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# Exit Problems for Spectrally Negative Lévy Processes and Applications to Russian, American and Canadized Options

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We consider spectrally negative Lévy process and determine the joint Laplace transform of the exit time and exit position from an interval containing the origin of the process reflected in its supremum. In the literature of fluid models, this stopping time can be identified as *the time to buffer-overflow*. The Laplace transform is determined in terms of the scale functions that appear in the two sided exit problem of the given Lévy process. The obtained results together with existing results on two sided exit problems are applied to solving optimal stopping problems associated with the pricing of American and Russian options and their Canadized versions.

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## 1 Introduction

In this paper we consider the class of spectrally negative Lévy processes. These are real valued random processes with stationary independent increments which have no positive jumps. Amongst others Emery [11], Suprun [23], Bingham [4] and Bertoin [3] have all considered fluctuation theory for this class of processes. Such processes are often considered in the context of the theories of dams, queues, insurance risk and continuous branching processes; see for example [6, 4, 5, 19]. Following the exposition on two sided exit problems in Bertoin [3] we study first exit from an interval containing the origin for spectrally negative Lévy processes reflected in their supremum (equivalently spectrally positive Lévy processes reflected in their infimum). In particular we derive the joint Laplace transform of the time to first exit and the overshoot. The aforementioned stopping time can be identified in the literature of fluid models as *the time to buffer overflow* (see for example [1, 13]). Together

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with existing results on exit problems we apply our results to certain optimal stopping problems that are now classically associated with mathematical finance.

In sections 2 and 3 we introduce notation and discuss and develop existing results concerning exit problems of spectrally negative Lévy processes. In section 4 an expression is derived for the joint Laplace transform of the exit time and exit position of the reflected process from an interval containing the origin. This Laplace transform can be written in terms of scale functions that already appear in the solution to the two sided exit problem. In Section 5 we outline two classes of optimal stopping problem which are are associated with the pricing of American and Russian options. Sections 6 and 7 are devoted to solving these optimal stopping problems in terms of scale functions that appear in the afore mentioned exit problems. In Section 8 we consider a modification of these optimal stopping problems known as Canadization (corresponding to the case that the expiry dates of option contracts are randomized with an independent exponential distribution) and show that explicit solutions are also available in terms of scale functions. Finally we conclude the paper with some explicit examples of the optimal stopping problems under consideration.

## 2 Spectrally negative Lévy processes

Let $X = \{X_t, t \geq 0\}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. Restricting ourselves to spectrally negative Lévy processes, the process $X$ may be represented as

$$X_t = \mu t + \frac{\sigma^2}{2} W_t + J_t^{(-)} \tag{1}$$

where $W = \{W_t, t \geq 0\}$ is standard Brownian motion and $J^{(-)} = \{J_t^{(-)}, t \geq 0\}$ is a pure jump spectrally negative Lévy processes. Both processes are independent. We exclude the case that $X$ is a subordinator (that is the case where $X$ has monotone paths).

The jumps of $J^{(-)}$ are all nonpositive and hence the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$. A standard property of Lévy processes, following from the stationarity of their increments, is that when the moment generating function of the process at time $t$ exists, it satisfies

$$\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\,\psi(\theta)} \tag{2}$$

for some function $\psi(\theta)$, the cumulant, which is well defined at least on the non-negative complex half plane and will be referred to as the Levy exponent of $X$. It can be checked that this function is strictly convex and tends to infinity as $\theta$ tends to infinity, see Bertoin [2, p188].

We restrict ourselves to the Lévy processes which have unbounded variation or have bounded variation and a Lévy measure of $X$ which is absolutely continuous with respect to the Lebesgue measure

$$\Lambda(\mathrm{d}x) \ll \mathrm{d}x. \tag{AC}$$

We conclude this section by introducing for any Lévy process having $X_0 = 0$ the family of martingales

$$\exp\left(cX_t - \psi(c)t\right),$$

defined for any $c$ for which $\psi(c) = \log \mathbb{E}[\exp cX_1]$ is finite. Further the corresponding family of measures with Radon Nikodym derivatives:

$$\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(cX_t - \psi(c)t\right). \tag{3}$$

For all such $c$ (including $c = 0$) the measure $\mathbb{P}^c_x$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$.

**Remark 1** *Under the measure $\mathbb{P}^c$ the characteristics of the process $X$ have changed. How they have changed can be found out by looking at the cumulant of $X$ under $\mathbb{P}^c$:*

$$\begin{aligned}\psi_c(\theta) &:= \log\left(\mathbb{E}^c[\exp(\theta X_1)]\right)\\ &= \log\left(\mathbb{E}[\exp\left((\theta + c)X_1 - \psi(c)\right)]\right)\\ &= \psi(\theta + c) - \psi(c), \qquad \theta \geq 0.\end{aligned}$$

# 3 Exit problems for Lévy processes

## 3.1 Scale functions

Bertoin [3] studies two sided exit problems of spectrally negative Lévy processes in terms of a class of functions known as $q$-scale functions. Here we give a slightly modified definition of these objects (Definition 2).

**Definition 1** *Let $q \geq 0$ and then define $\Phi_c(q)$ is the largest root of $\psi_c(\theta) = q$.*

**Definition 2** *For $q \geq 0$, the $q$-scale function $W^{(q)} : (-\infty, \infty) \to [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform*

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q)$$

*and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_c^{(q)}(x)$ to mean the $q$-scale function as defined above for $(X, \mathbb{P}^c)$.*

It is known that the $q$-scale function is increasing on $(0, \infty)$. Furthermore, if $X$ has unbounded variation or if $X$ has bounded variation and satisfies (AC), the restricted function $W_v^{(q)}|_{(0,\infty)}$ is continuously differentiable. See Lambert [15] and Bertoin [3]. For every $x \geq 0$, we can extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane by the identity

$$W_v^{(q)}(x) = \sum_{k \geq 0} q^k W_v^{\star(k+1)}(x) \tag{4}$$

where $W_v^{\star k}$ denotes the $k$-th convolution power of $W_v = W_v^{(0)}$. The convergence of this series is plain from the inequality

$$W_v^{\star k+1}(x) \leq x^k W_v(x)^{k+1}/k!, \qquad x \geq 0, k \in \mathbb{N},$$

which follows from the monotonicity of $W_v$.

**Remark 2** *By Corollary VII.1.5 in Bertoin [2]* $\lim_{x\downarrow 0} W_v(x) = 0$ *if and only if* $X$ *has unbounded variation. By the expansion (4) it also follows that, under the same condition,* $\lim_{x\downarrow 0} W_v^{(q)}(x) = 0$.

**Remark 3** *We have the following relationship between scale functions*

$$W^{(u)}(x) = \mathrm{e}^{vx} W_v^{(u-\psi(v))}(x)$$

*for* $v$ *such that* $\psi(v) < \infty$. *To see this, simply take Laplace transforms of both sides. By anatytical entension, we see that the identity remains valid for all* $u \in \mathbb{C}$.

Equally important as far as the following discussion is concerned is the function $Z^{(q)}$ which is defined as follows.

**Definition 3** *For* $q \geq 0$ *we define* $Z^{(q)} : \mathbb{R} \to [1, \infty)$ *by*

$$Z^{(q)}(x) = 1 + q \int_{-\infty}^{x} W^{(q)}(z)\mathrm{d}z. \tag{5}$$

*Keeping with our earlier convention, we shall use* $Z_c^{(q)}(x)$ *in the obvious way.*

Just like $W^{(q)}$,the function $Z^{(q)}$ may be characterised by its Laplace transform and continuity on $(0, \infty)$ . Indeed, we can check that

$$\int_0^\infty \mathrm{e}^{-\theta x} Z^{(q)}(x)\mathrm{d}x = \psi(\theta)/\theta(\psi(\theta) - q), \qquad \theta > \Phi(q).$$

Note that when $q > 0$ this function inherits some properties from $W^{(q)}(x)$. Specifically it is strictly increasing, is equal to the constant 1 for $x \leq 0$ and $Z^{(q)}\big|_{(0,\infty)} \in C^2(0, \infty)$. When $q = 0$ then $Z^{(0)}(x) = Z(x) = 1$. Also by working with the analytic extention of $q \mapsto W_v^{(q)}(x)$ we can define $q \mapsto Z_v^{(q)}(x)$ for all $q \in \mathbb{C}$.

We state the following result for the limit of $Z^{(q)}(x)/W^{(q)}(x)$ as $x$ tends to infinity. For the formulation of this result and in the sequel, we shall understand $0/\Phi(0)$ to mean $\lim_{\theta\downarrow 0} \theta/\Phi(\theta)$.

**Lemma 1** *For* $q \geq 0$, $\lim_{x\to\infty} Z^{(q)}(x)/W^{(q)}(x) = q/\Phi(q)$.

PROOF First suppose $q > 0$. The fact that $\theta \mapsto \psi(\theta)$ is increasing for $\theta \geq \Phi(0)$ in conjunction with equation (4) implies that $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) > 0$. Recalling that $\psi^{-1}_{\Phi(q)}$ is the Laplace transform of $W_{\Phi(q)}$, we now deduce from a Tauberian theorem (e.g. [2, p. 10]) that

$$0 < W_{\Phi(q)}(\infty) = \lim_{x\to\infty} W_{\Phi(q)}(x) = 1/\psi'_{\Phi(q)}(0) < \infty \tag{6}$$

By partial integration, we find

$$Z^{(q)}(x) = 1 + q(W^{(q)}(x) - W^{(q)}(0^+))/\Phi(q) - q\int_0^x \mathrm{e}^{\Phi(q)y} W'_{\Phi(q)}(y)\mathrm{d}y/\Phi(q),$$

where $W^{(q)}(0^+) = \lim_{x\downarrow 0} W^{(q)}(x)$. Recall from Remark 3 that $W^{(q)}(x)$ is equal to $\exp(\Phi(q)x)$ times $W_{\Phi(q)}(x)$. Next divide the last equality by $W_{\Phi(q)}(x)$. Equation (6) in conjuntion with dominated convergence implies that the integral $\int_0^x \mathrm{e}^{\Phi(q)(y-x)} W'_{\Phi(q)}(y)\mathrm{d}y/W_{\Phi(q)}(x)$ converges to zero as $x$ tends to $\infty$; hence $Z^{(q)}(x)/W^{(q)}(x)$ converges to $q/\Phi(q)$.

Consider now the case $q = 0$. We know from e.g. Bertoin [2] that $\Phi(0) > 0$ if and only if $X$ drifts to $-\infty$. Recalling that by Remark 3 $W(x) = \exp(\Phi(0)x)W_{\Phi(0)}(x)$, we see that, if $X$ drifts to $-\infty$, the limit $\lim_{x\to\infty} W(x)^{-1} = 0$. If $X$ does not drift to $-\infty$, we find by the same Tauberian theorem that $W(x)^{-1} \sim x\psi(x^{-1})$ as $x \to \infty$. We finish the proof by noting that $\psi'(0^+) = \Phi'(0^+)^{-1}$, since $\psi(\Phi(q)) = q$. □

## 3.2 Exit from a finite interval

The following Proposition gives a complete account of the two sided exit problem for the class of spectrally negative Lévy processes we are interested in. Before stating the result, we first introduce the following passage times.

**Definition 4** *We denote the passage times above and below $k$ for $X$ by*

$$T_k^- = \inf\{t > 0 : X_t \leq k\} \text{ and } T_k^+ = \inf\{t > 0 : X_t \geq k\}. \tag{7}$$

**Proposition 1** *Let $q \geq 0$. The Laplace transform of the two-sided exit time $T_a^- \wedge T_b^+$ on the part of the probability space where $X$, starting in $x \in (a,b)$, exits the interval $(a,b)$ above and below are respectively given by*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_b^+} I_{(T_b^+ < T_a^-)}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} \tag{8}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_a^-} I_{(T_b^+ > T_a^-)}\right] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}. \tag{9}$$

PROOF This result can be extracted directly out of existing litterature. See for example Bertoin [2, Thm. VII.8] for a proof of (8). Combining this with Bertoin [3, Cor 1], we find equation (9). Note, in Bertoin [3] there is a small typographic mistake so that in equation (9) the function $\int_0^{x-a} W^{(q)}(y)\mathrm{d}y$ is used instead of $Z^{(q)}(x-a) - 1$. □

**Remark 4** *The strong Markov property, in conjunction with equation (8), is enough to prove that*

$$\mathrm{e}^{-q(T_b^+\wedge T_a^-\wedge t)}W^{(q)}(X_{T_b^+\wedge T_a^-\wedge t}-a) \tag{10}$$

*is a martingale. To see this let* $\tau = T_b^+ \wedge T_a^-$ *and note that* $W^{(q)}(X_\tau - a)/W^{(q)}(b-a)$ *is another way of writing the indicator of* $\{T_b^+ < T_a^-\}$*. Thus, by (8)*

$$\begin{aligned}
&W^{(q)}(x-a)\\
&\quad= \mathbb{E}_x\left(\mathrm{e}^{-q\tau}W^{(q)}\left(X_\tau - a\right)\right)\\
&\quad= \mathbb{E}_x\left[\left(\mathbb{E}_x\left(\mathrm{e}^{-q\tau}W^{(q)}\left(X_\tau - a\right)\right)\middle|\,\mathcal{F}_t\right)\right]\\
&\quad= \mathbb{E}_x\left[I_{(t\le\tau)}\mathrm{e}^{-qt}\mathbb{E}_{X_t}\left(\mathrm{e}^{-q\tau}W^{(q)}\left(X_\tau - a\right)\right) + I_{(t>\tau)}\mathrm{e}^{-q\tau}W^{(q)}\left(X_\tau - a\right)\right]\\
&\quad= \mathbb{E}_x\left[I_{(t\le\tau)}\mathrm{e}^{-qt}W^{(q)}\left(X_t - a\right) + I_{(t>\tau)}\mathrm{e}^{-q\tau}W^{(q)}\left(X_\tau - a\right)\right]\\
&\quad= \mathbb{E}_x\left[\mathrm{e}^{-q(\tau\wedge t)}W^{(q)}\left(X_{\tau\wedge t} - a\right)\right].
\end{aligned}$$

*Now that this constant expectation has been established, the martingale property follows by a similar manipulaton of the expression (10). This technique can also be employed to prove that similar martingales exist when replacing* $W^{(q)}(x-a)$ *by*

$$Z^{(q)}(x-a) + \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}(1 - Z^{(q)}(x-a))$$

*and hence (by linearlity)* $Z^{(q)}(x-a)$*.*

## 3.3 Exit from a positive half-line

The purpose of this section is to evaluate the joint moment-generating function of the time $X$ exits $[k,\infty)$ and its position at that time. The result is not new and a proof of a variant of our proposition below is due to Emery [11].

**Proposition 2** *For* $u \ge 0$ *and* $v$ *such that* $\psi(v) < \infty$ *the joint Laplace transform of* $T_k^-$ *and* $X_{T_k^-}$ *is given by*

$$\mathbb{E}_x[\exp\{-uT_k^- + vX_{T_k^-}\}] = \mathrm{e}^{vx}\left(Z_v^{(p)}(x-k) - W_v^{(p)}(x-k)p/\Phi_v(p)\right),$$

*where* $p = u - \psi(v)$*.*

Emery's proof relies on Wiener-Hopf factorization in combination with complex-analytic arguments. Here we present a probabilistic proof, expressing the Laplace transform in the previously introduced $p$-scale function and its anti-derivative.

**Remark 5** *Note the formula in Proposition 2 is stated in terms of the functions* $W_v^{(p)}, Z_v^{(p)}$*. However, we can reformulate the formula in Theorem 1 entirely in terms of* $W^{(u)}$ *by using Remark 3. Furthermore* $\Phi_v(u-\psi(v)) = \Phi(u) - v$ *for* $u \ge 0$ *and* $v$ *such that* $\psi(v) < \infty$*. Indeed, Remark 1 implies for* $u > 0$

$$u - \psi(v) = \psi_v(\Phi_v(u-\psi(v))) = \psi(\Phi_v(u-\psi(v)) + v) - \psi(v);$$

*for $u = 0$ the identity follows by continuity. Using these facts, it can be checked that the Laplace transform $f_{u,v}(\theta)$ of $\mathbb{E}_x[\exp\{-uT_0^- + vX_{T_0^-}\}]$ is given by*

$$f_{u,v}(\theta) = (\psi(\theta) - u)^{-1}\left(\frac{\psi(\theta) - \psi(v)}{\theta - v} - \frac{u - \psi(v)}{\Phi(u) - v}\right),$$

*which agrees with Bingham [4, Thm. 6.5].*

Proof of Proposition 2 Suppose first $u \geq \psi(v) \vee 0$. Let $\widetilde{v}$ denote $\max\{v, 0\}$. By Fatou's lemma

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-uT_k^- + vX_{T_k^-}} I_{(T_k^- = \infty)}\right] &\leq \liminf_{n\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-uT_k^- + vX_{T_k^-}} I_{(T_k^- \geq n)}\right] \\ &\leq \liminf_{n\to\infty} \mathrm{e}^{-un}\mathbb{E}_x[\mathrm{e}^{\widetilde{v}X_n}] = \lim_{n\to\infty} \mathrm{e}^{-(u-\psi(\widetilde{v}))n}\end{aligned}$$

which is zero. It follows again by bounded convergence that

$$\begin{aligned}\mathbb{E}_x[\exp\{-uT_k^- + vX_{T_k^-}\}] &= \lim_{n\to\infty} \mathbb{E}_x\left[\exp\{-u(T_k^- \wedge T_{k+n}^+) + vX_{T_k^- \wedge T_{k+n}^+}\} I_{(T_k^- < T_{k+n}^+)}\right] \\ &= \lim_{n\to\infty} \mathbb{E}_x^v\left[\exp\{-(u - \psi(v))(T_k^- \wedge T_{k+n}^+)\} I_{(T_k^- < T_{k+n}^+)}\right] \mathrm{e}^{vx} \\ &= \mathrm{e}^{vx}(Z_v^{(p)}(x-k) - W_v^{(p)}(x-k)p/\Phi_v(p)), \qquad (11)\end{aligned}$$

where in the final line we used (9) and Lemma 1. Thus, we have the stated identity for the restricted class of $u, v$ which includes the case that $u \geq 0$ when $\psi(v) \leq 0$.

Now fix $v$ such that $0 < \psi(v) < \infty$. Since Theorem VII.1 in Bertoin [2] states that

$$\Phi(q) = -\log \mathbb{E}[\mathrm{e}^{-qT_1^+}],$$

we see that $\Phi$ can be analytically extended to $\{q \in \mathbb{C} : \mathrm{Re}\ q > 0\}$. Hence by Remark 5 and the properties of the $q$-scale function, we see that the right-hand side of (11) can be analytically extended to $\{u \in \mathbb{C} : \mathrm{Re}\ u > 0\}$. For every $n$ the coefficient $d_n$ of the corresponding powerseries in $u = \psi(v)$ is given in terms of the $n$-th (right-)derivative with respect to $u$ of the left-hand side in $u = \psi(v)$, that is

$$d_n = \mathbb{E}_x[(-T_k^-)^n \exp\{-uT_k^- + vX_{T_k^-}\}]/n!.$$

Since the series $\sum_n d_n(u - \psi(v))^n$ converges for $|u - \psi(v)| < |\psi(v)|$, the identity theorem implies that (11) is valid for $u > 0$. By taking limits on both sides of (11) with the help of the bounded convergence theorem for the left hand side, the proof is completed. □

**Remark 6** *Following an analogous reasoning as in Remark 4, we see that*

$$\exp\{-u(T_k^- \wedge t) + vX_{T_k^- \wedge t}\}\left(Z_v^{(p)}(X_{T_k^- \wedge t} - k) - W_v^{(p)}(X_{T_k^- \wedge t} - k)p/\Phi_v(p)\right),\ t \geq 0$$

*is a $\mathbb{P}$-martingale.*

# 4 Exit problems for reflected Lévy processes

Let

$$\overline{X}_t = \max\left\{s, \sup_{0\le u\le t} X_u\right\},$$

that is the non-decreasing process representing the current maximum of $X$ given that at time zero, the maximum from some arbitrary prior point of reference in time is $s$. Further, let us alter slightly our notation so that now $\mathbb{P}_{s,x}$ refers to the Lévy process $X$ which at time zero is given to have a current maximum $s$ and position $x$. The notation $\mathbb{P}^c_{s,x}$ is also used in the obvious way. Further in the sequel, we shall frequently exchange between $\mathbb{P}^c_{s,x}$, $\mathbb{P}^c_{(s-x),0}$ and $\mathbb{P}^c_{-(s-x)}$ as appropriate.

We can address similar questions to those of the previous section of the process $Y = \overline{X} - X$. In this case, two sided exit problems for the process $Y$ are essentially the same as for the process $X$. We are more interested however one sided exit problems centred around the stopping time

$$\tau_k := \inf\{t \ge 0 : Y_t \notin [0,k)\}$$

defined for $k > 0$.

**Theorem 1** *For $u \ge 0$ and $v$ such that $\psi(v) < \infty$, the joint Laplace transform of $\tau_k$ and $Y_{\tau_k}$ is given by*

$$\mathbb{E}_{s,x}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}] = \mathrm{e}^{-vz}\left(Z^{(p)}_v(k-z) - W^{(p)}_v(k-z)\frac{pW^{(p)}_v(k) + vZ^{(p)}_v(k)}{W^{(p)\prime}_v(k) + vW^{(p)}_v(k)}\right)$$

*where $z = s - x \ge 0$ and $p = u - \psi(v)$.*

Proof Suppose first that $u, v$ are such that $u \ge \psi(v) \vee 0$ and let $z = s - x$. Denote by $\tau_{\{0\}}$ the first time that $Y$ hits zero. An application of the strong Markov property of $Y$ at $\tau_{\{0\}}$ yields that $\mathbb{E}_{s,x}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}]$ is equal to

$$\mathbb{E}_{s,x}\left[\mathrm{e}^{-u\tau_k - vY_{\tau_k}} I_{(\tau_k < \tau_{\{0\}})}\right] + C\mathbb{E}_{s,x}\left[\mathrm{e}^{-u\tau_{\{0\}}} I_{(\tau_k > \tau_{\{0\}})}\right] \tag{12}$$

where $C = \mathbb{E}_{s,s}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}] = \mathbb{E}_{0,0}[\mathrm{e}^{-u\tau_k - vY_{\tau_k}}]$. Since

$$\{Y_t, t \le \tau_{\{0\}}, \mathbb{P}_{s,x}\} \stackrel{d}{=} \{-X_t, t \le T^+_0, \mathbb{P}_{-z}\} \tag{13}$$

and $\exp(vX_{T^-_{-k}\wedge T^+_0} - \psi(v)(T^-_{-k} \wedge T^+_0) + vz)$ is an equivalent change of measure under $\mathbb{P}_{-z}$ (since $T^-_{-k} \wedge T^+_0$ is almost surely finite), we can rewrite the first expectation on the right hand side of (12) as $\exp(-vz)$ times

$$\mathbb{E}^v_{-z}\left[\mathrm{e}^{(\psi(v)-u)T^-_{-k}} I_{(T^-_{-k} < T^+_0)}\right] = Z^{(p)}_v(k-z) - W^{(p)}_v(k-z)\frac{Z^{(p)}_v(k)}{W^{(p)}_v(k)} \tag{14}$$

from Propostition 1. By (13), Remark 3 and again Propostition 1 we find for the second expectation on the right-hand side of (12)

$$\mathbb{E}_{-z}\left[\mathrm{e}^{-uT_0^+} I_{(T_{-k}^- > T_0^+)}\right] = \frac{W^{(u)}(k-z)}{W^{(u)}(k)} = \mathrm{e}^{-vz}\frac{W_v^{(p)}(k-z)}{W_v^{(p)}(k)}. \tag{15}$$

We compute $C$ by excursion theory. To be more precise, we are going to make use of the compensation formula for excursion theory. For this we shall use standard notation (see Bertoin [2, Ch. 4]). We are interested in the excursion process $e = \{e_t, t \geq 0\}$ of $Y$, which takes values in the space of excursions $\mathcal{E}$ and is given by

$$e_t = \{Y_s, L^{-1}(t^-) \leq s < L^{-1}(t)\} \qquad \text{if } L^{-1}(t^-) < L^{-1}(t),$$

where $L^{-1}$ is the right inverse of a local time $L$ of $Y$ at 0. We take the running supremum of $X$ to be this local time (c.f. Bertoin [2, Ch. VII]). The space $\mathcal{E}$ is endowed with the Itô excursion measure $n$. A famous theorem of Itô now states that, if $Y$ is recurrent, $\{e_t, t \geq 0\}$ is a Poisson point process with characteristic measure $n$; if $Y$ is transient, $\{e_t, t \leq L(\infty)\}$ is a Poisson point process with the same characteristic measure, stopped at the first excursion of infinite lifetime. For an excursion $\epsilon \in \mathcal{E}$ with lifetime $\zeta$, we denote by $\overline{\epsilon}$ the supremum of $\epsilon$, that is, $\overline{\epsilon} = \sup_{s \leq \zeta} \epsilon(s)$. If $Y$ is recurrent $h = \{h_t : t \geq 0\}$ being the point process of maximum heights of excursions appearing in the process $e$ is a Poisson point process and if $Y$ is transient $h = \{h_t : t \leq L(\infty)\}$ is a stopped Poisson point process.

Following the line of reasoning in Bertoin [2] concerning Proposition 1 we can also deduce the characteristic measure of the process $h$. Suppose first $Y$ is recurrent. The event that $X$ starting in 0 exits the interval $(-x, y)$ at $y$ is equal to the event $A = \{h_t \leq t + x \forall\, t \leq x + y\}$. Hence from Theorem 1 we find by differentiation that

$$W(x)/W(x+y) = \exp\left(-\int_0^y n(\overline{\epsilon} \geq t + x)\mathrm{d}t\right) \Rightarrow n(\overline{\epsilon} \geq k) = W'(k)/W(k).$$

If $X$ is transient, we can replace the event $A$ by $A' = \{h_t \leq t + x \;\forall\, t \leq x + y, x + y \leq L(\infty)\}$ and recall that the stopped Poisson point process has the same characteristic measure $n$.

Now let $\rho_k = \inf\{t \geq 0 : \epsilon(t) \geq k\}$ and denote by $\epsilon_g$ the excursion starting in $g$. The promised calculation involving the compensation formula is as follows.

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-u\tau_k - vY_{\tau_k}}\right) &= \mathbb{E}\left(\sum_g \left\{\mathrm{e}^{-ug} I_{\left(\sup_{h<g}\overline{\epsilon}_h < k\right)}\right\}\left\{I_{(\overline{\epsilon}_g \geq k)}\mathrm{e}^{-u(\tau_k - g) - vY_{\tau_k}}\right\}\right) \\
&= \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-us} I_{\left(\sup_{h<s}\overline{\epsilon}_h < k\right)} L(ds)\right)\int_{\mathcal{E}} I_{(\overline{\epsilon} \geq k)}\mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon\right) \\
&= \int_0^\infty \mathbb{E}\left(\mathrm{e}^{-uL_t^{-1}} I_{\left(\sup_{l<L_t^{-1}} e_l < k, t < L(\infty)\right)}\right)\mathrm{d}t \\
&\quad \times \int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon | \overline{\epsilon} \geq k\right) n\left(\overline{\epsilon} \geq k\right).
\end{aligned}$$

The suprema and the sum are taken over left starting points of excursions. The desired expectation is now identified as the product of the two items in the last equality, say $I_1$ and

$I_2$ which can now be evaluated separately. For the first, note that $L_t^{-1}$ is a stopping time and hence an argument involving a change of measure yields

$$\begin{aligned} I_1 &= \int_0^\infty \mathbb{E}\left( \mathrm{e}^{-uL_t^{-1}+\Phi(u)t} I_{\left(\sup_{l<L_t^{-1}} e_l<k, t<L(\infty)\right)} \right) \mathrm{e}^{-\Phi(u)t}\mathrm{d}t \\ &= \int_0^\infty \mathbb{P}^{\Phi(u)}\left( \sup_{l<L_t^{-1}} e_l < k, t < L(\infty) \right) \mathrm{e}^{-\Phi(u)t}\mathrm{d}t. \end{aligned}$$

Furthermore the probability in this integral is the chance that in the Poisson point process of excursions (indexed by local time), the first excursion of height greater or equal to $k$ occurs after time $s$. The characteristic measure of this Poisson point process associated with measure $\mathbb{P}^{\Phi(u)}$ is $W'_{\Phi(u)}(x)/W_{\Phi(u)}(x)$ and hence

$$\mathbb{P}^{\Phi(u)}\left( \sup_{l<L_t^{-1}} e_l < k, t < L(\infty) \right) = \exp\left\{ -t\frac{W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)} \right\}$$

so that

$$\begin{aligned} I_1 &= \int_0^\infty \exp\left\{ -t\frac{\Phi(u)W_{\Phi(u)}(k)+W'_{\Phi(u)}(k)}{W_{\Phi(u)}(k)} \right\}\mathrm{d}t \\ &= \frac{W_{\Phi(u)}(k)}{\Phi(u)W_{\Phi(u)}(k)+W'_{\Phi(u)}(k)} = \frac{W^{(u)}(k)}{W^{(u)\prime}(k)} \end{aligned}$$

where the final identity follows from Remark 3. Note that $I_1 > 0$, since $W^{(u)}$ is an increasing non-negative on $(0,\infty)$. Now turning to $I_2$ we begin by noting that $n(\overline{\epsilon} \geq k)$ is the characteristic measure of the Poisson point process of excursions having height greater or equal than $k$. It follows from before that $n(\overline{\epsilon} \geq k) = W'(k)/W(k)$. Our aim is now to prove that

$$\int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon | \overline{\epsilon} \geq k\right) = \frac{Z_v^{(p)}(k)\, W_v^{(p)\prime}(k)\,/W_v^{(p)}(k) - pW_v^{(p)}(k)}{W'(k)\,/W(k)}. \tag{16}$$

and hence that

$$I_2 = Z_v^{(p)}(k)\, W_v^{(p)\prime}(k)\,/W_v^{(p)}(k) - pW_v^{(p)}(k) \tag{17}$$

To do this, we will prove that with $\rho_\theta = \inf\{t \geq 0 : \epsilon(t) \geq \theta\}$ for $\theta \in [0,k]$,

$$M_\theta = \mathrm{e}^{-u\rho_\theta - v\epsilon(\rho_\theta)} \left( \frac{Z_v^{(p)}(k-\epsilon(\rho_\theta)) - W_v^{(p)}(k-\epsilon(\rho_\theta))\, Z_v^{(p)}(k)\,/W_v^{(p)}(k)}{1 - W(k-\epsilon(\rho_\theta))\,/W(k)} \right)$$

is a martingale under the measure $n(\cdot|\overline{\epsilon} \geq k)$ with respect to the filtration $\{\mathcal{G}_\theta : \theta \in [0,k]\}$ where $\mathcal{G}_\theta = \sigma(\epsilon(t) : t \leq \rho_\theta)$. Once we have proved that $\{M_\theta : \theta \in [0,k]\}$ is a martingale,

(16) follows because

$$\begin{aligned}\int_{\mathcal{E}} \mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} n\left(d\epsilon | \overline{\epsilon} \geq k\right) &= n(M_k | \overline{\epsilon} \geq k) = M_0 \\ &= \lim_{z \downarrow 0} \frac{Z_v^{(p)}(k-z) - W_v^{(p)}(k-z) Z_v^{(p)}(k) / W_v^{(p)}(k)}{1 - W(k-z)/W(k)} \\ &= \frac{Z_v^{(p)}(k) W_v^{(p)\prime}(k) / W_v^{(p)}(k) - p W_v^{(p)}(k)}{W'(k)/W(k)}.\end{aligned}$$

Let $\eta(\cdot) = n(\cdot | \overline{\epsilon} \geq k)$. To show that the sequence $\{M_\theta : \theta \in [0,k]\}$ is a martingale consider first that

$$\eta\left(M_k | \mathcal{G}_\theta\right) = \frac{n\left(\mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} \mathbf{1}_{(\rho_k < \infty)} \middle| \mathcal{G}_\theta\right)}{n\left(\rho_k < \infty | \mathcal{G}_\theta\right)}.$$

Using the strong Markov property for excursions, we have that given $\mathcal{G}_\theta$ the law of the continuing excursion is that of $-X$ killed on entering $(-\infty, 0)$ with entrance law being that of $\epsilon(\rho_\theta)$. Thus, we find that

$$\begin{aligned}& n\left(\mathrm{e}^{-u\rho_k - v\epsilon(\rho_k)} \mathbf{1}_{(\rho_k < \infty)} \middle| \mathcal{G}_\theta\right) \\ &= \mathrm{e}^{-u\rho_\theta} \mathbb{E}_{-\epsilon(\rho_\theta)} \left( \mathrm{e}^{-uT_{-k}^- + vX_{T_{-k}^-}} \mathbf{1}_{(T_{-k}^- < \infty)} \mathbf{1}_{(T_{-k}^- < T_0^+)} \right) \\ &= \mathrm{e}^{-u\rho_\theta} \mathbb{E}^v_{-\epsilon(\rho_\theta)} \left( \mathrm{e}^{-pT_{-k}^-} \mathbf{1}_{(T_{-k}^- < T_0^+)} \right) \mathrm{e}^{v\epsilon(\rho_\theta)} \\ &= \mathrm{e}^{-u\rho_\theta - v\epsilon(\rho_\theta)} \left( Z_v^{(p)}(k - \epsilon(\rho_\theta)) - W_v^{(p)}(k - \epsilon(\rho_\theta)) \frac{Z_v^{(p)}(k)}{W_v^{(p)}(k)} \right)\end{aligned}$$

and choosing $u = v = 0$ in the above calculation

$$n\left(\rho_k < \infty | \mathcal{G}_\theta\right) = 1 - W(k - \epsilon(\rho_\theta))/W(k).$$

The martingale status of $\{M_\theta : \theta \in [0,k]\}$ is proved and hence (17) holds.

Putting the pieces together from $I_1$ and $I_2$ and noting Remark 3 implies

$$W^{(u)}(k)/W^{(u)\prime}(k) = W_v^{(p)}(k)/(W_v^{(p)\prime}(k) + v W_v^{(p)}(k)),$$

we find

$$C = -W_v^{(p)}(k) \frac{p W_v^{(p)}(k) + v Z_v^{(p)}(k)}{W_v^{(p)\prime}(k) + v W_v^{(p)}(k)} + Z_v^{(p)}(k) \tag{18}$$

and a weaker version of the Theorem is proved by substitution of (14),(15) and (18) in (12).

The result is now established for $u, v$ such that $u \geq \psi(v) \vee 0$. Fix now any $v$ such that $0 < \psi(v) < \infty$ and note that the right-hand side can be extended to an analytic function on $\{u \in \mathbb{C} : \mathrm{Re}\ u > 0\}$. By an argument, analogous to the one used in the proof of Proposition 2, we conclude that the identity is valid for $u \geq 0$ and $v$ such that $\psi(v) < \infty$ □

**Remark 7** *By following similar calculations to those mentioned in Remark 4 it is not difficult to show that for all $u, v$ as in Theorem 1*

$$\mathrm{e}^{-u(t\wedge\tau_k)-vY_{t\wedge\tau_k}}\left(Z_v^{(p)}(k-Y_{t\wedge\tau_k})-\frac{vZ_v^{(q)}(k)+pW_v^{(p)}(k)}{W_v^{(p)\prime}(k)+vW_v^{(p)}(k)}W_v^{(p)}(k-Y_{t\wedge\tau_k})\right)$$

*is a martingale, where, as before, $p = u - \psi(v)$.*

# 5 Russian and American options

Consider a financial market consisting of a riskless bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0\mathrm{e}^{rt} \qquad B_0 > 0, \qquad r \geq 0, \qquad t \geq 0.$$

The price of the risky asset is modeled as the exponential Lévy process

$$S_t = \exp(X_t), \qquad t \geq 0,$$

where $X = \{X_t, t \geq 0\}$ is a Lévy process. We assume that there exists a measure (equivalent with respect to the implicit measure of the risky asset) such that the process

$$\left\{\mathrm{e}^{-rt}S_t : t \geq 0\right\}$$

is a martingale. Let us assume further that $\mathbb{P}$ takes the role of this measure. Note that this necessarily implies that $\psi(1) = r$.

Russian options were originally introduced by Shepp and Shiryaev [21, 22] within the context of the Black-Scholes market (the case that the underlying Lévy process is a Brownian motion with drift). In this paper we shall consider perpetual Russian options under the given model of spectrally negative Lévy processes. This option gives the holder the right to exercise at any almost surely finite $\mathbf{F}$-stopping time $\tau$ yielding payouts

$$\mathrm{e}^{-\alpha\tau}\max\left\{M_0, \sup_{0\leq u\leq\tau} S_u\right\}, \qquad M_0 \geq S_0, \alpha > 0.$$

The constant $M_0$ can be viewed as representing the "starting" maximum of the stock price (say, the maximum over some previous period $(-t_0, 0]$). The discount factor $\alpha$ is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite.

Somewhat more studied are American Put options which give the holder again the right to exercise at any $\mathbf{F}$-stopping time $\tau$ yielding a payout

$$\mathrm{e}^{-\alpha\tau}(K - S_\tau)^+, \text{ where } \alpha \geq 0$$

can be regarded as the dividend rate of the underlying stock. The American option has been dealt with as early as McKean [16].

Standard theory of pricing American-type options in the original Black-Scholes market directs one to solving optimal stopping problems. For the Russian and American put, the analogy in this context takes the form of evaluating

$$V_r(M_0, S_0) := \sup_\tau \mathbb{E}_{\log S_0}\left[\mathrm{e}^{-(\alpha+r)\tau} \max\left\{M_0, \sup_{0\leq u\leq\tau} S_u\right\}\right]$$
$$V_a(S_0) := \sup_\tau \mathbb{E}_{\log S_0}\left[\mathrm{e}^{-(\alpha+r)\tau}(K - S_\tau)^+\right]$$

where the supremum is taken over all almost surely finite respectively all $\mathbf{F}$-stopping times. That is to find a stopping time which optimizes the expected discounted claim under the risk neutral measure. The real object of interest is of course the finite time version with the extra constraint $\tau \leq T$, where $T$ is a given expiration time (this is closely related to the lookback option). Note however Carr [7] has shown that a close relative of the perpetual version lies at the basis of a very efficient approximation for the finite time expiration option, justifying therefore the interest in perpetuals. We shall address this matter in more detail in Section 8.

When dealing with the first class of option, our method leans on the experience of Shepp and Shiryaev [21, 22], Duffie and Harrison [10], Graverson and Peškir [12] and Kyprianou and Pistorius [14] all of which deal with the perpetual Russian option within the standard Black-Scholes market. Following Shepp and Shiryaev's technique of performing a change of measure using the $\mathbb{P}_{s,x}$-martingale $\exp\{-rt-x\}S_t$, we can reduce the above optimal stopping problem to finding a function $w^R$ and an almost surely finite stopping time $\tau^*$ such that

$$w^R(s,x) = \mathrm{e}^x \times \sup_\tau \mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+r)\tau+Y_\tau}\right] = \mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+r)\tau^*+Y_{\tau^*}}\right] \tag{19}$$

where $Y_t = \overline{X}_t - X_t$ is the reflection of the Lévy process in its supremum. We refer to this optimal stopping problem as the *Russian optimal stopping problem.* Given the existing results for the standard Black-Scholes market, we should expect that the optimal stopping time will be an upcrossing time of the reflected process $Y$ at a certain constant (positive) level $k$. Indeed this turns out to be the case. For the American Put option, the associated optimal stoping problem is to find a function $w^A$ and a stopping time $\tau^*$ such that

$$w^A(x) = \sup_\tau \mathbb{E}_x[\mathrm{e}^{-(\alpha+r)\tau}(K - S_\tau)^+] = \mathbb{E}_x[\mathrm{e}^{-(\alpha+r)\tau^*}(K - S_{\tau^*})^+] \tag{20}$$

where the supremum is taken over all $\{\mathcal{F}_t\}_{t\geq 0}$-measurable stopping times. We shall refer to this as the *American optimal stopping problem.* As is the case with the classical Black-Scholes market, we shall show that the solution to the American optimal stopping problem lies with the downcrossing of an optimal constant level.

# 6 The Russian optimal stopping problem

In this section we offer the solution to the optimal stopping problem stated in Section 5. The result follows as an application of our study of exit problems for the reflected Lévy processes.

**Corollary 1** *Suppose $X$ is as in Theorem 1 with $\psi(1) = r$. Then, for $k \geq 0, s \geq x$, $\alpha > 0$ and $q = \alpha + r$*

$$\mathbb{E}^1_{s,x}\left(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}}\right) = \mathrm{e}^{(s-x)}\left(Z^{(q)}(k-s+x) + \frac{Z^{(q)}(k) - qW^{(q)}(k)}{W^{(q)\prime}(k) - W^{(q)}(k)} W^{(q)}(k-s+x)\right)$$

PROOF Noting that $\psi_1(-1) = \psi(0) - \psi(1) = -r$, we may apply Theorem 1 with $u = \alpha, v = -1$ and $\mathbb{P}$ replaced by $\mathbb{P}^1$. The proof is finished once we note that $p = \alpha + r = q$ and

$$\mathrm{e}^x \mathrm{e}^x (W_1)^{(\alpha+r)}_{-1} \equiv \mathrm{e}^x W_1^{(\alpha)} \equiv W^{(q)}$$

each time by Remark 3. □

The next theorem presents the solution to the Russian optimal stopping problem.

**Theorem 2** *Let $\kappa^* = \inf\{x : Z^{(q)}(x) \leq qW^{(q)}(x)\}$ and define $u : [0,\infty) \to [0,\infty)$ by*

$$u(z) = \mathrm{e}^z Z^{(q)}(\kappa^* - z).$$

*Then the solution to (19) is given by $w^R(s,x) = \mathrm{e}^x u(s-x)$ where $\tau^* = \tau_{\kappa^*}$ is the optimal stopping time.*

Before we start the proof we collect some useful facts:

**Lemma 2** *Define the function $f : [0,\infty) \to \mathbb{R}$ by $f(x) = Z^{(q)}(x) - qW^{(q)}(x)$ and let $\kappa^*$ be as in Theorem 2. Then*

*(i) For $q > r$, $f$ decreases monotonically to $-\infty$.*

*(ii) If $W^{(q)}(0^+) \geq q^{-1}$, $\kappa^* = 0$; otherwise $\kappa^* > 0$ is the unique root of $f(x) = 0$.*

PROOF (i) By Remark 3, the function $f$ has derivative in $x > 0$

$$f'(x) = qW^{(q)}(x) - qW^{(q)\prime}(x) = q\mathrm{e}^{\Phi(q)x}\left((1-\Phi(q))W_{\Phi(q)}(x) - W'_{\Phi(q)}(x)\right).$$

For $x > 0$ and $q \geq r$, this derivative is seen to be negative. Indeed, recall that $W_{\Phi(q)}$ is positive and increasing on $(0,\infty)$ and $\Phi(q) \geq \Phi(r) = 1$. Lemma 1 implies that the sign of the limit $f(\infty) = \lim_{x\uparrow\infty} f(x)$ is the same as the sign of

$$q^{-1}\lim_{x\to\infty} f(x)/W^{(q)}(x) = \Phi(q)^{-1} - 1.$$

It follows with the help of Remark 3 that $W^{(q)}(x) = \exp(\Phi(q)x)W_{\Phi(q)}(x)$ tends to infinity and the statement now follows.

(ii) If $W^{(q)}(0^+) \geq q^{-1}$, (i) implies that $\kappa^* = 0$, whereas if $W^{(q)}(0^+) < q^{-1}$, we have existence and uniqueness of a positive root of $Z^{(q)}(x) = qW^{(q)}(x)$. □

PROOF OF THEOREM 2 Suppose now first $W^{(q)}(0^+) = 0$ (that is $X$ has unbounded variation). From the properties of $Z^{(q)}$, we see that $u$ lives in $C^1(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{k_*\})$. Hence Itô's lemma implies that $\exp\{-\alpha t\}u(\overline{X}_t - X_t)$ can be written as the sum of stochastic and

Stieltjes integrals. The non-martingale component of these integrals can be expressed as $\exp\{-\alpha t\}$ times

$$(\widehat{\Gamma}_1 - \alpha)u(\overline{X}_t - X_t)\mathrm{d}t + u'(\overline{X}_t - X_t)\mathrm{d}\overline{X}_t = (\widehat{\Gamma}_1 - \alpha)u(Y_t)\mathrm{d}t \tag{21}$$

where $\widehat{\Gamma}_1$ is the infinitesimal generator corresponding to the process $-X$ under $\mathbb{P}^1$ and the equality follows from the fact that the process $\overline{X}_t$ only increments when $Y_t = 0$ (since $Y$ reaches zero always by creeping in the absence of positive jumps of $X$) and $u'(0) = 0$. From Remark 7 we know that $\exp\{-\alpha(t \wedge \tau_{\kappa^*})\}u(Y_{t\wedge\tau_{\kappa^*}})$ is a martingale, which implies that on $\{t \leq \tau_{\kappa^*}\}$, and hence on $\{X_t < \kappa^*\}$,

$$(\widehat{\Gamma}_1 - \alpha)u(z) = 0 \text{ for } z \in [0, \kappa_*).$$

Now recall that under the measure $\mathbb{P}^1_{s,x}$ the process $\exp\{-X_t + rt\}$ is a martingale. By a similar reasoning to the above, we can deduce that $(\widehat{\Gamma}_1 + r)(\exp\{x\}) = 0$. Specifically this implies for $z > \kappa_*$ that

$$(\widehat{\Gamma}_1 - \alpha)u(z) = \left(\widehat{\Gamma}_1 + r - (r + \alpha)\right)(\exp\{z\}) \leq 0.$$

By the expression (21) for the non-martingale part of $\mathrm{d}\left(\exp\{-\alpha t\}u(Y_t)\right)$ we deduce that

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha t + Y_t} Z^{(q)}(\kappa^* - Y_t)\right] \leq \mathrm{e}^{(s-x)} Z^{(q)}(\kappa^* - s + x)$$

A similar argument to the one presented in Remark 4 now shows that $\exp\{-\alpha t\}u(Y_t)$ is a $\mathbb{P}^1_{s,x}$-supermartingale.

Doob's optional stopping theorem for supermartingales together with the fact that $\exp\{z\} \leq u(z)$ implies that for all almost surely finite stopping times $\tau$,

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha\tau + Y_\tau}\right] \leq \mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha\tau} u(Y_\tau)\right] \leq u(s - x).$$

Since the inequalities above can be made an equalities by choosing $\tau = \tau_{\kappa^*}$, the proof is complete if $X$ has unbounded variation.

If $W^{(q)}(0^+) \in (0, q^{-1})$ ($X$ has bounded variation) we see that $u$ lives in $C^0(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{\kappa^*\})$. Itô's lemma for this case becomes nothing more than the change of variable formula for Stieltjes integrals (c.f. Protter [20]) and the rest of the proof follows exactly the same line of reasoning as above.

Finally the case $W^{(q)}(0^+) \geq q^{-1}$ (again $X$ has bounded variation). Recall from Lemma 2 that if $W^{(q)}(0^+) \geq q^{-1}$, then for $x > 0$

$$Z^{(q)}(x) - qW^{(q)}(x) < 0 \qquad \text{and} \qquad W^{(q)}(x) - W^{(q)\prime}(x) < 0.$$

Hence, recalling that $Z^{(q)}(x) = 1$ for $x < 0$, we see from Corollary 1 that for any $k > 0$

$$\begin{aligned}\mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}}) &= \mathbb{E}^1_{s,x}(\mathrm{e}^{-\alpha\tau_k + Y_{\tau_k}} Z(k - Y_{\tau_k})) \\ &\leq \mathrm{e}^{(s-x)} Z^{(q)}(k - s + x).\end{aligned}$$

As before we conclude that, for any $k > s - x$, $\{e^{-\alpha(\tau_k \wedge t)} u(Y_{\tau_k \wedge t})\}_{t \geq 0}$ is a supermartingale and hence using similar reasoning to the previous case it is still the case that $\{e^{-\alpha t} u(Y_t)\}_{t \geq 0}$ is a supermartingale. It follows that for all almost surely finite $\mathbf{F}$-stopping times $\tau$,

$$\mathbb{E}^1_{s,x}(e^{-\alpha\tau + Y_\tau}) \leq \mathbb{E}^1_{s,x}(e^{-\alpha\tau + Y_\tau} Z(k - Y_\tau)) \leq e^{s-x} Z^{(q)}(k - s + x).$$

Taking the limit of $k$ to $s - x$ we see that, by the continuity of $Z^{(q)}$ in zero, for any a.s. finite stopping time $\tau$,

$$\mathbb{E}^1_{s,x}(e^{-\alpha\tau + Y_\tau}) \leq e^{s-x}$$

with equality for $\tau = 0$, which completes the proof. □

**Remark 8** *One may feel curious about the fact that $k_*$ given by $Z^{(q)}(\kappa^*) = qW^{(q)}(\kappa^*)$ gives the optimal crossing level. Heuristically, if $W^{(q)}(0^+) > 0$, the optimal level can be found by* value matching

$$lim_{x \uparrow \kappa^*} w^R(x) = lim_{x \downarrow \kappa^*} w^R(x);$$

*if $W^{(q)}(0^+) = 0$ we find $\kappa^*$ by* smooth fit

$$lim_{x \uparrow \kappa^*} w^{R\prime}(x) = lim_{x \downarrow \kappa^*} w^{R\prime}(x).$$

*Both these conditions can be checked to result in $Z^{(q)}(\kappa^*) = qW^{(q)}(\kappa^*)$. We see that this choice of $\kappa^*$ guarantees that the value function is sufficiently regular for an application of Itô's formula in the case of unbounded variation.*

# 7 The American optimal stopping problem

Recall that we understand $0/\Phi_1(0)$ to mean $\lim_{\theta \downarrow 0} \theta/\Phi_1(\theta)$.

**Theorem 3** *Let $\alpha \geq 0$ and let the level $k^*$ be given by*

$$k^* = \log(K) + \log(q/\Phi(q)) + \log(\Phi_1(\alpha)/\alpha).$$

*Define the function $w : \mathbb{R} \to \mathbb{R}$ by*

$$w(x) = KZ^{(q)}(x - k^*) - e^x Z_1^{(\alpha)}(x - k^*). \tag{22}$$

*The solution to the American put optimal stopping problem (20) is given by $w_A = w$ where $\tau^* = T^-_{k^*}$ is the optimal stopping time.*

Before we go to the proof, we summarize some usefull results which we will need further on.

**Corollary 2** *For $\alpha \geq 0, s \geq 0, k \leq \log K, q = \alpha + \psi(1)$ we have*

$$\mathbb{E}_x\left[e^{-qT_k^-}\left(K - \exp\{X_{T_k^-}\}\right)^+\right] = K\left(Z^{(q)}(x-k) - W^{(q)}(x-k)\frac{q}{\Phi(q)}\right) - e^x\left(Z_1^{(\alpha)}(x-k) - W_1^{(\alpha)}(x-k)\frac{\alpha}{\Phi_1(\alpha)}\right)$$

PROOF Note that for $k \leq \log K$

$$\mathbb{E}_x\left[\mathrm{e}^{-(\alpha+r)T_k^-}(K-S_{T_k^-})^+\right] = K\mathbb{E}_x\left[\mathrm{e}^{-qT_k^-}\right] - \mathrm{e}^x\mathbb{E}_x\left[\mathrm{e}^{-qT_k^-+X_{T_k^-}}\right],$$

which gives the stated formula after invoking Proposition 2. □

**Lemma 3** *Consider the function $w$ and the level $k^*$ as defined in Theorem 3. Then $k^* < \log K$ and*

*(i)* $w(x) \geq (K-\mathrm{e}^x)^+$ *and* $w(x) = \mathbb{E}_x[\mathrm{e}^{-qT_{k^*}^-}(K-S_{T_{k^*}^-})^+]$.

*(ii)* $\left\{\mathrm{e}^{-q(T_{k^*}^-\wedge t)}w(X_{T_{k^*}^-\wedge t}); t \geq 0\right\}$ *is a* $\mathbb{P}$*-martingale.*

We postpone the proof of the lemma to the end of this subsection and first prove the theorem.

PROOF OF THEOREM 3 Let $\tau$ be any $\mathbf{F}$-stopping time. If $W^{(q)}(0^+) = 0$ ($X$ has quadratic variation), we see that, by the properties of $Z^{(q)}, Z_1^{(\alpha)}$, the function $w$ is $C^2$ everywhere except in $k^*$ where it is continuously differentiable. Hence, by applying Itô's lemma to $\mathrm{e}^{-q(T_{k^*}^-\wedge t)}w(X_{T_{k^*}^-\wedge t})$, and using Lemma 3(ii), it follows as before that $(\Gamma - q)w(x) = 0$ for all $x > k^*$. Moreover, for $x \leq \log K$, we find

$$(\Gamma - q)(K-\mathrm{e}^x) = q(\mathrm{e}^x - K) - \psi(1)\mathrm{e}^x \leq -r\mathrm{e}^x < 0.$$

Hence, $(\Gamma - q)w(x) \leq 0$ for all $x$. Combining with (21), we deduce that $\{\exp(-qt)w(X_t) : t \geq 0\}$ is a $\mathbb{P}$-supermartingale. Doob's optimal stopping theorem and Lemma 3 (i) imply that for all $\mathbf{F}$-stopping times $\tau$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau}(K-S_\tau)^+\right] \leq \mathbb{E}_x\left[\mathrm{e}^{-q\tau}w(X_\tau)\right] \leq w(x). \tag{23}$$

Choosing $\tau = T_{k^*}^-$ forces the inequalities (23) to be equalities (Lemma 3(i)) and the proof is done.

If $W^{(q)}(0^+) > 0$ ($X$ has bounded variation), we note that $X$ is a positive drift minus a jump process of bounded variation. The same argument as in the proof of Theorem 2 justifies the use of the Change of Variable Formula. Then we use a similar reasoning as above to finish the proof. □

PROOF OF LEMMA 3 By strict convexity of $\psi$ combined with $\psi(0) = 0$, we deduce that for any $\alpha > 0$

$$r = \psi(1) < \frac{r}{\alpha+r}\psi\left(\frac{\alpha+r}{r}\right) + \frac{\alpha}{\alpha+r}\psi(0) \iff \alpha + r < \psi\left(\frac{\alpha+r}{r}\right).$$

Recalling that $\Phi(\psi(v)) = v$ for $v > 0$ and $\Phi$ is increasing, we see that the last inequality is equivalent to

$$\Phi(\alpha+r)^{-1} > \frac{r}{\alpha+r} \iff \frac{(\Phi(\alpha+r)-1)}{\Phi(\alpha+r)} < \frac{\alpha}{\alpha+r}.$$

Hence using that for $\alpha > 0$

$$\alpha = \psi_1(\Phi_1(\alpha)) = \psi(\Phi_1(\alpha)+1) - r \iff \Phi(\alpha+r) = \Phi_1(\alpha)+1,$$

we deduce that $k^* < \log K$.

The first part of (i) follows by substituting the formula for $k^*$ in Proposition 2 and recalling that by Remark 3 $W^{(q)}(x) = \mathrm{e}^x W_1^{(\alpha)}(x)$. Since $w(x) = (K - \mathrm{e}^x)$ for $x \leq k^*$, we then find

$$\mathbb{E}_x \left[ \mathrm{e}^{-qT_{k^*}^-} w(X_{T_{k^*}^-}) \right] = w(x).$$

Combining this with the strong Markov property, we can prove, along the same line of reasoning as in Remark 4, that

$$\left\{ \mathrm{e}^{-q(T_{k^*}^- \wedge t)} w(X_{T_{k^*}^- \wedge t}) : t \geq 0 \right\}$$

is a $\mathbb{P}$-martingale. Finally part(ii). For $x \leq k^*$ we have $w(x) = (K - \mathrm{e}^x)^+$, for $x \geq \log K$ we see that $w(x) > 0 = (K - \mathrm{e}^x)^+$ whereas for $\log K > x > k^*$ we find that

$$\begin{aligned} w(x) &= KZ^{(q)}(x - k^*) - \mathrm{e}^x Z_1^{(\alpha)}(x - k^*) \\ &= K - \mathrm{e}^x + \int_0^{x-k^*} [KqW^{(q)}(y) - \alpha \mathrm{e}^x W_1^{(\alpha)}(y)] \mathrm{d}y \\ &= (K - \mathrm{e}^x)^+ + \int_0^{x-k^*} [Kq - \alpha \mathrm{e}^{(x-y)}] W^{(q)}(y) \mathrm{d}y > (K - \mathrm{e}^x)^+ \end{aligned}$$

where we used again $\mathrm{e}^x W^{(\alpha)}(x) = W^{(q)}(x)$ and the definition of $Z^{(q)}$. Note that the integrand is positive, since for $y > 0$ we have that $W^{(q)}(y)$ as well as $Kq - \alpha \mathrm{e}^{x-y}$ are positive. □

**Remark 9** *Just as in the Russian optimal stopping problem, we can find the optimal level $k^*$ heuristically by value matching (if $W^{(q)}(0^+) = W_1^{(\alpha)}(0^+) > 0$) or smooth fit (if $W^{(q)}(0^+) = W^{(\alpha)}(0^+) = 0$).*

**Remark 10** *Since $\Phi_1$ is the right-inverse of $\psi_1$, we note that $\lim_{\alpha \downarrow 0} \alpha / \Phi_1(\alpha) = \psi_1'(0)$, which is equal to $\psi'(1)$. Hence, for $\alpha = 0$, the expression for $k^*$ coincides with the one found by Chan [8].*

**Remark 11** *Denote by $I_t, I$ the infimum of $X$ up to $t$ and up to an independent exponential time $\eta(q)$ with parameter $q$ respectively,*

$$I_t = \inf\{X_s : 0 \leq s \leq t\} \qquad I = \inf\{X_s : 0 \leq s < \eta(q)\}$$

*where as before we write $q = \alpha + r$. The solution to the optimal stopping problem (20) in terms of the distribution of $I$ has been long in the literature Darling [9], in the case we replace $X$ by a random walk. Recently, Mordecki [17] extended this result to the case where $X$ is a general Lévy process. In our notation his result reads as follows.*
The solution to (20) is given by

$$w^A(x) = \mathbb{E}[K\mathbb{E}(\mathrm{e}^I) - \mathrm{e}^{x+I}]^+ / E(\mathrm{e}^I) \tag{24}$$

where the optimal stopping time is given by $\tau^* = T_{l^*}^-$ whit $l^* = \log K\mathbb{E}(\mathrm{e}^I)$.

*The result of Theorem 3 agrees with this result of Mordecki since the same optimal stopping time is found. Indeed, for spectrally negative Lévy processes the Laplace transform of $I$ is well known to be (e.g. [2])*

$$\mathbb{E}(\mathrm{e}^{I}) = \frac{q}{\Phi(q)} \cdot \frac{\Phi(q)-1}{q-\psi(1)} = \frac{q}{\Phi(q)} \cdot \frac{\Phi_1(\alpha)}{\alpha},$$

*where we used Remark 5, which implies $k^* = l^*$.*

# 8 Canadized options

Suppose now we consider a claim structure in which the holder again receives a payout like that of the Russian or American put option. However, we also impose the restriction that the holder must claim before some time $\eta(\lambda)$, where $\eta(\lambda)$ is an **F**- independent exponential random variable with parameter $\lambda$. If the holder has not exercised by time $\eta(\lambda)$ then they are forced a rebate equal to the claim evaluated at time $\eta(\lambda)$. This is what is known in the literature as Canadization (c.f. Carr [7]). In the next two subsections we will treat respectively the Canadized Russian and American put.

## 8.1 Canadized Russian options

We are thus interested in a solution to the optimal stopping problem

$$w^{CR}(s,x) = \mathrm{e}^{x} \sup \mathbb{E}^{1}_{s,x}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right] \tag{25}$$

where the supremum is taken over almost surely finite stopping times $\tau$. Using the fact that $\eta(\lambda)$ is independent of the Lévy process, we can rewrite this problem in the following form,

$$w^{CR}(s,x) = \mathrm{e}^{x} \sup_{\tau} \mathbb{E}^{1}_{s,x}\left[\mathrm{e}^{-(\alpha+\lambda)\tau+Y_\tau} + \lambda \int_0^\tau \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right].$$

Given the calculations in Kyprianou and Pistorius [14], one should again expect to see that the optimal stopping time is of the form $\tau_k$ for some $k > 0$.

From now we assume that $p = \alpha + \lambda + r$

**Lemma 4** *For each $k > 0$,*

$$\begin{aligned}\mathbb{E}^{1}_{s,x}\left[\mathrm{e}^{-\alpha(\tau_k\wedge\eta(\lambda))+Y_{(\tau_k\wedge\eta(\lambda))}}\right] &= \left(\frac{p-\lambda}{p}\right)\mathrm{e}^{(s-x)}Z^{(p)}(k-s+x) + \frac{\lambda}{p}\mathrm{e}^{(s-x)} \\ &\quad + \mathrm{e}^{(s-x)}\frac{(p-\lambda)\left(Z^{(p)}(k) - pW^{(p)}(k)\right)+\lambda}{p\left(W^{(p)\prime}(k) - W^{(p)}(k)\right)}W^{(p)}(k-s+x). \end{aligned} \tag{26}$$

PROOF Consider the Itô Lemma applied to the process $\exp\{-(\alpha+\lambda)t + Y_t\}$ on the event $\{t \le \tau_k\}$. Standard calculations making use of the fact that $(\Gamma_1 + r)(\exp\{-x\}) = 0$ yield

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\right) &= -(\alpha+\lambda)\,\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t - r\mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t \\ &\quad + \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}\overline{X}_t + \mathrm{d}M_t\end{aligned}$$

where $\mathrm{d}M_t$ is a martingale term. Taking expectations of the stochastic integral given by the above equalities we have

$$\begin{aligned} p\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right] &= \mathrm{e}^{(s-x)} - \mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+\lambda)\tau_k+Y_{\tau_k}}\right] \\ &\quad +\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t}d\overline{X}_t\right]. \end{aligned} \tag{27}$$

The last term in the previous expression can be dealt with by taking account of the fact that $\overline{X} = L$, the local time at the supremum of the process $X$. Recall that $\tau_{\{0\}}$ is the first time that $Y$ reaches 0 and note that $\mathrm{d}\overline{X}_t = 0$ on the set where $\{\tau_k < \tau_{\{0\}}\}$. Letting $A \in \mathcal{F}_t$ be the set

$$A = \left\{\sup_{0\le u\le L_t^{-1}} Y_u < k, t < L(\infty)\right\},$$

we have

$$\begin{aligned} &\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t}d\overline{X}_t I_{\left(\tau_k>\tau_{\{0\}}\right)}\right] \\ &= \mathbb{E}^1_{-(s-x)}\left[\mathrm{e}^{-(\alpha+\lambda)\tau_{\{0\}}} I_{\left(\tau_k>\tau_{\{0\}}\right)}\right]\mathbb{E}^1\left[\int_0^\infty I_{(t<\tau_k)}\mathrm{e}^{-(\alpha+\lambda)t+Y_t}dL_t\right] \\ &= \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)}\mathbb{E}^1\left[\int_0^\infty I_A \mathrm{e}^{-(\alpha+\lambda)L_t^{-1}}\mathrm{d}t\right] \\ &= \frac{W_1^{(\alpha+\lambda)}(k-s+x)}{W_1^{(\alpha+\lambda)}(k)}\int_0^\infty \mathrm{e}^{-\Phi_1(\alpha+\lambda)t}\mathbb{P}^{1+\Phi_1(\alpha+\lambda)}(A)\,\mathrm{d}t \end{aligned} \tag{28}$$

where in the last equality we have applied a change of measue with respect to $\mathbb{P}^1$ using the exponential density $\exp\{\Phi_1(\alpha+\lambda)X_t - (\alpha+\lambda)t\}$.

We can apply now familiar techniques from excursion theory. Excursions of $Y$ away from zero that exceed height $k$ form a Poisson process with characteristic measure given by $W'_{1+\Phi_1(\alpha+\lambda)}(k)/W_{1+\Phi_1(\alpha+\lambda)}(k)$. The probability in the last line of (28) can now be re-written as

$$\mathbb{P}^{1+\Phi_1(\alpha+\lambda)}\left(\sup_{0\le u\le L_t^{-1}} Y_u < k, t < L(\infty)\right) = \exp\left\{-t\frac{W'_{1+\Phi_1(\alpha+\lambda)}(k)}{W_{1+\Phi_1(\alpha+\lambda)}(k)}\right\}.$$

Completing the integral in (28) much in the same way the integral $I_1$ was computed in Theorem 1 we end up with

$$\mathbb{E}^1_{s,x}\left[\int_0^{\tau_k} \mathrm{e}^{-(\alpha+\lambda)t+Y_t}d\overline{X}_t\right] = \mathrm{e}^{(s-x)}\frac{W^{(p)}(k-s+x)}{W^{(p)\prime}(k) - W^{(p)}(k)}.$$

Substituting this term back in (27) and combining with Corollary 1 can be checked to result in the expression stated. □

**Theorem 4** *Let* $\kappa_* = \inf\{x \geq 0 : Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/q\}$ *and define* $h : [0,\infty) \to [0,\infty)$ *by*

$$h(z) = (p-\lambda)\mathrm{e}^z Z^{(p)}(\kappa_* - z)/p + \lambda \mathrm{e}^z/p.$$

*Then the solution to the optimal stopping problem (25) is* $w^{CR}(s,x) = \mathrm{e}^x h(s-x)$ *where* $\tau^* = \tau_{\kappa_*}$ *is the optimal stopping time.*

The proof of the theorem uses the following observation:

**Lemma 5** *Let* $h$ *and* $\kappa_*$ *be as in Theorem 4. If* $W^{(p)}(0^+) \geq p^{-1}$ *then* $\kappa_* = 0$. *If* $W^{(p)}(0^+) < p^{-1}$, $\kappa_* > 0$ *is the unique root of* $Z^{(p)}(x) - pW^{(p)}(x) = -\lambda/p$ *and for* $t \geq 0$

$$\mathrm{e}^{-(\alpha+\lambda)(\tau_{\kappa_*}\wedge t)} h(Y_{\tau_{\kappa_*}\wedge t}) + \lambda \int_0^{\tau_{\kappa_*}\wedge t} \mathrm{e}^{-(\alpha+\lambda)s+Y_s}\mathrm{d}s$$

*is a* $\mathbb{P}^1_{s,x}$*-martingale.*

PROOF The statements involving $\kappa_*$ follow from Lemma 2. Note that $h(s-x) = \exp\{s-x\}$ when $s - x \geq \kappa_*$. Let

$$U_t = \mathrm{e}^{-(\alpha+\lambda)t} h(Y_t) + \lambda \int_0^t \mathrm{e}^{-(\alpha+\lambda)s+Y_s}\mathrm{d}s.$$

It is a matter of checking that the special choice of $\kappa_*$ together with Lemma 4 imply that $h(s-x) = \mathbb{E}^1_{s,x}[U_{\tau_{\kappa_*}}]$ for all $s - x \geq 0$.

Starting from this fact and making use of the strong Markov property, we can prove that $h(s-x)$ is equal to $\mathbb{E}^1_{s,x}\left[U_{\tau_{\kappa_*}\wedge t}\right]$, in the same vein as Remark 4. The martingale property of $U_{t\wedge\tau_{\kappa_*}}$ will follow in a similar fashion to the proof of this fact. □

PROOF OF THEOREM 4 First suppose $W^{(p)}(0^+) = 0$. We know that $U_{\tau_{\kappa_*}\wedge t}$ is a $\mathbb{P}^1_{s,x}$-martingale from the previous lemma. As earlier seen, $Z^{(p)}$ is twice differentiable everywhere with continuous derivatives except in $\kappa_*$ where it is just continuously differentiable. The Itô formula applied to $U_{\tau_{\kappa_*}\wedge t}$ implies now that necessarily on $\{t \leq \tau_{\kappa_*}\}$, and hence on $\{Y_t < \kappa_*\}$,

$$\left(\widehat{\Gamma}_1 - (\alpha+\lambda)\right) h(Y_t)\mathrm{d}t + \lambda \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t + h'(Y_{t-})d\overline{X}_t = 0$$

$\mathbb{P}^1_{s,x}$-almost surely. It can be easily checked that $h'(0) = 0$ by simple differentiation and use of the definition of $\kappa_*$. Since, as before, we see that $\overline{X}_t$ only increments when $\overline{X}_{t-} = X_{t-}$ (and this when the process creeps) it follows that the integral with respect to $d\overline{X}_t$ above is zero.

Recall that $\left(\widehat{\Gamma}_1 + r\right)(\exp\{y\}) = 0$. Since in the regime $z \geq \kappa_*$, $h(z) = \exp\{z\}$ we have on $\{X_t \geq \kappa_*\}$

$$\begin{aligned}\left(\widehat{\Gamma}_1 - (\alpha+\lambda)\right) h(Y_{t-})\mathrm{d}t + \lambda \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t &= \lambda \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t - p\mathrm{e}^{Y_t}\mathrm{d}t \\ &\leq p\mathrm{e}^{Y_t}\left(\mathrm{e}^{-(\alpha+\lambda)t} - 1\right)\mathrm{d}t,\end{aligned}$$

which is non-positive. From these inequalities we now have, as before, that $\mathbb{E}^1_{s,x}(U_t) \leq h(s-x)$ for all $t \geq 0$ and $s - x \geq 0$. Computations along the lines in the previous Lemma show that this is sufficient to conclude that $U_t$ is a $\mathbb{P}^1_{s,x}$-supermartingale.

We finish the proof of optimal stopping as in the previous optimal stopping problem. Note that

$$h(z) = \mathrm{e}^z + (p-\lambda)\mathrm{e}^z \int_{-\infty}^{\kappa_* - z} W^{(p)}(y)\,dy \geq \mathrm{e}^z.$$

By the supermartingale property and Doob's optional stopping theorem, for all almost surely finite stopping times $\tau$

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-(\alpha+\lambda)\tau+Y_\tau} + \lambda\int_0^\tau \mathrm{e}^{-(\alpha+\lambda)t+Y_t}\mathrm{d}t\right] \leq \mathbb{E}^1_{s,x}(U_\tau) \leq h(s-x).$$

Since we can make these inequalities equalities by choosing $\tau = \tau_{\kappa_*}$, we are done.

If $W^{(p)}(0^+) \in (0, q^{-1})$, the use of the Change of Variable Formula is justified by the same arguments as used in the proofs of Theorema 2, 3. The proof then goes the same as above.

Finally, if $W^{(p)}(0^+) \geq q^{-1}$, we see from Lemma 5 that $Z^{(p)}(x) - pW^{(p)}(x) \leq -\lambda/p$ for all $x$ positive and the proof runs analogously as the one of Theorem 2. Indeed one should find for all $k > s - x$ and almost surely finite $\mathbf{F}$-stopping times $\tau$,

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right] \leq \frac{p-\lambda}{p}\mathrm{e}^{s-x}Z^{(p)}(k-s+x) + \frac{\lambda}{p}\mathrm{e}^{s-x}.$$

Taking the limit of $k$ down to $s-x$ and using the continuity of $Z^{(p)}$ in zero, we conclude that for all almost surely finite stopping times $\tau$

$$\mathbb{E}^1_{s,x}\left[\mathrm{e}^{-\alpha(\tau\wedge\eta(\lambda))+Y_{(\tau\wedge\eta(\lambda))}}\right] \leq \mathrm{e}^{s-x}$$

with equality for $\tau = 0$. □

## 8.2 Canadized American Put

As before, as an extension of the perpetual option and as a first approximation to the finite time counter part, we now consider the problem of finding a rational value for the American put option with time of expiration given by the independent exponential random variable $\eta(\lambda)$. We solve the corresponding optimal stopping problem by taking $\alpha = 0$ and finding a function $w^{AC}$ and a stopping time $\tau_*$ such that

$$w^{AC}(x) = \sup \mathbb{E}_x[\mathrm{e}^{-r(\tau\wedge\eta(\lambda))}(K - S_{\tau\wedge\eta(\lambda)})^+] = \mathbb{E}_x[\mathrm{e}^{-r(\tau^*\wedge\eta(\lambda))}(K - S_{\tau^*\wedge\eta(\lambda)})^+]$$

where the supremum is taken over all $\mathbf{F}$-stopping times $\tau$.

**Proposition 3** *Let $k \leq \log K$. Then,*

$$\begin{aligned}\mathbb{E}_x[\mathrm{e}^{-r(T_k^-\wedge\eta(\lambda))}(K - S_{T_k^-\wedge\eta(\lambda)})^+] &= \frac{Kr}{q}Z^{(q)}(x-k) + \frac{K\lambda}{q}Z^{(q)}(x-\log K) \\ &\quad - \mathrm{e}^x Z_1^{(\lambda)}(x-\log K) - CW^{(q)}(x-k) + \frac{\lambda}{\Phi_1(\lambda)}\mathrm{e}^x W_1^{(\lambda)}(x-k) \quad (29)\end{aligned}$$

*where* $q = r + \lambda$ *and*

$$C = \frac{K}{\Phi(q)}\left(r + \lambda \mathrm{e}^{\Phi(q)(k-\log K)}\right) + \frac{\lambda}{\Phi_1(\lambda)}\left(\mathrm{e}^k - K\mathrm{e}^{\Phi(q)(k-\log K}\right).$$

PROOF First note that since $\eta(\lambda)$ is independent of $X$ we can write

$$\mathbb{E}_x[\mathrm{e}^{-r(T_k^- \wedge \eta(\lambda))}(K - S_{T_k^- \wedge \eta(\lambda)})^+] = \mathbb{E}_x[\mathrm{e}^{-qT_k^-}(K - S_{T_k^-})^+] + \lambda \mathbb{E}_x\left[\int_0^{T_k^-} \mathrm{e}^{-qt}(K - \mathrm{e}^{X_t})^+ \mathrm{d}t\right]. \tag{30}$$

Rewriting the second expectation on the right hand side as

$$\mathbb{E}_x\left[\int_0^{T_k^-} \mathrm{e}^{-qt}(K - \mathrm{e}^{X_t})^+ \mathrm{d}t\right] = q^{-1}\int_k^{\log K} (K - \mathrm{e}^y)\mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_k^-),$$

we see it can be evaluated using the expression for the resolvent density of $X$ killed upon entering $(-\infty, k]$, which can be extracted directly from [3, Lemma 1]. After some calculations and combined with Corollary 2 we find the formula as stated. □

**Theorem 5** *Let* $q = \lambda + r$ *and*

$$k_* = \log K + \frac{1}{\Phi(q)}\log\left(r\frac{\Phi_1(\lambda)}{\lambda}\right)$$

*and define the function* $v : \mathbb{R} \to \mathbb{R}$ *by*

$$v(x) = KrZ^{(q)}(x - k^*)/q + K\lambda Z^{(q)}(x - \log K)/q - \mathrm{e}^x Z_1^{(\lambda)}(x - \log K).$$

*Then the solution of the Canadized American Put optimal stopping problem is given by* $w^{AC} = v$ *where* $\tau_* = T_{k_*}^-$ *is the optimal stopping time.*

The proof of this Theorem is left to the reader since it is in principle similar in nature to the case of the Canadized Russian optimal stopping problem. The following Lemma, whose proof is also left to the reader for the same reasons, serves as an interim step.

**Lemma 6** *Consider the function* $v$ *and the level* $k_*$ *as defined in Theorem 5. Then* $v(x) \geq (K - \mathrm{e}^x)^+$ *and*

$$\left\{\mathrm{e}^{-q(T_{k_*}^- \wedge t)}v(X_{T_{k_*}^- \wedge t}) + \lambda\int_0^{T_{k_*}^- \wedge t} \mathrm{e}^{-qt}(K - \mathrm{e}^{X_t})^+ \mathrm{d}t : t \geq 0\right\}$$

*is a* $\mathbb{P}$*-martingale.*

# 9 Examples

In this section we provide some explicit examples of the foregoing theory and check these against known results in the literature.

## 9.1 Exponential Brownian motion

In the case of the classical Black-Scholes geometric Brownian motion model the functions $W^{(q)}$ and $Z^{(q)}$ are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2 \epsilon} \mathrm{e}^{\gamma x} \sinh(\epsilon x), \;\; Z^{(q)}(x) = \mathrm{e}^{\gamma x} \cosh(\epsilon x) - \frac{\gamma}{\epsilon} \mathrm{e}^{\gamma x} \sinh(\epsilon x)$$

on $x \geq 0$ where $\epsilon = \epsilon(q) = \sqrt{(\frac{r}{\sigma^2} - \frac{1}{2})^2 + \frac{2q}{\sigma^2}}$ and $\gamma = \frac{r}{\sigma^2} - \frac{1}{2}$. Note $\gamma \pm \epsilon$ are the roots of $\frac{\sigma^2}{2}\theta^2 + (r - \frac{\sigma^2}{2})\theta - q = 0$. Let $\kappa^*$ be given by

$$\exp(\kappa^*) = \left( \frac{\epsilon - \gamma + 1}{\epsilon + \gamma - 1} \cdot \frac{\epsilon + \gamma}{\epsilon - \gamma} \right)^{1/2\epsilon}, \tag{31}$$

then after some algebra we find the value function for the Russian optimal stopping problem is

$$w^R(x,s) = \mathrm{e}^s \left[ \frac{\epsilon + \gamma}{2\epsilon} \left( \frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}} \right)^{\epsilon - \gamma} + \frac{\epsilon - \gamma}{2\epsilon} \left( \frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}} \right)^{-\epsilon - \gamma} \right]$$

for $s - x \in [0, \kappa^*)$ and $\mathrm{e}^s$ otherwise. This expression is the same as Shepp and Shiryaev [21, 22] found. In the same vein we find an expression for the Canadized Russian case. Indeed, let $\kappa_*$ be the unique positive root of

$$(\epsilon - \gamma + 1)(\epsilon + \gamma)\mathrm{e}^{-\epsilon x} - (\epsilon + \gamma - 1)(\epsilon - \gamma)\mathrm{e}^{\epsilon x} - 2q^{-1}\epsilon\lambda \mathrm{e}^{-\gamma x} = 0,$$

where $\epsilon = \epsilon(p)$ for $p = r + \alpha + \lambda$. Then we find the value function to be given by

$$w^{CR}(s,x) = \mathrm{e}^s \left[ \frac{q}{q+\lambda} \left( \frac{\epsilon + \gamma}{2\epsilon} \left( \frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa_*}} \right)^{\epsilon - \gamma} + \frac{\epsilon - \gamma}{2\epsilon} \left( \frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa_*}} \right)^{-\epsilon - \gamma} \right) + \frac{\lambda}{q+\lambda} \right]$$

for $s - x \in [0, \kappa_*)$ and $\mathrm{e}^s$ otherwise. Now we turn our attention to the perpetual American Put option. Note $Z_1^{(\lambda)}$ is equal to the expression for $Z^{(q)}$ but now with $\gamma$ replaced by $\gamma - 1$. Plugging in the formulas for the scale functions $Z^{(q)}$ and $Z_1^{(\lambda)}$ in Theorem 3 and reordering the terms the value function is seen to be equal to

$$w^A(x) = \frac{K}{\epsilon - \gamma + 1} \exp\left( (\gamma + \epsilon)(x - k^*) \right)$$

for $x \geq k^*$ and $K - \mathrm{e}^x$ otherwise. Here the optimal crossing level is $k^* = \log K \frac{q}{\lambda} \frac{\gamma + \epsilon - 1}{\gamma + \epsilon}$. Finally, we consider the case of the Canadized American put. Define the functions $\mathbb{R} \to \mathbb{R}$

$$b(x) = K \frac{r}{q} \cdot \frac{\epsilon + \gamma}{2\epsilon} \mathrm{e}^{(\gamma - \epsilon)x} \qquad c(x) = K \frac{\lambda}{q} \cdot \frac{\epsilon - \gamma}{2\epsilon} \cdot \frac{1}{\epsilon + \gamma - 1} \mathrm{e}^{(\gamma + \epsilon)x}$$

$$p(x) = K \mathrm{e}^{(\gamma - \epsilon)x} \left[ \frac{\lambda}{q} \frac{\gamma + \epsilon}{2\epsilon} - \frac{\gamma + \epsilon - 1}{2\epsilon} \right].$$

See Carr [7] for the special interpretation of $b, c, p$. For

$$k_* = \log K + \frac{1}{\epsilon + \gamma} \log(r(\epsilon + \gamma - 1)/\lambda),$$

we find after some algebra, that the value function for the Canadized American put problem can be represented by

$$w^{CA}(x) = \begin{cases} p(x - \log K) + b(x - k_*) & \text{if } x \geq \log K, \\ K\frac{\lambda}{q} - \mathrm{e}^x + b(x - k_*) + c(x - \log K) & \text{if } x \in (k_*, \log K), \\ K - \mathrm{e}^x & \text{if } x \leq k_*. \end{cases}$$

Recalling that $q = \lambda + r$ and taking $\lambda$ to be $T^{-1}$ this formula agrees with formula (8) in [7].

## 9.2 Jump-diffusion with hyperexponential jumps

Let $X$ be

$$X_t = (a - \sigma^2/2)t + \sigma W_t - \sum_{i=1}^{N_t} Y_i$$

where $N$ is a Poisson process and $\{Y_i\}$ is a sequence of i.i.d. random variables with distribution

$$F(y) = 1 - \sum_{i=1}^{n} A_i \mathrm{e}^{-\alpha_i y}, \qquad y \geq 0,$$

where $A_i > 0$; $\sum_i A_i = 1$; and $0 < \alpha_1 < \ldots < \alpha_n$. The processes $W, N, Y$ are independent. We claim that for $x \geq 0$ the function $Z^{(q)}$ of $X$ is given by

$$Z^{(q)}(x) = \sum_{i=0}^{n+1} D_i \mathrm{e}^{\theta_i x}$$

where $\theta_0 < \theta_1 < \ldots < \theta_n < 0 < \theta_{n+1}$ are the roots of $\psi(\theta) = q$ and

$$D_i = \prod_{k=1}^{n} (\theta_i/\alpha_k + 1) \Bigg/ \prod_{k=0, k\neq i}^{n+1} (\theta_i/\theta_k - 1).$$

Indeed, recall that $\psi(\lambda)/\lambda(\psi(\lambda) - q))$ is the Laplace transform of $Z^{(q)}$. Moreover,

$$\begin{aligned} D_i &= \frac{1}{\theta_i} \frac{\prod_{k=0}^{n+1}(-\theta_k)}{\prod_{k=0,k\neq i}^{n+1}(\theta_k - \theta_i)} \frac{\prod_{k=1}^{n}(\theta_i + \alpha_k)}{\prod_{k=1}^{n} \alpha_k} \\ &= \frac{q}{\theta_i} \frac{\prod_{k=1}^{n}(\theta_i + \alpha_k)}{\prod_{k=0,k\neq i}^{n+1}(\theta_k - \theta_i)} = \frac{\psi(\theta_i)}{\theta_i} \frac{1}{\psi'(\theta_i)} \end{aligned}$$

are seen to be the coefficients in the partial fraction expansion of $\psi(\lambda)/(\lambda(\psi(\lambda) - q))$. Hence we find for the value function of the Russian option

$$w^R(s, x) = \mathrm{e}^x \begin{cases} \sum_{i=0}^{n+1} D_i \left(\frac{\mathrm{e}^{s-x}}{\mathrm{e}^{\kappa^*}}\right)^{1-\theta_i} & s - x \in [0, \kappa^*), \\ \mathrm{e}^{s-x} & s - x > \kappa^*; \end{cases}$$

where $\kappa^* > 0$ is the root of $\sum_{i=0}^{n+1}(\theta_i - 1)D_i e^{x\theta_i} = 0$. In the case of the American put option, we find after some algebra that

$$w^A(x) = \begin{cases} K - e^x & x < k^*, \\ K \sum_{i=0}^{n} \frac{D_i}{\theta_i - 1} \frac{\theta_i - \theta_{n+1}}{\theta_{n+1}} e^{\theta_i (x - k^*)} & x \geq k^*, \end{cases}$$

where $k^* = \log\left(K \frac{q}{\lambda} \frac{\theta_{n+1} - 1}{\theta_{n+1}}\right)$. These two formulas can be checked to agree with the results in the literature [18, 17]. Finally, we consider the Canadized American put option. The value function can then be checked to be given by

$$w^{CA}(x) = \begin{cases} K - e^x & x < k_*, \\ K\frac{\lambda}{q} - e^x + K\frac{r}{q}\sum_{i=0}^{n+1} D_i e^{\theta_i (x - k_*)} & x \in (k_*, \log K), \\ \frac{K}{q}\sum_{i=0}^{n} D_i \left[r\left(\frac{r}{\lambda}(\theta_{n+1} - 1)\right)^{-\theta_i/\theta_{n+1}} - \lambda(\theta_i - 1)^{-1}\right] e^{\theta_i (x - \log K)} & x > \log K, \end{cases}$$

where $k_* = \log K + \log\left(\frac{r}{\lambda}(\theta_{n+1} - 1)\right)/\theta_{n+1}$.

## 9.3 Stable jumps

We model $X$ as

$$X_t = \sigma Z_t$$

where $Z$ is a standard stable process of index $\gamma \in (1, 2]$. Its cumulant is given by $\psi(\theta) = (\sigma\theta)^\gamma$. Note the martingale restriction amounts to $1 = \sigma^\gamma$. By inverting the Laplace transform $(\psi(\theta) - q)^{-1}$, Bertoin [3] found that the $q$-scale function is given by

$$W^{(q)}(x) = \gamma \frac{x^{\gamma - 1}}{\sigma^\gamma} E'_\gamma\left(q\frac{x^\gamma}{\sigma^\gamma}\right), \qquad x > 0$$

and hene $Z^{(q)}(x) = E_\gamma(q(x/\sigma)^\gamma)$ for $x > 0$, where $E_\gamma$ is the Mittag-Leffler function of index $\gamma$

$$E_\gamma(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1 + \gamma n)}, \qquad y \in \mathbb{R}.$$

From Theorem 2, 4 , 3 and 5 we can find closed formulas for the (Canadized) Russian and American put option. In particular, we note that for the American put and its Canadized version the optimal excercise levels $k^*, k_*$ are respectively given by

$$k^* = \log K \frac{q}{\alpha} \frac{q^{1/\gamma} - \sigma}{q^{1/\gamma}} \qquad k_* = \log K + \frac{\sigma}{q^{1/\gamma}} \log \frac{r(q^{1/\gamma} - \sigma)}{\sigma\lambda}.$$

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