# A simple proof of the exponential convergence of the modified Jacobi-Perron algorithm

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**Abstract**

It has recently been shown in Ito et al. (1993) that the modified Jacobi-Perron algorithm is strongly convergent (in the sense of Brentjes 1981) almost everywhere with exponential rate. Their proof relies on very complicated computations. In this paper we will show that the original paper of Podsypanin (1977) on the modified Jacobi-Perron algorithm almost contains a proof of this convergence with exponential rate. The only ingredients missing in that paper are some ergodic-theoretical facts about the transformation generating the approximations. This leads to a very simple proof of the beforementioned exponential convergence in the modified Jacobi-Perron algorithm.

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## 1 Background of the problem

The archetypal example of a multi-dimensional generalisation of the regular continued fraction expansion is the *Jacobi-Perron algorithm* (JPA), see Bernstein (1971), Brentjes (1981) and Schweiger (1973). This algorithm produces, for almost all points $(x, y)$ in the unit square, a sequence $(\frac{p_n}{q_n}, \frac{r_n}{q_n})$ of rational vectors with the same denominator such that

$$\frac{p_n}{q_n} \to x, \quad \text{and } \frac{r_n}{q_n} \to y, \tag{1}$$

for $n \to \infty$.

In Podsypanin (1977) a modification of this algorithm was introduced which is referred to as the *modified Jacobi-Perron algorithm* (MJPA). Podsypanin showed that this algorithm also produces a sequence of rational vectors with the convergence properties in (1). For a general account on convergence in this type of algorithms, see Kraaikamp and Meester (1997). We begin this paper with a description of the MJPA.

Let $E$ be the unit square and define the subsets $E_0 = E \cap \{(x, y); x \geq y, x > 0\}$ and $E_1 = E \cap \{(x, y); x < y\}$. For $(x, y) \in E$ we define the map $T : E \to E$ by

$$T(x, y) = \begin{cases} (\frac{y}{x}, \{\frac{1}{x}\}), & \text{if } (x, y) \in E_0, \\ (\{\frac{1}{y}\}, \frac{x}{y}), & \text{if } (x, y) \in E_1, \\ (0, 0), & \text{if } x = y = 0, \end{cases}$$

where $\{\cdot\}$ denotes fractional part. We write $(x_n, y_n) = T^n(x, y)$, $n \geq 0$, and define two sequences of digits $(a_n)_n$ and $(\varepsilon_n)_n$ by

$$(a_n, \varepsilon_n) = \begin{cases} ([\frac{1}{x_{n-1}}], 0), & \text{if } (x_{n-1}, y_{n-1}) \in E_0, \\ ([\frac{1}{y_{n-1}}], 1), & \text{if } (x_{n-1}, y_{n-1}) \in E_1, \end{cases}$$

where $[\cdot]$ denotes integer part. Podsypanin shows that

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \theta_n \prod_{k=1}^{n} \begin{pmatrix} a_k & \varepsilon_k & 1 - \varepsilon_k \\ 1 - \varepsilon_k & 0 & \varepsilon_k \\ \varepsilon_k & 1 - \varepsilon_k & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_n \\ y_n \end{pmatrix} \tag{2}$$

where

$$\theta_n = \prod_{k=0}^{n-1} \max(x_k, y_k) \ \in [0,1].$$

For $n \geq 1$, the matrix product

$$\prod_{k=1}^{n} \begin{pmatrix} a_k & \varepsilon_k & 1-\varepsilon_k \\ 1-\varepsilon_k & 0 & \varepsilon_k \\ \varepsilon_k & 1-\varepsilon_k & 0 \end{pmatrix}$$

is denoted by $B^{(n)} = B^{(n)}(x,y)$ and written as

$$B^{(n)} = \begin{pmatrix} q_n & q_n' & q_n'' \\ p_n & p_n' & p_n'' \\ r_n & r_n' & r_n'' \end{pmatrix}. \tag{3}$$

Note that $q_n = q_n(x,y)$ and similarly for the other quantities. For future use we define in addition $(q_{-2}, p_{-2}, r_{-2}) = (0,0,1)$, $(q_{-1}, p_{-1}, r_{-1}) = (0,1,0)$, $(q_0, p_0, r_0) = (1,0,0)$ and $\varepsilon_0 = \varepsilon_{-1} = 1$.

Among other things, Podsypanin showed that if $(x,y) \in E$ and at least one of $x$ and $y$ is irrational, then

$$\lim_{n\to\infty} \frac{p_n}{q_n} = x, \quad \lim_{n\to\infty} \frac{r_n}{q_n} = y. \tag{4}$$

In Ito, Keane and Ohtsuki (1993), the following result concerning the speed of the approximations in (4) was given. See also the corrected version of their proof in Fujita et al. (1996):

**Theorem 1** *For the MJPA there exists a constant $\delta > 0$ such that for almost every pair of numbers $(x,y)$ in the unit square there exists $n_0 = n_0(x,y)$ such that for any $n \geq n_0$,*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}}, \quad \left| y - \frac{r_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}},$$

*where the integers $p_n, r_n$ and $q_n$ are as in (3).*

The proof of Theorem 1 in Ito, Keane and Ohtsuki (1993) requires the explicit density of the invariant measure associated with $T$. (This density was given by Schweiger (1978)) The computations in their proof are very involved and complicated. Schweiger (1996) introduced a completely different approach to the problem; he was able to prove the analogue of Theorem 1 for the JPA. This is remarkable, since the invariant measure for the transformation generating the JPA is not known explicitly. In fact, Schweiger pointed out that the classical paper of Paley and Ursell (1930) already essentially contains a proof of the theorem; the only ingredient which was not available to Paley and Ursell was the ergodic theorem and some ergodic theory of multidimensional continued fractions.

In this paper, we will show that the original paper of Podsypanin on the MJPA contains all the ingredients for a proof of Theorem 1. The only additional ingredient will be some ergodic theory. The method used is similar in spirit to the method in Schweiger (1996), although the important estimates are based on completely different quantities. The next section reviews well known facts about the MJPA. The new proof of Theorem 1 is given in the last section. As in Ito et al. (1993), this new proof also in fact leads to a value for $\delta$ in the statement of Theorem 1.

I would like to thank Cor Kraaikamp for stimulating discussions, and Fritz Schweiger for explaining his result on the JPA.

## 2 Ingredients

In this section we collect some classical facts about the MJPA.

**Lemma 1** (Schweiger (1978, 1991)) *The transformation $T$ admits an invariant probability measure $\mu$ on $E$ such that $(E, T, \mu)$ is ergodic. The measure $\mu$ is absolutely continuous with respect to Lebesgue measure.*

**Corollary 1** (see also Ito et. al (1993)) *For $\mu$ almost all $(x, y) \in E$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = -\int_E \log \theta(x, y) d\mu(x, y),$$

*where $\theta(x, y) = \max\{x, y\}$.*

**Proof** From (2) we have

$$1 = \theta_n(q_n + x_n q_n' + y_n q_n''),$$

and hence

$$0 = \frac{1}{n}\log\theta_n + \frac{1}{n}\log q_n + \frac{1}{n}\log(1 + x_n\frac{q_n'}{q_n} + y_n\frac{q_n''}{q_n}).$$

Since $q_n \geq q_n'$ and $q_n \geq q_n''$, the last term tends to zero and we find

$$\lim_{n\to\infty}\frac{1}{n}\log q_n = -\lim_{n\to\infty}\frac{1}{n}\log\theta_n = -\int_E \log\theta d\mu,$$

where the last equality is just the ergodic theorem. □

We write $v_n$, $v_n'$ and $v_n''$ for the columns of $B^{(n)}$. Furthermore, define

$$j(n) = \max\{m < n\,;\, \varepsilon_n \neq \varepsilon_{n-1} \neq \cdots \neq \varepsilon_{m+2} = \varepsilon_{m+1}\}.$$

In words, $j(n)$ is found by looking backwards from time $n$: $j(n)$ is the first time after we have seen two equal consecutive $\varepsilon_k$'s. So for instance, if $\varepsilon_n = \varepsilon_{n-1}$, then $j(n) = n - 2$, which is the maximal value $j(n)$ can obtain.

**Lemma 2** (Podsypanin (1977)) *If $\varepsilon_n = 1$ then $v_n' = v_{n-1}$ and $v_n'' = v_{j(n)}$; if $\varepsilon_n = 0$ then $v_n' = v_{j(n)}$ and $v_n'' = v_{n-1}$.*

**Proof** This follows from straightforward inspection of the algorithm. □

**Remark** The fact that the relations in Lemma 2 involve indices arbitrarily far back from $n$ is one of the main differences between the JPA and the MJPA. In the JPA, the analogue of $j(n)$ is always equal to $n-2$.

Next we define

$$\Delta_n = xq_n - p_n,\ \Delta'_n = xq'_n - p'_n,\ \Delta''_n = xq''_n - p''_n.$$

**Lemma 3** (Podsypanin (1977))
*(i) For all $n \geq 1$ we have*

$$\begin{aligned} |\Delta_{n+1}| &\leq \max\{|\Delta_n|, |\Delta'_n|, |\Delta''_n|\} \\ &= \max\{|\Delta_n|, |\Delta_{n-1}|, |\Delta_{j(n)}|\}. \end{aligned}$$

*(ii) If $\varepsilon_{n+1} = 1$ then $\Delta_{n+1} = -x_{n+1}\Delta_n - y_{n+1}\Delta'_n$; If $\varepsilon_{n+1} = 0$ then $\Delta_{n+1} = -y_{n+1}\Delta_n - x_{n+1}\Delta''_n$. We therefore always have*

$$\begin{aligned} |\Delta_{n+1}| &\leq (x_{n+1} + y_{n+1}) \max\{|\Delta_n|, |\Delta'_n|, |\Delta''_n|\} \\ &= (x_{n+1} + y_{n+1}) \max\{|\Delta_n|, |\Delta_{n-1}|, |\Delta_{j(n)}|\}. \end{aligned}$$

**Proof** (i) The inequality is Lemma 6 in Podsypanin (1977) and the equality follows from Lemma 2. (ii) The inequality is obtained in the course of the proof of Lemma 6 in Podsypanin (1977) and the equality is again Lemma 2. □

## 3 A simple proof of Theorem 1

We only prove the first inequality, the second is proved similarly. From the definitions we have, for all $n$,

$$\left|x - \frac{p_n}{q_n}\right| = \frac{|\Delta_n|}{q_n}. \tag{5}$$

It follows from Corollary 1 that for some $R > 1$ we have

$$q_n \leq R^n. \tag{6}$$

Suppose now that we can show that $|\Delta_n|$ goes down exponentially fast, i.e. suppose we can show

$$|\Delta_n| \leq r^n \tag{7}$$

for some $0 < r < 1$. We can then choose $\delta > 0$ so small that $r < R^{-\delta}$. It then follows, using (6), that

$$|\Delta_n| \leq r^n \leq \left(\frac{1}{R^n}\right)^\delta \leq q_n^{-\delta}.$$

Hence the right hand side of (5) is bounded above by $q_n^{-(1+\delta)}$ and the proof is complete.

It remains to prove (7), and this is where Lemma 3 comes in. Exponential decay is not immediate from this lemma, although it is not hard to believe that it is not too far off either. The problem lies in the fact that we have no a priori control over $j(n)$ (and as mentioned before, this problem does not arise in the JPA). To deal with this problem, we will create 'blocks' of consecutive $\varepsilon_k$'s equal to 1, which will force the $j(n)$'s to be not too far away from $n$. Let

$$F = E_1 \cap \{(x,y); x + y \leq 1\}$$

and

$$A = \left( \bigcap_{i=0}^{2} T^{-i} E_0 \right) \cap \left( \bigcap_{i=3}^{5} T^{-i} F \right).$$

It is not hard to check that $\mu(A) > 0$. Define the function $f$ by

$$f(x, y) = \max\{(x_3 + y_3), (x_4 + y_4), (x_5 + y_5)\},$$

where $(x_n, y_n) = T^n(x, y)$ as before. Let

$$n_0 = n_0(x, y) = \min\{n \geq 0; T^n(x, y) \in A\},$$

and for $k \geq 1$,

$$n_k = n_k(x, y) = \min\{n > n_{k-1}; T^n(x, y) \in A\}.$$

Note that $n_0(x, y) = 0$ for $(x, y) \in A$. Also, by construction we have $n_{k+1} \geq n_k + 6$ which makes the recursion in the next lemma simple to state and to prove. Finally, let

$$M_k = M_k(x, y) = \max\{|\Delta_{n_k}|, |\Delta_{n_k+1}|, |\Delta_{n_k+2}|\}.$$

**Lemma 4** *For all* $n_k + 3 \leq n \leq n_{k+1} + 2$ *we have*

$$|\Delta_n| \leq f(T^{n_k}) M_k.$$

*In particular, we have*

$$M_{k+1} \leq f(T^{n_k}) M_k. \tag{8}$$

**Proof** Since $T^{n_k}(x, y) \in A$, we have that $\varepsilon_{n_k+1} = \varepsilon_{n_k+2} = 0$ and hence $j(n_k + 2) = n_k$. Therefore it follows from Lemma 3(ii) that

$$|\Delta_{n_k+3}| \leq (x_{n_k+3} + y_{n_k+3}) \max\{|\Delta_{n_k}|, |\Delta_{n_k+1}|, |\Delta_{n_k+2}|\}.$$

For $n_k + 4$ we obtain in a similar fashion,

$$\begin{aligned}
|\Delta_{n_k+4}| &\leq (x_{n_k+4} + y_{n_k+4}) \max\{|\Delta_{n_k+1}|, |\Delta_{n_k+2}|, |\Delta_{n_k+3}|\} \\
&\leq (x_{n_k+4} + y_{n_k+4}) \max\{|\Delta_{n_k+1}|, |\Delta_{n_k+2}|, \\
&\quad \max\{|\Delta_{n_k}|, |\Delta_{n_k+1}|, |\Delta_{n_k+2}|\}\} \\
&= (x_{n_k+4} + y_{n_k+4}) \max\{|\Delta_{n_k}|, |\Delta_{n_k+1}|, |\Delta_{n_k+2}|\},
\end{aligned}$$

and similarly for $n_k + 5$.

For the indices $n_k + 6 \leq n \leq n_{k+1} + 2$ we use induction: Suppose that for all $n_k + 3 \leq i \leq n$ we have $|\Delta_i| \leq f(T^{n_k})M_k$. Since by construction $j(n) \geq n_k + 3$ for all these values of $n$, it follows from Lemma 3(i) that also $|\Delta_{n+1}| \leq f(T^{n_k})M_k$. □

Define on $A$ the induced transformation $T_A$ by $T_A = T^{n_1}$ and more generally, $T_A^k = T^{n_k}$. Since $(E, T, \mu)$ is ergodic, so is the system $(A, T_A, \mu(\cdot)/\mu(A))$. From (8) we find, writing $g = T^{n_0}$,

$$\begin{aligned}
\frac{1}{k}\log M_{k+1} &\leq \frac{1}{k}\sum_{i=0}^{k}\log f(T^{n_i}) + \frac{1}{k}\log M_0 \\
&= \frac{1}{k}\sum_{i=1}^{k}\log f(T^{n_i}) + \frac{1}{k}\log M_0 + \frac{1}{k}\log(f \circ g) \\
&= \frac{1}{k}\sum_{i=1}^{k}\log f(T^{n_i - n_0} \circ g) + \frac{1}{k}\log M_0 + \frac{1}{k}\log(f \circ g) \\
&= \frac{1}{k}\sum_{i=1}^{k}\log f(T_A^i \circ g) + \frac{1}{k}\log M_0 + \frac{1}{k}\log(f \circ g).
\end{aligned}$$

Since $g$ cannot map a set of positive $\mu$ measure onto a set of $\mu$ measure zero, the point $g(x, y)$ is generic for $\mu$ almost all $(x, y) \in E$. Therefore the ergodic theorem (applied to $T_A$) tells us that for almost all $(x, y) \in E$, this last expression converges for $k \to \infty$ to

$$\mu(A)^{-1} \int_A \log f d\mu,$$

which is strictly negative since $f(x,y) \leq 1$ on $A$. It follows that $M_k$ goes down exponentially fast almost surely. From Lemma 4 we see that for all $n \geq n_k + 3$, $|\Delta_n| \leq M_k$. Finally, $n_k/k$ converges almost surely to $\mu(A)^{-1}$ as $n \to \infty$ by the ergodic theorem (applied to $T$). Therefore, $|\Delta_n|$ itself also goes down exponentially fast, and the proof is complete.

In fact, we even get an explicit bound on the rate at which $|\Delta_n|$ tends to zero: Since $M_k$ goes down at least as fast as $\exp\{k\mu(A)^{-1} \int_A \log f d\mu\}$, $|\Delta_n|$ goes down at least as fast as $\exp\{n \int_A \log f d\mu\}$. Since the behaviour of $q_n$ on the exponential scale is known exactly (see Corollary 1), this leads to a value for $\delta$ in the statement of Theorem 1. □

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