Equivariant Cohomology and Stationary Phase

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Preface

This is the text of a survey lecture given at the conference on "Symplectic Geometry and its Applications", Keio University, Yokohama, July 21, 1993. I have been stimulated by many people, but I would like to thank especially L. Jeffrey for her helpful explanations to me of [**17**].

1. Equivariant Cohomology

Equivariant cohomology is a structure which is attached to a smooth action of a Lie group G on a smooth manifold M. It can be defined as the cohomology of $\mathrm{E}\,G \times_G M$, in which $\mathrm{E}\,G \to \mathrm{B}\,G$ is the universal principal G-bundle; $\mathrm{B}\,G$ is the classifying space of the group G.

Although this explains several aspects of equivariant cohomology, cf. Atiyah and Bott [**1**], for our purposes it is more convenient to use the model of H. Cartan, introduced in [**5**], [**6**]. It is a variation of de Rham cohomology, in which the algebra $\Omega(M)$ of smooth differential forms on M is replaced by the algebra

$$A := (\mathrm{S}(\mathfrak{g}^*) \otimes \Omega(M))^G \tag{1.1}$$

of G-*equivariant* polynomial mappings

$$\omega : \mathfrak{g} \ni X \mapsto \omega(X) \in \Omega(M), \tag{1.2}$$

from the Lie algebra $\mathfrak{g}$ of G to $\Omega(M)$. (It will be convenient to allow complex valued differential forms, so all algebras are over $\mathbf{C}$.) The equivariance of ω means that

$$\omega(\mathrm{Ad}\, g(X)) = (g_M^*)^{-1}(\omega(X)), \quad g \in G,\ X \in \mathfrak{g}. \tag{1.3}$$

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Here Ad stands for the *adjoint* action of G on its Lie algebra $\mathfrak{g}$ and g_M^* denotes *pullback* of differential forms by means of the action $g_M : M \to M$ on M of the element $g \in G$.

In $\Omega(M)$ one has the derivations d and $\mathrm{i}(v)$, of exterior differentiation and contraction with a vectorfield v in M, respectively. These are related to the *Lie derivative* by means of the *homotopy formula*

$$\mathcal{L}(v) := \frac{d}{dt}_{t=0} (e^{tv})^* = \mathrm{d} \circ \mathrm{i}(v) + \mathrm{i}(v) \circ \mathrm{d}\,. \tag{1.4}$$

Here e^{tv} denotes the flow in M after time t with velocity field equal to v.

If, for each $X \in \mathfrak{g}$, the vectorfield X_M denotes the infinitesimal action of X in M, then the *equivariant exterior differentiation* D is defined by:

$$(\mathrm{D}\,\omega)(X) := \mathrm{d}(\omega(X)) - \mathrm{i}(X_M)(\omega(X)), \quad X \in \mathfrak{g},\ \omega : \mathfrak{g} \to \Omega(M). \tag{1.5}$$

Clearly $\mathrm{D} : A \to A$, and one also gets that $\mathrm{D} \circ \mathrm{D} = 0$. For the latter one uses that A consists of equivariant mappings : $\mathfrak{g} \to \Omega(M)$, which implies, substituting $g = \exp(tX)$ in (1.3) and differentiating with respect to t at $t = 0$, that

$$0 = \mathcal{L}(X_M)(\omega(X)) = (\mathrm{d} \circ \mathrm{i}(X_M) + \mathrm{i}(X_M) \circ \mathrm{d})(\omega(X)), \tag{1.6}$$

cf. (1.4). The quotient

$$\mathrm{H}_G^*(M) := \ker \mathrm{D} \,/\, \mathrm{im}\, \mathrm{D} \tag{1.7}$$

is called the *equivariant cohomology* of the G-action on M. It can be shown that if G is *compact*, which we assume from now on, then $\mathrm{H}_G^*(M)$ is canonically isomorphic to the topological equivariant cohomology, cf. [**1**].

In order to explain the *grading* in $\mathrm{H}_G^*(M)$, let $A^{k,\,l}$ denote the space of elements of A which are homogeneous polynomial mappings of degree k, from $\mathfrak{g}$ to $\Omega^k(M)$. If $\omega \in A^{k,\,l}$, then

$$X \mapsto \mathrm{d}(\omega(X)) \in A^{k,\,l+1} \tag{1.8}$$

and

$$X \mapsto \mathrm{i}(X_M)(\omega(X)) \in A^{k+1,\,l-1}. \tag{1.9}$$

So we get $\mathrm{D_p} : A^p \to A^{p+1}$, if we define

$$A^p := \bigoplus_{k,\,l | 2k+l=p} A^{k,\,l} \tag{1.10}$$

as the space of *equivariant forms of degree* p. We get

$$\mathrm{H}_G^*(M) = \bigoplus_{p \geq 0} \mathrm{H}_G^p(M), \tag{1.11}$$

in which

$$\mathrm{H}_G^p(M) := \ker \mathrm{D}_p \,/\, \mathrm{im}\, \mathrm{D}_{p-1} \tag{1.12}$$

is the cohomology in degree p. In Section 3 we will see another reason why it is natural to give the indeterminate X degree two.

If $G = \mathbf{R}/\mathbf{Z}$ is the circle, then $\mathfrak{g} = \mathbf{R}$ and we can write, for $\omega \in A$:

$$\omega(X) = \sum_{j \geq 0} X^j \omega_j , \tag{1.13}$$

in which the $\omega_j \in \Omega(M)^G$ form a sequence of G-invariant differential forms on M. The sum is finite: if $\omega \in A^p$, then $\omega_j \in \Omega^{p-2j}(M)^G$, which is equal to zero if $p - 2j < 0$ or $p - 2j > \dim M$. The equivariant exterior derivative is given by

$$(\mathrm{D}\,\omega)_j = \mathrm{d}\,\omega_j - \mathrm{i}(v)\omega_{j-1}, \tag{1.14}$$

in which the vectorfield $v = 1_M$ is the infinitesimal action of $1 \in \mathbf{R} = \mathfrak{g}$ on M. So the computation of the equivariant cohomology involves sequences of equations in $\Omega(M)^G$.

A similar remark holds true for torus actions, using a multi-index notation in (1.13). For nonabelian Lie algebras $\mathfrak{g}$, the choice of the basis is not so obvious. One also has that the monomials $X^j\,\omega_j$ need not be equivariant, so do not always belong to A.

2. Localization in the Orbit Space

Replacing M by G-invariant open subsets U, we get a sheaf of algebras $A(U)$. The G-invariant open subsets of M correspond to the open subsets of the orbit space M/G, so the $A(U)$ can be viewed as a sheaf over M/G. It is a fine sheaf, because of the existence of partitions of unity by means of G-invariant functions, obtained from arbitrary partitions of unity by averaging these over G. Using Mayer-Vietoris sequences as in Bott and Tu [**4**, Ch. II], one can think of the equivariant cohomology of M as being built up out of the local equivariant cohomology groups $\mathrm{H}^*_G(U)$.

Each $x \in M$ has a G-invariant open neighborhood U_x and a G-equivariant retraction of U_x to the orbit $G \cdot x \simeq G/G_x$ through x. This leads to

$$\mathrm{H}^*_G(U_x) \simeq \mathrm{H}^*_G(G/G_x) \simeq \mathrm{S}(\mathfrak{g}_x^*)^{G_x}, \tag{2.1}$$

the ring of $\mathrm{Ad}\,G_x$-invariant polynomials on $\mathfrak{g}_x$. Here

$$G_x := \{g \in G \mid g_M(x) = x\} \tag{2.2}$$

is the stabilizer of x in G and

$$\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\} \tag{2.3}$$

is its Lie algebra.

Formula (2.1) shows that the local cohomology is not trivial (as for the de Rham cohomology) if $\mathfrak{g}_x \neq 0$. It is even infinite-dimensional over $\mathbf{C}$; it is a polynomial algebra of rank equal to the rank of $\mathfrak{g}_x$. This rank is equal to the dimension of a maximal abelian subalgebra of $\mathfrak{g}_x$, or of the orbit space of the adjoint action of G_x in $\mathfrak{g}_x$.

This is most spectacular if x is a fixed point for the group action, in which case (2.1) is obvious and we get that the equivariant cohomology is equal to the ring

$$I := \mathrm{S}(\mathfrak{g}^*)^G \tag{2.4}$$

of $\mathrm{Ad}\, G$-invariant polynomials on $\mathfrak{g}$. Note that A and $\mathrm{H}^*_G(M)$ are algebras over I, because multiplication with $f \in I$ is a linear mapping : $A \to A$, which commutes with the algebra structure in A and also with d and $\mathrm{i}(X_M)$, hence with D.

3. Locally Free Actions

The other extreme occurs if the action is *locally free*, which means that $\mathfrak{g}_x = 0$ for all $x \in M$. In this case the quotient space is a manifold of dimension equal to $\dim(M) - \dim(G)$ with mild singularities, which locally are those of quotients of a manifold by a finite group action. The concept of such a manifold was introduced by Satake [**22**] under the name of *V-manifold*, but nowadays the name *orbifold* also has become popular. The point of [**22**] is that on such a manifold the de Rham theory goes through, practically without any change. One has for instance Poincaré duality defined by integration over the manifold, if the V-manifold is oriented. Note that if the action is free, that is $G_x = \{1\}$ for all $x \in M$, then M/G is a smooth manifold and $\pi : M \to M/G$ is a smooth fibration, known in the literature as a *principal fiber bundle*. Because of the many interesting examples, it is worthwile however to allow locally free actions which are not free.

Now $\mathfrak{g}_x = 0$ yields in view of (2.1) that the local cohomology is trivial, and we get that

$$\mathrm{H}^*_G(M) \xleftarrow[\pi^*]{\sim} \mathrm{H}^*(M/G). \tag{3.1}$$

In other words: *If the action is locally free, then the equivariant cohomology of M is canonically isomorphic to the de Rham cohomology of the quotient space M/G.*

More precisely, if $\pi : M \to M/G$ denotes the projection $\pi : x \mapsto G \cdot x$, which assigns to each $x \in M$ the G-orbit through x, then the pullback π^* by π is an isomorphism from $\Omega(M/G)$ onto the subspace $\Omega(M)_{\mathrm{basic}}$ of the so-called *basic* differential forms in M. These are defined as the $\beta \in \Omega(M)$ which are G-invariant and satisfy $\mathrm{i}(X_M)\beta = 0$ for all $X \in \mathfrak{g}$. As a constant map from $\mathfrak{g}$ to $\Omega(M)$, such a β belongs to A, and $\mathrm{D}\,\beta = 0$ if and only if $\mathrm{d}\,\beta = 0$. The isomorphism (3.1) now means that if $\omega \in A$ and $\mathrm{D}\,\omega = 0$, then there exists $\nu \in A$ and $\beta \in \Omega(M)_{\mathrm{basic}}$, such that

$$\omega(X) = \beta + (\mathrm{D}\,\nu)(X), \quad X \in \mathfrak{g}. \tag{3.2}$$

We have already observed before that A and $\mathrm{H}_G^*(M)$ are modules over the ring I of Ad-invariant polynomials : $\mathfrak{g} \to \mathbf{C}$. If the action is locally free, then each $f \in I$ corresponds via (3.1) to a cohomology class c_f in $\mathrm{H}^{\mathrm{even}}(M/G)$, these cohomology classes of M/G are called the *characteristic classes* of the fibration $M \to M/G$. In this way the cohomology of M/G, which is finite-dimensional over $\mathbf{C}$ if M is compact, can be viewed as a module over the ring of characteristic classes.

In [**6**, p. 63] an explicit construction of β and ν is indicated, using a *connection form* θ. That is, a $\mathfrak{g}$-valued one form in M, which is G-equivariant and which reproduces X when applied to X_M. In formula:

$$\theta \in (\mathfrak{g} \otimes \Omega^1(M))^G, \quad \mathrm{i}(X_M)\theta \equiv X, \quad X \in \mathfrak{g}. \tag{3.3}$$

Connection forms exist if (and only if) the action is locally free. They can be constructed first in tubular neighborhoods of orbits and then pieced together by means of G-invariant partitions of unity.

If θ is a connection form in M, then the corresponding *curvature form in M* is defined by

$$\Omega := \mathrm{d}\,\theta - [\theta,\, \theta] \in (\mathfrak{g} \otimes \Omega^2(M))^G. \tag{3.4}$$

Here $[\theta,\, \theta] \in (\mathfrak{g} \otimes \Omega^2(M))^G$ is defined by

$$[\theta,\, \theta]_x(v,\, w) = [\theta_x(v),\, \theta_x(w)], \quad v,\, w \in \mathrm{T}_x\, M. \tag{3.5}$$

The curvature form in M has the property that, for each $f \in I$, $f(\Omega)$ is a closed basic form, of even degree. So $f(\Omega) = \pi^*\gamma$ for a uniquely determined closed from γ in M/G. The corresponding class $[\gamma] \in \mathrm{H}^{\mathrm{even}}(M/G)$ is equal to the characteristic class c_f, so in particular it does not depend on the choice of θ. The form γ is called the *characteristic form in M/G*, defined by θ and f.

The relation $X \leftrightarrow \Omega$ explains why the indeterminate X has been given degree two; this is the choice which makes (3.1) into an *isomorphism of graded rings.*

For torus actions, the situation is considerably simpler. We then have

$$\Omega := \mathrm{d}\,\theta = \pi^* R \tag{3.6}$$

for a closed $\mathfrak{g}$-valued two-form R in M/G, called the *curvature form in M/G*. It defines the *Chern class* $c := [R] \in \mathfrak{g} \otimes \mathrm{H}^2(M)$, and we have $c_f = f(c)$.

In the case of the circle $G = \mathbf{R}/\mathbf{Z}$, $\mathfrak{g} = \mathbf{R}$, $\omega(X) = \sum X^j \omega_j$, the form β in (3.3) is given explicitly by:

$$\beta = \sum_{j \geq 0} (\mathrm{d}\,\theta)^j \wedge \omega_j - \sum_{j \geq 0} \theta \wedge (\mathrm{d}\,\theta)^j \wedge \mathrm{i}(v)\omega_j\,. \tag{3.7}$$

If $\omega = f \in I$, then $\beta = f(\Omega)$, confirming the description of the characteristic classes, which we gave above.

Combining (3.1) with the observation that the equivariant cohomology of a point is isomorphic to the ring of Ad-invariant polynomials on the Lie algebra, one can now also explain the second identity in (2.1). Indeed, if H is a closed Lie subgroup of G, then we can use the left-right action of $G \times H$ on G and write

$$\mathrm{H}_G^*(G/H) \xrightarrow{\sim} \mathrm{H}_{G\times H}^*(G) \xleftarrow{\sim} \mathrm{H}_H^*(\text{point}). \tag{3.8}$$

4. Integration

From now on, we assume that M is compact and oriented and that the G-action preserves the orientation. If $\omega \in A$, then the integral

$$(\int \omega)(X) := \int_M \omega(X)^{[\dim M]}, \quad X \in \mathfrak{g} \tag{4.1}$$

definies an $\mathrm{Ad}\, G$-invariant function on $\mathfrak{g}$, so $\int \omega \in \mathrm{S}(\mathfrak{g}^*)^G$. Here we have written

$$\omega(X) = \sum_{k=0}^{\dim M} \omega(X)^{[k]}, \quad \omega(X)^{[k]} \in \Omega^k(M). \tag{4.2}$$

Note that $\omega = \mathrm{D}\,\nu$ implies that that

$$\omega(X)^{[\dim M]} = \mathrm{d}(\nu(X))^{[\dim M]}, \tag{4.3}$$

because

$$(\mathrm{i}(X_M)\nu(X))^{[\dim M]} = \mathrm{i}(X_M)(\nu(X)^{[\dim M+1]}) = 0. \tag{4.4}$$

So Stokes' theorem yields that $\int \omega = 0$ if $\omega \in \operatorname{im} D$, which means that integration yields a map

$$\int : \mathrm{H}_G^*(M) \to I = \mathrm{S}(\mathfrak{g}^*)^G. \tag{4.5}$$

Because the ring I has no zero divisors, the map $\int$ can only be nonzero if the rank of $\mathrm{H}_G^*(M)$ is equal to the rank of $\mathfrak{g}$. That is, it is necessary for having $(\int \omega)(X) \neq 0$ for some $\omega \in A$ satisfying $\mathrm{D}\omega = 0$, that there exist $x \in M$ at which

$$\operatorname{rank} \mathfrak{g}_x = \operatorname{rank} \mathfrak{g}. \tag{4.6}$$

See [**1**, §3] for more about the rank of the module $\mathrm{H}_G^*(M)$. The localization of $\int \omega$ at the points where (4.6) holds is expressed in a more explicit way in the *localization formula* (4.13) *of Berline-Vergne* [**3**] *and Atiyah-Bott* [**1**]. For its formulation, we need some information about the action of a torus $T \subset G$ near its fixed points in M.

If $X \in \mathfrak{g}$, then the zeroset

$$Z = Z_X := \{x \in M \mid X_M(x) = 0\} \tag{4.7}$$

of X_M in M is equal to the fixed point set

$$M^T := \{x \in M \mid t_M(x) = x \text{ for all } \ t \in T\} \tag{4.8}$$

of the torus

$$T = T_X := \text{closure in } G \text{ of } \{\exp(\tau X) \mid \tau \in \mathbf{R}\}. \tag{4.9}$$

We write $\mathfrak{t}$ for the Lie algebra of T. For generic X, T is a maximal torus in G and $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

Using Bochner's local linearization theorem of actions of compact Lie groups near fixed points, one obtains that each connected component F is a smooth compact submanifold of M^T, and there are only finitely many F's. For each $x \in F$, the normal space $\mathrm{T}_x M / \mathrm{T}_x F$ splits into two-dimensional T-invariant planes P_j, on which the infinitesimal action of $Y \in \mathfrak{t}$ is equal to $\lambda_j(Y)$ times the standard infinitesimal rotation of a quarter turn. Here

$$\lambda_j \in \mathfrak{t}^*, \quad \lambda_j(\ker \exp \cap \mathfrak{t}) \subset 2\pi\mathbf{Z} \tag{4.10}$$

are (the real versions of) the *weights* of the torus action.

Because of the rigidity in (4.10), the weights do not depend on the choice of the point x in the connected manifold F. Also, writing the quarter turn in the plane P_j as multiplication with i, P_j can be viewed as a complex line bundle over F, with a curvature form in $\Omega^2(F)$ attached to a connection in P_j. If λ_j occurs with multiplicity, then we get a complex vector bundle over F and the Chern form has to be replaced by a curvature matrix. (One may also use the "splitting principle" as in [**4**, §21], in order to reduce the computations to the case of complex line bundles.) In this way the normal bundle $\mathrm{N}(F)$ of F in M may be provided with the structure of a Hermitian complex vector bundle, the infinitesimal action of X on the frame bundle $\mathrm{FN}(F)$ of $\mathrm{N}(X)$ will be denoted by $\mathrm{L}\, X$.

The *equivariant Euler form* of the normal bundle of F is now defined as

$$\varepsilon(X) := \det_{\mathbf{C}}[\frac{i}{2\pi}(\mathrm{L}\, X - \Omega)] \in \Omega^{\mathrm{even}}(F). \tag{4.11}$$

Here Ω denotes the curvature form in $\mathrm{FN}(F)$, defined by a connection form in the bundle $\mathrm{FN}(F) \to F$. Because the complex determinant is a conjugacy-invariant polynomial, the characteristic form (4.11) is well-defined.

If $\lambda_j(X) \neq 0$ for all j, then this is an invertible element in the commutative algebra $\Omega^{\mathrm{even}}(F)$, with inverse given by

$$\frac{1}{\varepsilon(X)} = \prod_j \frac{2\pi}{i\lambda_j(X)} \cdot \det_{\mathbf{C}}[\sum_{l \geq 0}((\mathrm{L}\, X)^{-1}\Omega)^l]. \tag{4.12}$$

Note that the terms in the right hand side can only be nonzero if $2l \leq \dim F$. In particular the sum is finite. If $F = \{x\}$ is an isolated point, or more generally if the normal bundle of F is trivial, then $1/\varepsilon(X)$ is just equal to the scalar $\prod_j (2\pi / i\lambda_j(X))$.

With these notations, the localization formula now reads:

$$(\int \omega)(X) = \sum_F \int_F (\mathrm{i}_F^* \, \omega(X)/\varepsilon(X))^{[\dim F]}. \tag{4.13}$$

On F the orientation is chosen such that it is compatible with the orientations of M and $\mathrm{N}(F)$. Note that the condition (4.6) just means that $\mathfrak{g}_x$ contains a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, so $x \in M^T$. It is also remarkable that the *polynomial* $(\int \omega)(X)$ is equal to a sum of *rational* functions of X, which in general may have quite high order poles. The vanishing of the sum over F of the coefficients of these poles is just one example of the many magic identities which follow from the localization formula.

The proof indicated in [**3**] uses Stokes' formula in the complement of a small tubular neighborhood of M^T. For the curvature computations, see [**10**, Sec. 2], which can be turned into a proof of (4.13), if the factor $(-1)^k \, e^{J_X} \, \sigma^{n-k}/(n-k)!$ is replaced by ω_k, if $\omega(X) = \sum_k X^k \, \omega_k$. Note that it is sufficient to prove (4.13) for circle actions, because the rays through the integral lattice $\ker \exp \cap \mathfrak{t}$ form a dense subset of $\mathfrak{t}$.

The proof of Berline, Getzler and Vergne in [**2**, Ch. 7] is based on an idea, which potentially has much wider applications. In a general form, due to Witten [**23**], it is the observation that

$$\int_M e^{s \, \mathrm{D}\, \lambda(X)} \, \omega(X) = \int_M \omega(X) \tag{4.14}$$

for all $s \in \mathbf{C}$, if $\omega, \lambda \in A$, $\mathrm{D}\,\omega = 0$ and λ is of odd degree. (This makes $\mathrm{D}\,\lambda(X)$ of even degree, so that its exponential, as a power series, is unambiguously defined.) Indeed, because

$$\frac{d}{ds} e^{s \, \mathrm{D}\, \lambda} \, \omega = \mathrm{D}\, \lambda \, e^{s \, \mathrm{D}\, \lambda} \, \omega = \mathrm{D}(\lambda \, e^{s \, \mathrm{D}\, \lambda} \, \omega), \tag{4.15}$$

its integral over M is equal to zero, which shows that the left hand side in (4.14) is constant as a function of s. Note that the non-polynomial part of $s \mapsto e^{s \, \mathrm{D}\, \lambda(X)}$ is given by the exponential function $s \mapsto e^{-s\, \varphi}$, in which

$$\varphi = \mathrm{i}(X_M) \, \lambda(X)^{[1]}. \tag{4.16}$$

Now we use a G-invariant Riemannian structure β on M and choose

$$\lambda(X) := \beta(X_M, \, \cdot) \in (\mathfrak{g}^* \otimes \Omega^1(M))^G. \tag{4.17}$$

Then $\varphi = \beta(X_M, X_M)$, and $e^{-s\, \varphi}$ gets a Gaussian concentration at the zeroset $Z_X = M^T$ of X_M. The right hand side in (4.13) now is equal to the constant term in the asymptotic expansion of (4.14) as $s \in \mathbf{R}$, $s \to +\infty$. This is easy to prove in the case of isolated fixed points. For the details of the proof in the general case, see [**2**, pp. 219-223].

5. Hamiltonian Actions

Now assume that M carries a *symplectic form* σ. That is, $\sigma \in \Omega^2(M)$, $\mathrm{d}\,\sigma = 0$, and, for each $x \in M$, σ_x is a nondegenerate antisymmetric bilinear form on $\mathrm{T}_x\, M$. This implies that $\dim M = 2m$ for some integer m. We assume that the action of G on M is *Hamiltonian*, which means that there exists

$$\mu \in (\mathfrak{g}^* \otimes \Omega^0(M))^G, \tag{5.1}$$

such that, for each $X \in \mathfrak{g}$, the vector field X_M is equal to the Hamiltonian vectorfield in M defined by the function $\mu(X)$:

$$\mathrm{i}(X_M)\sigma = -\,\mathrm{d}(\mu(X)), \quad X \in \mathfrak{g}. \tag{5.2}$$

This can be summarized in the statement that $\mathrm{D}\,\hat{\sigma} = 0$, if $\hat{\sigma} \in A$ is defined by

$$\hat{\sigma}(X) := \sigma - \mu(X), \quad X \in \mathfrak{g}. \tag{5.3}$$

An immediate consequence is that, for each equivariantly closed form ω, the form

$$\alpha(X) := e^{-i\hat{\sigma}(X)}\omega(X) \tag{5.4}$$

is also equivariantly closed. So the localization formula (4.13) can be applied to write its integral over M as a sum of contributions from the connected components F of Z_X, the zeroset of X_M:

$$\begin{aligned} I(X) := \int (e^{-i\hat{\sigma}}\omega)(X) &= \int_M e^{i\mu(X)} \sum_k \frac{(-i\sigma)^k}{k!}\, \omega(X)^{[2(m-k)]} \\ &= \sum_F e^{i\langle X,\, \mu(F)\rangle}\, r_F(X), \end{aligned} \tag{5.5}$$

in which

$$r_F(X) := \int_F \mathrm{i}_F^*(e^{-i\sigma}\omega(X))/\varepsilon(X). \tag{5.6}$$

Note that Z_X is equal to the set of critical points of the function $\mu(X)$. This also implies that $\mu(X)$ is constant on each connected component F of Z_X, its value on F has been denoted by $\langle X,\, \mu(F)\rangle$ in (5.5).

The integral on the left hand side of (5.5) is an oscillatory integral with phase function equal to $\mu(X)$. The terms (5.6) coincide with the leading terms of the asymptotic expansion of (5.5) for $X \to \infty$, given by the method of stationary phase. One says that in this case the method of stationary phase is *exact*. This was observed for $\omega(X) \equiv 1$ in [**9**]. However, in the next sections we will discuss how the generalization to arbitrary equivariantly closed forms ω can be used in the study of the ring structure of the cohomology of the reduced phase space.

Another observation is that Z_X, being equal to the set of critical points of $\mu(X)$, is always nonvoid. Actually, using the $\mu(X)$ as Morse functions, Ginzburg

[**11**] proved the very strong statement that integration over M defines a *Poincaré duality for* $\mathrm{H}_G^*(M)$, in the sense that

$$[\omega] \mapsto ([\nu] \mapsto \int \omega\, \nu) : \mathrm{H}_G^*(M) \to \mathrm{Hom}_I(\mathrm{H}_G^*(M),\, I) \tag{5.7}$$

is an *isomorphism of* I*-modules.* Recall that I stands for the ring of $\mathrm{Ad}\, G$-invariant polynomials on $\mathfrak{g}$. This is in extreme contrast with the case that the G-action is locally free, because then $\int_M \omega(X) \equiv 0$ for every equivariantly closed form ω.

6. The Reduced Phase Space

Writing

$$\mu(x) : X \mapsto \mu(X)(x) \in \mathfrak{g}^*, \quad x \in M, \tag{6.1}$$

μ can also be seen as an equivariant mapping from M to $\mathfrak{g}^*$, this is called the *momentum mapping* of the Hamiltonian action of G on M. We now assume that $0 \in \mathfrak{g}^*$ is a *regular value* of the momentum mapping $\mu : M \to \mathfrak{g}^*$. This implies that the level set $\mu^{-1}(0)$ is a smooth compact submanifold of M, of codimension equal to $\dim \mathfrak{g}$. It is G-invariant and G acts locally freely on $\mu^{-1}(0)$, so the orbit space

$$M_0 := \mu^{-1}(0)/G \tag{6.2}$$

is an orbifold.

We will write π_0 for the projection $x \mapsto G \cdot x$ from $\mu^{-1}(0)$ to M_0, and i_0 for the identity from $\mu^{-1}(0)$ to M. Then

$$\ker(\mathrm{T}_x\, \pi_0) = \mathrm{T}_x(G \cdot x) = \ker(\mathrm{i}_0^*\, \sigma_x), \tag{6.3}$$

and it follows that the unique two-form σ_0 in M_0, determined by

$$\mathrm{i}_0^*\, \sigma = \pi_0^* \sigma_0, \tag{6.4}$$

is a symplectic form on M_0. The symplectic orbifold $(M_0,\, \sigma_0)$ is called the *Marsden-Weinstein reduced phase space*, at the level 0. This name is inspired by classical mechanics. However, a wealth of examples occur in complex algebraic geometry, where M is a complex projective variety and $M_0 \simeq M//G^{\mathbf{C}}$ is Mumford's geometric quotient by action of the complexification $G^{\mathbf{C}}$ of G, which is a reductive complex algebraic group. See Ness [**21**, §2]. Also moduli spaces can sometimes be identified with reduced phase spaces.

Using the gradient flow of the function $x \mapsto ||\mu(x)||^2$ on M, Kirwan [**19**] proved the fundamental theorem that the first arrow in

$$\mathrm{H}_G^*(M) \xrightarrow[\mathrm{i}_0^*]{} \mathrm{H}_G^*(\mu^{-1}(0)) \xleftarrow[\pi_0^*]{\sim} \mathrm{H}^*(M_0) \tag{6.5}$$

is *surjective.*

The surjectivity of *Kirwan's homomorphism*

$$\kappa_0 := (\pi_0^*)^{-1} \circ \mathrm{i}_0^* : \mathrm{H}_G^*(M) \to \mathrm{H}^*(M_0) \tag{6.6}$$

raises the hope that the cohomology $\mathrm{H}^*(M_0)$ of the reduced phase space M_0 may be computed from the equivariant cohomology $\mathrm{H}_G^*(M)$ of M. (Not from the ordinary cohomology $\mathrm{H}^*(M)$ of M, which in examples can be much simpler than $\mathrm{H}^*(M_0)$.) In special cases, Kirwan [**19**] computed the Betti numbers of M_0 in this way.

7. Integration over the Reduced Phase Space

However, also the ring structure of $\mathrm{H}^*(M_0)$ often is very interesting, because the product corresponds to *intersection of cycles*. For any equivariantly closed form ω in M, write

$$I_0(\omega) := \int_{M_0} \kappa_0(\omega), \tag{7.1}$$

for the integral over the reduced phase space of $\kappa_0(\omega)$. Combining the facts that κ_0 is a ring homomorphism and surjective with Poincaré duality in M_0, we get

$$\ker \kappa_0 = \{\omega \in \mathrm{H}_G^*(M) \mid I_0(\omega\, \nu) = 0 \text{ for all } \nu \in \mathrm{H}_G^*(M)\}. \tag{7.2}$$

So the *ring*

$$\mathrm{H}^*(M_0) \simeq \mathrm{H}_G^*(M)/\ker \kappa_0 \tag{7.3}$$

can be described if the relation

$$I_0(\omega\, \nu) = 0, \;\; \omega,\, \nu \in \mathrm{H}_G^*(M), \tag{7.4}$$

is known.

In order to get hold of this, Witten [**23**] showed that (4.14), this time with

$$\lambda(X) = \mu(X)\, \beta(X_M\, , \cdot), \tag{7.5}$$

leads to a localization of $I_0(\omega)$ at the critical points of $x \mapsto \|\mu(x)\|^2$. This has been worked out by Wu [**24**] in the case of a circle action and for $\omega = e^{\hat{\sigma}}$. The result is a formula for the symplectic volume of the reduced phase space, in terms of the fixed points of the circle action.

With a somewhat different proof, Kalkman [**18**] obtained, also for circle actions but for any $\omega \in \mathrm{H}_G^{\dim M - 2}(M)$, the formula

$$\int_{M_0} \kappa_0(\omega) = \sum_{F \mid \mu(F) > 0} \int_F X\; \mathrm{i}_F^*\, \omega(X)/\varepsilon(X). \tag{7.6}$$

As an application, he computed the ring structure of $\mathrm{H}^*(M_0)$, for a circle action on $M = \mathbf{CP}^n$. (In the sum on the right hand side of (7.6), the condition $\mu(F) > 0$ for the fixed point components may also be replaced by $\mu(F) < 0$, adding a minus sign in front of the sum sign.)

In Kalkman's Ph. D. thesis, (7.6) is proved by observing that $\mu^{-1}(0)$ is the boundary of the domain where $\mu > 0$. Then Stokes' theorem is applied in the complement in this domain of a small tubular neighborhood of the fixed point set. A remarkable feature of this proof is that it works, with $\mu^{-1}(0)$ replaced by ∂M, for an arbitrary (not necessarily Hamiltonian) circle action on any compact oriented manifold with boundary ∂M.

The remainder of this section is an attempt to explain the results of Jeffrey and Kirwan [**17**]. It contains a generalization of (7.6) to Hamiltonian actions of *arbitrary compact Lie groups* G. See formula (7.18) below.

The starting point of Jeffrey and Kirwan is the $\mathfrak{g}$-Fourier transform

$$f(\xi) = (\mathcal{F}_{\mathfrak{g}}\, I)(\xi) = \int_{\mathfrak{g}} [\int_M e^{-i\langle X,\, \xi - \mu\rangle} e^{-i\sigma} \omega(X)] dX \tag{7.7}$$

of the temperate function $I(X)$ on $\mathfrak{g}$, which was introduced in (5.5). That is, f is a temperate distribution in $\mathfrak{g}^*$. Here dX is the Euclidean measure with respect to an $\mathrm{Ad}\, G$-invariant inner product in $\mathfrak{g}$, which in the sequel will also be used in order to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Its restriction to the maximal abelian subalgebra $\mathfrak{t}$ defines a Euclidean measure on $\mathfrak{t}$ and an identification of $\mathfrak{t}^*$ with $\mathfrak{t}$.

Let φ be a test function (smooth and with compact support) on $\mathfrak{g}^*$. Using the dual measure in $\mathfrak{g}^*$, interchanging the order of integration and writing

$$\omega(X) = \sum_j X^j\, \omega_j \tag{7.8}$$

with a multi-index j, we get

$$\int_{\mathfrak{g}^*} \varphi(\xi)\, f(\xi)\, d\xi = (2\pi)^n \sum_j \int_M (D^j \varphi \circ \mu)\, e^{-i\sigma}\, \omega_j. \tag{7.9}$$

Here $n = \dim \mathfrak{g}$. In other words,

$$f = (2\pi)^n \sum_{j,k} (-D)^j\, \mu_*(\frac{(-i\sigma)^k}{k!}\, \omega_j^{[2(m-k)]}). \tag{7.10}$$

Here μ_*, the transposed of μ^*, denotes the pushforward of measures in M to measures in $\mathfrak{g}^*$ by means of the momentum mapping $\mu : M \to \mathfrak{g}^*$. It follows that the distribution f is supported by the image of the momentum mapping, a set which is known to intersect $\mathfrak{t}^*$ in a convex polytope, if M is connected. If $\omega = 1$, then f is equal to $(2\pi)^n\, (-i)^m$ times the pushforward under μ of the canonical (Liouville) measure $\sigma^m/m!$ of M. In particular, it is a measure. For general ω it can be a distribution of arbitrarily high order.

If V is a sufficiently small open neighborhood of 0 in $\mathfrak{g}^*$, then there exists a G-equivariant retraction ρ from $\mu^{-1}(V)$ onto $\mu^{-1}(0)$ such that $\rho \times \mu$ is a diffeomorphism from $\mu^{-1}(V)$ onto $\mu^{-1}(0) \times V$, and moreover the symplectic form is given by

$$\sigma = \rho^* \pi_0^* \sigma_0 + \mathrm{d}\langle \rho^* \theta,\, \mu\rangle. \tag{7.11}$$

Here θ is a connection form for the locally free G-action on $\mu^{-1}(0)$. This result follows from the normal form of Hamiltonian group actions as obtained by Gotay [**12**], Marle [**20**], and Guillemin and Sternberg [**15**, §41].

Now assume that $\mathrm{supp}(\varphi) \subset V$. Using the normal form and the fact that in $\mu^{-1}(V)$ we may replace $\omega(X)$ by $\rho^* \pi_0^* \kappa_0(\omega)$, one obtains that $\langle \varphi, f \rangle$ is equal to a nonzero universal constant (which involves the volume of the π_0-fiber) times

$$\int_{M_0} \Big(\int_{\mathfrak{g}^*} \varphi(\xi)\, e^{-i\langle \xi,\, \Omega \rangle}\, d\xi \Big)\, e^{-i\sigma_0}\, \kappa_0(\omega). \tag{7.12}$$

Here Ω is the curvature form in $\mu^{-1}(0)$ of θ, and we take φ to be $\mathrm{Ad}\, G$-invariant in order to obtain that the integral over ξ is a well-defined characteristic form in $M_0 = \mu^{-1}(0)/G$.

It follows that f is equal to an $\mathrm{Ad}\, G$-invariant polynomial near the origin in $\mathfrak{g}^*$. For torus actions and $\omega = 1$, this was actually the way in which it was proved in [**9**], that the pushforward of the canonical density under the momentum mapping is a piecewise polynomial density in $\mathfrak{g}^*$. By letting the support of φ shrink to 0, one obtains that the integral of $e^{-i\sigma_0}\, \kappa_0(\omega)$ over M_0 is equal to a nonzero universal constant times $f(0)$.

The next step is that one would like to use the localization formula (5.5), in order to write $f(0)$ as the sum of contributions from the connected components F of the fixed point set M^T. Now (5.5) is an equation between functions on $\mathfrak{t}$, so we begin by expressing $f(0)$ in terms of the restriction of I to $\mathfrak{t}$. Let φ be an $\mathrm{Ad}\, G^*$-invariant smooth and compactly supported function in $\mathfrak{g}^*$ with integral equal to one. (Later we shall see that we also could take a Gaussian.) Let

$$\psi(X) = \int_{\mathfrak{g}^*} e^{-i\langle X,\, \xi \rangle}\, \varphi(\xi)\, d\xi \tag{7.13}$$

denote its $\mathfrak{g}^*$-Fourier transform. ψ is an $\mathrm{Ad}\, G$-invariant entire function on the complexification of $\mathfrak{g}$, satisfying the Paley-Wiener estimates. Note also that $\psi(0) = 1$. Then

$$\begin{aligned} f(0) &= \lim_{\epsilon \downarrow 0} \epsilon^{-n} \int_{\mathfrak{g}^*} \varphi(\epsilon^{-1}\, \xi)\, (\mathcal{F}_{\mathfrak{g}}\, I)(\xi)\, d\xi \\ &= \lim_{\epsilon \downarrow 0} \int_{\mathfrak{g}} \psi(\epsilon\, X)\, I(X)\, dX = c\, \lim_{\epsilon \downarrow 0} \int_{\mathfrak{t}} \psi(\epsilon\, X)\, I(X)\, \pi(X)\, dX. \end{aligned} \tag{7.14}$$

Here c is a universal positive constant and the polynomial $\pi(X) = \pi(-X)$ is equal to the product of all the roots of the Lie algebra $\mathfrak{g}$ with respect to the maximal abelian subalgebra $\mathfrak{t}$; these roots are regarded as linear forms on $\mathfrak{t}$.

The problem which arises now, is that the poles of the rational functions $r_F(X)$ which appear in (5.5) are not locally integrable, so we cannot substitute (5.5) in (7.14) right away. However, using that the integrand in (7.14) is a rapidly decreasing analytic function of X, we can apply Cauchy's integral theorem and replace X by $X + iY$ in the integrand, for any $Y \in \mathfrak{t}$. If Y lies in the complement

$\tilde{\mathfrak{t}}$ of the zeroset of all the weights λ_j, for all j and all F, then we get that $f(0)$ is equal to a nonzero universal constant times the sum over all F of

$$\int_{\mathfrak{t}} \psi(\epsilon\,(X+iY))\, e^{i\langle X+iY,\, \mu(F)\rangle}\, r_F(X+iY)\, \pi(X+iY)\, dX. \tag{7.15}$$

Because of the Cauchy integral theorem, (7.15) does not change if Y is replaced by any Z in the connected component $C_{F,Y}$ of Y in the complement $\tilde{\mathfrak{t}}_F$ of the weight hyperplanes for the action on the normal bundle of F. Note that $C_{F,Y}$ is an open polyhedral cone, determined by a choice of signs (the same as for Y) of the weights at F. Also, $C_{F,Y}$ does not depend on the choice of Y in the connected component $\wedge$ of $\tilde{\mathfrak{t}}$. For this reason, we write $C_{F,\,\wedge}$ instead of $C_{F,Y}$, this is just the connected component of $\tilde{\mathfrak{t}}_F$ which contains $\wedge$. Conversely, $\wedge$ is equal to the intersection of the chambers $C_{F,\,\wedge}$, where F ranges over the connected components of M^T. One might call $C_F = C_{F,\,\wedge}$ an *action chamber at* F. The choice of $\wedge$ corresponds to a choice $F \mapsto C_F$ of action chambers, such that the intersection of the C_F's is nonvoid.

If $\langle Z,\, \mu(F)\rangle > 0$, then the exponential decrease as $t \to \infty$, which occurs if Z is replaced by $t\, Z$, shows that the integral is equal to zero, unless F belongs to

$$\mathcal{F}_\wedge := \{F \mid \langle Z,\, \mu(F)\rangle \leq 0 \;\text{ for all } Z \in C_{F,\,\wedge}\}. \tag{7.16}$$

It will be argued below that (7.15) has an asymptotic expansion in integral powers of ϵ as $\epsilon \downarrow 0$; the *constant term* in this expansion will be called the *residue* $\mathrm{Res}_{\varphi,\,\wedge}$ of the meromorphic function

$$e^{i\langle X,\, \mu(F)\rangle}\, \pi(X)\, r_F(X) \tag{7.17}$$

of $X \in \mathfrak{t} \otimes \mathbf{C}$. With this notation, we arrive at the following version of the formula of Jeffrey and Kirwan [**17**, Th. 8.1]:

$$\int_{M_0} e^{-i\sigma_0}\, \kappa_0(\omega) = c \sum_{F \in \mathcal{F}_\wedge} \mathrm{Res}_{\varphi,\,\wedge} \;\left[e^{i\langle X,\, \mu(F)\rangle}\, \pi(X)\, r_F(X)\right] \;. \tag{7.18}$$

In order to further investigate the residues, we note that

$$X \mapsto r_F(X + itY) \tag{7.19}$$

converges for $t \downarrow 0$ in the space of temperate distributions on $\mathfrak{t}$, the limit will be denoted by $r_{F,\,\wedge}$. Its $\mathfrak{t}$-Fourier transform $\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge}$ is a temperate distribution in $\mathfrak{t}^*$.

In order to express (7.15) in terms of the distribution $\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge}$, it is convenient to write

$$\pi(X) = \varpi(X)\, \varpi(-X) = \pm\varpi(X)^2, \tag{7.20}$$

in which $\varpi(X)$ denotes the product of a choice of positive roots. We then have, modulo nonzero universal factors:

$$(7.21)\quad \varpi(X) \int_{\mathfrak{g}^*} \varphi(\xi)\, e^{-i\langle X,\,\xi\rangle}\, d\xi = \varpi(X) \int_{\mathfrak{g}^*} \varphi(\xi) \int_{G^0/T} e^{-i\langle X,\,\mathrm{Ad} g^*\,\xi\rangle}\, dg\, d\xi$$
$$= \varpi(X) \int_{\mathfrak{t}^*} \varphi(\xi) \int_{G^0/T} e^{-i\langle X,\,\mathrm{Ad} g^*\,\xi\rangle}\, dg\, \pi(\xi)\, d\xi = \int_{\mathfrak{t}^*} \varphi(\xi)\, e^{-i\langle X,\,\xi\rangle}\, \varpi(\xi)\, d\xi.$$

Here we have used the formula

$$(7.22)\qquad \int_{G^0/T} e^{-i\langle X,\,\mathrm{Ad} g^*\,\xi\rangle}\, dg = \mathrm{const} \sum_{s\in W} \frac{e^{-i\langle X,\,s^*\xi\rangle}}{\varpi(X)\,\varpi(s^*\xi)}$$

of Harish-Chandra [**16**, Corollary]. This can also be viewed as an application of the method of exact stationary phase, cf. Guillemin and Prato [**14**, Lemma 2.4].

Substituting (7.22) in (7.15), we get that (7.15) is equal to a nonzero universal constant times

$$(7.23)\qquad \epsilon^{-n} \int_{\mathfrak{t}^*} \varphi(\epsilon^{-1}\,\xi)\, \varpi(\xi)\, \varpi(\frac{\partial}{\partial\xi})\, (\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge})(\xi - \mu_{\mathfrak{t}}(F))\, d\xi.$$

Here $\mu_{\mathfrak{t}}$ denotes the momentum mapping for the action of T, so $\mu_{\mathfrak{t}}(F) \in \mathfrak{t}^*$ is equal to the restriction of $\mu(F) \in \mathfrak{g}^*$ to $\mathfrak{t}$. Note that (7.23), for arbitrary φ and $\epsilon = 1$, yields the F-contribution to the whole distribution f, not only to its value at the origin in $\mathfrak{g}^*$.

The distribution $\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge}$ can be described in terms of the convolutions $m_{F,\,\wedge}$ of the *halfline measures* m_j, defined by

$$(7.24)\qquad \langle \varphi,\, m_j\rangle = \int_0^\infty \varphi(t\,\lambda_{j,\,\wedge})\, dt,$$

where

$$(7.25)\qquad \lambda_{j,\,\wedge} = \mathrm{sign}\langle Y, \lambda_j\rangle \cdot \lambda_j,\quad Y \in C_{F,\,\wedge}.$$

and the λ_j range over the weights of the T-action on the normal bundle of F. In the convolution product, the factors m_j may appear with higher multiplicities, but each has to appear at least once. Such convolutions of halfline measures were introduced by Duflo, Heckman and Vergne [**8**]. The support of each such $m_{F,\,\wedge}$ is equal to the cone spanned by the $\lambda_{j,\,\wedge}$, which in turn is equal to the dual cone $\Gamma_{F,\,\wedge}$ of $C_{F,\,\wedge}$. It follows from the fact that μ has regular values, that the $\lambda_{j,\,\wedge}$ span $\mathfrak{t}^*$. This implies that $\Gamma_{F,\,\wedge}$ has a nonvoid interior and that the measure $m_{F,\,\wedge}$ is determined by a locally integrable density, cf. Guillemin, Lerman and Sternberg [**13**, Prop. 2.4]. Moreover, this density is piecewise polynomial, in the following sense. Let $\Gamma_{F,\,\wedge}^{\mathrm{reg}}$ denote the set of $\eta \in \Gamma_{F,\,\wedge}$ which do not belong to a cone spanned by less than $\dim \mathfrak{t}$ of the $\lambda_{j,\,\wedge}$. The statement then is that $m_{F,\,\wedge}$ is equal to a polynomial in each connected component of $\Gamma_{F,\,\wedge}^{\mathrm{reg}}$, cf. [**13**, Th. 2.7].

The distribution $\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge}$ now can be written as a finite linear combination of derivatives of the $m_{F,\,\wedge}$. It follows that the support of $\mathcal{F}_{\mathfrak{t}}\, r_{F,\,\wedge}$ is contained in

$\Gamma_{F,\wedge}$, and that $\mathcal{F}_{\mathfrak{t}}\, r_{F,\wedge}$ is equal to a polynomial in each connected component of $\Gamma_{F,\wedge}^{\mathrm{reg}}$.

If $-\mu_{\mathfrak{t}}(F) \in \Gamma_{F,\wedge}^{\mathrm{reg}}$, then we can write

$$\mathrm{Res}_{\varphi,\wedge}\left[e^{i\langle X,\,\mu(F)\rangle}\,\pi(X)\,r_F(X)\right] = (\pi(D)\,\mathcal{F}_{\mathfrak{t}}\,r_{F,\wedge})(-\mu_{\mathfrak{t}}(F)), \tag{7.26}$$

which is independent of the choice of the test function φ. However, in general the condition that $-\mu_{\mathfrak{t}}(F) \in \Gamma_{F,\wedge}^{\mathrm{reg}}$ need not hold, one can already find counterexamples for two-dimensional torus actions on $\mathbf{CP}^3$.

In general, near $-\mu_{\mathfrak{t}}(F)$ the distribution $\mathcal{F}_{\mathfrak{t}}\, r_{F,\wedge}$ is a linear combination of derivatives of piecewise polynomial densities. Substituting this in (7.23) and transposing all derivatives to $\varphi(\epsilon^{-1}\xi)\,\varpi(\xi)$ by means of partial integrations, we see that (7.23) has an asymptotic expansion in integral powers of ϵ as $\epsilon \downarrow 0$. The coefficients are equal to sums of integrals over cones $\tilde{\Gamma}$ of products of polynomials with derivatives of φ. Here the $\tilde{\Gamma}$ are the cones which near 0 are equal to $\mu_{\mathfrak{t}}(F) + \Gamma$, in which Γ is a connected component of $\Gamma_{F,\wedge}^{\mathrm{reg}}$. If $-\mu_{\mathfrak{t}}(F) \in \Gamma_{F,\wedge}^{\mathrm{reg}}$, then $\tilde{\Gamma} = \mathfrak{t}^*$ and the derivatives of φ can be transposed to the polynomials by means of partial integrations, but if $-\mu_{\mathfrak{t}}(F)$ belongs to the boundary of Γ then this procedure would lead to additional boundary terms.

In any case, this shows that the residue is always well-defined. It may depend on the choice of φ, although the sum over all F of the residues neither depends on φ, nor on $\wedge$. The description of $\mathcal{F}_{\mathfrak{t}}\, r_{F,\wedge}$ also shows that, instead of the compactly supported smooth function φ, we could have taken a Gaussian.

The formula (7.18) may be compared with the formula which Guillemin and Prato [**14**] obtained for f, in the case that $\omega = 1$, the T-fixed points are isolated and their $\mu_{\mathfrak{t}}$-images are not in the walls of the Weyl chambers in $\mathfrak{t}^*$.

Finally, if σ is replaced by $\delta\,\sigma$, $\delta > 0$, then μ gets replaced by $\delta\,\mu$ and σ_0 by $\delta\,\sigma_0$. The local contributions at each F in (7.18) is a polynomial in δ, cf. (5.6) and (7.23). This leads to a formula for $\int_{M_0} \kappa_0(\omega)$ as the sum over F of the constant terms of the local contributions, viewed as polynomials in δ.

One may also note that the topological equivariant cohomology can be defined over $\mathbf{Z}$. If G acts (locally) freely on $\mu^{-1}(0)$, then κ_0 maps to the integral (rational) cohomology of M_0, so the explicit computation of the universal factor should confirm that $\int_{M_0} \kappa_0(\omega)$ is integral (rational) for integral equivariant cohomology classes ω.

Further explorations might tell how efficient the formula really is for the computation of the ring structure of the cohomology the reduced phase space. For instance, a natural question is whether this can be used for the computation of the cohomology ring of an arbitrary toric variety, which is a reduced phase spaces for a torus action on a (noncompact) complex vector space. The result may then be compared with the formula of Danilov [**7**, §10]. In [**17**], examples have been worked out for the non-Abelian group $G = \mathrm{SU}(2)$.

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