Upper and lower space-time envelopes for oscillating random walks conditioned to stay positive.

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Abstract

We provide integral tests for functions to be upper and lower space time envelopes for random walks conditioned to stay positive. As a result we deduce a 'Hartman-Winter' Law of the Iterated Logarithm for random walks conditioned to stay positive under a third moment assumption. We also show that under a second moment assumption the conditioned random walk grows faster than $n^{1/2}(\log n)^{-(1+\varepsilon)}$ for any $\varepsilon > 0$. The results are proved using three key facts about conditioned random walks. The first is the step distribution obtained in Bertoin and Doney (1994), the second is the pathwise construction in terms of excursions in Tanaka (1989) and the third is a new Skorohod type embedding of the conditioned process in a Bessel-3 process.

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1 Introduction

Suppose that under the law $\mathbb{P}$, the process $S = \{S_n : n \geq 0\}$ is a random walk in $\mathbb{R}$ such that $S_0 = 0$ and the step distribution S_1 satisfies $\mathbb{E}(S_1) = 0$ and $\mathbb{E}(S_1^2) = \sigma^2 < \infty$. Suppose that H_k^- is the k-th strict ascending ladder height of the reflected random walk $-S$ and define the renewal function

$$V(x) = \sum_{k \geq 0} \mathbb{P}\left(H_k^- \leq x\right).$$

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It is a non-decreasing right-continuous function. Also define $\tau^- = \inf\{n \geq 1 : S_n \in (-\infty, 0)\}$ the first time that S enters the negative portion of the real line and let $\mathcal{F}_n$ be the natural σ-algebra generated by the first k steps of S. Bertoin and Doney (1994) show that for each $A \in \mathcal{F}_n$ the limiting probabilities

$$\lim_{n\uparrow\infty} \mathbb{P}\left(A|\tau^- > n\right) \tag{1}$$

are well defined and that they induce a new measure $\mathbb{P}^\uparrow$ on the paths of S, the law of the random walk conditioned to stay positive. Further, if $\mathbb{P}_x$ is the translation of the measure $\mathbb{P}$ for which $S_0 = x \geq 0$ and $\mathbb{E}_x$ is the associated expectation operator then

$$\mathbb{P}_x^\uparrow(A) = \frac{1}{V(x)}\mathbb{E}_x\left(\mathbf{1}_{(A\cap\{\tau^->k\})}V(S_k)\right). \tag{2}$$

is the law of the random walk conditioned to stay positive but with initial value $x \geq 0$.

The results of Bertoin and Doney provide an analogue for random walks of the relationship between standard Brownian motion and Bessel-3 processes. It was shown by McKean (1963) that, in a similar sense to (1) and (2), a standard Brownian motion conditioned to stay positive has the same law as a Bessel-3 process with state space $[0, \infty)$ started from the origin.

Our aim is to investigate the asymptotics of the random walk conditioned to stay positive. We find that the results for Bessel-3 processes obtained by Motoo (1959) have almost exact counterparts in the asymptotics for the random walk conditioned to stay positive. We will demonstrate integral tests for the conditioned random walk that determines an upper space time envelope under a third moment and a lower space time envelope under a second moment condition.

Previous work on the conditioned random walk showed in a weak sense that the random walk conditioned to stay positive prefers paths that follow space-time trajectories 'in the neighbourhood' of $n^{1/2}$. For example, Iglehart (1974) showed that, under a third moment condition, rescaling the random walk $S_{[nt]}$ by $\sigma n^{1/2}$ where $t \in [0, 1]$ and then considering the law of this process in t conditioned to stay positive as n tends to infinity, one recovers essentially a rescaled Brownian meander. In Ritter (1981) it was shown that, under a second moment condition, the process $(S, \mathbb{P}^\uparrow)$ grows no slower than $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ in a weak sense. In parallel with the writing of this paper, Biggins (2001) has also established results concerning the occupation of conditioned random walks below specified levels under second moment conditions. We now state our main theorem.

Theorem 1 ***(Upper Space-Time Envelope)*** *Let $\phi \uparrow \infty$ be slowly varying and suppose $\mathbb{E}|S_1|^3 < \infty$, then*

$$\mathbb{P}^\uparrow\left(S_n > \sqrt{n\sigma^2}\phi(n) \text{ i.o. as } n\uparrow\infty\right) = 0 \text{ or } 1$$

accordingly as

$$\int^{\infty} \frac{\phi(t)^3}{t} e^{-\frac{1}{2}\phi(t)^2} dt < \infty \ \ or \ = \infty.$$

(Lower Space-Time Envelope) *Suppose that $\psi \downarrow 0$, then*

$$\mathbb{P}^{\uparrow}\left(S_n < \sqrt{n\sigma^2}\psi\left(n\right) \ \ i.o \ as \ n \uparrow \infty\right) = 0 \ or \ 1$$

accordingly as

$$\int^{\infty} \frac{\psi\left(t\right)}{t} dt < \infty \ \ or \ = \infty.$$

In this Theorem, we use the terminology 'space-time envelope' to mean a function of time which eventually, with probability one, captures the path of the conditioned random walk completely above or below it. An upper space-time envelope contains the process from above and a lower space-time envelope from below.

From the upper space-time envelope tests the LIL follows in the usual way by considering the functions $\phi(t) = ((2 \pm \epsilon) \log\log t)^{1/2}$ for all $\epsilon > 0$.

Corollary 2 ***(Law of the Iterated Logarithm)*** *Under the moment condition $\mathbb{E}|S_1|^3 < \infty$*

$$\limsup_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2 \log\log n}} = 1 \quad \mathbb{P}^{\uparrow}\text{-}a.s.$$

In addition to this Corollary, it is also possible to deduce reasonably fine statements about a lower space-time envelope though there is no corresponding LIL. Our theorem shows that, under only a second moment, the functions

$$\psi_{k+1}\left(t\right) = c\left[\prod_{i=0}^{k+1} \log_{(i)} t\right]^{-1}, \qquad \psi^{\varepsilon}_{k+1}\left(t\right) = c\left[\left(\prod_{i=0}^{k} \log_{(i)} t\right)\left(\log_{(k+1)} t\right)^{1+\varepsilon}\right]^{-1}$$

where $\log_{(i)} t$ is the i-th iterate of $\log t$ ($\log_{(0)} t = 1$) and $n \geq 1$ and c is an arbitrary positive constant, serve to produce

$$\liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2}\psi_{k+1}(n)} < 1 \text{ and } \liminf_{n\uparrow\infty} \frac{S_n}{\sqrt{2n\sigma^2}\psi^{\varepsilon}_{k+1}(n)} > 1.$$

The proof of Theorem 1 is essentially a consequence of being able to reconstruct $(S, \mathbb{P}^{\uparrow})$ in two different ways. The first comes from Tanaka (1989) and uses independent time reversed excursions of $(S, \mathbb{P})$ below the origin glued end to end to give a pathwise construction of $(S, \mathbb{P}^{\uparrow})$. From Tanaka's construction, the lower space time boundary results are read off almost immediately from existing theory of the fluctuation of random walks. The second reconstruction is new and relies on piecing together appropriately stopped passages of the Bessel-3 process in the spirit of the Skorohod embedding problem. For this latter construction, knowing the distribution of S_1 under $\mathbb{P}_x^{\uparrow}$, as given by (2), will prove to be important in the calculations.

The paper is structured as follows. We begin with some preliminary results drawn from the literature on Bessel processes and random walks. Section 3 is devoted to Tanaka's decomposition and the lower space-time envelope results that follow. In Section 4 we discuss a new Skorohod type embedding which is the key tool for the upper envelope. Finally Section 5 finishes off the proof of Theorem 1.

2 Preliminary results

The Bessel-3 process has the special property that it is equal in law to the absolute value of a standard Brownian motion plus its local time at zero. Informally speaking we can think of the paths of Bessel-3 processes as the result of gluing end to end Brownian excursions away from the origin with a small 'nudge' upwards given by the increment in local time at the end of each excursion. When the Brownian motion experiences long periods away from the origin, the local time grows slowly. On the other hand, when local time is growing quickly, the Brownian motion is making lots of small excursions from the origin. The combined effect of these two types of behaviour should in principle motivate ideas about the maximal and minimal growth of the Bessel-3 process.

A description of the upper and lower envelopes for Bessel-3 processes is due originally to Motoo (1959), although he did not appeal to the decomposition referred to above.

Theorem 3 *Let* $X = \{X_t\}_{t\geq 0}$ *be a Bessel-3 process on* $(0,\infty)$ *started from* $x \geq 0$ *with respect to the law* $\mathcal{P}_x$.

(i) Suppose that $\phi \uparrow \infty$. *Then*

$$\mathcal{P}_x\left(X_t > \sqrt{t}\phi(t) \text{ i.o. as } t\uparrow\infty\right) = 0 \text{ or } 1$$

according to the integral test

$$\int^{\infty} \frac{\phi(t)^3}{t} e^{-\frac{1}{2}\phi(t)^2} dt < \infty \text{ or } = \infty. \tag{3}$$

(ii) Further suppose that $\psi \downarrow 0$,

$$\mathcal{P}_x\left(X_t < \sqrt{t}\psi(t) \text{ i.o. as } t\uparrow\infty\right) = 0 \text{ or } 1$$

according to the integral test

$$\int^{\infty} \frac{\psi(t)}{t} dt < \infty \text{ or } = \infty. \tag{4}$$

Note that the original version of this theorem was stated for $x = 0$. However, the decomposition of a Bessel-3 started from $x > 0$ on $(0, \infty)$ in terms of a stopped Brownian motion and a Bessel-3 started from 0 on $[0, \infty)$ given by Williams (1973) means that this theorem is equally valid for processes on $[0, \infty)$ starting from $x > 0$. Williams' decomposition says that a Brownian motion started from $x > 0$, stopped when it hits a uniformly chosen point in $(0, x)$ with the path of a Bessel-3 started from 0 and with state space $[0, \infty)$ glued on at this end point, has the same law as a Bessel-3 process started at $x > 0$ and with state space $(0, \infty)$. Since the afore mentioned hitting time is almost surely finite, a simple space-time translation in Motoo's theorem for $x = 0$ yields the version we have given above.

Despite the apparent gap in the literature for conditioned random walks, there are certain theorems for random walks which offer integral tests remarkably similar to those of Motoo. In particular, we refer to the collective results of Khinchin (1924), Kolmogorov (1929), Feller (1946), Hirsch (1965) and Csáki (1978) which we have summarized in the Theorem below.

Theorem 4 *Suppose that the random walk $(S, \mathbb{P})$ has step distribution such that*

$$\limsup_{x \uparrow \infty} \log \log x \int_{|y|>x} y^2 \mathbb{P}\left(S_1 \in dy\right) < \infty. \tag{5}$$

(It would suffice then for example that $\mathbb{E}\left(S_1^2 \left|\log \log |S_1|\right|\right) < \infty$).

(i) Let $\phi \uparrow \infty$ then

$$\mathbb{P}\left(S_n > \sqrt{n\sigma^2}\phi(n) \;\; i.o \; as \; n \uparrow \infty\right) = 0 \; or \; 1$$

accordingly as

$$\int^{\infty} \frac{\phi(t)}{t} e^{-\frac{1}{2}\phi(t)^2} dt < \infty \; or \; = \infty.$$

(ii) Now suppose that $\psi \downarrow 0$. Then

$$\mathbb{P}\left(\max_{k \leq n} S_k < \sqrt{n\sigma^2}\psi(n) \;\; i.o \; as \; n \uparrow \infty\right) = 0 \; or \; 1$$

accordingly as

$$\int^{\infty} \frac{\psi(t)}{t} dt < \infty \; or \; = \infty. \tag{6}$$

As we earlier mentioned, it will become apparent that the results we obtain for the conditioned random walk contain identicle integral tests to the previous two theorems simply becasue they are a result of translating (parts of) these theorems into the relevant context.

3 Tanaka's Decomposition and lower space-time envelopes

Tanaka (1989) gives the following fundamental construction of $(S, \mathbb{P}^{\uparrow})$ from $(S, \mathbb{P})$. Let $e_1, e_2, ...$ be independent copies of the time reversed negative excursion

$$e = (0, S_{\sigma^+} - S_{\sigma^+ -1}, S_{\sigma^+} - S_{\sigma^+ -2}, ..., S_{\sigma^+} - S_1, S_{\sigma^+})$$

under the measure $\mathbb{P}$ where $\sigma^+ = \inf\{k \geq 0 : S_k \in (0, \infty)\}$. We write $e_k = (e_k(1), ..., e_k(l_k))$. The process $S^{\uparrow} = \left\{S_n^{\uparrow} : n \geq 0\right\}$, constructed by gluing these time-reversed excursions end to end,

$$S_n^{\uparrow} = \begin{cases} e_1(n) & \text{for } 0 \leq n \leq l_1 \\ e_1(l_1) + e_2(n - l_1) & \text{for } l_1 < n \leq l_2 + l_1 \\ \vdots & \vdots \\ \sum_{j=1}^{k-1} e_j(l_j) + e_k\left(n - \sum_{j=1}^{k-1} l_j\right) & \text{for } \sum_{j=1}^{k-1} l_j < n \leq \sum_{j=1}^{k} l_j \\ \vdots & \vdots \end{cases}$$

forms a Markov chain whose transition function is that of $\left(S, \mathbb{P}^{\uparrow}\right)$. That is to say that the pair $(S^{\uparrow}, \mathbb{P})$ and $(S, \mathbb{P}^{\uparrow})$ are identical. Note that this construction is analogous to the description of a Bessel-3 process in terms of the absolute value of a standard Brownian motion plus its local time at zero.

Let $\{(M_k^+, \upsilon_k^+)\}_{k\geq 0}$ be the space-time points of increase of the 'future minimum' of $(S^{\uparrow}, \mathbb{P})$. That is, $\upsilon_0^+ = 0$,

$$\upsilon_k^+ = \inf\left\{n > \upsilon_{k-1}^+ : \min_{r\geq n} S_r = S_n\right\} \text{ and } M_k^+ = S_{\upsilon_k^+}$$

From this construction, we can deduce that path for path, the sequence $\{(M_k^+, \upsilon_k^+)\}_{k\geq 0}$ corresponds precisely to $\left\{(H_k^+, \sigma_k^+)\right\}_{k\geq 0}$, the sequence of strictly increasing ladder heights and times respectively of $(S, \mathbb{P})$. To see this note that the paths of the random walk $(S, \mathbb{P})$ can be reconstructed by gluing together end to end independent copies of the $(S, \mathbb{P})$-excursion (without time reversing),

$$(S_0, S_1, ..., S_{\sigma^+}),$$

(note that $\sigma^+ = \sigma_1^+$). The space-time points for the beginning and ends of these excursions are the same points as for the time reversed excursions used in Tanaka's construction of a conditioned random walk. (This can be best seen in a simple sketch where one can go from the one process to the other by systematically extracting the excursions, rotating them by 180° and then replacing them between the same end points).

Let $L = \{L_n\}_{n\geq 0}$ be the local time at the maximum in $(S, \mathbb{P})$, that is

$$L_n = \left|\left\{k \leq n : \max_{i\leq k} S_i = S_k\right\}\right|.$$

Equivalently, L is the local time at the future minimum of $(S^{\uparrow}, \mathbb{P})$. The hierarchy

$$S_n \leq H^+_{L_n} = M^+_{L_n} \leq S^{\uparrow}_n. \tag{7}$$

is now easy to see from the pathwise (excursion) constructions of $(S^{\uparrow}, \mathbb{P})$ and $(S, \mathbb{P})$.

Proof of Theorem 1 (Lower space time envelope). Suppose that $\psi \downarrow 0$ satisfies the divergent integral test in part (ii) of Theorem 4. Since $H^+_{L_n} = \max_{i \leq n} S_i$, it follows

$$\begin{aligned} 1 &= \mathbb{P}\left(\max_{i\leq n} S_i < \sqrt{\sigma^2 n}\psi(n) \text{ i.o. as } n \uparrow \infty\right) \\ &= \mathbb{P}\left(M^+_{L_n} < \sqrt{\sigma^2 n}\psi(n) \text{ i.o. as } n \uparrow \infty\right) \end{aligned}$$

which shows there exists a sequence of (random) times such that the previous future minimum of $(S^{\uparrow}, \mathbb{P})$ is below the space-time curve $\sqrt{n\sigma^2}\psi(n)$ infinitely often. Let the increasing sequence of random times at which this occurs be denoted by $\{n_i\}_{i\geq 1}$. That is, with probability one, $H^+_{L_{n_i}} < \sqrt{n_i\sigma^2}\psi(n_i)$ for each i. Define $\triangle H^+_k = H^+_k - H^+_{k-1}$ and n'_i the next time index after n_i at which the process $H^+_{L_n}$ increments (that is $\nu^+_{L_{n_i}+1} = \sigma^+_{L_{n_i}+1}$ the inverse local time of $L_{n_i}+1$). Note that

$$\begin{aligned} S^{\uparrow}_{n'_i} &= H^+_{L_{n_i}} + \triangle H^+_{L_{n_i}+1} \\ &< \sqrt{n'_i\sigma^2}\psi(n'_i) + \triangle H^+_{L_{n_i}+1}. \end{aligned} \tag{8}$$

Without loss of generality (as far as the interesting cases are concerned) we have assumed that $\sqrt{n}\psi(n)$ is an increasing function. Since $\triangle H^+_k$ has the same distribution as S_{σ^+} under $\mathbb{P}$, it follows from Feller (1971) that $\gamma := \mathbb{E}(\triangle H^+_k) = \sigma^{-1}\sqrt{2}e^{\chi^+}$ where

$$\chi^+ = \sum_{n\geq 1} \frac{1}{n}\left[\mathbb{P}(S_n > 0) - \frac{1}{2}\right]$$

and is finite. The strong law of large numbers implies that $\lim_{k\uparrow\infty} H^+_k/k = \gamma$ from which it can be deduced that $\limsup_{k\uparrow\infty} \triangle H^+_k/k = 0$, c.f. Lyons *et al.* (1995). Hence it follows that $\limsup_{k\uparrow\infty} \triangle H^+_k/H^+_k = 0$. We now have for sufficiently large i and any small $\varepsilon > 0$ that

$$\triangle H^+_{L_{n_i}+1} < \varepsilon\left(H^+_{L_{n_i}} + \triangle H^+_{L_{n_i}+1}\right) \text{ a.s.}$$

leading to (for sufficiently small $\varepsilon > 0$),

$$\triangle H^+_{L_{n_i}+1} < \frac{\varepsilon}{1-\varepsilon}\sqrt{n'_i\sigma^2}\psi(n'_i) \text{ a.s.}$$

Continuing from (8) we now have that there exist an increasing sequence of times such that with probability one,

$$S^{\uparrow}_{n'_i} < \frac{1}{1-\varepsilon}\sqrt{n'_i\sigma^2\psi(n'_i)}$$

for all i. Since the integral test which ψ satisfies is unaffected by multiplicative constants in front of ψ, we have shown that

$$\mathbb{P}^{\uparrow}\left(S_n < \sqrt{\sigma^2 n}\psi(n) \text{ i.o. as } n\uparrow\infty\right) = 1$$

Consider now under the same moment condition, the case that $\psi\downarrow 0$ satisfies the convergent integral test in part (ii) of Theorem 4, with the help of (7)

$$\begin{aligned} 1 &= \mathbb{P}\left(\max_{i\le n} S_i \ge \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right) \\ &= \mathbb{P}\left(S^{\uparrow}_n \ge \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right) \\ &= \mathbb{P}^{\uparrow}\left(S_n \ge \sqrt{\sigma^2 n}\psi(n) \text{ ev. as } n\uparrow\infty\right) \end{aligned}$$

where 'ev.' means eventually. ■

4 Skorohod embedding

Suppose now for convenience in this section and the next, we change the definition of $\mathbb{P}^{\uparrow}_x$ (and accordingly $\mathbb{E}^{\uparrow}_x$) to be the law of the process $V(S) = \{V(S_n)\}_{n\ge 0}$ under the assumption that $V(S_0) = x$. As V is the renewal function associated with the increasing ladder heights of the reflected random walk $(-S, \mathbb{P})$,

$$\lim_{x\uparrow\infty}\frac{V(x)}{x} = \frac{1}{\mathbb{E}(H_1^-)} = \frac{\sigma}{\sqrt{2}}e^{-\chi^-} \tag{9}$$

where, according to Feller (1971), χ^- is finite and given by

$$\chi^- = \sum_{n\ge 1}\frac{1}{n}\left[\mathbb{P}(S_n < 0) - \frac{1}{2}\right].$$

Thus, as our lower envelope shows that the process S drifts to infinity, S and $V(S)$ are asymptotically equivalent (up to multiplication by a constant).

For all $x\ge 0$, let

$$x' = \frac{1}{\mathbb{E}^{\uparrow}_x\left(\frac{1}{V(S_1)}\right)} = \frac{x}{\mathbb{P}_x(S_1 > 0)} \ge x. \tag{10}$$

Recall that $\mathcal{P}_y$ is the law of the Bessel-3 process on $(0,\infty)$ started at $y\ge 0$ and let $\mathcal{E}_y$ be the associated expectation operator.

Theorem 5 *For each $x \geq V(0)$, there exists a stopping time T with $\mathcal{E}_{x'}T < \infty$ such that X_T under $\mathcal{P}_{x'}$ has the same distribution as $V(S_1)$ under $\mathbb{P}_x^{\uparrow}$.*

We shall prove this theorem for the case that the step distribution of S has no atoms. This is purely for the sake of notational convenience and the proof is almost identical when this restriction is not in place.

Proof. The proof follows the standard proof of the Skorohod embedding for Brownian motion with some adaptations.

It is known (Revuz and Yor (1994)) that the Bessel-3 has scale function $s(z) = -1/z$. For this reason we shall work with $\rho_t = 1/X_t$, which is a local martingale. Note that under $\mathcal{P}_{x'}$, $\rho_0 = 1/x' = \mathbb{E}_x^{\uparrow}(1/V(S_1))$. Now write

$$\mu(dy) = \mathbb{P}_x^{\uparrow}\left(\frac{1}{V(S_1)} \in dy\right)$$

and define $F^+(db) = \mu(db)$ for $1/x' < b < \infty$ and $F^-(da) = \mu(da)$ for $0 \leq a \leq 1/x'$. It follows by symmetry that there exists a positive constant γ such that

$$\int_{1/x'}^{\infty} \left\{b - \frac{1}{x'}\right\} \mu(db) = \int_{0}^{1/x'} \left\{\frac{1}{x'} - a\right\} \mu(da) = \gamma.$$

Next we define a bivariate random variable (α, β) defined on $(0, 1/x'] \times (1/x', \infty]$ having distribution

$$\Pr((\alpha, \beta) \in da \times db) = \frac{1}{\gamma}(b - a) F^-(da) F^+(db).$$

Setting

$$T = \inf\{t \geq 0 : \rho_t \notin (\alpha, \beta)\}$$

it is straight forward to check that $\rho_T = 1/X_T$ has the same distribution under $\mathcal{P}_{x'}$ as $1/V(S_1)$ under $\mathbb{P}_x^{\uparrow}$. Indeed for $1/x' < b < \infty$, using the fact that the process ρ is in scale,

$$\begin{aligned}\mathcal{P}_{x'}(\rho_T \in db) &= \int_{0}^{1/x'} \frac{1/x' - a}{b - a} \frac{1}{\gamma}(b - a) F^-(da) F^+(db) \\ &= F^+(db).\end{aligned}$$

An identical calculation follows for $\mathcal{P}_{x'}(\rho_T \in da)$ where $0 \leq a \leq 1/x'$. In conclusion the distribution of X_T under $\mathcal{P}_{x'}$ is the same as $V(S_1)$ under $\mathbb{P}_x^{\uparrow}$.

To conclude, we show that $\mathcal{E}_{x'}T < \infty$ using the local martingale $U_t = X_t^2 - 3t$. As T is the hitting time of a two sided (albeit randomized) barrier, it can be shown in the usual way that, since U is bounded up to T, $U_{t\wedge T}$ is a $\mathcal{P}_{x'}$-martingale with mean $(x')^2$. Conditioning on the randomized barrier,

$$\begin{aligned}\mathcal{E}_{x'}T &= E\left(\mathcal{E}_{x'}\left(T|\alpha,\beta\right)\right) \\ &= \frac{1}{3}E\left(\mathcal{E}_{x'}\left(X_T^2|\alpha,\beta\right)-(x')^2\right) \\ &= \frac{1}{3}\left(\int y^2\mu\left(dy\right)-(x')^2\right), \qquad (11)\end{aligned}$$

which is finite for all $x > 0$. ■

In the obvious way we can now glue together samples of Bessel-3 processes so that the process $(S, \mathbb{P}^{\uparrow})$ is embedded in a Bessel-3 process and can be recovered by sampling the continuous time process with an appropriate sequence of stopping times.

Corollary 6 (Skorohod embedding) *There exists a sequence of stopping times $\{\eta_n\}_{n\geq 0}$, with $\mathcal{E}_0\eta_n < \infty$, such that the distribution of $V(S_n)$ under $\mathbb{P}^{\uparrow}$ is equal to X_{η_n} under $\mathcal{P}_0$ for all $n \geq 0$. Further, $\eta_n = \sum_{i=0}^{n}(\tau_i + T_i)$ where $\tau_0 = T_0 = 0$ and*

(i) *given $\mathcal{F}_{i-1} = \sigma(X_t : t \leq \eta_{i-1})$*

$$\tau_i = \inf\left\{t \geq 0 : X_{t+\eta_{i-1}} = x'_i\right\} \text{ where } x'_i = \frac{1}{\mathbb{E}^{\uparrow}_{X_{\eta_{i-1}}}\left(\frac{1}{V(S_1)}\right)}$$

(ii) *given $\mathcal{G}_{i-1} = \sigma(X_s : s \leq \eta_{i-1} + \tau_i)$*

$$T_i = \inf\left\{t \geq 0 : X_{t+\eta_{i-1}+\tau_i} \overset{d}{=} V(S_i)\right\}.$$

Proof. Note that the stopping time in (i) brings the Bessel-3 process to the point which is equal in distribution to the mean of $1/V(S_i)$ given $V(S_{i-1})$. It is straightforward to check (see (10)) that given $\mathcal{F}_{i-1}$, $x'_i \geq X_{\eta_{i-1}}$ and hence, by the upward drift of Bessel-3, τ_i conditionally has finite mean. The stopping time in (ii) is a version of the stopping time T defined in Theorem 5 and hence also has finite mean. ■

The times $\{\eta_n\}_{n\geq 0}$ have finite mean but they are not identically distributed. In order to use our Skorohod embedding we need a strong law of large numbers for the sequence $\{\eta_n\}_{n\geq 0}$.

Lemma 7 *Assume the moment condition $\mathbb{E}|S_1|^3 < \infty$, then*

$$\lim_{n\uparrow\infty}\frac{\eta_n}{n} = \frac{\sigma^4}{2}e^{-2\chi^-} \ \mathcal{P}_0\text{-}a.s.$$

Proof. To prove this strong law of large numbers we will show that

$$\limsup_{n\uparrow\infty}\frac{1}{n}\sum_{i=1}^{n}\tau_i=0 \text{ and } \lim_{n\uparrow\infty}\frac{1}{n}\sum_{i=1}^{n}T_i=\frac{\sigma^4}{2}e^{-2\chi^-} \tag{12}$$

$\mathcal{P}_0$-almost surely.

For the first of these two, note first of all that the conclusions concerning lower space-time envelopes show that $V(S_n)$ tends to infinity $\mathbb{P}^{\uparrow}$-almost surely at a rate faster than $n^{1/2-\varepsilon}$ ($\varepsilon>0$) for example. Let $\xi=\sigma e^{-\chi^-}/\sqrt{2}$. Using (10) with the asymptotics of (9),

$$0\le\frac{1}{\mathbb{E}_x^{\uparrow}\left(\frac{1}{V(S_1)}\right)}-x\sim\frac{x\mathbb{P}(S_1\le -x/\xi)}{\mathbb{P}(S_1>-x/\xi)}\to 0$$

as $x\uparrow\infty$. Since the $\mathcal{P}_x$-probability of a Bessel-3 process hitting $y\ge x$ is 1, the expected hitting time is finite and decreasing to zero as $y\downarrow x$ and $V(S_n)>n^{1/2-\varepsilon}$ $\mathbb{P}^{\uparrow}$-eventually it is easy to show that $\limsup_{n\uparrow\infty}(1/n)\sum_{i=1}^{n}\tau_i=0$.

To show the second of the limit results in (12) begin by recalling from (11) that

$$\begin{aligned}3\mathcal{E}_{x'_{i-1}}\left(T_i|\mathcal{G}_{i-1}\right)&=\mathcal{E}_{x'_{i-1}}\left(X_{T_i}^2\middle|\mathcal{G}_{i-1}\right)-\left(x'_{i-1}\right)^2\\&=\Gamma\left(X_{\eta_{i-1}}\right)\end{aligned}$$

where

$$\begin{aligned}\Gamma(x)&=\mathbb{E}_x^{\uparrow}\left(V(S_1)^2\right)-\frac{x^2}{\mathbb{P}_x(S_1>0)^2}\\&=\frac{\mathbb{E}_x\left(V(S_1)^3\mathbf{1}_{(S_1>0)}\right)}{x}-\frac{x^2}{\mathbb{P}_x(S_1>0)^2}.\end{aligned}$$

As $x\uparrow\infty$, by (9), the last equality is asymptotically equivalent to

$$\begin{aligned}&\frac{\xi^3\mathbb{E}\left((x/\xi+S_1)^3\mathbf{1}_{(S_1>-x/\xi)}\right)}{x}-\frac{x^2}{\mathbb{P}(S_1>-x/\xi)^2}\\&\quad=-x^2\frac{1-\mathbb{P}(S_1>-x/\xi)^3}{\mathbb{P}(S_1>-x/\xi)^2}+3x\xi^2\mathbb{E}\left(S_1\mathbf{1}_{(S_1>-x/\xi)}\right)\\&\quad+3\xi^2\mathbb{E}\left(S_1^2\mathbf{1}_{(S_1>-x/\xi)}\right)+\frac{\xi^3}{x}\mathbb{E}\left(S_1^3\mathbf{1}_{(S_1>-x/\xi)}\right).\end{aligned}$$

Label the four terms on the right hand side of the last equality as $a_1,..,a_4$. The idea is to show that all of them tend to zero with the exception of a_3 which, without further calculation, can be seen to converge to $3\sigma^2\xi^2$.

Starting from the first term,

$$
\begin{aligned}
|a_1| &= x^2 \left(1 - \mathbb{P}\left(S_1 > -x/\xi\right)\right) \frac{1 + \mathbb{P}\left(S_1 > -x/\xi\right) + \mathbb{P}\left(S_1 > -x/\xi\right)^2}{\mathbb{P}\left(S_1 > -x/\xi\right)^2} \\
&= x^2 \mathbb{P}\left(S_1 \le -x/\xi\right) \frac{1 + \mathbb{P}\left(S_1 > -x/\xi\right) + \mathbb{P}\left(S_1 > -x/\xi\right)^2}{\mathbb{P}\left(S_1 > -x/\xi\right)^2} \to 0,
\end{aligned}
$$

by Markov's inequality and the third moment assumption. For the second term, as $\sigma^2 < \infty$,

$$
\begin{aligned}
|a_2| &= \left| -3x\xi^2 \mathbb{E}\left(S_1 \mathbf{1}_{(S_1 \le -x/\xi)}\right) \right| \\
&\le 3\xi^3 \mathbb{E}\left(S_1^2 \mathbf{1}_{(S_1 \le -x/\xi)}\right) \to 0.
\end{aligned}
$$

Finally, with the third moment condition, it is clear that $a_4 \to 0$.

In conclusion we have proved that as i tends to infinity, since $x'_{i-1} \ge X_{\eta_{i-1}} \stackrel{d}{=} V\left(S_{i-1}\right)$ and $V\left(S_n\right) > n^{1/2-\varepsilon}$ $\mathbb{P}^{\uparrow}$-eventually,

$$
\mathcal{E}_{x'_{i-1}}\left(T_i | \mathcal{G}_{i-1}\right) \to \frac{\sigma^4}{2} e^{-2\chi^-}
$$

almost surely and hence the second limit result in (12) holds. ■

Remark 8 The third moment condition is needed for the convergence of a_4 to zero. We do not know if this is the best possible (see also Remark 11 later).

Remark 9 It is worth noting that we used (a conclusion deducible from) the results on lower space-time envelopes in order to prove this Skorohod type embedding. In the way we have derived our results here, it would not make sense to use the Skorohod embedding to translate the lower space-time envelope results of Motoo to the case of the conditioned random walk as we are intending to do for the upper space time envelopes.

5 Completing the proofs

Recall that we still have in force the slightly changed definition of $\mathbb{P}_x^{\uparrow}$. Assume also that $\mathbb{E}|S_1|^3 < \infty$. We begin by assuming the function $\phi \uparrow \infty$ is slowly varying at infinity and satisfies the integral test

$$
\int^{\infty} \frac{\phi(t)^3}{t} e^{-\frac{1}{2}\phi(t)^2} dt < \infty.
$$

We want to prove under these conditions that

$$
\limsup_{n \uparrow \infty} \frac{S_n}{\sqrt{n\sigma^2}\,\phi\left(n\right)} < 1 \qquad \mathbb{P}^{\uparrow}\text{-a.s.} \tag{13}
$$

From Motoo's Theorem for Bessel-3 processes, the fact that ϕ is slowly varying (and hence slowly varying uniformly on compacts) and Lemma 7 we can immediately say (as we are sampling the process X at a subsequence) that

$$\begin{aligned}\limsup_{n\uparrow\infty}\frac{X_{\eta_n}}{\sqrt{n\sigma^2}\phi(n)} &= \limsup_{n\uparrow\infty}\frac{X_{\eta_n}}{\sqrt{\eta_n}\phi(\eta_n)}\cdot\frac{1}{\sigma}\sqrt{\frac{\eta_n}{n}}\frac{\phi\left(\frac{\eta_n}{n}\cdot n\right)}{\phi(n)}\\ &< \frac{\sigma}{\sqrt{2}}e^{-\chi^-},\end{aligned}$$

$\mathcal{P}_0$-almost surely. Together with Corollary 6 and (9) the above inequality gives us (13).

Now, with the same conditions on ϕ and the step distribution as above, we assume

$$\int^{\infty}\frac{\phi(t)^3}{t}e^{-\frac{1}{2}\phi(t)^2}dt=\infty.$$

Once again we will use Motoo's result via the Skorohod embedding to prove

$$\limsup_{n\uparrow\infty}\frac{S_n}{\sqrt{n\sigma^2}\phi(n)}>1,\qquad \mathbb{P}^{\uparrow}\text{-a.s.}\tag{14}$$

The following modulus of continuity result will therefore be necessary.

Lemma 10 *Let $\kappa=\sigma^4e^{-2\chi^-}/2$ then the Bessel-3 process X satisfies*

$$\lim_{t\uparrow\infty}\frac{X_{\eta_{[t]}}-X_{\kappa t}}{\sqrt{2t\log\log t}}=0\ \mathcal{P}_0\text{-}a.s.$$

Proof. To prove this Lemma, it suffices to prove that

$$\lim_{r\downarrow 0}\limsup_{n\uparrow\infty}\sup_{r^n\le t\le r^{n+1}}\frac{|X_t-X_{r^n}|}{\sqrt{2r^n\log\log(r^n)}}=0\ \mathcal{P}_0\text{-a.s.}$$

as $\eta_{[t]}/t\to\kappa$ almost surely as $t\uparrow\infty$. The proof of this is similar to that of the corresponding result for Brownian motion, given as Theorem 12.6 of Kallenberg (1997). The key estimate is that the maximum of a Bessel-3 process over a given time interval has the same exponential tail estimate as that of the supremum of a Brownian motion over the same time interval. This is a standard result and can be found, for example, in Borodin and Salminen (1996). ∎

Continuing with the proof of (14) we have from Motoo's theorem and the slow variation of ϕ, that

$$\begin{aligned}\limsup_{n\uparrow\infty}\frac{X_{\eta_n}}{\sqrt{n\sigma^2}\phi(n)} &= \frac{\sqrt{\kappa}}{\sigma}\limsup_{t\uparrow\infty}\frac{X_{\eta_{[t]}}}{\sqrt{\kappa t}\phi(\kappa t)}\\ &= \frac{\sqrt{\kappa}}{\sigma}\limsup_{t\uparrow\infty}\left\{\frac{X_{\eta_{[t]}}-X_{\kappa t}}{\sqrt{\kappa t}\phi(\kappa t)}+\frac{X_{\kappa t}}{\sqrt{\kappa t}\phi(\kappa t)}\right\}.\end{aligned}$$

almost surely. We can assume that $\phi(t) \geq (2\log\log t)^{1/2}$ without loss of generality as it focuses on the interesting cases near to the transition of finiteness in the integral test giving the dichotomy. We have by the previous Lemma that

$$\limsup_{n\uparrow\infty} \frac{X_{\eta_n}}{\sqrt{n\sigma^2}\phi(n)} > \frac{\sigma}{\sqrt{2}} e^{-\chi^-}$$

The lower bound (14) follows from this last calculation together with the Skorohod embedding and (9).

Remark 11 Returning to the pathwise construction $(S^\uparrow, \mathbb{P})$ and the hierarchy in (7), we can conclude that when $\phi \uparrow \infty$ satisfies the divergent integral test of part (i) of Theorem 4 and (5) holds,

$$\begin{aligned}
\mathbb{P}^\uparrow\left(S_n > \sqrt{n\sigma^2}\phi(n) \text{ i.o. as } n\uparrow\infty\right) &= \mathbb{P}\left(S_n^\uparrow > \sqrt{n\sigma^2}\phi(n) \text{ i.o. as } n\uparrow\infty\right) \\
&\geq \mathbb{P}\left(S_n > \sqrt{n\sigma^2}\phi(n) \text{ i.o. as } n\uparrow\infty\right) \\
&= 1.
\end{aligned}$$

Thus we achieve a lower bound for the upper space-time envelope with a weaker moment condition but according to a slightly different integral test. Note however, that this result also takes care of the functions which are not covered by the assumption made in the previous paragraph.

To see what difference there is in these in the integral tests

$$\int^\infty \frac{\phi(t)^3}{t} e^{-\frac{1}{2}\phi(t)^2} dt = \infty \text{ and } \int^\infty \frac{\phi(t)}{t} e^{-\frac{1}{2}\phi(t)^2} dt = \infty,$$

we need to look at finer functions than $\phi(t) = (2\log\log t)^{1/2}$. Consider for example the function

$$\phi(t) = [2\log\log t + 4\log\log\log t]^{\frac{1}{2}},$$

which fulfils the former of the previous two integral tests but not the second. It would seem that the price of establishing finer space-time boundaries which just fail to be upper envelopes is the higher third moment condition.

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References

[1] Bertoin, J. and Doney, R.A. (1994) On conditioning a random walk to stay positive. *Ann. Probab.* **22**, 2152–2167.

[2] Biggins (2001) Random walk conditioned to stay positive. *Preprint.*

[3] Borodin, A.N. and Salminen, P. (1996) *Handbook of Brownian Motion - Facts and Formulae.* Probability and Its Applications. Birkhäuser.

[4] Csáki, E. On the lower limits of maxima and minima of Wiener processes and partial sums. *Z. Wahr. werw. Geb.* **43**, 205–221.

[5] Feller, W. (1946) The law of the iterated logarithm for indentically distributed random variables. *Ann. Math.* **47**, 631–638.

[6] Feller, W. (1971) *An Introduction to Probability Theory and its Applications.* Vol II. 2nd ed. Wiley, New York.

[7] Hirsch, W.M. (1965) A strong law for the maximum cumulative sum of independent random variables. *Comm. Pure Appl. Math.* **18**, 109–127.

[8] Iglehart, D. L. (1974) Functional central limit theorems for random walks conditioned to stay positive. *Ann Probab.* **2**, 608–619.

[9] Kallenberg, O. (1997) *Foundations of Modern Probability.* Springer.

[10] Khinchin, A. (1924) Ein Satz der Warscheinlichkeitsrechung. *Fund. Math.* **6**, 9–10.

[11] Kolmogorov, A. (1929) Uber das gesetz des iterierten logarithmus. *Math. Ann.* **101**, 126–135.

[12] Lyons, R., Pemantle, R. and Peres, Y. (1995) Conceptual proofs of LlogL criteria for mean behaviour of branching processes. *Ann. Probab.* **23**, 1125–1138.

[13] McKean, H.P. (1963) Excursions of a non-singular diffusion. *Z. Wahr. werw. Geb.* **1**, 230–239.

[14] Motoo (1959) Proof of the iterated logarithm through diffusion equation. *Ann. Inst. Statist. Math.* **10**, 21–28.

[15] Revuz, D. and Yor, M. (1994) *Continuous martingales and Brownain motion.* Springer-Verlag.

[16] Ritter, G. A. (1981) Growth of random walks conditioned to stay positive. *Ann. Probab.* **9**, 699–704.

[17] Shiga, T. and Watanabe, S. (1973) Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahr. werw. Geb.* **27**, 37–46

[18] Tanaka, H. (1989) Time reversal of random walks in one dimention. *Tokyo J. Math.* **12**, 159–174.

[19] Williams, D. (1974) Path decomposition and continuity of local time for one dimensional diffusions I. *Proc. London. Math. Soc.* **28**, 738–768