

# SCALING FOR A RANDOM POLYMER

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## Abstract

Let  $Q_n^\beta$  be the law of the  $n$ -step random walk on  $\mathbb{Z}^d$  obtained by weighting simple random walk with a factor  $e^{-\beta}$  for every self-intersection (Domb-Joyce model of ‘soft polymers’). It was proved by Greven and den Hollander (1993) that in  $d = 1$  and for every  $\beta \in (0, \infty)$  there exist  $\theta^*(\beta) \in (0, 1)$  and  $\mu_\beta^* \in \{\mu \in l^1(\mathbb{N}) : \|\mu\|_{l^1} = 1, \mu > 0\}$  such that under the law  $Q_n^\beta$  as  $n \rightarrow \infty$ :

- (i)  $\theta^*(\beta)$  is the limit empirical speed of the random walk;
- (ii)  $\mu_\beta^*$  is the limit empirical distribution of the local times.

A representation was given for  $\theta^*(\beta)$  and  $\mu_\beta^*$  in terms of a largest eigenvalue problem for a certain family of  $\mathbb{N} \times \mathbb{N}$  matrices. In the present paper we use this representation to prove the following scaling result as  $\beta \downarrow 0$ :

- (i)  $\beta^{-\frac{1}{3}}\theta^*(\beta) \rightarrow b^*$ ;
- (ii)  $\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \xrightarrow{L^1} \eta^*(\cdot)$ .

The limits  $b^* \in (0, \infty)$  and  $\eta^* \in \{\eta \in L^1(\mathbb{R}^+) : \|\eta\|_{L^1} = 1, \eta > 0\}$  are identified in terms of a Sturm-Liouville problem, which turns out to have several interesting properties.

The techniques that are used in the proof are functional analytic and revolve around the notion of epi-convergence of functionals on  $L^2(\mathbb{R}^+)$ . Our scaling result shows that the speed of soft polymers in  $d = 1$  is not right differentiable at  $\beta = 0$ , which precludes expansion techniques that have been used successfully in  $d \geq 5$  (Hara and Slade (1992a,b)). In simulations the scaling limit is seen for  $\beta \leq 10^{-2}$ .

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# 0 Introduction and main results

## 0.1 Model and motivation

A polymer is a long chain of molecules with two characteristic properties: (i) an irregular shape (due to entanglement); (ii) a certain stiffness (due to sterical hindrance). One way of describing such a polymer is the following model, which is based on a *random walk with self-repellence*.

Let  $(S_i)_{i \geq 0}$  be simple random walk on  $\mathbb{Z}^d$  ( $d \geq 1$ ), starting at the origin. Let  $P_n$  be its law on  $n$ -step paths and let  $E_{P_n}$  be expectation w.r.t.  $P_n$ . Define a new law  $Q_n^\beta$  on  $n$ -step paths by setting

$$\frac{dQ_n^\beta}{dP_n}((S_i)_{i=0}^n) = \frac{1}{Z_n^\beta} \exp \left[ \Leftrightarrow \beta \sum_{\substack{i,j=0 \\ i \neq j}}^n 1\{S_i = S_j\} \right], \quad (0.1)$$

where  $Z_n^\beta$  is the normalizing constant

$$Z_n^\beta = E_{P_n} \left( \exp \left[ \Leftrightarrow \beta \sum_{\substack{i,j=0 \\ i \neq j}}^n 1\{S_i = S_j\} \right] \right) \quad (0.2)$$

and  $\beta \in [0, \infty]$  is a parameter. The law  $Q_n^\beta$  is called the *n-polymer measure with strength of repulsion  $\beta$* <sup>1</sup>.

Eqs.(0.1-2) define what is called the Domb-Joyce model of ‘soft polymers’, where the weight factor gives a penalty  $e^{-\beta}$  for every self-intersection. The limiting cases  $\beta = 0$  and  $\beta = \infty$  correspond to simple random walk resp. self-avoiding random walk. For a recent guide to the literature on this model the reader is referred to Madras and Slade (1993) Section 10.1.

It is generally believed that for  $\beta \in (0, \infty]$  the mean square displacement behaves like

$$E_{Q_n^\beta}[|S_n|^2] \sim Dn^{2\nu} \quad (n \rightarrow \infty), \quad (0.3)$$

where  $D = D(\beta, d) > 0$  is some amplitude and  $\nu = \nu(d)$  is a *critical exponent*. The latter is believed to be *independent* of  $\beta$  and to assume the values<sup>2</sup>

$$\begin{aligned} \nu &= 1 & d &= 1 \\ &= \frac{3}{4} & d &= 2 \\ &= 0.588 \dots & d &= 3 \\ &= \frac{1}{2} & d &\geq 4. \end{aligned} \quad (0.4)$$

Note that  $\nu = \frac{1}{2}$  is the exponent for simple random walk ( $\beta = 0$ ) in any  $d \geq 1$  (with  $D = 1$ ). Apparently, the repulsion changes the qualitative behavior when  $d \leq 3$  but not when  $d \geq 4$ <sup>3</sup>. The fact that  $\nu$  is the same for all  $\beta \in (0, \infty]$  says that soft polymers are in the same universality class as self-avoiding walk.

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<sup>1</sup>Note that if  $\beta > 0$  then  $(Q_n^\beta)_{n \geq 0}$  is not a consistent family, i.e.,  $Q_n^\beta$  is not the projection on  $n$ -step paths of the law of some process evolving in time (like  $P_n$ ).

<sup>2</sup>The value in  $d = 3$  is well below  $\max\{\frac{3}{d+2}, \frac{1}{2}\}$ , the so-called Flory value (Madras and Slade (1993) Section 2.2).

<sup>3</sup>Actually,  $d = 4$  is a critical dimension where it is believed that  $E_{Q_n^\beta}[|S_n|^2] \sim Dn(\log n)^{\frac{1}{4}}$ , containing a logarithmic correction to (0.3-4).

Sofar a rigorous proof of (0.3-4) has only been given for  $d \geq 5$  (Hara and Slade (1992a,b)<sup>4</sup>) and for  $d = 1$  (Greven and den Hollander (1993)). In the latter work there is also a recipe for evaluating the amplitude  $D(\beta, 1)$  as a function of  $\beta$ , which we next describe.

## 0.2 Speed and local times in $d = 1$

Define the random variables

$$\theta_n = \frac{1}{n} |S_n| \quad (0.5)$$

$$\mu_n = \frac{1}{|R_n|} \sum_{x \in R_n} \delta_{\ell_n(x)}, \quad (0.6)$$

where

$$\begin{aligned} R_n &= \left( \min_{0 \leq i \leq n} S_i, \max_{0 \leq i \leq n} S_i \right) \cap \mathbb{Z} \\ \ell_n(x) &= \#\{0 \leq i < n : S_i = x\}. \end{aligned} \quad (0.7)$$

In words,  $\theta_n$  is the *empirical speed* and  $\mu_n$  is the *empirical distribution of local times* after  $n$  steps. Theorems 1-3 below are taken from Greven and den Hollander (1993) and are the starting point of the present paper.

**Theorem 1** *For every  $\beta \in (0, \infty)$  there exists  $\theta^*(\beta) \in (0, 1)$  such that*

$$\lim_{n \rightarrow \infty} Q_n^\beta \left( |\theta_n \Leftrightarrow \theta^*(\beta)| \leq \epsilon \right) = 1 \quad \text{for every } \epsilon > 0, \quad (0.8)$$

*with  $\beta \rightarrow \theta^*(\beta)$  analytic,  $\lim_{\beta \downarrow 0} \theta^*(\beta) = 0$  and  $\lim_{\beta \rightarrow \infty} \theta^*(\beta) = 1$ <sup>5</sup>.*

**Theorem 2** *For every  $\beta \in (0, \infty)$  there exists  $\mu_\beta^* \in \{\mu \in l^1(\mathbb{N}) : \|\mu\|_{l^1} = 1, \mu > 0\}$  such that*

$$\lim_{n \rightarrow \infty} Q_n^\beta \left( \|\mu_n \Leftrightarrow \mu_\beta^*\|_{l^1} \leq \epsilon \right) = 1 \quad \text{for every } \epsilon > 0, \quad (0.9)$$

*with  $\beta \rightarrow \mu_\beta^*$  analytic,  $\lim_{\beta \downarrow 0} \mu_\beta^* = 0$  and  $\lim_{\beta \rightarrow \infty} \mu_\beta^* = \delta_1$  pointwise.*

The limits  $\theta^*(\beta)$  and  $\mu_\beta^*$  in Theorems 1 and 2 can be found in terms of the following *largest eigenvalue problem*. Let  $A_{r,\beta}$  ( $r \in \mathbb{R}, \beta > 0$ ) be the matrix

$$A_{r,\beta}(i, j) = e^{r(i+j-1) - \beta(i+j-1)^2} P(i, j) \quad (i, j \in \mathbb{N}), \quad (0.10)$$

where  $P$  is the Markov matrix

$$P(i, j) = \begin{pmatrix} i+j \Leftrightarrow 2 \\ i \Leftrightarrow 1 \end{pmatrix} \left( \frac{1}{2} \right)^{i+j-1}. \quad (0.11)$$

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<sup>4</sup>The proof in Hara and Slade (1992a,b) is for  $\beta = \infty$ . However, the technique that is used (the so-called ‘lace expansion’) easily implies the same result for all  $\beta \in (0, \infty]$ . Brydges and Spencer (1985) earlier used the same technique to prove (0.3-4) for  $d \geq 5$  and  $\beta$  sufficiently small.

<sup>5</sup>Note that (0.5) and (0.8) imply (0.3) with  $\nu(1) = 1$  and  $D(\beta, 1) = [\theta^*(\beta)]^2$ .

Let  $(\lambda(r, \beta), \tau_{r, \beta})$  be the unique solution of the largest eigenvalue problem <sup>6</sup>

$$\begin{aligned} A_{r, \beta} \tau &= \lambda \tau \quad (\lambda > 0, \tau \in l^2(\mathbb{N})) \\ \|\tau\|_{l^2} &= 1, \tau > 0. \end{aligned} \tag{0.12}$$

**Theorem 3** Fix  $\beta \in (0, \infty)$ . Let  $r^*(\beta) \in (0, \infty)$  be the unique solution of

$$\lambda(r, \beta) = 1. \tag{0.13}$$

Then

$$\begin{aligned} \frac{1}{\theta^*(\beta)} &= \left[ \frac{\partial}{\partial r} \lambda(r, \beta) \right]_{r=r^*(\beta)} \\ \mu_\beta^*(k) &= \left[ \sum_{\substack{i, j \in \mathbb{N} \\ i+j-1=k}} \tau_{r, \beta}(i) A_{r, \beta}(i, j) \tau_{r, \beta}(j) \right]_{r=r^*(\beta)} \quad (k \in \mathbb{N}). \end{aligned} \tag{0.14}$$

The representation in Theorem 3 is not easy to manipulate, which is why precise analytical estimates of  $\theta^*(\beta)$  and  $\mu_\beta^*$  are hard to get. For instance, the intuitively appealing conjecture that  $\beta \rightarrow \theta^*(\beta)$  is increasing still remains open (see Greven and den Hollander (1993)). However, it is easy to get numerical estimates (see section 0.3). Moreover, we shall see that (0.13-14) provide a good starting point for carrying out a *scaling analysis* as  $\beta \downarrow 0$  (see sections 0.4-5), which is the main topic of the present paper.

### 0.3 Numerical estimates of $r^*(\beta)$ and $\theta^*(\beta)$

Table 1 below lists some numerical estimates of  $r^*(\beta)$  and  $\theta^*(\beta)$  obtained from (0.13-14), based on a  $300 \times 300$  truncation of  $A_{r, \beta}$  defined in (0.10). We have used a standard iteration method to estimate the largest eigenvalue and corresponding eigenvector for a range of  $r, \beta$ -values.

$\beta$	$\beta^{-\frac{2}{3}} r^*(\beta)$	$\beta^{-\frac{1}{3}} \theta^*(\beta)$
2	1.696	0.793
0.5	1.730	1.055
$10^{-2}$	2.011	1.10938
$10^{-3}$	2.098	1.10930
$10^{-4}$	2.144	1.10886
$10^{-5}$	2.168	1.10910
$10^{-6}$	2.179	1.10924

Table 1

There is ample evidence for the asymptotic behavior  $r^*(\beta) \sim a^* \beta^{\frac{2}{3}}$  and  $\theta^*(\beta) \sim b^* \beta^{\frac{1}{3}}$  ( $\beta \downarrow 0$ ), with estimates  $a^* = 2.19 \pm 0.01$  and  $b^* = 1.109 \pm 0.001$ .

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<sup>6</sup>  $A_{r, \beta} : l^2(\mathbb{N}) \mapsto l^2(\mathbb{N})$  is positive, self-adjoint and compact for all  $r \in \mathbb{R}, \beta > 0$ . Both  $(r, \beta) \mapsto \lambda(r, \beta)$  and  $(r, \beta) \mapsto \tau_{r, \beta}$  are analytic. Moreover,  $r \mapsto \lambda(r, \beta)$  is strictly increasing and log-convex,  $\lambda(0, \beta) < 1$  and  $\lambda(\infty, \beta) = \infty$  for every  $\beta > 0$  (see Greven and den Hollander (1993)).

The value of  $\theta^*(\beta)$  has been computed by making use of the identity

$$\begin{aligned} \frac{1}{\theta^*(\beta)} &= \sum_{k \in \mathbb{N}} k \mu_\beta^*(k) \\ &= 2 \left[ \sum_{i \in \mathbb{N}} i \tau_{r^*(\beta), \beta}^2(i) \right] \Leftrightarrow 1 \end{aligned} \tag{0.15}$$

(Greven and den Hollander (1993)). Since  $\tau_{r, \beta}$  is easier to estimate than  $\frac{\partial}{\partial r} \lambda(r, \beta)$ , the relation in (0.15) allows for better accuracy than (0.14).

## 0.4 Main results

The goal of this paper is to turn the numerical observations in section 0.3 into a mathematical statement. Our results are formulated in Theorems 4-7 below.

1. Our main scaling theorem reads:

**Theorem 4** *There exist  $a^*, b^* \in (0, \infty)$  and  $\eta^* \in \{\eta \in L^1(\mathbb{R}^+) : \|\eta\|_{L^1} = 1, \eta > 0\}$  such that as  $\beta \downarrow 0$*

$$\begin{aligned} \beta^{-\frac{2}{3}} r^*(\beta) &\rightarrow a^* \\ \beta^{-\frac{1}{3}} \theta^*(\beta) &\rightarrow b^* \\ \beta^{-\frac{1}{3}} \mu_\beta^*([\cdot \beta^{-\frac{1}{3}}]) &\rightarrow^{L^1} \eta^*(\cdot). \end{aligned} \tag{0.16}$$

2. The limits  $a^*, b^*$  and  $\eta^*$  in Theorem 4 can be identified in terms of the following *Sturm-Liouville problem*. For  $a \in \mathbb{R}$ , let  $\mathcal{L}^a$  be the differential operator defined by

$$(\mathcal{L}^a x)(u) = (2au \Leftrightarrow 4u^2)x(u) + x'(u) + ux''(u) \quad (x \in C^\infty(\mathbb{R}^+)). \tag{0.17}$$

In section 5 we shall show that the largest eigenvalue problem

$$\begin{aligned} \mathcal{L}^a x &= \rho x \quad (\rho \in \mathbb{R}, x \in L^2(\mathbb{R}^+) \cap C^\infty(\mathbb{R}^+)) \\ (i) \quad &\|x\|_{L^2} = 1, x > 0 \\ (ii) \quad &\int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\} du < \infty \end{aligned} \tag{0.18}$$

has a unique solution  $(x^a, \rho(a))$  with the following properties:

$$\begin{aligned} (i) \quad &a \rightarrow \rho(a) \text{ is analytic, strictly increasing and strictly convex on } \mathbb{R} \\ (ii) \quad &\rho(0) < 0, \lim_{a \uparrow \infty} \rho(a) = \infty \text{ and } \lim_{a \downarrow -\infty} \rho(a) = \Leftrightarrow \infty \\ (iii) \quad &a \rightarrow x^a \text{ is analytic as a map from } \mathbb{R} \text{ to } L^2(\mathbb{R}^+). \end{aligned} \tag{0.19}$$

The main part of our analysis to prove Theorem 4 will revolve around the following theorem, which is proved in sections 2-5:

**Theorem 5** *Fix  $a \in \mathbb{R}$ . As  $\beta \downarrow 0$*

$$\begin{aligned} \beta^{-\frac{1}{3}} [\lambda(a\beta^{\frac{2}{3}}, \beta) \Leftrightarrow 1] &\rightarrow \rho(a) \\ \beta^{-\frac{1}{3}} \tau_{a\beta^{\frac{2}{3}}, \beta}([\cdot \beta^{-\frac{1}{3}}]) &\rightarrow^{L^2} x^a(\cdot). \end{aligned} \tag{0.20}$$

We shall show in section 6 that (0.20) identifies the limits in Theorem 4 as follows:

**Theorem 6**  $a^*, b^*$  and  $\eta^*$  are given by

$$\begin{aligned} a^* & \text{ is the unique solution of } \rho(a) = 0 \\ \frac{1}{b^*} & = \rho'(a^*) \\ \eta^*(\cdot) & = \frac{1}{2}[x^{a^*}(\frac{1}{2}\cdot)]^2. \end{aligned} \tag{0.21}$$

**3.** The analysis in section 5 of the Sturm-Liouville problem will lead to the following additional properties:

**Theorem 7** (i)  $u \rightarrow x^{a^*}(u)$  is analytic and strictly decreasing on  $\mathbb{R}_0^+ = [0, \infty)$ .

(ii)  $u \rightarrow u \frac{d}{du} x^{a^*}(u)$  is unimodal with a minimum at  $u = \frac{1}{2}a^*$ .

(iii)

$$\lim_{u \rightarrow \infty} u^{-\frac{3}{2}} \log x^{a^*}(u) = \Leftrightarrow \frac{4}{3}. \tag{0.22}$$

(iv)

$$\frac{1}{b^*} = 2 \int_0^\infty u [x^{a^*}(u)]^2 du. \tag{0.23}$$

Theorems 4-7 are proved in sections 2-6. Section 1 contains preparations.

Our result  $\theta^*(\beta) \sim b^* \beta^{\frac{1}{3}}$  implies that the speed is not right-differentiable at  $\beta = 0$ . Thus the limit of weak repellence cannot be treated by perturbation type arguments (i.e., by doing an expansion of (0.1-2) for small  $\beta$ ).

### 0.5 Numerical estimates of $a^*, b^*$ and $\eta^*$

Let  $y^{a,\rho}$  be the unique power series solution of  $\mathcal{L}^a y = \rho y$  with  $y^{a,\rho}(0) = 1$ . We shall see in section 5 that this power series has infinite radius of convergence and has coefficients which satisfy a simple recurrence relation (see (5.23) below). Moreover, we shall see that:

(i)  $\rho(a)$  is simple

(ii)  $\mathcal{S}_a = \{\rho \in \mathbb{R} : y^{a,\rho} \in L^2(\mathbb{R}^+)\}$  is a countable set which has  $\rho(a)$  as a maximum

(iii)  $\rho \notin \mathcal{S}_a : \lim_{u \rightarrow \infty} y^{a,\rho}(u) = \pm\infty$

(iv)  $\rho \in \mathcal{S}_a, \rho \neq \rho(a) : y^{a,\rho}(u) < 0$  for some  $u > 0$

(v)  $y^{a,\rho(a)} = x^a$ , the monotone solution of (0.18).

Properties (i)  $\Leftrightarrow$  (v) give us a way to estimate  $a^*$  and  $x^{a^*}$ . Namely, put  $\rho = 0$  and consider  $y^{a,0}$ , the unique power series solution of  $\mathcal{L}^a y = 0$  ( $a \in \mathbb{R}$ ). Since  $a^*$  is the unique value of  $a$  for which  $y^{a,0} \in L^2(\mathbb{R}^+)$  and  $y^{a,0} \geq 0$ , we can vary  $a$  and tune into  $a^*$  by looking at the tail behavior and the sign of  $y^{a,0}$ . It turns out that this method is very sensitive indeed and that  $a^*$  can be estimated by  $a^* = 2.189 \pm 0.001$ . For  $a$  outside this interval it was found that either  $y^{a,0}(u) < 0$  for some  $u \in [0, 3]$ , or  $u \rightarrow y^{a,0}(u)$  not monotone on  $u \in [0, 3]$ .

[Figure 1]

Figure 1 compares  $x^{a^*}$  with the numerical estimates in section 0.3. The solid line is  $u \rightarrow y^{a,0}(u)/\|y^{a,0}\|_{L^2}$  for  $a = 2.189$ . The dots are the values of  $\beta^{-\frac{1}{6}} \tau_{r^*(\beta),\beta}(\lceil u\beta^{-\frac{1}{3}} \rceil)$  for  $\beta = 10^{-4}$  and  $\lceil u\beta^{-\frac{1}{3}} \rceil = 1, \dots, 64$ . The agreement is excellent. (For  $\beta = 10^{-5}$  and  $\beta = 10^{-6}$  all dots were found to lie on the solid line within printing precision.)

Pick  $a = 2.189$ . Since  $y^{a,0}$  is an approximation of  $x^{a^*}$ , we can estimate  $\frac{1}{b^*}$  by the integral  $2 \int_0^\infty u [y^{a,0}(u)]^2 du$  (recall (0.23)). However, we have only computed  $y^{a,0}(u)$  for  $u \in [0, 3]$  and it turns out that this is not enough to get a good estimate of  $b^*$  up to the third decimal. A better way is to use (0.15) and estimate

$$\frac{1}{b^*} \approx 2\beta^{\frac{2}{3}} \sum_{i \in \mathbb{N}} i [y^{a,0}(i\beta^{\frac{1}{3}})]^2 \Leftrightarrow \beta^{\frac{1}{3}}. \quad (0.24)$$

This gives  $b^* = 1.109 \pm 0.001$ .

## 0.6 The Edwards model

Westwater (1984) studies Brownian motion on  $\mathbb{R}$  with self-repulsion, i.e., the Edwards model where (0.1) is replaced by

$$\frac{d\nu_T^g}{d\mu_T}((W_t)_{0 \leq t \leq T}) = \frac{1}{Z_T^g} \exp \left[ \Leftrightarrow g \int_0^T ds \int_0^T dt \delta(W_s \Leftrightarrow W_t) \right]. \quad (0.25)$$

Here  $\mu_T$  is the Wiener measure on Brownian motion paths  $(W_t)_{0 \leq t \leq T}$ ,  $\delta$  the Dirac-function,  $g \in [0, \infty)$  the repulsion parameter and  $Z_T^g$  the normalizing constant.<sup>7</sup> We give two properties showing that the Edwards model arises as the weak interaction limit of the Domb-Joyce model.

**Property 1** For every  $g \in [0, \infty)$

$$Q_n^{gn^{-\frac{3}{2}}} \left( (n^{-\frac{1}{2}} S_{[tn]})_{0 \leq t \leq 1} \in \cdot \right) \Rightarrow \nu_1^g \left( (W_t)_{0 \leq t \leq 1} \in \cdot \right) \text{ as } n \rightarrow \infty. \quad (0.26)$$

**Proof.** See Brydges and Slade (1994) Theorem 1.3. The double sum in (0.1) equals  $\Leftrightarrow(n+1) + \sum_x \ell_n^2(x)$  (recall (0.7)), of which the first term may be absorbed into the normalizing constant  $Z_n^\beta$  in (0.2). The key point is that  $n^{-\frac{3}{2}} \sum_x \ell_n^2(x)$  under the law  $P_n$  converges to  $\int_{\mathbb{R}} \hat{\ell}_1^2(x) dx$  under the law  $\mu_1$  (recall footnote 7). This immediately implies (0.26). The analogous for  $T \neq 1$  is obvious.  $\square$

Westwater (1984) proves the following result which is analogous to Theorems 1 and 3:

For every  $g \in [0, \infty)$

$$\lim_{T \rightarrow \infty} \nu_T^g \left( \left| \frac{1}{T} |W_T| \Leftrightarrow \hat{\theta}^*(g) \right| \leq \epsilon \right) = 1 \text{ for every } \epsilon > 0, \quad (0.27)$$

where

$$\hat{\theta}^*(g) = \left[ \frac{\partial}{\partial \lambda} E(g, \lambda) \right]_{\lambda=0} \quad (0.28)$$

with  $E(g, \lambda)$  the smallest eigenvalue in  $L^2(\mathbb{R}^+)$  of the operator  $\hat{\mathcal{L}}^{g,\lambda}$  given by

$$\left( \hat{\mathcal{L}}^{g,\lambda} y \right)(v) = \left[ gv^2 + \lambda v^{-2} \Leftrightarrow \frac{1}{2} v^{-1} \left( \frac{d^2}{dv^2} + \frac{1}{4} v^{-2} \right) v^{-1} \right] y(v). \quad (0.29)$$

(The term between round brackets equals  $v^{\frac{1}{2}} \Delta_{rad}^{(2)} v^{-\frac{1}{2}}$  with  $\Delta_{rad}^{(2)}$  the 2-dimensional Laplace operator.)

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<sup>7</sup>The double integral in (0.25) should be read as  $\int_{\mathbb{R}} \hat{\ell}_T^2(x) dx$ , where  $\hat{\ell}_T(x) = \int_0^T dt \delta(W_t - x)$  is the density of the occupation time measure w.r.t. Lebesgue measure.

**Property 2** For every  $g \in [0, \infty)$

$$E(g, 0) = a^* g^{\frac{2}{3}} \tag{0.30}$$

$$\left[ \frac{\partial}{\partial \lambda} E(g, \lambda) \right]_{\lambda=0} = b^* g^{\frac{1}{3}},$$

with  $a^*, b^*$  the same constants as in Theorems 4 and 6.

**Proof.** Take the eigenvalue problem

$$\left( \hat{\mathcal{L}}^{g, \lambda} y \right)(v) = E(g, \lambda) y(v). \tag{0.31}$$

Substitute into (0.31) the following change of variables:

$$y(v) = v^{\frac{1}{2}} x \left( \frac{1}{2} g^{\frac{1}{3}} v^2 \right) \tag{0.32}$$

$$u = \frac{1}{2} g^{\frac{1}{3}} v^2.$$

Then, after a small computation, we obtain the Sturm-Liouville problem in (0.17-18)

$$\left( \mathcal{L}^a x \right)(u) = \rho x(u), \tag{0.33}$$

with

$$a = g^{-\frac{2}{3}} E(g, \lambda) \tag{0.34}$$

$$\rho = g^{-\frac{1}{3}} \lambda.$$

Think of (0.34) as a parametrization of the curve  $a \rightarrow \rho(a)$  in terms of  $\lambda$ . Recalling the definition of  $a^*, b^*$  in (0.21), we now get from (0.33-0.34) that

$$\rho(a^*) = 0 \Leftrightarrow a^* = g^{-\frac{2}{3}} E(g, 0) \tag{0.35}$$

and

$$\begin{aligned} \left[ \frac{\partial}{\partial \lambda} E(g, \lambda) \right]_{\lambda=0} &= g^{-\frac{1}{3}} \left[ \frac{\partial}{\partial \rho} E(g, \rho g^{\frac{1}{3}}) \right]_{\rho=0} \\ &= g^{-\frac{1}{3}} \left[ \frac{\partial}{\partial \rho} \left( a(\rho) g^{\frac{2}{3}} \right) \right]_{\rho=0} \\ &= g^{\frac{1}{3}} a'(0) \\ &= g^{\frac{1}{3}} \frac{1}{\rho'(a^*)} \\ &= g^{\frac{1}{3}} b^*, \end{aligned} \tag{0.36}$$

where  $\rho \rightarrow a(\rho)$  is the inverse function of  $a \rightarrow \rho(a)$ . □

Properties 1 and 2 show that Theorems 4 and 6 connect up nicely with the Edwards model.



We close this section with a *heuristic* explanation of the power  $\frac{1}{3}$  in our result  $\theta^*(\beta) \sim b^* \beta^{\frac{1}{3}}$  ( $\beta \downarrow 0$ ). First, by Brownian scaling

$$E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) = \frac{1}{T} E_{\nu_T^g}(W_T^2). \quad (0.37)$$

Since, according to (0.27),

$$[\hat{\theta}^*(g)]^2 = \lim_{T \rightarrow \infty} \frac{1}{T^2} E_{\nu_T^g}(W_T^2), \quad (0.38)$$

it follows that

$$\hat{\theta}^*(g) = g^{\frac{1}{3}} \hat{\theta}^*(1). \quad (0.39)$$

Next, according to Theorem 1,

$$[\theta^*(g)]^2 = \lim_{n \rightarrow \infty} \frac{1}{n^2} E_{Q_n^g}(S_n^2). \quad (0.40)$$

Moreover, by Property 1 we know that for  $g, T$  fixed

$$\frac{1}{n} E_{Q_n^{g(\frac{T}{n})^{\frac{3}{2}}}}(S_n^2) \sim E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) \quad (n \rightarrow \infty). \quad (0.41)$$

Now, if we assume that (0.41) continues to hold for  $g$  fixed and  $T = n$ , then by using (0.40-41) resp. (0.37-38) we arrive at

$$\begin{aligned} [\theta^*(g)]^2 &\sim \frac{1}{n^2} E_{Q_n^g}(S_n^2) \\ &\sim E_{\nu_1^{gT^{\frac{3}{2}}}}(W_1^2) \\ &= \frac{1}{T^2} E_{\nu_T^g}(W_T^2) \\ &\sim [\hat{\theta}^*(g)]^2 \quad (T = n \rightarrow \infty). \end{aligned} \quad (0.42)$$

The above argument has uniformity problems because (0.39) and (0.42) would imply  $\theta^*(g) = g^{\frac{1}{3}} \theta^*(1)$  for all  $g$ . However, this cannot be true because  $\theta^*(g) \leq 1$  for all  $g$ . Nevertheless, it explains the power  $\frac{1}{3}$  without using the explicit solution.

## 1 Preparations

In this section we formulate the functional analytic framework in which we are going to approach our scaling theorem. Section 1.1 shows that our key result, Theorem 5 in section 0.4, is equivalent to convergence of a variational problem involving a certain functional  $F_\beta^a$  to a variational problem involving some limit functional  $F^a$  (Lemma 1 and Proposition 1 below). Section 1.2 shows that this convergence holds when  $F_\beta^a$  epi-converges to  $F^a$  and certain compactness properties are satisfied (Proposition 2 below). In this section we also formulate the main steps that have to be checked in order to prove these facts (Proposition 3 below). In section 1.3 we collect some properties of the matrix  $P$ , defined in (0.11), that will be needed in the proofs.

## 1.1 A variational representation

Rayleigh's formula for the pair  $(\lambda(r, \beta), \tau_{r, \beta})$  defined in (0.12) reads

$$(i) \quad \lambda(r, \beta) = \max_{\substack{y \in l^2(\mathbb{N}), y \geq 0, \\ \|y\|_{l^2} \leq 1}} \langle y, A_{r, \beta} y \rangle_{l^2} \quad (1.1)$$

(ii)  $\tau_{r, \beta}$  is the unique maximizer.

In anticipation of the scaling suggested by Table 1, we pick  $r = a\beta^{\frac{2}{3}}$  ( $a \in \mathbb{R}$ ) and rewrite (1.1) in the following form. Define the functional  $F_\beta^a : L^2(\mathbb{R}^+) \rightarrow \mathbb{R}$  as

$$F_\beta^a(x) = \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv x(u)x(v) A_{a\beta^{\frac{2}{3}}, \beta}(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \Leftrightarrow \beta^{-\frac{1}{3}} \|x\|_{L^2}^2. \quad (1.2)$$

**Lemma 1** For all  $\beta > 0$

$$(i) \quad \beta^{-\frac{1}{3}} [\lambda(a\beta^{\frac{2}{3}}, \beta) \Leftrightarrow 1] = \max_{\substack{x \in L^2(\mathbb{R}^+), x \geq 0, \\ \|x\|_{L^2} = 1}} F_\beta^a(x) \quad (1.3)$$

(ii)  $\beta^{-\frac{1}{6}} \tau_{a\beta^{\frac{2}{3}}, \beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil)$  is the unique maximizer.

**Proof.** (i) Fix  $\beta > 0$ . For  $x \in L^2(\mathbb{R}^+)$  define

$$\hat{x}(i) = \beta^{-\frac{1}{6}} \int_{(i-1)\beta^{\frac{1}{3}}}^{i\beta^{\frac{1}{3}}} x(u) du \quad (i \in \mathbb{N}). \quad (1.4)$$

Then the first term in (1.2) equals  $\beta^{-\frac{1}{3}} \langle \hat{x}, A_{a\beta^{\frac{2}{3}}, \beta} \hat{x} \rangle_{l^2}$ . Hence using (1.1)(i) we may write

$$\beta^{-\frac{1}{3}} [\lambda(a\beta^{\frac{2}{3}}, \beta) \Leftrightarrow 1] = \max_{\substack{y \in l^2(\mathbb{N}), y \geq 0, \\ \|y\|_{l^2} \leq 1}} \max_{\substack{x \in L^2(\mathbb{R}^+), x \geq 0, \\ \|x\|_{L^2} = 1, \hat{x} = y}} F_\beta^a(x). \quad (1.5)$$

Note that, by Cauchy-Schwarz, we have  $\|\hat{x}\|_{l^2} \leq \|x\|_{L^2}$  and so the restrictions  $\|y\|_{l^2} \leq 1, \|x\|_{L^2} = 1, \hat{x} = y$  in (1.5) are compatible. Interchange the two maxima in (1.5) to get the claim.

(ii) Use that  $\|\hat{x}\|_{l^2} = \|x\|_{L^2}$  iff  $x(u) = \beta^{-\frac{1}{6}} \hat{x}(i)$  for  $u \in ((i-1)\beta^{\frac{1}{3}}, i\beta^{\frac{1}{3}}]$ .  $\square$

In sections 2-5 we shall prove:

**Proposition 1** As  $\beta \downarrow 0$

$$(i) \quad \max_{\substack{x \in L^2(\mathbb{R}^+), x \geq 0, \\ \|x\|_{L^2} = 1}} F_\beta^a(x) \rightarrow \max_{\substack{x \in L^2(\mathbb{R}^+), x \geq 0, \\ \|x\|_{L^2} = 1}} F^a(x) \quad (1.6)$$

(ii) unique maximizer l.h.s.  $\xrightarrow{L^2}$  unique maximizer r.h.s.,

where the limit functional  $F^a : L^2(\mathbb{R}^+) \rightarrow \overline{\mathbb{R}}$  is given by

$$F^a(x) = \int_0^\infty \left\{ (2au \Leftrightarrow 4u^2) [x(u)]^2 \Leftrightarrow u [x'(u)]^2 \right\} du, \quad (1.7)$$

with the understanding that  $F^a(x) = \Leftrightarrow \infty$  if the integral is not defined.

Note that  $F^a(x) = \langle x, \mathcal{L}^a x \rangle_{L^2}$  for all  $x$  where both sides are finite, with  $\mathcal{L}^a$  as defined in (0.17).

Lemma 1 and Proposition 1 imply Theorem 5. To prove Proposition 1, we shall need the notion of epi-convergence, which we next explain.

## 1.2 Epi-convergence

Let  $(X, \tau)$  be a metrizable topological space and let  $Y \subset X$  be dense in  $X$ . Let

$$\begin{aligned} G_\beta &: X \rightarrow \mathbb{R} \quad (\beta > 0) \\ G &: X \rightarrow \overline{\mathbb{R}}. \end{aligned} \tag{1.8}$$

**Definition 1** *The family  $(G_\beta)_{\beta>0}$  is said to be epi-convergent to  $G$  on  $Y$ , written*

$$e \Leftrightarrow \lim_{\beta \downarrow 0} G_\beta = G \text{ on } Y, \tag{1.9}$$

if the following properties hold:

$$\begin{aligned} (i) \quad & \forall x_\beta \xrightarrow{\tau} x \text{ in } Y : \limsup_{\beta \downarrow 0} G_\beta(x_\beta) \leq G(x) \\ (ii) \quad & \exists x_\beta \xrightarrow{\tau} x \text{ in } Y : \liminf_{\beta \downarrow 0} G_\beta(x_\beta) \geq G(x). \end{aligned} \tag{1.10}$$

The importance of the notion of epi-convergence is contained in the following proposition:

**Proposition 2** *Suppose that*

- (1)  $e \Leftrightarrow \lim_{\beta \downarrow 0} G_\beta = G$  on  $Y$
- (2)  $\forall \beta > 0 : G_\beta$  is continuous on  $X$  and has a unique maximizer  $\bar{x}_\beta \in X$
- (3)  $\exists K \subset Y$  such that
  - (i)  $K$  is  $\tau$ -relatively compact in  $X$
  - (ii)  $G$  has a unique maximizer  $\bar{x} \in \overline{K}$
  - (iii)  $\exists (x_\beta)_{\beta>0} \subset \overline{K}$  such that  $x_\beta \Leftrightarrow \bar{x}_\beta \xrightarrow{\tau} \bar{x}$  and  $G_\beta(x_\beta) \Leftrightarrow G_\beta(\bar{x}_\beta) \rightarrow 0$  as  $\beta \downarrow 0$ .

Then as  $\beta \downarrow 0$

$$\sup_{x \in X} G_\beta(x) \rightarrow \sup_{x \in X} G(x) \tag{1.11}$$

$$\bar{x}_\beta \xrightarrow{\tau} \bar{x}. \tag{1.12}$$

**Proof.** See Attouch (1984) Theorem 1.10 and Proposition 1.14.  $\square$

**Remark:** Epi-convergence differs from pointwise convergence:  $\lim_{\beta \downarrow 0} G_\beta(x) = G(x)$  for all  $x \in Y$ . Namely, (1.10)(i),(ii) are weaker in the sense that they require only inequalities, but stronger in the sense that they involve limits in neighborhoods rather than single points. Epi-convergence is a unilateral notion. We have chosen the direction that is suitable for suprema rather than infima.

Fix  $a \in \mathbb{R}$ . We are going to apply Proposition 2 with the following choices:

$$\begin{aligned} X &= \{x \in L^2(\mathbb{R}^+) : x \geq 0, \|x\|_{L^2} = 1\} \\ Y &= X \cap C^1(\mathbb{R}_0^+) \\ \tau &= \text{topology induced by } \|\cdot\|_{L^2} \\ K &= K_C^a = \{x \in Y : F^a(x) \geq C\} \\ G_\beta &= F_\beta^a \\ G &= F^a \end{aligned} \tag{1.13}$$

with  $F_\beta^a$  and  $F^a$  defined in (1.2) and (1.7) and with  $C$  large enough so that  $K_C^a \neq \emptyset$ . Our main result is:

**Proposition 3** *Assumptions (1)-(3) in Proposition 2 hold for the choice in (1.13).*

We prove Assumption (1) in section 2, (3)(i),(ii) in section 5 and (3)(iii) in section 3. We already know (2) to be true because of Lemma 1(ii).

Proposition 3 proves Proposition 1 in section 1.1.

### 1.3 Properties of $P$

We list a few identities and estimates for the matrix  $P$ , defined in (0.11), that will be needed later on.

**Lemma 2** For every  $i \geq 1, k \geq 0$

$$\sum_{j \geq 1} \frac{(i+j+k \Leftrightarrow 2)!}{(i+j \Leftrightarrow 2)!} P(i, j) = 2^k \frac{(i+k \Leftrightarrow 1)!}{(i \Leftrightarrow 1)!}. \quad (1.14)$$

**Proof.** Elementary. Use that the summands in the l.h.s. can be rewritten as  $P(i+k, j)$  times the r.h.s. Then use that  $\sum_{j \geq 1} P(i+k, j) = 1$ .  $\square$

**Lemma 3** (i) For  $i, j \rightarrow \infty$  such that  $i \Leftrightarrow j = o((i+j)^{\frac{2}{3}})$

$$P(i, j) = \left\{ \frac{1}{\sqrt{2\pi(i+j)}} \exp \left[ \Leftrightarrow \frac{(i \Leftrightarrow j)^2}{2(i+j)} \right] \right\} \left[ 1 + \mathcal{O}((i+j)^{-\frac{1}{3}}) \right]. \quad (1.15)$$

(ii) There exist  $0 < c_1 < c_2 < \infty$  such that

$$\exp \left[ \Leftrightarrow c_2 \frac{(i \Leftrightarrow j)^2}{(i+j)} \right] \leq P(i, j) \leq \exp \left[ \Leftrightarrow c_1 \frac{(i \Leftrightarrow j)^2}{(i+j)} \right] \text{ for all } i, j \geq 1. \quad (1.16)$$

**Proof.** Via Stirling's formula. See also Révész (1990) Theorem 2.8.  $\square$

Lemma 2 allows us to compute the following moments, which we shall need in section 2:

$$\begin{aligned} \sum_{j \geq 1} (i+j \Leftrightarrow 1)^n P(i, j) &= 2i && (n=1) \\ &4i^2 + 2i && (n=2) \\ &8i^3 + 12i^2 + 6i && (n=3) \\ &16i^4 + 48i^3 + 72i^2 + 32i && (n=4). \end{aligned} \quad (1.17)$$

Lemma 3(i) is a Gaussian approximation of  $P$ , while Lemma 3(ii) shows that  $P(i, j)$  is small away from the diagonal.

**Lemma 4** For all  $i, j \geq 0$  with  $(i, j) \neq (0, 0)$

$$P(i+1, j) + P(i, j+1) \Leftrightarrow 2P(i+1, j+1) = 0 \quad (1.18)$$

with the convention  $P(i, 0) = P(0, j) = 0$ .

**Proof.** Elementary.  $\square$

Lemma 4 will be needed in sections 2 and 3 to obtain monotonicity properties and estimates of  $\tau_{a\beta^{\frac{2}{3}}, \beta}$ , the eigenvector of  $A_{a\beta^{\frac{2}{3}}, \beta}$ .

## 2 $(F_\beta^a)_{\beta>0}$ is epi-convergent to $F^a$

In this section we prove Assumption (1) in Proposition 2 for the choice in (1.13).

This section is technically somewhat involved, as it consists of a chain of estimates and inequalities that are needed to handle the epi-convergence. The proof is contained in Lemmas 5-8 below. Throughout sections 2 and 3 we fix  $a \in \mathbb{R}$  and we write the abbreviations  $F_\beta = F_\beta^a$ ,  $F = F^a$ ,  $A_\beta = A_{a\beta^{\frac{2}{3}}, \beta}$ ,  $\lambda(\beta) = \lambda(a\beta^{\frac{2}{3}}, \beta)$ ,  $\tau_\beta = \tau_{a\beta^{\frac{2}{3}}, \beta}$ .

We begin by splitting  $F_\beta, F$  into two parts, namely (recall (1.2) and (1.7))

$$\begin{aligned} F_\beta &= F_\beta^1 + F_\beta^2 \\ F &= F^1 + F^2 \end{aligned} \tag{2.1}$$

with

$$F_\beta^1(x) = \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv x^2(u) [A_\beta \Leftrightarrow P](\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \tag{2.2}$$

$$F_\beta^2(x) = \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil)$$

and

$$\begin{aligned} F^1(x) &= \int_0^\infty du (2au \Leftrightarrow 4u^2) x^2(u) \\ F^2(x) &= \Leftrightarrow \int_0^\infty du u [x'(u)]^2. \end{aligned} \tag{2.3}$$

**Lemma 5**  $\forall x_\beta \rightarrow^{L^2} x$  in  $X$  :  $\limsup_{\beta \downarrow 0} F_\beta^1(x_\beta) \leq F^1(x)$ .

**Proof.** Abbreviate

$$e_\beta(i, j) = a\beta^{\frac{2}{3}}(i + j \Leftrightarrow 1) \Leftrightarrow \beta(i + j \Leftrightarrow 1)^2, \tag{2.4}$$

which is the exponent appearing in  $A_\beta(i, j)$ , i.e.,  $A_\beta = e^{e_\beta} P$  (see (0.10)). We note that  $e_\beta$  has the following properties:

$$\begin{aligned} (i) \quad & e_\beta(i, j) \leq 0 \text{ for } i \geq a\beta^{-\frac{1}{3}}, j \geq 1 \\ (ii) \quad & e_\beta(i, j) \leq \frac{1}{4} a^2 \beta^{\frac{1}{3}} \text{ for } i, j \geq 1. \end{aligned} \tag{2.5}$$

Hence, for small enough  $\beta$  and large enough  $N$

$$\begin{aligned} F_\beta^1(x_\beta) &\leq \beta^{-\frac{2}{3}} \int_0^N du \int_0^\infty dv x_\beta^2(u) \\ &\quad \times \left\{ e_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) + e_\beta^2(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \right\} P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \end{aligned} \tag{2.6}$$

(use that  $e^t \leq 1 + t + t^2$  for  $t \ll 1$  and  $t \leq 0$ ). The integral over  $v$  can be transformed into the following sum:

$$\beta^{\frac{1}{3}} \sum_{j \geq 1} \left\{ e_\beta(i, j) + e_\beta^2(i, j) \right\} P(i, j) \text{ with } i = \lceil u\beta^{-\frac{1}{3}} \rceil. \tag{2.7}$$

Using (1.17), we can carry out the summation. Namely,

$$\begin{aligned} \sum_{j \geq 1} e_\beta(i, j) P(i, j) &= a\beta^{\frac{2}{3}}(2i) \Leftrightarrow \beta(4i^2 + 2i) \\ \sum_{j \geq 1} e_\beta^2(i, j) P(i, j) &= a^2 \beta^{\frac{4}{3}}(4i^2 + 2i) \Leftrightarrow 2a\beta^{\frac{5}{3}}(8i^3 + 12i^2 + 6i) \\ &\quad + \beta^2(16i^4 + 48i^3 + 72i^2 + 32i). \end{aligned} \tag{2.8}$$

Since  $i = \lceil u\beta^{-\frac{1}{3}} \rceil \leq (N+1)\beta^{-\frac{1}{3}}$ , the contribution to (2.6) of the second sum can be estimated above by

$$\beta^{\frac{1}{3}}(6a^2(N+1)^2 + 168(N+1)^4) \int_0^N du x_\beta^2(u) = \mathcal{O}(\beta^{\frac{1}{3}}), \quad (2.9)$$

where we use that  $\|x_\beta\|_{L^2} = 1$ . The error term is uniform in  $x_\beta$  for fixed  $N$ . Hence we get

$$\begin{aligned} F_\beta^1(x_\beta) &\leq \beta^{-\frac{1}{3}} \int_0^N du x_\beta^2(u) \\ &\quad \times \left\{ a\beta^{\frac{2}{3}}(2\lceil u\beta^{-\frac{1}{3}} \rceil) \Leftrightarrow \beta(4\lceil u\beta^{-\frac{1}{3}} \rceil^2 + 2\lceil u\beta^{-\frac{1}{3}} \rceil) \right\} + \mathcal{O}(\beta^{\frac{1}{3}}) \\ &= \int_0^N du x_\beta^2(u)(2au \Leftrightarrow 4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \quad (2.10)$$

Now let  $\beta \downarrow 0$ . Then we obtain, recalling that  $x_\beta \xrightarrow{L^2} x$ ,

$$\begin{aligned} \limsup_{\beta \downarrow 0} F_\beta^1(x_\beta) &\leq \limsup_{\beta \downarrow 0} \int_0^N du x_\beta^2(u)(2au \Leftrightarrow 4u^2) \\ &= \int_0^N du x^2(u)(2au \Leftrightarrow 4u^2). \end{aligned} \quad (2.11)$$

Finally, let  $N \rightarrow \infty$  and note that the r.h.s. of (2.11) converges to  $F^1(x)$ .  $\square$

**Lemma 6**  $\forall x \in X : \liminf_{\beta \downarrow 0} F_\beta^1(x) \geq F^1(x)$ .

**Proof.** Estimate

$$F_\beta^1(x) \geq \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv x^2(u) e_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \quad (2.12)$$

(use that  $e^t \geq 1+t$  for all  $t$ ). The integral over  $v$  is  $\beta^{\frac{1}{3}}$  times the first sum computed in (2.8) with  $i = \lceil u\beta^{-\frac{1}{3}} \rceil$ . Hence

$$\begin{aligned} F_\beta^1(x) &\geq \beta^{-\frac{1}{3}} \int_0^\infty du x^2(u) \\ &\quad \times \left\{ a\beta^{\frac{2}{3}}(2u\beta^{-\frac{1}{3}}) \Leftrightarrow \beta(4(u\beta^{-\frac{1}{3}} + 1)^2 + 2(u\beta^{-\frac{1}{3}} + 1)) \right\} \\ &= \int_0^\infty du x^2(u)(2au \Leftrightarrow 4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \quad (2.13)$$

Now let  $\beta \downarrow 0$ . Then the claim follows.  $\square$

**Lemma 7**  $\forall x_\beta \xrightarrow{L^2} x$  in  $X$  with  $x \in Y : \limsup_{\beta \downarrow 0} F_\beta^2(x_\beta) \leq F^2(x)$ .

**Proof.** The proof is in Steps 1-3 below.

**STEP 1** For every  $\epsilon > 0$  and  $N, M$  finite

$$F_\beta^2(x_\beta) \leq \Leftrightarrow \frac{1}{2}(1 + \mathcal{O}(\beta^{\frac{1}{3}})) \int_\epsilon^N du \int_{-M}^M dw \left[ \frac{1}{\beta^{\frac{1}{3}}} \{x_\beta(u) \Leftrightarrow x_\beta(u + w\beta^{\frac{1}{3}})\} \right]^2 N_{2u}(w), \quad (2.14)$$

where  $N_{2u}$  is the Gaussian with mean zero and variance  $2u$ .

**Proof.** Pick  $\epsilon > 0$  and  $N, M$  finite. Then

$$F_\beta^2(x_\beta) \leq \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{-9N^2 \beta^{\frac{1}{3}}} \int_\epsilon^N du \int_{u-M\beta^{\frac{1}{6}}}^{u+M\beta^{\frac{1}{6}}} dv [x_\beta(u) \Leftrightarrow x_\beta(v)]^2 P([\!|u\beta^{-\frac{1}{3}}\!|, [\!|v\beta^{-\frac{1}{3}}\!|]), \quad (2.15)$$

where we use that  $A_\beta = e^{\epsilon\beta} P$  with  $e_\beta([\!|u\beta^{-\frac{1}{3}}\!|, [\!|v\beta^{-\frac{1}{3}}\!|]) \geq \Leftrightarrow 9N^2 \beta^{\frac{1}{3}}$  on the integration area (see (2.4)). Put  $w = \beta^{-\frac{1}{6}}(v \Leftrightarrow u)$ . Then by Lemma 3(i)

$$\begin{aligned} F_\beta^2(x_\beta) &\leq \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{-9N^2 \beta^{\frac{1}{3}}} \int_\epsilon^N du \int_{-M}^M dw \beta^{\frac{1}{6}} [x_\beta(u) \Leftrightarrow x_\beta(u + w\beta^{\frac{1}{6}})]^2 \\ &\quad \times \left\{ \frac{1}{\sqrt{2\pi 2u\beta^{-\frac{1}{3}}}} \exp\left[\Leftrightarrow \frac{w^2}{4u}\right] \right\} (1 + o(\beta^{\frac{1}{9}})), \end{aligned} \quad (2.16)$$

where the error term is uniform on the integration area. Collecting all the powers of  $\beta$ , we get the claim.  $\square$

To investigate the limit of the integral in (2.14) as  $\beta \downarrow 0$ , we proceed with a technical fact contained in Steps 2 and 3 below. Let  $T_h$  be the translation operator defined by  $T_h x_\beta(\cdot) = x_\beta(\cdot + h)$ .

**STEP 2** For every  $0 < a < b < \infty$

$$\liminf_{h \rightarrow 0, \beta \downarrow 0} \int_a^b \left\{ \frac{1}{h} [T_h x_\beta \Leftrightarrow x_\beta](u) \right\}^2 \geq \int_a^b [x'(u)]^2 du. \quad (2.17)$$

**Proof.** Since (2.17) is trivial when the liminf is infinite, we may assume that the liminf is finite, say  $L$ . Pick any subsequence  $h_n, \beta_n$  along which the liminf is reached, and put  $y_n = \frac{1}{h_n} [T_{h_n} x_{\beta_n} \Leftrightarrow x_{\beta_n}]$ . Then, because  $\|y_n\|_{L^2[a,b]} \leq L + 1 < \infty$  for  $n$  large enough, it follows from the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15) that there exists a subsequence  $(y_{n_k})$  and a  $y \in L^2[a, b]$  such that

$$y_{n_k} \rightarrow y \text{ weakly in } L^2[a, b] \quad (k \rightarrow \infty). \quad (2.18)$$

Thus, for any  $\phi \in C_c^1(a, b) = \{\phi \in C^1(a, b) : \text{supp}(\phi) \subset (a, b)\}$

$$\int_a^b y_{n_k}(u) \phi(u) du \rightarrow \int_a^b y(u) \phi(u) du \quad (k \rightarrow \infty). \quad (2.19)$$

Next, the l.h.s. of (2.19) can be rewritten as

$$\begin{aligned} \int_a^b y_{n_k}(u) \phi(u) du &= \int_a^b \frac{1}{h_n} [T_{h_n} x_{\beta_n} \Leftrightarrow x_{\beta_n}](u) \phi(u) du \\ &= \int_{a+h_n \mathbf{1}_{\{h_n < 0\}}}^{b+h_n \mathbf{1}_{\{h_n > 0\}}} x_{\beta_n}(u) \frac{1}{h_n} [T_{-h_n} \phi \Leftrightarrow \phi](u) du \\ &= \int_a^b x_{\beta_n}(u) \frac{1}{h_n} [T_{-h_n} \phi \Leftrightarrow \phi](u) du + o(1) \quad (n \rightarrow \infty). \end{aligned} \quad (2.20)$$

The last equality holds because  $\|x_{\beta_n}\|_{L^2(\mathbb{R}^+)} = 1$  and  $|\frac{1}{h_n} [T_{-h_n} \phi \Leftrightarrow \phi]| \leq \max_{u \in \mathbb{R}^+} |\phi'(u)| < \infty$ . Let  $n \rightarrow \infty$  and note that by the latter property

$$\frac{1}{h_n} [T_{-h_n} \phi \Leftrightarrow \phi] \rightarrow \Leftrightarrow \phi' \text{ pointwise and weakly in } L^2[a, b]. \quad (2.21)$$

Together with  $x_{\beta_n} \rightarrow^{L^2} x$ , (2.21) implies that the last integral in (2.20) tends to  $\int_a^b x(u)[\Leftrightarrow\phi'(u)]du = \int_a^b x'(u)\phi(u)du$  (recall from (1.13) that  $x \in Y \subset C^1(\mathbb{R}_0^+)$ ). Since  $C_c^1(a, b)$  is dense in  $L^2[a, b]$  in the weak topology, we thus have from (2.19)

$$y = x' \text{ a.e. on } [a, b]. \quad (2.22)$$

The claim now follows by combining (2.18) and (2.22), and noting that  $\|\cdot\|_{L^2[a, b]}$  is lower semicontinuous in the weak topology:  $L = \lim_{k \rightarrow \infty} \|y_{n_k}\|_{L^2[a, b]} \geq \|y\|_{L^2[a, b]} = \|x'\|_{L^2[a, b]}$ .  $\square$

**STEP 3** For every  $\epsilon > 0$  and  $N$  finite, every  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  bounded and continuous, and every  $w \in \mathbb{R}$

$$\liminf_{\beta \downarrow 0} \int_{\epsilon}^N du f(u) \left[ \frac{1}{\beta^{\frac{1}{6}}} \{x_{\beta}(u) \Leftrightarrow x_{\beta}(u + w\beta^{\frac{1}{6}})\} \right]^2 \geq \int_{\epsilon}^N du f(u) [wx'(u)]^2. \quad (2.23)$$

**Proof.** Pick any sequence  $(f_n)$  of functions on  $\mathbb{R}^+$  such that

$$\begin{aligned} (i) \quad & f_n(u) = f_{n,k} \text{ for } c_{n,k-1} < u \leq c_{n,k} \quad (k = 1, \dots, n; c_{n,0} = \epsilon, c_{n,n} = N) \\ (ii) \quad & f_n \leq f \\ (iii) \quad & f_n \uparrow f \text{ in sup-norm on } [\epsilon, N] \text{ as } n \rightarrow \infty. \end{aligned} \quad (2.24)$$

Then, by (i) and (ii),

$$\begin{aligned} l.h.s. (2.23) & \geq \liminf_{\beta \downarrow 0} \int_{\epsilon}^N du f_n(u) \left[ \frac{1}{\beta^{\frac{1}{6}}} \{x_{\beta}(u) \Leftrightarrow x_{\beta}(u + w\beta^{\frac{1}{6}})\} \right]^2 \\ & \geq \sum_{k=1}^n f_{n,k} \liminf_{\beta \downarrow 0} \int_{c_{n,k-1}}^{c_{n,k}} du \left[ \frac{1}{\beta^{\frac{1}{6}}} \{x_{\beta}(u) \Leftrightarrow x_{\beta}(u + w\beta^{\frac{1}{6}})\} \right]^2 \\ & \geq \sum_{k=1}^n f_{n,k} \int_{c_{n,k-1}}^{c_{n,k}} du [wx'(u)]^2 \\ & = \int_{\epsilon}^N du f_n(u) [wx'(u)]^2, \end{aligned} \quad (2.25)$$

where in the third inequality we use (2.17) with  $h = w\beta^{\frac{1}{6}}$  and  $a = c_{n,k-1}, b = c_{n,k}$  ( $k = 1, \dots, n$ ). Now let  $n \rightarrow \infty$  and use (iii) together with Fatou to get the claim in (2.23).  $\square$

Using (2.23) we can now finish the proof of Lemma 7. Indeed, continuing with (2.14), we get

$$\begin{aligned} \limsup_{\beta \downarrow 0} F_{\beta}^2(x_{\beta}) & \leq \Leftrightarrow \frac{1}{2} \int_{-M}^M dw \int_{\epsilon}^N du N_{2u}(w) [wx'(u)]^2 \\ & = \Leftrightarrow \frac{1}{2} \int_{\epsilon}^N du [x'(u)]^2 \int_{-M}^M dw w^2 N_{2u}(w). \end{aligned} \quad (2.26)$$

Finally, let  $M \rightarrow \infty$  and note that  $\int_{-\infty}^{\infty} dw w^2 N_{2u}(w) = 2u$ . Then let  $N \rightarrow \infty$  and  $\epsilon \downarrow 0$  to get the claim in Lemma 7.  $\square$

**Lemma 8**  $\forall x \in Y$  such that  $\int_0^{\infty} u^2 x^2(u) du < \infty : \liminf_{\beta \downarrow 0} F_{\beta}^2(x) \geq F^2(x)$ .

**Proof.** The double integral defining  $F_{\beta}^2(x)$  is split into three parts, which we estimate separately in Steps 1-3 below.



**STEP 1**

$$\lim_{\beta \downarrow 0} \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv 1_{\{u > \beta^{-\frac{1}{6}} \text{ or } v > \beta^{-\frac{1}{6}}\}} [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) = 0. \quad (2.27)$$

**Proof.** First consider the part where  $u > \beta^{-\frac{1}{6}}, v \geq 0$ . By (2.5)(ii) and Lemma 3(ii)

$$\begin{aligned} & \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_{\beta^{-\frac{1}{6}}}^\infty du \int_0^\infty dv [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \\ & \geq \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \left\{ \int_{\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv [x(u) \Leftrightarrow x(v)]^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \right. \\ & \quad \left. + \int_{\beta^{-\frac{1}{6}}}^\infty du \int_0^{\frac{1}{2}\beta^{-\frac{1}{6}}} dv [x(u) \Leftrightarrow x(v)]^2 e^{-c_1\beta^{-\frac{1}{3}}\frac{(u-v)^2}{(u+v)}} \right\} \quad (2.28) \\ & = \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \left\{ \int_{\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv [x(u) \Leftrightarrow x(v)]^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \right. \\ & \quad \left. + \mathcal{O}(e^{-\frac{1}{12}c_1\beta^{-\frac{1}{2}}}) \right\}, \end{aligned}$$

where  $c_1$  is the constant in Lemma 3(ii). To get the error term we have used that  $(u \Leftrightarrow v)^2/(u+v) \geq \frac{1}{3}(u \Leftrightarrow v)$  on the integration area. The double integral in the r.h.s. of (2.28) can be bounded above by

$$\begin{aligned} & \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv [x(u) \Leftrightarrow x(v)]^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \\ & \leq 2 \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du x^2(u) \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty dv P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \quad (2.29) \\ & \leq 2\beta^{\frac{1}{3}} \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du x^2(u). \end{aligned}$$

Hence

$$\begin{aligned} r.h.s. (2.28) & \geq \Leftrightarrow [1 + \mathcal{O}(\beta^{\frac{1}{3}})] \beta^{-\frac{1}{3}} \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du x^2(u) \\ & \geq \Leftrightarrow 4 [1 + \mathcal{O}(\beta^{\frac{1}{3}})] \int_{\frac{1}{2}\beta^{-\frac{1}{6}}}^\infty du u^2 x^2(u) \quad (2.30) \\ & = o(1). \end{aligned}$$

By symmetry, the same estimate holds for the part with  $u \geq 0, v > \beta^{-\frac{1}{6}}$ .  $\square$

**STEP 2**

$$\lim_{\beta \downarrow 0} \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv 1_{\{u, v \leq \beta^{-\frac{1}{6}}, |u-v| > \beta^{\frac{1}{24}}\}} [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) = 0. \quad (2.31)$$

**Proof.** By (2.5)(ii) and Lemma 3(ii), the integral in the l.h.s. of (2.31) can be bounded below by

$$\begin{aligned} & \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv 1_{\{|u-v| > \beta^{\frac{1}{24}}\}} [x(u) \Leftrightarrow x(v)]^2 e^{-c_1\beta^{-\frac{1}{3}}\frac{(u-v)^2}{(u+v)}} \\ & \geq \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} e^{-\frac{1}{2}c_1\beta^{-\frac{1}{12}}} \int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv [x^2(u) + x^2(v)] \quad (2.32) \\ & = \mathcal{O}(e^{-\frac{1}{4}c_1\beta^{-\frac{1}{12}}}), \end{aligned}$$

where  $c_1$  is the constant in Lemma 3(ii). □

**STEP 3**

$$\begin{aligned}
& \liminf_{\beta \downarrow 0} \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv \, 1_{\{|u-v| \leq \beta^{\frac{1}{24}}\}} [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \\
& = \liminf_{\beta \downarrow 0} F_\beta^2(x) \\
& \geq F^2(x).
\end{aligned} \tag{2.33}$$

**Proof.** By (2.5)(ii) and the mean value theorem we have

$$\begin{aligned}
& \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} \int_0^{\beta^{-\frac{1}{6}}} du \int_0^{\beta^{-\frac{1}{6}}} dv \, 1_{\{|u-v| \leq \beta^{\frac{1}{24}}\}} [x(u) \Leftrightarrow x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \\
& \geq \Leftrightarrow \frac{1}{2} \beta^{-\frac{2}{3}} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \int_0^\infty du \int_0^\infty dv \, 1_{\{|u-v| \leq \beta^{\frac{1}{24}}\}} [(u \Leftrightarrow v)x'(\xi_{uv})]^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil)
\end{aligned} \tag{2.34}$$

for some  $\xi_{uv}$  between  $u$  and  $v$ . Let, as in (2.34),

$$I_\beta(u) = \beta^{-\frac{2}{3}} \int_0^\infty dv \, 1_{\{|u-v| \leq \beta^{\frac{1}{24}}\}} (u \Leftrightarrow v)^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil). \tag{2.35}$$

Then, because  $x \in C^1(\mathbb{R}_0^+)$ , it follows that

$$r.h.s. (2.34) = \Leftrightarrow \frac{1}{2} e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \int_0^\infty du \, I_\beta(u) [x'(\xi_u)]^2 \tag{2.36}$$

for some  $\xi_u \in [u \Leftrightarrow \beta^{\frac{1}{24}}, u + \beta^{\frac{1}{24}}] \cap \mathbb{R}_0^+$ .

Next, using (1.17) we can estimate

$$\begin{aligned}
I_\beta(u) & \leq \beta^{-\frac{2}{3}} \int_0^\infty dv \, (u \Leftrightarrow v)^2 P(\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil) \\
& = \beta^{\frac{1}{3}} \sum_{j \geq 1} P(\lceil u\beta^{-\frac{1}{3}} \rceil, j) \left\{ (u\beta^{-\frac{1}{3}})^2 \Leftrightarrow (u\beta^{-\frac{1}{3}})(2j \Leftrightarrow 1) + (j^2 \Leftrightarrow j + \frac{1}{3}) \right\} \\
& = \beta^{\frac{1}{3}} \left\{ (u\beta^{-\frac{1}{3}})^2 \Leftrightarrow (u\beta^{-\frac{1}{3}})(2\lceil u\beta^{-\frac{1}{3}} \rceil + 1) + (\lceil u\beta^{-\frac{1}{3}} \rceil^2 + 3\lceil u\beta^{-\frac{1}{3}} \rceil + \frac{1}{3}) \right\} \\
& \leq 2u + \frac{13}{3}\beta^{\frac{1}{3}}.
\end{aligned} \tag{2.37}$$

Combining (2.34) and (2.36-37) with the estimates obtained in Steps 1 and 2, we now have

$$F_\beta^2(x) = l.h.s. (2.34) + o(1) \geq \Leftrightarrow (1 + \mathcal{O}(\beta^{\frac{1}{3}})) \int_0^\infty du \, (u + \frac{13}{6}\beta^{\frac{1}{3}}) [x'(\xi_u)]^2 + o(1). \tag{2.38}$$

Next, pick  $\delta > 0$  and define

$$z_k^\delta = \sup_{(k-1)\delta < u \leq k\delta} u [x'(u)]^2 \quad (k \geq 1). \tag{2.39}$$

Since  $\xi_u \in [u \Leftrightarrow \beta^{\frac{1}{24}}, u + \beta^{\frac{1}{24}}] \cap \mathbb{R}_0^+$ , it follows that for  $\beta$  small enough

$$\begin{aligned}
& \int_0^\infty du (u + \frac{13}{6}\beta^{\frac{1}{3}})[x'(\xi_u)]^2 \\
& \leq (1 + \delta) \int_\delta^\infty du \xi_u [x'(\xi_u)]^2 + 2\delta \int_0^\delta du [x'(\xi_u)]^2 \\
& \leq (1 + \delta) \sum_{k \geq 1} (\delta z_k^\delta + 2\beta^{\frac{1}{24}} \max\{z_k^\delta, z_{k+1}^\delta\}) + 2\delta \int_0^\delta du \sup_{v \leq 2\delta} [x'(v)]^2 \\
& \leq (1 + \delta)(1 + 4\delta^{-1}\beta^{\frac{1}{24}}) \sum_{k \geq 1} \delta z_k^\delta + 2\delta^2 \sup_{v \leq 2\delta} [x'(v)]^2.
\end{aligned} \tag{2.40}$$

Now let  $\beta \downarrow 0$  followed by  $\delta \downarrow 0$ . Because  $x \in C^1(\mathbb{R}_0^+)$ , we have

$$\lim_{\delta \downarrow 0} \sum_{k \geq 1} \delta z_k^\delta = \int_0^\infty u [x'(u)]^2 du = \Leftrightarrow F^2(x) \tag{2.41}$$

$$\lim_{\delta \downarrow 0} \sup_{v \leq 2\delta} [x'(v)]^2 = [x'(0+)]^2 < \infty \tag{2.42}$$

and so

$$\liminf_{\beta \downarrow 0} r.h.s. (2.38) \geq F^2(x). \tag{2.43}$$

□

Lemmas 5-8 show that  $F_\beta$  epi-converges to  $F$  on  $Y$ . To see why, recall (2.1) and note that if  $\int_0^\infty u^2 x^2(u) du = \infty$ , then  $F(x) \leq F^1(x) = \Leftrightarrow \infty$ . This proves Assumption (1) in Proposition 2 as was claimed in Proposition 3.

### 3 An approximate maximizer of $F_\beta^a$

Again we fix  $a \in \mathbb{R}$  and suppress it from the notation. Like section 2, this section is technically somewhat involved, as it consists of a chain of estimates and inequalities that are needed to handle the approximation.

Define the scaled form of the eigenvector  $\tau_\beta$  of  $A_\beta$  as

$$\bar{\tau}_\beta(u) = \beta^{-\frac{1}{6}} \tau_\beta(i) \text{ for } (i \Leftrightarrow 1)\beta^{\frac{1}{3}} < u \leq i\beta^{\frac{1}{3}} \quad (i \geq 1). \tag{3.1}$$

By Lemma 1,  $\bar{\tau}_\beta$  is the unique maximizer of  $F_\beta$ . However,  $\bar{\tau}_\beta$  is a step function and therefore  $F(\bar{\tau}_\beta)$  is not defined, i.e.,  $\bar{\tau}_\beta \notin \overline{K} = \{x \in X : F(x) \geq \Leftrightarrow C\}$  (recall (1.13)). Thus, to apply Proposition 2, we must find an approximation of  $\bar{\tau}_\beta$  that lies in  $\overline{K}$  and approximates  $F_\beta(\bar{\tau}_\beta)$  (i.e., we must prove Assumption 3(iii) in Proposition 2).

**Proposition 4**  $\exists (\tilde{\tau}_\beta) \subset \overline{K}$  such that as  $\beta \downarrow 0$

$$\begin{aligned}
& (i) \quad \|\tilde{\tau}_\beta \Leftrightarrow \bar{\tau}_\beta\|_{L^2} \rightarrow 0 \\
& (ii) \quad 0 \leq F_\beta(\tilde{\tau}_\beta) \Leftrightarrow F_\beta(\bar{\tau}_\beta) \rightarrow 0.
\end{aligned} \tag{3.2}$$

The proof of Proposition 4 is contained in Lemmas 9-13 below. We shall see that it suffices to pick for  $\tilde{\tau}_\beta$  the following *linear* and *renormed* interpolation of  $\bar{\tau}_\beta$ :

$$\begin{aligned}
& \tilde{\tau}_\beta = \hat{\tau}_\beta \|\hat{\tau}_\beta\|_{L^2}^{-1} \\
& \hat{\tau}_\beta(u) = \beta^{-\frac{1}{6}} \left\{ \tau_\beta(i) + (u\beta^{-\frac{1}{3}} \Leftrightarrow i)(\tau_\beta(i) \Leftrightarrow \tau_\beta(i \Leftrightarrow 1)) \right\} \text{ for } (i \Leftrightarrow 1)\beta^{\frac{1}{3}} < u \leq i\beta^{\frac{1}{3}} \quad (i \geq 1).
\end{aligned} \tag{3.3}$$

(put  $\tau_\beta(0) = \tau_\beta(1)$ ).

We begin with two lemmas showing what is needed about  $\tau_\beta$  in order to prove Proposition 4. Abbreviate  $\Delta\tau_\beta(i) = \tau_\beta(i) \Leftrightarrow \tau_\beta(i \Leftrightarrow 1)$  ( $i \geq 1$ ).

**Lemma 9** (i)  $\|\bar{\tau}_\beta \Leftrightarrow \tilde{\tau}_\beta\|_{L^2} \leq \|\Delta\tau_\beta\|_{l^2} + \tau_\beta^2(0)$ .

(ii)  $0 \leq F_\beta(\bar{\tau}_\beta) \Leftrightarrow F_\beta(\tilde{\tau}_\beta) \leq \lambda(\beta)\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2}^2[1 \Leftrightarrow \frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2 + \frac{1}{2}\tau_\beta^2(0)]^{-1}$ .

**Proof.** (i) From (3.1) and (3.3) we compute

$$\|\bar{\tau}_\beta \Leftrightarrow \hat{\tau}_\beta\|_{L^2}^2 = \frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2 \quad (3.4)$$

$$\|\hat{\tau}_\beta\|_{L^2}^2 = \|\tau_\beta\|_{l^2}^2 \Leftrightarrow \langle \tau_\beta, \Delta\tau_\beta \rangle_{l^2} + \frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2. \quad (3.5)$$

Using the relation  $\langle \tau_\beta, \Delta\tau_\beta \rangle_{l^2} = \frac{1}{2}\|\Delta\tau_\beta\|_{l^2}^2 \Leftrightarrow \frac{1}{2}\tau_\beta^2(0)$ , together with (3.4-5) and  $\|\tau_\beta\|_{l^2} = 1$ , we get

$$\begin{aligned} \|\bar{\tau}_\beta \Leftrightarrow \tilde{\tau}_\beta\|_{L^2} &\leq \|\bar{\tau}_\beta \Leftrightarrow \hat{\tau}_\beta\|_{L^2} + \|\hat{\tau}_\beta \Leftrightarrow \tilde{\tau}_\beta\|_{L^2} \\ &= \|\bar{\tau}_\beta \Leftrightarrow \hat{\tau}_\beta\|_{L^2} + \|\hat{\tau}_\beta\|_{L^2} \Leftrightarrow 1 \\ &= \left(\frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2\right)^{\frac{1}{2}} + \left|[1 \Leftrightarrow \frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2 + \frac{1}{2}\tau_\beta^2(0)]^{\frac{1}{2}} \Leftrightarrow 1\right| \\ &\leq \left(\frac{1}{3}\right)^{\frac{1}{2}}\|\Delta\tau_\beta\|_{l^2} + \frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2 + \frac{1}{2}\tau_\beta^2(0) \\ &\leq \left(\left(\frac{1}{3}\right)^{\frac{1}{2}} + \frac{1}{3}\right)\|\Delta\tau_\beta\|_{l^2} + \frac{1}{2}\tau_\beta^2(0) \end{aligned} \quad (3.6)$$

where we use that  $\|\Delta\tau_\beta\|_{l^2} \leq 2, \tau_\beta^2(0) \leq 1$ .

(ii) From the definition of  $F_\beta$  in (1.2) we get, after substitution of (3.1) and (3.3),

$$F_\beta(\bar{\tau}_\beta) = \beta^{-\frac{1}{3}}\langle \tau_\beta, A_\beta\tau_\beta \rangle_{l^2} \Leftrightarrow \beta^{-\frac{1}{3}}\|\tau_\beta\|_{l^2}^2 \quad (3.7)$$

$$F_\beta(\hat{\tau}_\beta) = \beta^{-\frac{1}{3}}\langle (\tau_\beta \Leftrightarrow \frac{1}{2}\Delta\tau_\beta), A_\beta(\tau_\beta \Leftrightarrow \frac{1}{2}\Delta\tau_\beta) \rangle_{l^2} \Leftrightarrow \beta^{-\frac{1}{3}}\|\hat{\tau}_\beta\|_{L^2}^2.$$

It follows from (3.7) that

$$\begin{aligned} F_\beta(\bar{\tau}_\beta) \Leftrightarrow F_\beta(\hat{\tau}_\beta) &= F_\beta(\bar{\tau}_\beta) \Leftrightarrow \frac{1}{\|\hat{\tau}_\beta\|_{L^2}^2} F_\beta(\hat{\tau}_\beta) \\ &= \beta^{-\frac{1}{3}} \frac{1}{\|\hat{\tau}_\beta\|_{L^2}^2} \left\{ \frac{1}{3}\lambda(\beta)\|\Delta\tau_\beta\|_{l^2}^2 \Leftrightarrow \frac{1}{4}\langle \Delta\tau_\beta, A_\beta\Delta\tau_\beta \rangle_{l^2} \right\}, \end{aligned} \quad (3.8)$$

where in the second equality we use the symmetry of  $A_\beta$  and the relations  $A_\beta\tau_\beta = \lambda(\beta)\tau_\beta$  and (3.5). Finally, observe that  $|\langle \Delta\tau_\beta, A_\beta\Delta\tau_\beta \rangle_{l^2}| \leq \langle |\Delta\tau_\beta|, A_\beta|\Delta\tau_\beta| \rangle_{l^2} \leq \lambda(\beta)\|\Delta\tau_\beta\|_{l^2}^2$  to get the claim.  $\square$

**Lemma 10**

$$\begin{aligned} F(\tilde{\tau}_\beta) &\geq \Leftrightarrow 2\sqrt{5}|a| \left( \beta^{\frac{2}{3}} \sum_{i \geq 1} i^2 \tau_\beta^2(i) \right)^{\frac{1}{2}} [1 \Leftrightarrow \frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2 + \frac{1}{2}\tau_\beta^2(0)]^{-\frac{1}{2}} \\ &\quad + \left\{ 20\beta^{\frac{2}{3}} \sum_{i \geq 1} i^2 \tau_\beta^2(i) + \beta^{-\frac{1}{3}} \sum_{i \geq 1} i \Delta\tau_\beta^2(i) \right\} [1 \Leftrightarrow \frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2 + \frac{1}{2}\tau_\beta^2(0)]^{-1}. \end{aligned} \quad (3.9)$$

**Proof.** According to (2.1) and (2.3)

$$F(\hat{\tau}_\beta) = \int_0^\infty du \left\{ (2au \Leftrightarrow 4u^2) \hat{\tau}_\beta^2(u) \Leftrightarrow u[\hat{\tau}'_\beta(u)]^2 \right\}. \quad (3.10)$$

Use (3.3) to obtain the estimates

$$\begin{aligned} \int_0^\infty du u^2 \hat{\tau}_\beta^2(u) &\leq \beta^{\frac{2}{3}} \sum_{i \geq 1} i^2 \max\{\tau_\beta^2(i), \tau_\beta^2(i \Leftrightarrow 1)\} \\ \int_0^\infty du u [\hat{\tau}'_\beta(u)]^2 &\leq \beta^{-\frac{1}{3}} \sum_{i \geq 1} i \Delta \tau_\beta^2(i). \end{aligned} \quad (3.11)$$

Since  $\int_0^\infty du u \hat{\tau}_\beta^2(u) \leq (\int_0^\infty du u^2 \hat{\tau}_\beta^2(u))^{\frac{1}{2}} \|\hat{\tau}_\beta\|_{L^2}$ , we get the claim because  $F(\tilde{\tau}_\beta) = \frac{1}{\|\hat{\tau}_\beta\|_{L^2}^2} F_\beta(\hat{\tau}_\beta)$ .  $\square$

Lemmas 9 and 10 set the stage for the proof of Proposition 4. Namely, we now see that it suffices to prove the following estimates:

**Lemma 11** *There exist  $C_1, C_2, C_3, C_4$  such that for  $\beta$  small enough*

$$\begin{aligned} (i) \quad &\sum_{i \geq 1} i^2 \tau_\beta^2(i) \leq C_1 \beta^{-\frac{2}{3}} \\ (ii) \quad &\sum_{i \geq 1} i \Delta \tau_\beta^2(i) \leq C_2 \beta^{\frac{1}{3}} \\ (iii) \quad &\tau_\beta^2(0) \leq C_3 \beta^{\frac{1}{3}} \log \frac{1}{\beta} \\ (iv) \quad &\|\Delta \tau_\beta\|_{l^2}^2 \leq C_4 \beta^{\frac{2}{3}} \log \frac{1}{\beta}. \end{aligned} \quad (3.12)$$

Indeed, Lemmas 11(iii-iv) and 9(i-ii) imply (3.2), while Lemmas 11(i-ii) and Lemma 10 imply that  $F(\tilde{\tau}_\beta) \geq \Leftrightarrow C$  for  $\beta$  small enough and  $C$  sufficiently large, which guarantees that  $\tilde{\tau}_\beta \in \overline{K} = \overline{K}_C^a$ .

In the proof of Lemma 11 we shall make use of the following two additional lemmas, the proof of which is deferred to section 4:

**Lemma 12**  $\forall \beta > 0 : \Delta \tau_\beta(i) = \tau_\beta(i) \Leftrightarrow \tau_\beta(i \Leftrightarrow 1) \leq 0$  for all  $i \geq \frac{a}{2} \beta^{-\frac{1}{3}}$ .

**Lemma 13**

$$\begin{aligned} \limsup_{\beta \downarrow 0} \beta^{-\frac{1}{3}} [\lambda(\beta) \Leftrightarrow 1] &\leq \frac{1}{4} a^2 \\ \liminf_{\beta \downarrow 0} \beta^{-\frac{1}{3}} [\lambda(\beta) \Leftrightarrow 1] &\geq \begin{cases} \frac{1}{2\pi} a^2 \Leftrightarrow \frac{2}{a} & (a > 1) \\ \frac{1}{\pi} a \Leftrightarrow \frac{1}{2\pi} \Leftrightarrow 2 & (a \leq 1). \end{cases} \end{aligned} \quad (3.13)$$

**Proof of Lemma 11(i).**

**STEP 1** *For every  $\epsilon > 0$  small enough there exists  $C_5$  such that*

$$\sum_{i \leq \epsilon \beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \leq C_5 \beta^{-\frac{2}{3}} \text{ for } \beta \text{ small enough.} \quad (3.14)$$

**Proof.** We start with the trivial inequality

$$\frac{1}{2} \sum_{i,j} [\tau_\beta(i) \Leftrightarrow \tau_\beta(j)]^2 A_\beta(i,j) \geq 0. \quad (3.15)$$

The l.h.s. of (3.15) can be written out and estimated from above as follows:

$$\begin{aligned}
& [1 \Leftrightarrow \lambda(\beta)] + \sum_{i,j} \tau_\beta^2(i) [A_\beta(i,j) \Leftrightarrow P(i,j)] \\
& \leq [1 \Leftrightarrow \lambda(\beta)] + \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \sum_j [e^{\epsilon\beta(i,j)} \Leftrightarrow 1] P(i,j) \\
& \leq [1 \Leftrightarrow \lambda(\beta)] + \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \sum_j [e_\beta(i,j) + \epsilon_\beta^2(i,j)] P(i,j).
\end{aligned} \tag{3.16}$$

For the two inequalities we refer to (2.4-5) (use that  $e^t \leq 1 + t + t^2$  for  $t \ll 1$  and  $t \leq 0$ ). The sum over  $j$  has been evaluated in (2.8). Using that  $i \leq \epsilon\beta^{-\frac{1}{2}}$ , we get

$$\begin{aligned}
& r.h.s. (3.16) \\
& \leq [1 \Leftrightarrow \lambda(\beta)] + \beta^{\frac{1}{3}} \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \{2a(i\beta^{\frac{1}{3}}) \Leftrightarrow (4 \Leftrightarrow 168\epsilon^2 \Leftrightarrow 6a\beta^{\frac{1}{3}})(i\beta^{\frac{1}{3}})^2\}.
\end{aligned} \tag{3.17}$$

Combining (3.15-17) we arrive at the following inequality:

$$\begin{aligned}
& (4 \Leftrightarrow 168\epsilon^2 \Leftrightarrow 6a\beta^{\frac{1}{3}}) \beta^{\frac{2}{3}} \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \\
& \leq \beta^{-\frac{1}{3}} [1 \Leftrightarrow \lambda(\beta)] + 2a\beta^{\frac{1}{3}} \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} i \tau_\beta^2(i).
\end{aligned} \tag{3.18}$$

Now,  $\beta^{-\frac{1}{3}} [1 \Leftrightarrow \lambda(\beta)] \leq C_6$  by Lemma 13. Moreover, by Cauchy-Schwarz

$$\sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} i \tau_\beta^2(i) \leq \left( \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \right)^{\frac{1}{2}}. \tag{3.19}$$

Hence the claim in (3.14) follows for  $\epsilon$  such that  $168\epsilon^2 < 4$ .  $\square$

**STEP 2** For every  $\epsilon > 0$  small enough there exists  $C_7$  such that

$$\sum_{i > \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \leq C_7 \beta^{\frac{1}{3}} \text{ for } \beta \text{ small enough.} \tag{3.20}$$

**Proof.** Rewrite (3.15) as (recall also (3.16))

$$\sum_{i > \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \sum_j [1 \Leftrightarrow e^{\epsilon\beta(i,j)}] P(i,j) \leq [1 \Leftrightarrow \lambda(\beta)] + \sum_{i \leq \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \sum_j [e^{\epsilon\beta(i,j)} \Leftrightarrow 1] P(i,j). \tag{3.21}$$

Since  $e_\beta(i,j) \leq \frac{1}{4}a^2\beta^{\frac{1}{3}}$  for  $i, j \geq 1$  and  $e_\beta(i,j) \leq \frac{1}{2}\epsilon^2$  for  $i > \epsilon\beta^{-\frac{1}{2}}, j \geq 1$  (see (2.4-5)), we get

$$(1 \Leftrightarrow e^{-\frac{1}{2}\epsilon^2}) \sum_{i > \epsilon\beta^{-\frac{1}{2}}} \tau_\beta^2(i) \leq C_6 \beta^{\frac{1}{3}} + (\epsilon^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \Leftrightarrow 1). \tag{3.22}$$

This implies the claim in (3.20).  $\square$

**STEP 3** For every  $\epsilon > 0$  there exists  $C_8$  such that

$$\sum_{i > \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \leq C_8 \beta^{-\frac{2}{3}} \text{ for } \beta \text{ small enough.} \tag{3.23}$$

**Proof.** Pick  $i > \epsilon\beta^{-\frac{1}{2}}$  and  $\delta > 0$  arbitrary. Then, using (2.4) and Lemma 3(ii), we see that there exists  $C(\delta) > 0$  such that

$$\begin{aligned}
\tau_\beta(i) &= \frac{1}{\lambda(\beta)} \sum_j A_\beta(i, j) \tau_\beta(j) \\
&= \frac{1}{\lambda(\beta)} \sum_j e^{\epsilon_\beta(i, j)} P(i, j) \tau_\beta(j) \\
&\leq \frac{1}{\lambda(\beta)} e^{-\frac{1}{2}\epsilon^2} \sum_j P(i, j) \tau_\beta(j) \\
&\leq (1 + \delta) e^{-\frac{1}{2}\epsilon^2} \sum_{j > (1-\delta)i} P(i, j) \tau_\beta(j) + \mathcal{O}(e^{-iC(\delta)}).
\end{aligned} \tag{3.24}$$

Using (3.24) we get

$$\begin{aligned}
\sum_{i > \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) &\leq (1 + \delta) e^{-\frac{1}{2}\epsilon^2} \sum_{i > \epsilon\beta^{-\frac{1}{2}}} i^2 \sum_{j > (1-\delta)i} \tau_\beta(i) P(i, j) \tau_\beta(j) + o(1) \\
&\leq \frac{(1+\delta)}{(1-\delta)^2} e^{-\frac{1}{2}\epsilon^2} \sum_{i > \epsilon\beta^{-\frac{1}{2}}} [(1 \Leftrightarrow \delta)i]^2 \tau_\beta^2([(1 \Leftrightarrow \delta)i]) + o(1) \\
&\leq \frac{(1+\delta)}{(1-\delta)^3} e^{-\frac{1}{2}\epsilon^2} \sum_{i > (1-\delta)\epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) + o(1) \\
&\leq \frac{(1+\delta)}{(1-\delta)^3} e^{-\frac{1}{2}\epsilon^2} \left\{ \sum_{i > \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) + \epsilon^2 \beta^{-1} C_7 \beta^{\frac{1}{3}} \right\} + o(1).
\end{aligned} \tag{3.25}$$

The second and the third inequality use Lemma 12, the fourth uses (3.20) with  $\epsilon$  replaced by  $(1 \Leftrightarrow \delta)\epsilon$ . Now pick  $\delta$  so small that  $\frac{(1+\delta)}{(1-\delta)^3} < e^{\frac{1}{4}\epsilon^2}$ . Then we obtain

$$(1 \Leftrightarrow e^{-\frac{1}{4}\epsilon^2}) \sum_{i > \epsilon\beta^{-\frac{1}{2}}} i^2 \tau_\beta^2(i) \leq e^{-\frac{1}{4}\epsilon^2} \epsilon^2 C_7 \beta^{-\frac{2}{3}} + o(1). \tag{3.26}$$

This proves the claim in (3.23).  $\square$

Steps 1-3 complete the proof of Lemma 11(i).  $\square$

**Proof of Lemma 11(ii).**

**STEP 4** For all  $\beta$

$$\sum_{i \geq 1} i \Delta \tau_\beta^2(i+1) = \frac{1}{\lambda(\beta)} \sum_{i, j \geq 1} (i+j \Leftrightarrow 1) [1 \Leftrightarrow e^{\epsilon_\beta(i+1, j) - \epsilon_\beta(i, j)}] \tau_\beta(i) A_\beta(i, j) \tau_\beta(j). \tag{3.27}$$

**Proof.** Write out

$$\begin{aligned}
\sum_i i \Delta \tau_\beta^2(i+1) &= \sum_i i [\tau_\beta(i+1) \Leftrightarrow \tau_\beta(i)]^2 \\
&= \sum_i i [\tau_\beta^2(i+1) + \tau_\beta^2(i)] \Leftrightarrow \frac{1}{\lambda(\beta)} \sum_{i, j} 2i \tau_\beta(i) A_\beta(i+1, j) \tau_\beta(j).
\end{aligned} \tag{3.28}$$

Now substitute the relation (see (0.10-11))

$$\begin{aligned}
A_\beta(i+1, j) &= e^{\epsilon_\beta(i+1, j)} P(i+1, j) \\
&= e^{\epsilon_\beta(i+1, j)} \frac{i+j-1}{2i} P(i, j) \\
&= e^{\epsilon_\beta(i+1, j) - \epsilon_\beta(i, j)} \frac{i+j-1}{2i} A_\beta(i, j).
\end{aligned} \tag{3.29}$$

This gives

$$\begin{aligned} \sum_i i \Delta \tau_\beta^2(i+1) &= r.h.s. (3.27) + \sum_i i [\tau_\beta^2(i+1) + \tau_\beta^2(i)] \\ &\Leftrightarrow \frac{1}{\lambda(\beta)} \sum_{i,j} (i+j \Leftrightarrow 1) \tau_\beta(i) A_\beta(i,j) \tau_\beta(j). \end{aligned} \quad (3.30)$$

Both sums in the r.h.s. are equal to  $\sum_i (2i \Leftrightarrow 1) \tau_\beta^2(i)$  and therefore cancel out.  $\square$

**STEP 5** For  $\beta$  small enough

$$\frac{1}{\lambda(\beta)} \sum_{i,j \geq 1} (i+j \Leftrightarrow 1) [1 \Leftrightarrow e^{e_\beta(i+1,j) - e_\beta(i,j)}] \tau_\beta(i) A_\beta(i,j) \tau_\beta(j) \leq C_9 \beta^{\frac{1}{3}}. \quad (3.31)$$

**Proof.** By (2.4) we have  $e_\beta(i+1, j) \Leftrightarrow e_\beta(i, j) = a\beta^{\frac{2}{3}} \Leftrightarrow \beta(2i+2j \Leftrightarrow 1)$ . Hence

$$\begin{aligned} l.h.s. (3.31) &\leq \frac{1}{\lambda(\beta)} \sum_{i,j \geq 1} (i+j \Leftrightarrow 1) [e_\beta(i, j) \Leftrightarrow e_\beta(i+1, j)] \tau_\beta(i) A_\beta(i, j) \tau_\beta(j) \\ &\leq \frac{1}{\lambda(\beta)} 2\beta \sum_{i,j} (i+j)^2 \tau_\beta(i) A_\beta(i, j) \tau_\beta(j) \\ &\leq 8\beta \sum_i i^2 \tau_\beta^2(i) \end{aligned} \quad (3.32)$$

(use that  $e^t \geq 1+t$  for all  $t$ ). In the third inequality we use the symmetry of  $A_\beta$  and the fact that  $\|A_\beta\|_{l^2} = \lambda(\beta)$ . The claim now follows from Lemma 11(i).  $\square$

Steps 4-5 complete the proof of Lemma 11(ii).  $\square$

**Proof of Lemma 11(iii).**

**STEP 6**

$$\tau_\beta^2(0) \leq C_{10} \beta^{\frac{1}{3}} \log \frac{1}{\beta} \text{ for } \beta \text{ small enough.} \quad (3.33)$$

**Proof.** By Cauchy-Schwarz, we have for every  $N$

$$\begin{aligned} \tau_\beta(0) &= \tau_\beta(N) \Leftrightarrow \sum_{i=1}^N \Delta \tau_\beta(i) \\ &\leq \tau_\beta(N) + \left( \sum_{i=1}^N \frac{1}{i} \right)^{\frac{1}{2}} \left( \sum_{i=1}^N i \Delta \tau_\beta^2(i) \right)^{\frac{1}{2}}. \end{aligned} \quad (3.34)$$

Pick  $N = \lceil \beta^{-\frac{1}{2}} \rceil$ . Lemma 11(i) gives  $\tau_\beta(\lceil \beta^{-\frac{1}{2}} \rceil) \leq C_1 \beta^{\frac{1}{3}}$ . Together with Lemma 11(ii) and the estimate  $\sum_{i=1}^{\lceil \beta^{-\frac{1}{2}} \rceil} \frac{1}{i} \leq \log \frac{1}{\beta}$ , the claim follows.  $\square$

Step 6 completes the proof of Lemma 11(iii).  $\square$

**Proof of Lemma 11(iv).**

**STEP 7** For all  $\beta$

$$\begin{aligned} \sum_{i \geq 1} \Delta \tau_\beta^2(i+1) &= \frac{2}{\lambda(\beta)} \sum_{\substack{i,j \\ (i,j) \neq (1,1)}} [1 \Leftrightarrow e^{e_\beta(i-1,j) - e_\beta(i,j)}] \tau_\beta(i) A_\beta(i, j) \tau_\beta(j) \\ &\Leftrightarrow \tau_\beta^2(1) [1 \Leftrightarrow \frac{2}{\lambda(\beta)} A_\beta(1, 1)]. \end{aligned} \quad (3.35)$$



**Proof.** By Lemma 4 we have the following relation:

$$A_\beta(i, j) \Leftrightarrow A_\beta(i \Leftrightarrow 1, j) = A_\beta(i, j \Leftrightarrow 1) \Leftrightarrow A_\beta(i, j) + 2A_\beta(i, j)[1 \Leftrightarrow e^{e_\beta(i-1, j) - e_\beta(i, j)}] \quad (3.36)$$

(note that  $e_\beta(i \Leftrightarrow 1, j) = e_\beta(i, j \Leftrightarrow 1)$ ). Hence

$$\begin{aligned} \sum_i \Delta \tau_\beta^2(i+1) &= \sum_i [\tau_\beta(i+1) \Leftrightarrow \tau_\beta(i)]^2 \\ &= \tau_\beta^2(1) + 2 \sum_{i \geq 2} \tau_\beta(i) [\tau_\beta(i) \Leftrightarrow \tau_\beta(i \Leftrightarrow 1)] \\ &= \tau_\beta^2(1) + \frac{2}{\lambda(\beta)} \sum_{i \geq 2} \sum_j \tau_\beta(i) [A_\beta(i, j) \Leftrightarrow A_\beta(i \Leftrightarrow 1, j)] \tau_\beta(j) \\ &= \tau_\beta^2(1) + \frac{2}{\lambda(\beta)} \sum_{i \geq 2} \sum_j \tau_\beta(i) [A_\beta(i, j \Leftrightarrow 1) \Leftrightarrow A_\beta(i, j)] \tau_\beta(j) \\ &\quad + \frac{4}{\lambda(\beta)} \sum_{i \geq 2} \sum_j [1 \Leftrightarrow e^{e_\beta(i-1, j) - e_\beta(i, j)}] \tau_\beta(i) A_\beta(i, j) \tau_\beta(j). \end{aligned} \quad (3.37)$$

The third term in the last expression is twice the sum in the r.h.s. of (3.35) except for the part with  $i = 1, j \geq 2$ . The second term can be rewritten by carrying out the sum over  $i$ , namely (use that  $A(i, 0) = 0$ )

$$\begin{aligned} j = 1: & \Leftrightarrow \frac{2}{\lambda(\beta)} \sum_{i \geq 2} \tau_\beta(i) A_\beta(i, 1) \tau_\beta(1) \\ &= \Leftrightarrow 2\tau_\beta^2(1) + \frac{2}{\lambda(\beta)} \tau_\beta^2(1) A_\beta(1, 1) \\ j \geq 2: & \frac{2}{\lambda(\beta)} \sum_{i \geq 2} \tau_\beta(i) [A_\beta(i, j \Leftrightarrow 1) \Leftrightarrow A_\beta(i, j)] \tau_\beta(j) \\ &= 2\tau_\beta(j) [\tau_\beta(j \Leftrightarrow 1) \Leftrightarrow \tau_\beta(j)] \Leftrightarrow \frac{2}{\lambda(\beta)} \tau_\beta(1) [A_\beta(1, j \Leftrightarrow 1) \Leftrightarrow A_\beta(1, j)] \tau_\beta(j). \end{aligned} \quad (3.38)$$

Thus, after carrying out the sum over  $j$ , we see that (3.37) becomes

$$\begin{aligned} \sum_i \Delta \tau_\beta^2(i+1) &= \Leftrightarrow \sum_j \Delta \tau_\beta^2(j+1) \\ &\quad + 2 \left\{ r.h.s. (3.35) \Leftrightarrow \frac{2}{\lambda(\beta)} \sum_{j \geq 2} [1 \Leftrightarrow e^{e_\beta(0, j) - e_\beta(1, j)}] \tau_\beta(1) A_\beta(1, j) \tau_\beta(j) \right. \\ &\quad \left. + \tau_\beta^2(1) [1 \Leftrightarrow \frac{2}{\lambda(\beta)} A_\beta(1, 1)] \right\} \\ &\quad + \frac{2}{\lambda(\beta)} \tau_\beta^2(1) A_\beta(1, 1) \Leftrightarrow \frac{2}{\lambda(\beta)} \sum_{j \geq 2} \tau_\beta(1) [A_\beta(1, j \Leftrightarrow 1) \Leftrightarrow A_\beta(1, j)] \tau_\beta(j). \end{aligned} \quad (3.39)$$

Now, by (3.36) for  $i = 1$ ,

$$2[1 \Leftrightarrow e^{e_\beta(0, j) - e_\beta(1, j)}] A_\beta(1, j) = \Leftrightarrow [A_\beta(1, j \Leftrightarrow 1) \Leftrightarrow A_\beta(1, j)] + A_\beta(1, j). \quad (3.40)$$

Hence (3.39) simplifies to

$$\begin{aligned} \sum_i \Delta \tau_\beta^2(i+1) &= \Leftrightarrow \sum_j \Delta \tau_\beta^2(j+1) + 2 \text{ r.h.s. (3.35)} \\ &\quad + \left\{ 2\tau_\beta^2(1) \Leftrightarrow \frac{2}{\lambda(\beta)} \tau_\beta^2(1) A_\beta(1, 1) \Leftrightarrow \frac{2}{\lambda(\beta)} \sum_{j \geq 2} \tau_\beta(1) A_\beta(1, j) \tau_\beta(j) \right\}. \end{aligned} \quad (3.41)$$

But the term between braces is zero.  $\square$

**STEP 8** For  $\beta$  small enough

$$r.h.s. (3.35) \leq C_{11}\beta^{\frac{2}{3}} \log \frac{1}{\beta}. \quad (3.42)$$

**Proof.** The first term in (3.35) is easy to bound. Indeed, we have  $e_\beta(i \Leftrightarrow 1, j) \Leftrightarrow e_\beta(i, j) = \Leftrightarrow a\beta^{\frac{2}{3}} + \beta(2i + 2j \Leftrightarrow 3)$  and hence we get

$$\begin{aligned} 1^{st} \text{ term in (3.35)} &\leq \frac{2}{\lambda(\beta)} \sum_{\substack{i,j \\ (i,j) \neq (1,1)}} a\beta^{\frac{2}{3}} \tau_\beta(i) A_\beta(i, j) \tau_\beta(j) \\ &\leq 2a\beta^{\frac{2}{3}} \end{aligned} \quad (3.43)$$

(in the first inequality use that  $e^t \geq 1 + t$  for all  $t$ ). For the second term in (3.35), use that  $P(1, 1) = \frac{1}{2}$  and  $e_\beta(1, 1) = a\beta^{\frac{2}{3}} \Leftrightarrow \beta$ . Together with  $\lambda(\beta) \geq 1 \Leftrightarrow C_6\beta^{\frac{1}{3}}$  (see below (3.18)) we get

$$2^{nd} \text{ term in (3.35)} \leq 2\tau_\beta^2(1)C_6\beta^{\frac{1}{3}} \text{ for } \beta \text{ small enough.} \quad (3.44)$$

Finally, use Step 6 to get the claim (recall that  $\tau_\beta(0) = \tau_\beta(1)$  in (3.3)).  $\square$

Steps 7-8 complete the proof of Lemma 11(iv).  $\square$

Lemma 11 completes the proof of Proposition 4. Lemmas 12 and 13 will be proved in section 4.

Proposition 4 shows that Assumption 3(iii) in Proposition 2 holds. We shall prove Assumptions 3(i),(ii) in section 5.

## 4 Proof of Lemmas 12 and 13

### 4.1 Proof of Lemma 12

Let  $(e_i)_{i \geq 1}$  be the canonical base of  $l^2(\mathbb{N})$ . Let  $s = (s(i))_{i \geq 1}$  be any sequence of numbers in  $(0, \infty)$  and let  $t = (t(i))_{i \geq 1}$  be given by  $t(1) = 1, t(i) = \prod_{k=1}^{i-1} s(k)$  ( $i \geq 2$ ). Define

$$\begin{aligned} B_s &= \{x \in l^2(\mathbb{N}) : x \geq 0, x(i+1) \leq s(i)x(i)\} \\ B_s^0 &= \{x = \sum_j c_j f_j : c_j \geq 0, c_j \neq 0 \text{ finitely often}\} \end{aligned} \quad (4.1)$$

where  $f_j \in l^2(\mathbb{N})$  is defined by

$$f_j = \sum_{i=1}^j t(i)e_i. \quad (4.2)$$

**Lemma 14** (i)  $B_s$  is a closed convex cone.

(ii)  $B_s$  is the closure of  $B_s^0$ .

**Proof.** Elementary.  $\square$

Recall footnote 6. Since, for every  $\beta > 0$ ,  $A_\beta$  is a continuous operator on  $l^2(\mathbb{N})$ , we have from Lemma 14(ii) that

$$A_\beta B_s \subset B_s \Leftrightarrow A_\beta f_j \in B_s \text{ for all } j \geq 1. \quad (4.3)$$

Since, for every  $\beta > 0$ ,  $A_\beta$  is symmetric and has a spectral gap, we also know that  $\lambda(\beta)^{-n} A_\beta^n x \rightarrow^{l^2} \langle x, \tau_\beta \rangle_{l^2} \tau_\beta$  ( $n \rightarrow \infty$ ) for any  $x \in l^2(\mathbb{N})$ . Pick any  $x \in B_s$  with  $x \neq 0$  to get that

$$A_\beta B_s \subset B_s \Rightarrow \tau_\beta \in B_s. \quad (4.4)$$

Below we shall prove the following:

**Lemma 15** *If  $s$  satisfies*

$$\frac{is(i) + j \frac{1}{s(j)}}{i + j} \geq e^{a\beta^{\frac{2}{3}} - \beta(2i+2j+1)} \text{ for all } i \geq 1, j \geq 0, \quad (4.5)$$

then  $A_\beta f_j \in B_s$  for all  $j \geq 1$ .

Lemma 15 combined with (4.3-4) shows that

$$\tau_\beta \in \bigcap_{\{s: s \text{ satisfies (4.5)}\}} B_s. \quad (4.6)$$

The r.h.s. of (4.5) is  $\leq 1$  when  $i + j \geq \frac{a}{2}\beta^{-\frac{1}{3}}$ . One therefore easily sees that the following choice of  $s$  satisfies (4.5):

$$\begin{aligned} s(i) &= 1 && \text{for } i > \frac{a}{2}\beta^{-\frac{1}{3}} \\ &= N\beta^{-\frac{1}{3}} && \text{for } i \leq \frac{a}{2}\beta^{-\frac{1}{3}}, N \text{ large enough.} \end{aligned} \quad (4.7)$$

This proves Lemma 12.

**Proof of Lemma 15.** We must show that for all  $i, j \geq 1$

$$\begin{aligned} 0 &\leq s(i)(A_\beta f_j)(i) \Leftrightarrow (A_\beta f_j)(i+1) \\ &= \sum_{k=1}^j \left[ s(i)A_\beta(i, k)t(k) \Leftrightarrow A_\beta(i+1, k \Leftrightarrow 1)t(k \Leftrightarrow 1) \right] \Leftrightarrow A_\beta(i+1, j)t(j) \end{aligned} \quad (4.8)$$

(recall from Lemma 4 that  $A_\beta(i+1, 0) = 0$  by convention). In order to do so, define

$$\psi_i(j) = \sum_{k=1}^j \left[ s(i)A_\beta(i, k)t(k) \Leftrightarrow A_\beta(i+1, k \Leftrightarrow 1)t(k \Leftrightarrow 1) \right] \Leftrightarrow 2A_\beta(i+1, j)t(j). \quad (4.9)$$

The following lemma gives a sufficient criterion for  $\psi_i(j) \geq 0$ , which implies (4.8):

**Lemma 16** *If*

$$s(i)A_\beta(i, j+1) + \frac{1}{s(j)}A_\beta(i+1, j) \Leftrightarrow 2A_\beta(i+1, j+1) \geq 0 \text{ for all } i \geq 1, j \geq 0, \quad (4.10)$$

then

$$\begin{aligned} (i) & j \rightarrow \psi_i(j) \text{ is nondecreasing for all } i \geq 1 \\ (ii) & \psi_i(1) \geq 0 \text{ for all } i \geq 1. \end{aligned} \quad (4.11)$$

**Proof.** (i) Write out

$$\begin{aligned}\psi_i(j+1) \Leftrightarrow \psi_i(j) &= s(i)A_\beta(i, j+1)t(j+1) + A_\beta(i+1, j)t(j) \Leftrightarrow 2A_\beta(i+1, j+1)t(j+1) \\ &= t(j+1)\left[s(i)A_\beta(i, j+1) + \frac{1}{s(j)}A_\beta(i+1, j) \Leftrightarrow 2A_\beta(i+1, j+1)\right] \geq 0.\end{aligned}\tag{4.12}$$

(ii) Similarly, by (4.10) with  $j = 0$  (since  $t(1) = 1$  and  $A_\beta(i+1, 0) = 0$ )

$$\psi_i(1) = s(i)A_\beta(i, 1) \Leftrightarrow 2A_\beta(i+1, 1) \geq 0.\tag{4.13}$$

□

To complete the proof of Lemma 15, it remains to rewrite (4.10) in the form of (4.5). Abbreviate  $f(i) = \exp[a\beta^{\frac{2}{3}}i \Leftrightarrow \beta i^2]$ . Then we have  $A_\beta(i, j) = f(i+j \Leftrightarrow 1)P(i, j)$ . Use Lemma 4 to write

$$\begin{aligned}l.h.s. (4.10) &= f(i+j)\left[s(i)P(i, j+1) + \frac{1}{s(j)}P(i+1, j)\right] \Leftrightarrow 2f(i+j \Leftrightarrow 1)P(i+1, j+1) \\ &= P(i, j+1)\left[s(i)f(i+j) \Leftrightarrow f(i+j+1)\right] \\ &\quad + P(i+1, j)\left[\frac{1}{s(j)}f(i+j) \Leftrightarrow f(i+j+1)\right].\end{aligned}\tag{4.14}$$

Next use that  $P(i, j+1)/P(i+1, j) = i/j$ . Then (4.10) is seen to be equivalent to

$$\frac{is(i) + j\frac{1}{s(j)}}{i+j} \geq \frac{f(i+j+1)}{f(i+j)}.\tag{4.15}$$

□

## 4.2 Proof of Lemma 13

To prove the upper bound in (3.13), use (2.5)(ii) to get

$$\begin{aligned}\lambda(\beta) &= \sum_{i,j} \tau_\beta(i)A_\beta(i, j)\tau_\beta(j) \\ &= \sum_{i,j} \tau_\beta(i)e^{\varepsilon_\beta(i,j)}P(i, j)\tau_\beta(j) \\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \sum_{i,j} \tau_\beta(i)P(i, j)\tau_\beta(j) \\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}},\end{aligned}\tag{4.16}$$

where the last inequality follows from  $\|P\|_{l^2} \leq 1$ . This immediately gives the claim.

To prove the lower bound in (3.13), use (1.3)(i) to get that for any  $x \in L^2(\mathbb{R}^+)$  with  $\|x\|_{L^2} = 1$

$$\beta^{-\frac{1}{3}}[\lambda(\beta) \Leftrightarrow 1] \geq F_\beta^a(x).\tag{4.17}$$

Pick for  $x$

$$x_\sigma(u) = \left(\frac{2}{\pi\sigma^2}\right)^{\frac{1}{4}} e^{-\frac{u^2}{4\sigma^2}} \quad (\sigma > 0).\tag{4.18}$$

Now, we know from Lemmas 5-8 that

$$\lim_{\beta \downarrow 0} F_\beta^a(x_\sigma) = F^a(x_\sigma). \quad (4.19)$$

Hence  $\liminf_{\beta \downarrow 0} \beta^{-\frac{1}{3}}[\lambda(\beta) \Leftrightarrow 1] \geq F^a(x_\sigma)$ . Compute

$$\begin{aligned} F^a(x_\sigma) &= \int_0^\infty \left\{ (2au \Leftrightarrow 4u^2)[x_\sigma(u)]^2 \Leftrightarrow u[x'_\sigma(u)]^2 \right\} du \\ &= \left( \frac{2}{\pi\sigma^2} \right)^{\frac{1}{2}} \int_0^\infty (2au \Leftrightarrow 4u^2 \Leftrightarrow \frac{u^3}{4\sigma^4}) e^{-\frac{u^2}{2\sigma^2}} du \\ &= \left( \frac{8}{\pi} \right)^{\frac{1}{2}} a\sigma \Leftrightarrow 4\sigma^2 \Leftrightarrow \frac{1}{(2\pi)^{\frac{1}{2}}\sigma}. \end{aligned} \quad (4.20)$$

Pick  $\sigma = \sigma(a) = \frac{a\sqrt{1}}{(8\pi)^{\frac{1}{2}}}$  to get the claim.  $\square$

## 5 Analysis of the limit variational problem

Recall the notation in (1.13)

$$\begin{aligned} X &= \{x \in L^2(\mathbb{R}^+) : x \geq 0, \|x\|_{L^2} = 1\} \\ Y &= X \cap C^1(\mathbb{R}_0^+) \\ K &= K_C^a = \{x \in Y : F^a(x) \geq \Leftrightarrow C\}. \end{aligned} \quad (5.1)$$

In this section we analyze the limit variational problem appearing in (1.6), i.e.,

$$\sup_{x \in X} F^a(x). \quad (5.2)$$

In section 5.1 we show that  $x \rightarrow F^a(x)$  is upper semicontinuous and  $K_C^a$  is relatively compact in  $X$  (in the  $L^2$ -topology). This implies that  $F^a$  achieves a maximum in  $\overline{K_C^a} = \{x \in X : F^a(x) \geq \Leftrightarrow C\}$  ( $\neq \emptyset$  for  $C$  large enough). In section 5.2 we show that all maxima of  $F^a$  in  $X$  are solutions of the Sturm-Liouville problem

$$\mathcal{L}^a x = \rho x \quad (\rho \in \mathbb{R}, x \in X \cap C^\infty(\mathbb{R}^+)), \quad (5.3)$$

where  $\mathcal{L}^a$  is defined in (0.17). In section 5.3 we analyze (5.3) and show that it has a unique solution  $x^a$  satisfying  $F^a(x^a) > \Leftrightarrow \infty$  and  $x^a > 0$ , with corresponding eigenvalue  $\rho(a)$ . This identifies  $x^a$  as the unique maximizer of (5.2) and  $\rho(a)$  as the maximum. We also study  $a \rightarrow x^a$  and  $a \rightarrow \rho(a)$  to prove the claims that were made in (0.19).

### 5.1 Existence of a maximizer of $F^a$ in $\overline{K_C^a}$

It will be expedient to transform  $F^a, \mathcal{L}^a, K_C^a$  as follows. Define (recall (1.7))

$$\begin{aligned} \hat{F}^a(x) &= \Leftrightarrow F^a(x) + \left(\frac{a^2}{4} + 1\right) \|x\|_{L^2}^2 \\ &= \int_0^\infty \{q(u)[x(u)]^2 + p(u)[x'(u)]^2\} du \end{aligned} \quad (5.4)$$

with

$$\begin{aligned} p(u) &= u \\ q(u) &= (2u \Leftrightarrow \frac{1}{2}a)^2 + 1. \end{aligned} \quad (5.5)$$

$\hat{F}^a$  is the “energy” functional corresponding to the Sturm-Liouville differential operator  $\hat{\mathcal{L}}^a$  defined by (recall (0.17))

$$\begin{aligned} (\hat{\mathcal{L}}^a x)(u) &= \Leftrightarrow(\mathcal{L}^a x)(u) + (\frac{a^2}{4} + 1)x(u) \\ &= q(u)x(u) \Leftrightarrow [p(u)x'(u)]'. \end{aligned} \quad (5.6)$$

Define (recall (1.13))

$$\begin{aligned} \hat{K}_C^a &= K_{C - (\frac{a^2}{4} + 1)}^a \\ &= \{x \in Y : \hat{F}^a(x) \leq C\}. \end{aligned} \quad (5.7)$$

**Lemma 17** For every  $a \in \mathbb{R}$

- (i)  $\hat{K}_C^a \neq \emptyset$  for  $C$  large enough
- (ii)  $\hat{K}_C^a$  is relatively compact in  $L^2(\mathbb{R}^+)$  for all  $C \in \mathbb{R}$
- (iii)  $x \rightarrow \hat{F}^a(x)$  is lower semicontinuous on  $X$ .

**Proof.** Standard.

(i) Trivial.

(ii) We check the conditions in Dunford and Schwartz (1964) Theorem IV.8.20.

(a)  $\hat{K}_C^a$  is bounded in  $L^2(\mathbb{R}^+)$ .

(b) By Cauchy-Schwarz

$$\begin{aligned} \int_0^\infty (x(u+v) \Leftrightarrow x(u))^2 du &= \int_0^\infty (\int_u^{u+v} x'(t) dt)^2 \\ &\leq \int_0^\infty du [\log(u+v) \Leftrightarrow \log u] \int_u^{u+v} dt t [x'(t)]^2 \\ &= \int_0^\infty dt t [x'(t)]^2 I(t, v) 1_{\{t \geq v\}}, \end{aligned} \quad (5.8)$$

where

$$I(t, v) = (t+v) \log(1 + \frac{v}{t}) + (t \Leftrightarrow v) \log(1 \Leftrightarrow \frac{v}{t}). \quad (5.9)$$

Since  $t \rightarrow I(t, v)$  is decreasing and  $I(v, v) = 2v \log 2$ , it follows that

$$\lim_{v \downarrow 0} \int_0^\infty (x(u+v) \Leftrightarrow x(u))^2 du = 0 \text{ uniformly for } x \in \hat{K}_C^a. \quad (5.10)$$

(c) From  $p(u) \geq 0$  and  $\lim_{u \rightarrow \infty} q(u) = \infty$  follows

$$\lim_{N \rightarrow \infty} \int_N^\infty x^2(u) du = 0 \text{ uniformly for } x \in \hat{K}_C^a. \quad (5.11)$$

Conditions (a)-(c) imply that  $\hat{K}_C^a$  is relatively compact.

(iii) Define

$$V^a = \{x \in L^2(\mathbb{R}^+) : \hat{F}^a(x) < \infty\}. \quad (5.12)$$

On  $V^a$  define the inner product

$$\langle x, y \rangle_{V^a} = \int_0^\infty \{q(u)x(u)y(u) + p(u)x'(u)y'(u)\} du. \quad (5.13)$$

Then  $(V^a, \langle \cdot, \cdot \rangle_{V^a})$  is a Hilbert space,  $\|x\|_{V^a} \geq \|x\|_{L^2}$  and

$$\hat{F}^a(x) = \langle x, x \rangle_{V^a} = \|x\|_{V^a}^2. \quad (5.14)$$

Thus we must prove that  $\liminf_{n \rightarrow \infty} \|x_n\|_{V^a} \geq \|x\|_{V^a}$  for any  $x_n \rightarrow^{L^2} x$ .

Let  $L = \liminf_{n \rightarrow \infty} \|x_n\|_{V^a}$ . The case  $L = \infty$  being trivial, assume  $L < \infty$ . Then, by the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15), there exists a subsequence  $(x_{n_k})$  and a  $y \in V^a$  such that  $L = \lim_{k \rightarrow \infty} \|x_{n_k}\|_{V^a}$  and  $x_{n_k} \rightarrow y$  weakly in  $V^a$  ( $k \rightarrow \infty$ ). Hence  $L \geq \|y\|_{V^a}$  by Fatou. But, by (ii), weak convergence in  $V^a$  implies strong convergence in  $L^2(\mathbb{R}^+)$ . Hence  $x_{n_k} \rightarrow^{L^2} y$ . Together with  $x_n \rightarrow^{L^2} x$  this implies  $y = x$  and hence the claim follows.

Incidentally, note from (5.4-5) that  $V^a$  does not depend on  $a$  because it is nothing other than the collection of  $x \in L^2(\mathbb{R}^+)$  for which  $\int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\} du < \infty$  (recall (0.18)).  $\square$

Lemma 17 implies that  $\hat{F}^a$  achieves a minimum in  $\overline{K_C^a}$  (for  $C$  large enough).

## 5.2 Characterization of the minimizer(s) of $\hat{F}^a$

**Lemma 18** *Any minimizer  $\bar{x}$  of  $\hat{F}^a$  in  $X$  is a solution of  $\hat{\mathcal{L}}^a x = \rho x$  for  $\rho = \hat{\rho}(a) \in \mathbb{R}$ , the minimal eigenvalue of  $\hat{\mathcal{L}}^a$  in  $V^a$ .*

**Proof.** Standard.

Define  $\hat{\rho}(a)$  by

$$\hat{\rho}(a) = \min_{x \in X} \hat{F}^a(x). \quad (5.15)$$

Let  $\bar{x} \in V^a$  be any minimizer. Then for any  $h \in L^2(\mathbb{R}^+)$  and  $\epsilon > 0$

$$\hat{F}^a(\bar{x} + \epsilon h) \geq \hat{\rho}(a) \|\bar{x} + \epsilon h\|_{L^2}^2. \quad (5.16)$$

Writing out both sides of (5.16) and using that  $\hat{F}^a(\bar{x}) = \hat{\rho}(a)$ , we obtain (see (5.13-14))

$$2\epsilon \langle \bar{x}, h \rangle_{V^a} + \epsilon^2 \|h\|_{V^a}^2 \geq \hat{\rho}(a) \{2\epsilon \langle \bar{x}, h \rangle_{L^2} + \epsilon^2 \|h\|_{L^2}^2\}. \quad (5.17)$$

Let  $\epsilon \downarrow 0$  to obtain

$$\langle \bar{x}, h \rangle_{V^a} \geq \hat{\rho}(a) \langle \bar{x}, h \rangle_{L^2} \text{ for all } h \in V^a. \quad (5.18)$$

Replace  $h$  by  $\Leftrightarrow h$  to get the reverse inequality. Thus

$$\langle \bar{x}, h \rangle_{V^a} = \hat{\rho}(a) \langle \bar{x}, h \rangle_{L^2} \text{ for all } h \in V^a. \quad (5.19)$$

Now note that we have from (5.6) and (5.13) after partial integration

$$\langle \bar{x}, h \rangle_{V^a} = \langle \bar{x}, \hat{\mathcal{L}}^a h \rangle_{L^2} \text{ for all } h \in C_c^2(\mathbb{R}^+) \quad (5.20)$$

It follows from (5.19-20) and the symmetry of  $\hat{\mathcal{L}}^a$  that  $\bar{x}$  is a weak solution of  $\hat{\mathcal{L}}^a x = \hat{\rho}(a)x$ . This in turn implies that  $\bar{x}$  is a strong solution.

To see that  $\hat{\rho}(a)$  is the minimal eigenvalue of  $\hat{\mathcal{L}}^a$  in  $V^a$ , note that if  $\hat{\mathcal{L}}^a x = \rho x$ , then by (5.6), (5.13-14) and integration by parts

$$\hat{F}^a(x) = \langle x, x \rangle_{V^a} = \langle x, \hat{\mathcal{L}}^a x \rangle_{L^2} = \rho \|x\|_{L^2}^2 = \rho. \quad (5.21)$$

$\square$

### 5.3 Analysis of the Sturm-Liouville problem

Lemmas 17-18 show that  $F^a$  has a maximizer in  $\overline{K_C^a}$  and that each maximizer is a solution of  $\mathcal{L}^a x = \rho x$  for  $\rho = \rho(a)$ , the maximal eigenvalue of  $\mathcal{L}^a$  in  $V^a$  (recall (5.4-7)).

**Lemma 19** (i) *All solutions of  $\mathcal{L}^a x = \rho x$  are of the form*

$$x^{a,\rho}(u) = f^{a,\rho}(u) + g^{a,\rho}(u) \log u, \quad (5.22)$$

where  $f^{a,\rho}$  and  $g^{a,\rho}$  are power series with infinite radius of convergence.

(ii)  $F^a(x^{a,\rho}) = \Leftrightarrow \infty$  if  $g^{a,\rho} \not\equiv 0$ .

**Proof.** (i) Formally substitute  $f^{a,\rho}(u) = \sum_{n \geq 0} f_n u^n$  and  $g^{a,\rho}(u) = \sum_{n \geq 0} g_n u^n$ . Then the coefficients are found to satisfy the recurrence relations

$$\begin{aligned} g_n &= \frac{1}{n^2}(\rho g_{n-1} \Leftrightarrow 2a g_{n-2} + 4g_{n-3}) \quad (n \geq 1) \\ f_n &= \frac{1}{n^2}(\rho f_{n-1} \Leftrightarrow 2a f_{n-2} + 4f_{n-3} \Leftrightarrow 2n g_n) \quad (n \geq 1) \end{aligned} \quad (5.23)$$

(with  $f_{-1} = f_{-2} = g_{-1} = g_{-2} = 0$ ). Note that  $g^{a,\rho}$  is a solution of (5.3) and that  $f^{a,\rho}$  depends on  $g^{a,\rho}$ . By induction on  $n$ , (5.23) is easily shown to give the following bounds:

$$\begin{aligned} |f_n| &\leq K_1^n (n!)^{-\frac{2}{3}} \quad (n \geq 1) \\ |g_n| &\leq K_2^n (n!)^{-\frac{2}{3}} \quad (n \geq 1) \end{aligned} \quad (5.24)$$

with  $K_1, K_2$  large enough (depending on  $\rho, a$  and  $f_0, g_0$ ). This implies that the formal solution exists everywhere.

(ii) Trivial, since  $\frac{d}{du} x^{a,\rho}(u) \sim g_0 u^{-1}$  ( $u \downarrow 0$ ) with  $g_0 \neq 0$  implies  $F^a(x^{a,\rho}) = \Leftrightarrow \infty$ , while  $g_0 = 0$  implies that  $g_n \equiv 0$ .  $\square$

At this stage we know from Lemma 19 that all maximizers of  $F^a$  are of the form  $x^{a,\rho}(u) = f^{a,\rho}(u)$  and, in particular, are analytic on  $\mathbb{R}_0^+$ .

Our next step is to find the asymptotic behavior of the solutions of (5.3) as  $u \rightarrow \infty$ . This will be needed to get uniqueness of the maximizer.

**Lemma 20**  $\mathcal{L}^a x = \rho x$  has two independent solutions  $x_-^{a,\rho}$  and  $x_+^{a,\rho}$  satisfying

$$\lim_{u \rightarrow \infty} u^{-\frac{3}{2}} \log x_{\pm}^{a,\rho}(u) = \pm \frac{4}{3}. \quad (5.25)$$

**Proof.** We use Coddington and Levinson (1955) Theorem 2.1 page 143-144. Define

$$\begin{aligned} w_1(u) &= x(u^2) \\ w_2(u) &= u^{-2} w_1'(u). \end{aligned} \quad (5.26)$$

Then (5.3) can be written as

$$w'(u) = u^{-r} B(u)w(u), \quad (5.27)$$

where  $r = 2$  and

$$\begin{aligned} w(u) &= \begin{pmatrix} w_1(u) \\ w_2(u) \end{pmatrix} \\ B(u) &= \begin{pmatrix} 0 & 1 \\ 16 \Leftrightarrow \frac{8a}{u^2} + \frac{4\rho}{u^4} & \Leftrightarrow \frac{3}{u^3} \end{pmatrix}. \end{aligned} \quad (5.28)$$



Note that  $B(u) = \sum_{n \geq 0} u^{-n} B_n$  ( $B_0 \neq 0$ ) is a convergent power series in  $u^{-1}$ , with  $B_0$  having eigenvalues  $\lambda_{1,2} = \pm 4$ . Therefore (5.27) has a formal solution of the form

$$w(u) = P(u)u^R e^{Q(u)}, \quad (5.29)$$

where  $P(u) = \sum_{n=0}^{\infty} u^{-n} P_n$  ( $\det P_0 \neq 0$ ) is a formal power series in  $u^{-1}$ ,  $R$  is a complex diagonal matrix and  $Q = \frac{u^{r+1}}{r+1} Q_0 + \dots + u Q_r$  is a matrix polynomial with  $Q_i$  diagonal and  $Q_0 = \text{diag}\{\lambda_1, \lambda_2\}$ . From the proof of the theorem it follows that  $P, Q, R$  can be chosen to be real because  $B, \lambda_{1,2}$  are real. On p.151 of Coddington and Levinson (1955) there is the further remark that for every formal solution there exists an actual solution with the same asymptotics.  $\square$

We see from Lemma 20 that  $x_+^{a,\rho} \notin L^2(\mathbb{R}^+)$  and so (5.3) has a unique solution in  $L^2(\mathbb{R}^+)$  up to multiplicative constants.

**Lemma 21** *Define*

$$\mathcal{S}_a = \{\rho \in \mathbb{R} : f^{a,\rho} \in L^2(\mathbb{R}^+), f^{a,\rho}(0) = 1\}. \quad (5.30)$$

*Then*

- (i)  $\mathcal{S}_a$  is countable, bounded from above and has a maximum
- (ii)  $\rho(a) = \max \mathcal{S}_a$  is geometrically simple
- (iii)  $f^{a,\rho(a)} > 0$
- (iv)  $\forall \rho \in \mathcal{S}_a, \rho < \max \mathcal{S}_a : f^{a,\rho}$  changes sign in  $\mathbb{R}^+$ .

**Proof.** Standard Sturm-Liouville theory.

(i),(ii) By Lemma 17(ii),  $V^a$  is compactly imbedded in  $L^2(\mathbb{R}^+)$  (compare (5.7) and (5.12)). Therefore the eigenfunctions of  $\mathcal{L}^a$  in  $V^a$  form an orthogonal basis of  $V^a$ . Since  $V^a$  is separable, this in turns implies that  $\mathcal{S}_a$  is countable. We know from Lemmas 19-20 that  $\mathcal{L}^a$  has a unique eigenvector in  $V^a$  with eigenvalue  $\rho(a)$ , i.e.,  $\rho(a)$  is geometrically simple. Since  $\rho(a) = \max_{x \in V^a} F^a(x) = \max \mathcal{S}^a$  by Lemma 18, we also know that  $\mathcal{S}_a$  is bounded from above and has a maximum.

(iii) From (1.7) one sees that  $F^a(|f^{a,\rho(a)}|) = F^a(f^{a,\rho(a)})$ . Therefore it follows from the uniqueness of the maximizer that  $f^{a,\rho} = |f^{a,\rho}| \geq 0$ . Let  $u_0 = \inf\{u > 0 : f^{a,\rho(a)}(u) = 0\} > 0$ . If  $u_0 < \infty$ , then we must have  $\frac{d}{du} f^{a,\rho(a)}(u_0) = 0$  and  $\frac{d^2}{du^2} f^{a,\rho(a)}(u_0) > 0$ . However, this contradicts  $(\mathcal{L}^a f^{a,\rho(a)})(u) = \rho(a) f^{a,\rho(a)}(u)$  at the point  $u = u_0$  (see (0.17)).

(iv) This follows from (iii) and the fact that the eigenfunctions of  $\mathcal{L}^a$  in  $V^a$  form an orthogonal basis.  $\square$

Lemmas 17-18 and 21 show that Assumptions 3(i), (ii) in proposition 2 hold.

## 5.4 Dependence on $a$

The maximal eigenvalue and eigenvector of (0.17-18) are

$$\begin{aligned} \rho(a) &= \max \mathcal{S}_a \\ x^a &= \frac{f^{a,\rho(a)}}{\|f^{a,\rho(a)}\|_{L^2}}. \end{aligned} \quad (5.31)$$

We can now prove the following properties:

**Lemma 22** (i)  $a \rightarrow \rho(a)$  and  $a \rightarrow x^a$  are analytic  
(ii)  $a \rightarrow \rho(a)$  is strictly increasing and strictly convex on  $\mathbb{R}$   
(iii)  $\rho(0) < 0$ ,  $\lim_{a \uparrow \infty} \rho(a) = \infty$  and  $\lim_{a \downarrow -\infty} \rho(a) = \Leftrightarrow \infty$ .

**Proof.** (i) We give the proof by applying Crandall and Rabinowitz (1973) Lemma 1.3 in the following setting. Pick  $a \in \mathbb{R}$  and consider the Hilbert space  $(V, \langle \cdot, \cdot \rangle_V)$  with  $V = V^0$ . Then, from (5.5-6) and (5.13),

$$\begin{aligned} \langle x^a, y \rangle_{V^a} &= \langle \mathcal{L}^a x^a, y \rangle_{L^2} = \rho(a) \langle x^a, y \rangle_{L^2} \\ \langle x^a, y \rangle_{V^a} &= \langle x^a, y \rangle_V \Leftrightarrow 2ab \langle x^a, y \rangle + \frac{a^2}{4} \langle x^a, y \rangle_{L^2}, \end{aligned} \tag{5.32}$$

where  $b : V \times V \rightarrow \mathbb{R}$  is the bilinear form defined by

$$b(x, y) = \int_0^\infty ux(u)y(u)du. \tag{5.33}$$

For every  $x \in V$  the functional  $y \rightarrow b(x, y)$  is continuous and linear. Hence it follows from the Riesz representation theorem (Rudin (1987) Theorem 6.19) that there exists a unique linear operator  $B : V \rightarrow V$  such that

$$b(x, y) = \langle Bx, y \rangle_V \text{ for all } x, y \in V. \tag{5.34}$$

$B$  is symmetric because  $b$  is.  $B$  is bounded because

$$\begin{aligned} \|Bx\|_V^2 &= b(x, Bx) \\ &\leq \left( \int_0^\infty u^2 x^2(u)du \right)^{\frac{1}{2}} \|Bx\|_{L^2} \\ &\leq \frac{1}{2} \|x\|_V \|Bx\|_{L^2} \\ &\leq \frac{1}{2} \|x\|_V \|Bx\|_V \end{aligned} \tag{5.35}$$

(see (5.5) and (5.13)), so that  $\|Bx\|_V \leq \frac{1}{2} \|x\|_V$ . To see that  $B$  is compact, let  $(x_n)$  be a bounded sequence in  $V$ . Then, by Lemma 17(ii), there exists a subsequence  $(x_{n_k})$  and an  $x \in V$  such that  $x_{n_k} \xrightarrow{L^2} x$  ( $k \rightarrow \infty$ ). Hence, as in (5.35),

$$\begin{aligned} \|Bx_{n_k} \Leftrightarrow Bx\|_V^2 &= b(x_{n_k} \Leftrightarrow x, B(x_{n_k} \Leftrightarrow x)) \\ &\leq \|x_{n_k} \Leftrightarrow x\|_{L^2} \frac{1}{2} \|B(x_{n_k} \Leftrightarrow x)\|_V \\ &\leq \|x_{n_k} \Leftrightarrow x\|_{L^2} \frac{1}{4} \|x_{n_k} \Leftrightarrow x\|_V \\ &\rightarrow 0 \text{ (} k \rightarrow \infty \text{)}. \end{aligned} \tag{5.36}$$

In the same manner we can prove that there exists a unique linear, symmetric and compact operator  $C : V \rightarrow V$  such that

$$\langle x, y \rangle_{L^2} = \langle Cx, y \rangle_V \text{ for all } x, y \in V. \tag{5.37}$$

Now rewrite (5.32) as follows

$$\langle [Id \Leftrightarrow 2aB \Leftrightarrow (\rho(a) \Leftrightarrow \frac{a^2}{4})C]x^a, y \rangle_V = 0 \text{ for all } y \in V. \quad (5.38)$$

Hence,  $(V, \langle \cdot, \cdot \rangle_V)$  being a Hilbert space, we have

$$\begin{aligned} x^a \text{ is a } C\text{-eigenfunction of } Id \Leftrightarrow 2aB \\ \text{with (largest) eigenvalue } \rho(a) \Leftrightarrow \frac{a^2}{4}. \end{aligned} \quad (5.39)$$

Next note that  $a \rightarrow Id \Leftrightarrow 2aB$  is analytic in the operator norm. Therefore, to get the claim from Crandall and Rabinowitz (1973) Lemma 1.3, it suffices to check that  $\rho(a) \Leftrightarrow \frac{a^2}{4}$  is a  $C$ -simple eigenvalue of  $Id \Leftrightarrow 2aB$ , i.e.,

- (a)  $\dim(N(A^a)) = \text{codim}(R(A^a)) = 1$
- (b)  $Cx^a \notin R(A^a)$ ,

where  $A^a = Id \Leftrightarrow 2aB \Leftrightarrow (\rho(a) \Leftrightarrow \frac{a^2}{4})C$  and  $N(A^a), R(A^a)$  denote the null space resp. range of  $A^a$ .

We have  $\dim(N(A^a)) = 1$  because of Lemma 21(ii). Moreover, because  $2aB + (\rho(a) \Leftrightarrow \frac{a^2}{4})C$  is compact we have  $\dim(N(A^a)) = \text{codim}(R(A^a))$  (Rudin (1991) Theorem 4.25). This proves (a). To prove (b), first use that  $A^a$  is symmetric and bounded to get that  $N(A^a) = R(A^a)^\perp$  (the orthogonal complement of  $R(A^a)$ ) and  $R(A^a) = \overline{R(A^a)}$  (Rudin (1991) Theorems 4.12 and 4.23). Since  $\overline{R(A^a)} = R(A^a)^{\perp\perp}$ , it follows that  $N(A^a)^\perp = R(A^a)$ . Hence (b) is equivalent to  $\langle Cx^a, x^a \rangle_V \neq 0$ . But  $\langle Cx^a, x^a \rangle_V = \langle x^a, x^a \rangle_{L^2} = 1$  by (5.37).

(ii) Because

$$\rho(a) = \sup_{x \in X} F^a(x) \quad (5.40)$$

with unique maximizer  $x = f^{a, \rho(a)}$ , we immediately see from (1.7) that

$$\frac{\rho(a + \epsilon) \Leftrightarrow \rho(a)}{\epsilon} \geq \int_0^\infty 2u [f^{a, \rho(a)}(u)]^2 du > 0 \quad (5.41)$$

(pick  $\|f^{a, \rho(a)}\|_{L^2} = 1$ ). This demonstrates that  $\rho'(a)$  is everywhere strictly positive. Moreover, since  $a \rightarrow F^a(x)$  is linear for every  $x$  we have from (5.40) that  $a \rightarrow \rho(a)$  is convex. Because of analyticity, it follows that either  $a \rightarrow \rho(a)$  is strictly convex or  $\rho(a) = C_1 a + C_2$ . However, the latter is impossible because of Lemma 13.

(iii) Trivial. Let  $\epsilon \rightarrow \pm\infty$  in (5.41) or else see (1.7). □

## 6 Proof of Theorems 4-7

We can now collect the results from sections 2-5 and give the proofs of our theorems in section 0.4.

**Proof of Theorem 5.** Combine Propositions 1-3 with (1.13). The proof of Proposition 3 was given in Lemma 1 and in sections 2, 3 and 5. □

**Proof of Theorems 4 and 6.**

1.  $r^*(\beta) \sim a^* \beta^{\frac{2}{3}}$ .

According to (0.13),  $r^*(\beta)$  is defined as the unique solution of

$$\lambda(r, \beta) = 1. \quad (6.1)$$

From (0.20) we know that for every  $a \in \mathbb{R}$

$$\beta^{-\frac{1}{3}}[\lambda(a\beta^{\frac{2}{3}}, \beta) \Leftrightarrow 1] \rightarrow \rho(a). \quad (6.2)$$

Let  $a^* > 0$  be the solution of  $\rho(a) = 0$  (see Lemma 22). Now, because  $r \rightarrow \lambda(r, \beta)$  is increasing (as is obvious from (0.10)), we have for every  $\epsilon > 0$

$$\begin{aligned} \lambda(r, \beta) &\geq 1 + \beta^{\frac{1}{3}}\rho(a^* + \epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \geq (a^* + \epsilon)\beta^{\frac{2}{3}} \\ \lambda(r, \beta) &\leq 1 + \beta^{\frac{1}{3}}\rho(a^* \Leftrightarrow \epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \leq (a^* \Leftrightarrow \epsilon)\beta^{\frac{2}{3}}. \end{aligned} \quad (6.3)$$

Since  $\rho(a^* \Leftrightarrow \epsilon) < 0 < \rho(a^* + \epsilon)$  for every  $\epsilon > 0$  (see Lemma 22(ii)), (6.1) combined with (6.3) implies

$$(a^* \Leftrightarrow \epsilon)\beta^{\frac{2}{3}} \leq r^*(\beta) \leq (a^* + \epsilon)\beta^{\frac{2}{3}} \text{ for } \beta \text{ small enough.} \quad (6.4)$$

Let  $\epsilon \downarrow 0$  to get the claim.

**2.**  $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ .

According to (0.14),  $\theta^*(\beta)$  is defined as

$$\frac{1}{\theta^*(\beta)} = \left[ \frac{\partial}{\partial r} \lambda(r, \beta) \right]_{r=r^*(\beta)}. \quad (6.5)$$

Define

$$\xi(r, \beta) = \frac{\frac{\partial}{\partial r} \lambda(r, \beta)}{\lambda(r, \beta)} = \frac{\partial}{\partial r} \log \lambda(r, \beta). \quad (6.6)$$

Because  $r \rightarrow \lambda(r, \beta)$  is increasing and log-convex (see footnote 6), we have that for all  $h, \beta > 0$  and  $a \in \mathbb{R}$

$$\begin{aligned} \xi(a\beta^{\frac{2}{3}}, \beta) &\leq \frac{1}{h\beta^{\frac{2}{3}}} \left[ \log \lambda((a+h)\beta^{\frac{2}{3}}, \beta) \Leftrightarrow \log \lambda(a\beta^{\frac{2}{3}}, \beta) \right] \\ \xi(a\beta^{\frac{2}{3}}, \beta) &\geq \frac{1}{h\beta^{\frac{2}{3}}} \left[ \log \lambda(a\beta^{\frac{2}{3}}, \beta) \Leftrightarrow \log \lambda((a \Leftrightarrow h)\beta^{\frac{2}{3}}, \beta) \right]. \end{aligned} \quad (6.7)$$

Together with (6.2) this gives

$$\begin{aligned} \limsup_{\beta \downarrow 0} \beta^{\frac{1}{3}} \xi(a\beta^{\frac{2}{3}}, \beta) &\leq \frac{\rho(a+h) \Leftrightarrow \rho(a)}{h} \\ \liminf_{\beta \downarrow 0} \beta^{\frac{1}{3}} \xi(a\beta^{\frac{2}{3}}, \beta) &\geq \frac{\rho(a) \Leftrightarrow \rho(a \Leftrightarrow h)}{h}. \end{aligned} \quad (6.8)$$

Let  $h \downarrow 0$  to get (use Lemma 22)

$$\lim_{\beta \downarrow 0} \beta^{\frac{1}{3}} \xi(a\beta^{\frac{2}{3}}, \beta) = \rho'(a). \quad (6.9)$$

Next, because  $r \rightarrow \xi(r, \beta)$  is increasing we have, via (6.4), for  $\beta$  small enough

$$\begin{aligned} \xi(r^*(\beta), \beta) &\leq \xi((a^* + \epsilon)\beta^{\frac{2}{3}}, \beta) = \beta^{-\frac{1}{3}}\rho'(a^* + \epsilon) + o(\beta^{-\frac{1}{3}}) \\ \xi(r^*(\beta), \beta) &\geq \xi((a^* \Leftrightarrow \epsilon)\beta^{\frac{2}{3}}, \beta) = \beta^{-\frac{1}{3}}\rho'(a^* \Leftrightarrow \epsilon) + o(\beta^{-\frac{1}{3}}). \end{aligned} \quad (6.10)$$

Since (recall that  $\lambda(r^*(\beta), \beta) = 1$ )

$$\frac{1}{\theta^*(\beta)} = \xi(r^*(\beta), \beta), \quad (6.11)$$

it follows that

$$\rho'(a^* \Leftrightarrow \epsilon) \leq \frac{1}{\beta^{-\frac{1}{3}}\theta^*(\beta)} \leq \rho'(a^* + \epsilon) \text{ for } \beta \text{ small enough.} \quad (6.12)$$

Let  $\epsilon \downarrow 0$  to get the claim with  $\frac{1}{b^*} = \rho'(a)$ .

**3.**  $\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \rightarrow^{L^2} x^{a^*}(\cdot)$ .

Put  $a^*(\beta) = \beta^{-\frac{2}{3}}r^*(\beta)$ . Then, similarly as in Lemma 1,

$$\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \text{ is the unique maximizer of } F_\beta^{a^*(\beta)}, \quad (6.13)$$

where the parameter  $a$  is replaced by  $a^*(\beta)$ .

**Lemma 23** *Assumptions (1)-(3) in Proposition 2 hold for the following choice replacing (1.13):*

$$\begin{aligned} K &= K_C^{a^*} \text{ (} C \text{ sufficiently large)} \\ G_\beta &= F_\beta^{a^*(\beta)} \\ G &= F^{a^*}. \end{aligned} \quad (6.14)$$

**Proof.** The point is that  $\lim_{\beta \downarrow 0} a^*(\beta) = a^*$ . It is trivial to check that all estimates in sections 2 and 3 remain valid when the fixed parameter  $a$  is replaced by  $a + o(1)$  ( $\beta \downarrow 0$ ). See, in particular, the proofs of Lemmas 5, 6, 11-13.  $\square$

The claim in **3** now follows from Proposition 2.

**4.**  $\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \rightarrow^{L^1} \frac{1}{2}[x^{a^*}(\frac{1}{2}\cdot)]^2$ .

The proof is in Steps 1-2 below.

Abbreviate  $A_\beta = A_{r^*(\beta),\beta}$  and  $\tau_\beta = \tau_{r^*(\beta),\beta}$ . According to (0.14)

$$\mu_\beta^*(k) = \sum_{i+j-1=k} \tau_\beta(i)A_\beta(i,j)\tau_\beta(j). \quad (6.15)$$

**STEP 1** *There exists  $c$  such that*

$$\int_N^\infty |\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil u\beta^{-\frac{1}{3}} \rceil) \Leftrightarrow \frac{1}{2}[x^{a^*}(\frac{1}{2}u)]^2| du \leq cN^{-2} \text{ for } \beta \text{ small enough.} \quad (6.16)$$

**Proof.** Estimate (recall that  $\lambda(r^*(\beta), \beta) = 1$ )

$$\begin{aligned}
\int_N^\infty \beta^{-\frac{1}{3}} \mu_\beta^*([u\beta^{-\frac{1}{3}}]) du &= \sum_{k \geq N\beta^{-\frac{1}{3}}} \mu_\beta^*(k) \\
&= \sum_{i,j: i+j-1 \geq N\beta^{-\frac{1}{3}}} \tau_\beta(i) A_\beta(i, j) \tau_\beta(j) \\
&\leq 2 \sum_{i \geq \frac{1}{2}N\beta^{-\frac{1}{3}}} \tau_\beta^2(i) \\
&\leq 8N^{-2} \beta^{\frac{2}{3}} \sum_{i \geq 1} i^2 \tau_\beta^2(i) \\
&\leq 8C_1 N^{-2}.
\end{aligned} \tag{6.17}$$

The last inequality is Lemma 11(i). Furthermore,

$$\int_N^\infty \frac{1}{2} [x^{a^*}(\frac{1}{2}u)]^2 du \leq 4N^{-2} \int_N^\infty u^2 [x^{a^*}(\frac{1}{2}u)]^2 du. \tag{6.18}$$

Since  $x^{a^*} \in K_C^{a^*}$ , the integral in the r.h.s. is finite and so the claim follows.  $\square$

**STEP 2**  $\lim_{\beta \downarrow 0} \int_0^N |\beta^{-\frac{1}{3}} \mu_\beta^*([u\beta^{-\frac{1}{3}}]) \Leftrightarrow \frac{1}{2} [x^{a^*}(\frac{1}{2}u)]^2| du = 0$  for every fixed  $N$ .

**Proof.** Use the triangle inequality to split the integral into three parts:

$$\int_0^N |\beta^{-\frac{1}{3}} \mu_\beta^*([u\beta^{-\frac{1}{3}}]) \Leftrightarrow \frac{1}{2} [x^{a^*}(\frac{1}{2}u)]^2| du \leq I_\beta^{1,N} + I_\beta^{2,N} + I_\beta^{3,N} \tag{6.19}$$

with

$$\begin{aligned}
I_\beta^{1,N} &= \beta^{\frac{1}{3}} \sum_{i,j: i+j-1 \leq N\beta^{-\frac{1}{3}}} |\bar{\tau}_\beta(i\beta^{\frac{1}{3}}) \Leftrightarrow x^{a^*}(i\beta^{\frac{1}{3}})| A_\beta(i, j) x^{a^*}(j\beta^{\frac{1}{3}}) \\
I_\beta^{2,N} &= \beta^{\frac{1}{3}} \sum_{i,j: i+j-1 \leq N\beta^{-\frac{1}{3}}} \bar{\tau}_\beta(i\beta^{\frac{1}{3}}) A_\beta(i, j) |\bar{\tau}_\beta(j\beta^{\frac{1}{3}}) \Leftrightarrow x^{a^*}(j\beta^{\frac{1}{3}})| \\
I_\beta^{3,N} &= \int_0^N |\sum_{i,j: i+j-1 = [u\beta^{-\frac{1}{3}}]} x^{a^*}(i\beta^{\frac{1}{3}}) A_\beta(i, j) x^{a^*}(j\beta^{\frac{1}{3}}) \Leftrightarrow \frac{1}{2} [x^{a^*}(\frac{1}{2}u)]^2| du.
\end{aligned} \tag{6.20}$$

Here  $\bar{\tau}_\beta$  is the scaled form of  $\tau_\beta$  given by the same relation as (3.1).

For  $I_\beta^{1,N}$  use Cauchy-Schwarz and (2.5)(ii) to estimate

$$\begin{aligned}
I_\beta^{1,N} &\leq \beta^{\frac{1}{3}} \left( \sum_{i,j: i+j-1 \leq N\beta^{-\frac{1}{3}}} [\bar{\tau}_\beta(i\beta^{\frac{1}{3}}) \Leftrightarrow x^{a^*}(i\beta^{\frac{1}{3}})]^2 A_\beta(i, j) \right)^{\frac{1}{2}} \\
&\quad \times \left( \sum_{i,j: i+j-1 \leq N\beta^{-\frac{1}{3}}} [x^{a^*}(j\beta^{\frac{1}{3}})]^2 A_\beta(i, j) \right)^{\frac{1}{2}} \\
&\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \left( \beta^{\frac{1}{3}} \sum_{i \leq N\beta^{-\frac{1}{3}}} [\bar{\tau}_\beta(i\beta^{\frac{1}{3}}) \Leftrightarrow x^{a^*}(i\beta^{\frac{1}{3}})]^2 \right)^{\frac{1}{2}} \\
&\quad \times \left( \beta^{\frac{1}{3}} \sum_{j \leq N\beta^{-\frac{1}{3}}} [x^{a^*}(j\beta^{\frac{1}{3}})]^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{6.21}$$

Define  $\bar{x}_\beta^{a^*}$  by  $\bar{x}_\beta^{a^*}(u) = x^{a^*}(i\beta^{\frac{1}{3}})$  for  $(i \Leftrightarrow 1)\beta^{\frac{1}{3}} \leq u \leq i\beta^{\frac{1}{3}}$  ( $i \geq 1$ ), in analogy with (3.1). Then (6.21) becomes

$$I_\beta^{1,N} \leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \|\bar{\tau}_\beta \Leftrightarrow \bar{x}_\beta^{a^*}\|_{L^2[0,N]} \|\bar{x}_\beta^{a^*}\|_{L^2[0,N]}. \tag{6.22}$$

Now let  $\beta \downarrow 0$  and use that  $\bar{\tau}_\beta \rightarrow^{L^2} x^{a^*}$  and  $\bar{x}_\beta^{a^*} \rightarrow^{L^2} x^{a^*}$  to get  $\limsup_{\beta \downarrow 0} I_\beta^{1,N} = 0$ . The same argument gives that  $\limsup_{\beta \downarrow 0} I_\beta^{2,N} = 0$ .

To estimate  $I_\beta^{3,N}$ , we use the mean value theorem to expand  $x^{a^*}(i\beta^{\frac{1}{3}})$  and  $x^{a^*}(j\beta^{\frac{1}{3}})$  around  $\frac{1}{2}u$ . Namely

$$\begin{aligned} I_\beta^{3,N} &= \int_0^N \left| \sum_{i,j: i+j-1=\lceil u\beta^{-\frac{1}{3}} \rceil} \left\{ x^{a^*}\left(\frac{1}{2}u\right) + (i\beta^{\frac{1}{3}} \Leftrightarrow \frac{1}{2}u) \frac{d}{du} x^{a^*}(\xi) \right\} \right. \\ &\quad \left. \times A_\beta(i,j) \left\{ x^{a^*}\left(\frac{1}{2}u\right) + (j\beta^{\frac{1}{3}} \Leftrightarrow \frac{1}{2}u) \frac{d}{du} x^{a^*}(\eta) \right\} \Leftrightarrow \frac{1}{2} [x^{a^*}\left(\frac{1}{2}u\right)]^2 \right| du \end{aligned} \quad (6.23)$$

with  $\xi, \eta$  between  $\frac{1}{2}u$  and  $i\beta^{\frac{1}{3}}$  resp.  $j\beta^{\frac{1}{3}}$ . Next note that  $x^{a^*}(u), \left| \frac{d}{du} x^{a^*}(u) \right| \leq M < \infty$  for all  $u \in \mathbb{R}^+$ . Hence

$$\begin{aligned} I_\beta^{3,N} &\leq M^2 \int_0^N \left| \sum_{i,j: i+j-1=\lceil u\beta^{-\frac{1}{3}} \rceil} A_\beta(i,j) \Leftrightarrow \frac{1}{2} \right| du \\ &\quad + 2M^2 \int_0^N \sum_{i,j: i+j-1=\lceil u\beta^{-\frac{1}{3}} \rceil} \left\{ |i\beta^{\frac{1}{3}} \Leftrightarrow \frac{1}{2}u| + |j\beta^{\frac{1}{3}} \Leftrightarrow \frac{1}{2}u| \right\} A_\beta(i,j). \end{aligned} \quad (6.24)$$

Next we insert  $A_\beta = e^{\epsilon\beta} P$  and use that (recall (2.4) and (0.11))

$$|e_\beta(i,j)| \leq (|a| + N) N \beta^{\frac{1}{3}} \text{ for } i, j \leq N \beta^{-\frac{1}{3}} \quad (6.25)$$

$$\sum_{\substack{i,j \\ i+j-1=k}} P(i,j) = \frac{1}{2} (k \geq 1) \quad (6.26)$$

$$\sum_{\substack{i,j \\ i+j-1=k}} \left( i \Leftrightarrow \frac{1}{2}k \right)^2 P(i,j) = \frac{1}{2} \left( \frac{1}{4}k + 1 \right) (k \geq 1) \quad (6.27)$$

Then (6.24) yields

$$I_\beta^{3,N} \leq M^2 N \frac{1}{2} (e^{(|a|+N)N\beta^{\frac{1}{3}}} \Leftrightarrow 1) + 2M^2 e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \int_0^N du \left\{ (z_\beta(u))^{\frac{1}{2}} + z_\beta(u) \right\} \quad (6.28)$$

where

$$z_\beta(u) = \beta^{\frac{2}{3}} \sum_{i,j: i+j-1=\lceil u\beta^{-\frac{1}{3}} \rceil} \left( i \Leftrightarrow \frac{1}{2}u\beta^{-\frac{1}{3}} \right)^2 P(i,j) = \mathcal{O}(\beta^{\frac{1}{3}}). \quad (6.29)$$

Let  $\beta \downarrow 0$  to get  $\limsup_{\beta \downarrow 0} I_\beta^{3,N} = 0$ . □

Steps 1-2 prove the claim in 4.

Results 1-4 complete the proof of Theorems 4 and 6. □

**Proof of Theorem 7.** The asymptotic behavior of  $x^{a^*}$  in (iii) was proved in Lemma 20 (pick  $a = a^*$  and  $\rho = 0$ ). To prove (i) and (ii), we recall that  $x^{a^*}$  solves (see (0.17))

$$0 = (\mathcal{L}^{a^*} x)(u) = (2a^*u \Leftrightarrow 4u^2)x(u) + [ux']'(u) \quad (6.30)$$

and has a power series representation (see (5.23))

$$\begin{aligned} x^{a^*}(u) &= \sum_{n \geq 0} x_n u^n \\ x_n &= \frac{1}{n^2} (\Leftrightarrow 2a^*x_{n-2} + 4x_{n-3}) \quad (n \geq 1) \\ x_{-1} &= x_{-2} = 0. \end{aligned} \quad (6.31)$$

We observe that  $u \rightarrow 2a^*u \Leftrightarrow 4u^2$  changes sign from positive to negative at  $u = \frac{1}{2}a^*$ . Since  $x^{a^*}(u) > 0$  for all  $u \geq 0$ , it follows from (6.30) that  $u \rightarrow u \frac{d}{du} x^{a^*}(u)$  is unimodal with a minimum at  $u = \frac{1}{2}a^*$ . It is clear that  $u \frac{d}{du} x^{a^*}(u) \rightarrow 0$  as  $u \downarrow 0$ . By the unimodality we must have that  $u \frac{d}{du} x^{a^*}(u) \rightarrow c$  as  $u \rightarrow \infty$ . However,  $c$  must be 0 otherwise  $\int_0^\infty u [\frac{d}{du} x^{a^*}(u)]^2 du = \infty$ , which is impossible since  $F^{a^*}(x^{a^*}) = \rho(a^*) = 0 > \Leftrightarrow \infty$  (see (1.7)). Thus we conclude that  $u \frac{d}{du} x^{a^*}(u) < 0$  for all  $u > 0$ , which implies that  $u \rightarrow x^{a^*}(u)$  is strictly decreasing.

To prove (iv), use (0.15) to write

$$\begin{aligned} \frac{\beta^{\frac{1}{3}}}{\theta^*(\beta)} &= \beta^{\frac{1}{3}} \sum_i (2i \Leftrightarrow 1) \tau_{r^*(\beta), \beta}^2(i) \\ &= \int_0^\infty 2u \{ \beta^{-\frac{1}{6}} \tau_{r^*(\beta), \beta}(\lceil u \beta^{-\frac{1}{3}} \rceil) \}^2 du. \end{aligned} \quad (6.32)$$

As  $\beta \downarrow 0$  the l.h.s. tends to  $\frac{1}{b^*}$ . Thus we must show that the r.h.s. tends to  $\int_0^\infty 2u [x^{a^*}(u)]^2 du$ . To prove this claim, first note that

$$\begin{aligned} \int_N^\infty 2u \{ \beta^{-\frac{1}{6}} \tau_{r^*(\beta), \beta}(\lceil u \beta^{-\frac{1}{3}} \rceil) \}^2 du &= \beta^{\frac{1}{3}} \sum_{i \geq N \beta^{-\frac{1}{3}}} (2i \Leftrightarrow 1) \tau_{r^*(\beta), \beta}^2(i) \\ &\leq \frac{2}{N} \beta^{\frac{2}{3}} \sum_{i \geq N \beta^{-\frac{1}{3}}} i^2 \tau_{r^*(\beta), \beta}^2(i) \\ &\leq \frac{2}{N} C_1 \text{ for } \beta \text{ sufficiently small} \end{aligned} \quad (6.33)$$

where we use Lemma 11(i). Similarly  $\int_N^\infty 2u [x^{a^*}(u)]^2 du \leq \frac{2}{N} \int_N^\infty u^2 [x^{a^*}(u)]^2 du = o(N^{-1})$  as  $N \rightarrow \infty$ . Next, recall **3** in the proof of Theorems 4 and 6 to see that

$$\lim_{\beta \downarrow 0} \int_0^N 2u \{ \beta^{-\frac{1}{6}} \tau_{r^*(\beta), \beta}(\lceil u \beta^{-\frac{1}{3}} \rceil) \}^2 du = \int_0^N 2u [x^{a^*}(u)]^2 du \text{ for all } N. \quad (6.34)$$

Let  $N \rightarrow \infty$  to get the claim. □

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