Travelling wave solutions to the K-P-P equation at supercritical wave speeds: a parallel to Simon Harris' probabilistic analysis.

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Abstract

Recently Harris (1999), using probabilistic methods alone, has given new proofs for the (known) existence, asymptotics and uniqueness of travelling wave solutions to the K-P-P equation. Following in this vein we outline alternative probabilistic proofs for wave speeds exceeding the critical (minimal) wave speed. Specifically the analysis is confined to the study of additive and multiplicative martingales and the construction of size biased measures on the space of (marked) trees generated by the branching process. This paper also acts as a prelude to its companion Kyprianou (2000b) which deals with the more difficult case of travelling waves at criticality. The importance of these new probabilistic proofs is their generic nature which in principle can be extended to study other types of spatial branching diffusions and associated travelling waves.

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1 Introduction

A branching Brownian motion is constructed as follows. An initial ancestor begins its existence at the origin of one-dimensional Euclidean space and time. This individual is immortal and moves according to an independent

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copy of standard Brownian motion B. The initial ancestor produces a random number of offspring, X, at times which form a Poisson process, n, with rate $\beta > 0$. We shall assume that X has distribution $(p_k : k \geq 0)$ such that $m := \sum_{k\geq 0} kp_k < \infty$. Starting from their point of creation on the path of their parent, each of these children moves and reproduces according to an independent copy of the triple (B, n, X). Let Z_t be the point process describing the number and positions of individuals alive at time t, $\{\Xi_k(t) : k = 1, ..., Z_t(\mathbb{R})\}$. In this text we shall use the Ulam-Harris labelling notation such that an individual u is identified by its line of decent from the initial ancestor. That is, if $u = (i_1, ..., i_{n-1}, i_n)$ then she is the i_nth child of the i_{n-1}th child of ....of the i_1th child of the initial ancestor. Thus uv refers to the individual who, from u's perspective, has line of descent expressed as v.

A natural martingale that arises in branching Brownian motion is of the form

$$W_t(\lambda) := \sum_{u\in N_t} e^{-\lambda(\Xi_u(t)+c_\lambda t)}$$

for t and λ positive, where N_t is the set of individuals alive at time t and $c_\lambda = \lambda/2 + \beta m/\lambda$. See Chauvin (1991), Kingman (1975), Biggins (1977) and Neveu (1988) for further details. Also from these references, it is known (or can be deduced) that $W(\lambda) := \lim_{t\uparrow\infty} W_t(\lambda)$ exists almost surely and in L^1 if $\lambda \in [0, \sqrt{2\beta m})$ and $E(X \log X) < \infty$, otherwise, its almost sure limit is identically zero.

Interest in the limit of this martingale is stimulated by its intimate connection with travelling wave solutions to the Kolmogorov-Petrovski-Piscounov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(f(u) - 1), \tag{1}$$

where $f(u) = E(s^X)$, taking solutions $u : \mathbb{R} \times \mathbb{R}^+ \to [0, 1]$. This reaction-diffusion equation has been studied by many authors, both probabilistically and analytically [see for example Kolmogorov *et al.* (1937), Fisher (1937), Skorohod (1964), McKean (1975), Bramson (1978, 1983), Neveu (1988), Uchiyama (1978), Aronson and Wienburger (1975), Biggins and Kyprianou (1996), Karpelevich *et al.* (1993) and Kelbert and Suhov (1995) to name but a few]. Of particular interest however is the recent exposition of Harris (1999) which, using probabilistic arguments *alone*, gives an excellent derivation of the existence, uniqueness and asymptotics of travelling wave solutions to (1).

By a travelling wave solution it is meant a twice differentiable, monotone increasing function $\Phi_c : \mathbb{R} \to [0,1]$ such that $\Phi_c(-\infty) = 0 = 1 - \Phi_c(\infty)$ with $u(x,t) = \Phi_c(x - ct)$ a solution to (1); $c \geq 0$ is the wave speed. Substituting into (1) shows that Φ_c solves the ordinary differential equation

$$\frac{1}{2}\Phi_c'' + c\Phi_c' + \Phi_c(f(\Phi_c) - 1) = 0. \tag{2}$$

We shall now give a brief account of the connection between these travelling waves and the limit $W(\lambda)$. For further information, one should consult Neveu (1988), McKean (1975), Chauvin (1993) and for a complete account, Harris (1999).

Travelling waves exist if and only if $c \geq \underline{c} = \sqrt{2\beta m}$. We can parameterize wave speeds such that $0 \leq c \in \{c_\lambda : \lambda \geq 0\}$, in which case the critical wave speed occurs when $\lambda = \sqrt{2\beta m} =: \underline{\lambda}$. When $c_\lambda > \underline{c}$ such that $\lambda \in [0, \underline{\lambda})$, (supercritical wave speeds) there exists a unique (modulo an additive constant in the argument) supercritical travelling wave which can be expressed as the exponentially rescaled Laplace transform of $W(\lambda)$. That is to say

$$\Phi_{c_\lambda}(x) = E\left(\exp\{-e^{-\lambda x}W(\lambda)\}\right).$$

At criticality, when $\lambda = \underline{\lambda}$ and $W(\underline{\lambda}) \equiv 0$, there is again a unique travelling wave (modulo an additive constant in the argument) which this time is the exponentially rescaled Laplace transform of

$$\partial W(\underline{\lambda}) := \lim_{t\uparrow\infty} -\frac{\partial}{\partial\lambda}W_t(\lambda)\bigg|_{\lambda=\underline{\lambda}}.$$

Thus

$$\Phi_{\underline{c}}(x) = E\left(\exp\left\{-e^{-\underline{\lambda}x}\partial W(\underline{\lambda})\right\}\right).$$

We shall offer in this paper a complete proof of the existence, asymptotics and uniqueness of the above mentioned travelling wave solutions at supercritical wave speeds using again purely probabilistic methods but none the less different to those of Harris (1999). For the case of the critical wave speed the author has also alternative probabilistic proofs for existence, uniqueness and characterization. Their complexity deserves a platform of its own and thus are presented in the companion paper Kyprianou (2000b). The method used for the critical case uses a non-homogenous branching tree to approximate the original process. This technique is a continuous time version of an idea originally developed in Kyprianou and Biggins (2000) which deals with similar issues but for the branching random walk.

The reason for pursuing probabilistic proofs of the existence uniqueness and asymptotics of these travelling waves goes deeper than pure aesthetics. It is anticipated that the probabilistic view will also shed more light on the problem of understanding the asymptotic behaviour of the position of right most particle in a spatial branching diffusions. Bramson (1978, 1983) has already treated branching Brownian motion in this respect, but the right-most particle issue remains unresolved for the branching random walk and indeed other types of spatial branching diffusions.

Recently there has been a lot of interest in the construction of so called size-biased probability measures on branching trees. These size-biased measures have been skilfully used in conjunction with a fundamental measure theoretic result by Lyons *et al.* (1995) and others to show, within the context of a variety of different branching processes, necessary and sufficient conditions for the convergence of additive martingales similar to $W_t(\lambda)$. For further references, see Lyons (1997), Kurtz *et al.* (1997), Olofsson (1998) and Athreya (1999). In the next section we shall use these ideas to give a new proof of the L^1-convergence of the martingale $W_t(\lambda)$ for $\lambda \in [0, \underline{\lambda})$ and almost sure zero limit for $\lambda \geq \underline{\lambda}$. The existence of solutions to (2) thus follows easily [c.f. Neveu (1988) and Harris (1999)].

By freezing particles in the branching Brownian motion who are first in their line of decent to hit the space time line $y + c_\lambda t = x$ (where y is the spatial variable and x is a positive constant) we produce sequences of subpopulations, indexed by x, known as stopping lines. It is known [Neveu (1988), Chauvin (1991), Kyprianou (1999)] that these stopping lines can be used to construct additive martingales similar in structure to $W_t(\lambda)$ as well as multiplicative martingales built from travelling wave solutions to the K-P-P equation. In Section 3, we shall study the limit of these two classes of martingales and show that they are, in some sense, equivalent. This equivalence induces a new proof of the result that, when $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$, any solution to (2) satisfies the asymptotic relation

$$1 - \Phi_{c_\lambda}(x) \sim k \times e^{-\lambda x} \tag{3}$$

as x tends to infinity, where k is a positive constant. This asymptotic gives us almost immediately uniqueness [c.f. Neveu (1988) and Harris(1999)].

Finally, the reader will note that only one Lemma is formally stated in this paper. Whist we have presented here new *proofs* of existence, asymptotics and uniqueness, only the included Lemma is in fact a new *result.*

2 Martingale convergence and existence of supercritical travelling waves

For future use we shall recall some standard Radon-Nikodym derivatives for measures we shall be interested in. Let $\mathbb{L}^{(\alpha)}$ be the law of a Poisson process $n = (\{\nu_i : i = 1, ..., n_t\} : t \geq 0)$ with rate $\alpha > 0$ and $\mathbb{L}_t^{(\alpha)}$ its restriction to $\sigma(n_s : s \leq t)$. We have

$$\frac{d\mathbb{L}_t^{(\beta(m+1))}}{d\mathbb{L}_t^{(\beta)}}(n) = e^{-\beta m t}(m+1)^{n_t} \tag{4}$$

for all $t > 0$. Define $\mathbb{P}^\lambda$ to be the law of a Brownian motion B with negative drift $\lambda > 0$ and $d\mathbb{P}_t^\lambda$ its restriction to $\sigma(B_s : s \leq t)$ so that

$$\frac{d\mathbb{P}_t^\lambda}{d\mathbb{P}_t^0}(B) = e^{-\lambda B_t - \frac{1}{2}\lambda^2 t} \tag{5}$$

for all $t > 0$. [Note that the last two changes of measure are essentially versions of the Girsanov Theorem for Lévy processes]. Finally let $(\widetilde{p}_k : k \geq 0)$ be the tilted distribution for X such that $\widetilde{p}_k = (k+1)p_k/(m+1)$ for all $k \geq 0$.

Now let $\mathcal{T}$ be the space of trees generated by the branching Brownian motion and $\mathcal{F}_t$ is the sigma algebra generated by the subspace of $\mathcal{T}$ consisting of trees truncated at time t. Suppose that μ is the natural probability measure on $\mathcal{T}$ corresponding to branching Brownian motion as outlined in the introduction and let μ_t be is its restriction to $\mathcal{F}_t$.

In following any line of decent from the origin of space and time we identify a process $\xi = (\xi_t : t \geq 0)$ embedded within branching Brownian motion which we shall refer to as a *spine.* Now let $\widetilde{\mathcal{T}}$ be the space of trees with distinguished spine ξ. For each $(\tau, \xi) \in \widetilde{\mathcal{T}}$, at the i-th fission point along the spine, there are $k_i \geq 0$ new trees growing, $\{\tau_j \in \mathcal{T} : 0 \leq j \leq X_i\}$. We construct the (non-probability) measure μ_t^* $(t \geq 0)$ on $\widetilde{\mathcal{T}}$ such that

$$d\mu_t^*(\tau, \xi) = d\mathbb{P}_t^0(\xi) d\mathbb{L}_t^{(\beta)}(n) \prod_{i=1}^{n_t} p_{X_i} \prod_{j=1}^{X_i} d\mu_{t-\nu_i}(\tau_j).$$

This new measure is a decomposition of μ_t over $\widetilde{\mathcal{T}}$ so that

$$d\mu_t(\tau) = \sum_{u\in N_t} I(\Xi_u(t) = \xi_t) d\mu_t^*(\tau,\xi) \tag{6}$$

for all $t > 0$. Consider also the bivariate probability measure π_t^* $(t \geq 0)$ on $\widetilde{\mathcal{T}}$ where

$$\begin{aligned} d\pi_t^*(\tau,\xi) &= e^{-\lambda(\xi_t + c_\lambda t)} \times d\mu_t^*(\tau,\xi) \\ &= e^{-\lambda\xi_t - \frac{1}{2}\lambda^2 t} \times e^{-\beta m t}(m+1)^{n_t} \\ &\qquad \times \prod_{i=1}^{n_t}\left(\frac{X_i+1}{m+1}\right) \times \frac{1}{X_i+1} \times d\mu_t^*(\tau,\xi) \\ &= d\mathbb{P}_t^\lambda(\xi) d\mathbb{L}_t^{(\beta(m+1))}(n) \\ &\qquad \times \prod_{i=1}^{n_t}\left(\frac{X_i+1}{m+1}\right) \times \frac{1}{X_i+1} \times \prod_{j=1}^{X_i} d\mu_{t-\nu_i}(\tau_j). \end{aligned} \tag{7}$$

Note that (6) can be used to check that π^* really is a probability measure. Marginalizing this measure to $\mathcal{T}$ (using (6)) we have a probability measure π_t $(t \geq 0)$ which, in view of (6), satisfies

$$\frac{d\pi_t}{d\mu_t} = W_t(\lambda) \tag{8}$$

for all $t > 0$.

In view of the Radon-Nikodym derivatives outlined at the beginning of this section, the construction in (7) suggests that the measure π^* corresponds to a branching Brownian motion having a distinguished spine ξ such that:

(i) the spine moves according to a Brownian motion with negative drift λ,

(ii) points of fission along the spine form a Poisson process with accelerated rate $(m+1)\beta$,

(iii) the distribution of offspring numbers at each point of fission on the spine has tilted measure $(\widetilde{p}_k : k \geq 0)$ and finally,

(iv) the spine is chosen randomly so that at each fission point the next individual to represent the spine is chosen with uniform probability from the current representative and its offspring.

The idea of spines and size-biasing in branching Brownian motion can also be seen for example in the work of Chauvin *et al.* (1991). It was considered there how to reconstruct a measure representing the distribution of the branching tree given that a specific point in space and time has been populated (i.e. given that a spine passes through a certain space-time point).

Using the change of measure (8) on the space of branching trees we can recover the known necessary and sufficient conditions on λ and X that imply L^1-convergence of $W_t(\lambda)$. Essential to the argument is the following fundamental measure theoretic result [see for example Durrett (1991) pp210 or Athreya (1999)]. Let $\overline{W}(\lambda) = \limsup_{t\uparrow\infty} W_t(\lambda)$ so that $\overline{W}(\lambda) = W(\lambda)$ μ-a.s., then

$$\overline{W}(\lambda) = \infty \qquad \pi\text{-a.s} \Longrightarrow \overline{W}(\lambda) = 0 \qquad \mu\text{-a.s.,} \tag{9}$$

$$\overline{W}(\lambda) < \infty \qquad \pi\text{-a.s} \Longrightarrow \int \overline{W}(\lambda)\, d\mu = 1. \tag{10}$$

In order to make use of (9) and (10) we shall take advantage of the properties that ξ and a sequence of independent copies of X have under π^*.

Consider the moment condition on $X \log X$. A sequence of simple calculations shows that $E(X\log X)$ is (in)finite if and only if

$$\sum_{k\geq 1} P_{\pi^*}(\log X > ck)$$

is (in)finite for any $c > 0$. Thus if $(X_k : k \geq 1)$ is a sequence of independent copies of X representing the numbers of offspring of ξ at each point of fission, then (by the Borel-Cantelli Lemma) $\limsup_{k\uparrow\infty} k^{-1}\log X_k$ is (infinite) zero according to whether the given moment is (in)finite. Consider also that for $\lambda \geq \underline{\lambda}$, $c_\lambda \leq \lambda$ so that $(\xi_t + c_\lambda t : t \geq 0)$ is a π^*-Brownian motion with non-positive drift. Similarly, if $\lambda \in [0, \underline{\lambda})$, then $(\xi_t + c_\lambda t : t \geq 0)$ is a π^*-Brownian motion with strictly positive drift. Consequently as $W_t(\lambda) \geq \exp\{-\lambda(\xi_t + c_\lambda t)\}$ and $W_{\nu_k}(\lambda) \geq X_k \exp\{-\lambda(\xi_{\nu_k} + c_\lambda \nu_k)\}$ we have respectively that if either $\lambda \geq \underline{\lambda}$ or $E(X\log X) = \infty$

$$\limsup_{t\uparrow\infty} W_t(\lambda) = \infty \qquad \pi\text{-a.s.}$$

and thus $W(\lambda) = 0$ μ-a.s. (Note that in the second case we use also the Renewal Theorem).

Now let $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$. Define $\mathcal{G}$ to be the sigma algebra generated by the diffusion on the spine ξ, the Poisson process representing the birth times along the spine n and $(X_k : k \geq 1)$. A brief computation, based on the decomposition of $W_t(\lambda)$ according to contributions from descendents of individuals born along the spine, yields

$$E_{\pi^*}\left(W_t(\lambda) \middle| \mathcal{G} \right) = \sum_{i=1}^{n_t} X_i e^{-\lambda(\xi_{\nu_i} + c_\lambda \nu_i)} + e^{-\lambda(\xi_t + c_\lambda t)}. \tag{11}$$

Within the specified regime of λ recall that $(\xi_t + c_\lambda t : t \geq 0)$ is a π-Brownian motion with strictly positive drift. [Note however that when $\lambda = 0$, the summands in (11) are simply $X_i e^{-\beta m t}$]. The moment condition ensures that extremes of the sequence of variables $(X_n : n \geq 1)$ have sub-exponential behaviour. Consequently (again using the Renewal Theorem)

$$\lim_{t \uparrow \infty} E_{\pi^*}\left(W_t(\lambda) \middle| \mathcal{G} \right) < \infty \qquad \pi\text{-}a.s.$$

Fatou's Lemma now tells us that $\liminf_{t\uparrow\infty} W_t(\lambda) < \infty$ π-a.s. In light of (8), $W_t(\lambda)^{-1}$ is a π-martingale with an almost sure limit and thus by the previous statement, $\lim_{t\uparrow\infty} W_t(\lambda) < \infty$ π-a.s. Thus we conclude that for $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$, $W_t(\lambda)$ converges almost surely and in mean.

As mentioned in the introduction, existence at supercritical wave speeds ($\lambda \in [0, \underline{\lambda})$) follows almost immediately. To see this it suffices to follow the reasoning of Harris (1999) as below.

We can easily make the decomposition for all $t > s > 0$,

$$W_t(\lambda) = \sum_{u \in N_s} e^{-\lambda(\Xi_u(s) + c_\lambda s)} W_{t-s}(\lambda, u), \tag{12}$$

where $W_{t-s}(\lambda, u)$ are independent copies of $W_{t-s}(\lambda)$ for each $u \in N_s$. Letting t tend to infinity and taking an exponentially rescaled Laplace transform of the resulting identity yields the functional equation

$$\Phi(x) = E\left[\prod_{u \in N_s} \Phi\left(x + \Xi_u(s) + c_\lambda s\right) \right]$$

for all $s > 0$, where $\Phi(x) = E\left[\exp\{-e^{-\lambda x} W(\lambda)\}\right]$. Theorem 8 of Kyprianou (1999) concludes that this Φ solves the above functional equation if and only if it is a travelling wave solution to the K-P-P equation with wave speed c_λ.

3 Martingales on stopping lines, asymptotics and uniqueness

On the space-time half plane $\{(y,t) : y \in \mathbb{R}, t \in \mathbb{R}^+\}$, consider the barrier $\Gamma^{(-x,c_\lambda)}$ described by the line $y + c_\lambda t - x = 0$ for $x > 0$. By arresting lines of descent the first time they hit this barrier we produce a random collection of individuals, C_x, which is a *stopping line.* For further information on stopping lines, their rigorous definition and properties, one should consult Neveu (1988) and Chauvin (1991). What is important to note for our purposes is that $\{C_x\}_{x\geq 0}$ is a sequence of dissecting stopping lines tending to infinity on which the branching property holds and whose cardinality, $\{|C_x|\}_{x\geq 0}$ forms a continuous time branching process (x plays the role of time). [Note that by a sequence of dissecting stopping lines tending to infinity we mean any line of decent from the initial ancestor will hit $\Gamma^{(-x,c_\lambda)}$ with probability one for all $x > 0$ and $\lim_{x\uparrow\infty} \inf\{|u| : u \in C_x\} = \infty$ almost surely]. Let $\{\sigma_u(x) : u \in C_x\}$ be the times at which individuals meet the barrier $\Gamma^{(-x,c_\lambda)}$. From the afore mentioned references, it is known that when $\lambda \in [0, \underline{\lambda})$,

$$\prod_{u\in C_x} \Phi_{c_\lambda}\left(\Xi_u\left(\sigma_u\left(z\right)\right) + c_\lambda \sigma_u\left(z\right)\right) = \Phi_{c_\lambda}\left(x\right)^{|C_x|}$$

is a martingale with expectation $\Phi_{c_\lambda}(x)$ that converges almost surely and in mean. It follows that

$$\lim_{x\uparrow\infty} -|C_x| \log \Phi_{c_\lambda}(x)$$

exists and has mass in $(0,\infty)$.

Define for $x > 0$

$$W_{C_x}(\lambda) = \sum_{u\in C_x} e^{-\lambda(\Xi_u(\sigma_u(z))+c_\lambda\sigma_u(z))} = e^{-\lambda x}|C_x|.$$

We shall show in the only Lemma of this paper (below) that for $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$, that this sequence of variables is a martingale which converges almost surely and in mean to $W(\lambda)$, the limit of $\{W_t(\lambda)\}_{t\geq 0}$. Since we have shown in the previous section that this limit has mass in $(0,\infty)$, we now have two sequences of (Seneta-Heyde) norming constants for the branching process $\{|C_x|\}_{x\geq 0}$. Consequently, these two norming sequences must be asymptotically equivalent. That is to say,

$$\lim_{x\uparrow\infty} \frac{-\log \Phi_{c_\lambda}(x)}{e^{-\lambda x}} = \lim_{x\uparrow\infty} \frac{1-\Phi_{c_\lambda}(x)}{e^{-\lambda x}} = k$$

where the second equality follows since $\Phi_{c_\lambda}(\infty) = 1$ and k is a positive constant. We have thus constructed an alternative proof of the asymptotic (3). Once this is known it is very easy show uniqueness of travelling waves with supercritical wave speeds as in Harris (1999). We include the argument here for completeness. From the previously mentioned references Neveu (1988) and Chauvin (1991), it is known that for any supercritical travelling wave Φ_{c_λ} that $(z > 0)$

$$M_t(z) := \prod_{u \in N_t} \Phi_{c_\lambda}(z + \Xi_u(t) + c_\lambda t)$$

is a multiplicative martingale with expectation $\Phi_{c_\lambda}(z)$, convergent almost surely and in mean as t tends to infinity. Since $W(\underline{\lambda}) \equiv 0$, it follows that the largest summand in $W_t(\underline{\lambda})$ tends almost surely to zero as t tends to infinity. Hence if $L_t = \min_{u \in N_t} \Xi_u(t)$, then $\lim_{t \uparrow \infty} L_t + \underline{\lambda} t = \infty$ almost surely. Assuming that $\lambda \in [0, \underline{\lambda})$ and hence $c_\lambda \geq \underline{\lambda}$, we have also that $\lim_{t \uparrow \infty} L_t + c_\lambda t = \infty$ almost surely. Combining these facts with our asymptotic for travelling waves with wave speed $c_\lambda > \underline{c}$, we have as t tends to infinity

$$\begin{aligned} -\log M_t(z) &= \sum_{u \in N_t} -\log \Phi_{c_\lambda}(z + \Xi_u(t) + c_\lambda t) \\ &\sim \sum_{u \in N_t} 1 - \Phi_{c_\lambda}(z + \Xi_u(t) + c_\lambda t) \\ &\sim k \sum_{u \in N_t} e^{-\lambda(z + \Xi_u(t) + c_\lambda t)} \\ &= k e^{-\lambda z} W_t(\lambda). \end{aligned}$$

Thus any travelling wave solution with supercritical wave speed satisfies

$$\Phi_{c_\lambda}(z) = E\left[\lim_{t \uparrow \infty} M_t(z)\right] = E\left(\exp\left\{-k e^{-\lambda z} W(\lambda)\right\}\right)$$

and therefore uniqueness (modulo an additive constant in the argument) follows.

Our work is thus concluded by verifying the previous claim that $W_{C_x}(\lambda)$ is a martingale. Similar ideas can be found in Kyprianou (2000a).

Lemma *Let $\mathcal{F}_{C_x}$ $(x \geq 0)$ be the natural filtration describing all ancestral paths receding from the stopping line C_x to the initial ancestor. Then* $W_{C_x}(\lambda)$

is an $\mathcal{F}_{C_x}$-martingale that converges almost surely and in mean to $W(\lambda)$ when $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$.

Proof. Let $A_t(C_x) = \{u \in N_t : v \notin C_z \ \forall v \leq u\}$ and define

$$W_{N_t \wedge C_x}(\lambda) = \sum_{u \in A_t(C_x)} e^{-\lambda(\Xi_u(t) + c_\lambda t)} + e^{-\lambda x} |C_{x,t}|$$

where $C_{x,t} = \{u \in C_x : \sigma_u(x) \leq t\}$. By decomposing members of N_t in accordance with their ancestors (if at all) in C_x, much as in (12), an easy calculation to shows that

$$E(W_t(\lambda) | \mathcal{F}_{C_x}) = W_{N_t \wedge C_x}(\lambda).$$

As C_x is a dissecting stopping line, $\lim_{t \uparrow \infty} |A_t(C_x)| = 0$ and $\lim_{t \uparrow \infty} |C_x \backslash C_{x,t}| = 0$ almost surely. When $\lambda \in [0, \underline{\lambda})$ and $E(X \log X) < \infty$, $W_t(\lambda)$ has an L^1 limit and hence with the previous remarks

$$\lim_{t \uparrow \infty} E(W_t(\lambda) | \mathcal{F}_{C_x}) = E(W(\lambda) | \mathcal{F}_{C_x}) = W_{C_x}(\lambda) \tag{13}$$

showing that $W_{C_x}(\lambda)$ is an $\mathcal{F}_{C_x}$-martingale.

As the sequence C_x is tending to infinity, then $\lim_{x \uparrow \infty} A_t(C_x) = N_t$ and $\lim_{x \uparrow \infty} |C_{x,t}| = 0$ almost surely. Talking the limit in (13) with respect to x instead thus gives us,

$$E(W_t(\lambda) | \mathcal{F}_\infty) = W_t(\lambda)$$

for all $t > 0$ where $\mathcal{F}_\infty = \sigma\left(\bigcup_{x \geq 0} \mathcal{F}_{C_x}\right)$. This implies that $W_t(\lambda)$ is $\mathcal{F}_\infty$-measurable for each $t > 0$ and thus so is its limit $W(\lambda)$. In conclusion $\lim_{x \uparrow \infty} W_{C_x}(\lambda) = W(\lambda)$. The Lemma is proved.

On a final note, the Lemma confirms that λ is the Malthusian parameter of the branching process $\{|C_x|\}_{x \geq 0}$.

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