LICHTENBAUM–TATE DUALITY FOR VARIETIES OVER *P*-ADIC FIELDS

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S. Lichtenbaum has proved in [L1] that there is a nondegenerate pairing

$$\mathrm{Pic}(C) \times \mathrm{Br}(C) \to \mathrm{Br}(K) = \mathbf{Q}/\mathbf{Z} \tag{1}$$

between the Picard group and the Brauer group of a nonsingular projective curve C over a p-adic field K (a finite extension of the p-adic numbers $\mathbf{Q}_p$). His proof consists of a reduction via explicit cocycle calculations in Galois cohomology to a combination of Tate duality for group schemes over p-adic fields and the autoduality of the Jacobian of a smooth curve. In this paper we will reconstruct the above duality as a purely formal combination of a generalized form of Tate duality over p-adic fields and a form of Poincaré duality for curves over arbitrary fields of characteristic zero. This gives a more conceptual proof of Lichtenbaum's result and an analogue in higher dimensions.

Let $\varphi\colon X \to \operatorname{Spec} K$ be a variety over a p-adic field, and consider the cohomological Brauer group

$$\mathrm{Br}(X) := H^2(X, \mathbf{G}_m),$$

or more generally the étale cohomology group $H^i(X, \mathbf{G}_m)$ for some $i \geq 0$. The group $\mathrm{Ext}^{2-i}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)$ is a natural candidate for its dual via the Yoneda pairing into $\mathrm{Br}(K)$. We will see that this Ext-group should not be computed on the étale site over K, but on the *smooth site* K_{sm}; see Section 1.2 for a definition and a motivation of this choice of topology. These groups turn out to give interesting homology groups for varieties over an arbitrary field k. For technical reasons we will require that the ground field k has characteristic zero and that $\varphi\colon X \to \operatorname{Spec} k$ is proper and smooth (see Remark 2.1). The analogy to étale homology with coefficients in $\mathbf{Z}/n$ prompts for the notation

$$`H_i(X, \mathbf{Z}) := \mathrm{Ext}^{-i}_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m),$$

with the quotes added in order to avoid confusion with motivic homology. Indeed, these groups can be regarded as intermediates between étale homology with coefficients in $\hat{\mathbf{Z}}$ (see Section 2.2) and motivic homology with coefficients in $\mathbf{Z}$. For example, when k is algebraically closed, we have for $i > 2$ that

$$`H_i(X, \mathbf{Z}) = H_i(X, \hat{\mathbf{Z}}),$$

whereas $`H_0(X, \mathbf{Z})$ is canonically isomorphic to (the k-points of) the *total Albanese variety* of X (see Sections 1.1, 2.2, and 3.2). On the other hand, the motivic homology group $H_0(X, \mathbf{Z})$ is the Chow group of zero-cycles. Therefore I will refer to the homology theory defined above as *pseudo-motivic homology*. The following result shows that for duality over a p-adic field these pseudo-motivic homology groups are just right.

Theorem 1. *Let X be a smooth proper variety over a finite extension K of $\mathbf{Q}_p$. For every $r \in \mathbf{Z}$ the Yoneda pairing*

$$‘H_r(X,\mathbf{Z}) \times H^{r+2}(X,\mathbf{G}_m) \to \mathrm{Br}(K) = \mathbf{Q}/\mathbf{Z} \tag{2}$$

is nondegenerate, inducing perfect pairings

$$\begin{aligned} &‘H_r(X,\mathbf{Z}) \times H^{r+2}(X,\mathbf{G}_m)^{\frown} \to \mathbf{Q}/\mathbf{Z} && \text{for } r=-2,-1,\\ &‘H_r(X,\mathbf{Z})^{\frown} \times H^{r+2}(X,\mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z} && \text{for } r=0,1, \end{aligned}$$

and

$$‘H_r(X,\mathbf{Z}) \times H^{r+2}(X,\mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z} \qquad \text{for } r \geq 2.$$

Here a pairing between topological groups $A \times B \to \mathbf{Q}/\mathbf{Z}$ is called *nondegenerate* if the induced homomorphisms from A to the Pontryagin dual of B and from B to the Pontryagin dual of A are monomorphisms and *perfect* if these induced maps are isomorphisms. The topology we choose implicitly on our groups is the discrete topology for torsion groups and the profinite topology on all other groups. The notation $A^{\frown}$ denotes the completion of A with respect to the profinite topology.

PROOF. For X geometrically irreducible, this is a special case of Theorem 4.3; removing the irreducibility condition is a straightforward generalization. ☐

Theorem 2 (Poincaré duality for curves)**.** *Let C be a smooth projective curve over a field of characteristic zero. For any $i \in \mathbf{Z}$ we have a natural isomorphism*

$$H^i(C,\mathbf{G}_m) \xrightarrow{\sim} ‘H_{1-i}(C,\mathbf{Z}).$$

PROOF. For C geometrically irreducible, this is a weak version of Theorem 3.7. Removing the irreducibility condition is straightforward. ☐

Note that when C has a k-rational point (or, more generally, a divisor defined over k of degree 1), then Theorem 2 is hardly more than a reformulation of the autoduality of the Jacobian of C. This more general result seems to be new. The proof follows a usual pattern: in Section 3.3 we will construct a pairing

$$R\varphi_*\mathbf{G}_m \overset{L}{\otimes} R\varphi_*\mathbf{G}_m \to \mathbf{G}_m[-1]$$

in the derived category of sheaves on k_{sm} that induces the above isomorphism. This pairing is constructed using the Suslin–Voevodsky cycle complexes, and it is shown to induce an isomorphism using Friedlander–Voevodsky duality for curves.

Remark. In view of the calculations of the pseudo-motivic homology groups in high degree (see Section 2.2), I do not expect that the above Poincaré duality generalizes to higher dimensions. To be precise, I do not think that for $d > 1$ there are (complexes of) sheaves $‘\mathbf{Z}(d)$ on the smooth site over $\mathbf{Q}$ such that for each proper smooth purely d-dimensional variety X over a field of characteristic zero we have $H^i(X, ‘\mathbf{Z}(d)) = ‘H_{2d-i}(X,\mathbf{Z})$.

Corollary 1 (Lichtenbaum–Tate duality [L1])**.** *Let C be a smooth projective curve over a p-adic field K. For every $i \in \mathbf{Z}$ we have a nondegenerate pairing*

$$H^i(X,\mathbf{G}_m) \times H^{3-i}(X,\mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z}.$$

These pairings satisfy the usual symmetry rules for cup products, and they induce perfect pairings

$$H^0(X,\mathbf{G}_m)^\frown \times H^3(X,\mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z},$$
$$`H^1(X,\mathbf{G}_m)^\frown \times H^2(X,\mathbf{G}_m) \to \mathbf{Q}/\mathbf{Z}.$$

PROOF. The existence, nondegeneracy and perfectness of the pairings follows immediately from the theorems above. The symmetry rules follow from the construction. The implicit claim that the pairings given here coincide with Lichtenbaum's pairings is justified by Lemma 3.1 and the construction of the Poincaré duality pairing in Section 3.3. □

In the course of proving Theorem 1 we will collect several other dualities. In particular, we get the following result. Recall that the *period* of a principal homogeneous space X for an abelian variety A over a field K is defined to be the order of the class of X in the Weil–Châtelet group $H^1(K,A)$. More generally, we define the period of an arbitrary nonsingular, complete, geometrically irreducible variety X to be the period of the Albanese torsor $\mathrm{Alb}^1(X)$, which is associated to zero-cycles of degree 1 (see Section 1.1 for the formal definition).

Theorem 3. *Let X be a smooth proper geometrically irreducible variety over a p-adic field K.*

(i) *The image of the mapping δ in the exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{K}})^{\mathrm{Gal}(\overline{K}/K)} \xrightarrow{\delta} \mathrm{Br}(k) \to \mathrm{Br}(X)$$

induced by the Hochschild-Serre spectral sequence is a finite group dual to the cokernel of the degree mapping $`H_0(X,\mathbf{Z}) \to \mathbf{Z}$.

(ii) *The image of the mapping δ^0 in the exact sequence*

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\overline{K}})^{\mathrm{Gal}(\overline{K}/K)} \xrightarrow{\delta^0} \mathrm{Br}(k)$$

induced by the above exact sequence is a finite group dual to the cokernel of the mapping $`H_0(X_{\overline{K}},\mathbf{Z})^{\mathrm{Gal}(\overline{K}/K)} \to \mathbf{Z}$ induced by the degree mapping. The order of this cokernel is the period of X.

PROOF. See Section 4.3. □

Note that for a curve X we have by Poincaré duality that $`H_0(X,\mathbf{Z}) = \mathrm{Pic}(X)$, so in that case the first part of the theorem is equivalent to Roquette's theorem ([Ro, Th. 1], see also [L1, p. 120]). The part of the theorem concerning the period of X was already mentioned in [vH, Rem. 5.4], with a sketch of a proof via cocycle calculations.

Corollary 2. *Let X be a principal homogeneous space for an abelian variety over a p-adic field K. The restriction map*

$$\mathrm{Br}(K) \to \mathrm{Br}(K(X))$$

from the Brauer group of K to the Brauer group of the function field of X is injective if and only if X is trivial.

PROOF. Immediate from the above theorem and the injectivity of the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(K(X))$ (see [Gr, II, Cor. 1.8] or [M1, Exa. 2.22]). □

For a smooth proper geometrically irreducible variety X over a number field k, we can now use class field theory in order to get a sufficient condition for the surjectivity of the map

$$\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \tag{3}$$

(which is in any case injective, since X is proper over k). In analogy with the terminology of [CM], where Coray and Manoil study the map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)},$$

we will say that $\mathrm{Pic}^0(X)$ is *big* if the map (3) is surjective. In other words, $\mathrm{Pic}^0(X)$ is big if every k-rational point on the Picard variety $\mathrm{Pic}^0(X/k)$ of X corresponds to a divisor class containing a divisor defined over k. Recall, that the *Tate–Shafarevich group* of an abelian variety A over the number field k is the subgroup of $H^1(k,A)$ consisting of classes that become trivial when restricted to $H^1(k_v, A_{k_v})$ for any completion k_v of k.

Corollary 3. *Let X be a smooth proper geometrically irreducible variety over a number field k. If the class of the Albanese torsor $\mathrm{Alb}^1(X)$ is contained in the Tate–Shafarevich group of $\mathrm{Alb}(X)$, then $\mathrm{Pic}^0(X)$ is big.*

PROOF. Consider the following diagram with well-known exact rows (see Section 1.1).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}^0(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} & \longrightarrow & \mathrm{Br}(k) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v \mathrm{Pic}^0(X_{k_v}) & \longrightarrow & \prod_v \mathrm{Pic}^0(X_{\bar{k}_v})^{\mathrm{Gal}(\bar{k}_v/k_v)} & \longrightarrow & \prod_v \mathrm{Br}(k_v)
\end{array}$$

Here v ranges over finite and infinite primes. The right hand vertical arrow is injective by class field theory, so the statement follows immediately from Theorem 3.ii and its analogue over the real numbers (see [vH, Cor. 5.3]). $\square$

1. Preliminaries

In this section we will fix some notation and terminology and we will briefly consider the cohomology of sheaves on the smooth site over a scheme. The notation and terminology in this paper concerning derived categories is all standard (see for example [GM]), except maybe the choice not to make a distinction in terminology or notation between cohomology and 'hypercohomology' (the classical term for the result of applying a higher derived functor to a complex, rather than a single object). A *sheaf* will be a sheaf of abelian groups, unless explicitly mentioned otherwise.

A *variety* over a field k will be a separated geometrically reduced (but not necessarily irreducible) scheme over k, and it will be of finite type unless explicitly mentioned otherwise (the group varieties we encounter will in general only be locally of finite type). When there is no danger of confusion, we will denote the scheme $\mathrm{Spec}\, k$ by k. The base change of a variety X over k to an extension field k' will be denoted by $X_{k'}$, and the base change to the separable closure $\bar{k}$ of k will be denoted by $\overline{X}$. A *curve* over k will be a variety over k of pure dimension 1.

1.1. Picard and Albanese variety

We will start by recalling some well-known results; the main reason for repeating them is to fix the notation and terminology, since there does not seem to be a well-established standard.

For a proper variety $\varphi\colon X \to k$ over a field k, the higher direct image sheaf $R^1\varphi_*\mathbf{G}_m$ on the *fpqc*-site over k is represented by a group scheme locally of finite type over k (see [Mur, II.15]), hence by a group variety locally of finite type if k is of characteristic zero. In that case we will denote the group variety representing $R^1\varphi_*\mathbf{G}_m$ by $\mathrm{Pic}(X/k)$ and call it the *total Picard variety* of X. Note that in general the *Picard group* $\mathrm{Pic}(X) = H^1(X,\mathbf{G}_m)$ $(= H^1_{\mathrm{fpqc}}(X,\mathbf{G}_m))$ does not coincide with the group of k-points of $\mathrm{Pic}(X/k)$: we have the well-known long exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X/k)(k) \to \mathrm{Br}(k) \to \mathrm{Br}(X), \tag{4}$$

where $\mathrm{Br}(X)$ denotes the cohomological Brauer group of X.

From now on we will assume that X is smooth and proper over a field k of characteristic zero. Then the connected component of $\mathrm{Pic}(X/k)$ containing zero is an abelian variety over k which we denote by $\mathrm{Pic}^0(X/k)$ and which we call the *Picard variety* of X. We have an exact sequence

$$0 \to \mathrm{Pic}^0(X/k) \to \mathrm{Pic}(X/k) \to NS(\overline{X}) \to 0,$$

where $NS(\overline{X})$ is the finitely generated group variety corresponding to the Néron–Severi group of $\overline{X}$, equipped with its natural Galois action. We denote by $\mathrm{Pic}^0(X)$ the inverse image of $\mathrm{Pic}^0(X/k)$ under the canonical injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X/k)(k)$ and we put

$$NS(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X).$$

In order to define the (total) Albanese variety, we consider the *fpqc*-sheaf $\mathscr{Z}_X$ on k associated to the presheaf that sends a scheme U to the free abelian group generated by the set $X(U)$ of maps from U to X. Still assuming X to be smooth and proper over a field k of characteristic zero, we have that the sheaf $\mathscr{Z}_X$ admits a homomorphism

$$\alpha\colon\ \mathscr{Z}_X \to \mathrm{Alb}^*(X)$$

into a sheaf represented by a group variety locally of finite type over k of which the connected component $\mathrm{Alb}(X)$ containing zero is an abelian variety. The map α is the universal homomorphism of $\mathscr{Z}_X$ to sheaves represented by group varieties of which the connected component containing zero is a semi-abelian variety (see for example [Ra, §2]). We will call $\mathrm{Alb}^*(X)$ the *total Albanese variety of* X. The abelian variety $\mathrm{Alb}(X)$ is the (classical) Albanese variety of X. When X is geometrically irreducible, we have a short exact sequence

$$0 \to \mathrm{Alb}(X) \to \mathrm{Alb}^*(X) \to \mathbf{Z} \to 0, \tag{5}$$

where the map to $\mathbf{Z}$ corresponds, via α, to the degree map $\mathscr{Z}_0(X/k) \to \mathbf{Z}$. The connected component of $\mathrm{Alb}^*(X)$ mapping to $n \in \mathbf{Z}$ will be denoted by $\mathrm{Alb}^n(X)$. In particular, $\mathrm{Alb}^0(X) = \mathrm{Alb}(X)$, and α induces a morphism from X to $\mathrm{Alb}^1(X)$ the *Albanese torsor* of X, which is a principal homogeneous space over $\mathrm{Alb}(X)$. Of course, any k-valued point $x \in X(k)$ induces, by subtraction, an isomorphism $\mathrm{Alb}^1(X) \to \mathrm{Alb}^0(X)$ of principal homogeneous spaces over $\mathrm{Alb}(X)$, hence a morphism $\alpha_x\colon X \to \mathrm{Alb}(X)$. This is the classical *Albanese map* for the pair (X,x), which is universal for maps of X into abelian varieties that send x to zero.

Remark 1.1. The terms 'Picard group', 'Picard variety' and 'Albanese variety' are traditional, and so is the notation $\mathrm{Pic}(X)$ and $\mathrm{Alb}(X)$. The notation $\mathrm{Pic}(X/k)$ and $\mathrm{Alb}^1(X)$ is a variation on notation introduced by Grothendieck in [Gr]. What I call here the 'total Picard variety' is often called the *Picard scheme*. By analogy, the term 'Albanese scheme' is used in [Ra] for what I call the 'total Albanese variety'. Indeed, when a variety is defined to be irreducible, a distinguishing feature of the 'Picard scheme' and the 'Albanese scheme' is that they are not varieties. However, in this paper a variety is not necessarily irreducible, since irreducibility does not behave well under base change, so the adjective 'total' seems a better way to make the distinction.

1.2. Smooth cohomology

For a scheme X the site X_{sm} has as underlying category the category of smooth schemes locally of finite type over X. The coverings are the smooth surjective morphisms.

The cohomology of sufficiently nice sheaves on X_{sm} is the same as the cohomology on other popular sites, like the (small) étale site $X_{\text{ét}}$ or the (big) flat site X_{fl}, for which the underlying category consists of schemes that are étale and of finite type over X (resp. locally of finite type over X) and the coverings are the surjective étale (resp. flat) morphisms. We will use that for each sheaf $\mathscr{G}$ represented by a smooth commutative group scheme over X we have equalities

$$H^i(X_{\mathrm{fl}},\mathscr{G}) = H^i(X_{\mathrm{sm}},\mathscr{G}) = H^i(X_{\text{ét}},\mathscr{G}) \tag{6}$$

for any i. This follows from the vanishing of higher direct images of the sheaf $\mathscr{G}$ for the mappings between the various topologies (see [Gr, III, Th. 11.7], or [M1, Th. III.3.9]). Note that this implies in particular that, with $\varphi\colon X \to k$ as in the previous section, the total Picard group $\mathrm{Pic}(X/k)$ also represents $R^1\varphi_*\mathbf{G}_m$ on the smooth site over k. The same holds when $\mathscr{G}$ is a direct limit of sheaves represented by smooth group schemes, or an inverse limit of sheaves represented by finite group schemes, or when $\mathscr{G}$ is a complex of sheaves for which all cohomology sheaves $\mathscr{H}^*(\mathscr{G})$ are of the above form.

In the above situation we will often omit the reference to the topology and write $H^i(X,\mathscr{G})$ for any of the above groups. We will not make any distinction in notation between commutative group schemes and the sheaves they represent. Also, we will freely use the equivalence of categories between étale sheaves on k and Galois modules. With these conventions, we have for example

$$H^i(k_{\mathrm{sm}},\mathbf{G}_m) = H^i(k_{\text{ét}},\mathbf{G}_m) = H^1(k,\mathbf{G}_m) = H^i(k,\bar{k}^*) = H^i(\mathrm{Gal}(\bar{k}/k),\bar{k}^*).$$

For our purposes here, the difference between the étale site and the smooth site lies in the internal and external Hom- and Ext-groups between sheaves represented by group schemes. The étale site turns out to be too small to give good results; consider for example $\mathscr{H}om_{k_{\text{ét}}}(\mathbf{G}_m,\mathbf{G}_m) = \mathrm{Hom}(\bar{k}^*,\bar{k}^*)$. We need a site with a bigger underlying category. In this respect the (big) flat site is actually as good as the smooth site, as we will see in Lemma 1.2. but the smooth site has the advantage that when X is a smooth variety over k, then all schemes in the underlying category of X_{sm} are smooth varieties over k. This is convenient for certain calculations (see Section 2.2) and also when we want to represent complexes of sheaves by Suslin–Voevodsky cycle complexes (see Section 3.3).

Lemma 1.2. *Let G_1, G_2 be smooth group schemes locally of finite type over a scheme X. Let $\alpha\colon X_{\mathrm{fl}} \to X_{\mathrm{sm}}$ be the canonical morphism of sites. Then we have a canonical isomorphism*

$$R\alpha_* R\mathscr{H}om_{X_{\mathrm{fl}}}(\mathscr{G}_1, \mathscr{G}_2) = R\mathscr{H}om_{X_{\mathrm{sm}}}(\mathscr{G}_1, \mathscr{G}_2).$$

PROOF. We have that α^* is exact(on the level of underlying categories α is a full embedding), so its right adjoint α_* sends injectives to injectives, and adjunction gives us an isomorphism

$$R\alpha_* R\mathscr{H}om_{X_{\mathrm{fl}}}(\alpha^*\mathscr{G}_1, \mathscr{G}_2) = R\mathscr{H}om_{X_{\mathrm{sm}}}(\mathscr{G}_1, R\alpha_*\mathscr{G}_2).$$

Now $\alpha^*\mathscr{G}_1$ is represented by $\mathscr{G}_1$ on X_{fl}, since $\mathscr{G}_1$ is a smooth group scheme. The complex $R\alpha_*\mathscr{G}_2$ is quasi-isomorphic to the sheaf represented by $\mathscr{G}_2$, since the higher direct images of $\mathscr{G}_2$ under α vanish, as we saw above. □

Corollary 1.3. *For M a finitely generated group scheme, T a torus, and A an abelian variety over a field k of characteristic zero we have*

$$\begin{aligned} R\mathscr{H}om_{k_{\mathrm{sm}}}(M, \mathbf{G}_m) &= \mathscr{H}om_{\mathbf{G}/k}(M, \mathbf{G}_m), \\ R\mathscr{H}om_{k_{\mathrm{sm}}}(T, \mathbf{G}_m) &= \mathscr{H}om_{\mathbf{G}/k}(T, \mathbf{G}_m), \\ R\mathscr{H}om_{k_{\mathrm{sm}}}(A, \mathbf{G}_m) &= \mathscr{E}xt^1_{\mathbf{G}/k}(A, \mathbf{G}_m)[-1] = A^t[-1], \end{aligned}$$

where $\mathscr{H}om_{\mathbf{G}/k}$ and $\mathscr{E}xt^1_{\mathbf{G}/k}$ are the internal Hom *and* Ext *in the category $\mathbf{G}/k$ of commutative group schemes over k and A^t is the dual abelian variety of A.*

Recall that the notation $[i]$ is used to indicate a shift by i in the indexing of a complex. In particular, we have here that $A^t[-1]$ denotes the complex consisting of the single object A^t in degree 1.

PROOF. By Lemma 1.2 this follows from [O, Th. 17.4, Th. 18.1], [Br2, §§7, 8, 10], and the vanishing of the higher direct images under $\alpha\colon k_{\mathrm{fl}} \to k_{\mathrm{sm}}$ of sheaves represented by smooth group schemes. □

2. Pseudo-motivic homology

Throughout this section, $\varphi\colon X \to k$ will be a smooth proper scheme over a field k of characteristic zero. We will establish the basic properties of the pseudo-motivic homology groups

$$'H_*(X, \mathbf{Z}) = \mathrm{Ext}^{-*}_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)$$

and do some calculations. The definition of these homology groups is completely analogous to the definitions à la Verdier of homology of locally compact spaces and of étale homology with finite coefficients:

$$H_*(X, \mathbf{Z}/n) = \mathrm{Ext}^{-*}_{k_{\text{ét}}}(R\varphi_*\mu_n, \mathbf{G}_m) \tag{7}$$

(see for example [DV, Exp. VIII], and recall that X is *proper* over k).

Remark 2.1. Of course, the above definition makes sense for arbitrary X over an arbitrary field k. Formally, this would give something that plays the role of homology with compact supports, but since the groups themselves might not be nice enough I would prefer not to use the notation $'H_*(X, \mathbf{Z})$, when X is not smooth and proper, or when k is of characteristic

$p > 0$. I should say that I do not know exactly what I mean by 'nice enough', but I would hope that at least we would have that the complex $R\mathscr{H}om_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)$ is concentrated in nonpositive degree, and that it admits a filtration for which the graded pieces are either complexes of group schemes, or profinite étale (compare Section 2.2). In view of the results and constructions in [Ra], it seems reasonable to expect that in characteristic zero these properties can be obtained for arbitrary X by taking a smooth hypercovering. In characteristic $p > 0$ the groups under consideration need not even be nice in the above sense when X is smooth and proper, due to the 'pathological' behaviour of the $\mathscr{E}xt_{k_{\mathrm{sm}}}$-functor (see for example [Br1]).

2.1. Basic properties

The dual Kummer sequence

Applying the right derived functors of $\mathrm{Hom}_{k_{\mathrm{sm}}}(R\varphi_* -, \mathbf{G}_m)$ to the Kummer sequence

$$0 \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to 0$$

gives a long exact sequence

$$\cdots \to {}^{\backprime}H_i(X, \mathbf{Z}) \xrightarrow{\times n} {}^{\backprime}H_i(X, \mathbf{Z}) \to H_i(X, \mathbf{Z}/n) \to \cdots \tag{8}$$

All basic constructions that follow below also exist for coefficients modulo n, and they are compatible with the Kummer exact sequences.

Functoriality

For a map $f\colon Y \to X$ of proper smooth schemes over k the adjunction morphism $\mathbf{G}_m \to Rf_*\mathbf{G}_m$ induces the *push-forward* homomorphism $f_*\colon {}^{\backprime}H_*(Y, \mathbf{Z}) \to {}^{\backprime}H_*(X, \mathbf{Z})$. If $f\colon Y \to X$ is finite étale, then the *trace map* $f_*\mathbf{G}_m \to \mathbf{G}_m$ (cf. [M1, Lemma V.1.12]) induces the *étale pull-back* $f^*\colon {}^{\backprime}H_*(X, \mathbf{Z}) \to {}^{\backprime}H_*(Y, \mathbf{Z})$. If f is of constant degree d, then $f_* \circ f^*$ is multiplication by d. If Y is Galois over X with Galois group G, then then $f^* \circ f_*$ sends a class β to the class $\sum_{g \in G} g \cdot \beta$.

Note that if k'/k is a finite extension and we have a base change diagram

$$\begin{array}{ccc} X_{k'} & \xrightarrow{\pi} & X_k \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \operatorname{Spec} k' & \xrightarrow{\pi} & \operatorname{Spec} k \end{array}$$

then the trace map induces an adjunction formula

$$\mathrm{Ext}^*_{k'_{\mathrm{sm}}}(R\varphi'_*\mathbf{G}_m, \mathbf{G}_m) = \mathrm{Ext}^*_{k_{\mathrm{sm}}}(R(\pi \circ \varphi')_*\mathbf{G}_m, \mathbf{G}_m)$$

(see [M1, Lemma V.1.12]). Therefore, the group ${}^{\backprime}H_*(X_{k'}, \mathbf{Z})$ does not depend on the question whether we consider $X_{k'}$ as variety over k or over k'. In particular, we have a push-forward map π_* and a pull-back map π^* between the homology of X and $X_{k'}$.

Product with cohomology

The pairing (2) is a special case of the *Yoneda pairing*

$$\begin{array}{ccccc} {}^{\backprime}H_i(X, \mathbf{Z}) & \times & H^j(X, \mathbf{G}_m) & \to & H^{j-i}(k, \mathbf{G}_m) \\ \gamma & \times & \omega & \mapsto & \gamma \cdot \omega \end{array}$$

which is defined for arbitrary i, j via the canonical map

$$\mathrm{Ext}^{-i}_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m,\mathbf{G}_m) = \mathrm{Ext}^{0}_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m,\mathbf{G}_m[-i]) \to \mathrm{Hom}(H^j(k,R\varphi_*\mathbf{G}_m),H^{j-i}(k,\mathbf{G}_m)).$$

From the definitions it is easy to check that for a morphism $f\colon Y \to X$ we have the *projection formula*

$$f_*\gamma\cdot\omega = \gamma\cdot f^*\omega.$$

Homology of a point

For any finite field extension k' of k, we have a canonical isomorphism

$$‘H_i(\operatorname{Spec} k',\mathbf{Z}) = H^{-i}(k',\mathbf{Z}). \tag{9}$$

Under this isomorphism the pushforward morphism

$$\pi_*\colon ‘H_i(\operatorname{Spec} k',\mathbf{Z}) \to ‘H_i(\operatorname{Spec} k,\mathbf{Z})$$

corresponds to the trace map (i.e., the corestriction map in Galois cohomology). Moreover, the Yoneda product defined above corresponds for $X = \operatorname{Spec} k'$ to the cup product

$$H^{-i}(k',\mathbf{Z}) \times H^j(k',\mathbf{G}_m) \to H^{j-i}(k',\mathbf{G}_m)$$

followed by the trace map

$$H^{j-i}(k',\mathbf{G}_m) \to H^{j-i}(k,\mathbf{G}_m).$$

Remark 2.2. The above connection to Galois *co*homology shows that the pseudo-motivic homology of $\operatorname{Spec} k$ is in general not equal to the Galois homology group $H_i(\mathrm{Gal}(\bar{k}/k),\mathbf{Z})$, so it seems better not to use the notation $‘H_i(k,\mathbf{Z})$, which might lead to misunderstandings.

2.2. Calculations

In this section $\varphi\colon X \to k$ will be a proper smooth geometrically irreducible variety of dimension d over a field of characteristic zero. The condition of geometric irreducibility is merely for ease of exposition.

A filtration on the derived direct image of $\mathbf{G}_m$

In order to compute the pseudo-motivic homology groups, we will first define a convenient filtration on $R\varphi_*\mathbf{G}_m$. Since we work in a derived category, where the notion of ‘subcomplex’ does not make sense, this filtration will simply be a sequence of morphisms

$$0 = \mathscr{F}_{-1} \to \mathscr{F}_0 \to \cdots \to \mathscr{F}_{2d+1} = \mathscr{F}_{2d+2} = \cdots = \mathscr{F}_\infty = R\varphi_*\mathbf{G}_m.$$

For every $i \geq 0$ we define the *ith graded piece* $\mathscr{G}_i$ to be the mapping cone of the map $\mathscr{F}_{i-1} \to \mathscr{F}_i$, giving a triangle

$$\mathscr{F}_{i-1} \to \mathscr{F}_i \to \mathscr{G}_i \to \mathscr{F}_i[1]. \tag{10}$$

Our filtration will have the property, that each graded piece consists of a sheaf concentrated in a single degree. It will almost be the canonical filtration $\mathscr{F}^{\mathrm{can}}_\bullet = \tau_{\leq i}R\varphi_*\mathbf{G}_m$, but not quite, since the graded piece of degree one for that filtration is $R^1\varphi_*\mathbf{G}_m = \mathrm{Pic}(X/k)$, which is an extension of the finitely generated group $NS(\overline{X})$ by the abelian variety $\mathrm{Pic}^0(X/k)$. It is better to separate these two parts. Therefore we take

$$\mathscr{F}_0 := \tau_{\leq 0}R\varphi_*G = \varphi_*\mathbf{G}_m = \mathbf{G}_m,$$

for $\mathscr{F}_1$ we take the mapping cone of the canonical map

$$\tau_{\leq 1} R\varphi_* \mathbf{G}_m \to NS(\overline{X}),$$

with the degree shifted by one, so that we have a triangle

$$\mathscr{F}_1 \to \tau_{\leq 1} R\varphi_* \mathbf{G}_m \to NS(\overline{X}) \to \mathscr{F}_i[1],$$

and for $i \geq 2$ we put

$$\mathscr{F}_i := \tau_{\leq i-1} R\varphi_* \mathbf{G}_m.$$

Using the standard notation of $\mathscr{G}[n]$ for a complex that consists of a sheaf concentrated in degree $-n$, we get

$$\mathscr{G}_i = \begin{cases} \mathbf{G}_m & \text{if } i = 0, \\ \mathrm{Pic}^0(X/k)[-1] & \text{if } i = 1, \\ NS(\overline{X})[-1] & \text{if } i = 2, \\ R^{i-1}\varphi_* \mathbf{G}_m[1-i] & \text{if } i \geq 3. \end{cases} \tag{11}$$

The sheaves $R^q\varphi_*\mathbf{G}_m$ are torsion for $q \geq 2$, since $H^q(X \times U, \mathbf{G}_m)$ is torsion for $q \geq 2$ and U smooth over over k by [Gr, II, Prop. 1,4]. In other words, we have for $i \geq 3$ that

$$\mathscr{G}_i = \varinjlim_n ({}_n\mathscr{G}_i)$$

where ${}_n\mathscr{G}_i$ is the complex consisting of the n-torsion of the sheaf $R^{i-1}\varphi_*\mathbf{G}_m$ in degree $i-1$. Using the Kummer sequence we see from the smooth specialization theorem for torsion coefficients (see [SGA4, Exp. XVI, Th. 2.1] or [M1, Cor. VI.4,2]) that the sheaf $R^q\varphi_*\mathbf{G}_m$ is isomorphic to the locally constant sheaf associated to the Galois module $H^q(\overline{X}, \mathbf{G}_m)$ for $q \geq 2$ and in fact to $H^q(\overline{X}, \mathbf{Q}/\mathbf{Z}(1))$ for $q > 2$, where $\mathbf{Q}/\mathbf{Z}(1) = \varinjlim_n \mu_n$. In other words,

$$\mathscr{G}_i = \begin{cases} \mathrm{Br}(\overline{X})[-2] & \text{if } i = 3, \\ H^{i-1}(\overline{X}, \mathbf{Q}/\mathbf{Z}(1))[i-1] & \text{if } i \geq 4. \end{cases}$$

In particular, we have by [SGA4, Exp. X, Cor. 4.3] (see also [M1, Th. VI.1.1]) that $\mathscr{G}_i = 0$ for $i > 2d+1$, hence that $\mathscr{F}_{2d+1} = \mathscr{F}_\infty$, as we claimed in the beginning. Below, we will also use the fact that by [D, Th. finitude] (see also [M1, Th. VI.2.1]) we have that ${}_n\mathscr{G}_i$ is finite for any $i > 2$ and any $n \in \mathbf{N}$.

Remark 2.3. I do not know whether $R^q\varphi_*\mathbf{G}_m$ is torsion for $q \geq 2$ when taken on sites for which the underlying category contains singular schemes, like the big flat site.

The dual filtration

For any complex $\mathscr{C}$ of sheaves on k_{sm} we define the *Cartier dual* of $\mathscr{C}$ to be the complex

$$\mathscr{C}^D := R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}, \mathbf{G}_m).$$

In particular, ${}^{\backprime}H_i(X, \mathbf{Z}) = H^{-1}(k_{\mathrm{sm}}, (R\varphi_*\mathbf{G}_m)^D)$. Dualizing the ascending filtration $\mathscr{F}_\bullet$ on $R\varphi_*\mathbf{G}_m$ we get a descending filtration

$$(R\varphi_*\mathbf{G}_m)^D = \mathscr{F}_\infty^D = \cdots = \mathscr{F}_{2d+2}^D = \mathscr{F}_{2d+1}^D \to \mathscr{F}_{2d}^D \to \cdots \to \mathscr{F}_0^D \to \mathscr{F}_{-1} = \cdots = 0,$$

and for every $i \in \mathbf{Z}$ we have a triangle

$$\mathscr{G}_i^D \to \mathscr{F}_i^D \to \mathscr{F}_{i-1}^D \to \mathscr{G}_i^D[1]. \tag{12}$$

In order give explicit descriptions of the $\mathscr{G}_i^D$ we will first consider the case $i \geq 3$ in greater detail. Since the $\mathscr{G}_i$ are torsion for $i \geq 3$, we have that

$$\mathscr{G}_i^D = R\mathscr{H}om_{k_{\mathrm{sm}}}(\varinjlim_n ({}_n\mathscr{G}_i), \mathbf{G}_m) = R\varprojlim_n R\mathscr{H}om_{k_{\mathrm{sm}}}({}_n\mathscr{G}_i, \mathbf{G}_m).$$

For $i \geq 4$ the surjections $H^{i-1}(\overline{X}, \mu_n) \to {}_n\mathscr{G}_i[i-1]$ and the isomorphisms $H_{i-1}(\overline{X}, \mathbf{Z}/n) = R\mathscr{H}om(H^{i-1}(\overline{X}, \mu_n), \mathbf{G}_m)$ induce an isomorphism

$$R\varprojlim_n R\mathscr{H}om_{k_{\mathrm{sm}}}({}_n\mathscr{G}_i, \mathbf{G}_m) = R\varprojlim_n H_{i-1}(\overline{X}, \mathbf{Z}/n)[i-1].$$

By [J, Th. 2.2] we have that

$$H^p(k, R\varprojlim_n H_q(\overline{X}, \mathbf{Z}/n)) = H^p_{\mathrm{cont}}(k, H_q(\overline{X}, \hat{\mathbf{Z}})),$$

where $H^p_{\mathrm{cont}}(k, -)$ denotes continuous Galois cohomology. Therefore we will write

$$\mathscr{H}_q^{\mathrm{cont}}(X/k, \hat{\mathbf{Z}}) := R\varprojlim_n H_q(\overline{X}, \mathbf{Z}/n).$$

Here we keep k in the notation, since it is important that the inverse limit is taken in the derived category of sheaves on $\operatorname{Spec} k$. For example, taking inverse limits does not commute with infinite field extensions (compare [K, §2]). In particular, when k is not algebraically closed, the complexes $\mathscr{H}_q^{\mathrm{cont}}(X, \hat{\mathbf{Z}})$ will in general not be concentrated in degree 0, whereas

$$\mathscr{H}_q^{\mathrm{cont}}(\overline{X}/\overline{k}, \hat{\mathbf{Z}}) = R\varprojlim_n H_q(\overline{X}, \mathbf{Z}/n) = H_q(\overline{X}, \hat{\mathbf{Z}}).$$

In the case $i = 3$ we have that $\mathscr{G}_3^D$ equals the complex $R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1))[2]$. As above, we have that

$$H^p(k_{\mathrm{sm}}, \mathscr{G}_3^D) = H^p_{\mathrm{cont}}(k, \mathrm{Hom}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1))).$$

Combined with the above calculations of the $\mathscr{G}_i$ for $i = 0$, 1, 2 and the results of Section 1, we get that

$$\mathscr{G}_i^D = \begin{cases} 0 & \text{if } i < 0 \\ \mathbf{Z} & \text{if } i = 0 \\ \mathrm{Alb}(X) & \text{if } i = 1 \\ \mathscr{H}om(NS(\overline{X}), \mathbf{G}_m)[1] & \text{if } i = 2 \\ R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1))[2] & \text{if } i = 3 \\ \mathscr{H}_{i-1}^{\mathrm{cont}}(X/k, \hat{\mathbf{Z}})[i-1] & \text{if } i \geq 4. \end{cases} \tag{13}$$

The modified Hochschild–Serre spectral sequence

Since the complexes $\mathscr{H}_{i-1}^{\mathrm{cont}}(X/k, \hat{\mathbf{Z}})$ are in general not concentrated in degree 0, the above calculations do not give a sensible description of the $E_2^{p,q}$-terms of the 'standard' Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(k_{\mathrm{sm}}, R^q\mathscr{H}om_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)) \Rightarrow {}^{\prime}H_{-p-q}(X, \mathbf{Z}).$$

Therefore it makes sense to modify this spectral sequence a little, replacing the degree filtration on $\mathscr{H}om_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)$, by the decreasing filtration

$$0 = \mathscr{F}^{HS}_{-2d-1} \to \mathscr{F}^{HS}_{-2d} \to \cdots \to \mathscr{F}^{HS}_0 = R\mathscr{H}om_{k_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{G}_m)$$

with $\mathscr{F}_i^{HS}$ for $-2d \le i \le -1$ determined by the requirement that we have a triangle

$$\mathscr{F}_i^{HS} \to (R\varphi_*\mathbf{G}_m)^D \to \mathscr{F}_{1-i}^D \to \mathscr{F}_i^{HS}[1].$$

Here $(R\varphi_*\mathbf{G}_m)^D \to \mathscr{F}_{1-i}^D$ is the canonical map associated to the filtration $\mathscr{F}_\bullet^D$ of $R\varphi_*\mathbf{G}_m$. As above, we obtain for every i the associated ith graded piece $\mathscr{G}_i^{HS}$, and we put

$$`\mathscr{H}_i(X,\mathbf{Z}) := \mathscr{G}_{-i}^{HS}[-i].$$

The filtration $\mathscr{F}_\bullet^{HS}$ gives rise to the *modified Hochschild–Serre spectral sequence*

$$E_2^{p,q} = H^p(k_{\mathrm{sm}}, `\mathscr{H}_{-q}(X,\mathbf{Z})) \Rightarrow `H_{-p-q}(X,\mathbf{Z}). \tag{14}$$

In this modified spectral sequence, the E_2-terms are easy to interpret, thanks to the calculations above. By construction we have that for $i > 0$ that

$$`\mathscr{H}_i(X,\mathbf{Z}) = \mathscr{G}_{i+1}^D[-i].$$

On the other hand, $`\mathscr{H}_0(X,\mathbf{Z})$ fits into an exact sequence

$$0 \to \mathrm{Alb}(X) \to `\mathscr{H}_0(X,\mathbf{Z}) \to \mathbf{Z} \to 0.$$

This suggests that $`\mathscr{H}_0(X,\mathbf{Z})$ is represented by the total Albanese variety $\mathrm{Alb}^*(X)$ defined in Section 1.1. We will see in Section 3.2 that this is indeed the case, and we will use it below to simplify the notation. The Albanese property of $`\mathscr{H}_0(X,\mathbf{Z})$ will not be used in an essential way before Section 3.3.

In terms of Galois cohomology, we get the following expression for the E_2-terms of the modified Hochschild–Serre spectral sequence.

$$H^p(k_{\mathrm{sm}}, `\mathscr{H}_{-q}(X,\mathbf{Z})) = \begin{cases} 0 & \text{if } q > 0, \\ H^p(k, \mathrm{Alb}^*(X)(\overline{k})) & \text{if } q = 0, \\ H^p(k, \mathrm{Hom}(NS(\overline{X}), \overline{k}^*)) & \text{if } q = -1, \\ H^p_{\mathrm{cont}}(k, \mathrm{Hom}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1))) & \text{if } q = -2, \\ H^p_{\mathrm{cont}}(k, H_{-q}(\overline{X}, \hat{\mathbf{Z}})) & \text{if } q < -2. \end{cases}$$

Remark 2.4. If all Galois cohomology groups of k with finite coefficient modules are finite, then

$$H^s_{\mathrm{cont}}(k, \mathrm{Hom}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1))) = \varprojlim_n H^s(k, \mathrm{Hom}({}_n\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1)))$$

and

$$H^s_{\mathrm{cont}}(k, H_t(\overline{X}, \hat{\mathbf{Z}})) = \varprojlim_n H^s_{\mathrm{cont}}(k, H_t(\overline{X}, \mathbf{Z}/n))$$

(see [J, Rem. 3.5]). The finiteness condition is fulfilled when k is a p-adic field.

Calculations over an algebraically closed field

For $\overline{X}$ over the algebraic closure $\overline{k}$ of k the above gives us:

$$`H_i(\overline{X},\mathbf{Z}) = \begin{cases} 0 & \text{if } i < 0, \\ \mathrm{Alb}^*(X)(\overline{k}) & \text{if } i = 0, \\ \mathrm{Hom}(NS(\overline{X}), \overline{k}^*) & \text{if } i = 1, \\ \mathrm{Hom}(\mathrm{Br}(\overline{X}), \mathbf{Q}/\mathbf{Z}(1)) & \text{if } i = 2, \\ H_i(\overline{X}, \hat{\mathbf{Z}}) & \text{if } i > 2. \end{cases}$$

High degree homology

Over an arbitrary field k of characteristic 0, we see from the modified Hochschild–Serre spectral sequence that the canonical map $R\varphi_*\mathbf{Q}/\mathbf{Z}(1) \to R\varphi_*\mathbf{G}_m$ induces for $i > 2$ an isomorphism

$$`H_i(X,\mathbf{Z}) = R^{-i}\mathrm{Hom}_{k_{\text{ét}}}(R\varphi_*\mathbf{Q}/\mathbf{Z}(1),\mathbf{G}_m).$$

The right hand side of this equation is canonically isomorphic to the ith *continuous étale homology* group $H_i^{\text{cont}}(X,\hat{\mathbf{Z}})$ as defined in [K, §3.2] (recall that X is proper over k).

Calculations in degree 0 over a p-adic field

Now let us assume k has cohomological dimension ≤ 2, which is the case when X is a p-adic field (see [S, Prop. II.15]). Then the $E_2^{s,t}$-terms of the modified Hochschild–Serre spectral sequence vanish for $s > 2$. We get an exact sequence

$$`H_0(X,\mathbf{Z}) \to \mathrm{Alb}^*(X)(k) \to H^2(k,\mathrm{Hom}(NS(\overline{X}),\bar{k}^*)).$$

The kernel $`H_0(X,\mathbf{Z})^{\mathrm{Alb}}$ of the Albanese map $`H_0(X,\mathbf{Z}) \to \mathrm{Alb}^*(X)(k)$ fits into an exact sequence

$$H^2_{\text{cont}}(k,\mathrm{Hom}(\mathrm{Br}(\overline{X}),\mathbf{Q}/\mathbf{Z}(1))) \to `H_0(X,\mathbf{Z})^{\mathrm{Alb}} \to H^1(k,\mathrm{Hom}(NS(\overline{X}),\bar{k}^*)) \to 0$$

When k is a p-adic field, we actually have

$$H^2(k,\mathrm{Hom}(NS(\overline{X}),\bar{k}^*)) = \mathrm{Hom}(NS(\overline{X})^{\mathrm{Gal}(\bar{k}/k)},\mathbf{Q}/\mathbf{Z})$$

and

$$H^2_{\text{cont}}(k,\mathrm{Hom}(\mathrm{Br}(\overline{X}),\mathbf{Q}/\mathbf{Z}(1))) = \mathrm{Hom}(\mathrm{Br}(\overline{X})^{\mathrm{Gal}(\bar{k}/k)},\mathbf{Q}/\mathbf{Z})$$

as we easily deduce from Tate duality for finitely generated groups (compare Proposition 4.1.

3. The cycle map, the Albanese property and Poincaré duality

In this section we will construct a cycle map for zero-cycles into the homology of degree zero, and check that this map satisfies the Albanese property. Then we prove Poincaré duality for curves (Theorem 2 from the introduction).

3.1. The cycle map for zero-cycles

Let k' be a finite extension of a field k of characteristic 0. The canonical isomorphism (9) gives in degree zero a canonical isomorphism

$$`H_0(\mathrm{Spec}\,k',\mathbf{Z}) = \mathbf{Z}.$$

The canonical generator of $`H_0(\mathrm{Spec}\,k',\mathbf{Z})$ will be called the *fundamental class* of $\mathrm{Spec}\,k'$. We will denote it by $[\mathrm{Spec}\,k'] \in `H_0(\mathrm{Spec}\,k',\mathbf{Z})$. We now define the cycle map

$$\mathrm{cl}\colon\ \mathscr{Z}_0(X) \to `H_0(X,\mathbf{Z}) \tag{15}$$

from the group of zero-cycles into homology by sending a closed point $x \in X$ to the image of $[x]$ under the mapping $i_*\colon \text{`}H_0(x,\mathbf{Z}) \to \text{`}H_0(X,\mathbf{Z})$, where i is the inclusion. By construction the cycle map commutes with the push-forward associated to a morphism of varieties $f\colon X \to Y$.

The following lemma implies that Lichtenbaum's pairing of $\mathscr{Z}_0(X)$ with $\mathrm{Br}(X)$, as defined in [L1, §3], factorizes via the cycle map and the Yoneda pairing.

Lemma 3.1. *With notations as above, we have that for any $r \geq 0$ and any $\omega \in H^r(X,\mathbf{G}_m)$ the image of $\mathrm{cl}(x) \times \omega \in H^r(k,\mathbf{G}_m)$ under the pairing* (2.1) *coincides with the image of* ω *under the composite mapping*

$$H^r(X,\mathbf{G}_m) \xrightarrow{i^*} H^r(x,\mathbf{G}_m) \xrightarrow{tr} H^r(k,\mathbf{G}_m),$$

where the mapping tr is induced by the trace map.

PROOF. Immediate from the definitions. $\square$

Later, it will be important that the cycle map for zero-cycles is already defined on the sheaf level. Let $\mathscr{Z}_X$ be the free sheaf on k_{sm} of abelian groups over X, i.e., the sheaf associated to the presheaf $U \mapsto \mathbf{Z}[X(U)]$. For every U smooth over k we have that a morphism $s\colon U \to X$ induces via pull-back a homomorphism from the complex of sheaves $R\varphi_*\mathbf{G}_m$ to the sheaf $\mathbf{G}_m$, both restricted to U. Thus we get a homomorphism

$$c\ell\colon\ \mathscr{Z}_X \to R^0\mathscr{H}om(R\varphi_*\mathbf{G}_m,\mathbf{G}_m) = \text{`}\mathscr{H}_0(X,\mathbf{Z}) \tag{16}$$

of sheaves on k_{sm}; it follows from the definitions that taking sections over k gives back the original cycle map (15).

Proposition 3.2. *Let X be a proper smooth geometrically irreducible variety over a field of characteristic zero. The cycle map factorizes via rational equivalence, giving a homomorphism*

$$\mathrm{cl}\colon\ CH_0(X) \to \text{`}H_0(X,\mathbf{Z})$$

PROOF. The group $\mathscr{Z}_0^{\mathrm{rat}}(X)$ of zero-cycles rationally equivalent to 0 is generated by zero-cycles of the form $\pi_*(f)$, where $\pi\colon C \to X$ is a morphism of a nonsingular projective curve C to X, and (f) is the divisor of a rational function f on C. Since $\mathrm{cl}(\pi_*(f)) = \pi_*\,\mathrm{cl}((f))$, it is sufficient to check the proposition for a nonsingular projective curve C.

Since $\text{`}\mathscr{H}_0(C,\mathbf{Z})$ is represented by a commutative group variety locally of finite type, the universal property of the total Albanese variety implies that the map $\mathscr{Z}_X \to \text{`}\mathscr{H}_0(X,\mathbf{Z})$ induced by (16) factorizes via the Albanese map. Taking sections over k and using the injectivity of the map

$$\text{`}H_0(C,\mathbf{Z}) \to \text{`}\mathscr{H}_0(C,\mathbf{Z})(k)$$

(Hilbert's Theorem 90), we get that the cycle map (15) factorizes as

$$\mathscr{Z}_0(C) \to \mathrm{Alb}^*(C)(k) \to \text{`}H_0(C,\mathbf{Z}).$$

The kernel of the first map is equal to $\mathscr{Z}_0^{\mathrm{rat}}(C)$ by the Abel–Jacobi theorem. $\square$

3.2. The Albanese map

In this section we will prove that the map

$$c\ell\colon \mathscr{Z}_X \to {}^{t}\mathscr{H}_0(X,\mathbf{Z}) \tag{17}$$

induced by the map (16) satisfies the Albanese property. In particular, ${}^{t}\mathscr{H}_0(X,\mathbf{Z})$ is represented by the total Albanese variety $\mathrm{Alb}^*(X)$, as was claimed and used in Section 2.2.

The covariant functoriality of ${}^{t}\mathscr{H}_0(-,\mathbf{Z})$, will enable us to reduce the proof to the case where X is a principal homogeneous space for an Abelian variety, and then the statement follows from Proposition 3.3 below, which claims that the cycle map induces for any abelian variety A an isomorphism between A itself and the connected component ${}^{t}\mathscr{H}_0(A,\mathbf{Z})^0$ of ${}^{t}\mathscr{H}_0(A,\mathbf{Z})$ containing zero.

Proposition 3.3. *Let A be an abelian variety over a field k of characteristic zero. The map*

$$\begin{array}{rcl} a\colon A & \to & {}^{t}\mathscr{H}_0(A,\mathbf{Z})^0 \\ x & \mapsto & c\ell([x]-[0]) \end{array}$$

is an isomorphism of (sheaves represented by) abelian varieties.

PROOF. The map a is a priori only a morphism of varieties, but since ${}^{t}\mathscr{H}_0(A,\mathbf{Z})^0$ is (represented by) an abelian variety, and 0 is mapped to 0, it is a *homomorphism* of abelian varieties. In order to prove that a is an isomorphism, it is sufficient to check that the induced map on n-torsion

$${}_nA \to {}_n{}^{t}\mathscr{H}_0(A,\mathbf{Z})^0$$

is an isomorphism for all $n \in \mathbf{N}$. This is equivalent to proving that the induced map of finite n-torsion groups

$$a_n\colon {}_nA(\bar{k}) \to {}_n{}^{t}H_0(\bar{A},\mathbf{Z})^0 \left(= {}_n{}^{t}H_0(\bar{A},\mathbf{Z})\right)$$

is an isomorphism.

Let $\varphi\colon A \to k$ be the structure map, and let $\mathbf{n}\colon A \to A$ be multiplication by n. We define $R\varphi_*\mathbf{G}_m/\mathbf{n}^*$ to be the cone of the induced map

$$R\varphi_*\mathbf{G}_m \xrightarrow{\mathbf{n}^*} R\varphi_*\mathbf{G}_m,$$

and we put

$$\begin{aligned} H^i(\bar{A},\mathbf{G}_m,\mathbf{n}^*) &:= H^i(\bar{k},R\varphi_*\mathbf{G}_m/\mathbf{n}^*) \\ {}^{t}H_i(\bar{A},\mathbf{Z},\mathbf{n}_*) &:= R^{-i}\mathrm{Hom}_{\bar{k}_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m/\mathbf{n}^*,\mathbf{G}_m). \end{aligned}$$

We get long exact sequences

$$H^0(\bar{A},\mathbf{G}_m) \xrightarrow{\mathbf{n}^*} H^0(\bar{A},\mathbf{G}_m) \to H^0(\bar{A},\mathbf{G}_m,\mathbf{n}^*) \to H^1(\bar{A},\mathbf{G}_m) \xrightarrow{\mathbf{n}^*} H^1(\bar{A},\mathbf{G}_m) \to \cdots$$

and

$$\cdots \to {}^{t}H_1(\bar{A},\mathbf{Z}) \xrightarrow{\mathbf{n}_*} {}^{t}H_1(\bar{A},\mathbf{Z}) \to {}^{t}H_0(\bar{A},\mathbf{Z},\mathbf{n}_*) \to {}^{t}H_0(\bar{A},\mathbf{Z}) \xrightarrow{\mathbf{n}_*} {}^{t}H_0(\bar{A},\mathbf{Z})$$

Recall that the pull-back $\mathbf{n}^*$ is the identity on $\varphi_*\mathbf{G}_m$, multiplication by n on $\mathrm{Pic}^0(A/k)$ and multiplication by n^2 on $NS(\bar{A})$, and that $NS(\bar{A})$ is torsion free (see [Mum, §8]. This implies that

$$H^0(\bar{A},\mathbf{G}_m,\mathbf{n}^*) = {}_n\mathrm{Pic}(\bar{A})$$

and

$${}^{\backprime}H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*) = {}_n{}^{\backprime}H_0(\overline{A}, \mathbf{Z})$$

It follows that we have a map

$$a_{\mathbf{n}_*} \colon {}_nA(\overline{k}) \to {}^{\backprime}H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*)$$

that fits into the following commutative diagram.

$$\begin{tikzcd}
& {}_nA(\overline{k}) \arrow[dl, "a_n"'] \arrow[dr, "a_{\mathbf{n}_*}"] & \\
{}_n{}^{\backprime}H_0(\overline{A}, \mathbf{Z}) \arrow[rr, equal] & & {}^{\backprime}H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*)
\end{tikzcd}$$

Hence it suffices to show that $a_{\mathbf{n}_*}$ is an isomorphism.

The cohomology sheaves $\mathscr{H}^i(R\varphi_*\mathbf{G}_m/\mathbf{n}^*)$ are torsion for every $i \in \mathbf{Z}$, as we see from the above expression of the endomorphism $\mathbf{n}^*$ as multiplication by powers of n. Therefore the comparison between smooth and étale cohomology gives us that the group ${}^{\backprime}H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*)$ is canonically isomorphic to the group $\mathrm{Hom}(R\varphi_*\mathbf{G}_m/\mathbf{n}^*, \overline{k}^*)$ computed in the derived category of étale sheaves on $\overline{k}$. We will now define a suitable complex of abelian groups that represents the complex of étale sheaves $R\varphi_*\mathbf{G}_m/\mathbf{n}^*$, in order to be able to compute $\mathrm{Hom}(R\varphi_*\mathbf{G}_m/\mathbf{n}^*, \overline{k}^*)$ in the derived category of abelian groups.

Let $\mathscr{C}$ be the complex of abelian groups

$$\mathscr{O}^*_{{}_n\overline{A}, \overline{A}} \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{A}, {}_n\overline{A})$$

where $\mathscr{O}^*_{{}_n\overline{A}, \overline{A}}$ is the multiplicative group of invertible functions on $\overline{A}$ having no poles or zeroes on the n-torsion points, and $\mathrm{Div}(\overline{A}, {}_n\overline{A})$ is the group of divisors on $\overline{A}$ with supports outside the n-torsion points. The moving lemma for divisors implies that $\mathscr{C}$ is quasi-isomorphic to the complex

$$\mathscr{K}(\overline{A})^* \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{A}),$$

hence we have a canonical map of complexes of groups (étale sheaves over $\overline{k}$)

$$\mathscr{C} \to R\varphi_*\mathbf{G}_m,$$

that induces an isomorphism in cohomology of degree ≤ 1.

Defining $\mathscr{O}^*_{{}_n\overline{A}, \overline{A}}/\mathbf{n}^*$ to be the cokernel of the injective map

$$\begin{array}{rcl} \mathscr{O}^*_{{}_n\overline{A}, \overline{A}} & \to & \mathscr{O}^*_{{}_n\overline{A}, \overline{A}} \\ f & \mapsto & f \circ \mathbf{n} \end{array}$$

and $\mathrm{Div}(\overline{A}, {}_n\overline{A})/\mathbf{n}^*$ to be the cokernel of the injective map

$$\begin{array}{rcl} \mathrm{Div}(\overline{A}, {}_n\overline{A}) & \to & \mathrm{Div}(\overline{A}, {}_n\overline{A}) \\ D & \mapsto & \mathbf{n}^{-1}(D), \end{array}$$

we see that the corresponding complex $\mathscr{C}/\mathbf{n}^*$ maps canonically to $R\varphi_*\mathbf{G}_m/\mathbf{n}^*$, inducing an isomorphism

$$H^0(\mathscr{C}/\mathbf{n}^*) \xrightarrow{\sim} H^0(\overline{A}, \mathbf{G}_m, \mathbf{n}^*) = {}_n\mathrm{Pic}(\overline{A})$$

that sends the class of a function g on $\overline{A}$ with $\mathrm{div}(g) = \mathbf{n}^{-1}(D)$ for some divisor D to the n-torsion divisor class $[D] \in \mathrm{Pic}(\overline{A})$. Since $\overline{k}^*$ is a divisible group, hence injective, we also see that

$$‘H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*) = \mathrm{Hom}(H^0(R\varphi_* \mathbf{G}_m/\mathbf{n}^*), \overline{k}^*) = \mathrm{Hom}(H^0(\mathscr{C}/\mathbf{n}^*), \overline{k}^*).$$

In particular, we obtain a perfect pairing between $‘H_0(\overline{A}, \mathbf{Z}, \mathbf{n}_*)$ and ${}_n\mathrm{Pic}(\overline{A})$ into $\overline{k}^*$, and the map $a_{\mathbf{n}_*}$ induces a pairing

$${}_nA(\overline{k}) \times {}_n\mathrm{Pic}(\overline{A}) \to \overline{k}^*.$$

From the above discussion and the definition of the cycle map we see that this pairing is given by the formula

$$(x, [D]) \mapsto f(x)/f(0),$$

where D is a divisor with support outside the n-torsion points of $\overline{A}$ and f is a function with $\mathrm{div}(f) = \mathbf{n}^{-1}(D)$. In other words, this pairing coincides up to sign with the Weil pairing, which is nondegenerate (see for example [Mum, §20]). We conclude that $a_{\mathbf{n}^*}$ is an isomorphism for every $n \in \mathbf{N}$. □

Remark 3.4. In the proof of the proposition we pass to torsion elements, since there would have been no point in considering the composite map

$$A(\overline{k}) \to ‘H_0(\overline{A}, \mathbf{Z}) \to \mathrm{Hom}(\mathscr{C}, \overline{k}^*),$$

which is zero: since $\overline{k}^*$ is an injective étale sheaf, the right hand map factorizes via $\mathrm{Hom}_{\overline{k}_{\mathrm{sm}}}(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}$.

Corollary 3.5. *Let $A^\bullet$ be an extension of $\mathbf{Z}$ by an abelian variety A^0 over a field k of characteristic zero. Let A^1 be the connected component of $A^\bullet$ mapping to $1 \in \mathbf{Z}$. We have an isomorphism of sheaves on k_{sm} represented by group varieties*

$$a^\bullet \colon A^\bullet \xrightarrow{\sim} ‘\mathscr{H}_0(A^1, \mathbf{Z})$$

such that $a^\bullet$ restricted to A^1 is the canonical map

$$A^1 \to ‘\mathscr{H}_0(A^1, \mathbf{Z})$$

of sheaves of sets induced by the cycle map $c\ell$.

PROOF. For a scheme T that is smooth over k with $x_1 \in A^1(T) \neq \emptyset$ we send $x \in A^i(T)$ to

$$c\ell([x - (i-1)x_1] + (i-1)[x_1]) \in ‘\mathscr{H}(A^1, \mathbf{Z})(T).$$

It follows from Proposition 3.3 that this map does not depend on the choice of x_1. Since the extension

$$0 \to A^0 \to A^\bullet \to \mathbf{Z} \to 0$$

is locally trivial on the smooth site over k, we have sections locally everywhere so the above defines a homomorphism

$$a^\bullet \colon A^\bullet \xrightarrow{\sim} ‘\mathscr{H}_0(X, \mathbf{Z}).$$

If we have a global section $x_1 \in A^1(k)$, it follows easily from Proposition 3.3 that $a^\bullet$ is an isomorphism; otherwise we make a base change to a finite extension of k such that we obtain a global section of A^1, and we apply Proposition 3.3. □

Theorem 3.6. *Let X be a proper smooth geometrically irreducible variety over a field k of characteristic zero. The homomorphism of sheaves*

$$c\ell\colon\ \mathscr{Z}_X \to {}^{\text{‘}}\mathscr{H}_0(X,\mathbf{Z})$$

is the universal homomorphism of $\mathscr{Z}_X$ into sheaves on k_{sm} represented by group varieties locally of finite type of which the connected component containing zero is an abelian variety. In particular, ${}^{\text{‘}}\mathscr{H}_0(X,\mathbf{Z})$ is represented by the total Albanese variety of X.

PROOF. Let A be a commutative group variety locally of finite type of which the connected component containing zero is an abelian variety. Let

$$f\colon\ \mathscr{Z}_X \to A$$

be a homomorphism of sheaves. We will show that f factorizes via the cycle map $c\ell$.

Let A^0 be the connected component of A containing zero. In order to be able to use Corollary 3.5, we need to replace A by an extension $A^\bullet$ of $\mathbf{Z}$ by A^0. Let $\mathscr{Z}_X^0$ be the subsheaf of $\mathscr{Z}_X$ of elements of degree zero. Since X is geometrically connected, we have that $\mathscr{Z}_X^0$ maps to A^0, so $\mathscr{Z}_X/\mathscr{Z}_X^0$ $(=\mathbf{Z})$ maps to A/A^0. We take the fibre product $A^\bullet = A\times_{A/A^0}\mathbf{Z}$, and we have a homomorphism

$$f^\bullet\colon\ \mathscr{Z}_X \to A^\bullet$$

defined by $f^\bullet(z) = (f(z),\deg(z))$. We denote by $\pi\colon\ A^\bullet \to A$ the canonical projection. In order to prove the theorem, it is sufficient to show that $f^\bullet$ factorizes via the cycle map $c\ell$ and a homomorphism from ${}^{\text{‘}}\mathscr{H}_0(X,\mathbf{Z})$ to $A^\bullet$, since $f = f^\bullet\circ\pi$.

Let $A^1 \subset A^\bullet$ be the connected component mapping to $1\in\mathbf{Z}$, and let

$$f^1\colon\ X \to A^1$$

be the morphism of varieties induced by $f^\bullet$. By Corollary 3.5 we have the following commutative diagram.

$$\begin{array}{ccc}
\mathscr{Z}_X & \xrightarrow{\quad f^\bullet \quad} & A^\bullet \\
\Big\downarrow{\scriptstyle c\ell} & & \Big\downarrow{\scriptstyle a^\bullet} \\
{}^{\text{‘}}\mathscr{H}_0(X,\mathbf{Z}) & \xrightarrow{\ f^1_*\ } & {}^{\text{‘}}\mathscr{H}_0(A^1,\mathbf{Z})
\end{array}$$

Since $a^\bullet$ is an isomorphism, the diagram gives the desired factorization of $f^\bullet$ via ${}^{\text{‘}}\mathscr{H}_0(X,\mathbf{Z})$. $\square$

3.3. Poincaré duality for curves

The proof of Poincaré duality for curves in the present setting is analogous to the proof of duality for cohomology with coefficients in the nth roots of unity μ_n of a smooth projective curve $\varphi\colon\ C\to \operatorname{Spec} k$ over a field of characteristic not dividing n. Writing $\mathbf{Z}/n(j) = \mu_n^{\otimes j}$, we have that geometric Poincaré duality consists in that case of a composite morphism

$$R\varphi_*\mathbf{Z}/n(1) \overset{L}{\underset{\mathbf{Z}/n}{\otimes}} R\varphi_*\mathbf{Z}/n(1) \to R\varphi_*\mathbf{Z}/n(2) \xrightarrow{\ \mathrm{tr}\ } \mathbf{Z}/n(1)[-2]$$

in the derived category of n-torsion sheaves on $\operatorname{Spec} k$ that induces an isomorphism

$$R\varphi_*\mathbf{Z}/n(1) \xrightarrow{\sim} R\mathscr{H}om_{k_{\text{ét}},\mathbf{Z}/n}(R\varphi_*\mathbf{Z}/n(1),\mathbf{Z}/n(1)[-2]) \tag{18}$$

in the derived category of $\mathbf{Z}/n$ modules on $k_{\text{ét}}$ (see for example [D, Dualité]).

A natural integral analogue of the above duality would be a pairing

$$R\varphi_*\mathbf{Z}(1) \overset{L}{\otimes} R\varphi_*\mathbf{Z}(1) \to R\varphi_*\mathbf{Z}(2) \xrightarrow{\text{tr}} \mathbf{Z}(1)[-2]$$

inducing an isomorphism

$$R\varphi_*\mathbf{Z}(1) \xrightarrow{\sim} R\mathscr{H}om(R\varphi_*\mathbf{Z}(1), \mathbf{Z}(1)[-2]).$$

Here the complex $\mathbf{Z}(1)$ is by definition quasi-isomorphic to the sheaf $\mathbf{G}_m$ in degree 1 (see [L2]), whereas there are several working definitions for the complex $\mathbf{Z}(2)$.

The pairing to be constructed is a kind of intersection pairing, hence we need representatives of the complex of sheaves $R\varphi_*\mathbf{Z}(1)$ on the smooth site over k with good intersection properties. For this we take the Suslin–Voevodsky complexes of equidimensional cycles. Once the pairing is constructed, we will prove that this pairing induces the desired isomorphism using Theorem 3.6; the comparison with the cycle map we need for this uses Friedlander–Voevodsky duality. We will first recall the necessary definitions and results. For consistency with the rest of the paper, the notation $\mathbf{Z}(i)$ will not be used in the rest of this section.

Sheaves of equidimensional cycles

Let $\varphi\colon X \to k$ be a variety over a field of characteristic zero. As in [FV] we denote by $z_{\text{equi}}(X,r)$ the presheaf on k_{sm} that associates to every smooth scheme U locally of finite type over k the group $z_{\text{equi}}(X,r)(U)$ of algebraic cycles on $X \times U$ that are equidimensional of relative dimension r over U. For a variety Y we denote by $z_{\text{equi}}(X,Y,r)(U)$ the presheaf

$$U \mapsto z_{\text{equi}}(Y,r)(X \times U)$$

Observe that when X is smooth over k, then

$$z_{\text{equi}}(X,Y,r) = \varphi_*\varphi^* z_{\text{equi}}(Y,r). \tag{19}$$

For a presheaf F we denote by F_{sm} the associated sheaf, and by $C_*(F)$ the associated simplicial complex of presheaves, i.e., the complex of presheaves associated to the simplicial presheaf

$$U \mapsto F(\Delta^\bullet \times U),$$

where $\Delta^\bullet$ is the standard cosimplicial scheme over k (see for example [FV, § 4]).

By [V, Th. 3.4.2 and Cor. 4.1.8] we have a canonical isomorphism

$$\mathbf{G}_m = C_*(z_{\text{equi}}(\mathbf{A}^1, 0))[-1]. \tag{20}$$

The right hand side is the complex we will use to define our pairing. When X and Y are smooth over k, the natural embedding of presheaves

$$z_{\text{equi}}(X,Y,r) \hookrightarrow z_{\text{equi}}(X \times Y, r + \dim Y)$$

induces by [FV, Th. 7.1] quasi-isomorphisms of the associated simplicial complexes of presheaves.

$$C_*(z_{\text{equi}}(X,Y,r)) \xrightarrow{\sim} C_*(z_{\text{equi}}(X \times Y, r + \dim Y)).$$

This is what we will call *Friedlander–Voevodsky duality*. From this duality and the homotopy invariance of complexes of presheaves of the form $C_*(-)$ (see [FV, Lemma 4.1]) we see that the pullback map

$$z_{\text{equi}}(X,r) \to z_{\text{equi}}(X \times \mathbf{A}^1, r+1)$$

induces a quasi-isomorphism of associated simplicial complexes of presheaves.

Theorem 3.7. *Let $\varphi \colon C \to \operatorname{Spec} k$ be a smooth projective geometrically irreducible curve over a field of characteristic zero. There is a pairing*

$$R\varphi_*\mathbf{G}_m \overset{L}{\otimes} R\varphi_*\mathbf{G}_m \to \mathbf{G}_m[-1]$$

that induces an isomorphism

$$R\varphi_*\mathbf{G}_m \xrightarrow{\sim} R\mathscr{H}om_{k_{\text{sm}}}(R\varphi_*\mathbf{G}_m, \mathbf{Z})[-1]$$

in the derived category of sheaves on the smooth site over k. In particular, we have for any $i \in \mathbf{Z}$ an isomorphism

$$H^i(C, \mathbf{G}_m) \xrightarrow{\sim} {}^{\text{`}}H_{1-i}(C, \mathbf{Z}).$$

PROOF. By (19) and (20) we have a natural homomorphism

$$C_*(z_{\text{equi}}(C, \mathbf{A}^1, 0)) \to R\varphi_*\mathbf{G}_m.$$

Since C is a smooth projective curve, this is a quasi-isomorphism after sheafifying for the smooth topology by [V, Th. 3.4.2]. We have an obvious symmetric pairing of presheaves

$$z_{\text{equi}}(C, \mathbf{A}^1, 0) \times z_{\text{equi}}(C, \mathbf{A}^1, 0) \to z_{\text{equi}}(C, \mathbf{A}^2, 0)$$

that takes closed subvarieties V, W on $\mathbf{A}^1 \times X \times U$ to the cycle associated to the fibre product $V \times_{X \times U} W \subset \mathbf{A}^2 \times X \times U$. This induces a pairing

$$C_*(z_{\text{equi}}(C, \mathbf{A}^1, 0)) \times C_*(z_{\text{equi}}(C, \mathbf{A}^1, 0)) \to C_*(z_{\text{equi}}(C, \mathbf{A}^2, 0)).$$

Composing with the push-forward map

$$C_*(z_{\text{equi}}(C, \mathbf{A}^2, 0)) \xrightarrow{\varphi_*} C_*(z_{\text{equi}}(\operatorname{Spec} k, \mathbf{A}^2, 1))$$

and the isomorphism

$$C_*(z_{\text{equi}}(\operatorname{Spec} k, \mathbf{A}^2, 1)) = C_*(z_{\text{equi}}(\mathbf{A}^2, 1)) \simeq C_*(z_{\text{equi}}(\mathbf{A}^1, 0)),$$

we obtain a pairing of complexes of presheaves that induces a pairing

$$R\varphi_*\mathbf{G}_m[1] \overset{L}{\otimes} R\varphi_*\mathbf{G}_m[1] \to \mathbf{G}_m[1].$$

Shifting the degrees gives the pairing we require.

In order to check that our pairing induces Poincaré duality, it is sufficient to check that we have isomorphisms

$$\mathscr{H}^i(C, \mathbf{G}_m) \to {}^{\text{`}}\mathscr{H}_{1-i}(C, \mathbf{Z})$$

of the homology sheaves in degrees $i = 0$ and 1, since the homology sheaves of the complex $R\varphi_*\mathbf{G}_m$ and its dual vanish in all other degrees. We first prove the case $i = 1$. Since both source and target are representable by an extension of $\mathbf{Z}$ by the Jacobian of C, it would by

Theorem 3.6 be sufficient to check that the sheaf-theoretic cycle map (17) factorizes via the map

$$\mathscr{H}^1(C,\mathbf{G}_m) \to {}^{\backprime}\mathscr{H}_0(C,\mathbf{Z}). \tag{21}$$

We will construct a candidate (23), but we will not quite need to prove that this composite map coincides with the cycle map (17). Consider the following diagram of pairings, in which all vertical arrows become quasi-isomorphisms after applying $C_*(-)$ and where all pairings are defined via intersection products or fibre products in the obvious way.

$$\begin{tikzcd}
z_{\mathrm{equi}}(C,\mathbf{A}^1,0) \otimes z_{\mathrm{equi}}(C,\mathbf{A}^1,0) \arrow[r] & z_{\mathrm{equi}}(C,\mathbf{A}^2,1) \arrow[r, "\varphi_*"] & z_{\mathrm{equi}}(\mathbf{A}^2,1) \\
z_{\mathrm{equi}}(C\times\mathbf{A}^1,1) \otimes z_{\mathrm{equi}}(C,\mathbf{A}^1,0) \arrow[u, hook] \arrow[r] & z_{\mathrm{equi}}(C\times\mathbf{A}^2,1) \arrow[u, hook] \arrow[r, "\varphi_*"] & z_{\mathrm{equi}}(\mathbf{A}^2,1) \arrow[u, equal] \\
z_{\mathrm{equi}}(C,0) \otimes z_{\mathrm{equi}}(C,\mathbf{A}^1,0) \arrow[u] \arrow[r] & z_{\mathrm{equi}}(C\times\mathbf{A}^1,0) \arrow[u] \arrow[r, "\varphi_*"] & z_{\mathrm{equi}}(\mathbf{A}^1,0) \arrow[u]
\end{tikzcd} \tag{22}$$

We have a natural map

$$\mathscr{Z}_C \to z_{\mathrm{equi}}(C,0),$$

and from the leftmost column we get a map

$$z_{\mathrm{equi}}(C,0) \to \mathscr{H}^1(C,\mathbf{G}_m).$$

Composing with (21) we get a map

$$\mathscr{Z}_C \to \mathscr{H}^1(C,\mathbf{G}_m) \to {}^{\backprime}\mathscr{H}_0(C,\mathbf{Z}). \tag{23}$$

Since ${}^{\backprime}\mathscr{H}_0(C,\mathbf{Z})$ is the total Albanese variety of C, the universal property gives a commutative diagram

$$\begin{tikzcd}
\mathscr{Z}_C \arrow[r] \arrow[d] & \mathscr{H}^1(C,\mathbf{G}_m) \arrow[r] & {}^{\backprime}\mathscr{H}_0(C,\mathbf{Z}) \\
{}^{\backprime}\mathscr{H}_0(C,\mathbf{Z}) \arrow[ur, dotted] \arrow[urr, dotted] & &
\end{tikzcd} \tag{24}$$

We will check that the right-hand diagonal arrow in this diagram is the identity. Since we deal with a morphism of sheaves represented by smooth group varieties, it is sufficient to check this at the global sections, provided we pass to the algebraic closure $\bar{k}$ of k. From the bottom row of diagram (22) we see that (23) sends the cycle associated to a closed point $i\colon x \hookrightarrow \overline{C}$ to the element in ${}^{\backprime}H_0(\overline{C},\mathbf{Z}) = \mathrm{Hom}_{\bar{k}_{\mathrm{sm}}}(R\varphi_*\mathbf{G}_m,\mathbf{G}_m)$ represented by the morphism

$$R\varphi_*\mathbf{G}_m \xrightarrow{i^*} R\psi_*\mathbf{G}_m = \mathbf{G}_m,$$

where $\psi\colon x \to \operatorname{Spec}\bar{k}$ is the structure map. This coincides with the cycle map (15), hence the right-hand diagonal arrow of (24) is the identity, so the homomorphism (21) is an isomorphism.

In order to prove that

$$\mathscr{H}^0(C,\mathbf{G}_m) \to {}^{\backprime}\mathscr{H}_1(C,\mathbf{Z})$$

is an isomorphism as well, we simply observe (using the calculations in Section 2.2 and the symmetry of the pairing) that this mapping can be obtained from the isomorphism (21) by applying the functor $\mathscr{H}om_{k_{\mathrm{sm}}}(-,\mathbf{G}_m)$. $\square$

4. Generalized Tate duality

In this section we will prove Theorems 1 and 3. These are actually straightforward consequences of Theorem 4.3 below, which asserts that the complexes $\mathscr{F}_i$ defined in Section 2.2 satisfy Tate duality when X is smooth and proper over a p-adic field. This in turn follows from the duality for the graded pieces $\mathscr{G}_i$ which is classical Tate duality, since the $\mathscr{G}_i$ consist of (direct limits of) étale finitely generated groups, tori and abelian varieties concentrated in a single degree.

A crucial role in the proof of Theorem 4.3 below will be played by the following collection indexed by $i \geq 0$ of compatible systems of pairings into $H^2(K, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}$ with long exact rows.

$$
\begin{array}{ccccccccc}
\cdots \leftarrow H^{r+1}(K, \mathscr{G}_i^D) & \longleftarrow & H^r(K, \mathscr{F}_{i-1}^D) & \longleftarrow & H^r(K, \mathscr{F}_i^D) & \longleftarrow & H^r(K, \mathscr{G}_i^D) & \longleftarrow \cdots \\
\times & & \times & & \times & & \times & \\
\cdots \longrightarrow H^{1-r}(K, \mathscr{G}_i) & \longrightarrow & H^{2-r}(K, \mathscr{F}_{i-1}) & \to & H^{2-r}(K, \mathscr{F}_i) & \to & H^{2-r}(K, \mathscr{G}_i) & \to \cdots
\end{array}
\tag{25}
$$

For $i \geq 0$ this system is constructed from the triangles (10) and (12) using the Yoneda pairing. It will allow us to glue the Tate duality for the $\mathscr{G}_i$ (see Proposition 4.1 in order to obtain duality for the $\mathscr{F}_i$. In the gluing process some caution is necessary, since some of the duality pairings for the $\mathscr{G}_i$ are only perfect after taking suitable completions. Lemma 4.2 provides the essential arguments that will allow us to proceed. In Section 4.3 we will prove Theorem 3. Throughout this section X is a smooth, proper and geometrically irreducible over a p-adic field K.

4.1. Classical Tate duality

Proposition 4.1. *Let X be a smooth and proper geometrically irreducible variety over a p-adic field K. Consider for $i \geq 0$, $r \in \mathbf{Z}$ the Yoneda pairing*

$$H^r(K, \mathscr{G}_i^D) \times H^{2-r}(K, \mathscr{G}_i) \to H^2(K, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}.$$

(i) *For every $i \geq 0$, $r \in \mathbf{Z}$ the pairing is nondegenerate.*
(ii) *The pairing is perfect if the pair (i, r) is not in the set $\{(0,0), (0,2), (2,1), (2,3)\}$.*
(iii) *The pairing induces perfect pairings*

$$
\begin{aligned}
H^2(K, \mathscr{G}_0^D) &\times H^0(K, \mathscr{G}_0)^\wedge \to \mathbf{Q}/\mathbf{Z}, \\
H^1(K, \mathscr{G}_2^D) &\times H^1(K, \mathscr{G}_2)^\wedge \to \mathbf{Q}/\mathbf{Z}, \\
H^0(K, \mathscr{G}_0^D)^\wedge &\times H^2(K, \mathscr{G}_0) \to \mathbf{Q}/\mathbf{Z}, \\
H^{-1}(K, \mathscr{G}_2^D)^\wedge &\times H^3(K, \mathscr{G}_2) \to \mathbf{Q}/\mathbf{Z}.
\end{aligned}
$$

(iv) *For $i = 0$ or 1, $r \in \mathbf{Z}$ the groups $H^r(K, \mathscr{G}_i)$ and $H^{2-r}(K, \mathscr{G}_i^D)$ vanish if the pair (i, r) is not in the set $\{(0,0), (0,2), (1,1), (1,2)\}$. For $i > 1$ the groups $H^r(K, \mathscr{G}_i)$ and $H^{2-r}(K, \mathscr{G}_i^D)$ vanish if r is not in the range $i-1, \ldots, i+1$.*

PROOF. For $i = 0, 2$ the proposition follows from Hilbert's Theorem 90 and Tate–Poitou duality for finitely generated groups (see [M2, Thm. I.2.1]). For $i = 1$ the proposition follows from Tate duality for abelian varieties (see [M2, Cor. 3.4]).

For $i > 2$ the proposition follows from Tate–Poitou duality for finite groups, since

$$\mathscr{G}_i = \varinjlim_n {}_n\mathscr{G}_i,$$

where the complex ${}_n\mathscr{G}_i$ consists of the (finite) n-torsion subgroup ${}_n\mathscr{H}^{i-1}(\mathscr{G}_i)$ of the Galois module $\mathscr{H}^{i-1}(\mathscr{G}_i) = H^{i-1}(\overline{X}, \mathbf{G}_m)$ placed in degree $i-1$. Hence for all $r \in \mathbf{Z}$ we have

$$H^{2-r}(K, \mathscr{G}_i) = \varinjlim_n H^{2-r}(K, {}_n\mathscr{G}_i),$$

and

$$H^r(K, \mathscr{G}_i^D) = H^r(K, R\varprojlim_n ({}_n\mathscr{G}_i)^D) = R\varprojlim_n H^r(K, ({}_n\mathscr{G}_i)^D) = \varprojlim_n H^r(K, ({}_n\mathscr{G}_i)^D),$$

since the groups $H^r(K, ({}_n\mathscr{G}_i)^D)$ are finite (see [M2, Th. I.2.1]). $\square$

4.2. Gluing dualities

As was mentioned in the introduction of this section, we will need the following technical result for the proof of Theorem 4.3.

Lemma 4.2. *Let X be a smooth and proper geometrically irreducible variety over a p-adic field K. Consider the compatible system of pairings* (25).

(i) *The boundary map $H^1(K, \mathscr{G}_2) \to H^2(K, \mathscr{F}_1)$ has finite image.*
(ii) *For $i = 1, 2$ the boundary map $H^0(K, \mathscr{F}_{i-1}^D) \to H^1(K, \mathscr{G}_i^D)$ has finite image.*
(iii) *The boundary map $H^{-1}(K, \mathscr{F}_2^D) \to H^0(K, \mathscr{G}_3^D)$ has finite image.*

PROOF. (i) The image of the boundary map is the cokernel of the map

$$\mathrm{Pic}(X) = H^1(K, \mathscr{F}_2) \to H^1(K, \mathscr{G}_2) = H^0(K, NS(\overline{X})),$$

which is well-known to be finite.

(ii) For $i = 1$ the image of the boundary map is the cokernel of the map

$$H^0(K, \mathscr{F}_1^D) \to H^0(K, \mathscr{F}_0^D).$$

The image of this map contains the image of the composite map

$$CH_0(X) \xrightarrow{\mathrm{cl}} {}^{\prime}H_0(X, \mathbf{Z}) \to H^0(K, \mathscr{F}_0^D) = \mathbf{Z},$$

which coincides with the degree map for zero-cycles, hence the cokernel under consideration is finite. For $i = 2$, we consider the commutative diagram

$$\begin{array}{ccc} H^0(K, \mathscr{F}_1^D) & \longrightarrow & H^1(K, \mathscr{G}_2^D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H^1(K, \mathscr{F}_1), \mathbf{Q}/\mathbf{Z}) & \to & \mathrm{Hom}(H^2(K, \mathscr{G}_2), \mathbf{Q}/\mathbf{Z}) \end{array}$$

obtained from the system of pairings (25). The right hand vertical arrow is an isomorphism by Proposition 4.1, and the image of the bottom arrow is finite by part (i) of this lemma.

(iii) The image of the map $H^{-1}(K, \mathscr{F}_2^D) \to H^0(K, \mathscr{G}_3^D)$ is the cokernel of the map

$$H^{-1}(K, \mathscr{F}_3^D) \to H^{-1}(K, \mathscr{F}_2^D),$$

hence a quotient of the cokernel of the natural map

$${}^{\prime}H_1(X, \mathbf{Z}) = H^{-1}(K, \mathscr{F}_\infty) \to H^{-1}(K, \mathscr{F}_2^D) = H^0(K, \mathrm{Hom}(NS(\overline{X}), \overline{K}^*)).$$

In order to show that the latter cokernel is finite, it is sufficient to prove that it has finite exponent, since K^*/K^{*n} is finite for any $n \in \mathbf{N}$. Therefore it is sufficient to prove that the cokernel of the map

$$\varepsilon\colon {}^{\backprime}H_1(X_L, \mathbf{Z}) \to H^0(L, \mathrm{Hom}(NS(\overline{X}), \overline{K}^*))$$

has finite exponent for some finite extension L of K in $\overline{K}$.

We will finish the proof by showing that this is a rather straightforward consequence of the fact that for divisors algebraic equivalence modulo torsion coincides with numerical equivalence. So let us take L large enough such that $NS(X_L) = NS(\overline{X})$ and such that we have a finite collection

$$f_i\colon C_i \to X_L$$

of smooth, projective, geometrically irreducible curves C_i over L mapping to X_L, that generates a subgroup of finite index in $\mathrm{Hom}(NS(\overline{X}), \mathbf{Z})$ via the intersection product. To be precise, taking $\mathscr{Z}_1(\bigcup C_i)$ to be the group of 1-dimensional cycles on the disjoint union of the C_i we have that the right kernel of the pairing

$$\begin{array}{ccccc} \mathscr{Z}_1(\bigcup C_i) & \times & NS(\overline{X}) & \to & \mathbf{Z} \\ \sum a_i[C_i] & \times & [D] & \mapsto & \sum a_i (f_i)_*[C_i] \cdot [D] \end{array}.$$

is precisely the (finite) torsion subgroup of $NS(\overline{X})$.

After tensoring with L^* we obtain a map

$$\mathscr{Z}_1(\bigcup C_i) \otimes L^* \to \mathrm{Hom}(NS(\overline{X}), L^*)$$

of which the cokernel has finite exponent. By Section 2.2 we have a canonical isomorphism

$$\mathscr{Z}_1(\bigcup C_i) \otimes L^* = {}^{\backprime}H_1(\bigcup C_i, \mathbf{Z}),$$

which fits into the following commutative diagram by the projection formula.

$$\begin{array}{ccc} \mathscr{Z}_i(\bigcup C_i) \otimes L^* & \longrightarrow & \mathrm{Hom}(NS(\overline{X}), \mathbf{Z}) \otimes L^* \\ \| & & \| \\ {}^{\backprime}H_1(\bigcup C_i, \mathbf{Z}) & & \mathrm{Hom}(NS(\overline{X}), L^*) \\ \downarrow{\scriptstyle (f_i)_*} & & \| \\ {}^{\backprime}H_1(X_L, \mathbf{Z}) & \xrightarrow{\ \varepsilon\ } & H^0(L, \mathrm{Hom}(NS(\overline{X}), \overline{K}^*)) \end{array}$$

Hence the cokernel bottom arrow has finite exponent. $\square$

Theorem 4.3. *Let X be a nonsingular complete variety over a p-adic field K For $i \geq 0$, let $\mathscr{F}_i$ be the complex defined in Section 2.2, and consider the Yoneda pairing*

$$H^r(K, \mathscr{F}_i^D) \times H^{2-r}(K, \mathscr{F}_i) \to H^2(K, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}.$$

(i) *For every $i \geq 0$ and $r \in \mathbf{Z}$ the above pairing is nondegenerate.*
(ii) *For every $i \geq 0$, $r \leq -2$ the pairing is perfect.*

(iii) *For every $i \geq 0$ the following induced pairings are perfect:*

$$\begin{aligned} H^2(K,\mathscr{F}_i^D) \ &\times H^0(K,\mathscr{F}_i)^\wedge \to \mathbf{Q}/\mathbf{Z}, \\ H^1(K,\mathscr{F}_i^D) \ &\times H^1(K,\mathscr{F}_i)^\wedge \to \mathbf{Q}/\mathbf{Z}, \\ H^0(K,\mathscr{F}_i^D)^\wedge &\times H^2(K,\mathscr{F}_i) \ \to \mathbf{Q}/\mathbf{Z}, \\ H^{-1}(K,\mathscr{F}_i^D)^\wedge &\times H^3(K,\mathscr{F}_i) \ \to \mathbf{Q}/\mathbf{Z}. \end{aligned}$$

(iv) *For every $i \geq 0$, $r > 2$, the cohomology groups in the pairing are zero.*

PROOF. In all four cases the proof will proceed by induction on the the level i of the 'filtration' $\mathscr{F}_i$, using Proposition 4.1 and Lemma 4.2 and the following commutative diagrams with exact rows that are obtained from the system of pairings (25).

$$\begin{array}{ccccccccc} H^{r-1}(\mathscr{F}_{i-1}^D) & \longrightarrow & H^r(\mathscr{G}_i^D) & \longrightarrow & H^r(\mathscr{F}_i^D) & \longrightarrow & H^r(\mathscr{F}_{i-1}^D) & \longrightarrow & H^{r+1}(\mathscr{G}_i^D) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{3-r}(\mathscr{F}_{i-1})^* & \to & H^{2-r}(\mathscr{G}_i)^* & \longrightarrow & H^{2-r}(\mathscr{F}_i)^* & \to & H^{2-r}(\mathscr{F}_{i-1})^* & \to & H^{1-r}(\mathscr{G}_i)^* \end{array} \tag{26}$$

and

$$\begin{array}{ccccccccc} H^{3-r}(\mathscr{F}_{i-1}) & \longrightarrow & H^{2-r}(\mathscr{G}_i) & \longrightarrow & H^{2-r}(\mathscr{F}_i) & \to & H^{2-r}(\mathscr{F}_{i-1}) & \longrightarrow & H^{1-r}(\mathscr{G}_i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{r-1}(\mathscr{F}_{i-1}^D)^* & \to & H^r(\mathscr{G}_i^D)^* & \longrightarrow & H^r(\mathscr{F}_i^D)^* & \longrightarrow & H^r(\mathscr{F}_{i-1}^D)^* & \to & H^{r+1}(\mathscr{G}_i^D)^* \end{array} \tag{27}$$

Here $H^q(-)$ is short for $H^q(K,-)$, and $-^*$ denotes the Pontryagin dual $\mathrm{Hom}_{\mathrm{cont}}(-,\mathbf{Q}/\mathbf{Z})$. The exactness of the bottom rows is clear at the duals of torsion groups (which are equipped with the discrete topology); at the duals of groups which are not purely torsion (which are equipped with the profinite topology), the exactness follows from [M2, Prop. 0.20] and Lemma 4.2.

(i) In order to show that $H^r(\mathscr{F}_i^D) \to H^{2-r}(\mathscr{F}_i)^*$ is injective, consider diagram (26). By the induction hypothesis and Proposition 4.1, we know that all vertical arrows but the middle one are injective. Moreover, the map $H^{r-1}(\mathscr{F}_{i-1}^D) \to H^{3-r}(\mathscr{F}_{i-1})^*$ is either an isomorphism, or the image of $H^{r-1}(\mathscr{F}_{i-1})^D \to H^r(\mathscr{G}_i^D)$ is a finite group I by Lemma 4.2, in which case we replace the left column of diagram (26) by the map $I \to I^*$. The injectivity follows by a diagram chase.

The map $H^{2-r}(\mathscr{F}_i) \to H^r(\mathscr{F}_i^D)^*$ is treated similarly.

(ii) For $r \leq -2$ the surjectivity of the maps $H^r(\mathscr{F}_i^D) \to H^{2-r}(\mathscr{F}_i)^*$ and $H^{2-r}(\mathscr{F}_i) \to H^r(\mathscr{F}_i^D)^*$ follows by a similar diagram chase, using Proposition 4.1 and the induction hypothesis, which give the surjectivity of the second and the fourth vertical arrows and the injectivity of the rightmost vertical arrow in diagrams (26) and (27).

(iii) The injectivity and surjectivity of the map

$$H^0(\mathscr{F}_i)^\wedge \to H^2(\mathscr{F}_i^D)^*$$

follows immediately from Proposition 4.1 and the above commutative diagrams, since $H^0(\mathscr{F}_i) = H^0(\mathscr{G}_0)$ and $H^2(\mathscr{F}_i^D) = H^2(\mathscr{G}_0^D)$ for all $i \geq 0$. The injectivity and surjectivity of

$$H^1(\mathscr{F}_i)^\wedge \to H^2(\mathscr{F}_i^D)^*$$

follows by induction from diagram (27), Proposition 4.1, and Lemma 4.2, since we may replace $H^2(\mathscr{F}_i)$ and its dual by finite groups in the diagram, and then the upper row remains exact after taking profinite completions. For the isomorphisms $H^0(\mathscr{F}_i^D)^\frown \to H^2(\mathscr{F}_i)^*$ and $H^{-1}(\mathscr{F}_i^D)^\frown \to H^3(\mathscr{F}_i)^*$ we use similar arguments.

For the arrows in the other direction, like

$$H^2(\mathscr{F}_i^D) \to (H^0(\mathscr{F}_i)^\frown)^* = H^0(\mathscr{F}_i)^*$$

we have injectivity by part (i) of this theorem, and the surjectivity follows by diagram chasing in (26) and (27) and induction on i from Proposition 4.1.

(iv) This follows from Proposition 4.1.iv by induction on i. □

4.3. Proof of Theorem 3

(i) Let $\varphi\colon X \to \operatorname{Spec} K$ be the structure morphism. Consider the triangle

$$\varphi_*\mathbf{G}_m \to R\varphi_*\mathbf{G}_m \to \tau_{\geq 1}R\varphi_*\mathbf{G}_m \to \varphi_*\mathbf{G}_m[1]$$

in the derived category of sheaves on the smooth site over K. Since

$$H^1(K, \tau_{\geq 1}R\varphi_*\mathbf{G}_m) = H^1(K, R^1\varphi_*\mathbf{G}_m[-1]) = \mathrm{Pic}(X/K)(K)$$

and $\varphi_*\mathbf{G}_m = \mathbf{G}_m$, the associated long exact sequence of cohomology groups contains the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X/K)(k) \xrightarrow{\delta} \mathrm{Br}(K) \xrightarrow{\varphi^*} \mathrm{Br}(X),$$

where the first term is zero by Hilbert's Theorem 90. This exact sequence coincides with the exact sequence in the first statement of the theorem. The Cartier dual of the map $\varphi_*\mathbf{G}_m \to R\varphi_*\mathbf{G}_m$ gives rise to the degree map

$${}^{\backprime}H_0(X, \mathbf{Z}) \xrightarrow{\deg} \mathbf{Z},$$

so the Yoneda pairing gives the following compatible system of pairings into $\mathbf{Q}/\mathbf{Z}$.

$$\begin{array}{ccc} \mathrm{Br}(K) & \xrightarrow{\varphi^*} & \mathrm{Br}(X) \\ \times & & \times \\ \mathbf{Z} & \xleftarrow{\deg} & {}^{\backprime}H_0(X, \mathbf{Z}) \end{array}$$

Since $\mathrm{Br}(K)$ is the Pontryagin dual of $\mathbf{Z}$, and $\mathrm{Br}(X)$ is the Pontryagin dual of ${}^{\backprime}H_0(X, \mathbf{Z})$ by Theorem 4.3.iii, we have that the kernel of φ_* is the dual of the cokernel of the degree map. Therefore these two (finite cyclic) groups have the same order.

(ii) Take $\mathscr{F}_1$ as in Section 2.2, and consider the triangle

$$\varphi_*\mathbf{G}_m \to \mathscr{F}_1 \to \mathrm{Pic}^0(X/K)[-1] \to \varphi_*\mathbf{G}_m[-1].$$

Now we proceed as in the proof of part (i) of the theorem, observing that

$$H^1(K, \mathscr{F}_1) = \mathrm{Pic}^0(X)$$

and

$$\mathrm{Ext}^0_{K_{\mathrm{sm}}}(\mathscr{F}_1, \mathbf{G}_m) = {}^{\backprime}H_0(X_{\overline{K}}, \mathbf{Z})^{\mathrm{Gal}(\overline{K}/K)},$$

as we see from the calculations in Section 2.2. The final statement follows from the fact that

$${}^{\backprime}H_0(X_{\overline{K}}, \mathbf{Z})^{\mathrm{Gal}(\overline{K}/K)} = \mathrm{Alb}^*(X)(K)$$

by Theorem 3.6. □

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