Stationary Configurations and Geodesic Description of Supersymmetric Black Holes

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Stationary Configurations and Geodesic Description of Supersymmetric Black Holes

Stationaire configuraties en geodetische beschrijving van supersymmetrische zwarte gaten

(met een samenvatting in het Nederlands)

Proefschrift

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Preface

Already in the late 18th century scientists speculated about the existence of stellar objects whose mass was confined to such a small space that the escape velocity of any other object would exceed the speed of light. For this to happen the mass density of the stellar object must be very large. This can occur if a star collapses under the pressure of its gravitational self-interaction. Our sun, for instance, would have to shrink to a ball with radius of approximately 3 km. The gravitational forces of such a collapsed star are so strong that nothing, not even light-rays, can emerge from it. Nowadays, these objects are called black holes, and they have, ever since their early theoretical discovery, fascinated physicists and non-physicists alike.

For many theoretical physicists black holes are a kind of laboratory, in which they put their ideas and theories to a test. In fact, many of the questions concerning black holes touch the fundamental open problem of contemporary theoretical physics, namely that of reconciling quantum theory with the theory of gravity (general relativity). When describing black holes elements of both theories become relevant. Black holes therefore serve as probes for the yet unknown theory of quantum gravity.

For general relativity, black holes are simply spacetimes that possess horizons. This means that any signal emanating from the region of spacetime within the horizon stays eternally trapped. In particular, the singularity at the center of a black hole cannot be seen from a distant observer. Such an observer can actually get to know very little about a black hole (without falling in). All information that is measurable is associated to long-range forces exerted by the black hole. Actually, from far away, a black hole looks very much like a particle with a certain mass and a certain charge. This means that, once the black hole has settled down in its final state, all the details of the in-falling matter and radiation, which formed the black hole during the period of collapse, have been averaged out. This is sometimes expressed by the one-liner: “a black hole has no hair”.

The theory of black holes is a well-developed subject in general relativity, and one of the cornerstones of this theory is formed by the laws of black hole mechanics. Remarkably, these laws share a close similarity to the laws of thermodynamics. One of the laws, for instance, states that the horizon area cannot decrease in any physical process. The same is true for the entropy of a thermodynamic system. This led Bekenstein to the conjecture that black holes are, in fact, thermodynamic ensembles and that their area is a measure for the entropy. In the context of the classical theory of black holes, however, this analogy is a purely formal one.

The study of quantum field theory in a spacetime containing a black hole supports the thermodynamic interpretation that Bekenstein gave to the black hole area. Under some very general assumptions, Hawing showed that black holes do emit particles due to a quantum effect. The effect responsible is called spontaneous pair production, with which the process is meant in which the vacuum spontaneously emits a particle and an anti-particle. If gravity is weak, the particle and the anti-particle enjoy only a very short lifetime as they almost immediately annihilate each other and the resulting energy is reabsorbed by the vacuum. On an average, energy therefore stays conserved. But when this particle/anti-particle-pair is subject to the strong gravitational forces just outside the black hole horizon, the anti-particle tends to be sucked in by the black hole, giving the particle a chance to escape before facing annihilation. The predominantly positive energy modes carried by the escaping particles are measured by a distant observer as radiation. This radiation was found to be that characteristic of a black body at the so-called Hawking temperature. This thermal radiation does not reveal anything of the inner structure of the black hole: it captures only the random fluctuations of the vacuum near the horizon, polarized by the strong gravitational forces. This is a rather disturbing conclusion, for it implies that a black hole is a sink for information: if particles in very particular quantum states fall into the black hole, all of the information concerning their states is lost, because the black hole radiates but thermally. Such an information loss seems to be in conflict with the quantum mechanical principle of unitary time-evolution.

On the other hand, Hawking's discovery that black holes radiate and hence have a temperature suggests that the analogy between the laws of black hole mechanics and the ones governing a thermodynamic system can be taken more literally. Recall that the thermodynamic properties, such as pressure, temperature, or entropy of an ideal gas, for instance, are explained in the context of statistical physics as averages of certain observables of an underlying quantum theory of microscopic degrees of freedom. The picture of treating a black hole as a thermodynamic system would become compelling if, in a similar way, the laws of black hole mechanics would result from the statistical treatment of the underlying degrees of freedom describing the black hole. What would these microscopic degrees of freedom be?

This question is one of the strongly debated issues in black hole physics. While the classical theory of gravitation is very successful in describing the large scale structure of the universe, its applicability is limited when it comes to the small scale structure of spacetime. Providing a description of the microscopic degrees of freedom of a black hole is a great challenge for any candidate theory of quantum gravity. String theory has provided some exciting insights into the microscopic nature of black holes, and much of this thesis is dedicated to the exploration of the consequences of this approach. String theory does not change the laws of quantum mechanics, but it delicately changes the way we think about spacetime and herewith gravity. According to string theory, the texture of spacetime is made up of vibrating strings, and their fusing and splitting captures all the possible ways spacetime can be deformed. These effects can be seen only at very small length scales, or equivalently, at very high energy scales. Spacetime then, as we perceive it, is something like the deep, collective rumbling of such vibrating strings. This spacetime is necessarily ten-dimensional. Six of the ten dimensions are wrapped on a tiny compact space, which is practicably invisible to us, such that the everyday laws of physics are still effectively defined in four-dimensional spacetime. String theory regards the microscopic degrees of freedom of black holes as the vibrational patterns produced by strings trapped on such tiny compactification spaces.

Outline of this thesis

This thesis is written for a specialist audience. I have decided not to introduce and review much of the material covered in this thesis in an elementary way. For most subjects there exists a vast amount of introductory texts and I provide references where this is appropriate. The first two chapters are, however, introductory in style. In chapter I an overview over some of the fascinating aspects of black hole physics is provided. In particular, I discuss the string theory approach to black hole entropy. One of the consequences of the string theory results is that black hole entropy can be understood within the context of an effective field theory only if one resorts to supergravity theories with higher-order curvature interactions. To this extent, I first introduce some relevant elements of $N=2$ supersymmetric theories and supergravity theories in chapter II. In chapter III $N = 2$ supergravity theories with higher-order curvature interactions are described. Chapter IV contains a classification of the fully supersymmetric vacua and a characterization of a large class of stationary BPS black hole configurations in the presence of higher-derivative interactions. In chapter V the derivation of the macroscopic entropy formula appropriate for theories with higher-order curvature interactions is reviewed and compared to the results of string theory. Furthermore, in the absence of higher-order curvature interactions, the metric on the moduli space of simple multi-centered black hole solutions is calculated. In chapter VI, finally, a formalism to derive the geodesic description of generic gravitational solitons is developed.

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I

Black holes and string theory

In this chapter an overview of some of the fascinating aspects of black hole physics is presented. Section 1 contains a discussion of the laws of black hole mechanics. These bear a striking similarity to the laws of thermodynamics. For quite some time, the consequences of this formal analogy remained opaque. One of the remarkable results of string theory is that it provides a microscopic description of a certain class of black holes. As a result, the thermodynamics of these black holes is derived from the statistical theory of the underlying microscopic degrees of freedom. The prototypical black holes described by string theory are the Reissner-Nordström black holes. Some of their properties are discussed in section 2. In sections 3 and 4 we turn to the microscopic description of these black holes and derive the entropy formula by microstate counting. In section 5 we discuss alternative approaches to black hole entropy.

1. The laws of black hole mechanics

The theory of black holes is a well-developed subject in general relativity. Two results form the cornerstone of this theory: the uniqueness theorems and the laws of black hole mechanics. The uniqueness theorems state that, while a black hole can form from an asymmetric gravitational collapse, the asymptotic equilibrium configurations of Einstein-Maxwell gravity are axisymmetric and characterized by just three parameters, the total mass M, the total charge Q, and the angular momentum J. All other details of the matter and radiation that form the black hole are dissipated off as gravitational and electromagnetic radiation in the process of collapse. The corresponding three-parameter class of equilibrium solutions is formed by the Kerr-Newman solutions. In this thesis we will be concerned mainly with the non-rotating subclass, the so-called Reissner-Nordström black holes. While the three parameters (M, Q, J) measure to the spatial asymptotic fall-off of the gravitational and electromagnetic fields there are also theorems that refer to the properties of the black hole horizons: one states that the surface gravity κ, which measures the acceleration of an object near the horizon, is constant on the horizon (see *e.g.* [1]), the other result [2] implies that the horizon area A of a black hole does not decrease in physical processes, $\delta A \geq 0$. These results are based on theorems of differential geometry and depend only on the geometrical definition of black hole horizons and on certain weak assumptions concerning the type of matter distribution.

The constancy of κ on the horizon and the non-decreasing horizon area are reminiscent of the zeroth and second law of thermodynamics, which state that the temperature is constant throughout a body in thermal equilibrium and that the entropy of such a system does not decrease in any physical process. This analogy is even more compelling in view of the differential mass formula derived in [3],

$$\delta M = \frac{\kappa}{8\pi}\delta A - \Omega\,\delta J + \Phi\,\delta Q\,.$$

This formula expresses the change in the total mass of the black hole under a small stationary perturbation of the solution. Here, the conjugate variables κ, Ω, and Φ are the surface gravity, the angular velocity at the horizon, and the co-rotating electric potential at the horizon, respectively. (There is also an analogue of the third law of thermodynamics, but we will not be concerned with this.) Taken together, these relations are called the laws of black hole mechanics. As stressed in [4], it is important to realize that at this point the similarity between the laws of thermodynamics and those of black hole mechanics is a purely formal one. The zeroth and second law of black hole mechanics are theorems of differential geometry and quite different in essence from the corresponding laws of thermodynamics, these being empirical laws describing the large scale approximation to a set of complicated underlying microscopic laws governing the equilibrium system. The analogy between mass M and energy E, surface gravity κ and temperature T, and horizon area A and entropy S seems particularly questionable when it comes to the temperature: by its very definition, a classical black hole does not radiate and there seems no way to run it as a heat-machine. It is therefore hard to understand why κ should have anything to do with the zero temperature of the classical black hole.

Quantum mechanically, black holes are not so cold after all. The spontaneous quantum particle creation in the immediate vicinity of the horizon results in so-called Hawking radiation [5]. In Hawking's approximation, the radiation is the perfectly thermal one of a black body. Its temperature is proportional to the surface gravity of the black hole,

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}\,. \tag{1}$$

This result supports the view that there is indeed more to the formal analogy sketched above but shows at the same time that a full understanding of these issues necessarily involves a quantum theory of black holes. The analogy could be taken literally, if the surface area of the horizon would in fact measure the entropy of some underlying microscopic degrees of freedom of the black hole, as conjectured first by Bekenstein [6],

$$\frac{S_{\mathrm{bh}}}{k_{\mathrm{B}}} = \frac{1}{4}\frac{A}{l_{\mathrm{P}}^2}\,.$$

Here, k_{B} is Boltzmann's constant and l_{P} is the Planck length. If the entropy could be understood in terms of statistical physics, where the entropy is associated with the logarithm of the degeneracy of states of the quantum black hole for given energy and charge, this picture would become compelling

It is here that string theory has made a remarkable breakthrough. We will discuss it in the following sections. For doing so we first review some properties of the Reissner-Nordström black holes. These black holes serves as a prototype for the string theory discussion. We will discuss more general black holes in chapter IV.

2. Reissner-Nordström black holes

Let us take a closer look at the class of static non-rotating charged black holes. The metric and electromagnetic field strength of these so-called Reissner-Nordström black holes are given by[a]

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\,, \tag{2}$$

$$F = \frac{Q}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r\,, \tag{3}$$

where $\Delta = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-)$ and $\mathrm{d}\Omega_2^2$ is the SO(3)-invariant metric on the unit two-sphere S^2. Here, M denotes the mass of the black hole, Q the charge, and r is the radial coordinate. This configuration is a solution of the Einstein and Maxwell equations, which derive from the bulk action[b]

$$(2\kappa_4^2)\,S_4 = \int \mathrm{d}^4x\,\sqrt{|g|}\,R - \frac{1}{2}\int *F \wedge F\,,$$

where R is the Ricci scalar and F is the U(1)-field strength. If $M < |Q|$ there are no real roots $r_\pm$, and there is a naked singularity, which is not hidden by a horizon. A classical argument shows that such a spacetime cannot have formed by gravitational collapse (see *e.g.* [7]). For $M \geq |Q|$ one has two real roots $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The outer radius r_+ defines the location of the future event horizon. The associated surface gravity κ is given by

$$\kappa = \frac{(r_+ - r_-)}{2r_+^2} = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}\,.$$

The surface gravity is independent of the angular variables and therefore indeed constant on the horizon. An interesting limit to consider is $M \to |Q|$, for which the horizon degenerates and the surface gravity vanishes, $\kappa \to 0$. The black hole is called extremal in this limit. It still describes a smooth geometry, the singularity of the black hole being hidden just behind the horizon at $r_\pm = M$. Semi-classically, extremal black holes are stable and do not evaporate, since their Hawking temperature (1) vanishes.

[a] Appendix A contains a summary of notations and conventions used in this thesis.

[b] The four-dimensional gravitational coupling constant κ_4 is related to four-dimensional Newton's constant by $\kappa_4^2 = 8\pi G_{\mathrm{N}}$. We will usually use Planck units for which $G_{\mathrm{N}} = \hbar = c = 1$. In these units all quantities, such M, Q, or the radius r are dimensionless.

Another interesting feature is that in the extremal limit there exist multi-center generalizations of this geometry, describing stationary configurations of multiple extremal holes placed at arbitrary relative positions. These configurations are possible due to the exact cancellation of the electric repulsion and the gravitational attraction. We discuss such configurations at length in chapter IV.

In a supersymmetric context, these properties (along with the mass bound $M \geq |Q|$ that guarantees the regularity of the solutions) can be understood by supersymmetry: the Reissner-Nordström black holes are solutions of the field equations derived from an $N = 2$ supersymmetric extensions of the Hilbert-Maxwell action, and as such subject to $N = 2$ supersymmetry transformations. The underlying supersymmetry algebra has a central charge Z. Generic asymptotically flat solutions do not preserve any global supersymmetries, and hence constitute long representations of the supersymmetry algebra. Their masses are subject to the $N = 2$ supersymmetric mass bound $M \geq |Z|$. The extremal Reissner-Nordström black holes, on the other hand, preserve one globally defined Killing spinor and hence constitute short representations of the $N = 2$ supersymmetry algebra. Consequently, these configurations saturate the mass bound, $M = |Z|$, where the central charge in the present case is given by $|Q|$. Such solutions are called BPS configurations. This interpretation of the mass bound (and its saturation) is much like the interpretation of the Bogomol'nyi mass bound in Yang-Mills gauge theory.

3. D-branes, *p*-branes, and microstate counting in string theory

Many curved string backgrounds are known, so-called p-branes, describing brane-like solutions of the equations of motion of one of the various ten-dimensional supergravity theories. The p-branes are extended in p spatial directions and describe non-trivial spacetime geometries carrying Ramond-Ramond (RR) fluxes or fluxes of the Neveu-Schwarz (NS) gauge fields. In the latter case these are the fundamental string, called the F-string, and the NS5-brane. There are also 2-branes (membranes) and 5-brane solutions of eleven-dimensional supergravity, termed M2-branes and M5-branes. We will give explicit M5-solutions in the next section.

The various sets of ten-dimensional supergravity field equations result from the requirement that the non-linear sigma model, describing the propagation of strings in some background, is at a critical point and hence corresponds to a conformal field theory. The conditions of criticality (the β-functions) are calculated in string perturbation theory by a double expansion: one is the loop expansion given in terms of the string coupling constant, which is related to the vacuum expectation value of the dilaton. The other is an expansion in the dimensionful parameter α' and describes the coupling of the string world-sheet to operators of higher mass dimension. Both these expansions modify the conditions of criticality and therefore the ten-dimensional actions these conditions are derived from. There is evidence that the various ten-dimensional field theories one obtains by considering different types of strings are all related to a single eleven-dimensional theory called M-theory.

The relevance of the p-brane solutions was fully appreciated when open string theories with Dirichlet boundary conditions, so-called Dp-branes, were studied. An open string theory on a Dp-brane is a string theory describing world-sheets whose boundary is held fixed on a p-dimensional spatial hypersurface. The ends of the strings in a D3-brane theory, for instance, sweep out world-lines in a $3+1$-dimensional spacetime. It is simple to quantize a Dp-brane string theory, if the p-dimensional hypersurfaces are flat and embedded in flat spacetime, since in this case the open strings are free. In these flat D-brane theories one finds modes in the spectrum (of the open strings attached to the D-brane) which deform these rigid hyperplanes. One identifies such modes with the fluctuation modes of these rigid hyperplanes themselves, suggesting that they are dynamical objects in their own right. This is much like the case of the closed perturbative string defined on a flat spacetime background. It contains massless gravitational modes which represent fluctuations of the background itself. More striking was the realization [8] that the D-branes actually carry charges of the RR gauge fields and are in one-to-one correspondence with the various RR p-brane solutions of the effective ten-dimensional field theories.

Many of the p-brane solitons are black, that is, they posses event horizons in the extended dimensions transverse to the branes. If one therefore wraps the p spatial directions, along which a black p-brane is stretched, on a tiny compact manifold, one is effectively left with the extended dimensions containing a horizon. String theory thinks about black holes as interacting strings trapped on tiny compact manifolds. More precisely: the string backgrounds, described by the lower-dimensional effective field theory backgrounds such as the Reissner-Nordström black holes, are viewed as the long-range fields produced by stable classical string sources of elementary (closed) strings, oscillating and wrapped around compact dimensions. String theory accounts for the entropy of black holes by considering the degeneracy of such oscillating and wrapped string configurations, which produce the same long-range fields and therefore the same asymptotic charges. Every one of these solitonic string configurations defines a consistent string background and hence a conformal sigma model. One way to account for this degeneracy is to analyze the spectrum of one such conformal sigma model. Certain states in its spectrum correspond to marginal operators that deform the reference sigma model to another nearby conformal theory defining a string background with the same long-range behavior. When speaking about supersymmetric black holes the relevant conformal field theories must possess a certain amount of (world-sheet) supersymmetry, and the degeneracy of these conformal theories is described by supersymmetric marginal deformations. For supersymmetric sigma models the space of such deformations is determined by the (cohomology of the) target space. The problem therefore often reduces to one of understanding the topological properties of the wrapped compactification manifold. In the next section we will present a simple example of such a microstate counting, for which the problem reduces to enumerating the different possible intersections of the branes on the compactification manifold.

In practice not every lower-dimensional black hole can be described by string theory, as it is not sufficient to only identify the relevant conformal sigma model describing the interacting strings trapped on the compactification manifolds. Quantitative results, say for the spectrum of the theory, can be given only if the string theory perturbation expansion is controllable. This means that the effective string coupling (measured by the dilaton in the p-brane background) must be small such that string loop corrections are subleading. At the same time, various curvatures and field strengths in the string frame must not blow up, such that world-sheet corrections (α'-corrections) are subleading. This is the case for many dyonic black holes in the limit of large charges, and we will discuss an example of such a black hole in the next section.

Lower-dimensional black holes can be realized by wrapping branes with NS charges or with RR charges. The first proposal [9] was to identify the microstates of extremal electric black holes with the excitations of the fundamental string. This was worked out in great detail in [10,11]. The realization of black holes in terms of RR-branes overwhelmed the discussion ever since the discovery of the D-brane technology. The method one resorts to in this context is the use of a weak-strong coupling duality. This amounts to swapping the conformal field theory description of the throat region of the RR p-brane (for the "electrically" charged backgrounds with $p > 3$ the effective string coupling is large in the limit of large charges), with the weakly coupled Dp-brane theory of the flat rigid p-hyperplanes corresponding to the asymptotic region of the curved p-brane solution. Of course, such a duality is not expected to be a symmetry of the full quantum theory. According to the lore, there is, however, a precise correspondence of the so-called BPS spectra of the dual theories, since the properties of so-called BPS states, such as their mass, do not change when smoothly changing the string coupling constant. Using D-brane techniques circumvents having to deal with strongly coupled strings. But since it crucially depends on the properties of BPS protected states, its applicability is in principle quite limited. We comment on some remarkable result for near-extremal black holes at the end of the following section. Interesting work was also performed in the context of NS-brane realizations. We comment on these approaches in section 5.

4. Microstate counting for the extremal Reissner-Nordström black hole

In this section we sketch the string theory microstate counting for the simple example of an extremal Reissner-Nordström black hole (2). We realize this configuration in terms of intersecting M5-branes. These are brane solutions of eleven-dimensional supergravity [12], the bosonic part of which reads

$$(2\kappa_{11}^2)\, S_{11} = \int \mathrm{d}^{11}x\, \sqrt{|G|}\, R - \frac{1}{2}\int *F_4 \wedge F_4 + \frac{1}{6}\int C_3 \wedge F_4 \wedge F_4\,.$$

Here G is the eleven-dimensional metric, and F_4 is the field strength of a three-form potential C_3. The Bianchi identity and the field equation are given by

$$\mathrm{d}F_4 = 0\,, \quad \mathrm{d}\,(*F_4 + F_4 \wedge C_3) = 0\,.$$

The combination $H_7 = *F_4 + F_4 \wedge C_3$ is the dual field strength to F_4. This couples to "electric" membrane charges (two-branes), while C_3 couples to "magnetic" five-brane charges (five-branes). These charges are conserved charges due to the Bianchi identity and the equation of motion,

$$Q' = \int_{\partial V_8} (*F_4 + F_4 \wedge C_3)\ , \quad P = \int_{\partial V_5} F_4\,.$$

where V_8 and V_5 are the volumes orthogonal to the $p = 2$ and $p = 5$ spatial directions of the branes sources.

The background describing three intersecting M5-branes is given by [13–15],

$$\begin{aligned}
\mathrm{d}s_{11}^2 &= (F_1F_2F_3)^{-2/3}\Big[F_1F_2F_3(\mathrm{d}u\mathrm{d}v + K\mathrm{d}u^2) + \mathrm{d}\vec{x}^{\,2} \\
&\qquad + F_2F_3(\mathrm{d}y_2^2 + \mathrm{d}y_3^2) + F_1F_3(\mathrm{d}y_4^2 + \mathrm{d}y_5^2) + F_1F_2(\mathrm{d}y_6^2 + \mathrm{d}y_7^2)\Big]\,, \\
F_4 &= 3\Big[{}^3{*}\,\mathrm{d}F_1^{-1} \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3 + {}^3{*}\,\mathrm{d}F_2^{-1} \wedge \mathrm{d}y_4 \wedge \mathrm{d}y_5 + {}^3{*}\,\mathrm{d}F_3^{-1} \wedge \mathrm{d}y_6 \wedge \mathrm{d}y_7\Big]\,.
\end{aligned}$$

Here $u = y_1 - t$ and $v = 2t$, and ${}^3{*}$ is the Hodge-duality with respect to the three coordinates $\vec{x}$ transverse to the three branes. The functions F_i^{-1} are harmonic functions, which in the simplest case have the form $F_i^{-1} = 1 + P_i/|\vec{x}|$, such that the corresponding branes have charges P_i and vanishing Q_i'. The y-coordinates label the directions along which the branes are stretched. We can visualize this schematically in the table 1. The direction y_1 is parallel to all the branes. The effect of the term

TABLE 1. Three intersection M5-branes: the directions along the brane are denoted by "—", the directions transverse to the brane by "X".

brane	charge	y_1	y_2	y_3	y_4	y_5	y_6	y_7	$\vec{x}$
$M5_1$	P_1	—	X	X	—	—	—	—	X
$M5_2$	P_2	—	—	—	X	X	—	—	X
$M5_3$	P_3	—	—	—	—	—	X	X	X

$K = 1 + Q/|\vec{x}|$ in the metric is to add momentum Q along the direction y_1. This is necessary if we want to compactify all internal radii y_i on circles. The momentum prohibits the y_1-circle from shrinking.[c] The metric is regular at $|\vec{x}| \to 0$ but

[c]Intuitively, this is quite simple to see: the metric component in the direction y_2, for example, is proportional to $[F_1^{-2}F_2F_3]^{1/3}$, corresponding to the fact that the second and third brane are extended in the y_2-direction, while the first brane is transverse to this direction. In fact, as one approaches a brane,

possesses a horizon, which is a surface in the $(t, \vec{x})$ subspace at $r = |\vec{x}| = 0$ and is extended in the seven dimensions of the branes, hiding their charges P_i. The four-dimensional Reissner-Nordström black hole geometry is obtained from this eleven-dimensional configuration by compactifying the seven dimensions along which the branes are stretched. We consider the simple case of a torus compactification. The compactification radius of the direction y_i is taken to be L_i. We remark that the volume of the six-torus spanned by the y_2 to y_7-direction is independent of the radial distance r from the horizon in the extended directions. The area of the horizon is consequently given by

$$\begin{aligned} A_9 &= V_6 \lim_{r\to 0} \int \sqrt{K(F_1F_2F_3)^{1/3}}\, \mathrm{d}y_1 \left[(F_1F_2F_3)^{-2/3} r^2 \mathrm{d}\Omega_2^2\right] \\ &= 4\pi\, V_7 \sqrt{QP_1P_2P_3}\,, \end{aligned}$$

where $V_7 = L_1 V_6 = \prod L_i$ and $\mathrm{d}\Omega_2^2$ is the SO(3)-invariant metric on the unit two-sphere S^2. Upon compactification the metric (in the Einstein frame) becomes [16]

$$\mathrm{d}s_4^2 = -\lambda^2(r)\mathrm{d}t^2 + \lambda^{-2}(r)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2)\,, \tag{4}$$

where

$$\lambda^2(r) = \sqrt{K^{-1}F_1F_2F_3} = \frac{r^2}{\sqrt{(r+Q)(r+P_1)(r+P_2)(r+P_3)}}$$

The area of the horizon in four dimensions is just

$$A_2 = (V_7)^{-1} A_9 = 4\pi\sqrt{QP_1P_2P_3}\,.$$

This black hole is a generalization of the (extremal) Reissner-Nordström black hole we presented in the previous section and includes both an electric charge Q as well as magnetic charges P_i. This comes from the fact that the reduction of eleven-dimensional supergravity on the seven circles produces several different U(1)-gauge fields which can carry the different charges Q and P_i of the black hole. We note, however, that certain characteristic features are maintained. The horizon area for the extremal Reissner-Nordström black hole (2) was given by $4\pi r_+^2 = 4\pi Q^2$. Our membrane realization of the black hole has the same feature, if we take, for instance, $Q = P_i$.

We can rewrite the four-dimensional electric and magnetic magnetic charges Q and P_i as the quantized momenta and the winding numbers on the M-branes. The precise discussion[d] of the quantization conditions and of charge normalizations can

$|\vec{x}| \to 0$, the volume perpendicular to the brane expands, while it shrinks in directions parallel to the brane as a result of the brane tension. This can be seen by comparing the different powers of F_i appearing in the metric. So, as far as the y_2-direction is concerned, the $M5_2$-$M5_3$-brane system is stabilized, by placing $M5_1$-branes perpendicular to them, all with comparable charges. Since all branes are parallel to the y_1-direction, on needs to add momentum along y_1 for stabilization.

[d] The quantization of the M5-brane charges follow from the reduction of the M-theory branes to D-branes of IIA string theory, for which the quantization conditions of the tensions are known.

be found in [15]. The result is

$$Q = \frac{\kappa_{11}^2}{V_7}\frac{N}{L_1}\,, \quad P_i = \frac{n_i}{2\pi X_i}\left(\frac{\pi\kappa_{11}}{2}\right)^{1/3} .$$

where N and n_i are the integer numbers of momentum and winding quanta on the branes: the n_i count the number of parallel M5-branes in the i-th orientation, while there is a quantum N of Kaluza-Klein momentum $2\pi N/L_1$ traveling along the y_1-direction. The X_i stand for the volumes of the compact transverse directions of the branes in the i-th orientation, hence $X_1 = L_2L_3$, $X_2 = L_4L_5$, and $X_3 = L_6L_7$. The entropy can be expressed directly in terms of these integers,[e]

$$S_{\rm bh} = \frac{2\pi A_2}{\kappa_4^2} = \frac{2\pi A_9}{\kappa_{11}^2} = \frac{8\pi^2 V_7}{\kappa_{11}^2}\sqrt{P_1P_2P_3Q} = 2\pi\sqrt{n_1n_2n_3N}\,. \tag{5}$$

In above formula we used the fact that upon compactification the four-dimensional gravitational constant κ_4^2 is related to the one of the eleven-dimensional theory according to $\kappa_{11}^2 = V_7\,\kappa_4^2$.

In the following we address the question of how to account for this entropy by microstate counting. In the microscopic picture the black hole is made up of the three clusters of n_i parallel and relatively displaced M5-branes wrapped on a six-torus times a circle. Looking at table 1 it is clear that the common intersections of the five-branes are all along the y_1-direction. These intersection form straight strings wrapping the circle in the y_1-direction. With respect to the remaining directions, the branes intersect on a total of $n_1n_2n_3$ different points of the six-torus and over a single point in the three extended directions $\vec{x}$.

The conjecture about the microstates of a black hole in this setup is the following [15]: the dominant contribution to the degeneracy of states is associated with the intersections of the brane configuration. From an M-theory perspective these intersections are seen as M2-branes connecting the M5-branes that have collapsed to strings on the mutual intersections. These collapsed M2-branes give rise to massless modes that are described by a 1+1-dimensional conformal nonlinear sigma model in the limit that the radii of the six-torus are much smaller than circle L_1. The massless modes deform the $n_1n_2n_3$ string-like defects within the $5+1$-dimensional world-volume of any of the five-branes. It is therefore suggestive to associate a central charge of $c_0 = 4(1+\frac{1}{2})$ to each of the intersections, which accounts for the four bosonic transverse modes and their superpartners. Here, we have assumed that the 1+1-dimensional model possesses a certain amount of supersymmetry. The total central charge is therefore $c = n_1n_2n_3c_0$. Of course, there are other modes of the M5-brane system, which are not accounted for by this sigma model. In the limit of large charges n_i such contributions are subleading as far as the degeneracy of states is concerned. This can be made more precise when working with D-branes, which are described by their open

[e] This is in units for which $\hbar = c = 1$. In these units all quantities are measured in units of the Planck length $l_{\rm P}^2 = G_{\rm N} = \kappa^2/8\pi$.

string excitations [17,18]. Furthermore, we have suppressed the fact that the branes are actually indistinguishable. As a consequence, one would need to factor out the permutation group, which would lead to an orbifold theory. For our simple geometry we can ignore this subtlety [17].

The degeneracy of states of supersymmetric black holes is associated with the different ways one can distribute N-quanta of momenta over the $n_1 n_2 n_3 c_0$ different oscillators describing the string-like defects, while preserving the corresponding amount of supersymmetry. In the present case this can be accomplished by exciting left-moving modes only. Since the oscillators of the string-like defects run along the y_1-direction, which is identified under $y_1 \equiv y_1 + L_1$, the modes are quantized in units of $2\pi/L_1$. The Cardy formula [19] gives the asymptotic degeneracy of states for large excitation levels N compared to the central charge,

$$S_{\mathrm{stat}} \approx \log d(c, N) = 2\pi\sqrt{\tfrac{1}{6}Nc} = 2\pi\sqrt{n_1 n_2 n_3 N}\,, \quad (N \gg c)\,.$$

We see that this corresponds exactly to the entropy S_{bh} given by the Bekenstein-Hawking area law (5)!

From a point of view of dualities choosing $N \gg n_1 n_2 n_4$ is somewhat unnatural. In the case $N \approx n_i$ there is another suggestion on how to count the microstates [20]. The n_i quanta of flux can also be realized by three single M5-branes wrapped n_i times around the circle y_1. There is only one string-like intersection of the three five-branes now, but it itself winds $n_1 n_2 n_3$ times, so the modes of the single string ($c = 6$) are quantized in units of $2\pi/(n_1 n_2 n_3 L_1)$. The Cardy formula yields the same result.

It should be noted that the details of the compactifications were not all that important in this analysis. The only information relevant in this calculation was the number of string-like intersections. The six-torus we considered as the compactification manifold possesses non-trivial four-cycles. The n_1 M5-branes, for instance, wrap the four cycle in the direction y_4 to y_7. The other branes wrapped other cycles of the six-torus. The cycles triply intersect over points along the y_1-direction and doubly intersect over two-cycles. Similar M5-brane setups have been studied, in which a six-dimensional Calabi-Yau manifold times a circle is utilized as a compactification manifold. Like in the above torus compactification, the Calabi-Yau spaces possess self-intersecting four-cycles, on which five-branes can be wrapped. Let us denote such a cycle by $\mathcal{P} = p^A \Sigma_A$, where Σ_A is a basis of the forth integer homology class of the Calabi-Yau manifold. The integers p^A correspond to the integers n_i of the torus compactification and count the number of times the M5-brane is wrapped around the cycle Σ_A. Like on the six-torus, the four-cycle $\mathcal{P}$ intersects over two-cycles and triply intersects over a point. The number of triple intersections is denoted by C_{ABC}. The study of the space of deformations of the cycle $\mathcal{P}$ within the Calabi-Yau space is quite involved and relies on certain technical assumptions on the cycle $\mathcal{P}$ that correspond to taking the large charge limit. We do not need the details here. The result [21] is that the low-energy dynamics of the cycle $\mathcal{P}$ is described by a sigma model with $(0, 4)$ chiral world-sheet supersymmetry. This supersymmetry is crucial for describing the black holes in four dimensions, which preserve four supersymmetries. Therefore, the degeneracy of states of the four-dimensional extremal black hole are accounted for by the left-moving excitation of the $(0,4)$ supersymmetric ground state. Calculating the central charge of the left-moving sector and using the Cardy formula gives the result for the microscopic entropy

$$S_{\text{stat}} = 2\pi\sqrt{\tfrac{1}{6}N(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A)}\,. \tag{6}$$

Here $c_{2A} = \int_{\Sigma_A} c_2(\mathrm{T}M)$, where $c_2(\mathrm{T}M)$ is the second Chern class of the tangent bundle of the Calabi-Yau manifold. The intriguing consequence of this result is that the microstate counting predicts a deviation from the Bekenstein-Hawking area law. The first term under the square root is the term that corresponds to the contribution of the Bekenstein-Hawking area law. The second term is a deviation and is subleading in the limit of large charges. This deviation was interpreted in [21,22] as resulting from R^4-corrections to the effective superstring action [23,24]. Such interactions lead to R^2-interactions in the effective four-dimensional field theory after compactification. In [25] it was shown, that this deviation predicted by microstate counting is indeed in agreement with the macroscopic entropy based on an effective field theory computation including higher-curvature interactions. One important ingredient of this analysis is the adoption of a more general definition of entropy, which is appropriate for gravity theories with higher-derivative interactions. We will discuss this issue in chapter V. The second important ingredient is the so-called fix-point behavior. This property is due to supersymmetry enhancement and expresses the fact that on the horizon of the black holes the various fields have to take fixed values, which are expressed solely in terms of the charges. That this property holds even in the presence of R^2-interactions is deduced in chapter IV.

There have been various generalizations of this microstate counting to other types of brane setups and other compactifications. Physically interesting are the attempts to generalize the techniques of microstate counting to non-extremal black holes. While it is simple to construct, *e.g.*, a system of non-extremal intersecting M5-branes [26] in supergravity and to derive the entropy that results from its compactification, a straightforward application of a perturbative string theory calculation to the near-extremal case does not, at first sight, seem appropriate. Nevertheless, even for non-extremal static [17,27] and extremal and near-extremal spinning black holes [28,29] microstate counting has reproduced the expected area law. In addition, near to extremality, phenomena such as Hawking radiation, are captured by perturbative string theory. In [17], *e.g.*, Hawking radiation is thought of as resulting from open-closed string interactions. In this picture, a near-extremal black hole is described by taking the same setup as for the extremal case, but putting, in addition to left-moving, also right-moving open strings along the common brane intersection. Left- and right moving string modes can interact and form closed string states. These can scatter off from the branes into the transversal directions. It is quite remarkable that to leading order such a simple picture correctly accounts for the thermal Hawking radiation and reproduces the expected Hawking temperature.

5. Near-horizon geometry, AdS/CFT, and black hole moduli spaces

Another, complementary approach is to describe the near-horizon degrees of freedom of a black hole directly in terms of the coupled string theory involving NS-branes. In fact, the D-brane and NS-brane description are on equal footing from an M-theory perspective. In the NS-brane picture the microscopic degrees of freedom of the interacting strings near the horizon are related to certain Wess-Zumino-Witten conformal field theories. In this approach, as well, the entropy of extremal and near-extremal black holes is successfully reproduced by microstate counting. The setup is particularly appealing, as the microstates are associated directly to string states at the horizon, and it does not involve any weak-strong coupling duality. We will, however, refrain of further comment and refer to the literature [26,30–33].

Another line of ideas is inspired by the conjectured AdS/CFT-correspondence principle [34,35], which proposes that there exists a conformal field theory dual to string theory on AdS spaces. The reasoning leading to this conjecture will not be repeated here. A good reference for this presentation is [36]. A phenomenologically interesting case, where this conjecture is expected to apply, is the near-horizon geometry of extremal Reissner-Nordström black holes (2). In isotropic coordinates $r = \rho(1 + Q/\rho)$ this metric is given by

$$\mathrm{d}s^2 = -\left(1 + \frac{Q}{\rho}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{Q}{\rho}\right)^2 \left[\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2\right],$$

In these coordinates the horizon is located at $\rho = 0$. If we restore length units, the near horizon limit is defined by $l_\mathrm{P} \to 0$ with the dimensionless Q and ρ/l_P held fixed,

$$\mathrm{d}s^2_{\text{n.h.}} = -\frac{\rho^2}{Q^2}\mathrm{d}t^2 + \frac{Q^2}{\rho^2}\mathrm{d}\rho^2 + Q^2\mathrm{d}\Omega_2^2 . \tag{7}$$

This is a metric of $\mathrm{AdS}_2 \times S^2$ with $\mathrm{SO}(1,2) \times \mathrm{SO}(3)$ isometry group and is known as the Bertotti-Robinson spacetime. The isometry group of the AdS_2-part can be made more explicit in coordinates where $q^2 = Q^3/\rho$. The near-horizon metric takes the form

$$\mathrm{d}s^2_{\text{n.h.}} = -\frac{Q^4}{q^4}\mathrm{d}t^2 + 4\frac{Q^2}{q^2}\mathrm{d}\rho^2 + Q^2\mathrm{d}\Omega_2^2 .$$

The isometry group is generated by the Killing vectors (see *e.g.* [37])

$$h = \partial_t \,, \quad d = t\partial_t + \frac{1}{2}q\partial_q \,, \quad k = (t^2 + q^4/Q^2)\partial_t + t\, q\, \partial_q \,,$$

which satisfy the algebra[f] of SL(2, $\mathbb{R}$) with respect to the Lie bracket,

$$[d, h] = -h\,, \quad [d, k] = k\,, \quad [h, k] = 2d\,.$$

As we will show explicitly in chapter IV the horizon preserves 8 supersymmetries, so in the spirit of the AdS/CFT-conjecture one expects that there exists a $SU(1, 1|2)$ superconformal mechanical dual. This observation renewed the interest in (super)conformal quantum mechanics [38–40], and various superconformal extensions were suggested and constructed [41–47]. An interesting proposal for the dual of the string theory on the AdS_2-geometry was presented in [37] who conjectured that the dual $SU(1, 1|2)$ superconformal quantum mechanical model is in fact an $N = 4$ superconformal extension of the Calogero model [48]. There are indications that the quantum mechanical ground state degeneracy in fact scales like the length squared of the system.

In the case of AdS_2, the AdS/CFT-correspondence is not yet fully understood. This is partly due to some peculiar features of AdS_2 not shared by its higher-dimensional cousins. For instance, it contains two disconnected timelike boundaries, and hence a holographic interpretation is not obvious. Another observation is that the near horizon geometry (7) is not the unique extremal ground state with charge Q. This is related to the existence of multi-centered black holes, which are discussed at length in chapter IV and V. Multi-centered black holes are extremal and described by metrics of the form (4), where the harmonic function has poles at multiple centers,

$$\lambda(\vec{x}) = 1 + \sum_A \frac{q_A}{|\vec{x} - \vec{x}_A|}\,.$$

It is interesting to discuss the regime, in which the centers approach each other to distances much smaller than the Planck length,

$$|\vec{x}_A - \vec{x}_B|/l_P = \delta \ll 1\,,$$

where we have restored length units. In the near horizon limit, $l_P \to 0$, we keep the distance between the centers δ small but fixed. Keeping the dimensionless $|\vec{x}|/l_P$ and q_A fixed, this limit amounts to dropping the constant term in the harmonic function λ. For large values of $|\vec{x}|/l_P$ compared to δ the near-horizon geometry looks like AdS_2 with metric l_P^{-2} times the expression (7) with radius $\sqrt{Q} = (\sum_A q_A)^{1/2}$. This is called the geometry of near-coincident black holes [44]. At shorter distances $|\vec{x}|/l_P \ll \delta$ the throat region branches up into a tree-like structure. Each of its branches ends on the familiar AdS_2 near-horizon geometry of one the centers. In fact, in this limit $l_P \to 0$ the asymptotically flat region decouples and one is describing coalescing black holes. In [44,49,50] arguments are put forward to suggest that the volume of moduli space of coinciding black holes becomes very large. Together this suggest that studying the cohomology of this decoupled region of moduli space may account for the degeneracy of quantum ground states of a single extremal black hole with charge $Q = \sum_A q_A$. One pictures that in the near-horizon limit the degrees of freedom of a black hole

[f] Note that the algebras of SL(2, $\mathbb{R}$), SO(1, 2), Sp(2), and SU(1, 1) are all isomorphic.

are accounted for by bound states of lighter oscillating black holes which are lumped together by velocity dependent forces [51]. Interestingly, the moduli space metric in the near-horizon limit exhibits conformal symmetry. The study of the moduli spaces of multi-centered black holes has been at the center of much attention and is the subject of chapters V and VI.

These are but a few of the possible approaches to the theory of the quantum black hole. So far, all of these approaches have relied on specifying some underlying degrees of freedom which are believed to describe the black hole. In particular, in the D-brane approach the analysis additionally makes use of supersymmetry. Nevertheless, as far as the entropy is concerned, we have seen that often specific details of the quantum gravity model do not play a too crucial role and one might suspect that in fact there is some underlying symmetry principle which gets inherited by the quantum theory from the underlying classical black hole background. This has been first investigated in the context of five-dimensional black hole in [52] and [53], the work of which is based on [54], who remarked that the asymptotic isometry group of AdS_3 is generated by (two copies of) the Virasoro algebra. On the other hand, AdS_3 is the asymptotic geometry of (2+1)-dimensional BTZ black holes [55], so the conclusion is that the states of a consistent quantum theory of gravity on this background geometry must fall into representations of the Virasoro algebra, and therefore constitute the states of a conformal field theory. In fact, the central charge of the corresponding Virasoro algebra can be calculated and, using the Cardy formula [19], can be successfully compared with the entropy of the BTZ black hole. There have been various attempts to generalize this argument to black holes with the same near-horizon geometry as the BTZ black hole. What remains unsatisfying is that the Virasoro algebra envisaged is the algebra of deformations of the asymptotic boundary of AdS_3 instead of the one of the horizon geometry as one might expect. To this extent the algebra of surface deformations of the horizon was analyzed in [56–59]. It was found that it contains a Virasoro algebra as well.

II

Supersymmetry and supergravity

In this chapter an introduction to $N = 2$ supersymmetry and supergravity in four spacetime dimensions is presented. In particular, we discuss vector multiplets and hypermultiplets in flat spacetimes and coupled to supergravity backgrounds. To this extent we introduce important elements of the superconformal approach to supergravity. The emphasis in this presentation is on the geometrical aspects of the target spaces of the matter multiplets. An important property of theories of abelian vector multiplets is that their action has the form of a generalized Maxwell Lagrangian, with terms that are at most quadratic in the field strengths. As a result, the field equations are subject to electric-magnetic duality transformations. We discuss these transformations in the supersymmetric context.

The outline of this chapter is as follows: in section 1 we discuss electric-magnetic duality transformations in a simple setting and discuss their physical relevance. In sections 2 to 4 we discuss various $N = 2$ supermultiplets, coupled to flat and rigidly superconformal backgrounds, and present their actions. In section 5 the coupling to supergravity is considered. In section 6 we discuss symplectic reparameterizations. Finally, section 7 contains further details on a coordinate independent formulation of the vector multiplet geometry. This is illustrated by considering Calabi-Yau compactifications.

1. Electric-magnetic duality

In standard electrodynamics a simultaneous rotation of (E, H) and (D, B) leave the Maxwell equations invariant, provided the electric and magnetic charge and current densities, $(\rho_{\rm e}, \rho_{\rm m})$ and $(J_{\rm e}, J_{\rm m})$, transform analogously [60]. These rotations give rise to an equivalence, not to a symmetry. Let us discuss electric-magnetic duality transformations for a generalized Maxwell action describing n different abelian field strengths, which we denote by F^I,

$$4\pi S[F] = -\frac{1}{2}\int i\left(\bar\tau_{IJ}F^{+I}\wedge *F^{+J} - \tau_{IJ}F^{-I}\wedge *F^{-J}\right) .$$

Here, we defined[a] the (anti-)selfdual two-forms $F^{\pm I} = \frac{1}{2}(F^I \pm i * F^I)$, which satisfy $*F^{\pm I} = \mp i F^{\pm I}$, and a constant complex coupling matrix

$$\tau_{IJ} = \frac{\theta_{IJ}}{2\pi} + \frac{4\pi i}{g^2_{IJ}} .$$

The Bianchi identities and the equations of motion are given by

$$\mathrm{d}(F^+ + F^-)^I = 0 , \quad \mathrm{d}(G^+ + G^-)_I = 0 , \tag{1}$$

respectively, where the field strength G_I is defined by

$$G^\pm_I = -4\pi \, \frac{\delta S}{\delta F^{\pm I}} . \tag{2}$$

In the language of macroscopic electrodynamics the fields $G_I = \bar{\tau}_{IJ} F^{+J} + \tau_{IJ} F^{-J}$ comprise the displacement and the magnetic fields, while the F^I contains the electric fields and the magnetic inductions. The couplings τ_{IJ} thus play the role of the permeability and permittivity. The Bianchi identities and field equations (1) are invariant under the following rotation,

$$\begin{pmatrix} F^\pm \\ G^\pm \end{pmatrix} \longrightarrow \begin{pmatrix} F'^\pm \\ G'^\pm \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^\pm \\ G^\pm \end{pmatrix} , \tag{3}$$

where $U^I{}_J$, Z^{IJ}, W_{IJ}, and $V_I{}^J$ are $n \times n$ submatrices. Demanding that the rotated field strengths (F', G') derive from an action $S'[F']$ using (2), implies that the transformation (3) is an element of $\mathrm{Sp}(2n, \mathbb{R})$. Using (2) one finds that the coupling constants τ'_{IJ} of the action $S'[F']$ must be related to the original ones by

$$\tau_{IJ} \longrightarrow \tau'_{IJ} = (V\tau + W)_{IL}[(U + Z\tau)^{-1}]^L{}_J . \tag{4}$$

That the coupling matrix τ'_{IJ} is symmetric if the transformation (3) is an element of $\mathrm{Sp}(2n, \mathbb{R})$. These transformations are referred to as symplectic reparameterizations.

If magnetic and electric currents are introduced as sources for the field equations,

$$\mathrm{d}(F^+ + F^-)^I = *{j_\mathrm{m}}^I , \quad \mathrm{d}(G^+ + G^-)_I = *j_{\mathrm{e}\,I} ,$$

then these currents too must transform, like the field strengths, as vectors under symplectic reparameterizations. In particular, the magnetic and electric charges, defined by integrating over a spatial volume surrounding the current densities

$$\begin{aligned} \int_V \mathrm{d}F^I &= \oint_{\partial V} \left(F^{+I} + F^{-I}\right) = 2\pi q^I_\mathrm{m} , \\ \int_V \mathrm{d}G_I &= \oint_{\partial V} \left(G^+_I + G^-_I\right) = -2\pi q_{\mathrm{e}\,I} , \end{aligned} \tag{5}$$

constitute a symplectic pair. The normalization of the charges is such that the magnetic induction and the electric field produced by a static point charge has the characteristic $1/4\pi r^2$ fall-off times $2\pi q^I_\mathrm{m}$ and $\frac{1}{2} g^2 (q_{\mathrm{e}\,I} + \theta_{IJ} q^J_\mathrm{m}/2\pi)$, respectively. This follows

[a] We use form notation. Details concerning the notation and conventions are found in appendix A.

directly from the Bianchi identities and the equations of motion. In the presence of a theta angle, the magnetic charge of a particle therefore contributes to the electric charge. This effect was first described in [61]. It is well known from semi-classical [62–64] and topological arguments that the (abelian) charges must form a lattice with an elementary cell of the size $2\hbar$. This lattice is left invariant only by the subgroup $\mathrm{Sp}(2n, \mathbb{Z})$ of symplectic reparameterizations.

Since the $F^I \wedge F^J$ term is a total derivative, the generalized theta angle θ_{IJ} can be shifted at will at the level of the classical Lagrangian. This is no longer the case non-perturbatively and is reflected, in the canonical treatment, in the appearance of the theta angles in above charge formulae. Furthermore, in an effective theory, the $F \wedge F$-term is proportional to the integer-valued Pontryagin index, which counts the total instanton number of the background. In the presence of instantons, therefore, the shift invariance of the theta angles is reduces to shifting by integer multiples[b] of 2π, since in that case the action gets shifted by an integer multiple of 2π. The theta parameters are therefore periodically identified.

One may wonder to which extent the reparameterizations of the field equations can be viewed as resulting from an operation on the Lagrangian. Indeed, some of these transformations can be seen as Legendre transformations at the level of the Lagrangian. For this, let us regard F^I as an unconstrained two-form and implement the Bianchi identities by introducing the one-form A'_I as a Lagrangian multiplier in the action. This one-form couples naturally to the magnetic charge and current density,

$$4\pi S[A', F] = 4\pi S[F] - \int A'_I \wedge \mathrm{d}F^I \,.$$

Replacing A'_I by $A'_I + \mathrm{d}\Lambda'_I$ changes the Lagrangian only by a total derivative. Hence, A'_I is subject to gauge transformations, $\delta A'_I = \mathrm{d}\Lambda'_I$, and therefore represents a gauge potential. Solving the equation of motion of A'_I implies the original Bianchi identity for F^I, and one is brought back to the original action. Alternatively, one can solve the field equations of the two-form F^I and reinsert the solution back into the action. One finds, up to total derivatives,

$$4\pi S'[F'] = -\frac{1}{2} \int i \left(\bar{\tau}'^{IJ} F'^{+}_I \wedge *F'^{+}_J - \tau'^{IJ} F'^{-}_I \wedge *F'^{-}_J \right) \,,$$

where $\tau'^{IJ} = -(\tau_{IJ})^{-1}$. This transformation, which amounts to a coupling inversion, corresponds to one specific element of the $\mathrm{Sp}(2n, \mathbb{R})$ transformation (4). The actions $S'[F']$ and $S[F]$ are two different but equivalent descriptions of the same theory. Symplectic reparameterizations do not present an invariance or a symmetry of the theory. This would be the case only if the couplings would not change, such that $S'[F']$ would correspond to $S[F']$. The transformations (3) must be conceived as reparameterizations: the same theory is described equivalently in terms of different coordinates (or

[b] The normalization of the effective abelian theory depends on the abelian projection and therefore on the embedding into the nonabelian microscopic theory. We chose an embedding for which the Pontryagin index $(16\pi^2)^{-1} \int F \wedge F$ takes integer values. This fixes the overall-normalization of the Lagrangian.

fields) and different coupling constants. This is what makes electric-magnetic duality interesting at all: for the case of the coupling inversion discussed above, it relates the strong and weak coupling regimes of a theory.

One must realize that our whole discussion, so far, relied on the fact that only the field strengths F^I appeared in the Lagrangian. This is not the case for theories that depend on the gauge potential itself, as is the case for minimal couplings to charged matter, Yang-Mills or Chern-Simons-like theories.

Let us conclude this section with a discussion of the physical relevance of electric-magnetic duality. One basic assumption of the present discussion is that electric and magnetic currents appear on footing. The magnetic currents couple to the dual field strength and describe classical, magnetically charged sources. Magnetic monopoles and dyons arise (as opposed to in quantum electrodynamics itself) in phases of spontaneously broken nonabelian gauge theories [65–67]. They appear as static solitonic field configurations with finite energy and carry magnetic charge with respect to the abelian projection of the spontaneously broken gauge group. Again, we emphasize that that electric-magnetic duality is generically not realized as a symmetry of the theory but relates different equivalent descriptions. This is also the case for the $N = 2$ supersymmetric gauge theories we discuss in the next section: magnetic monopoles (residing in hypermultiplets) have different quantum numbers as compared to the ones of electric excitations (these reside in vector multiplets). Nevertheless, electric-magnetic reparameterization is at the heart of, for instance, our present qualitative understanding of the confinement phenomena in terms of monopole condensation. The conceptual principles of using dualities as a tool for understanding non-perturbative phenomena were pioneered by 't Hooft [68–70]. The key idea is to utilize dualities for comparing various dual perturbative field theory descriptions, which appropriately capture the physics of the theory in the corresponding dual regimes. In the simple case of a strong-weak coupling duality, the perturbative description of electric excitations at weak coupling is related to the dual perturbative description in terms of the magnetic excitations of the strong coupling regime. This strategy stands at the beginning of many efforts to understanding non-perturbative aspects of gauge theories. For $N = 2$ supersymmetric Yang-Mills theories, this has led to the remarkable result of Seiberg and Witten [71–73]. In their analysis various dual descriptions are continuously patched together to form a complete characterization of the theory at low energies. The $N = 2$ theory they considered is perturbatively well controllable and nevertheless undergoes interesting dynamics. For standard gauge groups, these theories possess potentials with flat directions parameterized by so-called moduli fields. In the spontaneously broken phase, the gauge group is typically broken to its maximal abelian subgroup. The so-called Wilsonian action is constructed, in principle, by integrating out all massive modes of the theory and retaining the light massless modes (which are often associated to the moduli fields). Such an action is local but can contain infinitely many higher derivative terms (suppressed by the cut-off scale) describing the effective couplings to the massive modes that were integrated out. In principle, it is possible to integrate out whole supermultiplets of massive modes at once. Hence the procedure of obtaining the Wilsonian action does not break supersymmetry. We will therefore mainly discuss the properties of effective supersymmetric abelian gauge theories. Supersymmetry imposes strong restrictions on the possible coupling structure (at least at the two-derivative level) and provides non-renormalization theorems for certain quantities. One such relation, which is not affected by quantum corrections, is the mass-central charge relation of BPS states. For compact gauge groups, dyons are subject to the so-called Bogomol'nyi mass bound [74]. This mass bound has a natural interpretation when the theory is embedded into a theory with extended supersymmetry. In that case, the masses of massive multiplets are bound from below by the central charge of the extended supersymmetry algebra. If this bound is saturated one speaks of a BPS state. The properties of BPS states under electric-magnetic duality transformations have been decisive for the advances in the supersymmetric gauge theories.

In the following sections we will introduce some of the basic building blocks of $N = 2$ supergravity theories. We utilize the superconformal approach and present it here as a three-step program: first, the relevant multiplets transforming under rigid $N = 2$ supersymmetry are introduced and invariant actions are constructed. Then, the conditions are analyzed for which these actions are invariant under rigid superconformal symmetries. In a last step, these rigid symmetries are promoted to local ones, thereby coupling the various multiplets minimally to the gauge fields of the superconformal algebra. Certain of these multiplets act as compensators and render the superconformally invariant theory gauge-equivalent to a theory of Poincaré supergravity. For the rest of this chapter we focus mainly on vector multiplets and hypermultiplets, and on their couplings. The Weyl multiplet, which contains the gauge fields of the superconformal algebra, is presented in next chapter. In order to discuss vector multiplets (and chiral backgrounds in chapter III) it is useful to first introduce the $N = 2$ chiral multiplets.

2. Chiral multiplets

Chiral multiplets contain $16 + 16$ bosonic and fermionic off-shell degrees of freedom. In the superspace formulation, a chiral superfield Φ is defined by the condition that the chiral superspace derivative

$$D^i = \frac{\partial}{\partial \bar{\theta}_i} + \gamma^\mu \theta^i \frac{\partial}{\partial x^\mu}$$

vanishes when acting on it, $D^i \Phi = 0$. Here, the θ^i and θ_i constitute the chiral components of the two Majorana spinor coordinates of $N = 2$ superspace. Here and in the following we use the chiral SU(2) notation for Majorana spinors.[c] Written in

[c]The chiral projections of a Majorana spinor transform in conjugate representations of the automorphism group. The specific choice of assigning a certain chirality to one of the conjugate representations is

components, the chiral superfields have the following expansion [75],

$$\begin{aligned}\Phi(z,\theta) = A(z) + \bar\theta^i\Psi_i(z) + \tfrac12\bar\theta^i B_{ij}(z)\theta^j + \tfrac14\left(\varepsilon_{ij}\bar\theta^i\gamma^{ab}\theta^j\right)F^-_{ab}(z)\\ + \tfrac1{12}\left(\varepsilon_{ij}\bar\theta^i\gamma_{ab}\theta^j\right)\bar\theta^k\gamma^{ab}\Lambda_k(z) + \tfrac1{12}(\varepsilon_{ij}\bar\theta^i\gamma_{ab}\theta^j)^2C(z)\,,\end{aligned} \tag{6}$$

where the complex spacetime parameters z are defined by $z^\mu = x^\mu + \bar\theta^i\gamma^\mu\theta_i$. The chiral superfield contains two complex scalars, A and C, two $\mathrm{SU(2)_R}$-doublets of Majorana fermions, Ψ^i and Λ^i, an $\mathrm{SU(2)_R}$-triplet of complex scalars B_{ij}, as well as an anti-selfdual two-form F^-. In the superspace formulation it is clear that the chiral multiplets form a ring structure under superfield multiplication: the product of two chiral superfields is again a chiral superfield. Let us denote the components of two chiral superfields Φ^I, $I=1,2$, by $\Phi^I = (A^I, \Psi^I, B^I, F^{-I}, \Lambda^I, C^I)$. Then the product $\Phi^1\times\Phi^2$ has components A^1A^2 at the lowest level, $A^1\Psi^2 + A^2\Psi^1$, at the second, and so on. At the highest or C-level one finds [75]

$$\begin{aligned}C_{\Phi^1\times\Phi^2} = A^1C^2 + A^2C^1 - \tfrac12\varepsilon^{ik}\varepsilon^{jl}B^1_{ij}B^2_{ij}\\ + F^{1-}_{ab}F^{2-ab} + \varepsilon^{ij}\left(\bar\Psi^1_i\Lambda^2_j + \bar\Psi^2_i\Lambda^1_j\right)\,.\end{aligned} \tag{7}$$

Clearly, any holomorphic function $F(\Phi^I)$ of chiral superfields Φ^I, $I=1,\ldots,n$, is again a chiral superfield.

Rigid supersymmetry transformation rules, which are compatible with above ring structure, are induced by translation in superspace with constant parameters ϵ^i and ϵ_i. The components of Φ transforms as

$$\begin{aligned}\delta A &= \bar\epsilon^i\Psi_i\,,\\ \delta\Psi_i &= 2\partial\!\!\!/ A\epsilon_i + B_{ij}\epsilon^j + \tfrac12\gamma^{ab}F^-_{ab}\varepsilon_{ij}\epsilon^j\,,\\ \delta B_{ij} &= 2\bar\epsilon_{(i}\,\partial\!\!\!/\Psi_{i)} + 2\varepsilon_{k(i}\bar\epsilon^k\Lambda_{j)}\,,\\ \delta F^-_{ab} &= \tfrac12\varepsilon^{ij}\bar\epsilon_i\partial\!\!\!/\gamma_{ab}\Psi_j + \tfrac12\bar\epsilon^i\gamma_{ab}\Lambda_i\,,\\ \delta\Lambda_i &= -\tfrac12\gamma^{ab}\partial\!\!\!/\, F^-_{ab}\epsilon_i + \varepsilon^{kj}\partial\!\!\!/\, B_{ij}\epsilon_k + \varepsilon_{ij}C\epsilon^j\,,\\ \delta C &= -2\varepsilon^{ij}\bar\epsilon_i\partial\!\!\!/\,\Lambda_j\,.\end{aligned} \tag{8}$$

Note that the highest component of the chiral multiplet transforms into a total derivative and can therefore serve as a density for constructing rigid supersymmetric actions. In a superspace treatment, the C-component can be extracted by performing a chiral superspace integral. Hence, under supersymmetry transformations, the expression

$$4\pi\mathcal{L} = \mathrm{Im}\int \mathrm{d}^4\theta\, F(\Phi^I)\,,$$

indicated by upper and lower $\mathrm{SU(2)_R}$ indices $i, j, k, \ldots$. Details about these assignments are found in the tables of appendix B.

transforms into a total derivative for any holomorphic function F. Whether or not this expression defines a sensible action is not at stake at the moment. We will return to this issue in the next section.

The superconformal algebra contains, apart from the $N = 2$ supersymmetry algebra, the generators of dilatations D, special conformal transformations K, and S-supersymmetry. In order to discuss rigid superconformal transformations we therefore assign a Weyl weight w and a chiral weight c to the chiral multiplet. These weights determine the transformation rules under dilatations D and chiral $\mathrm{U}(1)_{\mathrm{R}}$ transformations. It follows from the superconformal algebra that these weights are related by $w = -c$. On chiral superfields these transformations are represented as

$$\Phi(z, \theta) \longrightarrow \mathrm{e}^{w\bar{\Lambda}}\Phi(z, \mathrm{e}^{-\Lambda/2}\theta) . \tag{9}$$

where $\Lambda = \Lambda_D + i\Lambda_{\mathrm{U}(1)}$. For the components this implies that A scales with weight w, Ψ_i with weight $w + 1/2$, and so on in half-integer steps. The C-component has Weyl weight $w + 2$. In order for an action, constructed from the C-component of a chiral function $F(\Phi^I)$, to posses rigid conformal symmetry, the chiral multiplet $F(\Phi^I)$ must therefore have weight $w = 2$ in four dimensions. We will discuss the consequences of this condition for the case of vector multiplets in the following section.

3. Vector multiplets

In this section we introduce the $N = 2$ vector multiplet and derive an action for it based on a chiral superspace integral. A vector multiplets can be regarded as a reduced chiral multiplet. The fields of the latter furnish a multiplet with 16 + 16 off-shell degrees of freedom, which is therefore reducible. There exists a set of 8 + 8 Lorentz covariant constraints which constitute a so-called linear multiplet. In the rigid abelian case, these constraints contain the integrability condition for the two-form F^- and a reality condition on B_{ij},

$$\partial^a(F^+_{ab} - F^-_{ab}) = 0 , \quad B_{ij} = \varepsilon_{ik}\varepsilon_{jl}B^{kl} , \tag{10}$$

where B^{ij} denotes the complex conjugate of B_{ij}. The other constraints express Λ_i and C in terms of derivatives of unconstrained fields,

$$\Lambda_i = -\varepsilon_{ij}\partial\!\!\!/\Psi^j , \quad C = -2\,\Box A^* . \tag{11}$$

The constraints are solved by considering only real triples B_{ij} and expressing the two-forms as the field strength of a gauge field, $F = \mathrm{d}W$. The reduced chiral multiplet therefore contains $8+8$ off-shell degrees of freedom. In superspace above constraints are expressed by [75],

$$(\varepsilon_{ij}\bar{D}^i\gamma_{ab}D^j)^2\Phi = -24\Box\Phi . \tag{12}$$

This shows that in general the product of two reduced chiral superfields is no longer reduced. Let us denote the independent fields of the constrained chiral superfield by $\Phi = (X, \Omega_i, W_\mu, Y_{ij})$, where X is a complex scalar, Ω_i and Ω^i denote $\mathrm{SU}(2)_\mathrm{R}$-doublets of chiral fermions, W_μ is a gauge field, and Y_{ij} denotes a $\mathrm{SU}(2)_\mathrm{R}$-triplet of real scalars. Together, they form the $8+8$ off-shell degrees of freedom of the so-called vector multiplet. Under rigid supersymmetry its components transform as [76],

$$\begin{aligned}
\delta X^I &= \bar{\epsilon}^i \Omega_i^I \, , \\
\delta \Omega_i^I &= 2\not{D}\, X^I \epsilon_i + Y_{ij}^I \epsilon^j + \tfrac{1}{2}\gamma^{ab} F(W)_{ab}^{I-} \varepsilon_{ij}\epsilon^j - 2g f_{JK}{}^I X^J \bar{X}^K \varepsilon_{ij}\epsilon^j \, , \\
\delta Y_{ij}^I &= 2\bar{\epsilon}_{(i} \not{D}\, \Omega_{j)}^I + 2\varepsilon_{ik}\varepsilon_{jl}\bar{\epsilon}^{(k} \not{D}\, \Omega^{l)I} - 4g f_{JK}^I \varepsilon_{k(i} \left(\bar{\epsilon}_{j)} X^J \Omega^{kK} - \bar{\epsilon}^k \bar{X}^I \Omega_{j)}^K \right) \, , \\
\delta W_\mu^I &= \bar{\epsilon}_i \gamma_\mu \varepsilon^{ij} \, \Omega_j^I \, .
\end{aligned}$$

As compared to above, we have incorporated nonabelian gauge transformations. The indices I, J, K run over the adjoint representation of the gauge group, and f_{JK}^I are the structure constants, $[t_I, t_J] = f_{IJ}{}^K t_K$. Accordingly, the derivatives D_μ are covariant with respect to nonabelian gauge transformations, for instance, $D_\mu X^I = \partial_\mu X^I - g f_{JK}^I W_\mu^J X^K$, where g is the coupling constant.

As for general chiral multiplets, Lagrangians for vector multiplets can be constructed using chiral superspace integrals. Comparing (11), (6) and (7) one realizes that for the quadratic function $\frac{1}{2} i \Phi^2$ one recovers the standard kinetic terms for a free complex scalar, fermion and gauge field. For a general function $F(\Phi^I)$ of reduced chiral multiplets $\Phi^I = (X^I, \Omega_i^I, W_\mu^I, Y_{ij}^I)$, one obtains the kinetic terms [77]

$$\begin{aligned}
4\pi \mathcal{L}_{\text{kin}} = \Big[\, & i D^\mu F_I \, D_\mu \bar{X}^I + \tfrac{1}{2} i F_{IJ} \bar{\Omega}_i^I \not{D} \Omega^{iJ} - \tfrac{1}{8} i F_{IJ} \, Y_{ij}^I Y^{Jij} \\
& + \tfrac{1}{4} i F_{IJ} F_{ab}^{-I} F^{-Jab} + \text{h.c.} \Big] \, .
\end{aligned}$$

In above formula, F_I and F_{IJ} denote partial derivatives of the holomorphic function $F(X)$ with respect to the vector multiplet scalars X^I. In the case of nonabelian vector multiplets, the index I runs over the adjoint representation. The gauge invariance of the chiral superspace integral imposes restrictions on the function $F(X)$. The invariance is guaranteed if $F(X)$ itself is invariant, $F_I f_{JK}^I X^K = 0$. (This condition can be relaxed [77].) The nonabelian vector multiplet Lagrangian also contains a scalar potential proportional to

$$-i g^2 (F_I f_{JK}{}^I \bar{X}^K)(f_{MN}{}^J \bar{X}^M X^N) + \text{h.c.} \, .$$

Depending on the gauge group, the scalars can acquire vacuum expectation values, in which case they induce spontaneous symmetry breaking. For the case of semisimple groups the vacuum expectation values of the scalars break the gauge group to its maximal abelian subgroup. (When coupling vector multiplets to hypermultiplets, further contributions to the scalar potential arise [78]. We have also suppressed the discussion of couplings involving fermion bilinears.)

The metric N_{IJ} of the target space, parameterized by the scalar fields X^I and $\bar{X}^I$ of the vector multiplets,

$$4\pi\mathcal{L}_{\text{kin}} = -N_{I\bar{J}} D^\mu X^I D_\mu \bar{X}^{\bar{J}} + \ldots$$

derives from a potential $K(X, \bar{X})$,

$$N_{I\bar{J}} = -i(F_{IJ} - \bar{F}_{IJ}) = \frac{\partial}{\partial X^I}\frac{\partial}{\partial \bar{X}^J} K(X, \bar{X}), \tag{13}$$

where

$$K(X, \bar{X}) = -i\bar{X}^I F_I(X) + i X^I \bar{F}_I(\bar{X}). \tag{14}$$

The target space is therefore a Kähler manifold. This property actually results already from requiring $N = 1$ supersymmetry. From the $N = 2$ superspace construction it is clear that the Kähler potential is subject to further restrictions, since the whole coupling structure is characterized by one single holomorphic function $F(X)$. The resulting geometry is referred to as special Kähler geometry. At this point, giving the precise definition of this geometry would be premature since we have not yet discussed the role of symplectic reparameterizations. This issue is addressed in section 6.

For the supersymmetry analysis it is natural to work with the scalars X^I and $\bar{X}^I$, $I = 1, \ldots, n$ of the n vector multiplets, as well as with the holomorphic function $F(X)$. Characterizing the target space geometry in terms of these fields is referred to utilizing preferential coordinates. To underline the geometrical features of the target space one can envisage a holomorphic coordinate transformation and proceed to a formulation in terms of holomorphic sections $X^I(z)$, where z^A and $\bar{z}^{\bar{A}}$, $A = 1, \ldots, n$, provide local holomorphic and anti-holomorphic coordinates of the target space. Note that $F_I(z) = F_I(X(z))$ is also holomorphic in z. It is straightforward to express the transformation rules and the Lagrangian entirely in terms of these sections and derivatives thereof. For instance, the kinetic term for the scalars takes the form

$$4\pi\mathcal{L}_{\text{kin}} = -N_{A\bar{B}}(z, \bar{z})\partial^\mu z^A \partial_\mu \bar{z}^{\bar{B}} + \ldots,$$

where the target space metric in (anti-)holomorphic coordinates is given by

$$N_{A\bar{B}}(z, \bar{z}) = \partial_A \partial_{\bar{B}} K(z, \bar{z}), \quad K(z, \bar{z}) = -i\Big[\bar{X}^I(z) F_I(\bar{z}) - X^I(z)\bar{F}_I(\bar{z})\Big]$$

and depends only on the section $(X^I(z), F_I(z))$. In section 6 we will argue that this pair transforms as a vector under symplectic reparameterizations.

Let us consider vector multiplet actions with rigid superconformal invariance. For simplicity, we will stick to the formulation in terms of the preferential coordinates X and $\bar{X}$. In order to construct scale invariant Lagrangians for reduced chiral multiplets there are two issues that must be reconsidered. First, the restriction (12) is consistent with Weyl and chiral invariance only if $w = 1$. Second, the holomorphic function $F(\Phi^I)$ must have Weyl weight $w = 2$, as we have argued in the previous section.

In the case of reduced chiral multiplets the holomorphic function must therefore be a homogeneous function of degree two. For vector multiplets this implies

$$F(\lambda X) = \lambda^2 F(X)\,,$$

for some real scale factor λ. From this property one derives many important relations, such as

$$F(X) = \tfrac{1}{2} F_I X^I\,, \quad F_I = F_{IJ} X^J\,, \quad F_{IJK} X^K = 0\,. \tag{15}$$

In the following we explain the consequences of these relations for the geometry of the target space. To this extent we note that the complex scalar fields of the vector multiplets naturally define local holomorphic and anti-holomorphic coordinates X^I and $\bar{X}^{\bar{J}}$, respectively. In these coordinates the associated almost complex structure J is constant and diagonal. In the X and $\bar{X}$ coordinates, its components read $J^I{}_J = i\delta^I{}_J$, $J^{\bar{I}}{}_{\bar{J}} = -i\delta^{\bar{I}}{}_{\bar{J}}$. From (13) on derives that the hermitian connection is given by

$$\Gamma_{JK}{}^I = N^{I\bar{L}} \partial_J N_{K\bar{L}} = -iN^{I\bar{L}} F_{JKL}\,, \quad \Gamma_{\bar{J}\bar{K}}{}^{\bar{I}} = N^{\bar{I}L} \partial_{\bar{J}} N_{L\bar{K}} = iN^{\bar{I}L} \bar{F}_{\bar{J}\bar{K}\bar{L}}\,.$$

This connection is metric compatible. In our coordinates this is the statement that $D_I N_{J\bar{K}} = \partial_I N_{J\bar{K}} - \Gamma_{IJ}{}^L N_{L\bar{K}} = 0$, and analogous for the anti-holomorphic indices. It can be verified that the complex structure is covariantly constant with respect to this connection. The non-vanishing components of the Kähler form in holomorphic coordinates are given by

$$\Omega_{I\bar{J}} = N_{K\bar{J}} J^K{}_I = iN_{I\bar{J}}\,, \quad \Omega_{\bar{I}J} = N_{\bar{K}J} J^{\bar{K}}{}_{\bar{I}} = -iN_{I\bar{J}}\,.$$

The dilatation and the chiral $\mathrm{U}(1)_\mathrm{R}$-transformations of the vector multiplet scalars define the vector fields χ^I and k^I,

$$\delta X^I = \Lambda_D \chi^I + \Lambda_{\mathrm{U}(1)} k^I\,. \tag{16}$$

From (9) one finds that the components of these vectors are given by $\chi^I = X^I$ and $k^I = -iX^I$. The anti-holomorphic components follow by complex conjugation. As noted above, it is convenient to characterize the geometry in terms of the preferential coordinates, but it is simple to reformulate the geometry in terms of local holomorphic section $X^I(z^A)$. As a result of Weyl and chiral $\mathrm{U}(1)_\mathrm{R}$ symmetry, the sections are defined projectively. The vector fields k^I and χ^I are expressed as $k^I(X(z)) = k^A(z)\partial_A X^I(z)$ and $\chi^I(X(z)) = \chi^A(z)\partial_A X^I(z)$. We will stick to the preferential coordinates X^I and $\bar{X}^{\bar{I}}$in the following, but remark that all expressions below can be written in terms of coordinates z^A and $\bar{z}^{\bar{A}}$ by using the matrix $\partial_A X^I(z)$ and its inverse.

From the explicit form of the hermitian connection and the homogeneity (15) one finds that

$$D_I \chi^J = \delta_I{}^J\,, \quad D_{\bar{I}} \bar{\chi}^{\bar{J}} = \delta_{\bar{I}}{}^{\bar{J}}\,. \tag{17}$$

The derivatives D contain the hermitian connection. This relation expresses the fact that χ^I is an exact homothetic Killing vector. Homothety is the statement that

$$D_I \chi_{\bar{J}} + D_{\bar{J}} \chi_I = 2N_{I\bar{J}} .$$

It is important to realize that therefore χ^I and $\bar{\chi}^{\bar{J}}$ are not Killing vectors. Exactness expresses the fact that the homothetic one-forms χ_I and $\bar{\chi}_{\bar{I}}$ derive from a potential. In fact, this potential is given by the Kähler potential $K(X, \bar{X})$ of the target space metric $N_{I\bar{J}}$ given in (14). This can be seen by noting that the K is written in terms of the homothetic Killing vectors as

$$\chi_I \, \chi^I = \bar{\chi}_{\bar{I}} \, \bar{\chi}^{\bar{I}} = K .$$

Using (17), one finds $\chi_I = \partial_I K$. The existence of an homothetic Killing vector is a general feature of conformal sigma models [79]. In general, spaces with a homothety (17) have a cone structure. The existence of a covariantly constant complex structure guarantees the existence of a U(1) isometry,

$$k^I = -J^I{}_J \chi^J , \quad \bar{k}^{\bar{I}} = -J^{\bar{I}}{}_{\bar{J}} \chi^{\bar{J}} . \tag{18}$$

The vectors k^I and $\bar{k}^{\bar{I}}$ are associated with the chiral $\mathrm{U(1)_R}$ transformations of the fields. They are Killing vectors of the target space metric,

$$D_I k_{\bar{J}} + D_{\bar{J}} k_I = 0 , \quad D_I k_J + D_J k_I = 0 ,$$

This can also be seen by noting that $D_I k_{\bar{J}} = -\Omega_{I\bar{J}}$, which is antisymmetric. The contraction of the Killing vectors is given by the Kähler potential, $K = k_I k^I = \bar{k}_{\bar{I}} \bar{k}^{\bar{I}}$. The isometry is holomorphic, as it leaves the complex structures invariant, $\mathcal{L}_k J = 0$. Likewise, the Kähler potential is invariant under the U(1)-isometry, since from (18) it follows that $k^I \chi_I + \bar{k}^{\bar{I}} \bar{\chi}_{\bar{I}} = 0$. The chiral $\mathrm{SU(2)_R}$-transformations act trivially on the scalar fields and therefore present a trivial isometry. In summary, the target space of the superconformal vector multiplets is a cone over a so-called Sasakian space [80]. The latter is a U(1)-fibration over the special Kähler manifold relevant for Poincaré supergravity. We come back to this in section 5.

Finally, we comment on the supersymmetry algebra and on central charges. The conserved supercurrent for the nonabelian vector multiplets reads [81],

$$J_{\mu i} = \frac{1}{4\pi} N_{IJ} \left[\not{D} \, \bar{X}^I \gamma_\mu \Omega_i^J - \varepsilon_{ij} \left(\tfrac{1}{4} \gamma^{ab} F_{ab}^{-I} - g f^I_{MN} \bar{X}^M X^N \right) \gamma_\mu \Omega^{jJ} \right] .$$

The other chirality components result from complex conjugation. We consider the abelian limit. Using the canonical quantization conditions[d] on the fields, the conserved

[d] The only relevant Dirac brackets for a bosonic solution are the ones involving two fermions,

$$\left\{ \Omega^I_j(x), \bar{\Omega}^{iJ}(y) \right\}_{x_0=y_0} = 8\pi\hbar \, [N^{-1}]^{IJ} \, \delta_j{}^i \left(\frac{1+\gamma_5}{2} \gamma_4 \right) \delta^3(\vec{x} - \vec{y}) ,$$

$$\left\{ \Omega^{iI}(x), \bar{\Omega}^J_j(y) \right\}_{x_0=y_0} = 8\pi\hbar \, [N^{-1}]^{IJ} \, \delta^i{}_j \left(\frac{1-\gamma_5}{2} \gamma_4 \right) \delta^3(\vec{x} - \vec{y}) .$$

charges $Q_i = \int d^3x\, J_i^0$ generate the supersymmetry algebra,

$$\begin{aligned}
\{Q_i, \bar{Q}^j\} &= -\tfrac{1}{2} i\hbar (1-\gamma_5)\delta_i{}^j \Big[\gamma_\mu P^\mu + \gamma_m Z^m\Big]\,, \\
\{Q_i, \bar{Q}_j\} &= i\hbar (1-\gamma_5)\epsilon_{ij} \Big[\bar{X}^I(\infty) q_{\mathrm{e}\,I} - \bar{F}(\infty)_I q_{\mathrm{m}}^I\Big]\,.
\end{aligned} \tag{19}$$

Here, P^μ is proportional to the spatial integral over $T^{\mu 0}$, where $T^{\mu\nu}$ denotes the energy-momentum tensor. The spatial vector Z^m is given by an integral involving the pullback of the Kähler form $\partial_m X^I \partial_n \bar{X}^J \Omega_{I\bar{J}}$, where m, n run over spatial indices. This contribution can be written as a boundary term and vanishes if the fields (X^I, F_I) approach constant values at spatial infinity. Details can be found in [81,82]. The right-hand side of the second anti-commutator defines the central charge of the algebra and represents the (anti-holomorphic) BPS mass. The charges $q_{\mathrm{e}\,I}$ and q_{m}^I are defined in (5). The observation that central charges in the supersymmetry algebra appear as surface integrals for non-trivial field configurations was noted in [83]. The BPS mass formula plays an important role in understanding the effective action of SU(2) Yang-Mills theory [71]. In section 6 we discuss the generalization of the symplectic reparameterizations to the context of $N = 2$ supersymmetric model. We will assert that the holomorphic BPS mass transforms as a scalar under electric-magnetic duality transformations. We will also encounter the (anti-)holomorphic BPS mass when discussing the ADM mass of BPS black holes in chapter IV.

4. Hypermultiplets

Hypermultiplets describe matter fields in supersymmetric theories. They contain two complex scalar fields and their fermionic superpartners. The scalars of the hypermultiplet parameterize a hyperkähler manifold [84] in the case of rigid, and a quaternionic manifold (with negative curvature) [85] in the case of local supersymmetry. In this section we follow the presentation in [86,87].

We consider r hypermultiplets containing $4r$ real scalars ϕ^A, $2r$ positive-chirality spinors ζ^α and $2r$ negative-chirality spinors $\zeta^{\bar{\alpha}}$. The spinors are related by complex conjugation, under which indices are converted according to $\alpha \leftrightarrow \bar{\alpha}$, while $\mathrm{SU}(2)_{\mathrm{R}}$-indices $i, j, \ldots$ are raised and lowered. They therefore make up $2r$ Majorana spinors. In contrast to vector multiplets and, as we shall see, the superconformal gravity multiplet, there does not exist an finite dimensional, unconstrained off-shell formulation of the $N = 2$ supersymmetry algebra for hypermultiplets. On-shell representations of the supersymmetry algebra close only up to the equations of motion. Therefore, the supersymmetry transformation rules and the form of the action, from which these equations of motion derive, are closely related. The on-shell transformation rules for the hypermultiplet are given by

$$\begin{aligned}
\delta_{\mathrm{Q}}\phi^A &= 2\left(\gamma^A_{i\bar\alpha}\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\bar\epsilon_i\zeta^{\alpha}\right) , \\
\delta_{\mathrm{Q}}\zeta^{\alpha} &= V_{Ai}{}^{\alpha}\not{D}\phi^A\epsilon^i - \delta_{\mathrm{Q}}\phi^A\Gamma_A{}^{\alpha}{}_{\beta}\zeta^{\beta} , \\
\delta_{\mathrm{Q}}\zeta^{\bar\alpha} &= \bar V_A{}^{i\bar\alpha}\not{D}\phi^A\epsilon_i - \delta_{\mathrm{Q}}\phi^A\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\zeta^{\bar\beta} .
\end{aligned}$$

Here, the fields $\gamma^A_{i\bar\alpha}$ and $V_{Ai}{}^{\bar\alpha}$ are real $(4r) \times (4r)$ matrices. The $\mathrm{SU}(2)_{\mathrm{R}} \cong \mathrm{Sp}(1)$ cannot be realized on the fields. The covariant derivatives D_μ contain the connection $\Gamma_A{}^{\alpha}{}_{\beta}$ associated with the reparameterizations $\zeta^\alpha \to S(\phi)^\alpha{}_\beta\zeta^\beta$, $D_\mu\zeta^\alpha = \partial_\mu\zeta^\alpha + \partial_\mu\phi^A\Gamma_A{}^\alpha{}_\beta\zeta^\beta$. The hypermultiplet Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{2}g_{AB}\partial_\mu\phi^A\partial^\mu\phi^B - G_{\bar\alpha\beta}\left(\bar\zeta^{\bar\alpha}\not{D}\zeta^\beta + \bar\zeta^\beta\not{D}\zeta^{\bar\alpha}\right) - \tfrac{1}{4}W_{\bar\alpha\beta\bar\gamma\delta}\bar\zeta^{\bar\alpha}\gamma_\mu\zeta^\beta\bar\zeta^{\bar\gamma}\gamma^\mu\zeta^\delta .$$

Here $G_{\bar\alpha\beta}$ is an hermitian target space metric for the fermions. Requiring that the supersymmetry transformations close up to equations of motion derived from above action, puts strong constraints on the objects introduced above. We refer to [86] for a full account. The curvature of the connection $\Gamma_A{}^\alpha{}_\beta$ must take values in $\mathrm{sp}(r) \cong \mathrm{usp}(2r,\mathbb{C})$. Therefore, the matrices $\gamma^A_{i\bar\alpha}$ and $V_{Ai}{}^{\bar\alpha}$, which turn out to be each others inverse, act as quaternionic vielbeins, converting tangent space indices into indices of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$-bundle. These vielbeins are related to the target space metric g_{AB} and fermion metric $G_{\bar\alpha\beta}$ and are covariantly constant with respect to the Christoffel connection of the target space metric g_{AB} and the connection $\Gamma_A{}^\alpha{}_\beta$. The curvature $W_{\bar\alpha\beta\bar\gamma\delta}$ is expressed in terms of the Riemann curvature contracted with the quaternionic vielbeins. The covariant constancy of the vielbeins allows the construction of various covariantly constant tensors. Important for our discussion are the three covariantly constant, hermitian matrices $\vec\Omega_{AB}$,

$$\vec\Omega_{AB}\cdot\vec\sigma^{\,i}{}_j = -2\varepsilon_{jl}\bar V_{[A}{}^{i\bar\alpha}\Omega_{\bar\alpha\bar\beta}\bar V_{B]}{}^{l\bar\beta} .$$

These antisymmetric tensors define the triplet of covariantly constant complex structures of the hyperkähler geometry, which satisfy the quaternionic algebra,

$$\vec J^A{}_B = g^{AC}\vec\Omega_{CB} , \quad J^\Pi J^\Sigma = -\delta^{\Pi\Sigma} - \varepsilon^{\Pi\Sigma}{}_\Lambda J^\Lambda .$$

In the following we consider rigid superconformal hypermultiplets [86]. One assumes that $\delta_S\phi^A = \delta_K\phi^A = \delta_K\zeta^\alpha = 0$. For the scalars, the transformations under the remaining symmetries of the superconformal algebra are parameterized by

$$\delta\phi^A = \Lambda_D\chi^A(\phi) + \Lambda_{\mathrm{U}(1)}k^A(\phi) + \tfrac{1}{2}\vec\Lambda_{\mathrm{SU}(2)}\vec k^A(\phi) . \tag{20}$$

Closure of the superconformal algebra implies that $\chi^A(\phi)$ must be an exact homothetic Killing vector,

$$D_A\chi^B = \delta_A{}^B .$$

Furthermore, the $\mathrm{U}(1)_\mathrm{R}$ acts trivially, $k^A(\phi) = 0$, and $\vec{k}^A$ is related to χ^A by the complex structure,

$$\vec{k}^A = \vec{J}^A{}_B \chi^B , \tag{21}$$

As remarked, the homothety condition already follows from requiring that the kinetic terms of the scalars are invariant under dilatations. Locally, the homothetic one-forms derive from the hyperkähler potential,

$$\chi_A = \partial_A \chi .$$

Given a hyperkähler metric g_{AB} and a homothety, the hyperkähler potential can be expressed by

$$\chi = \tfrac{1}{2} \chi^A g_{AB} \chi^B , \quad g_{AB} = D_A \partial_B \chi . \tag{22}$$

The hyperkähler metric therefore has a cone structure. It also possesses an Sp(1)-isometry, since from (21) one finds that $\vec{k}^A \chi_A = 0$, such that the hyperkähler potential is Sp(1)-invariant. The vectors $\vec{k}^A$ are in fact Killing vectors. This is seen by noting that $D_A \vec{k}_B = -\vec{\Omega}_{AB}$, which is antisymmetric. The Sp(1)-Killing vectors are not tri-holomorphic, *i.e.*, they do not leave the complex structures $\vec{J}$ invariant. Instead, the complex structures transform as an Sp(1)-triplet

$$\mathcal{L}_{\vec{\Lambda}\cdot\vec{k}}[J^\Gamma] = 2\, \varepsilon^\Gamma{}_{\Pi\Sigma}\, \Lambda^\Pi\, J^\Sigma .$$

The target space of superconformal hypermultiplets is therefore a cone over a so-called tri-Sasakian manifold [80]. This latter space is an Sp(1)-fibration over a quaternion-Kähler manifold relevant in Poincaré supergravity. This is described in the following section.

We close the present discussion with the remark that on the hyperkähler cone there is no set of preferential coordinates. On the cone one can, however, resort to $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$-sections defined in terms of the homothety and the quaternionic vielbeins,

$$A_i^\alpha = \chi^B V_{Bi}{}^\alpha .$$

In terms of these sections the quaternionic vielbeins are given by $V^\alpha_{B\,i} = D_B A_i{}^\alpha$. In the next chapter we will give the transformation rules and the action in terms of these sections. We will see that these sections can be read off from the S-supersymmetry transformation rules. For vector multiplets the analogous formula for the sections is $X^I(z) = \chi^A(z) V_A{}^I(z)$, where, of course, $V_A{}^I(z)$ is the holomorphic matrix $V_A{}^I(z) = \partial_A X^I(z)$ and $\chi^A(z)$ is the homothety in coordinates z^A. (Here, the index A should not be confused with the indices appearing in the hypermultiplet sector.)

5. Poincaré supergravity

In this section we discuss the coupling of vector multiplets and hypermultiplets to supergravity. Our main focus is on the geometry of the target spaces and on the coupling structure of the gauge fields of the vector multiplets. This will enable us to discuss symplectic reparameterizations in the next section. In order to couple the multiplets to a supergravity background one utilizes the superconformal approach. In this approach one gauges the symmetries of the superconformal group, which contains the usual Poincaré supersymmetry group as a subgroup. We emphasize that we do not intend to describe models with a superconformal invariance. Quite on the contrary: by carefully coupling multiplets of compensating degrees of freedom to the superconformal gauge theory, superconformal gravity becomes gauge equivalent to Poincaré supergravity.

One of the advantages of the superconformal approach is that a completely off-shell formulation of the theory is available. This simplifies the coupling to on-shell multiplets such as the hypermultiplets. The superconformal algebra contains general-coordinate, local Lorentz, dilatation, special conformal, chiral $[\mathrm{SU}(2)\times \mathrm{U}(1)]_{\mathrm{R}}$, supersymmetry ($Q$), and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations ($e_\mu^{\,a}$), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu^i$, A_μ), and Q-supersymmetry (ψ_μ^i) are realized by independent fields. The remaining gauge fields of Lorentz (ω_μ^{ab}), special conformal ($f_\mu^{\,a}$), and S-supersymmetry transformations (ϕ_μ^i) are dependent fields. They are composite objects, which depend in a complicated way on the independent fields [76,77,88,89]. The superconformal gauge fields reside in the so-called Weyl multiplet. Their corresponding curvatures and covariant fields are contained in a reduced chiral tensor multiplet, which comprises $24+24$ off-shell degrees of freedom. In addition to the independent superconformal gauge fields the Weyl multiplet also contains three auxiliary fields: a Majorana spinor doublet χ^i, a scalar D and a selfdual Lorentz tensor T_{abij} (where $i, j, \ldots$ are chiral SU(2) spinor indices). Many of the details are not relevant for the present discussion and are given in chapter III.

As compared to rigid supersymmetry there are two main differences in the Lagrangians and the transformation rules for the vector multiplets and hypermultiplets. First, the derivatives are covariantized with respect to the superconformal invariances. In addition, there are further couplings to the auxiliary matter fields of the Weyl multiplet in both the transformation rules and in the Lagrangian. We will give the details in chapter III. For the moment it suffices to consider the result for the kinetic terms of the bosonic Lagrangian describing abelian vector multiplets and hypermultiplets [77,86],

$$
\begin{aligned}
8\pi e^{-1}\mathcal{L}_{\mathrm{kin}} = &- N_{IJ}\mathcal{D}^\mu X^I\,\mathcal{D}_\mu \bar{X}^I - \tfrac{1}{2} g_{AB}\mathcal{D}_\mu\phi^A\mathcal{D}^\mu\phi^B \\
&+ K(X,\bar{X})(\tfrac{1}{6}R - D) + \tfrac{1}{2}\chi(\phi)(\tfrac{1}{3}R + D) \\
&- \Big[\tfrac{1}{4} i F_{IJ}\mathcal{F}^{+I}_{ab}\mathcal{F}^{+Jab} + \tfrac{1}{8} i F_I \mathcal{F}^{+I}_{ab} T^{+ab} + \tfrac{1}{32} i F (T^{+}_{ab})^2 + \text{h.c.}\Big] .
\end{aligned}
\tag{23}
$$

Here, R denotes the Ricci scalar of the spacetime metric. The covariant derivatives are given by

$$
\mathcal{D}_\mu X^I = \partial_\mu X^I - b_\mu \chi^I - A_\mu k^I\,, \quad \mathcal{D}_\mu\phi^A = \partial_\mu\phi^A - b_\mu\chi^A - \vec{\mathcal{V}}_\mu\cdot \vec{k}^A\,.
$$

They contain the homotheties χ^I and χ^A and the Killing vectors k^I and $\vec{k}^A$ of the Kähler and hyperkähler cones, respectively, and were introduced in (16) and (20). The Kähler potential $K(X,\bar{X})$ and and hyperkähler potential $\chi(\phi)$ have been defined in (14) and (22). The tensors $\mathcal{F}^I_{\mu\nu}$ contain, apart from the gauge field strength $F^I_{\mu\nu} = 2\partial_{[\mu} W^I_{\nu]}$, terms involving $T^-_{\mu\nu} = \varepsilon_{ij} T^{ij}_{\mu\nu}$ (we suppress further terms proportional to fermion bilinears),

$$\mathcal{F}^I_{ab} = F^I_{ab} - \left(\tfrac{1}{4}\bar{X}^I T^-_{ab} + \text{h.c.}\right) .$$

In order to make contact with Poincaré supergravity one needs to eliminate the gauge fields of the superfluous gauge invariances, such as the gauge fields of dilatation and the chiral $[\mathrm{SU}(2)\times\mathrm{U}(1)]_\mathrm{R}$-transformations. Note that when coupling vector multiplets and hypermultiplets to a superconformal background the $[\mathrm{SU}(2)\times\mathrm{U}(1)]_\mathrm{R}$-isometries of the rigid target spaces are gauged. The resulting target space geometries are therefore related to the cone geometries by a quotients. Let us explain this in the following. Since the Lagrangian is K-invariant and the gauge field b_μ is (as we will see in chapter III) the only independent field transforming under conformal boost, it must drop out of the Lagrangian and is from here on disregarded. The equations of motion for the $[\mathrm{SU}(2)\times\mathrm{U}(1)]_\mathrm{R}$ gauge fields $\vec{\mathcal{V}}_\mu$ and A_μ are algebraic and solved by [77,78]

$$\vec{\mathcal{V}}_\mu = \frac{1}{2\chi}\,\vec{k}_A \partial_\mu \phi^A\,, \quad A_\mu = \frac{1}{2K}\left(k_I\partial_\mu X^I + \bar{k}_{\bar{I}}\partial_\mu \bar{X}^{\bar{I}}\right) .$$

Let us stress that although these expressions are presented in terms of the preferred coordinates, it is simple to derive corresponding results in terms of sections. The field equation of the auxiliary $T^{\pm}_{ab}$-field of the Weyl multiplet yields the relation

$$N_{IJ}X^I X^J T^+_{ab} = 4N_{IJ}X^I F^{+J}_{ab} .$$

We have suppressed further contributions proportional to fermion bilinears in this discussion. These expressions are reinserted into the Lagrangian (23) with the result [78]

$$\begin{aligned}
8\pi e^{-1}\mathcal{L}_\text{kin} = &- K\mathcal{M}_{I\bar{J}}\partial_\mu X^I\,\partial^\mu \bar{X}^J - \tfrac{1}{2}\chi G_{AB}\partial_\mu\phi^A\partial^\mu\phi^B \\
&+ K\Big[\tfrac{1}{6}R - \tfrac{1}{4}(\partial_\mu \ln|K|)^2 - D\Big] + \chi\Big[\tfrac{1}{6}R - \tfrac{1}{4}(\partial_\mu \ln|\chi|)^2 + D\Big] \\
&- \tfrac{1}{4}i\mathcal{N}_{IJ}F^{+I}_{ab}F^{+Jab} + \tfrac{1}{4}i\bar{\mathcal{N}}_{IJ}F^{-I}_{ab}F^{-Jab} . \qquad (24)
\end{aligned}$$

The target space metric of the vector multiplets and hypermultiplets in the Poincaré frame read

$$\mathcal{M}_{I\bar{J}} = \frac{1}{K}\left[N_{IJ} - \frac{1}{2K}\chi_I\bar{\chi}_{\bar{J}} - \frac{1}{2K}k_I\bar{k}_{\bar{J}}\right] , \qquad (25)$$

$$G_{AB} = \frac{1}{\chi}\left[g_{AB} - \frac{1}{2\chi}\chi_A\chi_B - \frac{1}{2\chi}\vec{k}_A\cdot\vec{k}_B\right] . \qquad (26)$$

The metric (25) is that of the so-called special Kähler manifold that is relevant for Poincaré supergravity. The target space of the superconformal vector multiplets is a cone over a $U(1)$-fibration over this special Kähler manifold. The metric (26) of the hypermultiplet target space has a similar structure and describes the metric of the quaternion-Kähler manifold relevant for Poincaré supergravity theories. The target space of the superconformal hypermultiplets is a cone over an Sp(1)-fibration over this quaternion-Kähler manifold [90,91]. Note that both quotient metrics (25) and (26) possess null directions.

The gauge coupling functions of the vector multiplets in the Poincaré frame receive non-holomorphic contributions,

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i\,\frac{N_{IK}X^I N_{JL}X^L}{N_{KL}X^K X^L}\,. \tag{27}$$

These non-holomorphic contributions originate from integrating out the $T^{\pm}_{ab}$-field. The equation of motion for the D-field further relates the Kähler cone with the hyperkähler cone

$$\chi(\phi) = 2K(X,\bar{X})\,,$$

Note that the signs of the kinetic terms of χ and K in the Lagrangian (24) correspond to those of compensators.[e] Up to a total derivative, these terms can be scaled away by rescaling the vielbein, $e_\mu{}^a \to |K|^{-1/2}e_\mu{}^a = |\chi/2|^{-1/2}e_\mu{}^a$. The rescaling is such that the Lagrangian is canonically normalized (in Planck units),

$$\begin{aligned} 8\pi e^{-1}\mathcal{L}_{\text{kin}} &= -\tfrac{1}{2}R - \mathcal{M}_{I\bar{J}}\partial_\mu X^I\,\partial^\mu \bar{X}^I - G_{AB}\partial_\mu\phi^A\partial^\mu\phi^B \\ &\quad -\tfrac{1}{4}i\mathcal{N}_{IJ}F^{+I}_{ab}F^{+Jab} + \tfrac{1}{4}i\bar{\mathcal{N}}_{IJ}F^{-I}_{ab}F^{-Jab}\,. \end{aligned}$$

Of course, the superconformal quotient we sketched above becomes more complicated when taking fermionic couplings and possible gaugings into account. Furthermore, in above computations, we have extensively used the properties (15) which follow from the homogeneity of the function $F(X)$. In the presence of the chiral background describing R^2-interactions that we describe in chapter III, the transition to the Poincaré frame is much more subtle, since many of the previously auxiliary fields become dynamical.

We emphasize that we have not imposed any gauge choices. In particular, the coordinates X^I and $\bar{X}^I$, $I = 0,\ldots,n$, are still subject to $\mathrm{U}(1)_\mathrm{R}$-transformations and dilatations and are therefore projectively defined. Note that the $F_I(X)$ scales just as X^I. The quotient metric $\mathcal{M}_{I\bar{J}}$, on the other hand, is invariant under these rescalings and describes the geometry of the n (complex) dimensional quotient space. It is often convenient to choose holomorphic and anti-holomorphic coordinates z^A and $\bar{z}^{\bar{A}}$, $A = 1,\ldots,n$ and describe X^I and $F_I(X)$ as projectively defined sections $(X^I(z), F_I(z))$, where, at this point, $F_I(z) = F_I(X(z))$. In terms of these coordinates the metric is given by

$$\mathcal{M}_{A\bar{B}} = \partial_A\partial_{\bar{B}}\mathcal{K}(z,\bar{z})\,,\quad \mathcal{K}(z,\bar{z}) = -\ln\Big[i\bar{X}^{\bar{I}}(\bar{z})F_I(z) - iX^I(z)\bar{F}_{\bar{I}}(\bar{z})\Big] \tag{28}$$

[e] We work with mostly positive cone metrics N_{IJ} and g_{AB}. The cone metrics must have one negative eigenvalue in order to provide compensating multiplets. The quotient metric (25) and (26) are positive definite and describe physical vector multiplets and hypermultiplets.

These sections are chosen to be holomorphic in order to preserve the hermiticity of the metric $\mathcal{M}_{A\bar{B}}$. The vector multiplet sector therefore singles out preferred coordinates on the quotient geometry. In the next section we return to the subject of electric-magnetic duality transformations. This discussion will show that the sections $(X^I(z), F_I(z))$ are subject to symplectic reparameterizations.

We conclude this section with the remark that in contrast to the special Kähler geometry, the quaternionic-Kähler manifold does not come with preferred coordinates. This is a result of the fact that the $\mathrm{Sp}(1)$ isometries of the hyperkähler cone are not tri-holomorphic.

6. Symplectic reparameterizations

In this section we discuss symplectic reparameterizations in the various settings of supersymmetry we have sketched so far. The reason why we can discuss this issue in one go stems from the fact that the superconformal gravity multiplet is inert under these transformations. We follow the presentation given in [92].

As argued in section 1, symplectic reparameterizations are equivalence transformations at the level of the field equations of abelian gauge theories. We consider here the generalization of these $\mathrm{Sp}(2n+2,\mathbb{R})$-reparameterizations to the context of the $N=2$ supersymmetric theories of abelian vector multiplets. Using the definition (2), one finds the dual field strengths (up to a normalization and the metric determinant),

$$G_I^+ = \mathcal{N}_{IJ} F^{+J} + \mathcal{O}_I^+ \,, \quad G_I^- = \bar{\mathcal{N}}_{IJ} F^{-J} + \mathcal{O}_I^- \,. \tag{29}$$

The functions $\mathcal{N}_{IJ}$ are field dependent and are determined from varying the terms in the Lagrangian which are quadratic in field strengths. The tensor $\mathcal{O}_I^\pm$ comes from varying other terms in the Lagrangian which couple linearly to $F^{\pm I}$. These may come from couplings to fermions, to fields of the superconformal background, or to background chiral fields, which are discussed in chapter III. For rigid supersymmetric or rigid superconformal theories we have $\mathcal{N}_{IJ} = \bar{F}_{IJ}$, while $\mathcal{O}^\pm_{\mu\nu I}$ involves only gaugino bilinears. For local superconformal symmetry the precise values of $\mathcal{N}_{IJ}$ and $\mathcal{O}^\pm_{\mu\nu I}$ depend on whether one has integrated out the tensor $T^\pm_{\mu\nu}$. If one has not, then the couplings are still given by $\mathcal{N}_{IJ} = \bar{F}_{IJ}$, whereas the tensors $\mathcal{O}^\pm_{\mu\nu I}$ now read, up to fermionic contributions, $\mathcal{O}^+_{\mu\nu I} = \frac{1}{4}(F_I - \bar{F}_{IJ} X^J)\, T^+_{\mu\nu}$. In the Poincaré frame, in which the $T^\pm_{ab}$ fields have been integrated out, there are no bosonic fields that coupling linearly to the field strengths. The coupling matrix $\mathcal{N}_{IJ}$, however, contain the additional contributions given in (27).

The Bianchi identities and the field equations (1) can be written as

$$\partial^\mu \left[\sqrt{|g|}\left(F^{+I}_{\mu\nu} - F^{-I}_{\mu\nu}\right)\right] = 0\,, \quad \partial^\mu \left[\sqrt{|g|}\left(G^+_{I\mu\nu} - G^-_{I\mu\nu}\right)\right] = 0\,. \tag{30}$$

These equations are invariant under a general linear transformation (3) of the field strengths. From equations (3) and (29) one derives that the coupling functions must transform as

$$\mathcal{N}_{IJ} \longrightarrow \mathcal{N}'_{IJ} = (V\mathcal{N} + W)_{IL}[(U + Z\mathcal{N})^{-1}]^{L}{}_{J}\,.$$

The metric $\mathcal{N}_{IJ}$ remains symmetric if this transformation is a symplectic transformation. Furthermore, the matrix $\mathcal{O}^{\pm}_{I}$ must transform according to

$$\mathcal{O}^{\pm}_{I} \longrightarrow \mathcal{O}'^{\pm}_{I} = \mathcal{O}^{\pm}_{J}[(U + Z\mathcal{N})^{-1}]^{J}{}_{I}\,.$$

The coupling matrix $\mathcal{N}_{IJ}$ is a field dependent object, which is expressed in terms of the preferential coordinates X^I and $\bar{X}^I$, the function $F(X)$, and derivatives thereof. The transformation of $\mathcal{N}_{IJ}$ is induced if we transform X^I and $F_I(X)$ as components of a symplectic vector (X^I, F_I),

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} X'^I \\ F'_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}\,. \tag{31}$$

In fact, the components F'_I again derive from a holomorphic function $F'(X')$ according to $\partial/\partial X'^I F'(X')$ if the transformation is symplectic.

As remarked in section 1, the charges $(q^I_{\rm m}, -q_{{\rm e}I})$, defined in equation (5), constitute a symplectic vector. Therefore, the (anti-)holomorphic BPS mass given by (19) transforms as a scalar under symplectic reparameterizations and is therefore a duality invariant quantity. The same holds for the Kähler potentials $K(X, \bar{X})$ and $\mathcal{K}(z, \bar{z})$ given in equations (14) and (28), respectively, as well as the combination $F(X) - \frac{1}{2}X^I F_I(X)$. The holomorphic quantity $F(X)$, on the other hand, does not transform as a scalar under symplectic reparameterizations. If this were so electric-magnetic duality would be a symmetry of the theory.

7. Special geometry and Calabi-Yau compactifications

In this section we elaborate on the target space geometry of vector multiplets in Poincaré supergravity, so-called special Kähler geometry. Up to now this geometry has been described in coordinates, for which $F_I(X(z))$ derives from a holomorphic potential $F(X(z))$ according to $F_I(X(z)) = \partial/\partial X^I(z) F(X(z))$. While this is natural from a supergravity point of view (especially when dealing with chiral backgrounds) this characterization of the target space geometry does depend on the the special choice of coordinates X^I and a choice of function $F(X)$.

In the previous section it was shown that the pair $(X^I(z), F_I(z))$ are subject to constant $\mathrm{Sp}(2n+2,\mathbb{R})$ transformations. It is important to realize that in order to discuss these transformations one did not rely on the fact that $F_I(z)$ derived from a holomorphic function $F(X(z))$. In fact, it was shown in [93–95], that the the full supergravity action can be characterized in terms of the symplectic vector $(X^I(z), F_I(z))$, where $F_I(z)$ does not necessarily derive from a function. This vector is subject to certain constraints which define the special geometry. It is known [94] that there are representations of the theory for which no function $F(X)$ exists, although after a suitable electric-magnetic duality transformation the constraints can be solved in terms of a function $F(X)$. In the presence of a chiral background, which we discuss in chapter III, this feature does not seem to play a direct role.

As noted at the end of section 5, the symplectic vector $(X^I(z), F_I(z))$ is projectively defined. This means that it is defined up to multiplication by a function $\lambda(z)$

$$(X^I(z), F_I(z)) \longrightarrow \mathrm{e}^{\lambda(z)}(X^I(z), F_I(z))\,. \tag{32}$$

This transformation does not affect the metric $\mathcal{M}_{I\bar{J}}$ but changes the Kähler potential $\mathcal{K}$ (defined in equation 28) by a so-called Kähler transformation,

$$\mathcal{K}(z,\bar{z}) \longrightarrow \mathcal{K}(z,\bar{z}) - \lambda(z) - \bar{\lambda}(\bar{z})\,.$$

Geometrically, the starting point of the coordinate-independent construction is a globally defined, holomorphic section Ω of $\mathcal{L}\otimes\mathcal{H}$, where $\mathcal{L}$ denotes a complex line bundle and $\mathcal{H}$ is an $\mathrm{Sp}(2n+2,\mathbb{R})$ vector bundle over a (Hodge) Kähler manifold[f]. The symplectic pair $(X^I(z), F_I(z))$ are the components of Ω in a coordinate patch $\{z^A, \bar{z}^{\bar{B}}\}$ expressed in terms of a real basis of the flat sections (α_I, β^I) of $\mathcal{H}$,

$$\Omega = X^I(z)\alpha_I - F_I(z)\beta^I\,.$$

On the intersections of two local coordinate patches, $\{z^A, \bar{z}^{\bar{B}}\}$ and $\{z'^A(z), \bar{z}'^{\bar{B}}(\bar{z})\}$, the components $(X'^I(z'), F'_I(z'))$ of Ω may differ from $(X^I(z), F_I(z))$ by a symplectic reparameterization and by a multiplication with a holomorphic function (32).

For theories with n rigidly supersymmetry abelian vector multiplets one still works with local section $(X^I(z), F_I(z))$ of an $\mathrm{Sp}(2n,\mathbb{R})$ vector bundle, but contrary to the supergravity setting, this symplectic pair is not projectively defined. Therefore, the line bundle $\mathcal{L}$ is just a trivial $\mathbb{C}$ factor, which further degenerates to $\mathrm{U}(1)$, since the real part can be absorbed into the $\mathrm{Sp}(2n,\mathbb{R})$-transformations. Since the function $F(X)$ is of arbitrary degree in the rigid case, the symplectic transformations may also contain an inhomogeneous or translational part, which amount to the Kähler transformations.

The interesting point is that in many physically relevant situations the components of the vector $(X^I(z), F_I(z))$ can be written as period integrals of some corresponding Riemannian manifold. For rigid special geometry the probably best known examples are the genus one Seiberg-Witten curves, the period vectors of which provide local coordinates on the quantum moduli space of supersymmetric SU(2) Yang-Mills theory. Many of the properties of rigid special geometry follow directly from its geometrical representation: symplectic reparameterization correspond to transformations of the canonical homology basis and the metric is identified with the so-called period matrix. This matrix is guaranteed to be symmetric and positive definite as a result of what are called Riemann's relations.

[f] That the Kählerian base manifold must be Hodge (Kählerian of the restricted type) is understood from the supergravity point of view from the fact that the $\mathrm{U}(1)_\mathrm{R}$-field strength, when integrated out, becomes proportional to the Kähler two-form. The fluxes of the former are quantized, which implies integer cohomology of the Kähler form. Such manifolds are Hodge manifolds.

More related to the topic of this thesis is the geometrical realization of (local) special geometry in Calabi-Yau compactifications of type II string theory. For type IIB on a Calabi-Yau three-fold X, for example, the harmonic three-forms give rise to vector multiplets in four dimensions. These forms in turn parameterize the complex structure moduli space of the Calabi-Yau space. The special geometry of the vector multiplets is induced via period maps by the specialty of the complex structure moduli space $\mathcal{M}_X$. Let us explain this statement: one can make use of the fact that any Calabi-Yau three-fold X possesses a uniquely defined $(3,0)$-form Ω. This form provides the projectively defined section $(X^I(z), F_I(z))$ of the bundle $\mathcal{H}$ of harmonic three-forms $H^3(X,\mathbb{C})$ over the n-dimensional base manifold $\mathcal{M}_X$. The fiber of this bundle $\mathcal{H} \cong H^3(X,\mathbb{C})$ is $b_3 = 2n+2$-dimensional, where n is the number of (2,1)-forms. The symplectic pair of projective coordinates $(X^I(z), F_I(z))$ is represented as period integrals

$$X^I(z) = \int_X \Omega \wedge \beta^I \,, \quad F_I(z) = \int_X \Omega \wedge \alpha_I \,,$$

where α_I, β^J form a real, local basis[g] for the harmonic three-forms $H^3(X,\mathbb{C})$, $I = 0, \ldots, n$. The period integrals are expressed in terms of non-projective coordinates z^A. The choice of such a basis of harmonic three-forms is unique up to $\mathrm{Sp}(2n+2,\mathbb{R})$-transformations. These reparameterizations induce the symplectic transformation on the local sections $(X^I(z), F_I(z))$. The Kähler potential is given by[h]

$$\mathcal{K}(z,\bar{z}) = -\ln i \int_X \Omega \wedge \bar{\Omega} = -\ln(i\bar{X}^I(z)F_I(z) - iX^I(z)\bar{F}_I(z)) \,.$$

The Kähler transformations are a result of the fact that the overall scale of the compactification manifold is not fixed, $\Omega \longrightarrow \exp\lambda\,\Omega$.

The representation of the local sections in terms of period maps also emerges when performing a standard Kaluza-Klein reduction. The Kaluza-Klein ansatz for the type IIB self-dual five-form field strength, for instance, amounts to

$$F_5 = \sum_{I=0}^{n} \left(F^I \wedge \alpha_I - G_I \wedge \beta^I \right) \,, \tag{33}$$

where, due to charge quantization, α_I and β^J form a basis for the integral harmonic three-forms $H^3(X,\mathbb{Z})$. Consequently, they are subject to $\mathrm{Sp}(2n+2,\mathbb{Z})$-reparameterizations and the pair (F^I, G_I) forms a symplectic vector. The corresponding term in

[g] We adopt the following convention:

$$\int_X \alpha_I \wedge \beta^J = -\int_X \beta^J \wedge \alpha_I = \delta_I{}^J \,, \quad \int_X \alpha_I \wedge \alpha_J = \int_X \beta^I \wedge \beta^J = 0 \,.$$

[h] We note that Poincaré duality implies that Ω is anti-selfdual.

the action of ten-dimensional type IIB supergravity reads[i]

$$S^{10}_{\mathrm{IIB}} = -\frac{1}{4}\int F_5 \wedge *F_5 + \dots ,$$

The Kaluza-Klein ansatz for $*F_5$ can be obtained by calculating the Hodge dual of the cohomology basis (α_A, β^B). Explicit formulae were calculated in [96] and can be found in [97]. In special projective coordinates they involve the matrices $\mathcal{I}_{IJ} = \mathrm{Im}\,\mathcal{N}_{IJ}$ and $\mathcal{R}_{IJ} = \mathrm{Re}\,\mathcal{N}_{IJ}$, which are expressed in terms of the the holomorphic function $F(X(z))$ by (27). Integrating over the internal compactification manifold one finds (see for instance [97,98])

$$\begin{aligned} S^4_{\mathrm{IIB}}(F,G) \quad = \quad -\frac{1}{4}\int \Big[& F^I \wedge *F^J (\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{IJ} - 2F^I \wedge *G_J (\mathcal{R}\mathcal{I}^{-1})_I{}^J \\ & - G_I \wedge *G_J \mathcal{I}^{-1\,IJ} \Big] - \frac{1}{2}\int G_I \wedge F^I + \dots . \end{aligned}$$

The last term in the action acts as a Lagrangian multiplier and has been added to insure the selfduality of the five-form ansatz. Eliminating G_I as an independent field renders the familiar relation for the symplectic dual of the field strength

$$G_I = \mathrm{Im}\mathcal{N}_{IJ} * F^J + \mathrm{Re}\mathcal{N}_{IJ} F^J = \bar{\mathcal{N}}_{IJ} F^{+J} + \mathcal{N}_{IJ} F^{-J} .$$

This is the expression for the dual field strength given in (29). Inserted into the action we recover

$$\begin{aligned} S^{(4)}_{\mathrm{IIB}}(F) &= \frac{1}{2}\int F^I \wedge G_I + \dots \\ &= \frac{1}{2}\int \Big(\mathrm{Im}\mathcal{N}_{IJ} F^I \wedge *F^J + \mathrm{Re}\mathcal{N}_{IJ} F^I \wedge F^J \Big) + \dots \\ &= \frac{1}{2}\int \Big(i\bar{\mathcal{N}}_{IJ} F^{+I} \wedge *F^{+J} - i\mathcal{N}_{IJ} F^{-I} \wedge F^{-J} \Big) + \dots , \end{aligned}$$

which, up to the field-redefinition $F^\pm \to F^\mp$, is the vector multiplet action (24) with coupling matrix $\mathcal{N}_{IJ}$ given by (27). From (33) it follows that the $\mathrm{Sp}(2n+2, \mathbb{Z})$ reparameterizations of the basis of harmonic three-forms induce the electric-magnetic duality transformations of the field strengths in four dimensions. The special Kähler geometry of the effective vector multiplet action is inherited from that of the complex structure moduli space of the Calabi-Yau compactification manifold.

[i] Incidently, there does not exist a local action for the selfdual five-form field strength in ten dimensions. We can use the action as given in the text but have to impose selfduality by hand. Correspondingly, the two-from G_I in the Kaluza-Klein ansatz is considered independent at first. We impose self-duality by adding a Lagrangian multiplier. Actually the five-form field that appears in the effective action depends also on the NS and RR two-forms, $F_5 = \mathrm{d}C_4 - \mathrm{d}B_2 \wedge C_2$ and possibly other contributions if fluxes are switched on.

III

Supergravity theories with higher-order curvature interactions

In this chapter we give many of the more technical details one needs to describe supergravity theories with matter and with higher-order curvature interactions. After presenting the superconformal transformation rules of the Weyl multiplet and of the matter multiplets coupled to gravity we prepare many necessary formulae for the supersymmetry analysis of chapter IV. Most of this material has been presented in [99]. There are several introductions to the topics covered in this section [100–103]. We work with the superconformal off-shell formulation of $N = 2$ supergravity [76,77,88,89]. This is the only framework known in which it is possible to describe (holomorphic) R^2-couplings in a systematic fashion.

1. Superconformal gravity

The superconformal algebra contains general coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q), and special supersymmetry (S) transformations. The gauge fields associated with general coordinate transformations (e_μ^a), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu{}^i{}_j$, A_μ), and Q-supersymmetry (ψ_μ^i) are realized by independent fields. The remaining gauge fields of Lorentz (ω_μ^{ab}), special conformal (f_μ^a), and S-supersymmetry transformations (ϕ_μ^i) are dependent fields. They are composite objects, which depend in a complicated way on the independent fields [77,88,89] and are given in appendix B. The corresponding curvatures and covariant fields are contained in a reduced chiral tensor multiplet, which comprises $24 + 24$ off-shell degrees of freedom. In addition to the independent superconformal gauge fields it contains three auxiliary fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij} (where $i, j, \ldots$ are chiral SU(2) spinor indices)[a]. We summarize the transformation rules for some of the independent fields of the Weyl multiplet under Q- and S-supersymmetry and under special conformal

[a] By an abuse of terminology, T_{abij} is often called the graviphoton field strength. It is antisymmetric in both Lorentz indices a, b and chiral SU(2) indices i, j. Its complex conjugate is the anti-selfdual field T_{ab}^{ij}. Our conventions are such that SU(2) indices are raised and lowered by complex conjugation. The SU(2) gauge field $\mathcal{V}_\mu{}^i{}_j$ is anti-hermitian and traceless, i.e., $\mathcal{V}_\mu{}^i{}_j + \mathcal{V}_{\mu j}{}^i = \mathcal{V}_\mu{}^i{}_i = 0$.

transformations, with parameters ϵ^i, η^i and $\Lambda_{\rm K}^a$, respectively[b],

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i \gamma^a \psi_{\mu i} + \text{h.c.}\,,\\
\delta \psi_\mu^i &= 2\mathcal{D}_\mu \epsilon^i - \tfrac{1}{8} T^{ab\,ij} \gamma_{ab}\,\gamma_\mu \epsilon_j - \gamma_\mu \eta^i\,,\\
\delta b_\mu &= \tfrac{1}{2}\bar\epsilon^i \phi_{\mu i} - \tfrac{3}{4}\bar\epsilon^i \gamma_\mu \chi_i - \tfrac{1}{2}\bar\eta^i \psi_{\mu i} + \text{h.c.} + \Lambda_{\rm K}^a\, e_{\mu a}\,,\\
\delta A_\mu &= \tfrac{1}{2} i \bar\epsilon^i \phi_{\mu i} + \tfrac{3}{4} i \bar\epsilon^i \gamma_\mu \chi_i + \tfrac{1}{2} i \bar\eta^i \psi_{\mu i} + \text{h.c.}\,,\\
\delta \mathcal{V}_\mu{}^i{}_j &= 2\bar\epsilon_j \phi_\mu^i - 3\bar\epsilon_j \gamma_\mu \chi^i + 2\bar\epsilon^i \psi_{\mu i} - (\text{h.c.; traceless})\,,\\
\delta T_{ab}^{ij} &= 8\bar\epsilon^{[i} R(Q)_{ab}^{j]}\,,\\
\delta \chi^i &= -\tfrac{1}{12}\gamma_{ab} \not{D}\, T^{ab\,ij} \epsilon_j + \tfrac{1}{6} R(\mathcal{V})_{ab}{}^i{}_j\, \gamma^{ab} \epsilon^j - \tfrac{1}{3} i R(A)_{ab} \gamma^{ab} \epsilon^i\\
&\quad + D\,\epsilon^i + \tfrac{1}{12} T_{ab}^{ij} \gamma^{ab} \eta_j\,,
\end{aligned}
$$

where $\mathcal{D}_\mu$ are derivatives covariant with respect to Lorentz, dilatational, $[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\rm R}$ transformations, and D_μ are derivatives covariant with respect to all superconformal transformations. The quantities $R(Q)_{\mu\nu}^i$, $R(\mathcal{V})_{\mu\nu}{}^i{}_j$, and $R(A)_{\mu\nu}$ are the supercovariant curvatures related to Q-supersymmetry and $[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\rm R}$ transformations, respectively. Their precise definitions are given in appendix B.

The most important of the commutator relations that specify the superconformal algebra is the one between the supersymmetries,

$$
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\rm cov}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\rm G}(\theta)\,. \tag{1}
$$

The associated parameters are given by

$$
\begin{aligned}
\xi^\mu &= 2\,\bar\epsilon_2^i \gamma^\mu \epsilon_{1i} + \text{h.c.}\,,\\
\varepsilon^{ab} &= \bar\epsilon_1^i \epsilon_2^j\, T_{ij}^{ab} + \text{h.c.}\,,\\
\Lambda_K^a &= \bar\epsilon_1^i \epsilon_2^j\, D_b T_{ij}^{ba} - \tfrac{3}{2}\bar\epsilon_2^i \gamma^a \epsilon_{1i}\, D + \text{h.c.}\,,\\
\eta^i &= 3\,\bar\epsilon_{[1}^i \epsilon_{2]}^j\, \chi_j\,.
\end{aligned}
$$

In above formula, $\delta_{\rm cov}$ is a covariantized general coordinate transformation and is given in appendix B. It should be noted that this commutator is slightly changed as compared to the one of the $N = 2$, $d = 4$ superconformal algebra $\mathrm{SO}(4,2) \cong \mathrm{SU}(2,2|2)$. The reason is that the set of conventional constraints chosen to constrain the gauge fields of local Lorentz, special conformal, and S-supersymmetry transformations are not fully covariant. The extra terms on the right-hand side of (1) present compensating field-dependent gauge transformations, which must be added to preserve the constraints. The gauge transformations $\delta_{\rm G}$ may be present if additional supermultiplets are added to the superconformal theory and account for extra gauge invariances that may arise in such a situation. These may also include central charge transformations. This is discussed for the case of vector multiplets in the next section.

[b] To clarify our notation, for instance, $\bar\eta^i \epsilon_j - (\text{h.c.; traceless}) = \bar\eta^i \epsilon_j - \bar\eta_j \epsilon^i - \tfrac{1}{2}\delta^i{}_j (\bar\eta^k \epsilon_k - \bar\eta_k \epsilon^k)$.

We will make explicit use of the transformation rule of the S-supersymmetry gauge field,

$$\begin{aligned}\delta\phi_\mu^i =& -2\, f_\mu^a \gamma_a \epsilon^i + \tfrac{1}{4} R(\mathcal{V})_{ab}{}^i{}_j \gamma^{ab}\gamma_\mu \epsilon^j + \tfrac{1}{2} i\, R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i\,,\\ & - \tfrac{1}{8}\not{D}\, T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j + 2\mathcal{D}_\mu \eta^i\,.\end{aligned}$$

Here $f_\mu{}^a$ is the gauge field of the conformal boosts, defined up to fermionic terms by

$$f_\mu{}^a = \tfrac{1}{2} R(\omega, e)_\mu{}^a - \tfrac{1}{4}(D + \tfrac{1}{3} R(\omega, e))\, e_\mu^a - \tfrac{1}{2} i\, \tilde{R}_\mu{}^a(A) + \tfrac{1}{16} T^{ij}_{\mu b}\, T^{ab}_{ij}\,, \qquad (2)$$

with $R(\omega, e)_\mu{}^a = R(\omega)^{ab}_{\mu\nu}\, e^\nu_b$ the (non-symmetric) Ricci tensor and $R(\omega, e)$ the Ricci scalar. Here $R(\omega)^{ab}_{\mu\nu}$ is the curvature associated with the spin connection field ω^{ab}_μ. The spin connection is not the usual torsion-free connection, but contains the dilatational gauge field b_μ. Because of that the curvature satisfies the Bianchi identity

$$R(\omega)^{ab}_{[\mu\nu}\, e_{\rho]\,b} = 2\partial_{[\mu} b_\nu\, e^a_{\rho]}\,.$$

This leads to the modified pair-exchange property

$$R(\omega)_{ab}{}^{cd} - R(\omega)^{cd}{}_{ab} = 2\Big(\delta^{[c}_{[a} R(\omega, e)_{b]}{}^{d]} - \delta^{[c}_{[a} R(\omega, e)^{d]}{}_{b]}\Big)\,.$$

Further relations between various curvatures are given in appendix B.

Poincaré supergravity theories are obtained by coupling the Weyl multiplet to additional superconformal multiplets containing gauge and matter fields. This was sketched in section II.5. These multiplets act as compensators for the superfluous gauge invariances of the superconformal gauge group and bridge the deficit in degrees of freedom between the Weyl multiplet and the Poincaré supergravity multiplet. For instance, the graviphoton, represented by an abelian vector field in the Poincaré supergravity multiplet, is provided by one of the superconformal vector multiplets. These also provide the local central charge transformations. It turns out that a second compensating multiplet is required and that various choices are possible. We will utilize a hypermultiplet for this purpose.

2. Chiral multiplets and chiral backgrounds

In section II.2 we introduced the chiral multiplet, which transforms under rigid superconformal symmetries. In this section some details on the coupling of the chiral multiplet to superconformal gravity is presented. Of particular interest for the present discussion is the reduced chiral tensor multiplet W that contains the superconformal gauge fields through their covariant derivatives and curvature tensors. From this multiplet one may form an unreduced chiral multiplet W^2, which will be used to incorporate the holomorphic R^2-interaction.

Let us denote the bosonic component fields of the chiral multiplet by A, B_{ij}, F^-_{ab}, and C. Here A and C denote the complex scalar fields, appearing at the θ^0- and θ^4-level of the chiral background superfield, respectively, while the symmetric complex SU(2)-tensor B_{ij} and the anti-selfdual Lorentz tensor F^-_{ab} reside at the θ^2-level. The fermion fields at level θ and θ^3 are denoted by Ψ_i and Λ_i. The rigid transformation rules (given by equations 8 and 9 of chapter II) were generalized in [75] to include local superconformal transformations. As compared to the rigid transformation rules the derivatives appearing are covariantized with respect to all superconformal transformations. At the θ^3- and θ^4-level there are further modifications which involve the auxiliary fields of the Weyl multiplet. These transformation rules are consistent with the multiplication rule for chiral multiplets given by (7) of chapter II. We will need the transformation rules only for the first two components,

$$\begin{aligned} \delta A &= \bar{\epsilon}^i \Psi_i \,, \\ \delta \Psi_i &= 2 \not{D} A \epsilon_i + \tfrac{1}{2} \varepsilon_{ij} F_{ab} \gamma^{ab} \epsilon^j + B_{ij} \epsilon^j + 2 w A \eta_i \,, \end{aligned}$$

Here, $w = -c$ denotes the Weyl weight of the chiral multiplet. The C-component of a chiral multiplet has Weyl weight $w + 2$ and chiral weight $-c + 2$. Therefore, the C component of a $w = 2$ chiral multiplet may serve as a starting point for the construction of chiral density formula. The corresponding density formula which is covariant with respect to all superconformal transformations was found in [75]. In particular, the density formula can be applied to any holomorphic function $F(\Phi^I)$ of chiral multiplets Φ^I with Weyl weights w_I, provided that it is homogeneous of Weyl weight two, $F(\lambda^{w_I} \Phi^I) = \lambda^2 F(\Phi^I)$.

As in the rigid case there exist various types of reduced chiral multiplets. One of these is the vector multiplet and is discussed in the next section. Another is the reduced chiral tensor multiplet that contains the gauge fields of the Weyl multiplet through their covariant derivatives and curvature tensors,

$$W_{ab}^{ij} = T_{ab}^{ij} - \tfrac{1}{2} \mathcal{R}(M)_{ab}{}^{cd} \theta^i \gamma_{cd} \theta^j + \ldots \,. \tag{3}$$

Among various other curvatures it comprises $\mathcal{R}(M)_{ab}{}^{cd}$, which is the supersymmetrized curvature of the spin connection and contains the Riemann-tensor. From this multiplet one forms the unreduced chiral multiplet $W^2 = [W^{abij} \varepsilon_{ij}]^2$, which has Weyl and chiral weights $w = 2$ and $c = -2$, respectively [104]. This multiplet will be used to describe holomorphic R^2-interactions by including a W^2-dependence in the holomorphic function F. The components of W^2 are denoted with a caret and are given by [25,104]

$$\begin{aligned} \hat{A} &= (\varepsilon_{ij}\, T_{ab}^{ij})^2 \,, \\ \hat{\Psi}_i &= 16\, \varepsilon_{ij} R(Q)_{ab}^{j}\, T^{klab}\, \varepsilon_{kl} \,, \\ \hat{B}_{ij} &= -16\, \varepsilon_{k(i} R(\mathcal{V})^k{}_{j)ab}\, T^{lmab}\, \varepsilon_{lm} - 64\, \varepsilon_{ik} \varepsilon_{jl} \bar{R}(Q)_{ab}^{k} R(Q)^{l\,ab} \,, \\ \hat{F}^{-ab} &= -16\, \mathcal{R}(M)_{cd}{}^{ab}\, T^{klcd}\, \varepsilon_{kl} - 16\, \varepsilon_{ij}\, \bar{R}(Q)_{cd}^{i} \gamma^{ab} R(Q)^{j\,cd} \,, \end{aligned}$$

$$
\begin{aligned}
\hat{\Lambda}_i &= 32\,\varepsilon_{ij}\,\gamma^{ab} R(Q)^j_{cd}\,\mathcal{R}(M)^{cd}{}_{ab} + 16\,(\mathcal{R}(S)_{ab\,i} + 3\gamma_{[a} D_{b]}\chi_i)\,T^{klab}\,\varepsilon_{kl} \\
&\quad - 64\,R(\mathcal{V})_{ab}{}^k{}_i\,\varepsilon_{kl}\,R(Q)^{l\,ab}\,, \\
\hat{C} &= 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd} + 32\,R(\mathcal{V})^{-ab\,k}{}_l\,R(\mathcal{V})^{-l}_{ab\,k} \\
&\quad - 32\,T^{ab\,ij}\,D_a\,D^c T_{cb\,ij} + 128\,\bar{\mathcal{R}}(S)^{ab}_i\,R(Q)^i_{ab} + 384\,\bar{R}(Q)^{ab\,i}\gamma_a D_b \chi_i\,.
\end{aligned}
\tag{4}
$$

We refer to appendix B for the definitions of the various curvature tensors.

In rigid supersymmetry, chiral backgrounds are often introduced as spurion fields to describe coupling constants as superfields, which a frozen to a certain constant value. The complex coupling constant τ of the renormalizable supersymmetric Yang-Mills theory, for instance, can be incorporated by considering a holomorphic function of the form $F(S,\Phi) = S\Phi\Phi$, where S is a reduced chiral superfield frozen to the value τ, and Φ are chiral vector multiplets (*cf.* section II.1). Decisive in this context is that the way τ may appear in the Lagrangian, perturbatively and non-perturbatively, is fixed by the holomorphicity properties of the function F. This has been used to derive non-renormalization theorems. A similar example is the complex dilaton field of string theory, the vacuum expectation value of which measures the string coupling constant. It resides in a chiral multiplet as well, although in this case the chiral multiplet is dynamical.

In the context of type II string theories couplings of the form $\sum_g \mathcal{F}_g W^{2g}$ were studied in [24] by computing certain string amplitudes $\mathcal{F}_g$ for $c = 9$, $(2,2)$-superconformal field theory compactifications. The amplitudes capture the interaction of the universal sectors, which include the gravitational multiplet and the universal hypermultiplet, and were calculated for orbifolds and general Calabi-Yau compactifications. Remarkably, the contributions $\mathcal{F}_g$ from the g-th string-loop order are related to the topological partition functions F_g of a twisted two-dimensional non-linear sigma model defined on a Calabi-Yau target manifold [105,106]. In fact, for the case of Calabi-Yau compactifications this will be reflected in the fact that the form of the R^2-interactions are determined by topological invariants of the Calabi-Yau compactification manifold. Modular invariance of F_g implies the existence of certain non-holomorphic corrections, which can be determined recursively by so-called anomaly equations. From a supergravity point of view (part of) these non-holomorphic can be understood by studying symplectic reparameterizations in a chiral background [107]. Such non-holomorphic interactions are not taken into account in this analysis.

3. Vector multiplets and hypermultiplets

Many of the important elements of $N = 2$ vector multiplets were given in section II.3. There, we discussed the transformation rules of the vector multiplets for rigid superconformal symmetry. In this section, we promote these symmetries to local invariances, thereby coupling the vector multiplets to a superconformal background. The relevant result are reported in [75]. On chiral multiplets with Weyl weight $w = 1$

the following set of 8 + 8 constraints can be imposed (we denote the anti-selfdual two-form of the chiral multiplet by $\mathcal{F}_{ab}^{-}$),

$$\Lambda_i = -\varepsilon_{ij} \not{D}\Psi^j \,, \quad B_{ij} = \varepsilon_{ik}\varepsilon_{jl}B^{kl} \,,$$
$$C = -\Big(D^a D_a \bar{A} + \tfrac{1}{4}\varepsilon^{ij} T^{+}_{abij}\, \mathcal{F}^{+ab} + 3\, \bar{\chi}_i\, \Psi^i\Big) \,,$$

while the supersymmetrized field strengths satisfy the Bianchi identity

$$D^b\Big(\mathcal{F}^{+}_{ab} - \mathcal{F}^{-}_{ab} + \tfrac{1}{4} X T_{ab\,ij}\varepsilon^{ij} - \tfrac{1}{4}\bar{X} T^{ij}_{ab}\varepsilon_{ij}\Big) = \tfrac{3}{4}\Big(\bar{\chi}^i\gamma_a\Omega^j\varepsilon_{ij} - \bar{\chi}_i\gamma_a\Omega_j\varepsilon^{ij}\Big) \,. \quad (5)$$

The constraints are the generalizations of the constraints (10) and (11) of section II.3 and are covariant with respect to local superconformal transformations. They are solved in terms of the 8 + 8 off-shell degrees of freedom of the vector multiplet, which are a complex scalar X, a doublet of chiral fermions Ω_i, a vector gauge field W_μ, and a triplet of real scalars Y_{ij}. The Weyl and chiral weights and the fermion chirality of the vector multiplet component fields are listed in table 3 of the appendix B. Under Q- and S-supersymmetry these transform as follows,

$$\begin{aligned}
\delta X^I &= \bar{\epsilon}^i\Omega_i^I \,, \\
\delta\Omega_i^I &= 2\not{D} X^I\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}\mathcal{F}^{I\mu\nu-}\gamma_\mu\gamma_\nu\epsilon^j + Y^I_{ij}\epsilon^j - 2g f_{JK}{}^I X^J\bar{X}^K\varepsilon_{ij}\epsilon^j + 2X^I\eta_i \,, \\
\delta W^I_\mu &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega^I_j + 2\varepsilon_{ij}\bar{\epsilon}^i\bar{X}^I\psi^j_\mu + \text{h.c.} \,, \\
\delta Y^I_{ij} &= 2\bar{\epsilon}_{(i}\not{D}\Omega^I_{j)} + 2\varepsilon_{ik}\varepsilon_{jl}\bar{\epsilon}^{(k}\not{D}\Omega^{l)\,I} - 4g f_{JK}{}^I\,\varepsilon_{k(i}\Big(\bar{\epsilon}_{j)}X^J\Omega^{k\,K} - \bar{\epsilon}^k\bar{X}^J\Omega^K_{j)}\Big) \,. \quad (6)
\end{aligned}$$

We have generalized these transformation rules to include nonabelian gauge transformations. The index I runs over the adjoint representation of a given gauge group, and the component fields transform in the adjoint representation. Here, $f_{IJ}{}^K$ are the structure constants of the gauge group, $[t_I, t_J] = f_{IJ}{}^K\, t_K$, and g is a coupling constant. The field strengths $\mathcal{F}^I_{\mu\nu}$ are expressed in terms of the nonabelian field strengths

$$F^I_{\mu\nu} = 2\partial_{[\mu}W^I_{\nu]} - g f_{JK}{}^I\, W^J_\mu W^K_\nu \,,$$

according to

$$\mathcal{F}^I_{\mu\nu} = F^I_{\mu\nu} - \Big(\varepsilon_{ij}\bar{\psi}^i_{[\mu}\gamma_{\nu]}\Omega^{j\,I} + \varepsilon_{ij}\bar{X}^I\bar{\psi}^i_\mu\psi^j_\nu + \tfrac{1}{4}\varepsilon_{ij}\bar{X}^I T^{ij}_{\mu\nu} + \text{h.c.}\Big) \,.$$

This tensor satisfy the supersymmetrized Bianchi identities (5). As compared to the rigid case, the tensor $\mathcal{F}^I_{ab}$ contains various couplings to the superconformal background. Under supersymmetry it transform as follows,

$$\delta\mathcal{F}^I_{ab} = -2\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[a}D_{b]}\Omega^I_j - \varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\Omega^I_j + \text{h.c.} \,.$$

The transformation rules (6) satisfy the commutator relation (1), including a field-dependent gauge transformation on the right-hand side, which acts with the following parameter

$$\theta^I = 4\varepsilon^{ij}\bar{\epsilon}_{2i}\epsilon_{1j}\, X^I + \text{h.c.} \,.$$

Since at least one of the vector multiplets scalars is non-vanishing for Poincaré supergravity the last term in (1) will account for local central charge transformations.

Superconformal actions for vector multiplets can be derived by using the chiral density formula [75]. As compared to the rigid superconformal symmetry, this density formula contains various extra terms describing couplings to the superconformal background, which are necessary for local superconformal invariance. The density formula can be applied to any holomorphic function of chiral superfields, provided its Weyl weight is two. The vector multiplet Lagrangians are characterized, as in the rigid case, by holomorphic functions $F(X^I)$ that are homogeneous of degree two in X^I. An important observation is that this function can depend on any other chiral field, as long as its scale and chiral weights are properly accounted for. In particular, this means that we can make use of functions $F(X, \hat{A})$ that depend holomorphically on the scalar of a background chiral multiplet $\hat{A}$. The background chiral multiplet can be a reduced or general chiral multiplet. Under the rescaling with a complex factor λ, the holomorphic function scales as

$$F(\lambda X, \lambda^w \hat{A}) = \lambda^2 F(X, \hat{A}) \,.$$

Therefore, this function satisfies the relation

$$X^I F_I + w\hat{A}\, F_A = 2F \,.$$

Here F_I and F_A denote the derivatives of $F(X, \hat{A})$ with respect to X^I and $\hat{A}$, respectively, and w denotes the Weyl weight of the background field. Many more important relations can be derived from this by taking further derivatives. Eventually, we will identify $\hat{A}$ with the lowest component $(\varepsilon_{ij} T^{ij}_{ab})^2$ of the W^2 multiplet in order to incorporate R^2-terms.

Let us turn to the superconformal hypermultiplets and their action. For writing down actions and transformation rules with local superconformal covariance it is advisable to work with local sections of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$-bundle $A_i{}^\alpha$ instead of the coordinates ϕ^A. We introduced these section at the end of section II.4. While the indices A of the real coordinates ϕ^A run from 1 to $4r$, the indices of the $\mathrm{Sp}(r)$-bundle, α and $\bar{\alpha}$, run from 1 to $2r$. The existence of such a bundle is a consequence of the homothety and is read off from the S-supersymmetry transformation rules of the fermions, $\delta_S \zeta^\alpha = \chi^A V_{Ai}{}^\alpha \eta^i$,

$$A_i{}^\alpha(\phi) = \chi^B(\phi) V_{Bi}{}^\alpha(\phi) \,.$$

Comparison with the S-variation of the gaugino (6) shows that in the case of the vector multiplets the sections are given by the X^I themselves.

In terms of the hypermultiplet sections the quaternionic vielbeins are given by $V^\alpha_{B\,i} = D_B A_i{}^\alpha$ and the hyper-Kähler potential is expressed as

$$\varepsilon_{ij}\, \chi = \bar{\Omega}_{\alpha\beta} A_i{}^\alpha A_j{}^\beta \,. \tag{7}$$

When written in terms of sections the supersymmetry transformation rules linearize to a certain degree [77,86],

$$
\begin{aligned}
\delta A_i{}^\alpha &= 2\bar\epsilon_i\zeta^\alpha + 2\varepsilon_{ij}G^{\alpha\bar\beta}\Omega_{\bar\beta\bar\gamma}\bar\epsilon^j\zeta^{\bar\gamma} - \delta_Q\phi^B\Gamma_B{}^\alpha{}_\beta A_i{}^\beta\,,\\
\delta\zeta^\alpha &= \not{D}A_i{}^\alpha\varepsilon^i - \delta_Q\phi^B\Gamma_B{}^\alpha{}_\beta\zeta^\beta + A_i{}^\alpha\eta^i\,,\\
\delta\zeta^{\bar\alpha} &= \not{D}A^{i\bar\alpha}\varepsilon_i - \delta_Q\phi^B\Gamma_B{}^{\bar\alpha}{}_{\bar\beta}\zeta^{\bar\beta} + A^{i\bar\alpha}\eta_i\,,
\end{aligned}
$$

where $\delta_Q\phi^A = 2\left(\gamma^A_{i\bar\alpha}\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^A_{i\alpha}\bar\epsilon_i\zeta^\alpha\right)$. The derivatives D_μ are covariant with respect to all superconformal transformations. For later purposes we give the covariant derivative $\mathcal{D}_\mu$ with respect to the bosonic invariances,

$$
\mathcal{D}_\mu A_i{}^\alpha = \partial_\mu A_i{}^\alpha - b_\mu A_i{}^\alpha + \tfrac{1}{2}V_{\mu i}{}^j A_j{}^\alpha + \partial_\mu\phi^A\,\Gamma_A{}^\alpha{}_\beta\,A_i{}^\beta\,.
$$

The Weyl and chiral weights of these fields are given in table 3 of appendix B.

4. Action and symplectic reparameterizations

We have given parts of the $N=2$ action describing vector multiplets and hypermultiplets in chapter II. We give it here again for r hypermultiplets and $n+1$ abelian vector multiplets in the presence of a chiral background. The bosonic terms of the action are encoded in the function $F(X,\hat A)$, in the hypermultiplet sections $A_i{}^\alpha(\phi)$, and in the target space connections $\Gamma_A{}^\alpha{}_\beta$,

$$
\begin{aligned}
8\pi\,e^{-1}\,\mathcal{L} = {}& i\,\mathcal{D}^\mu F_I\,\mathcal{D}_\mu\bar X^I - i F_I\,\bar X^I(\tfrac16 R - D) - \tfrac18 i F_{IJ}\,Y^I_{ij}Y^{Jij} - \tfrac14 i\hat B_{ij}\,F_{AI}Y^{Iij}\\
&+ \tfrac14 i F_{IJ}(F^{-I}_{ab} - \tfrac14\bar X^I T^{ij}_{ab}\varepsilon_{ij})(F^{-Jab} - \tfrac14\bar X^J T^{ijab}\varepsilon_{ij})\\
&- \tfrac18 i F_I(F^{+I}_{ab} - \tfrac14 X^I T_{abij}\varepsilon^{ij})T^{ab}_{ij}\varepsilon^{ij} - \tfrac1{32} i F(T_{abij}\varepsilon^{ij})^2\\
&+ \tfrac12 i F_A\hat C - \tfrac18 i F_{AA}(\varepsilon^{ik}\varepsilon^{jl}\hat B_{ij}\hat B_{kl} - 2\hat F^-_{ab}\hat F^{-ab})\\
&+ \tfrac12 i\hat F^{-ab}\,F_{AI}(F^{-I}_{ab} - \tfrac14\bar X^I T^{ij}_{ab}\varepsilon_{ij}) + \text{h.c.}\\
&- \tfrac12\varepsilon^{ij}\,\bar\Omega_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta + \chi(\tfrac16 R + \tfrac12 D)\,.
\end{aligned}\tag{8}
$$

Recall that two compensating multiplets are needed if above theory is to be gauge equivalent to a theory of Poincaré supergravity. One of these multiplets is always a vector multiplet and for the second one we choose a hypermultiplet. This implies that the number of physical vector multiplets is equal to n and the number of physical hypermultiplets is equal to $r-1$.

Even in the presence of the chiral background the Lagrangian has the form of a generalized Maxwell Lagrangian with terms that are at most quadratic in the field strengths. This feature will change once we start eliminating auxiliary fields.[c] Hence

[c] Because the chiral background field given in (4) involves terms of higher order in derivatives, the Lagrangian will contain higher-derivative interactions. The most conspicuous ones are the interactions quadratic in the Riemann curvature. Such Lagrangians generically describe negative-metric states. However, they should not be regarded as elementary Lagrangians, but rather as effective Lagrangians. This implies

it is advisable to first solve the Maxwell equations, before eliminating the auxiliary fields. One distinguishes the Bianchi equations, which are expressed directly in terms of the field strengths $F^{\pm I}_{\mu\nu}$, and the equations for the electric and magnetic displacement fields $G^{\pm}_{\mu\nu I}$, which are proportional to the variation of the action with respect to the $F^{\pm I}_{\mu\nu}$. With suitable proportionality factors, these tensors read (we suppress fermion contributions),

$$G^{+}_{\mu\nu I} = \bar F_{IJ}\, F^{+J}_{\mu\nu} + \mathcal{O}^{+}_{\mu\nu I}\,, \qquad G^{-}_{\mu\nu I} = F_{IJ}\, F^{-J}_{\mu\nu} + \mathcal{O}^{-}_{\mu\nu I}\,,$$

where

$$\begin{aligned}\mathcal{O}^{+}_{\mu\nu I} &= \tfrac{1}{4}(F_I - \bar F_{IJ} X^J)\, T_{\mu\nu ij}\varepsilon^{ij} + \hat F^{+}_{\mu\nu}\, \bar F_{IA}\,,\\ \mathcal{O}^{-}_{\mu\nu I} &= \tfrac{1}{4}(\bar F_I - F_{IJ} \bar X^J)\, T^{ij}_{\mu\nu}\,\varepsilon_{ij} + \hat F^{-}_{\mu\nu}\, F_{IA}\,.\end{aligned}$$

We note that since we have not integrated out the field T^{ij}_{ab} at this point, the coupling functions are given simply by $\bar F_{IJ}$ and F_{IJ}. Observe that the tensors $\mathcal{O}^{\pm}_{\mu\nu I}$ therefore contain terms proportional to the field T^{ij}_{ab}, but also depends on the chiral background. The Maxwell equations in the absence of charges read (in the presence of the background), $\mathcal{D}^a(F^- - F^+)^I_{ab} = 0$, and $\mathcal{D}^a(G^- - G^+)_{ab\,I} = 0$. Eventually we will solve these equations for a given configuration of electric and magnetic charges in a stationary geometry. These charges will be denoted by (p^I, q_J) and are normalized such that for a stationary multi-centered solutions with charges at centers $\vec x_A$ Maxwell's equations read

$$\partial_\mu \begin{pmatrix} \sqrt{g}(F^- - F^+)^{I\,\mu t} \\ \sqrt{g}(G^- - G^+)^{\mu t}_I \end{pmatrix} = 4\,i\,\pi \sum_A \delta(\vec x - \vec x_A) \begin{pmatrix} p^I_A \\ q_{AI} \end{pmatrix}. \tag{9}$$

Observe that $\sqrt{g}\,(F^- - F^+)^{I\,\mu\nu}$ and $\sqrt{g}\,(G^- - G^+)^{\mu\nu}_I$ are Weyl invariant quantities. As discussed in chapter II, the field equations of the vector multiplets are subject to equivalence transformations corresponding to electric-magnetic duality. The electric-magnetic duality transformations cannot be performed at the level of the action, but only at the level of the equations of motion. After applying the transformations one can find the corresponding action. This is then characterized by a relation between two different functions $F(X, \hat A)$. We emphasize that these transformations do not affect the fields of the Weyl multiplet and of the chiral background. Accordingly, the background field $\hat A$ is inert. From the discussion on in section II.6 we recall that the two complex $(2n+2)$-component vectors $(X^I, F_I(X, \hat A))$ and $(F^{\pm I}_{ab}, G^{\pm}_{ab\,I})$ transform linearly under the $\mathrm{Sp}(2n+2;\mathbb{R})$-duality group, but more such vectors can be constructed. The first vector has weights $w = 1$ and $c = -1$, whereas the second one has zero Weyl and chiral weights. Although the background field $\hat A$ itself is inert under

that auxiliary fields that appear with derivatives, should still be eliminated. This leads to an infinite series of terms that corresponds to an expansion in terms of momenta divided by the Planck mass.

the dualities, it nevertheless enters in the explicit form of the transformations. Consequences of this are discussed in [92,108,109]. It follows from (9) that the charges (p^I, q_J), as well, comprise a symplectic vector. In the presence of these charges the symplectic transformations are restricted to an integer-valued subgroup that keeps the lattice of electric and magnetic charges invariant as discussed in chapter II.

The various transformation rules only take a symplectically invariant form when one solves the field equations for the auxiliary fields Y_{ij}^I,

$$Y_{ij}^I = i N^{IJ} \Big(F_{JA}\, \hat{B}_{ij} - \bar{F}_{JA}\, \varepsilon_{ik}\varepsilon_{jl}\, \hat{B}^{kl} \Big) .$$

With this result we can cast $\delta\Omega_i^I$ and $\delta\hat{\Psi}_i$ in a symplectically covariant form (we suppress fermionic bilinears),

$$\begin{aligned} \begin{pmatrix} \delta\Omega_i^I \\ \delta(F_{IJ}\Omega_i^J + F_{IA}\hat{\Psi}_i) \end{pmatrix} = & \\ & +2\not{D} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \epsilon_i + \tfrac{1}{2}\varepsilon_{ij}\gamma^{ab}\epsilon^j \left[\begin{pmatrix} F_{ab}^{-I} \\ G_{abI}^{-} \end{pmatrix} - \tfrac{1}{4}\varepsilon_{kl}T_{ab}^{kl} \begin{pmatrix} \bar{X}^I \\ \bar{F}_I \end{pmatrix} \right] \\ & + i\hat{B}_{ij}\,\epsilon^j \begin{pmatrix} N^{IJ} F_{JA} \\ \bar{F}_{IJ} N^{JK} F_{KA} \end{pmatrix} - i\varepsilon_{ik}\varepsilon_{jl}\hat{B}^{kl}\,\epsilon^j \begin{pmatrix} N^{IJ}\bar{F}_{JA} \\ F_{IJ} N^{JK}\bar{F}_{KA} \end{pmatrix} + 2\eta_i \begin{pmatrix} X^I \\ F_I \end{pmatrix} . \end{aligned}$$

In the above formulae, N^{IJ} is the inverse of the matrix $N_{IJ} = -iF_{IJ} + i\bar{F}_{IJ}$, which, as discussed in chapter II, is the target space metric of the Kähler cone.

5. Supersymmetry variations

For the rest of this section we will evaluate the supersymmetry variations of a number of spinors that are needed in the analysis in subsequent sections. Some of the spinors can act as suitable compensating fields with regard to S-supersymmetry. We also evaluate the supersymmetry variations of the supercovariant derivatives of the spinors belonging to one of the matter multiplets as well as the variation of the supersymmetry field strength $R(Q)_{ab}^i$. This analysis naturally leads us to the definition of a number of bosonic quantities that play a central role in what follows. It is important that one considers supersymmetry variations of objects, which transform as tensors under symplectic reparameterizations.

The first spinor we consider is expressed in terms of hypermultiplet fermions and reads

$$\zeta_i^{\mathrm{H}} \equiv \chi^{-1}\bar{\Omega}_{\alpha\beta}A_i{}^{\alpha}\,\zeta^{\beta} .$$

Its supersymmetry variation reads

$$\delta\zeta_i^{\rm H} = \chi^{-1}\bar\Omega_{\alpha\beta} A_i{}^\alpha \not{D} A_j{}^\beta\, \epsilon^j + \varepsilon_{ij}\eta^j \;,$$

where χ is the hyperkähler potential defined in (7) and where terms proportional to the fermion fields are suppressed. It is important to realize that one has the decomposition [86]

$$\chi^{-1}\bar\Omega_{\alpha\beta} A_i{}^\alpha D_\mu A_j{}^\beta = \tfrac{1}{2} k_\mu\, \varepsilon_{ij} + k_{\mu\, ij} \;,$$

where k_μ is real and given by

$$k_\mu = \chi^{-1}(\partial_\mu - 2\, b_\mu)\chi \;,$$

and $k_{\mu\, ij}$ is symmetric in i, j and pseudoreal so that it transforms as a vector under SU(2). Its explicit form is not important for us. Hence we write

$$\delta\zeta_i^{\rm H} = \tfrac{1}{2}\not{k}\, \varepsilon_{ij}\, \epsilon^j + \not{k}_{ij}\, \epsilon^j + \varepsilon_{ij}\, \eta^j \;.$$

In the vector multiplet sector there are two spinors that can be constructed which transform as scalars under electric-magnetic duality. One, denoted by $\zeta_i^{\rm V}$, transforms inhomogeneously under S-supersymmetry. It can be conveniently introduced from the variation of the symplectically invariant expression (with $w = 2$ and $c = 0$)

$$\mathrm{e}^{-\mathcal{K}} = i\Big[\bar X^I\, F_I(X,\hat A) - \bar F_I(\bar X,\bar{\hat A})\, X^I\Big] . \tag{10}$$

The object $\mathcal{K}$ resembles the Kähler potential of special geometry (*cf.* equation 28 of chapter II). Its supersymmetry variation leads to the spinor

$$\zeta_i^{\rm V} \equiv -\Big(\Omega_i^I\, \frac{\partial}{\partial X^I} + \hat\Psi_i\, \frac{\partial}{\partial \hat A}\Big)\mathcal{K} = -i\, \mathrm{e}^{\mathcal{K}}\Big[(\bar F_I - \bar X^J F_{IJ})\Omega_i^I - \bar X^I F_{IA}\, \hat\Psi_i\Big] .$$

It is obvious that $\zeta_i^{\rm V}$ transforms as a scalar under symplectic reparameterizations, because it follows from a symplectic scalar. This can also be seen by noting that $\zeta_i^{\rm V}$ is generated by the symplectic product $\bar F_I\, \delta X^I - \bar X^I\, \delta F_I$. This leads us to introduce yet another spinor ζ_i^0 generated by $F_I\, \delta X^I - X^I\, \delta F_I$,

$$\zeta_i^0 \equiv (F_I - X^J F_{IJ})\Omega_i^I - X^I F_{IA}\, \hat\Psi_i \;.$$

This spinor is invariant under S-supersymmetry and it vanishes in the absence of the chiral background. However, it does not play a useful role in what follows. Under Q- and S-supersymmetry $\zeta_i^{\rm V}$ transforms as

$$\begin{aligned}
\delta\zeta_i^{\rm V} = {}& \mathrm{e}^{\mathcal{K}} \not{D}\, \mathrm{e}^{-\mathcal{K}} \epsilon_i + 2i\, \not{A}\epsilon_i - \tfrac{1}{2} i\varepsilon_{ij}\, \mathcal{F}^-_{ab}\, \gamma^{ab}\epsilon^j \\
& + \mathrm{e}^{\mathcal{K}}\, N^{IJ}\Big[(\bar F_I - \bar F_{IK}\bar X^K)F_{JA}\, \hat B_{ij} - (\bar F_I - F_{IK}\bar X^K)\bar F_{JA}\, \varepsilon_{ik}\varepsilon_{jl}\hat B^{kl}\Big]\epsilon^j \\
& + 2\, \eta_i \;,
\end{aligned}$$

where we ignored higher-order fermionic terms. The quantity $\mathcal{A}_\mu$ resembles a covariantized (real) Kähler connection and $\mathcal{F}^-_{ab}$ is an anti-selfdual tensor,

$$\begin{aligned}\mathcal{A}_\mu &= \tfrac{1}{2}\mathrm{e}^{\mathcal{K}}\Big(\bar{X}^J \overset{\leftrightarrow}{\mathcal{D}}_\mu\, F_J - \bar{F}_J \overset{\leftrightarrow}{\mathcal{D}}_\mu\, X^J\Big)\,,\\ \mathcal{F}^-_{ab} &= \mathrm{e}^{\mathcal{K}}\, \Big(\bar{F}_I\, F^{-I}_{ab} - \bar{X}^I\, G^-_{ab\,I}\Big)\,.\end{aligned}$$

There is another symplectically invariant contraction of the field strengths,

$$\begin{aligned}\mathrm{e}^{\mathcal{K}}\, \Big(F_I\, F^{-I}_{ab} - X^I\, G^-_{ab\,I}\Big) + \tfrac{1}{4} i\,\varepsilon_{ij}\, T^{ij}_{ab} = \\ \mathrm{e}^{\mathcal{K}}\, F_{IA}\Big[w\hat{A}(F^{-I}_{ab} - \tfrac{1}{4}\bar{X}^I\, \varepsilon_{ij}\, T^{ij}_{ab}) - X^I\, \hat{F}^-_{ab}\Big]\,,\end{aligned}$$

which appears in the variation of ζ^0_i.

As it turns out we also need to consider the supersymmetry variations of derivatives of the fermion fields. However, one can restrict oneself to the variation of the supercovariant derivative of a single fermion field, as is discussed in appendix C. For this field we choose ζ^{H}_i. The Q- and S-supersymmetry variation of its covariant derivative reads

$$\begin{aligned}\delta(D_\mu\zeta^{\mathrm{H}}_i) = {}& \tfrac{1}{2}\mathcal{D}_\mu(\chi^{-1}\mathcal{D}_\nu\chi)\,\varepsilon_{ij}\,\gamma^\nu\epsilon^j + \mathcal{D}_\mu k_{\nu ij}\,\gamma^\nu\epsilon^j\\ &- \tfrac{1}{32}\chi^{-1/2}(\delta^j_i\mathcal{D}_\nu - k_{\nu ik}\varepsilon^{kj})(\chi^{1/2}T^{lm}_{ab}\,\varepsilon_{lm})\,\gamma^\nu\gamma^{ab}\gamma_\mu\epsilon_j\\ &+ \varepsilon_{ij}\,\Big[f_\mu{}^a\gamma_a\epsilon^j - \tfrac{1}{8}R(\mathcal{V})^j{}_{kab}\gamma^{ab}\gamma_\mu\epsilon^k - \tfrac{1}{4}i\,R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^j\Big]\\ &+ (\tfrac{1}{4}\chi^{-1}\not{\mathcal{D}}\,\chi\,\varepsilon_{ij} + \tfrac{1}{2}\not{k}_{ij})\,\gamma_\mu\eta^j\,.\end{aligned}$$

Finally we present the variation of the curvature tensor $R(Q)^i_{\mu\nu}$, defined by

$$R(Q)^i_{\mu\nu} = 2\mathcal{D}_{[\mu}\psi^i_{\nu]} - \gamma_{[\mu}\phi^i_{\nu]} - \tfrac{1}{8}T^{ij}_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]j}\,,$$

where ϕ^i_μ is the dependent gauge field associated with S-supersymmetry, defined in appendix B. The variation of this tensor reads,

$$\delta R(Q)^i_{ab} = -\tfrac{1}{2}\not{D}T^{ij}_{ab}\,\epsilon_j + R(\mathcal{V})^-_{ab}{}^i{}_j\,\epsilon^j - \tfrac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i + \tfrac{1}{8}T^{ij}_{cd}\,\gamma^{cd}\gamma_{ab}\,\eta_j\,,$$

where $\mathcal{R}(M)_{ab}{}^{cd}$ is defined in appendix B.

IV

Supersymmetric vacua and stationary BPS configurations

1. Introduction

This chapter is based on [99,110], where a broad class of stationary solutions of four-dimensional $N = 2$ supergravity theories with R^2-interactions is described. The solutions that are considered are BPS solutions, because they possess a residual $N = 1$ supersymmetry. Some of them describe extremal black holes that carry electric and/or magnetic charges or superpositions thereof. We also describe rotating solutions with one or several centers. The extremal black holes are solitonic interpolations between two fully supersymmetric ground-states. Without R^2-interactions these are flat Minkowski spacetime at spatial infinity and a Bertotti-Robinson geometry at the horizon. In that case, the moduli fields, which can take arbitrary values at infinity, must flow to specific values at the horizon which are determined in terms of the charges. This so-called fixed-point behavior explains why the black hole entropy depends only on the charges and not on the asymptotic values of the moduli. This is in contradistinction with the black hole mass which does depend on the values of the fields at spatial infinity. Owing to this fixed-point behavior the resulting expressions for the entropy, based on the effective low-energy action, can be compared successfully with microstate counting results from string and brane theory, which also depend exclusively on the charges.

Solutions based on supergravity actions without R^2-terms were analyzed some time ago [111–120]. In [25] it was shown that corrections to the black hole entropy associated with R^2-terms are in agreement with certain subleading corrections to the entropy (in the limit of large charges) that follow from the counting of microstates [21]. As mentioned in chapter I, the main ingredients of the derivation in [25] are the behavior of the solution at the horizon and the use of a definition of the black hole entropy that is appropriate when R^2-interactions are present. The latter point is addressed in chapter V.

This chapter contains a careful study of various supersymmetric black hole solutions in the presence of R^2-interactions. Our analysis thereby provides the complete proof underlying the results of [25]. We consider the full interpolating extremal black hole solution, multi-centered solutions, as well as general stationary solutions. All solutions known so far (in particular the ones of [121]) are contained as special cases. We begin our analysis by determining all spacetimes with $N = 2$ supersymmetry.

We prove that, in spite of the presence of R^2-terms, there is still a unique spacetime, which is of the Bertotti-Robinson type, whose radius as well as the values of the various moduli fields are determined by the electric and magnetic charges carried by the solution. Flat Minkowski spacetime can be viewed as a special case of such a solution, but here the moduli are constant and arbitrary and there are no electric and magnetic fields. Our analysis thus shows that the enhancement of supersymmetry at the horizon forces the moduli fields to take prescribed values. Consequently the uniqueness of the horizon geometry implies the existence of a fixed-point behavior even in the presence of R^2-interactions. Note that the fixed-point behavior is usually derived by invoking flow arguments based on the interpolating solutions (see, *e.g.*, [113,114,119,120]), but these arguments are much more difficult to derive in the presence of R^2-interactions.

Subsequently we turn to the analysis of spacetimes with residual $N = 1$ supersymmetry. A general analysis of the conditions for $N = 1$ supersymmetry turns out to be extremely cumbersome. We therefore restrict ourselves to a well-defined class of embeddings of residual supersymmetry and derive the corresponding restrictions on the bosonic background configurations. Our analysis is set up in such a way that the presence of the R^2-interactions hardly poses complications. This is so because the R^2-terms are incorporated into the Lagrangian by allowing the holomorphic function to depend on an extra holomorphic parameter. Furthermore, by stressing the underlying electric-magnetic duality of the field equations throughout the calculations, the dependence on the R^2-interactions remains almost entirely implicit and does not require much extra attention.

Using the restrictions posed by residual supersymmetry and assuming stationary field configurations we analyze the solutions. We prove that they are expressed in terms of harmonic functions associated with the electric and magnetic charges carried by the solutions, while the spatial dependence of the moduli is governed by so-called "generalized stabilization equations". The latter were first conjectured in [116–118] and in [122] for the case without and with R^2-interactions, respectively. The resulting stationary solutions include the case of multi-centered solutions of extremal black holes.

Our analysis of the restrictions imposed by $N = 2$ and $N = 1$ supersymmetry on the solutions is based on the existence of a full off-shell superconformal multiplet calculus for $N = 2$ supergravity theories [76,77,88,89], and was discussed in chapters II and III of this thesis. It turns out that the hypermultiplets play only a rather passive role. It proves advantageous to perform most of the analysis before writing the theory in its Poincaré form (by imposing gauge conditions or reformulating it in terms of fields that are invariant under the action of those superconformal symmetries that are absent in Poincaré supergravity). As a consequence we fix the stationary spacetime line element only at a relatively late stage of the analysis. An unusual complication is that, in order to determine the restrictions imposed by full or residual supersymmetry, it is not sufficient to consider the supersymmetry variation of the fermions only. One also needs to impose the vanishing of the supersymmetry variation of derivatives of the fermion fields. We present an argument that shows which of these variations are needed.

2. Fully supersymmetric vacua

From the supersymmetry variations presented in section III.5 one can determine the conditions on the bosonic fields imposed by the requirement of full $N = 2$ supersymmetry. These conditions follow from setting all Q-supersymmetry variations of the fermionic quantities to zero. However, these variations are determined up to an S-supersymmetry transformation. Thus one can either impose the vanishing of all Q-variations up to a uniform S-supersymmetry transformation, or one can restrict oneself to linear combinations that are invariant under S-supersymmetry and require their Q-supersymmetry variations to vanish. Examples of such S-invariant combinations are, for instance, $\Omega_i^I - X^I \zeta_i^{\rm V}$ and $\hat\Psi_i - w\hat A\,\zeta_i^{\rm V}$, while the spinor ζ_i^0 is S-invariant by itself. In this section we will include an arbitrary number of hypermultiplets.

We start by considering the S-supersymmetric linear combination of $\zeta_i^{\rm V}$ and $\zeta_i^{\rm H}$. Requiring its Q-supersymmetry variation to vanish for all supersymmetry parameters, we establish immediately that

$$\mathcal{F}_{ab}^- = \hat B_{ij} = k_{\mu\,ij} = \mathcal{A}_\mu = 0\,,$$

and

$$\mathcal{D}_\mu\Big(\mathrm{e}^{\mathcal{K}}\chi\Big) = 0\,. \tag{1}$$

Comparing the supersymmetry variations of the vector multiplet fermions to those of $\zeta_i^{\rm V}$ leads to

$$\begin{aligned} F_{ab}^{-I} &= \tfrac14 \varepsilon_{kl} T_{ab}^{kl}\,\bar X^I\,,\\ G_{abI}^- &= \tfrac14 \varepsilon_{kl} T_{ab}^{kl}\,\bar F_I\,,\\ \mathcal{D}_\mu\Big(\mathrm{e}^{\mathcal{K}/2}X^I\Big) &= \mathcal{D}_\mu\Big(\mathrm{e}^{\mathcal{K}/2}F_I\Big) = 0\,. \end{aligned} \tag{2}$$

These equations themselves again imply that $\mathcal{F}_{ab}^-$ and $\mathcal{A}_\mu$ vanish. Furthermore, by using the explicit expression of the tensors G_{abI}^-, one finds that $\hat F_{ab}^- = 0$. The last two equations imply that we also have

$$\mathcal{D}_\mu\Big(\mathrm{e}^{w\mathcal{K}/2}\hat A\Big) = 0\,.$$

From the supersymmetry variations of the hypermultiplets we find a similar result,

$$\mathcal{D}_\mu\Big(\chi^{-1/2}A_i{}^\alpha\Big) = 0\,. \tag{3}$$

Observe that all the above equations are K-invariant.

Subsequently we compare the supersymmetry variations of the spinors χ^i and ζ_i^{V}, which leads to the relations,

$$D = R(\mathcal{V})_{ab}{}^i{}_j = R(A)_{ab} = \mathcal{D}_a\Big(\mathrm{e}^{-\mathcal{K}/2}T^{abij}\Big) = 0\,.$$

With these results it follows that the vector field strengths satisfy the following equations,

$$\mathcal{D}^a F_{ab}^{-I} = \mathcal{D}^a G_{abI}^{-} = 0\,,$$

which imply (but are stronger than) the equations of motion and the Bianchi identities for the vector fields.

A similar calculation for the curvature $R(Q)^i_{ab}$ yields

$$\begin{aligned}\mathcal{D}_c T_{ab}^{ij} &= -\tfrac{1}{2}\mathcal{D}_d\mathcal{K}\left(\delta_c^d\, T_{ab}^{ij} - 2\delta_{[a}^d\, T_{b]c}^{ij} + 2\eta_{c[a}\, T_{b]}^{ij\,d}\right),\\ \mathcal{R}(M)_{ab}{}^{cd} &= 0\,.\end{aligned}$$

The first equation is consistent with the result found earlier. Because $D = 0$, the tensor $\mathcal{R}(M)_{ab}{}^{cd}$ is just the traceless part of the curvature tensor $R(\omega)_{ab}{}^{cd}$ associated with the spin connection field ω_μ^{ab} (which at this stage depends on the dilatational gauge field b_μ). Upon suppressing b_μ, this tensor becomes equal to the Weyl tensor. Hence the above condition will eventually lead to the conclusion that $N = 2$ supersymmetric solutions require a conformally flat spacetime. We stress again that all of the above conditions are K-invariant.

Before continuing, let us make a few remarks. First of all, we note that at this stage all equations are consistent with all the superconformal symmetries; in particular, we have not yet fixed a scale. All the above results are also manifestly consistent with electric-magnetic duality. Secondly we found a number of conditions on the chiral background field, namely $\hat{B}_{ij} = \hat{F}_{ab}^{-} = 0$ and the covariant constancy of $\exp(w\mathcal{K}/2)\hat{A}$. So far no conditions have been derived for its highest-θ component $\hat{C}$, but by considering the supersymmetry variation of the spinor $\hat{\Lambda}_i$ one can easily show that $\hat{C} = 0$. It is illuminating to verify whether these results hold for the chiral field starting with $\hat{A} = [T^{abij}\varepsilon_{ij}]^2$. It turns out that they are indeed satisfied on the basis of the above results, with the exception of the $\hat{C}$ component which contains a term proportional to the second derivative of $T_{ab\,ij}$. Also in view of later applications we consider this term in more detail and note that the bosonic contribution to the second derivative of T_{ab}^{ij} takes the form

$$D_\mu D_c T_{ab}^{ij} = \mathcal{D}_\mu \mathcal{D}_c T_{ab}^{ij} + f_{\mu c}\, T_{ab}^{ij} - 2 f_{\mu[a}\, T_{b]c}^{ij} + 2 f_\mu{}^d\, \eta_{c[a}\, T_{b]d}^{ij}\,.$$

Consequently

$$D_\mu D^a T_{ab}^{ij} = \mathcal{D}_\mu \mathcal{D}^a T_{ab}^{ij} - f_\mu{}^a\, T_{ab}^{ij}\,.$$

With this result we consider the relevant term in $\hat{C}$,

$$T^{ab\,ij}\, D_a D^c T_{cb\,ij} = T^{ab\,ij}\, \mathcal{D}_a \mathcal{D}^c T_{cb\,ij} - f_a{}^c\, T^{ab\,ij}\, T_{cb\,ij}\,, \tag{4}$$

where we note in passing that, in the first term on the right-hand side, we can symmetrize the derivatives as the antisymmetric part vanishes due to the (anti-)selfduality of the T-fields. By using the equations found above, we can work out the double derivative on the T-field, and verify whether it vanishes against the second term proportional to $f_\mu^{\ a}$.

Rather than determining $f_\mu^{\ a}$ in this way, we continue and consider the supersymmetry variation of the supercovariant derivatives of fermion fields. First we make the observation that the derivatives of S-invariant combinations of fields, whose Q-supersymmetric variations were already required to vanish in the bosonic background, will still vanish. But we can also compare the variation of the supercovariant derivative of a fermion field to the variation of a fermion field without derivatives. Consider for example the Q-variation of the following S-invariant expression

$$D_\mu \zeta_i^{\mathrm{H}} + (-\tfrac{1}{4}\chi^{-1} \not{D}\chi\, \delta_i^j + \tfrac{1}{2}\not{k}_{ik}\varepsilon^{kj})\, \gamma_\mu \zeta_j^{\mathrm{H}}\,. \tag{5}$$

The derivative of another fermion field can now be written as the derivative of an S-invariant linear combination of that fermion field with a bosonic expression times ζ_i^{H}, which is one of the previously considered linear combinations whose vanishing variation in the supersymmetric background has already been ensured, a term proportional to (5) and terms proportional to ζ_i^{H} without a derivative. Therefore, once we have imposed that the variation of (5) vanishes, then the variation of the derivative of every other fermion field is guaranteed to vanish against some bosonic term times the variation of ζ_i^{H}. Consequently variations of such linear combinations can be ignored and our only task is to require that the variation of (5) vanishes. Note that the above argument can be extended to variations of multiple derivatives as well, which therefore can also be ignored. For a more explicit proof of this statement we refer to appendix C.

Imposing the condition that the Q-supersymmetry variation of (5) vanishes, we find that most terms vanish already by virtue of previous results and we are left with just one more equation,

$$D_\mu\Big(\chi^{-1} D^a \chi\Big) = \tfrac{1}{2}\Big(\chi^{-1}\, D_\mu \chi\Big)\Big(\chi^{-1} D^a \chi\Big) - \tfrac{1}{4} e_\mu^{\ a}\, \Big(\chi^{-1}\, D_c \chi\Big)^2\,.$$

Note that we have superconformal derivatives here which involve the gauge field $f_\mu{}^a$ associated with conformal boosts. Upon using the previous results (1), (2) and (3), all equations coincide. Hence we are left with the following equation for $f_\mu^{\ a}$,

$$f_\mu{}^a = -\tfrac{1}{2}\mathcal{D}_\mu\Big(\mathrm{e}^{\mathcal{K}}\, \mathcal{D}^a \mathrm{e}^{-\mathcal{K}}\Big) + \tfrac{1}{4}\Big(\mathrm{e}^{\mathcal{K}}\, \mathcal{D}_\mu \mathrm{e}^{-\mathcal{K}}\Big)\Big(\mathrm{e}^{\mathcal{K}}\, \mathcal{D}^a \mathrm{e}^{-\mathcal{K}}\Big) - \tfrac{1}{8} e_\mu^{\ a}\, \Big(\mathrm{e}^{\mathcal{K}}\, \mathcal{D}_c \mathrm{e}^{-\mathcal{K}}\Big)^2\,, \tag{6}$$

which is K-invariant. With this result we can verify that the term (4) vanishes as well, so that we establish that the $\hat{C}$ component of the Weyl multiplet vanishes. The above equation (6) can be rewritten as

$$R(\omega,e)_\mu{}^a - \tfrac{1}{6} R(\omega,e)\, e_\mu^{\ a} = -\tfrac{1}{8} T_{\mu b}^{ij}\, T_{ij}^{ab} + \mathcal{D}_\mu \mathcal{D}^a \mathcal{K} + \tfrac{1}{2}\mathcal{D}_\mu \mathcal{K}\, \mathcal{D}^a \mathcal{K} - \tfrac{1}{4} e_\mu^{\ a}\, (\mathcal{D}_c \mathcal{K})^2\,.$$

So far the analysis is valid for any chiral background field. For the rest of this section we assume that the chiral multiplet is given by equation (4) of chapter III so that at this point we have identified all supersymmetric configurations in the presence of R^2-terms. The results obtained so far are in a manifestly conformally covariant form. We can now impose gauge choices and set $b_\mu = 0$ (because of the K-invariance the conditions found above are in fact independent of b_μ) and $\exp[\mathcal{K}]$ equal to a constant. (Alternative we may use $\exp[\mathcal{K}]$ as a compensator to make all quantities invariant under scale transformations, at which point the field b_μ will drop out.) The values of $\exp[-\mathcal{K}]$ and χ are related. With the choice that we made for the action we find that $\chi = -2\exp[-\mathcal{K}]$ as a result of the field equation for the field D. For future reference, we give both the field equations for the field D and for the U(1) gauge field A_μ,

$$\begin{aligned} 3\,\mathrm{e}^{-\mathcal{K}} + \tfrac{1}{2}\chi =&\ -192i\, D(F_A - \bar{F}_A) \\ &+ 4i\Big\{(\varepsilon_{ij}\, T^{ij}_{cd})^{-2}\, \varepsilon_{kl} T^{abkl}\, (F_I\, F^{-I}_{ab} - X^I\, G^-_{abI}) - \text{h.c.}\Big\}\,, \end{aligned} \tag{7}$$

$$\begin{aligned} \mathrm{e}^{-\mathcal{K}}\mathcal{A}_a =&\ 128i\,\mathcal{D}^b\Big(F_A\, R(A)^-_{ab} - \text{h.c.}\Big) - 8\,\mathcal{D}_c(F_A + \bar{F}_A)\, T_{ijab}\, T^{ijcb} \\ &+ 8(F_A - \bar{F}_A)\,\Big(T^{ij}_{ab}\mathcal{D}_c T^{cb}_{ij} - T_{ijab}\,\mathcal{D}_c T^{cbij}\Big) \\ &- 8\,\mathcal{D}^b\Big\{(\varepsilon_{ij}\, T^{ij}_{de})^{-2}\, \varepsilon_{kl} T^{kl\,c}{}_{[a}\, (F_I\, F^{-I}_{b]c} - X^I\, G^-_{b]cI}) + \text{h.c.}\Big\}\,. \end{aligned} \tag{8}$$

Observe that these field equations can only be found from the action, and cannot be obtained from requiring that the supersymmetry variations vanish, because the action consists of a linear combination of two actions that are separately invariant, corresponding to the vector multiplets and the hypermultiplets, respectively (we point out that the hypermultiplets contribute only fermionic terms to (8), which have been suppressed above). The coefficient of the Ricci scalar in the action is now equal to $-(16\pi)^{-1}\exp[-\mathcal{K}]$, so that Newton's constant equals $G_\mathrm{N} = \exp[\mathcal{K}]$, assuming a conventionally normalized flat metric. Furthermore we can put the gauge fields A_μ and $\mathcal{V}_\mu{}^i{}_j$ to zero, because their field strengths vanish.

The most general $N = 2$ supersymmetric background can now be characterized as follows. First of all the spacetime has zero Weyl tensor and is thus conformally flat. Its Ricci tensor is given by

$$R_{\mu\nu} = -\tfrac{1}{8} T^{ij}_{\mu\rho}\, T_{ij\nu}{}^{\rho}\,,$$

where $T_{ij\mu\nu}$ ($T^{ij}_{\mu\nu}$) is a covariantly constant (anti-)selfdual tensor. Furthermore we have a number of constants X^I. The electric/magnetic field strengths are also covariantly constant and given by

$$F^{-I}_{\mu\nu} = \tfrac{1}{4}\varepsilon_{kl} T^{kl}_{\mu\nu}\, \bar{X}^I\,, \qquad G^-_{\mu\nu I} = \tfrac{1}{4}\varepsilon_{kl} T^{kl}_{\mu\nu}\, \bar{F}_I\,. \tag{9}$$

By using relations for products of (anti-)selfdual tensors one can verify that the integrability condition that follows from the covariant constancy of the tensor fields $T^{ij}_{\mu\nu}$, is identically satisfied. In order to investigate explicit solutions one chooses coordinates such that the metric reads

$$g_{\mu\nu} = \mathrm{e}^{2f(x)+\mathcal{K}}\,\eta_{\mu\nu}\,, \tag{10}$$

with $\eta_{\mu\nu}$ the flat Minkowski metric (normalized in the standard way). We included the factor $\exp[\mathcal{K}]$, which we adjusted to a constant, so that the function f is independent of the scale. To have a vanishing Ricci scalar the function $\exp[f]$ must be harmonic,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\,\mathrm{e}^f = 0\,.$$

The remaining conditions are (here we raise and lower indices with the flat metric)

$$\begin{aligned} R_{\mu\nu} &= 2\partial_\mu\partial_\nu f - 2\partial_\mu f\,\partial_\nu f + \eta_{\mu\nu}\,(\partial_\rho f)^2 = -\tfrac{1}{8}T^{ij}_{\mu\rho}\,T_{ij\nu}{}^{\rho}\,\mathrm{e}^{-2f-\mathcal{K}}\,,\\ \partial_\mu T^{ij}_{\nu\rho} &= 2\partial_\mu f\,T^{ij}_{\nu\rho} - 2\partial_{[\nu} f\,T^{ij}_{\rho]\mu} + 2\eta_{\mu[\nu}\,T^{ij}_{\rho]\sigma}\,\partial^\sigma f\,. \end{aligned}$$

As a result of the second condition we derive

$$\partial_{[\mu}T^{ij}_{\nu\rho]} = \partial^\mu T^{ij}_{\mu\nu} = 0\,,$$

so that $T^{ij}_{\mu\nu}$ is a harmonic tensor.

We are interested in time-independent solutions so that we assume that f is independent of the time t. In that case we can express the tensor field in terms of a complex potential Φ. Denoting spatial world indices by $\hat a$, $\hat b$, $\hat c$, we may write

$$\varepsilon_{ij}T^{ij}_{\hat a\hat b} = \varepsilon_{\hat a\hat b\hat c}\,\partial_{\hat c}\Phi\,, \qquad \varepsilon_{ij}T^{ij}_{t\hat a} = i\,\partial_{\hat a}\Phi\,,$$

where Φ is a complex harmonic function. The equations are now solved for by

$$\Phi = 4\,z\,\mathrm{e}^{f+\mathcal{K}/2}\,,$$

with z a constant phase factor, and f satisfying

$$\mathrm{e}^f\,\partial_{\hat a}\partial_{\hat b}\mathrm{e}^f = 3\,\partial_{\hat a}\mathrm{e}^f\,\partial_{\hat b}\mathrm{e}^f - \delta_{\hat a\hat b}\,(\partial_{\hat c}\mathrm{e}^f)^2\,. \tag{11}$$

This system of differential equations can be integrated. Its solution is unique (up to translations) and is given by $\exp[f(r)] = c/r$, where c is a real constant. This leads to a Bertotti-Robinson spacetime, the geometry of which describes the near-horizon limit of an extremal black hole with horizon at $r = 0$. Thus there exist no fully supersymmetric multi-centered solutions, which is not surprising in view of the fact that the differential equations (11) are nonlinear in $\exp[f]$. The field $\hat A$ is now equal to

$$\hat A = (\varepsilon_{ij}T^{ij}_{ab})^2 = \frac{64\,\mathrm{e}^{-\mathcal{K}}}{\bar z^2\,\mathrm{e}^{2f(r)}}\,(\partial_{\hat a} f)^2\,.$$

From evaluating (9) it follows that the electric and magnetic charges are equal to

$$p^I = c\,\mathrm{e}^{\mathcal{K}/2}\,[\bar z\,X^I + z\,\bar X^I]\,, \qquad q_I = c\,\mathrm{e}^{\mathcal{K}/2}\,[\bar z\,F_I + z\,\bar F_I]\,. \tag{12}$$

With this result we consider the so-called BPS mass, which takes the form

$$Z = \mathrm{e}^{\mathcal{K}/2}(p^I\,F_I - q_I\,X^I) = -i\,z\,c\,, \tag{13}$$

so that we obtain the equations (sometimes called stabilization equations) [111,112, 114,115],

$$\bar{Z}\begin{pmatrix} X^I \\ F_I \end{pmatrix} - Z\begin{pmatrix} \bar{X}^I \\ \bar{F}_I \end{pmatrix} = i\,\mathrm{e}^{-\mathcal{K}/2}\begin{pmatrix} p^I \\ q_I \end{pmatrix} . \tag{14}$$

Observe that this result is covariant with respect to electric-magnetic duality.

Finally we note that the area in Planck units equals

$$\frac{\text{Area}}{G_{\mathrm{N}}} = 4\pi\, c^2 = 4\pi\, |Z|^2 .$$

This does not determine the black hole entropy, because the Bekenstein-Hawking area law is not applicable for these black holes [123–126]. After including an appropriate correction one obtains instead [25]

$$\mathcal{S} = \pi\Big[|Z|^2 - 256\,\mathrm{Im}[F_A(X^I, \hat{A})]\Big] , \tag{15}$$

where $\hat{A} = -64\,\bar{Z}^{-2}\,\mathrm{e}^{-\mathcal{K}}$. We will comment on this result in chapter V.

In section 3 we will be using another coordinate frame with line element given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2g}\,\mathrm{d}t^2 + \mathrm{e}^{-2g}\,\mathrm{d}\vec{x}^2 . \tag{16}$$

The conformal coordinates of this section are related to those of the above frame by

$$t \longrightarrow \frac{d}{c^2\,\mathrm{e}^{\mathcal{K}}}\,t , \quad \vec{x} \longrightarrow d\,\frac{\vec{x}}{|\vec{x}|^2} ,$$

where d is some real constant. The function e^{-2g} in (16) corresponding to the line element (10) is equal to

$$\mathrm{e}^{-2g} = \frac{c^2\,\mathrm{e}^{\mathcal{K}}}{|\vec{x}|^2} .$$

For later reference let us give the field strengths (9) in the frame (16),

$$F_{tm}^{-\,I} = iz\bar{X}^I\,\mathrm{e}^g\,\frac{x^m}{|\vec{x}|^2} , \qquad G^{-}_{tm\,I} = iz\bar{F}_I\,\mathrm{e}^g\,\frac{x^m}{|\vec{x}|^2} . \tag{17}$$

Here (t, m) denote world indices in the frame (16). For these expressions Maxwell's equations (*cf.* equation 9 of chapter III) are satisfied with the charges defined in (12). Observe that, when calculating Maxwell's equations directly in the frame (10), one encounters a different sign as compared to (12). This is related to the fact that a charge located at the origin in the frame (16) corresponds to a charge at infinity in the conformal coordinates used in this section. When evaluating Maxwell's equations in the latter coordinates one is considering the corresponding mirror charge placed at the origin. This explains the apparent sign discrepancy.

3. Stationary BPS configurations

A general analysis of the conditions for residual $N = 1$ supersymmetry is extremely cumbersome. Therefore we base ourselves on a given class of embeddings of the residual supersymmetry by imposing the following condition on the supersymmetry parameters,

$$h\,\epsilon_i = \varepsilon_{ij}\,\gamma_0\,\epsilon^j\,, \tag{18}$$

where h is some unknown phase factor which is in general not constant, and which transforms under U(1) with the same weight as the fields X^I. At the moment we proceed without imposing gauge choices. Therefore the choice of γ_0 is somewhat arbitrary, because it can be changed into any other gamma matrix by means of a local Lorentz transformation. However, we will eventually impose a gauge condition on the vierbein field, which restricts the local Lorentz transformations to the three-dimensional rotations[a]. It is clear that (18) is then consistent with spatial rotations and SU(2) transformations, although we will not require the solutions to be invariant under these symmetries. An embedding condition such as (18) was also used in the analysis presented in [116,118–120] of $N = 2$ theories without R^2-interactions.

Subject to this embedding we can now evaluate the conditions for $N = 1$ supersymmetry by following essentially the same steps as in the previous section. We start by considering the variations of the vector multiplet fermions and of the spinors ζ_i^{V} and ζ_i^{H}. They lead to the equations

$$\hat{B}_{ij} = k_{a\,ij} = 0\,, \tag{19}$$

and

$$\begin{aligned} \mathcal{A}_0 &= 0\,, & \mathcal{A}_p &= \mathrm{Re}[h\,\mathcal{F}^-_{0p}]\,, \\ \mathcal{D}_0(\chi\mathrm{e}^{\mathcal{K}}) &= 0\,, & \mathcal{D}_p(\chi\mathrm{e}^{\mathcal{K}}) &= 2\,\chi\mathrm{e}^{\mathcal{K}}\,\mathrm{Im}[h\,\mathcal{F}^-_{0p}]\,. \end{aligned} \tag{20}$$

where the indices $(0, p)$ with $p = 1, 2, 3$ refer to the tangent space. With this result we find that the variation of ζ_i^{V} simplifies considerably and reduces to

$$\delta\zeta_i^{\mathrm{V}} = \chi^{-1}\not{\mathcal{D}}\chi\,\epsilon_i + 2\,\eta_i\,. \tag{21}$$

For the hypermultiplets we find the same condition as for full supersymmetry,

$$\mathcal{D}_a(\chi^{-1/2}A_i{}^{\alpha}) = 0\,.$$

Returning to the vector multiplet spinors, we then establish the relations

$$\mathcal{D}_0(\chi^{-1/2}\,X^I) = \mathcal{D}_0(\chi^{-1/2}\,F_I) = 0\,,$$

[a] In view of this, Lorentz covariant derivatives should be applied with caution, as the various equations we are about to derive are not Lorentz covariant.

and

$$\begin{aligned}\mathcal{D}_p(\chi^{-1/2}\,X^I) &= -h\,\chi^{-1/2}(F^{-I}_{0p} - \tfrac{1}{4}\varepsilon_{kl}T^{kl}_{0p}\,\bar{X}^I)\,,\\ \mathcal{D}_p(\chi^{-1/2}\,F_I) &= -h\,\chi^{-1/2}(G^{-}_{0pI} - \tfrac{1}{4}\varepsilon_{kl}T^{kl}_{0p}\,\bar{F}_I)\,.\end{aligned} \tag{22}$$

These last two equations transform covariantly with respect to electric-magnetic duality. Taking their symplectically invariant product with $(\bar{X}^I, \bar{F}_I)$ leads to the previous equations (20).

Subsequently we consider the variations of the spinor χ^i, which lead to

$$\begin{aligned}R(\mathcal{V})_{ab}{}^i{}_j &= 0\,,\\ \mathcal{D}_c(\chi^{1/2}\,T^{ijc0}\,\varepsilon_{ij}) &= -6h\,\chi^{1/2}\,D\,,\\ \mathcal{D}_c(\chi^{1/2}\,T^{ijcp}\,\varepsilon_{ij}) &= 8ih\,\chi^{1/2}\,R(A)^{-0p}\,.\end{aligned} \tag{23}$$

Note that the first equation is consistent with the fact that $\hat{B}_{ij}$ vanishes (*cf.* equation 4 of chapter III). In view of the fact that the SU(2) field strengths vanish, we will set the SU(2) connections to zero in what follows.

The variations for the field strength $R(Q)^i_{ab}$ lead to

$$\begin{aligned}&\mathcal{D}_0T^{ij}_{ab} - \tfrac{1}{2}\chi^{-1}\mathcal{D}_d\chi\left(\delta^d_0\,T^{ij}_{ab} - 2\delta^d_{[a}\,T^{ij}_{b]0} + 2\eta_{0[a}\,T^{ij\,d}_{b]}\right) = 0\,,\\ &\mathcal{D}_pT^{ij}_{ab} - \tfrac{1}{2}\chi^{-1}\mathcal{D}_d\chi\left(\delta^d_p\,T^{ij}_{ab} - 2\delta^d_{[a}\,T^{ij}_{b]p} + 2\eta_{p[a}\,T^{ij\,d}_{b]}\right) = 4h\,\varepsilon^{ij}\,\mathcal{R}(M)^-_{ab\,0p}\,.\end{aligned} \tag{24}$$

We now consider the variation of derivatives of fermion fields. (The variation of the gravitini is considered later on.) The arguments presented below (5) and in appendix C about the fact that there is no need to consider more than one of these variations, apply also to residual supersymmetry. Hence we consider the Q-supersymmetry variation of (5), making use of the previously obtained results. This yields the following equation,

$$\begin{aligned}&D_\mu(\chi^{-1}D^a\chi) + \tfrac{1}{4}(\chi^{-1}D_c\chi)^2\,e_\mu{}^a - \tfrac{1}{2}(\chi^{-1}D_\mu\chi)(\chi^{-1}D^a\chi) =\\ &\qquad -\tfrac{3}{2}D(e_\mu{}^a - 2e_{\mu0}\,\eta^{a0}) - 2i[R(A)^+ - R(A)^-]_\mu{}^a - 4i\,R(A)^-_{\mu0}\,\eta^{a0}\,.\end{aligned} \tag{25}$$

All terms in this equation are real, with the exception of the last term, from which it follows that $R(A)^\pm_{a0}$ must be purely imaginary, so that

$$R(A)_{a0} = \tilde{R}(A)_{pq} = 0\,. \tag{26}$$

Just as before, (25) fixes the value of the gauge field $f_\mu{}^a$, which takes the (K-invariant) form

$$\begin{aligned}f_\mu{}^a = &-\tfrac{1}{2}\mathcal{D}_\mu(\chi^{-1}\mathcal{D}^a\chi) - \tfrac{1}{8}(\chi^{-1}\mathcal{D}_c\chi)^2\,e_\mu{}^a + \tfrac{1}{4}(\chi^{-1}\mathcal{D}_\mu\chi)(\chi^{-1}\mathcal{D}^a\chi)\\ &-\tfrac{3}{4}D(e_\mu{}^a - 2e_{\mu0}\,\eta^{a0}) - i[R(A)^+ - R(A)^-]_\mu{}^a - 2i\,R(A)^-_{\mu0}\,\eta^{a0}\,.\end{aligned} \tag{27}$$

Comparing with equation (2) of chapter III yields

$$
\begin{aligned}
&R(\omega,e)_\mu{}^a - \tfrac{1}{6}R(\omega,e)\,e_\mu^a = \\
&\qquad -\mathcal{D}_\mu(\chi^{-1}\mathcal{D}^a\chi) - \tfrac{1}{4}(\chi^{-1}\mathcal{D}_c\chi)^2\,e_\mu{}^a + \tfrac{1}{2}(\chi^{-1}\mathcal{D}_\mu\chi)(\chi^{-1}\mathcal{D}^a\chi) \\
&\qquad -\tfrac{1}{8}T_{\mu b}^{ij}\,T_{ij}^{ab} - D\,(e_\mu{}^a - 3e_{\mu 0}\,\eta^{a0}) + i[\tilde R(A)_{\mu 0}\,\eta^{a0} - \tilde R(A)_0{}^a\,e_\mu{}^0]\,. \quad (28)
\end{aligned}
$$

Let us briefly return to (23) and (24) and explore the consequences of (26). The first equation of (24) yields

$$
\mathcal{D}_0 T^{ij\,0p} - \tfrac{1}{2}\chi^{-1}\mathcal{D}_0\chi\;T^{ij\,0p} + \tfrac{1}{2}\chi^{-1}\mathcal{D}_q\chi\;T^{ij\,qp} = 0\,. \qquad (29)
$$

Making use of this, the last equation (23) leads to

$$
\mathcal{D}_q T^{ij\,qp} + \chi^{-1}\mathcal{D}_0\chi\;T^{ij\,0p} = -2ih\,\varepsilon^{ij}\;\tilde R(A)^{0p}\,,
$$

which can be rewritten as

$$
\mathcal{D}_{[p}T^{ij}_{q]0}\,\varepsilon_{ij} = 2i\,R(A)_{pq}h - \tfrac{1}{2}\chi^{-1}\mathcal{D}_0\chi\;T^{ij}_{pq}\,\varepsilon_{ij}\,. \qquad (30)
$$

Observe that so far we have not imposed any gauge conditions. In order to proceed we will now choose a gauge condition that eliminates the freedom of making (local) scale transformations and conformal boosts. This gauge condition amounts to choosing $b_\mu = 0$ and χ constant. Therefore the covariant derivative $\mathcal{D}_a$ contains only the spin connection fields and the U(1) connection, when appropriate.

In this gauge, (30) and the second equation of (23) read,

$$
\bar h\,\mathcal{D}_{[p}T^{ij}_{q]0}\,\varepsilon_{ij} = 2i\,R(A)_{pq}\,, \qquad \bar h\,\mathcal{D}^p T^{ij}_{p0}\,\varepsilon_{ij} = 6\,D\,. \qquad (31)
$$

Furthermore we establish from (28) that

$$
R(\omega,e) = -3D\,. \qquad (32)
$$

Then, from the second equation of (24), one derives the following expressions for the components of the curvature tensor $\mathcal{R}(M)_{ab\,cd}$,

$$
\begin{aligned}
\mathcal{R}(M)_{pq\,0r} &= \tfrac{1}{8}i\,\varepsilon_{pq}{}^s\,\bar h\,\mathcal{D}_r T^{ij}_{s0}\,\varepsilon_{ij} + \text{ h.c.}\,, \\
\mathcal{R}(M)_{0r\,pq} &= \tfrac{1}{8}i\,\varepsilon_{pq}{}^s\,\bar h\,\mathcal{D}_s T^{ij}_{r0}\,\varepsilon_{ij} + \text{ h.c.}\,, \\
\mathcal{R}(M)_{0p\,0q} &= -\tfrac{1}{8}\bar h\,\mathcal{D}_q T^{ij}_{p0}\,\varepsilon_{ij} + \text{ h.c.}\,, \\
\mathcal{R}(M)_{pq\,rs} &= \tfrac{1}{8}\varepsilon_{rs}{}^v\varepsilon_{pq}{}^u\,\bar h\,\mathcal{D}_v T^{ij}_{u0}\,\varepsilon_{ij} + \text{ h.c.}\,.
\end{aligned}
$$

These expressions satisfy all the constraints (1) listed in appendix B, provided one makes use of the relations (31) for $R(A)$ and D. Using (27) and the definition of $\mathcal{R}(M)$ allows us to find expressions for the components of the Riemann tensor. Making use of (31) we find

$$
\begin{aligned}
R(\omega)_{pq\,0r} &= R(\omega)_{0r\,pq} \\
&= \tfrac{1}{8}\varepsilon_{pq}{}^{s}\Big[i(\bar h\,\mathcal{D}_r T^{ij}_{s0}\,\varepsilon_{ij} - \tfrac{1}{2}T^{ij}_{r0}\,T_{ij\,s0}) + \text{ h.c.}\Big]\,, \\
R(\omega)_{0p\,0q} &= R(\omega)_{0q\,0p} \\
&= -\tfrac{1}{8}\Big[(\bar h\,\mathcal{D}_q T^{ij}_{p0}\,\varepsilon_{ij} + \tfrac{1}{2}T^{ij}_{q0}\,T_{ij\,p0}) + \text{ h.c.}\Big]\,, \\
R(\omega)_{pq\,rs} &= -\tfrac{1}{2}\delta_{[r[p}\Big[\bar h\,\mathcal{D}_{q]}T^{ij}_{s]0}\,\varepsilon_{ij} + \text{ h.c.}\Big] \\
&\quad + \tfrac{1}{4}\delta_{[r[p}\Big[T^{ij}_{q]0}\,T_{ijs]0} + T_{ijq]0}\,T^{ij}_{s]0} - \delta_{q]s]}\,T^{ij\,v}{}_{0}\,T_{ij\,v0}\Big]\,.
\end{aligned} \tag{33}
$$

Here we observe that, owing to (31), this result satisfies all the algebraic properties of a Riemann tensor, such as cyclicity and pair exchange. We also note that, by virtue of (31), (33) gives rise to (28) upon contraction.

At this point we adopt a gauge condition for local Lorentz invariance. We remind the reader that the supersymmetry embedding condition (18) is obviously inconsistent with local Lorentz invariance and presupposes that we would eventually impose such a gauge condition. Therefore we bring the vierbein field in block-triangular form by imposing $e_t{}^p = 0$, thereby leaving the SO(3) tangent-space rotations unaffected. Denoting world indices by (t, m), with $m = 1, 2, 3$, we parametrize the vierbein as follows,

$$
e_\mu{}^0 \mathrm{d}x^\mu = \mathrm{e}^g[\,\mathrm{d}t + \sigma_m\,\mathrm{d}x^m\,]\,, \qquad e_\mu{}^p \mathrm{d}x^\mu = \mathrm{e}^{-g}\,\hat e_m{}^p\,\mathrm{d}x^m\,,
$$

where $\hat e_m{}^p$ is the rescaled dreibein of the three-dimensional space. The corresponding inverse vierbein components are then given by

$$
e_0{}^t = \mathrm{e}^{-g}\,, \quad e_0{}^m = 0\,, \quad e_p{}^t = -\sigma_p\,\mathrm{e}^g\,, \quad e_p{}^m = \mathrm{e}^g\,\hat e_p{}^m\,,
$$

where, on the right-hand side, spatial tangent-space and world indices are converted by means of the dreibein fields $\hat e_m{}^p$ and its inverse.

Now we concentrate on stationary spacetimes, so that we can adopt coordinates such that the vierbein components are independent of the time coordinate t. In that case the spin connection fields take the following form,

$$
\begin{aligned}
\omega_{l\,pq} &= \mathrm{e}^g[\,\hat\omega_{l\,pq} + 2\delta_{l[p}\,\nabla_{q]}g\,]\,, \\
\omega_{0\,pq} &= \omega_{q\,p0} = -\tfrac{1}{2}\mathrm{e}^{3g}\,\varepsilon_{pql}\,R(\sigma)^l\,, \\
\omega_{0\,0p} &= \mathrm{e}^g\,\nabla_p g\,,
\end{aligned}
$$

where $\hat\omega_m{}^{pq}$ is the spin-connection field associated with the dreibein fields $\hat e$ in the standard way. We used the definition

$$
R(\sigma)^l = \varepsilon^{lpq}\,\nabla_p\sigma_q\,.
$$

Observe that $\nabla^p R(\sigma)_p = 0$. The covariant derivatives ∇_m refer to the three-dimensional space only. Hence they contain the three-dimensional spin connection $\hat\omega_m{}^{pq}$.

The corresponding curvature components take the following form (where we consistently use three-dimensional notation on the right-hand side),

$$
\begin{aligned}
R(\omega)_{pq\,0r} &= \tfrac{1}{2}\varepsilon_{pq}{}^{s}\,\mathrm{e}^{4g}\Big[\nabla_r R(\sigma)_s + 5R(\sigma)_s\,\nabla_r g + R(\sigma)_r \nabla_s g - 2\delta_{sr}\,R(\sigma)^u \nabla_u g\Big]\,,\\
R(\omega)_{0p\,0q} &= -\mathrm{e}^{2g}\Big[\nabla_p\nabla_q g + 3\,\nabla_p g\,\nabla_q g - \delta_{pq}\,(\nabla_r g)^2\Big]\\
&\quad + \tfrac{1}{4}\mathrm{e}^{6g}\Big[R(\sigma)_p\,R(\sigma)_q - \delta_{pq}R(\sigma)^2\Big]\,,\\
R(\omega)_{pq\,rs} &= \mathrm{e}^{2g}\,\hat{R}_{pq\,rs} - 4\,\mathrm{e}^{2g}\,\delta_{[p[r}\Big[\nabla_{s]}\nabla_{q]}g + \nabla_{s]}g\,\nabla_{q]}g - \tfrac{1}{2}\delta_{s]q]}\,(\nabla_u g)^2\Big]\\
&\quad + 3\,\mathrm{e}^{6g}\,\delta_{[p[r}\Big[R(\sigma)_{s]}\,R(\sigma)_{q]} - \tfrac{1}{2}\delta_{s]q]}\,R(\sigma)^2\Big]\,.
\end{aligned}
\tag{34}
$$

However, for stationary solutions also other quantities than those that encode the spacetime should be time-independent. Hence we infer that $\bar{h}X^I$, $\bar{h}F_I$, and $\bar{h}T_{p0}^{ij}$ are time-independent while $(\partial_t + iA_t)h = 0$.

Until now we have restricted our attention to quantities that are supercovariant with respect to full $N = 2$ supersymmetry. However, when considering residual supersymmetry, certain linear combinations of the gravitini will still transform covariantly. To see how this works, let us record the gravitini transformation rules in the restricted background. Here we make use of (21) to argue that there is no need for including compensating S-supersymmetry transformations. The result takes the form

$$
\begin{aligned}
\delta\psi_t^i &= 2\partial_t\epsilon^i + iA_t\,\epsilon^i + \mathrm{e}^{2g}\Big[T_p - \nabla_p g + \tfrac{1}{2}i\,\mathrm{e}^{2g}\,R(\sigma)_p\Big]\,\gamma^p\gamma_0\epsilon^i\,,\\
\delta\psi_m^i &= 2\nabla_m\epsilon^i - (T_m - iA_m)\epsilon^i\\
&\quad - i\,\hat{e}_m^{\,p}\,\varepsilon_p{}^{qr}\Big[T_r - \nabla_r g + \tfrac{1}{2}i\,\mathrm{e}^{2g}\,R(\sigma)_r\Big]\,\gamma_q\gamma_0\epsilon^i\\
&\quad + \sigma_m\,\mathrm{e}^{2g}\Big[T_p - \nabla_p g + \tfrac{1}{2}i\,\mathrm{e}^{2g}\,R(\sigma)_p\Big]\,\gamma^p\gamma_0\epsilon^i\,,
\end{aligned}
$$

where we have introduced a three-dimensional world vector T_m,

$$
T_m \equiv \tfrac{1}{4}\mathrm{e}^{-g}\,\hat{e}_m{}^p\,\bar{h}T_{p0}^{ij}\,\varepsilon_{ij}\,.
$$

Now we observe that the combinations $\psi_{\mu i} - \bar{h}\,\varepsilon_{ij}\gamma_0\psi_\mu^j$ transform covariantly under the residual supersymmetry. From the requirement that these covariant variations vanish we deduce directly that

$$
T_m = \nabla_m g - \tfrac{1}{2}i\,\mathrm{e}^{2g}\,R(\sigma)_m\,, \qquad \bar{h}\nabla_m h + iA_m = -\tfrac{1}{2}i\,\mathrm{e}^{2g}\,R(\sigma)_m\,.
$$

This leads to the following expressions for the gravitini variations,

$$
\delta\psi_t^i = 2\partial_t\epsilon^i + iA_t\epsilon^i\,, \qquad \delta\psi_m^i = 2\nabla_m\epsilon^i - (\nabla_m g + \bar{h}\nabla_m h)\epsilon^i\,. \tag{35}
$$

With these results we return to the previous identities and verify whether they are now satisfied. It is straightforward to see that this is the case for (29). For the other identities one needs the covariant derivative $\bar h\,\mathcal{D}_p T^{ij}_{q0}$, which, in three-dimensional notation, reads

$$\bar h\,\mathcal{D}_p T^{ij}_{q0}\varepsilon_{ij} = 4\,\mathrm{e}^{2g}\Big[\nabla_p T_q + 2\,T_p T_q - \delta_{pq}\,(T_r)^2\Big]\,.$$

It is now straightforward to prove (31) with D given by

$$D = \tfrac{2}{3}\mathrm{e}^{2g}\Big[\nabla_p^2 g - (\nabla_p g)^2 + \tfrac{1}{4}\mathrm{e}^{4g}\,(R(\sigma)_p)^2\Big]\,. \tag{36}$$

Furthermore, it turns out that (33) and (34) agree, provided that the curvature of the three-space is zero,

$$\hat R_{mn\,pq} = 0\,,$$

so that the three-dimensional space is flat. Observe that this result is consistent with the integrability condition corresponding to the Killing spinor equations that one obtains when setting the gravitino variations (35) to zero. The only remaining equations are now (22), which express the abelian field strengths in terms of the other fields,

$$\begin{aligned} F^{-I}_{0p} &= -\mathrm{e}^{g}\Big[\nabla_p(\bar h X^I) + (\nabla_p g)\,h\bar X^I - \tfrac{1}{2}i\,\mathrm{e}^{2g}R(\sigma)_p(\bar h X^I + h\bar X^I)\Big]\,,\\ G^{-}_{0pI} &= -\mathrm{e}^{g}\Big[\nabla_p(\bar h F_I) + (\nabla_p g)\,h\bar F_I - \tfrac{1}{2}i\,\mathrm{e}^{2g}R(\sigma)_p(\bar h F_I + h\bar F_I)\Big]\,, \end{aligned}$$

where on the right-hand side, we consistently use three-dimensional tangent space indices. With these results we derive the following expressions,

$$\begin{aligned} \mathcal{D}^a F^{-I}_{ap} &= i\mathrm{e}^{g}\,\varepsilon_p{}^{qr}\nabla_q F^{-I}_{0r} = \\ &= -\tfrac{1}{2}\mathrm{e}^{g}\,\varepsilon_p{}^{qr}\nabla_q\Big[\mathrm{e}^{3g}R(\sigma)_r(\bar h X^I + h\bar X^I)\Big] - i\,\mathrm{e}^{2g}\,\varepsilon_p{}^{qr}\,\nabla_q g\,\nabla_r(\bar h X^I - h\bar X^I)\,, \\ \mathcal{D}^a F^{-I}_{a0} &= \mathrm{e}^{g}\Big[\nabla^q F^{-I}_{q0} - 2(\nabla^q g - \tfrac{1}{2}i\,\mathrm{e}^{2g}R(\sigma)^q)\,F^{-I}_{q0}\Big] = \\ &= \mathrm{e}^{2g}\Big[\nabla_p^2(\bar h X^I) + (\nabla_p^2 g)\,h\bar X^I - (\nabla_p g)^2\,h\bar X^I + (\nabla_p g)\,\nabla^p(h\bar X^I - \bar h X^I) \\ &\qquad - \tfrac{1}{2}i\,\mathrm{e}^{3g}R(\sigma)^p\,\nabla_p[\mathrm{e}^{-g}(h\bar X^I - \bar h X^I)] + \tfrac{1}{2}\mathrm{e}^{4g}(R(\sigma)_p)^2(h\bar X^I + \bar h X^I)\Big]\,, \end{aligned}$$

and likewise for the electric-magnetic dual equations (*i.e.*, replacing F^{-I} by G^-_I and X^I by F_I). The imaginary parts of the above expressions correspond to Maxwell's equations for the abelian vector fields. Because the first expression is manifestly real, the corresponding Maxwell equation (and its electric-magnetic dual) is satisfied. The imaginary part of the second expression and its dual equation provide the remaining Maxwell equations, which read

$$\begin{aligned} \nabla_p^2\Big[\mathrm{e}^{-g}(\bar h X^I - h\bar X^I)\Big] &= 0\,,\\ \nabla_p^2\Big[\mathrm{e}^{-g}(\bar h F_I - h\bar F_I)\Big] &= 0\,, \end{aligned} \tag{37}$$

which shows that the functions in parentheses are harmonic. Furthermore we note the equations

$$F_{0p}^{-I} + F_{0p}^{+I} = -\nabla_p\Big[\mathrm{e}^{g}(\bar h X^I + h\bar X^I)\Big]\,,$$
$$G_{0pI}^{-} + G_{0pI}^{+} = -\nabla_p\Big[\mathrm{e}^{g}(\bar h F_I + h\bar F_I)\Big]\,,$$

so that the functions under the derivative can be regarded as electric and magnetic potentials.

So far our analysis is valid for any chiral background. Now we identify this background with (4) and note that the field $\hat A$ can be written as

$$\hat A = -64\mathrm{e}^{2g}\, h^2\, (T_p)^2\,.$$

With this choice for the background we now evaluate the field equations for the fields D and A_μ, which were given in (7) and (8), respectively. Using (22), (36) and the homogeneity properties of $F(X,\hat A)$, the first equation takes the form

$$\begin{aligned}\mathrm{e}^{-\mathcal{K}} + \tfrac12\chi = &- 128i\,\mathrm{e}^{3g}\,\nabla^p\Big[\mathrm{e}^{-g}\,\nabla_p g\,(F_A - \bar F_A)\Big] - 32i\,\mathrm{e}^{6g}\,(R(\sigma)_p)^2(F_A-\bar F_A)\\ &- 64\,\mathrm{e}^{4g} R(\sigma)_p\,\nabla^p(F_A+\bar F_A)\,. \end{aligned} \tag{38}$$

The second equation (8) comprises four equations. The one with $a = 0$ turns out to be identically satisfied, by virtue of of an intricate interplay of all the results that we obtained above. This constitutes a very subtle check upon the correctness of the results obtained so far. Using similar manipulations the equation (8) with $a = p$ can be written as

$$\begin{aligned}&(\bar h X^I - h\bar X^I)\,\overset{\leftrightarrow}{\nabla}_p\,(\bar h F_I - h\bar F_I) - \tfrac12\chi\,\mathrm{e}^{2g}\,R(\sigma)_p = \\ &\qquad 128\,\mathrm{e}^{2g}\,\nabla^q\Big[2\nabla_{[p}g\,\nabla_{q]}(F_A+\bar F_A) + i\nabla_{[p}\Big(\mathrm{e}^{2g}\,R(\sigma)_{q]}\,(F_A-\bar F_A)\Big)\Big]\,. \end{aligned} \tag{39}$$

To arrive at this concise expression requires an extensive usage of many of the previously obtained results, and in particular of (38).

This concludes our analysis. The solutions can now be expressed in terms of harmonic functions according to (37). The two field equations (38) and (39) then determine the function g and $R(\sigma)_p$, from which all other quantities of interest follow. We should point out that there are some equations of motion whose validity has not yet been verified. We claim that those are implied by the residual supersymmetry of our solutions. For instance, for the vector multiplets we have imposed the Maxwell equations. Therefore the $N = 1$ supersymmetry variation of the field equations of the vector multiplet fermions can only lead to the field equations of the vector multiplet scalars, which must thus be satisfied by supersymmetry. For the hypermultiplets a similar argument holds. Indeed, the result (32), which is crucial for the validity of the field equation for the hypermultiplet scalars, has already been established on the basis of the previous analysis. The field equations for the fields of the Weyl multiplet have been imposed, with the exception of those for the vierbein field and the tensor field

T_{ijab} (the field equations for the SU(2) gauge fields are trivially satisfied because of the SU(2) symmetry of our solutions). However, the field equations of the gravitino fields and of the fermion doublet χ^i transform into these two field equations, from which one may conclude that they are also satisfied by supersymmetry.

4. Summary of the BPS analysis

Let us discuss what we have found in the previous section. We have characterized all stationary solutions with a residual $N = 1$ supersymmetry embedded according to (18). In principle there may exist other solutions based on inequivalent embeddings of $N = 1$ supersymmetry. It should be interesting to apply our approach to more general embeddings of the residual supersymmetry.

By imposing the conditions for residual supersymmetry and a subset of the field equations we have obtained the full class of these solutions, albeit not explicitly because the equations depend on the holomorphic function $F(X, \hat{A})$ that characterizes both the vector multiplets and the R^2-interactions. A gratifying feature of our results is that the presence of the R^2-interactions gives rise to relatively minor complications, something that may seem rather surprising in view of the complicated structure of the higher-order derivative terms in the action. There are two reasons for the fact that these complications can remain so implicit in our analysis. The first is that the higher-order derivative interactions are nicely encoded in the holomorphic function $F(X, \hat{A})$. The second reason is that we have consistently used quantities that transform covariantly under electric-magnetic duality. Without this guidance there would be a multitude of ways to express our results and perform the analysis.

We have also shown that solutions with supersymmetry enhancement exhibit fixed-point behavior of the moduli fields, simply because the solutions with full $N = 2$ supersymmetry are unique. This result is relevant when calculating the horizon geometry of extremal black holes since it explains why the black hole entropy depends only on the electric and magnetic charges carried by the black hole.

Let us briefly summarize the solutions that we have found. Following [115] we introduce the rescaled U(1) and Weyl invariant variables,

$$Y^I = \mathrm{e}^{-g}\,\bar{h}\,X^I\ , \qquad \Upsilon = \mathrm{e}^{-2g}\,\bar{h}^2\,\hat{A}\,, \tag{40}$$

so that, using the homogeneity of $F(X, \hat{A})$, we can write

$$F(Y, \Upsilon) = \exp[-2g]\,\bar{h}^2 F(X, \hat{A})\,,$$

and

$$\begin{pmatrix} Y^I \\ F_I(Y, \Upsilon) \end{pmatrix} = \mathrm{e}^{-g}\,\bar{h} \begin{pmatrix} X^I \\ F_I(X, \hat{A}) \end{pmatrix}\,.$$

Observe that $F_A(X,\hat{A}) = F_\Upsilon(Y,\Upsilon)$. Henceforth we will always use the rescaled variables. The rescaled background field Υ is given by

$$\Upsilon = -64\Big(\nabla_m g - \tfrac{1}{2} i\,\mathrm{e}^{2g}\, R(\sigma)_m\Big)^2 . \tag{41}$$

Furthermore from (40) and equation (10) of chapter III and we infer that

$$\mathrm{e}^{-2g} = i\,\mathrm{e}^{\mathcal{K}}\Big[\bar{Y}^I F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon}) Y^I\Big]. \tag{42}$$

According to (37) we can express the imaginary part of (Y^I, F_J) in terms of a symplectic array of $2(n+1)$ harmonic functions $(H^I(\vec{x}), H_J(\vec{x}))$,

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon}) \end{pmatrix} = i \begin{pmatrix} H^I \\ H_I \end{pmatrix} . \tag{43}$$

These are the "generalized stabilization equations" which were conjectured in [116, 118] and [122] for the case without and with R^2-interactions respectively (a derivation for certain solutions without R^2-terms appeared in [121]). In principle these equations determine the full spatial dependence of Y^I in terms of the harmonic functions and the background field Υ. However, explicit solutions of the stabilization equations can only be obtained in a small number of cases and usually one has to solve the equations by iteration which is extremely cumbersome. We will discuss a few examples of explicit solutions in section 5.

We write the harmonic functions as a linear combination of several harmonic functions associated with multiple centers located at $\vec{x}_A$ with electric charges q_{AI} and magnetic charges p^I_A,

$$H^I(\vec{x}) = h^I + \sum_A \frac{p^I_A}{|\vec{x}-\vec{x}_A|} , \qquad H_I(\vec{x}) = h_I + \sum_A \frac{q_{AI}}{|\vec{x}-\vec{x}_A|} ,$$

where the (h^I, h_J) are constants and the charges are normalized according to (9) of chapter III. Furthermore, we recall

$$\begin{aligned} F^{-I}_{0p} &= -\mathrm{e}^{2g}\Big[\nabla_p Y^I + (\nabla_p g - \tfrac{1}{2} i\,\mathrm{e}^{2g} R(\sigma)_p)(Y^I + \bar{Y}^I)\Big], \\ G^-_{0pI} &= -\mathrm{e}^{2g}\Big[\nabla_p F_I + (\nabla_p g - \tfrac{1}{2} i\,\mathrm{e}^{2g} R(\sigma)_p)(F_I + \bar{F}_I)\Big], \end{aligned} \tag{44}$$

and hence

$$\begin{aligned} F^{-I}_{0p} + F^{+I}_{0p} &= -\nabla_p\Big[\mathrm{e}^{2g}(Y^I + \bar{Y}^I)\Big], \\ G^-_{0pI} + G^+_{0pI} &= -\nabla_p\Big[\mathrm{e}^{2g}(F_I + \bar{F}_I)\Big]. \end{aligned}$$

We also rewrite the expressions (38) and (39) in terms of the rescaled variables,

$$\begin{aligned}
\mathrm{e}^{-\mathcal{K}} + \tfrac{1}{2}\chi =& - 128i\, \mathrm{e}^{3g}\, \nabla^p\Big[\mathrm{e}^{-g}\, \nabla_p g\, (F_\Upsilon - \bar F_\Upsilon)\Big] - 32i\, \mathrm{e}^{6g}\, (R(\sigma)_p)^2 (F_\Upsilon - \bar F_\Upsilon) \\
&- 64\, \mathrm{e}^{4g} R(\sigma)_p\, \nabla^p (F_\Upsilon + \bar F_\Upsilon)\,, \qquad (45) \\
H^I \overset{\leftrightarrow}{\nabla}_p H_I =& - \tfrac{1}{2}\chi\, R(\sigma)_p \\
&- 128\, \nabla^q\Big[2\nabla_{[p} g\, \nabla_{q]} (F_\Upsilon + \bar F_\Upsilon) + i\, \nabla_{[p}\Big(\mathrm{e}^{2g}\, R(\sigma)_{q]}\, (F_\Upsilon - \bar F_\Upsilon)\Big)\Big]\,. \qquad (46)
\end{aligned}$$

We note that both sides of (46) are manifestly divergence-free away from the centers. Furthermore, in the one-center case where the solution has spherical symmetry and depends only on the radial coordinate, the terms involving F_Υ and its complex conjugate vanish in (46).

It is remarkable that apart from (43) the only other equations that must be solved to fully specify the stationary BPS solutions, are the ones for the spacetime line element. To this extent we eliminate $\mathrm{e}^{-\mathcal{K}}$ from above equations using (42).

$$\begin{aligned}
& i\, \Big[\bar Y^I F_I(Y,\Upsilon) - \bar F_I(\bar Y, \bar\Upsilon) Y^I\Big] + \tfrac{1}{2}\, \chi\, \mathrm{e}^{-2g} = \\
&\quad 128i\, \mathrm{e}^{g}\, \nabla^p\Big[\big(\nabla_p \mathrm{e}^{-g}\big)\, (F_\Upsilon - \bar F_\Upsilon)\Big] - 32i\, \mathrm{e}^{4g}\, (R(\sigma)_p)^2 (F_\Upsilon - \bar F_\Upsilon) \\
&\quad - 64\, \mathrm{e}^{2g} R(\sigma)_p\, \nabla^p (F_\Upsilon + \bar F_\Upsilon)\,, \\
& H^I \overset{\leftrightarrow}{\nabla}_p H_I + \tfrac{1}{2}\, \chi\, R(\sigma)_p = \\
&\quad -128i\, \nabla^q\Big[\nabla_{[p}\Big(\mathrm{e}^{2g}\, R(\sigma)_{q]}\, (F_\Upsilon - \bar F_\Upsilon)\Big)\Big] - 128\, \nabla^q\Big[2\nabla_{[p} g\, \nabla_{q]} (F_\Upsilon + \bar F_\Upsilon)\Big]\,, \qquad (47)
\end{aligned}$$

Let us first briefly discuss the solutions in the absence of R^2-interactions. Then (45) and (46) imply that

$$\mathrm{e}^{-\mathcal{K}} + \tfrac{1}{2}\chi = 0\,, \qquad R(\sigma)_m = -2\, \chi^{-1}\, H^I \overset{\leftrightarrow}{\nabla}_m H_I\,.$$

Once we have solved the stabilization equations, we have thus constructed the full solution in terms of the harmonic functions. For the static solutions, where $R(\sigma)_m = 0$, this implies that $H^I \overset{\leftrightarrow}{\nabla}_m H_I = 0$, which leads to the following condition on the charges [116,118],

$$h^I\, q_{AI} - h_I\, p^I_A = 0\,, \qquad p^I_A\, q_{BI} - q_{AI}\, p^I_B = 0\,. \qquad (48)$$

The second condition implies that the charges are mutually local, *i.e.*, the solution can be related to one carrying electric charges only by electric-magnetic duality. Moreover it implies that the total angular momentum of a dyon A in the field of a dyon B vanishes.

Asymptotically, at spatial infinity, the fields can be expanded in powers of $1/|\vec{x}|$,

$$Y^I(\vec{x}) = Y^I(\infty) + \frac{y^I}{|\vec{x}|} + \cdots\,, \qquad F_I(\vec{x}) = F_I(\infty) + \frac{f_I}{|\vec{x}|} + \cdots\,. \tag{49}$$

Inspection of (43) then shows that $Y^I(\infty) - \bar{Y}^I(\infty) = ih^I$, $F_I(\infty) - \bar{F}_I(\infty) = ih_I$ as well as $y^I - \bar{y}^I = ip^I$ and $f_I - \bar{f}_I = iq_I$, where p^I and q_I denote the (total) magnetic and electric charges, respectively. The homogeneity of the holomorphic function F implies $F_I\,\delta Y^I - Y^I\,\delta F_I = 0$, and therefore we conclude that $y^I F_I(\infty) - f_I Y^I(\infty) = 0$. The following results can then be obtained by explicit calculation,

$$\begin{aligned}
\vec{R}(\sigma)(\vec{x}) &= \mathrm{e}^{\mathcal{K}}\Big[h_I p^I - h^I q_I\Big]\,\frac{\vec{x}}{|\vec{x}|^3} + \cdots\,,\\
\mathrm{e}^{-\mathcal{K}-2g} &= \Big[\mathrm{e}^{-\mathcal{K}-2g}\Big]_\infty \Big\{1 + \Big[\mathrm{e}^{\mathcal{K}/2+g}\Big]_\infty \frac{2M_{\mathrm{ADM}}}{|\vec{x}|} + \cdots\Big\}\,,
\end{aligned}$$

where M_{ADM} denotes the ADM mass (in Planck units),

$$M_{\mathrm{ADM}} = \tfrac{1}{2}\Big[\mathrm{e}^{\mathcal{K}/2+g}\Big]_\infty \Big(p^I F_I(\infty) - q_I Y^I(\infty) + \mathrm{h.c.}\Big)\,. \tag{50}$$

Note that the M_{ADM} can be written as $M_{\mathrm{ADM}} = \frac{1}{2}[\bar{h}Z(\infty) + h\bar{Z}(\infty)]$, where Z was defined in (13). For static solutions $\bar{h}Z$ is real by virtue of the first condition in (48), so that $M_{\mathrm{ADM}} = \bar{h}Z(\infty)$ [116,118]. With these results one easily shows that the electric and magnetic fields (44) have the characteristic $1/r^2$ fall-off at spatial infinity.

We now discuss the solutions with R^2-interactions. In the presence of these interactions the equations (45) and (46) are more difficult to analyze. We note that, generically, multi-centered solutions satisfying $H^I \overset{\leftrightarrow}{\nabla}_p H_I = 0$ are not static, since (46) then reads

$$R(\sigma)_p = -256\,\chi^{-1}\,\nabla^q\Big[2\nabla_{[p}g\,\nabla_{q]}(F_\Upsilon + \bar{F}_\Upsilon) + i\,\nabla_{[p}\Big(\mathrm{e}^{2g}\,R(\sigma)_{q]}\,(F_\Upsilon - \bar{F}_\Upsilon)\Big)\Big]\,.$$

Examples of black holes exhibiting this feature will be discussed in section 5.

When a solution has a horizon with full supersymmetry, we can connect the results of this section to those of section 2. In doing so, it is important to keep in mind that we used a different parametrization of the metric in section 2 (*cf.* equation 10). The results can be connected through the following identifications, which are valid at the horizon (which we take to be located at $|\vec{x}| = 0$, for convenience),

$$\begin{aligned}
&Y^I \approx \frac{[\mathrm{e}^{\mathcal{K}/2}\bar{Z}X^I]_{\mathrm{hor}}}{|\vec{x}|}\,, \qquad F_I(Y) \approx \frac{[\mathrm{e}^{\mathcal{K}/2}\bar{Z}F_I(X)]_{\mathrm{hor}}}{|\vec{x}|}\,,\\
&\mathrm{e}^{-g} \approx \frac{[\mathrm{e}^{\mathcal{K}/2}|Z|]_{\mathrm{hor}}}{|\vec{x}|}\,, \qquad h \approx \frac{Z}{|Z|}\Big|_{\mathrm{hor}}\,, \qquad \Upsilon \approx -\frac{64}{|\vec{x}|^2}\,.
\end{aligned} \tag{51}$$

In particular, when approaching the horizon, the expressions for the field strengths (44) coincide with (17).

In the presence of R^2-interactions, the homogeneity of the holomorphic function $F(Y,\Upsilon)$ implies $F_I\,\delta Y^I - Y^I\,\delta F_I = 2\Upsilon\,\delta F_\Upsilon$. If we assume that, at spatial infinity, the fields Y^I, F_I and e^{-g} have an asymptotic expansion of the type (49), and if we furthermore assume that $\Upsilon\,\delta F_\Upsilon$ falls off to zero sufficiently rapidly so that we have $y^I F_I(\infty) - f_I Y^I(\infty) = 0$, then the ADM mass of the solution is still given by (50).

5. Examples of stationary BPS configurations

The equations (43), (47), and (41) determine the stationary BPS solutions. They can, however, be solved explicitly in only very few cases. Even in the absence of R^2-terms the solution to the equations (43), for instance, is often not known. Furthermore, equations (47) and (41) are coupled and can usually be solved only iteratively. For concreteness, let us consider a first simple example,

$$F(Y,\Upsilon) = -\tfrac{1}{2}\, i\, Y^I\, \eta_{IJ}\, Y^J + c\,\Upsilon\,, \tag{52}$$

where η is a real symmetric matrix and c a complex number. Since $F_I(Y,\Upsilon)$ does not depend on Υ it is simple to solve the equations (43),

$$Y^I = \tfrac{1}{2}\left(iH^I - \eta^{IJ} H_J\right)\,, \qquad F_I = \tfrac{1}{2}\left(iH_I + \eta_{IJ} H^J\right)\,, \tag{53}$$

where $\eta_{IJ}\eta^{JK} = \delta_I^K$. The dependence on the R^2-background will enter only when solving for the line element (as is done in the next section). This is rather the exception than the rule. Consider, for instance, the coupling function which arises in Calabi-Yau three-fold compactifications in the large volume limit,

$$F(Y,\Upsilon) = \frac{D_{ABC}\, Y^A\, Y^B\, Y^C}{Y^0} + d_A\,\frac{Y^A}{Y^0}\,\Upsilon\,, \tag{54}$$

with $A, B, C = 1, 2, \ldots, n$. We construct solutions to this model satisfying $H^0 = 0$ so that Y^0 is real. Introducing the matrix $D_{AB} = D_{ABC} H^C$ and assuming its invertibility $D_{AB} D^{BC} = \delta_A^C$, the stabilization equations can be solved to all orders in Υ,

$$\begin{aligned} Y^A &= \tfrac{1}{6} Y^0 D^{AB}\left(H_B + i d_B \frac{\Upsilon - \bar\Upsilon}{Y^0}\right) + \tfrac{1}{2}\, i\, H^A\,, \\ (Y^0)^2 &= \frac{D_{ABC} H^A H^B H^C - \frac{1}{3}(\Upsilon - \bar\Upsilon)^2 d_A D^{AB} d_B - 2(\Upsilon + \bar\Upsilon) d_A H^A}{4\left(H_0 + \frac{1}{12} H_A D^{AB} H_B\right)}\,. \end{aligned} \tag{55}$$

For more complicated $F(Y,\Upsilon)$, solving equations (43) can become very involved. It may be possible to cast $F(Y,\Upsilon)$ into a power series expansion (possibly after an electric-magnetic duality transformation) in which case the generalized stabilization equations (43) can be solved iteratively in powers of Υ,

$$Y^I = \sum_{n,m} Y^I_{(n,m)}(H^J, H_K)\,\Upsilon^n \bar\Upsilon^m\,. \tag{56}$$

Clearly, $F_I(Y,\Upsilon)$ and $F_\Upsilon(Y,\Upsilon)$ will have corresponding expansions in Υ and $\bar\Upsilon$ once the solutions $Y^I(H^J, H_K, \Upsilon, \bar\Upsilon)$ are inserted, and one could in principle, by treating

Υ as a formal expansion parameter, proceed to solve (47) and (41). Since such a procedure is not feasible in practice, the question arises whether it makes sense to solve the equations (47) and (41) iteratively and truncate at some suitable order. To address this question we recall that the function $F(Y,\Upsilon)$ is homogeneous of degree two,

$$F(\lambda Y, \lambda^2\,\Upsilon) = \lambda^2\,F(Y,\Upsilon)\,.$$

It follows that

$$Y^I F_I(Y,\Upsilon) + 2\Upsilon\,F_\Upsilon(Y,\Upsilon) = 2F(Y,\Upsilon)\,,$$

and in particular we have $F_I(\lambda Y, \lambda^2\,\Upsilon) = \lambda\,F_I(Y,\Upsilon)$. This shows that the equations (43) are invariant under this rescaling if we let the harmonic functions H^I and H_I scale with weight one. Therefore the coefficient functions $Y^I_{(n,m)}(H^J, H_K)$ in the expansion (56) will scale with weight $1-2(n+m)$, such that every power of Υ is accompanied by a net amount of two inverse powers of harmonic functions H^{-2}. The expressions (55) illustrate this feature. In a similar way homogeneity is reflected at the level of the Lagrangian. Therefore, corrections due to R^2-interactions become subleading whenever $|\Upsilon| \ll H^2$. One can pinpoint such a hierarchy when one encounters supersymmetry enhancement in the immediate neighborhood of a charged center. There, $|\Upsilon|$ has a $1/r^2$-fall-off proportional to a charge-independent constant, while the harmonic functions fall off as Q/r, where Q is the charge carried by the center. This is the reason why the corrections to the entropy of BPS black hole configurations [25], for instance, are subleading in the limit of large charges. In fact, due to homogeneity, the entropy will have an expansion of the form $S = \pi\,\sum_{n\geq 0} S^{(n)}\,Q^{2-2n}$, where the coefficients $S^{(n)}$ are independent of the charges. As argued above, homogeneity implies that the expansion of the line element takes the schematic form

$$\mathrm{e}^{-2g} \sim \sum_{n\geq 0} \alpha_{(n)} |\Upsilon|^n H^{2-2n}\,, \qquad R(\sigma) \sim \sum_{n\geq 0} \beta_{(n)} |\Upsilon|^n H^{2-2n}\,, \tag{57}$$

where $\alpha_{(n)}$ and $\beta_{(n)}$ are independent of the harmonic functions. This shows that a truncation at some finite order may be sensible only in situations in which Υ falls off much stronger than the harmonic functions H^2 when moving away from the centers.

Let us reconsider the holomorphic coupling function (52). In this simple example we can use c as an expansion parameter, since by homogeneity every power of c will always be accompanied by two inverse powers of harmonic functions. After solving the generalized stabilization equations (53) the remaining equations (47) reduce to

$$\begin{aligned}\Big(H^I\eta_{IJ}H^J + H_I\eta^{IJ}H_J\Big) + \chi\,\mathrm{e}^{-2g} &= 256\,i\,(c-\bar c)\Big[\mathrm{e}^g\,\nabla_p^2\mathrm{e}^{-g} - (\tfrac{1}{2}\mathrm{e}^{2g}R(\sigma)_p)^2\Big]\,,\\ 2H^I \overset{\leftrightarrow}{\nabla}_p H_I + \chi\,R(\sigma)_p &= -256\,i\,(c-\bar c)\,\nabla^q\Big[\nabla_{[p}\Big(\mathrm{e}^{2g}\,R(\sigma)_{q]}\Big)\Big]\,.\end{aligned} \tag{58}$$

The case where c is real corresponds to adding a total derivative term to the action. Above formulae show that the line element stays unaltered in this case, while the field

Υ is given by

$$\Upsilon = 64\left[\frac{\left(H^I - i\eta^{IJ}H_J\right)\nabla_p H_I - \left(H_I + i\eta_{IJ}H^J\right)\nabla_p H^I}{H^I\eta_{IJ}H^J + H_I\eta^{IJ}H_J}\right]^2 .$$

There are other less obvious situations where the dependency on c drops out of the equations (58). This is the case, for instance, when $R(\sigma)_p = 0$ and all the harmonic functions are proportional to e^{-g},

$$H^I = a^I\,\mathrm{e}^{-g}\,, \quad H_I = a_I\,\mathrm{e}^{-g}\,,$$

where (a^I, a_I) are constants.

Generically, however, the right-hand sides of (58) do not vanish and the equations must be solved iteratively. For mutually local charges, $H^I \overset{\leftrightarrow}{\nabla}_p H_I = 0$, static solutions with $R(\sigma)_p = 0$ are possible. This is what we want to investigate in the following. The remaining equation,

$$\mathrm{e}^{-2g}\Big(\chi - 256\,i(c-\bar{c})\,\mathrm{e}^{3g}\nabla_p^2\mathrm{e}^{-g}\Big) = -\Big(H^I\eta_{IJ}H^J + H_I\eta^{IJ}H_J\Big)\,,$$

is a non-linear differential equation for e^{-g}. To zeroth order in c we find

$$[\mathrm{e}^{-2g}]^{(0)} = -\chi^{-1}\,\Big(H^I\eta_{IJ}H^J + H_I\eta^{IJ}H_J\Big)\,.$$

Making the ansatz $\mathrm{e}^{-g} = \sum_{n\geq 0}[\mathrm{e}^{-g}]^{(n)}\big(256 i(c-\bar{c})/\chi\big)^n$ the line element is determined iteratively by

$$[\mathrm{e}^{-g}]^{(n)} = \tfrac{1}{2}\,[\mathrm{e}^{2g}]^{(0)}\Big(\nabla_p^2\,[\mathrm{e}^{-g}]^{(n-1)} - \sum_{i,j,k}{}^{'}[\mathrm{e}^{-g}]^{(i)}[\mathrm{e}^{-g}]^{(j)}[\mathrm{e}^{-g}]^{(k)}\Big)\,, \qquad (59)$$

where the truncated sum $\sum'_{i,j,k}$ runs over all $0 \leq i, j, k < n$ subject to $i+j+k = n$. The presence of the overall factor $[\mathrm{e}^{2g}]^{(0)} \sim H^{-2}$ on the right-hand side of (59) indeed induces the expansion indicated by (57).

Let us return to the more complicated example (54). In this case we can use d_A as the expansion parameters. The exact solution to the generalized stabilization equations for the case $H^0 = 0$ are given in (55). We find by direct calculation that

$$i\Big[\bar{Y}^I F_I - Y^I\bar{F}_I\Big] = -\frac{D_{ABC}H^AH^BH^C}{Y^0} + d_AH^A\frac{\Upsilon+\bar{\Upsilon}}{Y^0}\,,$$

and

$$F_\Upsilon - \bar{F}_{\bar{\Upsilon}} = \frac{i d_A H^A}{Y^0}\,, \quad F_\Upsilon + \bar{F}_{\bar{\Upsilon}} = \tfrac{1}{3}\,d_A D^{AB}\left(H_B + i d_B\frac{\Upsilon - \bar{\Upsilon}}{Y^0}\right)\,. \qquad (60)$$

To leading order the solutions to (47) are given by

$$\begin{aligned} \tfrac{1}{2}\,\chi\,\mathrm{e}^{-2g} &= \frac{D_{ABC}H^AH^BH^C}{[Y^0]^{(0)}} + \mathcal{O}(d_A) \\ \tfrac{1}{2}\,\chi\,R(\sigma)_p &= -H^I\overset{\leftrightarrow}{\nabla}_p H_I + \mathcal{O}(d_A)\,, \end{aligned} \qquad (61)$$

where $[Y^0]^{(0)}$ is the tree-level expression for Y^0,

$$([Y^0]^{(0)})^2 = \frac{D_{ABC}\,H^A H^B H^C}{4(H_0 + \frac{1}{12}H_A D^{AB} H_B)}\,.$$

Keeping track of the terms coming from (60) the expressions for the line element to second order in d_A are readily read off from (47),

$$\begin{aligned}
\tfrac{1}{2}\,\chi\,\mathrm{e}^{-2g} ={}& \frac{D_{ABC}\,H^A H^B H^C}{[Y^0]^{(0)}} - 128\gamma^{-1}\nabla^p\left[(\nabla_p\gamma)\,\frac{d_A H^A}{[Y^0]^{(0)}}\right] \\
&+ 32\gamma^{-4}(\rho_p)^2\frac{d_A H^A}{[Y^0]^{(0)}} - \tfrac{64}{3}\gamma^{-2}\,\rho_p\nabla^p\Big(d_A D^{AB} H_B\Big) + \mathcal{O}(d_A\, d_B)\,, \\
\tfrac{1}{2}\,\chi\; R(\sigma)_p ={}& -H^I \overset{\leftrightarrow}{\nabla}_p H_I + \frac{256}{3}\nabla^q\Big[\gamma^{-1}(\nabla_{[p}\gamma)\nabla_{q]}(d_A D^{AB} H_B)\Big] \\
&+ 128\nabla^q\nabla_{[p}\left(\gamma^{-2}\rho_{q]}\,\frac{d_A H^A}{[Y^0]^{(0)}}\right) + \mathcal{O}(d_A\, d_B)\,,
\end{aligned}$$

where γ^2 and ρ_p are abbreviations for the leading order results as given in (61) for e^{-2g} and $R(\sigma)_p$ respectively. Again, the first order approximation to the line element is cast into the form (57), the first correction from the R^2-terms being suppressed by one order in H^{-2}.

6. Static versus stationary extremal solutions

In section 4 we pointed out that higher-order curvature interactions induce non-static pieces in the line element even for extremal configurations with mutually local charges, $H^I \overset{\leftrightarrow}{\nabla}_p H_I = 0$. This effect can be observed in the previously discussed example. This issue does not arise when the solution has only one center because the right-hand side of the second equation (47) vanishes due to rotational symmetry. Therefore the question arises whether R^2-interactions still allow for regular multi-centered black hole solutions. Again, it is difficult to address this question in general. Due to (57) we expect e^{-2g} generically to diverge as $|\vec{x} - \vec{x}_A|^{-2}$ as one approaches one of the charge centers $\vec{x}_A$. If the charges are not mutually local one expects $R(\sigma)_p$ to behave as $|\vec{x} - \vec{x}_A|^{-3}$. For mutually local charges, on the other hand, the singularity of $R(\sigma)_p$ is, in fact, milder. Inspection of the expressions for the curvature components given in (34) show that every term involving $R(\sigma)_p$ is accompanied by a sufficient amount of factors of e^g such that the effect of the non-static pieces of the line element vanishes near the center. This indicates that multi-centered extremal solutions still possess an $\mathrm{AdS}_2 \times S_2$ near-horizon geometry. The following example illustrates this.

Let us consider a simple model describing pure supergravity with the particular R^2-interactions given by

$$F(Y,\Upsilon) = -\tfrac{1}{2}\,i\,(Y^0)^2 + b\,\frac{\Upsilon^2}{(Y^0)^2}\,.$$

We assume b to be a real constant. By the same arguments as above, we use b as an expansion parameter. We solve the generalized stabilization equations for the purely electric situation $H^0 = 0$, $H_0 \equiv H$. To zeroth order in b we find

$$e^{-2g} = -\frac{H^2}{\chi} + \mathcal{O}(b)\,, \quad R(\sigma)_p = 0 + \mathcal{O}(b)\,, \quad \Upsilon = -64\,\frac{(\nabla_p H)^2}{H^2} + \mathcal{O}(b)\,.$$

To calculate the leading corrections to the line element one needs $F_\Upsilon + \bar{F}_\Upsilon$ to first order in b,

$$F_\Upsilon + \bar{F}_\Upsilon = -1024\,b\,\frac{(\nabla_p H)^2}{H^4} + \mathcal{O}(b^2)\,.$$

We consider a harmonic function H with multiple centers located at $\vec{x}_A$. For simplicity let us assume that center 1 is at $\vec{x}_1 = 0$ and calculate $R(\sigma)_p$ around this center. Therefore we expand the harmonic function appearing in above expression in powers of $|\vec{x}|$,

$$\begin{aligned}
H &= h + \sum_A \frac{q_A}{|\vec{x}-\vec{x}_A|} = \frac{q_1}{|\vec{x}|} + h + \sum_{A\neq 1}\frac{q_A}{|\vec{x}_A|} + \mathcal{O}(|\vec{x}|)\,,\\
\nabla_p H &= -q_1\frac{x_p}{|\vec{x}|^3} + \sum_{A\neq 1} q_A\frac{x_{Ap}}{|\vec{x}_A|^3} + \mathcal{O}(|\vec{x}|)\,,\\
\nabla_p\nabla_q H &= q_1\left(3\frac{x_p\,x_q}{|\vec{x}|^5} - \frac{\delta_{pq}}{|\vec{x}|^3}\right) + \sum_{A\neq 1} q_A\left(3\frac{x_{Ap}\,x_{Aq}}{|\vec{x}_A|^5} - \frac{\delta_{pq}}{|\vec{x}_A|^3}\right) + \mathcal{O}(|\vec{x}|)\,.
\end{aligned} \tag{62}$$

In the limit $|\vec{x}| \to 0$ one finds that $R(\sigma)_p$ to first order in b is given by,

$$\tfrac{1}{2}\,\chi\,R(\sigma)_p = 3\cdot(512)^2\,b\,q_1^{-3}\left(\frac{\delta_{pq}+\hat{x}_p\hat{x}_q}{|\vec{x}|}\right)\sum_{A\neq 1} q_A\,\frac{x_{Aq}}{|\vec{x}_A|^3} + \mathcal{O}(b^2)\,,$$

where $\hat{x}_p$ denote the components of the unit vector. Thus, according to the above arguments, we recover in this simple example the usual $\mathrm{AdS}_2 \times S_2$ geometry typical for extremal configurations.

V

On entropy and moduli spaces of black holes

In the previous chapter we constructed stationary BPS black hole configurations and presented a general entropy formula, which applies to black hole solutions that arise in theories containing higher-order curvature interactions. This macroscopic entropy formula is not the usual Bekenstein-Hawking area law, but it contains additional contributions, which are derived using the more general definition suggested by Wald. This definition is based on the notion of a conserved Noether charge, and it is reviewed in sections 1 and 2. We return to the example of the self-intersecting M5-brane discussed in chapter I and address the question of whether the additional contributions in the macroscopic entropy formula correctly account for the deviation from the area law found by microstate counting.

Section 3 contains a discussion of the moduli space of multi-centered black hole solutions. While this subject is interesting in its own right, the study of these moduli spaces could shed more light on black hole entropy. This perspective is based on the idea that at least a part of the degeneracy of black hole states is accounted for by the degeneracy of bound states of coalescing black holes. We review the method proposed by Ferrell and Eardley for deriving the moduli space metric for the particular case of extremal Reissner-Nordström black holes and discuss the limit of coalescence. Interestingly, the metric on the moduli space derives from a potential. This feature remains true when calculating the moduli space metrics for more complicated multi-centered black holes. In section 4 we point to a possible connection between black hole entropy and this moduli space potential.

1. Entropy as a Noether charge

The first law of black hole mechanics, which we discussed in the chapter I, was originally derived using the Einstein equations for gravity. When discussing gravity with higher-order curvature interactions, these equations are modified, and the derivation of the first law breaks down when using the Bekenstein-Hawking area law as the definition for black hole entropy. In [123], Wald proposed an improved derivation of the first law applicable to any diffeomorphism invariant theory. In this construction, the entropy of a black hole is identified with the conserved Noether surface charge $\mathcal{S}$ associated to the diffeomorphisms generated by the horizon-generating Killing vector field. In the absence of higher-curvature interactions, the Noether charge entropy coincides with the area law, while there are explicit corrections in the presence of R^2-terms.

In field theory there are well-known procedures to derive the Noether currents for global symmetries of the Lagrangian. Less familiar is the fact that Noether currents can be constructed for local invariances of the action as well. We elaborate on this in the following. Consider an action given in terms of a Lagrangian density $\mathcal{L}$, which depends arbitrarily on a set of fields we collectively denote by ψ. These fields may include the metric $g_{\mu\nu}$, as well as matter or gauge fields. The variation of the action with respect to a general variation of the dynamical field variables ψ produces the usual split,

$$\delta\Big(\sqrt{|g|}\,\mathcal{L}\Big) = \sqrt{|g|}\,E\cdot\delta\psi + \partial_\mu\Big(\sqrt{|g|}\,\theta^\mu(\psi,\delta\psi)\Big)\,,$$

where "$\cdot$" is a contraction of the variation of the dynamical fields $\delta\psi$ with the tensors E that constitute the field equations $E=0$. The vector $\theta^\mu(\psi,\delta\psi)$ denotes the usual boundary terms one encounters in the variational principle.

Let us construct a Noether current associated with an arbitrary invariance of the action. Under such an infinitesimal symmetry transformation, which we parameterize by ξ, the Lagrangian density is left invariant up to a total derivative,

$$\delta_\xi\Big(\sqrt{|g|}\,\mathcal{L}\Big) = \partial_\mu\Big(\sqrt{|g|}\,N^\mu(\psi,\xi)\Big)\,,$$

where $N^\mu(\psi,\xi)$ is determined uniquely up to a divergence-free vector field. The conserved Noether current associated with the invariance is defined by

$$J^\mu(\psi,\xi) = \theta^\mu(\psi,\delta_\xi\psi) - N^\mu(\psi,\xi)\,.$$

Due to the ambiguity in the definition of $N^\mu(\psi,\xi)$, the Noether current too is determined up to the addition of a divergence-free vector field. It is simple to show that the current $J^\mu(\psi,\xi)$ is indeed conserved for any ξ,

$$\nabla_\mu J^\mu = -E\cdot\delta_\xi\psi\,.$$

Therefore, for field configurations satisfying the field equations, the current can be written as the divergence of the so-called Noether potential,

$$J^\mu = \nabla_\nu Q^{\mu\nu}\,.$$

The Noether potential $Q^{\mu\nu}(\psi,\xi)$ is antisymmetric in μ and ν. It is determined up to the addition of a divergence-free term. Since it generically depends on the transformation parameter ξ itself, it therefore does not necessarily vanish for invariant field configurations $\delta_\xi\psi = 0$. An important application of this formalism is the construction of a Noether potential associated to diffeomorphism invariance. With respect to diffeomorphisms, $\delta_\xi\psi = \mathcal{L}_\xi\psi$, a covariant Lagrangian density transforms into a total derivative, $\delta_\xi(\sqrt{|g|}\mathcal{L}) = \partial_\mu(\sqrt{|g|}\,\xi^\mu\mathcal{L})$. According to above definition we have $N^\mu(\psi,\xi) = \xi^\mu\mathcal{L}$, and hence the corresponding conserved current reads $J^\mu(\psi,\xi) = \theta^\mu(\psi,\delta_\xi\psi) - \xi^\mu\mathcal{L}$.

Integrating the Noether potential over the boundary ∂V of a spacelike hypersurface V,

$$\int_V \mathrm{d}\Sigma_\mu J^\mu = \int_{\partial V} \mathrm{d}\Sigma_{\mu\nu} Q^{\mu\nu},$$

defines a unique conserved surface charge. Here $\mathrm{d}\Sigma_\mu$ is the volume element of V, and $\mathrm{d}\Sigma_{\mu\nu}$ is the induced volume element on its boundary. If ξ corresponds to a symmetry transformation of the field configuration (in the case of diffeomorphisms these are given by global Killing vectors) the corresponding conserved surface charge is the Noether charge in the ordinary sense.

The notion of a Noether charge provides a generalized definition of black hole entropy that is consistent with the first law of black hole mechanics in the presence of higher-order curvature interactions. The central observation of Wald [123] is that if one considers field variations $\Delta\psi$, such that ψ and $\psi + \Delta\psi$ are configurations of a continuous variety of solutions, and focuses on the residual diffeomorphism $\delta_\chi \psi$ of this solution space generated by the horizon-generating Killing vector field χ, the following quantity must vanish

$$\Delta H = \int_{\partial V} \mathrm{d}\Sigma_{\mu\nu} \Big(\Delta Q^{\mu\nu}(\psi, \chi) - 2\chi^{[\mu}\theta^{\nu]}(\psi, \Delta\psi)\Big) = 0\,.$$

Here, H is the Hamiltonian that generates the flow along χ. We refer to [123,127] for details. The boundary ∂V has two disconnected pieces, asymptotic spatial infinity and the Killing horizon, and the expression therefore relates the variation of quantities defined at spatial infinity to quantities defined at the horizon. It is shown in [123,125] that the contribution of the former is proportional to the variation of the mass M and of the angular momentum J, defined in terms of Komar integrals. The contribution of the integral over the horizon, on the other hand, is proportional to the surface gravity and is associated with the variation of the black hole entropy (we use the conventions of [128]),

$$\mathcal{S} = -\pi \int_{\mathrm{hor}} \epsilon_{\mu\nu} Q^{\mu\nu}\, \mathrm{d}\Sigma\,, \tag{1}$$

where $\epsilon_{\mu\nu}$ is the binormal on the horizon normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. We have $\mathrm{d}\Sigma_{\mu\nu} = \epsilon_{\mu\nu}\mathrm{d}\Sigma$, where $\mathrm{d}\Sigma$ is the surface element on the horizon. (We refer to the literature for a discussion of the subtlety that arises when defining entropy in the case of extremal black holes, for which the surface gravity vanishes at the horizon.) With these identifications, the vanishing of ΔH gives rise to the generalized first law of black hole mechanics.

2. Macroscopic entropy formula

Let us employ the Noether charge formula (1) and calculate the black hole entropy for theories described by Lagrangians of the form

$$\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}; \phi, \nabla_\mu\phi)\,,$$

which depend on the metric $g_{\mu\nu}$, the Riemann curvature $R_{\mu\nu\rho\sigma}$, on matter fields ϕ, and covariant derivatives thereof. The approach is generalizable to Lagrangians that depend on derivatives of the Riemann tensor [125,126], but this will not be relevant for the present discussion. It is shown in [123,124] that the corresponding Noether potential, evaluated on the horizon, is given by

$$Q^{\mu\nu}\Big|_{\rm hor} = -2\epsilon_{\rho\sigma}\,\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\Big|_{\rm hor}\,. \tag{2}$$

We first evaluate this expression for the case of classical general relativity, $\mathcal{L}_{\rm GR} = -\frac{1}{16\pi}R$,

$$\frac{\delta\mathcal{L}_{\rm GR}}{\delta R_{\mu\nu\rho\sigma}} = -\frac{1}{16\pi}g^{\mu[\rho}g^{\sigma]\nu}\,.$$

Using the normalization of the binormal one immediately obtains the Bekenstein-Hawking area law $\mathcal{S} = A/4$.

We now turn to the supergravity theories discussed in chapters III and IV and present the entropy formula for the supersymmetric black holes we constructed. Since these theories depend on the Riemann tensor but not on derivatives thereof we can apply formulae (1) and (2) to the action (8) of chapter III in the Poincaré frame and substitute the values the various fields take at the horizon. Recall that the values of these fields are determined by the fixed-point behavior derived in section IV.2. Apart from the usual Einstein-Hilbert term there are further dependencies on the Riemann tensor hidden in the background chiral multiplet describing the R^2-interactions. According to (4) of chapter III, there are two places where this can occur: in the $\hat{F}^-_{ab}$-component and in the $\hat{C}$-component of the background chiral multiplet,

$$\begin{aligned}\hat{F}^{-ab} &= -16\,\mathcal{R}(M)_{cd}{}^{ab}\,T^{klcd}\,\varepsilon_{kl} + \dots\,,\\ \hat{C} &= 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd} - 32\,T^{ab\,ij}\,D_a\,D^c T_{cb\,ij} + \dots\,.\end{aligned}$$

The bosonic terms of the double derivative are given in expression (4) of chapter IV,

$$T^{ab\,ij}\,D_a D^c T_{cb\,ij} = T^{ab\,ij}\,\mathcal{D}_a\mathcal{D}^c T_{cb\,ij} - f_a{}^c\,T^{ab\,ij}\,T_{cb\,ij}\,.$$

Recall that the gauge field of conformal boosts $f_a{}^c$ is not independent but is given by an expression that involves the Ricci tensor and Ricci scalar (*cf.* equation 2 of chapter III).

When evaluated at the horizon, there are fewer terms that contribute to the Noether potential than one might expect. This results from the fact that many tensors vanish at the horizon due to the fixed-point behavior. This is the case, in particular, for the supersymmetrized curvature $\mathcal{R}(M)_{ab}^{-cd}$, the field strength $\hat{F}_{ab}^{-}$, and the combination $F_{ab}^{I} - \frac{1}{4}T_{ab}^{-}\bar{X}^{I}$. Since all terms involving $\hat{F}_{ab}^{-}$ in the action are multiplied by terms proportional to such vanishing tensors, the variation of the $\hat{F}_{ab}^{-}$-terms with respect to the Riemann tensor does not contribute to the entropy formula. Likewise, since $\mathcal{R}(M)_{ab}^{-cd}$ vanishes at the horizon, the only contribution from the variation of the term $\frac{1}{2}i F_A \hat{C}$ + h.c. in the action stems from variations of the gauge field $f_a{}^c$. Straightforward calculation of the variation and substitution of the fixed, non-vanishing value that T_{ab}^{ij} takes at the horizon yield the following result [25] for the macroscopic entropy,

$$\mathcal{S} = \pi \Big[|Z|^2 - 256\,\mathrm{Im}[F_A(X^I, \hat{A})] \Big] . \tag{3}$$

We referred to this expression in (15) of chapter IV. The first term $|Z|^2 = A/4$ is the area of the black hole and arises from the contribution of the Einstein-Hilbert term in the effective action. The quantity $|Z|^2$ is determined in terms of the charges of the black hole through the stabilization equations (*cf.* equation 14 of chapter IV). In these equations, the higher-derivative interactions enter through the dependence of $F_I(X, \hat{A})$ on the background $\hat{A} = \big(\varepsilon_{ij} T_{ab}^{ij}\big)^2$. The near-horizon solution, and hence the area $|Z|^2$, is therefore modified in the presence of higher-order curvature interactions. The second term in the entropy formula originates form the variation of the $f_a{}^c$-term we discussed above. It presents an explicit deviation from the area law. It is important to stress that both contributions separately are scalars under electric-magnetic duality transformations. It is convenient to rewrite the entropy formula in terms of the rescaled variables Y^I and Υ used in section IV.4. Using the homogeneity of the function $F(X, \hat{A})$ and the relations (51) of section IV.4, one finds

$$\mathcal{S} = \pi \lim_{|\vec{x}| \to 0} |\vec{x}|^2 \Big(i \Big[\bar{Y}^I F_I(Y, \Upsilon) - \bar{F}_I(Y, \Upsilon) Y^I \Big] + 4\,\mathrm{Im}\,\Upsilon F_\Upsilon \Big) . \tag{4}$$

We now have all the ingredients together to address the question raised in chapter I: can the deviation from the Bekenstein-Hawking area law, predicted by microstate counting, be understood in terms of higher-order curvature interactions in the effective field theory? The corresponding effective Lagrangian is discussed in section IV.5 and is characterized in terms of the rescaled variables by the homogeneous function

$$F(Y, \Upsilon) = -\frac{1}{6} \frac{C_{ABC}\, Y^A\, Y^B\, Y^C}{Y^0} - \frac{1}{24 \cdot 64} c_{2A} \frac{Y^A}{Y^0} \Upsilon .$$

This model describes the effective action of the M5-brane wrapped on a self-intersecting cycle $\mathcal{P} = p^A \Sigma_A$ of a Calabi-Yau three-fold [21]. We briefly discussed this setup in section I.4. The coefficients C_{ABC} are the triple intersection numbers of the cycle $\mathcal{P}$ and c_{2A} are the second Chern class numbers. The effective Lagrangian associated with this homogeneous function contains terms proportional to $c_{2A}\mathrm{Im}(z^A \hat{C})$, where $z_A = Y^A/Y^0$ and $\hat{C}$ contains, among others, the square of the anti-selfdual part of the Weyl tensor. In section IV.5 the equations that determine the fields Y^I and Υ

in terms of the harmonic functions were solved for the case $p^0 = 0$. For $q^A = 0$ one finds, using (4), the macroscopic entropy [128],

$$\mathcal{S} = 2\pi\sqrt{\tfrac{1}{6}|q_0|(C_{ABC}\,p^A p^B p^C + c_{2A}\,p^A)}\,.$$

This is in perfect agreement with the result found by microstate counting (*cf.* equation 6 of chapter I. The charge q^0 corresponds to the N quanta of Kaluza-Klein momentum.) Both the macroscopic and the microscopic analysis can be generalized to incorporate electric charges $q^A \neq 0$, but we refrain from giving these details. The agreement provides a highly non-trivial check on the consistency of both the microscopic and macroscopic approaches to black hole entropy. Note that the modifications of the macroscopic entropy due to R^2-terms are subleading in the limit of large charges. This is also the limit, for which the result based on microstate counting is valid. The entropy formulae of extremal black hole solutions that arise in supergravity theories based on various other homogeneous functions have been worked out. We refer to [103] for a summary.

3. Moduli space and dynamics of multi-centered black holes

The perfect agreement of the microscopic and the macroscopic entropy of the black holes we considered is an encouraging result, especially in view of the fact that the methods applied in the two approaches are of completely different nature. The extremal black holes with large charges, however, are rather special, and it is therefore desirable to develop other approaches to black hole physics. We have sketched some strategies on how to describe more general black hole in chapter I. The approach we want to focus on in the following is the study of the moduli spaces of multi-centered black holes. The motivation for studying this subject are manifold. As far as black hole entropy is concerned, the quantum mechanics on the moduli space of coalescing black holes is of prime interest. The states described by this quantum mechanical model are believed to correspond to the low-energy quantum states of configurations of black holes bound states, and it is therefore conceivable that (at least part of) the entropy of a black hole can be associated with the degeneracy of such bound states. So far, this approach has not lead to compelling results. One of the reasons is that the calculation of the moduli space geometry can become rather complicated and is not simple to control. Furthermore, the quantization of the resulting mechanical models, even for the simplest moduli spaces, turns out to be quite involved and so far the results [129,130], while promising, are not conclusive. Our interest in the question originates from the desire to understand the effects of R^2-terms on the moduli space geometry. The consequences of higher-derivative effective interactions on the geometry of moduli spaces are still largely unexplored. (We note in passing that as a first step in this program one could envisage studying the moduli space of monopoles of Seiberg-Witten theory [131].)

Moduli space calculations for black holes have been presented in various contexts: in [132] the asymptotic moduli space metric was calculated by deriving the motion of a test black hole in the static multi-centered black hole background. More complete calculations were performed in [133,134], and later generalized in [135,136] to dilaton-coupled Einstein-Maxwell systems in general dimensions and to p-branes. The moduli spaces of black holes in pure supergravity in four and five dimensions were discussed in [45,137] and [44,138], respectively, the moduli space of black holes in supergravity coupled to U(1)-vector multiplets is discussed in [139]. In chapter VI a more extensive list of references is given. These calculations resort to the methods developed by Ferrell and Eardley [133,134] in the context of multi-centered extremal Reissner-Nordström black hole solutions. Since this approach differs from the one developed in chapter VI, we present this analysis here in some detail.

We choose a slightly different normalization of the gauge fields as compared to section I.2, such the extremal Reissner-Nordström black hole solution reads

$$\mathrm{d}s^2 = -\psi^{-2}\mathrm{d}t^2 + \psi^2(\mathrm{d}\vec{x})^2 \,, \quad A = -(1-\psi^{-1})\mathrm{d}t \,,$$

where ψ denotes a harmonic function determined by the source ρ according to

$$\rho = -\frac{1}{4\pi}\Delta\psi \,, \quad \rho = \sum_A m_A \delta^{(3)}(\vec{x}-\vec{x}_A) \,.$$

The Laplace operator is with respect to the flat Euclidean metric, $\Delta = \sum_m \partial_m\partial_m$. In the following we use the letters $m, n, \ldots$ to denote spatial indices, whereas $A, B, \ldots$ denote the centers of the black holes. Above configuration is a solution of the equations of motion corresponding to the action

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{|g|}\,\big(R(g) - F_{\mu\nu}F^{\mu\nu}\big) - \sum_A m_A \int_A (\mathrm{d}s - A) \,.$$

This action is the bosonic part of the supergravity action in the Poincaré frame described by the holomorphic function

$$F(X) = -\tfrac{1}{2}\,i\,X^2 \,.$$

The program for calculating the geodesic approximation (sometimes called the moduli space approximation) to the dynamics of black holes, proposed by Ferrell and Eardley, consists of the following steps:

(i) The centers $\vec{x}_A$ parameterize the space of static solutions. These parameters are viewed as collective coordinates. In the geodesic approximation one promotes these collective coordinates to arbitrary time dependent functions, $\vec{x}_A \to \vec{x}_A(t)$. This implies that the harmonic function ψ that characterizes the static solution, becomes implicitly time dependent,

$$\psi = 1 + \sum_A \frac{m_A}{|\vec{x}-\vec{x}_A(t)|} \,.$$

(ii) All fields of the solution acquire explicit and undetermined field perturbations. Certain perturbations, however, are not included for symmetry reasons. For instance, the perturbation in the t-component of the gauge field and in the g_{tt}-component of the metric are set to zero. Furthermore, there is no perturbation of the geometry of the conformally flat three-space. For the case at hand, the perturbed metric and the perturbed gauge connection are parameterized by two vectors N^m and R_m, respectively,

$$\begin{aligned} \mathrm{d}s^2 &= -\psi^{-2}\mathrm{d}t^2 + 2N_m \mathrm{d}x^m \mathrm{d}t + \psi^2 (\mathrm{d}\vec{x})^2 \,, \\ A &= -(1-\psi^{-1})\mathrm{d}t + R_m \mathrm{d}x^m \,. \end{aligned} \tag{5}$$

(iii) The action is expanded to second order in velocities $\dot{\vec{x}}_A(t)$. The field perturbations N^m and R_m are considered to be first-order in velocities.

(iv) The action is varied with respect to the field perturbations. This yields a set of constraint equations, which determine the perturbations in terms of the velocities and the static, implicitly time dependent, solutions.

(v) The solutions for the perturbations are reinserted into the action and the three-dimensional integral is performed to obtain an effective Lagrangian for the collective coordinates.

(vi) The effective mechanical model has the form of a non-linear sigma model. The target space metric of the model is the metric on the moduli space of solutions.

The outline of the following sections is as follows: in sections 3.1 to 3.4 we implement step (iii) of above program. In section 3.5 we solve for the field perturbations and read off the metric on the moduli space (steps iv to vi). In section 3.6 we discuss the result.

3.1. Gravity in ADM variables. In view of the perturbation ansatz (5) it is convenient to make use of the standard decomposition of the Einstein-Hilbert action in terms of the ADM-variables.[a] The metric is parameterized by the three-metric, the shift- and the lapse-function,

$$\mathrm{d}s^2 = -(N^2 - N^m h_{mn} N^n)\mathrm{d}t^2 + 2h_{mn}\, N^m\, \mathrm{d}t\, \mathrm{d}x^n + h_{mn}\, \mathrm{d}x^m \mathrm{d}x^n \,.$$

The inverse metric is given by

$$g^{tt} = -N^{-2}\,, \quad g^{tm} = N^{-2}N^m\,, \quad g^{mn} = h^{mn} - N^{-2}N^m N^n \,.$$

In particular, it is convenient to make use of the Gauss-Codazzi relation and expresses the Ricci scalar in terms of Ricci scalar ${}^3R(h)$ of the three-metric h and the extrinsic curvature K_{mn},

$$\int \mathrm{d}^4x \sqrt{-g}\, {}^4R = \int \mathrm{d}^4x \sqrt{h} N \left({}^3R(h) + K_{mn}K^{mn} - K^2 \right) + (\text{b.t.})\,,$$

[a] In the following we use conventions of [1], which differs from the ones of appendix B by a sign in the definition of the curvature tensor.

where $K = h^{mn} K_{mn}$ and (b.t.) is a boundary term, which will be ignored in the present discussion. In view of (5), we factor out a conformal factor $\mathrm{e}^{-2\omega}$ from the spatial metric, $h_{mn} = \mathrm{e}^{-2\omega} \hat{g}_{mn}$. Note that

$$\sqrt{-g} = N\sqrt{h} = N\mathrm{e}^{-3\omega}\sqrt{\hat{g}}\,.$$

We express everything with respect to the covariant derivative $\hat{\nabla}$ belonging to $\hat{g}$. The indices of hatted objects are raised and lowered with the metric $\hat{g}$. In above coordinates, the extrinsic curvature is given by

$$\begin{aligned} K_{mn} &= \frac{1}{2N}\left(\partial_t h_{mn} - 2\nabla_{(m} N_{n)}\right) \\ &= \frac{\mathrm{e}^{-2\omega}}{N}\left\{\tfrac{1}{2}\partial_t \hat{g}_{mn} - \dot{\omega}\,\hat{g}_{mn} - \hat{\nabla}_{(m}\hat{N}_{n)} + \hat{g}_{mn}\hat{N}^s\hat{\nabla}_s\,\omega\right\}\,. \end{aligned}$$

Using above decomposition, the Einstein-Hilbert term is written with respect to $\hat{\nabla}$ as follows,

$$\begin{aligned} \int \mathrm{d}^4x\sqrt{-g}\;{}^4R &= \int \mathrm{d}^4x\sqrt{\hat{g}}\,\mathrm{e}^{-g}\,N\left({}^3\hat{R}(\hat{g}) + 4\,\hat{\Delta}\,\omega - 2(\hat{\nabla}\omega)^2\right) + (\text{b.t.}) + \\ &+ \int \mathrm{d}^4x\sqrt{\hat{g}}\,\frac{\mathrm{e}^{-3\omega}}{N}\Big[-6\,\dot{\omega}^2 - 4\,\dot{\omega}\,(\hat{\nabla}^m\hat{N}_m - 3\hat{\nabla}^m\omega\hat{N}_m) + \hat{\nabla}_{(m}\hat{N}_{n)}\hat{\nabla}^{(m}\hat{N}^{n)} \\ &- \hat{N}^m\hat{\nabla}_m\omega\left(2\hat{\nabla}^n\hat{N}_n - 3\hat{\nabla}^n\omega\hat{N}_n\right) - (\hat{\nabla}^m\hat{N}_m - 3\hat{\nabla}^m\omega\hat{N}_m)^2 + U - V\Big]\,. \end{aligned} \tag{6}$$

The expressions U and V are proportional to $\partial_t\hat{g}_{mn}$,

$$\begin{aligned} U &= \partial_t\hat{g}_{mn}\left(\tfrac{1}{4}\hat{g}^{ms}\hat{g}^{nv}\partial_t\hat{g}_{sv} - \dot{\omega}\,\hat{g}^{mn} - \hat{\nabla}^{(m}\hat{N}^{n)} + \hat{g}^{mn}\hat{N}^s\hat{\nabla}_s\omega\right)\,, \\ V &= \hat{g}^{mn}\partial_t\hat{g}_{mn}\left(\tfrac{1}{4}\hat{g}^{sv}\partial_t\hat{g}_{sv} - 3\,\dot{\omega} - (\hat{\nabla}^s\hat{N}_s - 3\hat{\nabla}^s\omega\hat{N}_s)\right)\,. \end{aligned}$$

These terms do not play a role in the later calculation, since the three-metric $\hat{g}_{mn}$ of (5) is flat and constant. We also suppress the boundary term in the following. This section has been completely general and did not involve any approximation whatsoever. Variation of this action with respect to N and N^m yields the so-called super-Hamiltonian and super-momentum constraints of the ADM-formulation of general relativity.

3.2. Gravity perturbations. The ansatz (5) corresponds to keeping the three-metric conformally flat and not including a perturbation for g_{tt}, hence, in ADM-variables, $N^2 - \vec{N}^2 = \psi^{-2}$. We make the following substitutions in (6),

$$N^2 = \psi^{-2}(1 + \psi^4\hat{N}_s\hat{N}_s)\,, \quad \mathrm{e}^{-\omega} = \psi\,,$$

and expand the result to second order in velocities and perturbations. It is convenient to express the result in terms of the rescaled variable $\hat{Q}_m = \psi N_m = \psi^3\hat{N}_m$. Note that the indices of hatted objects are raised and lowered by $\hat{g}$, *e.g.*, $\hat{Q}^m = \psi^3 \hat{N}^m = \psi^3 N^m$. For the Einstein-Hilbert action,

$$16\pi S_{\mathrm{G}} = \int \mathrm{d}^4x \,\sqrt{-g}\; {}^4R(g)\,,$$

one finds, dropping the hats on $\hat{\nabla}$ and $\hat{Q}$ in the following, the zeroth-order contribution in perturbations and velocities,

$$16\pi S_{\mathrm{G}}^{(0)} = \int \mathrm{d}^4x \left[-4\psi^{-1}\Delta\psi + 2\psi^{-2}(\vec{\nabla}\psi)^2\right].$$

There are no first-order contributions, since only squares of the extrinsic curvature appear in the action. The second-order contribution is found after partially integrating several times and completing squares,

$$\begin{aligned}16\pi S_{\mathrm{G}}^{(2)} = \int \mathrm{d}^4x\Big[&- 6\dot{\psi}^2\psi^2 - 4\vec{Q}\vec{\nabla}\dot{\psi} - 4\psi^{-2}\left(\nabla_{[m}Q_{n]} - \psi^{-1}\nabla_{[m}(\psi Q_{n]})\right)^2 \\ &+ \psi^{-4}\nabla_{[m}(\psi Q_{n]})\nabla^{[m}(\psi Q^{n]}) + \psi^{-4}\vec{Q}^2(\vec{\nabla}\psi)^2\Big].\end{aligned}$$

The term proportional to $\vec{Q}^2(\vec{\nabla}\psi)^2$ arises from the expansion of the factor N in the metric determinant and will be canceled by a corresponding term that appears in the expansion of the Maxwell action.

3.3. Electromagnetic perturbations. The normalization of the Maxwell fields is chosen such that the kinetic term is given by

$$16\pi S_{\mathrm{EM}} = -\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}\,.$$

Inserting the perturbation ansatz (5) into the definition of the field strength yields,

$$F_{tm} = \dot{R}_m + \psi^{-2}\hat{\nabla}_m\psi\,, \qquad F_{nm} = 2\hat{\nabla}_{[n}R_{m]}\,.$$

Expanding the Maxwell action to second order in perturbations and velocities one finds

$$\begin{aligned}16\pi S_{\mathrm{EM}} = \int \mathrm{d}^4x\sqrt{\hat{g}}\Big[&\frac{2\psi}{N}F_{tm}\hat{g}^{mn}F_{tn} - \frac{N}{\psi}\hat{g}^{nm}\hat{g}^{sr}F_{ns}F_{mr} \\ &- \frac{4\psi}{N}\hat{g}^{sm}N^nF_{tm}F_{ns} + \mathcal{O}(3)\Big].\end{aligned}$$

After partial integration one finds (dropping hats in the following),

$$\begin{aligned}16\pi S_{\mathrm{EM}}^{(0)} &= 2\int \mathrm{d}^4x\,\psi^{-2}(\vec{\nabla}\psi)^2\,,\\ 16\pi S_{\mathrm{EM}}^{(2)} &= \int \mathrm{d}^4x\Big[- \psi^{-4}\vec{Q}^2(\vec{\nabla}\psi)^2 - 4(\vec{\nabla}\dot{\psi})\vec{R} - 4\psi^{-2}\nabla_{[n}R_{m]}\nabla^{[n}R^{m]} \\ &\qquad + 8\psi^{-3}\nabla^{[n}(\psi Q^{s]})\nabla_{[n}R_{s]} - 8\psi^{-2}\nabla^{[n}Q^{s]}\nabla_{[n}R_{s]}\Big].\end{aligned}$$

3.4. Source terms. The current and matter source actions can be expressed in terms of a dust density $\rho = -\frac{1}{4\pi}\Delta\psi$. At first we leave this dust density unspecified. At the end, we consider the black hole limit, $\rho \longrightarrow \sum_A m_A \delta^{(3)}(\vec{x} - \vec{x}_A(t))$. The matter source action reads

$$S_{\text{matter}} = -\int \mathrm{d}s \left(\rho\, \mathrm{d}^3x\right) = -\int \mathrm{d}^4x\, \rho\, \left(N^2 - \psi^2 \hat{g}_{mn}(N+\dot{x})^m (N+\dot{x})^n\right)^{1/2} .$$

Expanding to second order in perturbations and velocities one finds

$$S^{(0)}_{\text{matter}} = -\int \mathrm{d}^4x\, \psi^{-1}\rho = \frac{1}{4\pi}\int \mathrm{d}^4x\, \psi^{-1}\Delta\psi\, ,$$

$$S^{(2)}_{\text{matter}} = \int \mathrm{d}^4x\, \rho\, \left(\vec{Q}\dot{\vec{x}} + \frac{\psi^3}{2}\dot{\vec{x}}\dot{\vec{x}}\right) .$$

The current source action is simple and is given by

$$S_{\text{current}} = \int \mathrm{d}^4x\, \rho A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = -\int \mathrm{d}^4x\rho + \int \mathrm{d}^4x\, \rho\left(\psi^{-1} + \vec{R}\dot{\vec{x}}\right) .$$

3.5. Effective Action. It is convenient to introduce the quantity $\vec{P} = \vec{Q} + \vec{R}$ and express the resulting combined action in terms of the velocities, $\vec{P}$ and $\vec{Q}$. The zeroth-order action is related to the energy density of the static configuration,

$$S^{(0)}_{\text{G}} + S^{(0)}_{\text{EM}} + S^{(0)}_{\text{matter}} + S^{(0)}_{\text{current}} = -\int \mathrm{d}^4x\, \rho\, .$$

The combined second-order expressions nicely combine and reproduce the result [133, 134],

$$\begin{aligned} &S^{(2)}_{\text{G}} + S^{(2)}_{\text{EM}} + S^{(2)}_{\text{matter}} + S^{(2)}_{\text{current}} = \\ &\int \mathrm{d}^4x \left[-\frac{3}{8\pi}\dot{\psi}^2\psi^2 + \vec{P}\left(\rho\dot{\vec{x}} - \frac{1}{4\pi}\vec{\nabla}\dot{\psi}\right) + \frac{1}{2}\rho\,\psi^3\dot{\vec{x}}\dot{\vec{x}} + \right. \\ &\left. + \frac{1}{16\pi}\psi^{-4}\nabla_{[m}(\psi Q_{n]})\nabla^{[m}(\psi Q^{n]}) - \frac{1}{4\pi}\psi^{-2}\left(\nabla_{[m}P_{n]} - \psi^{-1}\nabla_{[m}(\psi Q_{n]})\right)^2\right] . \end{aligned}$$

Following [133,134], one introduces the vector $\vec{K}$ defined by $\Delta K^m = -4\pi\rho\,\dot{x}^m$, where Δ is the Laplacian of the Euclidean three-metric. From current conservation, $\vec{\nabla}(\rho\,\dot{\vec{x}}) + \dot{\rho} = 0$, one derives

$$\rho\,\dot{x}_n - \frac{1}{4\pi}\nabla_n\dot{\psi} = \frac{1}{2\pi}\nabla_m\nabla_{[n}K_{m]}\, .$$

The perturbations $\vec{P}$ and $\vec{Q}$ are constrained by imposing their field equations, which are given by

$$\nabla_m \nabla_{[m} K_{n]} = \nabla_m \left(\psi^{-2} \Big[\nabla_{[m} P_{n]} - \psi^{-1} \nabla_{[m} (\psi Q_{n]}) \Big] \right) ,$$
$$0 = \nabla_m \left(\psi^{-3} \Big[\nabla_{[m} P_{n]} - \frac{3}{4} \psi^{-1} \nabla_{[m} (\psi Q_{n]}) \Big] \right) .$$

These equations can be solved by

$$\begin{aligned} \nabla_{[m} P_{n]} &= -3\, \psi^2 \nabla_{[m} K_{n]} - 3\, \psi^2 \varepsilon_{mns} \nabla_s \Lambda_2 + 4\, \psi^3 \varepsilon_{mns} \nabla_s \Lambda_1 , \\ \nabla_{[m} (\psi Q_{n]}) &= -4\, \psi^3 \nabla_{[m} K_{n]} - 4\, \psi^3 \varepsilon_{mns} \nabla_s \Lambda_2 + 4\, \psi^4 \varepsilon_{mns} \nabla_s \Lambda_1 , \end{aligned} \tag{7}$$

where $\Lambda_{1,2}$ are two independent integration functions. Reinserting these expressions into the effective action one finds

$$\begin{aligned} S_{\text{approx}} = \int \mathrm{d}^4 x \Big[&- \frac{3\psi^2}{8\pi} \dot{\psi}^2 - \frac{3\psi^2}{4\pi} \left(\nabla_{[m} K_{n]} \right)^2 - \rho + \frac{1}{2} \rho\, \dot{x}^2 \psi^3 \\ &- \frac{\psi^2}{4\pi} (\vec{\nabla} \Lambda_2)^2 + \frac{\psi^2}{\pi} (\vec{\nabla} \Lambda_2 - \psi \vec{\nabla} \Lambda_1)^2 \Big] . \end{aligned} \tag{8}$$

It is important to realize that the functions of integration do not drop out automatically in the action and therefore potentially contribute to the metric on the moduli space. In principle, the integration functions $\Lambda_{1,2}$ are determined by the integrability conditions derived from (7). In [133,134] it is claimed that in the black hole limit, the integration functions in fact do not contribute to the effective actions. To this extent, it is useful to relate the integration functions $\Lambda_{1,2}$ to the ones introduced in [133,134],

$$\Lambda_1 = \psi^{-3} \alpha \, , \quad \Lambda_2 = \nu + 2 \psi^{-2} \alpha \, .$$

In fact, one of the integrability conditions expresses ν in terms of α,

$$\frac{1}{2} \psi^3 \Delta \nu + \alpha \Delta \psi = 0 \, .$$

This can be inverted for ν with the following result in the black hole limit $\rho = \sum m_A \delta(\vec{x} - \vec{x}_A)$,

$$\nu(\vec{x}) = -4\pi \sum_A \frac{1}{|\vec{x} - \vec{x}_A|} \psi^{-3}(\vec{x}_A)\, \alpha(\vec{x}_A) \, .$$

If $\alpha(\vec{x})$ is not too singular, as $\vec{x} \to \vec{x}_A$, then $\nu = 0$. In this case, $\Lambda_1 = \frac{1}{2} \psi^{-1} \Lambda_2$ and the contribution of the integration functions to the action (8) is proportional to the Laplacian of ψ,

$$S_{\text{approx}} = \ldots + \frac{1}{4\pi} \int \mathrm{d}x^4 \left[-2 \psi^{-1} \Delta \psi\, \alpha^2 \right] ,$$

and vanishes if α is not too singular. In principle, analyzing the second integrability condition allows one to estimate the degree of divergence of α as one approaches a center, but we refrain from giving these details here. It is clear, however, that generically such integration functions do contribute to the effective moduli action. Using $\dot{\psi} = -\vec{\nabla}\vec{K}$ we find (suppressing the issue of the integration functions)

$$\begin{aligned} S_{\text{approx}} = &- \int \mathrm{d}^4x\, \rho - \frac{1}{8\pi}\int \mathrm{d}^4x\, \Delta\psi\, \dot{x}^2 \\ &- \frac{3}{8\pi}\int \mathrm{d}^4x\, \psi^2 \Big[(\vec{\nabla}\vec{K})^2 + \nabla_m K_n \nabla^m K^n - \nabla_n K_m \nabla^m K^n\Big] . \end{aligned}$$

In the black hole limit, $\rho \to \sum m_A \delta(\vec{x} - \vec{x}_A)$, we can replace the term involving the Laplacian by $\sum_A \dot{x}_A^2\, \partial_A^2 \psi$ under the integral. Furthermore, we have

$$K_m = \sum_A \frac{m_A\, \dot{x}_{Am}}{|\vec{x} - \vec{x}_A(t)|} , \quad \nabla_m K_n = -\sum_A \dot{x}_{Am} \nabla_{An} \psi .$$

Using these expressions, the action for the collective coordinates can be written as a non-linear sigma model,

$$S_{\text{approx}} = -\sum_A m_A + \frac{1}{2}\int \mathrm{d}t \sum_{AB} g_{AmBn}\ \dot{x}_A^m\ \dot{x}_B^n , \tag{9}$$

where $g_{AmBn}(\vec{x}_C)$ is identified with the the metric on the moduli space of static solutions and is given in terms of a derivative of a potential [44,45],

$$\begin{aligned} g_{AmBn}(\vec{x}_C) &= \Big(\delta_m{}^i \delta_n{}^j + \varepsilon_m{}^{il} \varepsilon_{ln}{}^j\Big) \nabla_{Ai} \nabla_{Bj} \Theta(\vec{x}_A) , \\ \Theta(\vec{x}_A) &= -\frac{1}{16\pi}\int \mathrm{d}^3x\, \psi^4 . \end{aligned} \tag{10}$$

Varying the action (9) with respect to $\vec{x}_A(t)$ yields the equation for geodesic motion on the moduli space of solutions.

3.6. Discussion. The reason why the method used to derive the geodesic description of the multi-centered black hole dynamics works hinges on an intricate interplay between the restricted perturbation ansatz (5) and, as a consequence, on the existence of additional constraint equations when working to first order in velocities and perturbations. Recall that we imposed field equations for $\vec{P}$ and $\vec{Q}$ to first order in velocities and perturbations. These equations are constraint equations as they involve at most first time derivatives on implicitly time dependent functions. These constraint equations are in fact linear combinations of the original field equations expanded to first order in velocities and perturbations,

$$\nabla_\mu F^{\mu m} = 4\pi J^m , \quad G^{tm} = 8\pi T^{tm} .$$

The first equation is the spatial component of Maxwell's equation, the second one is the spatial component of the initial value constraint (sometimes called super-momentum constraint) of general relativity. The latter is a constraint equation to any order, since it does not involve second time derivatives and no first time derivatives on the shift- and lapse-function. The spatial components of Maxwell's equation, on the other hand, do contain a second time derivative. It is only when working to first order in velocities and perturbations and when restricting the perturbations to the ansatz (5) that one recovers a constraint equation, namely Ampère's law. The Gauss constraint and the so-called super-Hamiltonian constraint,

$$\nabla_\mu F^{\mu t} = 4\pi J^t \,, \quad G^{tt} = 8\pi T^{tt} \,.$$

on the other hand, are satisfied automatically to first order by the ansatz (5). It is clear that due to the mismatch between the number of genuine constraint equations and the number of explicit perturbations incorporated in (5) the method proposed by [133,134] is very much tailored to this specific application and approximation scheme. It is quite remarkable that the method can be applied to the more complicated black hole solutions of $N = 2$ supergravity coupled to U(1)-vector multiplets [139], though here it is necessary to adopt a scheme including many more field perturbations. On the other hand, generalizing the calculation of Ferrell and Eardley to black holes of supergravity theories with R^2-interactions has proven to be very arduous. We stress that R^2-interactions potentially affect to the metric on the moduli space of solutions. We discuss these issues in the chapter VI.

The effective action (9) is the starting point for investigating the classical slow-motion of extremal black holes and for addressing questions of scattering and coalescence. Since Θ is a polynomial of degree four in the black hole centers, there are only two-body, three-body, and four-body interactions. Furthermore, due to translational invariance, the center of mass motion decouples. Generically, the system is not integrable and one has to resort to numerical methods.

An interesting property of the moduli space metric is that it possesses an SL(2, $\mathbb{R}$) conformal symmetry in the limit of small black hole separations [45,139]. We mentioned this scaling limit in section I.5 in the context of the AdS/CFT-correspondence. In this limit, the moduli space potential reduces to

$$\Theta(\vec{x}_A) = -\frac{1}{16\pi}\int \mathrm{d}^3x \left[\sum_A \frac{m_A}{|\vec{x}-\vec{x}_A|}\right]^4 .$$

The corresponding metric in fact admits an exact homothetic Killing vector,

$$\chi^{Am} = -2\,x^{Am} \,, \quad \chi_{Am} = \partial_{Am}\chi \,,$$

with homothetic Killing potential [45]

$$\chi = 2\sum_{A\neq B} m_A^3 m_B \frac{1}{|\vec{x}_A - \vec{x}_B|} \,.$$

In [45] the supersymmetric completion of (9) was studied. The Lagrangian is based on a superspace integral utilizing $\Theta(\Phi)$ as the holomorphic function, where Φ denote constrained $N = 4$ superfields in one dimension, each containing $3 + 4$ physical degrees of freedom [140]. The superconformal extension of the supersymmetric models presented in [45] have a D(2, 1; 0) superconformal symmetry. It is still unclear as to whether this superconformal mechanical model bears any relation to the superconformal models we mentioned in section I.5 in the context of the AdS/CFT-correspondence.

4. Entropy formula and moduli space potential

The multi-centered solutions we have studied in chapter IV can be used as a starting point for computing the metric on the moduli space of extremal black holes in the presence of R^2-interactions. In [139] the result of [133] was generalized to supergravity theories of the type we considered in chapter IV, but in the absence of R^2-interactions. It was found that the metric on the moduli space of electrically charged BPS black holes is still determined in terms of the moduli space potential Θ given in equation (10), where ψ^2 is the function that appears in the line-element and is identical to the function e^{-2g} in the notation of chapter IV,

$$\psi^2 = \mathrm{e}^{-2g} .$$

For these more complicated theories, the function $\psi = \mathrm{e}^{-g}$ is not just a simple harmonic function, but depends on the scalar fields of the vector multiplets, which in turn are expressed by harmonic functions. In the presence of R^2-interactions, e^{-2g} receives additional modifications (we use Planck units),

$$\begin{aligned}\mathrm{e}^{-2g} = {}& i\left[\bar{Y}^I F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon})Y^I\right] - 128i\,\mathrm{e}^{g}\,\nabla^p\Big[\left(\nabla_p \mathrm{e}^{-g}\right)\,(F_\Upsilon - \bar{F}_\Upsilon)\Big] \\ &+ 32i\,\mathrm{e}^{4g}\,(R(\sigma)_p)^2(F_\Upsilon - \bar{F}_\Upsilon) + 64\,\mathrm{e}^{2g} R(\sigma)_p\,\nabla^p(F_\Upsilon + \bar{F}_\Upsilon) .\end{aligned}$$

How do the R^2-terms affect the moduli space potential? We expect that the moduli space potential receives corrections as a result of the R^2-dependence of e^{-2g}. On the other hand, there may also be additional modifications of Θ, which originate from the various additional couplings of the supergravity action in the presence of higher-order curvature interactions. In the following, we focus only on the modifications encoded in e^{-2g}. Using above expression we rewrite Θ as follows,

$$\begin{aligned}-16\,\pi\,\Theta &= \int \mathrm{d}^3x\;\psi^4 = \int \mathrm{d}^3x\;\mathrm{e}^{-4g} \\ &= \int \mathrm{d}^3x\;\mathrm{e}^{-2g}\left(i\Big[\bar{Y}^I F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon})Y^I\Big] - 4\,|\Upsilon|\,\mathrm{Im}\,F_\Upsilon\right), \qquad (11)\end{aligned}$$

where we integrated by parts, used the fact that $R(\sigma)_p$ is divergence free, and inserted the expression for the rescaled chiral background, $\Upsilon = -64(\nabla_m g - \frac{1}{2} i\,\mathrm{e}^{2g}\,R(\sigma)_m)^2$. Observe that the combination $i\big[\bar{Y}^I F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon})Y^I\big] - 4\,|\Upsilon|\,\mathrm{Im}\,F_\Upsilon$, when evaluated at the horizon of a BPS black hole, is precisely equal to $\pi^{-1}|\vec{x}|^{-2}$ times the expression for its macroscopic entropy (4) of the black hole! This intriguing feature may indicate that there are in fact no additional explicit modifications of Θ due to the R^2-interactions. In establishing (11) we dropped certain boundary terms when integrating by parts. Some of those are known to be proportional to $|\vec{x}_A - \vec{x}_B|^{-1}$ (for two non-coincident centers A and B) and therefore do not contribute to the metric on the moduli space [45]. We close with the remark that, just as the macroscopic entropy, the moduli space potential is a scalar under electric-magnetic duality transformations.

VI

Moduli spaces and geodesic description

This chapter contains an introduction to the geodesic description of soliton dynamics for theories with gauge invariance and/or diffeomorphism invariance. The motivation for this work is to address the question of how the higher-order curvature corrections, which we have discussed at length in chapter IV, affect the geometry of the moduli space of stationary multi-centered solutions. As mentioned in chapter V, a good understanding of this geometry is relevant for the understanding of the (quantum) mechanics of black holes. It is, however, not yet possible to present definite answers to these last questions. This chapter therefore differs from the previous ones in that it deals with the conceptual basis of the geodesic approximation rather than with its concrete applications. Some results of this chapter have been reported in [141,142].

1. Introduction

The study of the time evolution of (multi-)solitonic field configurations was initiated by the work of Manton [143]. The principal idea is to approximate the full classical dynamics of solitons by their geodesic motion in the space of static (or stationary) configurations (moduli space). In the case of SU(2) magnetic monopoles the 2-monopole moduli space is known exactly [144,145] and questions concerning monopole scattering can be addressed in this geodesic approximation. However, the n-monopole solution space is understood only asymptotically [146,147]. Nevertheless, a general (albeit implicit) formula for the metric on the moduli space has been given [148]. Certain elements of this chapter are based on the fairly general setup presented in [149]. For applications along these lines, we refer to [150,151]. Moduli space calculations have also appeared in the context of lump solutions in CP^1-models [152,153] and for the CP^1-model coupled to gravity [154]. The moduli spaces of abelian vortices were discussed in [155,156] and a nice discussion of string soliton scattering appeared in [157] (see also [158]). The dynamics of Kaluza-Klein monopoles was discussed in [159].

The moduli spaces of BPS black holes are known explicitly in a number of cases. The first references on this subject are [132–134]. In the latter two papers the mechanics of four-dimensional multi-centered Reissner-Nordström black holes was derived. We reviewed this approach in detail in chapter V. This work was generalized in [135,136] to dilaton-coupled Einstein-Maxwell systems in general dimensions and to p-branes. The moduli spaces of black holes in pure supergravity in four and five dimension were discussed in [45,137] and [44,138], respectively; the moduli space of black holes in supergravity coupled to abelian vector multiplets was discussed in [139].

In the above references many diverse, seemingly unrelated, and often context-adapted methods are used. The goal of this chapter is to present a conceptual basis from a more unified perspective. The construction of the geodesic approximation we propose is directed towards the moduli spaces of stationary solutions of generic field theories with arbitrary invariance groups. In particular, it is shown how the well-developed methods used in the gauge theory setting [148] can be carried over to theories with diffeomorphism invariance. While the geodesic description is by itself an approximation based on the restricted motion in the space of static solutions, there is no compelling mathematical reason why one has to subsequently resort to a low-velocity approximation. Indeed, our methods can in principle deal with field theories that contain higher powers in the time derivative. The terms of higher order in the time derivatives of the fields may still influence the standard kinetic terms in the moduli action that are quadratic in the velocities. There are general arguments that indicate that this is indeed the case, at least for the solutions discussed in section IV.

1.1. Geodesic approximation for the nonlinear sigma model. Let us first remind the reader how the geodesic approximation (sometimes also called the moduli space approximation) appears in its simplest form for a theory without gauge invariance. Consider a nonlinear sigma model in $d+1$ spacetime dimensions with a potential

$$L = \int \mathrm{d}^d x \left(\tfrac{1}{2} g_{IJ}\, \partial_t \phi^I\, \partial_t \phi^J\right) - V[\phi, \partial_m \phi]\,.$$

We assume that this theory has static solutions that can be parametrized by a number of continuous integration constants X^a, which we call collective coordinates. These solutions are encoded in time-independent functions $\phi^I(\vec{x}, X^a)$, which characterize a continuous variety of extrema of the potential. We have a situation in mind where the field theory has solutions that describe localized lumps so that the collective coordinates X^a typically denote their positions. Often, the total number of lumps is fixed by topological constraints. In that case $\phi^I(\vec{x}, X^a)$ describes a family of degenerate static multi-lump solutions. The collective coordinates X^a parameterize the moduli space of these solutions and two neighboring solutions are related through a variation of the collective coordinates, $X^a \to X^a + \delta X^a$, which induces a variation on the fields $\delta_{\rm cov}\phi^I = \delta X^a \partial_a \phi^I(\vec{x}, X^a)$. The reason for introducing the notation $\delta_{\rm cov}$ refers to the situation with gauge invariance, as will be explained in due course.

When we let the collective coordinates depend on the time t, the field equations induce equations of motion for the collective coordinates. These describe the (approximate) dynamics of the solitons as a geodesic motion in the moduli space of static configurations, parametrized by the X^a. To see this, let us first write the equations of motion for $\phi^I(t,\vec{x})$,

$$g_{IJ}(\phi)\left(\partial_t{}^2\phi^J + \Gamma^J_{KL}(\phi)\,\partial_t\phi^K\,\partial_t\phi^L\right) + \left(\frac{\partial V}{\partial\phi^I} - \partial_m\frac{\partial V}{\partial(\partial_m\phi^I)}\right) = 0\,, \tag{1}$$

where, Γ^J_{KL} denotes the Christoffel connection associated with g_{IJ}. When substituting the static solutions $\phi^I(\vec{x}, X^a(t))$ with time dependent collective coordinates into this equation, the last terms referring to the potential will vanish because the continuous class of static solutions $\phi^I(\vec{x}, X^a(t))$ has a fixed potential energy and corresponds to a subspace of the space of field configurations for which the potential is stationary with respect to arbitrary field variations.

The field equation (1) follows from requiring the action to be stationary with respect to arbitrary variations of the fields. But these are not the equations that are appropriate in the geodesic approximation, since those refer to field variations that are restricted to the extremal potential energy surface. Instead we should thus restrict ourselves to variations induced by the change of the collective parameters, *i.e.*, to $\delta_{\rm cov}\phi^I = \delta X^a(t)\,\partial_a\phi^I(\vec{x}, X^a(t))$, where the $\delta X^a(t)$ are arbitrary functions of time t. The relevant dynamical equations in the geodesic description then follow from multiplying (1) with $\delta_{\rm cov}\phi^I$ and integrating over the d-dimensional space. This leads to the following geodesic equation,

$$G_{ab}\left(\partial_t{}^2X^b + \Gamma^b_{cd}\,\partial_tX^c\,\partial_tX^d\right) = 0\,, \tag{2}$$

where the moduli metric and the corresponding Christoffel connection are given by,

$$\begin{aligned} G_{ab}(X) &= \int \mathrm{d}^dx\; g_{IJ}\,\partial_a\phi^I\,\partial_b\phi^J\,,\\ \Gamma^a_{bc}(X) &= G^{ad}\int \mathrm{d}^dx\; g_{IJ}\left(\partial_b\partial_c\phi^I + \Gamma^I_{KL}\,\partial_b\phi^K\,\partial_c\phi^L\right)\partial_d\phi^J\,, \end{aligned} \tag{3}$$

where everywhere we put $\phi^I = \phi^I(\vec{x}, X^a(t))$. Obviously, one can consider diffeomorphisms of the moduli space coordinates X^a, under which G_{ab} transforms as a symmetric tensor and the velocities $\dot{X}^a = \partial_tX^a$ transform as covariant vectors.

Here, we have chosen to effect the geodesic approximation through the equations of motion, but for the example at hand we could easily have worked at the level of the action: after substituting $\phi^I(t,\vec{x}) = \phi^I(\vec{x}, X^a(t))$ into the action and performing the integration over spatial coordinates, one obtains the action for a particle moving through the target space parametrized by the collective coordinates X^a and described by the metric G_{ab} given above. This target space is just the moduli space of the static solutions. The geodesic equation (2) results from varying this action with respect to $X^a(t) \to X^a(t) + \delta X^a(t)$.

In this paper the term “geodesic approximation” refers to the description where one restricts the fields to evolve solely within the space of static field configurations. By itself, this does not imply that one resorts to an approximation in terms of low velocities. The only reason why the particle action is quadratic in velocities is that the underlying field theory is at most quadratic in the time derivatives. Terms of higher order in the time derivatives of the fields lead to terms of higher order in the velocities (which couple to symmetric target-space tensors $U_{a_1\cdots a_n}(X)$) in the moduli space action, and can be evaluated in the same way. As long as there are no higher-order time derivatives, the moduli action will be a function of coordinates and velocities, and not of the accelerations or derivatives thereof. Of course, one expects the geodesic approximation to break down at high velocities on physical grounds, because the motion will no longer be confined to the static field configurations.

1.2. Global symmetries and moduli space isometries. The symmetry features of the underlying field theory can lead to corresponding features at the level of the geodesic description. Here we should make a distinction between rigid and local symmetries and symmetries of spacetime itself. Local symmetries will be discussed at length later in this chapter. Spacetime symmetries that involve the time coordinate cannot be preserved in the geodesic description, which is based on identifying a time coordinate from the start. We return to this point in due course. Rigid spacetime symmetries that do not involve the time coordinate and rigid target space symmetries induce an action on the collective coordinates, provided they act non-trivially on the static solutions $\phi(\vec{x}, X^a)$. Here we briefly discuss the consequences of such rigid symmetries.

Assuming the $\phi(\vec{x}, X^a)$ completely characterize the static solutions, they will transform among themselves under the rigid symmetry transformations. This implies that the action of the symmetry on the fields induces a transformation on the collective coordinates. For the model at hand, the symmetries of the field theory take the form of isometries of the target space (possibly further restricted by the requirement that they also leave the potential energy term invariant). They correspond to the infinitesimal transformations $\phi^I \rightarrow \phi^I + k^I(\phi)$, where the $k^I(\phi)$ are Killing vectors associated with the metric g_{IJ}. Because both $\phi^I(\vec{x}, X)$ and $\phi^I(\vec{x}, X) + k^I(\phi(\vec{x}, X))$ are static solutions, it follows that a moduli space vector $K^a(X)$ must exist such that,

$$K^a(X)\,\partial_a\phi^I(\vec{x}, X) = k^I(\phi(\vec{x}, X))\,. \tag{4}$$

Contracting this equation with $g_{IJ}(\phi(\vec{x}, X))\partial_b\phi^J(\vec{x}, X)$ and integrating over space one finds the inverse relation,

$$K_a(X) = \int \mathrm{d}^d x\; k_I(\phi(\vec{x}, X))\,\partial_a\phi^I(\vec{x}, X)\,.$$

We have lowered the index of K^a by contracting with the moduli metric $G_{ab}(X)$. From the fact that k^I is a Killing vector associated with g_{IJ} it follows that K^a generates an isometry of the moduli space metric, *i.e.*, $\mathcal{L}_K G_{ab} = 0$. To see this one conveniently makes use of (3) and (4).

Similarly, spacetime symmetries that do not involve the time coordinate induce Killing symmetries on moduli space. As an example consider the case of translational symmetry, *i.e.*, invariance of the field theory under $\vec{x} \rightarrow \vec{x} + \vec{r}$, where $\vec{r}$ is constant. Obviously translated solutions remain within the variety of solutions that we are considering, so that there must be d vectors $\vec{R}^a(X)$ in the moduli space subject to $\partial_m \phi^I(\vec{x}, A) = R^a_m(X)\, \partial_a \phi^I(\vec{x}, X)$. Performing the same steps as before, we find

$$\vec{R}_a(X) = \int \mathrm{d}^d x \ \vec{\partial}_x \phi^I(\vec{x}, X)\, g_{IJ}(\phi(\vec{x}, X))\, \partial_a \phi^J(\vec{x}, X)\,.$$

The vectors $\vec{R}_a$ satisfy

$$\mathcal{L}_{\vec{R}}\, G_{ab}(X) = \int \mathrm{d}^d x \ \vec{\partial}_x \Big(\partial_a \phi^I\, g_{IJ}\, \partial_b \phi^J \Big)\,.$$

Dropping the surface term it follows that we have d (abelian) isometries of the target space metric. Obviously these isometries are associated with the center-of-mass motion of the static solutions.

1.3. Outline. So far we have discussed a field theory without local gauge symmetries. Gauge invariances and diffeomorphisms are more subtle to deal with, since they, generically, do not represent symmetries but redundancies in the field representation. In order to deal with gauge degeneracies, one can either gauge-fix all gauge invariances and unambiguously determine the stationary solution space in that gauge, or one can proceed gauge covariantly by working with gauge equivalence classes of solutions. The non-covariant approach tends to be problematic because one will have to show at the end that the result does not depend on the chosen gauge. Therefore we will follow a covariant approach, which will be introduced in the next section for a gauge theory.

The construction of the geodesic approximation proceeds in three, logically distinct steps. First, the space of static/stationary soliton configuration is determined and analyzed with respect to its residual gauge symmetries. The notion of a covariant variation (*i.e.* parallel transport in the moduli space) is introduced in order to compare two neighboring equivalence classes. In a second step the time dependence is re-introduced by letting the collective coordinates depend on time. Here the residual gauge symmetries are lifted to the the time-dependent situation. In this thesis the re-introduction of time into the problem is referred to as the "geodesic lift". This lift requires the introduction of terms that depend explicitly on the velocities $\dot{X}^a(t)$, to preserve the invariance under the residual gauge transformations. We will establish that these velocity-dependent modifications follow from gauge invariance. In this way one defines a consistent mapping of the time evolution of the collective coordinates to the time evolution in the field theory configuration space. With this relation, the action principle for the dynamics of the collective coordinates is induced by the action principle of the underlying field theory. In particular, the covariant variation of the fields, which is the representation in field configuration space induced by the change of the collective coordinates, must satisfy the same properties as in the original field theory variational principle. Finally, one has to deal with the modding out of the gauge redundancy in order to ensure that the geodesic motion is orthogonal to the gauge orbits in the underlying field theory. This requirement is related to imposing the correct initial-value constraints in order to obtain a well-posed Cauchy problem in the underlying field theory.

The outline of the chapter is as follows: in section 2 we discuss ordinary gauge theories and perform the three steps mentioned above for scalar and vector fields, stressing the role of the residual gauge symmetries and of the geodesic lift. In section 3 we explain how to generalize the construction to theories with diffeomorphism invariance. In section 4 we summarize our findings and give an outlook.

2. Geodesic description for gauge theories

In this section we discuss the geodesic approximation for gauge theories in flat spacetime. We concentrate, for concreteness, on a gauge theory minimally coupled to a scalar field ϕ in the adjoint representation of the gauge group,

$$S = \int \mathrm{d}^{d+1}x \, \left(\mathrm{Tr}\Big[\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} + \frac{1}{2} D_\mu\phi\, D^\mu\phi\Big] - V(\phi) \right) . \tag{5}$$

Throughout this section we will write the fields in Lie algebra valued form. We assume that this field theory has a continuous variety of static solutions parametrized by a number of collective coordinates X^a. More general Lagrangians can be studied within the framework that we will explain, including Lagrangians that contain higher powers of the covariant derivatives and the fields strengths. But the emphasis will be on the conceptual framework, rather than on specific applications.

2.1. Static solution space. In the case of a gauge theory the static solutions are in general subject to a class of residual gauge transformations that do not involve the time variable. This implies that these solutions are still ambiguous and the corresponding gauge degeneracy has to be modded out when extracting the correct moduli space description. As mentioned previously, one could adopt a gauge condition that would result in a class of unique solutions, depending again on collective coordinates X^a (which themselves are gauge invariant). However, it is unclear whether this description will lead to a gauge invariant and gauge independent moduli space metric. This question is hard to answer, also in view of the fact that it is difficult to respect the gauge conditions when re-introducing time. See, for instance, the examples discussed in [148,149], where the initial gauge conditions are modified by velocity-dependent terms. Therefore we will pursue a covariant approach, in which none of the residual (*i.e.* time-independent) gauge transformations are fixed. We will then argue that the above mentioned velocity-dependent modifications follow from gauge covariance and are uniquely determined within the geodesic approximation.

The residual gauge transformations, which act on the static configurations, are parametrized by parameters Λ that depend on $\vec{x}$. In addition they can depend on the collective coordinates X^a, so that inequivalent solutions (characterized by different values for the X^a) may be subject to different gauge transformations. The residual transformations take the form,

$$\delta\phi = [\Lambda, \phi], \quad \delta A_t = [\Lambda, A_t], \quad \delta A_m = \partial_m \Lambda - [A_m, \Lambda],$$

where the gauge connections and the scalar fields that appear in (5) are denoted by $A_m(\vec{x}, X)$, $A_t(\vec{x}, X)$ and $\phi(\vec{x}, X)$, and belong to the class of static solutions parametrized by the collective coordinates.

The fact that we are dealing with fields and transformations that depend on both $\vec{x}$ and X^a, implies that we are dealing with an extended base space parametrized by the coordinates (x^m, X^a). To define parallel transport in this extended bundle, we need connections $A_M = (A_m, A_a)$, where $A_a(\vec{x}, X)$ is a new connection field, which for the moment is left undetermined. This suggests that the new connection field will transform under residual gauge transformations according to

$$\delta A_a = \partial_a \Lambda - [A_a, \Lambda]. \tag{6}$$

The implications of this are discussed below.

When comparing two neighboring solutions $\phi(\vec{x}, X)$ and $\phi(\vec{x}, X + \delta X)$ (and correspondingly for the connections A_m and A_t) one must account for the fact that the solutions, as one explicitly writes them down, are but representatives of a whole class of gauge degenerate solutions. Comparing two solutions characterized by collective coordinates X^a and $X^a + \delta X^a$, one must therefore allow for residual gauge transformations that differ at these two values of the collective coordinates. This difference is reflected in an infinitesimal gauge transformation parametrized by the new connections A_a with parameter $\Lambda = \delta X^a A_a$, so that the resulting field variations induced by shifts of the collective coordinates take the form [148,149],

$$\begin{aligned}
\delta_{\rm cov}\phi &= \delta X^a\, \partial_a \phi - [\delta X^a A_a, \phi] = \delta X^a\, D_a \phi\,, \\
\delta_{\rm cov} A_t &= \delta X^a\, \partial_a A_t - [\delta X^a A_a, A_t] = \delta X^a\, D_a A_t = \delta X^a F_{at}\,, \\
\delta_{\rm cov} A_m &= \delta X^a\, \partial_a A_m - D_m(\delta X^a A_a) = \delta X^a\, F_{am}\,.
\end{aligned} \tag{7}$$

where $F_{mt} = -F_{tm} = D_m A_t$ and $F_{at} = -F_{ta} = D_a A_t$ are the nonabelian field strengths in the static geometry and $F_{am} = \partial_a A_m - \partial_m A_a - [A_a, A_m]$. Assuming the gauge transformation (6) for the connection A_a, we conclude that the combined variations (7) are covariant. This result can now be extended straightforwardly to obtain the expression for $\delta_{\rm cov} A_a$. We give the combined result for A_M,

$$\delta_{\rm cov} A_M = \delta X^b\, \partial_b A_M - D_M(\delta X^b A_b) = \delta X^b\, F_{bM}\,, \tag{8}$$

where F_{MN} denotes the field strength in the extended space,

$$F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N]. \tag{9}$$

As indicated above, the letters $M, N, \ldots$ refer to indices that run both over spatial $(m, n, \ldots)$ and moduli space $(a, b, \ldots)$ indices. Obviously these field strengths satisfy corresponding Bianchi identities. It is straightforward to determine the action of

$\delta_{\rm cov}$ for other covariant quantities. We give two examples,

$$\delta_{\rm cov} D_M\phi = \delta X^a D_a D_M\phi = D_M(\delta_{\rm cov}\phi) - (\delta_{\rm cov}A_M)\phi\,,$$
$$\delta_{\rm cov} F_{MN} = \delta X^a D_a F_{MN} = -2\,\delta X^a\, D_{[M}F_{N]a} = 2D_{[M}(\delta_{\rm cov}A_{N]})\,,$$

where we used the Ricci identity for the commutator of two covariant derivatives and the Bianchi identity for the field strength in the extended base space. The above formulae take the form of a generalized Leibniz rules for $\delta_{\rm cov}$ whose relevance will be explained later. There are analogous formulae for expressions such as $F_{mt} = D_m A_t$ and $D_t\phi = -[A_t,\phi]$, which follow analogously but become non-trivial when performing the geodesic lift. This issue is addressed below.

2.2. Geodesic lift. We now proceed and reintroduce a time dependence through the collective coordinates, $X^a \to X^a(t)$, which implies that both the fields and the parameters of the residual gauge transformations will implicitly depend on time. It is clear that the residual gauge invariance, which is crucial for the moduli space geometry, should be preserved when introducing the time dependence in this way. For many quantities this is trivial and the time dependence of the collective coordinates does not play a role. But as soon as one is dealing with time derivatives, this is no longer so, and this is why the geodesic lift is subtle. In the field theory the derivatives appear already in covariantized form, such as in $D_t\phi = \partial_t\phi - [A_t,\phi]$. In the static case this derivative transforms covariantly under the residual gauge transformations, just because the time derivative vanishes and A_t and ϕ transform covariantly. To ensure that the derivative remains covariant in the geodesic lift, we must modify the connection A_t,

$$A'_t = A_t + \dot X^a A_a\,,$$

so that A'_t transforms under residual gauge transformations as

$$\begin{aligned}\delta A'_t &= \dot X^a D_a\Lambda(\vec x, X) - [A_t, \Lambda(\vec x, X)]\\ &= \mathrm{d}_t\Lambda(\vec x, X) - [A'_t, \Lambda(\vec x, X)] \equiv D'_t\Lambda(\vec x, X)\,.\end{aligned}$$

Here $\mathrm{d}_t = \mathrm{d}/\mathrm{d}t$ denotes the total time derivative that acts on both the explicit time coordinate and on the implicit time dependence contained in the collective coordinates. Hence, for the parameters $\Lambda(\vec x, X)$, we have $\partial_t\Lambda = 0$ and $\mathrm{d}_t\Lambda = \dot X^a\partial_a\Lambda$. Observe that the covariant derivative D_t is defined with the explicit time derivative ∂_t, which vanishes on static quantities, while D'_t contains the total time derivative d_t. On $\phi(\vec x, X(t))$ we thus have $D_t\phi = -[A_t,\phi]$, while

$$D'_t\phi = \dot X^a\partial_a\phi - [A'_t,\phi] = \dot X^a D_a\phi - [A_t,\phi]\,. \tag{10}$$

Obviously $D'_t\phi$ transforms covariantly under the (lifted) gauge transformations with parameters $\Lambda(\vec x, X(t))$. With this modified connection we can thus preserve the gauge invariance under residual gauge transformations in the presence of time derivatives. This is an obvious necessity in view of the fact that the geodesic description deals with the time evolution in the space of static configurations, whose nature depends crucially on modding out the corresponding residual gauge transformations.

Not only the connection A_t is modified in the geodesic lift, but also corresponding covariant field strengths are affected. As usual they follow from taking commutators of the appropriate covariant derivatives. We have already defined F_{MN}; the field strengths F_{tM} are modified and denoted by $F'_{tM} = -[D'_t, D_M]$, so that

$$F'_{tM} = -F'_{Mt} = F_{tM} + \dot{X}^b F_{bM} ,$$

where we used $\partial_M \dot{X}^a = 0$. Observe that the field strengths are only linear in velocities; there are no $\ddot{X}^a$ or $\dot{X}^a \dot{X}^b$ terms, reflecting that the field-strength is linear in first-order (time)derivatives. Of course, when applying higher-order time derivatives, as in the field equations, one does obtain terms proportional to powers of accelerations and derivatives thereof. The curvatures F_{MN} remain as in (9). Also the new curvatures satisfy corresponding Bianchi identities,

$$D'_t F_{MN} + 2\, D_{[M} F'_{N]t} = 0 \,. \tag{11}$$

As before we must determine the covariant variations induced by the variation of the collective coordinates, $X^a(t) \to X^a(t) + \delta X^a(t)$. The covariantization is effected by including a gauge transformation with parameter $-\delta X^a(t)\, A_a(\vec{x}, X(t))$. This parameter depends on time both through the collective coordinates $X^a(t)$ as well as through their variations $\delta X^a(t)$ with arbitrary functions, which implies

$$\begin{aligned}
\delta_{\rm cov} A'_t &= \delta X^a(t)\, \partial_a A'_t + \delta \dot{X}^a A_a - D'_t(\delta X^a(t) A_a) \\
&= \delta X^a\, \partial_a (A_t + \dot{X}^b A_b) - \delta X^a\, D'_t A_a \\
&= \delta X^a\, F'_{at} \,.
\end{aligned} \tag{12}$$

where we used $\mathrm{d}_t \delta X^a = \delta \dot{X}^a$. Observe that this covariant result is in line with the results derived previously in (7). Furthermore, the variation is proportional to δX^a and not to $\delta \dot{X}^a$.

Secondly, we verify that the generalized Leibniz rules still hold on the covariant quantities. For the time derivative of ϕ, we combine the terms proportional to δX^a and $\delta \dot{X}^a$ and find after proper covariantization,

$$\begin{aligned}
\delta_{\rm cov}(D'_t \phi) &= \delta X^a\, D_a (D'_t \phi) + \delta \dot{X}^a D_a \phi \\
&= \delta X^a D'_t D_a \phi + \delta \dot{X}^a D_a \phi - \delta X^a\, [F'_{at}, \phi] \\
&= D'_t (\delta_{\rm cov} \phi) - [\delta_{\rm cov} A'_t, \phi] \,.
\end{aligned}$$

A similar calculation reveals

$$\begin{aligned}
\delta_{\rm cov} F'_{tM} &= \delta X^a\, D_a F'_{tM} + \delta \dot{X}^a\, F_{aM} \\
&= D'_t (\delta X^a\, F_{aM}) - \delta X^a\, D_M F'_{at} \\
&= D'_t (\delta_{\rm cov} A'_M) - D'_M (\delta_{\rm cov} A'_t) \,.
\end{aligned}$$

where we used the Bianchi identity (11). Hence we see that the lifted expressions satisfy the generalized Leibniz rules, provided we identify the covariant variation of A'_t with (12). These generalized Leibniz rules are the same as for the underlying field theory, which is crucial for a proper identification of the covariant field variations $\delta_{\rm cov}$ with the variations associated with the moduli action principle. It is a welcome feature that these variations are proportional to the variation of the collective coordinates and not of the velocities. Furthermore they involve contributions that are at most linear in the velocities.

The fact that we had to include a velocity-dependent correction into the connection A'_t shows that maintaining the residual gauge covariance in the geodesic lift dictates the form that these correction terms should have. To exhibit this feature for general field theories with gauge and diffeomorphism invariance is the central theme of this chapter. Purely based on covariance, we could have separated the two terms in the modified connection, but this would have affected the generalized Leibniz rules formulated above.

2.3. Effective moduli action. To demonstrate the use of the results obtained so far, we consider the action (5) and replace $F_{\mu\nu}$, D_μ and ϕ by the appropriate quantities in the geodesic lift, namely F_{mn} F'_{tm}, $D_m\phi$, $D'_t\phi$ and $\phi(\vec{x}, X(t))$. Dropping the (constant) contribution from the potential, one obtains the following action for the collective coordinates,

$$S[X(t)] = \int \mathrm{d}t \, \left(\frac{1}{2} G_{ab}(X)\, \dot{X}^a \dot{X}^b - J_a(X)\, \dot{X}^a\right), \tag{13}$$

where

$$\begin{aligned} G_{ab}(X) &= -\int \mathrm{d}^d x \; \mathrm{Tr}\Big[F_{am}\, F_{bm} + D_a\phi\, D_b\phi\Big]\,, \\ J_a(X) &= -\int \mathrm{d}^d x \; \mathrm{Tr}\Big[A_t\, (D_m F_{ma} + [\phi, D_a\phi])\Big]\,. \end{aligned} \tag{14}$$

This result is invariant under residual gauge transformations and covariant under moduli space diffeomorphisms (similar results were obtained in [148,149]).

The equations of motion one obtains by varying the action (13) with respect to the collective coordinates, $X^a(t) \to X^a(t) + \delta X^a(t)$, can be written, after partial integrations, as convolutions of the original field equations with the covariant variations of the fields as defined previously. These are the geodesic equations belonging to the particle Lagrangian given by (13) and (14) written in terms of the solutions to the original field equations,

$$\begin{aligned} \mathrm{Tr} \int \mathrm{d}t \; \delta X^a(t) \int \mathrm{d}^d x \; \Bigg\{ &-\frac{\delta_{\rm cov} A'_\mu}{\delta X^a(t)} \Big(D'_\nu F'^{\mu\nu} + \big[D'_\mu\phi, \phi\big]\Big) + \\ &+ \frac{\delta_{\rm cov}\phi}{\delta X^a(t)}\, D'_\mu D'^\mu \phi \Bigg\} = 0\,. \end{aligned} \tag{15}$$

The various expressions for the fields and their derivatives are the ones defined in the geodesic lift, and therefore it is important that the generalized Leibniz rules apply. Observe that there was no need for adopting a gauge condition, but, on the other hand, the result still depends on the newly introduced but so far undetermined connection A_a. This connection can be eliminated in a gauge invariant fashion by use of its equation of motion,

$$\dot{X}^a \dot{X}^b \int \mathrm{d}^d x \ \mathrm{Tr}\Big[\delta A_a \Big(D_m F_{mb} + [\phi, D_b\phi]\Big)\Big] = 0\,, \tag{16}$$

valid for any δA_a. Note that the term $\int J_a \dot{X}^a$ does not contribute to (16), because its variation is just proportional to the (static) A_t-equation of motion. Moreover, its contribution vanishes also in the effective action, once the constraint (16) on A_a is imposed. In fact, upon partial integration and comparison with (8), one observes that (16) is the well-known orthogonality condition [143,160],

$$\int \mathrm{d}^d x \ \mathrm{Tr}\Big[(\delta_{\mathrm{cov}} A_m)\,(\delta_{\mathrm{gauge}} A_m) + (\delta_{\mathrm{cov}}\phi)\,(\delta_{\mathrm{gauge}}\phi)\Big] = 0\,, \tag{17}$$

which ensures that the geodesic motion corresponding to δ_{cov} is orthogonal to the gauge orbits. In this connection observe that the moduli space metric $G_{ab}(X)$ can be written as

$$G_{ab}(X) = -\int \mathrm{d}^d x \ \mathrm{Tr}\left[\frac{\delta_{\mathrm{cov}} A_m}{\delta X^a}\,\frac{\delta_{\mathrm{cov}} A_m}{\delta X^b} + \frac{\delta_{\mathrm{cov}}\phi}{\delta X^a}\,\frac{\delta_{\mathrm{cov}}\phi}{\delta X^b}\right]. \tag{18}$$

Since the constraint (16) is a covariant equation for A_a, we may solve for A_a and reinsert the result into the expression for the metric G_{ab} without affecting gauge invariance. In practice it is quite complicated to obtain explicit expressions for the moduli metric. Notwithstanding these practical difficulties, the above framework can in principle be used for more general Lagrangians, including Lagrangians that contain terms of higher order in the field strengths. The latter case would lead to an effective moduli action that contains terms of higher order in the velocities. While we refrain from discussing this in any detail, it should be noted that this aspect is important for us in view of the fact that we are interested in obtaining results for supergravity Lagrangians that contain interactions quadratic in the Riemann curvature.

2.4. Different spacetime coordinate frames. In many cases one is dealing with a Lorentz invariant field theory, so that the identification of the time coordinate is somewhat arbitrary and depends on the frame that has been adopted. Of course, choosing another Lorentz frame will lead to identical results. Formally we can set up the formulation in such a way that the choice of the Lorentz frame is encoded in a time evolution vector k^μ, which is a constant timelike vector that can be chosen at will. A time coordinate τ can be defined by $k^\mu \partial_\mu = \partial_\tau$ and one also derives that $\mathrm{d}\tau = k^{-2} k_\mu \,\mathrm{d}x^\mu$. Stationary solutions are characterized by their independence on the coordinate τ, so that *e.g.*, $k^\mu \partial_\mu \phi = 0$. Of course, any frame can be brought into standard form by means of a suitable Lorentz transformation, with adapted coordinates for which $k^\mu = (1, \vec{0})$.

Keeping k^μ general one can postpone the definition of the time coordinate until the end. But one can also go further and formulate the previous results in the context of a more general coordinate frame in order to pave the way for the discussion in the next section. Such a coordinate system will no longer constitute a Lorentz frame, although it still describes flat Minkowski spacetime. Both the metric $g_{\mu\nu}$ and the time evolution vector k^μ take a more complicated form and are no longer constant. The time τ remains integrable and k^μ is a Killing vector. This is reflected in the following conditions,

$$k^\mu \partial_\mu = \partial_\tau \,, \quad \mathrm{d}\tau = k^{-2} k_\mu \, \mathrm{d}x^\mu \,, \quad \partial_{[\mu}(k^{-2} k_{\nu]}) = \mathcal{L}_k k = \mathcal{L}_k g = 0 \,. \tag{19}$$

Clearly k^μ remains timelike in the new coordinate system ($k^\mu k_\mu < 0$).

The static fields and the corresponding residual gauge transformations are now defined by the condition,

$$\mathcal{L}_k \phi = \mathcal{L}_k A = \mathcal{L}_k \Lambda = 0 \,. \tag{20}$$

and the moduli space of static configurations can be defined as before. The fields are functions of the spacetime coordinates and the collective coordinates, x^μ and X^a, respectively. The gauge fields and the metric may be regarded as vectors and tensors in this extended space. As explained before we have to introduce extra connection fields A_a, so that altogether we are dealing with gauge fields A_Ω, where the index Ω runs over both spacetime indices $\mu, \nu, \ldots$ and moduli space indices $a, b, \ldots$. In view of the condition (20), the solutions depend on one coordinate less (corresponding to the time coordinate τ).

For the gauge field the residual gauge transformations can now be written as,

$$\delta_\Lambda A_\Omega = D_\Omega \Lambda = \partial_\Omega \Lambda - [A_\Omega, \Lambda] \,,$$

where $D_\Omega \Lambda$ refers to all the covariant derivatives in the extended space. The transformation of the gauge field along the time component, $k^\Omega A_\Omega$, remains covariant in view of the condition (20). Here we extended the timelike Killing vector k^μ to a vector of the extended space, but this extension is trivial because the components k^a are zero: $k^\Omega = (k^\mu, 0)$. At this point we do not introduce a metric for the extended space and the spacetime metric $g_{\mu\nu}$ does not depend on the collective coordinate but is just a given background metric.

A variation of the collective coordinates $X^a \to X^a + \delta X^a$ induces a motion on the configuration space of static solutions, just as before. For the case at hand the covariant variations take the compact form,

$$\delta_{\rm cov} \phi = \delta X^a \, D_a \phi \,, \quad \delta_{\rm cov} A_\Omega = \delta X^a \, F_{a\Omega} \,.$$

Here we used the covariant field strengths,

$$F_{\Omega\Sigma} = \partial_\Omega A_\Sigma - \partial_\Sigma A_\Omega - [A_\Omega, A_\Sigma] \,.$$

The geodesic approximation is effected by promoting the collective coordinates to time-dependent coordinates, $X^a \to X^a(\tau)$. All gauge transformations and diffeomorphisms now carry an implicit time dependence through the collective coordinates.

As before, we introduce total derivatives $\mathrm{d}_\mu = \mathrm{d}/\mathrm{d}x^\mu$, which act both on the explicit spacetime coordinates and on the implicit τ-dependence of the collective coordinates. These derivatives are now generalized to

$$\mathrm{d}_\Omega X^a = k^{-2} k_\Omega \, \dot{X}^a \, ,$$

so that on static solutions, say on $\phi(x, X)$, the derivative d_Ω acts as,

$$\mathrm{d}_\Omega \phi = \partial_\Omega \phi + k^{-2} k_\Omega \, \dot{X}^a \, \partial_a \phi \, , \tag{21}$$

where we assumed $k_\Omega = (g_{\mu\nu} k^\nu, 0)$, so that for $\Omega = a$ we have $\mathrm{d}_a \phi = \partial_a \phi$.

As before, we should modify the connection in order to maintain the gauge covariance in the geodesic lift. In the new coordinate frame, this modification takes the form,

$$A'_\Omega = A_\Omega + k^{-2} k_\Omega \, \dot{X}^a A_a \, .$$

Under residual gauge transformations the new connections A'_Ω transform as

$$\delta A'_\Omega = D_\Omega \Lambda + k^{-2} k_\Omega \, \dot{X}^a \, D_a \Lambda = \mathrm{d}_\Omega \Lambda - [A'_\Omega, \Lambda] \, ,$$

where $k^\Omega D_\Omega \Lambda = -k^\Omega [A_\Omega, \Lambda]$. The lifted covariant derivative of the scalar field reads

$$D'_\Omega \phi = D_\Omega \phi + k^{-2} k_\Omega \, \dot{X}^a D_a \phi \, .$$

In adapted coordinates, this reduces to the expression (10). This derivative transforms covariantly under the residual gauge transformations so that we remain within the context of the space of static configurations, just as before.

What remains is to calculate the field strengths in the context of the geodesic lift. As before we consider the commutator of two covariant derivatives,

$$F'_{\Omega\Sigma} = 2 \, \mathrm{d}_{[\Omega} A'_{\Sigma]} - [A'_\Omega, A'_\Sigma] = F_{\Omega\Sigma} + 2 \, k^{-2} k_{[\Omega} \dot{X}^a F(A)_{a\Sigma]} \, ,$$

where we have made use of

$$\mathrm{d}_\Omega \dot{X}^a = k^{-2} k_\Omega \ddot{X}^a \, .$$

Moreover, we suppressed a term proportional to $\mathrm{d}_{[\Omega} \, \mathrm{d}_{\Sigma]}$ which vanishes in view of the fact that the curl of the vector $k^{-2} k_\Omega$ vanishes according to (19). When nonzero, this term would give rise to torsion in the extended space. This issue will play a role in the following section.

Also the definition of the covariant variation induced by $X^a \to X^a + \delta X^a$ follows from the same reasoning as before,

$$\delta_{\mathrm{cov}} A'_\Omega = \delta X^a \, \partial_a A'_\Omega + k^{-2} k_\Omega \, \delta \dot{X}^a A_a - D'_\Omega (\delta X^a A_a) = \delta X^a \, F'_{a\Omega} \, ,$$

where we used $D'_\Omega(\delta X^a) = k^{-2} k_\Omega \delta \dot{X}^a$. These field strengths satisfy corresponding Bianchi identities,

$$D'_{[\Omega} F'_{\Sigma\Xi]} = 0 \, .$$

Owing to this and the Ricci identity, the generalized Leibniz rules hold in the geodesic lift,

$$\delta_{\rm cov}(D'_\Omega\phi) = D'_\Omega(\delta_{\rm cov}\phi) - [\delta_{\rm cov}A'_\Omega, \phi]\,, \qquad \delta_{\rm cov}F'_{\Omega\Sigma} = 2\,D'_{[\Omega}(\delta_{\rm cov}A'_{\Sigma]})\,.$$

In adapted coordinates, all these results coincide with the results derived in the previous subsection. At this point one may wonder whether the metric $g_{\mu\nu}$ could not also depend on the collective coordinates. In that case the time evolution vector k_Ω would have nonzero components for $\Omega = a$, and the above formulae would follow for that case as well. Nevertheless we would not go beyond the case of a flat metric and the curl of $k^{-2}k_\Omega$ would still vanish.

In the next section we will go beyond flat spacetime and consider a gauge theory in a nontrivial stationary spacetime. The metric is dynamical and determined by, for example, Einstein's equations. This implies that the metric depends non-trivially on the collective coordinates. The residual gauge transformations will then include certain diffeomorphisms of the extended space characterized by parameters ξ^Ω. When performing the geodesic lift, one obtains formulae that are quite analogous to the ones obtained above, except that in that case the appropriate description involves torsion. Here the formulae are motivated by the necessity of preserving the residual gauge and diffeomorphism invariance in the geodesic lift.

3. Gauge theory and spacetime diffeomorphisms

We now couple the gauge theory to a nontrivial dynamical metric $g_{\mu\nu}$. This metric is determined by certain field equations, such as Einstein's equations, but we refrain from being more specific at this point and concentrate first on the geometric features. The corresponding action reads

$$S = \int \mathrm{d}^{d+1}x\,\sqrt{|g|}\,\left(\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}D_\mu\phi\,D^\mu\phi\right] - V(\phi)\right)\,. \tag{22}$$

As before, we assume the existence of a continuous variety of stationary solutions, parameterized by collective coordinates X^a. The stationary solutions are characterized by the existence of a (globally defined) timelike Killing vector field k^μ, which generates diffeomorphisms that leave the solutions invariant,

$$\mathcal{L}_k\phi = \mathcal{L}_kA = \mathcal{L}_kg = 0\,.$$

As alluded to above, we now assume the metric to be dynamical, so that it will depend on both spacetime and collective coordinates. As before, we trivially extend the Killing vector k^μ to a vector over spacetime and moduli space, $k^\Omega = (k^\mu, 0)$, hence $k^a = 0$. The indices $\Omega, \Sigma, \ldots$ run over the indices $\mu, \nu, \ldots$ of spacetime and $a, b, \ldots$ of moduli space. In the previous section, where we treated a flat background, the vector k^Ω was linked to the specification of a global coordinate systems, and the extension of k_μ to $k_\Omega = (k_\mu, 0)$ enabled us to compare the result for the geodesic lift in different but equivalent coordinate systems. In the present case, however, we are dealing with a theory that is diffeomorphism invariant. The stationary solutions are invariant under diffeomorphisms generated by the timelike Killing vector k^Ω. The first objective here is to extend $k_\mu = g_{\mu\nu}k^\nu$ to a covariant vector k_Ω of the extended space. This covariant vector is subsequently needed for the construction of the geodesic lift while preserving the residual diffeomorphism invariance. The resulting expressions are in agreement with the findings in [141], where adapted coordinates were used.

In general, the stationary solutions are subject to a class of residual gauge transformations and residual spacetime diffeomorphisms, which are parameterized by the functions $\Lambda(x, X)$ and $\xi^\mu(x, X)$, respectively, and are subject to the condition,

$$\mathcal{L}_k\xi = 0\,, \quad \mathcal{L}_k\Lambda = 0\,.$$

The residual transformations leave the vector field k^Ω invariant and hence preserve the stationarity of the solution. In addition, we consider reparametrizations of the collective coordinates, encoded in the functions $\delta X^a = \xi^a(X)$. The associated diffeomorphisms may only depend on X^a and not on the spacetime coordinates x^μ. This implies that the components of the extended contravariant and covariant vectors, V^Ω and W_Ω, respectively, transform as follows under the residual diffeomorphisms of the extended space, characterized by $\xi^\Omega = (\xi^\mu(x, X), \xi^a(X))$,

$$\begin{aligned}
\delta V^\mu &= \partial_\nu\xi^\mu\, V^\nu + \partial_b\xi^\mu\, V^b - \xi^\Omega\partial_\Omega V^\mu\,,\\
\delta V^a &= \partial_b\xi^a\, V^b - \xi^\Omega\partial_\Omega V^a\,,\\
\delta W_\mu &= -\partial_\mu\xi^\nu\, W_\nu - \xi^\Omega\partial_\Omega W_\mu\,,\\
\delta W_a &= -\partial_a\xi^\nu\, W_\nu - \partial_a\xi^b\, W_b - \xi^\Omega\partial_\Omega W_a\,.
\end{aligned} \tag{23}$$

The vectors V^Ω and W_Ω can be restricted in a way that is consistent with these residual diffeomorphisms, by setting $V^a = 0$ and/or $W_\mu = 0$. In particular, this shows that the condition on the time evolution field, $k^a = 0$, is preserved by the diffeomorphisms of the extended space, so that we can indeed assume that k^Ω transforms consistently as a contravariant vector. We note that from (23) it follows that generically $k^\mu(x, X)$ depends on both spacetime and collective coordinates. Assuming that the spacetime metric $g_{\mu\nu}$ belongs to a covariant tensor of the extended space, it follows that there must be other tensor components which transform into $g_{\mu\nu}$ under the residual diffeomorphisms. On the other hand, we may introduce a (reference) metric $g_{ab}(X)$ at this point for the collective coordinates (to be distinguished from the dynamical metric that we have to determine later) which depends exclusively on the collective coordinates and not on the spacetime coordinates. The metric $g_{ab}(X)$ constitutes a covariant tensor in the extended space. We will further clarify this issue shortly.

For the sake of clarity let us illustrate the present set-up in adapted coordinates: in these coordinates the coordinate $x^0 = t$ is singled out as the time coordinate, hence $k^\Omega = (1, 0, \ldots, 0)$. This implies that the metric, the gauge fields and the scalar fields do not depend on the coordinate t. Furthermore, the residual diffeomorphisms leave the vector k^Ω invariant. We can parameterize the line element by

$$\mathrm{d}s^2 = g_{tt}(\vec{x}, X)\,(\mathrm{d}t + \sigma_m(\vec{x}, X)\,\mathrm{d}x^m)^2 + h_{mn}(\vec{x}, X)\,\mathrm{d}x^m\mathrm{d}x^n\,, \tag{24}$$

where the functions g_{tt}, σ_m, and h_{mn} are all time independent. Obviously, $k^2 = g_{tt}$ and

$$k^{-2}k_t = 1 \,, \qquad k^{-2}k_m = \sigma_m(\vec{x}, X) \,.$$

However, if k_Ω is to transform as a vector in the extended space, there must also exist components $k_a(\vec{x}, X)$. Before introducing these, we will present a variety of arguments why the various fields have indeed an interpretation in terms of the extended space parametrized by the coordinates (x^μ, X^a).

3.1. Stationary solution space. Our first task is to determine the transformation properties for $A_\Omega = (A_\mu, A_a)$ with respect to the residual gauge transformations and diffeomorphisms. Naturally, we expect that the combined transformation rules for the residual transformations are given by

$$\delta A_\Omega = -\xi^\Sigma \partial_\Sigma A_\Omega - \partial_\Omega \xi^\Sigma \, A_\Sigma + D_\Omega \Lambda \,.$$

The last term is the gauge transformation and was discussed already for the flat background. We note that this term transforms as a vector in the extended space, hence, the same must be true for the A_Ω. Indeed, the consistency of the combined algebra of residual gauge transformations and diffeomorphism implies that A_Ω must itself transform as a covariant vector.

When comparing two neighboring stationary solutions, one must again account for the fact that the actual solutions are representatives of a class of solutions that are degenerate with respect to the residual gauge transformations and diffeomorphisms. This was the argument that originally forced us to introduce the extra connection fields A_a, and in the context of the residual diffeomorphisms it now forces us to introduce a new field $V_a{}^\mu$. When comparing two solutions parametrized by collective coordinates X^a and $X^a + \delta X^a$, one must again allow for residual transformations. For the scalar $\phi(x, X)$, *e.g.*, the stationary covariant variation is given by

$$\delta_{\text{cov}}\phi = \delta X^a (D_a - V_a{}^\mu D_\mu)\phi \,. \tag{25}$$

The second term on the right-hand side is the accompanying spacetime diffeomorphism with parameter $\delta X^a V_a{}^\mu$. We have covariantized the derivatives with respect to gauge transformations. In order for the second term to covariantize the variation with respect to the residual spacetime diffeomorphisms the field $V_a{}^\mu$ must transform in the following inhomogeneous way,

$$\delta_\xi V_a{}^\mu = -\partial_a \xi^\mu + V_a{}^\nu \partial_\nu \xi^\mu - \partial_a \xi^b V_b{}^\mu - \xi^\Omega \, \partial_\Omega V_a{}^\mu \,. \tag{26}$$

The field $V_a{}^\mu$ has a natural interpretation in terms of a universal vielbein. Adopting the notation that underlined indices refer to the tangent space we introduce a vielbein of the extended space, which we restrict to a block-triangular form,

$$E_{\underline{\Omega}}{}^\Sigma(x, X) = \begin{pmatrix} e_{\underline{\mu}}{}^\nu & 0 \\ e_{\underline{a}}{}^\nu & e_{\underline{a}}{}^b \end{pmatrix} .$$

As argued earlier, the restriction to a block-triangular form is consistent with residual diffeomorphisms. Here, $e_{\underline{\mu}}{}^{\nu}(x, X)$ denotes the spacetime vielbein, and $e_{\underline{a}}{}^{b}(X)$ is a spacetime-independent reference frame for moduli space. With the exception of $e_{\underline{a}}{}^{b}(X)$ all components of the vielbein depend on both x^{μ} and X^{a} and all are subject to $\mathcal{L}_k E_{\underline{\Omega}}{}^{\Sigma} = 0$. The standard transformation rules for the vielbein under the residual diffeomorphisms read,

$$\delta_\xi E_{\underline{\Omega}}{}^{\Lambda} = -\xi^{\Sigma}\,\partial_\Sigma E_{\underline{\Omega}}{}^{\Lambda} + E_{\underline{\Omega}}{}^{\Sigma}\,\partial_\Sigma \xi^{\Lambda}\,.$$

Comparing this with (26) leads to the following identification,

$$e_{\underline{a}}{}^{\mu} = -e_{\underline{a}}{}^{b}\, V_b{}^{\mu}\,. \tag{27}$$

Therefore we conclude that the geometry of the extended space plays a natural role.

It is straightforward to construct the inverse universal vielbein, which again takes a block-triangular form,

$$E_{\Omega}{}^{\underline{\Sigma}} = \begin{pmatrix} e_{\mu}{}^{\underline{\nu}} & 0 \\ e_{a}{}^{\underline{\nu}} & e_{a}{}^{\underline{b}} \end{pmatrix}, \qquad e_{a}{}^{\underline{\nu}} = -e_{a}{}^{\underline{b}}\, e_{\underline{b}}{}^{\rho}\, e_{\rho}{}^{\underline{\nu}} = V_a{}^{\mu} e_{\mu}{}^{\underline{\nu}}\,.$$

where the matrices $e_{\mu}{}^{\underline{\nu}}(x, X)$ and $e_{a}{}^{\underline{b}}(X)$ are the inverse of $e_{\underline{\mu}}{}^{\nu}$ and $e_{\underline{a}}{}^{b}$, respectively.

The tangent-space group acting on these universal vielbein components decomposes into two groups in order to preserve the block-triangular form. The local Lorentz group acts on the indices $\underline{\mu}$ in the usual way, while the tangent-space group of the moduli space is simply the orthogonal group. This implies that there are two independent invariant tensors in the tangent space, which we denote by $\eta_{\underline{\mu}\,\underline{\nu}}$ and $\eta_{\underline{a}\,\underline{b}}$. From these various covariant symmetric tensors can be formed by contraction with components of the universal vielbein,

$$\begin{aligned}
\widetilde{g}_{\Omega\Lambda} &= \eta_{\underline{\mu}\,\underline{\nu}}\, e_{\Omega}{}^{\underline{\mu}} e_{\Lambda}{}^{\underline{\nu}} = \begin{pmatrix} g_{\mu\nu} & g_{\mu\rho} V_b{}^{\rho} \\ V_a{}^{\rho} g_{\rho\nu} & V_a{}^{\rho} V_b{}^{\sigma} g_{\rho\sigma} \end{pmatrix}, \\
\widehat{g}_{\Omega\Lambda} &= \eta_{\underline{a}\,\underline{b}}\, e_{\Omega}{}^{\underline{a}} e_{\Lambda}{}^{\underline{b}} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ab} \end{pmatrix}, \\
\widetilde{g}^{\Omega\Lambda} &= \eta^{\underline{a}\,\underline{b}}\, e_{\underline{a}}{}^{\Omega}\, e_{\underline{b}}{}^{\Lambda} = \begin{pmatrix} g^{cd} V_c{}^{\mu} V_d{}^{\nu} & -V_c{}^{\mu} g^{cb} \\ -g^{ac} V_c{}^{\nu} & g^{ab} \end{pmatrix}, \\
\widehat{g}^{\Omega\Lambda} &= \eta^{\underline{\mu}\,\underline{\nu}}\, e_{\underline{\mu}}{}^{\Omega}\, e_{\underline{\nu}}{}^{\Lambda} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{28}$$

The tensors $\widetilde{g}_{\Omega\Lambda}$ and $\widehat{g}_{\Omega\Lambda}$ are the covariant extensions of the spacetime metric $g_{\mu\nu} = e_{\mu\underline{\sigma}}\, e_{\nu}{}^{\underline{\sigma}}$ and of the reference frame $g_{ab} = e_{a\underline{c}}\, e_{b}{}^{\underline{c}}$, respectively. Similarly, the tensors

$\widehat{g}^{\Omega\Lambda}$ and $\widetilde{g}^{\Omega\Lambda}$ are the extensions of the inverse spacetime metric $g^{\mu\nu}$ and the inverse frame g^{ab}, respectively. This is in line with the fact that the restrictions $V^{ab\cdots} = 0$ and/or $W_{\mu\nu\cdots} = 0$ on higher-rank tensor fields are covariant with respect to residual diffeomorphisms, as was argued previously for the case of contravariant and covariant vector fields. The matrices $\widetilde{g}_{\Omega\Lambda}$ and $\widehat{g}_{\Omega\Lambda}$ are not invertible. From above expressions it follows directly that

$$\widetilde{g}_{\Omega\Sigma}\,\widetilde{g}^{\,\Sigma\Lambda} = \widehat{g}_{\Omega\Sigma}\,\widehat{g}^{\,\Sigma\Lambda} = 0\,,$$

$$\widetilde{g}_{\Omega\Sigma}\,\widehat{g}^{\,\Sigma\Lambda} = \begin{pmatrix} \delta_\mu{}^\nu & 0 \\ 0 & 0 \end{pmatrix}\,, \quad \widehat{g}_{\Omega\Sigma}\,\widetilde{g}^{\,\Sigma\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_a{}^b \end{pmatrix}\,.$$

As a result of the product structure of the tangent space there is no unique choice for the covariant metric tensor of the extended space associated to the universal vielbein. On the other hand, the extension of the normalized vector $k^{-2}g_{\mu\nu}k^\nu$ to the extended space is unique and given by

$$k^{-2}k_\Omega = k^{-2}(k_\mu, V_a{}^\nu k_\nu)\,, \quad k_\mu = g_{\mu\nu}k^\nu\,, \quad k^2 = k_\mu k^\mu\,.$$

As opposed to the flat spacetime background, the curl of $k^{-2}k_\Omega$ does not vanish. An important consequence of the triangular form of the vielbein is that for contravariant vectors satisfying $V^a = 0$ and covariant vectors subject to $W_\mu = 0$ one finds,

$$V_{\underline{a}} = V^{\underline{a}} = 0\,, \quad W_{\underline{\mu}} = W^{\underline{\mu}} = 0\,.$$

Note in particular that $k_{\underline{a}} = 0$. For completeness, let us give the expressions for $k^{-2}k_\Omega$ in adapted coordinates (24),

$$k^{-2}k_\Omega = (1, \sigma_m, \sigma_a)\,, \quad \sigma_a = V_a{}^t + V_a{}^n\sigma_n\,.$$

We now define the covariant variation of arbitrary fields. In (25) the covariant variation of the scalar field $\phi(x, X)$ was given. Using the identification (27) the covariant variation can be written as

$$\delta_{\rm cov}\phi = \delta X^\Omega \partial_\Omega\phi - [\delta X^\Omega A_\Omega, \phi]\,,$$

where,

$$\delta X^\Omega = \delta X^b e_b{}^{\underline{a}} e_{\underline{a}}{}^\Omega = (-\delta X^a V_a{}^\mu, \delta X^a)\,.$$

It follows from (26) that δX^Ω is the covariant extension of the moduli space vector δX^a. It satisfies

$$\delta X^\Omega k_\Omega = 0\,.$$

The covariant variation of the scalar field consists of a diffeomorphism with parameter $-\delta X^\Omega$, accompanying gauge transformation with parameter $-\delta X^\Omega A_\Omega$. This definition of the covariant variation is applicable for general tensor fields,

$$\delta_{\rm cov}(\delta X) = -\mathcal{L}_{\delta X} - \delta_{\rm gauge}(\delta X \cdot A)\,. \tag{29}$$

This is a coordinate and background independent definition of the covariant variation. In particular, it does not involve an extended metric tensor (or affine connection). This is essential when passing to the geodesic lift. Using this definition, the covariant variation of the gauge field A_Ω reads

$$\begin{aligned}\delta_{\text{cov}} A_\Omega &= \delta X^\Lambda \partial_\Lambda A_\Omega + \partial_\Omega \left(\delta X^\Lambda\right) A_\Lambda - D_\Omega \left(\delta X^\Lambda A_\Lambda\right) \\ &= \delta X^\Lambda F_{\Lambda\Omega} \end{aligned} \tag{30}$$

where the field strength is defined as for the flat background, $F_{\Omega\Lambda} = 2\partial_{[\Omega} A_{\Lambda]} - [A_\Omega, A_\Lambda]$. It satisfies the Bianchi identities $D_{[\Omega} F_{\Sigma\Xi]} = 0$. Definition (29) has the desired property that it implies generalized Leibniz rules,

$$\begin{aligned}\delta_{\text{cov}} D_\Omega \phi &= D_\Omega \left(\delta_{\text{cov}} \phi\right) - [\delta_{\text{cov}} A_\Omega, \phi]\,, \\ \delta_{\text{cov}} F_{\Omega\Lambda} &= 2\, D_{[\Omega} \left(\delta_{\text{cov}} A_{\Lambda]}\right)\,,\end{aligned}$$

where we used the Ricci identity for the commutator of two covariant derivatives and the Bianchi identity for the field strengths.

3.2. Geodesic lift. As in section 2.4, the dynamics in the geodesic approximation is characterized by the timelike Killing vector field k^μ and a time parameter τ defined by $k^\mu \partial_\mu = \partial_\tau$. The collective coordinates are promoted to time dependent coordinates, $X^a \to X^a(\tau)$. We note that the moduli space vector $\dot{X}^a$ of the geodesic lift is extended to the contravariant vector $\dot{X}^\Omega = (-\dot{X}^b V_b{}^\mu, \dot{X}^a)$ on the extended space. On the other hand, since partial derivatives ∂_Ω act only on an explicit coordinate dependence, $D_\Omega \phi = \partial_\Omega \phi - [A_\Omega, \phi]$ remains a covariant vector with respect to the implicitly time-dependent residual transformations in the geodesic lift. The geodesic lift for the covariant derivative of the scalar field therefore reads

$$D'_\Omega \phi = D_\Omega \phi + k^{-2} k_\Omega\, \dot{X}^a (D_a - V_a{}^\mu D_\mu)\phi = D_\Omega \phi + k^{-2} k_\Omega\, \dot{X}^\Lambda D_\Lambda \phi\,.$$

This is a direct generalization of the result found for the flat spacetime background. As stressed in the beginning of this section, the vector $k^{-2} k_\Omega$ is a covariant vector on the extended space. Therefore, the $\dot{X}$-dependent terms multiplying this vector must transform as a scalar under the residual invariances, as is indeed the case for the contraction $\dot{X}^\Lambda D_\Lambda \phi$. We can project this covariant derivative onto directions along the vector field k^μ,

$$k \cdot D'\phi = k \cdot D\phi + \dot{X}^\Lambda D_\Lambda \phi\,.$$

The first term on the right-hand side contains only the connection piece, $k \cdot D\phi = -[k \cdot A, \phi]$. The second term accounts for the time dependence induced by the collective coordinates in the geodesic lift. The covariant derivatives projected onto directions along to the spatial hypersurface orthogonal to k^μ are unaltered in the geodesic lift. This is the same result as the one found in section 2.4.

Since the covariant derivatives contain the gauge connection, the geodesic lift of the gauge field is given by

$$A'_\Omega = A_\Omega + k^{-2}k_\Omega \dot{X}^a(A_a - V_a{}^\mu A_\mu) = A_\Omega + k^{-2}k_\Omega \dot{X}^\Sigma A_\Sigma\,. \tag{31}$$

Under residual gauge transformations parameterized by $\Lambda(x, X(\tau))$ the geodesically lifted connection transforms as

$$\delta_\Lambda A'_\Omega = D_\Omega\Lambda + k^{-2}k_\Omega \dot{X}^\Sigma D_\Sigma\Lambda = \mathrm{d}_\Omega\Lambda' - [A'_\Omega, \Lambda]\,. \tag{32}$$

This last equation follows directly from the expression (31). Here we have introduced the derivatives d_Ω along same lines as in the previous section. On functions with implicit τ-dependence only, the derivatives d_Ω are given by

$$\mathrm{d}_\Omega\phi(x, X(\tau)) = \Big[\partial_\Omega + k^{-2}k_\Omega \dot{X}^\Lambda\partial_\Lambda\Big]\phi(x, X(\tau))\,.$$

It is important to realize that, as opposed to (21), this derivative does not correspond to a (total) coordinate derivative. On functions that depend exclusively on the time parameter, such as the velocities $\dot{X}^a(\tau)$, the total derivatives are defined by

$$\mathrm{d}_\Omega\dot{X}^a(\tau) = k^{-2}k_\Omega\,\ddot{X}^a(\tau)\,.$$

While suggestive, the derivation of this last equation is subtle and the equation, as we have written it, is not covariant. For the present discussion it is sufficient to remark that it allows for the construction of covariant field strengths. The commutator of two derivatives, acting on any function $\phi(x, X^a(\tau))$ with implicit time dependence, ceases to vanish and is given by

$$[\mathrm{d}_\Omega, \mathrm{d}_\Lambda]\,\phi(x, X(\tau)) = t_{\Omega\Lambda}{}^\Sigma\,\partial_\Sigma\phi(x, X(\tau))\,, \tag{33}$$

where we defined the covariant tensor

$$t_{\Omega\Lambda}{}^\Sigma = 2\mathrm{d}_{[\Omega}\left(k^{-2}k_{\Lambda]}\dot{X}^\Sigma\right) = 2\dot{X}^a\mathrm{d}_{[\Omega}\left(k^{-2}k_{\Lambda]}e_a{}^{\underline{c}}e_{\underline{c}}{}^\Sigma\right)\,.$$

For a flat spacetime the tensor $t_{\Omega\Lambda}{}^\Sigma$ vanishes as $k^{-2}k_\Omega$ is related to a coordinate transformation and is therefore curl-free (19). This is no longer the case for a diffeomorphism invariant theory and one must deal with a geometry that involves torsion. This has various consequences: first, special care must be taken when identifying covariant field strengths in the geodesic lift and checking their Bianchi identities; second, defining the covariant variation for tensors in the geodesic lift becomes more intricate. As remarked previously, these two issues are related. For instance, the covariant variation of the gauge field in the geodesic lift is related to the geodesic lift of the field strength in view of (30). Furthermore, it is necessary to enforce generalized Leibniz rules for the covariant variation in the geodesic lift, which make usage of the Ricci and Bianchi identities.

In order to develop the geometry of the geodesic lift systematically, one must, in principle, discuss the role of an affine connection $\Gamma_{\Omega\Lambda}{}^\Sigma$. So far, this complication was avoided by focusing only on the field strengths and not on the covariant derivatives of tensors. Moreover, the definition of the covariant variation involves a Lie derivative and is hence independent of an affine connection. For the present discussion it is sufficient to realize that the affine connection must contain non-vanishing torsion components in the geodesic lift. There may, of course be further modifications to the affine connection in the geodesic lift. These will become relevant when discussing the gravity sector of the theory.

Let us identify the gauge field strength by considering the Ricci identity for covariant derivatives acting on the scalar field in the geodesic lift,

$$\big[\mathcal{D}'_\Omega, \mathcal{D}'_\Lambda\big]\,\phi = \big[\mathrm{d}_\Omega, \mathrm{d}_\Lambda\big]\phi - \Big[\big(2\mathrm{d}_{[\Omega}A'_{\Lambda]} - \big[A'_\Omega, A'_\Lambda\big]\big), \phi\Big] - 2\,\Gamma'_{[\Omega\Lambda]}{}^{\Sigma} D'_\Sigma\phi\,. \quad (34)$$

Here we use $\mathcal{D}'$ to denote the fully covariant derivative in the geodesic lift which contains both the gauge and the affine connection. Comparison with (33) reveals that the antisymmetric part $T'_{\Omega\Lambda}{}^{\Sigma}$ of the affine connection $\Gamma'_{\Omega\Lambda}{}^{\Sigma}$ in the geodesic lift reads

$$T'_{\Omega\Lambda}{}^{\Sigma} = 2\,\Gamma'_{[\Omega\Lambda]}{}^{\Sigma} = t_{\Omega\Lambda}{}^{\Pi}\big(\delta_\Pi{}^{\Sigma} - k^{-2}k_\Pi\dot{X}^{\Sigma}\big)\,, \quad (35)$$

such that the commutator $[\mathrm{d}_\Omega, \mathrm{d}_\Lambda]\phi$ is absorbed. Here, and in the following, we assume that the affine connection $\Gamma_{\Omega\Lambda}{}^{\Sigma}$ of the stationary geometry is torsion-free, but this is not essential. The matrix $(\delta_\Pi{}^{\Sigma} - k^{-2}k_\Pi\dot{X}^{\Sigma})$ on the right-hand side of above equation is chosen such that the derivative $D'_\Sigma\phi$ in the torsion term of (34) is converted to a covariant derivative $D_\Pi\phi$ of the stationary geometry in view of (33). Note that this result relies crucially on the property

$$\dot{X}^\Omega k_\Omega = 0\,.$$

Consequently, the field strength is modified according to

$$\begin{aligned} F'_{\Omega\Lambda} &= 2\mathrm{d}_{[\Omega}A'_{\Lambda]} - [A'_\Omega, A'_\Lambda] - T'_{\Omega\Lambda}{}^{\Sigma}A'_\Sigma \\ &= 2\mathrm{d}_{[\Omega}A'_{\Lambda]} - [A'_\Omega, A'_\Lambda] - t_{\Omega\Lambda}{}^{\Sigma}A_\Sigma\,, \end{aligned} \quad (36)$$

such that the Ricci identity for the covariant derivatives on the scalar fields indeed takes the form,

$$\big[\mathcal{D}'_\Omega, \mathcal{D}'_\Lambda\big]\,\phi = -[F'_{\Omega\Lambda}, \phi]\,. \quad (37)$$

The field strength (36), with the term $t_{\Omega\Lambda}{}^{\Sigma}A_\Sigma$, is in fact gauge covariant with respect to the residual gauge transformations (32). This is made more explicit by rewriting the expression for the field strength, using the explicit expression for the derivative d_Ω and the tensor $t_{\Omega\Lambda}{}^{\Sigma}$,

$$F'_{\Omega\Lambda} = F_{\Omega\Lambda} - 2k^{-2}k_{[\Omega}\dot{X}^{\Sigma}F_{\Lambda]\Sigma}\,.$$

This is the generalization of the result that was derived in [141] by requiring gauge and diffeomorphism covariance. Furthermore, the field strength $F'_{\Omega\Lambda}$ satisfies the Bianchi identities

$$\mathcal{D}'_{[\Omega}F'_{\Lambda\Sigma]} = D'_{[\Omega}F'_{\Lambda\Sigma]} + T'_{[\Omega\Lambda}{}^{\Pi}F'_{\Sigma]\Pi} = 0\,. \quad (38)$$

We note in passing that the various result of the geodesic lift can be related to the stationary geometry by

$$\mathrm{d}_\Omega\phi = \Phi_\Omega{}^\Lambda\,\partial_\Lambda\phi\,, \quad A'_\Omega = \Phi_\Omega{}^\Lambda\,A_\Lambda\,, \quad F'_{\Omega\Lambda} = \Phi_\Omega{}^\Sigma\,\Phi_\Lambda{}^\Pi\,F_{\Sigma\Pi}\,, \tag{39}$$

where

$$\Phi_\Omega{}^\Lambda = \left(\delta_\Omega{}^\Lambda + k^{-2}k_\Omega\dot{X}^\Lambda\right)\,, \quad (\Phi^{-1})_\Lambda{}^\Omega = \Phi^\Omega{}_\Lambda = \left(\delta^\Omega{}_\Lambda - k^{-2}k_\Lambda\dot{X}^\Omega\right)\,.$$

The torsion is related to the curl of this matrix

$$t_{\Omega\Lambda}{}^\Sigma = 2\,\mathrm{d}_{[\Omega}\Phi_{\Lambda]}{}^\Sigma\,.$$

Using this relation the Bianchi identity for the field strength in the geodesic lift can be written in terms of the one of the stationary geometry,

$$D'_{[\Omega}F'_{\Lambda\Sigma]} + T'_{[\Omega\Lambda}{}^\Pi F'_{\Sigma]\Pi} = \Phi_\Omega{}^\Pi\,\Phi_\Lambda{}^\Xi\,\Phi_\Sigma{}^\Theta\,D_{[\Pi}F_{\Xi\Theta]} = 0\,. \tag{40}$$

It is amusing to realize that the torsion (35) of the affine connection $\Gamma'_{\Omega\Lambda}{}^\Sigma$ is obtained by considering the antisymmetric part of the expression

$$(\Gamma_{\Omega\Lambda}{}^\Sigma)' = \Phi_\Pi{}^\Sigma\left(\mathrm{d}_\Omega\Phi_\Lambda{}^\Pi + \Phi_\Omega{}^\Xi\Phi_\Lambda{}^\Theta\Gamma_{\Xi\Theta}{}^\Pi\right)\,, \tag{41}$$

where $\Gamma_{\Lambda\Sigma}{}^\Pi$ is the torsion-free affine connection of the stationary geometry. This expression takes the form of a general basis rotation on tangent space (which is to be distinguished from a general coordinate transformation). Above formulae might indicate that the geodesic lift can be related to an uniform velocity-dependent tangent-frame rotation encoded in the matrix $\Phi_\Omega{}^\Lambda$. Obviously, the consequences of this interpretation deserve further study.

3.3. Covariant variation in the geodesic lift. In the following we discuss the covariant variation of various fields in the geodesic lift. Clearly, the covariant variation of the scalar remains unaltered,

$$\delta_{\mathrm{cov}}\phi = \delta X^\Omega D_\Omega\phi = \delta X^\Omega D'_\Omega\phi\,,$$

where we have used the fact that δX^Ω is unchanged in the geodesic lift and that $\delta X^\Omega k_\Omega = 0$. This expression is in accord with the prescription (29) for the covariant variation. Deriving the covariant variation of general tensor fields, such as the gauge connection A'_Ω, is more involved. In the previous section we have seen that the differential calculus in the geodesic lift is somewhat intricate and that consequently the definition of the field strength, for instance, depends on the torsion-component of an affine connection. One might therefore question how the definition of the covariant variation (29), which involves the Lie derivative and hence is independent of the affine connection, can be reconciled with the definition of the field strength in view of (30). The reason is that the Lie derivative, as well, contains certain torsion-like modifications if one works with derivatives that do not commute. Recall that the Lie derivatives of higher-rank tensors are found inductively. The starting point is the commutator of two vector fields $V = V^\Omega \mathrm{d}_\Omega$ and $W = W^\Omega \mathrm{d}_\Omega$, acting on functions, such as $\phi(x, X(\tau))$,

$$\begin{aligned}\big[V, W\big]\phi &= V^\Omega \mathrm{d}_\Omega (W^\Lambda \mathrm{d}_\Lambda \phi) - (V \leftrightarrow W)\\ &= \Big[V^\Lambda \mathrm{d}_\Lambda W^\Sigma - W^\Lambda \mathrm{d}_\Lambda V^\Sigma + V^\Omega W^\Lambda t_{\Omega\Lambda}{}^\Pi \big(\delta_\Pi{}^\Sigma - k^{-2} k_\Pi \dot{X}^\Sigma\big)\Big]\mathrm{d}_\Sigma \phi\,.\end{aligned}$$

The bracket on the last line defines the components of the Lie derivative $\mathcal{L}_W V$ when working with the derivatives d_Ω. The last term in this bracket is exactly the torsion-like contribution (35) which entered the definition of the field strength. It arises from the commutator of the derivatives d_Ω. The presence of these torsion-like contributions pertains when passing to the Lie derivative of higher-rank tensors. The details are not necessary here and we only give the results. The covariant variation of the gauge connection is given by

$$\begin{aligned}\delta_{\mathrm{cov}} A'_\Omega &= \delta X^\Lambda \mathrm{d}_\Lambda A'_\Omega + \mathrm{d}_\Omega \left(\delta X^\Lambda\right) A'_\Lambda - \delta X^\Lambda t_{\Lambda\Omega}{}^\Pi \big(\delta_\Pi{}^\Sigma - k^{-2} k_\Pi \dot{X}^\Sigma\big) A'_\Sigma\\ &\quad - D'_\Omega \left(\delta X^\Lambda A'_\Lambda\right)\\ &= \delta X^\Lambda \left(\mathrm{d}_\Lambda A'_\Omega - D'_\Omega A'_\Lambda - t_{\Lambda\Omega}{}^\Sigma A_\Sigma\right) = \delta X^\Lambda F'_{\Lambda\Omega}\,,\end{aligned}$$

where we used $\dot{X}^\Omega k_\Omega = \delta X^\Omega k_\Omega = 0$ and the definition (36). The last term on the first line arises from the modifications to the Lie derivative in the presence of non-commuting derivatives. The geodesic lift therefore preserves the basic relations (30) of the stationary solution space. The same modifications of the Lie derivative is relevant to verify the generalized Leibniz rule

$$\begin{aligned}\delta_{\mathrm{cov}} D'_\Omega \phi &= \delta X^\Lambda D'_\Lambda D'_\Omega \phi + \mathrm{d}_\Omega (\delta X^\Lambda) D'_\Lambda \phi\\ &\quad - \delta X^\Lambda t_{\Lambda\Omega}{}^\Pi \big(\delta_\Pi{}^\Sigma - k^{-2} k_\Pi \dot{X}^\Sigma\big) D'_\Sigma \phi\\ &= D'_\Omega (\delta X^\Lambda D'_\Lambda \phi) - [\delta X^\Lambda F'_{\Lambda\Omega}, \phi]\\ &= D'_\Omega (\delta_{\mathrm{cov}} \phi) - [\delta_{\mathrm{cov}} A'_\Omega, \phi]\,,\end{aligned}$$

where we used the Ricci identity (37) and (35). Finally, for the field strength we find

$$\begin{aligned}\delta_{\mathrm{cov}} F'_{\Omega\Lambda} &= \delta X^\Sigma D'_\Sigma F'_{\Omega\Lambda} - 2\mathrm{d}_{[\Omega} \left(\delta X^\Sigma\right) F'_{\Lambda]\Sigma}\\ &\quad + 2\delta X^\Sigma\, t_{\Sigma[\Omega}{}^\Pi \big(\delta_\Pi{}^\Xi - k^{-2} k_\Pi \dot{X}^\Xi\big) F'_{\Lambda]\Xi}\\ &= -2 D'_{[\Omega} F'_{\Lambda]\Sigma} \delta X^\Sigma - T'_{\Omega\Lambda}{}^\Sigma \delta X^\Pi F'_{\Pi\Sigma}\\ &= 2 D'_{[\Omega} \delta_{\mathrm{cov}} A'_{\Lambda]} - T'_{\Omega\Lambda}{}^\Sigma \delta_{\mathrm{cov}} A'_\Sigma\,.\end{aligned}$$

We used the Bianchi identities (38) and (35) for going from the first to the second line. The last line is the appropriate Leibniz rule for geometries that involve torsion. We discussed the relevance of these relations for the action principle in the previous section.

One easily verifies the these expressions are related to the corresponding relations of the stationary geometry,

$$\delta_{\rm cov} D'_\Omega\phi = \Phi_\Omega{}^\Lambda \big(\delta_{\rm cov} D_\Lambda\phi\big)\,, \quad \delta_{\rm cov} F'_{\Omega\Lambda} = \Phi_\Omega{}^\Sigma \Phi_\Lambda{}^\Pi \big(\delta_{\rm cov} F_{\Sigma\Pi}\big)\,.$$

3.4. Effective moduli action. The derivation of the effective action proceeds along the same lines as in section 2. In the action (22) we substitute the expressions $F'_{\mu\nu}$ and $D'_\mu\phi$ we found for the field strength and the covariant derivative in the geodesic lift, respectively. These expressions are contracted with the inverse metric $g^{\mu\nu}$. The contributions that do not involve any $\dot X$-dependence are equal to the constant stationary potential and are therefore dropped. In addition, there are terms that are linear and utmost quadratic in $\dot X$,

$$S[X(\tau)] = \int \mathrm{d}\tau\, \left(\frac{1}{2}\, G_{ab}\dot X^a \dot X^b + J_a \dot X^a\right)\,. \tag{42}$$

The moduli space metric G_{ab} and the vector J_a are read off directly from expanding

$$\frac{1}{4}\, F'_{\mu\nu}\, g^{\mu\rho} g^{\nu\sigma}\, F'_{\rho\sigma} + \frac{1}{2}\, D'_\mu\phi\, g^{\mu\nu}\, D'_\nu\phi\,,$$

and extracting the first-order and second-order terms in $\dot X^a$. The resulting expressions are direct generalizations of the results found in section 2,

$$\begin{aligned} G_{ab} = -\int \mathrm{d}V\, \mathrm{Tr}\Big[&(F_{\mu a} - V_a{}^\rho F_{\mu\rho})\, h^{\mu\nu}\, (F_{\nu b} - V_b{}^\sigma F_{\nu\sigma}) \\ &+ (D_a\phi - V_a{}^\mu D_\mu\phi)(D_b\phi - V_b{}^\nu D_\nu\phi)\Big]\,, \\ J_a = -\int \mathrm{d}V\ \mathrm{Tr}\Big[&k^\mu F_{\mu\nu}\, h^{\nu\rho}\, (F_{\rho a} - V_a{}^\sigma F_{\rho\sigma}) + (k\cdot D\phi)(D_a\phi - V_a{}^\mu D_\mu\phi)\Big]\,. \end{aligned}$$

Here, $\mathrm{d}V$ is the d-dimensional volume form of the induced metric $h_{\mu\nu} = g_{\mu\nu} - k^{-2}k_\mu k_\nu$ on the spacelike hypersurface, $\mathrm{d}V = \frac{\sqrt{h}}{|k|d!} k^\sigma \varepsilon_{\sigma\mu_1\cdots\mu_d} \mathrm{d}x^{\mu_1}\cdots \mathrm{d}x^{\mu_d}$, where $\sqrt{h}$ is the determinant of the non-trivial d-dimensional submatrix of $h_{\mu\nu}$. The volume form is invariant under residual diffeomorphisms. In above expressions we used the fact that k^2 is negative definite. The matrix $h^{\mu\nu} = g^{\mu\nu} - k^{-2}k^\mu k^\nu$ is the hypersurface inverse of $h_{\mu\nu}$. In adapted coordinates (24) the non-vanishing parts of the matrices $h_{\mu\nu}$ and $h^{\mu\nu}$ are h_{mn} and h^{mn}, respectively. The expressions for the metric G_{ab} and the vector J_a are covariant with respect to the extended residual diffeomorphisms and gauge transformations. This follows, as discussed in section 3.1, from the fact that the hypersurface inverse can be extended trivially to the covariant tensor

$$h^{\Omega\Lambda} = \begin{pmatrix} h^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}\,, \quad k_\Omega h^{\Omega\Lambda} = k^\Omega h_{\Omega\Lambda} = 0\,.$$

Similarly, terms such as $D_a\phi - V_a{}^\mu D_\mu\phi$ are components of a covariant vector defined on the extended space, $(0,\, D_a\phi - V_a{}^\mu D_\mu\phi)$.

We next turn to the constraints that follow from setting the variation of the action with respect to A_a to zero. To a large extent, the discussion of the constraints carry over from the case of a flat spacetime background. Varying the action with respect to A_a yields the constraint

$$\dot{X}^a \dot{X}^b \int \mathrm{d}V \ \mathrm{Tr}\Big[\, \delta A_a \Big(\mathcal{D}_\mu\{h^{\mu\nu}(F_{b\nu} - V_b{}^\sigma F_{\sigma\nu})\} + \big[\phi, (D_b\phi - V_b{}^\sigma D_\sigma\phi)\big]\Big)\Big] = 0\,.$$

The derivative $\mathcal{D}_\mu$ refers to the stationary geometry and contains both the affine and the gauge connection. In above formulae we assumed that the affine connection is the Christoffel connection, and we have integrated partially along the directions of the spatial hypersurface. Just as for flat spacetime backgrounds, the variation of the term $J_a\dot{X}^a$ is proportional to the stationary equations of motion. Moreover, this term vanishes in the action once the constraint is imposed. The constraint can be written in terms of the generalized orthogonality condition

$$\int \mathrm{d}V \ \mathrm{Tr}\Big[(\delta_{\text{gauge}}A_\mu)h^{\mu\nu}(\delta_{\text{cov}}A_\nu) + (\delta_{\text{gauge}}\phi)(\delta_{\text{cov}}\phi)\Big] = 0\,.$$

Likewise, the moduli metric takes the form

$$G_{ab}(X) = -\int \mathrm{d}V \ \mathrm{Tr}\left[\frac{\delta_{\text{cov}}A_\mu}{\delta X^a}\, h^{\mu\nu}\, \frac{\delta_{\text{cov}}A_\nu}{\delta X^b} + \frac{\delta_{\text{cov}}\phi}{\delta X^a}\, \frac{\delta_{\text{cov}}\phi}{\delta X^b}\right].$$

These formulae are the generalization of the expressions (17) and (18) to the case of a curved spacetime background. This demonstrates that the methodology for dealing with gauge invariance can be carried over to a treatment of theories with diffeomorphism invariance. In practice, solving the constraint equations for the connection components A_a in terms of the stationary solution is quite involved.

It is clear that due to the covariance of the underlying theory, the variation of the action (42) with respect to the collective coordinates, $X^a(\tau) \to X^a(\tau) + \delta X^a(\tau)$, can be related to the covariant variation of the fields, convoluted with the original field equations in the geodesic lift just as in (15). However, since we have not yet accounted for the dynamics of gravitation, a discussion of the corresponding expressions is premature.

4. Summary and outlook

In the previous sections we identified the crucial ingredients relevant for deriving the geodesic description of solitons arising in theories with gauge and diffeomorphism invariance. To this extent we carefully analyzed the geodesic description of gauge theory solitons and pinpointed the constructional principles in section 2. In section 3 we demonstrated that these elements carry over to the study of the moduli spaces of solitons arising in theories with diffeomorphism invariance. To complete this discussion we still need to take the dynamics of the gravitational background into account. This issue is currently under investigation.

The following features of this formalism are worth emphasizing. First, the form of the velocity-dependent modifications of the various fields are determined completely by gauge and diffeomorphism covariance and the requirement that covariant field variations $\delta_{\rm cov}$ are identified with the variations of the collective coordinates in the moduli action principle. This feature is in contradistinction to the methodology advocated by [133–135,139,161], where, in principle, arbitrary velocity-dependent field perturbations are included. We discussed this issue in chapter V. Second, even for curved spacetime backgrounds, there is a one-to-one correspondence between the (gauge) invariances and the (initial-value) constraint equations. The constraints imply that the motion in moduli space, represented on the space of field configurations, is orthogonal to the gauge orbits. More work is needed to establish this for gravitation and diffeomorphism invariance. Third, there is no mathematically compelling reason for resorting to a low-velocity approximation. The formalism therefore lends itself to analyzing solitons that arise in theories with higher-order derivative interactions.

There are various issues that must still be addressed before turning to applications. Of immediate interest is the geodesic description of the gravity sector. Depending on the application one has in mind, a metric formulation or a formulation in terms of vielbeins is more appropriate. Certain elements have already been uncovered. For instance, the spacetime metric $g_{\mu\nu}$, appropriately enlarged, constitutes a covariant tensor in the extended space even in the geodesic lift. On the other hand, just like the gauge connection, the affine connection is modified by velocity-dependent terms and involves torsion. In view of (39), (40), and (41) it is tempting to conclude that the geodesic lift for the Riemann tensor and Ricci tensor are given by analogous formulae, which involve the matrix $\Phi_\Omega{}^\Lambda$, but we prefer to refer to future analysis.

An important further issue is the discussion of source terms. For the discussion of the lump-solutions of section 1 and Yang-Mills monopoles section 2 this was not necessary, since these solutions carry topological charge. On the other hand, source terms play a crucial role in the moduli space description of electrically charged, extremal black holes as explained in chapter V.

Once these remaining issues have been understood, many interesting applications can be worked out. We have mentioned several of these earlier. Of central importance is the application of the formalism to the geodesic description of extremal black holes and the comparison with the results of [133,134]. Reverting to the discussion at the end of chapter V, one might envisage addressing, in a second step, the question of the effects of R^2-terms on the moduli space geometry, although the corresponding calculations are probably very involved. In this context it is important to keep in mind that much of the symmetry structure of the moduli space metric is induced by the symmetries of solutions of the underlying field theory. For the nonlinear sigma model, we briefly discussed this in section 1. Having a fully covariant framework at our disposal, we expect that much about the moduli space geometry can be deduced from symmetry properties of the underlying solution spaces.

A

Notations and conventions

$d = 4$ supergravity

In chapters I through IV we denote spacetime indices by $\mu, \nu, \cdots$, and Lorentz indices by $a, b, \cdots = 0, 1, 2, 3$. Indices $i, j, k, \ldots$ are usually reserved for SU(2)-indices. Our conventions for (anti-)symmetrization are

$$[ab] = \tfrac{1}{2}(ab - ba)\,, \quad (ab) = \tfrac{1}{2}(ab + ba)\,.$$

We take

$$\gamma_a \gamma_b = \eta_{ab} + \gamma_{ab}\,, \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,,$$

where η_{ab} is of signature $(-+++)$. The complete antisymmetric tensor satisfies

$$\varepsilon^{abcd} = e^{-1}\varepsilon^{\mu\nu\lambda\sigma} e_\mu^a e_\nu^b e_\lambda^c e_\sigma^d\,, \qquad \varepsilon^{0123} = i\,,$$

which implies

$$\gamma_{ab} = -\tfrac{1}{2}\varepsilon_{abcd}\gamma^{cd}\gamma_5\,.$$

The dual of an antisymmetric tensor field F_{ab} is given by

$$\tilde{F}_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F^{cd}\,,$$

and the (anti-)selfdual part of F_{ab} reads

$$F_{ab}^{\pm} = \tfrac{1}{2}(F_{ab} \pm \tilde{F}_{ab})\,.$$

We note the following useful identities for (anti-)selfdual tensors in 4 dimensions:

$$\begin{aligned}
G^{\pm}_{[a[c}\, H^{\pm}_{d]b]} &= \pm\tfrac{1}{8} G^{\pm}_{ef}\, H^{\pm ef}\, \varepsilon_{abcd} - \tfrac{1}{4}(G^{\pm}_{ab}\, H^{\pm}_{cd} + G^{\pm}_{cd}\, H^{\pm}_{ab})\,,\\
G^{\pm}_{ab}\, H^{\mp cd} + G^{\pm cd}\, H^{\mp}_{ab} &= 4\delta^{[c}_{[a} G^{\pm}_{b]e}\, H^{\mp d]e}\,,\\
\tfrac{1}{2}\varepsilon^{abcd}\, G^{\pm e}_{[c}\, H^{\pm}_{d]e} &= \pm G^{\pm[a}{}_{e}\, H^{\pm b]e}\,,\\
G^{\pm ac}\, H_c^{\pm b} + G^{\pm bc}\, H_c^{\pm a} &= -\tfrac{1}{2}\eta^{ab}\, G^{\pm cd}\, H^{\pm}_{cd}\,,\\
G^{\pm ac}\, H_c^{\mp b} &= G^{\pm bc}\, H_c^{\mp a}\,,\\
G^{\pm ab}\, H^{\mp}_{ab} &= 0\,.
\end{aligned}$$

Note that under hermitian conjugation (h.c.) selfdual becomes anti-selfdual and vice-versa. Any SU(2) index i or any quaternionic index α changes position under h.c., for instance

$$(T_{ab\,ij})^* = T_{ab}^{ij}\,, \qquad (A_i^\alpha)^* = A_\alpha^i\,.$$

Form notation

When using form notation we follow the conventions of [162]. The normalization of the p-form components is given by

$$A_p = \frac{1}{p!} A_{\mu_1\ldots\mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\,.$$

In accordance with the usual convention for differential forms, the Hodge dual $*$ is calculated with a real totally antisymmetric tensor. (When working in supergravity we define the dual field strength $\tilde{F}$ with an extra factor of i.) We have

$$(*\,A_p)_{\mu_1\ldots\mu_{d-p}} = \frac{1}{p!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_{(d-p)}}{}^{\nu_1\cdots\nu_p}(A_p)_{\nu_1\cdots\nu_p}\,,$$

and $*\,*\,A_p = (-1)^{p(d-p)+\sigma} A_p$, where $\sigma = 1$ for Minkowskian, and $\sigma = 0$ for Euclidean signature. The invariant volume form is denoted by

$$*\,1 = \frac{\sqrt{|g|}}{d!}\,\varepsilon_{\mu_1\cdots\mu_d}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d =: \sqrt{|g|}\,\mathrm{d}^d x\,.$$

The exterior derivative is given by

$$(\mathrm{d}A_{p-1})_{\mu_1\cdots\mu_p} = p\,\partial_{[\mu_1}(A_{p-1})_{\mu_2\cdots\mu_p]}\,.$$

The Hodge dual is defined such that for two p-forms η_p and ω_p one has

$$\omega_p \wedge *\,\eta_p = \eta_p \wedge *\,\omega_p = \frac{\sqrt{|g|}}{p!}\,\eta_{\mu_1\cdots\mu_p}\,\omega^{\mu_1\cdots\mu_p}\,\mathrm{d}^d x\,.$$

This follows from the identity

$$\varepsilon_{a_1\cdots a_r\,c_{r+1}\cdots c_d}\,\varepsilon^{b_1\cdots b_r\,c_{r+1}\cdots c_d} = r!(d-r)!\,\delta_{[a_1}{}^{[b_1}\cdots\delta_{a_r]}{}^{b_r]}\,. \tag{1}$$

For even dimensional spacetimes, $d = 2n$, one can define the (anti-)selfdual n-forms according to

$$F_n^{\pm} = \tfrac{1}{2}(1 \pm c_n *)F_n\,, \tag{2}$$

for some number c_n satisfying $c_n^2 * * = 1$. For Lorentz signature metrics this implies $c_n = \pm i$, for $d = 2n = 4, 8, 12, \ldots$, and $c_n = \pm 1$ for $d = 2n = 2, 6, 10, \ldots$, whereas it is the other way around for Euclidean signature. Note that $(c_n * F_n^{\pm}) = \pm F_n^{\pm}$. Specializing to $d = 4$ we note that this definition is identical to the one given in components previously.

B

Supersymmetry conventions

The superconformal algebra consists of general coordinate, local Lorentz, dilatation, special conformal, chiral U(1) and SU(2), and Q- and S-supersymmetry transformations. The fully supercovariant derivatives are denoted by D_a. We use $\mathcal{D}_\mu$ to denote a covariant derivative with respect to Lorentz, dilatation, chiral U(1), SU(2), and gauge transformations. The component fields of the various superconformal multiplets carry certain Weyl and chiral weights. Those of the Weyl multiplet and of the supersymmetry transformation parameters are listed in table 1, whereas those of the vector and of hypermultiplets are given in table 3. These tables also list the fermion chirality of the various component fields. The gauge fields are normalized as

$$\begin{aligned}
h_\mu^{ab}(M) &= \omega_\mu^{ab}\,, & h_\mu(D) &= b_\mu\,,\\
h_\mu(\mathrm{U}(1)) &= A_\mu\,, & h_\mu{}^i{}_j(\mathrm{SU}(2)) &= -\tfrac{1}{2}\mathcal{V}_\mu{}^i{}_j\,,\\
h_\mu^i(Q) &= \tfrac{1}{2}\psi_\mu^i\,, & h_\mu^i(S) &= \tfrac{1}{2}\phi_\mu^i\,,\\
h_\mu^a(K) &= f_\mu^a\,,
\end{aligned}$$

such that the supercovariant derivative reads,

$$D_\mu = \partial_\mu - \sum_A \delta_A(h_\mu(A))\,,$$

where $h_\mu(A)$ are the gauge fields associated with δ_A. The covariantized general coordinate transformation is given by given by

$$\delta_{\mathrm{cov}}(\xi) = \mathcal{L}_\xi - \sum_{A\neq P_a} \delta_A(\xi^\mu h_\mu(A)),$$

where $\mathcal{L}_\xi$ is the Lie-derivative. We use the the symbol $\mathcal{D}_\mu$ to denote a covariant derivative with respect to M, D, U(1), SU(2), and gauge transformations. To exhibit the form of the derivatives $\mathcal{D}_\mu$ and the normalization of the gauge fields contained in them, we give the derivative of the chiral spinor ϵ^i,

$$\mathcal{D}_\mu\epsilon^i = \partial_\mu\epsilon^i - \tfrac{1}{4}\omega_\mu^{ab}\,\gamma_{ab}\epsilon^i + \tfrac{1}{2}(b_\mu + iA_\mu)\epsilon^i + \tfrac{1}{2}\mathcal{V}_\mu{}^i{}_j\epsilon^j\,.$$

TABLE 1. Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the Weyl multiplet component fields and of the supersymmetry transformation parameters.

	Weyl multiplet											parameters	
	${e_\mu}^a$	ψ^i_μ	b_μ	A_μ	${\mathcal{V}_\mu}^i{}_j$	T^{ij}_{ab}	χ^i	D	ω^{ab}_μ	${f_\mu}^a$	ϕ^i_μ	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$	$+$	$-$

The gauge fields for Lorentz and S-supersymmetry transformations are composite objects and given by

$$\begin{aligned}
\omega^{ab}_\mu =& -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu \\
& -\tfrac{1}{4}(2\bar\psi^i_\mu\gamma^{[a}\psi^{b]}_i + \bar\psi^{ai}\gamma_\mu\psi^b_i + \text{h.c.})\,, \\
\phi^i_\mu =& \tfrac{1}{2}(\gamma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\rho\sigma})\Big(\mathcal{D}_\rho\psi^i_\sigma - \tfrac{1}{16}T^{ab\,ij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\Big) \\
=& -\tfrac{1}{3}(4\delta^{[\rho}_\mu\gamma^{\sigma]} + \varepsilon_{\mu\lambda}{}^{\rho\sigma}\gamma^\lambda)\Big(\mathcal{D}_\rho\psi^i_\sigma - \tfrac{1}{16}T^{ab\,ij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\Big)\,,
\end{aligned}$$

respectively. The gauge field for special conformal transformations is also a composite object and was already given in (2) of chapter III, up to fermionic terms. The composite gauge fields given above transform as follows

$$\begin{aligned}
\delta\omega^{ab}_\mu =& -\tfrac{1}{2}\bar\epsilon^i\gamma^{ab}\phi_{\mu i} - \tfrac{1}{2}\bar\epsilon^i T^{ab}_{ij}\psi^j_\mu + \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\gamma^{ab}\chi_i \\
& + \bar\epsilon^i\gamma_\mu R(Q)^{ab}_i - \tfrac{1}{2}\bar\eta^i\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\Lambda_K^{[a}\,e_\mu^{b]}\,, \\
\delta\phi^i_\mu =& -2f^a_\mu\gamma_a\epsilon^i - \tfrac{1}{8}\not{D}T^{ij}_{ab}\gamma^{ab}\gamma_\mu\epsilon_j + \tfrac{3}{2}\big[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi^i_\mu - (\bar\chi_j\gamma^a\psi^j_\mu)\gamma_a\epsilon^i\big] \\
& + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac{1}{2}i\,R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i + 2\mathcal{D}_\mu\eta^i + \Lambda^a_K\gamma_a\psi^i_\mu\,, \\
\delta f^a_\mu =& -\tfrac{1}{2}\bar\epsilon^i\psi^j_\mu\,D_bT^{ba}_{ij} - \tfrac{3}{4}e_\mu{}^a\bar\epsilon^i\not{D}\chi_i - \tfrac{3}{4}\bar\epsilon^i\gamma^a\psi_{\mu i}\,D \\
& + \bar\epsilon^i\gamma_\mu D_bR(Q)^{ba}_i + \tfrac{1}{2}\bar\eta^i\gamma^a\phi_{\mu i} + \text{h.c.} + \mathcal{D}_\mu\Lambda^a_K\,.
\end{aligned}$$

Throughout this thesis we need certain supercovariant curvature tensors. They are listed in table 2. The following modified curvature tensors appear in the component fields of the chiral multiplet W^2 (*cf.* equation 4, chapter III),

$$\begin{aligned}
\mathcal{R}(M)_{ab}{}^{cd} =& R(M)_{ab}{}^{cd} + \tfrac{1}{16}\Big(T^{ijcd}\,T_{ijab} + T^{ij}_{ab}\,T^{cd}_{ij}\Big)\,, \\
\mathcal{R}(S)^i_{ab} =& R(S)^i_{ab} + \tfrac{3}{4}T^{ij}_{ab}\chi_j\,.
\end{aligned}$$

TABLE 2. Curvatures of the superconformal algebra

$$
\begin{aligned}
R(Q)^i_{\mu\nu} &= 2\mathcal{D}_{[\mu}\psi^i_{\nu]} - \gamma_{[\mu}\phi^i_{\nu]} - \tfrac{1}{8}T^{ab\,ij}\gamma_{ab}\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]} - i\Big(\tfrac{1}{2}\bar\psi^i_{[\mu}\phi_{\nu]i} + \tfrac{3}{4}\bar\psi^i_{[\mu}\gamma_{\nu]}\chi_i - \text{h.c.}\Big)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\partial_{[\mu}\mathcal{V}^i_{\nu]j} + \mathcal{V}^i_{[\mu k}\mathcal{V}^k_{\nu]j}\\
&\quad + \Big(2\bar\psi^i_{[\mu}\phi_{\nu]j} - 3\bar\psi^i_{[\mu}\gamma_{\nu]}\chi_j - (\text{h.c.}\,;\ \text{traceless})\Big)\,,\\
R(M)^{ab}_{\mu\nu} &= R(\omega)^{ab}_{\mu\nu} - 4f_{[\mu}^{[a}e_{\nu]}^{b]} + \Big(\tfrac{1}{2}\bar\psi^i_{[\mu}\gamma^{ab}\phi_{\nu]i} + \text{h.c.}\Big)\\
&\quad + \Big(\tfrac{1}{2}\bar\psi^i_{[\mu}T^{ab}_{ij}\phi^i_{\nu]} - \tfrac{3}{4}\bar\psi^i_{[\mu}\gamma_{\nu]}\gamma^{ab}\chi^i - \bar\psi^i_{[\mu}\gamma_{\nu]}R_{\mu\nu}(Q)_i + \text{h.c.}\Big)\,,\\
R(S)^i_{\mu\nu} &= 2\mathcal{D}_{[\mu}\phi^i_{\nu]} - 2f^a_{[\mu}\gamma_a\psi^i_{\nu]} - \tfrac{1}{8}\not{D}T^{ij}_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - 3\bar\chi_j\gamma^a\psi^j_{[\mu}\gamma_a\psi^i_{\nu]}\\
&\quad + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi^j_{\nu]} + \tfrac{1}{2}i\,R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi^i_{\nu]}\,,\\
R(D)_{\mu\nu} &= 2\partial_{[\mu}b_{\nu]} - 2f^a_{[\mu}\,e_{\nu]a} - \Big(\tfrac{1}{2}\bar\psi^i_{[\mu}\phi_{\nu]i} + \tfrac{3}{4}\bar\psi^i_{[\mu}\gamma_{\nu]}\chi_i + \text{h.c.}\Big)\,.
\end{aligned}
$$

The T^2-modification cancels exactly the T^2-terms in the contribution to $\mathcal{R}(M)$ from $f_\mu^{\ a}$. The curvature $\mathcal{R}(M)_{ab}{}^{cd}$ satisfies the following relations,

$$
\begin{aligned}
\mathcal{R}(M)_{\mu\nu}{}^{ab}\,e^\nu{}_b &= i\,\tilde R(A)_\mu{}^a + \tfrac{3}{2}D\,e_\mu{}^a\,,\\
\tfrac{1}{4}\varepsilon_{ab}{}^{ef}\,\varepsilon^{cd}{}_{gh}\,\mathcal{R}(M)_{ef}{}^{gh} &= \mathcal{R}(M)_{ab}{}^{cd}\,,\\
\varepsilon_{cdea}\,\mathcal{R}(M)^{cd\,e}{}_b = \varepsilon_{becd}\,\mathcal{R}(M)_a{}^{e\,cd} &= 2\tilde R_{ab}(D) = 2i\,R_{ab}(A)\,. \qquad (1)
\end{aligned}
$$

The first one is the constraint that determines the field $f_\mu{}^a$ while the remaining equations are Bianchi identities. Note that the modified curvature does not satisfy the pair exchange property,

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{R}(M)^{cd}{}_{ab} + 4i\,\delta^{[c}_{[a}\,\tilde R(A)_{b]}{}^{d]}\,.$$

From these equations one determines for instance

$$\mathcal{R}(M)^{\pm}_{0[p}{}^{0}{}_{q]} = \pm\tfrac{1}{2}i\,R(A)^{\pm}_{pq}\,.$$

We note that $R(Q)^i_{ab}$ satisfies the constraint

$$\gamma^\mu R(Q)^i_{\mu\nu} + \tfrac{3}{2}\gamma_\nu\chi^i = 0\,, \qquad (2)$$

which must therefore hold for its variation as well. This constraint implies that the tensor $R(Q)^i_{\mu\nu}$ is anti-selfdual, as follows from contracting it with $\gamma^\nu\,\gamma_{ab}$.

The curvature $\mathcal{R}(S)^i_{ab}$ satisfies

$$\gamma^a\tilde{\mathcal{R}}(S)^i_{ab} = 2\,D^a\,\tilde R(Q)^i_{ab}\,,$$

TABLE 3. Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the vector and hypermultiplet component fields.

	vector multiplet				hypermultiplet	
	X^I	Ω_i^I	W_μ^I	Y_{ij}^I	A_i^α	ζ^α
w	1	$\frac{3}{2}$	0	2	1	$\frac{3}{2}$
c	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
γ_5		$+$				$-$

as a result of the Bianchi identities and of the constraint (2). This identity (upon contraction with $\gamma^b\gamma_{cd}$) leads to

$$\mathcal{R}(S)^i_{ab} - \tilde{\mathcal{R}}(S)^i_{ab} = 2\boldsymbol{D}(R(Q)^i_{ab} + \tfrac{3}{4}\gamma_{ab}\chi^i)\,.$$

C

Supersymmetric bosonic backgrounds

In this appendix we discuss the restrictions imposed on a bosonic background by setting the supersymmetry variations of fermionic quantities to zero. In particular we want to answer the question, to which extent the supersymmetry variations of the covariant derivatives of fermions must be considered. Our statement is true for full as well as residual supersymmetry.

Let us consider all fermionic fields (except the gravitino ψ^i_μ and the S-gauge field ϕ^i_μ), including also all fermionic composites and fermions with any number of covariant derivatives. Let us denote such fermionic quantities collectively by Ψ. Under S-supersymmetry their transformation rules have the form

$$\delta_S(\Psi) = F_\Psi(\phi)\eta\,,$$

where $F_\Psi(\phi)$ is a matrix depending on the bosonic fields ϕ. Let us denote a field Ψ by ζ_n, if the matrix $F_\Psi(\phi)$ is a non-trivial constant, and call this constant α_n. The ζ_n-fields act as S-compensators. This is the case, for instance, for ζ^{H}_i or ζ^{V}_i discussed in section III.5. From any Ψ and any compensator ξ_n one constructs the S-invariant supersymmetry constraint

$$\delta_Q\left(\Psi - \frac{1}{\alpha_m}F_\Psi(\phi)\zeta_m\right) = 0\,. \tag{1}$$

For $\Psi = \zeta_n$ this means

$$\delta_Q\left(\zeta_n - \frac{\alpha_n}{\alpha_m}\zeta_m\right) = 0\,. \tag{2}$$

Let us denote by D the superconformal covariant derivative. We note the following property: if we satisfy (1) for some ζ_n then (1) is satisfied for any choice of ζ_m if we impose (2). This means that we can limit ourselves to using always the same compensator if we ensure (2). Let us consider $\Psi = D\zeta_n$. Due to the connections contained in the covariant derivative, $D\zeta_n$ in general does not transform like a ζ-field under S-supersymmetry. The condition (1) is therefore non-trivial,

$$\delta_Q\left(D\zeta_n - \frac{1}{\alpha_m}F_{D\zeta_n}(\phi)\zeta_m\right) = 0\,, \tag{3}$$

for any choice of m and n. We wish to argue for the following statement: if we ensure (1) for all fermions without derivatives, in particular thus (2) for all compensators, and we impose (3) for one particular choice of compensator, then no additional constraints are found by considering covariant derivatives of other fermions. In formulae the statement is that if we have satisfied (3) for one specific choice of n and m, it follows from $\delta_Q(\Psi - \frac{1}{\alpha_m} F_\Psi(\phi)\zeta_m) = 0$ that for the covariant derivative $D\Psi$ one automatically has $\delta_Q(D\Psi - \frac{1}{\alpha_m} F_{D\Psi}(\phi)\zeta_m) = 0$. The reason is quite simple. If $\Psi - \frac{1}{\alpha_n} F_\Psi \zeta_n$ is an S-invariant combination then it follows that $\delta_S[D(\Psi - \frac{1}{\alpha_n} F_\Psi \zeta_n)] = 0$ in a bosonic background. This is because S-variations on the covariant derivative do not survive in a bosonic background. Working out the covariant derivative and using that the combination $D\zeta_n - \frac{1}{\alpha_m} F_{D\zeta_n}(\phi)\zeta_m$ is S-invariant by construction, we arrive to the conclusion that $\delta_S[D\Psi - \frac{1}{\alpha_n} DF_\Psi \zeta_n - \frac{1}{\alpha_n^2} F_\Psi F_{D\zeta_n}\zeta_n] = 0$ in the bosonic background. Thus up to fermionic terms, which are not relevant for a bosonic background, one finds that the uniquely determined bosonic part of $F_{D\Psi}$ is given by $F_{D\Psi} = DF_\Psi + \frac{1}{\alpha_n} F_\Psi F_{D\zeta_n}$ for any n. Using this relation it follows, by the same line of reasoning, that from

$$\delta_Q \left[D\Big(\psi - \tfrac{1}{\alpha_n} F_\psi \zeta_n\Big)\right] = 0 \tag{4}$$

one has

$$0 = \delta_Q \left[D\Psi - \frac{1}{\alpha_n} \left(DF_\Psi + \frac{1}{\alpha_n} F_\Psi F_{D\zeta_n} \right) \zeta_n \right] = \delta_Q \left[D\Psi - \frac{1}{\alpha_n} F_{D\Psi} \zeta_n \right]$$

for any n in a bosonic background. Thus granted we have imposed the constraints (1) and, in particular, all constraints (2) for fermionic fields without extra covariant derivatives, as well as the constraints (3) for any one choice of n and m, no new constraints will be obtainable from considering the variations of covariant derivatives of these fermions.

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Samenvatting

(Summary in Dutch)

In het volgende geven we een samenvatting van dit proefschrift in het Nederlands. We beginnen met een samenvattende vertaling van het voorwoord. Daarna bespreken we de inhoud van de hoofdstukken I–VI.

Dit proefschrift gaat over zwarte gaten. Sinds hun vroege theoretische ontdekking spelen zwarte gaten een belangrijke rol in het theoretische onderzoek. Veel van de opgeroepen vragen hebben te maken met het centrale open probleem van de hedendaagse theoretische natuurkunde, namelijk met de vraag hoe de quantumtheorie met de theorie van gravitatie (de algemene relativiteitstheorie) in één, mathematisch consistente formulering kan worden verenigd.

De theorie van zwarte gaten is een veel bestudeerd onderwerp in de algemene relativiteitstheorie. Eén van de centrale resultaten wordt gevormd door de wetten van de mechanica van zwarte gaten. Deze wetten lijken opvallend veel op die van de thermodynamica. Eén wet, bijvoorbeeld, zegt dat de oppervlakte van de horizon van een zwart gat in een fysisch proces niet afneemt. Hetzelfde is het geval voor de entropie van een thermodynamisch systeem, en men zou zich kunnen afvragen in hoeverre een zwart gat als een thermodynamisch systeem opgevat zou kunnen worden, ondanks het feit dat, in eerste instantie, de analogie alleen van formele aard is. De opvatting van een zwart gat als thermodynamisch systeem werd onderbouwd door de ontdekking dat quantumeffecten ertoe leiden dat zwarte gaten tóch stralen: ze zijn niet helemáál zwart, maar stralen volgens een patroon dat karakteristiek is voor een thermisch ensemble bij een bepaalde temperatuur.

Thermodynamische eigenschappen, zoals temperatuur, druk of entropie, worden in de statische fysica afgeleid uit de verwachtingswaarden van bepaalde grootheden in de onderliggende quantumtheorie, welke de microscopische vrijheidsgraden van het thermodynamische systeem beschrijft. De thermodynamische interpretatie van de wetten van de mechanica van zwarte gaten zou kunnen worden bevestigd als men deze wetten, op soortgelijke manier, door de statistische studie van de microscopische vrijheidsgraden zou kunnen afleiden. Wat zouden deze onderliggende, microscopische vrijheidsgraden van zwarte gaten kunnen zijn?

Deze vraag is een van de veel bediscussieerde onderwerpen in de fysica van zwarte gaten. Terwijl de algemene relativiteitstheorie zeer succesvol is in het beschrijven van het heelal op grote afstandsschalen is haar toepasbaarheid beperkt als het om de structuur van de ruimtetijd op kleine afstandsschalen gaat. Een belangrijke toets voor iedere kandidaattheorie van quantumgravitatie is het correct beschrijven van de microscopische vrijheidsgraden van zwarte gaten. De snaartheorie heeft in dit opzicht veelbelovende resultaten geleverd en in grote gedeelten van dit proefschrift worden de consequenties van deze aanpak uitgewerkt.

In dit proefschrift komen de volgende onderwerpen ter sprake:

Hoofdstuk I – Black holes and string theory. Dit eerste, introducerende hoofdstuk geeft een overzicht van verschillende aspecten van de fysica van zwarte gaten. In het bijzonder worden de wetten van de mechanica van zwarte gaten besproken, en wordt de klasse van zwarte gaten beschreven, die in latere hoofdstukken in detail wordt bestudeerd. Vervolgens wordt de nadruk gelegd op de snaartheoretische beschrijving van de microscopische vrijheidsgraden van zwarte gaten. Voor een bepaalde klasse van zwarte gaten slaagt deze aanpak er inderdaad in, een formule voor de entropie af te leiden, die tot op laagste orde in overeenstemming is met de (macroscopische) wetten van de mechanica van zwarte gaten, en dus diens thermodynamische interpretatie te bevestigen. Verder wordt erop gewezen dat in sommigen gevallen de statistische beschrijving deviaties van de horizonoppervlakte-wet voorspelt.

Hoofdstuk II – Supersymmetry and supergravity. Een belangrijk ingrediënt in de voorafgaande analyse van de entropie in hoofdstuk I is supersymmetrie. In dit tweede introducerende hoofdstuk worden daarom belangrijke elementen van $N = 2$-supersymmetrie en supergravitatie besproken. De nadruk ligt daarbij op de geometrische aspecten van de constructie en op de rol van symplectische herparametrisatie van deze theorieën.

Hoofdstuk III – Supergravity theories with higher-order curvature interactions. In dit hoofdstuk worden de meer technische aspecten van de constructie van de supergravitatie-theorieën beschreven. In het bijzonder worden de interacties van materievelden met hogere machten van de kromming (R^2-interacties) geïncorporeerd. De complete bosonische actie wordt gegeven en verschillende supersymmetrie-transformaties worden uitgewerkt.

Hoofdstuk IV – Supersymmetric vacua and stationary BPS configurations. Dit hoofdstuk bevat een gedetailleerde analyse van supersymmetrische zwarte gaten in de aanwezigheid van R^2-interacties. Eerst worden ruimtetijden die volledige $N = 2$-supersymmetrie bewaren onderzocht. Het wordt afgeleid dat supersymmetrische ruimtetijden uniek en van het Bertotti-Robinson type zijn. Daarbij zijn alle velden van deze configuraties bepaald door de elektrische en magnetische lading van de oplossingen. Een belangrijke conclusie van dit resultaat is dat voor zwarte gaten met een volledig supersymmetrische horizon alle velden vaste waarden aannemen op de horizon, zelfs in aanwezigheid van R^2-interacties. Vervolgens wordt een grote klasse van ruimtetijden die de helft van de $N = 2$-supersymmetrie bewaren bepaald. Deze

BPS-configuraties beschrijven onder meer stationaire zwarte gaten, die tussen de volledig supersymmetrische ruimtetijden interpoleren en mogelijk meerdere centra hebben. Tenslotte worden verschillende expliciete voorbeelden besproken.

Hoofdstuk V – On entropy and moduli spaces of black holes. In dit hoofdstuk wordt uitgelegd hoe de deviatie van de microscopische entropie van zwarte gaten ten opzichte van de horizonoppervlakte-wet in overeenstemming kan worden gebracht met een macroscopische beschrijving in termen van een gravitatietheorie met R^2-interacties. Een belangrijk ingrediënt hiertoe is een entropieformule die algemener is dan de horizonoppervlakte-wet en die toepasbaar is voor theorieën die interacties bevatten met hogere afgeleiden. De derivatie van zo'n entropieformule wordt uitgelegd. Vervolgens gaat de aandacht uit naar de moduli-ruimte van zwarte gaten. Voor het meest simpele geval wordt de metriek op deze ruimte volgens een standaard methode bepaald. Deze metriek kan worden beschreven als een afgeleide van een potentiaal. Er wordt gewezen op een mogelijk verband tussen dit potentiaal en de entropie van zwarte gaten.

Hoofdstuk VI – Moduli spaces and geodesic description. Het laatste hoofdstuk is een studie van de moduli-ruimte en de geodetische beschrijving van solitonische oplossingen. Er wordt een algemeen formalisme ontwikkeld, dat voor zowel solitonen van niet-abelse ijktheorieën als voor gravitationele solitonen (zoals extremale zwarte gaten) van toepassing is. Er wordt afgeleid dat ijk- en/of diffeomorfisme-invariantie onder meer de effectieve moduli-actie, en hiermee de metriek van de moduli-ruimte, bepalen.

Curriculum Vitae

Ik ben geboren op 7 maart 1974 te Niederbipp in Zwitserland. Tot mijn twintigste jaar heb ik in Rumisberg aan de zuidelijke voet van de Jura gewoond, van waaruit ik vanaf 1986 het gymnasium te Solothurn bezocht. In de herfst 1994 ben ik naar Bern getrokken om aan de universiteit natuurkunde, filosofie en wiskunde te studeren. Eind 1998 ben ik afgestudeerd op een scriptie in de theoretische natuurkunde onder begeleiding van Heinrich Leutwyler. In april 1999 ben ik begonnen als promovendus aan het Instituut voor Theoretische Natuurkunde en aan het Spinoza Instituut van de Universiteit Utrecht onder begeleiding van Bernard de Wit. Tijdens mijn aanstelling als onderzoeker in opleiding heb ik geassisteerd bij verschillende colleges. Verder heb ik deelgenomen aan talrijke scholen, workshops en conferenties in Nederland en in het buitenland, en heb ik op verschillende plaatsen voordrachten over mijn werk gegeven. In de herfst 2003 zal ik mijn onderzoek voortzetten als postdoctoraal onderzoeker aan het Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut) te Golm bij Potsdam.