# On the approximation by Lüroth Series

Karma Dajani and Cor Kraaikamp

**Abstract**

Let $x \in (0,1]$ and $p_n/q_n$, $n \geq 1$ be its sequence of Lüroth Series convergents. Define the approximation coefficients $\theta_n = \theta_n(x)$ by $\theta_n = q_n x - p_n$, $n \geq 1$. In [BBDK] the limiting distribution of the sequence $(\theta_n)_{n\geq 1}$ was obtained for a.e. $x$ using the natural extension of the ergodic system underlying the Lüroth Series expansion. Here we show that this can be done without the natural extension. We also will get a bound on the speed of convergence. Using the natural extension we will study the distribution for a.e. $x$ of the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n\geq 1}$. It turns out that for a.e. $x$ the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ is distributed according to a continuous singular distribution function $G$. Furthermore we will see that two consecutive $\theta$'s are positively correlated.

AMS classification : 11K55, 28D05

## 1 Introduction

Let $x \in (0,1]$, then

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots , \tag{1}$$

where $a_n \geq 2$, $n \geq 1$. J. Lüroth, who introduced the series expansion (1) in 1883, showed (among other things) that every irrational number $x$ has a unique infinite expansion (1) and that each rational either has a finite or an infinite periodic expansion, see also [L] and [Pe]. The series expansion (1) of $x$ is called the *Lüroth Series* of $x$.

Dynamically the Lüroth series expansion (1) of $x$ is generated by the operator $T : [0,1] \to [0,1]$, defined by

$$Tx := \lfloor \frac{1}{x} \rfloor \left( \lfloor \frac{1}{x} \rfloor + 1 \right) x - \lfloor \frac{1}{x} \rfloor \ , \ x \neq 0; \ T0 := 0, \tag{2}$$

(see also figure 1), where $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding $\xi$. For $x \in [0,1]$ we define $a(x) := \lfloor \frac{1}{x} \rfloor + 1$, $x \neq 0$; $a(0) := \infty$ and $a_n(x) = a(T^{n-1}x)$ for $n \geq 1$. From (2) it follows that $Tx = a_1(a_1-1)x - (a_1-1)$, and therefore

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)} Tx = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{T^n x}{a_1(a_1-1)\cdots a_n(a_n-1)} .$$

Putting

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{a_1(a_1-1)\cdots a_k(a_k-1)a_{k+1}} \ , \ n \geq 1, \tag{3}$$

where $q_1 := a_1$; $q_n = a_1(a_1 \Leftrightarrow 1)\cdots a_{n-1}(a_{n-1} \Leftrightarrow 1)a_n$, $n \geq 2$, it follows from (3) that

$$x \Leftrightarrow \frac{p_n}{q_n} = \frac{T^n x}{q_n(a_n \Leftrightarrow 1)}\ , \quad n \geq 1. \tag{4}$$

¿From $a_n \geq 2$ and $0 \leq T^n x \leq 1$ it follows that the series from (1) converges to $x$. We will write

$$x = < a_1, a_2, \cdots, a_n, \cdots > \quad \text{and} \quad \frac{p_n}{q_n} = < a_1, a_2, \cdots, a_n > \ . \tag{5}$$

In [JdV], H. Jager and C. de Vroedt showed that the stochastic variables $a_1(x), \ldots, a_n(x), \ldots$ are independent[1] with $\lambda_1(a_n = k) = \frac{1}{k(k-1)}$ for $k \geq 2$, and that $T$ is measure preserving and ergodic with respect to Lebesgue measure. From the ergodicity of $T$ and Birkhoff's Individual Ergodic Theorem a number of results were obtained, analogous to classical results on continued fractions, e.g.

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = e^c, \text{ a.e. where } c \approx 1.25\,,$$

$$\lim_{n\to\infty} \log(x \Leftrightarrow \frac{p_n}{q_n}) = \Leftrightarrow d, \text{ a.e., where } d \approx 2.03\,.$$

Here and in the following a.e. will be with respect to Lebesgue measure.

Figure 1

In view of (4) it is natural to define and study the so-called *approximation coefficients* $\theta_n = \theta_n(x)$, $n \geq 1$, defined by

$$\theta_n = \theta_n(x) := q_n \left| x \Leftrightarrow \frac{p_n}{q_n} \right|, \quad n \geq 1.$$

As in the case of the regular continued fraction these $\theta$'s give an indication of "the quality of approximation of $x$ by its $n$-th convergent[2] $p_n/q_n$", see also [JK]. Note that the absolute value signs are in fact superfluous here. In view of (4) one has

$$\theta_n = \frac{T^n x}{a_n \Leftrightarrow 1}\ , \quad n \geq 1. \tag{6}$$

[1] Here and in the following $\lambda_n$ will denote Lebesgue measure on $\mathbf{R}^n$.

[2] In case of the regular continued fraction one defines $\Theta_n := q_n|q_n x - p_n|$, $n \geq 1$, where $p_n/q_n$ is the $n$-th regular convergent of $x$.

Putting $T_n := T^n x$ it follows from (2) and (5) that

$$T_n \ = < \ a_{n+1}, \, a_{n+2}, \cdots \ > \ .$$

We say that $T_n$ is *the future of $x$ at time $n$*. Similarly is

$$V_n \ = < \ a_n, \, a_{n-1}, \cdots \, a_1 \ > = \ \frac{1}{a_n} + \frac{1}{a_n(a_n \Leftrightarrow 1)a_{n-1}} + \cdots + \frac{1}{a_n(a_n \Leftrightarrow 1)\cdots a_2(a_2 \Leftrightarrow 1)a_1}$$

*the past of $x$ at time $n$*. Putting $V_0 := 0$, from (6) one sees that $\theta_n$ is expressed in terms of both the past (viz. $a_n$) and the future. Therefore, in order to obtain the distribution of the sequence $(\theta_n)_{n\geq 1}$ for a.e. $x$ the natural extension of the ergodic system ((0,1], $\mathcal{B}_1$, $\lambda_1$, $T$) (here $\mathcal{B}_1$ is the collection of Borel sets of (0,1]) was constructed in [BBDK].

**Theorem 1** *([BBDK]) Let $\Omega \ := \ [0,1] \times [0,1]$ and $\mathcal{B}_2$ be the collection of Borel sets of $\Omega$. Let $\mathcal{T} \ : \Omega \to \Omega$ be defined by*

$$\mathcal{T}(x,y) \ := \ ( \ Tx \ , \ \frac{1}{a(x)} + \frac{y}{a(x)(a(x) \Leftrightarrow 1)} \ ) \ , \ \ (x,y) \in \Omega,$$

*then the system*

$$([0,1] \times [0,1], \mathcal{B}_2, \, \lambda_2, \, \mathcal{T}_\varepsilon)$$

*is the natural extension of* $([0,1], \mathcal{B}_1, \, \lambda_1, \, T_\varepsilon)$. *Moreover,* $([0,1] \times [0,1], \mathcal{B}_2, \, \lambda_2, \, \mathcal{T}_\varepsilon)$ *is Bernoulli.*

¿From this theorem we have the following lemma.

**Lemma 1** *([BBDK]) For almost all $x$ the two-dimensional sequence*

$$\mathcal{T}^n(x,0) \ = \ (T_n, V_n) \ , \ \ n \geq 1,$$

*is uniformly distributed over $\Omega = [0,1] \times [0,1]$.*

The distribution of the sequence $(\theta_n)_{n\geq 1}$ now follows from lemma 1.

**Theorem 2** *([BBDK]) For almost all $x$ and for every $z \in (0,1]$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{ \, 1 \leq j \leq N \ : \ \theta_j(x) < z \, \}$$

*exists and equals $F(z)$, where*

$$F(x) \ = \ \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \ \frac{z}{k} \ + \ \frac{1}{\lfloor \frac{1}{z} \rfloor + 1} \ , \, 0 < z \leq 1 \, . \tag{7}$$

Taking the first moment, theorem 2 yields that for a.e. $x$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \theta_n \ = \ \frac{\zeta(2) \Leftrightarrow 1}{2} \ = \ 0.322467\cdots \ \ , \tag{8}$$

where $\zeta(s)$ is the zeta-function.

In fact one needs not use the natural extension to study the distribution of the sequence $(\theta_n)_{n\geq 1}$. Since

$$T_n \;=\; \frac{1}{a_{n+1}} + \frac{T_{n+1}}{a_{n+1}(a_{n+1} \Leftrightarrow 1)}\;,$$

see also (2), it follows that

$$a_{n+1}T_n \;=\; 1 + \frac{T_{n+1}}{a_{n+1} \Leftrightarrow 1}$$

and therefore (6) yields that

$$\theta_{n+1} \;=\; a_{n+1}T_n \Leftrightarrow 1\,,\;\; n \geq 1, \tag{9}$$

i.e. the distribution of the sequence $(\theta_n)_{n\geq 1}$ can be obtained from $([0,1], \mathcal{B}, \lambda, T_\varepsilon)$.

In general the ergodic theorem does not yield any information on the speed of convergence, see also [P], section 3.2, where examples are given to show that convergence can be arbitrarily slow. Here we are however in the situation that we can apply theorem 5.8 from [JdV], which yields the following much stronger result.

**Theorem 3** *For almost all $x$ and for every $z \in (0,1]$ one has for every $\varepsilon > 0$*

$$\frac{1}{N}\#\{\,1 \leq j \leq N\;:\;\theta_j(x) < z\,\} \Leftrightarrow F(z) \;=\; o(N^{-\frac{1}{2}}\log^{\frac{3+\varepsilon}{2}} N),\;\; n \to \infty.$$

In this paper we will study the distribution for a.e. $x$ of the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n\geq 1}$. We will show that two consecutive $\theta$'s are positively correlated.

## 2 On the relation between $\theta_n$ and $\theta_{n+1}$

¿From (7) and (9) it is natural to define the map $\Psi : \Omega \to \Omega$, given by

$$\Psi(x,y) \;:=\; \left(\frac{x}{a(y) \Leftrightarrow 1},\, a(x)x \Leftrightarrow 1\right)\;,\;\; (x,y) \in \Omega\,.$$

Obviously one has

$$\Psi(T_n, V_n) \;=\; (\theta_n, \theta_{n+1})\,,\;\; n \geq 1. \tag{10}$$

Putting

$$V_{A,B} \;:=\; \{(x,y) \in \Omega :\, a(x) = A,\, a(y) = B\}\,,\;\; A, B \geq 2,$$

one finds

$$V_{A,B} \;=\; (\frac{1}{A}, \frac{1}{A \Leftrightarrow 1}] \times (\frac{1}{B}, \frac{1}{B \Leftrightarrow 1}]\,.$$

For $(x,y) \in V_{A,B}$ one has $\Psi(x,y) = (\frac{x}{B-1}, Ax \Leftrightarrow 1)$ (where $1/A < x \leq 1/(A \Leftrightarrow 1)$). Hence putting

$$\begin{cases} \alpha \;:=\; \frac{x}{B-1} \Leftrightarrow x \;=\; (B \Leftrightarrow 1)\alpha \\ \beta \;:=\; Ax \Leftrightarrow 1 \end{cases}$$

yields

$$\beta \;=\; A(B \Leftrightarrow 1)\alpha \Leftrightarrow 1\,,\;\; \alpha \in \left(\frac{1}{A(B \Leftrightarrow 1)}, \frac{1}{(A \Leftrightarrow 1)(B \Leftrightarrow 1)}\right]\,. \tag{11}$$

Thus we see that $\Psi$ maps the rectangle $V_{A,B}$ onto the line segment $L_{A,B}$, which has endpoints $(\frac{1}{A(B-1)}, 0)$ and $(\frac{1}{(A-1)(B-1)}, \frac{1}{A-1})$. Notice that from (10) and (11) one has

$$\theta_{n+1} \;=\; a_{n+1}(a_n \Leftrightarrow 1)\theta_n \Leftrightarrow 1,\;\; n \geq 1, \tag{12}$$

and $(\theta_n, \theta_{n+1}) \in \Xi$, where

$$\Xi := \bigcup_{A,B \geq 2} L_{A,B}\,,$$

see also figure 2.

Figure 2

Notice, that from (7) it follows that always

$$0 \leq \theta_n < 1\,,\;\; n \geq 1.$$

Note that figure 2 shows that a Vahlen-type theorem as one has for the continued fraction (see [JK]) is not possible for Lüroth Series. That is, there does not exist a constant $c < 1$, such that for every $x$ one has

$$\min(\theta_n(x),\, \theta_{n+1}(x)) \;<\; c$$

(recall that for continued fractions always $0 \leq \Theta_n(x) < 1$ and $\min(\Theta_n(x), \Theta_{n+1}(x)) < 1/2$). However, it is also clear from figure 2, that

$$(T_n, V_n) \notin V_{2,2} \;\Leftrightarrow\; \theta_n < \frac{1}{2}$$

and

$$(T_n, V_n) \in (\frac{1}{2}, \frac{3}{4}) \times (\frac{1}{2}, 1] \;\Rightarrow\; \theta_{n+1} < \frac{1}{2}\,.$$

We have the following proposition, which follows directly from lemma 1 (see also theorem 2 with $z = 1/2$).

**Proposition 1** *For almost all $x$ one has with probability $3/4$ that $\theta_n < \frac{1}{2}$ and with probability $7/8$ that*

$$\min(\theta_n(x),\, \theta_{n+1}(x)) \;<\; \frac{1}{2}.$$

*Furthermore, given that $\theta_n < 1/2$ one has with probability $5/6$ that $\theta_{n+1} < 1/2$. The same holds when $\theta_n$ and $\theta_{n+1}$ are interchanged.*

**Remarks** In view of (12) it is obvious that for a.e. $x$ two consecutive $\theta$'s are NOT independent. In fact proposition 1 suggests that two consecutive $\theta$'s are positively correlated. That this is the case almost surely is shown in section 3.2. The situation here is similar to that for the regular continued fraction; there Vahlen's theorem suggests that two consecutive $\Theta$'s are negatively correlated. This is indeed the case as was shown by Vincent Nolte in an unpublished document, see also [N].

# 3 On the distribution of $(\theta_n, \theta_{n+1})_{n\geq 1}$

In this section we will show in 3.1 that for almost all $x$ the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ is distributed according to a continuous singular distribution function $G$. Before stating the result we first recall the definition of a continuous distribution function, see also [T], p. 20. In 3.2 we will study for a.e. $x$ the distribution of the sequence $(\theta_n + \theta_{n+1})_{n\geq 1}$, which will then be used to show that two consecutive $\theta$'s are positively correlated.

## 3.1 A continuous singular distribution functon

**Definition 1** *A distribution function $G$ is said to be continuous singular if it is continuous and if there exists a Borel set $S$ with Lebesgue measure zero such that $\mu_G(S) = 1$. Here $\mu_G$ denotes the Lebesgue-Stieltjes measure determined by $G$.*

We have the following theorem.

**Theorem 4** *For almost all $x$ and for all $(z_1, z_2) \in [0,1] \times [0,1]$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \leq j \leq N \,:\; \theta_j(x) < z_1 \,,\, \theta_{j+1} < z_2 \,\}$$

*exists and equals $G(z_1, z_2)$, where $G$ is given by*

$$G(\xi, \eta) := \sum_{A,B\geq 2} \lambda_2(V^*_{A,B}(\xi,\eta))\,, \qquad (\xi,\eta) \in \Omega, \tag{13}$$

*where*

$$V^*_{A,B}(\xi,\eta) := \{(\alpha,\beta) \in V_{A,B} : \alpha < \min((B \Leftrightarrow 1)\xi, \frac{1+\eta}{A})\}\,. \tag{14}$$

*Finally, $G$ is a continuous singular distribution function with support $\Xi$.*

**Proof** The first assertion follows from (7), (9) and lemma 1. In order to show that $G$ is a continuous distribution function we have to show, see also [T], section 2.2 :

(i) $G(x_1, x_2) \to 1$ as $\min(x_1, x_2) \to \infty$.

(ii) For each $i \in \{1,2\}$, $G(x_1, x_2) \to 0$ as $x_i \to \Leftrightarrow\infty$.

(iii) $G(x_1, x_2)$ is continuous.

(iv) Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, where $a_i < b_i$, $i \in \{1,2\}$ and put

$$(\mathbf{a},\, \mathbf{b}] := \{\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i \leq b_i,\, i \in \{1,2\}\,\}.$$

Then for each cel $(\mathbf{a},\, \mathbf{b}] \subset \mathbf{R}^2$ we must have

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G \,\geq\, 0\,,$$

where

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G \,=\, G(b_1, b_2) \Leftrightarrow G(a_1, b_2) \Leftrightarrow G(b_1, a_2) + G(a_1, a_2)\,.$$

Notice that (i) and (ii) follow from the definition of $G$; clearly $G$ is monotone in each of its coordinates, and in case $x_i < 0$ (for $i \in \{1,2\}$) one has that $G(x_1, x_2) = 0$. In case $\min(x_1, x_2) \geq 1$ it follows that $G(x_1, x_2) = 1$. That $G$ is continuous clearly follows from (13). In order to prove (iv) we introduce for $A, B \geq 2$ a function $G_{A,B} : \Omega \to \mathbf{R}$, given by

$$G_{A,B}(\xi, \eta) := \lambda_2(V^*_{A,B}(\xi, \eta)), \qquad (\xi, \eta) \in \Omega,$$

where $V^*_{A,B}(\xi, \eta)$ is as in (14). Notice that

$$G(\xi, \eta) := \sum_{A,B \geq 2} G_{A,B}(\xi, \eta), \qquad (\xi, \eta) \in \Omega.$$

It is now sufficient to show that for all $A, B \geq 2$ and each cel $(\mathbf{a}, \mathbf{b}] \subset \Omega$ one has

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} \geq 0 .$$

Fix $A, B \geq 2$ and let

$$m(\xi, \eta) = m_{A,B}(\xi, \eta) := \min\left((B \Leftrightarrow 1)\xi, \frac{1+\eta}{A}\right)$$

and $\pi_1(\xi, \eta) = \pi_{(A,B),1} := (B \Leftrightarrow 1)\xi$, $\pi_2(\xi, \eta) = \pi_{(A,B),2} := \frac{1+\eta}{A}$, one has the following, possibly overlapping, cases.

**(I)** $m(a_1, b_2) < m(b_1, b_2)$ and

(Ia) $m(a_1, a_2) < m(b_1, a_2)$.
Notice that the monotonicity of $\pi_2$ as a function of its first coordinate yields that

$$\pi_1(a_1, a_2) < \pi_2(a_1, a_2)$$

and therefore $m(a_1, b_2) = m(a_1, a_2)$, from which it follows, by definition of $G_{A,B}$ :

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = G_{A,B}(b_1, b_2) \Leftrightarrow G_{A,B}(b_1, a_2) \geq 0 .$$

(Ib) $m(a_1, a_2) = m(b_1, a_2)$.
In this case one has

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = G_{A,B}(b_1, b_2) \Leftrightarrow G_{A,B}(a_1, b_2) > 0 .$$

**(II)** $m(a_1, b_2) = m(b_1, b_2)$, which implies that $\pi_2(a_1, b_2) \leq \pi_1(a_1, b_2)$, which in turn yields that

$$\pi_2(a_1, a_2) \leq \pi_1(a_1, a_2) .$$

But then we only can have that

$$m(a_1, a_2) = m(b_1, a_2) ,$$

from which it at once follows that

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G_{A,B} = 0 .$$

**(III)** $m(b_1, a_2) < m(b_1, b_2)$ : see case **(I)**.

**(IV)** $m(b_1, a_2) = m(b_1, b_2)$ : see case **(II)**.

In order to show that $\mu_G(\Xi) = 1$, or equivalently that $\mu_G(\Xi^c) = 0$, it is sufficient to show that for each cel $(\mathbf{a},\, \mathbf{b}] \subset \Omega$, for which

$$\mathrm{card}(\,(\mathbf{a},\, \mathbf{b}] \cap \Xi\,) \;\leq\; 2,$$

one has that $\mu_G((\mathbf{a},\, \mathbf{b}]) = 0$, which is equivalent with

$$\Delta_{\mathbf{a}}^{\mathbf{b}} G \;=\; 0\,.$$

Notice that we may assume that $(\mathbf{a},\, \mathbf{b}]$ is contained in $\mathcal{S}_k$ for some $k \geq 2$, where

$$\mathcal{S}_k \;:=\; (\frac{1}{k},\, \frac{1}{k \Leftrightarrow 1}] \times [0, 1]\,, \qquad \text{for } k \geq 2.$$

Obviously there are only finitely many values of $A$ and $B$ such that $L_{A,B} \cap \mathcal{S}_k \neq \emptyset$. Let $A$ and $B$ two such values, then $(\mathbf{a},\, \mathbf{b}]$ either "lies above" $L_{A,B}$ or "below" $L_{A,B}$. Let $\mathcal{U} = \mathcal{U}(\mathbf{a}, \mathbf{b})$ be the collection of all pairs $(A, B)$ for which $(\mathbf{a},\, \mathbf{b}]$ "lies above" $L_{A,B}$.

Clearly one has

$$\mu_G((\mathbf{a},\, \mathbf{b}]) \;=\; \mu_G((\mathbf{a}^*,\, \mathbf{b}]) \Leftrightarrow \mu_G((\mathbf{a}^*,\, \mathbf{b}^*])\,,$$

where $\mathbf{a}^* := (a_1, 0)$ and $\mathbf{b}^* := (b_1, a_2)$. For $(A, B) \in \mathcal{U}$ we now define $L^*_{A,B}$ by

$$L^*_{A,B} \;:=\; \{(x, y) \in L_{A,B} : a_1 \leq x \leq b_1\}\,,$$

then

$$\mu_G((\mathbf{a}^*,\, \mathbf{b}]) \;=\; G(b_1, b_2) \Leftrightarrow G(a_1, b_2) \;=\; \lambda_2(\bigcup_{(A,B)\in\mathcal{U}} \Psi^{-1} L^*_{A,B}\,)$$

$$=\; G(b_1, a_2) \Leftrightarrow G(a_1, a_2) \;=\; \mu_G((\mathbf{a}^*,\, \mathbf{b}^*])\,,$$

from which the theorem follows.□

### 3.2 On the correlation between $\theta_n$ and $\theta_{n+1}$

In section 2 we saw that it is likely that $\theta_n$ and $\theta_{n+1}$ are positively correlated. In order to show this, we first give some definitions.[3]

**Definition 2** *The correlation-coefficient $\rho(\theta_n, \theta_{n+1})$ of $\theta_n$ and $\theta_{n+1}$ is defined by*

$$\rho(\theta_n, \theta_{n+1}) \;:=\; \frac{E(\theta_n \theta_{n+1}) \Leftrightarrow E(\theta_n) E(\theta_{n+1})}{\sqrt{V(\theta_n)}\sqrt{V(\theta_{n+1})}}\,,$$

*where $E(\theta_n)$ is the expectation of $\theta_n$, as given in (8) and $V(\theta_n)$ is the variance of $\theta_n$, defined by*

$$V(\theta_n) \;:=\; E(\theta_n^2) \Leftrightarrow (E(\theta_n))^2\,.$$

The nominator of $\rho(\theta_n, \theta_{n+1})$ equals the covariance $\mathrm{C}(\theta_n, \theta_{n+1})$ of $\theta_n$ and $\theta_{n+1}$.

[3]In section 3.1 we assume that the stochastic variables $\theta_n$ for $n \geq 1$ are all identically distributed with distribution function $F$ as given in (7).

**Definition 3** *Let $X$ and $Y$ be two stochastic variables. Then the covariance $C(X,Y)$ of $X$ and $Y$ is given by*

$$C(X,Y) \;:=\; E(\,(X \Leftrightarrow E(X))(Y \Leftrightarrow E(Y))\,)\;.$$

Notice that $\mathrm{C}(X,Y) > 0$ indicates that whenever $X$ (resp. $Y$) is bigger/smaller than its mean $\mathrm{E}(X)$ (resp. $\mathrm{E}(Y)$), the same is likely to hold for $Y$ (resp. $X$), i.e. $X$ and $Y$ are positively correlated. Similarly $\mathrm{C}(X,Y) < 0$ tells us that $X$ and $Y$ are negatively correlated. In case $\mathrm{C}(X,Y) = 0$ we say that $X$ and $Y$ are uncorrelated. One has that

$$X \text{ and } Y \text{ are independent } \Rightarrow X \text{ and } Y \text{ are uncorrolated,}$$

but the converse does not hold in general.

**Theorem 5** *For almost all $x$ one has that*

$$\rho(\theta_n,\theta_{n+1}) \;=\; \frac{(\zeta(2)\Leftrightarrow 1)((1\Leftrightarrow 3\zeta(2)+4\zeta(3))}{4\zeta(3)\Leftrightarrow(\zeta(2)\Leftrightarrow 1)(1+3\zeta(2))} \;=\; 0.5744202\ldots$$

*and therefore $\theta_n$ and $\theta_{n+1}$ are positively correlated a.s.*

Since $\mathrm{E}(\theta_n) = \mathrm{E}(\theta_{n+1})$ and $\mathrm{E}(\theta_n^2) = \mathrm{E}(\theta_{n+1}^2)$ one has

$$\rho(\theta_n,\theta_{n+1}) \;=\; \frac{\mathrm{E}(\theta_n\theta_{n+1})\Leftrightarrow \mathrm{E}(\theta_n)^2}{\mathrm{V}(\theta_n)}\;.$$

Notice, that from theorem 2 one has that $F$ has density $f$, where

$$f(x) \;=\; \sum_{\ell=2}^{k}\frac{1}{\ell}\,, \quad \text{for } x \in (\frac{1}{k},\frac{1}{k\Leftrightarrow 1}]\,,\; k\geq 2.$$

Taking second moments thus yields

$$\mathrm{E}(\theta_n^2) \;=\; \sum_{k=2}^{\infty}\int_{\frac{1}{k}}^{\frac{1}{k-1}} z^2 f(z)dz \;=\; \sum_{k=2}^{\infty}\frac{1}{3k}\frac{1}{(k\Leftrightarrow 1)^3} \;=\; \frac{1\Leftrightarrow\zeta(2)+\zeta(3)}{3} \;=\; 0.185708\ldots.$$

But then it follows from (8) that

$$\mathrm{V}(\theta_n) \;=\; \frac{4\zeta(3)\Leftrightarrow(\zeta(2)\Leftrightarrow 1)(1+3\zeta(2))}{12} \;=\; 0.0817226\ldots.$$

In order to find $\mathrm{E}(\theta_n\theta_{n+1})$ we will determine $\mathrm{E}((\theta_n+\theta_{n+1})^2)$, since

$$\mathrm{E}((\theta_n+\theta_{n+1})^2) \;=\; 2\mathrm{E}(\theta_n^2)+2\mathrm{E}(\theta_n\theta_{n+1})\,.$$

Hence we find for Lüroth series that

$$\mathrm{E}(\theta_n\theta_{n+1}) \;=\; \frac{1}{2}\mathrm{E}((\theta_n+\theta_{n+1})^2)\Leftrightarrow \mathrm{E}(\theta_n^2)\,.$$

¿From (7) and (9) one has

$$\theta_n+\theta_{n+1} \;=\; (a_{n+1}+\frac{1}{a_n\Leftrightarrow 1})T_n\Leftrightarrow 1\,,\; n\geq 1, \tag{15}$$

and from the definition of $L_{A,B}$ it follows that

$$\frac{1}{a_{n+1}(a_n\Leftrightarrow 1)} \;<\; \theta_n+\theta_{n+1} \leq \frac{a_n}{(a_{n+1}\Leftrightarrow 1)(a_n\Leftrightarrow 1)}\,,\; n\geq 1. \tag{16}$$

¿From lemma 1, (16) and (17) we have the following theorem.

**Theorem 6** *For almost all $x$ and for every $z \in (0,2]$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \le j \le N \;:\; \theta_j(x) + \theta_{j+1} < z \,\}$$

*exists and equals $S(z)$, where $S$ is a continuous distribution function with density $s$, given by*

$$s(z) \;=\; \sum_{A,B\ge 2} \frac{B-1}{A(B-1)+1} \mathbf{1}_{\left(\frac{1}{A(B-1)},\, \frac{B}{(A-1)(B-1)}\right]}(z)\,.$$

*Here $\mathbf{1}_{(\alpha,\beta]}(z)$ is the indicator fuction of the interval $(\alpha,\beta]$, i.e.*

$$\mathbf{1}_{(\alpha,\beta]}(z) \;:=\; \begin{cases} 1 & , z \in (\alpha,\beta], \\ 0 & , z \notin (\alpha,\beta]. \end{cases}$$

Theorem 6 now at once yields that

$$\mathrm{E}((\theta_n + \theta_{n+1})^2) \;=\; \sum_{B=2}^{\infty} \sum_{A=2}^{\infty} \frac{1}{B} \int_{\frac{1}{A(B-1)}}^{\frac{B}{(A-1)(B-1)}} \frac{1}{A(B-1)+1}\, z^2 dz$$

$$= \sum_{A,B\ge 2} \frac{A^2B^2 + AB(A-1) + (A-1)^2}{3BA^3(A-1)^3(B-1)^3} \;=\; \frac{3 - 3\zeta(2) + 2\zeta(2)\zeta(3)}{3} \;=\; 0.6737\ldots.$$

But then

$$\mathrm{E}(\theta_n\theta_{n+1}) - \mathrm{E}(\theta_n)\mathrm{E}(\theta_{n+1}) \;=\; \frac{(\zeta(2)-1)((1-3\zeta(2)+4\zeta(3))}{12}\,,$$

and therefore the first assertion of theorem 5 is immediate. It follows that $\theta_n$ and $\theta_{n+1}$ are indeed positively correlated.□

Notice that the proof of theorem 6 can easily be adapted to derive the distribution for a.e. $x$ of the sequence $(\theta_n - \theta_{n+1})_{n\ge 1}$. We leave this to the reader, but mention one - surprising - case : one has that the probability $\mathrm{P}(\theta_n < \theta_{n+1})$ that $\theta_n$ is smaller than $\theta_{n+1}$ is $0.391\cdots$, i.e. $\theta_{n+1}$ has the tendency to be smaller than its predecessor. To be more precise, we have the following proposition.

**Proposition 2** *For almost all $x$ one has that*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \le j \le N \;:\; \theta_j(x) < \theta_{j+1} \,\} \;=\; 0.391\cdots.$$

**Proof** From (6) and (9) it follows that $\theta_n < \theta_{n+1}$ is equivalent with

$$T_n \;>\; \frac{a_n - 1}{a_{n+1}(a_n - 1) - 1}\,.$$

But then lemma 1 yields that $\lim_{N\to\infty} \frac{1}{N}\#\{\, 1 \le j \le N \;:\; \theta_j(x) < \theta_{j+1} \,\}$ exists for a.e. $x$, and equals $\lambda(\mathcal{D})$, where $\mathcal{D}$ is given by

$$\mathcal{D} \;=\; \bigcup_{A,B\ge 2} \left[\frac{B}{AB-1}, \frac{B}{A(B-1)-1}\right) \times \left[0, \frac{1}{B}\right]\,,$$

see also figure 3. The proposition now follows from

$$\lambda(\mathcal{D}) \;=\; \sum_{A=2}^{\infty} \sum_{B=2}^{\infty} \frac{1}{(AB \Leftrightarrow A \Leftrightarrow 1)(AB \Leftrightarrow 1)B} \;=\; 0.391\cdots . \square$$

Figure 3

**Final remarks** Recently Jose Barrionuevo, Bob Burton and the present authors generalized the whole concept of Lüroth Series, see also [BBDK]. A new class of series expansions, the so-called *Generalized Lüroth Series* (or GLS), was introduced and their ergodic properties were studied. Examples of these GLS are the recent *alternating Lüroth Series*, as introduced by S. Kalpazidou and A. and J. Knopfmacher, but also familiar expansions like $r$-adic expansions (for $r \in \mathbf{Z}$, $r \geq 2$). Although $\beta$-expansions are not in this class, it turned out that many important ergodic properties of these expansions can be obtained using the appropriate GLS-expansion, see also [DKS].
Here we only want to mention that the entire approach of this paper can be carried over to GLS-expansions. In order to keep the exposition clear and easy we only dealt with the "classical" Lüroth expansion. Details are left to the reader, see also section 3.1 of [BBDK].

# References


[BBDK] Barrionuevo, Jose, Robert M. Burton, Karma Dajani and Cor Kraaikamp – *Ergodic Properties of Generalized Lüroth Series,* to appear in Acta Arithmetica.

[DKS] Dajani, Karma, Cor Kraaikamp and Boris Solomyak – *The natural extension of the $\beta$-transformation,* Preprint **909**, March 1995, Department of Math., University Utrecht.

[JK] Jager, H. and C. Kraaikamp – *On the approximation by continued fractions,* Indag. Math., **51** (1989), 289-307.

[JdV] Jager, H. and C. de Vroedt – *Lüroth series and their ergodic properties,* Indag. Math. **31** (1968), 31-42.

[L] Lüroth, J. – *Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe,* Math. Annalen **21** (1883), 411-423.

[N] Nolte, Vincent N. – *Some probabilistic results on continued fractions,* Doktoraal scriptie Universiteit van Amsterdam, Amsterdam, August 1989.

[Pe] Perron, O. – *Irrationalzahlen,* Walter de Gruyter & Co., Berlin, 1960.

[P] Petersen, K. – *Ergodic Theory,* Cambridge Studies in Advanced Mathematics, Vol. 2, Cambridge Univ. Press, Cambridge, 1983.

[T] Tucker, H. G. – *A Graduate Course in Probability,* Academic Press, New Yorh, 1967.

Karma Dajani
Universiteit Utrecht
Dept. of Mathematics
Budapestlaan 6,
P.O. Box 80.000
3508TA Utrecht
the Netherlands
dajani@math.ruu.nl

Cor Kraaikamp
Technische Universiteit Delft
and Thomas Stieltjes Institute for Mathematics
TWI (SSOR)
Mekelweg 4
2628 CD Delft
the Netherlands
C.Kraaikamp@twi.tudelft.nl