Routh's sphere

R. Cushman*

Abstract
In this paper we show that the integral map of Routh's sphere has monodromy when the sphere becomes gyroscopically unstable. This uses the non-Hamiltonian monodromy of [3]

Routh's sphere has center of mass not at its geometrical center and moment of inertia tensor with two equal principal moments of inertia. Moreover, it rolls on a horizontal plane under the influence of a constant vertical gravitational force.

1 The equations of motion

In this section we derive the equations of motion of Routh's sphere.

To set up the equations of motion, consider a reference sphere of radius r and mass m with center of mass at C, which is a distance α $(0 < \alpha < r)$ from its geometric center that lies at the origin, see figure 1. The position of the moving sphere is given by applying the element (A, a) of the 3-dimensional Euclidean group $\mathcal{E}(3) \subseteq \mathrm{SO}(3) \times \mathbf{R}^3$ to a position of the reference sphere. The center of mass C' of the moving sphere is a.

The tangent of left translation in $\mathcal{E}(3)$ gives the trivialization

$$L : \mathcal{E}(3) \times \mathrm{e}(3) \to T\,\mathcal{E}(3) : (A, a, \Omega, b) = \big(A, a, A^{-1}\dot{A}, A^{-1}\dot{a}\big) \to (\dot{A}, \dot{a}).$$

Here $\mathrm{e}(3)$ is the Lie algebra of $\mathcal{E}(3)$. Using the map L, we pull back the Lagrangian of the *unconstrained* moving sphere to $\mathcal{E}(3) \times \mathrm{e}(3)$ and obtain the Lagrangian

$$\mathcal{L} = T_{\mathrm{rot}} + T_{\mathrm{trans}} - V. \tag{1}$$

$T_{\mathrm{rot}} = \frac{1}{2} m \langle I(\omega), \omega \rangle$ is the rotational kinetic energy of the unconstrained moving sphere about its center of mass with $I = \mathrm{diag}(I_1, I_1, I_3)$ its moment of inertia tensor with respect to its principal axes at the center of mass and $\langle\, ,\, \rangle$ is the Euclidean inner product on $\mathbf{R}^3$. Here we have identified the matrix $\Omega = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathrm{so}(3)$ with the angular velocity vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbf{R}^3$. The translational energy of the center of mass of the unconstrained moving sphere is $T_{\mathrm{trans}} = \frac{1}{2} m \langle b, b \rangle$ and its potential energy is $V = mg\langle a, e_3 \rangle$.

*Mathematics Institute, Budapestlaan 6, University of Utrecht, 3508TA Utrecht, the Netherlands

The Lagrangian derivative of $\mathcal{L}$ is

$$\delta\mathcal{L} = \left(\frac{d}{dt}(I\omega) - (I\omega) \times \omega,\ m\,\frac{db}{dt} - m\,(b \times \omega) - mg\,A^{-1}e_3 \right). \tag{2}$$

Figure 1. Routh's sphere. (a) The reference sphere.
(b) The moving sphere.

The moving sphere is subjected to two kinds of constraint: a holonomic constraint of moving on a horizontal plane and a nonholonomic constraint of rolling. To treat these constraints let

$$u = u(A) \ = \ -A^{-1}e_3 \tag{3}$$

and let $s = s(A)$ be the vector in the reference sphere from the center of mass C to a point Q on the sphere. Thus the vector

$$n = \frac{1}{r}\,(s - \alpha\,e_3)$$

is a unit normal vector to the reference sphere at Q. In order that the point of contact P of the moving sphere with the horizontal plane be equal to $As + a$, the rotation A must map the unit normal vector n to the unit normal vector $-e_3$ to the horizontal plane at P. Hence

$$-e_3 = An \ = \ \frac{1}{r}\,A(s - \alpha\,e_3),$$

which gives

$$s = \ -r\,A^{-1}e_3 + \alpha\,e_3 \ = \ r\,u + \alpha\,e_3. \tag{4}$$

Because the point of contact P lies on the horizontal plane, it follows that

$$0 = \langle As + a, e_3 \rangle \ = \ \langle s, A^{-1}e_3 \rangle + \langle a, e_3 \rangle,$$

that is,

$$a_3 = \langle a, e_3 \rangle \; = \; \langle s, u \rangle, \tag{5}$$

which is the holonomic constraint. The condition that the sphere rolls without slipping is equivalent to requiring that the velocity of the point of contact P of the moving sphere with the horizontal plane is zero. In other words,

$$0 = \dot{A}s + \dot{a},$$

which is equivalent to,

$$0 = A^{-1}\dot{A}s + A^{-1}\dot{a} \; = \; \omega \times s + b. \tag{6}$$

Equations (5) and (6) define the 8-dimensional constraint manifold

$$\mathcal{C} = \Big\{ (A, a, \omega, b) \in \mathcal{E}(3) \times \mathbf{R}^6 \Big| b = -\omega \times s(A) \,\&\, \langle a, e_3 \rangle = \langle s(A), u(A) \rangle \Big\}. \tag{7}$$

2 Reduction of the $\mathcal{E}(2)$ symmetry

We now reduce the $\mathcal{E}(2)$ symmetry in Routh's sphere which is generated by translations of the horizontal plane and rotations about a vertical axis.

More formally, consider the 2-dimensional Euclidean group

$$\mathcal{E}(2) = \Big\{ (R_\varphi, x) \in \mathcal{E}(3) \big| R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\&\, x = (x_1, x_2, 0) \Big\}.$$

Viewed as a subgroup of $\mathcal{E}(3)$, the group $\mathcal{E}(2)$ acts on the *left* on the constraint manifold $\mathcal{C}$ by

$$\Phi : \mathcal{E}(2) \times \mathcal{C} \to \mathcal{C} : \Big((R_\varphi, x), (A, a, \omega, b) \Big) \to (R_\varphi A, R_\varphi a + x, \omega, b). \tag{8}$$

Note that Φ maps $\mathcal{C}$ into itself because

$$u(R_\varphi A) = -A^{-1} R_\varphi^{-1} e_3 \; = \; -A^{-1} e_3 \; = \; u(A)$$

and consequently,

$$s(R_\varphi A) = s(A),$$

using (4). Since the action Φ is free and proper, the orbit space $\mathcal{C}/\,\mathcal{E}(2)$ is a smooth manifold, which, after a little thought, is seen to be diffeomorphic to $S^2 \times \mathbf{R}^3$. The action Φ preserves the Lagrangian $\mathcal{L}$ (1), because

$$\langle R_\varphi a + x, e_3 \rangle = \langle a, e_3 \rangle.$$

Thus $\mathcal{E}(2)$ is a symmetry of Routh's sphere.

We now derive the classical equations of motion of Routh's sphere (which in fact are the equations of motion on $S^2 \times \mathbf{R}^3$ obtained after reducing $\mathcal{E}(2)$ symmetry). We apply the d'Alembert principle, which states that the Lagrangian derivative $\delta\mathcal{L}$ (2) of $\mathcal{L}$ (1) is perpendicular to all vectors $(\widetilde{\omega}, \widetilde{b}) \in \mathbf{R}^3 \times \mathbf{R}^3$ that satisfy the nonholonomic constraint (6). In other words,

$$\begin{aligned} 0 &= \Big\langle \frac{d}{dt}(I\omega) - I(\omega) \times \omega, \widetilde{\omega} \Big\rangle + \Big\langle m \frac{db}{dt} - m\,(b \times \omega) + mg\,u, \widetilde{b} \Big\rangle \\ &= \Big\langle \frac{d}{dt}(I\omega) - I(\omega) \times \omega \\ &\qquad + \Big\{ -m \frac{d}{dt}(\omega \times s) + m\,(\omega \times s) \times \omega - mg\,u \Big\} \times s, \widetilde{\omega} \Big\rangle \end{aligned}$$

for every $\widetilde{\omega} \in \mathbf{R}^3$. Therefore

$$\begin{aligned} \frac{d}{dt}(I\omega) &= I(\omega) \times \omega - m\,\langle s, s\rangle \frac{d\omega}{dt} + m\,\langle \frac{d\omega}{dt}, s\rangle\, s - m\,\omega\,\langle \frac{ds}{dt}, s\rangle \\ &\quad + m\,\langle s, \omega\rangle \frac{ds}{dt} + m\,\langle \omega, s\rangle\, \omega \times s + mg\,u \times s \end{aligned} \tag{9}$$

and

$$\frac{du}{dt} = u \times \omega, \tag{10}$$

which is obtained by differentiating (3). Equations (9) and (10) may be rewritten as

$$\frac{d}{dt}\Big(I\omega + m\,s \times (\omega \times s)\Big) = I\omega \times \omega + m\frac{ds}{dt} \times (\omega \times s) + m\,\langle \omega, s\rangle\, \omega \times s + mg\,u \times s \tag{11}$$

and

$$\frac{du}{dt} = u \times \omega, \tag{12}$$

where $(u, \omega) \in S^2 \times \mathbf{R}^3$. Since $s = ru + \alpha e_3$ (4), we see that solutions of (11) and (12) are integral curves of a vector field $\widetilde{V}$ on $S^2 \times \mathbf{R}^3$. A straightforward calculation shows that the energy of the sphere,

$$\widetilde{\mathcal{E}}(u, \omega) = \tfrac{1}{2}\,\langle I\omega, \omega\rangle + \tfrac{1}{2}\,m\,\langle \omega \times s, \omega \times s\rangle + mg\,\langle s, u\rangle, \tag{13}$$

is an integral of $\widetilde{V}$.

3 Reduction of the S^1 symmetry

Next we reduce the S^1 symmetry of Routh's sphere, which is generated by rotations about its e_3 principal axis.

More formally, define a *right* S^1 action on the constraint manifold $\mathcal{C}$ (7) by

$$\Psi : \mathcal{C} \times S^1 \to \mathcal{C} : \Big((A, a, \omega, b), \varphi\Big) \to (AR_\varphi^{-1}, R_\varphi a, R_\varphi \omega, R_\varphi b). \tag{14}$$

Since

$$u(AR_\varphi^{-1}) = -R_\varphi A^{-1} e_3 \;=\; R_\varphi u(A), \tag{15}$$

$R_\varphi e_3 = e_3$, and $s(A) = ru(A) + \alpha e_3$ (4), it follows that $s(AR_\varphi^{-1}) = R_\varphi s(A)$. Hence

$$R_\varphi \omega \times s(AR_\varphi^{-1}) = R_\varphi(\omega \times s(A)) \; = \; R_\varphi b$$

and

$$\begin{aligned} \Big\langle s(AR_\varphi^{-1}), u(AR_\varphi^{-1}) \Big\rangle &= \Big\langle R_\varphi s(A), R_\varphi u(A) \Big\rangle \; = \; \langle s(A), u(A) \rangle \\ &= \langle a, e_3 \rangle \; = \; \langle R_\varphi a, R_\varphi e_3 \rangle \; = \; \langle R_\varphi a, e_3 \rangle. \end{aligned}$$

Therefore Ψ maps $\mathcal{C}$ into itself. Since $R_\varphi \circ I = I \circ R_\varphi$, the action Ψ preserves the Lagrangian $\mathcal{L}$ (1). Thus S^1 is a symmetry of Routh's sphere.

Because $\mathcal{E}(2)$ acts on the constraint manifold $\mathcal{C}$ on the left and S^1 on the right, the actions Φ (8) and Ψ (14) commute. Hence Ψ induces an action ψ on the Φ-orbit space $S^2 \times \mathbf{R}^3$. From (14) and (15), we see that ψ is given by

$$\psi : S^1 \times (S^2 \times \mathbf{R}^3) \to S^2 \times \mathbf{R}^3 : \Big(\varphi, (u, \omega)\Big) \to (R_\varphi u, R_\varphi \omega). \tag{16}$$

A straightforward calculation shows that the $\mathcal{E}(2)$-reduced vector field $\widetilde{V}$ ((11) and (12)) and energy $\widetilde{\mathcal{E}}$ (13) are invariant under the action ψ.

We now reduce the S^1 symmetry using invariant theory. The algebra of S^1-invariant polynomials on $S^2 \times \mathbf{R}^3$ is generated by

$$\begin{aligned} \sigma_1 &= u_3 = \langle u, e_3 \rangle & \sigma_4 &= \omega_3 \\ \sigma_2 &= u_1\omega_2 - u_2\omega_1 = \langle u \times \omega, e_3 \rangle & \sigma_5 &= \omega_1^2 + \omega_2^2 \\ \sigma_3 &= u_1\omega_1 + u_2\omega_2 & \sigma_6 &= u_1^2 + u_2^2 \end{aligned} \tag{17}$$

subject to the relations

$$\begin{aligned} \sigma_2^2 + \sigma_3^2 &= \sigma_5\sigma_6, \qquad \sigma_5 \geq 0 \quad \& \quad \sigma_6 \geq 0 \\ \sigma_1^2 + \sigma_6 &= 1. \end{aligned} \tag{18}$$

Thus the ψ-orbit space $(S^2 \times \mathbf{R}^3)/S^1 = M$ is the semialgebraic subvariety of $\mathbf{R}^5$ defined by

$$\sigma_2^2 + \sigma_3^2 = (1 - \sigma_1^2)\sigma_5, \quad |\sigma_1| \leq 1 \quad \& \quad \sigma_5 \geq 0. \tag{19}$$

M is *not* smooth, because ψ leaves the lines

$$L_\pm = \Big\{(0, 0, \pm 1, 0, 0, \omega_3) \in S^2 \times \mathbf{R}^3 \Big| \omega_3 \in \mathbf{R}\Big\}$$

pointwised fixed. Thus the half planes

$$\Pi_\pm = \Big\{(\pm 1, 0, 0, \sigma_4, \sigma_5) \in \mathbf{R}^5 \Big| \sigma_4 \in \mathbf{R} \quad \& \quad \sigma_5 \geq 0\Big\} \tag{20}$$

form the set of singular points of M.

We now determine the "vector field" V on M obtained by reducing the S^1 symmetry of the vector field $\widetilde{V}$. To do this we need only compute the Lie derivatives of the invariants $\sigma_i,\ i = 1, \ldots, 5$ with respect to $\widetilde{V}$. Consider the following inner products

$$\begin{cases} \tau_1 &= \Big\langle I\omega + m\, s \times (\omega \times s), e_3 \Big\rangle \\ \tau_2 &= \Big\langle I\omega + m\, s \times (\omega \times s), u \Big\rangle \\ \tau_3 &= \Big\langle I\omega + m\, s \times (\omega \times s), s \Big\rangle \\ \tau_4 &= \Big\langle I\omega + m\, s \times (\omega \times s), u \times e_3 \Big\rangle \\ \tau_5 &= \Big\langle I\omega + m\, s \times (\omega \times s), \omega \Big\rangle. \end{cases}$$

It is easy to see that they are invariant under the action ψ (16). A calculation shows that

$$\begin{cases} \tau_1 &= (I_3 + mr^2 - mr^2\,\sigma_1)\sigma_4 - (mr\alpha + mr^2\,\sigma_1)\sigma_3 \\ \tau_2 &= (I_1 + m\alpha^2 + mr\alpha\,\sigma_1)\sigma_3 + (-mr\alpha + I_3\,\sigma_1 + mr\alpha\,\sigma_1^2)\sigma_4 \\ \tau_3 &= rI_1\,\sigma_3 + rI_3\,\sigma_1\sigma_4 + \alpha I_3\,\sigma_4 \\ \tau_4 &= -(I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1)\sigma_2 \\ \tau_5 &= \Big(I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1\Big)\sigma_5 + (I_3 + mr^2)\sigma_4^2 \\ &\qquad - mr^2(\sigma_3 + \sigma_1\sigma_4)^2 - 2mr\alpha\,\sigma_3\sigma_4. \end{cases} \tag{21}$$

Using the definition of the vector field $\widetilde{V}$, we compute the Lie derivative of $\tau_i,\ i = 1, \ldots, 5$ with respect to $\widetilde{V}$ and obtain

$$\begin{cases} \dot\tau_1 &= L_{\widetilde{V}}\tau_1 \;=\; -mr^2\,\sigma_2\sigma_3 - mr^2\,\sigma_1\sigma_2\sigma_4 \\ \dot\tau_2 &= mr\alpha\,\sigma_2\sigma_3 + mr\alpha\,\sigma_1\sigma_2\sigma_3 \\ \dot\tau_3 &= 0 \\ \dot\tau_4 &= -\Big(I_3 + mr^2 + mr\alpha\,\sigma_1\Big)\sigma_3\sigma_4 - 2mr\alpha\,\sigma_2^2 + mg\alpha\,(1 - \sigma_1^2) \\ &\qquad + \sigma_5\Big(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\,\sigma_1^2\Big) \\ \dot\tau_5 &= mr\alpha\,\sigma_2\sigma_5 - mg\alpha\,\sigma_2 + (I_3 + mr^2)\sigma_4\dot\sigma_4 \\ &\quad - \tfrac{1}{2}\,mr^2\dfrac{d}{dt}(\sigma_3 + \sigma_1\sigma_4)^2 - mr\alpha\,(\sigma_3\dot\sigma_4 + \sigma_4\dot\sigma_3) \\ &\qquad + \tfrac{1}{2}\Big(I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1\Big)\dot\sigma_5. \end{cases} \tag{22}$$

Differentiating the equations in (21) and equating the results to (22) gives

$$\begin{aligned} \dot\sigma_1 &= \sigma_2 \\ T(\sigma_1)\dot\sigma_2 &= \Big(I_3 + mr^2 + mr\alpha\,\sigma_1\Big)\sigma_3\sigma_4 - mg\alpha\,(1 - \sigma_1^2) \\ &\qquad - \sigma_5\Big(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\,\sigma_1^2\Big) \\ \dot\sigma_3 &= -I_3\,\frac{\sigma_2\sigma_4}{P(\sigma_1)}\Big(I_3 + mr^2 + mr\alpha\,\sigma_1\Big) \end{aligned} \tag{23}$$

$$
\begin{aligned}
\dot\sigma_4 &= -mr\,\frac{\sigma_2\sigma_4}{P(\sigma_1)}\Big(I_3\alpha + r(I_3-I_1)\sigma_1\Big)\\
T(\sigma_1)\dot\sigma_5 &= -2mr\alpha\,\sigma_2\sigma_5 - 2mg\alpha\,\sigma_2\\
&\qquad\qquad - 2mr^2(I_3-I_1)\frac{(I_3+mr^2+mr\alpha\,\sigma_1)}{P(\sigma_1)}\sigma_2\sigma_3\sigma_4,
\end{aligned}
$$

where

$$
\left\{\begin{aligned}
P(\sigma_1) &= I_1I_3 + mr^2I_1(1-\sigma_1^2) + mI_3(\alpha+r\sigma_1)^2\\
T(\sigma_1) &= I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1
\end{aligned}\right. \tag{24}
$$

The solutions of (23) which lie on the S^1 orbit space M are integral curves of the "vector field" V on M.

Since the energy $\widetilde{\mathcal{E}}$ (13) is invariant under the S^1 action ψ, $\widetilde{\mathcal{E}}$ induces a "smooth" function $\mathcal{E}$ on the orbit space M given by

$$
\mathcal{E}(\sigma_1,\ldots,\sigma_5) = \tfrac{1}{2}\,T(\sigma_1)\sigma_5 + \tfrac{1}{2}\,(I_3+mr^2)\sigma_4^2 - \tfrac{1}{2}\,mr^2(\sigma_3+\sigma_1\sigma_4)^2 - mr\alpha\,\sigma_3\sigma_4 + mg\alpha\,\sigma_1. \tag{25}
$$

$\mathcal{E}$ is an integral of the $\mathcal{E}(2)\times S^1$-reduced "vector field" V. From (22) we see that

$$
\mathcal{J}(\sigma_1,\ldots,\sigma_5) = rI_1\,\sigma_3 + I_3(\alpha + r\,\sigma_1)\sigma_4 \;=\; \tau_3 \tag{26}
$$

is an integral of V. The following calculation shows that

$$
\mathcal{K}(\sigma_1,\ldots,\sigma_5) = \sigma_4\sqrt{P(\sigma_1)} \tag{27}
$$

is also an integral of V:

$$
\begin{aligned}
\dot{\mathcal{K}} &= \Big(\dot\sigma_4\sqrt{P(\sigma_1)} + \sigma_4\frac{\frac{1}{2}\,P'(\sigma_1)}{\sqrt{P(\sigma_1)}}\Big)\dot\sigma_1\\
&= \left(-mr\,\frac{\sigma_2\sigma_4}{\sqrt{P(\sigma_1)}}(I_3\alpha + r(I_3-I_1)\sigma_1) + \sigma_2\sigma_4\frac{\frac{1}{2}\,P'(\sigma_1)}{\sqrt{P(\sigma_1)},}\right)\dot\sigma_1,\\
&\qquad\qquad \text{using (22) and (24)}\\
&= 0, \qquad\qquad \text{since } \tfrac{1}{2}\,P'(\sigma_1) = mr\alpha\,I_3 + mr^2(I_3-I_1).
\end{aligned}
$$

Therefore the functions

$$
\widetilde{\mathcal{J}}(u,\omega) = rI_1(u_1\omega_1 + u_2\omega_2) + I_3\omega_3(\alpha + ru_3) \tag{28}
$$

and

$$
\widetilde{\mathcal{K}}(u,\omega) = \omega_3\sqrt{I_1I_3 + mr^2I_1(1-u_3^2) + mI_3(\alpha+ru_3)^2} \tag{29}
$$

are integrals of the vector field $\mathcal{E}(2)$-reduced $\widetilde{V}$ on $S^2\times\mathbf{R}^3$. $\widetilde{\mathcal{J}}$ is Jellet's integral.

4 Invariant varieties $\mathcal{E}(2) \times S^1$ reduced "vector field"

In this section we study the geometry of the invariant varieties $M_{j,k} = M \cap \mathcal{J}^{-1}(j) \cap \mathcal{K}^{-1}(k)$ of the $\mathcal{E}(2) \times S^1$ reduced "vector field" V on the orbit space M defined by the j and k level sets of the integrals $\mathcal{J}$ and $\mathcal{K}$.

CASE 1: $j = k = 0$.
In this subsection we study the invariant variety $M_{0,0}$.

From the defining equation (19) of the orbit space M, we see that $M_{0,0}$ is given by

$$\begin{aligned} (1-\sigma_1^2)\sigma_5 - \sigma_2^2 - \sigma_3^2 &= 0, \qquad |\sigma_1| \le 1 \ \&\ \sigma_5 \ge 0 \\ rI_1\,\sigma_3 + I_3\,\sigma_4(\alpha + r\sigma_1) &= 0 \\ \sigma_4\sqrt{P(\sigma_1)} &= 0. \end{aligned} \tag{30}$$

Since $P(\sigma_1) > 0$ when $|\sigma_1| \le 1$, (30) is equivalent to

$$\sigma_3 = \sigma_4 = 0 \ \&\ (1-\sigma_1^2)\sigma_5 - \sigma_2^2 - \sigma_3^2 = 0, \qquad |\sigma_1| \le 1 \ \&\ \sigma_5 \ge 0.$$

Thus $M_{0,0}$ is the semialgebraic variety in $\mathbf{R}^3$ defined by

$$f(\sigma) = f(\sigma_1.\sigma_2,\sigma_5) = \sigma_2^2 - (1-\sigma_1^2)\sigma_5 = 0, \qquad |\sigma_1| \le 1 \ \&\ \sigma_5 \ge 0. \tag{31}$$

The only singular points of $M_{0,0}$ are $p_\pm = (\pm 1, 0, 0) = (\eta, 0, 0)$, since

$$0 = df(\sigma) = \left(2\sigma_1\sigma_5, 2\sigma_2, 1-\sigma_1^2\right)$$

implies that $\sigma_1 = \pm 1$, $\sigma_2 = \sigma_5 = 0$, and $f(p_\pm) = 0$. Because the Taylor polynomial of f about $p_\pm$ to second order is $\sigma_2^2 + 2\eta(\eta - \sigma_1)\sigma_5$, the tangent cone to $M_{0,0}$ at $p_\pm$ is nondegenerate, see figure 3b.

CASE 2: $j \ne 0$ or $k \ne 0$.
Consider the integral variety $M_{j,k} = M \cap \mathcal{J}^{-1}(j) \cap \mathcal{K}^{-1}(k)$, where j and k are not both zero. Eliminating σ_3 from the defining equation of the orbit space M (19) shows that $M_{j,k}$ is defined by

$$0 = F(\sigma_1,\sigma_2,\sigma_5) = (1-\sigma_1^2)\sigma_5 - \sigma_2^2 - \frac{1}{r^2 I_1^2}\Big(j - kI_3\,G(\sigma_1)\Big)^2, \tag{32}$$

where $G(\sigma_1) = (\alpha + r\sigma_1)/\sqrt{P(\sigma_1)}$ and $|\sigma_1| \le 1$ & $\sigma_5 \ge 0$. The point $\sigma^0 = (\sigma_1^0, \sigma_2^0, \sigma_5^0)$ is a singular point of $M_{j,k}$ if and only if $0 = F(\sigma^0)$ and

$$0 = DF(\sigma^0) = \Big(-2\sigma_1^0\sigma_5^0 + \frac{2kI_3}{r^2 I_1^2}(j - kI_3\,G(\sigma_1^0))G'(\sigma_1^0), -2\sigma_2^0, 1 - (\sigma_1^0)^2\Big). \tag{33}$$

From (33) we find that $\sigma_1^0 = \pm 1$ and $\sigma_2^0 = 0$. Therefore the condition $F(\sigma^0) = 0$ becomes $0 = \Big(j - kI_3 G(\sigma_1^0)\Big)^2$. Consequently, if $M_{j,k}$ has singular points, then (j,k) lies on the locus

$\Sigma^\pm : j = kI_3\, G(\pm 1)$, which is two lines intersecting at the origin. The defining equation of $\Sigma^\pm$ is equivalent to

$$k = \frac{\sqrt{P(\pm 1)}}{I_3(\alpha + r)}\, j. \tag{34}$$

Using the first component of $DF(\sigma^0)$, $\sigma_1^0 = \pm 1$, and (34), we obtain $\sigma_5^0 = 0$. Conversely, if $(j,k) \in \Sigma^\pm - \{(0,0)\}$, then $M_{j,k}$ has exactly one singular point $\sigma^0 = (\pm 1, 0, 0)$, see figure 2a and 2b. At the singular point σ^0 it follows that $\sigma_3^0 = 0$ and $\sigma_4^0 = k/\sqrt{P(\varepsilon)}$. When $(j,k) \notin \Sigma^- \cup \Sigma^+$, the variety $M_{j,k}$ is a smooth manifold, which is the graph of the smooth function

$$\sigma_5 = \frac{1}{1 - \sigma_1^2}\left[\sigma_2^2 + \frac{1}{r^2 I_1^2}(j - kI_3\, G(\sigma_1))^2\right], \qquad |\sigma_1| < 1,$$

see figure 2c. Next we determine the tangent cone to $M_{j,k}$ at the singular point σ^0. To do this we need only calculate $D^2F(\sigma_1^0)$. We obtain

$$D^2F(\sigma) = \begin{pmatrix} -2\sigma_5 + \frac{2kI_3}{r^2 I_1^2}\Big(-kI_3(G'(\sigma_1))^2 + (j - kI_3 G(\sigma_1))G''(\sigma_1)\Big) & 0 & -2\sigma_1 \\ 0 & -2 & 0 \\ -2\sigma_1 & 0 & 0 \end{pmatrix}.$$

Since $\sigma^0 = (\pm 1, 0, 0)$, $\sigma_5 = 0$, and $j - kI_3\, G(\pm 1) = 0$ (because σ^0 is a singular point of $M_{j,k}$), we find that

$$D^2F(\sigma^0) = \begin{pmatrix} -\frac{2k^2 I_3^2}{r^2 I_1^2} G'(\pm 1)^2 & 0 & \mp 2 \\ 0 & -2 & 0 \\ \mp 2 & 0 & 0 \end{pmatrix}.$$

Figure 2. The integral variety $M_{j,k}$. (a) $(j,k) \in \Sigma^- - \{(0,0)\}$. (b) $(j,k) \in \Sigma^+ - \{(0,0)\}$. (c) $(j,k) \notin \Sigma^- \cup \Sigma^+$.

$D^2F(\sigma^0)$ is nondegenerate. Thus the equation of the tangent cone $\mathcal{C}$ to $M_{j,k}$ at $\sigma^0 = (\pm 1, 0, 0)$ is

$$0 = \frac{2k^2 I_3^2}{r^2 I_1^2} G'(\pm 1)^2 (\sigma_1 - \varepsilon)^2 + 2\sigma_2^2 \pm 4(\sigma_1 \mp 1)\sigma_5. \tag{35}$$

5 The "vector field" V on $M_{j,k}$

In this section we carry out an analysis of the $\mathcal{E}(2) \times S^1$-reduced "vector field" V (23) on the invariant variety $M_{j,k}$ after discarding its singular points. Our discussion follows Routh [1] and shows that this vector field is a 2-parameter family of one degree of freedom *Hamiltonian* systems.

We start off by removing the singular half planes $\Pi_\pm$ (20) from M (19). What remains is a smooth submanifold $M^* = M - \{\Pi_- \cup \Pi_+\}$ of $\mathbf{R}^5$ which is the product of $\mathbf{R}$ and the graph of the smooth function

$$\sigma_5 = \frac{\sigma_2^2 + \sigma_3^2}{1 - \sigma_1^2}, \qquad |\sigma_1| < 1. \tag{36}$$

Note that the orbit map

$$\pi : S^2 \times \mathbf{R}^3 - \{L_+ \cup L_-\} \to M : (u, \omega) \to (\sigma_1, \ldots, \sigma_5) \tag{37}$$

is a proper submersion with fiber a unique S^1 orbit. Projecting V onto $(-1, 1) \times \mathbf{R}^3$ gives the vector field $\mathcal{V}$ whose integral curves satisfy

$$\left\{ \begin{array}{rcl} \dot\sigma_1 & = & \sigma_2 \\ T(\sigma_1)\dot\sigma_2 & = & \sigma_3\sigma_4\Big(I_3 + mr^2 + mr\alpha\,\sigma_1\Big) - mg\alpha\,(1 - \sigma_1^2) \\ & & -\dfrac{(\sigma_2^2 + \sigma_3^2)}{1 - \sigma_1^2}\Big(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\,\sigma_1^2\Big) \\ \dot\sigma_3 & = & -I_3\,\dfrac{\sigma_2\sigma_4}{P(\sigma_1)}\Big(I_3 + mr^2 + mr\alpha\,\sigma_1\Big) \\ \dot\sigma_4 & = & -mr\,\dfrac{\sigma_2\sigma_4}{P(\sigma_1)}\Big(I_3\alpha + r(I_3 - I_1)\sigma_1\Big). \end{array} \right. \tag{38}$$

Suppose that $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ lies on $M^*_{j,k} = \mathcal{J}^{-1}(j) \cap \mathcal{K}^{-1}(k) \cap M^*$. Eliminating σ_3 and σ_4 from the first two equations in (38) using

$$\sigma_4 = k/\sqrt{P(\sigma_1)} \quad \text{and} \quad \sigma_3 = \frac{1}{rI_1}\Big(j - kI_3\,G(\sigma_1)\Big), \tag{39}$$

where $G(\sigma_1) = (\alpha + r\sigma_1)/\sqrt{P(\sigma_1)}$, we see that the integral curves of $\mathcal{V}$ on the invariant manifold $M^*_{j,k}$ satisfy the equations

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 \\
T(\sigma_1)\dot{\sigma}_2 &= -\frac{R(\sigma_1)}{1-\sigma_1^2}\,\sigma_2^2 - j^2\left[\frac{1}{r^2 I_1^2}\frac{R(\sigma_1)}{1-\sigma_1^2}\right] \\
&+ jk\left[\frac{1}{r^2 I_1^2}\frac{1}{\sqrt{P(\sigma_1)}}\left(rI_1(I_3+mr^2+mr\alpha\,\sigma_1) + 2I_3\frac{(\alpha+r\sigma_1)R(\sigma_1)}{1-\sigma_1^2}\right)\right] \\
&- k^2\left[\frac{I_3}{r^2 I_1^2}\frac{(\alpha+r\sigma_1)(r+\alpha\sigma_1)}{1-\sigma_1^2}\right], \qquad (40)
\end{aligned}$$

where

$$R(\sigma_1) = mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\,\sigma_1^2.$$

Use (39) and (36) to eliminate σ_3, σ_4, and σ_5 from the $\mathcal{E}(2)\times S^1$-reduced energy $\mathcal{E}$ (25). After some remarkable cancellations we obtain

$$H_{j,k}(\sigma_1,\sigma_2) = \tfrac{1}{2}\,\frac{T(\sigma_1)}{1-\sigma_1^2}\,\sigma_2^2 + U_{j,k}(\sigma_1), \qquad |\sigma_1| < 1 \qquad (41)$$

where

$$U_{j,k}(\sigma_1) = \tfrac{1}{2}\,\frac{1}{r^2 I_1^2}\frac{\left(k\sqrt{I_3}(\alpha+r\sigma_1) - \frac{j}{\sqrt{I_3}}\sqrt{P(\sigma_1)}\right)^2}{1-\sigma_1^2} + mg\alpha\,\sigma_1 \qquad (42)$$

up to the additive constant $\frac{1}{2I_1}k^2 - \frac{m}{2I_1^2 I_3}j^2$. A calculation shows that the vector field $\mathcal{V}|M^*_{j,k}$ (40) is in Hamiltonian form on $(-1,1)\times\mathbf{R}$ with Hamiltonian $H_{j,k}$ (41) and symplectic form $\frac{T(\sigma_1)}{1-\sigma_1^2}\,d\sigma_1\wedge d\sigma_2$.

6 Homoclinic orbits

In this section we study the invariant varieties of the $\mathcal{E}(2)\times S^1$-reduced "vector field" V (23) which correspond to homoclinic trajectories.

We consider the restriction of V to the integral variety $M_{j,k} = M \cap J^{-1}(j) \cap \mathcal{K}^{-1}(k)$ when $k = k_j = \frac{\sqrt{P(\pm 1)}}{I_3(\alpha+r)}\,j$, because only then does $M_{j,k}$ contain the point $(\sigma_1^0,\ldots,\sigma_5^0) = (1,0,0,0,0)$ which corresponds to the sphere having its center of mass vertically above its geometric center. Let

$$M_j = M_{j,k_j}. \qquad (43)$$

Removing the singular points of M_j gives a smooth manifold M_j^*. Restricting the "vector field" V (38) to M_j^* gives the Hamiltonian vectorfield on $(-1,1)\times\mathbf{R}$ whose special Hamiltonian is

$$H_j(\sigma_1,\sigma_2) \;=\; \tfrac{1}{2}\,\frac{T(\sigma_1)}{1-\sigma_1^2}{\sigma_2}^2 + U_j(\sigma_1) \qquad |\sigma_1| < 1, \qquad (44)$$

where U_j is the special effective potential

$$U_j(\sigma_1) = U_{j,k_j}(\sigma_1) \;=\; -\tfrac{1}{2}\,\frac{P(1)}{r^2 I_1^2 I_3} j^2 \frac{1}{1+\sigma_1}\left(\frac{F(\sigma_1)-F(1)}{\sigma_1 - 1}\right)^2 (\sigma_1 - 1) + mg\alpha\,\sigma_1 \tag{45}$$

and

$$F(\sigma_1) = \frac{\alpha + r\sigma_1}{\alpha + r} - \sqrt{\frac{P(\sigma_1)}{P(1)}}. \tag{46}$$

We want to study the graph of the special effective potential U_j and to see how many times and with what multiplicities it intersects the horizontal line $mg\alpha$ when $|\sigma_1| \le 1$. This will give us information about asymptotic motions of Routh's sphere.

6.1 The special effective potential

In this subsection we prove some general facts about the special effective potential U_j.

First we show that the graph of the special effective potential U_j crosses the horizontal line $mg\alpha$ at least once in $(-1,1)$ when $|j| < j_0$. We argue as follows. From (45) we find that

$$U_j(1) = \lim_{\sigma_1 \to 1^-} U_j(\sigma_1) \;=\; mg\alpha, \qquad \lim_{\sigma_1 \to -1^+} U_j(\sigma_1) \;=\; \begin{cases} +\infty, & \text{if } j \neq 0 \\ -mg\alpha, & \text{if } j = 0, \end{cases}$$

and

$$U_j'(1) = \lim_{\sigma_1 \to 1^-} U_j'(\sigma_1) \;=\; -\tfrac{1}{4}\,\frac{P(1)F'(1)^2}{r^2 I_1^2 I_3} j^2 + mg\alpha \;=\; mg\alpha\left(1 - \frac{j^2}{j_0^2}\right),$$

where

$$j_0 = \frac{2rI_1}{F'(1)\sqrt{P(1)}}\sqrt{mg\alpha I_3}. \tag{47}$$

Thus $U_j'(1) < 0$ for $|j| < j_0$ and $U_{j_0}'(1) = 0$. This shows that $\widehat{U}_j(\sigma_1) = U_j(\sigma_1) - mg\alpha$ has at least one zero in $(-1,1]$ when $j = j_0$ and at least two zeros when $|j| < j_0$.

Next we show that when $|\sigma_1| < 1$ the graph of U_j crosses the horizontal line $mg\alpha$ at most four times if $|j| \le j_0$. From the definition of U_j (45) we see that if σ_1 is a zero of $\widehat{U}_j$ then

$$2C_j D(\alpha + r\sigma_1)\sqrt{P(\sigma_1)} = C_j\Big(D^2(\alpha + r\sigma_1)^2 + P(\sigma_1)\Big) - mg\alpha(1+\sigma_1)(1-\sigma_1)^2,$$

where $C_j = \frac{1}{2}\frac{j^2}{r^2 I_1^2 I_3}$ and $D = \frac{\sqrt{P(1)}}{\alpha + r}$. Squaring both sides of the above equation and simplifying gives

$$0 = Q_j(\sigma_1)(1-\sigma_1)^2, \tag{48}$$

where

$$
\begin{aligned}
Q_j(\sigma_1) &= \frac{C_j^2 r^2 I_1^2}{m^2 g^2 \alpha^2 (\alpha+r)^4}\Big(2\alpha I_3 + r(I_3 + m(\alpha+r)^2)(1+\sigma_1)\Big)^2 \\
&\quad -\frac{2C_j}{mg\alpha}\Big(D^2(\alpha + r\sigma_1)^2 + P(\sigma_1)\Big)(1+\sigma_1) + (1-\sigma_1^2)^2. \qquad (49)
\end{aligned}
$$

Clearly Q_j is a quartic polynomial in σ_1. If $\widehat{U}_j$ has more than four zeros in $(-1,1)$, then the above argument shows that Q_j would have to have more than four zeros, which is impossible.

For future reference we note that

$$
Q_j(1) = 4C_j\Big(C_j r^2 [(\alpha+r)(D^2 - mI_3) + mrI_1]^2 - 2mg\alpha P(1)\Big) \qquad (50)
$$

is 0 if and only if $j = 0$ or $j = j_0$. Moreover

$$
Q_j(-1) = \frac{2\alpha r I_1 I_3}{(\alpha+r)} C_j^2. \qquad (51)
$$

In the next two subsections we will discuss two special cases. The first case occurs when the Routh sphere is spinning very slowly about an vertical axis which passes through the center of mass and the geometric center. The second occurs when the axis is in the same vertical position as in the first case but the spin is less than j_0 (47). In both cases we want to describe the qualitative features of special integral curves of the "vector field" V near the singularities of the invariant variety M_j. These integral curves correspond to homoclinic trajectories. Knowing these homoclinic trajectories is essential in §7 where we verify the hypotheses of the non-Hamiltonian monodromy theorem.

6.2 The case when $|j|$ is small

When $j = 0$ the invariant variety $M_0 (= M_{0,0})$ has two singular lines $\{(\pm 1, 0, 0, 0, \sigma_5) \in \mathbf{R}^5 \big| \sigma_5 \geq 0\}$. If we remove these singularities it becomes a smooth submanifold M_0^* of $\mathbf{R}^4$ which is the graph of the function

$$
f(\sigma_1, \sigma_2, \sigma_5) = \sigma_2^2 - (1-\sigma_1^2)\sigma_5 \;=\; 0 \quad |\sigma_1| \leq 1 \;\&\; \sigma_5 \geq 0. \qquad (52)
$$

In this case, the Hamiltonian vector field on $(-1,1) \times \mathbf{R}$ has special Hamiltonian

$$
H_0(\sigma_1, \sigma_2) \;=\; \tfrac{1}{2}\frac{T(\sigma_1)}{1-\sigma_1^2}{\sigma_2}^2 + U_0(\sigma_1) \qquad |\sigma_1| < 1, \qquad (53)
$$

where $U_0(\sigma_1) = mg\alpha\,\sigma_1$. Because H_0 is an integral of $\mathcal{V}|M_0^*$ and $\sigma_2 = \dot{\sigma}_1$ we know that

$$
h_0 = H_0(\sigma_1, \dot{\sigma}_1) \;=\; \tfrac{1}{2}\frac{T(\sigma_1)}{1-\sigma_1^2}{\dot{\sigma}_1}^2 + U_0(\sigma_1), \qquad |\sigma_1| < 1 \qquad (54)
$$

is constant throughout the motion.

We now look at motion on the h_0-level set of the special Hamiltonian H_0. Then (54) becomes

$$\dot{\sigma_1}^2 = \frac{2(h_0 - mg\alpha\,\sigma_1)(1-\sigma_1^2)}{T(\sigma_1)}. \tag{55}$$

Since the left hand side of (55) is nonnegative, motion on $H_0^{-1}(h_0)$ can take place only on $(\sigma^-, \sigma^+) \subseteq (-1, 1)$, where $\sigma^- = -1$ and

$$\sigma^+ = \begin{cases} 1, & \text{if } h_0 \geq mg\alpha \\ h_0/mg\alpha, & \text{if } -mg\alpha \leq h_0 < mg\alpha. \end{cases}$$

Since $\lim_{\sigma_1 \to \sigma^\pm} T(\sigma_1) = I_1 + m(\alpha + r\sigma^\pm)^2 > 0$, using (55) we obtain $\lim_{\sigma_1 \to \sigma^\pm} \dot{\sigma}_1 = 0$. Thus motion in $H_0^{-1}(h_0)$ has two limit points $(\sigma^\pm, 0)$ in the σ_1–$\dot{\sigma}_1$ plane. We investigate these limit points further. Separating variables in (55) and integrating gives

$$I^+ = \int_{c^+}^{\sigma^+} \frac{d\sigma_1}{\sqrt{2(h_0 - mg\alpha\,\sigma_1)(1-\sigma_1^2)}} \quad \text{and} \quad I^+ = \int_{c^-}^{\sigma^-} \frac{d\sigma_1}{\sqrt{2(h_0 - mg\alpha\,\sigma_1)(1-\sigma_1^2)}},$$

where $c^\pm \in (\sigma^-, \sigma^+)$. $I^\pm$ is the time it takes a motion in $H_0^{-1}(h_0)$ to reach the limit point $(\sigma^\pm, 0)$. Since I^- is always finite and I^+ is finite when $h_0 \neq mg\alpha$, we see that $(\sigma^\pm, 0)$ is *not* an equilibrium point of the motion when $h_0 \neq \pm mg\alpha$. (When $h_0 = -mg\alpha$, the motion is an equilibrium point because $H_0^{-1}(-mg\alpha) = \{(\sigma^-, 0)\}$). When $(\sigma^-, 0)$ is not an equilibrium point, the motion on $H_0^{-1}(h_0)$ passes through it. To see this we argue as follows. From (55) we see that

$$\lim_{\sigma_1 \to -1^+} \sigma_2^2/(1-\sigma_1^2) = 2(h_0 + mg\alpha)/T(-1)$$

is nonzero when $h_0 \neq \pm mg\alpha$. From (36) and (17) it follows that $\omega_3 = \sigma_5$ is nonzero when $\sigma_1 = -1$. Therefore Routh's sphere is *not* in equilibrium. When $h_0 = mg\alpha$, the limit point $(\sigma^+, 0) = (1, 0)$ is an equilibrium point because motion in $H_0^{-1}(h_0) - \{(1, 0)\}$ takes an infinite time to reach it.

Since the special effective potential U_0 has no critical points on $(-1, 1)$ and is strictly decreasing, motion in $H_0^{-1}(h_0)$ proceeds directly from $(\sigma^+, 0)$ through $(\sigma^-, 0)$ and back to $(\sigma^+, 0)$ when $h_0 \neq -mg\alpha$. Thus we obtain the phase portrait given in figure 3a. Note that the level set $H_0^{-1}(mg\alpha)$ is not smooth at $(1, 0)$ whereas it is smooth at $(-1, 0)$. All other level sets of H_0 are smooth at $(\sigma^\pm, 0)$.

To understand the above phase portrait better we look at the zero level sets of the integrals $\mathcal{J}$ (26) and $\mathcal{K}$ (27) on the orbit space M (19). In other words, we use the space $M_0 (= M_{0,0})$ (31). We now look at the level curves of the reduced energy $\mathcal{E}$ (25) on M_0. Let $\mathcal{E}_0 = \mathcal{E}|M_0$. Since $\mathcal{E}_0(\eta, 0, 0) = \eta\, mg\alpha$ and $p_\pm = (\eta, 0, 0)$ is a singular point of M_0, $\eta\, mg\alpha$ are critical values of $\mathcal{E}_0$ and $p_\pm$ are critical points. To see if $\mathcal{E}_0$ has any other critical values, we use Lagrange multipliers on the smooth manifold $M_0^* = M_0 - \{p_\pm\}$ (52). If $\sigma \in M_0^*$ is a critical point then $0 = d\mathcal{E}_0(\sigma) + \lambda\, df(\sigma)$. This is equivalent to

$$\begin{aligned} mr\alpha\,\sigma_5 + mg\alpha + 2\lambda\,\sigma_1\sigma_5 &= 0 \\ 2\lambda\,\sigma_2 &= 0 \\ I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1 - \lambda(1-\sigma_1^2) &= 0 \\ \sigma_2^2 - (1-\sigma_1^2)\sigma_5 &= 0, \qquad |\sigma_1| \le 1 \ \&\ \sigma_5 \ge 0. \end{aligned} \tag{56}$$

Suppose that $\lambda = 0$. Then

$$0 = I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1 \ \ge\ I_1 + m(r-\alpha)^2 > 0,$$

where the first inequality follows because $\sigma_1 \ge -1$. This is a contradiction. Therefore $\lambda \ne 0$. Consequently, $\sigma_2 = 0$ and hence $0 = (1-\sigma_1^2)\sigma_5$. If $\sigma_5 = 0$, then $mg\alpha = 0$, which is a contradiction. Therefore $\sigma_5 > 0$. Hence $\sigma_1^2 = 1$. But then

$$0 = I_1 + mr^2 + m\alpha^2 + 2mr\alpha\,\sigma_1 = I_1 + m(r \pm \alpha)^2 > 0.$$

Figure 3. (a) Level curves of the special Hamiltonian $H_0 = \frac{1}{2}\frac{T(\sigma_1)}{1-\sigma_1^2}\sigma_2^2 + mg\alpha\,\sigma_1$ on $(-1,1) \times \mathbf{R}$. The points $(\pm 1, 0)$ are limit points of the motion on $H_0^{-1}(h_0)$. (b) Level curves of $\mathcal{E}_0 = \frac{1}{2}T(\sigma_1)\sigma_5 + mg\alpha\sigma_1$ on $M_0 : 0 = \sigma_2^2 - (1-\sigma_1^2)\sigma_5 \quad |\sigma_1| \le 1 \ \&\ \sigma_5 \ge 0$.

This is a contradiction. Thus $\lambda \neq 0$ is also false. Hence (56) has no solutions, that is, $\mathcal{E}_0$ has no critical points on the smooth manifold M_0^*. To sum up, we have shown that if $h_0 \neq \pm mg\alpha$, then h_0 is a regular value of $\mathcal{E}_0$ on M_0^* and hence $\mathcal{E}_0^{-1}(h_0)$ is a *smooth* submanifold of M_0^*.

Note that because the closure of the $mg\alpha$-level set of the special Hamiltonian H_0 (53) passes through $(-1,0)$, the $mg\alpha$-level set of $\mathcal{E}_0$ intersects the half line $\{(-1,0,\sigma_5) \in \mathbf{R}^3 \,|\, \sigma_5 > 0\}$. At this intersection the level set $\mathcal{E}_0^{-1}(mg\alpha)$ is smooth. Hence the motion in the σ_1–$\dot{\sigma}_1$ plane passes through $(-1,0)$. Since the motion of $H_0^{-1}(h_0)$ proceeds from $(\sigma^+,0)$ and goes to $(\sigma^-,0)$, the level sets $\mathcal{E}_0^{-1}(h_0)$ are topological circles when $h_0 \neq -mg\alpha$, see figure 3b.

The motion on $\mathcal{E}_0^{-1}(mg\alpha)$ (or on $H_0^{-1}(mg\alpha)$) has the following physical interpretation. The point $(1,0,0)$ corresponds to the position of the Routh sphere when its center of mass is vertically above its geometric center. Since $j=0$, the sphere is not spinning. Giving the sphere a small push, it rolls over the position $(-1,0,0)$, which corresponds to the center of mass being vertically below its geometric center, and then returns very near its initial position. Thus $\mathcal{E}_0^{-1}(mg\alpha) - \{(1,0,0)\}$ corresponds to an orbit of the $\mathcal{E}(2) \times S^1$-reduced "vector field" $V|M_0$ which is homoclinic to $(1,0,0)$.

We now suppose that $|j|$ is small. Note that when $j=0$ the quartic polynomial Q_j (49) becomes $(1-\sigma_1^2)^2$, which has zeros of multiplicity two at ± 1. Let $j^2 = \varepsilon$ and $C_\varepsilon = \frac{1}{2}\frac{\varepsilon}{r^2 I_1^2 I_3}$ where ε is positive and small. Then we may write Q_j (49) as

$$\widetilde{Q}_\varepsilon(x) = x^4 + A(\varepsilon)x^3 + B(\varepsilon)x^2 + C(\varepsilon)x + D(\varepsilon),$$

where $A(\varepsilon) = \rho\varepsilon + O(\varepsilon^2)$, $B(\varepsilon) = -2 + \mu\varepsilon + O(\varepsilon^2)$, $C(\varepsilon) = \lambda\varepsilon + O(\varepsilon^2)$, and $D(\varepsilon) = 1 + \beta\varepsilon + O(\varepsilon^2)$. When ε is small and positive, from (50) and (51) we see that $\widetilde{Q}_\varepsilon(1) < 0$ and $\widetilde{Q}_\varepsilon(-1) > 0$. Because

$$\widetilde{Q}_\varepsilon(\eta) = (\eta\,\rho + \mu + \eta\,\lambda + \beta)\varepsilon + O(\varepsilon^2),$$

with $\eta^2 = 1$ and ε sufficiently small and positive, it follows that $x_\eta = \eta\,\rho + \mu + \eta\,\lambda + \beta$ is negative, when $\eta = 1$, and is positive otherwise. Suppose that $\widetilde{x} = \eta + d_\eta\sqrt{\varepsilon} + O(\varepsilon)$ is a zero of $\widetilde{Q}_\varepsilon$. Then

$$0 = \widetilde{Q}_\varepsilon(\widetilde{x}) \;=\; (4d_\eta^2 + x_\eta)\varepsilon + O(\varepsilon^{3/2}),$$

which implies $d_\eta = \eta\,\frac{1}{2}\sqrt{-x_\eta}$. Therefore $\widetilde{Q}_\varepsilon$ has two *real* zeros near 1, and two zeros near -1, which are purely imaginary. Consequently, for every ε sufficiently small and positive, $\widetilde{Q}_\varepsilon$ has exactly one zero in $(-1,1)$ and this zero is simple. Therefore $\widehat{U}_j$ has at most one real root in $(-1,1)$ for every j with $|j|$ sufficiently small, for if it had two such roots, then so would Q_j, which is a contradiction. If this root of $\widehat{U}_j$ were multiple, then Q_j would have a multiple root. This is a contradiction. From §6.1 we know that $\widehat{U}_j$ has at least one root in $(-1,1)$. Hence $\widehat{U}_j$ has exactly one root in $(-1,1)$ and this root is simple for every j with $|j|$ sufficiently small.

An important consequence of the above argument is that $mg\alpha$ is a regular value of the special Hamiltonian H_j for every j with $|j|$ sufficiently small. Moreover, $H_j^{-1}(mg\alpha)$ is a connected smooth one dimensional submanifold of $(-1,1) \times \mathbf{R}$, whose closure in $\mathbf{R}^2$ is the one point compactification obtained by adding the point $(1,0)$. This is the homoclinic orbit which we will need in §7 when we verify the hypotheses of the non-Hamiltonian monodromy theorem.

This completes our discussion of the homoclinic orbits of the "vector field" V on M_j when $|j|$ is small.

6.3 The case when $|j|$ is near j_0

In this subsection we see if small homoclinic orbits exist when $|j|$ is close to but less than the critical value j_0. When $|j|$ decreases through j_0, Routh's sphere becomes gyroscopically unstable.

To do this we look at the Taylor polynomial of $\widehat{U}_j$ at $\sigma_1 = 1$. From the defintion (45) of the special effective potential U_j we obtain

$$\begin{aligned}
\widehat{U}_j(\sigma_1) &= -\tfrac{1}{2}\frac{P(1)}{r^2 I_1^2 I_3} j^2 \frac{1}{1+\sigma_1}\left(\frac{F(\sigma_1)-F(1)}{\sigma_1 - 1}\right)^2 (\sigma_1 - 1) + mg\alpha(\sigma_1 - 1) \\
&= \left[E_j \frac{1}{(1+\frac{1}{2}(\sigma_1 - 1))}\left(\frac{F(\sigma_1)-F(1)}{\sigma_1 - 1}\right)^2 + mg\alpha\right](\sigma_1 - 1), \\
&\qquad \text{where } E_j = -\tfrac{1}{4}\tfrac{P(1)}{r^2 I_1^2 I_3} j^2 \text{ and } F(\sigma_1) = \tfrac{\alpha + r\sigma_1}{\alpha + r} - \sqrt{\tfrac{P(\sigma_1)}{P(1)}} \\
&= \widehat{F}(\sigma_1)(\sigma_1 - 1) \qquad (57)
\end{aligned}$$

A straightforward calculation shows that

$$\widehat{F}(\sigma_1) = (E_j F'(1)^2 + mg\alpha) + E_j F'(1)\Big(F''(1) - \tfrac{1}{2}F'(1)\Big)(\sigma_1 - 1) + O((\sigma_1 - 1)^2),$$

where

$$F'(1) = \frac{rI_1}{I_3}\,\frac{I_3 + mr(\alpha + r)}{I_1 + (\alpha + r)^2}$$

and

$$F''(1) = -\frac{mr^2 I_1\Big((I_3 - I_1)(I_3 + mr^2) - m\alpha^2 I_3\Big)}{I_3^2\Big(I_1 + m(\alpha + r)^2\Big)^2}.$$

When $|j| = j_0$ the first order term of the Taylor polynomial of $\widehat{U}_{j_0}$ at $\sigma_1 = 1$ vanishes identically whereas the second order term is

$$\tfrac{1}{2}E_{j_0} F'(1)\Big(F''(1) - \tfrac{1}{2}F'(1)\Big). \qquad (58)$$

From now on we assume that $\frac{1}{2}F'(1) > F''(1)$, that is,

$$2mr^2 I_1\Big((I_3 - I_1)(I_3 + mr^2) - m\alpha^2 I_3\Big) + rI_1 I_3(I_3 + mr(\alpha + r))(I_1 + m(\alpha + r)^2) > 0. \qquad (59)$$

Then $\widehat{U}_{j_0}$ is a Morse function with a nondegenerate critical point of index 0 at $\sigma_1 = 1$. Note that (59) is an open condition on the parameters m, r, α, I_1, and I_3 defining Routh's sphere. Since $U_j'(1) < 0$ for every j with $|j|$ close to j_0, we see that $\widehat{U}_j$ has a simple zero in $(-1,1)$ which is close to 1 when $|j|$ close to j_0. In other words, the $mg\alpha$-level set of the special reduced Hamiltonian H_j is smooth (except at $\sigma_1 = 1$ and is connected). This level set is the desired small homoclinic orbit of the "vector field" V on M_j when $|j|$ close to j_0.

To understand the phase portrait of the special reduced Hamiltonian H_j when $|j|$ is close to but less than j_0, we analyze the behavior of the integral curves of the "vector field" V near the singular point $\sigma^0 = (1,0,0)$ of the invariant variety M_j (43). We will look at the level sets of the reduced energy $\mathcal{E}_j = \mathcal{E}|M_j$ near σ^0. Explicitly, $\mathcal{E}_j$ is restriction to M_j of the function on $\mathbf{R}^3$ given by

$$\begin{aligned}\widehat{\mathcal{E}}_j(\sigma) &= \tfrac{1}{2}\,T(\sigma_1)\sigma_5 + \tfrac{1}{2}\,(I_3 + mr^2)\sigma_4(\sigma_1)^2 - \tfrac{1}{2}\,mr^2(\sigma_3(\sigma_1) + \sigma_1\sigma_4(\sigma_1))^2 \\ &\quad - mr\alpha\,\sigma_3(\sigma_1)\sigma_4(\sigma_1) + mg\alpha\,\sigma_1 \end{aligned} \tag{60}$$

with $\sigma_3(\sigma_1) = (j - kI_3\,G(\sigma_1))/rI_1$ and $\sigma_4(\sigma_1) = k/\sqrt{P(\sigma_1)}$. Let $e_0 = \widehat{\mathcal{E}}_j(\sigma^0)$. To determine the tangent plane Π_{e_0} to the e_0-level set of $\widehat{\mathcal{E}}_j$ we calculate $\mathrm{d}\widehat{\mathcal{E}}_j(\sigma^0)$ and obtain

$$\mathrm{d}\widehat{\mathcal{E}}_j(\sigma^0) = \tfrac{1}{2}\,T(\varepsilon)\,\mathrm{d}\sigma_5 + mg\alpha\,\mathrm{d}\sigma_1.$$

Therefore the equation of Π_{e_0} is

$$0 = mg\alpha(\sigma_1 - \varepsilon) + \tfrac{1}{2}\Big(I_1 + m(\alpha + \varepsilon r)^2\Big)\sigma_5. \tag{61}$$

Next we find the intersection of the plane Π_{e_0} with the tangent cone $\mathcal{C}$ (35). Eliminating σ_5 from (35) and (61) gives

$$0 = \left[\frac{2k^2I_3^2}{r^2I_1^2}G'(\varepsilon)^2 - \frac{8mg\alpha\,\varepsilon}{I_1 + m(\alpha+\varepsilon r)^2}\right](\sigma_1 - \varepsilon)^2 + 2\sigma_2^2 \;=\; Q(\sigma_1, \sigma_2). \tag{62}$$

If $\varepsilon = -1$, the quadratic form Q is *positive definite*. Hence $\sigma_1 + 1 = \sigma_2 = 0$. This implies $\sigma_5 = 0$. In other words, the plane Π_{e_0} intersects the tangent cone $\mathcal{C}$ only at its vertex. Therefore for e slightly larger than e_0, the tangent plane to $\widehat{\mathcal{E}}_j^{-1}(e)$ intersects $\mathcal{C}$ in a smooth circle. Thus the singular point σ^0 is an "elliptic" equilibrium point of the "vector field" V on M_j when $(j,k) \in \Sigma^-$. If $\varepsilon = +1$, then the quadratic form Q is *indefinite* when

$$|k| < \frac{2rI_1}{I_3G'(1)}\sqrt{\frac{mg\alpha}{\Big(I_1 + m(\alpha+r)^2\Big)}} \;=\; k_0 \tag{63}$$

and *positive definite* otherwise. Therefore Π_{e_0} intersects the tangent cone $\mathcal{C}$ only at its vertex if (63) does *not* hold, whereas it intersects $\mathcal{C}$ in a topological circle (with only one singular point σ^0) when (59) and (63) hold. In this latter case, σ^0 is a "hyperbolic" equilibrium point of V. This is very reminiscent of the Hamiltonian Hopf bifurcation in the Lagrange top, see [2, chapter V]. However, there is a fundamental difficulty with this analogy: we are not in a Hamiltonian situation, that is, V is not a Hamiltonian vector field.

7 Monodromy

In this section we verify the hypotheses the non-Hamiltonian monodromy theorem, which is proved in [3]. The non-Hamiltonian monodromy theorem allows us to conclude that geometry of the integral map

$$F_j : \widetilde{\mathcal{J}}^{-1}(j) \to \mathbf{R}^2 : m \to \Big((\tilde{\mathcal{E}}|\mathcal{J}^{-1}(j))(m), (\tilde{\mathcal{K}}|\mathcal{J}^{-1}(j))(m) \Big) \tag{64}$$

of Routh's sphere is complicated. More precisely, let p be a hyperbolic equilibrium point of the $\mathcal{E}(2)$-reduced vector field $\widetilde{V}$ on the j-level set of Jellet's integral $\widetilde{\mathcal{J}}$ and let $(e_j, k_j) = F_j(p)$. Then, for every fixed value of j for which $|j|$ is sufficiently small or close to but less than j_0 and every (e, k) near but not equal to (e_j, k_j) in $\mathbf{R}^2$, the 2-torus fibration defined by the (e, k)-level sets of the integral map F_j is *nontrivial*.

In the next few paragraphs we give a precise statement of the non-Hamiltonian monodromy theorem.

Assumptions: Let v and w be two smooth vector fields on a smooth 4-dimensional manifold M. Let $p \in M$ and let f be a smooth mapping from M to $\mathbf{R}^2$ with $f(p) = 0$. For every $c \in \mathbf{R}^2$ we write $F_c = \{x \in M \mid f(x) = c\}$ for the fiber of f over c. We assume:

a) $v(p) = 0$ and $Dv(p)$ has no real eigenvalues. Moreover, one complex conjugate pair of eigenvalues has negative real part while the other has positive real part.

b) $[v, w] = 0$.

c) $L_v f = 0$ and $L_w f = 0$.

d) At each point $x \in F_0 \setminus \{p\}$, $v(x)$ and $w(x)$ are linearly independent and $Df(x)$ has rank equal to two.

e) F_0 is a compact connected subset of M.

Remark: If F_0 is not connected, but is equal to the union of two disjoint closed subsets K and L, where $p \in K$ and K is compact and connected, then one can replace M by an open neighborhood $\widetilde{M}$ of K such that $\widetilde{M} \cap F_0 = K$. One then requires that assumptions a)–e) hold with M replaced by $\widetilde{M}$.

Conclusions:

a) $F_0 \setminus \{p\}$ is diffeomorphic to the cylinder $(\mathbf{R}/2\pi\,\mathbf{Z}) \times \mathbf{R}$ and F_0 is homeomorphic to the one point compactification of this cylinder. Near p, F_0 is equal to the union two two-dimensional submanifolds of M which intersect transversally at the point p. If $S_\pm$ denotes the set of $x \in M$ such that $\phi^t(x) \to p$ as $t \to \pm\infty$, then $F_0 = S_+ = S_-$.

b) There is an open neighborhood $\widetilde{M}$ of F_0 in M and a simply connected open neighborhood U of 0 in $\mathbf{R}^2$ such that the restriction of f to $\widetilde{M} \setminus F_0$ defines a locally trivial 2-torus fibration over $U \setminus \{0\}$.

c) Let $\alpha : [0, 1] \to U \setminus \{0\}$ be a smooth closed curve in $U \setminus \{0\}$ which winds once around the origin in the positive direction. The 2-torus bundle $F_{\alpha(\varphi)}$, $\varphi \in [0, 1]$, over the loop α has monodromy $\mathcal{M} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ with respect to a suitable basis of generators of the two-dimensional lattice $\mathrm{H}_1 = \mathrm{H}_1 \left(F_{\alpha(0)} \cap \widetilde{M}, \mathbf{Z} \right)$.

In the course of proving the non-Hamiltonian monodromy theorem the following propositions are proved.

Proposition 1: There exist unique smooth functions σ and τ on U such that $\sigma(0) > 0$ and the flow of the vector field $u = (\sigma \circ f)\, w + (\tau \circ f)\, v$ defines a free action of the circle group $\mathbf{R}/2\pi\mathbf{Z}$ on $\widetilde{M} \setminus \{p\}$.

If $c \in U \setminus \{0\}$, then a u-circle in F_c defines a generator $\delta_1 = \delta_1(c)$ of the group of elements of $\mathrm{H}_1(F_c, \mathbf{Z})$ which are fixed under the monodromy operator $\mathcal{M} = \mathcal{M}_c$. For the second generator $\delta_2 = \delta_2(c)$ we can take a v-solution curve, starting and ending on a u-circle in F_c, followed by a part of the u-circle in order to close it up. The linear transformation $\mathrm{D}u(p)$ in T_pM defines a complex structure in T_pM, which in turn defines an orientation in T_pS_-, the eigenspace of T_pv on which the real part of T_pv is positive. This orientation extends in a continuous fashion to an orientation of $\mathrm{T}_xM/\mathrm{T}_xS_+$, for $x \in S_+$, $x \neq p$.

Proposition 2: If, for $x \in S_+$, $x \neq p$, the orientation of $\mathrm{T}_xM/\mathrm{T}_xS_+$ defined by the complex structure in T_pM agrees with the pullback of the orientation of $\mathbf{R}^2$ by means of T_xf, then we can take (δ_1, δ_2) as the ordered basis of H_1 in the theorem. If the orientations do not agree, then we get the inverse monodromy matrix on this basis of H_1.

In the non-Hamiltonian case the orientations in proposition 2 need no longer agree, because the mapping f is no longer determined by the vector fields v and w as in the Hamiltonian case. More precisely, if Φ is a local diffeomorphism near the origin in $\mathbf{R}^2$ such that $\Phi(0) = 0$, then the assumptions a)—e) remain satisfied with f replaced by $\Phi \circ f$. If Φ reverses the orientation of the plane, then the case where the orientations in proposition 2 agree is turned into a case where they don't.

In outline our verification of the hypotheses of the non-Hamiltonian monodromy theorem proceeds as follows. The $\mathcal{E}(2)$-reduced vector field $\widetilde{V}$ on $S^2 \times \mathbf{R}^3$ has three integrals $\widetilde{\mathcal{J}}$, $\widetilde{\mathcal{K}}$, and $\widetilde{\mathcal{E}}$. These integrals and the vector field $\widetilde{V}$ are invariant under the S^1 action ψ, whose infinitesimal generator is the vector field $\widetilde{X}$. In step 1 we show that every value j in the range of $\widetilde{\mathcal{J}}$ is a regular value. Thus the j-level set $\widetilde{\mathcal{J}}^{-1}(j)$ is a smooth 4-dimensional submanifold of $S^2 \times \mathbf{R}^3$, which we denote by $\widetilde{M}_j$. From the fact that $\widetilde{\mathcal{J}}$ is an integral of the S^1-invariant vector fields $\widetilde{X}$ and $\widetilde{V}$, it follows that $\widetilde{M}_j$ is an invariant manifold of $\widetilde{X}$ and $\widetilde{V}$. Thus $\widetilde{X}_j = \widetilde{X}|\widetilde{M}_j$ and $\widetilde{V}_j = \widetilde{V}|\widetilde{M}_j$ are S^1-invariant vector fields on $\widetilde{M}_j$. Because $\widetilde{X}_j$ is the infinitesimal generator of the S^1 action on $\widetilde{M}_j$, the vector fields $\widetilde{X}_j$ and $\widetilde{V}_j$ commute. On $\widetilde{M}_j$ the S^1 action ψ has two fixed points $p_\pm = \big(0, 0, \pm 1, 0, 0, j/(I_3(\alpha \pm r))\big)$. Therefore $p_\pm$ is an equilibrium point of $\widetilde{X}_j$ and $\widetilde{V}_j$. In what follows we only consider the equilibrium point $p = p_+$. In step 2 we show that $D\widetilde{V}_j(p)$ has two pairs of nonzero complex conjugate complex eigenvalues which are not real or purely imaginary when $0 < |j| < j_0$. In step 3 we show that the derivative of F_j has rank 2 on $F_j^{-1}(f_j) - \{p\}$ where $f_j = F(p)$. Moreover, we verify that the fiber $F_j^{-1}(f_j)$ of the integral map F_j is compact. In step 4 we show that the vector fields $\widetilde{X}_j$ and $\widetilde{V}_j$ are linearly independent along the connected component $F_j^{-1}(f_j)^0$ of $F_j^{-1}(f_j)$ which contains p. $F_j^{-1}(f_j)^0$ is the unstable manifold of $\widetilde{V}_j$ corresponding to the hyperbolic equilibrium point p. Note that the orbits of $\widetilde{V}_j$ lie on the fibers of the integral map F_j (64), since $\mathcal{E}_j = \widetilde{\mathcal{E}}|\widetilde{M}_j$ and $\mathcal{K}_j = \widetilde{\mathcal{K}}|\widetilde{M}_j$ are S^1-invariant integrals of $\widetilde{V}_j$.

We now verify all the unproved assertions in the above outline.

STEP 1. By construction, the $\mathcal{E}(2)$-reduced vector field $\widetilde{V}$ (11) and (12) on $S^2 \times \mathbf{R}^3$ is invariant under the S^1 action ψ (16). Consider the S^1-invariant functions

$$\widetilde{\mathcal{J}}(u, \omega) = rI_1 (u_1\omega_1 + u_2\omega_2) + I_3\omega_3(ru_3 + \alpha) \tag{65}$$

and

$$\widetilde{\mathcal{K}}(u, \omega) = \omega_3\sqrt{I_1 I_3 + mr^2 (1 - u_3^2) + mI_3 (\alpha + ru_3)^2} \tag{66}$$

on $S^2 \times \mathbf{R}^3$. We have show that $\widetilde{\mathcal{J}}$ and $\widetilde{K}$ are integrals of $\widetilde{V}$. Note that the infinitesimal generator $\widetilde{X}$ of the S^1 action ψ is the restriction of the vector field

$$X(u, \omega) = -u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - \omega_2\frac{\partial}{\partial \omega_1} - \omega_1\frac{\partial}{\partial \omega_2}$$

on $\mathbf{R}^3 \times \mathbf{R}^3$ to $S^2 \times \mathbf{R}^3$.

Using Lagrange multipliers we verify that $\widetilde{\mathcal{J}}$ has no critical points as follows. Suppose that $(u, \omega) \in S^2 \times \mathbf{R}^3$ is a critical point of $\widetilde{\mathcal{J}}$. Then

$$0 = \Big(rI_1\omega_1, rI_1\omega_2, rI_3\omega_3, rI_1u_1, rI_1u_2, I_3(ru_3 + \alpha)\Big) + 2\lambda\,(u_1, u_2, u_3, 0, 0, 0).$$

Therefore $u_1 = u_2 = 0$ and $u_3 = -\alpha/r$. But $1 = u_1^2 + u_2^2 + u_3^2 = \alpha^2/r^2$, which implies that $\alpha = r$. This contradicts the asssumption that $0 < \alpha < r$. Therefore $\widetilde{\mathcal{J}}$ has no critical points on $S^2 \times \mathbf{R}^3$. Hence every j is a regular value of $\widetilde{\mathcal{J}}$. From now on we will assume that j is in the range of $\widetilde{\mathcal{J}}$. □

STEP 2. If we linearize the vector field $\widetilde{V}$ at the equilibrium point p, we obtain

$$D\widetilde{V}_j(p) = \begin{pmatrix} 0 & -\alpha & 0 & -\beta \\ \alpha & 0 & \beta & 0 \\ 0 & -\gamma & 0 & -\delta \\ \gamma & 0 & \delta & 0 \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= -\frac{j}{I_3(\alpha + r)}, & \beta &= 1, \\ \gamma &= -\frac{mg\alpha}{I_1 + m(r+\alpha)^2}, & \delta &= -\frac{j(I_1 - I_3 + m\alpha^2 + mr\alpha)}{I_3(\alpha + r)(I_1 + m(r+\alpha)^2)}. \end{aligned} \tag{67}$$

Since the characteristic polynomial of $D\widetilde{V}_j(p)$ is

$$\Big(\lambda^2 + \tfrac{1}{2}\,(\delta^2 + 2\beta\gamma + \alpha^2)\Big)^2 + \Big((\alpha\delta - \beta\gamma)^2 - \tfrac{1}{4}\,(\delta^2 + 2\beta\gamma + \alpha^2)^2\Big),$$

$D\widetilde{V}_j(p)$ has a pair of complex conjugate complex eigenvalues which are not purely imaginary if and only if

$$(\alpha\delta - \beta\gamma)^2 > \tfrac{1}{4}\,(\delta^2 + 2\beta\gamma + \alpha^2)^2,$$

that is,

$$-\,\beta\gamma > \tfrac{1}{4}\,(\alpha - \delta)^2. \tag{68}$$

Using (67), we see that (68) is equivalent to requiring that

$$0 < |j| < j_0 = \frac{2I_3(\alpha + r)}{I_3 + mr(\alpha + r)}\,\sqrt{mg\alpha(I_1 + m(\alpha + r)^2)}.$$

As a check, if we substitute the above value of j_0 into $k_0 = \frac{1}{I_3 G(1)}\,j_0$, a short calculation gives the value of k_0 obtained in (63). □

On $\widetilde{M}_j$, the vector field $\widetilde{V}_j$ has two integrals $\widetilde{\mathcal{E}}_j$ and $\widetilde{\mathcal{K}}_j$ which are invariant under the S^1 action generated by the flow of the vector field $\widetilde{X}_j$. In step 2 we have shown that the equilibrium point p is hyperbolic when $0 < |j| < j_0$. Consider the integral map F_j (64) with $F_j(p) = (e_j, k_j) = f_j$. The stable and unstable manifolds of $\widetilde{V}_j$ associated to p lie in the connected component $F_j^{-1}(f_j)^0$ of the level set $F_j^{-1}(f_j) = \widetilde{\mathcal{E}}_j^{-1}(e_j) \cap \widetilde{\mathcal{K}}_j^{-1}(k_j)$ containing p. Since the integrals $\widetilde{\mathcal{E}}_j$ and $\widetilde{\mathcal{K}}_j$ and the vector fields $\widetilde{V}_j$ and $\widetilde{X}_j$ are invariant under the S^1 action, we may reduce the S^1 symmetry on $\widetilde{M}_j \cap \widetilde{\mathcal{K}}_j^{-1}(k_j)$ by passing to the orbit space M_j (43). After reducing the S^1 symmetry, the level set $F_j^{-1}(f_j)$ becomes the subvariety $\mathcal{E}_j^{-1}(e_j)$ of M_j. Here $\mathcal{E}_j$ (60) is the function obtained from $\widetilde{\mathcal{E}}_j$ by reducing the S^1 symmetry. Note that M_j is smooth except at Σ which is $(\pm 1, 0, 0)$, if $j = 0$ and $(1, 0, 0)$, if $|j| = j_0$.

Restricted to $\widetilde{M}_j - \{p_\pm\}$ the vector field $\widetilde{V}_j$ pushes down to the vector field $\mathcal{V}$ (38). On the invariant manifold $M_j^* = M_j - \Sigma$, the vector field $\mathcal{V}$ has integral curves which satisfy (40). These equations are in Hamiltonian form on the symplectic manifold $\Big((-1,1) \times \mathbf{R}, \frac{T(\sigma_1)}{1-\sigma_1^2} \mathrm{d}\sigma_1 \wedge \sigma_2\Big)$ with special Hamiltonian H_j (44).

Step 3. We now show that the derivative of the integral map F_j (64) has rank 2 on $F_j^{-1}(f_j) - \{p\}$. First, note that the orbit map π (16) is a proper submersion when restricted to the smooth manifold

$$\Big((S^2 \times \mathbf{R}^3) - \{L_+ \cup L_-\}\Big) \cap \widetilde{\mathcal{J}}^{-1}(j) = \widetilde{M}_j - \{p_\pm\} \;=\; \widetilde{M}_j^*.$$

Since M_j^*, is a smooth manifold contained in the orbit space M (19), we see that

$$\pi^{-1}(M_j^*) = \widetilde{\mathcal{K}}^{-1}(k_j) - \{p\}$$

is a smooth submanifold of $\widetilde{M}_j$. The subvariety $\mathcal{E}_j^{-1}(e_j)-\Sigma$ of M_j is a smooth 1-dimensional manifold, because it corresponds to the $mg\alpha$-level set of the special Hamiltonian H_j. Therefore

$$\pi^{-1}(\mathcal{E}_j^{-1}(e_j)-\Sigma)=(\widetilde{\mathcal{E}}_j^{-1}(e_j)-\{p\})\cap(\widetilde{\mathcal{K}}^{-1}(k_j)-\{p\}) \;=\; F_j^{-1}(f_j)-\{p\}$$

is a smooth 2-dimensional submanifold of M_j. Consequently, for every $m\in F_j^{-1}(f_j)-\{p\}$, we find that $\dim\ker DF_j(m) = \dim T_m(F_j^{-1}(f_j)-\{p\}) = 2$, which implies that $\operatorname{rank} DF_j(m)=\dim T_m M_j-\dim\ker DF_j(m)=2$.

Since the closure of $H_j^{-1}(mg\alpha)$ is compact and connected, we deduce that $\mathcal{E}_j^{-1}(e_j)$ is compact and connected. Therefore $F_j^{-1}(f_j)^0=\pi^{-1}(\mathcal{E}_j^{-1}(e_j))^0$ is compact and connected. □

STEP 4. Let $\mathcal{E}_j^{-1}(e_j)^0$ be the connected component of $\mathcal{E}_j^{-1}(e_j)$ which contains σ^0. We now show that the vector fields $\widetilde{V}_j$ and $\widetilde{X}_j$ are linearly independent on the smooth submanifold $F_j^{-1}(f_j)^0-\{p\}$ of $\widetilde{M}_j$. Suppose not. Then at some point $q\in F_j^{-1}(f_j)^0-\{p\}$ there is a nonzero real number λ such that $\lambda\widetilde{V}_j(q)+\widetilde{X}_j(q)=0$. After reducing the S^1 symmetry, this equation becomes $\mathcal{V}(\widehat{\sigma_1})=0$, where $\widehat{\sigma_1}$ lies in $\mathcal{E}_j^{-1}(e_j)^0$. In other words, $\widehat{\sigma_1}$ is an equilibrium point of the vector field $\mathcal{V}|M_j$ on $\mathcal{E}_j^{-1}(e_j)^0$. This is equivalent to saying that the special Hamiltonian H_j has a critical point on the connected component of the closure $H_j^{-1}(mg\alpha)$ which contains $(1,0)$. But this contradicts the fact that $mg\alpha$ is a regular value of H_j for all j with $|j|$ sufficiently small or $|j|$ close to but less than j_0. □

Thus we have verified that the hypotheses of the non-Hamiltonian monodromy hold for Routh's sphere when its center of mass is vertically above its geometric center and it is rotating slowly about a vertical axis or when it is rotating near the onset of gyroscopic instability.

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