Second Order Contact of Minimal Surfaces

J.J. Duistermaat

Abstract

The minimal surface equation Q in the second order contact bundle of $\mathbb{R}^3$, modulo translations, is provided with a complex structure and a canonical vector-valued holomorphic differential form Ω on $Q \setminus 0$. The minimal surfaces M in $\mathbb{R}^3$ correspond to the complex analytic curves C in Q, where the derivative of the Gauss map sends M to C, and M is equal to the real part of the integral of Ω over C. The complete minimal surfaces of finite topological type and with flat points at infinity correspond to the algebraic curves in Q.

Introduction

In Section 1.1 we introduce the second order contact bundle modulo translations Q of the minimal surface equation. Q is a two-dimensional vector bundle over the unit sphere S, and carries a canonical $\mathbb{R}^3$-valued one-form ω which has a pole type of singularity along the zero section of Q. If M is a minimal surface, then the assignment to each $x \in M$ of the second order contact element of M at x defines a mapping $n' : M \to Q$, which can be viewed as the derivative of the Gauss map $n : M \to S$. The image $n'(M)$ is a two-dimensional submanifold C of Q such that the restriction to C of ω is closed.

In Section 1.2 we show that conversely, if C is a two-dimensional submanifold of $Q \setminus 0$ such that $\mathrm{d}\omega|_C = 0$, then the submanifold M of $\mathbb{R}^3$, which is obtained from C by means of integration of ω over C, is a minimal surface in $\mathbb{R}^3$ such that $n'(M) = C$. The integral of ω over closed curves in C leads to a group $\mathcal{P}$ of periods of M, which is discussed in Section 1.3.

The next observation is that Q carries a complex structure, unique up to its sign, such that the condition $\mathrm{d}\omega|_C = 0$ is equivalent to the condition that C is a complex analytic curve in Q, with respect to this complex structure. In order to describe the complex structure, and at the same time the automorphism group of Q, we begin in Section 2.1 by exhibiting Q as an associated vector bundle $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{R}^2$. In Section 2.1 this bundle is identified with the associated complex line bundle $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$, where B is a suitable Borel subgroup of $\mathrm{SO}(3, \mathbb{C})$. Because all the spaces here are complex analytic, this provides Q with a complex structure. The subgroup B is chosen in such a way that ω is equal to the real part of a $\mathbb{C}^3$-valued holomorphic $(1, 0)$-form Ω on Q, equivariant for the action of $\mathrm{SO}(3, \mathbb{C})$ on Q.

This leads to the characterization in Section 2.3 of the $C = n'(M)$ as the complex analytic curves in Q. In Section 2.4 we discuss the relation with the so-called isotropic complex analytic curves in $\mathbb{C}^3$, which are used in the Weierstrass type of representation formulas for minimal surfaces. The discussion of the structure of Q is concluded in Section 2.5 with the introduction of the two-fold covering $\mathrm{SL}(2, \mathbb{C})$ of $\mathrm{SO}(3, \mathbb{C})$ as a convenient computational tool. It leads to the identification Q with the complex line bundle $\mathcal{O}(4)$ over $\mathbb{CP}^1$ and of its compactification $\overline{Q}$ with the fourth Hirzebruch surface Σ_4, a complex projective algebraic variety. It also leads to local coordinates in which Ω takes a relatively simple explicit form, cf. (2.18).

In the applications, one has to pay special attention to the flat points of the minimal surface M. These correspond to certain intersection points of C with the zero section 0_Q

of Q, where the one-forms ω and Ω are singular. These points are discussed in Section 3.1, whereas the other points of $C \cap 0_Q$, which correspond to the flat points of M at infinity, are analyzed in Section 3.2. The condition of flatness of the point at infinity is equivalent to the condition that the total curvature of the end of M is finite. Because we allow periodicities, the integral of the Gaussian curvature has to be taken over the quotient of the end by the translation symmetry.

In Section 3.3 we prove that $M/\mathcal{P}$, the minimal surface modulo its periods, is of finite topological type and has only flat points at infinity, if and only if C is a complex algebraic curve in the complex projective variety $\overline{Q}$, where C has to satisfy some additional conditions in order to ensure that the minimal surface M is smoothly immersed. If the Gauss map $n : M/\mathcal{P} \to S$ has degree d, then the number of flat points in $M/\mathcal{P}$ plus the number of flat points at infinity, each counted with multiplicity, is equal to $4d$. We obtain the Jorge-Meeks formula as a consequence. In Section 3.4 we study the holomorphic sections of Q, several of which correspond to known minimal surfaces.

In Section 3.5, we find that $M/\mathcal{P}$ has finite topological type, no points at infinity, and a degree two Gauss map, if and only if C is a hyperelliptic curve of genus three, lying in the usual way as a twofold branched covering over S. The group $\mathcal{P}(\Omega|_C)$ of the periods of $\Omega|_C$ is a lattice in $\mathbb{C}^3$, and the quotient J, which is compact, is isomorphic to the Jacobian variety $\mathrm{Jac}(C)$ of C. The integration of Ω over C defines an embedding from C onto a closed complex analytic curve in J. The isotropic complex curve , in $\mathbb{C}^3$ such that $M = \mathrm{Re}\,$, as in Section 2.4, is equal to the pull-back under the projection $\mathbb{C}^3 \to J$ of the image of C in J. Therefore , is a closed and smooth one-dimensional complex analytic submanifold of $\mathbb{C}^3$. A result of Pirola [19] implies that the hyperelliptic curves C, for which the period group $\mathcal{P}$ of the minimal surface M is a discrete subgroup of $\mathbb{R}^3$, are dense. In an open dense subset they form a countable union of families of hyperelliptic curves which depend on essentially five real parameters.

There is an enormous literature on minimal surfaces, among which the beautiful surveys of Nitsche and those of Dierkes, Hildebrandt, Küster and Wohlrab. We will use DHKW [4] as our main reference.

1 Second order Contact

In this section we introduce the real two-dimensional vector bundle Q over the unit sphere S such that the combination of the Gauss map and its derivative sends each minimal surface M in $\mathbb{R}^3$ to a surface C in Q. Moreover, there is a canonically defined $\mathbb{R}^3$-valued one-form ω on Q such that the minimal surface M is reconstructed from C by means of integration of ω over curves in C.

1.1 The Derivative of the Gauss Map

Let M be an oriented two-dimensional smooth submanifold of $\mathbb{R}^3$ and $S = \{s \in \mathbb{R}^3 \mid \|s\| = 1\}$ the sphere in $\mathbb{R}^3$ with radius equal to one and center at the origin. The mapping $n : M \to S$ which assigns to $x \in M$ the oriented normal $n(x)$ to the tangent space $\mathrm{T}_x\, M$, is called the *Gauss map* of M. The tangent map $\mathrm{T}_x\, n$ of n at x is a linear mapping from $\mathrm{T}_x\, M$ to $\mathrm{T}_{n(x)}\, S$. The translation $\tau_{x-n(x)}$ from $n(x)$ to x leads to an identification of $\mathrm{T}_{n(x)}\, S$ with $\mathrm{T}_x\, M$. Let $n'(x) = \mathrm{T}_x\, n \circ \tau_{x-n(x)}$ denote the linear mapping from $\mathrm{T}_{n(x)}\, S$ to $\mathrm{T}_{n(x)}\, S$ which is induced by $\mathrm{T}_x\, n$.

The linear endomorphism $n'(x)$ of $\mathrm{T}_{n(x)}\, S$ is symmetric with respect to the restriction to $\mathrm{T}_{n(x)}\, S$ of the standard inner product of $\mathbb{R}^3$. In fact, by means of a translation we can arrange that a given special point of M is at the origin, and by rotation in $\mathbb{R}^3$ we can subsequently arrange that $n(0) = e_3$, the vertical standard basis vector in $\mathbb{R}^3$. Then, near 0, M can be written as the graph of a smooth function f of two variables, the third coordinate as a function of the first two ones. If we identify $\mathrm{T}_{e_3}\, S$ with $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\}$, then a short calculation shows that the matrix of $n'(0)$ is equal to minus the Hessian matrix $f''(0)$ of f at 0. Because $f''(0)$ describes the second order contact of M at 0 with its tangent plane, we have in general that $n'(x)$ represents the second order contact of M at x with $\mathrm{T}_x\, M$.

The surface M is a *minimal surface* if and only if, for every $x \in M$, the trace of $n'(x)$ is equal to zero, cf. DHKW [4, (24) on p. 16 and p. 53]. For every $s \in S$, let

$$Q_s := \{q \in \mathrm{Lin}\,(\mathrm{T}_s\, S,\, \mathrm{T}_s\, S) \mid q^* = q,\ \mathrm{trace}\, q = 0\} \tag{1.1}$$

be the space of all traceless symmetric linear mappings from $\mathrm{T}_s\, S$ to itself. Q_s is a two-dimensional real vector space. The Q_s, $s \in S$, form a smooth rank two real vector bundle Q over S. If M is a minimal surface then, for every $x \in M$, the element $n'(x) \in Q_{n(x)}$ represents the second order contact of M at x. This defines a smooth mapping $n' : M \to Q$ such that the Gauss mapping n is equal to $\pi \circ n'$, if $\pi : Q \to S$ denotes the projection which assigns to $q \in Q_s$ the base point s.

Let M be a minimal surface in $\mathbb{R}^3$. A traceless symmetric two by two matrix is of the form $q = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ in which $a, b \in \mathbb{R}$. Its determinant is equal to $-a^2 - b^2$ and it follows that if $q \neq 0$, then q is invertible. This implies that, for every $x \in M$ the *Gauss curvature*

$$K(x) = \det n'(x) \quad \Big(\text{equal to } -a^2 - b^2 \text{ when } n(x) = e_3\Big) \tag{1.2}$$

of M at x is nonpositive. Moreover, if x is an *umbilic point of* M, i.e. $K(x) = 0$, then x is a *flat point of* M, i.e. $n'(x) = 0$, which is equivalent to the condition that M osculates its tangent plane at the point x. In other words, the following conditions i)—v) are equivalent.

i) x is not a flat point of M.

ii) $n'(x) \neq 0$.

iii) $n'(x)$ is invertible.

iv) The Gauss map n is a diffeomorphism from some open neighborhood M_0 of x in M onto an open neighborhood S_0 of $n(x)$ in S.

v) There is an open neighborhood M_0 of x in M such that the restriction to M_0 of n' is a smooth embedding from M_0 to a smooth local section $C_0 = n'(M_0)$ of the vector bundle Q over S.

Here the equivalence between iv) and v) follows from the fact that $n = \pi \circ n'$. Indeed, this implies that if n is a diffeomorphism, then n' is an embedding. Conversely if n' is a diffeomorphism from M_0 onto a smooth submanifold C_0 of W, then C_0 is a smooth local section of $\pi : Q \to S \iff$ the restriction to C_0 of π is a diffeomorphism from C_0 onto an open subset S_0 of $S \iff \pi \circ n'$ is a diffeomorphism from M_0 onto S_0.

1.2 Reconstruction of the Minimal Surface by Integration

If $q \in Q_s \setminus \{0\}$, then $q^{-1} : \mathrm{T}_s S \to \mathrm{T}_s S$ can be viewed as a linear mapping from $\mathrm{T}_s S$ to $\mathbb{R}^3$, if we identify the target tangent plane as a linear subspace of $\mathbb{R}^3$. In this way we obtain an $\mathbb{R}^3$-valued one-form (= differential form of degree one) on $Q \setminus 0$, which we denote by $\omega = q^{-1}\, \mathrm{d}s$. Here $0 = 0_Q$ denotes the zero section of Q. More precisely, if $q \in Q_s$, then ω is the linear mapping from $\mathrm{T}_q Q$ to $\mathbb{R}^3$ which is equal to the linear mapping $\mathrm{T}_q \pi$ from $\mathrm{T}_q Q$ to $\mathrm{T}_s S$, followed by the linear mapping q^{-1} from $\mathrm{T}_s S$ to $\mathbb{R}^3$. Note that ω has a pole-type of singularity at 0_Q.

Suppose that M is a connected immersed minimal surface and n' is a diffeomorphism from M onto a smooth submanifold C of Q. Let $\iota : M \to \mathbb{R}^3$ denote the identity on M, viewed as a mapping from M to $\mathbb{R}^3$. Then, for every $x \in M$ and $v \in \mathrm{T}_x M$, we have

$$\left((n')^*\omega\right)_x (v) = \omega_{n'(x)} \left(\mathrm{T}_x\, n'(v)\right) = q^{-1} \circ \mathrm{T}_{n'(x)}\, \pi \circ \mathrm{T}_x\, n'(v) = q^{-1} \circ \mathrm{T}_x\, n(v) = v,$$

where $q = n'(x) = \mathrm{T}_x\, n$ is viewed as a linear mapping from $\mathrm{T}_x M \simeq \mathrm{T}_{n(x)} S$ to $\mathbb{R}^3$. Here we have applied the chain rule to $\pi \circ n' = n$ in the third identity. This proves that

$$(n')^*\omega = \mathrm{d}\iota \quad \text{on} \quad M. \tag{1.3}$$

In turn this means that $\omega|_C = \mathrm{d}f$ if f is equal to the $\mathbb{R}^3$-valued function $\left((n')^{-1}\right)^* \iota = (n')^{-1}$ on C. It follows that M can be reconstructed from C in the following way. Choose, for any base point $q_0 \in C$, a corresponding base point $x_0 \in \mathbb{R}^3$ as $(n')^{-1}(q_0)$. Then, for every $q_1 \in V$, the point of M corresponding to q_1 is given by

$$x_1 = \phi(q_1) := (n')^{-1}(q_1) = x_0 + \int_{q_0}^{q_1} q^{-1}\, \mathrm{d}s, \tag{1.4}$$

in which the integral of the $\mathbb{R}^3$-valued one-form $\omega = \underline{q}^{-1}\, \mathrm{d}s$ is taken along any smooth curve γ in C which runs from q_0 to q_1. Note that the integral does not depend on the choice of γ.

The fact that $\omega|_C$ is exact, is locally equivalent to the condition that $(\mathrm{d}\omega)|_C = \mathrm{d}(\omega|_C) = 0$, which means that the restriction to C of $\mathrm{d}\omega$ is equal to zero. This condition means that, for every $q \in C$ and every pair $v, w \in \mathrm{T}_q C$ of tangent vectors to C, we have that $(\mathrm{d}\omega)_q(v, w) = 0$, or that $\mathrm{T}_q C$ is an isotropic linear subspace of $\mathrm{T}_q Q$ with respect to the $\mathbb{R}^3$-valued antisymmetric bilinear form $(\mathrm{d}\omega)_q$. Because it is clear that $\mathrm{d}\left(q^{-1}\, \mathrm{d}s\right) = \mathrm{d}\left(q^{-1}\right) \wedge \mathrm{d}s$ is far from zero as a two-form on the four-dimensional manifold $Q \setminus 0$, the equation $(\mathrm{d}\omega)|_C = 0$ is strong restriction on the two-dimensional real linear subspace $\mathrm{T}_q C$ of $\mathrm{T}_q Q$.

1.3 Surfaces in the Bundle and their Periods

Suppose that C is any smooth two-dimensional immersed submanifold of $Q \setminus 0$ such that $(\mathrm{d}\omega)|_C = 0$, which means that $\omega|_C$ is closed. Also assume that C is transversal to the fibers of the projection $\pi : Q \to S$, which means that the $\mathbb{R}^3$-valued linear form $q^{-1}\, \mathrm{d}s$ is injective on $\mathrm{T}_q C$. Then, after a choice of base points $q_0 \in C$, $x_0 \in \mathbb{R}^3$, (1.4) defines an embedding ϕ from each simply connected open subset C_0 of C onto a smooth two-dimensional submanifold M_0 of $\mathbb{R}^3$. For each $q \in C$ we have, with the notation $s = \pi(q)$, $x = \phi(q)$, that

$$\mathrm{T}_x M_0 = \mathrm{image}\, \mathrm{T}_q\, \phi = q^{-1}\left(\mathrm{T}_s S\right) = \mathrm{T}_s S,$$

where $\mathrm{T}_{\phi(q)} M_0$ and $\mathrm{T}_s S$ both are regarded as two-dimensional linear subspaces of $\mathbb{R}^3$. It follows that s is orthogonal to $\mathrm{T}_x M_0$, and we have a unique orientation of M_0 such that

$s = n(x)$, the normal of M_0 at the point x. Furthermore the equation $\mathrm{d}x = q^{-1}\,\mathrm{d}s$ implies that $\mathrm{d}s = q\,\mathrm{d}x$. Therefore the inverse of ϕ is equal to n'. The fact that n' maps M_0 into Q, where the fiber Q_s is equal to the space of traceless symmetric linear mappings from $\mathrm{T}_s\, S$ to $\mathrm{T}_s\, S$, finally implies that M_0 is a minimal surface. In this way *the mapping n' establishes a local equivalence between the minimal surfaces in $\mathbb{R}^3$ with nonvanishing curvature, modulo translations, and the two-dimensional smooth submanifolds C of $Q \setminus 0$ such that $\omega|_C$ is closed and C is transversal to the fibers of the projection $\pi : Q \to S$.*

Let $[\omega]$ denote the de Rham cohomology class of ω, which is an element of $\mathrm{H}^1_{\text{de Rham}}(C) \otimes \mathbb{R}^3$. Globally, the equation (1.4) in general defines a multi-valued immersion ϕ from C to $\mathbb{R}^3$, where the multi-valuedness is caused by the fact that for every closed loop γ in C, meaning that $q_1 = q_0$, we obtain that $x_1 - x_0 = \langle [\gamma], [\omega] \rangle$ need not be equal to zero. The $\langle [\gamma], [\omega] \rangle$, where $[\gamma]$ denotes the homology class of the closed loop γ in the image of $\mathrm{H}_1(C, \mathbb{Z})$ in $\mathrm{H}_1(C, \mathbb{R})$, are called the *periods* of ω. They form an additive subgroup $\mathcal{P}$ of $\mathbb{R}^3$, generated by the vectors $\langle [\gamma_j]\,, [\omega] \rangle$ if the $[\gamma_j]$ generate the image of $\mathrm{H}_1(C, \mathbb{Z})$ in $\mathrm{H}_1(C, \mathbb{R})$. The multi-valuedness of ϕ means that ϕ assigns to every q a coset of the form $x + \mathcal{P}$ in $\mathbb{R}^3$. This implies that the image M of C is *$\mathcal{P}$-periodic* in the sense that $M + p = M$ for every $p \in \mathcal{P}$. Conversely, if $M + p = M$, then $n'(x + p) = n'(x)$ for every $x \in M$, and it follows that the set of periods of M is equal to $\mathcal{P}$.

In Section 2 we will show that Q has a complex structure, unique up to sign, such that if $q \in Q \setminus 0$ then a two-dimensional real linear subspace P of $\mathrm{T}_q\, Q$ is isotropic with respect to $(\mathrm{d}\omega)_q$ if and only if P is equal to a one-dimensional complex linear subspace of the two-dimensional complex vector space $\mathrm{T}_q\, Q$. This implies that a surface C is of the form $n'(M)$ for a minimal surface M with nonvanishing curvature if and only if C is a complex analytic curve in the two-dimensional complex analytic manifold $Q \setminus 0$ which is transversal to the fibers.

Remark 1.1 Any second order partial differential equation for surfaces in $\mathbb{R}^3$ can be identified with a 7-dimensional hypersurface in the 8-dimensional second order contact bundle of $\mathbb{R}^3$. If the equation is translation invariant, then we can pass to its quotient by the translation group $\mathbb{R}^3$, which is a 4-dimensional manifold Q. If the equation is elliptic, then there is an almost complex structure J on Q such that the solution surfaces, modulo translations, are locally in a bijective correspondence with the complex analytic curves in (Q, J). For the minimal surface equation the almost complex structure J is integrable. I actually arrived at the description of the complex structure in Q from the application of the theory of second order contact structure to the minimal surface equation. However, in order to emphasize the especially nice features of the minimal surface equation and not to burden the presentation with the generalities about second order contact structures, I have chosen to present the direct description of the complex structure of Section 2. $\oslash$

Remark 1.2 The considerations in this section have been predominantly of a local nature. Avoiding the flat points, the mapping n' is an immersion from M to $Q \setminus 0$ which need not be injective, let alone that it is an embedding. One cause for non-injectivity could be the occurrence of nonzero periods, which can be remedied by passing to the quotient $M/\mathcal{P}$. However, there can also be self-intersections of the image $C = n'(M)$ which are not caused by periodicities; at such multiple points C has a singularity of multiple point type. Finally the mapping $n' : M/\mathcal{P} \to Q \setminus 0$ need not be proper, meaning that one can have a non-converging sequence of points in M such that the image points converge to a point in $Q \setminus 0$.

In the other direction, the immersion $(n')^{-1} : C \to \mathbb{R}^3$ can have self-intersections which generically occur along curves. In this paper we will not discuss the problem of avoiding such self-intersections of M or $M/\mathcal{P}$. $\oslash$

2 The Complex Structure

In this section, a complex structure is introduced on Q by identifying Q with an associated complex line bundle over $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$, where B is the group of all $b \in \mathrm{SO}(3, \mathbb{C})$ such that $b\,(\mathbb{C}\,(e_1 + \mathrm{i}\, e_2)) = \mathbb{C}\,(e_1 + \mathrm{i}\, e_2)$. Here $b \in B$ acts on $\mathbb{C}$ by means of multiplication by $\chi(b)$, in which $\chi : B \to \mathbb{C} \setminus \{0\}$ is the unique character (= Lie group homomorphism) such that $\chi(b) = \mathrm{e}^{\mathrm{i}\, 2\phi}$ if b is equal to the rotation abouth the vertical axis through the angle ϕ. In this description the base space S of Q, the unit sphere in $\mathbb{R}^3$, is identified with $\mathrm{SO}(3, \mathbb{C})/B$, which in turn is identified with the quadric N in the complex projective plane, defined by the homogeneous quadratic equation $\langle z, z\rangle = 0$, $z \in \mathbb{C}^3$.

The $\mathbb{R}^3$-valued one-form ω on $Q \setminus 0$ is equal to the real part of an $\mathrm{SO}(3, \mathbb{C})$-equivariant $\mathbb{C}^3$-valued holomorphic $(1, 0)$-form Ω. For a real two-dimensional submanifold C of $Q \setminus 0$, we have that $\omega|_C$ is closed if and only if C is a complex analytic curve in the complex two-dimensional manifold $Q \setminus 0$. If M is the minimal surface in $\mathbb{R}^3$ which is obtained by means of integration of $\omega|_C$, and , is the complex analytic curve in $\mathbb{C}^3$ which is obtained by means of integration of $\Omega|_C$, then $M = \mathrm{Re}$, , and , is the isotropic complex anaytic curve corresponding to M in the Weiestrass type of representation formulas.

The computations simplify considerably if one replaces the group $\mathrm{SO}(3, \mathbb{C})$ by the group $\mathrm{SL}(2, \mathbb{C})$. The adjoint representation (= action on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of $\mathrm{SL}(2, \mathbb{C})$ by means of conjugation) leads to a two-fold covering $\mathrm{Ad} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$, in which we use a suitable identification of $\mathfrak{sl}(2, \mathbb{C})$ with $\mathbb{C}^3$. Under this identification, B corresponds to the group L of all lower triangular matrices in $\mathrm{SL}(2, \mathbb{C})$, and $Q \simeq \mathrm{S0}(3, \mathbb{C}) \times_B \mathbb{C}$ to the associated line bundle $\mathrm{SL}(2, \mathbb{C}) \times_L \mathbb{C}$, where $\begin{pmatrix} z & 0 \\ v & 1/z \end{pmatrix} \in L$ acts on $\mathbb{C}$ by means of multiplication by z^4. In this description, the sphere $S \simeq N$ is identified with the complex projective line $\mathbb{CP}^1$, of which $\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$ is the automorphism group. It follows that the automorphism group of Q is equal to $(\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}) \times (\mathbb{C} \setminus \{0\}) \simeq \mathrm{SO}(3, \mathbb{C}) \times (\mathbb{C} \setminus \{0\})$. Here $c \in \mathbb{C} \setminus \{0\}$ acts on Q by multiplication with c in each fiber.

2.1 The Rotation Group

The vector bundle Q is homogeneous with respect to the group $\mathrm{SO}(3)$ of rotations in $\mathbb{R}^3$, in the sense that the mapping $(g, q) \mapsto (g\,(e_3), g\,q\,g^{-1})$ is surjective from $\mathrm{SO}(3) \times Q_{e_3}$ to Q. We have that (g, q) and (g', q') have the same image if and only if there exists a rotation r about the vertical axis, such that $g' = g\,r^{-1}$ and $q' = r\,q\,r^{-1}$. Therefore, if we denote by $\mathrm{SO}(2)$ the group of all rotations about the vertical axis, then Q is isomorphic to the space $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} Q_{e_3}$ of $\mathrm{SO}(2)$-orbits in $\mathrm{SO}(3) \times Q_{e_3}$, where $r \in \mathrm{SO}(2)$ acts on $\mathrm{SO}(3) \times Q_{e_3}$ by sending (g, q) to $(g\,r^{-1}, r\,q\,r^{-1})$. The projection $(g, q) \mapsto g$ factorizes to a projection from $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} Q_{e_3} \simeq Q$ onto $\mathrm{SO}(3)/\mathrm{SO}(2) \simeq S$, which corresponds to the projection $\pi : Q \to S$. Here $\mathrm{SO}(3)/\mathrm{SO}(2)$ is identified with S by means of the mapping $g \mapsto g\,(e_3)$, which is surjective from $\mathrm{SO}(3)$ onto S and has the $\mathrm{SO}(2)$-orbits as its fibers. The element $h \in \mathrm{SO}(3)$ acts on Q by sending (s, q) to $(h\,s, h\,q\,h^{-1})$, which corresponds to sending (g, q)

to $(h\,g,\,q)$. For this reason this action is called the *left action* of h on Q and denoted by L_h. The vector bundle $Q \simeq \mathrm{SO}(3) \times_{\mathrm{SO}(2)} Q_{e_3}$ is also called the *associated* $\mathrm{SO}(3)$*-bundle for the representation* $r \mapsto (q \mapsto r\,q\,r^{-1})$ *of* $\mathrm{SO}(2)$ *on the vector space* Q_{e_3}, cf. [5, Sec. 2.4].

Note that if $r = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ is the rotation in the horizontal plane through the angle ϕ, and $q = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, then $g\,q\,g^{-1} = \begin{pmatrix} a' & b' \\ b' & -a' \end{pmatrix}$, in which $a' = (\cos 2\phi)\,a - (\sin 2\phi)\,b$ and $b' = (\sin 2\phi)\,a + (\cos 2\phi)\,b$. In other words, the vector $(a',\,b')$ is obtained from the vector $(a,\,b)$ by applying the rotation in $\mathbb{R}^2$ through the angle 2ϕ.

The $\mathbb{R}^3$-valued differential form $\omega = q^{-1}\,\mathrm{d}s$ is equivariant for the left action of $\mathrm{SO}(3)$ on Q, in the sense that if $h \in \mathrm{SO}(3)$, then

$$\begin{aligned} ((\mathrm{L}_h)^*\,\omega)_{(s,\,q)}\,(\delta s,\,\delta q) &= \omega_{(h\,s,\,h\,q\,h^{-1})}\,(h\,\delta s,\,\ldots) \\ &= \left(h\,q\,h^{-1}\right)^{-1}\,h\,\delta s = h\,q^{-1}\,\delta s = h\,\omega_{(s,\,q)}(\delta s,\,\delta q). \end{aligned}$$

2.2 The Complex Rotation Group

At $s = e_3$ and a nonzero element q of Q_{e_3}, $\omega_{(e_3,\,q)}$ sends $(\delta s,\,\delta q)$ to the vector $y = q^{-1}\,\delta s \in \mathbb{R}^3$, such that $y_3 = 0$. If $q = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ then $q^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and therefore

$$\begin{aligned} y_1 &= \frac{1}{a^2+b^2}\,(a\,\delta s_1 + b\,\delta s_2) = \frac{1}{a^2+b^2}\,\mathrm{Re}\,[(a - \mathrm{i}\,b)\,(\delta s_1 + \mathrm{i}\,\delta s_2)] = \mathrm{Re}\,\frac{\delta s_1 + \mathrm{i}\,\delta s_2}{a + \mathrm{i}\,b}, \\ y_2 &= \frac{1}{a^2+b^2}\,(b\,\delta s_1 - a\,\delta s_2) = \frac{1}{a^2+b^2}\,\mathrm{Re}\,[\mathrm{i}\,(a - \mathrm{i}\,b)\,(\delta s_1 + \mathrm{i}\,\delta s_2)] = \mathrm{Re}\;\mathrm{i}\;\frac{\delta s_1 + \mathrm{i}\,\delta s_2}{a + \mathrm{i}\,b}. \end{aligned}$$

If we identify Q_{e_3} with $\mathbb{C}$ by identifying $q = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $a + \mathrm{i}\,b$, and if we write $\delta s = \delta g\,(e_3)$ with $\delta g \in \mathfrak{so}(3)$, then we obtain that

$$\widetilde{\omega}_{(g,\,q)}(\delta g,\,\delta q) = \mathrm{Re}\left[\widetilde{\Omega}_{(g,\,q)}(\delta g,\,\delta q)\right], \tag{2.1}$$

in which

$$\widetilde{\Omega}_{(g,\,q)}(\delta g,\,\delta q) = \frac{1}{q}\,\langle g^{-1} \circ \delta g\,(e_3)\,,\, e_1 + \mathrm{i}\,e_2\rangle \cdot g\,(e_1 + \mathrm{i}\,e_2)\,. \tag{2.2}$$

Here $\widetilde{\Omega}$ is an equivariant $\mathbb{C}^3$-valued one-form on $\mathrm{SO}(3) \times \mathbb{C} \setminus \{0\}$, which moreover is equal to the pull-back of a one-form Ω on $Q \setminus 0 \simeq \mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{C} \setminus \{0\}$ by means of the projection from $\mathrm{SO}(3) \times \mathbb{C} \setminus \{0\}$ onto $Q \setminus 0$. The rotation r about the vertical axis through the angle ϕ acts on $\mathbb{C}$ by means of multiplication by $\mathrm{e}^{\mathrm{i}\,2\phi}$, because $q \mapsto r\,q\,r^{-1}$ corresponds to applying r^2 to the first column of $q \in Q_{e_3}$.

If V is a complex vector space, then a V-valued one-form Θ on a complex manifold P is called *a* $(1,\,0)$*-form* if, for every $p \in P$, Θ_p is a complex linear mapping from $\mathrm{T}_p\,P$ to V. If moreover in local holomorphic coordinates the coefficients of Θ_p depend holomorphically on p, then Θ is called a V-valued holomorphic $(1,\,0)$-form on P. If we extend the standard inner product of $\mathbb{R}^3$ to the corresponding complex bilinear form on $\mathbb{C}^3 \times \mathbb{C}^3$, then (2.2) extends to a $\mathbb{C}^3$-valued holomorphic $(1,\,0)$-form on the complex analytic manifold $\mathrm{SO}(3,\,\mathbb{C}) \times \mathbb{C} \setminus \{0\}$, which we also denote by $\widetilde{\Omega}$. The idea is to introduce a closed Lie subgroup B of $\mathrm{SO}(3,\,\mathbb{C})$ with the following properties:

i) $\mathrm{SO}(2) \subset B$.

ii) The injections $\mathrm{SO}(3) \to \mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SO}(2) \to B$ induce an isomorphism from $Q \simeq \mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{C}$ onto $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$.

iii) $\widetilde{\Omega}$ is equal to the pull-back of a differential form Ω on $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$ by means of the projection $\psi : \mathrm{SO}(3, \mathbb{C}) \times \mathbb{C} \to \mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$.

As a consequence, $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$ will be a holomorphic complex line bundle over the complex one-dimensional complex analytic manifold $\mathrm{SO}(3, \mathbb{C})/B$, which must be isomorphic to the complex projective line $\mathbb{CP}^1$ because it is diffeomorphic to the sphere $\mathrm{SO}(3)/\mathrm{SO}(2) \simeq S$. Furthermore, Ω then is an $\mathrm{SO}(3, \mathbb{C})$-equivariant $\mathbb{C}^3$-valued holomorphic $(1, 0)$-form on $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$, such that $\mathrm{Re}\, \Omega = \omega$ in the identification $Q \xrightarrow{\sim} \mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$.

The condition that $\widetilde{\Omega} = \psi^* \Omega$ for some Ω implies that $\widetilde{\Omega}$ is equal to zero in the direction of the B-orbit, which means in view of (2.2) that

$$\langle X\, (e_3)\, ,\, e_1 + \mathrm{i}\, e_2 \rangle = 0$$

for every element X of the Lie algebra $\mathfrak{b}$ of B. This means that X must be of the form

$$X = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -\mathrm{i}\, b \\ b & \mathrm{i}\, b & 0 \end{pmatrix}, \quad a,\, b \in \mathbb{C}. \tag{2.3}$$

which are precisely the $X \in \mathfrak{so}(3, \mathbb{C})$ such that $X\, (e_1 + \mathrm{i}\, e_2)$ is equal to a complex multiple of $e_1 + \mathrm{i}\, e_2$, where for (2.3) the factor is equal to $-\mathrm{i}\, a$. For this reason, we define B as the set of all $b \in \mathrm{SO}(3, \mathbb{C})$ such that

$$b\, (\mathbb{C}\, (e_1 + \mathrm{i}\, e_2)) = \mathbb{C}\, (e_1 + \mathrm{i}\, e_2)\, . \tag{2.4}$$

B is a closed complex Lie subgroup of $\mathrm{SO}(3, \mathbb{C})$, with Lie algebra $\mathfrak{b}$ equal to the set of X as in (2.3).

Let $\widetilde{N}$ denote the *isotropic cone in* $\mathbb{C}^3$, which consists of the $z \in \mathbb{C}^3$, such that $\langle z,\, z \rangle = 0$. Let N denote the corresponding quadric in the complex projective plane, which consists of the $\mathbb{C} z$ such that $z \in \widetilde{N} \setminus \{0\}$. Because $e_1 + \mathrm{i}\, e_2 \in \widetilde{N} \setminus \{0\}$ and $\mathrm{SO}(3, \mathbb{C})$ leaves $\widetilde{N} \setminus \{0\}$ invariant, we have a mapping $g \mapsto g\, (\mathbb{C}\, (e_1 + \mathrm{i}\, e_2))$ from $\mathrm{SO}(3, \mathbb{C})$ to N, which in view of the definition (2.4) of B induces an injective mapping from $\mathrm{SO}(3, \mathbb{C})/B$ to N.

In order to find a natural diffeomorphism σ from N onto S, we write $z \in \mathbb{C}^3$, $z \neq 0$, as $z = x + \mathrm{i}\, y$ with $x,\, y \in \mathbb{R}^3$. Then $z \in \widetilde{N}$ if and only if $\langle x,\, x \rangle = \langle y,\, y \rangle$ and $\langle x,\, y \rangle = 0$, which implies that $\|x \wedge y\| = \|x\|^2$, where $x \wedge y \in \mathbb{R}^3$ denotes the exterior product of x and y. If $a,\, b \in \mathbb{R}$, then $(a + \mathrm{i}\, b)\, (x + \mathrm{i}\, y) = a\, x - b\, y + \mathrm{i}\, (b\, x - a\, y)$ and $(a\, x - b\, y) \wedge (b\, x + a\, y) = (a^2 + b^2)\;x \wedge y$. For every $z \in \widetilde{N} \setminus \{0\}$, we write

$$\sigma(z) = \langle x,\, x \rangle^{-1}\, x \wedge y. \tag{2.5}$$

Then $\sigma(z) \in S$ and $\sigma(c\, z) = \sigma(z)$ for every $c \in \mathbb{C} \setminus \{0\}$, and therefore $\sigma : \widetilde{N} \to S$ induces a mapping from N to S, which we also denote by σ. This mapping is $\mathrm{SO}(3)$-equivariant, and therefore surjective because $\mathrm{SO}(3)$ acts transitively on S. The mapping is injective. Proof: if $\sigma(x + \mathrm{i}\, y) = \sigma(x' + \mathrm{i}\, y')$ and $x + \mathrm{i}\ y,\, x' + \mathrm{i}\, y' \in \widetilde{N}$, then both $\|x\|^{-1}\, x$, $\|y\|^{-1}\, y$ and $\|x'\|^{-1}\, x'$, $\|y'\|^{-1}\, y'$ form an orthonormal basis of the same twodimensional linear subspace of

$\mathbb{R}^3$ with the same orientation. Therefore these are obtained from each other by means of a rotation in this plane, which in turn implies that $x' + \mathrm{i}\, y' \in \mathbb{C}(x + \mathrm{i}\, y)$. The conclusion is that $\sigma : N \to S$ is a diffeomorphism.

As a consequence, $\mathrm{SO}(3)$ acts transitively on N, which implies that $\mathrm{SO}(3, \mathbb{C})$ acts transitively on N. We conclude that the mapping $g \mapsto g\,(\mathbb{C}\,(e_1 + \mathrm{i}\, e_2))$ induces an isomorphism from $\mathrm{SO}(3, \mathbb{C})/B$ onto $N \simeq S$. Because $\sigma\,(\mathbb{C}\,(e_1 + \mathrm{i}\, e_2)) = e_3$ and the action of $g \in \mathrm{SO}(3)$ commutes with σ, it follows also that $B \cap \mathrm{SO}(3) = \mathrm{SO}(2)$.

Like $\mathrm{O}(3)$, the group $\mathrm{O}(3, \mathbb{C})$ has two connected components, corresponding to the sign of the determinant, cf. Chevalley [2, p. 16]. Therefore $\mathrm{SO}(3, \mathbb{C})$ is connected. Because $\mathrm{SO}(3, \mathbb{C})/B \simeq N \simeq S$ is simply connected, it follows that B is connected as well. The definition of B in (2.4) implies that, for every $b \in B$, $b\,(e_1 + \mathrm{i}\, e_2) = \rho(b)\,(e_1 + \mathrm{i}\, e_2)$, in which ρ is a character of B, a Lie group homomorphism from B to the muliplicative group of the nonzero complex numbers. Because B is connected, the character ρ is determined by the infinitesimal character $\rho' : \mathfrak{b} \to \mathbb{C}$, where $\rho'(X) = -\mathrm{i}\, a$ if X is as in (2.3). In the definition of $Q \simeq \mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{C}$ the action of the rotation r about the vertical axis through the angle ϕ on $\mathbb{C}$ is by means of multiplication by $\mathrm{e}^{\mathrm{i}\, 2\phi}$. Therefore the inclusions $\mathrm{SO}(3) \to \mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SO}(2) \to B$ lead to a well-defined mapping from $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{C}$ to $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$, if and only if we let act $b \in B$ act on $\mathbb{C}$ by means of multiplication by $\rho(b)^{-2}$. The isomorphism $\mathrm{SO}(3)/\mathrm{SO}(2) \to \mathrm{SO}(3, \mathbb{C})/B$ then implies that the mapping from $\mathrm{SO}(3) \times_{\mathrm{SO}(2)} \mathbb{C}$ to $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$ is an isomorphism of complex line bundles.

The last property of $\widetilde{\Omega}$ which has to verified in order that $\widetilde{\Omega} = \psi^* \Omega$ for some Ω, is that $\widetilde{\Omega}$ is invariant under the right action R_b of $b \in B$. We have in view of (2.2) that

$$\begin{aligned}
{\mathrm{R}_b}^* \widetilde{\Omega}_{(g,\, q)}(\delta g,\, \delta q) &= \left(\rho(b)^{-2}\, q\right)^{-1} \langle b \circ g^{-1} \circ \delta g \circ b^{-1}\,(e_3)\,,\, e_1 + \mathrm{i}\, e_2 \rangle \cdot g \circ b^{-1}\,(e_1 + \mathrm{i}\, e_2) \\
&= \frac{1}{q}\, \rho(b)^2\, \langle g^{-1} \circ \delta g \circ b^{-1}\,(e_3)\,,\, b^{-1}\,(e_1 + \mathrm{i}\, e_2) \rangle \cdot g \circ b^{-1}\,(e_1 + \mathrm{i}\, e_2) \\
&= \frac{1}{q}\, \langle g^{-1} \circ \delta g \circ b^{-1}\,(e_3)\,,\, e_1 + \mathrm{i}\, e_2 \rangle \cdot g\,(e_1 + \mathrm{i}\, e_2) \\
&= \frac{1}{q}\, \langle g^{-1} \circ \delta g \circ b^{-1}\,(e_3)\,,\, e_1 + \mathrm{i}\, e_2 \rangle \cdot g\,(e_1 + \mathrm{i}\, e_2) = \widetilde{\Omega}_{(g,\, q)}\,(\delta g, \delta q)\,,
\end{aligned}$$

which proves the desired invariance. Here we have used in the first equality that $\left(g \circ b^{-1}\right)^{-1} = b \circ g^{-1}$ and that $\delta\,\left(g \circ b^{-1}\right) = \delta g \circ b^{-1}$. In the third equality we have used that $b^{-1}\,(e_1 + \mathrm{i}\, e_2) = \rho(b)^{-1}\,(e_1 + \mathrm{i}\, e_2)$. In the fourth equality we have used that $b\,(e_3)$ is equal to e_3 plus a multiple of $e_1 + \mathrm{i}\, e_2$, because the elements of $\mathfrak{b}$ map e_3 into $\mathbb{C}\,(e_1 + \mathrm{i}\, e_2)$, cf. (2.3). Note also that $\langle g^{-1} \circ \delta g\,(e_1 + \mathrm{i}\, e_2)\,,\, e_1 + \mathrm{i}\, e_2 \rangle = 0$ because $g^{-1} \circ \delta g$ is antisymmetric.

Remark 2.1 We have, for every $g \in \mathrm{SO}(3, \mathbb{C})$, that $g\,(e_1 + \mathrm{i}\, e_2) \in \widetilde{N}$, which implies in view of (2.2) that $\widetilde{\Omega}$, and therefore Ω as well, takes its values in the isotropic cone $\widetilde{N}$ rather than in $\mathbb{C}^3$. Let C be a complex analytic curve in $Q \setminus 0$. Let , be the complex analytic curve in $\mathbb{C}^3$ which is obtained by integration of Ω over curves in C, as in (1.4) with $\omega = q^{-1}\, \mathrm{d}s$ replaced by Ω. Then , is an *isotropic curve* in the sense that, for every $z \in$, , we have that T_z , $\in N$. (Note however that this does not imply that $\widetilde{z} - z \in \widetilde{N}$ for every $z, \widetilde{z} \in$, .) $\oslash$

Remark 2.2 The group $\mathrm{SO}(3, \mathbb{C})$ acts transitively on $Q \setminus 0 \simeq \mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C} \setminus \{0\}$ and $\mathbb{C}(e_1 + \mathrm{i}\ e_2)$ is the unique one-dimensional complex linear subspace of $\mathbb{C}^3$ which is fixed by

B. It follows that every SO(3, $\mathbb{C}$)-equivariant $\mathbb{C}^3$-valued (1, 0)-form on $Q \setminus 0$ is equal to a constant multiple of Ω. I owe this observation to Erik van den Ban. $\oslash$

2.3 Complex Analytic Curves in Q

We now determine the real two-dimensional (immersed) submanifolds C of Q such that $\mathrm{d}\omega|_C = 0$, where $\omega = \mathrm{Re}\,\Omega$.

Because of the SO(3)-equivariance of ω, we only need to investigate the problem which real two-dimensional linear subspaces P of $\mathrm{T}_q\,Q$ are isotropic with respect to $(\mathrm{d}\omega)_q$ at points $q \in Q_{e_3}$, which corresponds to $g = I$ in the identification of q with the B-orbit of $(g,\, q)$ in SO(3, $\mathbb{C}$) $\times\, \mathbb{C}$. It follows from (2.2) that $\mathrm{d}\widetilde{\Omega}$ is nowhere zero on SO(3, $\mathbb{C}$) $\times\, \mathbb{C}$, which implies that $\mathrm{d}\Omega$ is nowhere zero on Q, because $\mathrm{d}\widetilde{\Omega} = \mathrm{d}(\psi^*\Omega) = \psi^*(\mathrm{d}\Omega)$, if ψ denotes the projection from SO(3, $\mathbb{C}$) $\times\, \mathbb{C}$ onto $Q \simeq \mathrm{SO}(3,\, \mathbb{C}) \times_B \mathbb{C}$. Therefore, $(\mathrm{d}\Omega)_q = \Theta\,(1,\, \mathrm{i},\, 0)$, in which Θ is a nonzero antisymmetric complex bilinear form on $\mathrm{T}_q\,Q$.

If $a,\, b \in \mathrm{T}_q\,Q$, then

$$0 = \omega_q(a,\, b) = \mathrm{Re}\,\Omega_q(a,\, b) = \mathrm{Re}\,(\Theta(a,\, b)(1,\, \mathrm{i},\, 0))$$

$\Longleftrightarrow (\mathrm{Re}\,\Theta(a,\, b) = 0 \,\&\, \mathrm{Re}\,(\Theta(a,\, b)\,\mathrm{i}) = 0) \Longleftrightarrow \Theta(a,\, b) = 0.$

Because $\mathrm{T}_q\,Q$ is a two-dimensional complex vector space and Θ is a nonzero antisymmetric complex bilinear $\mathbb{C}$-valued form on $\mathrm{T}_q\,Q$, we have that $\Theta(a,\, b) = 0$ if and only if a and b are linearly dependent over $\mathbb{C}$.

Proposition 2.1 *If C is a real two-dimensional submanifold of Q, then $\omega|_C$ is closed if and only if C is a complex analytic curve in Q.*

If M is a smoothly immersed minimal surface in $\mathbb{R}^3$, provided with the complex structure which makes the Gauss map $n : M \to S$ complex analytic, then the mapping $n' : M \to Q$ is also complex analytic.

Proof Let P be a real two-dimensional linear subspace of $\mathrm{T}_q\,Q$. Then $\mathrm{d}\omega_p$ is equal to zero on $P \times P$ if and only if $\Theta|_{P\times P} = 0$, which is certainly the case if P is a one-dimensional complex-linear subspace of $\mathrm{T}_q\,Q$. Conversely, if $\Theta_{P\times P} = 0$ and $a \in P$, $a \neq 0$, then every $b \in P$ is a complex multiple of b, which implies that P is a complex-linear subspace of $\mathrm{T}_q\,Q$. Therefore, $\mathrm{d}\omega_p$ is equal to zero on $P \times P$ if and only if P is a one-dimensional complex-linear subspace of $\mathrm{T}_q\,Q$.

The second statement follows from the "only if" part of the first statement. q.e.d.

Remark 2.3 The (almost) complex structure on Q with the above property is unique up to its sign. Indeed, suppose that J is a complex structure in $\mathrm{T}_q\,Q$, which means that J is a real linear mapping from $\mathrm{T}_q\,Q$ to itself, such that $J^2 = -I$. Suppose moreover that every one-dimensional complex linear subspace P of $\mathrm{T}_q\,Q$ is also a complex-linear subspace with respect to J, which means that $J(P) = P$. This implies that for every nonzero $a \in \mathrm{T}_q\,Q$ there exist unique real numbers $\alpha = \alpha(a)$ and $\beta = \beta(a)$, depending smoothly on a, such that $J(a) = \alpha(a)\,a + \beta(a)\,\mathrm{i}\,a = \gamma(a)\,a$, where we have written $\gamma(a) = \alpha(a) + \beta(a)\mathrm{i}$. Replacing a by $a+b$ such that a and b are linearly independent over $\mathbb{C}$, we obtain from $J(a+b) = J(a)+J(b)$ that $\gamma(a+b)\,(a+b) = \gamma(a)\,a + \gamma(b)\,b$, hence $\gamma(a) = \gamma(a+b) = \gamma(b)$. Because these $(a,\, b)$

are dense in $\mathrm{T}_q\, Q \times \mathrm{T}_q\, Q$, it follows that γ is a constant. But then $J^2 = -I$ is equal to multiplication by means of the constant γ^2, which implies that $\gamma = \pm\,\mathrm{i}$. This proves that the complex structure J on $\mathrm{T}_q\, Q$, such that the set of the $(\mathrm{d}\omega)_q$-isotropic two-dimensional real linear subspaces of $\mathrm{T}_q\, Q$ is equal to the set of the complex one-dimensional linear subspaces with respect to J, is uniquely determined up to its sign. $\oslash$

2.4 The Weierstrass Representation

Let M be an oriented two-dimensional real submanifold of $\mathbb{R}^3$ and $x \in M$. Then $\mathrm{T}_x\, M$ has a unique complex structure J_x such that J_x is antisymmetric with respect to the Euclidean inner product in $\mathrm{T}_x\, M$ and such that, for every nonzero $v \in \mathrm{T}_x\, M$, the pair $v,\, J_x(v)$ is positively oriented. (In view of ${J_x}^2 = -I$, the antisymmetry of J_x is equivalent to the condition that J_x is an orthogonal linear transformation.)

Recall the identity mapping $\iota : M \to \mathbb{R}^3$, viewed as an $\mathbb{R}^3$-valued smooth function on M, which had been used in (1.3). For each $x \in M$, we view the $\mathbb{R}^3$-valued one-form $a := \mathrm{d}\iota_x$ on $\mathrm{T}_x\, M$ as a linear mapping from $\mathrm{T}_x\, M$ to $\mathbb{R}^3$. There is a unique complex-linear mapping $\Phi_x : \mathrm{T}_x\, M \to \mathbb{C}^3$ such that, for every $v \in \mathrm{T}_x\, M$, $\mathrm{d}\iota_x(v) = \mathrm{Re}\, \Phi_x(v)$. Indeed, if we write $b(v) = \mathrm{Im}\, \Phi_x(v)$, then $v \mapsto a(v) + \mathrm{i}\, b(v)$ is complex-linear if and only if $\mathrm{i}\,(a(v) + \mathrm{i}\, b(v))$ is equal to $a\,(J_x(v)) + \mathrm{i}\, b\,(J_x(v))$, which in turn is equivalent to $a(v) = b\,(J_x(v))$ and $-b(v) = a\,(J_x(v)$. The second equation implies the first in view of ${J_x}^2 = -I$, and we arrive at the conclusion that

$$\Phi_x := \mathrm{d}\iota_x - \mathrm{i}\ \mathrm{d}\iota_x \circ J_x \tag{2.6}$$

is the unique complex-linear mapping $\Phi_x : \mathrm{T}_x\, M \to \mathbb{C}^3$ such that $\mathrm{d}\iota_x = \mathrm{Re}\, \Phi_x$. In other words, there is a unique $\mathbb{C}^3$-valued $(1,\, 0)$-form Φ on M, such that

$$\mathrm{d}\iota = \mathrm{Re}\, \Phi. \tag{2.7}$$

Remark 2.4 It follows from (2.6) that, for each $v \in \mathrm{T}_x\, M$, the real and imaginary part of $\Phi_x(v)$ have the same length and are orthogonal to each other, which is equivalent to the statement that $\langle \Phi_x(v),\, \Phi_x(v) \rangle = 0$. In other words, Φ *takes its values in the isotropic cone* $\widetilde{N}$ rather than in $\mathbb{C}^3$. In turn this implies that, for each $x \in M$, $\Phi_x\,(\mathrm{T}_x\, M) \in N$, where N is the quadric in the complex projective plane defined by the equation $\langle z,\, z \rangle = 0$. $\oslash$

If M is a minimal surface with nonzero curvature, then M has a unique complex structure J such that the Gauss map $n : M \to S$ is complex analytic, where we have provided S with the complex structure via the diffeomorphism $\sigma : N \to S$ defined by (2.5). Note that $q = n'(x)$ reverses the orientation if we identify $\mathrm{T}_x\, M$ and $\mathrm{T}_{n(x)}\, S$ by means of a translation, which means that $n(x) = -\langle v,\, v \rangle^{-1}\, v \wedge J_x(v)$, if $v \in \mathrm{T}_x\, M$ and $v \neq 0$. The definitions (2.6) and (2.5) therefore imply that the Gauss map can be expressed in terms of Φ by means of

$$n(x) = \sigma\left(\Phi_x\left(\mathrm{T}_x\, M\right)\right), \quad x \in M, \tag{2.8}$$

cf. DHKW [4, (22) on p. 94].

It follows from (2.7) and (1.3) that $\mathrm{Re}\, \Phi = \mathrm{d}\iota = (n')^*\omega = (n')^*\, \mathrm{Re}\, \Omega = \mathrm{Re}\left((n')^*\Omega\right)$, which in view of the uniqueness of the complexification of $\mathrm{d}\iota$ implies that

$$\Phi = (n')^*\Omega. \tag{2.9}$$

Note that both Φ and Ω are $\tilde{N}$-valued, cf. Remark 2.1. Because n' is a holomorphic mapping from M to Q and Φ is holomorphic on $Q \setminus 0$, we recover in this way the classical fact that *M is a minimal surface if and only if the $\mathbb{C}^3$-valued* $(1, 0)$*-form Φ on M is holomorphic*, cf. DHKW [4, Sec. 2.6 and 3.1]. The equation (2.9) implies that *all the one-forms Φ on the different minimal manifolds M are pull-backs of one and the same canonical $\mathbb{C}^3$-valued holomorphic* $(1, 0)$*-form Ω on the fixed space Q*. Only the mapping n', with which Ω is pulled back, varies with the minimal surface.

Any $(1, 0)$-form on a one-dimensional complex analytic manifold is holomorphic if and only if it is closed. Suppose that, for any base point $x_0 \in M$, we have chosen a point $z_0 \in \mathbb{C}^3$ such that $\mathrm{Re}\, z_0 = x_0$. Then we define, for every $x \in M$, the point $z = z(x) \in \mathbb{C}^3$ by

$$z = z_0 + \int_{x_0}^{x} \Phi, \tag{2.10}$$

in which the integral of the $\mathbb{C}^3$-valued one-form Φ is taken along any smooth curve γ in M which runs from x_0 to x. Note that the point z is only unique modulo periods $\langle [\gamma], [\Phi] \rangle$, where $[\gamma]$ denotes the homology class, in the image of $\mathrm{H}_1(M, \mathbb{Z})$ in $\mathrm{H}_1(M, \mathbb{R})$, of a closed loop γ in M and $[\Phi]$ denotes the de Rham cohomolgy class of Φ in $\mathrm{H}^1_{\text{de Rham}}(M) \otimes \mathbb{C}^3$. It follows from (2.7) and $\mathrm{Re}\, z_0 = x_0$ that

$$\mathrm{Re}\, z(x) = x, \quad x \in M, \tag{2.11}$$

which in turn implies that the periods of (2.10) are purely imaginary. The periods of (2.10) form an additive subgroup of $\mathrm{i}\,\mathbb{R}^3$, generated by the $\langle [\gamma_j], [\Phi] \rangle$, if the $[\gamma_j]$ generate the image of $\mathrm{H}_1(M, \mathbb{Z})$ in $\mathrm{H}_1(M, \mathbb{R})$.

The $z(x)$, $x \in M$, form a complex analytic curve , in $\mathbb{C}^3$, such that $M = \mathrm{Re}\,$, . Here $\mathrm{Re} : \mathbb{C}^3 \to \mathbb{R}^3$ is the projection which sends $z = x + \mathrm{i}\, y \in \mathbb{C}^3$, with $x, y \in \mathbb{R}^3$, to its real part $x = \mathrm{Re}\, z$. We have $\mathrm{T}_{z(x)}$, $= \Phi_x (\mathrm{T}_x M)$. It follows from Remark 2.4 that the curve , is *isotropic* in the sense that, for every $z \in$, , we have that T_z , $\in N$.

Conversely, for every isotropic complex analytic curve , in $\mathbb{C}^3$, the real two-dimensional manifold $M = \mathrm{Re}\,$, is a minimal surface. This leads to an identification between minimal surfaces in $\mathbb{R}^3$ and isotropic complex analytic curves in $\mathbb{C}^3$. The various *representation formulas of Weierstrass type*, as discussed in DHKW [4, Sec. 3.1 and 3.3], consist of constructions of isotropic complex analytic curves in $\mathbb{C}^3$. These are obtained by the integration of a suitable complex analytic function $\phi : D \to \mathbb{C}^3$, where D is an open subset of $\mathbb{C}$, such that, for every $\zeta \in D$, $\phi(\zeta) \neq 0$ and $\langle \phi(\zeta), \phi(\zeta) \rangle = 0$.

Let C be a complex analytic curve in $Q \setminus 0$. Let , be the complex analytic curve in $\mathbb{C}^3$ which is obtained by integration of $\Omega|_C$, and let $M = \mathrm{Re}\,$, be the minimal surface in $\mathbb{R}^3$, which is obtained by integration of $\omega|_C = \mathrm{Re}\, \Omega|_C$. That is, , is defined by (1.4), with $\omega = q^{-1}\, \mathrm{d}s$ replaced by Ω. It then follows from $C = n'(M)$ and (2.9) that , is equal to the complex curve in $\mathbb{C}^3$ which is obtained from the minimal surface M in $\mathbb{R}^3$ by integration of the $\mathbb{C}^3$-valued holomorphic $(1, 0)$-form Φ on M.

2.5 SL(2, C)

For computations, the group $\mathrm{SL}(2, \mathbb{C})$, of all complex 2×2-matrices with determinant equal to one, is easier to work with than the complex rotation group $\mathrm{SO}(3, \mathbb{C})$. The element $g \in \mathrm{SL}(2, \mathbb{C})$ acts on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of traceless complex 2×2-matrices by means of the conjugation $\mathrm{Ad}\, g : X \mapsto g\, X\, g^{-1}$.

If we use the identification

$$\Xi(x) = \begin{pmatrix} \mathrm{i}\, x_3 & \mathrm{i}\, x_1 - x_2 \\ \mathrm{i}\, x_1 + x_2 & -\mathrm{i}\, x_3 \end{pmatrix}, \quad x \in \mathbb{C}^3. \tag{2.12}$$

of $\mathbb{C}^3$ with $\mathfrak{sl}(2, \mathbb{C})$, then

$$\mathrm{trace}(\Xi(x)\,\Xi(y)) = -2\langle x,\, y\rangle, \quad x,\, y \in \mathbb{C}^3. \tag{2.13}$$

Because the trace is invariant under conjugation, it follows that the adjoint representation (= the action on the Lie algebra by means of conjugation) defines a Lie group homomorphism $\mathrm{Ad} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$.

Because $\mathrm{ad} = \mathrm{T}_I \mathrm{Ad}$ is injective, the adjoint representation is a covering map from $\mathrm{SL}(2, \mathbb{C})$ onto a subgroup of $\mathrm{SO}(3, \mathbb{C})$ of the same dimension as $\mathrm{SL}(2, \mathbb{C})$. Because both $\mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$ are complex 3-dimensional and $\mathrm{SO}(3, \mathbb{C})$ is connected, it follows that $\mathrm{Ad}\,(\mathrm{SL}(2, \mathbb{C})) = \mathrm{SO}(3, \mathbb{C})$. Because the kernel of the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$ is equal to $\{\pm I\}$, the adjoint representation defines a two-fold covering from $\mathrm{SL}(2, \mathbb{C})$ onto $\mathrm{SO}(3, \mathbb{C})$, or equivalently an isomorphism

$$\mathrm{Ad} : \mathrm{SL}(2, \mathbb{C})/\{\pm I\} \xrightarrow{\sim} \mathrm{SO}(3, \mathbb{C}). \tag{2.14}$$

Lemma 2.2 *$\mathrm{Ad}^{-1}(B)$ is equal to the group L of all lower triangular matrices in $\mathrm{SL}(2, \mathbb{C})$. The homomorphisms $\mathrm{Ad} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$ and $\mathrm{Ad} : L \to B$ induce an isomorphism*

$$\mathrm{SL}(2, \mathbb{C}) \times_L \mathbb{C} \xrightarrow{\sim} \mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C} \simeq Q, \tag{2.15}$$

if we let act the element $g = \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix}$ on $\mathbb{C}$ by means of multiplication by a^4. The pull-back of Ω under (2.15) is an equivariant $\mathbb{C}^3 \simeq \mathfrak{sl}(2, \mathbb{C})$-valued $(1, 0)$-form on $\mathrm{SL}(2, \mathbb{C}) \times_L \mathbb{C} \setminus \{0\}$, which we also denote by Ω. Its pull-back to $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C} \setminus \{0\}$ is given by

$$\widetilde{\Omega}_{(g,\, q)}(\delta g,\, \delta q) = \frac{\mathrm{i}}{q}\, \mathrm{trace}\left(g^{-1}\, \delta g\, Y\right) \cdot g\, Y\, g^{-1}. \tag{2.16}$$

Proof We have $Y := \Xi\,(e_1 + \mathrm{i}\, e_2) = \begin{pmatrix} 0 & 0 \\ 2\mathrm{i} & 0 \end{pmatrix}$. Furthermore, if $g \in \mathrm{SL}(2, \mathbb{C})$ then $g\, Y\, g^{-1} = \rho\, Y$ for some $\rho \in \mathbb{C}$ if and only if $g = \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix}$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, in which case $\rho = a^{-2}$. This proves the first statement.

In the definition of $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C}$, the element $b \in B$ acts on $\mathbb{C}$ by means of multiplication by ρ^{-2}, if $b\,(e_1 + \mathrm{i}\, e_2) = \rho\,(e_1 + \mathrm{i}\, e_2)$. This implies the second statement.

For the proof of (2.16), we recall the definition (2.2) of Ω on $\mathrm{SO}(3, \mathbb{C}) \times_B \mathbb{C} \setminus \{0\}$. The infinitesimal action of $X = g^{-1}\, \delta g = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ on $\Xi\,(e_3) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$ is equal to the commutator $X \circ \Xi\,(e_3) - \Xi\,(e_3) \circ X = \begin{pmatrix} 0 & -2\mathrm{i}\, b \\ 2\mathrm{i}\, c & 0 \end{pmatrix}$ of X and $\Xi\,(e_3)$. Its product with Y is equal to $\begin{pmatrix} 4b & 0 \\ 0 & 0 \end{pmatrix}$, of which the trace divided by -2, corresponding to the inner product in (2.2), is equal to $-2b$. On the other hand the trace of $X\,Y$ is equal to $2\,\mathrm{i}\,b$, which multiplied by i yields $-2b$, which completes the proof of (2.16). q.e.d.

Remark 2.5 In view of (2.13), the isotropic cone $\widetilde{N}$ of the $x \in \mathbb{C}^3$ such that $\langle x,\, x\rangle = 0$ corresponds to the set of $X \in \mathfrak{sl}(2,\, \mathbb{C})$ such that $\mathrm{trace}\,(X^2) = 0$, which are precisely the nilpotent elements in $\mathfrak{sl}(2,\, \mathbb{C})$. Note that the element Y in (2.16) is nilpotent. In other words, the $\mathfrak{sl}(2,\, \mathbb{C})$-valued one-form Ω on $\mathrm{SL}(2,\, \mathbb{C}) \times_L \mathbb{C}$ actually takes its values in the *nilpotent cone* in $\mathfrak{sl}(2,\, \mathbb{C})$. $\oslash$

The element $g \in \mathrm{SL}(2,\, \mathbb{C})$ acts on the complex projective line $\mathbb{CP}^1$ by sending the one-dimensional complex linear subspace l of $\mathbb{C}^2$ to $g(l)$. If e_2 denotes the second standard basis vector in $\mathbb{C}^2$ and $l = \mathbb{C}\,e_2$, then $g(l) = l$ if and only if $g \in L$. Because the action of $\mathrm{SL}(2,\, \mathbb{C})$ on $\mathbb{CP}^1$ is transitive, the mapping $g \mapsto g(l)$ induces a isomorphism from $\mathrm{SL}(2,\, \mathbb{C})/L$ onto $\mathbb{CP}^1$. This leads to an identification of

i) the unit sphere $S \simeq \mathrm{SO}(3)/\,\mathrm{SO}(2)$ in $\mathbb{R}^3$,

ii) the quadric $N \simeq \mathrm{SO}(3,\, \mathbb{C})/B$ in the complex projective plane, and

iii) the complex projective line $\mathbb{CP}^1 \simeq \mathrm{SL}(2,\, \mathbb{C})/L$

with each other.

The standard projective coordinate in the neighborhood $\mathbb{CP}^1 \setminus \{\mathbb{C}e_1\}$ of $\mathbb{C}\,e_2$ in $\mathbb{CP}^1$, is obtained by sending $u \in \mathbb{C}$ to the coset $g\,B \in \mathrm{SL}(2,\mathbb{C})/L$ of g, where $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. The mapping which sends $(u,\, q) \in \mathbb{C}^2$ to the L-orbit of $(g,\, q) \in \mathrm{SL}(2,\, \mathbb{C}) \times \mathbb{C}$ is the corresponding trivialization of the complex line bundle Q.

We obtain that $g^{-1}\,\delta g = \begin{pmatrix} 0 & \delta u \\ 0 & 0 \end{pmatrix}$, the product of which with Y has trace equal to $2\,\mathrm{i}\,\delta u$. Furthermore

$$g\,Y\,g^{-1} = \begin{pmatrix} 2\mathrm{i}\,u & -2\mathrm{i}\,u^2 \\ 2\mathrm{i} & -2\mathrm{i}\,u \end{pmatrix} = \Xi(x), \quad x = \left(1 - u^2,\, \mathrm{i}\,\left(1 + u^2\right),\, 2u\right). \tag{2.17}$$

It follows that, in the $(u,\, q)$-coordinates on Q,

$$\Omega = \frac{-2}{q}\,\mathrm{d}u\,\left(1 - u^2,\, \mathrm{i}\,\left(1 + u^2\right),\, 2u\right). \tag{2.18}$$

Recall that ω is equal to the real part of (2.18).

Remark 2.6 Suppose that the complex curve C is locally equal to the graph of a complex analytic function $q = q(u)$, in which u is the coordinate on $\mathbb{CP}^1$ which has been used in (2.18). If we write $\mathcal{F}(u) = -2/q(u)$, then (2.18) coincides with the *representation formula of Weierstrass*, which is given in DHKW [4, p.113].

However, the formula (2.18) is more explicit, because of the interpretation of the variable u as an analytic coordinate of the unit sphere, a coordinate for the image of the Gauss map, whereas q represents the second order contact of the minimal surface M, if $C = n'(M)$ and M is obtained from C by means of integration of $\omega|_C = \Omega|_C$. $\oslash$

Remark 2.7 It follows from (2.17) that

$$\sigma\left(\mathbb{C} g\, Y\, g^{-1}\right) = \left(1 + {u_1}^2 + {u_2}^2\right)^{-1} \left(-2u_1,\, -2u_2,\, 1 - {u_1}^2 - {u_2}^2\right), \quad u = u_1 + \mathrm{i}\, u_2,$$

in which σ denotes the isomorphism from the quadric N in the complex projective plane to the unit sphere S as defined in (2.5). The right hand side is the formula for the stereographic projection from the plane to the unit sphere minus the south pole $-e_3$, but with a reversed orientation. The reversal of the orientation corresponds to the reversal of orientation of the derivative of the Gauss map. $\oslash$

The standard projective coordinate in the coordinate neighborhood of the missing point is obtained by sending $v \in \mathbb{C}$ to the coset $h\, B \in \mathrm{SL}(2,\, \mathbb{C})/L$ of g, where $h = \begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$. If $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $k = \begin{pmatrix} a & 0 \\ b & 1/a \end{pmatrix}$, then $h\, k = g$ if and only if $a = 1/u$ and $b = u$. Because $(g,\, q)$ and $(h,\, a^4\, q)$ are in the same L-orbit in $\mathrm{SL}(2,\, \mathbb{C}) \times \mathbb{C}$, they define the same point in $\mathrm{SL}(2,\, \mathbb{C}) \times_L \mathbb{C} \simeq Q$, which means that

$$(u,\, q) \mapsto \left(u^{-1},\, u^{-4}\, q\right) \tag{2.19}$$

is the corresponding coordinate transformation (= retrivialization) of Q. The complex line bundle Q can be constructed by glueing two copies of $\mathbb{C}^2$ together on $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ by means of the mapping (2.19).

As a consequence, the global holomorphic sections of Q are of the form $q = f_0(u)$ where f_0 is a polynomial of degree fours, cf. Section 3.4. In other words, the Chern number or degree of the holomorphic line bundle Q over $\mathbb{CP}^1$ is equal to four, or $Q \simeq \mathcal{O}(4)$, cf. Griffiths and Harris [8, pp. 144, 145].

Remark 2.8 The description with $\mathrm{SL}(2,\, \mathbb{C})$ leads to an easy determination of the automorphism group of the complex line bundle B. Let Φ be an automorphism of the complex line bundle Q, in the sense that Φ is a complex analytic diffeomorphism on Q which maps each fiber to another fiber by means of a complex linear mapping. Viewing the fibers as the points of the base space $\mathbb{CP}^1$, this leads to a complex analytic diffeomorphism Ψ on $\mathbb{CP}^1$. It is known that each such Ψ is equal to the action of an element g of $\mathrm{SL}(2,\, \mathbb{C})$, where g is uniquely determined up to it sign. See Griffiths and Harris [8, p. 64]. Let Φ_0 be equal to the composition of Φ and the left action of g^{-1} on $Q \simeq \mathrm{SL}(2,\, \mathbb{C}) \times_L \mathbb{C}$. Then Φ_0 maps each fiber of Q to itself by means of multiplication by a nonzero complex number, which depends in a complex analytic fashion on the base point in $\mathbb{CP}^1$. Because $\mathbb{CP}^1$ is compact, it follows from the maximum principle that this function is a constant.

We conclude that the automorphism group of the complex line bundle Q is equal to $(\mathrm{SL}(2,\, \mathbb{C})/\pm I) \times (\mathbb{C} \setminus \{0\})$, where the action of the element c of the multiplicative group $\mathbb{C} \setminus \{0\}$ is equal to multiplication by c in the fibers. Via the isomorphisms (2.14) and (2.15), we obtain the equivalent statement that *the automorphism group of the complex line bundle Q is equal to the Cartesian product of the left action of* $\mathrm{SO}(3,\, \mathbb{C})$ *and the multiplications by nonzero complex numbers on the fibers.*

Recall that the equivariance of Ω under the left action of $\mathrm{SO}(3,\,\mathbb{C})$ says that if L_g denotes the left action of $g \in \mathrm{SO}(3,\,\mathbb{C})$ on $Q \setminus 0$, then $\mathrm{L}_g^*\,\Omega = g\,\Omega$, where in the right hand side we let act g on the values in $\mathbb{C}^3$ of Ω. On the other hand the multiplication by $c \in \mathbb{C} \setminus \{0\}$ in the fibers of $Q \setminus \{0\}$ sends Ω to $c^{-1}\,\Omega$, cf. (2.2). $\oslash$

Remark 2.9 Let M be a minimal surface in $\mathbb{R}^3$ and let $C = n'(M)$ be the corresponding complex analytic curve in Q. The left action of $g \in \mathrm{SO}(3,\,\mathbb{C})$ maps C to the analytic curve $\mathrm{L}_g(C)$, and we obtain a corresponding a minimal surface ${}_gM$ in $\mathbb{R}^3$ by integrating ω over $\mathrm{L}_g(C)$. If $g \in \mathrm{SO}(3)$, then it follows from the $\mathrm{SO}(3)$-equivariance of ω that ${}_gM = g(M)$, which is obtained from M by applying the rotation g in $\mathbb{R}^3$. (As usual, we work modulo translations of the minimal surface.) If $g \in \mathrm{SO}(3,\,\mathbb{C})$ but $g \notin \mathrm{SO}(3)$, then the relation between M and ${}_gM$ is less straighforward. The best one can say is that if , is the isotropic complex analytic curve in C^3 such that $M = \mathrm{Re}\,,$, where , is obtained by means of integration of $\Omega|_C$, then the $\mathrm{SO}(3,\,\mathbb{C})$-equivariance of Ω implies that ${}_gM = \mathrm{Re}(g(,\,))$.

The action of multiplication by $c \in \mathbb{C} \setminus \{0\}$ on the fibers of Q maps the complex analytic curve C in Q to the complex analytic curve $c\,C$ in Q. If $c \in \mathbb{R}$, then the minimal surface which is obtained by integrating $\omega = q^{-1}\,\mathrm{d}s$ over $c\,C$ is equal to the $c^{-1}\,M$, a homothetic or reflected homothetic image of the minimal surface M if $c > 0$ or $c < 0$, respectively.

If on the other hand $c = \mathrm{e}^{\mathrm{i}\,\phi}$, $\phi \in \mathbb{R}$, then the minimal surfaces obtained by integrating ω over $c\,C$ form a circle of minimal surfaces, which is called the *family of minimal surfaces which is associated to* M, cf. DHKW [4, p. 96, 97]. If , is the isotropic complex analytic curve in $\mathbb{C}^3$ such that $M = \mathrm{Re}\,,$, then the minimal surfaces associated to M are the $\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\,\phi}\;,\right)$, $\phi \in \mathbb{R}$. The minimal surface $\mathrm{Re}(\mathrm{i}\;,\,)$ is called the *adjoint surface* of M, cf. DHKW [4, p. 91]. $\oslash$

Remark 2.10 A natural compactification $\overline{Q}$ (in algebraic geometry called completion) of Q arises as the $\mathbb{CP}^1$ bundle over $\mathrm{SL}(2,\,\mathbb{C})/L \simeq \mathbb{CP}^1$, which is defined by

$$\overline{Q} = \mathrm{SL}(2,\,\mathbb{C}) \times_L \mathbb{CP}^1, \tag{2.20}$$

in which $\begin{pmatrix} a & 0 \\ b & 1/a \end{pmatrix} \in L$ acts on $\mathbb{CP}^1$ by sending $\mathbb{C}(q,\,r)$ to $\mathbb{C}\,(a^4\,q,\,r)$, which is the projective action of the linear transformation $\begin{pmatrix} a^4 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$. Actually, the second order contact bundle of minimal surfaces in $\mathbb{R}^3$, modulo translations, is naturally defined as $\overline{Q}$ rather than as Q. The addition to Q of the points of $\overline{Q} \setminus Q$ at infinity corresponds to the allowance of singularities of the minimal surface at which the Gauss curvature tends to $-\infty$.

The bundle $\overline{Q}$ can be identified with the *fourth Hirzebruch surface*, cf. Barth, Peters and Van de Ven [1, p. 141]. In Hirzebruch [9], the complex surface Σ_n has been introduced as the algebraic subvariety of $\mathbb{CP}^1 \times \mathbb{CP}^2$, which is defined by the equation $q_1\,{u_1}^n - q_2\,{u_2}^n = 0$, if $[u_1 : u_2]$ and $[q_0 : q_1 : q_2]$ denote the projective coordinates in $\mathbb{CP}^1$ and $\mathbb{CP}^2$, respectively. The projection to $[u_1 : u_2]$ exhibits Σ_n as a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$. We may regard the subset where $q_1 = q_2 = 0$ as the section at infinity. If we delete this, then we obtain a complex line bundle over $\mathbb{CP}^1$. Over the $[u_1 : u_2]$ with $u_1 \neq 0$, this complex line bundle has the trivialization $(u,\,q) \mapsto ([1 : u],\,[q : u^n : 1])$, and over the $[u_1 : u_2]$ with $u_2 \neq 0$, it has the trivialization $(v, r) \mapsto ([v:1], [r:1:v^n])$. The image points are the same if and only if $v = u^{-1}$ and $r = u^{-n}\, q$, which is the retrivialization (2.19) if $n = 4$.

Because there is a polynomial embedding from $\mathbb{CP}^1 \times \mathbb{CP}^2$ onto an algebraic subvariety of $\mathbb{CP}^4$, cf. Shafarevich [20, p. 43], it follows that $\overline{Q} \simeq \Sigma_4$ is a complex projective variety. $\oslash$

Remark 2.11 The identification (2.12) of $\mathbb{C}^3$ with $\mathfrak{sl}(2, \mathbb{C})$ has been chosen such that we have $x \in \mathbb{R}^3$ if and only if $\Xi(x)$ is an anti-selfadjoint 2×2-matrix. These matrices form the Lie algebra of the special unitary group $\mathrm{SU}(2)$, the group of unitary 2×2-matrices with determinant equal to one. The group $\mathrm{SU}(2)$ is a so-called compact real form of $\mathrm{SL}(2, \mathbb{C})$. We have that $\mathrm{SU}(2)$ is diffeomorphic to the unit sphere in $\mathbb{R}^4$, hence simply connected, and the two-fold covering $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the classical way of exhibiting $\mathrm{SU}(2)$ as the universal covering of $\mathrm{SO}(3)$, cf. [5, Sec. 1.2.B].

$L \cap \mathrm{SU}(2)$ is equal to the group $\mathrm{U}(1)$ of the matrices $g = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ such that $a \in \mathbb{C}$ and $|a| = 1$. If $a = \mathrm{e}^{\mathrm{i}\,\psi}$, then $\mathrm{Ad}\, g$ is equal to the rotation about the vertical axis through the angle $\phi = 2\psi$. It follows that $\mathrm{Ad}\, \mathrm{U}(1) = \mathrm{SO}(2)$, where the doubling of the angle expresses that the adjoint representation is a two-fold covering. The homomorphisms $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ and $\mathrm{Ad} : \mathrm{U}(1) \to \mathrm{SO}(2)$ induce an isomorphism from $\mathrm{SU}(2)/\,\mathrm{U}(1)$ onto $\mathrm{SO}(3)/\,\mathrm{SO}(2) \simeq S$, which leads to still another model for the unit sphere S. $\oslash$

3 Some Applications

In this section we discuss some applications of the correspondence between minimal surfaces M in $\mathbb{R}^3$ and complex analytic curves C in Q. We will restrict our attention to the case that M is smoothly immersed in $\mathbb{R}^3$, although it would have been natural to also allow branch points, even with infinite curvature (in which case C is a curve in the compactification $\overline{Q}$ of Q).

3.1 Flat Points

Let M be a smoothly immersed, connected and non-flat minimal surface in $\mathbb{R}^3$. Then the set M_0 of points in M where the curvature vanishes is a closed subset of M, consisting of isolated points. Because the Gauss map $n = \pi \circ n$ is a local diffeomorphism on $M \setminus M_0$, it follows that the mapping n' is an immersion from $M \setminus M_0$ to a smooth (immersed) analytic curve C in $Q \setminus 0$ which intersects the fibers of the projection $\pi : Q \to S$ transversally.

If we use the orientation on M which makes the Gauss map orientation preserving and provide M with the corresponding complex structure, cf. Section 2.4, then n' is a complex analytic mapping from M to Q, where $x_0 \in M_0$ if and only if $n'\,(x_0)$ belongs to the zero section of Q. Let $x_0 \in M_0$ and let $z \mapsto x(z)$, $z \in D$ be a holomorphic (= isothermal) coordinate in an open neighborhood of x_0 in M. Here $D = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ is a disk around the origin in the complex plane and $x(0) = x_0$, which means that $z = 0$ corresponds to the point x_0. By means of a rotation in $\mathbb{R}^3$ we can arrange that $n\,(x_0) = e_3$. If we use the (u, q)-trivialization of Q over $S \setminus \{-e_3\}$ as in (2.18), cf. also Remark 2.7, then $z \mapsto n'(x(z))$ corresponds to a pair of complex analytic functions $z \mapsto u(z)$, $z \mapsto q(z)$ on D, where $u(z)$ is the u-coordinate of the

Gauss map. Note that $u(0) = 0$ because $n(x_0) = e_3$ and $q(0) = 0$ because the curvature of M is equal to zero at x_0. The pull-back of Ω under the mapping $z \mapsto n'(x(z))$ is equal to

$$\Omega|_C = \frac{-2}{q(z)}\, u'(z)\ \mathrm{d}z\ \left(1 - u(z)^2,\ \mathrm{i}\ \left(1 + u(z)^2\right),\ 2u(z)\right), \tag{3.1}$$

cf. (2.18). Recall that M is obtained from C by means of integration of $\omega = \operatorname{Re}\Omega$ over C, hence by means of integration of the real part of (3.1) over the disc D in the z-plane.

Remark 3.1 The formula (3.1) can be identified with the *Enneper-Weierstrass representation formula* as in DHKW [4, (7) on p. 108], if we take $\nu(z) = u(z)$ and $\mu(z) = -4u'(z)/q(z)$. However in contrast with our situation, it is assumed in the Enneper-Weierstrass representation formula that μ is holomorphic, ν is meromorphic, and $\mu\,\nu^2$ is holomorphic on a simply connected domain in $\mathbb{C}$. $\oslash$

Because M is not flat, the function $u(z)$ is not equal to a constant and also q is not constantly equal to zero. It follows that there are positive integers k and l such that

$$0 \leq i < k \Rightarrow u^{(i)}(0) = 0,\ u^{(k)}(0) \neq 0 \quad \text{and} \quad 0 \leq j < l \Rightarrow q^{(j)}(0) = 0,\ q^{(l)}(0) \neq 0. \tag{3.2}$$

An equivalent condition is that $u(z)/z^k$ and $q(z)/z^l$ extend to an analytic function $\widetilde{u}$ and $\widetilde{q}$ on D, respectively, such that $\widetilde{u}(0) = u^{(k)}(0)/k! \neq 0$ and $\widetilde{q}(0) = q^{(l)}(0)/l! \neq 0$.

Because for finite nonzero $\mathrm{d}z$ the correponding tangent vector $\mathrm{d}x$ of M has a finite and nonzero limit as $z \to 0$, we obtain from (3.1) that necessarily $l = k-1$. Conversely, if $l = k-1$, then the pull-back of Ω by means of the mapping $z \mapsto n'(x(z))$ is holomorphic and nonzero at $z = 0$, which implies that (3.1) defines a smooth piece of a minimal surface.

The number k in (3.2) is the local mapping degree of the Gauss map at the point x_0. For $s \in S$ close to e_3 there are k points x close to x_0 such that $n(x) = s$, and the curvature is not equal to zero at each of these points x.

It also follows from (3.2) that the local piece of the curve C intersects the zero section 0 of Q at $(e_3,\, 0)$ with multiplicity equal to l, and the fiber with multiplicity equal to k. In this case the intersection multiplicity with the fiber is one more than the intersection multiplicity with the zero section, which implies that

3.2 Flat Points at Infinity

It follows from the Puiseux theory that a subset C in the $(u,\, q)$-space near the origin is equal to the image of a complex analytic map $z \mapsto (u(z),\, q(z))$ which satisfies (3.2), if and only if it is equal to the zeroset of an analytic function $f(u,\, q)$ such that $f(0,\, 0) = 0$ and f has neither u nor q as a factor, cf. Lojasiewicz [15, p. 173]. In other words, if and only if C in a neighborhood U of $(e_3,\, 0)$ in Q is equal to a *one-dimensional complex analytic subset of* Q, which contains $(e_3,\, 0)$ but does not contain the intersection with U of the zero section or the fiber Q_{e_3}.

Note that, at $(e_3,\, 0)$, the intersection number of C with the fiber, which is equal to the local mapping degree of the restriction to C of the projection $\pi : Q \to S$, is equal to k, whereas the intersection number of C with the zero section is equal to l. C is smooth at $(e_3,\, 0)$ if $k = 1$ or $l = 1$. We have concluded at the end of Section 3.1 that $(e_3,\, 0)$ corresponds to a finite limit point x_0 of a minimal surface M if and only if $l \leq k-1$, with equality if and only if M is smooth at x_0. Note that $l = k - 1$ implies that C is tangent to the fiber, even if $k \geq 3$, when C has a singular point at $(e_3,\, 0)$.

We assume from now on that $l \geq k$, and study the behaviour of the immersed minimal surface M in $\mathbb{R}^3$ which is obtained by integration of $\omega = \mathrm{Re}\,\Omega$ over $C \setminus \{(e_3,\, 0)\}$. If $l \geq k$, then (3.1) has a convergent $\mathbb{C}^3$-valued Laurent series of the form $\sum_{i \geq 0} z^{-l+k-1+i}\, \mathrm{d}z\, c_i$, with $\mathbb{C}^3$-valued coefficients c_r and

$$c_0 = -\frac{2k\, \widetilde{u}(0)}{\widetilde{q}(0)}\, (1,\, \mathrm{i},\, 0) \neq 0. \tag{3.3}$$

Its integral therefore has a convergent expansion of the form

$$a + \sum_{i=0}^{l-k-1} \frac{z^{-l+k+i}}{-l+k+i}\, c_i + (\log z)\, c_{l-k} + \sum_{j=l-k+1}^{\infty} \frac{z^{-l+k+j}}{-l+k+j}\, c_j, \tag{3.4}$$

where $a \in \mathbb{C}^3$ is a constant, and the finite sum of the poles is absent if $l = k$. In particular the immersed minimal surface M, obtained by integrating the real part of (3.1), tends to infinity when $l \geq k$.

The multi-valuedness of the logarithm leads to a period vector

$$p = \mathrm{Re}\,(2\pi\, \mathrm{i}\ c_{l-k}) \in \mathbb{R}^3, \tag{3.5}$$

which is a nonzero vector in the horizontal plane when $l = k$. If $l > k$, then it can happen that $p = 0$.

Let $l > k$. If z remains in a sector $0 < |z| < \epsilon$, $\alpha_1 < \arg z < \alpha_2$, then the term in (3.4) with $i = 0$ dominates, and it follows that the image is asymptotically equal to the graph of a function $x_3 = f(y)$ over a domain in the $y = (x_1,\, x_2)$-plane which asymptotically (for $\epsilon \downarrow 0$) is close to a sector with $\|y\|$ larger than a constant times $(1/\epsilon)^{l-k}$ and angle $(l-k)\,(\alpha_2 - \alpha_1)$. If $(l-k)\,(\alpha_2 - \alpha_1) > 2\pi$, then the function f has to be interpreted as a multi-valued function on the overlap. The gradient of f converges to zero if $\|y\| \to \infty$ in the sector.

If the period vector p in (3.5) is equal to zero, then $f(y)$ returns to the same value after y has made $l - k$ turns in the sector, which forces M to have self-intersections along real one-dimensional curves if $l - k > 1$. If $p \neq 0$, then a local piece U of M will return to $U + p$ when y, with large $\|y\|$, has made $l - k$ turns in the plane. The number $l - k$ is called the *spinning number of the end at infinity* of M if $p = 0$, cf. Hoffman and Karcher [10, (2.16) on p. 20]. We will use this name for $l - k$ also in the case that the period vector p is not equal to zero. Note that *if there is no periodicity, then the end at infinity is embedded, if and only if the spinning number is equal to one.*

The leading term in the asymptotic expansion for $f(y)$ as y runs to infinity in the sector is equal to a nonzero constant times $\log \|y\|$, if for each $0 < i < l - k$ the third component of c_r is equal to zero, but the real part of the third component of c_{l-k} is not equal to zero. In all other cases, $f(y)$ converges to a constant as y runs to infinity in the sector. In the literature, where one assumes that the period vector p is equal to zero, the end is called *flat* or *planar* if $f(y)$ converges to a constant, and *of catenoid type* if $f(y)$ has logarithmic growth, cf. DHKW [4, p. 198].

If $l = k$, (when the period vector p in (3.5) is always nonzero) then the minimal surface image of the sector $0 < |z| < \epsilon$, $\alpha_1 < \arg z < \alpha_2$ lies over a subset of the y-plane which is asymptotic to a half strip with width equal to $|c_0|\,(\alpha_2 - \alpha_1)$, at a distance to the origin equal to $|c_0| \log(1/\epsilon)$. Over it, the function $f(y)$ converges to the real part of the third component of a as y runs to infinity.

Now assume conversely that M is a smoothly immersed nonflat minimal surface in $\mathbb{R}^3$. For M, we introduce the following conditions.

i) M is given by means of a smooth conformal immersion from the punctured disc $D_\rho \setminus \{0\}$ to $\mathbb{R}^3$, where $D_\rho = \{z \in \mathbb{C} \mid |z| < \rho\}$. The orientation of M can be chosen such that, if $n(x)$ denotes the corresponding normal to M at x, the pull-back $z \mapsto n(\epsilon(z))$ of the Gauss mapping by means of ϵ is a complex analytic mapping from $D_\rho \setminus \{0\}$ to the Riemann sphere S. It follows that $n' \circ \epsilon$ is a complex analytic mapping from $D_\rho \setminus \{0\}$ to Q, cf. Proposition 2.1.

ii) In order to allow for periodicity, as should be done according to DHKW [4, p. 196], we allow that ϵ is multi-valued. However, we will assume that if $x = \epsilon(z)$ has moved to $\widetilde{x}$ when z has run around the origin once in $D_\rho \setminus \{0\}$, in the positive direction, then $n(x) = n(\widetilde{x})$ and $n'(\widetilde{x}) = n'(x)$. In other words, if we write $p = \widetilde{x} - x$, then the translate $M + p$ of M osculates M at $\widetilde{x}$. This assumption is equivalent to the condition that $z \mapsto n'(\epsilon(z))$ is a single-valued complex analytic mapping from $D_\rho \setminus \{0\}$ to Q. In the latter case the integral of ω over the image has p as a period, which means that actually $M + p = M$.

iii) There exists $0 < \sigma \leq \tau$ such that the image $\{n(\epsilon(z)) \mid z \in D_\sigma \setminus \{0\}\}$ of the Gauss map misses at least three points of S.

iii') For any $0 < \sigma < \rho$, the integral of the Gaussian curvature K over $\epsilon(D_\sigma)/\mathbb{Z}\, p$ is finite.

iv) If $\gamma : [0, \infty[\to D_\rho \setminus \{0\}$ is a C^1 curve such that $\lim_{t\to\infty} \gamma(t) = 0$, then the curve $t \mapsto \epsilon(\gamma(t))$ in $\mathbb{R}^3$ has infinite Euclidean length.

Lemma 3.1 *The following conditions* a), b), *and* c) *are equivalent.*

a) *Let C_σ denote the image $n' \circ \epsilon(D_\sigma \setminus \{0\})$ of the minimal surface $M_\sigma = \epsilon(D_\sigma \setminus \{0\})$ under the mapping $n' : M_\sigma \to Q$. Then there exists $0 < \sigma < \rho$ such that C_σ, after adding a point q_0 on the zero section of Q, is equal to the germ at q_0 of a complex one-dimensional analytic subset C of Q, where the intersection number k of C at q_0 with the fiber is less than or equal to the intersection number l of C at q_0 with the zero section.*

b) i) & ii) & iii') & iv).

c) i) & ii) & iii) & iv).

Proof For the a) $\Longrightarrow$ b) we only need to prove that (iii') follows from the description of C. Note that M is obtained by means of integration of ω over $C \setminus \{q_0\}$, which implies that $n(\epsilon(z))$ converges to q_0 when $z \to 0$. The Gaussian curvature K is equal to minus the Jacobi determinant of the Gauss map $n : M \to S$, if we provide M and S with the Euclidean non-oriented area forms $\mathrm{d}_2 x$ and $\mathrm{d}_2 s$, respectively. Furthermore the restriction to C of the projection $\pi : Q \to S$ has local mapping degree equal to k. It therefore follows that

$$\int_{n^{-1}(U)\cap M/\mathbb{Z}p} K(x)\ \mathrm{d}_2 x = -k\ \mathrm{area}(U), \tag{3.6}$$

if U is a small open neighborhood of q_0 in S and we take the pre-image $n^{-1}(U)$ of U under the Gauss map in M. If the period vector p in (3.5) is nonzero, then M is taken modulo the translates over integral multiples of p. Because $\lim_{z\to 0} n(\epsilon(z)) = q_0$, there exists $0 < \tau < \sigma$ such that $D_\tau \setminus \{0\}(n \circ \epsilon)^{-1}(U)$, which in view of (3.6) implies that the integral of K over $\epsilon\,(D_\tau \setminus \{0\})\,/\mathbb{Z}\,p$ is finite. Because the integral of K over $\epsilon\,(D_\sigma \setminus D_\tau)\,/\mathbb{Z}\,p$ is obviously finite, the conclusion iii') follows.

If iii') holds then the integral of K over $\epsilon\,(D_\sigma \setminus \{0\})\,/\mathbb{Z}\,p$ converges to zero as $\sigma \to 0$. The argument for (3.6) implies that the area of $n \circ \epsilon\,(D_\sigma \setminus \{0\})$ is less than or equal to $-k$ times the integral of K over $\epsilon\,(D_\sigma \setminus \{0\})\,/\mathbb{Z}\,p$, which implies that there exists $0 < \sigma \leq \tau$ such that the area of $n \circ \epsilon\,(D_\sigma \setminus \{0\})$ is strictly smaller than 4π, the area of S. This implies iii), and therefore we have proved that b) $\Longrightarrow$ c).

Now assume that c) holds. Using the great Picard theorem, cf. Conway [3, p. 300], we obtain from iii) that $n(\epsilon(z))$ extends to a complex analytic mapping from D to S. (I learned this argument from Osserman [17, p. 397].) By means of a rotation in $\mathbb{R}^3$, we can arrange that the limit point $\lim_{z\to 0}\ n(\epsilon(z))$ is equal to e_3. Using the trivialization of the bundle Q over $S\setminus\{-e_3\}$ by means of the $(u,\, q)$-coordinates in (2.18), the mapping $z \mapsto n'(\epsilon(z)) : D\setminus\{0\} \to Q$ is represented by two complex analytic functions $u(z)$ and $q(z)$ on $D\setminus\{0\}$, where $u(z)$ extends to a complex analytic function on D such that $u(0) = 0$.

If $0 < \sigma < \rho$, then there is a $\delta > 0$ such that, for each $z \in D_\sigma \setminus \{0\}$, the distance $d(\epsilon(z))$ from $x = \epsilon(z)$ to the boundary of M is larger than or equal to δ. Here $d(x)$ is defined as the infinmum of the Euclidean length of smooth curves in M which start at x and leave every compact subset of M. A theorem of Osserman [16] implies that if in a part M_0 of a smoothly immersed minimal surface the Gauss map avoids a neighborhood of a point on S, and the distance to the boundary of M is bounded in M_0, then the Gaussian curvature K is bounded on M_0. In view of (1.2), it follows therefore from iii) and iv) that the function $q(z)$ is bounded on $D_\sigma \setminus \{0\}$, and therefore it extends to a complex analytic function on D, which we denote with the same letter.

The minimal surface immersion $\epsilon(z)$ is found back from $(u(z),\, q(z)) \in Q$ by means of integration of the real part of (3.1). Because $\epsilon(z)$ cannot have a finite limit as $z \to 0$ in view of iv), it follows that $q(0) = 0$ and that (3.2) holds with $l \geq k$. This shows that c) $\Longrightarrow$ a) and we have proved the equivalence of a), b) and c). q.e.d.

We will say that M has a *flat point at infinity* if any of the equivalent conditions a), b), c) in Lemma 3.1 is satisfied. (We apologize for the collision of this terminology with the distinction between flat and catenoid ends at infinity. In our terminology both types of ends correspond to flat points at infinity.) Lemma 3.1 says that the condition iii) can be replaced by iii'). Moreover, if M satisfies i), ii), iii) and iv), then Lemma 3.1 implies that we have the seemingly much stronger asymptotic expansions for M as described after (3.4).

3.3 Algebraic Curves

We now turn to a global version of the conditions i), ii) in Section 3.2. We define a smoothly immersed minimal surface in $\mathbb{R}^3$ *of finite topological type* as a multi-valued smooth minimal surface immersion $\epsilon : D \setminus D_\infty \to \mathbb{R}^3$, with the following properties.

i) D is a compact Riemann surface and D_∞ is a finite subset of D, possibly empty. If we provide the immersed minimal surface M with the complex structure such that the Gauss mapping is a complex analytic mapping n from M to the Riemann sphere S, then $n \circ \epsilon$ is a complex analytic mapping from $D \setminus D_\infty$ to S. In turn this implies that $n' \circ \epsilon$ is a complex analytic mapping from $D \setminus D_\infty$ to Q, cf. Proposition 2.1.

ii) In order to allow for periodicity, the mapping ϵ may be multi-valued in the following way. If $\gamma : [0, 1] \to D \setminus D_\infty$ is a curve in $D \setminus D_\infty$ such that $\gamma(1) = \gamma(0)$ and $\epsilon(\gamma(t))$ has moved from x to $\widetilde{x}$ as t runs from 0 to 1, then $n(\widetilde{x}) = n(x)$ and $n'(\widetilde{x}) = n'(x)$. This assumption is equivalent to the condition that $n' \circ \epsilon$ is a single-valued mapping from $D \setminus D_\infty$ to Q. The image $n' \circ \epsilon(D \setminus D_\infty)$ therefore is a complex analytic curve C in Q and the minimal surface M is obtained by means of integration of $\omega|_C = \mathrm{Re}\,\Omega|_C$. If $\mathcal{P}$ denotes the corresponding set of periods, which is an additive subgroup of $\mathbb{R}^3$, and $\phi = \int_C \omega$ is the reconstruction map : $C \to M/\mathcal{P}$ of Section 1.2, then $\phi \circ (n' \circ \epsilon)$ is a single-valued mapping from $D \setminus D_\infty$ to $\mathbb{R}^3/\mathcal{P}$, which is the single-valued realization of ϵ. Note that $\mathcal{P}$ is generated by the $\langle [\gamma_i]\,, [(n' \circ \epsilon)^* \omega] \rangle$, where $[\gamma_i]$ runs over a set of generators of the image of $\mathrm{H}_1(D \setminus D_\infty, \mathbb{Z})$ in $\mathrm{H}_1(D \setminus D_\infty, \mathbb{R})$.

If C is a germ at q of an analytic subset of Q, then a *component of C at q* is defined as a set of the form $A \cup \{q\}$, in which A is a connected component of $B \cap (C \setminus \{q\})$ and B is a small ball around q in Q. We also recall the compactification $\overline{Q}$ of Q, the $\mathbb{CP}^1$-bundle Σ_4 over $\mathbb{CP}^1$, introduced in Remark 2.10.

Proposition 3.2 *Let $\epsilon : D \setminus D_\infty \to M/\mathcal{P}$ be a smoothly immersed nonflat minimal surface in $\mathbb{R}^3$, modulo its periods, of finite topological type. Assume moreover that, for every $e \in D_\infty$ and small disc $D(e)$ around e in D, $\epsilon|_{D(e)}$ is a flat point at infinity. Then, if we add to $n' \circ \epsilon(D \setminus D_\infty)$ the finitely many limit points $q(e)$, $e \in D_\infty$, on the zero section $0 = 0_Q$ of Q, we obtain a complex* algebraic *curve C in $\overline{Q}$ with the following properties.*

a) $C \subset Q$.

b) *For every $q \in C \setminus 0_Q$, every component of C at q is smooth and transversal to the fiber through q of $\pi : Q \to S$.*

c) *If $q \in C \cap 0_Q$, then each component of C at q has intersection number k with the fiber through q and intersection number l with 0_Q, where $l \geq k - 1$.*

The components of C at points of $C \cap 0_Q$ such that $l \geq k$ correspond bijectively to the points of D_∞, or the flat points at inifinity. The others correspond to the finite flat points of $M/\mathcal{P}$.

Conversely, if C is an algebraic curve in $\overline{Q}$ such that a), b), c) *hold, then it arises from a minimal surface $M/\mathcal{P}$ of finite topological type and with flat points at infinity as above, where $M/\mathcal{P}$ is obtained from C by means of integration of $\omega|_C = \mathrm{Re}\,\Omega|_C$.*

Proof The assumptions on ϵ imply that $n' \circ \epsilon$ extends to a complex analytic mapping from D to Q. Its image C is a compact subset of Q, and it follows from Lemma 3.1 that C is an analytic subset of Q. Because $\overline{Q}$ is a complex projective variety, cf. Remark 2.10, and Chow's theorem states that every complex analytic subset of a complex projective space is algebraic, cf. Griffiths and Harris [8, p. 167], the conclusion is that C is a complex algebraic curve in $\overline{Q}$, which moreover satisfies a). The properties b) and c) follow from the discussion in Section 3.1 and 3.2.

For the converse, let $\eta : D \to C$ be the normalization of C, as defined in for instance Łojasiewicz [15, p. 343, 344]. It is obtained by replacing, for every nonsmooth point q of C and small ball B in Q around q, the set $C \cap B$ by the disjoint union of the components of C at q, where each component is parametrized by means of a complex analytic map $z \mapsto (u(z),\, q(z))$, with z in a small disc in the complex plane, as in (3.2). Then D is a compact Riemann surface and $\eta : D \to C$ is a complex analytic mapping such that, for each $q \in C$, the number of elements in the pre-image $\eta^{-1}(\{q\})$ is equal to the number of components at q of C. Define D_∞ as the finite set of the points $e \in D$ such that $q = \eta(e) \in C \cap 0_Q$ and, for a small disc $D(e)$ around e in D, $\eta(D(e))$ is equal to a component at q of C as in c), with $l \geq k$.

If $\phi = \int_C \omega$ denotes the integration map from $C \setminus \eta(D_\infty)$ to the minimal surface $M/\mathcal{P}$, then $\epsilon = \phi \circ \eta$ is a minimal surface of finite topological type with flat points at infinity, and $n' \circ \epsilon$ maps $D \setminus D_\infty$ to C because n' is equal to the inverse of ϕ. q.e.d.

Remark 3.2 It is a theorem of Osserman [18, p.81, 82], that a complete immersed minimal surface M with finite total curvature is of finite topological type. The condition that M itself (not some quotient by non-trivial translations) has finite total curvature implies that $\mathcal{P} = \{0\}$. It follows from Proposition 3.2 that these M correspond to the complex algebraic curves C in $\overline{Q}$ which satisfy a), b), c), and moreover have the property that all the integrals of $\omega = \mathrm{Re}\,\Omega$ over closed curves in $C \setminus 0_Q$ are equal to zero. The latter is a very severe restriction on the algebraic curves C. It remains to be seen in how far a systematic search in the parameter range of the complex algebraic curves can be performed in order to find (or classify) the examples in which one is interested. $\oslash$

Remark 3.3 If one of the conditions a), b), c) for the complex algebraic curve C is not satisfied, then the corresponding minimal surface is no longer smoothly immersed and has singularities of branch point type. If C intersects the section $\infty_Q := \overline{Q} \setminus Q$ at infinity, then the curvature tends to infinity if one approches the singular point, in all other cases the curvature has a finite limit at the branch point. $\oslash$

Let $(u,\, q)$ be the trivialization of $Q \setminus Q_{-e_3}$ over $S \setminus \{-e_3\}$ as in (2.18). Because of the polynomial embedding of $\overline{Q}$ in a complex projective space as in Remark 2.10, any complex algebraic curve C in $\bar{Q}$ is equal, within $Q \setminus Q_{-e_3}$, to the zeroset of a polynomial $f(u,\, q)$ in the two complex variables u, q, which we can write in the form

$$f(u,\, q) = \sum_{j=0}^{d} f_j(u)\, q^j. \tag{3.7}$$

Here, for each $0 \leq j \leq d$, f_j is a polynomial in one variable and $f_d(u)$ is not identically equal to zero.

The condition that $C \subset Q$ implies that $f_d(u)$ has no zeros, which means that it is equal to a nonzero constant, which we can take equal to -1, because multiplication of f with a nonzero constant does not change its zeroset. However, the condition $C \subset Q$ implies that the same is true for the polynomial equation which is obtained after the retrivialization (2.19). We have

$$f\left(v^{-1},\, v^{-4}\, r\right) = \sum_{j=0}^{d} f_j\left(v^{-1}\right)\, v^{-4j}\, r^j = v^{-m} \sum_{j=0}^{d} v^{m-4j}\, f_j\left(v^{-1}\right)\, r^j, \tag{3.8}$$

in which m denotes the maximum of the numbers $4j + \text{degree}\, f_j$, $0 \leq j \leq d$. Because it is required that $m - 4d = 0$, it follows that the closure in $\overline{Q}$ of the zeroset of f is contained in Q, if and only if (3.7) holds with

$$f_d = -1, \quad \deg f_j \leq 4(d-j) \quad \text{for} \quad 0 \leq j \leq d-1. \tag{3.9}$$

We have the identities $d =$ the intersection number of C with each fiber of $\pi : Q \to S$ $=$ the degree of the mapping $\pi|_C : C \to S$ $=$ the degree of the Gauss map from $M/\mathcal{P}$ to S. (Strictly speaking, here $M/\mathcal{P}$ should be replaced by its compactification, obtained by the addition of a finite set of points at infinity. Note that d is also equal to the degree of the mapping $n \circ \epsilon$ from D to S.) In view of (3.6), we obtain that the total curvature of the minimal surface $M/\mathcal{P}$ modulo $\mathcal{P}$ is equal to

$$\int_{M/\mathcal{P}} K(x)\, \mathrm{d}_2 x = -4\pi\, d, \tag{3.10}$$

because the area of S is equal to 4π. As observed in Lemma 3.1, the condition that the total curvature of $M/\mathcal{P}$ is finite is equivalent to the condition of having only flat points at infinity. Therefore, *if the complete minimal surface modulo $\mathcal{P}$ is of finite topological type and has finite total curvature, then its total curvature is equal to $-4\pi\, d$, where d is equal to the degree of the Gauss map from $M/\mathcal{P}$ to S.*

The intersection points of $C \setminus Q_{-e_3}$ with the zero section 0_Q of Q correspond to the zeros u of the polynomial f_0 of degree less than or equal to $4d$, whereas (3.8) shows that at $(-e_3,\, 0)$ C has an intersection with 0_Q of multiplicity equal to $4d - \text{degree}\, f_0$. It follows that *if the degree of the Gauss map is equal to d, then the intersection number of C with the zero section of Q is equal to $4d$.* In other words, *the number of flat points of $M/\mathcal{P}$ plus its number of flat points at infinity, each counted with multiplicity, is equal to four times the degree of the Gauss map $n : M/\mathcal{P} \to S$.*

Remark 3.4 A topological explanation for the last statement can be given as follows. The Chern number of the complex line bundle Q over $S \simeq \mathbb{CP}^1$ is equal to 4, which implies that the self-intersection number $0_Q \cdot 0_Q$ of 0_Q is equal to 4.

The self-intersection number $F \cdot F$ of a fiber F of $\pi : \overline{Q} \to S$ is equal to zero, because two distinct fibers are disjoint. The argument of Griffiths and Harris [8, p. 518] yields that every real two-dimensional cycle $[C] \in \mathrm{H}_2(\overline{Q},\, \mathbb{Z})$ can be written as $c\, F + d\, 0_Q$, for some integers c and d. We then obtain from

$$C \cdot F = c\, F \cdot F + d\; 0_Q \cdot F = d$$

that $C \cdot F = d =$ the degree of the mapping $\pi|_C : C \to S$. On the other hand

$$C \cdot 0_Q = c\, F \cdot 0_Q + d\; 0_Q \cdot 0_Q = c + 4d.$$

Let $\infty_Q = \overline{Q} \setminus Q$ denote the section at infinity of $\pi : \overline{Q} \to S$, which has been added to Q in order to compactify Q. Then $0_Q \cdot \infty_Q = 0$ and $F \cdot \infty_Q = 1$, and therefore

$$C \cdot \infty_Q = c\, F \cdot \infty_Q + d\, 0_Q \cdot \infty_Q = c,$$

and we arrive at the conclusion that

$$C \cdot 0_Q = C \cdot \infty_Q + 4\, C \cdot F \tag{3.11}$$

for every real two-dimensional cycle C in $\overline{Q}$. In particular, the intersection number of C with the zero section is equal to four times the degree of the mapping $\pi|_C : C \to S$, if and only if the intersection number of C with the section at infinity is equal to zero.

Applying (3.11) to $C = \infty_Q$, it also follows that $\infty_Q \cdot \infty_Q = -4$. The negativity of this self-intersection number implies that ∞_Q is rigid, in the sense that ∞_Q can not be deformed to a nearby complex algebraic curve, because the intersection number between complex algebraic curves is always ≥ 0. $\oslash$

Let C be the algebraic curve in $\overline{Q}$ which corresponds to the minimal surface $\epsilon : D \setminus D_\infty \to M/\mathcal{P}$ of finite topological type and flat points at infinity, as in Proposition 3.2. Let $\eta : D \to S$ be the complex analytic extension to D of the Gauss map $n \circ \epsilon : D \setminus D_\infty \to S$. Then η exhibits the compact Riemann surface D as a branched covering of S, where the branch points in $S \simeq 0_Q$ correspond to the points $q \in C \cap 0_Q$. For each component A at q of C, we have a corresponding branch point $b \in D$, where the b are all distinct. We write $k(b)$ and $l(b)$ for the intersection number of A with the fiber through q and the zero section, respectively.

Let B denote the set of all the branch points $b \in D$, note that B is a finite subset of D and that D_∞ is equal to the set of $b \in B$ such that $l(b) \geq k(b)$. Also recall that $\mathrm{spin}(b) = l(b) - k(b)$ is the spinning number of the flat point at infinity, as introduced in the second paragraph after (3.5). Let d denote the degree of the Gauss map $n \circ \epsilon : D \to S$, $r := \#\,(D_\infty)$ the number of flat points at infinity and $s := \sum_{b \in D_\infty} \mathrm{spin}(b)$ the *total spin at infinity* of the minimal surface. We then have the following formula for the genus g of D:

$$g = 1 + d - r/2 - s/2. \tag{3.12}$$

Proof Because the genus of S is equal to zero, the Riemann-Hurwitz formula, cf. Farkas and Kra [6, p. 21], yields that

$$g = 1 - d + \sum_{b \in B} \frac{k(b) - 1}{2}. \tag{3.13}$$

Now we split the sum in the right hand side into a sum over the $b \in D \setminus D_\infty$ and the $b \in D_\infty$. If $b \in D \setminus D_\infty$, which corresponds to a finite flat point, then $k(b) - 1 = l(b)$, cf. the end of Section 3.1. On the other hand, if $b \in D_\infty$ then $k(b) - 1 = l(b) - \mathrm{spin}(b) - 1$, because of the definition of $\mathrm{spin}(b)$. The formula (3.12) now follows from (3.13), because $\sum_{b \in B} l(b) = C \cdot 0_Q = 4d$. q.e.d.

The formula (3.12) agrees with the Jorge-Meeks formula, cf. Jorge and Meeks [12, p. 210], Hoffman and Karcher [10, (2.16) on p. 20], where one has made the assumption that M itself has finite total curvature, not some quotient by a nonzero group of translations, cf. Remark

3.2. In [12] the Jorge-Meeks formula has been proved with the help of the Gauss-Bonnet formula for the total curvature, whereas our argument is based on the topology of the bundle Q which encodes the second order contact modulo translations.

If $\mathcal{P} = \{0\}$, then $\chi(M) = 2 - 2g - r$ is the *Euler characteristic* of M, where g is the genus of D and $r = \#(D_\infty)$. Combination of (3.10) and (3.12) therefore leads to the formula

$$\int_M K(x)\, \mathrm{d}_2 x = 2\pi\,(\chi(M) - s) \tag{3.14}$$

for the total curvature, where $s \geq r$ and $s = r$ if and only if all ends at infinity are embedded, cf. Jorge and Meeks [12, p. 210]. The inequality that the total curvature is $\leq 2\pi(\chi(M) - r)$ is due to Gackstatter [7].

3.4 Sections

The simplest complex algebraic curves in Q, apart from the fibers, are the sections, i.e. the curves C which have intersection number with the fibers equal to one. This means that $d = 1$ in (3.7), (3.9), or that in the (u, q)-trivialization C is equal to the graph $q = f_0(u)$ of a polynomial f_0 of degree ≤ 4. It follows that the holomorphic sections of Q form a 5-dimensional complex vector space. Of the 10 real parameters which determine the corresponding minimal surface, 6 are "essential", because we may substract the 3 real dimensions of SO(3) and the parameter of the dilations.

Remark 3.5 The space of holomorphic sections of Q can be recognized as the representation space for the irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ with highest weight equal to four times the fundamental highest weight of $\mathrm{SL}(2, \mathbb{C})$, where Q is the complex line bundle over $\mathrm{SL}(2, \mathbb{C})/L \simeq \mathbb{CP}^1$ in the Borel-Weil picture, cf. [5, (4.12.5), (4.12.7)]. $\oslash$

Because the degree d of the projection $\pi_C : C \to S$ is equal to one, we have for every $s \in S \setminus \pi\left(C \cap 0_Q\right)$ a unique $x \in M/\mathcal{P}$ such that $n(x) = s$, whereas the is no such x if $s \in \pi\left(C \cap 0_Q\right)$. Also note that $d = 1$ is equivalent to the condition that the total curvature of $M/\mathcal{P}$ is equal to -4π.

The topological intersection number of C with the zero section is equal to 4, where the intersection points of C with 0_Q correspond to the zeros of f_0, counted with multiplicity, and an intersection at $(-e_3, 0)$ with multiplicity $4 - \deg f_0$, if $\deg f_0 < 4$. If q is an intersection point of C and 0_Q, then the numbers k and l in (3.2) satisfy $k = 1$, whereas l is equal to the multiplicity of the intersection (= the order of the zero of f_0 if $\pi(q) \neq -e_3$). In particular, because $l \geq k$, each $q \in C \cap 0_Q$ corresponds to a single flat point at infinity, where $\pi(q)$ is equal to the limiting position of the normal. Also note that $l \geq k$ for every $q \in C \cap 0_Q$ implies that *the minimal surface has no finite flat points.*

Because C is diffeomorphic to the sphere, the sum of homology classes of the positively oriented small loops around the points of $C \cap 0_Q$ is equal to zero, which implies that the sum of the corresponding period vectors, the integrals of ω over these loops, is also equal to zero. It follows that we have no periodicity, i.e. $\mathcal{P} = \{0\}$, if we have only one intersection point q of C with 0_Q, which then necessarily has multiplicity equal to 4. By means of a rotation in $\mathbb{R}^3$, we can arrange that $\pi(q) = -e_3$, which implies that $f_0(u)$ is a nonzero constant. The corresponding minimal surface in $\mathbb{R}^3$ is *Enneper's surface*, cf. DHKW [4, pp. 144-149].

We have already observed in (3.3) and (3.5) that if $l = k(= 1)$, then the integral of ω over a small loop in C around q is equal to a nonzero period vector p. If $\pi(q) \neq -e_3$ and q has the coordinates $(a, 0)$ in the (u, q)-trivialization of $Q \setminus Q_{-e_3}$, then $f_0(u) = (u - a)\, g(u)$, where g is a polynomial with $\deg g = \deg f_0 - 1$ and $g(a) \neq 0$. Taking $u(z) = u$, $q(z) = f_0(u)$ in (3.1), we arrive at the formula

$$p = 4\pi \;\mathrm{Im} \left(\frac{1}{g(a)} \left(1 - a^2,\, \mathrm{i}\left(1 + a^2\right),\, 2a\right)\right) \tag{3.15}$$

for the period p.

Therefore the only remaining case without periodicity is that we have two intersection points q_1 and q_2 of C with 0_Q, each of which of multiplicity two. By means of a rotation in $\mathbb{R}^3$ we can arrange that $\pi(q_2) = -e_3$, which implies that $f_0(u) = c\,(u - a)^2$ for some $u \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$. A straighforward calculation shows that in this case the period vector p is equal to 4π times the imaginary part of $c^{-1}\,(-2a,\, 2\,\mathrm{i}\, a,\, 2)$, (The period vector of q_2 is equal to $-p$, because the sum of the period vectors is equal to zero.) We have that $p = 0$ if and only if $a = 0$ and $c \in \mathbb{R}$. The corresponding minimal surface in $\mathbb{R}^3$ is the *catenoid*, cf. DHKW [4, pp. 135-138]. The *helicoid* is the adjoint surface of the catenoid, for which $f_0(u) = \mathrm{i}\, u^2$, which belongs to the *associated family of the catenoid*, for which $f_0(u) = \mathrm{e}^{\mathrm{i}\,\theta}\; u^2$, cf. DHKW [4, pp. 138-140]. The period vector is equal to $p = -4\pi\,(\sin\theta)\, e_3$ for the members of this family.

Because the sum of the period vectors is equal to zero, we obtain at most three linearly independent ones, and this actually occurs for the generic section C (which necessarily has four simple intersection points with 0_Q). In this case $\mathcal{P}$ is a discrete additive subgroup of $\mathbb{R}^3$, $\mathbb{R}^3/\mathcal{P}$ is diffeomorphic to the three-dimensional torus, in which $M/\mathcal{P}$ is smoothly immersed. A case at hand is *Scherk's surface*, for which $f_0(u) = u^4 - 1$, cf. DHKW [4, p. 153].

On the other hand, it can also happen that the subgroup $\mathcal{P}$ of $\mathbb{R}^3$ is not discrete. For instance, one can have nonzero p_1 and p_2 in $\mathcal{P}$ such that $p_2 = \alpha\, p_1$, where α is an irrational real number. In such a case the minimal surface will be dense in $\mathbb{R}^3$, which makes it rather senseless to make a picture of the full minimal surface.

Bour's surfaces are given by $q = c\, u^m$, where $m \in \mathbb{R}$ and $c \in \mathbb{C} \setminus \{0\}$, cf. DHKW [4, p. 149]. These correspond to minimal surfaces as described in Proposition 3.2, if and only if $m \in \mathbb{Q}$ and $0 \leq m \leq 4$. In view of the retrivialization (2.19), the replacement of m by $4 - m$ corresponds to a switch in the role of e_3 and $-e_3$, which means that we may restrict our attention to the cases that $0 \leq m \leq 2$. For $m = 0$ we have Enneper's surface, and for $m = 2$ the catenoid family. If $m = a/b$ with $a, b \in \mathbb{Z}$, $b > 0$ and $\gcd(a, b) = 1$, then the complex algebraic curve C is defined by the equation $q^b = c\, u^a$. The degree of the Gauss mapping is equal to b, but $M/\mathcal{P}$ is isomorphic to the Riemann sphere with two points deleted. (According to (3.13), the genus of D is equal to zero.) In this sense, although C is not a section, Bour's surface is akin to the minimal surfaces defined by a section. The period p can be computed from (3.1) with $u(z) = z^b$, $q(z) = \widetilde{c}\, z$, where $\widetilde{c}^b = c$. It turns out that *the period vector is equal to zero, unless $a = 2$, $b = 1$, or $a = b = 1$.* In both of these exceptional cases, C is a section with two zeros at opposite points of S. The first case is the catenoid family. In the second case, where one zero is simple and the other has multiplicity 3, we have for every nonzero value of c that p is a nonzero vector in the horizontal plane.

3.5 Hyperelliptic Curves

It follows from (3.12) that a minimal surface as in Proposition 3.2 has *no flat points at infinity*, i.e. $D_\infty = \emptyset$, if and only if the genus g of C is equal to $d + 1$, if d denotes the degree of the Gauss map (= the degree of $\pi|_C : C \to S$).

Remark 3.6 In Meeks [14, Thm 7.1] the formula $d = g - 1$ was obtained as a consequence of the Gauss-Bonnet formula

$$-4\pi\, d = \int_{M/\mathcal{P}} K(x)\, \mathrm{d}_2 x = 2\pi\, \chi(M/\mathcal{P}) = 2\pi\, (2 - 2g).$$

He also observed that $g = 2$ cannot occur: if $g = 2$ then $d = 1$, hence the Gauss map is an isomorphism from $M/\mathcal{P}$ onto S, which implies that $g = 0$. $\oslash$

More explicitly, we have no flat points at infinity if and only if, for each $q \in C \cap 0_Q$ and each component A at q of C, we have that $l = k - 1$, if k is the intersection number of A with the fiber and l is the intersection number of A with the zero section 0_Q. This implies that $d \geq k \geq 2$. If we take $d = 2$, which implies that $g = 3$, then it follows that C is equal to the zeroset of a polynomial f as in (3.7), (3.9), with $d = 2$, $f_1(u) \equiv 0$, and f_0 a polynomial of degree 7 or 8 with only simple zeros. (The degree is equal to 7 if and only if one of the points of $C \cap 0_Q$ lies over $-e_3$, which always can be arranged by means of a rotation in $\mathbb{R}^3$.) In this case C is equal to the *hyperelliptic curve of genus 3* defined by the equation

$$q^2 = f_0(u), \tag{3.16}$$

in which the degree of f_0 is equal to 7 or 8 and f_0 has only simple zeros. C is a smooth complex algebraic curve in $\overline{Q}$ and the projection $\pi|_C : C \to S$ is the usual branched covering of $S \simeq \mathbb{CP}^1$ by the hyperelliptic curve C. It is a cyclic covering as described in Barth, Peters and Van de Ven [1, p. 42].

The minimal surface $M/\mathcal{P}$ modulo the periods has eight finite flat points, corresponding to the eight intersection points of C with the zero section 0_Q of Q. Via the coefficients of the polynomial f_0 in (3.16), the set $\mathcal{C}$ of these hyperelliptic curves C can be identified with a nonvoid open subset of $\mathbb{C}^9$.

Let $\oint \Omega : [\gamma] \mapsto \oint_\gamma \Omega$ be the mapping from $\mathrm{H}_1(C, \mathbb{Z})$ to $\mathbb{C}^3$, which is obtained by means of integration of the $\mathbb{C}^3$-valued one-form Ω over closed loops in C. The image $\mathcal{P}(\Omega|_C)$ of $\mathrm{H}_1(C, \mathbb{Z})$ in $\mathbb{C}^3$ is an additive subgroup of $\mathbb{C}^3$ with 6 generators, because $\mathrm{H}_1(C, \mathbb{Z}) \simeq \mathbb{Z}^6$, cf. Farkas and Kra [6, I.2.5]. If Re denotes the real linear mapping from $\mathbb{C}^3$ onto $\mathbb{R}^3$ which assigns to $z \in \mathbb{C}^3$ its real part, then $\mathrm{Re}\, \mathcal{P}(\Omega|_C) = \mathcal{P}$, the group of the integrals of $\omega = \mathrm{Re}\, \Omega$ over closed loops γ in C. Note that the fact that $\Omega|_C$ is holomorphic implies that the restriction of ω to C is smooth (real analytic).

Lemma 3.3 *$\mathcal{P}(\Omega|_C)$ is a lattice in $\mathbb{C}^3$, which means that it has a $\mathbb{Z}$-basis which is also an $\mathbb{R}$-basis of $\mathbb{C}^3 \simeq \mathbb{R}^6$. This implies that $\mathbb{C}^3/\mathcal{P}(\Omega|_C)$ is compact, a real 6-dimensional torus.*

The mapping $\int \Omega : q \mapsto \int_{q_0}^q \Omega$, where the integration is over a curve in C which runs from q_0 to q, defines a complex analytic embedding from C into $\mathbb{C}^3/\mathcal{P}(\Omega|_C)$. The image $(\int \Omega)(C)$ is a compact complex analytic curve in $\mathbb{C}^3/\mathcal{P}(\Omega|_C)$ and $\int \Omega$ is a complex analytic diffeomorphism from C onto its image.

Proof Let $\mathcal{H}^1(C)$ denote the space of holomorphic $(1, 0)$-forms on C, also called the abelian differentials on C. For any compact Riemann surface, the complex dimension of $\mathcal{H}^1(C)$ is equal to the genus g of C, cf. Farkas and Kra [6, p. 62], which in our case is equal to 3. For each closed curve γ in C we have the complex linear form $\oint_\gamma : \theta \mapsto \oint_\gamma \theta$ on $\mathcal{H}^1(C)$. The mapping $\oint : [\gamma] \to \oint_\gamma$ is injective from $\mathrm{H}_1(C, \mathbb{Z})$ to the dual space $\mathcal{H}^1(C)^*$ of $\mathcal{H}^1(C)$, and its image Λ is a lattice in $\mathcal{H}^1(C)^*$, cf. Farkas and Kra [6, III.2.8]. Λ is called the *period lattice of C*. This implies that $\mathrm{Jac}(C) := \mathcal{H}^1(C)^*/\Lambda$ is a real 6-dimensional torus, called the *Jacobi variety of C*.

The components of $\Omega|_C$ are equal to

$$\Omega_1 = -2\left(1-u^2\right)\,\mathrm{d}u/q, \quad \Omega_2 = -2\,\mathrm{i}\left(1+u^2\right)\,\mathrm{d}u/q, \quad \text{and} \quad \Omega_3 = -4u\,\mathrm{d}u/q,$$

cf. (2.18). Because the one-forms $u^j\,\mathrm{d}u/q$ with $j = 0, 1, 2$ form a basis of $\mathcal{H}^1(C)$, cf. Farkas and Kra [6, p. 104], the Ω_i also form a basis of $\mathcal{H}^1(C)$. It follows that $\underline{\Omega} : z \mapsto \sum_{i=1}^{3} z_i\,\Omega_i$ is a linear isomorphism from $\mathbb{C}^3$ onto $\mathcal{H}^1(C)$.

The adjoint $\underline{\Omega}^*$ is a linear isomorphism from $\mathcal{H}^1(C)^*$ onto $\left(\mathbb{C}^3\right)^* \simeq \mathbb{C}^3$. For every standard basis vector e_i of $\mathbb{C}^3$ and closed curve γ in C we have that

$$\langle \underline{\Omega}^*\left(\oint_\gamma\right), e_i\rangle = \langle \oint_\gamma, \underline{\Omega}\,(e_i)\rangle = \langle \oint_\gamma, \Omega_i\rangle = \oint_\gamma \Omega_i,$$

which shows that $\underline{\Omega}^* \circ \oint$ is equal to the mapping $\oint \Omega : \mathrm{H}_1(C, \mathbb{Z}) \to \mathbb{C}^3$. It follows that the image $\mathcal{P}\,(\Omega|_C)$ of $\oint \Omega$ is equal to $\underline{\Omega}^*(\Lambda)$, which implies that $\mathcal{P}\,(\Omega|_C)$ is a lattice in $\mathbb{C}^3$, because $\underline{\Omega}^*$ is a linear isomorphism and Λ is a lattice in $\mathcal{H}^1(C)^*$.

The second statement follows from the fact that the mapping

$$q \mapsto \left(\theta \mapsto \int_{q_0}^{q} \theta\right) : C \to \mathcal{H}^1(C)^*/\Lambda$$

is an embedding from C into $\mathrm{Jac}(C)$, cf. Farkas and Kra [6, III.6.4]. q.e.d.

The pre-image of $(\int \Omega)(C)$ in $\mathbb{C}^3$ under the projection from $\mathbb{C}^3$ to $\mathbb{C}^3/\mathcal{P}\,(\Omega|_C)$ is a closed and smooth complex one-dimensional analytic submanifold of $\mathbb{C}^3$, which we denote by , . Note that , is equal to the isotropic complex analytic curve which is associated to the minimal surface M as in Section 2.4. The restriction to , of the projection $\mathrm{Re} : \mathbb{C}^3 \to \mathbb{R}^3$ is a smooth immersion from , onto the minimal surface M in $\mathbb{R}^3$ which is obtained by means of integration of ω over C.

We also have that $\mathrm{Re}\,\mathcal{P}\,(\Omega|_C) = \mathcal{P}$, the group of periods of M. Because $\mathcal{P}\,(\Omega|_C)$ contains an $\mathbb{R}$-basis of $\mathbb{C}^3$ and the real linear mapping Re is surjective, it follows that $\mathcal{P}$ contains an $\mathbb{R}$-basis of $\mathbb{R}^3$. However, it can easily happen that $\mathcal{P}$ is not closed in $\mathbb{R}^3$, in which case the minimal surface M is dense in $\mathbb{R}^3$.

The period group $\mathcal{P}$ is closed, i.e. a lattice, in $\mathbb{R}^3$, if and only if the kernel $\left(\mathrm{i}\,\mathbb{R}^3\right)$ of the projection $\mathrm{Re} : \mathbb{C}^3 \to \mathbb{R}^3$ has an $\mathbb{R}$-basis which consists of elements of $\mathcal{P}\,(\Omega|_C)$. In other words, if and only if $\mathcal{P}\,(\Omega|_C)$ contains three linearly independent purely imaginary elements. In this case the minimal surface M is properly immersed in $\mathbb{R}^3$, and is triply periodic.

Extend the mapping $\oint \Omega : \mathrm{H}_1(C, \mathbb{Z}) \to \mathbb{C}^3$ to an $\mathbb{R}$-linear mapping from $\oint \Omega : \mathrm{H}_1(C, \mathbb{R}) \to \mathbb{C}^3$, which we denote with the same letter. Then $\mathcal{P} = \mathcal{P}(C)$ is a lattice in $\mathbb{R}^3$, if and only if the 3-dimensional real linear subspace

$$L(C) = (\oint \Omega)^{-1}\left(\mathrm{i}\,\mathbb{R}^3\right)$$

of the 6-dimensional real vector space $\mathrm{H}_1(C, \mathbb{R})$ contains three linearly independent elements of the lattice $\mathrm{H}^1(C, \mathbb{Z})$. Let G be the Grassmann manifold of all 3-dimensional real linear subspaces of $\mathrm{H}_1(C, \mathbb{R})$, and let G_0 denote the set of all $L \in G$ such that $L \cap \mathrm{H}_1(C, \mathbb{Z})$ contains a basis of L. Then G_0 is dense in G. Because of the discrete nature of the homology groups with values in $\mathbb{Z}$, there is a canonical identification, for any hyperelliptic curve C' near C, of $\mathrm{H}_1(C', \mathbb{Z})$ with $\mathrm{H}_1(C, \mathbb{Z})$, and therefore also of $\mathrm{H}_1(C', \mathbb{R})$ with $\mathrm{H}_1(C, \mathbb{R})$, and of the Grassmann manifold of $\mathrm{H}_1(C', \mathbb{R})$ with G.

Let $\mathcal{C}_0$ denote the set of $C' \in \mathcal{C}$ such that $\mathcal{P}(C')$ is a lattice in $\mathbb{R}^3$. If the mapping $C' \mapsto L(C')$ from the real 18-dimensional space $\mathcal{C}$ to the real 9-dimensional manifold G has surjective derivative at C, then the conclusion is that $\mathcal{C}_0$ is dense in a neighborhood of C in $\mathcal{C}$. Moreover, near C the set $\mathcal{C}_0$ then is equal to the union of countably many smooth real analytic submanifolds of $\mathcal{C}$, each of real dimension $18 - 9 = 9$. Of these 9 dimensions, 5 are essential, if we substract the 4 dimensions of the real rotations and the dilations.

Pirola [19] has proved that for the generic hyperelliptic curves C as above the derivative of the mapping $C' \to L(C')$ surjective at C. It follows that the set of C, such that the period group $\mathcal{P}$ is a lattice in $\mathbb{R}^3$, is dense in $\mathcal{C}$. Moreover, in the open dense subset where $\mathrm{D}L(C)$ is surjective, it is equal to the union of countably many smooth manifolds as described above.

There are many examples known of triply periodic minimal surfaces, among which those constructed by H.A. Schwarz and Alan Schoen, cf. DHKW [4, pp. 212-217]. These minimal surfaces are not only properly immersed, but even embedded in $\mathbb{R}^3$.

The embedded minimal surfaces in the real five-dimensional family of Meeks [14, Thm. 7.1] correspond to the genus three hyperelliptic curves C as above, which satisfy the additonal condition that C is invariant under the anti-holomorphic involution

$$A : (u,\, q) \mapsto \left(-1/\overline{u},\, \overline{q}/\overline{u}^4\right)$$

of Q which is induced by the antipodal mapping $s \mapsto -s$ on the sphere S. The $[\gamma] \in \mathrm{H}_1(C, \mathbb{Z})$ such that $A([\gamma]) = [\gamma]$ form a three-dimensional sublattice L of $\mathrm{H}_1(C, \mathbb{Z})$. Furthermore, because $A^*\Omega = -\overline{\Omega}$, it follows that $(\oint \Omega)(L) \subset \mathrm{i}\,\mathbb{R}^3$, which implies that $\mathcal{P}$ is a lattice in $\mathbb{R}^3$. The condition for C means that

$$f_0(u) = c \prod_{j=1}^{4} (u - a_j)\, (u + 1/\overline{a_j}),$$

where $c \in \mathbb{C}$, $a_j \in \mathbb{C}$, $c \prod_{j=1}^4 a_j \in \mathbb{R}$, and $a_j \neq a_k$, $a_j \neq -1/\overline{a_k}$ when $j \neq k$. This condition implies that the branch points consist of four arbitrary pairs of antipodal points on S, and there is a free nonzero real factor in f_0.

For many of the known examples, Karcher and Wohlgemut in [13, pp. 317-347] found explicit Weierstrass data. With the substitutions $g = u$, $\frac{\mathrm{d}h}{g} = \frac{-2\,\mathrm{d}u}{q}$, $\mathrm{d}h = \mu\, \frac{\mathrm{d}g}{g}$, cf. Karcher [13, p. 315 and 318], which implies that $\mu = -u^2/q$, we find that the examples in Karcher [13, pp. 319-329] belong to the family of Meeks [14, Thm. 7.1]. (In the case of (H) in Karcher [13, p.329], the quantity μ^2 probably should be replaced by μ^{-2}.) The first example of Schwarz corresponds to the hyperelliptic curve $q^2/4 = u^8 - 14\,u^2 + 1$, cf. Remark 2.6 and DHKW [4, p. 175]. On the other hand, the examples in Karcher [13, pp. 330-347] do not look like hyperelliptic curves of genus three.

3.6 The Costa Surface

The famous *Costa surface* is a nonperiodic, embedded minimal surface of genus one and with three flat points at infinity. Furthermore the degree of its Gauss map is equal to three, cf.

DHKW [4, p. 198].

An explicit Weierstrass representation has been given in Hoffman and Meeks [11, (1.1), (3.1), (3.5)]. If, in the case $k = 1$, we substitute $g = u$, $w = c/u$, $\eta = -2\,\mathrm{d}u/q$ and eliminate the variable z, then we arrive at the equation

$$c^5\, q^3 - c^2\, u^4\, q^2 - 27\, c^4\, u^4 + 4u^8 = 0 \tag{3.17}$$

for the curve C in $(u,\, q)$-coordinates. Therefore the degree of the Gauss map is equal to three. With the substituion $u = v^{-1}$, $q = v^{-4}\, r$, this equation is equivalent to

$$c^5\, r^3 - c^2\, r^2 - 27\, c^4\, v^8 + 4v^4 = 0.$$

According to Hoffman and Meeks [11, prop. 3.1], there is a unique positive value of c such that the corresponding minimal surface has no periods.

At $u = q = 0$, which lies over $e_3 \in S$, C has intersection number with the fiber and the zero section equal to three and four respectively, and therefore $u = q = 0$ corresponds to a flat point at infinity of M. At $v = r = 0$, which lies over $-e_3 \in S$, C has two components, each of which has intersection number with the fiber and the zero section equal to one and two, respectively. Therefore $v = r = 0$ corresponds to two flat points at infinity of M. The other intersection points with the zero section correspond to the equations $q = 0$, $u^4 = 27\, c^4$. At each of these four points the intersection number with the fiber and the zero section is equal to two and one, respectively, which means that these points correspond to the four finite flat points of M.

According to (3.12), the genus of D is equal to $g = 1 + 3 - \frac{3}{2} - \frac{3}{2} = 1$. In this way we have recovered the properties mentioned in the beginning of this section from the equation (3.17) which defines C.

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Adress of the author
Department of Mathematics, Utrecht University
Postbus 80 010, 3508 TA Utrecht, The Netherlands
e-mail: duis@math.uu.nl