A Gauss-Kusmin Theorem for Optimal Continued Fractions

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1 Introduction

One of the first ⇔ and still one of the most important ⇔ results in the metrical theory of continued fractions is the so-called Gauss-Kusmin theorem. Let $\xi \in [0,1)$, and let

$$\xi \;=\; \cfrac{1}{d_1 + \cfrac{1}{d_2 + \ddots + \cfrac{1}{d_n + \ddots}}} \;=\; [\,0;\, d_1,\, d_2,\, \cdots,\, d_n,\, \cdots] \tag{1}$$

be the *regular continued fraction* (RCF) expansion of ξ, then it was observed by Gauss [G] in 1800 that for $z \in [0,1]$

$$\lim_{n\to\infty} \lambda(\{\xi \in [0,1);\, T^n\xi \le z\}) \;=\; \frac{\log(1+z)}{\log 2}\,. \tag{2}$$

Here λ is the Lebesgue measure and the RCF-operator $T : [0,1) \to [0,1)$ is defined by

$$T\xi \;:=\; \frac{1}{\xi} \Leftrightarrow \lfloor \frac{1}{\xi} \rfloor\,,\, \xi \neq 0\,;\, T0 \;:=\; 0,$$

where $\lfloor\,.\,\rfloor$ denotes the *floor* - or *entier* function. It is not known how Gauss found (2), but later, in a letter dated January 30, 1812, Gauss asked Laplace to give an estimate of the error term $r_n(z)$, defined by

$$r_n(z) \;:=\; \lambda(T^{-n}[0,z]) \Leftrightarrow \frac{\log(1+z)}{\log 2}\,,\;\; n \ge 1.$$

It was Kusmin [Kus] in 1928 who was the first to prove (2) and at the same time to answer Gauss' question. Kusmin showed that

$$r_n(z) \;=\; \mathcal{O}(q^{\sqrt{n}})\,,$$

with $q \in (0,1)$, uniform in z. Independently, Paul Lévy [L] showed one year later that

$$r_n(z) \;=\; \mathcal{O}(q^n)\,,$$

with $q = 0.7\ldots$, uniform in z. Lévy's result, but with a better constant, was obtained by P. Szüsz in 1961 using Kusmin's approach. From that time on, a great number of such *Gauss-Kusmin theorems* followed. To mention a few: F. Schweiger (1968) [Sch1,2], P. Wirsing (1973) [Wir], K.I. Babenko (1978) [Ba], and more recently M. Iosifescu (1992) [Ios].

Gauss-Kusmin theorems for other continued fraction expansions were independently obtained by G.J. Rieger (1978) [Rie1] and A.M. Rockett (1980) [Roc]. Both Rieger and Rockett obtained a Gauss-Kusmin theorem for the *nearest integer continued fraction* (NICF). Rieger also obtained a Gauss-Kusmin theorem for the closely related *Hurwitz' singular continued fraction* (SCF), and other continued fraction expansions like the *continued fraction with odd partial quotients*.

Both the NICF as well as the SCF are examples of α-*expansions*, which were introduced and studied by H. Nakada in [N]. Let $\alpha \in [\frac{1}{2}, 1]$ be fixed, then the operator $T_\alpha : [\alpha \Leftrightarrow 1, \alpha) \rightarrow [\alpha \Leftrightarrow 1, \alpha)$ is defined by

$$T_\alpha \xi \;:=\; |\frac{1}{\xi}| \Leftrightarrow \lfloor |\frac{1}{\xi}| + 1 \Leftrightarrow \alpha \rfloor\,,\ \xi \neq 0\,;\ T_\alpha 0 \;:=\; 0. \tag{3}$$

Putting

$$\varepsilon_{\alpha,n}(\xi) := \operatorname{sgn}(T_\alpha^{n-1}\xi);\ a_{\alpha,n}(\xi) := \lfloor |\frac{1}{T_\alpha^{n-1}\xi}| + 1 \Leftrightarrow \alpha \rfloor\,,\ n \geq 1,$$

in case $T_\alpha^{n-1}\xi \neq 0$, and $\varepsilon_{\alpha,n}(\xi) := 0$; $a_{\alpha,n}(\xi) := \infty$ in case $T_\alpha^{n-1}\xi = 0$, one easily sees that every irrational $\xi \in [\alpha \Leftrightarrow 1, \alpha)$ has a unique α-expansion

$$\xi \;=\; \cfrac{\varepsilon_{\alpha,1}}{a_{\alpha,1} + \cfrac{\varepsilon_{\alpha,2}}{a_{\alpha,2} + \ddots + \cfrac{\varepsilon_{\alpha,n}}{a_{\alpha,n} + \ddots}}} \;=\; [\,0;\, \varepsilon_{\alpha,1}a_{\alpha,1},\, \cdots,\, \varepsilon_{\alpha,n}a_{\alpha,n},\, \cdots]. \tag{4}$$

In case $\alpha = 1$, (4) is simply the RCF-expansion of ξ; in case $\alpha = \frac{1}{2}$, (4) is the NICF-expansion of ξ and in case $\alpha = g := \frac{1}{2}(\sqrt(5) \Leftrightarrow 1) = 0.61\cdots$ one has that (4) is Hurwitz' SCF-expansion of ξ.

It should be noted that the methods of Rieger and Rockett can be easily adapted to obtain a Gauss-Kusmin theorem for any α-expansion, where $\alpha \in [\frac{1}{2}, 1]$.

Nakada's α-expansions are examples of *semi-regular continued fraction* (SRCF) expansions. In general a SRCF is a finite or infinite fraction

$$b_0 \;+\; \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \ddots + \cfrac{\varepsilon_n}{b_n + \ddots}}} \;=\; [\,b_0;\, \varepsilon_1 b_1,\, \varepsilon_2 b_2,\, \cdots,\, \varepsilon_n b_n,\, \cdots]\,, \tag{5}$$

with $\varepsilon_n = \pm 1$; $b_0 \in \mathbf{Z}$; $b_n \in \mathbf{N}$, for $n \geq 1$, subject to the condition

$$\varepsilon_{n+1} + b_n \geq 1, \text{ for } n \geq 1,$$

and with the restriction that in the infinite case

$$\varepsilon_{n+1} + b_n \geq 2, \text{ infinitely often.}$$

Moreover we demand that $\varepsilon_n + b_n \geq 1$ for $n \geq 1$.

Remark In case $\alpha = \frac{1}{2}$ one has that

$$b_n \geq 2 \text{ and } b_n + \varepsilon_{n+1} \geq 2,\ n \geq 1, \tag{6}$$

and conversely, if (5) is a SRCF-expansion of ξ which satisfies (6), then (5) is the NICF-expansion of ξ. In the same way the SCF-expansion of ξ is characterized by

$$b_n \geq 2 \text{ and } b_n + \varepsilon_n \geq 2,\ n \geq 1, \tag{7}$$

see also Section 3 or Perron's classical book [Pe].

Taking finite truncations in (5) yields a finite or infinite sequence of rational numbers A_n/B_n, $n \geq 1$, where

$$\frac{A_n}{B_n} = b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \ddots + \frac{\varepsilon_n}{b_n}}} = [\, b_0;\, \varepsilon_1 b_1,\, \varepsilon_2 b_2,\, \cdots,\, \varepsilon_n b_n \,]\,.$$

A SRCF-expansion (5) is a SRCF-expansion of ξ if

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \xi\,.$$

A *fastest* expansion of ξ is an expansion for which the growth rate of the denominators B_n is maximal; it turns out that this means that these denominators grow asymptotically as fast as the denominators of the NICF (or SCF) convergents of that ξ, see e.g. [Bos] or [K1]. *Closest* expansions are those for which $\sup\{\theta_k : \theta_k := B_k|B_k\xi \Leftrightarrow A_k|\}$ is minimal. Since in general the NICF does not provide closest expansions, and closest expansions (like Minkowski's *diagonal continued fraction* (DCF)) do not provide fastest expansions, a natural question arises whether exist a SRCF which is both fastest and closest. In [Ke] it was shown that such an algorithm does exist, and Selenius [Se] showed how such a SRCF of ξ can be obtained, given the RCF of ξ. In 1987, W. Bosma introduced a new continued fraction expansion which yields for every $\xi \in \mathbf{R}$ a SRCF-expansion of ξ which is both fastest and closest, without using the RCF-expansion of ξ. This new continued fraction algorithm, the so-called *optimal continued fraction* (OCF) expansion turned out to have approximation properties superior to any other SRCF-expansion, see also [BK1,2].

The OCF-expansion of an irrational number $\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2})$ is defined recursively as follows. Put

$$\begin{array}{ll} r_{-1} = 1; & r_0 = 0; \\ s_{-1} = 0; & s_0 = 1; \\ t_0 = \xi; & \varepsilon_1 = \operatorname{sgn}(t_0) \end{array}$$

and let for $k \geq 1$

$$\begin{array}{l} b_k = \lfloor |t_{k-1}^{-1}| \rfloor \\ v_k = b_k s_{k-1} + \varepsilon_k s_{k-2} \quad \text{ and } \; u_k = b_k r_{k-1} + \varepsilon_k r_{k-2}, \\ \alpha_k = \frac{v_k + s_{k-1}}{2v_k + s_{k-1}}\,. \end{array}$$

The partial quotients $a_k = a_k(\xi)$ are given by

$$a_k = \lfloor |t_{k-1}^{-1}| + 1 \Leftrightarrow \alpha_k \rfloor\,,$$

and the convergents r_k/s_k by

$$r_k = a_k r_{k-1} + \varepsilon_k r_{k-2} \text{ and } s_k = a_k s_{k-1} + \varepsilon_k s_{k-2}\,.$$

Next put

$$t_k \;=\; |t_{k-1}^{-1}| \Leftrightarrow a_k \;\text{ and }\; \varepsilon_{k+1} \;=\; \operatorname{sgn}(t_k).$$

For arbitrary (irrational) numbers ξ we define $\mathrm{OCF}(\xi) = [\,a_0;\, \varepsilon_1 a_1,\, \varepsilon_2 a_2,\, \cdots]$, where $a_0 \in \mathbf{Z}$ is such that $x \Leftrightarrow a_0 \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2})$ and $[\,0;\, \varepsilon_1 a_1,\, \varepsilon_2 a_2,\, \cdots]$ is the OCF-expansion of ξ.

Notice that the OCF behaves like an α-expansion, where at every stage of the algorithm the value of α (which is α_k) is adjusted. For more details on this, see [Bos], Section 4. An equivalent way of generating OCF-expansions ⇔ or any of the above mentioned continued fraction algorithms ⇔ is via the mechanism of S-expansions, which is dicussed to some detail in Section 3. This approach enables us to use ergodic theory in order to analyse the dynamical, metrical and number theoretical properties of these expansions.

In contrast with most continued fraction algorithms the OCF-algorithm is "two-dimensional" (there are some exceptions, e.g. the afore mentioned diagonal continued fraction (DCF), see [K1]); In order to apply the OCF-algorithm "one needs to know where one has been". It is exactly this aspect of the OCF which makes it very difficult ⇔ if not impossible ⇔ to obtain a Gauss-Kusmin theorem for the OCF in the same vein as those obtained for the NICF, SCF or for the RCF (it should be noticed that the approach from [Wir] and [Ba] cannot be used for the NICF or the SCF, see also [Rie1], p. 444).

The aim of this paper is to obtain a Gauss-Kusmin theorem for the OCF. To be more precise, we will show ⇔ among many other things ⇔ that for $z \in [\Leftrightarrow\frac{1}{2}, g]$

$$(*) \qquad \lambda\{\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2}) \,:\, T^n_{\mathrm{ocf}}\xi \le z\} \;=\; \mu_{\mathrm{ocf}}([\Leftrightarrow\frac{1}{2}, z]) + \mathcal{O}(g^n),$$

where μ_{ocf} is a probability measure on $[\Leftrightarrow\frac{1}{2}, g)$ with density $d_{\mathrm{ocf}}(x)$, given by

$$d_{\mathrm{ocf}}(x) \;=\; \begin{cases} \frac{1}{\log G}\frac{2x+1}{2x^2+2x+1} & \text{if } \Leftrightarrow\frac{1}{2} \le x < \Leftrightarrow g^2, \\[2ex] \frac{1}{\log G}\frac{x+1}{x^2+2x+2} & \text{if } \Leftrightarrow g^2 \le x < \frac{1}{2}, \\[2ex] \frac{3}{\log G}\frac{1-x-x^2}{(x^2+2x+2)(2x^2-2x+1)} & \text{if } \frac{1}{2} \le x < g, \end{cases} \tag{8}$$

and where $T^n_{\mathrm{ocf}}\xi$ is given by

$$T^n_{\mathrm{ocf}}\xi \;=\; [\,0;\, \varepsilon_{n+1}b_{n+1},\, \varepsilon_{n+2}b_{n+2},\, \cdots]\,,$$

in case

$$\xi \;=\; [\,0;\, \varepsilon_1 b_1,\, \cdots,\, \varepsilon_n b_n,\, \cdots]$$

is the OCF-expansion of ξ.

This paper is organized as follows. In Section 2 a "two-dimensional Gauss-Kusmin theorem" for Hurwitz' SCF will be discussed. Also a generalization of a Knuth-type theorem for the SCF will be obtained. Proofs in this section will follow those from [DK], where similar results for the RCF were obtained.

All these continued fraction expansion, that is, the NICF, SCF and OCF, are examples of a very large class of SRCF-expansions, the so-called S-expansions. In Section 3 these S-expansions will be briefly discussed.

In Section 4 we will recall a result from [K2] which states that maximal (i.e. *fastest*) S-expansions like the NICF, SCF or OCF, are metrically isomorphic. This isomorphism will then be used to carry over the results from section 2 to any maximal S-expansion, in particular to the OCF, from which the above mentioned result $(*)$ then follows.

2 A Two Dimensional Gauss-Kusmin Theorem

In this section we will derive a "two-dimensional" Gauss-Kusmin theorem, and also the analog of a theorem by D.E. Knuth [Kn] for the SCF. To be more precise, let

$$(X_g, \mathcal{B}_g, \mu_g, T_g)$$

be the dynamical system underlying Hurwitz' SCF, where $X_g = [\Leftrightarrow g^2, g)$, $\mathcal{B}_g$ is the collection of Borel sets on X_g, μ_g is a probability measure on X_g with density[1] $(\log G)^{-1}(2+x)^{-1}$ and T_g is defined as in (3). Then a Gauss-Kusmin theorem related to the natural extension

$$(\Omega_g, \overline{\mathcal{B}}_g, \bar{\mu}_g, \mathcal{T}_g)$$

of $(X_g, \mathcal{B}_g, \mu_g, T_g)$ will be derived. Here $\Omega_g = [\Leftrightarrow g^2, g) \times [0, \frac{1}{2}]$, $\overline{\mathcal{B}}_g$ is the collection of Borel sets on Ω_g, $\bar{\mu}_g$ is a probability measure with density $(\log G)^{-1}(1+xy)^{-1}$ on Ω_g and finally $\mathcal{T}_g$ is defined by

$$\mathcal{T}_g(\xi, \eta) := \left(T_g \xi, \frac{1}{\lfloor |\xi^{-1}| + g^2 \rfloor + \operatorname{sgn}(\xi) \cdot \eta} \right), \ (\xi, \eta) \in \Omega_g, \ \xi \neq 0.$$

For further reference we will mention here a slightly modified version of Rieger's 1978 version of the Gauss-Kusmin theorem for the SCF, see also in [Rie1] the proof of Satz 2 and (7.1).

Theorem 1 *For every Borel set $E \subset X_g$ one has*

$$|\lambda(T_g^{-n} E) \Leftrightarrow \mu_g(E)| < C \lambda(E) (\frac{3}{5})^n,$$

where λ is Lebesgue measure on $X_g = [\Leftrightarrow g^2, g)$ and where μ_g is defined as before, i.e.,

$$\mu_g(E) := \frac{1}{\log G} \int_E \frac{dx}{2+x}, \ E \in \mathcal{B}_g,$$

and C is a universal constant.

Remarks

1. A similar theorem can be formulated for the NICF, see [Rie1], Satz 2, and also [Roc]. In this paper we choose to work with the SCF instead of the NICF only because the natural extension of the SCF is "slightly nicer" than the one for the NICF, see also [Na], [K1]; one simply needs to discern less cases in the proofs of the various results in case one uses the SCF.

[1] Here and in the following $G := g + 1 = \frac{1}{2}(\sqrt{5} + 1)$. Also notice that $g^2 = 1 - g = 0.38\cdots$ and $gG = 1$.

2. The constant $\frac{3}{5}$ in Rieger's theorem is not best possible, see also [Rie1], p. 446 and the remarks after [Rie1], Satz 2.

Set

$$m_n(x) := \lambda(\{\xi \in X_g;\, T_g^n \xi \le x\}), \quad \text{for} \quad x \in [-g^2, g]\,. \tag{9}$$

Since for $-g^2 \le x \le g$

$$\{\xi : T_g\xi \le x\} = \bigcup_{k=2}^{\infty} [\frac{1}{k+x}, \frac{1}{k-g^2}] \cup \bigcup_{k=3}^{\infty} [\frac{-1}{k-g^2}, \frac{-1}{k+x}]\,, \tag{10}$$

the relation

$$m_{n+1}(x) = \sum_{k=2}^{\infty} (m_n(\frac{1}{k-g^2}) - m_n(\frac{1}{k+x})) + \sum_{k=3}^{\infty} (m_n(\frac{-1}{k+x}) - m_n(\frac{-1}{k-g^2})) \tag{11}$$

follows, which is fundamental in any proof of a Gauss-Kusmin theorem for the SCF.

In fact, the measure μ_g is an eigenfunction of (11); viz. if we put $m_n(x) := \log(2+x)$, then a simple calculation shows that $m_{n+1}(x) = \log(2+x)$. The factor $1/\log G$ is a normalizing constant.

Relation (10) easily follows from Figure 1.

Figure 1

(The map T_g)

Let $\xi \in [-g^2, g) \setminus \mathbf{Q}$, with SCF-expansion (4) (with $\alpha = g$). Finite truncation in (4) yields the sequence of SCF-convergents A_n/B_n of ξ

$$\frac{A_n}{B_n} = [\,0;\, \varepsilon_1 b_1,\, \cdots,\, \varepsilon_n b_n\,],\ n \ge 1.$$

One easily shows that

$$\begin{cases} A_{-1}(\xi) = 1; \quad A_0(\xi) = 0; \quad A_n(\xi) = b_n A_{n-1}(\xi) + \varepsilon_n A_{n-2}(\xi),\ n \ge 1 \\ B_{-1}(\xi) = 0; \quad B_0(\xi) = 1; \quad B_n(\xi) = b_n B_{n-1}(\xi) + \varepsilon_n B_{n-2}(\xi),\ n \ge 1. \end{cases} \tag{12}$$

For $(\xi, \eta) \in \Omega_g$, put

$$(T_0, V_0^*) := (\xi, \eta) \text{ and } (T_n, V_n^*) := \mathcal{T}_g^n(\xi, \eta),\, n \geq 1,$$

then

$$T_n = [\,0;\, \varepsilon_{n+1}b_{n+1},\, \varepsilon_{n+2}b_{n+2},\, \cdots];\;\; V_n^* = [\,0;\, b_n,\, \varepsilon_n b_{n-1},\, \cdots, \varepsilon_2(b_1+\eta)\,].$$

Of course, for $n \geq 0$ we have that $[\,0;\, \varepsilon_{n+1}b_{n+1},\, \varepsilon_{n+2}b_{n+2},\, \cdots]$ is the SCF-expansion of the number $T_n = T_g^n\xi \in [-g^2, g) \setminus \mathbf{Q}$; it satisfies (7) for every $n \geq 0$. Notice also that the first n digits of V_n^* satisfy (6). In particular we see that if $\eta = 0$ one has that

$$[\,0;\, b_n,\, \varepsilon_n b_{n-1},\, \cdots, \varepsilon_2 b_1\,]$$

is the NICF-expansion of the (rational) number V_n^*. In case $\eta = 0$ we will write V_n instead of V_n^*.

Now define

$$m_n(x,y) := \bar{\lambda}\{(\xi,\eta) \in \Omega_g \,:\, \mathcal{T}_g^n(x,y) \in [-g^2, x] \times [0,y]\}\,, \tag{13}$$

here (and in the rest of this paper) $\bar{\lambda}$ is normalized Lebesgue measure on Ω_g.

In this section we will obtain the following two theorems.

Theorem 2 *For all $n \geq 2$ and all $(x,y) \in \Omega_g$ one has*

$$m_n(x,y) = \frac{\log\left(\frac{1+xy}{1-g^2y}\right)}{\log G} + \mathcal{O}(g^n)\,,$$

the constant of the big $\mathcal{O}$-symbol is uniform.

Theorem 3 *Let K be a simply connected subset of Ω_g, such that*

$$\partial K = \ell_1 \cup \ldots \cup \ell_m,$$

where $m \in \mathbf{N}$ and each ℓ_i is given by either

$$\ell_i := \{(\xi, f_i(\xi))\,;\, \beta_i \leq \xi \leq \gamma_i\},$$

where $-g^2 \leq \beta_i < \gamma_i \leq g$ and $f_i : [\beta_i, \gamma_i] \to [0, \frac{1}{2}]$ is continuous and monotone, or by

$$\ell_i := \{(\beta_i, \eta)\,;\, \kappa_i \leq \eta \leq \tau_i\},$$

where $\beta_i \in [-g^2, g]$ and $0 \leq \kappa_i < \tau_i \leq \frac{1}{2}$, $i = 1, \ldots, m$.

Put

$$E_n(K) := \{\xi \in [-g^2, g)\,;\, (T_n, V_n) := \mathcal{T}_g^n(\xi, 0) \in K\}.$$

Then one has

$$\lambda(E_n(K)) = \bar{\mu}_g(K) + \mathcal{O}(g^n),$$

where the constant in the big-$\mathcal{O}$ symbol is uniform.

Clearly

$$\mathcal{T}_g^{n+1}(\xi,\eta) \in [\Leftrightarrow g^2, x] \times [0, y]$$

is equivalent to

$$T_g^{n+1}\xi \in [\Leftrightarrow g^2, x] \text{ and } 0 \leq V_{n+1} = \frac{1}{a_{n+1} + \varepsilon_{n+1} V_n} \leq y.$$

¿From (10) it follows that the former expression is equivalent to

$$T_g^n \xi \in \bigcup_{k=2}^{\infty} [\frac{1}{k+x}, \frac{1}{k \Leftrightarrow g^2}] \cup \bigcup_{k=3}^{\infty} [\frac{\Leftrightarrow 1}{k \Leftrightarrow g^2}, \frac{\Leftrightarrow 1}{k+x}]\,.$$

The latter expression can be understood as follows. Let $\ell := \lfloor \frac{1}{y} + \frac{1}{2} \rfloor$, then if $y \leq 1/\ell$, one has $\mathcal{T}_g^{n+1}(\xi,\eta) \in \mathcal{I}_{x,y} := [\Leftrightarrow g^2, x] \times [0, y]$ is equivalent to

$$\begin{aligned} \mathcal{T}_g^n(\xi,\eta) \;\; &\in \;\; [\frac{1}{\ell+x}, \frac{1}{\ell \Leftrightarrow g^2}] \times [\frac{1}{y} \Leftrightarrow \ell, \frac{1}{2}] \cup \\ & \quad \bigcup_{k=\ell+1}^{\infty} [\frac{1}{k+x}, \frac{1}{k \Leftrightarrow g^2}] \times [0, \frac{1}{2}] \\ & \quad \bigcup_{k=\ell+1}^{\infty} [\frac{\Leftrightarrow 1}{k \Leftrightarrow g^2}, \frac{\Leftrightarrow 1}{k+x}] \times [0, \frac{1}{2}] \end{aligned}$$

and if $y > 1/\ell$, then $\mathcal{T}_g^{n+1}(\xi,\eta) \in \mathcal{I}_{x,y}$ is equivalent to

$$\begin{aligned} \mathcal{T}_g^n(\xi,\eta) \;\; &\in \;\; [\frac{\Leftrightarrow 1}{\ell \Leftrightarrow g^2}, \frac{\Leftrightarrow 1}{\ell+x}] \times [0, \ell \Leftrightarrow \frac{1}{y}] \cup \\ & \quad \bigcup_{k=\ell}^{\infty} [\frac{1}{k+x}, \frac{1}{k \Leftrightarrow g^2}] \times [0, \frac{1}{2}] \\ & \quad \bigcup_{k=\ell+1}^{\infty} [\frac{\Leftrightarrow 1}{k \Leftrightarrow g^2}, \frac{\Leftrightarrow 1}{k+x}] \times [0, \frac{1}{2}]\,. \end{aligned}$$

¿From this and (13) one gets the following recursion formula

$$m_{n+1}(x,y) \; = \; \sum_{k=\ell}^{\infty} (m_n(\frac{1}{k \Leftrightarrow g^2}, \frac{1}{2}) \Leftrightarrow m_n(\frac{1}{k+x}, \frac{1}{2})) \tag{14}$$

$$\begin{aligned} &+ \quad \sum_{k=\ell+1}^{\infty} (m_n(\frac{\Leftrightarrow 1}{k+x}, \frac{1}{2}) \Leftrightarrow m_n(\frac{\Leftrightarrow 1}{k \Leftrightarrow g^2}, \frac{1}{2})) \\ &+ \quad m_n(\frac{\epsilon}{\ell+x}, \epsilon(\frac{1}{y} \Leftrightarrow \ell)) \Leftrightarrow m_n(\frac{\epsilon}{\ell \Leftrightarrow g^2}, \epsilon(\frac{1}{y} \Leftrightarrow \ell))\,, \end{aligned}$$

where

$$\epsilon \; = \; \begin{cases} 1 & \text{if } y < \frac{1}{\ell}\,, \\ \Leftrightarrow 1 & \text{if } y \geq \frac{1}{\ell}\,. \end{cases}$$

Lemma 1 *Let* $n \in \mathbf{N}$, $n \geq 2$ *and let* y *be a rational number from the interval* $[0, \frac{1}{2}]$ *with NICF-expansion*

$$y \;=\; [\,0; \ell_1, \epsilon_1 \ell_2, \cdots, \epsilon_{d-1}\ell_d\,]\,,\;\; \ell_i \geq 2,\; \epsilon_i \in \{\Leftrightarrow 1, 1\}\,,$$

where $d \leq \lfloor \frac{n}{2} \rfloor + 1$. *Then for all* x, $x^* \in [\Leftrightarrow g^2, g)$ *with* $x^* < x$ *one has*

$$\left| (m_n(x,y) \Leftrightarrow m_n(x^*,y)) \Leftrightarrow \frac{1}{\log G} \log \frac{1+xy}{1+x^*y} \right| \;<\; C\overline{\lambda}(\mathcal{I}_{x,y} \setminus \mathcal{I}_{x^*,y})\,(\frac{3}{5})^{n-d}\,.$$

Proof Let $y = y_0$ and for $i = 1, \cdots, d$ write

$$\begin{aligned} y_i &= [\,0; \ell_{i+1}, \epsilon_{i+1}\ell_{i+2}, \cdots, \epsilon_{d-1}\ell_d\,] \\ &= \begin{cases} \frac{1}{y_{i-1}} \Leftrightarrow \ell_i & \text{if } y_{i-1} < \frac{1}{\ell_i}, \\ \ell_i \Leftrightarrow \frac{1}{y_{i-1}} & \text{if } y_{i-1} \geq \frac{1}{\ell_i} \end{cases} \\ &= \epsilon_i(\frac{1}{y_{i-1}} \Leftrightarrow \ell_i)\,. \end{aligned}$$

Note that $\epsilon_i = 1$ if $y_{i-1} < \frac{1}{\ell_i}$ and $\epsilon_i = \Leftrightarrow 1$ else.

Applying the above recursion formula (14) one gets

$$\begin{aligned} m_n(x,y) \Leftrightarrow m_n(x^*,y) &= \sum_{k=\ell_1}^{\infty} \left(m_{n-1}(\frac{1}{k+x^*}, \frac{1}{2}) \Leftrightarrow m_{n-1}(\frac{1}{k+x}, \frac{1}{2}) \right) \\ &+ \sum_{k=\ell_1+1}^{\infty} \left(m_{n-1}(\frac{\Leftrightarrow 1}{k+x}, \frac{1}{2}) \Leftrightarrow m_{n-1}(\frac{\Leftrightarrow 1}{k+x^*}, \frac{1}{2}) \right) \\ &+ m_{n-1}(\frac{\epsilon_1}{\ell_1+x}, y_1) \Leftrightarrow m_{n-1}(\frac{\epsilon_1}{\ell_1+x^*}, y_1)\,. \end{aligned}$$

For any $D \in \overline{\mathcal{B}}_g$,

$$\frac{1}{\log G}\frac{2}{(1+G)^2}\bar{\lambda}(D) \;\leq\; \bar{\mu}_g(D) \;\leq\; \frac{1}{\log G}\frac{2}{G^2}\bar{\lambda}(D)\,. \tag{15}$$

For each $\bar{b} = (\,b_1, \epsilon_1 b_2, \cdots, \epsilon_{n-1}b_n)$, where $b_i \geq 2$ and $\epsilon_i \in \{\Leftrightarrow 1, +1\}$ satisfy (6), let

$$Z(\bar{b}) \;=\; \{x \in [0, \frac{1}{2}];\ \mathrm{NICF}(x) = [\,0; b_1, \epsilon_1 b_2, \cdots, \epsilon_{n-1}b_n, \underbrace{\cdots\cdots\cdots}_{\text{"free"}}\,]\,,$$

i.e., $Z(\bar{b})$ is a *cylinder set* (or: *fundamental interval*) for the nearest integer continued fraction.

Now from (15) and the fact that $\mathcal{T}_g$ is $\bar{\mu}_g$-invariant

$$\begin{aligned} &\sum_{k=\ell_1}^{\infty} (\frac{1}{k+x^*} \Leftrightarrow \frac{1}{k+x}) + \sum_{k=\ell_1+1}^{\infty} (\frac{\Leftrightarrow 1}{k+x} \Leftrightarrow \frac{\Leftrightarrow 1}{k+x^*}) \\ &= \bar{\lambda}\left[(\frac{1}{\ell_1+x}, \frac{1}{\ell_1+x^*}) \times [0, \frac{1}{2}] \right] \\ &\quad + \sum_{\ell_1+1}^{\infty} \bar{\lambda}\left[((\frac{\Leftrightarrow 1}{k+x^*}, \frac{\Leftrightarrow 1}{k+x}) \bigcup (\frac{1}{k+x}, \frac{1}{k+x^*})) \times [0, \frac{1}{2}] \right] \\ &\leq \frac{1}{2}(1+G)^2 \log G \left[\bar{\mu}_g((x^*, x) \times Z(\ell_1)) + \sum_{k=\ell_1+1}^{\infty} \bar{\mu}_g((x^*, x) \times Z(k)) \right] \\ &\leq 2G^2\bar{\lambda}(\mathcal{I}_{x,y} \setminus \mathcal{I}_{x^*,y})\,. \end{aligned}$$

A similar analysis leads to

$$\begin{aligned}
&\sum_{k=\ell_i}^{\infty} \qquad (|[0;k,\epsilon_{i-1}\ell_{i-1},\cdots,\epsilon_1(\ell_1+x^*)] \Leftrightarrow [0;k,\epsilon_{i-1}\ell_{i-1},\cdots,\epsilon_1(\ell_1+x)]|) \\
&\qquad + \textstyle\sum_{k=\ell_i+1}^{\infty} \quad (|[0;\Leftrightarrow k,\epsilon_{i-1}\ell_{i-1},\cdots,\epsilon_1(\ell_1+x)] \Leftrightarrow [0;\Leftrightarrow k,\epsilon_{i-1}\ell_{i-1},\cdots,\epsilon_1(\ell_1+x^*)]|) \\
&\qquad\qquad\qquad \leq 2G^2\bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y}),
\end{aligned}$$

see also [DK] where the case of the RCF was dealt with.

¿From the above discussion and Theorem 1 we get, since $m_n(x,\frac{1}{2}) = m_n(x)$

$$\begin{aligned}
&\sum_{k=\ell_1}^{\infty}\left(m_{n-1}(\frac{1}{k+x^*},\frac{1}{2}) \Leftrightarrow m_{n-1}(\frac{1}{k+x},\frac{1}{2})\right) \\
&\qquad + \sum_{k=\ell_1+1}^{\infty}\left(m_{n-1}(\frac{\Leftrightarrow 1}{k+x},\frac{1}{2}) \Leftrightarrow m_{n-1}(\frac{\Leftrightarrow 1}{k+x^*},\frac{1}{2})\right) \\
&= \sum_{k=\ell_1}^{\infty}\mu_g(\frac{1}{k+x},\frac{1}{k+x^*}) + \sum_{k=\ell_1}^{\infty}\lambda(\frac{1}{k+x},\frac{1}{k+x^*})\mathcal{O}((\frac{3}{5})^{n-1}) \\
&\qquad + \sum_{k=\ell_1+1}^{\infty}\mu_g(\frac{\Leftrightarrow 1}{k+x^*},\frac{\Leftrightarrow 1}{k+x}) + \sum_{k=\ell_1+1}^{\infty}\lambda(\frac{\Leftrightarrow 1}{k+x^*},\frac{\Leftrightarrow 1}{k+x})\mathcal{O}((\frac{3}{5})^{n-1}) \\
&= \frac{1}{\log G}\log\left(\frac{2\ell_1+2x^*+1}{2\ell_1+2x+1}\,\frac{\ell_1+x}{\ell_1+x^*}\right) \\
&\qquad + \frac{1}{\log G}\lim_{n\to\infty}\sum_{k=\ell_1+1}^{n}\log\left(\frac{2k+2x^*+1}{2k+2x+1}\,\frac{2k+2x\Leftrightarrow 1}{2k+2x^*\Leftrightarrow 1}\right) \\
&\qquad + 2G^2\bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y})\mathcal{O}((\frac{3}{5})^{n-1}) \\
&= \frac{1}{\log G}\log\left(\frac{\ell_1+x}{\ell_1+x^*}\right) + 2G^2\bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y})\mathcal{O}((\frac{3}{5})^{n-1}).
\end{aligned}$$

Thus we see that

$$\begin{aligned}
m_n(x,y) \Leftrightarrow m_n(x^*,y) &= \frac{1}{\log G}\log\left(\frac{\ell_1+x}{\ell_1+x^*}\right) + 2G^2\bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y})\mathcal{O}((\frac{3}{5})^{n-1}) \\
&+ m_{n-1}(\frac{\epsilon_1}{\ell_1+x},y_1) \Leftrightarrow m_{n-1}(\frac{\epsilon_1}{\ell_1+x^*},y_1).
\end{aligned}$$

Applying (14) d-times one gets

$$\begin{aligned}
&m_n(x,y) \Leftrightarrow m_n(x^*,y) = \\
&= \frac{1}{\log G}\left[\frac{\ell_1+x}{\ell_1+x^*}\,\frac{[\ell_2;\epsilon_1(\ell_1+x)]}{[\ell_2;\epsilon_1(\ell_1+x^*)]}\cdots\frac{[\ell_d;\epsilon_{d-1}\ell_{d-1},\cdots,\epsilon_2\ell_2,\epsilon_1(\ell_1+x)]}{[\ell_d;\epsilon_{d-1}\ell_{d-1},\cdots,\epsilon_2\ell_2,\epsilon_1(\ell_1+x^*)]}\right] \\
&\quad + \bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y})\mathcal{O}((\frac{3}{5})^{n-1}) + \cdots + \bar{\lambda}(\mathcal{I}_{x,y}\setminus\mathcal{I}_{x^*,y})\mathcal{O}((\frac{3}{5})^{n-d}).
\end{aligned}$$

Let

$$\begin{cases} P_{-1}=1; \quad P_0=0; \quad P_i=\alpha_iP_{i-1}+\epsilon_iP_{i-2},\ i=i,\cdots,d \\ \\ Q_{-1}=0; \quad Q_0=1; \quad Q_i=\alpha_iQ_{i-1}+\epsilon_iQ_{i-2},\ i=i,\cdots,d, \end{cases} \tag{16}$$

where $\alpha_1 = \ell_1 + x$, $\alpha_2 = \ell_2, \cdots, \alpha_d = \ell_d$. Then

$$\frac{Q_{i-1}}{Q_i} = [\,0;\, \ell_i,\, \epsilon_{i-1}\ell_{i-1},\, \ldots\,,\, \epsilon_1(\ell_1 + x)]$$

for $i = 1, \cdots, d$, from which it follows that

$$(\ell_1 + x)[\ell_2; \epsilon_1(\ell_1 + x)] \cdots [\ell_d; \epsilon_{d-1}\ell_{d-1}, \cdots, \epsilon_2\ell_2, \epsilon_1(\ell_1 + x)] = \frac{Q_1}{Q_0}\frac{Q_2}{Q_1} \cdots \frac{Q_d}{Q_{d-1}}$$

$$= \frac{Q_d}{Q_0} = Q_d.$$

Let P_i^* and Q_i^* be defined as in (16), with α_1 replaced by $\alpha_1^* = \ell_1 + x^*$.

Now

$$\begin{aligned} \frac{P_d}{Q_d} &= [\,0;\, \ell_1 + x,\, \epsilon_1\ell_2,\, \cdots\,,\, \epsilon_{d-1}\ell_d]\,, \\ \frac{P_d^*}{Q_d^*} &= [\,0;\, \ell_1 + x^*,\, \epsilon_1\ell_2,\, \cdots\,,\, \epsilon_{d-1}\ell_d] \end{aligned}$$

and

$$P_d = P_d^*.$$

Thus we find that

$$\begin{aligned} &\frac{\ell_1 + x}{\ell_1 + x^*}\frac{[\ell_2; \epsilon_1(\ell_1 + x)]}{[\ell_2; \epsilon_1(\ell_1 + x^*)]} \cdots \frac{[\ell_d; \epsilon_{d-1}\ell_{d-1}, \cdots, \epsilon_2\ell_2, \epsilon_1(\ell_1 + x)]}{[\ell_d; \epsilon_{d-1}\ell_{d-1}, \cdots, \epsilon_2\ell_2, \epsilon_1(\ell_1 + x^*)]} = \\ &= \frac{Q_d}{Q_d^*} = \frac{Q_d}{P_d}\frac{P_d^*}{Q_d^*} = \frac{x + [\ell_1; \epsilon_1\ell_2, \ldots, \epsilon_{d-1}\ell_d]}{x^* + [\ell_1; \epsilon_1\ell_2, \ldots, \epsilon_{d-1}\ell_d]} \\ &= \frac{x + \frac{1}{y}}{x^* + \frac{1}{y}} = \frac{1 + xy}{1 + x^*y}\,. \end{aligned}$$

Therefore,

$$m_n(x, y) \Leftrightarrow m_n(x^*, y) = \frac{1}{\log G}\log\left(\frac{1 + xy}{1 + x^*y}\right) + \bar{\lambda}(\mathcal{I}_{x,y} \setminus \mathcal{I}_{x^*,y})\mathcal{O}\left((\frac{3}{5})^{n-d}\right) .\Box$$

Remarks The proof of Theorem 2 now follows from Lemma 1 and (15). It is similar to the proof of [DK, Theorem 2], the essential difference being the fact that now the NICF-expansion of y is considered, instead of the RCF-expansion of y. As is well-known (and this follows directly from the fact that the NICF is an S-expansion, see also the next section), the sequence of NICF-convergents $(p_k/q_k)_{k \geq -1}$ forms a subsequence of the sequence of RCF-convergents of y. Thus it is possible to obtain sharper bounds, e.g., one has that

$$\left| y \Leftrightarrow \frac{p_k}{q_k} \right| < \frac{g}{q_k^2}\,.$$

Theorem 3 also follows from Lemma 1. Since Theorem 3 plays a key role in the proof of our main result, Theorem 6, and Theorem 2 is just a nice result along the way, we will leave the proof of Theorem 2 to the reader.

Proof of Theorem 3 Let $\bar{b} = (\, b_1,\, \epsilon_1 b_2, \ldots,\, \epsilon_{n-1} b_n)$ be some arbitrary admissible sequence of length n for the NICF, i.e., ϵ_i and b_i satisfy (6), and let $Z(\bar{b})$ be defined as before. For each $i = 1, \cdots, m$, let

$$Z^i(\bar{b}) \;=\; Z(\bar{b}) \cap \{y \in [0, \frac{1}{2}];\, (x, y) \in l_i \text{ for some } x \in [\Leftrightarrow g^2, g]\}\,,$$

and define $L^i_n(\bar{b})$, $R^i_n(\bar{b})$ as follows

$$[L^i_n(\bar{b}), R^i_n(\bar{b})] \;:=\; f_i^{-1}(Z^i(\bar{b}))\,.$$

Set

$$U_n \;:=\; \bigcup_{i=1}^{m} \bigcup_{\bar{b}} B^i_n(\bar{b})\,,$$

where

$$B^i_n(\bar{b}) \;:=\; \begin{cases} [L^i_n(\bar{b}), R^i_n(\bar{b})] \times Z(\bar{b}) & \text{if } f_i([\beta_i, \gamma_i]) \cap Z(\bar{b}) \neq \emptyset\,, \\ \\ \emptyset & \text{otherwise,} \end{cases}$$

see also Figure 2. Let

$$\beta \;:=\; \min_{1 \le i \le m} \beta_i \;\text{ and }\; \gamma \;:=\; \max_{1 \le i \le m} \gamma_i\,,$$

and define a partition $\mathcal{P}(n)$ of $[\beta, \gamma]$ by

$$\mathcal{P}(n) \;:=\; \bigvee_{i=1}^{m} \Big\{[L^i_n(\bar{b}), R^i_n(\bar{b})],\, [\beta, \beta_i],\, [\gamma_i, \gamma] \;:\; \bar{b} \text{ is NICF-admissible of lenght } n\Big\}\,.$$

Figure 2

Let $d = \lfloor \frac{n}{2} \rfloor + 1$ and $\overline{\mathcal{P}}_d = \mathcal{P}(n) \times \mathcal{F}_d$, with

$$\mathcal{F}_d = \{Z(\bar{b}) : \bar{b} \text{ is NICF-admissible of lenght } d\}\,,$$

and let $\bar{a} = (\epsilon_1 a_1,\, \epsilon_2 a_2,\, \cdots,\, \epsilon_n a_n)$ be a SCF-admissible sequence, i.e., (7) is satisfied. Define for $\bar{a}$ the sequence $\tilde{a}$ by $\tilde{a} := (a_n,\, \epsilon_n a_{n-1},\, \cdots,\, \epsilon_2 a_1)$. Then $\tilde{a}$ is a NICF-admissible sequence, i.e., (6) is satisfied. We denote by

$$\Delta(\bar{a}) = \{x \in [\Leftrightarrow g^2, g);\, \mathrm{SCF}(x) = [\,0;\, \epsilon_1 a_1,\, \epsilon_2 a_2,\, \cdots,\, \epsilon_n a_n,\, \underbrace{\cdots\cdots\cdots}_{\text{"free"}}\,]\,,$$

a *cylinder set* (or *fundamental interval*) for Hurwitz' singular continued fraction.

Note that

$$\mathcal{T}_g^n\left(\bigcup_{\epsilon_1\in\{-1,1\}} \Delta(\epsilon_1 a_1, \epsilon_2 a_2, \cdots, \epsilon_n a_n) \times [0, \frac{1}{2}]\right) = [\Leftrightarrow g^2, g] \times Z(a_n, \epsilon_n a_{n-1}, \cdots, \epsilon_2 a_1)\,,$$

with the convention that $\Delta(\Leftrightarrow 2, \epsilon_2 a_2, \cdots, \epsilon_n a_n) = \emptyset$.

Thus,

$$\begin{aligned}
&\mathcal{T}_g^n(E_n(K) \times [0, \frac{1}{2}]) \\
&= \mathcal{T}_g^n(\bigcup_{\substack{\text{all SCF-}\\ \text{admissible}\\ (\epsilon_1 a_1, \cdots, \epsilon_n a_n)}} (E_n(K) \cap \Delta(\epsilon_1 a_1, \cdots, \epsilon_n a_n)) \times [0, \frac{1}{2}]) \\
&= \mathcal{T}_g^n(\bigcup_{\substack{\text{all SCF-}\\ \text{admissible}\\ (a_1, \cdots, \epsilon_n a_n)}} \bigcup_{\epsilon_1\in\{-1,1\}} (E_n(K) \cap \Delta(\epsilon_1 a_1, \cdots, \epsilon_n a_n)) \times [0, \frac{1}{2}]) \\
&= \bigcup_{\substack{\text{all SCF-}\\ \text{admissible}\\ (a_1, \cdots, \epsilon_n a_n)}} \left(\mathcal{T}_g^n(E_n(K) \cap \bigcup_{\epsilon_1\in\{-1,1\}} \Delta(\epsilon_1 a_1, \cdots, \epsilon_n a_n))\right) \times Z(\tilde{a})\,.
\end{aligned}$$

Since K is simply connected

$$K \setminus U_d \subset K \setminus U_n \subset \mathcal{T}_g^n \left(E_n(K) \times [0, \frac{1}{2}] \right) \subset K \cup U_n \subset K \cup U_d \,,$$

where

$$K \setminus U_d \;=\; \bigcup \{ W \in \overline{\mathcal{P}}_d : \, W \subset K \setminus U_d \} \,,$$

and similarly for $K \cup U_d$. By Lemma 1 one has

$$\bar{\lambda} \left(\mathcal{T}_g^{-n} (K \setminus U_d) \right) \;=\; \bar{\mu}_g (K \setminus U_d)) + \mathcal{O}((\frac{3}{5})^{n-d}) \,,$$

and a similar statement for $K \cup U_d$. Using techniques from [K1], Section 1, one has for $\bar{b}$ an NICF-admissible sequence of length d, corresponding to a positive rational number p_d/q_d

$$Z(\bar{b}) \;=\; \begin{cases} \left(\frac{2p_d - p_{d-1}}{2q_d - q_{d-1}}, \, \frac{2p_d + p_{d-1}}{2q_d + q_{d-1}} \right) & \text{if } b_d > 2, \\ \\ \left(\frac{p_d}{q_d}, \, \frac{2p_d + p_{d-1}}{2q_d + q_{d-1}} \right) & \text{if } b_d = 2, \end{cases}$$

where[2] p_{d-1}/q_{d-1} and p_d/q_d are the last two NICF-convergents of p_d/q_d, and b_d is the last *partial quotient* (i.e., digit) of $\bar{b}$.

Since $|p_{d-1}q_d \Leftrightarrow p_d q_{d-1}| = 1$ and any sequence of NICF-convergents is a subsequence of a sequence of RCF-convergents,

$$\lambda(Z(\bar{b})) \,\le\, \frac{4}{(2q_d \Leftrightarrow q_{d-1})(2q_d + q_{d-1})} \,\le\, \frac{4}{\mathcal{F}_d \mathcal{F}_{d+1}} \,,$$

where $\mathcal{F}_n$, $n \geq 0$, is the Fibonacci sequence $0, 1, 1, 2, 3, \cdots$. ¿From this and (15) one obtains

$$\begin{aligned} \bar{\mu}_g (\bigcup_{\bar{b}} B_d^i(\bar{b})) \;\; &\le \;\; \frac{1}{\log G} \frac{2}{G^2} \sum_{\bar{b}} \bar{\lambda}(B_d^i(\bar{b})) \\ &\le \;\; \frac{1}{\log G} \frac{2}{G^2} \frac{4(\gamma_i \Leftrightarrow \beta_i)}{\mathcal{F}_d \mathcal{F}_{d+1}} \,. \end{aligned}$$

Since

$$\frac{G^{2d}}{5} \,\le\, \mathcal{F}_d \mathcal{F}_{d+1}$$

it follows that

$$\bar{\mu}_g(U_d) \,\le\, \left(\frac{5}{G^2} \sum_{i=1}^{m} (\gamma_i \Leftrightarrow \beta_i) \right) g^n \,.$$

The desired result now follows from the above and the observations that $\frac{3}{5} < g$ and $\lambda(E_n(K)) = \bar{\lambda}(E_n(K) \times [0, \frac{1}{2}])$. $\square$

Remark It should be clear that Theorem 3 remains correct if K is a finite union of simply connected subsets K_i of Ω_g, each satisfying the conditions of Theorem 3 imposed upon K.

[2] If $\alpha < \beta$, (β, α) is understood to be the interval (α, β).

We finish this section with a number of direct corollaries of Theorem 3. Let $\xi \in [-g^2, g)$ be an irrational number, with SCF-expansion (4) (where $\alpha = g$), sequence of SCF-convergents $(A_n/B_n)_{n\geq -1}$, and let $(T_n, V_n)_{n\geq -1}$ be defined as before, i.e.,

$$(T_n, V_n) \;=\; \mathcal{T}_g^n(\xi, 0).$$

Then we define the *approximation coefficients* $\Theta_n = \Theta_n(\xi)$ by

$$\Theta_n(\xi) \;=\; B_n^2 \left| \xi - \frac{A_n}{B_n} \right| , \; n \geq 1,$$

and one has that $\Theta_n < g$. We have the following corollaries.

Corollary 1 *Let* $K_n(z_1, z_2) = \{\xi \in [-g^2, g) \setminus \mathbf{Q} : \Theta_{n-1} \leq z_1, \Theta_n \leq z_2\}$ *for* $0 \leq z_1, z_2 \leq g$. *Furthermore, let* Γ_1 *denote the interior of the quadrangle with vertices* $(0,0)$, $(\frac{1}{2}, 0)$ $(\frac{1}{2+g}, \frac{2g}{2+g})$ *and* $(0, g)$, *and* Γ_{-1} *the interior of the quadrangle with vertices* $(0,0)$, $(\frac{1}{2}, 0)$ $(g, 2g^3)$ *and* $(0, g^2)$. *Then*

$$\lambda(K_n(z_1, z_2)) \;=\; H_g(z_1, z_2) + \mathcal{O}(g^n)\,,$$

where H_g *is the distribution function with density* h_g *given by*

$$h_g(\alpha, \beta) \;=\; \begin{cases} \frac{1}{\log G}\frac{1}{\sqrt{1-4\alpha\beta}} & \text{if } (\alpha,\beta) \in \Gamma_1 \setminus \Gamma_{-1}\,, \\[2ex] \frac{1}{\log G}\left(\frac{1}{\sqrt{1-4\alpha\beta}} + \frac{1}{\sqrt{1+4\alpha\beta}}\right) & \text{if } (\alpha,\beta) \in \Gamma_1 \cap \Gamma_{-1}\,, \\[2ex] \frac{1}{\log G}\frac{1}{\sqrt{1+4\alpha\beta}} & \text{if } (\alpha,\beta) \in \Gamma_{-1} \setminus \Gamma_1\,, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

The proof of Corollary 1 follows directly from Theorem 3 and the fact that

$$\Theta_{n-1} \;=\; \frac{V_n}{1 + T_n V_n} \quad \text{and} \quad \Theta_n \;=\; \frac{\epsilon_{n+1} T_n}{1 + T_n V_n}, \; n \geq 2.$$

Notice that we moreover have that for all ξ the sequence (Θ_{n-1}, Θ_n), $n \geq 1$, is a sequence in $\Gamma_1 \cup \Gamma_{-1}$, see also [K1], Section 6, and [J].
Choosing in Corollary 1 z_1 to be equal to g yields the following corollary, which is analogous to a theorem by D.E. Knuth [Kn] for the RCF-expansion.

Corollary 2 *Let* $J_n(z) = \{\xi \in [-g^2, g) \setminus \mathbf{Q} : \Theta_n \leq z\}$ *for* $0 \leq z \leq g$. *Then*

$$\lambda(J_n(z)) \;=\; F_g(z) + \mathcal{O}(g^n)\,,$$

where F_g *is the distribution function given by*

$$F_g(z) \;=\; \begin{cases} \frac{z}{\log G} & \text{if } 0 \leq z \leq g^2, \\[2ex] \frac{1}{\log G}\left(z - G^2 z + \log(G^2 z) + 1\right) & \text{if } g^2 \leq z \leq g, \\[2ex] 1 & \text{if } g \leq z \leq 1. \end{cases}$$

In 1983, W. Bosma, H. Jager and F. Wiedijk [BJW] obtained the "counterpart" of Corollary 2. They showed that for almost all $\xi \in [-g^2, g)$ (with respect to the Lebesgue measure) and $z \in [0, \frac{1}{2}]$ one has that the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N;\, \Theta_n \le z\}$$

exists, and equals $F_g(z)$. We speak here of counterpart because the two theorems are like the two faces of the same coin. One face deals with the pointwise convergence of ergodic averages, the other with weak convergence of probability measures with a given speed of convergence.

In [J], H. Jager showed that for a generic $\xi \in [0,1)$ the sequence $\mathcal{T}^n(\xi, 0)$ is distributed over Ω according to the density of the invariant measure $(\log 2)^{-1}(1+xy)^{-2}$. Due to the way S-expansions in general - and the SCF-expansion in particular - are defined it now at once follows that for a generic $\xi \in [-g^2, g)$ the sequence $\mathcal{T}_g(\xi, 0) = (T_n, V_n)$ is distributed over Ω_g according to the density function $(\log G)^{-1}(1+tv)^{-2}$, which is the density of the invariant measure of $\mathcal{T}_g$. From this and Birkhoff's Ergodic Theorem it follows that for any $K \subset \Omega_g$ satisfying the hypothesis of Theorem 3 and for almost every ξ (in the sence of Lebesgue) the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N;\, (T_n, V_n) \in K\}$$

exists, and equals $\bar{\mu}_g(K)$.

3 S-expansions

In this section we will recall some facts on S-expansions, which have been dealt with in [K1].

Let ξ be an irrational number, and let (4) be some SRCF-expansion of ξ. Suppose that we have for a certain $k \ge 0:\ a_{k+1} = 1\,,\ \varepsilon_{k+1} = \varepsilon_{k+2} = 1$. The operation by which the continued fraction (2) is replaced by[3]

$$[\, a_0;\, \varepsilon_1 a_1,\, \ldots\,,\, \varepsilon_{k-1} a_{k-1},\, \varepsilon_k (a_k + 1),\, -(a_{k+2} + 1),\, \varepsilon_{k+3} a_{k+3},\, \ldots \,]\ ,$$

which again is a SRCF-expansion of x, with convergents, say, $(c_n/d_n)_{n\ge -1}$, is called *the singularisation of the partial quotient* a_{k+1} *equal to 1.* One easily shows that $(c_n/d_n)_{n\ge -1}$ is obtained from $(r_n/s_n)_{n\ge -1}$ by skipping the term r_k/s_k. See also [K1], sections 2 and 4.

A simple way to derive a strategy for singularization is given by a *singularization area* S. Here we will choose S to be a subset of the natural extension

$$(\Omega, \overline{\mathcal{B}}, \bar{\mu}, \mathcal{T})$$

of the RCF. Here $\Omega := [0,1) \times [0,1]$, $\overline{\mathcal{B}}$ is the collection of Borel sets of Ω, and the two-dimensional RCF-operator $\mathcal{T}$ is given by

$$\mathcal{T}(\xi, \eta) := \left(T\xi,\, \frac{1}{\lfloor \xi^{-1} \rfloor + \eta} \right)\,,\ (\xi, \eta) \in \Omega,\, \xi \ne 0.$$

Finally, $\bar{\mu}$ is the invariant measure with density $(\log 2)^{-1}(1+xy)^{-2}$. It is well-known that the dynamical system $(\Omega, \overline{\mathcal{B}}, \bar{\mu}, \mathcal{T})$ is Bernoulli.

[3]In case $k = 0$ this comes down to replacing (4) by $[\, a_0 + 1;\, -(a_2+1),\, \varepsilon_3 a_3,\, \varepsilon_4 a_4,\, \ldots]$.

Definition 1 *A subset S from Ω is called a singularisation area if it satisfies*

(I) $S \in \mathcal{B}$ *and* $\mu(\partial S) = 0$;

(II) $S \subset ([\frac{1}{2}, 1) \setminus \mathbf{Q}) \times [0,1]$;

(III) $\mathcal{T}(S) \cap S = \emptyset$.

Remark It easily follows from Definition 1 and Figure 3 that

$$0 \leq \bar{\mu}(S) \leq 1 \Leftrightarrow \frac{\log G}{\log 2} = 0.3057\ldots,$$

see also [K1], Theorem (4.7). A singularisation area is called *maximal* in case

$$\bar{\mu}(S) = 1 \Leftrightarrow \frac{\log G}{\log 2} = 0.3057\cdots.$$

Figure 3

Definition 2 *Let S be a singularisation area and let ξ be a real irrational number. The S-expansion of ξ is that semi-regular continued fraction expansion converging to ξ, which is obtained from the RCF-expansion (1) of ξ by singularizing d_{n+1} if and only if $\mathcal{T}^n(\xi, 0) \in S$, $n \geq 0$.*

Some examples of singularisation areas are[4]

1. $S_{\frac{1}{2}} := [\frac{1}{2}, 1) \times [0, g]$ yields the *nearest integer continued fraction* (NICF). The area $S_{\frac{1}{2}}$ is maximal;

2. $S_g := \{(T, V) \in \Omega;\ (g, 1) \times [0, 1]\}$; this area yields Hurwitz' *singular continued fraction* (SCF); it is maximal, see [K2].

3. $S_{\text{ocf}} := \{(T, V) \in \Omega;\ V < \min(T, \frac{2T-1}{1-T})\}$; this area yields the OCF and is also maximal.

4. $S_{\text{dcf}} := \{(T, V) \in \Omega;\ \frac{T}{1+TV} > \frac{1}{2}\}$; this area yields the *diagonal continued fraction* (DCF) *of Minkowski;* it is not maximal, see [K3].

[4] All these areas need some minor modifications in order to satisfy the above definition 1, see [K1], (4.6)ii).

Remark Let $\xi \in [0,1)$ be some irrational number, with RCF-expansion (1). From Definition 2 and the above examples one easily sees that the following algorithm yields the NICF-expansion of ξ

> $>>$singularize in each block of m consecutive partial quotients $d_{n+1} = 1, \cdots, a_{n+m} = 1$, where $m \in \mathbf{N} \cup \{\infty\}$, $a_{n+m+1} \neq 1$ and $a_n \neq 1$ in case $n > 0$, the first, third, fifth, etc. partial quotient$<<$

while doing the same in case m is odd, and in case m is even

> $>>$singularize the first, third, fifth, etc. partial quotient$<<$

yields Hurwitz' SCF. The OCF "combines" both algorithms; first one singularizes the first and last 1's in every block of m consecutive 1's, and then "move in".
That the NICF, SCF and OCF-algorithms singularize blocks of odd length in the same way reflects the fact that these expansions are maximal; There is only one way to "throw out" (= to singularize) as many 1's as possible in a block of odd length. In a block of even length a "jump" has to be made somewhere, see also [K2]. E.g. for the NICF one makes this jump at the end, and for the SCF at the beginning. The OCF chooses the jump in such a way, that one is left with the smallest possible θ_k's. One can show, see [BK2], that for the OCF the jump takes place in the middle of the block.

That for a maximal S-expansion one always makes the maximal number of "throw-outs" in any block of consecutive 1's has several nice consequences. One is, that maximal S-expansions are metrically isomorphic, a fact we will use in Section 4. Another consequence is, that a *Heilbronn-theorem* for maximal S-expansions follows trivially from Rieger's 1978 Heilbronn-theorem for the NICF [Rie2]. In order to see this, recall that each rational number $p/q \in [0,1)$ has a unique finite RCF-expansion $p/q = [\,0; d_1, \cdots, d_\ell]$, with $d_\ell \neq 1$ (clearly $[\,0; d_1, \cdots, d_\ell] = [\,0; d_1, \cdots, d_\ell \Leftrightarrow 1, 1]$, but the latter expansion cannot be obtained via $T \Leftrightarrow$ and is therefore considered "illegal"). Thus the length of the S-expansion of p/q is the same as the length of the NICF-expansion of p/q in case S is maximal.

Proposition 1 *Let S be a maximal singularization area (with*[5] *$\bar{S}^o = S^o$ and $(\xi,\eta) \in \partial S \setminus S$ implies $\mathcal{T}(\xi,\eta) \in S$ or $\mathcal{T}^{-1}(\xi,\eta) \in S$). Let a and N be positive integers, such that $(a,N) = 1$. Denote by $\ell(a) = \ell(a,N)$ the length of the S-expansion of a/N, i.e., if*

$$\frac{a}{N} = [\,b_0; \varepsilon_1 b_1, \cdots, \varepsilon_\ell b_\ell\,]$$

is the S-expansion of a/N, then $\ell(a) = \ell$. Finally, let φ denote the Euler φ-function and let $\sigma_{-1}(N) := \sum_{d|N} 1/d$. Then

$$\sum_{\substack{1 \leq a \leq N \\ (a,N)=1}} \ell(a) = \frac{12 \log G}{\pi^2} \varphi(N) \log N + \mathcal{O}\left(N \sigma_{-1}^3(N)\right).$$

[5]This to prevent the existence of an exceptional subset of $\bar{S}$ of measure 0 where one does not singularize as many 1's as possible.

Let S be a singularization area and let ξ be a real irrational number, with RCF-expansion (1) and RCF-convergents $(P_n/Q_n)_{n\geq -1}$. Furthermore, let $[\,a_0;\,\varepsilon_1 a_1,\,\ldots\,,\,\varepsilon_k a_k,\,\ldots]$ be the S-expansion of ξ, with convergents r_k/s_k, $k \geq \Leftrightarrow 1$. Define the shift t by

$$t(\xi \Leftrightarrow a_0) \;:=\; [\,0;\,\varepsilon_2 a_2,\,\ldots,\,\varepsilon_k a_k,\,\ldots]\;.$$

For a fixed ξ and for $k \geq 0$ we put

$$t_k \;:=\; t^k(\xi \Leftrightarrow a_0) \;=\; [\,0;\,\varepsilon_{k+1}a_{k+1},\,\varepsilon_{k+2}a_{k+2},\,\ldots]\quad\text{and}\quad v_k \;:=\; s_{k-1}/s_k\,,$$

where

$$v_k \;=\; [\,0;a_k,\,\varepsilon_k a_{k-1},\,\ldots\,,\,\varepsilon_2 a_1\,],\; k\geq 1;\;\; v_0 = 0\,.$$

see also [K1], (1.4) and (5.1).
We have the following theorem.

Theorem 4 *Let S be a singularization area and put $\Delta_S := \Omega \setminus S$, $\Delta_S^- := \mathcal{T}S$ and $\Delta_S^+ := \Delta_S\setminus\Delta_S^-$. Let ξ be a real number, with RCF-expansion (1) and RCF-convergents $(P_n/Q_n)_{n\geq -1}$. Then one has*

1. *The system $(\Delta_S,\,\mathcal{B},\,\rho_S,\,\mathcal{O}_S)$ forms an ergodic system. Here ρ_S is the probability measure on $(\Delta_S,\mathcal{B})$ with density $((1\Leftrightarrow\mu(S))\log 2)^{-1}(1+xy)^{-2}$ and the map $\mathcal{O}_S$ is induced by $\mathcal{T}$ on Δ_S.*
2. *$\mathcal{T}^n(\xi,0)\in S \;\Leftrightarrow\; P_n/Q_n$ is not an S–convergent;*
3. *P_n/Q_n is not an S–convergent $\;\Rightarrow\;$ both P_{n-1}/Q_{n-1} and P_{n+1}/Q_{n+1} are S-convergents;*
4. *$\mathcal{T}^n(\xi,0)\in\Delta_S^+ \;\Leftrightarrow\; \exists k : \begin{cases} r_{k-1} = P_{n-1}, & r_k = P_n \\ s_{k-1} = Q_{n-1}, & s_k = Q_n \end{cases}$ and $\mathcal{T}^n(\xi,0) \;=\; (t_k,\,v_k)$;*
5. *$\mathcal{T}^n(\xi,0)\in\Delta_S^- \;\Leftrightarrow\; \exists k : \begin{cases} r_{k-1} = P_{n-2}, & r_k = P_n \\ s_{k-1} = Q_{n-2}, & s_k = Q_n \end{cases}$ and $\mathcal{T}^n(\xi,0) \;=\; (\frac{-t_k}{1+t_k},\,1\Leftrightarrow v_k)$;*

(See also [K1], Theorem (5.3)).

In view of Theorem 2 we define the map $\mathcal{M} : \Delta_S \to \mathbf{R}^2$ by

$$\mathcal{M}(T,V) \;:=\; \begin{cases} (T,V) & (T,V)\in\Delta_S^+\,; \\ (\frac{-T}{1+T},1\Leftrightarrow V) & (T,V)\in\Delta_S^-\,. \end{cases}$$

We have the following theorem.

Theorem 5 *Let S be a singularization area and put $\Omega_S := \mathcal{M}(\Delta_S)$. Let $\mathcal{B}$ be the collection of Borel subsets of Ω_S and let μ_S be the probability measure on $(\Omega_S,\,\mathcal{B})$, defined by*

$$\mu_S(E) \;:=\; \rho_S(\mathcal{M}^{-1}(E)),\; E\in\mathcal{B}\,.$$

Furthermore, if we define the map $\mathcal{T}_S : \Omega_S \to \Omega_S$ by

$$\mathcal{T}_S(t,v) \;:=\; \mathcal{M}(\mathcal{O}_S(\mathcal{M}^{-1}(t,v))),\; (t,v)\in\Omega_S\,,$$

then $\mathcal{T}_S$ is conjugate to $\mathcal{O}_S$ by $\mathcal{M}$ and $(\Omega_S,\,\mathcal{B},\,\mu_S,\,\mathcal{T}_S)$ forms an ergodic system with density $((1\Leftrightarrow\mu(S))\log 2)^{-1}(1+tv)^{-2}$. Finally, for almost all $x\in[0,1)$ the[6] sequence $(t_k,v_k)_{k\geq 0}$ is

[6] All almost sure statements in this paper are with respect to the Lebesgue measure.

distributed over Ω_S according to this density.

Remarks

(I) From Theorem 4 and Theorem 5 it follows that $(\Omega_S, \mathcal{B}, \mu_S, \mathcal{T}_S)$, which is the two-dimensional ergodic system underlying the corresponding S-expansion, is isomorphic (via the $\mathcal{M}$-map) to an induced system of (Ω, T) with return-time bounded by 2.

(II) One can show that $\mathcal{T}_S$ can be written in the following way

$$\mathcal{T}_S(t,v) \;=\; \left(\,|\frac{1}{t}| - f_S(t,v)\,,\; \frac{1}{\operatorname{sgn}(t)\cdot v + f_S(t,v)}\,\right), \quad \text{for } (t,v) \in \Omega_S.$$

Furthermore one has

$$a_{k+1} \;=\; f_S(t_k, v_k),\; k \geq 0, \;\text{ where } (t_0, v_0) = (x - a_0, 0).$$

Thus we see that the S-expansion is the process associated with $\mathcal{T}_S$ and f_S.

For the afore mentioned first three examples we have

$$f_{\frac{1}{2}}(t,v) \;=\; \lfloor |\frac{1}{t}| + \frac{1}{2} \rfloor \;\text{ (NICF) },\;\; f_g(t,v) \;=\; \lfloor |\frac{1}{t}| + g^2 \rfloor \;\text{ (SCF)}$$

and

$$f_{\mathrm{ocf}}(t,v) \;=\; \lfloor |\frac{1}{t}| + \frac{\lfloor |\frac{1}{t}| \rfloor + \operatorname{sgn}(t)v}{2(\lfloor |\frac{1}{t}| \rfloor + \operatorname{sgn}(t)v) + 1} \rfloor \;\text{ (OCF)}.$$

(III) In case of the OCF the last statement of Theorem 5 says that for a.e. $\xi \in [-\frac{1}{2}, \frac{1}{2})$ the sequence $(\mathcal{T}^n_{\mathrm{ocf}})_{n \geq 0}$ is distributed according to the density function $(\log G)^{-1}(1+tv)^{-2}$, i.e., it behaves like the orbit of a generic point.

4 Gauss-Kusmin for maximal S-expansions

Now we concentrate on maximal singularization areas S (like those for the NICF, SCF and OCF), i.e., $\mu(S) = 1 - \frac{\log G}{\log 2} = 0.3057\cdots$. In [K2] it was shown that for such singularization areas the systems $(\Delta_S, \mathcal{B}, \rho_S, \mathcal{O}_S)$ and $(\Delta_g, \mathcal{B}, \rho_g, \mathcal{O}_g)$ are isomorphic via a map $\psi\,:\,\Delta_S \to \Delta_g$, given by

$$\psi(\xi,\eta) := \begin{cases} (\xi,\eta) & (\xi,\eta) \in G_1 \;:=\; \Delta_S \cap \Delta_g, \\ \\ \mathcal{T}^{-1}(\xi,\eta) & (\xi,\eta) \in G_2 \;:=\; \Delta_S \setminus \Delta_g, \end{cases} \tag{17}$$

and define moreover $G_3 \;:=\; \Delta_g \setminus \Delta_S\,,\; G_4 \;:=\; S \cap S_g$ (in Figure 4 we have depicted $G_1, \ldots, G_4$ in case $S = S_{\mathrm{ocf}}$).

Figure 4

We now will prove the following theorem, which is the main result of this paper.

Theorem 6 *Let $\mathcal{K} \subset \Omega_{\rm ocf}$ be a simply connected subset of $\Omega_{\rm ocf}$, satisfying the conditions of Theorem 3. Putting*

$$D_n(\mathcal{K}) := \{\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2});\, \mathcal{T}^n_{ocf}(\xi, 0) \in \mathcal{K}\}\,,$$

one has

$$\lambda(D_n(\mathcal{K})) = \bar{\mu}_{ocf}(\mathcal{K}) + \mathcal{O}(g^n)\,,$$

where the constant in the big-$\mathcal{O}$ symbol is uniform.

Remark It should be mentioned that the same result holds (with the same proof) for any maximal S-expansion, see also the final remarks at the end of this section.

Let $\mathcal{K} \subset \Omega_{\rm ocf}$ be as in Theorem 6, and define

$$\begin{aligned}
\mathcal{U}_{\mathcal{K}} &:= \{\xi \in [\Leftrightarrow\frac{1}{2}, \Leftrightarrow g^2) : \mathcal{T}^n_{\rm ocf}(\xi, 0) \in \mathcal{K}\}\,,\\
\mathcal{V}_{\mathcal{K}} &:= \{\xi \in [\Leftrightarrow g^2, 0) : \mathcal{T}^n_{\rm ocf}(\xi, 0) \in \mathcal{K}\}\,,\\
\mathcal{W}_{\mathcal{K}} &:= \{\xi \in [0, \frac{1}{2}) : \mathcal{T}^n_{\rm ocf}(\xi, 0) \in \mathcal{K}\}\,.
\end{aligned}$$

Lemma 2 *Let $\mathcal{K} \subset \Omega_{\rm ocf}$ be a simply connected subset of $\Omega_{\rm ocf}$, satisfying the conditions of Theorem 3, then*

$$\lambda(\{\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2}) : \mathcal{T}^n_{ocf}(\xi, 0) \in \mathcal{K}\}) = \lambda(\{\xi \in [\Leftrightarrow g^2, g) : \mathcal{T}^n_g(\xi, 0) \in \mathcal{H}_{\mathcal{K}}\})\,,$$

where

$$\mathcal{H}_{\mathcal{K}} := \mathcal{M}(\psi(\mathcal{M}^{-1}(\mathcal{K})))\,.$$

Proof From the definitions of $\mathcal{M}$ and ψ, and by the S-mechanism (applied to $S_{\rm ocf}$ and S_g) it follows that

$$\begin{aligned}
\xi \in \mathcal{U}_{\mathcal{K}} \;&\Leftrightarrow\; \mathcal{O}^n_{\rm ocf}(\frac{\Leftrightarrow\xi}{1+\xi}, 1) \in \mathcal{M}^{-1}(\mathcal{K}) \;\text{ and }\; 1+\xi \in [\frac{1}{2}, g)\\
&\Leftrightarrow\; \mathcal{O}^n_g(1+\xi, 0) \in \psi(\mathcal{M}^{-1}(\mathcal{K})) \;\text{ and }\; 1+\xi \in [\frac{1}{2}, g)\\
&\Leftrightarrow\; \mathcal{T}^n_g(1+\xi, 0) \in \mathcal{H}_{\mathcal{K}} \;\text{ and }\; 1+\xi \in [\frac{1}{2}, g)\,,
\end{aligned}$$

where we used that

$$\mathcal{T}(1+\xi, 0) = \left(\frac{\Leftrightarrow\xi}{1+\xi}, 1\right)\,,$$

in case $1+\xi \in [\frac{1}{2}, g)$. Furthermore,

$$\begin{aligned}
\xi \in \mathcal{V}_{\mathcal{K}} \;&\Leftrightarrow\; \mathcal{O}^n_{\rm ocf}(\frac{\Leftrightarrow\xi}{1+\xi}, 1) \in \mathcal{M}^{-1}(\mathcal{K}) \;\text{ and }\; 1+\xi \in [g, 1)\\
&\Leftrightarrow\; \mathcal{T}^n_g(\xi, 0) \in \mathcal{H}_{\mathcal{K}} \;\text{ and }\; \xi \in [\Leftrightarrow g^2, 0)\,,
\end{aligned}$$

and

$$\xi \in \mathcal{W}_{\mathcal{K}} \quad \Leftrightarrow \quad \mathcal{T}_g^n(\xi,0) \in \mathcal{H}_{\mathcal{K}} \ \text{ and } \ \xi \in [0,\frac{1}{2}) \,.$$

Now the lemma follows from the above relations. $\Box$

Proof of Theorem 6 First note that due to the fact that the density function $(\log G)^{-1}(1+tv)^{-2}$ is invariant under $\mathcal{M}$, $\mathcal{T}_{\mathrm{ocf}}$, $\mathcal{T}$ and $\mathcal{T}_g$, one has

$$\bar{\mu}_{\mathrm{ocf}}(\mathcal{K}) \ = \ \bar{\mu}_g(\mathcal{H}_{\mathcal{K}}) \,.$$

Next, for $n \geq 1$ one has, due to Lemma 2

$$\lambda(D_n(\mathcal{K})) \Leftrightarrow \bar{\mu}_{\mathrm{ocf}}(\mathcal{K}) \ = \ \lambda(E_n(\mathcal{H}_{\mathcal{K}})) \Leftrightarrow \bar{\mu}_g(\mathcal{H}_{\mathcal{K}}) \,,$$

where $E_n(\mathcal{H}_{\mathcal{K}})$ is defined as in Theorem 3, viz.

$$E_n(\mathcal{H}_{\mathcal{K}}) \ := \{\xi \in [\Leftrightarrow g^2, g);\, \mathcal{T}_g(\xi,0) \in \mathcal{H}_{\mathcal{K}}\} \,.$$

The theorem now follows from Theorem 3, as soon as we have established that $\mathcal{H}_{\mathcal{K}}$ is a finite union of simply connected subsets of Ω_g, each satisfying the conditions from Theorem 3.

Let $G_1, \cdots, G_4$ be defined as in (17), and put

$$\mathcal{K}_1 \ := \ \mathcal{K} \cap G_1,\, \mathcal{K}_2 \ := \ \mathcal{K} \cap \mathcal{M}(G_2), \ \text{ and } \ \mathcal{K}_3 \ := \ \mathcal{K} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2) \,,$$

see Figure 5.

Figure 5

¿From the definitions of $\mathcal{M}$ and ψ it now follows that

$$\mathcal{M}^{-1}(\mathcal{K}_1) = \mathcal{K}_1, \ \mathcal{M}^{-1}(\mathcal{K}_2) \subset G_2 \ \text{ and } \ \mathcal{M}^{-1}(\mathcal{K}_3)$$

and

$$\psi(\mathcal{M}^{-1}(\mathcal{K}_1)) = \mathcal{K}_1, \ \psi(\mathcal{M}^{-1}(\mathcal{K}_2)) \subset G_3 \ \text{ and } \ \psi(\mathcal{M}^{-1}(\mathcal{K}_3)) = \mathcal{M}^{-1}(\mathcal{K}_3)$$

are simply connected subsets of Δ_{ocf} resp. Δ_g, all satisfying the conditions of Theorem 3 (Figure 6).

Putting

$$\begin{aligned} \mathcal{H}_1^d &:= \mathcal{K}_1 \cap \Omega_g; & \mathcal{H}_1^u &:= \mathcal{K}_1 \setminus \mathcal{H}_1^d; \\ \mathcal{H}_2^d &:= \psi(\mathcal{M}^{-1}(\mathcal{K}_2)) \cap \Omega_g; & \mathcal{H}_2^u &:= \psi(\mathcal{M}^{-1}(\mathcal{K}_2)) \setminus \mathcal{H}_2^d \end{aligned}$$

and

$$\mathcal{H}_3 := \mathcal{M}^{-1}(\mathcal{K}_3),$$

it follows that

$$\mathcal{H}_{\mathcal{K}} = \mathcal{H}_1^d \cup \mathcal{M}(\mathcal{H}_1^u) \cup \mathcal{H}_2^d \cup \mathcal{M}(\mathcal{H}_2^u) \cup \mathcal{K}_3 .$$

Figure 6

Thus it seems that $\mathcal{H}_{\mathcal{K}}$ is the union of at most five simply connected subsets of Ω_g (with disjoint interiors), each satisfying the conditions from Theorem 3. In fact, since $\mathcal{K} \cap \Omega_g = \mathcal{H}_1^d \cup \mathcal{K}_3$, we see that $\mathcal{H}_{\mathcal{K}}$ is the union of at most 4 of such subsets. This proves Theorem 6. $\square$

Figure 7

Let $z \in [\Leftrightarrow\frac{1}{2}, g)$, and choosing $\mathcal{K} = \mathcal{K}_z$ in Theorem 6, where

$$\mathcal{K}_z := \{(t, v) \in \Omega_{\mathrm{ocf}} : t \le z\}$$

at once yields $(*)$ as a corollary.

Corollary 3 *For $z \in [\Leftrightarrow\frac{1}{2}, g]$ one has*

$$\lambda\{\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2}) \,:\, T_{\mathrm{ocf}}^n \xi \leq z\} \;=\; \mu_{\mathrm{ocf}}([\Leftrightarrow\frac{1}{2}, z]) + \mathcal{O}(g^n),$$

where μ_{ocf} is a probability measure on $[\Leftrightarrow\frac{1}{2}, g)$ with density $d(x)$, given by (8).

Let $\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2})$ be an irrational number with OCF-expansion $[\,0; \varepsilon_1 a_1,\, \varepsilon_2 a_2,\, \cdots]$, sequence of OCF-convergents $(r_k/s_k)_{k \geq -1}$ and

$$(t_k, v_k) \;=\; \mathcal{T}_{\mathrm{ocf}}^k(\xi, 0),\; k \geq 0.$$

Then we define the *optimal approximation coefficients* $\theta_k \;=\; \theta_k(\xi)$ by

$$\theta_k(\xi) \;=\; s_k^2 \left| \xi \Leftrightarrow \frac{r_k}{s_k} \right| \;,\;\; k \geq 1.$$

That these θ's are indeed *optimal* in many respects was shown in [BK2].

¿From the definition of $\mathcal{T}_{\mathrm{ocf}}$ one easily finds, see e.g. [K1], that

$$\theta_{k-1} \;=\; \frac{v_n}{1 + t_k v_k} \quad \text{and} \quad \theta_k \;=\; \frac{\varepsilon_{k+1} t_n}{1 + t_k v_k},\; k \geq 2. \tag{18}$$

The following corollary is a consequence of (18) and Theorem 6.

Corollary 4 *Let $J_n(z) \;=\; \{\xi \in [\Leftrightarrow\frac{1}{2}, \frac{1}{2}) \setminus \mathbf{Q} : \theta_n \leq z\}$ for $0 \leq z \leq \frac{1}{2}$. Then*

$$\lambda(J_n(z)) \;=\; F_{ocf}(z) + \mathcal{O}(g^n)\,,$$

where F_{ocf} is the distribution function given by

$$F_{ocf}(z) \;=\; \begin{cases} \frac{z}{\log G} & \textit{for } 0 \leq z \leq \frac{1}{\sqrt{5}}\,, \\ \\ \frac{1}{\log G}\left(\sqrt{1 \Leftrightarrow 4z^2} + \log\left(G^{\frac{1-\sqrt{1-4z^2}}{2z}}\right)\right) & \textit{for } \frac{1}{\sqrt{5}} \leq z \leq \frac{1}{2}\,. \end{cases}$$

Final remarks

1. Corollary 4 is the "counterpart" of Theorem 5.13 from [BK1], which states that for a.e. ξ and for every $z \in [0, \frac{1}{2}]$ the sequence $(\theta_k)_{k \geq 1}$ is distributed over $[0, \frac{1}{2}]$ according to the distribution function F_{ocf}, so for almost all ξ and for all z

$$\lim_{k \to \infty} \frac{1}{k} \#\{j;\, 1 \leq j \leq k \;\text{ and }\; \theta_j(\xi) \leq z\} \;=\; F_{\mathrm{ocf}}(z)\,,$$

see also Corollary 2. Similar counterparts for many more theorems, e.g. from [BK1], can easily be obtained in the same manner by choosing the sets $\mathcal{K}$ appropriately.

2. As we mentioned before, all the result of this section can be obtained for any maximal singularization area S; there is no need (except clarity of exposition?) to stick to S_{ocf}. For instance, replacing S_{ocf} by $S_{\frac{1}{2}}$ illuminates the relation between the Gauss-Kusmin theorems for the NICF and the SCF, as found by [Rie1]. That this close relation between NICF and SCF not only follows from Rieger's result, but also from the way these continued fraction expansions are obtained via singularization, is illustrated by the following. The analog of Corollary 1 for the NICF is obtained by interchanging $,_1$ with $,_{-1}$, i.e., by reflecting them in the line $\alpha = \beta$. The analog of Corollary 2 for the NICF is complete identical to Corollary 2.

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