A NOTE ON ROTATIONS AND INTERVAL EXCHANGE TRANSFORMATIONS ON 3-INTERVALS

KARMA DAJANI

ABSTRACT. We prove the conjecture that an interval exchange transformation on 3-intervals with corresponding permutation $(1,2,3) \to (3,2,1)$, and rationally independent discontinuity points, is never measure theoretically isomorphic to an irrational rotation.

1. INTRODUCTION

Interval exchange transformations were first introduced by Keane in [K1], and are defined as follows. Let $I = [0,1)$, $n \geq 2$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ a probability vector with $\alpha_i > 0$. Define $\beta_0 = 0$ and $\beta_i = \sum_{k=1}^{i} \alpha_k$, and set $I_i = [\beta_{i-1}, \beta_i)$. Let τ be a permutation of $\{1, 2, \cdots, n\}$, and consider the probability vector $\alpha^\tau = (\alpha_{\tau^{-1}(1)}, \cdots, \alpha_{\tau^{-1}(n)})$. Note that $\alpha_{\tau^{-1}(i)} > 0$ for all i. Let $\beta_0^\tau = 0$ and $\beta_i^\tau = \sum_{k=1}^{i} \alpha_{\tau^{-1}(k)}$, and set $I_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)$.

Define $T : I \to I$ by

$$Tx = x - \beta_{i-1} + \beta^\tau_{\tau(i)-1}$$

if $x \in I_i$. T is called an (α, τ) interval exchange transformation on n intervals. It is clear that T is invertible, $T\beta_{i-1} = \beta^\tau_{\tau(i)-1}$ and T maps I_i isometrically onto $I^\tau_{\tau(i)}$. Further, T is continuous except possibly at $\{\beta_1, \cdots, \beta_{n-1}\}$. At these points T is right continuous. Note that T is continuous at β_i if and only if $\tau(i+1) = \tau(i) + 1$. In other words, T is discontinuous at β_i if and only if $T\beta_{i-1}, T\beta_i$ do not appear in this order as consecutive terms in the ordered set $\{\beta_0^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$. We say T is in *standard form* if T is discontinuous at β_i for all $i = 1, 2, \cdots, n-1$ or equivalently, if $\tau(i+1) \neq \tau(i) + 1$ for all $i = 1, 2, \cdots, n-1$. Notice that any interval exchange transformation on n intervals can be written in standard form as an interval exchange transformation on m intervals with $m \leq n$. Since if T is not in standard form, then T is continuous at β_i for some i, then $\tau(i+1) = \tau(i) + 1$, and so T maps the interval $[\beta_{i-1}, \beta_{i+1})$ isometrically onto $[\beta^\tau_{\tau(i)-1}, \beta^\tau_{\tau(i)+1})$. Thus, we can redefine T on intervals with end points

$$\{\beta_0, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_n\}.$$

We repeat this process until all the remaining β's are discontinuity points of T.

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The permutation τ corresponding to T is said to be irreducible if

$$\tau(\{1,2,\cdots,k\}) \neq \{1,2,\cdots,k\}, \text{ for all } k=0,1,\cdots,n-1.$$

Note that if τ is reducible, then T can be decomposed into two interval exchange transformations, one on $[0,\beta_k)$ and the other on $[\beta_k,1)$. We assume throughout this paper that T is irreducible.

Interval exchange transformations have been studied by several authors. Here we mention few of the known results. In [K1], Keane studied the minimality of such transformations, and in [K2] questions concerning unique ergodicity were investigated. It is easy to see that if $n=2$, T corresponds to a rotation and if $n=3$, then T can be seen as an induced transformation of a rotation. Thus, if the β's are rationally independent, then in both cases T is uniquely ergodic. Keynes and Newton [KN], and also Keane [K2] gave examples of interval exchange transformations that are not uniquely ergodic. Masur [M], and independently Veech [V1, V2, V3, V4, V5] showed that almost every minimal interval exchange transformation is uniquely ergodic. Later Boshernitzan [B] gave another proof of this result by more elementary means. Some of the spectral properties were studied by Veech in a series of papers [V3,V4,V5]. Oseledets [O] and Goodson [G] constructed ergodic interval exchange transformations with simple spectrum. Recently, Berthé, Chekhova and Ferenczi [BCF] proved that every ergodic interval exchange transformation on three intervals has simple spectrum. The first interval exchange transformation with continuous spectrum was given by Katok and Stepin [KS], their example is also an exchange on three intervals. In [BCF], the authors gave other examples of exchanges on three intervals with continuous spectrum, and they conjectured that no non-trivial exchange on three intervals is measure theoretically isomorphic to an irrational rotation. In section 2 we prove this conjecture as a corollary of a recent result by Simin Li [S], where he gave necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

2. Non-trivial exchanges on 3-intervals

Let $0<l<m<1$ with $1,l,m$ rationally independent. Consider the interval exchange transformation T given by

$$Tx = \begin{cases} x+1-l & x\in[0,l), \\ x+1-l-m & x\in[l,m), \\ x-m & x\in[m,1). \end{cases}$$

T corresponds to the permutation $(1,2,3)\to(3,2,1)$. Notice that T is the only interval exchange transformation on 3-intervals which is irreducible and in standard form. Moreover, by a result of Keane [K1], T is minimal. We call T a non-trivial exchange transformation on 3-intervals. It is well known that T is an induced transformation of the interval exchange transformation S defined on $[0,1-l+m)$ by

$$Sx = \begin{cases} x+1-l & x\in[0,m), \\ x-m & x\in[m,1-l+m). \end{cases}$$

Since after normalization S is isomorphic to an irrational rotation, S is minimal and uniquely ergodic, and hence so is T.

Let $\alpha = \frac{1-l}{1-l+m}$ and $\beta = \frac{1}{1-l+m}$. In [KS], the authors proved that if α has unbounded partial quotients and if for some subsequence q_n of denominators of convergents of α, we have

$$|\alpha - \frac{p_n}{q_n}| < o(\frac{1}{q_n^2}), \text{ and } |\beta - \frac{r}{q_n}| > \frac{c}{q_n}$$

for all r and some constant $c > 0$, then T is not measure theoretically isomorphic to an irrational rotation. In [BCF], it is proved that when α has bounded partial quotients, and $\beta \in K(\alpha)$ for some Cantor set $K(\alpha)$, then T is not measure theoretically isomorphic to an irrational rotation.

Simin Li [Li] gave recently necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation.

Theorem 1 (Li). *Let T be an interval exchange transformation, and let $d(T^n)$ be the number of discontinuities of T^n. Then, T is conjugate to an irrational rotation if and only if (i) T^n is minimal for all $n \geq 1$, (ii) $\{d(T^n)\}$ is bounded by some integer $N > 0$ and (iii) there exists $k > 0$ and $M \geq 2^{N^3+3N^2}$ such that $d(T^k) = d(T^{2k}) = \cdots = d(T^{Mk})$.*

Since a non-trivial interval exchange on 3-intervals is uniquely ergodic, to show that it is not measure theoretically isomorphic to an irrational rotation, we prove that $\{d(T^n)\}$ is an unbounded sequence.

Theorem 2. *Let T be a non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points. Let $D(T^n)$ be the set of discontinuity points of T^n, and let $d(T^n)$ denote the cardinality of $D(T^n)$. Then*

$$D(T^n) = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq n-1\},$$

and hence, $d(T^n) = 2n$.

Proof: The proof is done by induction on n. The result is true for $n = 1$. Suppose

$$D(T^k) = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq k-1\},$$

for $k = 1, 2, \cdots, n$. We prove the result for $k = n+1$. Let

$$0 < \beta_1 < \beta_2 < \cdots < \beta_{2n} < 1$$

be the discontinuities of T^n written in increasing order. By the induction hypothesis,

$$D(T^n) = \{\beta_i : 1 \leq i \leq 2n\} = \{T^{-i}l, T^{-j}m : 0 \leq i, j \leq n-1\}.$$

Let $\beta_0 = 0$ and $\beta_{2n+1} = 1$. The underlying partition of T^n is given by

$$\mathcal{P}(T^n) = \{[\beta_i, \beta_{i+1}) : i = 0, 1, \cdots 2n\}.$$

Let τ_n be the permutation corresponding to T^n (notice that T^n is an interval exchange transformation). Then,

$$T^n\{\beta_0, \beta_1, \cdots, \beta_{2n}\} = \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$$

with $\beta_0 = \beta_0^{\tau_n} = 0$, and $T^n\beta_i = \beta^{\tau_n}_{\tau_n(i+1)-1}$ for $i = 0, 1, \cdots, 2n$. Furthermore, since $1, l$ and m are rationally independent, and each $\beta_i^{\tau_n}$ is a linear combination of $1, l$ and m with integer coefficients, it follows that $l, m \notin \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$. Now invertibility of T implies that $T\beta_0^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tm, Tl$ are all distinct.

Suppose $l \in (\beta_{r-1}^{\tau_n}, \beta_r^{\tau_n})$, and $m \in (\beta_{s-1}^{\tau_n}, \beta_s^{\tau_n})$. We consider three cases.

Case 1. If $r = s$, then $T^{-n}l, T^{-n}m \in (\beta_{p-1}, \beta_p)$ where $p = {\tau_n}^{-1}(r)$. Since T is an order preserving isometry on $[\beta_{p-1}, \beta_p)$, it follows that $T^{-n}l < T^{-n}m$. The underlying partition of T^{n+1} is then given by

$$\mathcal{P}_1(T^{n+1}) = \{[\beta_0, \beta_1), \cdots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, T^{-n}m), [T^{-n}m, \beta_p),$$
$$[\beta_p, \beta_{p+1}), \cdots, [\beta_{2n}, \beta_{2n+1})\}.$$

To prove the result, we need to show that

$$\{\beta_1, \cdots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \cdots, \beta_{2n}\}$$

is the set of discontinuity points of T^{n+1}. Let

$$D_1 = \{\beta_0, \cdots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \cdots, \beta_{2n}\}$$

and

$$E_1 = \{\beta_0^{\tau_n}, \cdots, \beta_{r-1}^{\tau_n}, l, m, \beta_r^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\},$$

both considered as ordered sets. Then $TD_1 = E_1$, and by discontinuity of T^n at β_p, we have $T^n\beta_p \neq \beta_r^{\tau_n}$. Further, $\beta_i^{\tau_n} \in (0, l)$ for $1 \leq i \leq r-1$, and $\beta_i^{\tau_n} \in (m, l)$ for $r \leq i \leq 2n$,. Hence,

$$\begin{aligned} T^{n+1}D_1 &= TE_1 \\ &= \{Tm = 0, T\beta_r^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tl = 1-m, \\ &\quad T\beta_0^{\tau_n} = 1-l, T\beta_1^{\tau_n}, \cdots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

Here, the elements of TE_1 are listed in increasing order.

We first show that T^{n+1} is discontinuous at β_i for $i \neq p$. To do this, we need to prove that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ do not appear in this order as consecutive terms in TE_1. By assumption, T^n is discontinuous at β_i, hence $T^n\beta_{i-1}$ and $T^n\beta_i$ do not appear as consecutive terms of the form $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$ in E_1. Let $I_0 = [0, l)$, $I_1 = [l, m)$ and $I_2 = [m, 1)$. If $T^n\beta_{i-1}, T^n\beta_i \in I_j$ for some $j = 0, 2$, then since T maps I_j isometrically onto TI_j, it follows that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ cannot appear as consecutive terms in TE_1. If $T^n\beta_{i-1} \in I_j$ and $T^n\beta_i \in I_k$ for $j \neq k$, then either $T^n\beta_{i-1} \in I_0$ and $T^n\beta_i \in I_2$, or $T^n\beta_{i-1} \in I_2$ and $T^n\beta_i \in I_0$. In the first case we get $T^{n+1}\beta_i < 1-m < T^n\beta_{i-1}$, and in the second case, we get $T^{n+1}\beta_{i-1} < 1-m < T^{n+1}\beta_i$. Hence, $T^n\beta_{i-1}$ and $T^n\beta_i$ do not appear as consecutive terms of the form $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$ in E_1, and so T^{n+1} is discontinuous at β_i.
Now, the discontinuity of T^n at β_p implies that $T^{n+1}\beta_p \neq T\beta_r^{\tau_n}$, and

$$Tm = 0 < T\beta_r^{\tau_n} < T^{n+1}\beta_p.$$

Hence $T^{n+1}(T^{-n}m) = Tm = 0$ and $T^{n+1}\beta_p$ do not appear as consecutive terms in TE_1. So T^{n+1} is discontinuous at β_p.
The discontinuity of T^{n+1} at $T^{-n}l$ follows from the fact that $T^{n+1}\beta_{p-1}$ is an interior point of TI_2, while $T^{n+1}(T^{-n}l) = 1-m$ is the left end-point of TI_1. Finally, $T^{n+1}(T^{-n}m) = 0 < T\beta_r^{\tau_n} < 1-m = T^{n+1}(T^{-n}l)$ implies that T^{n+1} is discontinuous at $T^{-n}m$. Therefore, $D_1 = D(T^{n+1})$.

Case 2: If $r < s$ and $p = \tau_n^{-1}r < \tau_n^{-1}s = q$, then $T^{-n}l \in (\beta_{p-1}, \beta_p)$ and $T^{-n}m \in (\beta_{q-1}, \beta_q)$. The discontinuity of T^n at β_p and β_q implies $T^n\beta_p \neq \beta_r^{\tau_n}$ and $T^n\beta_q \neq \beta_s^{\tau_n}$. The underlying partition of T^{n+1} is easily seen to be

$$\mathcal{P}_2(T^{n+1}) = \{[\beta_0, \beta_1), \cdots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, \beta_p),$$

$$[\beta_p,\beta_{p+1}),\cdots[\beta_{q-1},T^{-n}m),[T^{-n}m,\beta_q),[\beta_q,\beta_{q+1}),\cdots,[\beta_{2n},1)\}.$$

To show the discontinuity of T^{n+1} at $\beta_1,\cdots,\beta_{2n},T^{-n}l,T^{-n}m$, we consider the ordered sets

$$D_2=\{\beta_0,\cdots,\beta_{p-1},T^{-n}l,\beta_p,\cdots,\beta_{q-1},T^{-n}m,\beta_q,\cdots,\beta_{2n}\}$$

and

$$E_2=\{\beta_0^{\tau_n},\cdots,\beta_{r-1}^{\tau_n},l,\beta_r^{\tau_n},\cdots,\beta_{s-1}^{\tau_n},m,\beta_s^{\tau_n},\cdots,\beta_{2n}^{\tau_n}\}.$$

Then, $T^nD_2=E_2$. Notice that $\beta_1^{\tau_n},\cdots,\beta_{r-1}^{\tau_n}$ are interior points of I_0, $\beta_r^{\tau_n},\cdots,\beta_{s-1}^{\tau_n}$ are interior points of I_1 and $\beta_s^{\tau_n},\cdots,\beta_{2n}^{\tau_n}$ are interior points of I_2. Thus,

$$\begin{aligned}T^{n+1}D_2 &= TE_2\\ &= \{Tm=0,T\beta_s^{\tau_n},\cdots,T\beta_{2n}^{\tau_n},Tl=1-m,\\ &\quad T\beta_r^{\tau_n},\cdots,T\beta_{s-1}^{\tau_n},T\beta_0^{\tau_n}=1-l,\cdots,T\beta_{r-1}^{\tau_n}\}.\end{aligned}$$

Here, the elements of TE_2 are listed in increasing order. We first prove that T^{n+1} is discontinuous at β_i for $i\neq p,q$. If $T^n\beta_{i-1},T^n\beta_i\in I_j$, then since $T^n\beta_{i-1},T^n\beta_i$ do not appear as consecutive terms in E_1 and since T is an isometry on I_j, we have that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ are not consecutive terms of E_2, and thus T^{n+1} is discontinuous at β_i. If $T^n\beta_i\in I_j$ and $T^n\beta_{i-1}\in I_k$ for $k\neq j$, then we consider several cases.

- If $T^n\beta_i\in I_2$ and $T^n\beta_{i-1}\in I_0$ or I_1, then since $T^n\beta_{i-1}\neq l$ we have $T^{n+1}\beta_i<1-m<T^{n+1}\beta_{i-1}$.
- If $T^n\beta_i\in I_1$ and $T^n\beta_{i-1}\in I_2$, then since $T^n\beta_i\neq l$ it follows that $T^{n+1}\beta_{i-1}<1-m<T^{n+1}\beta_i$.
- If $T^n\beta_i\in I_1$ and $T^n\beta_{i-1}\in I_0$, then $T^{n+1}\beta_i<T^{n+1}\beta_{i-1}$.
- If $T^n\beta_i\in I_0$ and $T^n\beta_{i-1}\in I_1$, then since $i\neq q$ we have $T^{n+1}\beta_{i-1}<T\beta_{s-1}^{\tau_n}<T\beta_0^{\tau_n}\leq T^{n+1}\beta_i$.
- If $T^n\beta_i\in I_0$ and $T^n\beta_{i-1}\in I_2$, then $T^{n+1}\beta_{i-1}<1-m<T^{n+1}\beta_i$.

In all the above cases we see that T^{n+1} is not continuous at β_i.
The discontinuity of T^{n+1} at β_p and β_q follows from the fact that $T^{n+1}\beta_p\neq T\beta_r^{\tau_n}$ and $T^{n+1}\beta_q\neq T\beta_s^{\tau_n}$, so that neither $T^{n+1}\beta_p$ and $T^{n+1}(T^{-n}l)$ nor $T^{n+1}\beta_q$ and $T^{n+1}(T^{-n}m)$ appear as consecutive terms in TE_2. Finally, from $T^{n+1}(T^{-n}l)=1-m<1-l<T\beta_{r-1}^{\tau_n}$ and $T^{n+1}(T^{-n}m)=0<1-m<T\beta_{s-1}^{\tau_n}$ we have that T^{n+1} is discontinuous at $T^{n+1}(T^{-n}l)$ and $T^{n+1}(T^{-n}m)$. Thus, $D_2=D(T^{n+1})$.

Case 3: If $r<s$ and $p=\tau_n^{-1}r>\tau_n^{-1}s=q$, then the underlying partition of T^{n+1} is given by

$$\begin{aligned}\mathcal{P}_3(T^{n+1})=\{&[\beta_0,\beta_1),\cdots,[\beta_{q-2},\beta_{q-1}),[\beta_{q-1},T^{-n}m),[T^{-n}m,\beta_q),[\beta_q,\beta_{q+1}),\cdots,\\ &[\beta_{p-2},\beta_{p-1}),[\beta_{p-1},T^{-n}l),[T^{-n}l\beta_p),[\beta_p,\beta_{p+1}),\cdots,[\beta_{2n},1)\}.\end{aligned}$$

Let

$$D_3=\{\beta_0,\cdots,\beta_{q-1},T^{-n}m,\beta_q,\cdots,\beta_{p-1},T^{-n}l,\beta_p,\cdots,\beta_{2n}\}$$

and

$$E_3=\{\beta_0^{\tau_n},\cdots,\beta_{r-1}^{\tau_n},l,\beta_r^{\tau_n},\cdots,\beta_{s-1}^{\tau_n},m,\beta_s^{\tau_n},\cdots,\beta_{2n}^{\tau_n}\}.$$

Then,

$$\begin{aligned}T^{n+1}D_3 &= TE_3\\ &= \{Tm=0,T\beta_s^{\tau_n},\cdots,T\beta_{2n}^{\tau_n},Tl=1-m,\\ &\quad T\beta_r^{\tau_n},\cdots,T\beta_{s-1}^{\tau_n},T\beta_0^{\tau_n}=1-l,\cdots,T\beta_{r-1}^{\tau_n}\}.\end{aligned}$$

The elements of D_3, E_3 and TE_3 are listed in increasing order. A similar argument as in the above two cases shows that $D_3 = D(T^{n+1})$. Thus, the theorem is proved.

Theorem 3. *Any non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points is not measure theoretically isomorphic to an irrational rotation.*

Proof: By theorem 2 and unique ergodicity, the result follows from Li's theorem.

In [BCF], the authors proved that every ergodic interval exchange transformation on three intervals has simple spectrum. Using this result and theorem 3, we have the following corollary.

Corollary 1. *Every non-trivial interval exchange transformation on three intervals with rationally independent discontinuity points has either rational or continuous spectrum.*

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Universiteit Utrecht, Fac. Wiskunde en Informatica and MRI, Budapestlaan 6, P.O. Box 80.000, 3508 TA Utrecht, the Netherlands

E-mail address: dajani@math.uu.nl