Snellius versneld

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1 Introduction

Practically all computations of the value of π before 1600 were done using Archimedes' method. As is well-known, this method consists of approximation of the circle with diameter 1 by inscribed and circumscribed regular polygons. Denote the circumference of the inscribed and circumscribed regular N-gon by P_N and Q_N respectively. Then $P_N < \pi < Q_N$ and

$$\lim_{N\to\infty} P_N = \lim_{N\to\infty} Q_N = \pi$$

A little trigonometry shows us that

$$Q_N = N \tan\frac{\pi}{N} \quad , \quad P_N = N\sin\frac{\pi}{N} \tag{1}$$

from which the duplication formulae

$$Q_{2N} = \frac{2P_N Q_N}{P_N + Q_N} \quad , \quad P_{2N} = \sqrt{P_N Q_{2N}} \tag{2}$$

follow readily. As is well-known, Archimedes, started with the values $Q_6 = 2\sqrt{3}$ and $P_6 = 3$ and calculated $Q_{12}, P_{12}, Q_{24}, \ldots, Q_{96}, P_{96}$ consecutively using the duplication formulae (2). See [A]. We also know that Ludolph van Ceulen, around 1600, continued this procedure until he obtained 35 decimal places of π. To get an idea of the accuracy of the approximation Q_N to π we use the Taylor series expansion of $\tan x$. We get,

$$\begin{aligned} Q_N &= N\left(\frac{\pi}{N} + \frac{1}{3}\left(\frac{\pi}{N}\right)^3 + \frac{2}{15}\left(\frac{\pi}{N}\right)^5 + \cdots\right) \\ &= \pi + \frac{1}{3}\frac{\pi^3}{N^2} + \frac{2}{15}\frac{\pi^5}{N^4} + \cdots \end{aligned}$$

In other words, $Q_N - \pi$ has order of magnitude $O(\frac{1}{N^2})$. More precisely, $Q_N - \pi$ is equal to $\frac{\pi^3}{3N^2}$ up to order $O(\frac{1}{N^4})$. Similarly, $P_N - \pi$ equals $-\frac{\pi^3}{6N^2}$ up to order $O(\frac{1}{N^4})$. From these two facts it follows immediately that $\frac{1}{3}(Q_N - \pi) + \frac{2}{3}(P_N - \pi) = \frac{1}{3}Q_N + \frac{2}{3}P_N - \pi$ has order $O(\frac{1}{N^4})$. So we see that $\frac{1}{3}Q_N + \frac{2}{3}P_N$ gives an approximation of π having approximately twice

as many correct digits as P_N or Q_N. This was discovered by the Dutch natural scientist Willibrord Snellius in 1621, about ten years after Van Ceulen's death. In fact, Snellius was Van Ceulen's successor at the University of Leiden. Snellius used geometrical observations to find his approximations. Only very much later Christiaan Huygens delivered a complete proof of the correctness of these observations.
The conclusion is that Van Ceulen could have stopped halfway through his calculations, compute $\frac{1}{3}Q_N + \frac{2}{3}P_N$ for the value of N then reached, and obtain 35 decimal places of π. Such a speedup of calculation makes one wonder if Snellius' discovery can be generalised in its turn. It is the purpose of this article to give a number of such generalisations. In our considerations we assume that we carry out a number of steps of the Archimedean algorithm, followed by addition of a few terms of one of the series expansions in this article. During the Archimedean steps we assume that we keep track of the latest values of P_N, Q_N as well as $P_N/N, Q_N/N$.
Suppose we wish to compute π to L decimal places. As a time unit we may take the time to perform one operation (addition, multiplication, division) of two L-digit numbers. Then the Archimedean algorithms gives us the answer with L-digit precision in $O(L)$ steps. However, if we combine the Archimedean steps with the Snellius' type acceleration we require only $O(\sqrt{L})$ steps. We have not made any effort to make this very precise, since modern day methods are far better suited for the high precision calculation of π.
Of course the possiblity of improvements, like the ones discussed in this paper, has been considered by many others, professional mathematicians and amateurs alike. See for example [Ph]. Unfortunately it is hard to get a good overview concerning publications on this subject. So we do not claim any originality in the results here. We simply consider it an amusing aside to π-folklore.
By the way, the second author of this article has tried to make an estimate of the time required for Van Ceulen's calculations by doing a few sample calculations on 35-digit numbers. It does not seem to be as bad as people usually think.

2 Accelerations based on arctan

The first improvement is obtained by using the arctangent series. From $Q_N = N\tan\frac{\pi}{N}$ it follows immediately that

$$\frac{\pi}{Q_N} = \frac{\arctan(Q_N/N)}{Q_N/N}.$$

Using the well-known Taylor series for arctan we obtain

Theorem 2.1

$$\frac{\pi}{Q_N} = 1 - \frac{1}{3}\left(\frac{Q_N}{N}\right)^2 + \frac{1}{5}\left(\frac{Q_N}{N}\right)^4 - \frac{1}{7}\left(\frac{Q_N}{N}\right)^6 + \cdots$$

So, by subtracting $Q_N \cdot \frac{1}{3}\left(\frac{Q_N}{N}\right)^2$ from Q_N Van Ceulen could have doubled the precision of his calculations in one stroke. By adding the next term the precision could have been tripled.

There is a nice variation on the above formula which does not use P_N/N or Q_N/N, but only the value $t_N = \frac{Q_N - P_N}{2Q_N}$. To explain this we need a few facts on hypergeometric functions. Let a, b, c be real numbers and $c \neq 0, -1, -2, \ldots$. Then the Gauss' hypergeometric function with parameters a, b, c is defined by the power series

$$F_{\text{Gauss}}(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

Here, $(x)_k$ is the so-called Pochhammer symbol defined by $(x)_k = x(x+1)\cdots(x+k-1)$. The series converges for all complex z with $|z| < 1$. There is an extensive and beautiful theory around such functions, which we cannot dwell upon here. Instead we simply like to state one of the many transformation formulas between hypergeometric series,

$$F_{\text{Gauss}}\left(\frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) = (1-2t)^a \, F_{\text{Gauss}}\left(a, b, \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}; t\right) \quad (3)$$

This formula is basically due to Kummer and can be found in [AS,p 561]. As a result of this formula we find the following application which is useful for us.

Proposition 2.2

$$F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) = 1 - \sum_{k=1}^{\infty} \frac{(2)_k}{(3/2)_k} \frac{t^k}{k}.$$

To see this, apply formula (3) with $a = b = 1$ to get

$$\begin{aligned} F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) &= (1-2t) F_{\text{Gauss}}\left(1, 1, \frac{3}{2}; t\right) \\ &= (1-2t) \sum_{k=0}^{\infty} \frac{k!}{(3/2)_k} t^k \end{aligned}$$

For any $k \geq 1$ the coefficient of t^k in the last product is of course equal to

$$\frac{k!}{(3/2)_k} - 2\frac{(k-1)!}{(3/2)_{k-1}}.$$

A straightforward calculation shows that this is equal to

$$\frac{(k+1)!}{(3/2)_k} \frac{1}{k} = \frac{(2)_k}{(3/2)_k} \frac{1}{k}.$$

Our Proposition now follows immediately. **qed**

We note that the arctangent series is an example of a hypergeometric series. One easily checks that

$$\frac{\arctan z}{z} = F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; -z^2\right)$$

We like to substitute $z = Q_N/N$ here. Now observe that $t_N = (Q_N - P_N)/Q_N = \frac{1}{2}(1 - \cos\frac{\pi}{N})$. Using this we find that

$$\begin{aligned}
-\left(\frac{Q_N}{N}\right)^2 &= -\left(\frac{\sin\frac{\pi}{N}}{\cos\frac{\pi}{N}}\right)^2 \\
&= 1 - \frac{1}{(\cos\frac{\pi}{N})^2} \\
&= 1 - \frac{1}{(1-2t_N)^2} = \frac{4t_N^2 - 4t_N}{(1-2t_N)^2}
\end{aligned}$$

Using this observation and Proposition 2.2 we find

Theorem 2.3

$$\frac{\pi}{Q_N} = 1 - \sum_{k=1}^{\infty} \frac{(2)_k}{(3/2)_k} \frac{t_N^k}{k}.$$

Note that if we take the first two terms of this series, we get

$$\pi \approx Q_N - \frac{4}{3}Q_N t_N = Q_N - \frac{2}{3}(Q_N - P_N) = \frac{1}{3}Q_N + \frac{2}{3}P_N,$$

which is precisely Snellius' improvement. Finally we like to point out that $t_N = 2(P_{2N}/2N)^2$.

3 Accelerations based on arcsin

Just as with the arctan series we can also play with the arcsin series, which reads

$$\begin{aligned}
\frac{\arcsin z}{z} &= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \frac{z^{2k}}{2k+1} \\
&= F_{\text{Gauss}}\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right).
\end{aligned}$$

As an immediate consequence it follows from $\frac{\pi}{P_N} = \frac{\arcsin(P_N/N)}{P_N/N}$ that

Theorem 3.1

$$\frac{\pi}{P_N} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \frac{1}{2k+1} \left(\frac{P_N}{N}\right)^{2k}.$$

There is a small variation based on the following Proposition.

Proposition 3.2

$$(1-z^2)^{1/2}\frac{\arcsin z}{z} = 1 + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}z^{2k}$$

This result follows from the another well-known formula in hypergeometric functions, which reads

$$(1-z)^{a+b-c}F_{\text{Gauss}}(a,b,c;z) = F_{\text{Gauss}}(c-a,c-b,c;z).$$

The formula can be found in [AS, p559]. Apply this with $a = b = 1/2, c = 3/2$ and z replaced by z^2 to get

$$(1-z^2)^{-1/2}\frac{\arcsin z}{z} = F_{\text{Gauss}}(1,1,\frac{3}{2};z^2).$$

Multiply on both sides by $1-z^2$ and notice that

$$\begin{aligned}(1-z^2)F_{\text{Gauss}}(1,1,\frac{3}{2};z^2) &= (1-z^2)\sum_{k=0}^{\infty}\frac{k!}{(3/2)_k}z^{2k} \\ &= 1+\sum_{k=1}^{\infty}\left(\frac{k!}{(3/2)_k}-\frac{(k-1)!}{(3/2)_{k-1}}\right)z^{2k} \\ &= 1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}z^{2k}\end{aligned}$$

qed

We apply our Proposition with $z = P_N/N = \sin\frac{\pi}{N}$. Notice that $(1-z^2)^{1/2} = \cos\frac{\pi}{N}$. Hence the Proposition implies the following.

Theorem 3.3

$$\frac{\pi}{Q_N} = 1 + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}\left(\frac{P_N}{N}\right)^{2k}.$$

Looking back we see that we expressed $\frac{\pi}{Q_N}$ as a power series in $\frac{P_N}{N}$ and in $\frac{Q_N}{N}$. We also expressed $\frac{\pi}{P_N}$ as a power series in $\frac{P_N}{N}$. Although there is certainly a power series for $\frac{\pi}{P_N}$ in terms of $\frac{Q_N}{N}$, the shape of this series does not seem to be as simply as the other three.

4 Gain of the acceleration

In this section we indicate briefly how much Archimedes' calculation can be speeded up using our Taylor series. As we said before, we use the time taken for one operation on two L-digit numbers as a unit of time. Suppose we wish to calculate π to L decimal places. We first carry out $\sqrt{L}$ steps of Archimedes' algorithm. This gives us $\sqrt{L}\cdot\log_{10}(4)$ correct decimal places. To increase this precision by a factor $\sqrt{L}$ we have to take $O(\sqrt{L})$ terms of any of the power series given in the previous sections. So the total number of steps is again $O(\sqrt{L})$. As we already indicated in the introduction, modern methods like the Gauss-Salamin algorithm or its speedup by the Borweins provide a much faster scheme of computation. The number of steps required for the latter methods is $O(\log L)$.

5 References

[A] Archimedes, Measurement of the circle, reprinted in [BBB].

[AS] M.Abramowitz, I.Stegun, *Handbook of mathematical functions*, Dover Publications 1972 (9th edition).

[BBB] L.Berggren, J.Borwein, P.Borwein, *Pi: a source book*, Springer Verlag 1997.

[Ph] G.M.Philips, Archimedes, the numerical analyst, American math. Monthly 88 (1981), 165-169. Reprinted in [BBB].