BASIC HARMONIC ANALYSIS ON PSEUDO–RIEMANNIAN SYMMETRIC SPACES

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Abstract. We give a survey of the present knowledge regarding basic questions in harmonic analysis on pseudo–Riemannian symmetric spaces G/H, where G is a semisimple Lie group: The definition of the Fourier transform, the Plancherel formula, the inversion formula and the Paley–Wiener theorem.

Key words: Harmonic analysis, Symmetric space

1. Introduction

The rich and beautiful theory of harmonic analysis on $\mathbb{R}^n$ and $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ has become a powerful tool, widely used in other branches of mathematics, in physics, engineering etc. From our point of view all the basic questions are completely and explicitly solved: The Fourier transform is defined, there exists a Plancherel formula and an inversion formula for it, and (for $\mathbb{R}^n$) there is a Paley–Wiener theorem, describing the image of the space of compactly supported functions.

There exist many generalizations of this theory. Let us mention a few directions, based on various ways of viewing $\mathbb{R}^n$ and $\mathbb{T}^n$.

$\mathbb{R}^n$ and $\mathbb{T}^n$ are locally compact groups:

- Fourier analysis on locally compact Abelian groups.
- The Peter–Weyl theory for Fourier analysis on compact groups.

- Representation theory for locally compact groups and rings of operators on Hilbert spaces (C^*–algebras, von Neumann algebras, etc.).

$\mathbb{R}^n$ and $\mathbb{T}^n$ are Lie groups:

- The representation theory for compact Lie groups (the Cartan–Weyl classification, Weyl's character formula etc.).
- Representation theory for general Lie groups (semisimple, reductive, nilpotent, solvable etc.).

$\mathbb{R}^n$ and $\mathbb{T}^n$ are smooth homogeneous manifolds:

- Harmonic analysis related to homogeneous spaces and their transformation groups.

Here we take the last mentioned viewpoint. We claim that inside the class of smooth manifolds the class of (not necessarily Riemannian) symmetric spaces constitutes an appropriate framework for generalization of harmonic analysis: On the one hand this class of manifolds is wide enough to contain very many important spaces of relevance in other branches of mathematics and in physics. On the other hand it is restrictive enough to make feasible a theory of harmonic analysis, with explicit parametrizations and descriptions of representations, explicit Plancherel formulae, etc.

2. Symmetric spaces

2.1. Definition and structure

We define a (affine) symmetric space as follows:

Definition. *A connected smooth manifold M with an affine connection is called a* symmetric space *if for every x in M the local reflection in x along geodesics extends to a global affine diffeomorphism, S_x, of M.*

Without going into technicalities we shall need a few facts about symmetric spaces (for details, see [21], [37], [38] and the references cited there):

The group $G = G(M)$ generated by the transformations $S_x \circ S_y$, $(x, y \in M)$, is a connected Lie group acting transitively on M. Therefore, choosing a base point $x_o \in M$, we may identify M with the homogeneous space G/H, where H is the stabilizer of x_o. If we define $\sigma(g) = S_{x_o} \circ g \circ S_{x_o}$ for $g \in G$, then σ is an involution of G, i.e. an automorphism whose square is the identity. It easily follows that H is an open subgroup of the group G^σ of σ–fixed points in G. On the other hand, if G is a connected Lie group with an involution σ, and H is an open subgroup of G^σ, then the homogeneous space G/H is a symmetric space on which G acts by affine transformations (but G differs in general from $G(G/H)$).

Let r be the Ricci curvature tensor on M. This is a covariant tensor of degree 2, which is canonically associated with the affine connection on M, and therefore G-invariant. A symmetric space has the special feature that its affine connection is torsion free, and that the Ricci curvature tensor r is covariantly constant. In particular, if r is symmetric and non-degenerate, then it defines a pseudo-Riemannian structure on M whose associated connection is the original affine connection.

Theorem 1. *Let $M = G/H$ be a symmetric space with $G = G(M)$.*

(i) *M is flat if and only if G is Abelian.*
(ii) *(M, r) is pseudo–Riemannian (that is, r is symmetric and non–degenerate) if and only if G is semisimple.*
(iii) *(M, r) is Riemannian (that is, r is symmetric and definite) if and only if G is semisimple and H is compact.*
(iv) *If M is irreducible then either $\dim(M) = 1$, or G is simple, or M is a simple Lie group G_1.*

In the last mentioned case, when $M = G_1$, we have that G is the product $G_1 \times G_1$ with the left times right action on G_1. In this case, the reflection $S_x\colon G_1 \to G_1$ in an element $x \in G_1$ is given by $S_x(g) = xg^{-1}x$. Choosing the identity element of G_1 as our base point we get that H is the diagonal $d(G_1)$ and that the involution of G is given by $\sigma(x, y) = (y, x)$. We call this the *group case*.

Our goal in this paper is to describe the state of the art for harmonic analysis on *semisimple symmetric spaces*, i.e. the spaces of case (ii) above. From now on we assume that $M = G/H$ is such a space, with $G = G(M)$ semisimple. Notice that this is a stronger assumption than just requiring M to be equipped with some pseudo-Riemannian structure which is compatible with the given affine connection ($\mathbb{R}^n$ with any pseudo–norm is an example – here $r = 0$). However by (iv), if M is irreducible and of dimension at least 2 then G is semisimple.

For simplicity of exposition we assume (which we may up to coverings of M) that G is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some n, and that G is stable under transposition. Let $K = G \cap \mathrm{SO}(n)$, or equivalently $K = G^\theta$, where $\theta(x) = {}^t x^{-1}$, then K is a maximal compact subgroup of G. We may choose the base point such that $\theta(H) = H$, or equivalently, such that $\sigma \circ \theta = \theta \circ \sigma$.

We shall distinguish between the following 3 types of irreducible semisimple symmetric spaces:

- M is of the *compact type* if $G = K$, or equivalently if all geodesic curves have compact closures.
- M is of the *non–compact type* if $H = K$, or equivalently if all geodesic curves have non–compact closures.
- M is of the *non–Riemannian type* if $G \neq K$ and $K \neq H$, or equivalently if there exist geodesic curves of both types.

If M is of one of the first two types we say that it is of the Riemannian type (cf. Thm. 1(iii)). Notice that a simple group G_1, considered as a symmetric space, is either of the compact type or of the non–Riemannian type.

2.2. Examples

The irreducible symmetric spaces have been classified by M. Berger [10]. Compared with the list of Riemannian symmetric spaces (see [27, Ch.X]), Berger's list is considerably longer.

There is (up to coverings) one two–dimensional space of each of the three types:

- The compact type: The *2–sphere* $\mathrm{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$.
- The non–compact type: The *hyperbolic 2–space* $M = \mathrm{H}^2$. This has several isomorphic realizations: As $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, as $\mathrm{SU}(1,1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1))$, or as $\mathrm{SO}_e(2,1)/\mathrm{SO}(2)$, corresponding to, respectively, the upper half plane in $\mathbb{C}$, the unit disk in $\mathbb{C}$, or a sheet of the two–sheeted hyperboloid in $\mathbb{R}^3$.
- The non–Riemannian type: The *one–sheeted hyperboloid* in $\mathbb{R}^3$, $H^{1,1} = \mathrm{SO}_e(2,1)/\mathrm{SO}_e(1,1)$, which can also be realized as $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. It has the two–fold cover $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}_e(1,1)$.

In higher dimensions there exist several 'families' of symmetric spaces, many of which have one of the spaces above as their lowest dimensional member. For example we could mention:

The *n–spheres*: $\mathrm{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

The spaces of *positive definite quadratic forms* in $\mathbb{R}^n$: $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

The spaces of *quadratic forms of signature* (p,q) in $\mathbb{R}^n$, (where $n = p+q$): $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(p,q)$.

The *hyperboloids* in $\mathbb{R}^{n+1}$: $\mathrm{H}^{p,q} = \{x \in \mathbb{R}^{n+1} \mid x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q+1}^2 = -1\}$ where $p+q=n$ (if $q=0$ one must take a connected component). Here $M = \mathrm{SO}_e(p,q+1)/\mathrm{SO}_e(p,q)$.

Similarly one can take the corresponding spaces over the complex numbers or over the quaternions.

2.3. Some basic notation

Let G, H, K, σ and θ be as above. Let $\mathfrak{g}$ be the (real) Lie algebra of G, and let $\mathfrak{h}$ and $\mathfrak{k}$ be the subalgebras corresponding to H and K, and $\mathfrak{q}$ and $\mathfrak{p}$ their respective orthocomplements with respect to the Killing form. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}$$

is the decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces for σ and θ respectively. Since θ and σ commute we also have the joint decomposition

$$\mathfrak{g} = \mathfrak{h}\cap\mathfrak{k} \,\oplus\, \mathfrak{h}\cap\mathfrak{p} \,\oplus\, \mathfrak{q}\cap\mathfrak{k} \,\oplus\, \mathfrak{q}\cap\mathfrak{p}. \tag{1}$$

Notice that there is a natural identification of $\mathfrak{q}$ with the tangent space $T_{x_o}(M)$. We denote by $\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}$ etc. the complexifications of $\mathfrak{g}$, $\mathfrak{h}$ etc.

A *Cartan subspace* $\mathfrak{b}$ for G/H is a maximal Abelian subspace of $\mathfrak{q}$, consisting of semisimple elements. (If we assume, as we may in the following, that $\mathfrak{b}$ is θ–invariant, then all its elements are automatically semisimple, once $\mathfrak{b}$ is maximal Abelian). All Cartan subspaces have the same dimension, which we call *the rank of* M. The number of H–conjugacy classes of Cartan subspaces is finite. Geometrically, a Cartan subspace is the tangent space of a maximally flat regular subsymmetric space.

We say that a Cartan subspace $\mathfrak{b}$ is *fundamental* if the intersection $\mathfrak{b} \cap \mathfrak{k}$ is maximal Abelian in $\mathfrak{q} \cap \mathfrak{k}$, and that it is *split* if the intersection $\mathfrak{b} \cap \mathfrak{p}$ is maximal Abelian in $\mathfrak{q} \cap \mathfrak{p}$. There is, up to conjugation by $K \cap H$, a unique fundamental and a unique split Cartan subspace. If the fundamental Cartan subspace is contained in $\mathfrak{k}$ it is called a *compact* Cartan subspace. The dimension of the $\mathfrak{p}$–part of a split Cartan subspace is called the *split rank* of M.

Let $\mathbb{D}(G/H)$ denote the algebra of G–invariant differential operators on G/H. There is a natural isomorphism (the *Harish–Chandra isomorphism*) χ of this algebra with the algebra $S(\mathfrak{b})^W$ of W–invariant elements in the symmetric algebra of any Cartan subspace $\mathfrak{b}_{\mathbb{C}}$. Here W is the reflection group of the root system of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. In particular, $\mathbb{D}(G/H)$ is commutative, and its characters are parametrized up to W–conjugation by $D \mapsto \chi_\lambda(D) = \chi(D)(\lambda) \in \mathbb{C}$. It is known (see [2]) that the symmetric elements of $\mathbb{D}(G/H)$ have self–adjoint closures as operators on $L^2(G/H)$.

3. Basic harmonic analysis

3.1. Harmonic analysis on $\mathbb{R}^n$

We want to generalize the basic notions and results from harmonic analysis on $\mathbb{R}^n$. These are:

The *Fourier transform:* $f \mapsto f^\wedge(\lambda) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(t) e^{-i\lambda \cdot t}\, dt$, $f \in C_c^\infty(\mathbb{R}^n)$.

The *Plancherel theorem:* $f \mapsto f^\wedge$ extends to an isometry of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

The *inversion formula:* If $f \in C_c^\infty(\mathbb{R}^n)$ then

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f^\wedge(\lambda) e^{i\lambda \cdot x}\, d\lambda.$$

The *Paley–Wiener theorem:* $f \mapsto f^\wedge$ is a bijection of $C_c^\infty(\mathbb{R}^n)$ onto $\mathrm{PW}(\mathbb{R}^n)$, where $\mathrm{PW}(\mathbb{R}^n)$ is the space of rapidly decreasing entire functions of exponential type. More precisely, a complex function ψ on $\mathbb{R}^n$ belongs to $\mathrm{PW}(\mathbb{R}^n)$ if and only if it extends to an entire function on $\mathbb{C}^n$ for which there exists $R > 0$ such that the following holds for all $N \in \mathbb{N}$:

$$\sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^N e^{-R|\operatorname{Im} \lambda|} |\psi(\lambda)| < +\infty. \tag{2}$$

The aim of the basic harmonic analysis on G/H is to obtain analogues of these notions and results.

3.2. The 'abstract' harmonic analysis on a semisimple symmetric space

Let G and H be as above, then $M = G/H$ has an invariant measure, and the action of G by translations gives a unitary representation $\mathcal{L}$ in the associated Hilbert space $L^2(G/H)$. From general representation theory it is known (since G is 'type 1') that this representation can be decomposed as a direct integral of irreducible unitary representations:

$$\mathcal{L} \simeq \int_{G^\wedge}^{\oplus} m_\pi \, \pi \, d\mu(\pi), \tag{3}$$

where the measure $d\mu$ (whose class is uniquely determined) is called the *Plancherel measure*, and m_π (which is unique almost everywhere) the *multiplicity* of π. Moreover, only the so–called *H–spherical representations* can occur in this decomposition. By definition, an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G is H–spherical if the space $(\mathcal{H}_\pi^{-\infty})^H$ of its H–fixed distribution vectors is non–trivial. Here we denote by $\mathcal{H}_\pi^\infty$ and $\mathcal{H}_\pi^{-\infty}$, respectively the C^∞ and the distribution vectors for $\mathcal{H}_\pi$, such that $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi \subset \mathcal{H}_\pi^{-\infty}$. We write

$$\mathcal{V}_\pi = (\mathcal{H}_\pi^{-\infty})^H.$$

It is known (see [2]) that $m_\pi \leq \dim \mathcal{V}_\pi < +\infty$, in particular, all multiplicities are finite. Denote by $G_H^\wedge$ the set of (equivalence classes) of H–spherical representations, then it follows that the Plancherel measure $d\mu$ is carried by $G_H^\wedge$.

The *'abstract' Fourier transform* $f \mapsto f^\wedge(\pi)$ for G/H is now defined by

$$f^\wedge(\pi)(\eta) = \pi(f)\eta = \int_{G/H} f(x)\pi(x)\eta \, dx \in \mathcal{H}_\pi^\infty$$

for $\pi \in G_H^\wedge, \eta \in \mathcal{V}_\pi$ and $f \in C_c^\infty(G/H)$. Thus

$$f^\wedge(\pi) \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{V}_\pi, \mathcal{H}_\pi^\infty) \simeq \mathcal{H}_\pi^\infty \otimes \mathcal{V}_\pi^*$$

(notice that the integral over G/H only makes sense because η is H–invariant). One can prove (using [35] and [40]) that there exists for almost all $\pi \in G_H^\wedge$ a subspace $\mathcal{V}_\pi^o$ (of dimension m_π) of $\mathcal{V}_\pi$, equipped with the structure of a Hilbert space, such that if $f^\wedge(\pi)$ is restricted to $\mathcal{V}_\pi^o$ for almost all π, then $f \mapsto f^\wedge$ extends to an isometry of $L^2(G/H)$ onto $\int_{G_H^\wedge}^{\oplus} \operatorname{Hom}_{\mathbb{C}}(\mathcal{V}_\pi^o, \mathcal{H}_\pi) d\mu(\pi)$. Here the norm on $\operatorname{Hom}_{\mathbb{C}}(\mathcal{V}_\pi^o, \mathcal{H}_\pi)$ is given by

$$\|\varphi\|_\pi^2 = \operatorname{Tr}(\varphi^* \circ \varphi) = \sum_i \|\varphi(v_i)\|^2, \qquad \varphi \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{V}_\pi^o, \mathcal{H}_\pi),$$

where φ^* is the adjoint of φ and $\{v_i\}_{i=1,\dots,m_\pi}$ is an orthonormal basis in $\mathcal{V}_\pi^o$.

We thus have the *Plancherel formula*

$$\|f\|_2^2 = \int_{G_H^\wedge} \|f^\wedge(\pi)\|_\pi^2 \, d\mu(\pi), \qquad f \in L^2(G/H).$$

Similarly, there is the *inversion formula* (for suitably nice functions f)

$$f(e) = \int_{G^\wedge_H} \sum_{i=1}^{m_\pi} \langle f^\wedge(\pi) v_i | v_i \rangle \, d\mu(\pi). \tag{4}$$

(Here $\langle \cdot | \cdot \rangle$ denotes the inner product on $\mathcal{H}_\pi$, as well as the naturally associated pairing $\mathcal{H}_\pi^\infty \times \mathcal{H}_\pi^{-\infty} \to \mathbb{C}$.) Consequently we also have, for suitable f

$$f(x) = \int_{G^\wedge_H} \sum_{i=1}^{m_\pi} \langle f^\wedge(\pi) v_i | \pi(x) v_i \rangle \, d\mu(\pi).$$

The basic problems are now

(a) Describe (parametrize) $G^\wedge_H$, or at least μ–almost all of it.
(b) For μ–almost all $\pi \in G^\wedge_H$ describe (parametrize) $\mathcal{V}_\pi^o$ and its Hilbert space structure.
(c) Determine $d\mu$ explicitly.

A *Paley–Wiener theorem* would amount to an intrinsic description of the Fourier image of $C_c^\infty(G/H)$ in terms of $G^\wedge_H$. We add this as a fourth basic problem:

(d) Describe $C_c^\infty(G/H)^\wedge$ in terms of the parametrizations and possible holomorphic extensions.

For each $\pi \in G^\wedge_H$ we have that $\mathcal{V}_\pi$ is a $\mathbb{D}(G/H)$–module in a natural way. Using that the symmetric elements of $\mathbb{D}(G/H)$ are essentially selfadjoint operators on $L^2(G/H)$ one can show (with the arguments in [40]) that $\mathcal{V}_\pi^o$ can be chosen to be invariant and diagonalizable for this action. Thus $\mathcal{V}_\pi^o$ is spanned by its joint eigenvectors for $\mathbb{D}(G/H)$. Let $\mathfrak{b} \subset \mathfrak{q}$ be a Cartan subspace. Then such an eigenvector satisfies

$$\pi(D)v = \chi_\lambda(D)v, \qquad D \in \mathbb{D}(G/H),$$

for some $\lambda \in \mathfrak{b}^*_\mathbb{C}$. We say that v is a *spherical vector of type* λ, and that the orthonormal basis $\{v_i\}_{i=1,\dots,m_\pi}$ in $\mathcal{V}_\pi^o$ is spherical if its members are spherical.

The maps $\xi_{\pi,i}\colon f \mapsto \langle f^\wedge(\pi) v_i | v_i \rangle$ in (4) are H–invariant distributions on G/H. As distributions on G they are positive definite and extreme (see [40]). With a spherical basis $\{v_i\}$ each $\xi_{\pi,i}$ is also a *spherical distribution*, that is an H–invariant eigendistribution for $\mathbb{D}(G/H)$. The solution of Problem (b) is then closely related to the study of the spherical distributions.

3.3. Results valid for specific classes of symmetric spaces

Here we give some brief remarks concerning the status of the above problems for some specific classes of semisimple symmetric spaces.

3.3.1. *The compact type.* For a homogeneous space G/H with a compact group G the abstract formulation above follows easily from the Peter–Weyl theorem and the

Schur orthogonality relations. In particular, $\mathcal{V}_\pi^o = \mathcal{V}_\pi = \mathcal{H}_\pi^H$, and if we give $\mathcal{V}_\pi^o$ the subspace norm from $\mathcal{H}_\pi$, we have $d\mu(\pi) = \dim(\pi)$. For the symmetric spaces of compact type we then have the following explicit solutions to the above problems (see [28, § V.4]):

(a) $G_H^\wedge$ is parametrized by a subset of the set of dominant weights.
(b) $\dim \mathcal{V}_\pi^o = 1$ for $\pi \in G_H^\wedge$.
(c) $d\mu$ is given by Weyl's dimension formula.
(d) The smooth functions are determined by a certain growth condition on the Fourier transforms (see [39]).

3.3.2. *The non–compact type.* We write M as G/K. The four questions are settled beautifully by the work of Harish–Chandra, Helgason and others. See [28, § IV.7]. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$.

(a) A sufficient subset of $G_K^\wedge$ is parametrized (up to conjugacy by the Weyl group W of $\mathfrak{a}$ in $\mathfrak{g}$) by means of the spherical functions $\varphi_\lambda, \lambda \in i\mathfrak{a}^*$ (see (23)) and the corresponding spherical principal series representations $(\pi_\lambda, \mathcal{H}_\lambda)$.
(b) For $\pi = \pi_\lambda \in G_K^\wedge$ we have $\mathcal{V}_\pi^o = \mathcal{H}_\lambda^K$ and $\dim(\mathcal{V}_\pi^o) = 1$. We can then use the subspace norm from $\mathcal{H}_\lambda$.
(c) The Plancherel measure is given by $d\mu(\pi_\lambda) = |\mathbf{c}(\lambda)|^{-2} d\lambda$ on $i\mathfrak{a}^*/W$. Here $\mathbf{c}(\lambda)$ is Harish–Chandra's c–function, which is explicitly given in terms of the structure of G/K by the formula of Gindikin–Karpelevic.
(d) We have $C_c^\infty(K\backslash G/K)^\wedge = \mathrm{PW}(\mathfrak{a})^W$. Here $\mathrm{PW}(\mathfrak{a})^W$ is the space of W–invariant functions in the image space $\mathrm{PW}(\mathfrak{a})$ for the Fourier transform

$$f \mapsto f^\wedge(\lambda) = \int_{\mathfrak{a}} f(X) e^{-\lambda(X)} dX, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, f \in C_c^\infty(\mathfrak{a}), \tag{5}$$

that is, the space of rapidly decreasing entire functions of exponential type on $\mathfrak{a}_{\mathbb{C}}^*$ (see Sect. 3.1, but note that since the imaginary unit i is not present in the exponent in (5) one has to replace $\operatorname{Im}\lambda$ by $\operatorname{Re}\lambda$ in (2)). Helgason has extended the Paley–Wiener theorem to the space $C_c^\infty(K; G/K)$ of K–finite functions in $C_c^\infty(G/K)$, and also to the full space $C_c^\infty(G/K)$.

3.3.3. *The group case, $M = G_1$.* This case is almost completely settled by the work of Harish–Chandra ([23]) and others.

(a) The map $\pi_1 \mapsto \pi_1 \otimes \pi_1^*$ is a bijective correspondence from the unitary dual $G_1^\wedge$ onto $G_H^\wedge$. A sufficient subset of $G_1^\wedge$ is described by the discrete series and different families of (cuspidal) principal series.
(b) For $\pi_1 \in G_1^\wedge$ and $\pi = \pi_1 \otimes \pi_1^*$ we have $\mathcal{V}_\pi = (\mathcal{H}_\pi^{-\infty})^H = \mathbb{C} 1_{\pi_1}$, where 1_{π_1} is the identity operator on $\mathcal{H}_{\pi_1}$. Notice however that in this case $\mathcal{V}_\pi \not\subset \mathcal{H}_\pi$, since the latter space can be identified with the space of Hilbert–Schmidt operators on $\mathcal{H}_{\pi_1}$. We take $\mathcal{V}_\pi^o = \mathcal{V}_\pi$, and use on it the Hilbert space structure obtained from the identification with $\mathbb{C}$ in which $1_{\pi_1} = 1$.

(c) With the above choice one can give $d\mu$ explicitly in terms of the formal degrees of discrete series and certain c–functions.

(d) A Paley–Wiener theorem for the K–finite functions on G_1 has been established in [14] (in split rank one) and [1] (in general). In particular, the Paley–Wiener space is determined by the minimal principal series only. For the full space $C_c^\infty(G_1)$ a Paley–Wiener theorem has not been established.

3.3.4. *The non–Riemannian type, rank one.* There is a vast literature dealing with the questions (a)–(c) on specific classes of rank one symmetric spaces of the non–Riemannian type. See for example [19], [40], [31]. Common for all these spaces is that the decomposition of $L^2(G/H)$ contains a discrete series as well as a continuous part. Problem (d) is solved in [9] (see below) for the K–finite functions, under the more general assumption that the *split* rank is one.

3.3.5. $G/H = \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(p,q)$. See [12].

4. A survey of results valid for general semisimple symmetric spaces

Even though the basic problems have been solved for many specific classes of semisimple symmetric spaces, there are still few final answers known which hold in complete generality. On the other hand, very much is known about the representations connected with these problems, and there is hope for the general answers in a not too distant future.

By analogy with the group case one expects in general that the left regular representation $\mathcal{L}$ on $L^2(G/H)$ can be decomposed in several 'series' of representations, one series for each H–conjugacy class of Cartan subspaces for $\mathfrak{q}$. The most extreme of these would then be the 'most continuous' part, corresponding to the conjugacy class of Cartan subspaces with maximal $\mathfrak{p}$–part (the split Cartan subspaces) and the 'most discrete' part (sometimes called the fundamental series), corresponding to the conjugacy class of Cartan subspaces with maximal $\mathfrak{k}$–part (the fundamental Cartan subspaces). If the fundamental Cartan subspaces are compact, then this 'most discrete' part is in fact the *discrete series*, that is, the irreducible subrepresentations of $\mathcal{L}$. For both of these 'extreme' parts of $L^2(G/H)$ very much is known about the Problems (a), (b) and (c); below (in Subsections 4.1, 4.3 and 4.4) we shall review some details and give precise references.

With respect to Problem (d) we want to mention two results of a general nature: One ([17], see Subsect. 4.2) which exhibits a large class (though too small to be 'the Paley–Wiener space' in general) of functions which are Fourier transforms of K–finite functions in $C_c^\infty(G/H)$, and another ([9], see Subsect. 4.5) which shows that the Fourier transform of a function in $C_c^\infty(G/H)$ is determined by its restriction to the meromorphic extension of the unitary principal series (the 'most continuous' part mentioned above). Along with the latter result goes a conjectural description of the K–finite Paley–Wiener space. The conjecture can be confirmed in the above mentioned cases 3.3.1, 3.3.2 and 3.3.3, and it also holds when G/H has split rank one.

4.1. The discrete series

The basic existence theorem is the following, where we preserve the notions from above. Let $L^2_d(G/H) \subset L^2(G/H)$ be the closed linear span of the irreducible subrepresentations of $\mathcal{L}$.

Theorem 2, [20], [33]. *Let G/H be a semisimple symmetric space. Then the discrete series space $L^2_d(G/H)$ is non–zero if and only if*

$$\operatorname{rank}(G/H) = \operatorname{rank}(K/K \cap H). \tag{6}$$

The condition (6) means that G/H has a compact Cartan subspace. An equivalent more geometric formulation is that it has a compact maximally flat subsymmetric space.

We shall now discuss Problems (a), (b) and (c) for the discrete series. Assume (as we may by the above theorem) that (6) holds, and let $\mathfrak{t}$ be a compact Cartan subspace of $\mathfrak{q}$. Let Σ be the root system of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ and Σ_c the subsystem of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$. Let W and W_c be the corresponding reflection groups.

A rough classification of the discrete series is obtained by means of the commutative algebra $\mathbb{D}(G/H)$. Recall that the characters of $\mathbb{D}(G/H)$ are parametrized by $\mathfrak{t}^*_{\mathbb{C}}/W$ via the Harish–Chandra isomorphism $\chi\colon \mathbb{D}(G/H) \to S(\mathfrak{t})^W$. Let $\mathcal{E}_\lambda(G/H)$ denote the joint eigenspace for $\mathbb{D}(G/H)$ in $C^\infty(G/H)$ corresponding to the character χ_λ, where $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$. Then $\mathcal{E}_{w\lambda}(G/H) = \mathcal{E}_\lambda(G/H)$ for all $w \in W$. Since $\mathbb{D}(G/H)$ is commutative and its symmetric elements act as essentially selfadjoint operators on $L^2(G/H)$, there is a joint spectral resolution of $L^2(G/H)$ for this algebra. The resulting decomposition is G–invariant because of the invariance of the elements in $\mathbb{D}(G/H)$. It follows (see [2]) that $L^2_d(G/H)$ admits an orthogonal G–invariant decomposition

$$L^2_d(G/H) = \widehat{\bigoplus_\lambda} L^2_\lambda(G/H),$$

where $L^2_\lambda(G/H)$ is the closure in $L^2(G/H)$ of $L^2(G/H) \cap \mathcal{E}_\lambda(G/H)$, and where the sum extends over the W–orbits in the set of those $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ for which $L^2_\lambda(G/H)$ is non–trivial. In order to parametrize the discrete series we must then determine this set of λ's, and for each λ therein the irreducible subrepresentations of $L^2_\lambda(G/H)$.

Let $\Lambda \subset i\mathfrak{t}^*$ denote the set of elements $\lambda \in i\mathfrak{t}^*$ satisfying the following conditions (i)–(iii).

(i) $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma$.

Given that (i) holds, let

$$\Sigma^+ = \{\alpha \in \Sigma \mid \langle \lambda, \alpha \rangle > 0\}, \tag{7}$$

then this is a positive system for Σ. Put $\Sigma_c^+ = \Sigma^+ \cap \Sigma_c$, and let ρ, resp. ρ_c, be defined as half the sum of the Σ^+–roots, resp. Σ_c^+–roots, counted with multiplicities.

(ii) $\lambda + \rho$ is a weight for T_H, i.e. $e^{\lambda+\rho}$ is well defined on T_H. Here T_H denotes the torus in G/H corresponding to $\mathfrak{t}$ (that is, $T_H = T/(T \cap H)$ where $T = \exp \mathfrak{t}$).

(iii) $\langle \lambda - \rho, \beta \rangle \geq 0$ for each compact simple root β in Σ^+.

(that β is compact means that the root space $\mathfrak{g}_{\mathbb{C}}^{\beta}$ is contained in $\mathfrak{k}_{\mathbb{C}}$). Notice that (ii) implies that Λ is a discrete subset of $i\mathfrak{t}^*$.

Under the assumption that $\lambda \in \Lambda$ there is a rather simple construction (which we shall outline below) of a $\mathfrak{g}$–invariant subspace $\mathcal{U}_{\lambda,K}$ of $C^\infty(K; G/H)$ (the space of K–finite functions in $C^\infty(G/H)$), which can be shown to be contained in $L^2_\lambda(G/H)$. Let $\mathcal{U}_\lambda$ denote the closure of $\mathcal{U}_{\lambda,K}$ in $L^2(G/H)$, then $\mathcal{U}_\lambda$ is a subrepresentation of $L^2_\lambda(G/H)$. Let π_λ denote this subrepresentation.

For 'large' $\lambda \in \Lambda$, or more precisely if $\langle \lambda + \rho - 2\rho_c, \alpha \rangle \geq 0$ for all $\alpha \in \Sigma_c^+$, it can be shown by elementary methods that $\mathcal{U}_\lambda \neq \{0\}$. For the remaining λ's one has to add a more technical assumption in order to ensure that $\mathcal{U}_\lambda \neq \{0\}$. We shall not state this condition here (the condition is stated in [30] together with a proof of its necessity for the non–vanishing of $\mathcal{U}_\lambda$).

Theorem 3, [33], [41]**.** *The discrete series space $L^2_d(G/H)$ is spanned by the $\mathcal{U}_\lambda$'s with $\lambda \in \Lambda$. Moreover for each $\lambda \in \Lambda$ either the representation π_λ is irreducible or $\mathcal{U}_\lambda$ is zero, and if $\lambda, \lambda' \in \Lambda$ we have $\mathcal{U}_{\lambda'} = \mathcal{U}_\lambda$ if and only if $\lambda' = w\lambda$ for some $w \in W_c$.*

It follows that if $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ then $L^2_\lambda(G/H)$ is the sum of those $\mathcal{U}_{w\lambda}$ for which $w \in W$ and $w\lambda \in \Lambda$. In particular it has at most as many components as the order of the quotient W/W_c.

With this result, Problem (a) is solved as regards to the discrete series. It is conjectured that $\pi_{\lambda'}$ is unitarily equivalent to π_λ if and only if $\mathcal{U}_{\lambda'} = \mathcal{U}_\lambda$, or equivalently in view of the above, that the discrete series have multiplicity one in the Plancherel formula. The conjecture is proved for all classical groups G, and is only open for a few exceptional cases for very special values of λ (see [11]).

Evaluation at the base point in G/H gives rise to an H–fixed distribution vector η_λ for $\mathcal{U}_\lambda$, for which it is easily seen that we have

$$f^\wedge(\pi_\lambda)(\eta_\lambda) = \mathrm{P}_\lambda f, \qquad f \in C_c^\infty(G/H),$$

where P_λ is the orthogonal projection of $L^2(G/H)$ onto $\mathcal{U}_\lambda$. It follows that if we take $\mathcal{V}^o_{\pi_\lambda} = \mathbb{C}\,\eta_\lambda$ and use on it the Hilbert space structure obtained from the identification with $\mathbb{C}$ in which $\eta_\lambda = 1$, then $d\mu(\pi_\lambda) = 1$. In other words, the Plancherel measure restricts to the counting measure on the discrete series. This provides the solution to Problems (b) and (c) for the discrete series.

At this point it is however interesting to note the following. Though the discrete series has been parametrized as above, it seems to be an open problem to determine an explicit expression for the spherical distribution $\xi_\lambda\colon f \mapsto \langle f^\wedge(\pi_\lambda)\eta_\lambda|\eta_\lambda\rangle$ on G/H associated to η_λ (or equivalently, for the projection operator P_λ, which is given by convolution with ξ_λ). In the group case one knows that ξ_λ is given by $d_\lambda\Theta_\lambda$, where d_λ is the formal degree and Θ_λ the character of π_λ (see [22, §5]), but there is no obvious generalization of this formula.

We shall not try to describe the proof of the above theorems. However as the construction of $\mathcal{U}_{\lambda,K}$ can be described by quite elementary methods we would like to indicate it.

Let the notation be as above, and recall the decomposition (1) of $\mathfrak{g}$. Let $\mathfrak{g}^d$ be the real form of $\mathfrak{g}_{\mathbb{C}}$ given by

$$\mathfrak{g}^d = \mathfrak{h}\cap\mathfrak{k} \oplus i(\mathfrak{h}\cap\mathfrak{p}) \oplus i(\mathfrak{q}\cap\mathfrak{k}) \oplus \mathfrak{q}\cap\mathfrak{p},$$

where i is the imaginary unit. Assume (again for simplicity of exposition) that G is a real form of a linear complex Lie group $G_{\mathbb{C}}$, and let G^d be the real form of $G_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}^d$. Then the subgroup $K^d = G^d \cap H_{\mathbb{C}}$ is a maximal compact subgroup. The symmetric space G^d/K^d is called the *non–compact Riemannian form* of G/H. The subgroup $H^d = G^d \cap K_{\mathbb{C}}$ of G^d is a (in general non–compact) real form of $K_{\mathbb{C}}$. Let $(G\cap G^d)_e$ denote the identity component of $G\cap G^d$. Then both G and G^d are contained in the set $K_{\mathbb{C}}(G\cap G^d)_e H_{\mathbb{C}}$. The K–finite functions on G/H extend naturally to left $K_{\mathbb{C}}$–finite and right $H_{\mathbb{C}}$–invariant functions on this set (and so do the H^d–finite functions on G^d/K^d, provided the H^d–action admits a holomorphic extension to $K_{\mathbb{C}}$). We call this *partial holomorphic extension.* Let $C^\infty(K;G/H)$ and $C^\infty(H^d;G^d/K^d)$ be the spaces of K–finite, resp. H^d–finite smooth functions on G/H, resp. G^d/K^d. There is a natural action of $\mathfrak{g}_{\mathbb{C}}$ on both of these spaces.

Proposition 4, [20]**.** *Partial holomorphic extension defines a $\mathfrak{g}_{\mathbb{C}}$–equivariant linear injection $f \to f^r$ of $C^\infty(K;G/H)$ into $C^\infty(H^d;G^d/K^d)$, the image of which is the set of functions in $C^\infty(H^d;G^d/K^d)$ for which the H^d–action extends holomorphically to $K_{\mathbb{C}}$. Moreover, f is a joint eigenfunction for $\mathbb{D}(G/H)$ if and only if f^r is a joint eigenfunction for $\mathbb{D}(G^d/K^d)$.*

The construction of G^d/K^d and this proposition hold independently of (6). However, this assumption is crucial for the following construction.

Since G^d/K^d is a Riemannian symmetric space the joint eigenfunctions for the algebra $\mathbb{D}(G^d/K^d)$ can be described by means of the so–called generalized Poisson transform. This is defined as follows. It follows from the fact that $\mathfrak{t}$ is a maximal Abelian subspace of $\mathfrak{q}$, that $\mathfrak{t}^r = i\mathfrak{t}$ is a maximal Abelian split subspace for $\mathfrak{g}^d$. Hence there is an Iwasawa decomposition

$$G^d = K^d T^r N^d \tag{8}$$

of G^d with $T^r = \exp \mathfrak{t}^r$, which corresponds to a given Σ^+. Let $P^d = M^d T^r N^d$ be the corresponding minimal parabolic subgroup in G^d, and for $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ let $\mathcal{D}'_\lambda =$

$\mathcal{D}'_\lambda(G^d/P^d)$ be the space of $(\lambda-\rho)$–homogeneous distributions on G^d/P^d, that is the space of generalized functions f on G^d satisfying

$$f(gman) = a^{\lambda-\rho} f(g), \qquad g \in G^d, m \in M^d, a \in T^r, n \in N^d.$$

The group G^d acts from the left on this space. The Poisson transform $\mathcal{P}_\lambda\colon \mathcal{D}'_\lambda \to C^\infty(G/H)$ is defined by

$$\mathcal{P}_\lambda f(x) = \int_{K^d} f(xk)\,dk = \int_{K^d} p_\lambda(x,k) f(k)\,dk, \quad x \in G^d.$$

Here the 'Poisson kernel' $p_\lambda \in C^\infty(G^d \times K^d)$ is defined by $p_\lambda(x,k) = a^{-\lambda-\rho}$, where $a \in T^r$ is the T^r–part of $x^{-1}k$ in the decomposition (8). It is known that $\mathcal{P}_\lambda$ is a G^d–equivariant transformation into a joint eigenspace for $\mathbb{D}(G^d/K^d)$ in $C^\infty(G^d/K^d)$, and that it is injective if Σ^+ is given by (7) (see for example [7, Thm. 12.2]).

Let $\mathcal{D}'_{\lambda,H^d}$ be the set of H^d–finite elements in $\mathcal{D}'_\lambda$, and let $\mathcal{D}'_{\lambda,H^d}(H^dP^d)$ denote the subset of elements supported on the H^d–orbit H^dP^d in G^d/P^d (which is closed, cf. [37, Prop. 7.1.8]). Let now $\lambda \in \Lambda$. Then condition (ii) implies that the H^d–finite action on $\mathcal{D}'_{\lambda,H^d}(H^dP^d)$ extends to a holomorphic $K_\mathbb{C}$–action. The space $\mathcal{U}_{\lambda,K}$ is now defined by

$$\mathcal{U}_{\lambda,K} = \{f \in C^\infty(K;G/H) \mid f^r \in \mathcal{P}_\lambda(\mathcal{D}'_{\lambda,H^d}(H^dP^d))\}.$$

The proof that $\mathcal{U}_{\lambda,K} \subset L^2_\lambda(G/H)$ can be found in [33] (see also [7, Thm. 19.1]).

4.2. A partial Paley–Wiener theorem

We now return to the general case, where condition (6) is not necessarily fulfilled. We shall see that a variation of the ideas going into Prop. 5 provides us with a construction of the inverse Fourier transform for a large family of 'nice' functions on $G^\wedge_H$.

Recall that for $f \in C_c^\infty(G/H)$ we have defined the Fourier transform $f^\wedge$ on $G^\wedge_H$ such that

$$f^\wedge(\pi) \in \mathrm{Hom}(\mathcal{V}^o_\pi, \mathcal{H}_\pi) = \mathcal{H}_\pi \otimes (\mathcal{V}^o_\pi)^*.$$

Let $\mathfrak{b} \subset \mathfrak{q}$ be a θ–invariant Cartan subspace. Let $\{v_i\}_{i=1,\dots,m_\pi}$ be a spherical basis for $\mathcal{V}^o_\pi$ (for a given π), and let $\lambda_i \in \mathfrak{b}^*_\mathbb{C}$ be the type of v_i (determined up to conjugation by W).

As in the previous section let G^d/K^d be the non–compact Riemannian form of G/H. In analogy with the definition of $\mathfrak{t}^r$ we define

$$\mathfrak{b}^r = \mathfrak{b} \cap \mathfrak{p} + i(\mathfrak{b} \cap \mathfrak{k}) = \mathfrak{b}_\mathbb{C} \cap \mathfrak{g}^d, \tag{9}$$

then $\mathfrak{b}^r$ is a maximal Abelian split subspace for $\mathfrak{g}^d$. Hence the roots of $\mathfrak{b}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ are real valued on $\mathfrak{b}^r$, and $\mathfrak{b}^r$ is W–invariant.

Let $PW(\mathfrak{b}^r)^W$ be the space of W–invariant entire rapidly decreasing functions of exponential type on $\mathfrak{b}^*_{\mathbb{C}}$. By the classical Paley–Wiener theorem this is the image of the space $C_c^\infty(\mathfrak{b}^r)^W$ under the Fourier transform on the Euclidean space $\mathfrak{b}^r$ (defined as in (5)), and by Helgason's Paley–Wiener theorem it is also the image of $C_c^\infty(K^d\backslash G^d/K^d)$ under the spherical Fourier transform (see 3.3.2 (d)).

Let $K^\wedge_{K\cap H}$ be the set of (equivalence classes) of irreducible representations of K with non–trivial $K \cap H$–fixed vectors. For any $\psi \in PW(\mathfrak{b}^r)^W$, $\mu \in K^\wedge_{K\cap H}$ and $\pi \in G^\wedge_H$ we define $F_{\psi,\mu}(\pi) \in \mathrm{Hom}(\mathcal{V}^o_\pi, \mathcal{H}_\pi)$ by

$$F_{\psi,\mu}(\pi)v_i = \psi(\lambda_i)P_\mu v_i, \qquad (i = 1, \ldots, m_\pi),$$

where $P_\mu \colon \mathcal{H}_{-\infty} \to \mathcal{H}_\infty$ is the K–equivariant extension to $\mathcal{H}_{-\infty}$ of the orthogonal projection of $\mathcal{H}_\pi$ onto its μ–component (given by the convolution with the normalized character of $\mu^\vee$). Notice that $F_{\psi,\mu}$ is independent of the choice of the spherical basis $\{v_i\}$ for $\mathcal{V}^o_\pi$.

Theorem 5, [17]**.** *Let $\psi \in PW(\mathfrak{b}^r)^W$ and $\mu \in K^\wedge_{K\cap H}$ be given, and let $F_{\psi,\mu}$ be as above. There exists a unique function $f = f_{\psi,\mu}$ in $C_c^\infty(G/H)$ such that $f^\wedge = F_{\psi,\mu}$, or equivalently, for any $\pi \in G^\wedge_H$ and any spherical vector $v \in \mathcal{V}^o_\pi$ of type $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$ we have*

$$f^\wedge(\pi)v = \psi(\lambda)P_\mu v. \tag{10}$$

Moreover, the function f is $K\cap H$–invariant and K–finite of type μ, and the equation (10) *holds more generally with v a spherical vector of type λ in $\mathcal{V}_\pi$.*

Notice that it follows from (10) that the spherical distributions given by $\xi_{\pi,i} \colon \varphi \mapsto \langle \varphi^\wedge(\pi)v_i|v_i\rangle$, $i = 1, \ldots, m_\pi$, satisfy

$$\xi_{\pi,i}(f) = \psi(\lambda_i)\langle P_\mu v_i|v_i\rangle$$

for all $\pi \in G^\wedge_H$.

In order to indicate the proof of Thm. 5 we shall need the following proposition, which is closely related to Prop. 4. Let the spaces $K\backslash G/H$ and $H^d\backslash G^d/K^d$ be given the measures inherited from the invariant measures on G/H and $H^d\backslash G^d$, respectively.

Proposition 6, [20]**.** *Partial holomorphic extension defines a norm–preserving isomorphism $f \to f^r$ of $L^1(K\backslash G/H)$ onto $L^1(H^d\backslash G^d/K^d)$.*

Indication of the proof of Thm. 5. The uniqueness of f follows easily from the abstract Plancherel theory discussed earlier. The existence is established as follows.

Let $\psi \in PW(\mathfrak{b}^r)^W$ and $\mu \in K^\wedge_{K\cap H}$ be given, and let V_μ be the representation space for μ, equipped with an inner product for which μ is unitary. By the Paley–Wiener theorem for the spherical Fourier transform on G^d/K^d there exists a K^d–invariant function $F \in C_c^\infty(G^d/K^d)$ such that

$$\int_{G^d/K^d} F(x)\varphi_\lambda(x)\,dx = \psi(\lambda)$$

for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, where φ_λ is the elementary spherical function on G^d/K^d. Let $e_o \in V_\mu$ be a $K \cap H$–fixed unit vector and define

$$F_\mu(x) = \dim(\mu) \int_{H^d} F(hx)\langle \mu(h)e_o|e_o\rangle dh, \qquad x \in G^d/K^d.$$

Here μ is defined on H^d by the holomorphic extension from K to $K_{\mathbb{C}}$. It follows that F_μ is H^d–finite of this type. Let $f \in C^\infty(K; G/H)$ be the element such that $f^r = F_\mu$ by Prop. 4.

It follows easily from this construction that f has compact support. To finish we must calculate $f^\wedge(\pi)v$ for all spherical $v \in \mathcal{V}_\pi$. Let $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ be the type of v. Since f is K–finite of type μ and $K \cap H$–fixed it suffices to calculate $\langle f^\wedge(\pi)v|v'\rangle$, where v' is a $K \cap H$–fixed vector in V_μ. Now $\langle f^\wedge(\pi)v|v'\rangle$ can be written as an integral over K followed by an integral over $K\backslash G/H$, and by Prop. 6 the latter can be transferred to an integral over $H^d\backslash G^d/K^d$ involving F_μ. After some rewriting one ends up by finding

$$\langle f^\wedge(\pi)v|v'\rangle = \int_{G^d/K^d} F(x)\varphi_\lambda(x)\, dx\, \langle P_\mu v|v'\rangle = \psi(\lambda)\langle P_\mu v|v'\rangle,$$

from which the result follows.

4.3. A Plancherel formula for the most continuous part of $L^2(G/H)$

In this subsection we discuss Problems (a), (b) and (c) for the 'most continuous part' of $L^2(G/H)$ (to be defined below). The main reference is [9].

Let notation be as in Sect. 2 (in [9] the assumptions on G/H are somewhat more general, but we shall not discuss that here). The representations $\pi_{\xi,\lambda}$ that occur in the most continuous part of $L^2(G/H)$ are constructed as follows. Let $P = MAN$ be a parabolic subgroup of G, with the indicated Langlands decomposition, satisfying $\sigma\theta P = P$ and being minimal with respect to this condition. Then M and A are σ–stable. Let $\mathfrak{a}_{\mathrm{q}} = \mathfrak{a} \cap \mathfrak{q}$, where $\mathfrak{a}$ is the Lie algebra of A, then it follows that $\mathfrak{a}_{\mathrm{q}}$ is a maximal Abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, and that the Levi part MA of P is the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in G. Let $(\xi, \mathcal{H}_\xi) \in M_{\mathrm{fu}}^\wedge$, the set of (equivalence classes of) finite dimensional irreducible unitary representations of M, and let $\lambda \in i\mathfrak{a}^*$. We require that $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$, that is that λ vanishes on $\mathfrak{a} \cap \mathfrak{h}$. Then by definition $\pi_{\xi,\lambda}$ is the induced representation $\pi_{P,\xi,\lambda} = \mathrm{Ind}_{P=MAN}^G\, \xi \otimes e^\lambda \otimes 1$ (the 'principal series' for G/H), that is, the representation space $\mathcal{H}_{\xi,\lambda}$ consists of (classes of) $\mathcal{H}_\xi$–valued measurable functions f on G, square integrable on K and satisfying

$$f(gman) = a^{-\lambda-\rho}\xi(m)^{-1}f(g), \qquad (g \in G, m \in M, a \in A, n \in N), \tag{11}$$

and G acts from the left. Here $\rho = \frac{1}{2}\,\mathrm{Tr}\,\mathrm{Ad}_{\mathfrak{n}} \in \mathfrak{a}_{\mathrm{q}}^*$. (The convention in (11) differs from the above cited references: The induction takes place on the opposite side.)

The Plancherel decomposition for the most continuous part of $L^2(G/H)$ is obtained by realizing the abstract Fourier transform explicitly for the principal series. This realization is then a partial isometry of $L^2(G/H)$ onto the direct integral

$$\int_{\xi,\lambda}^{\oplus} m_\xi \, \pi_{\xi,\lambda} \, d\mu(\xi,\lambda). \tag{12}$$

The multiplicities m_ξ (which happen to be independent of λ) and the measure $d\mu(\xi,\lambda)$ are explicitly described below. The most continuous part of $L^2(G/H)$, denoted $L^2_{\mathrm{mc}}(G/H)$, is then by definition the orthocomplement of the kernel of this partial isometry. Its Plancherel decomposition is exactly given by (12).

In order to realize the Fourier transform we must first discuss the space $\mathcal{V}_{\xi,\lambda} = (\mathcal{H}^{-\infty}_{\xi,\lambda})^H$. Let $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a fixed set of elements such that $w \mapsto HwP$ parametrizes the open $H \times P$ orbits on G (it is known (see [36] or [29]) that any set of representatives for the double quotient $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})\backslash N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$ can be used as $\mathcal{W}$ – in particular, $\mathcal{W}$ is finite). Viewing an element $f \in \mathcal{H}^{-\infty}_{\xi,\lambda}$ as an $\mathcal{H}_\xi$–valued distribution on G, satisfying appropriate conditions of homogeneity for the right action of P, it is easily seen that if f is H–invariant then f must restrict to a smooth function on each open $H \times P$ orbit. Hence it makes sense to evaluate f in the elements of $\mathcal{W}$, and in fact its restriction to the open orbit HwP will be uniquely determined from the value at w. We denote this value by $\mathrm{ev}_w(f)$. It is easily seen that ev_w maps $\mathcal{V}_{\xi,\lambda}$ into the space $\mathcal{H}_\xi^{w^{-1}(M\cap H)w}$ of $w^{-1}(M\cap H)w$–fixed elements in $\mathcal{H}_\xi$ (note that $w^{-1}Mw = M$, but $w^{-1}Hw$ may differ from H). Let $V(\xi)$ denote the formal direct sum

$$V(\xi) = \bigoplus_{w\in\mathcal{W}} \mathcal{H}_\xi^{w^{-1}(M\cap H)w}, \tag{13}$$

provided with the direct sum inner product (thus, by definition the summands are mutually orthogonal, even though this may not be the case in $\mathcal{H}_\xi$). Furthermore, let

$$\mathrm{ev}\colon \mathcal{V}_{\xi,\lambda} \to V(\xi)$$

denote the direct sum of the maps ev_w. The construction of the induced representations $\pi_{\xi,\lambda}$ and of the map ev makes sense for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, the complex linear dual of $\mathfrak{a}_{\mathrm{q}}$ (though the representations need not be unitary for λ outside $i\mathfrak{a}^*_{\mathrm{q}}$). We now have

Theorem 7, [3]. *The map* ev *is bijective for generic* $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

For generic λ, let

$$j(\xi,\lambda)\colon V(\xi) \to \mathcal{V}_{\xi,\lambda}$$

be the inverse of ev, then by definition we have for $\eta \in V(\xi)$ that the restriction of the distribution $j(\xi,\lambda)(\eta)$ to the open $H \times P$ orbit HwP, $w \in \mathcal{W}$, is the smooth $\mathcal{H}_\xi$–valued function given by

$$j(\xi,\lambda)(\eta)(hwman) = a^{-\lambda-\rho}\xi(m^{-1})\eta_w. \tag{14}$$

(Here η_w denotes the w–component of η, viewed as an element of $\mathcal{H}_\xi$.) Notice that if G/H is a Riemannian symmetric space, so that $H = K$, then we have $G = HP$ by the Iwasawa decomposition. Hence we can take $\mathcal{W} = \{e\}$, and since $M \subset K = H$ we have $V(\xi) = \{0\}$ unless ξ is the trivial representation $\mathbf{1}$, in which case $V(\mathbf{1}) = \mathbb{C}$. Then $j(\mathbf{1},\lambda)$ is completely determined by (14); in fact we have

$$j(\mathbf{1},\lambda)(x) = e^{-(\lambda+\rho)H(x)},$$

where $H\colon G \to \mathfrak{a}$ is the Iwasawa projection (since $V(\mathbf{1}) = \mathbb{C}$ we can omit η). Thus the kernel $P_\lambda(x,k) = j(\mathbf{1},\lambda)(x^{-1}k)$ on $G/K \times K$ is the generalized Poisson kernel. For general G/H we can supplement (14) as follows: If $\mathrm{Re}\langle\lambda+\rho,\alpha\rangle < 0$ for all α in the set Σ^+ of positive roots (the $\mathfrak{a}$–roots of $\mathfrak{n} = \mathrm{Lie}(N)$), then $j(\xi,\lambda)(\eta)$ is the continuous function on G given by (14) on HwP for all $w \in \mathcal{W}$ and vanishing on the complement of these sets (the condition on λ ensures the continuity). For elements λ outside the above region the distribution $j(\xi,\lambda)$ can be obtained from the above by meromorphic continuation. (See [34], [32], [3]. These results have been generalized to other principal series representations in [13], [15].)

Having constructed the H–invariant distribution vectors $j(\xi,\lambda)\eta$ as above we can now attempt to write down a Fourier transform for the principal series. For $f \in C_c^\infty(G/H)$ we consider the map

$$(\xi,\lambda) \mapsto f^\wedge(\pi_{\xi,\lambda})j(\xi,\lambda) = \pi_{\xi,\lambda}(f)\,j(\xi,\lambda) \in \mathcal{H}^\infty_{\xi,\lambda} \otimes V(\xi)^*. \tag{15}$$

In the Riemannian case this is exactly the Fourier transform, as defined by Helgason (see [24]). However when G/H is not Riemannian a new phenomenon may occur: by the above definitions (15) is a meromorphic function in λ, which may have singularities on the set $i\mathfrak{a}_\mathrm{q}^*$ of interest for the Plancherel decomposition, and thus it may not make sense for some singular $\lambda \in i\mathfrak{a}_\mathrm{q}^*$. This unpleasantness is overcome by a suitable normalization of $j(\xi,\lambda)$, which removes the singularities. The normalization is carried out by means of the standard intertwining operators $A(\bar{P},P,\xi,\lambda)$ from $\pi_{P,\xi,\lambda}$ to $\pi_{\bar{P},\xi,\lambda}$, where $\bar{P}$ is the parabolic subgroup opposite to P. Let

$$j^\circ(\xi,\lambda) = A(\bar{P},P,\xi,\lambda)^{-1}j(\bar{P},\xi,\lambda),$$

where $j(\bar{P},\xi,\lambda)$ is constructed as $j(\xi,\lambda)$ above, but with P replaced by $\bar{P}$. Since the intertwining operator $A(\bar{P},P,\xi,\lambda)$ is bijective for generic λ, it follows that

$$j^\circ(\xi,\lambda)\colon V(\xi) \to \mathcal{V}_{\xi,\lambda}$$

is again a bijection, for generic λ. Moreover, we now have

Theorem 8, [9]**.** *The meromorphic function* $\lambda \mapsto j^\circ(\xi,\lambda)$ *is regular on* $i\mathfrak{a}_\mathrm{q}^*$.

We can now define the Fourier transform $f \mapsto f^\wedge$ for the principal series properly by (15), but with j replaced by j°:

$$f^\wedge(\xi,\lambda) = \pi_{\xi,\lambda}(f)\, j^\circ(\xi,\lambda) \in \mathcal{H}^\infty_{\xi,\lambda} \otimes V(\xi)^*.$$

Notice that when G/H is Riemannian the normalization makes our Fourier transform different from that of Helgason – in this case the normalization amounts to a division by Harish–Chandra's c–function $\mathbf{c}(\lambda)$. See [8] for the determination of j° in the group case.

We can now give the solution to Problem (b) for this part of $L^2(G/H)$: We take $\mathcal{V}^o_{\xi,\lambda} = \mathcal{V}_{\xi,\lambda}$, and give it the Hilbert space structure that makes $j^\circ(\xi,\lambda)$ an isometry. The solution to Problem (c) is as follows. Let $\mathcal{H}$ be the Hilbert space given by

$$\mathcal{H} = \int_{\xi,\lambda}^{\oplus} \mathcal{H}_{\xi,\lambda} \otimes V(\xi)^*\, d\mu(\xi,\lambda), \tag{16}$$

with the measure $d\mu(\xi,\lambda) = \dim(\xi)\, d\lambda$, where $d\lambda$ is Lebesgue measure on $i\mathfrak{a}_\mathrm{q}^*$ (suitably normalized). Here ξ runs over $M^\wedge_{\mathrm{fu}}$ (notice however that some of them may disappear because $V(\xi)$ is trivial), and λ runs over an open chamber $i\mathfrak{a}_\mathrm{q}^{*+}$ in $i\mathfrak{a}_\mathrm{q}^*$ for the Weyl group $W_\mathrm{q} = N_K(\mathfrak{a}_\mathrm{q})/Z_K(\mathfrak{a}_\mathrm{q})$.

Theorem 9, [9]**.** *Let* $f \in C_c^\infty(G/H)$. *Then* $f^\wedge \in \mathcal{H}$ *and* $\|f^\wedge\| \le \|f\|_2$. *Moreover, the map* $f \mapsto f^\wedge$ *extends to an equivariant partial isometry* $\mathfrak{F}$ *of* $L^2(G/H)$ *onto* $\mathcal{H}$. *In particular, we thus have the multiplicities* $m_\xi = \dim V(\xi)$.

We define the most continuous part $L^2_{\mathrm{mc}}(G/H)$ of $L^2(G/H)$ as the orthocomplement of the kernel of $\mathfrak{F}$. Then $\mathfrak{F}$ restricts to an isometry of this space onto $\mathcal{H}$. In [9] it is shown that $L^2_{\mathrm{mc}}(G/H)$ is 'large' in $L^2(G/H)$ in a certain sense – in particular its orthocomplement (the kernel of $\mathfrak{F}$) has trivial intersection with $C_c^\infty(G/H)$ (thus $f \mapsto f^\wedge$ is injective, even though the extension $\mathfrak{F}$ need not be). Moreover, if G/H has split rank one, that is if $\dim \mathfrak{a}_\mathrm{q} = 1$, then there are at most two conjugacy classes of Cartan subspaces, and hence one expects from the analogy with the group case as mentioned earlier that only the corresponding two 'series' of representations will be present. Indeed this is the case; it is shown in [9] that the kernel of $\mathfrak{F}$ decomposes discretely when the split rank is one. Thus, in this case the Plancherel decomposition of $L^2(G/H)$ can be determined from Thm. 9 together with the description of the discrete series (see Sect. 4.1 above), except for the explicit determination of the Hilbert space structure on $\mathcal{V}^o_\pi$ for the discrete series representations π.

On the other hand, when G/H is Riemannian then $\mathfrak{F}$ is injective and Thm. 9 gives the complete Plancherel decomposition of $L^2(G/H)$ (in the formulation of Harish–Chandra and Helgason the Plancherel measure is $|\mathbf{c}(\lambda)|^{-2}\, d\lambda$, but here the factor $|\mathbf{c}(\lambda)|^{-2}$ disappears because of the normalization of j°).

A further discussion of the multiplicities m_π can be found in [8].

4.4. The K–finite case

The isomorphism of (16) onto $L^2_{\rm mc}(G/H)$ (the 'inverse Fourier transform') can be given more explicitly when one restricts to K-finite functions. In this subsection we shall discuss this restriction, which happens to be crucial in the proofs of Thms. 8 and 9.

4.4.1. *Eisenstein integrals.* Let (μ, V_μ) be a fixed, irreducible unitary representation of K. Taking μ–components in (16) we have

$$\mathcal{H}^\mu = \int_{\xi,\lambda}^{\oplus} \mathcal{H}^\mu_{\xi,\lambda} \otimes V(\xi)^* \, d\mu(\xi, \lambda). \tag{17}$$

Moreover, by Frobenius reciprocity we have

$$\mathcal{H}^\mu_{\xi,\lambda} \simeq \mathrm{Hom}_{M\cap K}(V_\mu, \mathcal{H}_\xi) \otimes V_\mu \tag{18}$$

as K–modules (where K acts on the second component in the tensor product), for all $\xi \in M^\wedge_{\rm fu}, \lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Note that since each representation $\xi \in M^\wedge_{\rm fu}$ is trivial on the non–compact part of M, we have that $\xi|_{M\cap K}$ is irreducible, and that $\mathrm{Hom}_{M\cap K}(V_\mu, \mathcal{H}_\xi)$ is non–trivial if and only if this restriction occurs as a subrepresentation of $\mu|_{M\cap K}$. We use the notation $\xi \uparrow \mu$ to indicate this occurrence; it happens only for finitely many ξ. Thus by taking K–types the integral over ξ in (17) becomes a finite sum, hence more manageable. In analogy with the earlier definition of the space $V(\xi)$ we now define the space $\mathcal{V}(\mu)$ to be the formal direct sum

$$\mathcal{V}(\mu) = \bigoplus_{w\in\mathcal{W}} V_\mu^{w^{-1}(K\cap M\cap H)w}.$$

It is easily seen from the above that

$$\mathcal{V}(\mu) \simeq \bigoplus_{\xi\uparrow\mu} \mathrm{Hom}_{M\cap K}(\mathcal{H}_\xi, V_\mu) \otimes V(\xi). \tag{19}$$

Hence in view of (18) we have

$$\mathcal{V}(\mu)^* \otimes V_\mu \simeq \bigoplus_{\xi\uparrow\mu} \mathcal{H}^\mu_{\xi,\lambda} \otimes V(\xi)^* \tag{20}$$

for all $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. From (17) and (20) we finally obtain

$$\mathcal{H}^\mu \simeq \int_\lambda^\oplus \mathcal{V}(\mu)^* \otimes V_\mu \, d\lambda \simeq L^2(i\mathfrak{a}^{*+}_{\mathrm{q}}) \otimes \mathcal{V}(\mu)^* \otimes V_\mu. \tag{21}$$

This isomorphism indicates that the Fourier transform, when restricted to K–finite functions of type μ, can be considered as a map into the $\mathcal{V}(\mu)^* \otimes V_\mu$–valued functions on $i\mathfrak{a}^*_{\mathrm{q}}$.

Instead of working with K–finite scalar–valued functions on G/H, it is convenient to consider 'μ–spherical' functions f on G/H, that is, V_μ–valued functions satisfying

$$f(kx) = \mu(k)f(x), \qquad k \in K,\, x \in G/H.$$

Let $L^2(G/H;\mu)$ denote the space of square integrable such functions, then by contraction we have a K–equivariant isomorphism

$$\gamma_\mu \colon L^2(G/H;\mu^\vee) \otimes V_\mu \xrightarrow{\sim} L^2(G/H)^\mu. \tag{22}$$

(Again K acts on the second component in the tensor product. The map $\dim(\mu)\gamma_\mu$ is an isometry.) Notice that when passing from K–finite functions to spherical functions one must also pass from μ to its contragradient $\mu^\vee$. Since $\mathcal{V}(\mu)^* = \mathcal{V}(\mu^\vee)$ we are led to the search, for each μ, of a Fourier transform, which is a partial isometry of $L^2(G/H;\mu)$ onto $L^2(i\mathfrak{a}_{\mathrm{q}}^{*+}) \otimes \mathcal{V}(\mu)$. Going through the above isomorphisms in detail, we are led to the following construction culminating in (26), which essentially is the 'projection' of the construction of $f \mapsto f^\wedge$ to functions of type μ.

For $\psi \in \mathcal{V}(\mu)$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with $\mathrm{Re}\langle \lambda + \rho, \alpha\rangle < 0$ for all $\alpha \in \Sigma^+$, let $\tilde{\psi}_\lambda$ be the V_μ–valued function on G defined by

$$\tilde{\psi}_\lambda(x) = \begin{cases} a^{-\lambda-\rho}\mu(m^{-1})\psi_w & \text{if } x = hwman \in Hw(M\cap K)AN, w \in \mathcal{W}, \\ 0 & \text{if } x \notin \cup_{w\in\mathcal{W}} HwP. \end{cases}$$

(It is to be noted that $M = w^{-1}(M\cap H)w(M\cap K)$, and hence $Hw(M\cap K)AN = HwMAN$.) It can be shown that $\tilde{\psi}_\lambda$ is continuous as a function of x, and has a distribution–valued meromorphic continuation in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Let $E_\mu(\psi,\lambda)$ be the smooth μ–spherical function on G/H defined by

$$E_\mu(\psi,\lambda)(x) = \int_K \mu(k)\tilde{\psi}_\lambda(x^{-1}k)\,dk.$$

(Even when $\tilde{\psi}_\lambda$ is only a distribution, the convolution with μ makes $E_\mu(\psi,\lambda)$ smooth.) We call these functions *Eisenstein integrals* for G/H. When G/H is Riemannian and μ is the trivial K–type $\mathbf{1}$, the construction produces the spherical functions

$$\varphi_\lambda(x) = \int_K e^{-(\lambda+\rho)H(x^{-1}k)}dk, \tag{23}$$

and for other K–types we get the generalized spherical functions of [26]. In the group case the Eisenstein integrals defined in this manner coincide, up to normalization, with Harish–Chandra's Eisenstein integrals associated to the minimal parabolic subgroup. It can be seen that the vector components of the Eisenstein integral $E_\mu(\psi,\lambda)$ are linear combinations of generalized matrix coefficients formed by the $j(\xi,\lambda)\eta$, $(\eta \in V(\xi),\ \xi \uparrow \mu^\vee)$, with K–finite vectors of type μ.

The spherical functions are eigenfunctions for the invariant differential operators on G/K – in analogy we have

$$DE_\mu(\psi,\lambda) = E_\mu(\chi_\mu(D,\lambda)\psi,\lambda) \tag{24}$$

for all $D \in \mathbb{D}(G/H)$. Here $\chi_\mu(D)$ is an $\mathrm{End}(\mathcal{V}(\mu))$–valued polynomial in λ. Just as it is the case for the spherical functions, one can derive an asymptotic expansion from this 'eigenequation'. Here we have to recall the 'KAH'–decomposition of G,

$$G = \mathrm{cl} \bigcup_{w\in\mathcal{W}} KA_{\mathrm{q}}^+ w^{-1}H,$$

where A_{q}^+ is the exponential of the positive chamber in $\mathfrak{a}_{\mathrm{q}}$ corresponding to Σ^+, and where the union inside the closure operator cl is disjoint. Since the Eisenstein integrals are K–spherical, we have to consider their behavior on $A_{\mathrm{q}}^+w^{-1}$, for all $w \in \mathcal{W}$. Notice that when G/H is Riemannian there is only one 'direction' to control, since the KAH–decomposition then specializes to the Cartan decomposition $G = \mathrm{cl}\, KA^+K$. The expansion is essentially as follows (see [4] and the remark below):

$$E_\mu(\psi,\lambda)(aw^{-1}) = \sum_{s\in W_{\mathrm{q}}} a^{s\lambda-\rho}[C(s,\lambda)\psi]_w + \text{ lower order terms in } a, \tag{25}$$

for $a \in A_{\mathrm{q}}^+$, $w \in \mathcal{W}$, where W_{q} is as defined above Thm. 9, and the 'c–function' $\lambda \mapsto C(s,\lambda)$ is a meromorphic function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with values in $\mathrm{End}(\mathcal{V}(\mu))$ (it follows easily from the μ–sphericality that we have $E_\mu(\psi,\lambda)(aw^{-1}) \in V_\mu^{w^{-1}(K\cap M\cap H)w}$ for $a \in A_{\mathrm{q}}$). The expansion converges for $a \in A_{\mathrm{q}}^+$; the 'lower order terms' involve powers of the form $a^{s\lambda-\rho-\nu}$ where ν is a sum of positive roots.

Remark. The definition of the Eisenstein integral given here does require μ to be finite dimensional, but not necessarily irreducible, and therefore remains valid for an arbitrary finite dimensional unitary representation of K. In fact such an Eisenstein integral E_τ is introduced in [4] for a representation τ defined as follows. Let $\mu \in K^\wedge$, and let $C(K)_{\mu^\vee}$ denote the space of continuous functions $K \to \mathbb{C}$ which are finite and isotypical of type $\mu^\vee$ for the right regular representation R of K. Put $V_\tau = C(K)_{\mu^\vee}$ and let τ be the restriction of the right regular representation R to V_τ. Then by the Peter-Weyl theorem we have natural isomorphisms $V_\tau \simeq V_{\mu^\vee} \otimes V_\mu$, and $\tau \simeq \mu^\vee \otimes I_{V_\mu}$. It now follows easily that the Eisenstein integrals E_τ and $E_{\mu^\vee}$ are related as follows. One has $\mathcal{V}(\tau) \simeq \mathcal{V}(\mu^\vee) \otimes V_\mu$, and accordingly, for $\psi \in \mathcal{V}(\mu^\vee)$, $v \in V_\mu$:

$$E_\tau(\psi \otimes v, \lambda)(x) = E_{\mu^\vee}(\psi,\lambda)(x) \otimes v.$$

It follows from this that the corresponding c–functions are related by $C_\tau(s,\lambda) = C_{\mu^\vee}(s,\lambda) \otimes I$. From these remarks it should be clear how the results of [4] carry over to the present situation.

4.4.2. *The Fourier transform.* It would now be natural to define the Fourier transform $\mathcal{F}_\mu f$ of a function $f \in C_c^\infty(G/H;\mu)$, the space of compactly supported and smooth μ–spherical functions on G/H, as the $\mathcal{V}(\mu)$–valued function φ on $\mathfrak{a}_{q\mathbb{C}}^*$ given by

$$\langle \varphi(\lambda)|\psi\rangle = \int_{G/H} \langle f(x)|E_\mu(\psi, -\bar{\lambda})(x)\rangle \, dx, \qquad \psi \in \mathcal{V}(\mu),$$

where the inner products $\langle\cdot|\cdot\rangle$ are the sesquilinear Hilbert space inner products on $\mathcal{V}(\mu)$ and V_μ, respectively. Via the isomorphisms in (21) and (22) this would essentially correspond to the Fourier transform in (15). However, as with $j(\xi, \lambda)$ we have the problem that $E_\mu(\psi, \lambda)$, which is meromorphic in λ, may have singularities on $i\mathfrak{a}_q^*$. Again we have to carry out a normalization: the *normalized Eisenstein integral* is defined by

$$E_\mu^\circ(\psi, \lambda) = E_\mu(C(1, \lambda)^{-1}\psi, \lambda).$$

In other words, the Eisenstein integral is normalized by its asymptotics, so that we have $E_\mu^\circ(\psi, \lambda)(aw^{-1}) \sim a^{\lambda-\rho}\psi_w$ for $a \in A^+$, $w \in \mathcal{W}$ and $\mathrm{Re}\,\lambda$ strictly dominant. It can be shown that this normalization corresponds to the one on $j(\xi, \lambda)$, in the sense that the vector components of $E_\mu^\circ(\psi, \lambda)$ are linear combinations of matrix coefficients formed by the $j^\circ(\xi, \lambda)\eta$, ($\eta \in V(\xi)$, $\xi \uparrow \mu^\vee$), with K–finite vectors of type μ. Moreover, it can be shown that the statement of Thm. 8 is equivalent with the following 'K–finite version':

Theorem 10, [9]**.** *The meromorphic function $\lambda \mapsto E_\mu^\circ(\psi, \lambda)$ is regular on $i\mathfrak{a}_q^*$, for every $\mu \in K^\wedge$ and $\psi \in \mathcal{V}(\mu)$.*

With this in mind we define the *μ–spherical Fourier transform* $\mathcal{F}_\mu f$ as above, but with E_μ replaced by E_μ°, that is, by

$$\langle \mathcal{F}_\mu f(\lambda)|\psi\rangle = \int_{G/H} \langle f(x)|E_\mu^\circ(\psi, -\bar{\lambda})(x)\rangle \, dx, \qquad \psi \in \mathcal{V}(\mu). \tag{26}$$

Then $\mathcal{F}_\mu f$ corresponds to $f^\wedge$ via the isomorphisms in (20) and (22). For completeness the precise correspondence is given as follows. Let $\gamma_\mu \colon C_c^\infty(G/H; \mu^\vee) \otimes V_\mu \to C_c^\infty(G/H)^\mu$ be the contraction (as in (22)) and let $\mathrm{pr}_{\xi,\lambda} \colon \mathcal{V}(\mu)^* \otimes V_\mu \to \mathcal{H}_{\xi,\lambda} \otimes V(\xi)^*$ be the projection corresponding to (20). Then for all $f \in C_c^\infty(G/H)^\mu$ we have

$$\dim(\xi)\, f^\wedge(\xi, \lambda) = \mathrm{pr}_{\xi,\lambda}\left[\left((\mathcal{F}_{\mu^\vee} \otimes I_{V_\mu})\left(\gamma_\mu^{-1} f\right)\right)(-\lambda)\right], \tag{27}$$

for $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$, $\xi \uparrow \mu$, and $f^\wedge(\xi, \lambda) = 0$ for all other ξ.

When G/H is Riemannian and $\mu = \mathbf{1}$, the normalization again amounts to division by $\mathbf{c}(\lambda)$, and thus $\mathcal{F}_\mu f$ is in this case related to the spherical Fourier transform of f as follows:

$$\mathcal{F}_\mu f(\lambda) = \mathbf{c}(-\lambda)^{-1} \int_{G/K} f(x)\varphi_{-\lambda}(x)\, dx,$$

where φ_λ is the elementary spherical function in (23). If G/H is Riemannian and μ is non–trivial there is a similar relation, also involving $\mathbf{c}(\lambda)^{-1}$, to the Fourier transform in [26].

Let $C^\circ(s,\lambda) = C(s,\lambda)C(1,\lambda)^{-1}$, then we have from (25)

$$E^\circ_\mu(\psi,\lambda)(aw) = \sum_{s\in W_{\mathrm{q}}} a^{s\lambda-\rho}[C^\circ(s,\lambda)\psi]_w + \text{ lower order terms in } a. \tag{28}$$

The following theorem generalizes results of Helgason and Harish–Chandra for the Riemannian case and the group case, respectively (see [25, Thm. 6.6], [23, Lemma 17.6]).

Theorem 11, [4]**,** [5]**.** *For every $s \in W_{\mathrm{q}}$ we have the following identity of meromorphic functions:*

$$C^\circ(s,\lambda)C^\circ(s,-\bar\lambda)^* = I_{\mathcal{V}(\mu)} \qquad (\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}).$$

In particular, for $\lambda \in i\mathfrak{a}^*_{\mathrm{q}}$, the endomorphism $C^\circ(s,\lambda)$ of $\mathcal{V}(\mu)$ is unitary.

Notice that by Riemann's boundedness theorem it follows from the above result that the meromorphic function $\lambda \mapsto C^\circ(s,\lambda)$ has no singularities on $i\mathfrak{a}^*_{\mathrm{q}}$. Therefore the possible singularities of $E^\circ_\mu(\psi,\lambda)$ must occur in the lower order terms of (28). This observation plays a crucial role in the proof of Thm. 10.

On G/K the spherical functions satisfy the functional equation $\varphi_{s\lambda} = \varphi_\lambda$, for all $s \in W_{\mathrm{q}}$. The analog for the normalized Eisenstein integral on G/H is

$$E^\circ_\mu(C^\circ(s,\lambda)\psi, s\lambda) = E^\circ_\mu(\psi,\lambda) \tag{29}$$

(see [4, Prop. 16.4]. For the group case, see also [23, Lemma 17.2]).

Though $E^\circ_\mu(\psi,\lambda)$ by Thm. 10 is regular on $i\mathfrak{a}^*_{\mathrm{q}}$, it will in general have singularities elsewhere on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. It is remarkable, though, that in a certain direction only finitely many singularities occur. To be more precise, one has the following. Let

$$(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_+ = \{\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \mid \mathrm{Re}\langle\lambda,\alpha\rangle \geq 0,\ \alpha \in \Sigma^+\},$$

and put $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_- = -(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_+$.

Theorem 12, [4]**.** *There exists a polynomial π' on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, which is a product of linear factors of the form $\lambda \mapsto \langle\lambda,\alpha\rangle$+constant, with α a root, such that $\pi'(\lambda)E^\circ_\mu(\psi,\lambda)$ is holomorphic on a neighborhood of $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_+$.*

Notice that π' depends on the K–type μ. Notice also that when G/H is Riemannian we actually have that $E^\circ_\mu(\psi,\lambda)$ itself is holomorphic on $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_+$. Indeed, the spherical functions are everywhere holomorphic, and the normalizing divisor $\mathbf{c}(\lambda)$ has no zeros on this set. Thus, for this case one can take $\pi' = 1$.

It follows from Thm. 12 and (26) that if we put

$$\pi(\lambda) = \overline{\pi'(-\bar{\lambda})} \tag{30}$$

then $\lambda \mapsto \pi(\lambda)\mathcal{F}_\mu f(\lambda)$ is holomorphic on a neighborhood of $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$.

4.4.3. *Wave packets.* For the μ–spherical Fourier transform a 'partial inversion formula' is given in [9] as follows. For a $\mathcal{V}(\mu)$–valued function φ on $i\mathfrak{a}^*_{\mathrm{q}}$ of suitable decay one can form a 'wave packet', which is the superposition of normalized Eisenstein integrals, with amplitudes given by φ, that is

$$\mathcal{J}_\mu\varphi(x) = \int_{i\mathfrak{a}^*_{\mathrm{q}}} E^\circ_\mu(\varphi(\lambda), \lambda)(x)\, d\lambda.$$

It is easily seen that the transform $\mathcal{J}_\mu$ is the transposed of $\mathcal{F}_\mu$. For Euclidean Fourier transform (and more generally for the spherical Fourier transform on a Riemannian symmetric space) this transform is also the inverse of $\mathcal{F}_\mu$; the inversion formula states that $\mathcal{J}_\mu\mathcal{F}_\mu$ is the identity operator (when measures are suitably normalized). In the non–Riemannian generality of G/H this cannot be expected, because of the possible presence of discrete series. However we do have

Theorem 13, [9]**.** *There exists an invariant differential operator D (depending on μ) on G/H satisfying the following:*

(i) *As an operator on $C^\infty_c(G/H)$, D is injective and symmetric.*
(ii) *$\mathcal{J}_\mu\mathcal{F}_\mu f = f$ for all $f \in D(C^\infty_c(G/H;\mu))$.*

From (24) one can derive that $\mathcal{J}_\mu\mathcal{F}_\mu D = \mathcal{J}_\mu\chi_\mu(D)\mathcal{F}_\mu = D\mathcal{J}_\mu\mathcal{F}_\mu$. Hence it follows from (ii) that $D(\mathcal{J}_\mu\mathcal{F}_\mu f - f) = 0$ for all $f \in C^\infty_c(G/H;\mu)$. Nevertheless, one cannot then conclude from (i) that in fact $\mathcal{J}_\mu\mathcal{F}_\mu f = f$ because $\mathcal{J}_\mu\mathcal{F}_\mu f$ is not compactly supported in general. The presence of D is important, for example it annihilates all the discrete series in $L^2(G/H;\mu)$.

The proof of Thm. 13 is very much inspired by Rosenberg's proof (see [28, Ch. IV, §7]) of the inversion formula for the spherical Fourier transform on G/K (in which case one can take $D = 1$). A key step in both proofs is the use of a 'shift argument', originally used by Helgason for the proof of the Paley–Wiener theorem, where the integration in $\mathcal{J}_\mu$ (after use of (28)) is moved away from $i\mathfrak{a}^*_{\mathrm{q}}$ in the direction of $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$, using Cauchy's theorem. It can be seen that one only meets a finite number of singular hyperplanes in this shift. The purpose of the operator D is to remove these singularities (among other things this means that π should be a divisor in $\chi_\mu(D)$), so that no residues are present. The shift allows one to conclude that $\mathcal{J}_\mu\mathcal{F}_\mu Df$ is compactly supported whenever f is, which is an important step in the proof of the theorem.

Thm. 13 is crucial in the proof of Thm. 9. Via the isomorphism (22) one obtains with $\mathcal{J}_{\mu^\vee}$ an explicit formula for the restriction to $\mathcal{H}^\mu$ of the isomorphism of $\mathcal{H}$ onto $L^2_{\mathrm{mc}}(G/H)$.

4.5. A Paley–Wiener theorem for G/H

Let π' be the minimal polynomial satisfying the conclusion of Thm. 12, and as before let π be given by (30). We define the *pre–Paley–Wiener space*, $\mathcal{M}_\mu$ as the space of $\mathcal{V}(\mu)$–valued meromorphic functions φ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, satisfying the following conditions:

(i) $\varphi(s\lambda) = C^\circ(s,\lambda)\varphi(\lambda)$, *for all* $s \in W_{\mathrm{q}}$, $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.
(ii) $\pi(\lambda)\varphi(\lambda)$ *is holomorphic on a neighborhood of* $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$.
(iii) *There exists a constant* $R > 0$ *and for every* $n \in \mathbb{N}$ *a constant* $C > 0$ *such that*

$$\|\pi(\lambda)\varphi(\lambda)\| \le C(1 + |\lambda|)^{-n} e^{R|\operatorname{Re}\lambda|}$$

for all $\lambda \in (\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$.

It can be seen that $\mathcal{F}_\mu$ maps $C_c^\infty(G/H;\mu)$ into $\mathcal{M}_\mu$ (properties (i) and (ii) are straightforward consequences of (29) and Thms. 11 and 12, whereas (iii) requires a more difficult estimate for $E^\circ_\mu(\psi,\lambda)$). It follows from the Paley–Wiener theorem of Helgason and Gangolli (see [28, Ch. IV, §7]), that when G/H is Riemannian and μ the trivial K–type then $\mathcal{F}_\mu$ is a surjection onto the pre–Paley–Wiener space, as defined above for this special case. However in general one has to require further conditions on a function $\varphi \in \mathcal{M}_\mu$ before it belongs to $\mathcal{F}_\mu(C_c^\infty(G/H;\mu))$. Briefly put, the extra condition is that any existing relation between the normalized Eisenstein integrals and their derivatives (with respect to λ) should be reflected by a similar condition on φ. More precisely, we require that:

For all finite collections of $\partial_1,\dots,\partial_k \in S(\mathfrak{a}^*_{\mathrm{q}})$ *(that is, constant coefficient differential operators on* $\mathfrak{a}^*_{\mathrm{q}}$*),* $\psi_1,\dots,\psi_k \in \mathcal{V}(\mu)$ *and* $\lambda_1,\dots,\lambda_k \in (\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$*, for which the relation*

$$\sum_{i=1}^{k} \partial_i\big[\,\pi(\lambda)\,\langle\psi|E^\circ_\mu(\psi_i,-\bar{\lambda})(x)\rangle\big]_{\lambda=\lambda_i} = 0 \tag{31}$$

holds for every $\psi \in \mathcal{V}(\mu)$*,* $x \in G/H$*, we also have the relation*

$$\sum_{i=1}^{k} \partial_i\big[\,\pi(\lambda)\,\langle\varphi(\lambda)|\psi_i\rangle\big]_{\lambda=\lambda_i} = 0. \tag{32}$$

The space of functions $\varphi \in \mathcal{M}_\mu$ satisfying this requirement is denoted PW_μ. It is clear from the definition (26) of $\mathcal{F}_\mu f$, that $\mathcal{F}_\mu f$ belongs to this space for $f \in C_c^\infty(G/H;\mu)$.

Theorem 14, [9]**.** *The* μ*–spherical Fourier transform* $\mathcal{F}_\mu$ *maps* $C_c^\infty(G/H;\mu)$ *into the Paley–Wiener space* PW_μ*. Moreover*

(a) $\mathcal{F}_\mu$ *is injective.*
(b) *If* $\dim\mathfrak{a}_{\mathrm{q}} = 1$ *then* $\mathcal{F}_\mu$ *is surjective.*

The injectivity of $\mathcal{F}_\mu$ is an immediate corollary of Thm. 13: If $\mathcal{F}_\mu f = 0$ then $\mathcal{F}_\mu Df = \chi_\mu(D)\mathcal{F}_\mu f = 0$, hence $Df = 0$ by (ii), and hence $f = 0$ by (i). The injectivity of $f \mapsto f^\wedge$ asserted earlier (below Thm. 9) is a consequence, by density of the K–finite functions in $C_c^\infty(G/H)$. The surjectivity statement in (b) is a by–product of the proof of Thm. 13.

For the Riemannian symmetric spaces the surjectivity of $\mathcal{F}_\mu$ (with an arbitrary K–type μ) is a consequence of the Paley–Wiener theorem in [26], and for the group G itself considered as a symmetric space it follows from the results in [15] and [1], as mentioned earlier. In [9] it is conjectured that $\mathcal{F}_\mu$ is surjective for general G/H as well.

We are now going to extend this theory to distributions, or more precisely, to generalized functions. Let $C_c^{-\infty}(G/H)$ denote the space of compactly supported generalized functions on G/H. Multiplication with the invariant measure dx induces an isomorphism of this locally convex space with the topological linear dual of $C^\infty(G/H)$, i.e. with the space of compactly supported distributions on G/H. If $u \in C_c^{-\infty}(G/H)$, and $f \in C^\infty(G/H)$, then we shall write accordingly:

$$\langle\, u \,,\, f \,\rangle = \int_{G/H} u(x)\, f(x)\, dx = u\, dx(f).$$

We have a natural embedding $C_c^\infty(G/H) \to C_c^{-\infty}(G/H)$; accordingly there is a natural extension of the Fourier transform $f \mapsto f^\wedge$ to the space of compactly supported generalized functions.

Let $C_c^{-\infty}(G/H;\mu)$ denote the space of compactly supported μ–spherical generalized functions $G/H \to V_\mu$. The μ–spherical Fourier transform $\mathcal{F}_\mu$ allows a natural extension to the space $C_c^{-\infty}(G/H;\mu)$ with values in the space of meromorphic functions $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \to \mathcal{V}(\mu)$.

A classical extension of the Paley–Wiener theorem for $\mathbb{R}^n$ states that the Fourier transform maps the space $C_c^{-\infty}(\mathbb{R}^n)$ of compactly supported generalized functions bijectively onto the space $\mathrm{PW}^*(\mathbb{R}^n)$ of entire functions on $\mathbb{C}^n$ for which there exists $R > 0$ such that (2) holds for some $N \in \mathbb{Z}$ (such functions are said to have *slow growth of exponential type*). We shall now state a conjectural analog of this result for $C_c^{-\infty}(G/H;\mu)$.

Let $\mathcal{M}^*_\mu$ be the pre–Paley–Wiener space of meromorphic functions $\varphi\colon \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \to \mathcal{V}(\mu)$ satisfying conditions (i) and (ii) of the definition of $\mathcal{M}_\mu$ and moreover the following condition:

(iii)' *There exist constants $R, C > 0$ and $n \in \mathbb{N}$ such that*

$$\|\pi(\lambda)\varphi(\lambda)\| \leq C\,(1 + |\lambda|)^n e^{R|\operatorname{Re}\lambda|}$$

for all λ in $(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}})_-$.

Furthermore, let PW^*_μ be the space of functions $\psi \in \mathcal{M}^*_\mu$ satisfying the Paley–Wiener relations given in (32).

It can be seen that $\mathcal{F}_\mu$ maps $C_c^{-\infty}(G/H;\mu)$ into PW_μ^* (properties (i) and (ii) are obtained by the same arguments that were used to establish these facts for smooth u, and (iii)' follows from the estimates for the derivatives of $E_\mu^\circ(\psi,\lambda)$ obtained in [4]). In analogy with Thm. 14 we now have:

Theorem 15, [6]**.** *The μ–spherical Fourier transform $\mathcal{F}_\mu$ maps $C_c^{-\infty}(G/H;\mu)$ into the Paley–Wiener space PW_μ^*. Moreover*

(a) *$\mathcal{F}_\mu$ is injective.*
(b) *If $\dim \mathfrak{a}_\mathrm{q} = 1$ then $\mathcal{F}_\mu$ is surjective.*

Moreover, we conjecture that the surjectivity of $\mathcal{F}_\mu$ holds in general. When G/H is Riemannian the surjectivity is established in [18].

4.6. A multiplier theorem

A linear operator

$$M\colon C_c^\infty(K;G/H) \to C_c^\infty(K;G/H)$$

is called a *multiplier* if it is equivariant for the actions of $\mathfrak{g}$, K and $\mathbb{D}(G/H)$ on this space, and has a continuous restriction to $C_c^\infty(G/H)^\mu$ for each $\mu \in K^\wedge$. If M is a multiplier, then it can be seen from the Fourier theory discussed in Sect. 4.4 that for almost every principal series representation $\pi = \pi_{\xi,\lambda}$ there exists an endomorphism Ψ_π of the space $\mathcal{V}_\pi^o$ such that

$$(Mf)^\wedge(\pi) = f^\wedge(\pi) \circ \Psi_\pi. \tag{33}$$

Moreover, Ψ_π will respect the eigenspace decomposition of $\mathcal{V}_\pi^o$ for $\mathbb{D}(G/H)$.

Simple examples of multipliers are the elements of $\mathbb{D}(G/H)$. If M is given by such an operator $D \in \mathbb{D}(G/H)$, then (33) can be written as follows:

$$(Mf)^\wedge(\pi)v = \chi(D)(\lambda) f^\wedge(\pi)v,$$

for any spherical vector $v \in \mathcal{V}_\pi^o$ of type $\lambda \in \mathfrak{b}_\mathbb{C}^*$. Of course an operator M thus defined extends to $C_c^\infty(G/H)$, but this will not be the case in general.

In [6] we give a simple construction of a large algebra of multipliers, containing the algebra $\mathbb{D}(G/H)$. The result is stated below. The existence of these multipliers is a generalization of Arthur's result [1, Thm. III.4.2] for the group case. Arthur's proof rests on his Paley–Wiener theorem (the generalization of which was conjectured in Sect. 4.5); a simpler construction using the correspondence $\varphi \mapsto \varphi^r$ in Prop. 4 was later given in [16]. Our construction for the general case is similar in that it also uses this correspondence.

Let $\mathfrak{b}$ be a θ–invariant maximally split Cartan subspace of $\mathfrak{q}$, and recall from Sect. 4.2 that $\mathrm{PW}(\mathfrak{b}^r)^W$ is the space of W–invariant entire rapidly decreasing functions of exponential type on $\mathfrak{b}_\mathbb{C}^*$. Let $\mathrm{PW}^*(\mathfrak{b}^r)^W$ be the space of W–invariant entire functions with slow growth of exponential type on $\mathfrak{b}_\mathbb{C}^*$ (see Sect. 4.5).

Theorem 16, [6]**.** *For every $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$ there exists a unique linear operator*

$$M_\psi \colon C_c^\infty(K; G/H) \to C_c^\infty(K; G/H)$$

such that for any $\pi \in G_H^\wedge$ and any spherical vector $v \in \mathcal{V}_\pi^o$ of type $\lambda \in \mathfrak{b}_\mathbb{C}^*$ we have

$$(M_\psi f)^\wedge(\pi)v = \psi(\lambda) f^\wedge(\pi)v, \qquad f \in C_c^\infty(K; G/H). \tag{34}$$

If $D \in \mathbb{D}(G/H)$, then $M_{\chi(D)} = D|C_c^\infty(K; G/H)$. Moreover, the map $\psi \mapsto M_\psi$ is an algebra homomorphism from $\mathrm{PW}^*(\mathfrak{b}^r)^W$ into the algebra of multipliers. Finally, for every $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$ the equation (34) holds more generally with v a spherical vector of type λ in $\mathcal{V}_\pi$.

Indication of the proof. The uniqueness of M_ψ follows easily from the abstract Plancherel theory discussed in Sect. 3.2. The existence is established as follows.

Let $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$ be given, and let $F \in C_c^{-\infty}(G^d/K^d)$ be the K^d–invariant generalized function such that

$$\int_{G^d/K^d} F(x)\varphi_\lambda(x)\, dx = \psi(\lambda)$$

for all $\lambda \in \mathfrak{b}_\mathbb{C}^*$. Then for $f \in C_c^\infty(K; G/H)$ it is easily seen that the convolution product

$$f^r * F(x) = \int_{G^d/K^d} f^r(y) F(y^{-1}x)\, dy, \qquad x \in G^d/K^d, \tag{35}$$

is smooth and H^d–finite, and that Prop. 4 allows us to define $M_\psi f \in C_c^\infty(K; G/H)$ by

$$(M_\psi f)^r = f^r * F.$$

By [21, Cor. II.4] there exists a natural isomorphism of algebras $D \mapsto D^r$ from $\mathbb{D}(G/H)$ onto $\mathbb{D}(G^d/K^d)$, such that $(Df)^r = D^r f^r$ for all $f \in C_c^\infty(K; G/H)$. Moreover, we have $\chi^r(D^r) = \chi(D)$, where $\chi^r \colon \mathbb{D}(G^d/K^d) \to S(\mathfrak{b}^r)^W$ is the Harish–Chandra isomorphism. Let now $D \in \mathbb{D}(G/H)$, and let F be associated to $\psi = \chi(D)$ as above. Then for all $g \in C^\infty(G^d/K^d)$ we have $g*F = D^r g$. It follows from this that $(M_{\chi(D)} f)^r = f^r * F = D^r f^r = (Df)^r$, for $f \in C_c^\infty(K; G/H)$. Hence $M_{\chi(D)} = D$.

It is easily seen that the map $\psi \mapsto M_\psi$ is additive and multiplicative. Hence, if $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$ and $D \in \mathbb{D}(G/H)$, then $M_\psi \circ D = M_\psi \circ M_{\chi(D)} = M_{\psi\chi(D)} = M_{\chi(D)} \circ M_\psi = D \circ M_\psi$, and one readily checks that M_ψ is a multiplier.

Finally (34) is seen by an argument similar to the proof of Thm. 5. □

Remark. It follows from the injectivity statements in Theorem 14 that M_ψ is already uniquely determined by the requirement that equation (34) should hold for all principal series representations $\pi = \pi_{\xi,\lambda}$, where $\xi \in M_{\mathrm{fu}}^\wedge$ and $\lambda \in i\mathfrak{a}_\mathrm{q}^*$.

The multipliers of Thm. 16 do actually extend to the space $C^{-\infty}(K; G/H)$ of K–finite generalized functions on G/H. Let this space be equipped with the direct sum of the usual strong dual topologies on the subspaces $C^{-\infty}(G/H)^\mu$, $\mu \in K^\wedge$.

Theorem 17, [6]. *Let* $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$. *Then the operator* M_ψ *of Theorem 16 extends to a continuous linear operator*

$$M_\psi\colon C^{-\infty}(K;G/H) \to C^{-\infty}(K;G/H).$$

This extension is equivariant for the actions of $\mathfrak{g}$, K *and* $\mathbb{D}(G/H)$ *and preserves the subspace* $C_c^{-\infty}(K;G/H)$ *of compactly supported generalized functions. Moreover, for any* $\pi \in G_H^\wedge$ *and any spherical vector* $v \in \mathcal{V}_\pi$ *of type* $\lambda \in \mathfrak{b}_\mathbb{C}^*$ *we have*

$$(M_\psi f)^\wedge(\pi)v = \psi(\lambda) f^\wedge(\pi)v, \qquad f \in C_c^{-\infty}(K;G/H). \tag{36}$$

Finally, if $\psi \in \mathrm{PW}(\mathfrak{b}^r)^W$ *then* M_ψ *maps* $C^{-\infty}(K;G/H)$ *into* $C^\infty(K;G/H)$.

Remark. Notice that (36) for $f \in C_c^{-\infty}(K;G/H)$ is an equation of elements in $\mathcal{H}_\pi^{-\infty}$. Notice also that the existence statement in Thm. 5 can be obtained from the final statement of Thm. 17, by applying M_ψ to the K–isotypical component $P_\mu\delta$ of type μ of the Dirac function δ supported at the origin.

Let $\mu \in K^\wedge$ and $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$ be given. We shall now discuss the relation of the multiplier M_ψ to the μ–spherical Fourier transform $\mathcal{F}_\mu$. We first note that the construction of M_ψ is easily extended to μ–spherical functions $f \in C_c^\infty(G/H;\mu)$: The partial holomorphic extension $\varphi \mapsto \varphi^r$ makes sense for vector valued functions, and so does the convolution product in (35). We denote the resulting linear operator $C_c^\infty(G/H;\mu) \to C_c^\infty(G/H;\mu)$ by M_ψ^μ. It is easily seen that we have the following relation between the operators M_ψ and M_ψ^μ:

$$\gamma_\mu \circ (M_\psi^{\mu^\vee} \otimes I_{V_\mu}) = M_\psi \circ \gamma_\mu,$$

where $\gamma_\mu\colon C_c^\infty(G/H;\mu^\vee) \otimes V_\mu \to C_c^\infty(G/H)^\mu$ as earlier is the contraction isomorphism.

Since $\mathfrak{b}$ is maximally split, we may as well assume that $\mathfrak{a}_\mathrm{q} = \mathfrak{b} \cap \mathfrak{p}$. Put $\mathfrak{b}_\mathrm{k} = \mathfrak{b} \cap \mathfrak{k}$, then (9) becomes

$$\mathfrak{b}^r = \mathfrak{a}_\mathrm{q} \oplus i\mathfrak{b}_\mathrm{k}.$$

Via this direct sum decomposition we identify $\mathfrak{a}_\mathrm{q}^*$ and $\mathfrak{b}_\mathrm{k}^{r*} := i\mathfrak{b}_\mathrm{k}^*$ with subspaces of $\mathfrak{b}^{r*}$.

Let $\xi \in M_\mathrm{fu}^\wedge$, and recall from Sect. 4.3 that $j^\circ(\xi,\lambda)$ for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ is a linear bijection of the space $V(\xi)$ onto the space $\mathcal{V}_{\xi,\lambda} = (\mathcal{H}_{\xi,\lambda}^{-\infty})^H$. Via j° we transfer the $\mathbb{D}(G/H)$–module structure of $\mathcal{V}_{\xi,\lambda}$ to $V(\xi)$. Thus for every $D \in \mathbb{D}(G/H)$ we define $\chi_\xi(D,\lambda) \in \mathrm{End}(V(\xi))$ by

$$Dj^\circ(\xi,\lambda) = j^\circ(\xi,\lambda)\,\chi_\xi(D,\lambda) \tag{37}$$

for generic λ. It is known that $\chi_\xi(D,\lambda)$ is in fact an $\mathrm{End}(V(\xi))$–valued polynomial in λ (cf. [4, Sect. 4]). It allows an eigenspace decomposition which is independent of λ. More precisely, if $\nu \in \mathfrak{b}_{\mathrm{k}}^{r*}$, define

$$V(\xi)_\nu = \{\eta \in V(\xi) \mid \chi_\xi(D,\lambda)\eta = \chi(D,\nu+\lambda)\eta,\ D \in \mathbb{D}(G/H),\ \lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*\}$$

(as before χ denotes the Harish–Chandra isomorphism from $\mathbb{D}(G/H)$ onto $S(\mathfrak{b}^r)^W$). Then for $\nu_1,\nu_2 \in \mathfrak{b}_{\mathrm{k}}^{r*}$ with $V(\xi)_{\nu_1} \neq 0$ we have $V(\xi)_{\nu_1} = V(\xi)_{\nu_2}$ if and only if ν_1 and ν_2 are conjugate under the centralizer W_{M} of $\mathfrak{a}_{\mathrm{q}}$ in W. Moreover, let $\mathcal{N}(\xi)$ denote the set of $\nu \in \mathfrak{b}_{\mathrm{k}}^{r*}/W_{\mathrm{M}}$ for which $V(\xi)_\nu \neq 0$. Then we have the direct sum decomposition

$$V(\xi) = \bigoplus_{\nu\in\mathcal{N}(\xi)} V(\xi)_\nu$$

(for details, see [4]). Notice that it follows from the above that $j^\circ(\xi,\lambda)$ maps $V(\xi)_\nu$ onto the space of $\mathbb{D}(G/H)$–spherical vectors of type $\nu+\lambda$ in $\mathcal{V}_{\xi,\lambda}$, for generic λ.

We now define, for a given W_{M}–invariant complex function ψ on $\mathfrak{b}_{\mathbb{C}}^*$ an endomorphism $\mathrm{M}(\psi,\xi,\lambda)$ of $V(\xi)$ by

$$\mathrm{M}(\psi,\xi,\lambda) = \psi(\nu+\lambda)\, I \quad \text{on} \quad V(\xi)_\nu \tag{38}$$

for $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. It follows that we have

$$(M_\psi f)^\wedge(\xi,\lambda) = f^\wedge(\xi,\lambda)\circ \mathrm{M}(\psi,\xi,\lambda)$$

for all $\xi \in M_{\mathrm{fu}}^\wedge$, $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$.

Recall the orthogonal decomposition (19) of $\mathcal{V}(\mu)$. According to this decomposition we define for each $\nu \in \mathfrak{b}_{\mathrm{k}}^{r*}$ a subspace $\mathcal{V}(\mu)_\nu$ of $\mathcal{V}(\mu)$ by

$$\mathcal{V}(\mu)_\nu = \bigoplus_{\xi\uparrow\mu} \mathrm{Hom}_{M\cap K}(V_\mu,\mathcal{H}_\xi)\otimes V(\xi)_\nu.$$

Then $\mathcal{V}(\mu)_\nu$ only depends on the W_{M}–conjugacy class of ν and writing $\mathcal{N}(\mu) = \cup_{\xi\uparrow\mu}\mathcal{N}(\xi)$ we have the following finite direct sum of non–trivial vector spaces:

$$\mathcal{V}(\mu) = \bigoplus_{\nu\in\mathcal{N}(\mu)} \mathcal{V}(\mu)_\nu. \tag{39}$$

The maps $\chi_\xi(\lambda)\colon \mathbb{D}(G/H) \to \mathrm{End}(V(\xi))$ and $\chi_\mu(\lambda)\colon \mathbb{D}(G/H) \to \mathrm{End}(\mathcal{V}(\mu))$ are closely related; in fact it follows from [4, Sect. 4] that $\chi_\mu(D,\lambda)$ corresponds to the direct sum of the maps $I\otimes\chi_\xi(D,\lambda)$ in the decomposition (19), or equivalently, to the direct sum of the maps $\chi(D,\nu+\lambda)I_{\mathcal{V}(\mu)_\nu}$ in the decomposition (39), for all $D\in\mathbb{D}(G/H)$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Let $\mathrm{M}_\mu(\psi,\lambda) \in \mathrm{End}(\mathcal{V}(\mu))$ be defined by the requirement

$$\mathrm{M}_\mu(\psi,\lambda) = I\otimes \mathrm{M}(\psi,\xi,\lambda) \quad \text{on} \quad \mathrm{Hom}_{M\cap K}(\mathcal{H}_\xi,V_\mu)\otimes V(\xi)$$

in the direct sum decomposition (19). We shall view $\mathrm{M}_\mu(\psi)$ as a multiplication operator on $\mathcal{V}(\mu)$–valued functions on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. It follows from the remarks made above that

$$\mathcal{F}_\mu(M^\mu_\psi f) = \mathrm{M}_\mu(\psi^\vee)\mathcal{F}_\mu f \tag{40}$$

for all $f \in C_c^\infty(G/H;\mu)$.

If the surjectivity conjectures for $\mathcal{F}_\mu$ stated in Sect. 4.5 are valid for G/H, then it follows from (40) that multiplication by $\mathrm{M}_\mu(\psi)$ leaves the spaces PW_μ and PW^*_μ invariant. This is indeed true in general:

Proposition 18, [6]**.** *Let* $\psi \in \mathrm{PW}^*(\mathfrak{b}^r)^W$. *Then multiplication by* $\mathrm{M}_\mu(\psi)$ *leaves the spaces* PW_μ *and* PW^*_μ *invariant. Moreover, if* $\psi \in \mathrm{PW}(\mathfrak{b}^r)^W$ *then multiplication by* $\mathrm{M}_\mu(\psi)$ *maps* PW^*_μ *to* PW_μ.

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