Realizability Toposes and Ordered PCA's

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1 Introduction

Partial combinatory algebras (pca's, for short), are well-known to form the basic ingredient for the construction of various realizability toposes, of which the Effective Topos is undoubtedly the most famous. There is more than one way to present the realizability topos associated to a pca; one may take the exact completion of the category of partitioned assemblies (see [7]), or one can use tripos theory. Triposes built from pca's are, together with those from locales, the most important and most extensively studied instances of triposes, but from a structural point of view, there are important differences between the two; whereas locales are organized in a well-behaved category, which is a reflective subcategory of the category of toposes, it is not immediately clear what an appropriate category for pca's may look like. Moreover, there are various nice properties in the localic case, such as the fact that there is a one-to-one correspondence between maps of locales and geometric morphisms between the corresponding sheaf toposes, and also the fact that this correspondence preserves epi-mono factorizations; such an intimate connection is absent for pca's.

As it turns out, we are in a better position to formulate a reasonable answer to the question how pca's should be organized from a categorical point of view, when we consider a weakening of the notion of a pca. This weakening will enable us to carry out constructions (which are impossible if we restrict ourselves to ordinary pca's) that render a clear connection between maps on the level of our combinatorial structures and on the topos/tripos- theoretic level.

The paper is organized in the following way: we start by defining the notion of an ordered pca, and exhibit some properties of the associated tripos and realizability topos. This section does hardly contain new and/or surprising information. The basic definitions were laid down in [10], and the facts that we mention are often an uncomplicated generalization of well-known properties of the Effective Topos. In section 3, we provide a categorical framework for ordered pca's. We first define an appropriate notion of a morphism of ordered pca's, and then we investigate which morphisms induce geometric morphisms between the triposes. We also exhibit a monad on the category of ordered pca's, an it turns out that the 2-category of realizability triposes over ordered pca's (with geometric morphisms geometric morphisms as arrows) is equivalent (in a suitable 2-categorical sense) to the Kleisli category of this monad. The last part of this section is devoted to a brief study of the occurrence of local localic maps between realizability toposes. This is essentially a translation of the work done in [1] into our own framework.

Finally, section 4 deals with an application to chains of inclusions of realizability toposes. In his thesis [7], Menni discusses hierarchies of toposes of the form $(\mathcal{C}_{reg(n)})_{ex}$, where $\mathcal{C}$ is a category satisfying some conditions, $(-)_{reg(n)}$ denotes the n-fold regular completion, and $(-)_{ex}$ denotes the exact completion. He conjectured, that there should be a tripos-theoretic presentation of this hierarchy, making use of a certain operation on ordered pca's. We explain how this works and show the conjecture to be true.

2 Ordered PCA's

In this section we present a generalization of the notion of a pca, which we will call an ordered partial combinatory algebra, ordered pca for short. We investigate some basic properties of those objects, and some special cases. Then we show that we can easily adapt the construction of a tripos for a pca, so that we can associate a realizability topos with a given ordered pca. We show that a lot of properties of the Effective Topos generalize to these realizability toposes. In particular, such toposes are exact completions.

2.1 Basics

Definition 2.1 An *ordered pca* is a triple $\mathbb{A} = (A, \leq, \bullet)$, where $\leq$ partially orders the set A, and where $\bullet$ is a partial function from $A \times A$ to A. We write $a \bullet b{\downarrow}$ or $ab{\downarrow}$ if (a, b) is in the domain of $\bullet$, in which case $a \bullet b$ or ab denote the value. We require that the following conditions are satisfied:

1. For all $a, b \in A$: if $ab{\downarrow}$, $a' \leq a$ and $b' \leq b$, then $a'b'{\downarrow}$ and $a'b' \leq ab$.
2. There are elements k and s of A that satisfy
   - for all $a, b \in A$: $ka{\downarrow}$ and $kab{\downarrow}$ and $kab \leq a$,
   - for all $a, b, c \in A$: $sa{\downarrow}$ and $sab{\downarrow}$ and if $(ac)(bc){\downarrow}$, then $sabc{\downarrow}$ and $sabc \leq (ac)(bc)$.

The first remark is, that every ordinary pca can be seen as an ordered pca, namely by taking the discrete ordering. The definition of an ordinary pca is motivated by the fact that the combinators k and s ensure that the structure is combinatorially complete. Now it is essential for our purposes that the way we weakened the definition of a pca does not seriously affect this property:

Proposition 2.2 (Combinatorial completeness) *Let $\mathbb{A}$ be an ordered pca. For every term t composed of elements of A, application and the variable x, there is an element $[\Lambda x.t]$ in A, such that for all $a \in A$: if $t[a/x]{\downarrow}$ then $[\Lambda x.t]a{\downarrow}$ and $[\Lambda x.t]a \leq t[a/x]$.*

As was already remarked in [10], the proof is an easy adaptation of the case that $\mathbb{A}$ is an ordinary pca. From this proposition it follows that there are pairing operations, written j, j_0, j_1 that satisfy

$$j_0(j(a, b)) \leq a, \qquad j_1(j(a, b)) \leq b.$$

It is well-known that every pca is either infinite or consists of only one element. (One way of understanding this is to observe first that, using k and s one can construct all the *numerals* $\bar{0}, \bar{1}, \ldots$, and then to remark that these all have to be distinct.) For ordered pca's there are somewhat more possible variations, so let us introduce the following terminology:

Definition 2.3 An ordered pca is called *trivial* if it has a least element, and it is called *pseudo-trivial* if there is an element that serves both as k and as s.

An example of a pseudo-trivial ordered pca that is not trivial is provided by a meet-semilattice (without a least element, of course). We have the following characterization:

Lemma 2.4 *For any ordered pca $\mathbb{A}$ the following statements are equivalent:*

1. *$\mathbb{A}$ is pseudo-trivial,*
2. *there is an element u such that $u \leq k = true$ and $u \leq sk = false$,*
3. *any two elements have a lower bound (not necessarily a meet),*
4. *there are natural numbers n, m such that $n \neq m$, but $\overline{n}$ and $\overline{m}$ have a lower bound ($\overline{n}$ denotes the element that corresponds to n for some coding of the natural numbers).*

Proof. $(1) \Rightarrow (3)$: consider the element $u = skkk = kskk$. We have $skkk \leq kk(kk) \leq k$, but also $kskk \leq sk$. Now $kxy \leq x$, so $(skkk)xy \leq x$. And $skxy \leq y$, so $(kskk)xy \leq y$, and we have found that $(skkk)xy = (kskk)xy = uxy$ is a lower bound of any x and y.

$(2) \Rightarrow (1)$: take u with $u \leq k$ and $u \leq sk$. Then uks is a lower bound for k and s, and this lower bound serves both as k and as s.

$(3) \Rightarrow (1), (2), (4)$ are trivial.

$(4) \Rightarrow (2)$: suppose $m > n$ and $x \leq \overline{m}$ and $x \leq \overline{n}$. We have, by combinatorial completeness, terms $zero$ and $pred$, that test for zero and take the predecessor. To be more precise: $zero \bullet \overline{p} \leq k$ if $p = 0$, and $zero \bullet \overline{p} \leq sk$ if $p \neq 0$, $pred \bullet \overline{p} \leq \overline{p-1}$. Now we find that $zero(pred^n \bullet \overline{m}) \leq sk$ and $zero(pred^n \bullet \overline{m}) \leq k$. So for x this implies $zero(pred^n \bullet x) \leq sk$ and $zero(pred^n \bullet x) \leq k$.

□

2.2 Triposes for ordered pca's

By now, the construction of a tripos, and hence of a realizability topos out of a partial combinatory algebra is standard. (The reference [4] is just as standard.) We give the straightforward generalization to ordered pca's.

So given a ordered pca $\mathbb{A} = (A, \leq, \bullet)$, define $I(\mathbb{A})$ as the set of all *downsets* in A, that is,

$$I(\mathbb{A}) = \{\alpha \subseteq A \mid \forall a \in \alpha, \forall a' \in A(a' \leq a \rightarrow a' \in \alpha)\}.$$

This downset $I(\mathbb{A})$, of course, is just the powerset if the order on $\mathbb{A}$ is discrete.

A downset α is called a *principal downset* iff it is of the form $\alpha = \{a \in A \mid a \leq b\}$ for some element $b \in A$.

Next, consider the following operations, where $\alpha, \beta \in I(\mathbb{A})$:

$$\begin{aligned}
\alpha \times \beta &= \downarrow(\{j(a,b) \mid a \in \alpha, b \in \beta\}) \\
\alpha \Rightarrow \beta &= \{a \in A \mid \forall b \in \alpha : a \bullet b\downarrow \ \& \ a \bullet b \in \beta\} \\
\prod_{x \in X} \alpha_x &= \bigcap_{x \in X} (A \Rightarrow \alpha_x) \\
\sum_{x \in X} \alpha_x &= \bigcup_{x \in X} \alpha_x
\end{aligned}$$

Now we get a preorder on $I(\mathbb{A})^X$ by putting (for $\phi, \psi \in I(\mathbb{A})^X$)

$$\phi \vdash \psi \quad \text{iff} \quad \exists a \in A \ \forall x \in X \ \forall b \in \phi(x) : ab\downarrow \ \& \ ab \in \psi(x)$$

The Heyting algebra operations on this preorder are given by pointwise application of the operations defined above, for example $(\phi \wedge \psi)(x) = \phi(x) \times \psi(x)$. For a function $f : X \to Y$, the quantifiers are given by

$$(\exists_f \phi)(y) = \sum_{f(x)=y} \phi(x), \qquad (\forall_f \phi)(y) = \prod_{f(x)=y} \phi(x)$$

This defines a tripos $I(\mathbb{A})^{(-)}$, and hence a topos, call it $\mathbf{RT}[\mathbb{A}]$.
Remark. This certainly looks like the most natural generalization of the ordinary construction of a tripos from a pca, but in those cases where the following condition is satisfied we have an interesting alternative (see [10]). We say that an ordered pca $\mathbb{A}$ has the *pasting property* iff any two elements a, b that have a lower bound also have a join (this amounts to saying that the underlying poset has pushouts) and if application preserves this join in both variables, i.e. $c(a \vee b) \simeq ca \vee cb$ and $(a \vee b)c \simeq ac \vee bc$. If our ordered pca has this pasting property then we can define $J(\mathbb{A}) \subseteq I(\mathbb{A})$ as those downsets in A that are closed under pushouts. Note that $J(\mathbb{A}) = I(\mathbb{A})$ if the ordering of $\mathbb{A}$ is discrete.

There is an inclusion map $i : J(\mathbb{A}) \hookrightarrow I(\mathbb{A})$, which induces an indexed map of preorders $i : J(\mathbb{A})^X \hookrightarrow I(\mathbb{A})^X$. Left adjoint to this map is composition with the operation Cl_p, which takes a downset to its closure under pushouts. From this it is not hard to establish that there is a geometric inclusion of triposes $J(\mathbb{A})^{(-)} \hookrightarrow I(\mathbb{A})^{(-)}$, and hence an inclusion of toposes (denote the topos represented by the tripos $J(\mathbb{A})^{(-)}$ by $\mathbf{RT}'[\mathbb{A}]$), $\mathbf{RT}'[\mathbb{A}] \hookrightarrow \mathbf{RT}[\mathbb{A}]$.

A tripos equivalent to $J(\mathbb{A})^{(-)}$ was already used by Pitts [8] in order to create a topos for extensional realizability, taking $\mathbb{A}$ to be the nonempty subsets of the natural numbers.
Remark. It is easily seen that $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{Set}$ if $\mathbb{A}$ is trivial. Moreover, if $\mathbb{A}$ is pseudo-trivial, then $\mathbf{RT}[\mathbb{A}]$ is a filter-quotient of the presheaf topos $\mathbf{Set}^{\mathbb{A}^{op}}$ (see [10]).

2.3 Categorical properties

This section contains some straightforward generalizations of well-known facts about the Effective Topos. Since the proofs involve only minor adaptations, we omit them, providing the reader only with the definitions and constructions involved.

Lemma 2.5 *Let* $\mathbf{RT}[\mathbb{A}]$ *refer to the realizability topos introduced in the previous section. Then* $\mathbf{Set} \simeq (\mathbf{RT}[\mathbb{A}])_{\neg\neg}$.

Proof. The geometric inclusion $\Gamma \dashv \nabla$ is defined exactly as for the Effective Topos (see [3]).

□

Definition 2.6 An object $(X, =)$ of $\mathbf{RT}[\mathbb{A}]$ is called *canonically separated* iff the equality on X satisfies $[\![x = x']\!] \neq \emptyset$ implies $x = x'$.

An object is separated for the double negation topology iff it is isomorphic to some canonically separated object. Later on, we shall use the fact, that every separated object embeds into a sheaf.

The next thing we mention is that the topos $\mathbf{RT}[\mathbb{A}]$ is an exact completion. The method used to show this is directly taken from [9]. We now briefly outline this procedure.

Definition 2.7 An object $(X, =)$ of $\mathbf{RT}[\mathbb{A}]$ is called *canonically projective* iff it is canonically separated and $[\![x = x]\!]$ is a *principal downset* for each $x \in X$.

Lemma 2.8 *Every object in* $\mathbf{RT}[\mathbb{A}]$ *can be covered by a canonically projective object.*

Proof. Given $(X, =)$, define $Q = \{(x, a) \mid a \in [\![x = x]\!]\}$, together with

$$[\![(x, a) = (x', a')]\!] = \begin{cases} \{b \mid b \leq a\} & \text{if } \ x = x' \text{ and } \ a = a' \\ \emptyset & \text{otherwise.} \end{cases}$$

This clearly is a canonically projective object. The projection

$$Pr((x, a), x') = \{b \mid b \leq a\} \times [\![x = x']\!]$$

is easily seen to be an epimorphism.

□

Lemma 2.9 *Consider the full subcategory of* $\mathbf{RT}[\mathbb{A}]$ *on the objects that are isomorphic to some canonically projective object. This category is closed under finite limits.*

Lemma 2.10 *The canonically projective objects are, up to isomorphism, precisely the projective objects of* $\mathbf{RT}[\mathbb{A}]$.

From these lemmas it follows that $\mathbf{RT}[\mathbb{A}]$ is the exact completion of its full subcategory on the projectives. We also get the following:

Corollary 2.11 $(\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}]})_{Reg} \simeq \mathbf{Sep}_{\mathbf{RT}[\mathbb{A}]}$.

Proof. Every separated object embeds into a sheaf, and sheaves are projective. Combined with the fact that there are enough projectives and that $\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}]}$ is left exact, we have fulfilled the necessary and sufficient conditions for a regular category to be a regular completion.

□

3 A 2-Category for ordered pca's

In Longley's thesis [6], we find a description of a 2-category of pca's. The definition of a morphism between two pca's is chosen in such a way, that there is a correspondence between such morphisms and certain exact functors between the associated realizability toposes. One could, of course, generalize these constructions to ordered pca's, but we think that some adaptations are desirable. First of all, in Longley's framework, a morphism $\phi : \mathbb{A} \to \mathbb{B}$ of pca's is defined to be a total relation from $\mathbb{A}$ to $\mathbb{B}$, for which there is an element $r \in \mathbb{B}$ such that $\phi(a,b)\ \&\ \phi(a',b')\ \&\ aa'\downarrow \Rightarrow rbb'\downarrow\ \ \&\ \phi(aa', rbb')$. Instead of relations, we rather work with functions, since these are often easier to deal with.

Now the succes of Longley's definition is easily seen to depend crucially on the following theorem by Pitts ([8], section 4.9):

Theorem 3.1 *There is a one-to-one correspondence between*

1. **Set**-*indexed functors from* $P(\mathbb{A})$ *to* $P(\mathbb{B})$ *that preserve* $T, \wedge$ *and* $\exists$ *and*

2. *functions* $f : \mathbb{A} \to P(\mathbb{B})$ *such that* $f(a) \neq \emptyset$ *for all* a*, and moreover* $\bigcap_{a,a' \in Dom(\bullet)} f(a) \to (f(a') \to f(aa')) \neq \emptyset$.

We will also base our definition on this theorem ourselves, but we are more interested in geometric morphisms than in exact functors commuting with $\exists$, so an important part of our approach will be a characterization of those functions between ordered pca's that induce geometric morphisms between the realizability toposes.

3.1 The category OPCA

We first present a suitable category for ordered pca's, that is, as said before, both an adaptation and a generalization of Longley's 2-category for pca's. The objects are, of course, ordered pca's. For morphisms, we introduce the following definition:

Definition 3.2 Let $\mathbb{A}$ and $\mathbb{B}$ be ordered pca's, and let $f : \mathbb{A} \to \mathbb{B}$ be a function. We say that f is a *morphism of ordered pca's* if:

- f is order-preserving,
- there exists an element $r \in \mathbb{B}$ such that $aa'\downarrow \Rightarrow (r \bullet f(a)) \bullet f(a')\downarrow$ and $(r \bullet f(a)) \bullet f(a') \leq f(aa')$.

We will refer to the element r in the definition as the witness of f. It is easily verified that if r witnesses $f : \mathbb{A} \to \mathbb{B}$ and r' witnesses $g : \mathbb{B} \to \mathbb{C}$ then $g \circ f$ has a witness $\lambda uv.r' \bullet (r' \bullet g(r) \bullet u) \bullet v$. Therefore, composition is well-defined. We will write **OPCA** for this category.

Next, we observe that the Hom-sets of this category are pre-ordered sets if we define, for $f, g : \mathbb{A} \to \mathbb{B} : f \leq g$ iff $\exists b \in \mathbb{B} : b \bullet f(a)\downarrow\ \ \&\ b \bullet f(a) \leq g(a)$

for all $a \in \mathbb{A}$. Since composition of morphisms preserves this ordering, in the sense that $f \leq g \Rightarrow fh \leq gh$ and $kf \leq kg$, we see that **OPCA** is a pre-order enriched category. We write $f \sim g$ for $f \leq g$ & $g \leq f$, and we say that f and g are *equivalent* as morphisms.

It is good to observe that a map $f : \mathbb{A} \to \mathbb{B}$ provides us with a description of $\mathbb{A}$ as an *internal* ordered pca in the topos $\mathbf{RT}[\mathbb{B}]$. The underlying set of this (canonically projective) object is the underlying set of $\mathbb{A}$, and the existence predicate is given by $E_f(a) = {\downarrow}(f(a))$. Moreover, if we have $f, g : \mathbb{A} \to \mathbb{B}$, then $f \leq g$ iff, internally in $\mathbf{RT}[\mathbb{B}]$, the identity on $\mathbb{A}$ is a map $(\mathbb{A}, E_f) \to (\mathbb{A}, E_g)$.

Remarks. The structure of the category **OPCA** is not particularly impressive. We mention the following:

1. (This was already observed by Longley.) The terminal object in **OPCA** is the one-point ordered pca. For any other trivial $\mathbb{A}$, there is, for any $\mathbb{B}$, always a morphism $f : \mathbb{B} \to \mathbb{A}$. This f is unique up to equivalence of morphisms. Trivial ordered pca's are also pseudo-initial, in the sense that for any other ordered pca $\mathbb{B}$, there is always a map into $\mathbb{B}$, and any two such maps are equivalent. To see why this must be the case, suppose that $\mathbb{A}$ has least element $\perp$, and consider two maps $f, g : \mathbb{A} \to \mathbb{B}$. Then for all $a \in \mathbb{A}$ we see that $(\lambda x.g(\perp)) \bullet f(a) \leq g(a)$, and $(\lambda x.f(\perp)) \bullet g(a) \leq f(a)$.

   Apart from this, we can observe that any constant function between ordered pca's is a morphism, and that any two constant maps are equivalent.

2. The category **OPCA** has products: given $\mathbb{A}$ and $\mathbb{B}$, we define $\mathbb{A} \times \mathbb{B}$ as $\mathbb{A} \times \mathbb{B} = (A \times B, \bullet, \leq)$ with $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$, $(a, b) \bullet (a', b'){\downarrow}$ iff $aa'{\downarrow}$ and $bb'{\downarrow}$, in which case $(a, b) \bullet (a', b') = (aa', bb')$. The pairs $(k_A, k_B), (s_A, s_B)$ serve as k and s in the product.

3. Monos and epis are just injective and surjective maps, respectively. For, consider a map $f : \mathbb{A} \to \mathbb{B}$ that is not injective, say $f(a) = f(a')$. Then we take two (different) maps $1 \to \mathbb{A}$ sending the unique element to a and a', respectively. Their composites with f are obviously equal.

   If $f : \mathbb{A} \to \mathbb{B}$ is not surjective, then there is some element $b_0 \in \mathbb{B}$ that is outside the image of f. Consider the trivial structure $\mathbb{P}$ consisting of two elements p, q with $p \leq q$. Now define maps $g, h : \mathbb{B} \to \mathbb{P}$ by

$$g(b) = \begin{cases} q & \text{if } b_0 < b \\ p & \text{otherwise,} \end{cases} \qquad h(b) = \begin{cases} q & \text{if } b_0 \leq b \\ p & \text{otherwise.} \end{cases}$$

   It is not hard to verify that these are indeed morphisms in our category, and that $gf = hf$, but not $g = h$.

4. If $f : \mathbb{A} \to \mathbb{B}$ is any map, witnessed by some $r \in \mathbb{B}$, then we define a structure $Im(f) = ({\downarrow}(Im(f)), \leq, \bullet')$, by taking $\leq$ to be the ordering on $\mathbb{B}$ restricted to the set ${\downarrow}(Im(f)) = \{b \in \mathbb{B} \mid \exists a \in \mathbb{A} : b \leq f(a)\}$, and by defining

$$b \bullet' b' \simeq r \bullet b \bullet b'$$

for any two elements $b, b' \in {\downarrow}(Im(f))$. Now f factors as a map $e : \mathbb{A} \to Im(f)$ followed by $m : Im(f) \to \mathbb{B}$. The map m is necessarily injective, and hence mono. Unfortunately, the map e is in general not surjective, so this procedure need not yield an epi-mono factorization of f. If, however, the set $Im(f)$ already is downwards closed in $\mathbb{B}$, then e is an epimorphism.

5. Equalizers do not exist in **OPCA**. The reason is simple: if we have two structures $\mathbb{A}$, $\mathbb{B}$, then we can take two different constant maps. There equalizer would have to have the empty set as underlying set, but no such ordered pca exists.

6. (Generalizing an old theorem by Pitts:) As said before, a map $f : \mathbb{A} \to \mathbb{B}$ gives a way to consider $\mathbb{A}$ as an internal ordered pca in the topos $\mathbf{RT}[\mathbb{B}]$. This gives rise to a $\mathbf{RT}[\mathbb{B}]$- tripos, call it $P_{\mathbb{A}}$, in the evident way, so we obtain another topos $\mathbf{RT}[\mathbb{B}](P_{\mathbb{A}})$. Pitts showed that, if $\mathbb{A} = \mathbb{N}$, the natural number object in $\mathbf{RT}[\mathbb{B}]$, then the topos $\mathbf{RT}[\mathbb{B}](P_{\mathbb{A}})$ also comes from an ordered pca. It turns out that this is true for arbitrary $\mathbb{A}$. Given $f : \mathbb{A} \to \mathbb{B}$, we construct a new ordered pca, called $\mathbb{A} \times_f \mathbb{B}$. The underlying partial order is the same as that of $\mathbb{A} \times \mathbb{B}$, but application is defined as:

$$(a, b) \bullet_f (a', b') \simeq (aa', b \bullet j(f(a'), b')).$$

Now there is a projection map $\pi_{\mathbb{A}} : \mathbb{A} \times_f \mathbb{B} \to \mathbb{A}$. For every fixed $b_0 \in \mathbb{B}$, the map $a \mapsto (a, b_0)$ is a section of $\pi_{\mathbb{A}}$. For the proof that the topos $\mathbf{RT}[\mathbb{B}](P_{\mathbb{A}})$ is equivalent to $\mathbf{RT}[\mathbb{A} \times_f \mathbb{B}]$, we refer to [8]. There is no difficulty in adapting the proof presented there to the more general case.

3.2 The Downset-monad

Now we describe a monad $(I^*, \delta, \cup)$ on **OPCA**. On objects, we define

$$I^*\mathbb{A} = (\{\alpha \mid \alpha \in I\mathbb{A}, \alpha \neq \emptyset\}, \subseteq, \bullet).$$

So the underlying set of $I^*\mathbb{A}$ consists of all nonempty downsets in $\mathbb{A}$. It is ordered by inclusion, and partial application is defined by $\alpha \bullet \beta{\downarrow}$ iff $\forall a \in \alpha\ \forall b \in \beta\ ab{\downarrow}$, and if $\alpha \bullet \beta{\downarrow}$ then $\alpha \bullet \beta = {\downarrow}\{ab \mid a \in \alpha, b \in \beta\}$. It is not hard to verify that this gives again a ordered pca, with ${\downarrow}(k)$ and ${\downarrow}(s)$ serving as combinators. Also, there is a map $\delta : \mathbb{A} \to I^*\mathbb{A}$, given by $\delta(a) = {\downarrow}(a)$.

For a morphism $f : \mathbb{A} \to \mathbb{B}$, we put $I^*f(\alpha) = \bigcup_{a \in \alpha} {\downarrow}(f(a))$. If f has a witness r, then I^*f has ${\downarrow}(r)$ as a witness, and if $\alpha \subseteq \alpha'$ then also $I^*f(\alpha) \subseteq I^*f(\alpha')$, so I^*f is a morphism. Finally, it is clear that composition and identities are preserved, so I^* is indeed an endofunctor. Actually, it is an endo-2-functor, since it preserves (and reflects) the ordering on morphisms.

Now let $\cup : I^*I^*\mathbb{A} \to I^*\mathbb{A}$ be the map given by union: $\cup\xi = \{a \in \mathbb{A} \mid \exists \alpha \in \xi : a \in \alpha\}$. The verifications that both δ and $\cup$ are natural transformations, and that the monad identities are satisfied are left to the reader.

We observe in passing, that if $f : I^*\mathbb{A} \to \mathbb{A}$ is an algebra for this monad then f is necessarily the supremum map (so $\mathbb{A}$ has suprema of all nonempty subsets). Conversely an opca $\mathbb{A}$ admitting all nonempty suprema is an algebra if there is some $a \in \mathbb{A}$ such that for all $\alpha, \alpha' \in I^*\mathbb{A} : \alpha\alpha'\downarrow \Rightarrow a \bullet \bigvee \alpha \bullet \bigvee \alpha'\downarrow$ & $a \bullet \bigvee \alpha \bullet \bigvee \alpha' \leq \bigvee \alpha\alpha'$.

The theorem by Pitts that we stated at the beginning of this section can now be strengthened as follows: let $\mathbf{Kl(I^*)}$ denote the Kleisli category for the monad $(I^*, \delta, \cup)$ (this is a 2-category, since the pre-ordering of the arrows is inherited from **OPCA**). Let **RTripExact** denote the 2-category of realizability triposes of the form $I(\mathbb{A})^{(-)}$, with exact functors as arrows, and natural transformations pre-ordering those exact functors. Then we obtain:

Theorem 3.3 *Every map $f : \mathbb{A} \to I^*\mathbb{B}$ induces a Set-indexed functor from $I(\mathbb{A})^{(-)}$ to $I(\mathbb{B})^{(-)}$, that commutes with $\wedge, \top$ and $\exists$. Moreover, every such Set-indexed functor is, up to isomorphism, induced by a map $f : \mathbb{A} \to I^*\mathbb{B}$. Hence we have a 2-functor from the Kleisli category $\mathbf{Kl(I^*)}$ to* **RTripExact**. *This 2-functor is full, and faithful up to isomorphism.*

Proof. Given $f : \mathbb{A} \to I^*\mathbb{B}$, define $\bar{f}(\alpha) = \bigcup_{a \in \alpha} f(a)$. Conversely, take $\phi : I(\mathbb{A}) \to I(\mathbb{B})$ with the mentioned properties. By Pitts' theorem it follows that there is a map $\lambda : \mathbb{A} \to I^*\mathbb{B}$ such that ϕ is naturally isomorphic to $\bar{\lambda}$, and $\bigcap_{a,a' \in Dom(\bullet)} \lambda(a) \to (\lambda(a') \to \lambda(aa')) \neq \emptyset$. This map λ does not necessarily preserve the ordering (i.e. $a' \leq a \Rightarrow \lambda(a') \subseteq \lambda(a)$ need not hold), but it does so up to a realizer: consider the object $X = \{(a', a) | a' \leq a\}$, and the two projections $\pi_1, \pi_2 \in I(\mathbb{A})^X$. Clearly $\pi_1 \vdash \pi_2$. Hence also $\lambda \circ \pi_1 \vdash \lambda \circ \pi_2$, so there is a realizer $c \in \bigcap_{a' \leq a}(\lambda(a') \to \lambda(a))$. Now we define $\lambda'(a) = \bigcup_{a' \leq a} \lambda(a')$. It is now easily seen that λ' is a morphism of ordered pca's, and that the map $\overline{\lambda'}$ is naturally isomorphic to $\bar{\lambda}$.

□

This theorem shows, in effect, that our approach is an extension of Longley's, because Longley's 2-category of pca's is a subcategory of $\mathbf{KL}(I^*)$.

A final observation for this section: just as a map $f : \mathbb{A} \to \mathbb{B}$ presents $\mathbb{A}$ as a projective internal ordered pca in $\mathbf{RT}[\mathbb{B}]$, a map $g : \mathbb{A} \to I^*\mathbb{B}$ presents $\mathbb{A}$ as a *separated* internal ordered pca in $\mathbf{RT}[\mathbb{B}]$.

3.3 Geometric Morphisms

For reasons that are about to become transparent, we now concentrate on morphisms $f : \mathbb{B} \to \mathbb{A}$ that satisfy the following property:

$$\forall a \in \mathbb{A} \exists b \in \mathbb{B} \forall b' \in \mathbb{B} : a \bullet f(b')\downarrow \Rightarrow bb'\downarrow \;\; \& \; f(bb') \leq a \bullet f(b') \qquad (\dagger)$$

It is evident that the composition of two maps satisfying (†) is again such a map, and that the identity map also has (†), so that we can form the subcategory **OPCA**†. Moreover, the structure maps of the monad δ and $\cup$ both satisfy (†), and if f satisfies (†), then so does I^*f. Therefore, the monad $(I^*, \delta, \cup)$ restricts to a monad on **OPCA**†. From now on, we will only be interested in the monad on the smaller category **OPCA†**, so when we refer to the monad $I^*, \delta, \cup$, or to anything related to this monad, we always mean the monad on **OPCA†**.

Let us now explain what the relevance of this property (†) is. Consider a morphism $f : \mathbb{B} \to I^*\mathbb{A}$ in **OPCA†**. First we will show that this induces a geometric morphism of triposes:

$$I(\mathbb{A}) \underset{f^{-1}}{\overset{\bar{f}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} I(\mathbb{B})$$

where the arrows $\bar{f}$ and f^{-1} are defined as

$$\bar{f}(\beta) = \bigcup_{b \in \beta} f(b), \qquad f^{-1}(\alpha) = \{b \in \mathbb{B} \mid f(a) \subseteq \alpha\}.$$

First, let us see why $\bar{f}$ is order-preserving (now, of course, we refer to the pre-order on $I(\mathbb{B})$, considered as a Heyting pre-algebra). Suppose $b \in \phi \to \psi$, for $\phi, \psi \in I(\mathbb{B})$. The morphism f has a witness, say r, satisfying $aa'\downarrow \Rightarrow r \bullet f(a) \bullet f(a')\downarrow \;\; \& \; r \bullet f(a) \bullet f(a') \subseteq f(aa')$. Now take any element $d \in f(b)$. Then $\lambda x.r \bullet d \bullet x \in \bar{f}(\phi) \to \bar{f}(\psi)$.

Second, $\bar{f}(\alpha \wedge \beta) = \cup\{f(j(a,b)) \mid a \in \alpha, b \in \beta\}$, whereas $\bar{f}(\alpha) \wedge \bar{f}(\beta) = \cup\{f(a) \mid a \in \alpha\} \wedge \cup\{f(b) \mid b \in \beta\}$. If we are given $x \in f(j(a,b))$, then we use that $r \bullet f(j_0) \bullet f(j(a,b)) \subseteq f(a)$ and $r \bullet f(j_1) \bullet f(j(a,b)) \subseteq f(b)$ and pairing to find an element of $f(a) \wedge f(b)$. Conversely, given $y \in f(a), z \in f(b)$, use the fact that $r \bullet (r \bullet f(j) \bullet f(a)) \bullet f(b) \subseteq r \bullet f(ja) \bullet f(b) \subseteq f(j(a,b))$, to obtain an element in $f(j(a,b))$. Combined with the fact that $\bar{f}$ preserves the top element (trivial) we have shown that finite meets are preserved.

Third, f^{-1} is order-preserving. Suppose $a \in \phi \to \psi$. Use (†) to find $b \in \mathbb{B}$ with $\forall b' \in \mathbb{B} : \downarrow(a) \bullet f(b')\downarrow \Rightarrow bb'\downarrow \;\; \& \; f(bb') \subseteq \downarrow(a) \bullet f(b')$. This b realises $f^{-1}(\phi) \to f^{-1}(\psi)$, since $f(b') \subseteq \phi \Rightarrow \downarrow(a) \bullet f(b')\downarrow$, so $bb'\downarrow \;\; \& \; f(bb') \subseteq \downarrow(a) \bullet f(b') \subseteq \psi$.

Finally, we have $\bar{f} \vdash f^{-1}$. The verification of this fact goes along the same lines as that of the previous facts. This completes the proof of the claim that we have an induced geometric morphism of triposes. Note in particular that for any map $g : \mathbb{B} \to \mathbb{A}$ in **OPCA†**, composition with the structure map $D : \mathbb{A} \to I^*\mathbb{A}$ of the monad induces a geometric morphism.

The next step is to show, that, up to isomorphism, any geometric morphism of realizability triposes is induced by a morphism in **OPCA†**.

Lemma 3.4 *Suppose we have a geometric morphism*

$$I(\mathbb{A}) \underset{f^*}{\overset{f_*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} I(\mathbb{B}).$$

Then there is a map $f : \mathbb{B} \to I^*\mathbb{A}$ such that $\bar{f} \dashv\vdash f^*, f^{-1} \dashv\vdash f_*$.

Proof. As has already been shown by Pitts, putting $f(b) = f^*(\downarrow(b))$ is the only choice we have, since this gives $f^*(\beta) \dashv\vdash \bigcup_{b\in\beta} f(b) = \bar{f}(\beta)$, because f^*, as a left adjoint, preserves unions.

We know that $\bar{f}f^{-1}(\alpha) \subseteq \alpha$ and $\beta \subseteq f^{-1}\bar{f}(\beta)$. So we get $f^{-1}(\alpha) \vdash f_*(\alpha)$. Also, we find $f_*(\alpha) \vdash f^{-1}\bar{f}f_*(\alpha) \vdash f^{-1}(\alpha)$, hence $f^{-1} \dashv\vdash f_*$.

Next, we show that this f is a morphism in **OPCA**†. Suppose that it isn't, that is, there is $\alpha \in I^*\mathbb{A}$ for which we have

$$\forall b \in \mathbb{B} \exists b' \in \mathbb{B} : \alpha \bullet f(b')\downarrow \;\; \& \; \neg(b \bullet b'\downarrow \;\; \& \; f(bb') \subseteq \alpha \bullet f(b').$$

We may take a choice function $k : \mathbb{B} \to \mathbb{B}$, that satisfies

$$\forall b \in \mathbb{B} : \alpha \bullet f(k(b))\downarrow \;\; \& \; \neg(b \bullet k(b)\downarrow \;\; \& \; f(b \bullet k(b)) \subseteq \alpha \bullet f(k(b)).$$

Now define $D_\alpha = \{b \in \mathbb{B} \mid \alpha \bullet f(b)\downarrow\}$. Consider the functions $\phi, \psi : D_\alpha \to I\mathbb{A}$, given by $\phi(b) = f(b)$, $\psi(b) = \alpha \bullet f(b)$. Clearly, we have that any $a \in \alpha$ satisfies $a \in \bigcap_{b\in D_\alpha} \phi(b) \to \psi(b)$. Now f^{-1} preserves the ordering, from which it follows that there is an element $x \in \bigcap_{b\in D_\alpha} f^{-1}\phi(b) \to f^{-1}\psi(b)$. We find in particular that, taking $b = k(x)$, $\forall y \in \mathbb{B} : f(y) \subseteq f(k(x)) \Rightarrow xy\downarrow \;\; \& \; f(xy) \subseteq \alpha \bullet f(k(x))$. If we take $y = g(x)$ we obtain a contradiction.

□

This establishes, that geometric morphisms $I(\mathbb{B})^{(-)} \to I(\mathbb{A})^{(-)}$, are, up to isomorphism, the same as ordered pca morphisms $\mathbb{A} \to I^*\mathbb{B}$ that satisfy the (†) property. But the latter are precisely the morphisms from $\mathbb{A}$ to $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(I^*)$ for the monad on **OPCA**†.

Let **RTrip** denote the 2-category with as objects triposes of the form $I(\mathbb{A})^{(-)}$ for some ordered pca $\mathbb{A}$, and as arrows geometric morphisms of triposes. For two geometric morphisms $(f^*, f_*), (g^*, g_*)$ from $I(\mathbb{B})^{(-)}$ to $I(\mathbb{A})^{(-)}$, we say that $(f^*, f_*) \leq (g^*, g_*)$ iff for every set X and any $\phi : X \to I\mathbb{A}$, $f^*\phi \vdash g^*\phi$. This makes **RTrip** into a preorder-enriched category. Moreover, let **RTop** be the 2-category of toposes of the form $\mathbf{RT}[\mathbb{A}]$ for some ordered pca $\mathbb{A}$, with geometric morphisms commuting with the inclusion of **Set**, and natural transformations between them. It is known that these categories are equivalent when we forget about the 2-categorical structure. The following lemma shows that there is also a correspondence between natural transformations on the tripos-level and on the topos-level.

Lemma 3.5 *Let $\mathbb{A}, \mathbb{B}$ be ordered pca's, an let $f, g : \mathbb{A} \to I^*\mathbb{B}$ be two maps in* **OPCA**†. *Then $\bar{f} \leq \bar{g}$ in* **RTrip** *iff there is a (necessarily unique) natural transformation $\eta : \bar{f} \to \bar{g}$ in* **RTop**.

Proof. First assume that $\bar{f} \leq \bar{g}$ in **RTrip**. This amounts to the existence of an element $b \in \mathbb{B}$ with the property that for all $a \in \mathbb{A} : b \bullet f(a)\downarrow \;\; \& \; b \bullet f(a) \subseteq g(a)$. Now consider an object $(X, =)$ in $\mathbf{RT}[\mathbb{B}]$. The functor $\bar{f}$ sends this object to $(X, =_1)$ with $[\![x =_f x']\!] = \bar{f}([\![x = x']\!])$, and similarly, $\bar{g}$ sends it to $(X, =_g)$. Define $\eta_X(x_1, x_2) = [\![x_1 =_f x_2]\!]$. This is easily seen to represent a map from $(X, =_f)$ to $(X, =_g)$. The η_X form a natural transformation from $\bar{f}$ to $\bar{g}$.

Next, assume that $\eta_X : (X, =_f) \to (X, =_g)$ is a natural transformation. Taking $(X, =)$ to be the separated object $(I^*\mathbb{B}, \exists_\delta(Id))$, we find that $\bar{f} \leq \bar{g}$ in **RTrip**.

Finally, suppose that $\theta_X, \eta_X : (X, =_f) \to (X, =_g)$ are both natural transformations. Observe that if an object $(Q, =)$ is separated, then $\eta_Q : (Q, =_f) \to (Q, =_g)$ is uniquely determined by $=_f$, and so $\theta_Q = \eta_Q$. Recall that $(X, =)$ can always be covered by a separated object $(Q, =)$, so that we have the following commutative diagram in $\mathbf{RT}[\mathbb{A}]$:

$$\begin{array}{ccc} (Q, =_f) & \xrightarrow{\theta_Q = \eta_Q} & (Q, =_g) \\ \downarrow & & \downarrow \\ (X, =_f) & \overset{\theta_X}{\underset{\eta_X}{\rightrightarrows}} & (X, =_g). \end{array}$$

Because the cover is an epimorphism, it follows that $\theta_X = \eta_X$.

□

Now we relate the preorder on Hom-sets in **OPCA**† to the one on the Hom-Sets in **RTrip**.

Lemma 3.6 *Let $f, g : \mathbb{A} \to I^*\mathbb{B}$ be two maps in* **OPCA**†*, inducing two geometric morphisms of triposes, $(\bar{f}, f^{-1})$ and $(\bar{g}, g^{-1})$. Then $f \leq g$ iff $(\bar{f}, f^{-1}) \leq (\bar{g}, g^{-1})$.*

Proof. If $f \leq g$ then there is an element $b \in \mathbb{B}$ with the property that $b \in \bigcap_{a \in \mathbb{A}} f(a) \to g(a)$. This implies that $b \in \bigcap_{\alpha \in I\mathbb{A}} \bar{f}(\alpha) \to \bar{g}(\alpha)$. Therefore $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$.

Conversely, assume $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$. In particular, taking X to be $\mathbb{A}$ and $\phi(a) = \downarrow(a)$, we find $\bar{f}(\phi)(a) = f(a)$, $\bar{g}(\phi)(a) = g(a)$, and there is an element $b \in \mathbb{B}$ such that $b \in \bigcap_{a \in \mathbb{A}} f(a) \to g(a)$, proving $f \leq g$.

□

We can wrap up by saying that there is a 2-functor from the Kleisli 2-category $\mathbf{Kl}(I^*)$ to the 2-category **RTrip** of realizability triposes. This functor is full, essentially surjective on objects and yields, for any pair of ordered pca's $\mathbb{A}$, $\mathbb{B}$, an equivalence of categories $Hom(\mathbb{A}, \mathbb{B}) \simeq Hom(I(\mathbb{B})^{(-)}, I(\mathbb{A})^{(-)})$, where the first pre-ordered Hom-set is taken in the Kleisli category, and the second in **RTrip**.

From this we obtain that there is, up to isomorphism, a one-one correspondence between maps from $\mathbb{A}$ to $\mathbb{B}$ in $\mathbf{Kl}(I^*)$, and geometric morphisms from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$ over **Set**.

3.4 Local maps

Let $\mathbb{B}$ be some pca and let $\mathbb{A}$ be a sub-pca of $\mathbb{B}$, that is, $\mathbb{A}$ is a subset containing k and s that is closed under the partial application. In [1] the toposes $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[\mathbb{B}]$ are compared. In the previous section we saw that a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$ is, up to isomorphism, the same as a map $f : \mathbb{A} \to I^*\mathbb{B}$ that satisfies the property (†). Note, however, that this property implies that $\forall b \in \mathbb{B} \exists a \in \mathbb{A} : f(a) \leq b$. Now for ordinary pca's this requirement reduces to surjectivity of the map f, and from this it readily follows that there will never be a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$, except for the trivial case where $\mathbb{A} = \mathbb{B}$. There is, however, a topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$, called the *relative realizability topos*, that has the property that there is a local localic geometric morphism $\mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{A}]$, and a logical functor $L : \mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{B}]$. (For more on local maps we refer to [5].) In a picture:

$$\mathbf{RT}[\mathbb{A}] \; \begin{array}{c} \xrightarrow{\quad \bar{i} \quad} \\ \perp \\ \xleftarrow{\quad i^{-1} \quad} \\ \perp \\ \xrightarrow[\quad i_* \quad]{} \end{array} \; \mathbf{RT}[\mathbb{B}, \mathbb{A}] \xrightarrow{\;L\;} \mathbf{RT}[\mathbb{B}]$$

The intermediate topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$ is constructed by taking the tripos $I(\mathbb{B})^{(-)}$ and taking the following preorder: $\phi \vdash' \psi$ iff $\exists a \in \mathbb{A} : a \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. (All the other structure is exactly as in the tripos $I(\mathbb{B})^{(-)}$.) Now the maps $\bar{i}, i_*$ and i^{-1} are defined on the tripos-level, as follows (for $\phi : X \to I(\mathbb{A}), \psi : X \to \mathbb{B}$):

$$\bar{i}(\phi)(x) = \downarrow(\phi(x)), \qquad i^{-1}(\psi)(x) = \psi(x) \cap \mathbb{A},$$

$$i_*(\phi)(x) = \bigcup_{\alpha \in I(\mathbb{B})} (\alpha \wedge (\mathbb{A} \cap \alpha \to \downarrow(\phi(x)))).$$

Remarks.

1. First of all, we have given this definition in such a way, that it also applies to ordered pca's. It is completely straightforward to check that this still gives a local geometric morphism: one can copy the proof of theorem 3.1 in [1] almost literally.

2. Second, note that the functors $\bar{i}$ and i^{-1} are precisely the maps that are induced by the inclusion $\mathbb{A} \hookrightarrow \mathbb{B} \hookrightarrow I^*\mathbb{B}$ as in the previous section.

3. We also mention that the counit of the adjunction $i^{-1} \dashv i_*$ is an isomorphism, just as the unit of $\bar{i} \dashv i^{-1}$ is, so that $\mathcal{T}[\mathbb{A}]$ is actually a retract of $\mathcal{T}[\mathbb{B}, \mathbb{A}]$.

Now for our purposes it will be interesting to know when the functor L is an equivalence.

Proposition 3.7 *If $\mathbb{A}$ is a sub-ordered pca of $\mathbb{B}$, the functor L is an equivalence if and only if $\forall b \in \mathbb{B} \exists a \in \mathbb{A} \forall b' \in \mathbb{B} : bb'\downarrow \Rightarrow i(a) \bullet b'\downarrow$ and $i(a) \bullet b' \leq bb'$.*

Note that this requirement is actually a strengthening of saying that the inclusion i satisfies (†).

Proof. ($\Rightarrow$:) If L is an equivalence, then i^{-1}, considered as a map from $I(\mathbb{B})^X$ to $I(\mathbb{A})^X$ is order-preserving, from which it follows that $\forall b \in \mathbb{B} \exists a \in \mathbb{A} : i(a) \leq b$. This implies the condition, by definition of ordered pca.

($\Leftarrow$:) Take $\phi, \psi : X \to I(\mathbb{B})$, and assume that we have $b \in \mathbb{B}$ with $b \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. We must show that $\exists a \in \mathbb{A} : i(a) \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. Pick $a \in \mathbb{A}$ as in the proposition. If $b' \in \phi(x)$, then $bb' \downarrow$, therefore $i(a) \bullet b' \downarrow$. Moreover $bb' \in \psi(x)$, and $i(a) \bullet b' \leq bb'$, so $i(a) \bullet b' \in \psi(x)$, because $\psi(x)$ is downward closed.

□

Remarks.

1. In our opinion, this proposition can be taken as providing some evidence for the claim that ordered pca's really are a useful generalization of ordinary pca's, because it shows us that there are non-trivial inclusions of ordered pca's that induce topos morphisms, something that is impossible for pca's.

2. If we have such a local localic map, induced by an inclusion $\mathbb{A} \hookrightarrow \mathbb{B}$ of ordered pca's, then it follows that $\mathbb{A}$ is actually a retract of $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(I^*)$. The converse need not be true.

3. We said before, that an inclusion of ordinary pca's would never yield a geometric morphism between the associated realizability toposes. It must be stressed, however, that the proof of this fact relies on classical logic, and does not remain true when we switch to an arbitrary base topos instead of **Set**. In fact, in [2] the notion of an *elementary subobject* is introduced. This definition is chosen in such a way, that if $\mathbb{B}$ is now a pca-object in an arbitrary topos $\mathcal{S}$, and $\mathbb{A}$ is a sub-pca of $\mathbb{B}$, then the requirement that $\mathbb{A}$ is an elementary subobject (rather than the maximal subobject) of $\mathbb{B}$ is enough to guarantee that there is a local map between the realizability toposes.

4 Application

In this section we study iteration of the endofunctor I^*. This gives rise to a sequence of ordered pca's, and, as we will see, to a sequence of the corresponding realizability toposes. It turns out, that these are not merely inclusions, but in fact local maps of toposes. It was already predicted by Menni that certain chains of realizability toposes could be obtained in this fashion.

4.1

Let us fix a ordered pca $\mathbb{A}$. In the category **OPCA**†, we have a diagram

$$\mathbb{A} \xrightarrow{\delta} I^*\mathbb{A} \xrightarrow{\delta} I^*I^*\mathbb{A} \xrightarrow{\cup} I^*\mathbb{A}$$

This composition equals the map $\delta : \mathbb{A} \to I^*\mathbb{A}$ (this is one of the monad identities), so in the category $Kl(I*)$, $\mathbb{A}$ is a retract of $I^*\mathbb{A}$. Now the inclusion of $\mathbb{A}$ in $I^*\mathbb{A}$ is easily seen to satisfy the condition of proposition (3.5) of the previous section. This means that there is an induced local localic geometric morphism. On the tripos level, it looks like this:

$$I(\mathbb{A}) \;\begin{array}{c} \xrightarrow{\quad D \quad} \\ \bot \\ \xleftarrow{\quad P \quad} \\ \bot \\ \xrightarrow[\quad U \quad]{} \end{array}\; I(I^*\mathbb{A}).$$

Let us give a direct description of the functors in this diagram (take $\alpha \in I(\mathbb{A})$ and $\xi \in I(I^*\mathbb{A})$):

$$D(\alpha) = {\downarrow}(\{ {\downarrow}(a) \mid a \in \alpha\}), \qquad P(\alpha) = {\downarrow}(\alpha),$$

$$U(\xi) = \bigcup_{\alpha \in \xi} \{a \mid a \in \alpha\}.$$

We used the notation U, D, and P as to remind the reader of the words "union", "discrete" and "principal", respectively.

On the level of toposes, we get the following, similar picture:

$$\mathbf{RT}[\mathbb{A}] \;\begin{array}{c} \xrightarrow{\quad D \quad} \\ \bot \\ \xleftarrow{\quad P \quad} \\ \bot \\ \xrightarrow[\quad U \quad]{} \end{array}\; \mathbf{RT}[I^*\mathbb{A}].$$

4.2

The next thing we wish to establish is a relation between the separated objects in $\mathbf{RT}[\mathbb{A}]$ and the projective objects in $\mathbf{RT}[I^*\mathbb{A}]$.

Theorem 4.1 *There is an equivalence of categories* $\mathbf{Proj}_{\mathbf{RT}[I^*\mathbb{A}]} \simeq \mathbf{Sep}_{\mathbf{RT}[\mathbb{A}]}$.

Proof. We have already seen that the maps U and P have the property that for principal downsets $\alpha \in I(I^*\mathbb{A}), UP(\alpha) = \alpha$, and that for an arbitraty downset $\beta \in I(\mathbb{A})$, $PU(\beta) = \beta$. Now for separated objects, the direct image functor P can be described by $P(X, =) = (X, P(=)$ (in general, the direct image functor that comes from a tripos morphism is not so easily described). This description establishes at once a one-one correspondence between the canonically separated objects of $\mathbf{RT}[\mathbb{A}]$ and the canonically projective objects of $\mathbf{RT}[I^*\mathbb{A}]$.

Now a morphism between separated objects can be viewed as a function that is realized by some element. That is, a function $f : X \to Y$ represents a morphism in $\mathbf{Sep}_{\mathbf{RT}[\mathbb{A}]}$ iff there exists an element $a \in A$ such that for each $x \in X, \forall d \in E(x) : a \bullet d\downarrow$ & $a \bullet d \in E(f(x))$. Similarly, a function $f : X \to Y$ represents a morphism in $\mathbf{Proj}_{\mathbf{RT}[I^*\mathbb{A}]}$ iff there is some $\alpha \in A'$ such that for all $x \in X, \alpha \bullet' \alpha_x\downarrow$ & $\alpha \bullet' \alpha_x \leq' \alpha_{f(x)}$. Now it is not hard to see that if a realizes f as morphism of separated objects, then $\downarrow\{a\}$ serves as α, and conversely that if α is given, then any $a \in \alpha$ works.

□

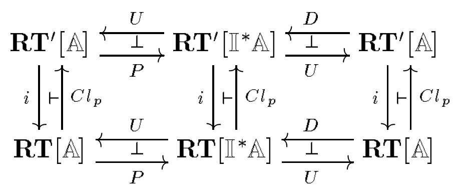
We also have the following:

Theorem 4.2 *There is an equivalence* $\mathbf{RT}[I^*\mathbb{A}] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}]})_{reg})_{ex}$.

Proof. We know that each $\mathbf{RT}[I^*\mathbb{A}]$ is the exact completion of its category of projectives, which is the same as the category of separated objects in $\mathbf{RT}[\mathbb{A}]$. But this latter category is the regular completion of the category of projectives of $\mathbf{RT}[\mathbb{A}]$.

□

Remember from section 2.2 that in case the ordered pca has the pasting property, we can also make use of the tripos $J(\mathbb{A})^{(-)}$ of nonempty downsets that are closed under pushouts. To complete the picture, we remark that the local localic map between $\mathbf{RT}[I^*\mathbb{A}]$ and $\mathbf{RT}[\mathbb{A}]$ restricts:



It is easiest to see why the functors U, P and D restrict if we consider them on the tripos-level (again, we use the same notation for the functors on the tripos- and on the topos-level). Note first that $P(\alpha)$ is trivially closed under pushouts, since it is principal. Second, if $\alpha \in I(\mathbb{A})$ is closed under pushouts, then the same holds for $D(\alpha)$, since if $\downarrow\{a\}, \downarrow\{b\} \in D(\alpha)$, then $\downarrow\{a\} \cup \downarrow\{b\} \subseteq \downarrow\{a \vee b\}$. Third, the map U also preserves the property of being closed under pushouts. Now the adjointness is immediate, and so is the commutation of the diagram.

4.3

We can iterate the downset-construction: starting with an arbitrary ordered pca $\mathbb{A} = \mathbb{A}_0$, we get a sequence $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ when we put $\mathbb{A}_{n+1} = (I^*\mathbb{A}_n)$.

This immediately gives us a sequence $I(\mathbb{A}_0)^{(-)}, I(\mathbb{A}_1)^{(-)}, \ldots$ of triposes, and hence a sequence $\mathbf{RT}[\mathbb{A}_0], \mathbf{RT}[\mathbb{A}_1], \ldots$ of toposes.

On the other hand, the results in [7] show that there are sequences of toposes of the form $(\mathcal{C}_{reg(n)})_{ex}$, (for appropriate categories $\mathcal{C}$). With the previous results in mind, the following theorem should not be all too surprising:

Theorem 4.3 *For each $n \in \mathbb{N}$, there is an equivalence of categories* $\mathbf{RT}[\mathbb{A}_n] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}_0]})_{reg(n)})_{ex}$.

Proof. This goes by induction and is an immediate consequence of the facts that we established concerning $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[I^*\mathbb{A}]$.

□

As a last observation, we mention the fact that there is also a chain of toposes coming from the hierarchy $J(\mathbb{A}), J(I^*\mathbb{A}), \ldots$. This chain is included in the one coming from $I(\mathbb{A}), I(I^*\mathbb{A}), \ldots$.

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