\,\Gamma_1\,] \ \vdash\ C}{\Delta[\,\Gamma_2 \odot_i \Gamma_1\,] \ \vdash\ C}\ \textsc{LWeakening}_i$$

$$\frac{\Delta[\,\Gamma_1\,] \ \vdash\ C}{\Delta[\,\Gamma_1 \odot_i \Gamma_2\,] \ \vdash\ C}\ \textsc{RWeakening}_i$$

does not conform to this definition. Neither does the following rule, though it is linear:

---

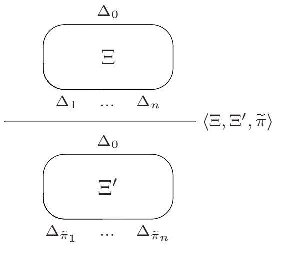
[2] By Ξ^* we mean Ξ in reversed order; hence in $\mathbf{CNL}_2$-notation the rule reads

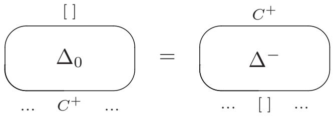
$$\frac{C^+ \odot \big(\Delta[\,\Xi^*[\,\Gamma_1, \ldots, \Gamma_n\,]\,]\big)^-}{C^+ \odot \big(\Delta[\,(\Xi')^*[\,\Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n}\,]\,]\big)^-}\ \langle \Xi, \Xi', \pi \rangle$$

i.e. (cf. Subsection 5.8.4)

$$\frac{C^+ \odot \Delta^-[\,\Xi[\,\Gamma_n^-, \ldots, \Gamma_1^-\,]\,]}{C^+ \odot \Delta^-[\,\Xi'[\,\Gamma_{\pi_n}^-, \ldots, \Gamma_{\pi_1}^-\,]\,]}\ \langle \Xi, \Xi', \pi \rangle$$

which is an instance (viz. with Δ_i equal to Γ^-_{n+1-i} and with Δ_0 equal to Δ^-, up to the choice of the root $[\,]$ respectively C^+) of the $(n+1)$-ary structural $\mathbf{CNL}_2$-rule (see Section 5.4):





Identity rules

$$\frac{}{A \vdash A}\ \textsc{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Delta[A] \vdash C}{\Delta[\Gamma] \vdash C}\ \textsc{Cut}$$

Logical rules for the $\otimes$-like connectives

$$\frac{\Gamma[\langle A \rangle^j] \vdash C}{\Gamma[\Diamond_j A] \vdash C}\ \mathrm{L}\Diamond_j \qquad \frac{\Gamma \vdash A}{\langle \Gamma \rangle^j \vdash \Diamond_j A}\ \mathrm{R}\Diamond_j$$

$$\frac{\Gamma[A \odot_i B] \vdash C}{\Gamma[A \otimes_i B] \vdash C}\ \mathrm{L}\otimes_i \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma \odot_i \Delta \vdash A \otimes_i B}\ \mathrm{R}\otimes_i$$

Logical rules for the $\parr$-like connectives

$$\frac{\Delta[B] \vdash C}{\Delta[\langle \Box_j B \rangle^j] \vdash C}\ \mathrm{L}\Box_j \qquad \frac{\langle \Gamma \rangle^j \vdash B}{\Gamma \vdash \Box_j B}\ \mathrm{R}\Box_j$$

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[\Gamma \odot_i A \multimap_i B] \vdash C}\ \mathrm{L}\multimap_i \qquad \frac{A \odot_i \Gamma \vdash B}{\Gamma \vdash A \multimap_i B}\ \mathrm{R}\multimap_i$$

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[B \multimapinv_i A \odot_i \Gamma] \vdash C}\ \mathrm{L}\multimapinv_i \qquad \frac{\Gamma \odot_i A \vdash B}{\Gamma \vdash B \multimapinv_i A}\ \mathrm{R}\multimapinv_i$$

Structural rules (for all $\langle \Xi, \Xi', \pi \rangle \in \mathcal{R}$)

$$\frac{\Delta[\ \Xi^*[\ \Gamma_1, \ldots, \Gamma_n\ ]\ ] \vdash C}{\Delta[\ (\Xi')^*[\ \Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n}\ ]\ ] \vdash C}\ \langle \Xi, \Xi', \pi \rangle$$

Figure 6.1. The sequent calculus $\mathbf{NL}\Diamond_{\mathcal{R}}$.

$$\frac{\Delta[\ \Gamma_1\ ] \odot_i \Gamma_2 \vdash C}{\Delta[\ \Gamma_2\ ] \odot_i \Gamma_1 \vdash C}$$

However, it may be admissible, depending on $\mathcal{R}$.

Given a set $\mathcal{R}$ of structural rules, we define the sequent calculus $\mathbf{NL}\Diamond_{\mathcal{R}}$ by the inference rules of Figure 6.1. From each n-ary structural rule

$$\frac{\Delta[\ \Xi^*[\ \Gamma_1, \ldots, \Gamma_n\ ]\ ]\ \vdash\ C}{\Delta[\ (\Xi')^*[\ \Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n}\ ]\ ]\ \vdash\ C}\ \langle \Xi, \Xi', \pi \rangle$$

we can derive — for every n-tuple of formulas $A_1, \ldots, A_n$ — the sequent $(\Xi')^*[\ A_{\pi_1}, \ldots, A_{\pi_n}\ ]^{\otimes} \vdash \Xi^*[\ A_1, \ldots, A_n\ ]^{\otimes}$, where by $\Gamma^{\otimes}$ we mean the formula obtained from Γ by replacing all $\langle\ \rangle^j$- and $\odot_i$-occurrences by $\Diamond_j$- and $\otimes_i$-occurrences respectively. This means that the axiom rule

$$\frac{}{(\Xi')^*[\ A_{\pi_1}, \ldots, A_{\pi_n}\ ]^{\otimes}\ \vdash\ \Xi^*[\ A_1, \ldots, A_n\ ]^{\otimes}}$$

is admissible. In fact, adding the structural rule to the calculus is equivalent to adding (all instances of) the corresponding axiom rule to the calculus.

Let $\mathcal{R}_{\max}$ be the following set of structural rules, where j, i and i' vary over the modes:

$$\frac{\Delta[\ \Gamma_1\ ]\ \vdash\ C}{\Delta[\ \langle \Gamma_1 \rangle^j\ ]\ \vdash\ C}\ [\mathrm{LTriv}_j] \qquad \frac{\Delta[\ \langle \Gamma_1 \rangle^j\ ]\ \vdash\ C}{\Delta[\ \Gamma_1\ ]\ \vdash\ C}\ [\mathrm{RTriv}_j]$$

$$\frac{\Delta[\ \Gamma_1 \odot_i \big(\Gamma_2 \odot_i \Gamma_3\big)\ ]\ \vdash\ C}{\Delta[\ \big(\Gamma_1 \odot_i \Gamma_2\big) \odot_i \Gamma_3\ ]\ \vdash\ C}\ [\mathrm{LAss}_i] \qquad \frac{\Delta[\ \big(\Gamma_1 \odot_i \Gamma_2\big) \odot_i \Gamma_3\ ]\ \vdash\ C}{\Delta[\ \Gamma_1 \odot_i \big(\Gamma_2 \odot_i \Gamma_3\big)\ ]\ \vdash\ C}\ [\mathrm{RAss}_i]$$

$$\frac{\Delta[\ \Gamma_1 \odot_i \Gamma_2\ ]\ \vdash\ C}{\Delta[\ \Gamma_2 \odot_i \Gamma_1\ ]\ \vdash\ C}\ [\mathrm{Com}_i] \qquad \frac{\Delta[\ \Gamma_1 \odot_i \Gamma_2\ ]\ \vdash\ C}{\Delta[\ \Gamma_1 \odot_{i'} \Gamma_2\ ]\ \vdash\ C}\ [\mathrm{Eq}_{i,i'}]$$

Lemma 6.2.1. *Any possible structural rule is admissible in* $\boldsymbol{NL}\Diamond_{\mathcal{R}_{max}}$. $\Diamond$

Let $\lceil - \rceil$ be the following translation from $\mathbf{NL}\Diamond$-formulas to $\mathfrak{L}_{2,\mathrm{i}}$-formulas, deleting the unary connectives and the mode indices and identifying both implications:

$$\begin{aligned} \lceil p_k \rceil &:= p_k & \lceil A \otimes_i B \rceil &:= \lceil A \rceil \otimes \lceil B \rceil \\ \lceil \Diamond_j A \rceil &:= \lceil A \rceil & \lceil A \multimap_i B \rceil &:= \lceil A \rceil \multimap \lceil B \rceil \\ \lceil \Box_j A \rceil &:= \lceil A \rceil & \lceil B \multimapinv_i A \rceil &:= \lceil A \rceil \multimap \lceil B \rceil \end{aligned}$$

For any structure tree Γ, let $\|\Gamma\|$ be the multiset of elements in Γ. We write $\lceil \|\Gamma\| \rceil$ for the multiset $\{\, \lceil A \rceil \mid A \in \|\Gamma\| \}$. Let $\mathbf{iMLL}^{>0}$ stand for the calculus $\mathbf{iMLL}$ restricted to the requirement that the antecedent multiset of all sequents in a derivation be non-empty.

Corollary 6.2.2. *The following maps between collections of sequents:*

$$\mathcal{S\varepsilon Q}(\boldsymbol{NL}\Diamond)\ =\ \mathcal{S\varepsilon Q}(\boldsymbol{NL}\Diamond_{\mathcal{R}})\ =\ \mathcal{S\varepsilon Q}(\boldsymbol{NL}\Diamond_{\mathcal{R}_{max}})\ \overset{\lceil\|-\|\rceil}{\to}\ \mathcal{S\varepsilon Q}(\boldsymbol{iMLL}^{>0})\ =\ \mathcal{S\varepsilon Q}(\boldsymbol{iMLL})$$

restrict to the collections of derivable sequents:

$$\mathcal{DS\varepsilon Q}(\boldsymbol{NL}\Diamond)\ \hookrightarrow\ \mathcal{DS\varepsilon Q}(\boldsymbol{NL}\Diamond_{\mathcal{R}})\ \hookrightarrow\ \mathcal{DS\varepsilon Q}(\boldsymbol{NL}\Diamond_{\mathcal{R}_{max}})\ \overset{\lceil\|-\|\rceil}{\twoheadrightarrow}\ \mathcal{DS\varepsilon Q}(\boldsymbol{iMLL}^{>0})\ \hookrightarrow\ \mathcal{DS\varepsilon Q}(\boldsymbol{iMLL})$$

Moreover, $\mathcal{DS\varepsilon Q}\big(\boldsymbol{iMLL}^{>0}\big)$ *is the image of* $\mathcal{DS\varepsilon Q}\big(\boldsymbol{NL}\Diamond_{\mathcal{R}_{max}}\big)$ *under the map* $\lceil\|-\|\rceil$. $\Diamond$

From the previous corollary we conclude that adding structural rules to $\mathbf{NL}\Diamond$ will never move us outside $\mathbf{MLL}$, whence Contraction$_i$ or L/RWeakening$_i$ are never admissible.

Lemma 6.2.3. *The left rules for the* $\otimes$*-like connectives (*$L\Diamond_j$*,* $L\otimes_i$*) and the right rules for the* $\parr$*-like connectives (*$R\Box_j$*,* $R\multimap_i$*,* $R\multimapinv_i$*) are reversible.* $\Diamond$

This is proved by means of their respective counterparts and Cut. The reversibility of $\mathrm{L}\Diamond_j$ and $\mathrm{L}\otimes_i$ means that the role of $\langle\ \rangle^j$ and $\odot_i$ in the antecedent structure trees actually coincides with that of $\Diamond_j$ respectively $\otimes_i$. However, this does not mean we can forget about the constructors, since the occurrence of a formula A as a leaf of a structure tree $\Gamma[A]$ guarantees A occurs positively (and not negatively) in the formula $\Gamma[A]^{\otimes}$, which is needed in order to have meaningful inference rules.

As an immediate consequence of the previous lemma we have

Lemma 6.2.4. *This calculus satisfies the following adjunctions:*

$$\begin{array}{rcll} A\otimes_i(-) & \dashv & A\multimap_i(-) & \textit{(for all formulas } A\textit{)}\\ (-)\otimes_i A & \dashv & (-)\multimapinv_i A & \textit{(for all formulas } A\textit{)}\\ \Diamond_j(-) & \dashv & \Box_j(-) & \end{array}$$

i.e.

$$\begin{array}{c} A\otimes_i B \vdash C \\ \hline\hline B\vdash A\multimap_i C\end{array}\Updownarrow \qquad \begin{array}{c} B\otimes_i A \vdash C \\ \hline\hline B\vdash C\multimapinv_i A\end{array}\Updownarrow \qquad \begin{array}{c} \Diamond_j B \vdash C \\ \hline\hline B\vdash \Box_j C\end{array}\Updownarrow$$

◇

We divide the logical rules in two parts:

- the *tensor rules* are the right rules for the $\otimes$-like connectives and the left rules for the $\parr$-like connectives (R$\Diamond_j$, R$\otimes_i$, L$\Box_j$, L$\multimap_i$, L$\multimapinv_i$);
- the *par rules* are the left rules for the $\otimes$-like connectives and the right rules for the $\parr$-like connectives (L$\Diamond_j$, L$\otimes_i$, R$\Box_j$, R$\multimap_i$, R$\multimapinv_i$).

In the derivations we will indicate par rules by dashed horizontal lines. Lemma 6.2.3 now can be reformulated as: all par rules are reversible.

In the sequel we will introduce square bracketed abbreviations like [Q] for structural rules.

Example 6.2.5. Let $\mathcal{R}$ consist of

$$\frac{\Delta[\,\Gamma_1 \odot_0 \big(\Gamma_2\odot_0\Gamma_3\big)\,]\vdash C}{\Delta[\,\big(\Gamma_1\odot_0\Gamma_2\big)\odot_0\Gamma_3\,]\vdash C}\,[\mathrm{LAss}_0]$$

Then we can derive:

$$\begin{array}{cl} \dfrac{\dfrac{}{A\vdash A}\qquad \dfrac{\dfrac{}{B\vdash B}\quad\dfrac{}{C\vdash C}}{B\odot_0 C\vdash B\otimes_0 C}}{A\odot_0\big(B\odot_0 C\big)\vdash A\otimes_0(B\otimes_0 C)} & \\ \cline{1-1} \big(A\odot_0 B\big)\odot_0 C\vdash A\otimes_0(B\otimes_0 C) & [\mathrm{LAss}_0] \\ \cdashline{1-1} (A\otimes_0 B)\odot_0 C\vdash A\otimes_0(B\otimes_0 C) & \mathrm{L}\otimes_0 \\ \cdashline{1-1} (A\otimes_0 B)\otimes_0 C\vdash A\otimes_0(B\otimes_0 C) & \mathrm{L}\otimes_0 \end{array}$$

$$\begin{array}{cl} \dfrac{\dfrac{}{C\vdash C}\qquad \dfrac{\dfrac{}{A\vdash A}\quad\dfrac{}{B\vdash B}}{A\odot_0 A\multimap_0 B\vdash B}}{A\odot_0\big((A\multimap_0 B)\multimapinv_0 C\odot_0 C\big)\vdash B} & \\ \cline{1-1} \big(A\odot_0 (A\multimap_0 B)\multimapinv_0 C\big)\odot_0 C\vdash B & [\mathrm{LAss}_0] \\ \cdashline{1-1} A\odot_0 (A\multimap_0 B)\multimapinv_0 C\vdash B\multimapinv_0 C & \mathrm{R}\multimapinv_0 \\ \cdashline{1-1} (A\multimap_0 B)\multimapinv_0 C\vdash A\multimap_0(B\multimapinv_0 C) & \mathrm{R}\multimap_0 \end{array}$$

$$\begin{array}{c}
\dfrac{\dfrac{}{B \vdash B} \qquad \dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{C \vdash C}}{A \odot_0 A \multimap_0 C \vdash C}}{A \odot_0 \big(B \odot_0 B \multimap_0 (A \multimap_0 C)\big) \vdash C} \\
\hline
\big(A \odot_0 B\big) \odot_0 B \multimap_0 (A \multimap_0 C) \vdash C \quad [\mathrm{LAss}_0] \\
\hdashline
(A \otimes_0 B) \odot_0 B \multimap_0 (A \multimap_0 C) \vdash C \quad \mathrm{L}\otimes_0 \\
\hdashline
B \multimap_0 (A \multimap_0 C) \vdash (A \otimes_0 B) \multimap_0 C \quad \mathrm{R}\multimap_0
\end{array}$$

$\Diamond$

EXAMPLE 6.2.6. **(Illustration: wh-extraction in English)** To give an indication of how we can use the calculus described in the previous section to give an account of linguistic phenomena, we will look at what is often called *wh*-extraction.

We will, for the purpose of the current discussion, look at only two *wh* words, 'which' and 'whom'. Both are noun modifiers which select a sentence from which a noun phrase is missing, the difference being that with 'whom' the missing noun phrase cannot occur in subject position, as indicated by the following examples. The * in sentence 4 denotes this sentence is ungrammatical.

(1) agent which [$[\,]_{np}$ read National Enquirer $]_s$
(2) agent which [Mulder liked $[\,]_{np}$ $]_s$
(3) agent which [Skinner considered $[\,]_{np}$ dangerous $]_s$

(4) *agent whom [$[\,]_{np}$ read National Enquirer $]_s$
(5) agent whom [Mulder liked $[\,]_{np}$ $]_s$
(6) agent whom [Skinner considered $[\,]_{np}$ dangerous $]_s$

To account for this different behaviour, we give a very simple grammar fragment with one binary mode 0 and two unary modes 0 and 1. An extracted *np* is marked as $\Diamond_0\Box_0 np$ if subject extraction is allowed and as $\Diamond_1\Box_1 np$ if it isn't. The fact that $\Diamond_j\Box_j A \vdash A$ is a theorem of the base logic for all j and A allows these constituents to function as an *np*. What is crucial is that the $\mathrm{L}\Box_j$ rule, read from premiss to conclusion, introduces unary brackets, which makes the following structural rules available for $\langle - \rangle^0$.

$$\frac{\Delta[\,\Gamma_1 \odot_0 \big(\Gamma_2 \odot_0 \langle\,\Gamma_3\,\rangle^0\big)\,] \vdash C}{\Delta[\,\big(\Gamma_1 \odot_0 \Gamma_2\big) \odot_0 \langle\,\Gamma_3\,\rangle^0\,] \vdash C}\,[\mathrm{Ass}_{0,0}]$$

$$\frac{\Delta[\,\big(\Gamma_1 \odot_0 \langle\,\Gamma_2\,\rangle^0\big) \odot_0 \Gamma_3\,] \vdash C}{\Delta[\,\big(\Gamma_1 \odot_0 \Gamma_3\big) \odot_0 \langle\,\Gamma_2\,\rangle^0\,] \vdash C}\,[\mathrm{MxCom}_{0,0}]$$

$$\frac{\Delta[\,\langle\,\Gamma_1\,\rangle^0 \odot_0 \Gamma_2\,] \vdash C}{\Delta[\,\Gamma_2 \odot_0 \langle\,\Gamma_1\,\rangle^0\,] \vdash C}\,[\mathrm{Com}_{0,0}]$$

The $[\mathrm{Ass}_{0,0}]$ and $[\mathrm{MxCom}_{0,0}]$ rules allow us to move out an embedded $\langle\,\Gamma\,\rangle^0$ constituent, whereas $[\mathrm{Com}_{0,0}]$ moves a $\langle\,\Gamma\,\rangle^0$ constituent from a left branch to a right branch after which any of the two other structural rules can apply.

Formulas marked with $\Diamond_1$, however, can only move from a right branch of a structure to another right branch. As a subject would appear on a left branch, this prevents subject extraction as desired.

$$\dfrac{\Delta[\,\Gamma_1 \odot_0 \big(\Gamma_2 \odot_0 \langle\,\Gamma_3\,\rangle^1\big)\,] \;\vdash\; C}{\Delta[\,\big(\Gamma_1 \odot_0 \Gamma_2\big) \odot_0 \langle\,\Gamma_3\,\rangle^1\,] \;\vdash\; C}\,[\mathrm{Ass}_{0,1}]$$

$$\dfrac{\Delta[\,\big(\Gamma_1 \odot_0 \langle\,\Gamma_2\,\rangle^1\big) \odot_0 \Gamma_3\,] \;\vdash\; C}{\Delta[\,\big(\Gamma_1 \odot_0 \Gamma_3\big) \odot_0 \langle\,\Gamma_2\,\rangle^1\,] \;\vdash\; C}\,[\mathrm{MxCom}_{0,1}]$$

The lexicon, with which we can derive all well-formed example sentences given above, is the following:

$\mathrm{lex}(\textit{agent}) = n$

$\mathrm{lex}(\textit{dangerous}) = n \multimapinv_0 n$

$\mathrm{lex}(\textit{Mulder}) = np$

$\mathrm{lex}(\textit{Skully}) = np$

$\mathrm{lex}(\textit{Skinner}) = np$

$\mathrm{lex}(\textit{National Enquirer}) = np$

$\mathrm{lex}(\textit{liked}) = (np \multimap_0 s) \multimapinv_0 np$

$\mathrm{lex}(\textit{read}) = (np \multimap_0 s) \multimapinv_0 np$

$\mathrm{lex}(\textit{considered}) = ((np \multimap_0 s) \multimapinv_0 (n \multimapinv_0 n)) \multimapinv_0 np$

$\mathrm{lex}(\textit{which}) = (n \multimap_0 n) \multimapinv_0 (s \multimapinv_0 \Diamond_0\Box_0 np)$

$\mathrm{lex}(\textit{whom}) = (n \multimap_0 n) \multimapinv_0 (s \multimapinv_0 \Diamond_1\Box_1 np)$

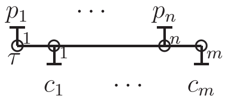
We can, for example, derive sentence 2 as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{np \;\vdash\; np}\,\mathrm{Ax} \qquad \dfrac{\dfrac{}{np \;\vdash\; np}\,\mathrm{Ax} \quad \dfrac{}{s \;\vdash\; s}\,\mathrm{Ax}}{np \odot_0 np \multimap_0 s \;\vdash\; s}\,\mathrm{L}{\multimap_0}}{np \odot_0 \big((np \multimap_0 s) \multimapinv_0 np \odot_0 np\big) \;\vdash\; s}\,\mathrm{L}{\multimapinv_0}}{np \odot_0 \big((np \multimap_0 s) \multimapinv_0 np \odot_0 \langle\,\Box_0 np\,\rangle^0\big) \;\vdash\; s}\,\mathrm{L}\Box_0}{\big(np \odot_0 (np \multimap_0 s) \multimapinv_0 np\big) \odot_0 \langle\,\Box_0 np\,\rangle^0 \;\vdash\; s}\,[\mathrm{Ass}_{0,0}]}{\big(np \odot_0 (np \multimap_0 s) \multimapinv_0 np\big) \odot_0 \Diamond_0\Box_0 np \;\vdash\; s}\,\mathrm{L}\Diamond_0}{np \odot_0 (np \multimap_0 s) \multimapinv_0 np \;\vdash\; s \multimapinv_0 \Diamond_0\Box_0 np}\,\mathrm{R}{\multimapinv_0} \qquad \dfrac{\dfrac{}{n \;\vdash\; n}\,\mathrm{Ax} \quad \dfrac{}{n \;\vdash\; n}\,\mathrm{Ax}}{n \odot_0 n \multimap_0 n \;\vdash\; n}\,\mathrm{L}{\multimap_0}}{n \odot_0 \big((n \multimap_0 n) \multimapinv_0 (s \multimapinv_0 \Diamond_0\Box_0 np) \odot_0 \big(np \odot_0 (np \multimap_0 s) \multimapinv_0 np\big)\big) \;\vdash\; n}\,\mathrm{L}{\multimapinv_0}$$

$\Diamond$

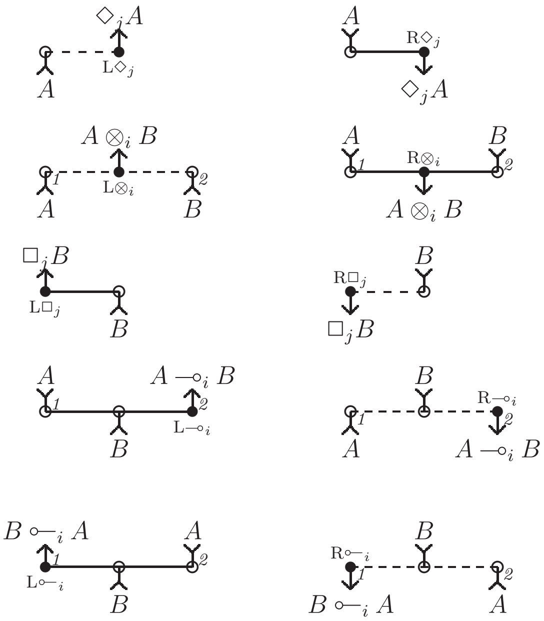
6.3. Proof structures

Given a set S, we generally define a *link* L in S to be an ordered pair $\langle P, C\rangle_\tau$ of sequences of elements of S (called the premisses and conclusions of L) labeled by a certain type τ. It will be represented by a horizontal bar, also labeled by τ, together with the elements of P above it and the elements of C below it:



Let S be a multiset of formulas, i.e. a set of formula occurrences. We will restrict to links L where one of the formulas (called the *main* formula or the *output* formula of L) is obtained as a connective applied to the other formulas (called the *active* formulas or the *input* formulas of L). Depending on whether the main formula is a premiss or a conclusion, and moreover on which connective is applied, we distinguish $6\,|I| + 4\,|J|$ types (where I and J are the sets of modes):

Definition 6.3.1. *A* proof structure $\langle S, \mathcal{L}\rangle$ *consists of a finite set S of formulas together with a set $\mathcal{L}$ of links in S of the following forms:*

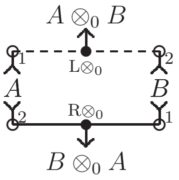


such that the following holds:

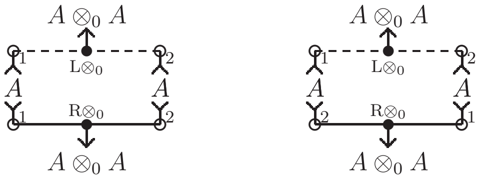
- *every formula of S is at most once a conclusion of a link;*
- *every formula of S is at most once a premiss of a link.*

◊

Here we have ordered the premisses/conclusions of the links in the picture from left to right. However, in general this is not always possible. E.g. in



the R$\otimes_0$ link has first premiss B and second premiss A. Observe that the following two proof structures are different as the first conclusion of the L$\otimes_0$ link is the first respectively second premiss of the R$\otimes_0$ link:



The formulas which are not the conclusion of a link are the *hypotheses* H_k of $\mathcal{S} = \langle S, \mathcal{L} \rangle$, while those that are not the premiss of a link are the *conclusions* Q_l of $\mathcal{S}$. This is also expressed by saying that $\mathcal{S}$ is a proof structure from $\{H_1, H_2, \ldots\}$ to $\{Q_1, Q_2, \ldots\}$.

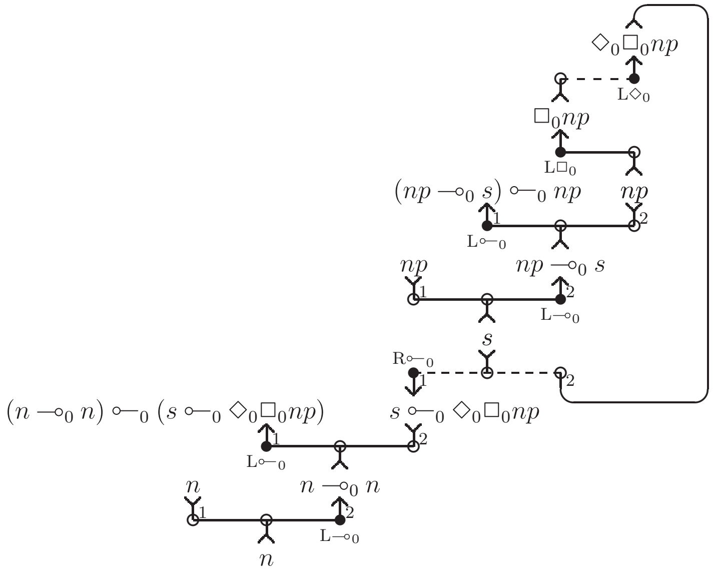
We divide the links in two parts:

- the *tensor links* are the right links for the $\otimes$-like connectives and the left links for the $\parr$-like connectives (R$\Diamond_j$, R$\otimes_i$, L$\Box_j$, L$\multimap_i$, L$\multimapinv_i$);
- the *par links* are the left links for the $\otimes$-like connectives and the right links for the $\parr$-like connectives (L$\Diamond_j$, L$\otimes_i$, R$\Box_j$, R$\multimap_i$, R$\multimapinv_i$).

We graphically indicate tensor vs. par links by solid vs. dashed horizontal lines.

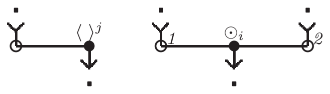
Note that there are no links corresponding to the identity rules. Instead we will have axiomatic and cut formulas. An *axiomatic formula* is a formula which is not the main formula of any link, whereas a *cut formula* is a formula which is the main formula of two links.

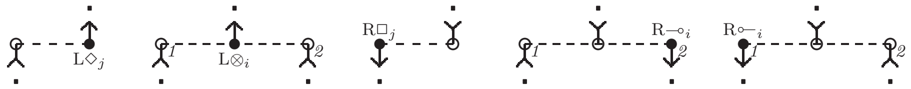
Example 6.3.2. A proof structure corresponding to the sequent derivation on page 193 is shown below.



◊

Definition 6.3.3. *A* correction structure $\langle N, \mathcal{L} \rangle$ *consists of a finite set* N *of nodes together with a set* $\mathcal{L}$ *of links in* N *of the following forms:*



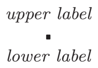


such that the following holds:

- *every node of S is at most once a conclusion of a link;*
- *every node of S is at most once a premiss of a link.*

$\Diamond$

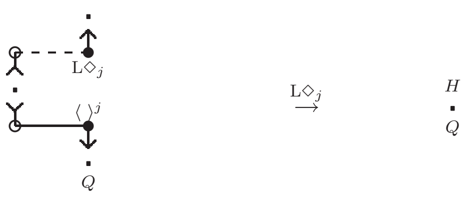
Definition 6.3.4. *An* hypothesis structure $\langle N, \mathcal{L}, \lambda\rangle$ *consists of a correction structure* $\langle N, \mathcal{L}\rangle$ *and a labeling* λ *of its nodes: to every node there are assigned a (perhaps empty)* upper label *and a* lower label

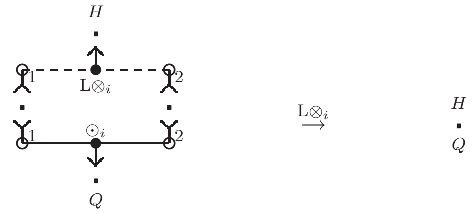


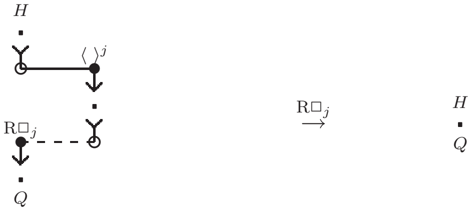
each one consisting of at most one formula. This labeling is such that exactly each hypothesis node h has a non-empty upper label $\{H_h\}$, and exactly each conclusion node q has a non-empty lower label $\{Q_q\}$, and we say that $\langle N, \mathcal{L}, \lambda\rangle$ is an hypothesis structure from $\{H_1, H_2, \ldots\}$ to $\{Q_1, Q_2, \ldots\}$. $\Diamond$

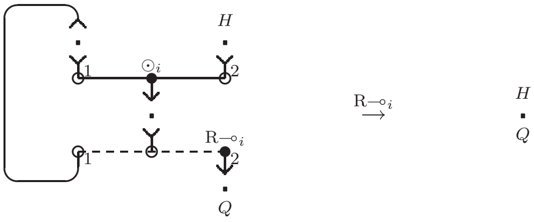
Next we will define *conversion steps* on hypothesis structures. One easily checks that these conversion steps preserve the labels: if $\mathcal{H} = \langle N, \mathcal{L}, \lambda\rangle$ is an hypothesis structure from $\{H_1, H_2, \ldots\}$ to $\{Q_1, Q_2, \ldots\}$, then so is $\mathcal{H}'$ which is obtained from $\mathcal{H}$ by applying a conversion step. There are two kinds of conversion steps: contractions and structural conversions. Every conversion step works on a number of links, constituting the so called *redex*, which is a correction structure itself. Below, all nodes of each depicted redex are distinct. Hence the redex of a contraction has one hypothesis node and one conclusion node, while the redex of an n-ary structural conversion has n hypothesis nodes and one conclusion node.

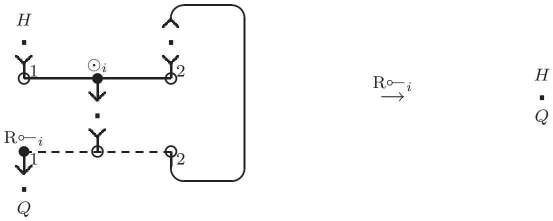
By a *contraction* we mean the replacement of one of the following pairs of links by a single node, which will be labeled as indicated (H and Q are labels, so each of them consists of zero or one formulas). The contraction will be named after the par link (L◇$_j$, L⊗$_i$, R□$_j$, R—∘$_i$, R∘—$_i$).











By a *structural conversion* we mean the following: for an n-ary structural rule

$$\frac{\Delta[\ \Xi^*[\ \Gamma_1,\ldots,\Gamma_n\ ]\ ]\ \vdash\ C}{\Delta[\ (\Xi')^*[\ \Gamma_{\pi_1},\ldots,\Gamma_{\pi_n}\ ]\ ]\ \vdash\ C}\ \langle \Xi,\Xi',\pi\rangle$$

both formal trees Ξ^* and $(\Xi')^*$ may be represented by a correction structure with n hypotheses and one conclusion. Ordering the premisses of all our $\mathrm{R}\odot_i$ links in the picture from left to right yields an order on the hypotheses of both correction structures. Now, if Ξ^* is part of $\mathcal{H}$ — the hypothesis nodes being $x_1,\ldots,x_n$ (in this order) — the structural conversion consists of replacing Ξ^* by $(\Xi')^*$ and permuting the nodes to get them in the order $x_{\pi_1},\ldots,x_{\pi_n}$. The conversion is denoted by

$$\Xi^*[\ x_1,\ldots,x_n\ ] \to (\Xi')^*[\ x_{\pi_1},\ldots,x_{\pi_n}\ ].$$

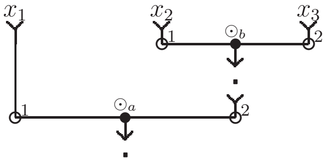
Example 6.3.5. Let us consider the structural rule [Q]:

$$\frac{\Delta[\ \Gamma_1\ \odot_a\ (\Gamma_2\ \odot_b\ \Gamma_3)\ ]\ \vdash\ C}{\Delta[\ (\Gamma_3\ \odot_d\ \langle\Gamma_1\rangle^e)\ \odot_c\ \Gamma_2\ ]\ \vdash\ C}\ [\mathrm{Q}]$$

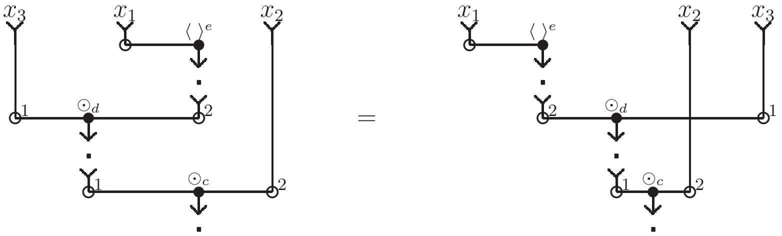
where a, b, c, d are binary modes and e is a unary mode. The corresponding structural conversion

$$x_1 \odot_a (x_2 \odot_b x_3) \rightarrow (x_3 \odot_d \langle x_1 \rangle^e) \odot_c x_2$$

consists in the replacement of



by



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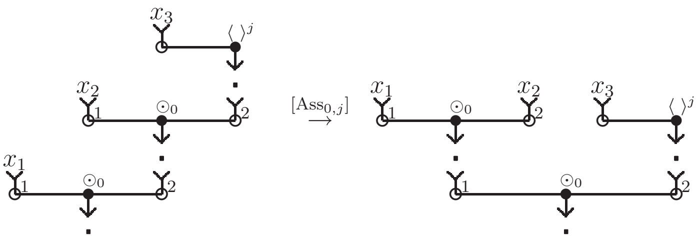
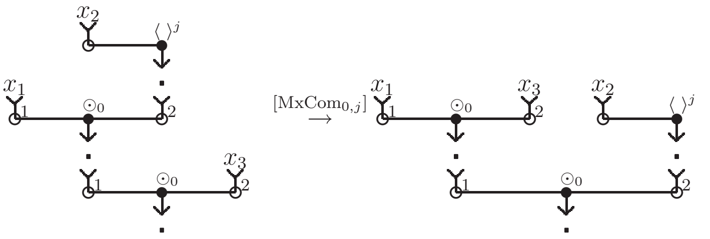
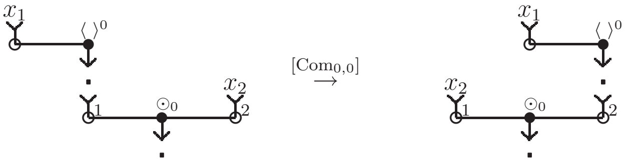
Observe that a structural conversion is a local operation; it does not influence the other links of $\mathcal{L}$, nor the other nodes of N. The last example shows how to see this in our graphical representation: we need not permute the hypothesis nodes, if we represent $(\Xi')^*$ in the appropriate way.

Example 6.3.6. The structural rules for our English fragment correspond to the following structural conversions, where j ranges over $\{0, 1\}$.







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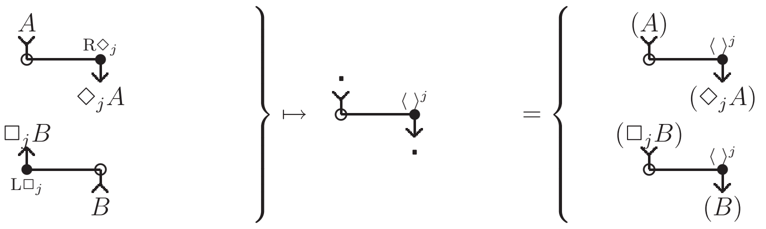
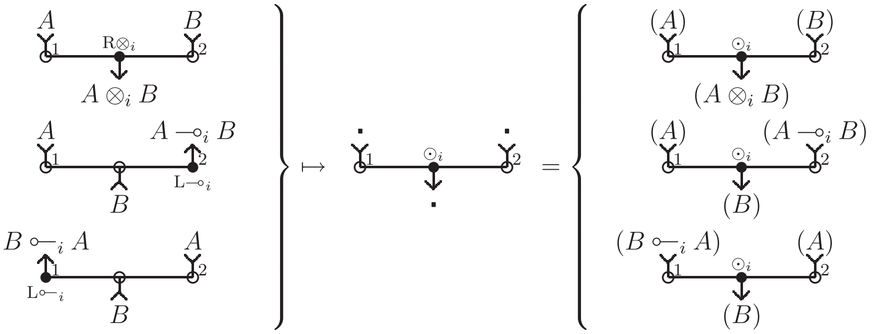
To any proof structure $\mathcal{S}$ from $\{H_1, H_2, \ldots\}$ to $\{Q_1, Q_2, \ldots\}$ we assign an hypothesis structure $\widehat{\mathcal{S}}$ from $\{H_1, H_2, \ldots\}$ to $\{Q_1, Q_2, \ldots\}$ by a replacement of the link types $\mathrm{R}\Diamond_j$ and $\mathrm{L}\Box_j$ by the new link type $\langle\ \rangle^j$, and by a replacement of the link types $\mathrm{R}\otimes_i$, $\mathrm{L}\multimap_i$ and $\mathrm{L}\multimapinv_i$ by the new link type $\odot_i$. The formulas A become the nodes (A) of $\widehat{\mathcal{S}}$, and the label H (resp. Q) of a node

$$(A)\overset{H}{\underset{Q}{\cdot}}$$

is chosen $\{A\}$ precisely if A is an hypothesis (resp. a conclusion), and empty otherwise. $\widehat{\mathcal{S}}$ is called the *underlying* hypothesis structure of $\mathcal{S}$.





For any structure tree Γ and formula C, let $\|\Gamma\|$ be the multiset of elements in Γ; let $\langle\!\langle\Gamma\rangle\!\rangle$ be the sequence of elements in Γ obtained by left to right traversal of the tree; let Γ_C be the obvious hypothesis structure from $\|\Gamma\|$ to $\{C\}$ with conclusion node (lower) labeled by C. Any hypothesis structure of this form will be called an *hypothesis tree*. Let $\twoheadrightarrow_{\mathcal{R}}$ be the transitive, reflexive closure of $\rightarrow_{\mathcal{R}}$, by which we mean the contractions as well as the structural conversions belonging to $\mathcal{R}$.

In the next sections we will prove our main theorem:

THEOREM 6.3.7.
$\Gamma \vdash C$ *is derivable in* $\boldsymbol{NL}\Diamond_{\mathcal{R}}$ *if and only if there is a proof structure* $\mathcal{S}$ *(from* $\|\Gamma\|$ *to* $\{C\}$*) such that* $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$*.* ◊

Note that any proof structure $\mathcal{S}$ such that $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$ is automatically from $\|\Gamma\|$ to $\{C\}$.

DEFINITION 6.3.8. *Let* $\mathcal{S}$ *be a proof structure.*

(1) $(\mathcal{S}, \rho)$ *is a* $\mathcal{R}$-conversion sequence of $\Gamma \vdash C$ *iff* $\widehat{\mathcal{S}} \overset{\rho}{\twoheadrightarrow}_{\mathcal{R}} \Gamma_C$*;*

(2) $\mathcal{S}$ *is a* $\mathcal{R}$-proof net of $\Gamma \vdash C$ *iff* $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$*, i.e. iff* $\widehat{\mathcal{S}} \overset{\rho}{\twoheadrightarrow}_{\mathcal{R}} \Gamma_C$ *for some* ρ*;*

(3) Let Σ *be a multiset of formulas. We say* $\mathcal{S}$ *is an* $\mathcal{R}$-proof net from Σ to $\{C\}$ *iff* $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$ *for some* Γ *such that* $\|\Gamma\| = \Sigma$*, or equivalently: iff* $\mathcal{S}$ *is a proof structure from* Σ *to* $\{C\}$ *on which all contractions can be applied (together with the necessary structural conversions) such that we end with an hypothesis tree.*

◊

With this definition Theorem 6.3.7 can be reformulated as:
There is an $\mathbf{NL}\Diamond_{\mathcal{R}}$ derivation of $\Gamma \vdash C$ if and only if there is a $\mathcal{R}$-proof net of $\Gamma \vdash C$.

For many applications one is interested in derivability of $\Gamma \vdash C$ for some structure tree Γ subject to certain constraints. Instead of checking the RHS condition for each Γ, we get the witnessing Γ as a result of the conversion steps, as stated in the following corollary. This fact shows the computational strength of our condition (see Section 6.7).

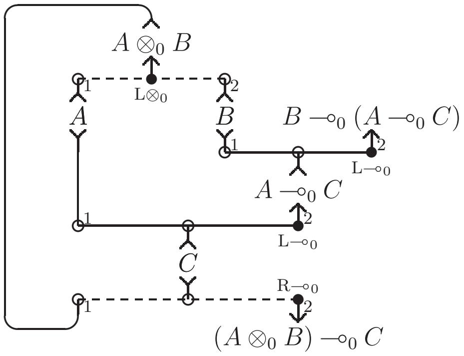
COROLLARY 6.3.9. ***(Abstraction of*** Γ***)***

(1) Let us call a sequence Σ *of formulas a* C-sequence *if, for some* Γ *such that* $\langle\!\langle\Gamma\rangle\!\rangle = \Sigma$*, the sequent* $\Gamma \vdash C$ *is derivable. Then* Σ *is a* C*-sequence iff there is a proof structure from* $\|\Sigma\|$ *to* $\{C\}$*, on which all contractions can be applied (together with the necessary structural conversions) such that we end with an hypothesis tree in which the order of the hypotheses equals* Σ*.*

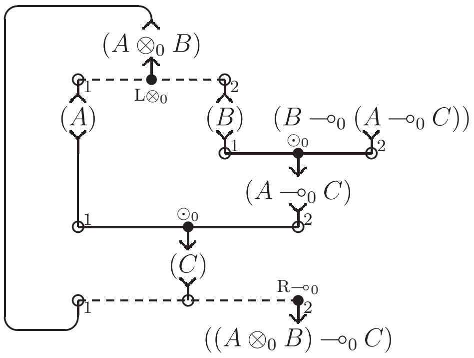
*(2) Let us call a multiset Σ of formulas a C-*multiset *if, for some Γ such that $\|\Gamma\| = \Sigma$, the sequent $\Gamma \vdash C$ is derivable. Then Σ is a C-multiset iff there is a proof structure from Σ to $\{C\}$, on which all contractions can be applied (together with the necessary structural conversions) such that we end with an hypothesis tree, i.e. iff there is a proof net from Σ to $\{C\}$.*

$\Diamond$

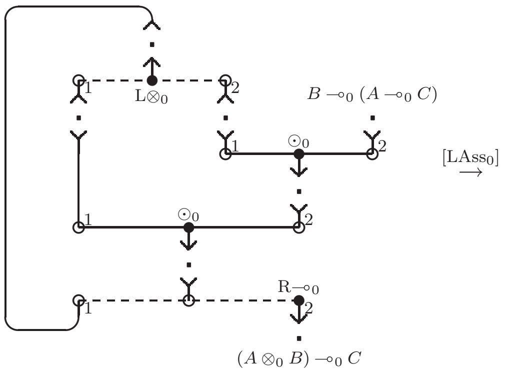
Example 6.3.10. The following proof structure

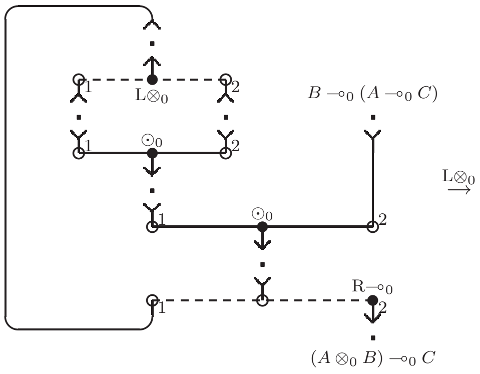


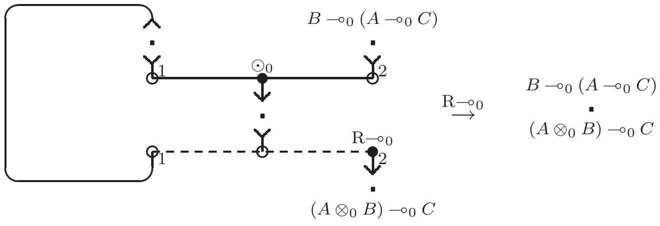
has as hypothesis structure



which converts (under $\mathcal{R} = \{[\mathrm{LAss}_0]\}$) as follows:

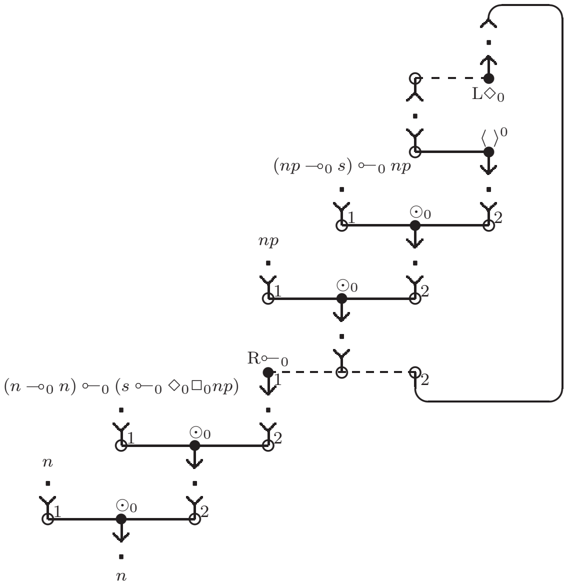




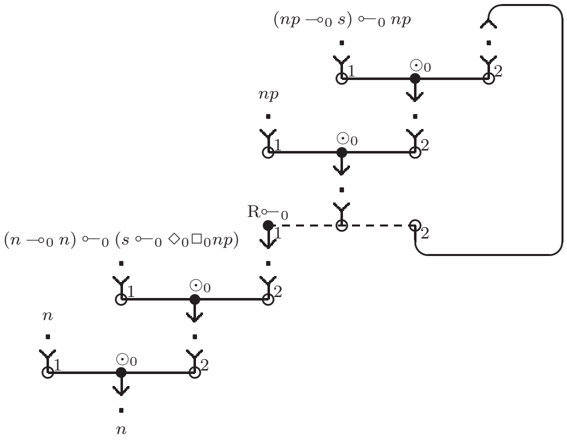


◊

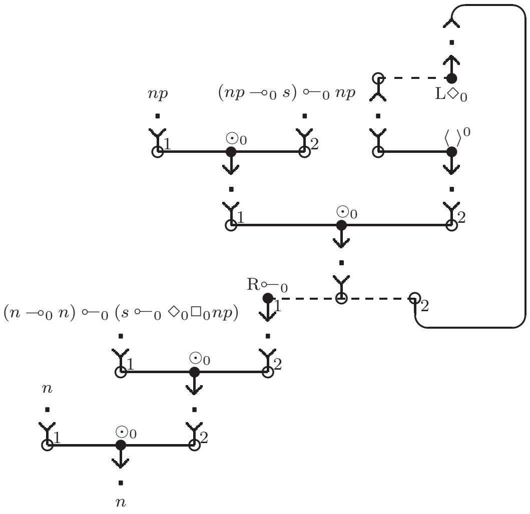
Example 6.3.11. The hypothesis structure corresponding to the proof structure of Example 6.3.2 is the following:



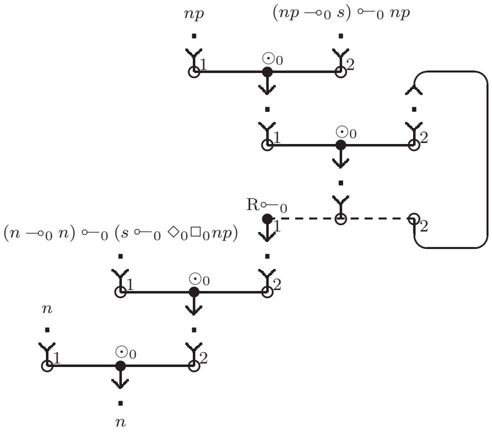
We can convert this hypothesis structure to an hypothesis tree using the following conversions. From the initial hypothesis structure we have two choices: we can apply either the L$\Diamond_0$ contraction or the [Ass$_{0,0}$] conversion. The L$\Diamond_0$ contraction gives us the following hypothesis structure:



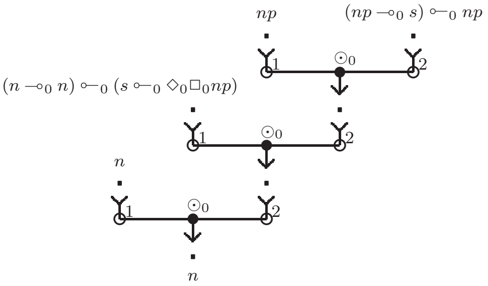
At this stage, none of the conversions apply and we still have to remove the R$\circ\!-_0$ link to get an hypothesis tree. This does not mean our proof structure is not a proof net, however, as we can try the other possibility:



after which we apply the L$\Diamond_0$ contraction.



Finally, we can apply the R$\circ\!\!-_0$ contraction to obtain the following hypothesis tree:



$\Diamond$

Remark 6.3.12. Lambek Calculus (**L**) as introduced in [**Lambek 58**] is defined in Section 4.9. It is related to the following special case of $\mathbf{NL}\Diamond_{\mathcal{R}}$:

- zero unary modes ($J = \emptyset$), implying no unary connectives;
- only one binary mode;
- the structural rules[3] [LAss] and [RAss], mimicking the fact that each sequent has a list instead of a structure tree as antecedent part;
- no other structural rules.

In fact, this calculus is equivalent to $\mathbf{L}^{>0}$: **L** restricted to the requirement that the antecedent list of all sequents in a derivation be non-empty. In this way Theorem 6.3.7 provides us with a correctness criterium for $\mathbf{L}^{>0}$. $\Diamond$

6.4. Soundness

Theorem 6.4.1. ***($\boxed{\Longrightarrow}$ part of Theorem 6.3.7)***
If $\Gamma \vdash C$ is derivable in $\mathbf{NL}\Diamond_{\mathcal{R}}$, then there is a proof structure $\mathcal{S}$ (from $\|\Gamma\|$ to $\{C\}$) such that $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$. $\Diamond$

Proof: We apply induction on the derivation of $\Gamma \vdash C$. By 'applying a conversion step to a proof structure' we will mean 'applying a conversion step to its underlying hypothesis structure'.

The identity rules

For an axiom

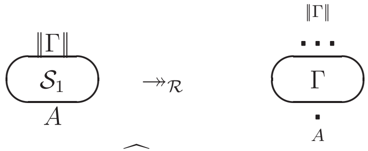
$$\frac{}{A \vdash A}\ \textsc{Ax}$$

take the trivial proof structure consisting of one formula A and no links. Its hypothesis structure $\overset{A}{\underset{A}{\cdot}}$ converts into A_A in zero steps.

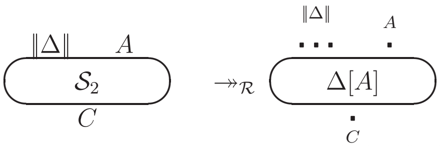
For a Cut inference

$$\frac{\Gamma \vdash A \qquad \Delta[A] \vdash C}{\Delta[\Gamma] \vdash C}\ \textsc{Cut}$$

by induction hypothesis we know that there are proof structures $\mathcal{S}_1$ from $\|\Gamma\|$ to $\{A\}$ such that $\widehat{\mathcal{S}_1} \twoheadrightarrow_{\mathcal{R}} \Gamma_A$:



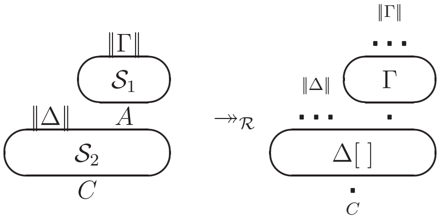
and $\mathcal{S}_2$ from $\|\Delta\| \cup \{A\}$ to $\{C\}$ such that $\widehat{\mathcal{S}_2} \twoheadrightarrow_{\mathcal{R}} \Delta[A]_C$:



Pasting $\mathcal{S}_1$ and $\mathcal{S}_2$ in A yields a proof structure from $\|\Gamma\| \cup \|\Delta\|$ to $\{C\}$ which converts into Γ_A pasted to $\Delta[A]_C$, i.e. it converts into $\Delta[\Gamma]_C$, as desired:

---

[3] See page 190 for the definition of the mentioned structural rules.

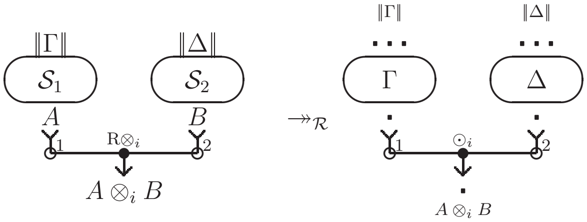


The tensor rules

For a $\mathrm{R}\otimes_i$ rule

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma \odot_i \Delta \vdash A \otimes_i B}\,\mathrm{R}\otimes_i \quad ,$$

assuming the appropriate induction hypothesis, we find:



The unary version

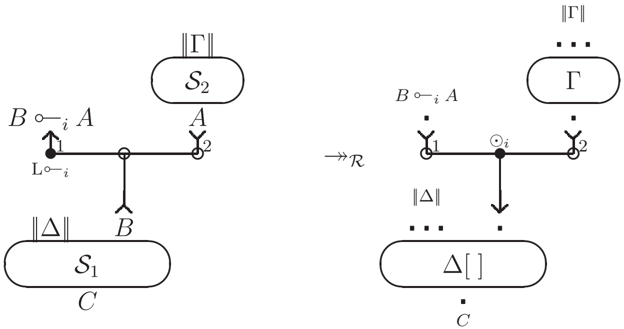
$$\frac{\Gamma \vdash A}{\langle\, \Gamma \,\rangle^j \vdash \Diamond_j A}\,\mathrm{R}\Diamond_j$$

is proved similarly.

For a $\mathrm{L}\circ\!\!-_i$ rule

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[B \circ\!\!-_i A \odot_i \Gamma] \vdash C}\,\mathrm{L}\circ\!\!-_i \quad ,$$

assuming the appropriate induction hypothesis, we find:



The $\mathrm{L}\multimap_i$ case is the symmetric counterpart, while the unary version

$$\frac{\Delta[B] \vdash C}{\Delta[\langle\, \Box_j B \,\rangle^j] \vdash C}\,\mathrm{L}\Box_j$$

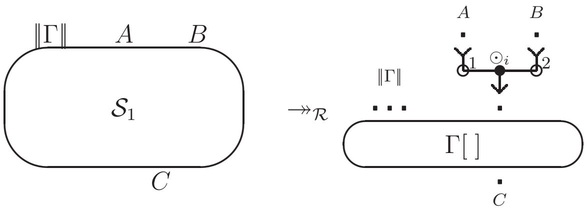
is proved analogously by deleting $\mathcal{S}_2$ in the diagram above.

The par rules

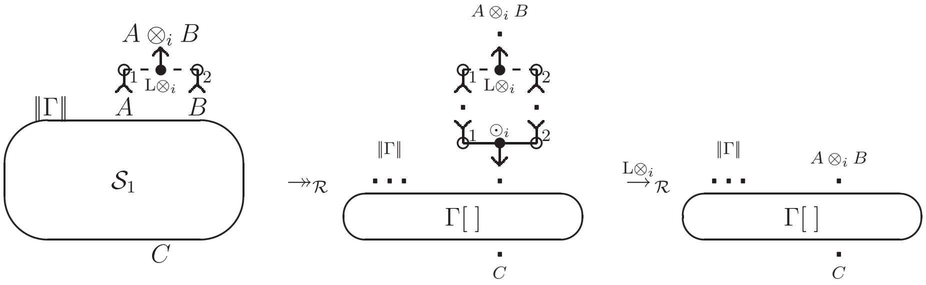
For a $\mathrm{L}\otimes_i$ rule

$$\frac{\Gamma[A \odot_i B] \vdash C}{\Gamma[A \otimes_i B] \vdash C}\ \mathrm{L}\otimes_i$$

we know by induction hypothesis that



whence



The unary version

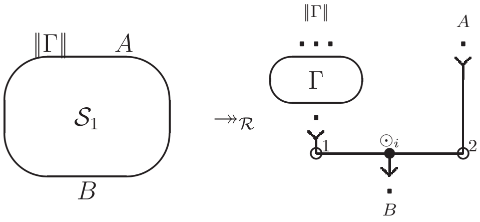
$$\frac{\Gamma[\langle A \rangle^j] \vdash C}{\Gamma[\Diamond_j A] \vdash C}\ \mathrm{L}\Diamond_j$$

is proved analogously, by an extension of the original conversion sequence by a $\mathrm{L}\Diamond_j$ contraction.

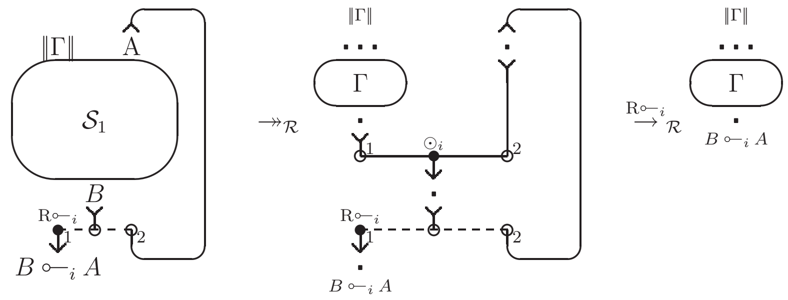
For a $\mathrm{R}\circ\!\!-_i$ rule

$$\frac{\Gamma \odot_i A \vdash B}{\Gamma \vdash B \circ\!\!-_i A}\ \mathrm{R}\circ\!\!-_i$$

we know by induction hypothesis that



whence



The R$\multimap_i$ case is the symmetric counterpart, while the unary version

$$\frac{\langle\, \Gamma \,\rangle^j \;\vdash\; B}{\Gamma \;\vdash\; \Box_j B}\ \mathrm{R}\Box_j$$

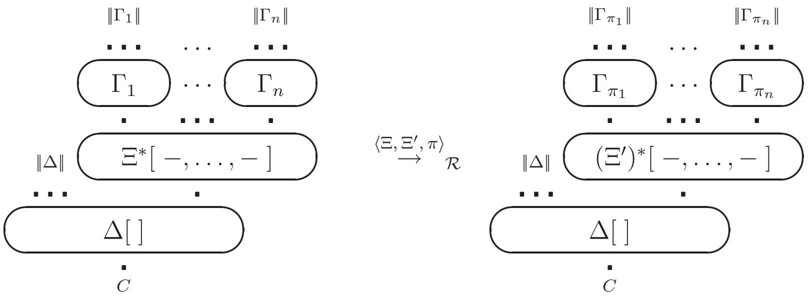
is proved analogously, by an extension of the original conversion sequence by a R$\Box_j$ contraction.

The structural rules

For a structural rule

$$\frac{\Delta[\ \Xi^*[\ \Gamma_1,\dots,\Gamma_n\ ]\ ]\ \vdash\ C}{\Delta[\ (\Xi')^*[\ \Gamma_{\pi_1},\dots,\Gamma_{\pi_n}\ ]\ ]\ \vdash\ C}\ \langle \Xi, \Xi', \pi \rangle$$

belonging to $\mathcal{R}$, assuming that $\widehat{\mathcal{S}_1} \twoheadrightarrow_{\mathcal{R}} \Delta[\ \Xi^*[\ \Gamma_1,\dots,\Gamma_n\ ]\ ]_C$, we can start with the same proof structure $\mathcal{S}_1$ and extend this conversion sequence by



////

Given a derivation $\mathcal{D}$, the construction in this proof yields exactly one proof structure and at least one conversion sequence. Actually it yields a non-empty collection of conversion sequences in the following way.

Observe that for every structural rule and for every par rule we have to extend an inductively obtained conversion sequence by the corresponding conversion, whereas no other inference rule induces a new conversion. Hence for every conversion sequence there is a bijective correspondence between the set of structural rules and par rules on the one hand, and the set of conversion steps on the other hand.

A priori there is no unique order of executing these conversions. For a binary tensor rule (R$\otimes_i$, L$\multimap_i$, L$\multimapinv_i$) as well as for a Cut inference, the conversion steps in the components $\mathcal{S}_1$ and $\mathcal{S}_2$ may be executed in a parallel way; i.e. independently of each other. Now we define the collection of conversion sequences of $\mathcal{D}$ by all possible ways of interleaving a conversion sequence of $\mathcal{D}_1$ and a conversion sequence of $\mathcal{D}_2$. This yields

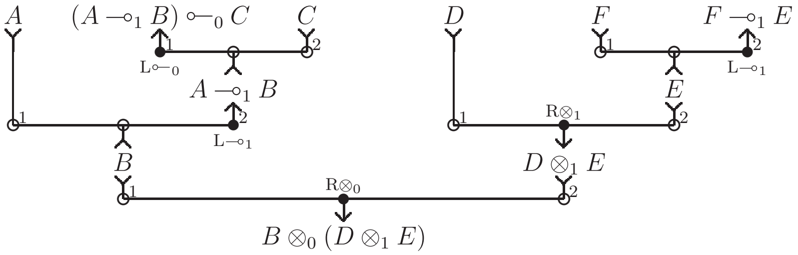
$$|\mathcal{D}| := \binom{k+l}{k} |\mathcal{D}_1| \, |\mathcal{D}_2|$$

conversion sequences, where k is the number of structural rules and par rules in $\mathcal{D}_1$, l is the number of structural rules and par rules in $\mathcal{D}_2$, and $|{-}|$ counts the number of conversion sequences in the inductively defined collection.

Example 6.4.2. 1. Let $\mathcal{D}$ be the following derivation:

$$\dfrac{\dfrac{C \vdash C \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \odot_1 A \multimap_1 B \vdash B}\,\mathrm{L}{\multimap_1}}{A \odot_1 \big((A \multimap_1 B) \multimapinv_0 C \odot_0 C\big) \vdash B}\,\mathrm{L}{\multimapinv_0} \qquad \dfrac{F \vdash F \qquad \dfrac{D \vdash D \qquad E \vdash E}{D \odot_1 E \vdash D \otimes_1 E}\,\mathrm{R}{\otimes_1}}{D \odot_1 \big(F \odot_1 F \multimap_1 E\big) \vdash D \otimes_1 E}\,\mathrm{L}{\multimap_1}}{\Big(A \odot_1 \big((A \multimap_1 B) \multimapinv_0 C \odot_0 C\big)\Big) \odot_0 \Big(D \odot_1 \big(F \odot_1 F \multimap_1 E\big)\Big) \vdash B \otimes_0 (D \otimes_1 E)}\,\mathrm{R}{\otimes_0}$$

The proof structure of this derivation reads:



and we find only one conversion sequence: the empty one.

2. The last derivation in Example 6.2.5 has the proof structure shown in Example 6.3.10. The conversion sequence given there again is the only one our procedure yields. ◊

6.5. Sequentialisation

Lemma 6.5.1. *If $\mathcal{S}$ is a non-trivial proof structure such that the underlying hypothesis structure $\widehat{\mathcal{S}}$ is actually an hypothesis tree Γ_C (for some structure tree Γ and formula C), then at least one of the leaves (conclusion and hypotheses) of $\mathcal{S}$ is the main formula of its link.* ◊

Proof: As $\mathcal{S}$ is not trivial (i.e. a singleton), it is clear that every leaf is connected to exactly one link, so the formulation of the lemma is well-defined.

To prove: if every hypothesis is an active formula of its link, then the conclusion is the main formula of its link. We proceed by induction on Γ.

The trivial case $\Gamma = \overset{A}{\cdot}$ cannot occur.

In case $\Gamma = \Gamma_1 \odot_i \Gamma_2$, assume every hypothesis is an active formula of its link. We write L for the final $\odot_i$ link, connecting Γ_1 and Γ_2. If Γ_1 is trivial, the assumption entails that the corresponding formula in $\mathcal{S}$ is an active formula of L. If Γ_1 is non-trivial, by induction hypothesis we know that its conclusion is the main formula of the link above, whence of the form $\Diamond_j A$ or $A \otimes_i B$. This implies that it is not the main formula of L, which would be $\Box_j B$, $A \multimap_i B$ or $B \multimapinv_i A$. Hence it is an active formula of L. The same holds for the second premiss of L. As both premisses are active, the conclusion of L must be main, as desired.

The case $\Gamma = \langle\, \Gamma_1 \,\rangle^j$ is proved analogously. ////

Alternatively, we can obtain this result as a corollary of the following lemma.

Lemma 6.5.2. *If $\mathcal{S}$ is a proof structure such that the underlying hypothesis structure $\widehat{\mathcal{S}}$ is actually an hypothesis tree, then $\lambda = \alpha$, where λ denotes the number of hypotheses of $\mathcal{S}$, and α the number of axiomatic formulas of $\mathcal{S}$.* $\Diamond$

Proof: By induction on Γ.

If $\Gamma = \overset{A}{\bullet}$, then $\mathcal{S}$ is singleton A, which has one hypothesis ($\lambda = 1$) and contains one axiomatic formula ($\alpha = 1$).

In case $\Gamma = \Gamma_1 \odot_i \Gamma_2$ we have $\lambda = \lambda_1 + \lambda_2$. In order to calculate α we differentiate on the following subcases:

- $\mathcal{S} = \mathcal{S}_1\ \mathrm{R}\otimes_i\ \mathcal{S}_2$: In this case $\alpha = \alpha_1 + \alpha_2 + 0$;
- $\mathcal{S} = \mathcal{S}_1\ \mathrm{L}\!\multimap_i\ \mathcal{S}_2$: Now $\alpha = \alpha_1 + (\alpha_2 - 1) + 1$, since the second premiss of the final $\odot_i$ link L becomes non-axiomatic, whereas the conclusion of L is a new axiomatic formula;
- $\mathcal{S} = \mathcal{S}_1\ \mathrm{L}\!\multimapinv_i\ \mathcal{S}_2$: Similarly we find $\alpha = (\alpha_1 - 1) + \alpha_2 + 1$.

Hence in all subcases $\alpha = \alpha_1 + \alpha_2$. Using the induction hypothesis we find

$$\lambda = \lambda_1 + \lambda_2 = \alpha_1 + \alpha_2 = \alpha$$

In case $\Gamma = \langle\, \Gamma_1 \,\rangle^j$ we have $\lambda = \lambda_1$, while $\alpha = \alpha_1 + 0$ (in the subcase that $\mathcal{S} = \mathrm{R}\Diamond_j \mathcal{S}_1$) or $\alpha = (\alpha_1 - 1) + 1$ (in the subcase that $\mathcal{S} = \mathrm{L}\Box_j \mathcal{S}_1$). So by induction hypothesis $\lambda = \alpha$. ////

Now suppose $\mathcal{S}$ is as in Lemma 6.5.1. Besides the hypotheses there is one other leaf: the conclusion. This yields

$$\text{the number of leaves} = \text{the number of axiomatic formulas} + 1.$$

Since a leaf is active exactly if it is an axiomatic formula of $\mathcal{S}$, subtracting the number of active leaves on both sides of this equation gives:

$$\text{the number of main leaves} = \text{the number of internal axiomatic formulas} + 1.$$

This implies that there is at least one main leaf, which gives an alternative proof of Lemma 6.5.1.

Example 6.5.3. The proof structure in Example 6.4.2.1 has $\lambda = 6$ hypotheses and $\alpha = 6$ axiomatic formulas. Neglecting the axiomatic leaves, we count 3 main leaves and 2 internal axiomatic formulas. $\Diamond$

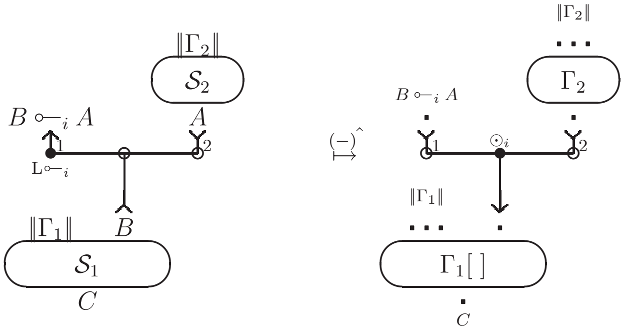
Theorem 6.5.4. ***($\boxed{\Longleftarrow}$ part of Theorem 6.3.7)***
If there is a proof structure $\mathcal{S}$ (from $\|\Gamma\|$ to $\{C\}$) such that $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$, then $\Gamma \vdash C$ is derivable in $\boldsymbol{NL}\Diamond_{\mathcal{R}}$. $\Diamond$

Proof: We apply induction on the length l of the conversion sequence $\widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$.

In case $l = 0$ we have $\widehat{\mathcal{S}} = \Gamma_C$. We proceed by induction on the size of Γ.

If $\Gamma = \overset{A}{\bullet}$, then A equals C, since they originate from the same formula constituting the whole proof structure $\mathcal{S}$. Now $A \vdash A$ is derivable by Ax.

If Γ is not trivial, by Lemma 6.5.1 we know $\mathcal{S}$ has at least one main leaf D. In case D is the main formula of a L$\circ\!\!-_i$ link, D is of the form $B \circ\!\!-_i A$ and must be the first premiss of this link. Now $\mathcal{S}$ and Γ are of the form



By induction hypothesis there are derivations $\mathcal{D}_2$ of $\Gamma_2 \vdash A$ and $\mathcal{D}_1$ of $\Gamma_1[B] \vdash C$, which may be combined to

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma_1[B] \vdash C\end{array}}{\Gamma_1[B \circ\!\!-_i A \odot_i \Gamma_2] \vdash C}\,\mathrm{L}\circ\!\!-_i$$

which is a derivation of $\Gamma \vdash C$.

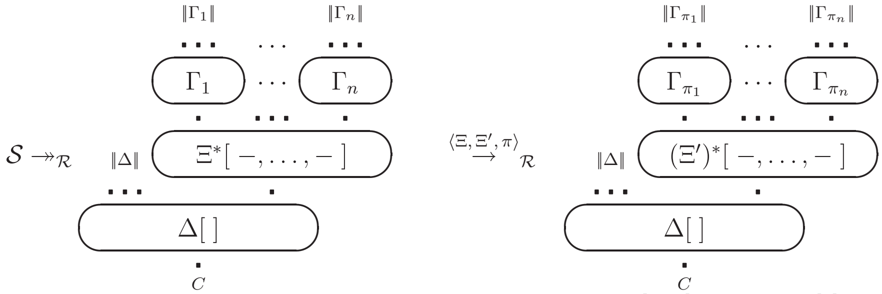
The remaining subcases in which D is the main formula of a R$\Diamond_j$, R$\otimes_i$, L$\Box_j$ or L$-\!\!\circ_i$ link are proved similarly.

Now assume $l > 0$.

If the last conversion step is a structural conversion

$$\Xi^*[\, x_1, \ldots, x_n \,] \rightarrow (\Xi')^*[\, x_{\pi_1}, \ldots, x_{\pi_n} \,].$$

our conversion sequence is

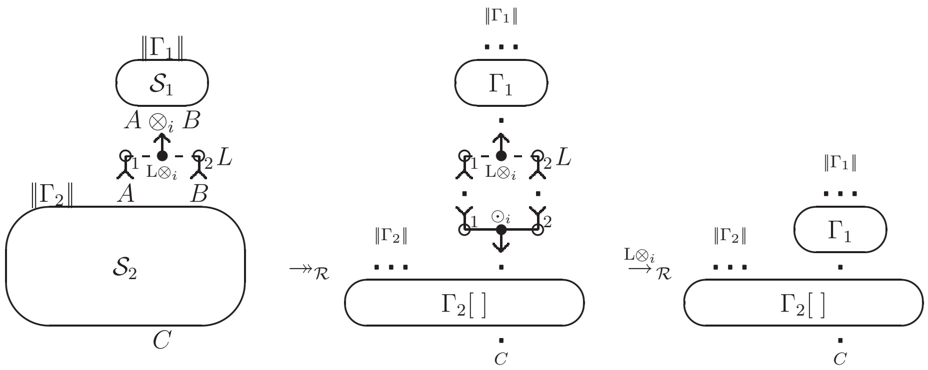


By induction hypothesis we know there is a derivation $\mathcal{D}_1$ of $\Delta[\, \Xi^*[\, \Gamma_1, \ldots, \Gamma_n \,] \,] \vdash C$, yielding

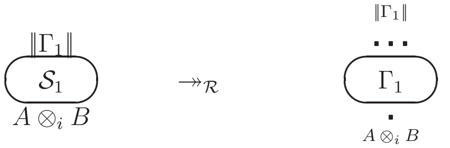
$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta[\, \Xi^*[\, \Gamma_1, \ldots, \Gamma_n \,] \,] \vdash C\end{array}}{\Delta[\, (\Xi')^*[\, \Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n} \,] \,] \vdash C}\,\langle \Xi, \Xi', \pi \rangle$$

which is a derivation of $\Gamma \vdash C$.

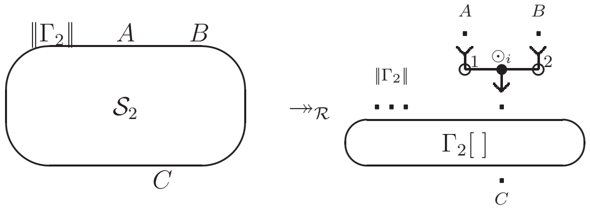
If the last conversion step is a L$\otimes_i$ contraction, we can split Γ in the node by which the pair of links is replaced. This yields two trees Γ_1 and $\Gamma_2[\,]$, satisfying $\Gamma = \Gamma_2[\Gamma_1]$.



Reading this sequence from right to left, the L$\otimes_i$ link (L) serves as a boundary: any structural rule is applied strictly over L or beneath L. Obviously also each point that is blown up is over or beneath L, whence our proof structure $\mathcal{S}$ splits as indicated. We can partition the conversion sequence into two conversion sequences for each of the substructures:



and



Now the total length of both sequences is $l-1$, whence each sequence is of length at most $l-1$, and applying the induction hypothesis we find derivations $\mathcal{D}_1$ of $\Gamma_1 \vdash A \otimes_i B$ and $\mathcal{D}_2$ of $\Gamma_2[A \odot_i B] \vdash C$. These may be combined into

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \vdash A \otimes_i B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma_2[A \odot_i B] \vdash C\end{array}}{\Gamma_2[A \otimes_i B] \vdash C}\,\mathrm{L}\otimes_i}{\Gamma_2[\Gamma_1] \vdash C}\,\textsc{Cut}$$

which is a derivation of $\Gamma \vdash C$, as desired.

The other four contractions are treated similarly. ////

For fixed $\mathcal{S}$ and $\rho : \widehat{\mathcal{S}} \twoheadrightarrow_{\mathcal{R}} \Gamma_C$, the construction given by this proof may yield several derivations of $\Gamma \vdash C$. This non-uniqueness lies in the possibility of several main leaves in case $l = 0$ and Γ is non-trivial.

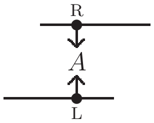
However, any such derivation $\mathcal{D}$ satisfies the property that there is a bijective correspondence between the links of $\mathcal{S}$ and the logical rules of $\mathcal{D}$. Observe that each par link of $\mathcal{S}$ also corresponds to a contraction in ρ. The remaining conversions, i.e. the structural conversions, correspond to structural rules of $\mathcal{D}$. What about the identity rules of $\mathcal{D}$?

We claim that is it possible to adapt the proof of Theorem 6.5.4 in such a way that all found derivations $\mathcal{D}$ of $\Gamma \vdash C$ satisfy the following, in addition to the above property:

- there is a bijective correspondence between axiomatic formulas and Ax rules, such that

  corresponds with an Ax rule on A;
- there is a bijective correspondence between cut formulas and Cut rules, such that

  corresponds with a Cut rule on A.

For this purpose we first state the following lemma.

Lemma 6.5.5. ***(Substitution)*** *Let $\mathcal{D}_1$ be a derivation of $\Gamma_1 \vdash C_1$ and $\mathcal{D}_2$ be a derivation of $\Gamma_2[C_1] \vdash C_2$.*
1. If $C_1 \vdash C_1$ is an axiom of $\mathcal{D}_1$, the succedent formula of which coincides with the succedent formula of $\Gamma_1 \vdash C_1$, then we can substitute $\mathcal{D}_2$ into $\mathcal{D}_1$ in order to get a derivation $\mathcal{D}_1[\mathcal{D}_2]$ of $\Gamma_2[\Gamma_1] \vdash C_2$.

$$\begin{array}{cc}
\begin{array}{c} \overline{C_1 \vdash \boldsymbol{C_1}} \quad \dots \\ \mathcal{D}_1 \quad \ddots \quad \vdots \quad \iddots \\ \Gamma_1 \vdash \boldsymbol{C_1} \end{array} & \begin{array}{c} \mathcal{D}_2 \\ \Gamma_2[C_1] \vdash C_2 \end{array} \\ \hdashline \multicolumn{2}{c}{\Gamma_2[\Gamma_1] \vdash C_2}
\end{array}
\quad \textit{becomes} \quad
\begin{array}{c} \mathcal{D}_2 \\ \boldsymbol{\Gamma_2}[C_1] \vdash \boldsymbol{C_2} \quad \dots \\ \mathcal{D}_1 \quad \ddots \quad \vdots \quad \iddots \\ \boldsymbol{\Gamma_2}[\Gamma_1] \vdash \boldsymbol{C_2} \end{array}$$

2. If $C_1 \vdash C_1$ is an axiom of $\mathcal{D}_2$, the antecedent formula of which coincides with the occurrence in $\Gamma_2[C_1] \vdash C_2$, then we can substitute $\mathcal{D}_1$ into $\mathcal{D}_2$ in order to get a derivation $\mathcal{D}_2[\mathcal{D}_1]$ of $\Gamma_2[\Gamma_1] \vdash C_2$.

$$\begin{array}{cc}
\begin{array}{c} \mathcal{D}_1 \\ \Gamma_1 \vdash C_1 \end{array} & \begin{array}{c} \dots \quad \overline{\boldsymbol{C_1} \vdash C_1} \\ \ddots \quad \vdots \quad \iddots \quad \mathcal{D}_2 \\ \Gamma_2[\boldsymbol{C_1}] \vdash C_2 \end{array} \\ \hdashline \multicolumn{2}{c}{\Gamma_2[\Gamma_1] \vdash C_2}
\end{array}
\quad \textit{becomes} \quad
\begin{array}{c} \mathcal{D}_1 \\ \dots \quad \boldsymbol{\Gamma_1} \vdash C_1 \\ \ddots \quad \vdots \quad \iddots \quad \mathcal{D}_2 \\ \Gamma_2[\boldsymbol{\Gamma_1}] \vdash C_2 \end{array}$$

$\Diamond$

Proof: In general, every leaf of a tree determines a path to the root. In particular every axiom rule of a derivation determines a path of sequents from that axiom to the conclusion of the derivation. Let $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ be two successive sequents in a certain path β, i.e. $\Gamma' \vdash \Delta'$ is the conclusion of an inference rule with $\Gamma \vdash \Delta$ among its hypotheses. For a binary inference rule R we say that β passes R via the left (right) hypothesis if $\Gamma \vdash \Delta$ is the first (second) hypothesis of R.

1. As the occurrence $\boldsymbol{C_1}$ is preserved in the path β in $\mathcal{D}_1$ between $C_1 \vdash \boldsymbol{C_1}$ and $\Gamma_1 \vdash \boldsymbol{C_1}$, the possible inference rules that β passes are Cut (via the right hypothesis),

the left logical rules (if binary, then via the right hypothesis), or a structural rule. Each of these rules have the property that if

$$\frac{(\quad \Gamma_0 \vdash C_0 \quad) \qquad \Gamma_1 \vdash \boldsymbol{C_1}}{\Gamma_3 \vdash \boldsymbol{C_1}}$$

is an instance, then so is

$$\frac{(\quad \Gamma_0 \vdash C_0 \quad) \qquad \boldsymbol{\Gamma_2}[\Gamma_1] \vdash \boldsymbol{C_2}}{\boldsymbol{\Gamma_2}[\Gamma_3] \vdash \boldsymbol{C_2}}$$

2. As the occurrence $\boldsymbol{C_1}$ is preserved in the path β in $\mathcal{D}_2$ between $\boldsymbol{C_1} \vdash C_1$ and $\Gamma_2[\boldsymbol{C_1}] \vdash C_2$, it will never be an active formula in any inference rule β passes. Hence, if

$$\frac{(\quad \Gamma_0 \vdash C_0 \quad) \qquad \Gamma_2[\boldsymbol{C_1}] \vdash C_2}{\Gamma_3[\boldsymbol{C_1}] \vdash C_3}$$

is an instance of a rule, then so is

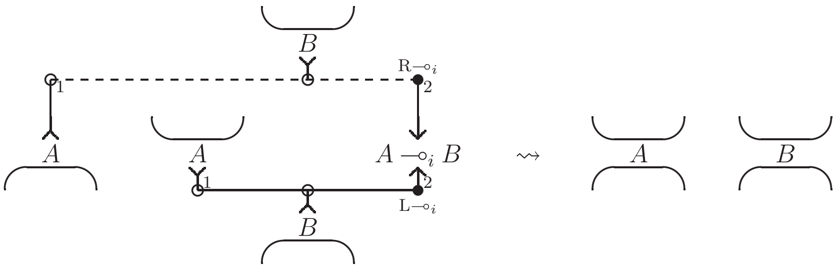
$$\frac{(\quad \Gamma_0 \vdash C_0 \quad) \qquad \Gamma_2[\boldsymbol{\Gamma_1}] \vdash C_2}{\Gamma_3[\boldsymbol{\Gamma_1}] \vdash C_3}$$

////

Now extend the proof of Theorem 6.5.4 by simultaneously showing that every axiomatic formula corresponds to an Ax rule, and moreover that every axiomatic conclusion corresponds to an axiom as in Lemma 6.5.5.1 and every axiomatic hypothesis corresponds to an axiom as in Lemma 6.5.5.2. We adapt the proof in the case that $l > 0$ and the last conversion step is a contraction: If the main formula of L (say: D) is a cut formula of $\mathcal{S}$ we proceed as described earlier. However, if D is not a cut formula, then D is an axiomatic leaf of one of the two substructures. Hence we can apply the par rule followed by the appropriate substitution.

6.6. Cut elimination

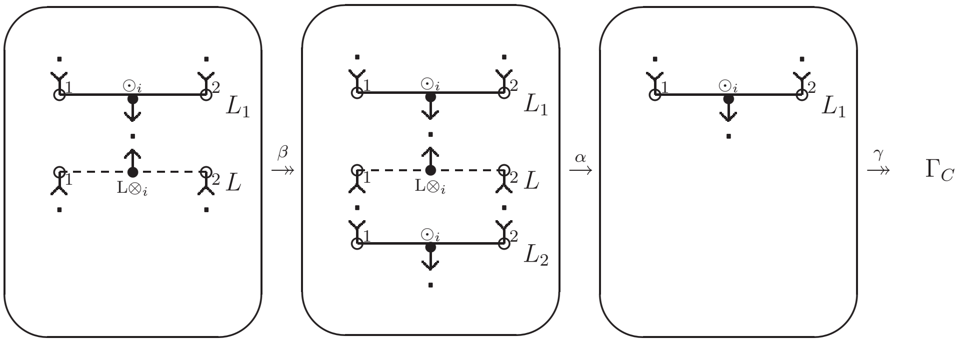
Recall that a cut formula is a formula which is the main formula of two dual links. A cut reduction step is defined by deleting these links and the cut formula, while pairwisely identifying the active formulas in case they are different (as occurrence of the same formula), or deleting them if they are identical.



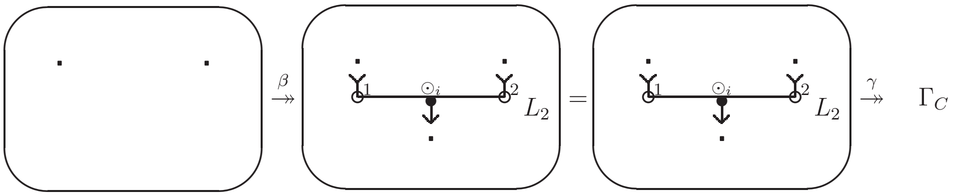
Let D be a cut formula and L the corresponding par link. We will show that, if $(\mathcal{S}, \rho)$ is a conversion sequence of $\Gamma \vdash C$, then so is $(\mathcal{S}', \rho')$, where $\mathcal{S} \rightsquigarrow \mathcal{S}'$, and ρ' consists of the same set of conversion steps as ρ, with the exception of the contraction α of L, in a sense to be made precise shortly.

THEOREM 6.6.1. *If $\mathcal{S}$ is a proof net of $\Gamma \vdash C$, and $\mathcal{S} \rightsquigarrow \mathcal{S}'$ by a cut reduction step, then $\mathcal{S}'$ is a proof net of $\Gamma \vdash C$ as well.* ♢

Example 6.6.2. 1. In order to get an idea of the proof, let us consider the following conversion sequence in which we assume L_1 remains untouched during β.

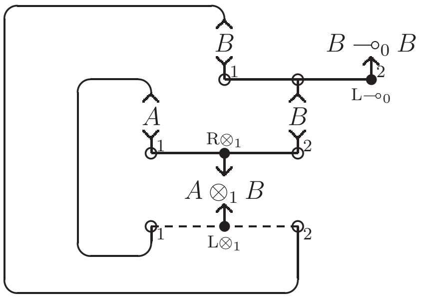


Executing a cut reduction step yields a proof structure to which we can apply the same conversion steps as before and in the same order, except the contraction α:

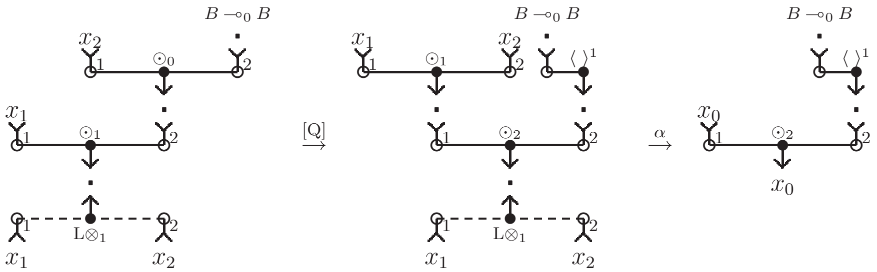


Observe that L_2 plays the role of L_1 in γ.

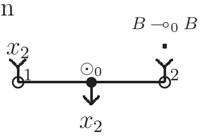
2. If L_1 does not remain untouched, we cannot in general transform an arbitrary conversion sequence $\widehat{\mathcal{S}} \overset{\beta}{\twoheadrightarrow} \mathcal{H}_1 \xrightarrow{\alpha} \mathcal{H}$ into $\widehat{\mathcal{S}'} \overset{\beta'}{\twoheadrightarrow} \mathcal{H}$. This is shown by the following proof structure in the calculus $\mathbf{NL}\Diamond_{\{[\mathrm{Q}]\}}$, where [Q] may be read off from the conversion step below.



This proof structure may be converted as follows:



but after cut elimination we obtain

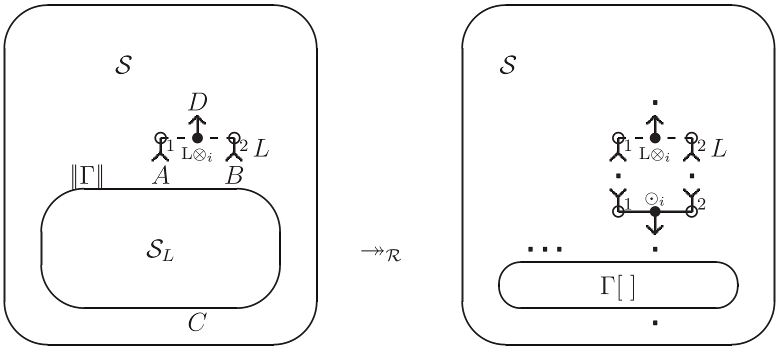
 ,

and no conversion step applies anymore, whence we cannot arrive at the same hypothesis structure as before. $\Diamond$

Let $(\mathcal{S}, \rho)$ be a conversion sequence of $\Gamma \vdash C$:

$$\widehat{\mathcal{S}} \overset{\rho}{\twoheadrightarrow}_{\mathcal{R}} \Gamma_C$$

For each par link L with main formula D and active formula(s) A (and B) we will define a substructure $\mathcal{S}_L$ of $\mathcal{S}$ (called the *block* of L in $\mathcal{S}$ w.r.t. ρ) and a subsequence ρ_L satisfying the following properties:



- $\mathcal{S}_L$ does not contain D. Consequently it does not contain L either.
- $\mathcal{S}_L$ has A (and B) among its leaves.
- The conversion sequence ρ_L acts completely within $\mathcal{S}_L$, and this restriction to $\mathcal{S}_L$ yields a *nice* hypothesis tree with respect to L. By this we mean that attaching L enables its contraction α.
- Our original conversion sequence may be replaced by

$$\widehat{\mathcal{S}} \overset{\rho_L}{\twoheadrightarrow}_{\mathcal{R}} \mathcal{H} \xrightarrow{\alpha} \mathcal{H}' \overset{\rho'_L}{\twoheadrightarrow}_{\mathcal{R}} \Gamma_C.$$

We will only sketch the idea; the formal definition and proof may be given simultaneously by induction on the length l of ρ.

First of all, deleting all p par links (but not their nodes) yield $p+1$ hypothesis trees, called the *components* of $\mathcal{S}$. This even holds for all intermediate hypothesis structures between $\widehat{\mathcal{S}}$ and Γ_C: Reasoning backwards from the hypothesis tree, we start with one component ($p = 0$). After a number of structural conversions, a contraction $\mathcal{H} \overset{\alpha}{\to} \Gamma'$ splits this component Γ' into two parts and replaces one node by a redex. The par link L of this redex now serves as a boundary between the two new components Γ_1 (attached to A (and B)) and Γ_2 (attached to D), while (at this moment) Γ_1 is a *nice* hypothesis tree w.r.t. L. All next structural conversions completely take place within one of the two components, and the next contraction takes place in exactly one of the two components as well. In this way every par link replaces one component by two new components, yielding $p+1$ components in each hypothesis structure, and $2p+1$ distinct components in the whole conversion sequence so far (read from right to left).

The block of a par link L is Γ_1 at stage $\mathcal{H}$, and further on it grows with the remaining conversions in ρ; it is clear that every conversion is completely inside the block, or completely outside the block, which proves our properties.

This shows that we can reorder our original conversion sequence ρ by ρ_1 in which L_1 remains untouched until α (cf. Example 6.6.2.1). Executing a cut reduction step yields a proof structure to which ρ_1 applies until α, and further on after α. (We refer to Example 4.5.2 for further details.)

6.7. Automated deduction

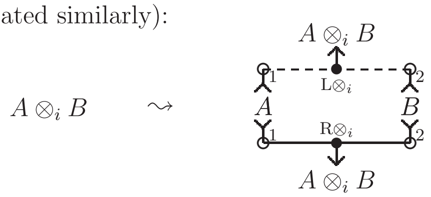
One of the attractive aspects of our formalism is that it lends itself well to automated proof search. First of all, in the previous section we saw that we could eliminate cut formulas from proof nets, making it unnecessary to consider cut formulas in our proof search. Secondly, we can restrict ourselves to proof nets where all our axiomatic formulas are atomic, as indicated by the following lemma:

Lemma 6.7.1. *Given a proof structure $\mathcal{S}$ we can construct a proof structure $\mathcal{S}'$ with the same hypotheses and conclusions where all axiomatic formulas are atomic and where $\widehat{\mathcal{S}'} \twoheadrightarrow_{\mathcal{R}} \widehat{\mathcal{S}}$, for an arbitrary set of structural conversions $\mathcal{R}$. We will call such a proof structure η-*expanded*. ◊

Proof: By induction on the total complexity of the axiomatic formulas.

If there are no complex axiomatic formulas in the proof structure, we take $\mathcal{S}' = \mathcal{S}$ and an empty conversion sequence.

If we have a proof structure $\mathcal{S}_0$ where the axiomatic formulas have $n+1$ total connectives, we can expand a complex axiomatic $A \otimes_i B$ formula in the following way (the other connectives are treated similarly):



The resulting proof structure $\mathcal{S}_1$ will have two new axiomatic formulas, and the total number of connectives of axiomatic formulas will be n.

By induction hypothesis we know that $\widehat{\mathcal{S}}_1' \twoheadrightarrow_{\mathcal{R}} \widehat{\mathcal{S}}_1$, so we can suffix a L$\otimes_i$ conversion producing $\widehat{\mathcal{S}}_1' \twoheadrightarrow_{\mathcal{R}} \widehat{\mathcal{S}}_1 \xrightarrow{\mathrm{L}\otimes_i} \widehat{\mathcal{S}}_0$. As we use only contractions, the theorem holds regardless of the structural rules. ////

As an immediate consequence of cut elimination and Lemma 6.7.1 we get the following corollary:

COROLLARY 6.7.2. *For every $\mathcal{R}$-proof net $\mathcal{P}$ of $\Gamma \vdash C$ there exists a $\mathcal{R}$-proof net $\mathcal{P}'$, also of $\Gamma \vdash C$, which is cut-free and η-expanded.* $\Diamond$

So we can, without loss of generality, restrict ourselves to proof structures where all complex formulas are neither axiomatic nor cut formulas. A simple algorithm for the enumeration of cut-free, η-expanded proof nets is the following.

Input – sequence $w_1, \ldots, w_n$ of words

– lexicon l, which assigns formulas to words

– goal formula Q

– set $\mathcal{R}$ of structural rules

Output set of cut-free, η-expanded proof nets from $\{l(w_1), \ldots, l(w_n)\}$ to $\{Q\}$

Usually we want to restrict this set of proof nets to those satisfying some additional constraints, like the one that left to right traversal of the hypothesis tree yields the formulas in the order indicated by the input sequence.

(1) For each of the words w_i in the input sequence, select one of the formulas assigned to this word from the lexicon.
(2) Decompose the formulas according to the links of Definition 6.3.1 until we reach the atomic subformulas. The disjoint union of these proof structures is a proof structure itself, though it will have several hypotheses in addition to those from the lexicon and several conclusions in addition to the goal formula.
(3) Identify each atomic premiss with an atomic conclusion to produce a proof structure from $\{l(w_1), \ldots, l(w_n)\}$ to $\{Q\}$.
(4) Convert the hypothesis structure corresponding to this proof structure to an hypothesis tree using only the structural conversions of $\mathcal{R}$ and the contractions.
(5) Check if this hypothesis tree satisfies our constraints.

We assume computation is nondeterministic, i.e. the steps of our algorithm can produce a number of solutions: the lexicon can produce different formulas for each word, and there can be many different ways of identifying the atomic formulas and many hypothesis trees to which we can convert our hypothesis structure. When one step in our algorithm fails to produce a solution, we backtrack to a previous step and try the next solution at this step until we have found all solutions.

As the connecting of step 3 and the conversions of step 4 are computationally expensive, it is desirable to do some static tests on the set of proof structures we get from the lexical formulas after step 2 of the algorithm, to make sure we at least have a chance of ultimately converting to a hypothesis tree. The following are two quick tests to reject proof structures which can never satisfy our correctness criterion.

First, by our definition of hypotheses and conclusions of proof structures, all atomic formulas, different from the lexical formulas or the conclusion Q, must be both a premiss and a conclusion of some link in a proof structure from $\{l(w_1), \ldots, l(w_n)\}$ to $\{Q\}$. So by counting we can determine whether each of these atomic formulas occurs as many times as a conclusion as it occurs as a premiss.

Secondly, the following lemma gives us a condition on the number of binary links occurring in a proof net.

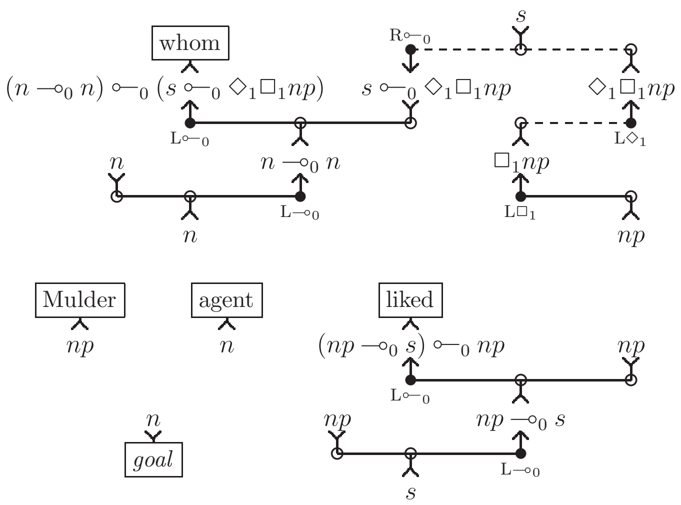
Lemma 6.7.3. *Suppose we have a proof structure $\mathcal{S}$ with h hypotheses, t binary tensor links, p binary par links and a single conclusion. Then the following holds if $\mathcal{S}$ is a proof net:*

$$t + 1 = p + h$$

$\Diamond$

Proof: Reasoning backwards from the hypothesis tree to the initial hypothesis structure we see that it holds for the hypothesis tree (with $p = 0$), that the structural conversions and the unary contractions preserve t, p and h, and that the contractions for the binary links increase t and p simultaneously. ////

Example 6.7.4. The proof structure we would get after lexical lookup and formula decomposition of 'agent whom Mulder liked' is the following:



We have marked the lexical hypotheses (resp. the conclusion) with an arrow leaving the formula from above (resp. below). This just is a reminder they should not be used as the conclusion (resp. premiss) of a link.

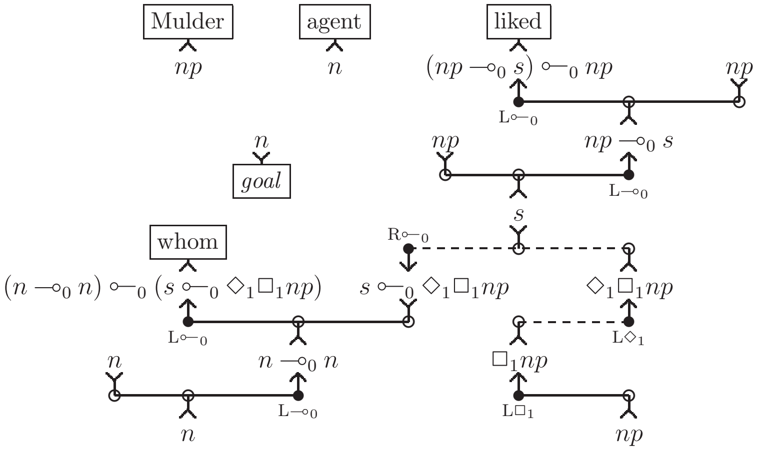
Now we count the atomic premisses and conclusions which are available. We have two formulas n as premisses and two as conclusions, we have one formula s as a premiss and one as a conclusion and we have two formulas np as premisses and two as conclusions. So it should at least be possible to find a way to identify these, and produce a proof structure from $\{n, (n \multimap_0 n) \multimapinv_0 (s \multimapinv_0 np), np, (np \multimap_0 s) \multimapinv_0 np\}$ to $\{n\}$.

Next we count 4 lexical hypotheses, 4 binary tensor links, and 1 binary par link, satisfying our equation $t + 1 = p + h$, and proceed to the identification phase.

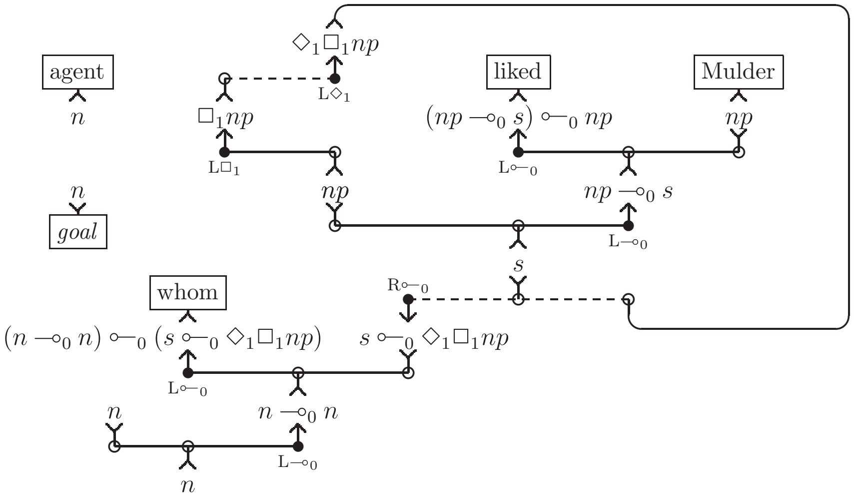
The proof structure for '*agent whom Mulder liked Skully' would fail both tests. There would be one more np occurrence as a conclusion, and there would be one hypothesis too many. $\Diamond$

For the identification of the atomic formulas we will generally have a number of possible choices. Given a proof structure in which an atomic formula A occurs k times as a premiss (and k times as a conclusion, according to our count check), there will be $k!$ ways of performing these identifications. In the proof structure above we count 1 formula s, 2 formulas np, and 2 formulas n, giving us a total of $1! \times 2! \times 2! = 4$ proof structures that we have to consider. For larger examples, this will quickly lead to an unacceptably long computation time. However, we will see that a smart algorithm can in many cases perform better than the worst case complexity might suggest, by exploiting the different kinds of information present in the proof structure.

Example 6.7.5. As a first step we identify the premiss s with the conclusion, resulting in the following proof structure:



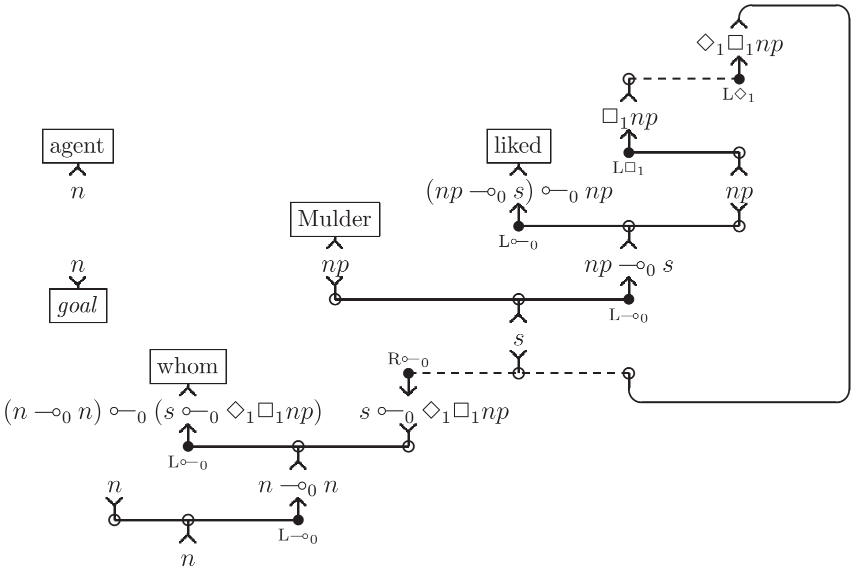
Next, we decide to identify the np formulas. Here we have two choices: either the lexical np formula is a premiss of the L$\multimap_0$ link, or it is a premiss of the L$\circ\!\!-_0$ link. We choose first to explore the second possibility, and get the following proof structure:



Now, when we look at the hypothesis structure of this proof structure, we see that no structural conversions apply to it and that the only contraction we can apply is the one for L$\Diamond_1$, after which we will be unable to contract the R$\multimapinv_0$ link or apply any structural conversion. Furthermore, this will apply to any hypothesis structure we get after identifying more formulas of the proof structure, because the R$\multimapinv_0$ link remains splitting in the sense of our sequentialization theorem, regardless of which additional formulas we identify in this proof structure.

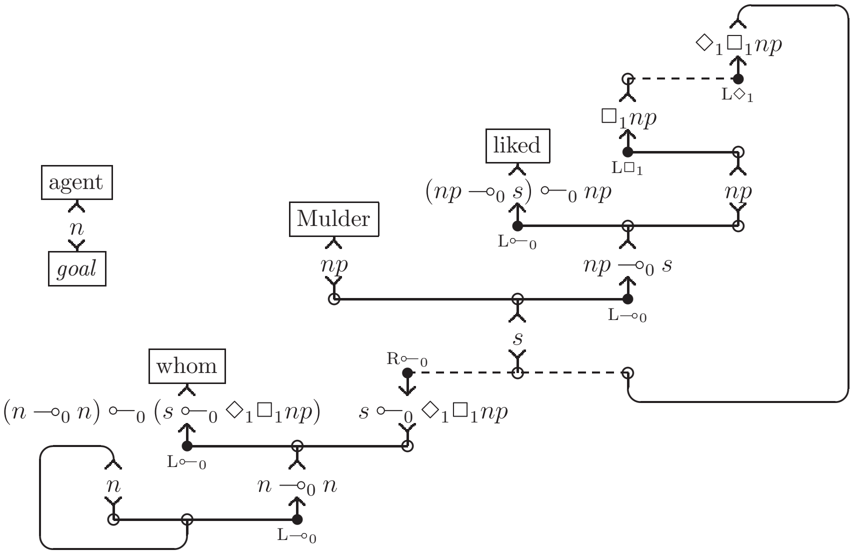
Observe also that if we would have used ‘which’ instead of ‘whom’ for this example, we could have applied the [Com$_{0,0}$] conversion at this point and continued to produce a proof net of ‘agent which liked Mulder’.

We return to the other way of identifying the np formulas, resulting in the following proof structure:



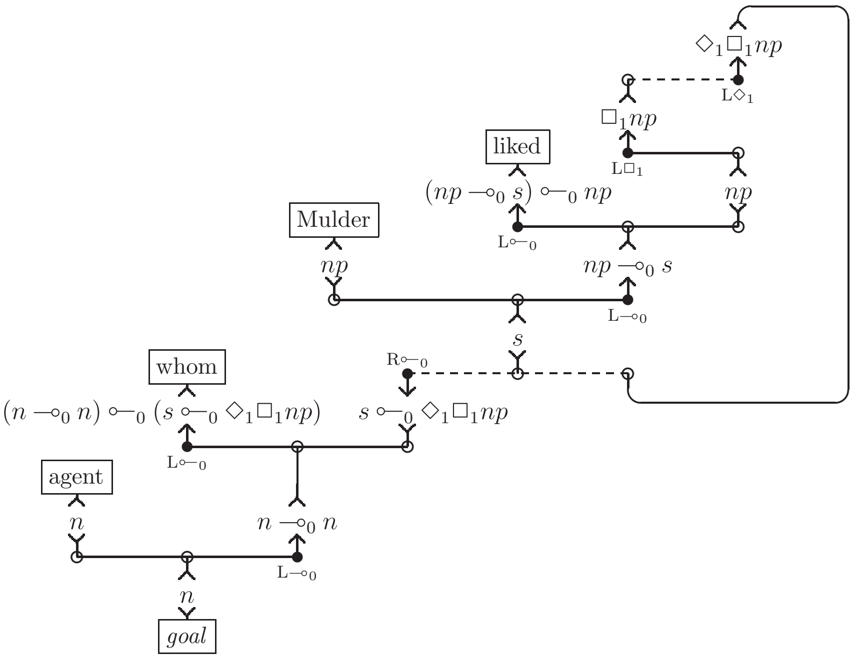
Reasoning as before, we see that at this point we *can* contract the R$\multimapinv_0$ link after both the [Ass$_{0,1}$] structural conversion and the L$\Diamond_1$ contraction, so continuing is warranted.

For the n formulas we again have two choices: we either identify the n hypothesis with the conclusion of the proof structure or with the conclusion of the L$\multimap_0$ link. Selecting the first option will give us the following proof structure:



Now we can immediately see we will be unable to contract the hypothesis structure of this proof structure to a tree, as none of our conversions allow us to connect the two separate components of the hypothesis structure and therefore we can never convert to a tree.

Fortunately, the final proof structure of this sequent



can be converted to an hypothesis tree (almost) as shown in Example 6.3.11. We can also see that this hypothesis tree has all formulas in the desired order.

This completes the enumeration of all possible cut-free, η-expanded proof nets from $\{n, (n \multimap_0 n) \circ\!\!-_0 (s \circ\!\!-_0 \Diamond_1 \Box_1 np), np, (np \multimap_0 s) \circ\!\!-_0 np\}$ to $\{n\}$ for the given fragment. $\Diamond$

APPENDIX A

Systems

In this appendix we review the calculi that are central to this thesis.

A.1. MLL

One-sided Multiplicative Linear Logic ($\mathbf{MLL}_1$) is concerned with finite multisets of formulas, e.g.

$$\Gamma = \{X, Y^\perp \otimes Y^\perp, (X \multimap Y)^\perp, X\}.$$

It is easy to see that Γ is not derivable when X and Y are two distinct atoms. Indeed, any derivable sequent (multiset) contains every atom an even number of times (since the positive occurrences are equinumerous to the negative ones). If $X = Y$ this multiset is derivable, however. Let us mention the derivation rules and show the derivation of $\Gamma = \{X, X^\perp \otimes X^\perp, (X \multimap X)^\perp, X\}$.

$\mathbf{MLL}_1$

$$\frac{}{\vdash X, X^\perp}\,\textsc{Ax}$$

$$\frac{\vdash \Gamma, X \qquad \vdash \Delta, X^\perp}{\vdash \Gamma, \Delta}\,\textsc{Cut}$$

$$\frac{\vdash \Gamma, X \qquad \vdash \Delta, Y}{\vdash \Gamma, \Delta, X \otimes Y}\,\text{R}\otimes \qquad \frac{\vdash\ \Gamma, X, Y}{\vdash\ \Gamma, X \parr Y}\,\text{R}\parr$$

Now a derivation of Γ is given by

$$\dfrac{\dfrac{}{\vdash X, X^\perp}\,\textsc{Ax} \qquad \dfrac{\dfrac{}{\vdash X, X^\perp}\,\textsc{Ax} \qquad \dfrac{}{\vdash X, X^\perp}\,\textsc{Ax}}{\vdash X, X^\perp \otimes X^\perp, X}\,\text{R}\otimes}{\vdash X, X, X^\perp \otimes X^\perp, X^\perp \otimes X}\,\text{R}\otimes$$

which equals Γ since the order does not matter, and since $(X \multimap X)^\perp = (X^\perp \parr X)^\perp = X^\perp \otimes X$.

The two-sided version ($\mathbf{MLL}_2$) is obtained by allowing "negative" occurrences of formulas, so that we can distinguish hypotheses and conclusions in a sequent. A sequent corresponding to the one-sided sequent Γ above, is given by

$$A \parr A, A \multimap A \ \vdash\ A, A$$

but also by

$$\vdash A, A^\perp \otimes A^\perp, (A \multimap A)^\perp, A$$

where $(-)^\perp$ and $- \multimap -$ should be seen as primitive connectives, and A is a formula in the two-sided language corresponding to X.

Finally, in the intuitionistic calculus (**IMLL**) we restrict to two-sided sequents with only one conclusion $\Gamma \vdash C$. The derivability question becomes rather: can we infer C from the open assumptions in Γ (using each assumption exactly once, as usual in linear logic)?

A.2. NCLL

In Non-commutative Cyclic Linear Logic (**NCLL**) sequents are cyclic lists rather than multisets. The one-sided sequent calculus is given by the rules

NCLL$_1$

$$\frac{}{\vdash \left(\!\left[X, X^\perp\right]\!\right)}\ \textsc{Ax}$$

$$\frac{\vdash (\![\Gamma, X]\!) \qquad \vdash \left(\!\left[\Delta, X^\perp\right]\!\right)}{\vdash (\![\Gamma, \Delta]\!)}\ \textsc{Cut}$$

$$\frac{\vdash (\![\Gamma, X]\!) \qquad \vdash (\![\Delta, Y]\!)}{\vdash (\![\Gamma, X \otimes Y, \Delta]\!)}\ \mathrm{R}\otimes \qquad \frac{\vdash (\![\Gamma, X, Y]\!)}{\vdash (\![\Gamma, X \parr Y]\!)}\ \mathrm{R}\parr$$

When we allow positive and negative occurrences of formulas, our sequents become cyclic lists of polarized formulas. It would be a severe restriction to require that all hypotheses be next to each other. Indeed, we will see natural examples violating this requirement. Hence we cannot represent a general sequent by $\Gamma \vdash \Delta$ (Γ and Δ lists), because it only represents a sequent of the form $(\![\Gamma^-, \Delta^+]\!)$. However, in the intuitionistic version all hypotheses *are* next to each other, as there is only one conclusion. So now our sequents $(\![C^+, \Gamma^-]\!)$ can be represented as $\Gamma \vdash C$ (Γ^- being Γ in reversed order and formula-wise provided with a negative sign), and we obtain Lambek calculus (**L**). As the order of the formulas counts, the derivability question asks if C is derivable from the open assumptions Γ, *in that order*. This question becomes of linguistic interest if we ask whether we can make a sentence out of some words. Of course, the answer depends on the order of the words. Lambek calculus is given by the following rules.

L

$$\frac{}{A \vdash A}\,\textsc{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash C}{\Delta_1, \Gamma, \Delta_2 \vdash C}\,\textsc{Cut}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\,\mathrm{L}\otimes \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\mathrm{R}\otimes$$

$$\frac{\Gamma \vdash A \qquad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, \Gamma, A \multimap B, \Delta_2 \vdash C}\,\mathrm{L}\!\multimap \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\,\mathrm{R}\!\multimap$$

$$\frac{\Gamma \vdash A \qquad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, B \multimapinv A, \Gamma, \Delta_2 \vdash C}\,\mathrm{L}\!\multimapinv \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B \multimapinv A}\,\mathrm{R}\!\multimapinv$$

A.3. CNL

We now want to take one further step: i.e. to abolish associativity. For the Lambek calculus the system we obtain is, not very surprisingly, known as Non-associative Lambek calculus (**NL**), and the sequents are rooted trees, the leaves representing the hypotheses and the root standing for the unique conclusion. Mimicking the tree structure in the antecedent part, our sequents are of the form $\Gamma \vdash C$ where Γ is a tree (constructed by means of a binary tree constructor $\odot$), and the rules are

NL

$$\frac{}{A \vdash A}\,\textsc{Ax}$$

$$\frac{\Gamma \vdash A \qquad \Delta[A] \vdash C}{\Delta[\Gamma] \vdash C}\,\textsc{Cut}$$

$$\frac{\Gamma[A \odot B] \vdash C}{\Gamma[A \otimes B] \vdash C}\,\mathrm{L}\otimes \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma \odot \Delta \vdash A \otimes B}\,\mathrm{R}\otimes$$

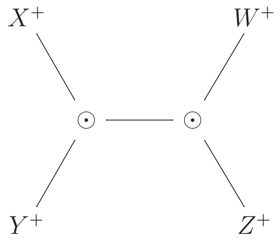
$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[\Gamma \odot A \multimap B] \vdash C}\,\mathrm{L}\!\multimap \qquad \frac{A \odot \Gamma \vdash B}{\Gamma \vdash A \multimap B}\,\mathrm{R}\!\multimap$$

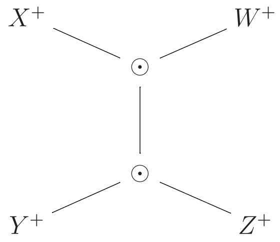
$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[B \multimapinv A \odot \Gamma] \vdash C}\,\mathrm{L}\!\multimapinv \qquad \frac{\Gamma \odot A \vdash B}{\Gamma \vdash B \multimapinv A}\,\mathrm{R}\!\multimapinv$$

One of our aims is to obtain this calculus as the intuitionistic fragment of a calculus which we will call Classical Non-associative Lambek calculus (**CNL**). In the one-sided version, if our sequents would be rooted trees, we immediately encounter the problem which rule for R$\otimes$ we should choose.

$$\frac{\vdash \Gamma \odot X \qquad \vdash Y \odot \Delta}{\vdash \big(\Gamma \odot X \otimes Y\big) \odot \Delta} \qquad \frac{\vdash \Gamma \odot X \qquad \vdash Y \odot \Delta}{\vdash \Gamma \odot \big(X \otimes Y \odot \Delta\big)}$$

So we should take a quotient on trees which make both inferred sequents equivalent. The solution is: cyclic trees. Paradoxically, on the outermost level they do satisfy commutativity as well as associativity. E.g., there are only two different cyclic trees on a 4-element cyclic list $(\!\![W^+, X^+, Y^+, Z^+]\!\!)$, viz.





where we have used geometrical means to specify the two equivalence classes.

This calculus **CNL** (and consequently **NL**) turns out to provide us with a basis for several extensions.

First of all, it is possible to introduce unary connectives $\Diamond$ (related to $\otimes$) and $\Box$ (related to $\parr$) as well, which play the role of modalities; this calculus is called the *base logic* (see [**Moortgat 97**]). Their structural counterpart is a unary tree constructor, viz. $\langle - \rangle$. For **NL**, the additional rules are

additional rules for the base logic **NL**$\Diamond$

$$\frac{\Gamma[\langle A \rangle] \vdash C}{\Gamma[\Diamond A] \vdash C}\,\text{L}\Diamond \qquad \frac{\Gamma \vdash A}{\langle \Gamma \rangle \vdash \Diamond A}\,\text{R}\Diamond$$

$$\frac{\Delta[B] \vdash C}{\Delta[\langle \Box B \rangle] \vdash C}\,\text{L}\Box \qquad \frac{\langle \Gamma \rangle \vdash B}{\Gamma \vdash \Box B}\,\text{R}\Box$$

Secondly, we can define different modes of connectives, e.g. two tensors $\otimes_0$, $\otimes_1$, implying two corresponding pars $\parr_0$, $\parr_1$ and two corresponding tree constructors $\odot_0$, $\odot_1$. The fine structure of our sequents prevents the two to become equivalent. So, a priori, there is no relation between them, except that they play a symmetric role and hence are indistinguishable.

However, we can add structural rules to the calculus, which may depend on the different modes. Adding rules enables us to consider a connective as "structurally strong" as we desire. So **CNL**, being of only little computational power, has the attractive quality of being easily extendible to the calculus of our interest, by "plugging in" modes and appropriate structural rules. For **NL**, given a set $\mathcal{R}$ of defining tuples $\langle \Xi, \Xi', \pi \rangle$ where Ξ and Ξ' are trees and π a permutation, the additional rules are additional rules for $\mathbf{NL}_{\mathcal{R}}$

$$\frac{\Delta[\ \Xi[\ \Gamma_1, \ldots, \Gamma_n\ ]\ ]\ \vdash\ C}{\Delta[\ \Xi'[\ \Gamma_{\pi_1}, \ldots, \Gamma_{\pi_n}\ ]\ ]\ \vdash\ C}\ \langle \Xi, \Xi', \pi \rangle$$

Example A.3.1. Let us consider two modes, and denote the connectives by

	structural node	conjunction	disjunction
mode 0	$\odot$	$\odot$ (next)	∇ (sequential)
mode 1	$\circledast$	$\otimes$ (times)	⅋ (par)

We postulate the following rules:

- mode 0 is associative;
- mode 1 is associative and commutative;
- $\odot \to \circledast$

We can represent the sequents of the thus obtained calculus by graphs with two types of links, viz. $\odot$ (defining a cyclic list of ends) and $\circledast$ (defining a multiset of ends). As soon as we replace a $\odot$ by $\circledast$, we introduce chaos (i.e. forget the order), and the link cannot interact with ∇ anymore, in a sense to be made precise in Section 6.3. It is an open question how this calculus relates to the Abrusci-Ruet calculus of order varieties ([**AR 98**]), but at least there are many correspondences. ◊

Among the sets of sequents of the different calculi we have the following maps:

$$\begin{tikzcd}
 & \mathfrak{L}_1 & \mathfrak{L}_2 & \mathfrak{L}_{2,\mathrm{i}} \\
\mathbf{MLL} & \mathbf{MLL}_1 \arrow[r, tail, shift right=1] \arrow[r, tail, shift right=3, "\nu{,}\psi"'] & \mathbf{MLL}_2 \arrow[l, two heads, shift right=2, "\pi"'] & \mathbf{IMLL} \arrow[l, hook', "\kappa"'] \\
\mathbf{NCLL} & \mathbf{NCLL}_1 \arrow[u, "\theta"] \arrow[r, tail, shift right=1] \arrow[r, tail, shift right=3, "\nu{,}\psi"'] & \mathbf{NCLL}_2 \arrow[u, "\theta"] \arrow[l, two heads, shift right=2, "\pi"'] & \mathbf{L} \arrow[u, "\theta"] \arrow[l, hook', "\kappa"'] \\
\mathbf{CNL} & \mathbf{CNL}_1 \arrow[u, "\theta"] \arrow[r, tail, shift right=1] \arrow[r, tail, shift right=3, "\nu{,}\psi"'] & \mathbf{CNL}_2 \arrow[u, "\theta"] \arrow[l, two heads, shift right=2, "\pi"'] & \mathbf{NL} \arrow[u, "\theta"] \arrow[l, hook', "\kappa"']
\end{tikzcd}$$

where the vertical maps are structure forgetting maps turning a cyclic tree into its underlying cyclic list, and turning a cyclic list into its underlying multiset. The horizontal maps are the continuations of the corresponding maps on formulas. The projection π actually computes the "De Morgan quotient" (treating A^- as $A^\perp$), while ν and ψ are two canonical injections, ν mapping $\alpha^\perp$ to $(\alpha)^\perp$ and commuting with α, $\otimes$ and ⅋ (and preserving the positive sign of X), and ψ assigning to X the unique $(-)^\perp$-free two-sided formula that — rightly polarized — equals X under π. We easily verify that θ commutes with all horizontal maps, and moreover we will see that $\pi\nu = \mathrm{id}$ and $\pi\psi = \mathrm{id}$.

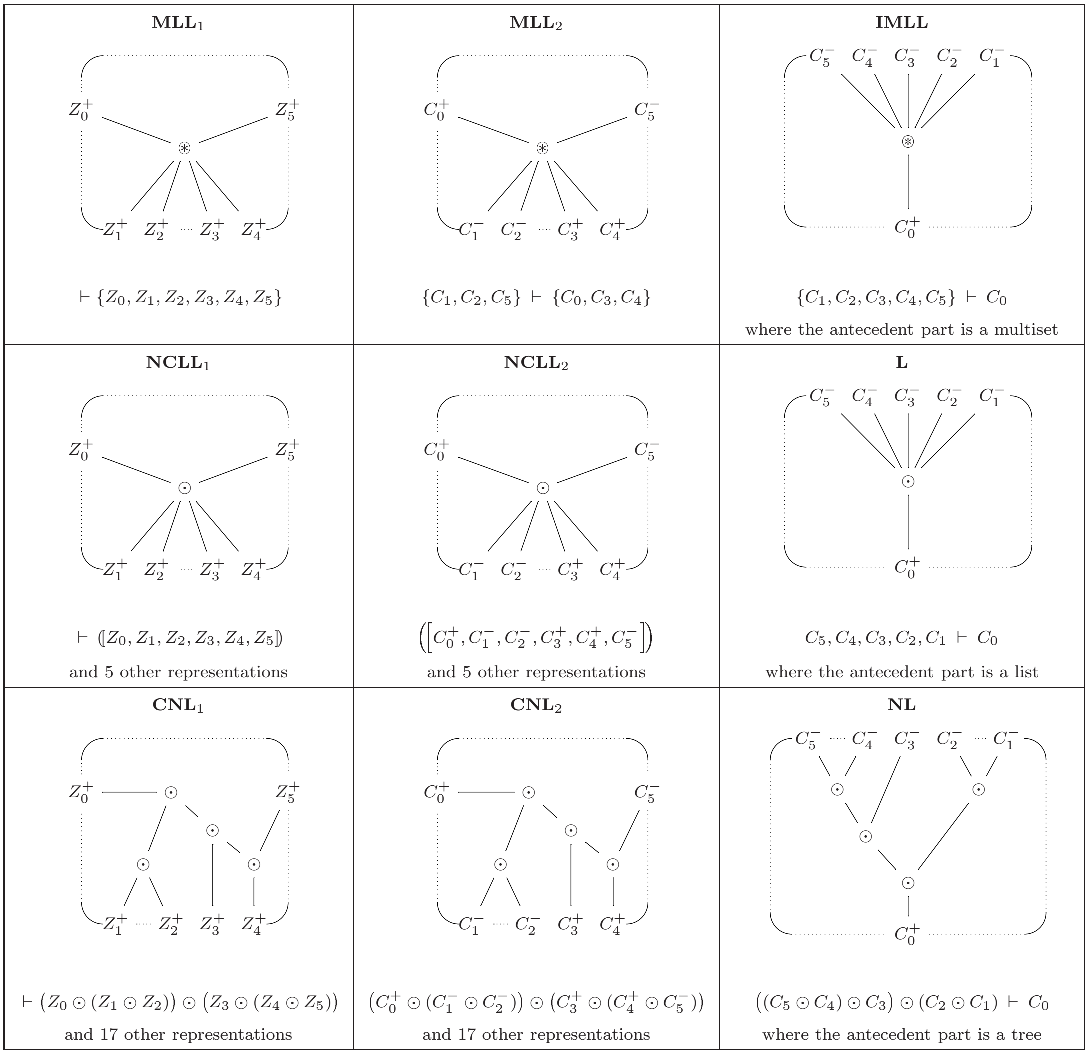


FIGURE A.1. The sequents of the different multiplicative calculi.

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Samenvatting

Lineaire logica is eind jaren tachtig ontsproten aan de observatie dat bepaalde logische connectieven beschouwd kunnen worden als de samenstelling van andere, meer primitieve, connectieven [**Girard 87**]. Dit heeft interessante gevolgen. Een "klassiek" connectief als $\wedge$ ('en') blijkt te bestaan in (minstens) twee gedaanten: de zogenaamde additieve ($\wedge_\mathrm{a}$) en de multiplicatieve ($\wedge_\mathrm{m}$) versie. Uit $A \wedge_\mathrm{a} B$ kunnen we zowel A als B concluderen, *maar niet beide*. Uit $A \wedge_\mathrm{m} B$ kunnen we ook zowel A als B concluderen, *maar niet slechts een van beide*. Lineaire logica heet wel *resource sensitive* omdat de multipliciteit van een formule (het aantal malen dat de formule voorkomt) van belang is en een "logische status" krijgt. 'A' staat voor één exemplaar van A en gedraagt zich net als $A \wedge_\mathrm{a} A$. Daarentegen staat $A \wedge_\mathrm{m} A$ voor twee exemplaren van A. Er is ook een manier om een willekeurig aantal exemplaren aan te duiden: $!A$. Deze zogenaamde exponent, '!', vormt de brug met de klassieke logica. Als voorbeeld van de bovengenoemde decompositie beschouwen we de ontbinding van de klassieke implicatie:

$$X \to Y = (!X) \multimap Y$$

Het is inderdaad intuïtief duidelijk dat Y uit X volgt precies dan als Y een, in principe willekeurig, aantal exemplaren van X benut: bijvoorbeeld in geval van de klassiek geldige formule $A \to (A \wedge A)$ benut $A \wedge A$ twee exemplaren van A; in $A \to (B \to B)$ maakt $B \to B$ helemaal geen gebruik van A. In dit proefschrift beschouwen we alleen varianten van het multiplicatieve fragment van lineaire logica.

De afleidbare objecten zijn *sequenten* van formules van de vorm

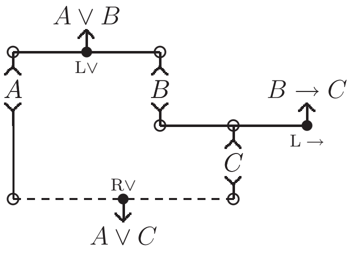
$$H_1, H_2, \ldots \vdash C_1, C_2, \ldots,$$

waarin een bepaalde formule meerdere keren mag voorkomen. Zo een sequent staat intuïtief voor de enkele formule $(H_1 \wedge H_2 \wedge \ldots) \to (C_1 \vee C_2 \vee \ldots)$.

[**Girard 87**] introduceerde eveneens het elegante begrip *bewijsnet*. Een bewijsnet staat voor een collectie van afleidingen, die enkel op "niet-essentiële" onderdelen verschillen en daarom in zekere zin "gelijk" zijn. Zo kan men tijdens het maken van een sequenten-afleiding voor de keuze komen te staan welke van een aantal mogelijke volgende stappen als *eerste* uit te voeren. In het corresponderende bewijsnet worden dan beide mogelijkheden *parallel* uitgevoerd. Als voorbeeld beschouwen we de volgende twee afleidingen, die slechts verschillen in de volgorde waarin de afleidingsregels L$\to$ en L$\vee$ zijn toegepast.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}}{A \vee B \vdash A, B}\,\mathrm{L}\vee \qquad \dfrac{}{C \vdash C}}{A \vee B, B \to C \vdash A, C}\,\mathrm{L}\!\to}{A \vee B, B \to C \vdash A \vee C}\,\mathrm{R}\vee \qquad\qquad \dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{}{B \vdash B} \qquad \dfrac{}{C \vdash C}}{B, B \to C \vdash C}\,\mathrm{L}\!\to}{A \vee B, B \to C \vdash A, C}\,\mathrm{L}\vee}{A \vee B, B \to C \vdash A \vee C}\,\mathrm{R}\vee$$

Hun respectievelijke bewijsnetten zijn inderdaad aan elkaar gelijk, en wel



Laat nu een sequent S gegeven zijn, die we willen onderzoeken op afleidbaarheid. Aan S kunnen we simpelweg alle mogelijke kandidaat-bewijsnetten toekennen, en deze checken op correctheid. Zo een kandidaat-bewijsnet, *bewijsstructuur* genaamd, is in feite een soort graaf, die alleen afhangt van S, en van bepaalde verbindingen tussen de atomaire subformules. Een criterium dat vaststelt of een bewijsstructuur daadwerkelijk een bewijsnet is, heet een correctheidscriterium. Er zijn verschillende van dergelijke criteria bekend voor multiplicatieve lineaire logica, die in aard variëren van graaftheoretisch en geometrisch tot algebraïsch. In dit proefschrift definiëren we een contractie-criterium: een bewijsstructuur is een bewijsnet dan en slechts dan als hij contraheert naar een bepaalde normaalvorm. De bedoelde contractie-relatie is gedefinieerd op de ruimte van zogenaamde *linkgrafen*, een begrip dat algemeen genoeg is om zowel bewijsstructuren als sequenten te omvatten.

Ons criterium geldt voor diverse *structurele verfijningen* van multiplicatieve lineaire logica. Voor een structurele verfijning is, naast het aantal malen dat een formule voorkomt, ook de onderlinge samenhang tussen de formules van belang. Dat wil zeggen, we maken onderscheid tussen $\vdash A, B, \Gamma$ en $\vdash B, A, \Gamma$ (wat leidt tot systemen als in [**Lambek 58, Yetter 90, Roorda 91, Retoré 93, Lambek 95, Abrusci 95, AR 98**]). Het specifieke systeem Niet-commutatieve Cyclische Lineaire Logica (**NCLL**) dat correspondeert met [**Yetter 90**] is onderwerp van Hoofdstuk 4. Wanneer we zelfs $\vdash \ldots, (A, B), C, \ldots$ en $\vdash \ldots, A, (B, C), \ldots$ onderscheiden, verkrijgen we het systeem Klassieke Niet-associatieve Lambekcalculus (**CNL**) van Hoofdstuk 5 dat correspondeert met [**dGL 00**]. Het gecontroleerd herinvoeren van dergelijke structurele regels (cf. [**Morrill 96, Moortgat 97**]) is het onderwerp van Hoofdstuk 6. De aldus verkregen grote variëteit aan substructurele systemen maakt het mogelijk de toegestane structurele regels vrij te kiezen afhankelijk van de beoogde toepassing. Automatische zinsontleding is een voorbeeld van zo een toepassing in de linguistiek.

In dit proefschrift voeren we de notie van een tweezijdig bewijsnet in: een bewijsnet mag open hypothesen hebben, en een link mag van een algemenere vorm zijn dan normaliter het geval is. Dit geeft onder andere aanleiding tot linken (zoals de hierboven toegepaste link L→) voor nieuwe connectieven, die — eenzijdig — alleen gedefinieerd zijn als operaties in de formuletaal. De zo verkregen theorie is in vele opzichten geschikter dan de eenzijdige theorie, bijvoorbeeld met betrekking tot dualiseerbaarheid, subnetten, de contractie-relatie, en de beperking tot intuïtionistische fragmenten. Onze tweezijdige benadering maakt bovendien duidelijk hoe en waarom bewijsnetten voor lineaire logica zijn wat natuurlijke deductie voor intuïtionistische logica is.

Het bewijs van de correctheid van snede-eliminatie ten opzichte van ons contractie-criterium vergt een grondige analyse van de afhankelijkheid van de opeenvolgende contractie-stappen in een gegeven conversie-rij. We menen dat dit bewijs kan dienen als een sleutel tot verdere resultaten inzake het parallel uitvoeren van de conversie-stappen.

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Curriculum vitae Quintijn Puite

Ik ben op 28 september 1971 geboren te Bennekom, gemeente Ede. Ik groeide op in Wageningen en haalde in 1990 het VWO-diploma aan de Rijksscholengemeenschap "Het Wagenings Lyceum" te Wageningen. Ik nam deel aan de 31st International Mathematical Olympiad in Beijing (China) in juli 1990, alwaar ik een eervolle vermelding behaalde. In september van hetzelfde jaar ging ik studeren aan de Universiteit Utrecht, en in juni 1991 haalde ik mijn propedeuse Wiskunde (cum laude) en Natuurkunde (cum laude). In augustus 1996 studeerde ik af in de Wiskunde (cum laude) bij Dr. H.A.J.M. Schellinx, met de scriptie "Correctness Criteria based on a Homology of Proof Structures in Multiplicative Linear Logic". Tijdens mijn studie gaf ik als studentassistent werkcolleges. Daarnaast was ik lid van het Utrechts Studenten Koor en Orkest (USKO).

In september 1996 werd ik aangesteld als assistent in opleiding (AIO) bij het Mathematisch Instituut van de faculteit Wiskunde en Informatica van de Universiteit Utrecht voor onderzoek in het aandachtsveld "The Geometry of Logic", onder begeleiding van Dr. H.A.J.M. Schellinx en met Prof.dr. I. Moerdijk als promotor. Dat onderzoek heeft geresulteerd in dit proefschrift. In het kader van mijn onderzoek bezocht ik diverse conferenties en instituten, onder andere in l'Aquila, Frascati en Rome (Italië); Edinburgh (Schotland, V.K.); Nancy en Rennes (Frankrijk). Bovendien was ik, als co-chair van de ESSLLI'99 Student Session Programme Committee, verantwoordelijk voor de reviewprocedure van de inzendingen op het gebied van de logica. In november 1999 nam ik deel aan de Nederlandse Studiegroep Wiskunde met de Industrie (tevens de 36th European Study Group with Industry). Hier werkte ik in groepsverband aan een probleem van KPN Research over efficiënt gebruik van geheugenruimte in WWW-caches.

Daarnaast heb ik gedurende mijn promotietijd met plezier en enthousiasme werkcolleges gegeven, begeleidde ik een groepje studenten bij hun 'Kaleidoscoop-werkstuk' en een student tijdens de voorbereiding van zijn 'kleine scriptie'. Tevens heb ik genoten van het meedoen aan muziekprojecten van diverse ensembles, waaronder Consort "de kleine Johannes", het Nederlands Studenten Kamerkoor (NSK) en PA'dam (Projectkoor Amsterdam).

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Notation

$\otimes, \parr$	the connectives of $\mathfrak{L}_1$; the connectives of $\mathfrak{L}_2$
$\multimap, \multimapinv, [-]^{\perp}$	the operations of $\mathfrak{L}_1$
$\multimap, \multimapinv, (-)^{\perp}$	the additional connectives of $\mathfrak{L}_2$
$\square$	a connective or an operation
$F \setminus G; F \xrightarrow{+} G$	$F \multimap G$
$G \mathbin{/} F; F \xrightarrow{-} G$	$G \multimapinv F$

$\{F\}$	the singleton multiset consisting of F
$\llparenthesis F \rrparenthesis$	the singleton list consisting of F
F	the singleton tree consisting of F
$\langle\!\vert F \vert\!\rangle$	the singleton multiset (list) (tree) consisting of F
Γ, Δ	the multiset union of the multisets Γ and Δ
Γ, Δ	the list concatenation of the lists Γ and Δ
$\Gamma \odot \Delta$	the binary tree with subtrees Γ and Δ
$\Gamma \diamond \Delta$	the multiset union (list concatenation) (binary tree) of Γ and Δ

$\equiv$	the De Morgan equivalence on $\mathfrak{L}_2$
$\simeq$	the equivalence on $\mathfrak{L}_2$ generated by De Morgan equivalence and associativity
$\simeq_{\mathrm{i}}$	the equivalence on $\mathfrak{L}_{2,\mathrm{i}}$ generated by De Morgan equivalence and associativity

$\#_{\otimes}(-), \#_{⅋}(-)$	the number of $\otimes$-symbols (⅋-symbols) (see Subsection 2.1.6)
$[\![-]\!]$	the list of positive occurrences of atoms (see Subsection 2.1.6)
$\langle\![-]\!\rangle$	the multiset of positive occurrences of atoms (see Subsection 2.1.6)
$\natural(-)$	the "number of negations" (see Subsection 2.1.6)

α_i	the atoms of $\mathfrak{L}_1$; the atoms of $\mathfrak{L}_2$
$\alpha_i^{\perp}$	the formal negation of an α_i: an additional atomic formula of $\mathfrak{L}_1$
$\alpha, \beta, \gamma, \ldots$	an atom of $\mathfrak{L}_1$; an atom of $\mathfrak{L}_2$
ζ, η	an edge of a link graph
$\hat{\eta}, \check{\eta}$	the two ends of an edge η
θ	a link type, e.g. $\otimes$, ⅋, $\perp$, $\odot$, $\circledast$
ι	the inclusion $\mathfrak{L} \hookrightarrow \mathfrak{L}^{\pm} : F \mapsto F^{+}$
κ	the inclusion $\mathfrak{L}_{2,\mathrm{i}} \hookrightarrow \mathfrak{L}_2$
λ	a labeling $\lambda : \widetilde{E} \rightarrowtail \mathfrak{L}^{\pm}$ (of a link graph)
μ	the map $\mathfrak{L}_2^{\pm} \to \mathfrak{L}_2 : \begin{cases} A^{+} \mapsto A \\ A^{-} \mapsto (A)^{\perp} \end{cases}$
ν	the injection $\mathfrak{L}_1 \overset{\nu}{\rightarrowtail} \mathfrak{L}_2$
$\xi^{\pm} : \mathfrak{L}^{\pm} \to \mathfrak{K}^{\pm}$	the extension of a map $\xi : \mathfrak{L} \to \mathfrak{K}^{\pm} : F \mapsto (F^{\bullet})^{\overline{F}}$
$\xi^{+}, \xi^{-} : \mathfrak{L}^{\pm} \to \mathfrak{K}^{\pm}$	the extensions of a map $\xi : \mathfrak{L}^{\pm} \to \mathfrak{K} : F^{\sigma} \mapsto \xi(F^{\sigma})$
π	the surjection $\mathfrak{L}_2 \overset{\pi}{\twoheadrightarrow} \mathfrak{L}_1$
π	the surjection $\mathfrak{L}_2^{\pm} \overset{\pi}{\twoheadrightarrow} \mathfrak{L}_1$
ρ, σ	a sign (i.e. + or $-$)
τ	the sign alternation map $\tau : \mathfrak{L}^{\pm} \to \mathfrak{L}^{\pm} : \begin{cases} F^{+} \mapsto F^{-} \\ F^{-} \mapsto F^{+} \end{cases}$
ϕ	the (decreasing) measure $\phi(\mathcal{P}) := \lvert\mathcal{E}\rvert + \lvert\mathcal{L}\rvert + 2\,\lvert\mathcal{L}'\rvert$
χ	the injection $\mathfrak{L}_1 \overset{\psi}{\rightarrowtail} \mathfrak{L}_2^{\pm} \overset{\mu}{\rightarrowtail} \mathfrak{L}_2$
ψ	the injection $\mathfrak{L}_1 \overset{\psi}{\rightarrowtail} \mathfrak{L}_2^{\pm}$
ω	a switching for a link graph

Γ, Δ, Π	a finite multiset (list) (tree) of formulas
Γ^+	Γ, formula-wise provided with a positive sign
Γ^-	Γ, formula-wise provided with a negative sign and in reversed order
$\Gamma \vdash \Delta$	the multiset (list) (tree) $\Delta^+ \diamond \Gamma^-$
Λ	the unit of the free group $\langle \mathcal{A} \rangle$

$\mathcal{A}$	the set of atoms $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$
$\mathcal{B}(l)$	the block of l in $\mathcal{P}$ w.r.t. δ
$\mathcal{C}$	a cluster
$\mathcal{D}$	a derivation
$\mathcal{E}$	a set of edges (of a link graph)
$\mathcal{G}$	a collection of link graphs (see Section 4.4)
$\mathcal{I}(A)$	the identity proof structure of $A \vdash A$
$\mathfrak{K}, \mathfrak{L}$	a language
$\mathfrak{L}_1$	the one-sided classical language
$\mathfrak{L}_2$	the two-sided classical language
$\mathfrak{L}_{2,\mathrm{i}}$	the intuitionistic language
$\mathfrak{L}^\pm$	$\mathfrak{L}^+ \cup \mathfrak{L}^-$
$\mathcal{L}$	a set of context links (of a link graph)
$\mathcal{L}'$	a set of connector links (of a link graph)
$\mathcal{P}$	a link graph
$\mathfrak{S}_n$	the symmetric group of degree n

$\mathsf{c}(A)$	the complexity of a formula A
e	an end of a link graph
l	a link of a link graph
$\mathsf{l}(A)$	the length of a formula A
i, j, m, n	a natural number

$A, B, C, D, \dots$	an $\mathfrak{L}_2$-formula
C_n	the nth Catalan number $\dfrac{1}{n+1}\dbinom{2n}{n}$; the number of rooted binary trees with $n+1$ leaves
$F, G, H, \dots$	a formula of a language
$N(A)$	the negative atomic subformulas of a formula A
$P(A)$	the positive atomic subformulas of a formula A
T_A	the upper construction tree of a formula A
T^A	the lower construction tree of a formula A
$X, Y, Z, W, \dots$	an $\mathfrak{L}_1$-formula
$X^\bullet$	the formula component of $\psi(X) = (X^\bullet)^{\overline{X}} \in \mathfrak{L}_2^{\pm}$
$\overline{X}$	the sign component of $\psi(X) = (X^\bullet)^{\overline{X}} \in \mathfrak{L}_2^{\pm}$