On a long range particle system with unbounded flip rates

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August 24, 2000

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Abstract

We consider an interacting particle system on $\{0,1\}^{\mathbb{Z}}$ with non-local, unbounded flip rates. Zeroes flip to one at a rate that depends on the number of ones to the right until we see a zero (the flip rate equals λ times one plus this number). The flip rate of the ones equals μ. We give motivation for models like this in general, and this one in particular. The system turns out not to be Feller, and we construct it using monotonicity. We show that for $\lambda < \mu$ the system has a unique non-trivial stationary distribution, which is ergodic, stationary, and has a density of ones of $\frac{\lambda}{\mu}$. For $\lambda \geq \mu$ the limit is degenerate at $\{1\}^{\mathbb{Z}}$. Our main tool is an explicit formula for the density of ones at any given moment.

Keywords: long range particle system, construction, monotonicity, unique stationary distribution, non-locality.

AMS classification: 60K35

*Research financially supported by the Dutch Organisation for Scientific Research NWO.

1 Introduction

Particle systems with local (and therefore bounded) rates have been studied extensively over the last twenty years or so. Excellent entries to this field are the two books of Liggett (1985, 2000). These particle systems are always Feller. More recently, there has been a growing interest, especially in the physics literature, in systems with long range dependencies and non-local flip rates. Some of these systems have attracted attention under the name self-organised criticality (Bak (1996), Jensen (1998)). The physical literature emphasizes the 'critical' behaviour of such systems, that is, power law decay in time and space of various quantities. In the pure mathematical sense, critical classical thermodynamic systems are not very well understood, and this makes it clear that mathematicians have other priorities when it comes to long range interaction particle systems.

The first obstacle for mathematicians is the very construction of such models in infinite volume. The classical construction techniques break down under non-locality. In the cases where an explicit construction can in fact be carried out, mathematicians are primarily interested in stationary distributions and their properties. In the light of the remarks above, it is not surprising that mathematicians try to get a feeling for this new class of models by looking at concrete examples which are simple enough to allow rigorous analysis, but which do have the required non-local flip rates.

In Maes et al. (2000) an infinite volume one-dimensional sandpile model is constructed. The resulting Markov process is not Feller and the only stationary distribution is the trivial one in which the system is completely full. In the current paper we introduce a new long range particle system which can be constructed with similar ideas as in Maes et al. (2000) but which turns out to have a unique non-trivial stationary distribution, various properties of which can in fact be established.

Informally, our system can be described as follows. The state space is $\Omega = \{0,1\}^{\mathbb{Z}}$. Let $\lambda > 0$, $\mu \geq 0$. Typically, we denote a state of the system by $\eta \in \Omega$. If $\eta(x)$ equals one, it flips to zero at rate μ. If $\eta(x)$ equals zero, it flips to one at rate λ times one plus the number of indices larger than x, until the first index larger than x with a zero.

The 'global' reason for studying this system is that is about the simplest non-local particle system for which we can expect a nontrivial stationary distribution. More specifically, a number of interpretations is possible, and we mention two such interpretations:

1. One can think of a toy model for a sandpile with dissipation. Grains of sand fall down on each site $i \in \mathbb{Z}$ according to a Poisson process with intensity λ. All these Poisson processes are independent of each other. If a grain falls down on some site i at a moment that site i is occupied by another grain, the falling grain slides to the nearest site on the left (i.e. a site with a lower number) where no grain is present. We suppose that the grain arrives at that site instantanously. Grains of sand dissapear independently of each other after an exponentially distributed time with parameter μ.

2. One can also interpret this system as a queueing system with impatient customers, where each site $i \in \mathbb{Z}$ is associated to a Poisson arrival process with intensity λ. There is a server at each site. The arrival processes are independent of each other. If there is an arrival of the Poisson process associated to some site i, we assign a service place to this customer in the following way. If the server at site x is not busy at the moment the customer arrives (i.e. there is no customer present at site i), the customer takes the place at site i. If the server at site i is busy, the customer is not allowed to take the place at site i. He must go to the nearest server on his left who is not busy, and is served there. We assume that customers arrive at their service place instantaneously and that service times are independent and exponentially distributed with parameter μ. After a customer is served, he leaves the system.

Because of the first interpretation, we shall call the system a sandpile model with dissipation (SMD). Because of the dissipation of sand, we do not expect genuine SOC behaviour (whatever that may be).

As anticipated above, it is not immediately clear that the above description gives rise to a well defined process in infinite volume. In Section 2 we shall construct a Markov semigroup $S(t)$, which is the semigroup of the SMD. The construction uses the monotonicity of the process and is in the same spirit as the constructions of the one-dimensional sandpile process in Maes et al. (2000) and the long range exclusion processes in Liggett (1980). We shall also show that the SMD is not a Feller process. In Section 3 we see that although the system is not Feller, there is, for some 'special' functions and configurations, a relation between the Markov semigroup of the SMD and its formal generator. This relation is used in Section 4 where we shall prove the following result. Here $\nu S(t)$ is the distribution of the SMD at time t if its initial configuration has distribution ν.

Theorem 1.1 *Let $\lambda > 0$ and $\mu \geq 0$ be given and let ν be a probability measure on Ω. Then for all λ and μ, the weak limit $\nu_\infty = \lim_{t\to\infty} \nu S(t)$ exists and is an ergodic stationary measure on Ω, with $\nu_\infty(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}$.*

When we think of our interpretation of the system as a queueing system, we see that the system has a non-trivial stationary distribution for exactly those parameter values λ and μ for which a $M(\lambda)/M(\mu)/1$ queueing system is stable. At first sight this might be surprising, since there is no 'waiting room' available in the SMD. On the other hand, when $\lambda < \mu$ there is globally enough service capacity and generally speaking, it seems reasonable that, if one allows interactions between queues, the time of the servers can be used more efficiently which decreases the waiting time (in this case there is no waiting time).

In order to understand the proof of Theorem 1.1, it is not necessary to read the appendix and all of Section 3. We shall indicate which part can be omitted.

2 Construction of the SMD

We start with some notation. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ be the state space of the SMD. The space Ω is equipped with the product topology and the Borel σ-algebra $\mathcal{B}$. Initial configurations will be denoted by $\eta, \xi \in \Omega$ and the (random) configuration of the system at time t if the initial configuration was η or ξ will be denoted by η_t or ξ_t respectively. We call a site $i \in \mathbb{Z}$ *occupied* in η iff $\eta(i) = 1$, we interpret this as the presence of a particle at site i in the SMD. When we write $\eta \leq \xi$, we mean that $\eta(i) \leq \xi(i)$ for all $i \in \mathbb{Z}$. Let Ω_F be the set of all configurations in $\{0,1\}^{\mathbb{Z}}$ which have only finitely many occupied sites. We define $l_\eta(i) \in \mathbb{N}$ to be the number of occupied sites in configuration η to the right of site i until the nearest site to the right of site i that is not occupied:

$$l_\eta(i) := \#\{j \in \mathbb{Z}\colon j > i \text{ and for all } i < j' \leq j\colon \eta(j') = 1\}.$$

Define the following flipping transformation T_i which changes the configuration at site i and leaves all other sites unchanged:

$$T_i(\eta)(x) := \begin{cases} \eta(x) & \text{if } x \neq i, \\ 1 - \eta(x) & \text{if } x = i. \end{cases}$$

In this section we shall define a Markov semigroup $S(t)$ acting on bounded measurable functions f, which will be the semigroup of the SMD: $S(t)f(\eta) := E^\eta(f(\eta_t))$. In Section 3, it will turn out that for some 'special' functions f and some 'special' $\eta \in \Omega$, $\lim_{t\downarrow 0} \frac{S(t)f(\eta)-f(\eta)}{t}$ exists and is equal to $Gf(\eta)$, where G is the formal generator:

$$\begin{aligned} Gf(\eta) \quad := \quad & \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ & + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right). \end{aligned}$$

We shall now construct the SMD. The construction proceeds in five steps and is very similar to the construction of the one-dimensional sandpile process as carried out in Maes et al. (2000). We shall first outline the procedure briefly and work out the details in the appendix.

Step 1. We define an interacting particle system with state space Ω_F in the following way. Choose $n \in \mathbb{N}$. To each site $i \in \mathbb{Z} \cap [-n, n]$ we associate a Poisson process with parameter $\lambda > 0$; these processes are independent of each other. Particles enter the system according to the following mechanism. If the system is in state $\eta \in \Omega_F$ and a Poisson arrival occurs of the Poisson process associated to site $i \in [-n, n]$ then:

- In case $\eta(i) = 0$, $\eta(i)$ changes to 1, i.e. the particle is placed at site i.

- In case $\eta(i) = 1$, then the particle is placed at the nearest site with a number smaller than i which is not occupied.

Particles leave the system, independently of each other and of the arrival processes, after a period which is exponentially distributed with parameter $\mu \geq 0$. We call the Markov process described above the n-process (because of the restriction on the arrival processes). The associated semigroup is denoted by $S_n(t)$ and is defined for bounded measurable functions on Ω_F. The state of the n-process at time t if its initial state was $\eta \in \Omega_F$ is denoted by $\eta_{n,t}$.

Step 2. We show that the n-process (defined on Ω_F) is monotone, i.e. we show that for $\xi \leq \eta$ there is a coupling $(\hat{\xi}_{n,t}, \hat{\eta}_{n,t})_{(t\geq 0)}$ of ${\xi_{n,t}}_{(t\geq 0)}$ and ${\eta_{n,t}}_{(t\geq 0)}$ such that $\hat{\xi}_{n,t} \leq \hat{\eta}_{n,t}$ for all $t \geq 0$.

Step 3. The monotonicity of the n-process on Ω_F makes it possible to extend the n-process to a process with state space Ω in the following way. Let $\mathcal{M}$ be the space of bounded Borel measurable increasing functions on Ω. For $f \in \mathcal{M}$, the semigroup of the extension of the n-process will be given by

$$S_n(t)f(\eta) := \lim_{\xi \in \Omega_F,\ \xi \uparrow \eta} S_n(t)f(\xi).$$

This is well defined, as we show in the appendix.

Step 4. We show that the semigroups $S_n(t)$ are monotone in n.

Step 5. Since the semigroups $S_n(t)$ are monotone in n we can define a 'limiting' process with semigroup $S(t)$, which is for $f \in \mathcal{M}$ and $\eta \in \Omega$ defined by

$$S(t)f(\eta) := \lim_{n\uparrow\infty} S_n(t)f(\eta)$$

Observe that it suffices to define $S(t)$ only for $f \in \mathcal{M}$, since the distribution of the 'limiting' process at time t is completely determined by the outcomes of $S(t)f(\eta)$ for $f \in \mathcal{M}$.

Finally we define the SMD to be the Markov process that corresponds to the semigroup $S(t)$.

The details which are left open in the above description can be found in the appendix.

The behaviour of the SMD is somewhat strange for configurations which have a finite number of unoccupied sites, as is the case in the one-dimensional sandpile process in Maes et al. (2000). As a consequence we have:

Proposition 2.1 *The Markov process associated to $S(t)$ is not a Feller process.*

Proof: Let $C(\Omega)$ be the space of all continuous functions on Ω. If the process were Feller, then for all $f \in C(\Omega)$ and all $\eta \in \Omega$:

$$\lim_{t \downarrow 0} S(t) f(\eta) = f(\eta). \tag{1}$$

We shall show that this does not hold for some special choice of f and η. Let η^* be given by

$$\eta^*(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and let η^*_m for $m \in \mathbf{N}$ be defined by

$$\eta^*_m(x) := \begin{cases} \eta^*(x) & \text{if } x \in [-m, m] \cap \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $f : \Omega \to \mathbf{R}$ by $f(\eta) = \eta(0)$. We shall show that if $\lim_{t \downarrow 0} S(t) f(\eta^*)$ exists, the limit cannot be smaller than $\frac{\lambda}{\lambda + \mu}$ and since $f(\eta^*) = 0$, this implies that (1) does not hold and that the process is not Feller. Define the random variable $\eta^*_{m,t}$ to be the state of the n-process at time t, if the initial configuration was η^*_m. Let $A_{m,n}$ be the event (in the n-process with initial configuration η_m) that during the time interval $[0, t]$ a customer takes place at site 0 and that his service does not end before time t. Let $B_{m,n}$ be the event (again in the n-process with initial configuration η_m) that there is an arrival during $[0, t]$ in at least one of the Poisson processes associated to the sites in $[1, n]$ *before* any of the sites in $[1, n]$ becomes unoccupied. Then for $m \geq n$:

$$\begin{aligned} S_n(t) f(\eta^*_m) &= P(\eta^*_{m_{n,t}}(0) = 1) \\ &\geq P(A_{m,n}) \\ &\geq P(B_{m,n}) e^{-\mu t} \\ &= \frac{\lambda n}{\lambda n + \mu n} (1 - e^{-n(\lambda + \mu) t}) e^{-\mu t}. \end{aligned}$$

So if $\lim_{t\downarrow 0} S(t)f(\eta^*)$ exists, then

$$\begin{aligned}\lim_{t\downarrow 0} S(t)f(\eta^*) &= \lim_{t\downarrow 0}\lim_{n\to\infty}\lim_{m\to\infty} S_n(t)f(\eta_m^*)\\ &\geq \lim_{t\downarrow 0}\lim_{n\to\infty}\lim_{m\to\infty} \frac{\lambda n}{\lambda n+\mu n}(1-e^{-n(\lambda+\mu)t})e^{-\mu t}\\ &= \frac{\lambda}{\lambda+\mu}.\end{aligned}$$

This proves Proposition 2.1. □

3 The relation between $S(t)$ and G

We saw in the previous section that the Markov process associated to $S(t)$ is not Feller. However, as in the one-dimensional sandpile model in Maes et al. (2000), we have that for some class of 'nice' functions and configurations:

$$\lim_{t\downarrow 0} \frac{S(t)f(\eta)-f(\eta)}{t}$$

exists, and is equal to $Gf(\eta)$. To achieve this, we need the concepts of N-local functions and decent configurations as introduced in Maes et al. (2000). We repeat the definitions here. Let Ω_1 be the set of configurations with an infinite number of unoccupied sites at either side of the origin,

$$\Omega_1 := \{\eta\in\Omega : \sum_{i<0}(1-\eta(i)) = \sum_{i>0}(1-\eta(i)) = \infty\}.$$

We write the ordered indices i with $\eta(i)=0$ as $(\ldots, X_{-1}(\eta), X_0(\eta), X_1(\eta),\ldots)$, where $X_0(\eta) := \min\{i\geq 0 : \eta(i)=0\}$. Let (for $\eta\in\Omega_1$)

$$I_n(\eta) = (X_{n-1}(\eta), X_n(\eta)]\cap\mathbb{Z},$$

be a partition of $\mathbb{Z}$ into finite sets. We write

$$K_N(\eta) := \bigcup_{j=-N}^{N} I_j(\eta),$$

and $|\cdot|$ for cardinality.

A function $f:\Omega\to\mathbb{R}$ is called *N-local* if for all η, ξ in Ω_1 with $K_N(\eta) = K_N(\xi)$ and $\eta(i)=\xi(i)$ for all $i\in K_N(\eta)=K_N(\xi)$, we have $f(\eta)=f(\xi)$. We shall also use this notion for functions which are only defined on a subset of Ω which contains Ω_1. A configuration η is called *decent* if $\eta \in \Omega_1$ and

$$a(\eta) := \limsup_{n\to\infty} \frac{|I_{-n}(\eta)| + \cdots + |I_n(\eta)|}{2n+1} < \infty.$$

If η has a positive density $\rho(\eta)$ of zeroes, then $a(\eta) = \frac{1}{\rho(\eta)}$, and hence η is decent. The set of decent configurations is called Ω_{dec}.

Theorem 3.1 *Let $f \in \mathcal{M}$ be N-local for some $N \in \mathbb{N}$ and let $\eta \in \Omega_{dec}$. Then $Gf(\eta)$ is well defined and for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,*

$$S(t)f(\eta) = \sum_{n=0}^{\infty} \frac{t^n G^n f(\eta)}{n!}$$

and therefore,

$$\lim_{t\downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}$$

exists and is equal to $Gf(\eta)$.

Since the details of the proof are different from the proof of the corresponding result in Maes et al. (2000), we shall include the proof of Theorem 3.1, but it is possible to skip this part and continue reading at Section 4.

Lemma 3.2 *Let $f : D \subset \Omega \to \mathbb{R}$ be N-local, with $\Omega_1 \subset D$. Then Gf is $(N+1)$-local.*

Proof: We show first that if f is N-local, $Gf(\eta)$ is finite on a subset of Ω which contains Ω_1. Remember that $Gf(\eta)$ was defined by

$$\begin{aligned} Gf(\eta) \quad := \quad & \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\ & + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right). \end{aligned}$$

Let $\eta \in \Omega_1$ and let f be N-local. It follows that for $i \in \mathbb{Z}\backslash\{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}$, $f(T_i(\eta)) - f(\eta) = 0$. This implies that the above sum converges.

Let us assume now that f is N-local and show that it follows that Gf is $(N+1)$-local. Assume that $\eta, \xi \in \Omega_1$ with $K_{N+1}(\eta) = K_{N+1}(\xi)$ and $\eta(i) = \xi(i)$ for all $i \in K_{N+1}(\eta) = K_{N+1}(\xi)$. We saw already that the sums in $Gf(\eta)$ and

$Gf(\xi)$ run over $i \in \{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}$. Observe that it follows from our assumptions that $f(\eta) = f(\xi)$ and that for $i \in \{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}$, $\mathbf{1}_{\{\eta(i)=1\}} = \mathbf{1}_{\{\xi(i)=1\}}$, $f(T_i(\eta)) = f(T_i(\xi))$ and $l_i(\eta) = l_i(\xi)$. So $Gf(\eta) = Gf(\xi)$ and we conclude that Gf is $(N+1)$-local. $\square$

Lemma 3.3 *Let $f : \Omega \to \mathbb{R}$ be N-local and bounded and let $\eta \in \Omega_1$. Then*

$$|G^n f(\eta)| \leq (2(\lambda + \mu))^n ||f||_\infty (|I_{-(N+n)}(\eta)| + \cdots + |I_{N+n}(\eta)|)^n. \tag{2}$$

Proof: We use induction on n. Suppose $f : \Omega \to \mathbb{R}$ is N-local and bounded and $\eta \in \Omega_1$. For $n = 1$ we saw in the proof of Lemma 3.2 that only the terms where $i \in \{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}$ contribute to the sum, so

$$\begin{aligned} |Gf(\eta)| &\leq \sum_{i \in \{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}} \mathbf{1}_{\{\eta(i)=0\}} \lambda(1 + l_\eta(i)) |(f(T_i(\eta) - f(\eta))| \\ &\quad + \sum_{i \in \{X_{-(N+1)}(\eta), \ldots, X_N(\eta)\}} \mathbf{1}_{\{\eta(i)=1\}} \mu |\left(f\left(T_i(\eta)\right) - f(\eta)\right)| \\ &\leq 2||f||_\infty \lambda (|I_{-N}(\eta)| + \cdots + |I_{N+1}(\eta)|) \\ &\quad + 2||f||_\infty \mu \left(|I_{-N}(\eta)| + \cdots + |I_N(\eta)|\right) \\ &\leq 2(\lambda + \mu) ||f||_\infty \left(|I_{-(N+1)}(\eta)| + \cdots + |I_{N+1}(\eta)|\right). \end{aligned}$$

So for $n = 1$, statement (2) in Lemma 3.3 is true. Assume that we know that (2) holds for all $n \leq k$ (induction hypothesis) and consider

$$\begin{aligned} |G^{k+1} f(\eta)| &= \left| \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda (1 + l_\eta(i)) \left(G^k f\left(T_i(\eta)\right) - G^k f(\eta) \right) \right. \\ &\qquad \left. + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(G^k f\left(T_i(\eta)\right) - G^k f(\eta) \right) \right|. \end{aligned}$$

If f is N-local, then $G^k f$ is $(N+k)$-local (this follows from Lemma 3.2), so for $i \in \mathbb{Z} \setminus \{X_{-(N+k+1)}(\eta), \ldots, X_{N+k}(\eta)\}$ we have that $G^k f\left(T_i(\eta)\right) - G^k f(\eta) = 0$. From this and the induction hypothesis we conclude that

$$|G^{k+1} f(\eta)| \leq$$
$$\sum_{i \in \{X_{-(N+k+1)}(\eta), \ldots, X_{N+k}(\eta)\}} (1 + l_\eta(i)) \lambda \mathbf{1}_{\{\eta(i)=0\}} \left(\left| G^k f(\eta) \right| + \left| G^k f(T_i(\eta)) \right| \right) +$$

$$\begin{aligned}
&\sum_{i\in\{X_{-(N+k+1)}(\eta),\ldots,X_{N+k}(\eta)\}} \mu \mathbf{1}_{\{\eta(i)=1\}}\left(\left|G^k f(\eta)\right| + \left|G^k f(T_i(\eta))\right|\right) \\
\leq\ & [(2(\lambda+\mu))^k ||f||_\infty (|I_{-(N+k)}(\eta)| + \cdots + |I_{N+k}(\eta)|)^k + \\
& (2(\lambda+\mu))^k ||f||_\infty (|I_{-(N+k+1)}(\eta)| + \cdots + |I_{N+k+1}(\eta)|)^k] \times \\
& [\lambda\left(|I_{-(N+k)}(\eta)| + \cdots + |I_{N+k+1}|\right) + \\
& \mu\left(|I_{-(N+k)}(\eta)| + \cdots + |I_{N+k}|\right)] \\
\leq\ & (2(\lambda+\mu))^{k+1} ||f||_\infty (|I_{-(N+k+1)}(\eta)| + \cdots + |I_{N+k+1}(\eta)|)^{k+1}.
\end{aligned}$$

This proves Lemma 3.3. Observe that the statement of the lemma also holds for $\eta \in \Omega_F$ and G replaced by G_m, the generator of the m-process on Ω_F. $\Box$

Finally we need the following lemma from Maes et al. (2000):

Lemma 3.4 *Let* $\{a_n : n \geq 0\}$ *be a sequence of positive real numbers such that* $\limsup_{n\to\infty} a_n/n = a < \infty$. *Then the series* $\sum_{n=0}^\infty t^n a_n^n/n!$ *converges for* $|t| < \frac{1}{ae}$.

Proof of Theorem 3.1: Let $f \in \mathcal{M}$ be N-local. For $\eta \in \Omega_F$, $f \in \mathcal{M}$ we have that for all t,

$$S_n(t)f(\eta) = \sum_{i=0}^\infty \frac{t^i G_n^i f(\eta)}{i!}.$$

So by definition we get that for $\eta \in \Omega$,

$$S(t)f(\eta) = \lim_{n\to\infty} \lim_{\eta'\in\Omega_F, \eta'\uparrow\eta} \sum_{i=0}^\infty \frac{t^i G_n^i f(\eta')}{i!}.$$

Suppose now that $\eta \in \Omega_{dec}$. We have from the remark at the end of the proof of Lemma 3.3 that when $\eta' \in \Omega_F$, $\eta' \leq \eta$,

$$\begin{aligned}
|G_n^i f(\eta')| &\leq (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta')| + \cdots + |I_{N+i}(\eta')|)^i \\
&\leq (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta)| + \cdots + |I_{N+i}(\eta)|)^i.
\end{aligned}$$

From Lemma 3.4 it follows that for decent configurations η we have, for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,

$$\sum_{i=0}^\infty \frac{t^i (2(\lambda+\mu))^i ||f||_\infty (|I_{-(N+i)}(\eta)| + \cdots + |I_{N+i}(\eta)|)^i}{i!} < \infty,$$

so using dominated convergence we obtain that for $t < \frac{1}{4(\lambda+\mu)ea(\eta)}$,

$$S(t)f(\eta) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{t^i G_n^i f(\eta)}{i!}.$$

We can deal with the limit for $n \to \infty$ in the same way, which leads to the desired result. □

4 The stationary distribution of the SMD

In this section we shall prove Theorem 1.1. The strategy will roughly be as follows:

Proposition 4.1 states that if the initial configuration of the SMD is chosen according to some stationary measure, then the distribution of the configuration at time t is also a stationary measure. Observe that there is really something to prove here, since the construction of the process is not stationary. Then we prove Proposition 4.2 which states that if the initial configuration is chosen according to an ergodic stationary measure, then also the distribution of the configuration at time t is an ergodic stationary measure. We shall need these results when we show that the stationary distribution of the SMD is an ergodic stationary measure.

Lemma 4.3 and Lemma 4.4 say that if the initial configuration of the system is chosen according to an ergodic stationary measure with a strictly positive density of empty sites, then there is a strictly positive density of sites that have not been occupied during a small time period. We need this result for the rather technical Lemmas 4.5 and 4.7. These lemmas are used in the proof of Lemma 4.8 and Proposition 4.9 to get a differential equation for the density of occupied sites, which makes it possible to compute the density of occupied sites at time t explicitly, if the starting configuration was chosen according to an ergodic stationary measure with a stricly positive density of empty sites. This is a key ingredient in the proof of Theorem 1.1.

Proposition 4.1 *If ν is a stationary measure on Ω then $\nu S(t)$ is a stationary measure on Ω.*

Proof: Let ν be a stationary measure on Ω and let the left-shift $T : \Omega \to \Omega$ be given by $T\eta(x) = \eta(x+1)$ for all $x \in \mathbb{Z}$. It suffices to show that

$$\nu S(t)(A) = \nu S(t)(T^{-1}A), \tag{3}$$

for cylinder events A. We first observe that for all $\eta \in \Omega$,

$$S(t)\mathbf{1}_A(\eta) = S(t)\mathbf{1}_{T^{-1}A}(T^{-1}\eta), \tag{4}$$

since if A depends only on coordinates in $[-n, n]$ and if $(\eta_m)_{m=1}^{\infty}$ is an increasing sequence in Ω_F and $\eta_m \uparrow \eta$ then $S_n(t)\mathbf{1}_A(\eta_m) = S_{n+1}(t)\mathbf{1}_{T^{-1}A}(T^{-1}\eta_m)$. Sending first $m \to \infty$ and then $n \to \infty$ leads to (4).

From this (3) follows easily, since

$$\begin{aligned}
\nu S(t)(A) &= \int_\Omega S(t)\mathbf{1}_A(\eta)\, d\nu(\eta) = \int_\Omega S(t)\mathbf{1}_{T^{-1}A}(T^{-1}\eta)\, d\nu(\eta) \\
&= \int_\Omega S(t)\mathbf{1}_{T^{-1}A}(\eta)\, d\nu(T\eta) = \int_\Omega S(t)\mathbf{1}_{T^{-1}A}(\eta)\, d\nu(\eta) \\
&= \nu S(t)(T^{-1}A).
\end{aligned}$$

□

Proposition 4.2 *If ν is an ergodic stationary measure on Ω, then $\nu S(t)$ is an ergodic stationary measure on Ω.*

Proof: Let $X_i(t)$, $i \in \mathbb{Z}$ be independent Poisson processes with parameter λ and let $D_i(t)$, $i \in \mathbb{Z}$ be independent Poisson processes with parameter μ. Let, for $\xi \in \Omega_F$, $\hat{\xi}_{n,t}$ be the state of the n-process at time t if the initial configuration is ξ, the arrivals take place according to the Poisson processes $X_i(t)$ and the departures according to the processes $D_i(t)$.

Define for $\eta \in \Omega$, $\hat{\eta}_{n,t}$ by

$$\hat{\eta}_{n,t}(i) := \lim_{\xi \in \Omega_F, \xi \leq \eta} \hat{\xi}_{n,t},$$

and define $\hat{\eta}_t$ by

$$\hat{\eta}_t(i) := \lim_{n \to \infty} \hat{\eta}_{n,t}(i).$$

Then $\hat{\eta}_t$ and η_t are identically distributed.

Now $\hat{\eta}_t$ is a function of the initial configuration and the arrival and departure processes, which commutes with the shift. So for ν ergodic, we conclude that $\nu S(t)$ is a factor of an ergodic stationary measure, and is therefore an ergodic stationary measure itself. $\square$

The next lemma gives a condition which ensures emptyness of a site in the n-process during a period of length t. We need this for the proof of Lemma 4.4, which says that if we start with a positive density of unoccupied sites, for some amount of time, this density remains positive.

Lemma 4.3 *Let $\eta \in \Omega$ and let $X_k(t)$, $k \in \mathbb{Z}$ be a sequence of independent Poisson arrival processes with parameter λ. Let $\hat{\eta}_{n,t}$ be defined as in the proof of Lemma 4.2. Then*

$$\eta(i) = 0, \;\; X_i(t) = 0$$

and

$$(j-i) - \sum_{k=i+1}^{j} (X_k(t) + \eta(k)) \geq 0,$$

for all $j \in [i+1, n] \cap \mathbb{Z}$ together imply that $\hat{\eta}_{n,s}(i) = 0$, for all $s \leq t$. (Here $[i+1, n] := \emptyset$, for $i \geq n$).

Strictly speaking, we can use a weaker condition for the sites with a number larger than n or smaller than $-n$, but the stated lemma suffices for our purposes.

Proof of Lemma 4.3: It suffices to prove the theorem for the case $\mu = 0$, since the state of the n-process with $\mu = 0$ cannot be larger than the state of the n-process where $\mu > 0$, if we use the same sequence of arrival processes in both cases. Furthermore, we shall only consider the case $i = 0$, the general statement can be proved analogously.

So assume that $\mu = 0$ and that $\eta(0) = 0$, $X_0(t) = 0$, and

$$j - \sum_{k=1}^{j} (X_k(t) + \eta(k)) \geq 0$$

for all $j \in [1, n] \cap \mathbb{Z}$. These conditions ensure that until time t, none of the particles that arrived in the arrival processes associated to sites $1, \ldots, n$ had to go to site 0 to be served there. Together with the conditions $\eta(0) = 0$ and $X_0(t) = 0$ it follows that for all $s \leq t$, $\hat{\eta}_{n,s}(0) = 0$. This can be made precise by an elementary induction argument on n. $\square$

Lemma 4.4 *Let ν_0 be an ergodic stationary measure on Ω. Suppose that*

$$\nu_0(\eta(0) = 0) = \gamma_0,$$

for some $\gamma_0 > 0$. Then for $t < \frac{\gamma_0}{2\lambda}$,

$$\nu_0 S(t)(\eta(0) = 0) > 0.$$

Proof: As in the proof of Lemma 4.3, it is enough to consider the case where $\mu = 0$. Let $\hat{\eta}_t$ and $X_i(t)$, $i \in \mathbb{Z}$, be defined as in the proof of Lemma 4.2.

Assume now that η has distribution ν_0. We call i a special empty point at time t (s.e.p. for short) if

$$\eta(i) + X_i(t) = 0$$

and

$$(j - i) - \sum_{k=i+1}^{j} (X_k(t) + \eta(k)) \geq 0$$

for all $j \geq (i + 1)$. This name is chosen because when i is a s.e.p, it follows from Lemma 4.3 that $\hat{\eta}_{n,t'}(i) = 0$ for all n and for all $t' < t$, which implies that $\hat{\eta}_{t'} = 0$ for all $t' < t$.

We shall prove that for $t < \frac{\gamma}{2\lambda}$ there is a strictly positive density of special empty points almost surely. By ergodicity of the stationary sequence $X_i(t)+\eta(i)$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{\{i \text{ is s.e.p.}\}} = P(0 \text{ is s.e.p.}),$$

almost surely. Define for $l \in \mathbf{N}$,

$$S_l(t) := \sum_{i=0}^{l} (1 - X_i(t) - \eta(i)).$$

Observe that $P(0 \text{ is s.e.p.}) = P(S_n(t) > 0, \forall n \geq 0)$. Since for $t < \frac{\gamma_0}{2\lambda}$,

$$E(1 - X_i(t) - \eta(i)) \geq \frac{\gamma_0}{2} > 0,$$

we know that $P(S_n = 0 \text{ i.o.}) = 0$ which implies that $P(S_n(t) > 0, \forall n \geq 0) > 0$, so $P(0 \text{ is s.e.p.}) > 0$. This implies that $\nu_0 S(t)(\eta(0) = 0) > 0$, for all $t < \frac{\gamma}{2\lambda}$. $\square$

Lemma 4.5 *Let ν be an ergodic stationary measure on Ω, with*

$$\nu(\eta(0) = 0) > 0,$$

ν-a.s. Then

$$\int_{\Omega} \mathbf{1}_{\{\eta(0)=0\}}(\eta)(1 + l_{\eta}(0))d\nu(\eta) = 1.$$

Proof: This follows from Theorem 4.6 (p. 46) of Peterson (1983). $\square$

We will now prove a relation as in Theorem 3.1, for a special function which is neither bounded nor monotone. We need this in the proofs of Lemma 4.7 and Lemma 4.8. We use the following subset of Ω_{dec},

$$\Omega_{dec}^{\gamma} := \left\{ \eta : \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} (1 - \eta(i)) = \gamma \right\}.$$

Lemma 4.6 *Let $\gamma > 0$ and let g be defined by $g(\eta) := \mathbf{1}_{\{\eta(0)=0\}}(\eta)(1 + l_{\eta}(0))$. Then for $\eta \in \Omega_{dec}^{\gamma}$ and $t < \frac{\gamma}{4(\lambda+\mu)e}$,*

$$S(t)g(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i g(\eta)}{i!}$$

Proof: Let $\eta \in \Omega_{dec}^{\gamma}$ and $t < \frac{\gamma}{4(\lambda+\mu)e}$. Define h by

$$h(\eta) := 1 + l_{\eta}(0).$$

We will show that

$$S(t)h(\eta) = \sum_{i=0}^{\infty} \frac{t^i G^i h(\eta)}{i!} < \infty, \tag{5}$$

and that

$$S(t)(\mathbf{1}_{\{\eta(0)=1\}}(\eta)h(\eta)) = \sum_{i=0}^{\infty} \frac{t^i G^i(\mathbf{1}_{\{\eta(0)=1\}}(\eta)h(\eta))}{i!} < \infty. \tag{6}$$

This suffices, since if (5) and (6) hold, we have that

$$\begin{aligned} S(t)g(\eta) &= S(t)(h(\eta) - \mathbf{1}_{\{\eta(0)=1\}}(\eta)h(\eta)) \\ &= \sum_{i=0}^{\infty} \frac{t^i G^i h(\eta)}{i!} - \sum_{i=0}^{\infty} \frac{t^i G_n^i(\mathbf{1}_{\{\eta(0)=1\}}(\eta)h(\eta))}{i!} \\ &= \sum_{i=0}^{\infty} \frac{t^i G^i(h(\eta) - \mathbf{1}_{\{\eta(0)=1\}}(\eta)h(\eta))}{i!} \\ &= \sum_{i=0}^{\infty} \frac{t^i G_n^i g(\eta)}{i!}. \end{aligned}$$

We will now prove (5). The proof of (6) proceeds analogously and is omitted. Let G_n be the generator of the n-process and define h_M $(M \in \mathbb{N})$ by

$$h_M(\eta) := \min\{h(\eta), M\}.$$

Observe that

$$S(t)h(\eta) = \lim_{M\to\infty} S(t)h_M(\eta).$$

Since $h_M \in \mathcal{M}$, we find that

$$\begin{aligned} S(t)h(\eta) &= \lim_{M\to\infty}\lim_{n\to\infty}\lim_{\eta'\in\Omega_F,\eta'\uparrow\eta} S_n(t)h_M(\eta') \\ &= \lim_{M\to\infty}\lim_{n\to\infty}\lim_{\eta'\in\Omega_F,\eta'\uparrow\eta} \sum_{i=0}^{\infty}\frac{t^iG_n^ih_M(\eta')}{i!} \\ &= \lim_{n\to\infty}\lim_{\eta'\in\Omega_F,\eta'\uparrow\eta}\lim_{M\to\infty} \sum_{i=0}^{\infty}\frac{t^iG_n^ih_M(\eta')}{i!}, \end{aligned}$$

where the third equality holds since all limits are increasing. We can now apply the dominated convergence theorem three times, to bring the limits into the sum. For the limit $M \to \infty$, observe that for $\eta' \in \Omega_F$ and all M,

$$|G_n^ih_M(\eta')| \le (2(\lambda+\mu))^i(|I_{-i}(\eta')| + \cdots + |I_i(\eta')|)^{i+1},$$

(this can be proved in the same way as Lemma 3.3) and that for $\eta' \in \Omega_F$,

$$\sum_{i=0}^{\infty}\frac{t^i(2(\lambda+\mu))^i(|I_{-i}(\eta')| + \cdots + |I_i(\eta')|)^{i+1}}{i!} < \infty,$$

which follows from Lemma 3.4. For the second limit, we use that for all $\eta' \in \Omega_F$ with $\eta' \le \eta$,

$$|G_n^ih(\eta')| \le (2(\lambda+\mu))^i(|I_{-i}(\eta)| + \cdots + |I_i(\eta)|)^{i+1},$$

and that

$$\sum_{i=0}^{\infty}\frac{t^i(2(\lambda+\mu))^i(|I_{-i}(\eta)| + \cdots + |I_i(\eta)|)^{i+1}}{i!} < \infty.$$

For the last limit we observe that the bound for $|G_n^ih(\eta')|$ is also valid for $|G_n^ih(\eta)|$, so that we are done. $\square$

Lemma 4.7 *Let ν be an ergodic stationary measure on Ω, with $\nu(\eta(0)=0) = \gamma > 0$. Then for $t < \frac{\gamma}{5(\lambda+\mu)e}$,*

$$\frac{d}{dt}\int_{\Omega_{dec}} \sum_{n=0}^{\infty} \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} d\nu(\eta) = \int_{\Omega_{dec}} \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} d\nu(\eta).$$

Proof: We shall denote the semigroup of the SMD with parameters λ and μ by $S_{\lambda,\mu}(t)$. Observe that $\nu(\Omega^{\gamma}_{dec}) = 1$. We shall show that there exists a ν-integrable function g such that for $t \in \left[0, \frac{\gamma}{5(\lambda+\mu)}e\right)$ and $\eta \in \Omega^{\gamma}_{dec}$:

$$\left| \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} \right| \leq g(\eta),$$

which suffices.

Recall from Theorem 3.1 that $\sum_{n=0}^{\infty} \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!}$ converges for $\eta \in \Omega^{\gamma}_{dec}$ and $t < \frac{\gamma}{4(\lambda+\mu)e}$. So for $t < \frac{\gamma}{5(\lambda+\mu)e}$ and $\eta \in \Omega^{\gamma}_{dec}$ we get (using Lemma 4.6), that

$$\begin{aligned}
&\left| \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} \right| \\
&= \left| \sum_{n=1}^{\infty} \frac{n t^{n-1} G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} \right| \\
&= \left| \sum_{n=0}^{\infty} \frac{t^n G^n (G\mathbf{1}_{\{\eta(0)=1\}}(\eta))}{n!} \right| \\
&= \left| \sum_{n=0}^{\infty} \frac{t^n G^n (\mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0)) - \mu \mathbf{1}_{\{\eta(0)=1\}}(\eta))}{n!} \right| \qquad (7) \\
&\leq S_{\lambda,\mu}(t)\mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0)) + \mu S_{\lambda,\mu}(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta) \\
&\leq S_{\lambda,\mu}(t)\mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0)) + \mu \leq S_{\lambda,0}(t)\mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0)) + \mu \\
&\leq S_{\lambda,0}\left(\frac{\gamma}{5(\lambda+\mu)e}\right)\mathbf{1}_{\{\eta(0)=0\}}\lambda(1+l_\eta(0)) + \mu
\end{aligned}$$

This is a ν integrable function, since

$$\int_{\Omega_{dec}} S_{\lambda,0}\left(\frac{\gamma}{5(\lambda+\mu)e}\right)\mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0))d\nu = \int_{\Omega_{dec}} \mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1+l_\eta(0))d\nu S_{\lambda,0}\left(\frac{\gamma}{5(\lambda+\mu)e}\right),$$

which is finite by Lemma 4.5 since $\nu S_{\lambda,0}\left(\frac{\gamma}{5(\lambda+\mu)e}\right)$ is an ergodic stationary measure with

$$\nu S_{\lambda,0}\left(\frac{\gamma}{5(\lambda+\mu)e}\right)(\eta(0)=0) > 0,$$

by Proposition 4.2 and Lemma 4.4. □

Lemma 4.8 *Let $\lambda > 0, \mu > 0$ and let ν be an ergodic stationary measure on Ω with*

$$\nu(\eta(0) = 0) = \gamma > 0.$$

Let $\gamma_t := \nu S(t)(\eta(0) = 0)$. Then for $t \leq \frac{\gamma}{6(\lambda+\mu)e}$,

$$\gamma_t = (1 - \frac{\lambda}{\mu}) - (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}.$$

Proof: We use that

$$\begin{aligned}\gamma_t &= 1 - \int_\Omega \mathbf{1}_{\{\eta(0)=1\}}(\eta) d\nu S(t) \\ &= 1 - \int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta) d\nu\end{aligned}$$

and derive a differential equation for $\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu$. Let $t \leq \frac{\gamma}{6(\lambda+\mu)e}$. Since ν concentrates on Ω^γ_{dec}, we may write

$$\begin{aligned}\frac{d}{dt}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu &= \frac{d}{dt}\int_{\Omega^\gamma_{dec}} \sum_{n=0}^\infty \frac{t^n G^n \mathbf{1}_{\{\eta(0)=1\}}(\eta)}{n!} d\nu \\ &= \int_{\Omega^\gamma_{dec}} \mathbf{1}_{\{\eta(0)=0\}}(\eta)\lambda(1 + l_\eta(0)) d\nu S(t) \\ &\quad -\mu \int_{\Omega^\gamma_{dec}} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta) d\nu \\ &= \lambda - \mu \int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu.\end{aligned}$$

Here we used Theorem 3.1, Lemma 4.5, Lemma 4.6, Lemma 4.7 and (7). We conclude that

$$\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu = ce^{-\mu t} + \frac{\lambda}{\mu},$$

for some $c \in \mathbb{R}$. Since

$$\int_\Omega S(0)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu = 1 - \gamma,$$

we get that

$$\gamma_t = 1 - \frac{\lambda}{\mu} - (1 - \gamma - \frac{\lambda}{\mu})e^{-\mu t}$$

and Lemma 4.8 is proved. □

Proposition 4.9 *Let ν be an ergodic stationary measure on Ω with*

$$\nu(\eta(0) = 0) = \gamma > 0.$$

Then for $\lambda > 0, \mu > 0$ and all $t \geq 0$ we have,

$$\nu S(t)(\eta(0) = 0) = \max\left\{(1 - \frac{\lambda}{\mu}) - (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}, 0\right\}.$$

For $\lambda > 0, \mu = 0$ and all $t \geq 0$,

$$\nu S(t)(\eta(0) = 0) = \max\left\{\gamma - \lambda t, 0\right\}.$$

Proof: Let $\lambda > 0$, $\mu > 0$ be given and let ν be as in the proposition, γ_t as above. Write $t^* = \frac{\gamma}{6(\lambda+\mu)e}$. We already know from Lemma 4.8 that the statement of the proposition is true for $t \leq t^*$, and that $\nu S(t^*)$ is an ergodic stationary measure with

$$\nu S(t^*)(\eta(0) = 0) = \gamma_{t^*} > 0.$$

This means that the differential equation which we derived in the proof of Lemma 4.8 also holds for $t \in \left[t^*, t^* + \frac{\gamma_{t^*}}{6(\lambda+\mu)e}\right]$, and that the expression for γ_t in Lemma 4.8 is also true for $t \in \left[t^*, t^* + \frac{\gamma_{t^*}}{6(\lambda+\mu)e}\right]$. Applying the same trick again and again leads to the conclusion that

$$\gamma_t = (1 - \frac{\lambda}{\mu}) - (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t},$$

for all t for which this expression is positive. When $\lambda \leq \mu$, this is the case for all t and we are done. When $\lambda > \mu$ we have in this way that for

$$t < \frac{\log(\gamma - 1 + \frac{\lambda}{\mu}) - \log(\frac{\lambda}{\mu} - 1)}{\mu} := T(\lambda),$$

$$\gamma_t = (1 - \frac{\lambda}{\mu}) - (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}.$$

We claim that $\gamma_t = 0$ for all $t \geq T(\lambda)$. To achieve this, we use the monotonicity of the process in the parameter λ (which can easily be proved using the basic coupling for the n-processes on Ω_F and taking limits). If we consider γ_t as a function of λ, we have that for $\lambda \leq \lambda'$, $\gamma_t(\lambda) \geq \gamma_t(\lambda')$. For $\alpha > 0$, we claim that it is impossible that $\gamma_t = \alpha$ for some $t \geq T(\lambda)$, since there exists a unique

$\lambda'' < \lambda$ such that the process with parameters λ'' and μ has $\gamma_t(\lambda'') = \frac{\alpha}{2}$. So we have that

$$\nu S(t)\,(\eta(0) = 0) = \max\left\{(1 - \frac{\lambda}{\mu}) - (1 - \frac{\lambda}{\mu} - \gamma)e^{-\mu t}, 0\right\}.$$

The proof for the case $\lambda > 0, \mu = 0$ proceeds analogously. □

Proof of Theorem 1.1: It follows immediately from Proposition 4.9 that the theorem is true when $\mu = 0$, hence we suppose that $\mu > 0$. Let η_0 be the configuration in which all sites are unoccupied and η_1 be the configuration in which all sites are occupied. Let the measures δ_0 and δ_1 be defined by $\delta_0(\{\eta_0\}) = 1$ and $\delta_1(\{\eta_1\}) = 1$. The proof is based on the following observations:

Observation 1:

By monotonicity of the process, for $f \in \mathcal{M}$ and η arbitrary, $S(t)f(\eta_1) \geq S(t)f(\eta) \geq S(t)f(\eta_0)$, so for all ν we have, $\delta_0 S(t) \leq \nu S(t) \leq \delta_1 S(t)$.

Observation 2:

The limits $\lim_{n\to\infty} \delta_0 S(t)$ and $\lim_{n\to\infty} \delta_1 S(t)$ exist and are stationary measures. We get existence by the fact that $\delta_0 S(t)$ is increasing in t and $\delta_1 S(t)$ is decreasing in t. We see this as follows: Since $\eta_{0_\epsilon} \geq \eta_0$ for all ϵ, we get that for $f \in \mathcal{M}$:

$$S(t+\epsilon)f(\eta_0) = S(t)S(\epsilon)f(\eta_0) \geq S(t)f(\eta_0).$$

So $\delta_0 S(t) \leq \delta_0 S(t+\epsilon)$. Similarly, since for all ϵ $\eta_{1_\epsilon} \leq \eta_1$,

$$S(t+\epsilon)f(\eta_1) = S(t)S(\epsilon)f(\eta_1) \leq S(t)f(\eta_1),$$

$\delta_1 S(t) \geq \delta_1 S(t+\epsilon)$. We conclude that $\lim_{n\to\infty} \delta_0 S(t)$ and $\lim_{n\to\infty} \delta_1 S(t)$ exist and denote the limiting measures by ν_0 and ν_1 respectively. Since by Proposition 4.1, $\delta_0 S(t)$ and $\delta_1 S(t)$ are stationary measures for all t, ν_0 and ν_1 are also stationary measures.

Observation 3: We claim that

$$\nu_1(\eta(0) = 1) = \nu_0(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}.$$

To see this, use Proposition 4.9, to obtain

$$\nu_0(\eta(0) = 1) = \lim_{t\to\infty} \delta_0 S(t)(\eta(0) = 1) = \min\left\{\frac{\lambda}{\mu}, 1\right\}.$$

For ν_1, things are a bit more subtle.

$$\nu_1(\eta(x)=1) = \lim_{t\to\infty} S(t)\mathbf{1}_{\{\eta(x)=1\}}(\eta_1), \tag{8}$$

but since δ_1 does not satisfy the assumptions of Proposition 4.9 we cannot use this proposition directly as was the case for ν_0. Let δ_p be the Bernoulli measure on Ω, with $\delta_p(\eta(x)=1)=p$, and let ξ_m be defined by $\xi_m(x)=1$ if $x\in[-m,m]$ and $\xi_m=0$ otherwise. We claim that

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_p = S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1). \tag{9}$$

To prove (9), observe that

$$\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_p \le S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1),$$

so it remains to prove that

$$\lim_{p\uparrow 1}\int S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_p \ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1).$$

By definition and by monotonicity,

$$\begin{aligned} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1) &= \lim_{n\to\infty}\lim_{m\to\infty} S_n(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \\ &= \lim_{m\to\infty}\lim_{n\to\infty} S_n(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \\ &= \lim_{m\to\infty} S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m) \end{aligned}$$

Now let $\epsilon>0$ and let $p(\epsilon,m) := (1-\epsilon)^{\frac{1}{2m+1}}$. Then

$$\delta_{p(\epsilon,m)}(\eta(-m)=1,\ldots,\eta(m)=1) = 1-\epsilon$$

and

$$\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_{p(\epsilon,m)} \ge (1-\epsilon)S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m),$$

so we get that for all m,

$$\begin{aligned} \lim_{p\uparrow 1}\int S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_p &= \lim_{\epsilon\downarrow 0}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_{p(\epsilon,m)} \\ &\ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\xi_m). \end{aligned}$$

Sending $m\to\infty$ leads to

$$\lim_{p\uparrow 1}\int S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\delta_p \ge S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta_1)$$

and (9) is proved. Putting (8), (9) and Proposition 4.9 together yields that

$$\begin{aligned}
\nu_1(\eta(0)=1) &= \lim_{t\to\infty}\lim_{p\uparrow 1}\int_\Omega S(t)\mathbf{1}_{\{\eta(0)=1\}}(\eta)d\nu_p \\
&= \lim_{t\to\infty}\lim_{p\uparrow 1}\min\left\{(1-(1-p)-\frac{\lambda}{\mu})e^{-\mu t}+\frac{\lambda}{\mu},1\right\} \\
&= \lim_{t\to\infty}\min\left\{(1-\frac{\lambda}{\mu})e^{-\mu t}+\frac{\lambda}{\mu},1\right\} \\
&= \min\left\{\frac{\lambda}{\mu},1\right\}
\end{aligned}$$

Conclusion: From Observation 1 and Observation 2 we conclude that $\nu_0 = \lim_{t\to\infty}\delta_0 S(t) \le \lim_{t\to\infty}\delta_1 S(t) = \nu_1$, with ν_0 and ν_1 stationary measures. If we combine this with Observation 3 and Corollary 2.8 (page 75) of Liggett (1985), we get that $\nu_0 = \nu_1$. So the process has an unique invariant measure $\lim_{t\to\infty}\nu S(t)$, which equals ν_0 and ν_1, and which is stationary.

Finally we show that ν_1 is ergodic. Observe that we cannot use method of Proposition 4.2, since we cannot write the state of the system in the stationary distribution as a function of the initial state and the arrival and departure processes. We use the monotonicity of the process and the fact that $\delta_0 S(t)$ and $\delta_1 S(t)$ are ergodic stationary measures for all t.

Recall that T is the left shift on Ω. To prove ergodicity of ν_1, it suffices to show that for all $A, B \in \mathcal{B}$ for which $\mathbf{1}_A, \mathbf{1}_B \in \mathcal{M}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\nu_1(T^k A\cap B) = \nu_1(A)\nu_1(B). \tag{10}$$

In the proof of (10) we use the same technique as Liggett (1985) for spin systems. Let A, B be as indicated above. Observe that it follows from Observation 1 that

$$\nu_1(T^k A\cap B) = \int_\Omega \mathbf{1}_{T^k A\cap B}(\eta)d\nu_1 \le \int_\Omega \mathbf{1}_{T^k A\cap B}(\eta)d\delta_1 S(t) \tag{11}$$

and that

$$\nu_1(T^k A\cap B) \ge \int_\Omega \mathbf{1}_{T^k A\cap B}(\eta)d\delta_0 S(t). \tag{12}$$

From (11) and (12) we conclude that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega \mathbf{1}_{T^k A\cap B}(\eta)d\delta_0 S(t) &\le \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\nu_1(T^k A\cap B) \\
&\le \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega \mathbf{1}_{T^k A\cap B}(\eta)d\delta_1 S(t).
\end{aligned}$$

Since $\delta_0 S(t)$ and $\delta_1 S(t)$ are ergodic stationary measures, we get that

$$\begin{aligned}
\int_\Omega \mathbf{1}_A(\eta) d\delta_0 S(t) \int_\Omega \mathbf{1}_B(\eta) d\delta_0 S(t) &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu_1(T^k A \cap B) \\
&\leq \int_\Omega \mathbf{1}_A(\eta) d\delta_1 S(t) \int_\Omega \mathbf{1}_B(\eta) d\delta_1 S(t).
\end{aligned}$$

Taking limits for $t \to \infty$ leads to (10), so ν_1 is an ergodic stationary measure.

It follows from Observation 3 that for $\lambda < \mu$,

$$\nu_1\left(\eta(0) = 1\right) = \frac{\lambda}{\mu},$$

and that for $\lambda \geq \mu$, ν_1 is degenerate at $\{1\}^{\mathbb{Z}}$. This proves the theorem. $\square$

Appendix

We give the details of the construction described in Section 2.

Step 1. We must to show that $\eta \in \Omega_F$ implies that $P(\eta_{n,s} \in \Omega_F,\ \forall s \leq t) = 1$, for all t. This is obvious, since the total arrival rate in this process is bounded by $\lambda(2n+1)$.

We can compute the generator of the n-process. Define

$$l^n_\eta(i) := \#\{j \in \mathbb{Z} \cap [-n, n]\colon\ j > i \text{ and for all } i < j' \leq j\colon\ \eta(j') = 1\}$$

and let f be a bounded function on Ω_F, $\eta \in \Omega_F$. The generator of the n-process is given by

$$\begin{aligned}
G_n f(\eta) &= \lim_{t \downarrow 0} \frac{S_n(t) f(\eta) - f(\eta)}{t} \\
&= \sum_i \mathbf{1}_{\{\eta(i)=0\}} \lambda l^n_\eta(i) \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\
&\quad + \sum_i \mathbf{1}_{\{\eta(i)=1\}} \mu \left(f\left(T_i(\eta)\right) - f(\eta)\right) \\
&\quad + \sum_{i=-n}^{n} \mathbf{1}_{\{\eta(i)=0\}} \lambda \left(f\left(T_i(\eta)\right) - f(\eta)\right).
\end{aligned}$$

Step 2. We shall prove that the n-process (with state space Ω_F) is monotone. We show that for $\xi \leq \eta$, the basic coupling (see Lindvall (1992) p. 177) $(\hat{\xi}_{n,t}, \hat{\eta}_{n,t})_{(t\geq 0)}$ of $\xi_{n,t\,(t\geq 0)}$ and $\eta_{n,t\,(t\geq 0)}$ has the property that $\hat{\xi}_{n,t} \leq \hat{\eta}_t$ for all t

with probability 1. In this coupling, we use for both processes the same sequence of Poisson processes and if both processes have a customer at the same site, we let these corresponding customers leave at the same time. This is possible since the exponential distribution has no memory.

Observe that if the starting configurations $\xi, \eta \in \Omega_F$ have the property that if both $\xi \leq \eta$ and $\xi(i) = \eta(i) = 0$, then the flipping rate of $\xi(i)$ is not larger than the flipping rate of $\eta(i)$ since $l^n_\xi(i) \leq l^n_\eta(i)$. Also, if both $\xi \leq \eta$ and $\xi(i) = \eta(i) = 1$, then the flipping rate of $\eta(i)$ is the same as the flipping rate of $\xi(i)$. From this we can conclude that the coupling has the property that for $\xi \leq \eta$, $\hat{\xi}_{n,t} \leq \hat{\eta}_{n,t}$ for all t with probability 1 (see Lindvall (1992), p. 178).

Step 3. Because of the monotonicity of the n-process we can extend the n-process to a process with state space Ω by defining its semigroup (for $f \in \mathcal{M}$) by

$$S_n(t)f(\eta) := \lim_{\xi \in \Omega_F,\ \xi \uparrow \eta} S_n(t)f(\xi).$$

(The fact that $S_n(t)$ is a semigroup follows from the construction). We show that $S_n(t)$ is well defined, that is, we show that the limit of $S_n(t)f(\xi_m)$ is independent of the sequence $(\xi_m)_{m \in \mathbb{N}}$ with elements in Ω_F that increases to η. Suppose that there exist an $\eta \in \Omega$ and two sequences $(\xi_m)_{m \in \mathbb{N}}$ and $(\xi'_m)_{m \in \mathbb{N}}$ with $\xi_m, \xi'_m \in \Omega_F$ for all $m \in \mathbb{N}$, $\xi_m \uparrow \eta$, $\xi'_m \uparrow \eta$ and

$$\lim_{m \to \infty} S_n(t)f(\xi_m) \neq \lim_{m \to \infty} S_n(t)f(\xi'_m).$$

Without loss of generality we may assume that

$$l_2 := \lim_{m \to \infty} S_n(t)f(\xi'_m) > \lim_{m \to \infty} S_n(t)f(\xi_m) =: l_1.$$

Let $\epsilon := \frac{1}{2}(l_2 - l_1)$. Then there exists an $N \in \mathbb{N}$ such that for all $m > N$, $S_n(t)f(\xi_m) \in [l_1 - \epsilon, l_1]$ and there exists an $N' \in \mathbb{N}$ such that for all $m > N'$, $S_n(t)f(\xi'_m) \in [l_2 - \epsilon, l_2]$ Observe that these intervals are disjoint, which implies that for $m > N$ and $m' > N'$

$$S_n(t)f(\xi'_{m'}) > S_n(t)f(\xi_m). \tag{13}$$

Take some number $k' > N'$. Then there exist a $k > N$ with $\xi'_{k'} \leq \xi_k$, so by the monotonicity of the n-process we get that

$$S_n(t)f(\xi'_{k'}) \leq S_n(t)f(\xi_k). \tag{14}$$

(13) and (14) contradict each other, so the assumption that $l_1 \neq l_2$ cannot be right. This implies that $S_n(t)f(\eta)$ is uniquely defined for all $\eta \in \Omega$ and $f \in \mathcal{M}$.

Step 4. To prove that $S_n(t)$ is monotone in n, we show that there is an appropriate coupling of the processes associated to S_n and S_{n+1}. Let $\eta, \xi \in \Omega_F$, let $\xi \leq \eta$ and $f \in \mathcal{M}$.

Again the basic coupling $(\hat{\eta}_{n,t}, \hat{\xi}_{n+1,t})_{(t\geq 0)}$ is a coupling of the processes $\eta_{n,t}{}_{(t\geq 0)}$ and $\xi_{n+1,t}{}_{(t\geq 0)}$ with the property that if $\xi \leq \eta$, then $\hat{\xi}_{n,t} \leq \hat{\eta}_{n+1,t}$ for all t with probability 1. This coupling shows that for $\eta \in \Omega_F$, $f \in \mathcal{M}$ we have that $S_n(t)f(\eta) \leq S_{n+1}(t)f(\eta)$.

Now let $\eta \in \Omega, f \in \mathcal{M}$ and take an increasing sequence ξ_k, $\xi_k \in \Omega_F$ with $\xi_k \uparrow \eta$. We get that for all k, $S_n(t)f(\xi_k) \leq S_{n+1}(t)f(\xi_k)$ and taking the limit $k \to \infty$ yields:

$$S_n(t)f(\eta) \leq S_{n+1}(t)f(\eta).$$

Step 5. By monotonicity of the semigroup $S_n(t)$ in n, we can define for $\eta \in \Omega, f \in \mathcal{M}$

$$S(t)f(\eta) = \lim_{n\to\infty} S_n(t)f(\eta).$$

We know that for all n, $S_n(t)$ is a semigroup on bounded functions on Ω_F. Because of monotonicity this implies that $S(t)$ is also a Markov semigroup on $\mathcal{M}$ (as in Maes et al. (2000)) and we can extend the definition of $S(t)f$ to all bounded Borel measurable functions as described in Liggett (1980). So there exists a unique Markov process η_t such that $S(t)f(\eta) = E^\eta f(\eta_t)$, this process is the SMD.

Acknowledgements We would like to thank Frank Redig for useful comments and discussions.

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