Some Hopf Algebras of Trees

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1 Introduction

In the literature several Hopf algebras that can be described in terms of trees have been studied. This paper tries to answer the question whether one can understand some of these Hopf algebras in terms of a single mathematical construction.

The starting point is the Hopf algebra of rooted trees as defined by Connes and Kreimer in [3] (section 2). Apart from its physical relevance, it has a universal property in Hochschild cohomology. We generalize the operadic construction by Moerdijk [12] of this Hopf algebra to more general trees (with colored edges), and prove a universal property in coalgebra Hochschild cohomology (sections 3, 4, and 5). For a Hopf operad P, the construction is based on the operad $\mathsf{P}[\lambda_n]$ obtained from P by adjoining a free n-ary operation. The difference with respect to [12] is that we also prove that the operad $\mathsf{P}[\lambda_n]$ is under suitable restrictions a Hopf operad. This presentation simplifies some proofs and assures that relevant functorial properties come for free. Section 6 shows that we can do with less restrictions if we are only interested in the the initial algebra $\mathsf{P}[\lambda_n](0)$.

For a specific multi-parameter family of Hopf algebras obtained in this way, we give explicit formulas for the a priori inductively defined comultiplication and for the Lie bracket underlying the dual (section 7). Some special cases yield natural examples of pre-Lie and dendriform (and thus associative) algebras (sections 8 and 10). We apply these results to explain some known Hopf algebras of trees from this point of view. Notably, the Loday-Ronco Hopf algebra of planar binary trees [10], and the Brouder-Frabetti pruning Hopf algebra [1] (sections 11 and 9).

The author plans to discuss the simplicial algebra structure on the set of initial algebras with free n-ary operation P_n for a Hopf operad P with multiplication in future work, together with some relations of the $\mathsf{P}[\lambda_n]$ to the algebra Hochschild complex.

The author is grateful to Ieke Moerdijk for suggesting the Hopf P-algebras P_n (cf. section 3 and 6) and their simplicial structure, and for motivating as well as illuminating discussions.

2 Preliminaries

This section fixes notation on trees and describes some results of Connes and Kreimer[3], and Chapoton and Livernet [2] that motivated this paper.

Rooted trees t are isomorphism classes of finite partially ordered sets which (i) have a minimal element r $(\forall x \neq r : r < x)$; we call r the **root**, and (ii) satisfy the tree condition: $(y \neq z) \wedge (y < x) \wedge (z < x)$ implies $(y < z) \vee (z < y)$. The elements of a tree are called **vertices**. A pair of vertices $v < w$ is called an **edge** if there is no vertex x such that $v < x < w$. The number of vertices of a tree t is denoted by $|t|$. A **path** from x to y in a tree is a sequence $(x_i)_i$ of elements $x = x_n > x_{n-1} > \ldots > x_1 > x_0 = y$ of maximal length. We will say that x is above y in a tree if there is a path from x to y. Thus we may depict a rooted tree as a finite directed graph, with one terminal vertex, the root. A vertex is called a **leaf** if there is no other vertex above it. A vertex is an **internal vertex** if it is not a leaf. We draw trees the natural way, with the root downwards.

A **forest** is a finite, partially ordered set satisfying only property (ii) above. A forest always is a disjoint union of trees, the connected components of the forest. A rooted tree can be pictured as a tree with unlabeled vertices and uncoloured edges. A forest as a collection of these.

In the sequel we need trees with colored edges. That is, there is a function from the set of edges to a fixed set of colours. Vertices can be labeled as well (i.e. there is a function from the set of vertices to a fixed set of labels). Also trees with a linear ordering on the incoming edges at each vertex are used.

We consider two operations on trees: cutting and grafting. Let s and t be trees and let $v \in t$ be a vertex of t. Define $s \circ_v t$ to be the tree that as a set is the disjoint union of s and t endowed with the ordering obtained by adding the edge $v < r_s$. This operation is called **grafting** s onto v. When using trees with colored edges, we write $s \circ_v^i t$ for grafting s onto v by an edge of colour i. For trees with labeled vertices, grafting preserves the labels. The operation of grafting s on t is visualised below.

A **cut** in a tree t is a subset of the set of edges of t. A cut is **admissible**

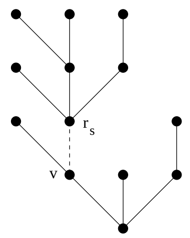

Figure 1: grafting s on t

if for each leaf m of t the unique path $m > \ldots > x_2 > x_1 > r$ to the root contains at most one edge of the cut. When we remove the edges in a cut we obtain a forest. For a cut c of t we denote by $R^c(t)$ the connected component of this forest containing the root and by $P^c(t)$ the direct union of the other components. We denote the set of admissible cuts of t by $C(t)$. For colored and labeled trees, cutting preserves the colours of edges that are not cut and labels of edges. The left picture below shows an admissible cut, whereas the cut on the right is not admissible.

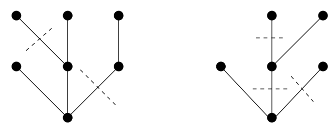

Figure 2: an admissible cut (left) and a non-admissible cut (right)

We describe the **Hopf algebra** H_R **of rooted trees** (defined in Connes-Kreimer [3]) as follows. Let k be a field. The set of rooted trees and the empty set generate a Hopf algebra over k. As an algebra it is the polynomial algebra in formal variables t, one for each rooted tree. Thus H_R is spanned by forests. Multiplication corresponds to taking the direct union. The algebra H_R is of course commutative. The unit is the empty tree.

Comultiplication Δ is defined on rooted trees t and extended as a algebra homomorphism:

$$\Delta(t) = \sum_{c \in C(t)} P^c(t) \otimes R^c(t),$$

where the sum is over admissible cuts. The counit $\varepsilon : H_R \to k$ takes value 1 on the empty tree and 0 on other trees.

The Hopf algebra H_R is $\mathbb{Z}$-graded with respect to the number of vertices in forests. The homogeneous elements of degree m are products $t_1 \cdot \ldots \cdot t_n$ of trees, such that $\sum |t_i| = m$. Both the product and the coproduct preserve the grading.

Denote by V^* the graded dual of a graded vector space V:

$$V^* = \bigoplus_n (V_n)^*,$$

the direct sum of the duals of the spaces of homogeneous elements. A basis for $(H_R)_n^*$ is $\{D_{t_1 \cdot \ldots \cdot t_k} | \sum_i |t_i| = n\}$, the dual basis to the basis for $(H_R)_n$ given by products of trees. Of course, H_R^* is a cocommutative Hopf algebra, with comultiplication

$$\Delta(D_{t_1 \cdot \ldots \cdot t_k}) = \sum_{i=0}^{k} \sum_{\sigma \in S(i,k-i)} D_{t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(i)}} \otimes D_{t_{\sigma^{-1}(i+1)}, \ldots, t_{\sigma^{-1}(k)}},$$

where $S(i, k-i)$ is the set of $(i, k-i)$-shuffles in S_k. We sum over these to avoid repetition of terms in the right hand side. The primitive elements are those dual to (single) rooted trees.

As proved by Connes and Kreimer [3], the Lie algebra which has H_R^* as its universal enveloping algebra is the linear span of rooted trees with the Lie bracket

$$[D_t, D_s] = \sum_{v \in s} D_{t \circ_v s} - \sum_{w \in t} D_{s \circ_w t}.$$

Chapoton and Livernet [2] show that this is the Lie algebra associated to the free pre-Lie algebra on one generator (cf. section 8 for details). The pre-Lie algebra structure $D_t * D_s$ is given by $D_t * D_s = \sum_{v \in s} D_{t \circ_v s}$.

3 Operads

3.1 CONVENTION For the rest of this paper, we restrict to the category of vector spaces over a fixed field k, but the general theory carries over to any symmetric monoidal category with countable coproducts and quotients of actions by finite groups on objects (cf. Moerdijk [12]).

Let kS_n denote the group algebra of the permutation group S_n on n objects. An **operad** P in this paper will mean an operad with unit. Thus an operad consists of a collection of right kS_n-modules $\mathsf{P}(n)$, together with an associative, kS_n-equivariant composition

$$\gamma : \mathsf{P}(n) \otimes \mathsf{P}(m_1) \otimes \ldots \otimes \mathsf{P}(m_n) \longrightarrow \mathsf{P}(m_1 + \ldots + m_n),$$

such that there exists an identity $\mathrm{id} : k \to \mathsf{P}(1)$ with the obvious property and a unit $u : k \to \mathsf{P}(0)$, which need not be an isomorphism. The existence of the unit map is the only change with respect to the usual definition of an operad (consult Kriz and May [7], or Ginzburg and Kapranov [6], or Getzler and Jones [5]). Following Getzler and Jones [5] we define an **operad with multiplication** as an operad together with an associative element $\mu \in \mathsf{P}(2)$.

The category of **pointed vector spaces** has as objects vector spaces V with a base point $u : k \to V$. Morphisms are base point preserving k-linear maps. We use notation $1 := u(1)$. There are some canonical functors: The forgetful functor to vector spaces will be used implicitly. The free associative algebra functor T on a pointed vector space V is left adjoint to the forgetful functor from unital associative algebras to pointed vector spaces.

For any pointed vector space V, there is an operad End_V such that $\mathsf{End}_V(n) = \mathrm{Hom}_k(V^{\otimes n}, V)$. The pointed structure of V is only used to to define $u : k \to \mathsf{End}(V)(0)$, the elements of $\mathrm{Hom}_k(V^{\otimes n}, V)$ need not preserve the base point. A **P-algebra** structure on V is a map of operads (i.e. preserving relevant structure) from P to End_V. Equivalently, a P-algebra is a vector space V together with linear maps

$$\gamma_V : \mathsf{P}(n) \otimes_{kS_n} (V^{\otimes m_1} \otimes \ldots \otimes V^{\otimes m_n}) \longrightarrow V^{\otimes m_1 + \ldots m_n},$$

compatible with composition in P and unit in the natural sense.

3.2 Example Let k be the operad, with $\mathsf{k}(0) = k = \mathsf{k}(1)$ and $\mathsf{k}(n) = 0$, otherwise (with $u = \mathrm{id}$). Algebras for k are pointed vector spaces. The operad k is the initial operad in our setting.

We denote by Com the operad with as algebras unital commutative algebras. This operad satisfies $\mathsf{Com}(n) = k$ for all n (where $u : k \to k$ is the identity map).

Likewise Ass is the operad satisfying $\mathsf{Ass}(n) = kS_n$, the group algebra of S_n, as a right S_n-module. The Ass-algebras are unital associative algebras. An operad with multiplication is an operad P together with a map of operads from Ass to P.

A map of operads $\varphi : \mathsf{P} \to \mathsf{Q}$ induces an obvious functor $\varphi^* : \mathsf{Q}\text{-Alg} \to \mathsf{P}\text{-Alg}$. This functor has a left adjoint $\varphi_! : \mathsf{P}\text{-Alg} \to \mathsf{Q}\text{-Alg}$. Let P be any operad and let $i : \mathsf{k} \to \mathsf{P}$ be the unique inclusion of operads. Then $i_!$ is the unitary free P-algebra functor.

Let P be an operad. We can form the operad $\mathsf{P}[\lambda_n]$ by adjoining a free n-ary operation to $\mathsf{P}(n)$. Algebras for $\mathsf{P}[\lambda_n]$ are just P-algebras A endowed with a linear map $\alpha : A^{\otimes n} \to A$. The reader familiar with the collections (cf. Getzler and Jones [5], or Ginzburg and Kapranov [6]), will recognize $\mathsf{P}[\lambda_n]$ as the coproduct of operads

$$\mathsf{P}[\lambda_n] = \mathsf{P} \oplus_{\mathsf{k}} FE_n$$

of P and the free operad FE_n on the collection E_n defined by $E_n(n) = kS_n$ and $E_n(m) = 0$ for $m \neq n$.

Denote the initial $\mathsf{P}[\lambda_n]$-algebra $\mathsf{P}[\lambda_n](0)$ by P_n. For any $\mathsf{P}[\lambda_n]$-algebra (A, α), there is by definition an unique P-algebra morphism γ such that the following diagram commutes.

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda_n} & P_n \\ {\scriptstyle \gamma^{\otimes n}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A^{\otimes n} & \xrightarrow{\alpha} & A \end{array}$$

The operad $\mathsf{P}[\lambda_n]$ can be described in terms of planar trees. A planar tree is understood to have a linear ordering on the incoming edges at each vertex and might have external edges. There is a linear ordering on all external edges. An element of $\mathsf{P}[\lambda_n](m)$ can be represented by a tree with m external edges and each vertex v with n incoming edges labeled either by an element of $\mathsf{P}(n)$ or by λ_n. We identify two such representing trees if they can be reduced to the same tree using edge contractions implied by composition in P, equivariance with respect to the kS_n-action at each internal vertex v and kS_m-equivariance at the external edges. The initial $\mathsf{P}[\lambda_n]$-algebra $\mathsf{P}[\lambda_n](0)$ is described by such trees without external edges.

3.3 EXAMPLE Let $\mathsf{P} = \mathsf{Com}$. The initial algebra $C_n = \mathsf{Com}[\lambda_n](0)$ can be described as the free commutative algebra on trees with edges colored by $\{1, \ldots, n\}$ (and no ordering on the edges). A bijection T is given by induction on the number of applications of λ as follows. $T(1) = 1 = \emptyset$, and $T(\lambda(1, \ldots, 1)) = r$ (the one vertex tree), and $T(\lambda(x_1, \ldots, x_n)$ is the tree obtained from the forest $x_1 \cdot \ldots \cdot x_n$ by adjoining a new root and connecting the roots of each tree in x_i to the new root by an edge of colour i. Figure 3 shows (twice) the tree $T(\lambda(x, \lambda(xx, 1)\lambda(x, 1)))$ in C_2, where $x = \lambda(1, 1)$.

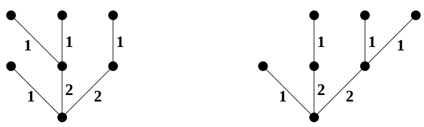

Figure 3: trees not equal in A_2, but equal in C_2

3.4 EXAMPLE Let $\mathsf{P} = \mathsf{Ass}$. The initial algebra $A_n = \mathsf{Ass}[\lambda_n](0)$ can be described as the free associative algebra on trees with edges colored by $\{1, \ldots, n\}$ and at each vertex the incoming edges of each color endowed with a linear ordering. Equivalently, we could say, a linear ordering on all incoming edges extending the ordering on the colours. A bijection is given by the map T with the same inductive definition as in the previous example, but using associative multiplication. In the associative case, the two trees in figure 3 are not identified. The tree to the right in figure 3 is $T(\lambda(x, \lambda(x,1)\lambda(xx,1)))$ in A_2, while the tree to the left is $T(\lambda(x, \lambda(xx,1)\lambda(x,1)))$ (where $x = \lambda(1,1)$).

3.5 EXAMPLE Let $\mathsf{P} = \mathsf{k}$. Then $k_n = \mathsf{k}[\lambda_n](0)$ can be identified with the free vectorspace on trees (not forests, but possibly empty) with edges labeled by $\{1, \dots, n\}$ and at most one incoming edge of each colour at each vertex. For the relation to n-ary trees, see section 9.

4 Hopf Operads

A **Hopf operad** (cf. Getzler and Jones [5], and also Moerdijk [12]) is an operad in the category of coalgebras over k. Thus, a **Hopf operad** P is a (k linear) operad together with a morphisms

$$k \xleftarrow{\varepsilon} \mathsf{P}(n) \xrightarrow{\Delta} \mathsf{P}(n) \otimes \mathsf{P}(n),$$

such that Δ and ε satisfy the usual axioms of a coalgebra: Δ is coassociative and ε is a counit for Δ (cf. Sweedler [14]), and such that the composition γ of the operad is a coalgebra morphism. Moreover, Δ should be compatible with the kS_n-action, where $\mathsf{P}(n) \otimes \mathsf{P}(n)$ is a kS_n- module via the diagonal coproduct. For any Hopf operad P, the tensor product of two P-algebras is a P-algebra again.

A Hopf P-algebra A is a P-algebra A in the category of (counital!) coalgebras. Note that we can not use the description of algebras in terms of the endomorphism operad in the category of coalgebras, since coalgebra homomorphisms do not form a linear space. In this generality it is not natural to consider antipodes. A Hopf Ass-algebra is just a bialgebra.

Let $\varphi : \mathsf{Q} \to \mathsf{P}$ be a map of Hopf operads. The map φ induces functors

$$\varphi^* : \mathsf{P}\text{-Alg} \to \mathsf{Q}\text{-Alg} \qquad \text{and} \qquad \bar{\varphi}^* : \mathsf{P}\text{-HopfAlg} \to \mathsf{Q}\text{-HopfAlg}.$$

The map φ^* has a left adjoint $\varphi_!$ (we work with k-vector spaces). Note that $\varphi^*(k) = k$ and for P-algebras A and B we have $\varphi^*(A \otimes B) = \varphi^*(A) \otimes \varphi^*(B)$ (the map φ is compatible with Δ). Using this observation, conclude that the adjunction induces algebra maps $\varphi_!(k) = \varphi_!\varphi_*(k) \to k$ and $\varphi_!(A \otimes B) \to \varphi_!(\varphi^*\varphi_! A \otimes \varphi^*\varphi_! B) \to \varphi_!(A) \otimes \varphi_!(B)$. These maps serve to show that $\varphi_!$ lifts to a left adjoint

$$\bar{\varphi}_! : \mathsf{Q}\text{-HopfAlg} \longrightarrow \mathsf{P}\text{-HopfAlg}$$

of $\bar{\varphi}^*$.

Let P be a Hopf operad. The aim of this section is to state a general result on Hopf operad structures on $\mathsf{P}[\lambda_n]$. For convenience we use the notation $\lambda = \lambda_n$ and $P_n = \mathsf{P}[\lambda_n](0)$ when no confusion can arise.

Let (A, α) be a $\mathsf{P}[\lambda_n]$-algebra and let $\sigma_1, \sigma_2 : A^{\otimes n} \to A$ be a pair of linear maps and define for each such pair a map $(\sigma_1, \sigma_2) : (A \otimes A)^{\otimes n} \to A \otimes A$ by

$$(\sigma_1, \sigma_2) = (\sigma_1 \otimes \alpha_n + \alpha_n \otimes \sigma_2) \circ \tau,$$

where τ is the n-fold twist map which identifies $(A \otimes A)^{\otimes n}$ with $A^{\otimes n} \otimes A^{\otimes n}$ by the symmetry of the tensor product. For vector spaces $V_1, \ldots, V_{2n}$, the map τ is the natural map

$$\tau : V_1 \otimes \ldots \otimes V_{2n} \mapsto V_1 \otimes \ldots V_{2n-1} \otimes V_2 \otimes V_4 \ldots \otimes V_{2n}$$

induced by the symmetry (in the dg case this involves some signs).

Let $\mathsf{P} \to \mathsf{Q}$ be a morphism of operads. Denote the P-algebra $\bigoplus_n \mathsf{Q}(n)$ by $(\mathsf{Q})_{\mathsf{P}}$. Denote by $\mathsf{Q}^{\otimes 2}$ the operad with n-ary operations $\mathsf{Q}(n)^{\otimes 2}$ and composition $\gamma_{\mathsf{Q}} \otimes \gamma_{\mathsf{Q}}$.

4.1 THEOREM *For each $m \in \mathbb{N}$ and $i = 1,\ 2$, let $\sigma_i(m) : P[\lambda_n](m)^{\otimes n} \to P[\lambda_n](m)$ be linear maps. If σ_i for $i = 1, 2$ satisfies*

$$\varepsilon \circ \sigma_i = \varepsilon^{\otimes n}, \quad \textit{and}$$
$$(\sigma_i \otimes \sigma_i) \circ \tau \circ \Delta^{\otimes n} = \tau \circ \Delta \circ \sigma_i;$$

then there exists an unique Hopf operad structure on $P[\lambda_n]$, such that comultiplication and counit extend Δ and ε of P, and we have $\Delta\lambda = (\sigma_1, \sigma_2) \circ \tau \Delta^{\otimes n}$ and $\varepsilon\lambda = \varepsilon^{\otimes n}$.

PROOF Let

$$\Delta : ((\mathsf{P}[\lambda_n])_{\mathsf{P}}, \lambda_n) \longrightarrow ((\mathsf{P}[\lambda_n]^{\otimes 2})_{\mathsf{P}}, (\sigma_1, \sigma_2))$$
$$\varepsilon : ((\mathsf{P}[\lambda_n])_{\mathsf{P}}, \lambda_n) \longrightarrow (k, 0)$$

be the unique $\mathsf{P}[\lambda_n]$-algebra morphisms which coincide with the comultiplication and the counit on P.

Then Δ is compatible with composition in $\mathsf{P}[\lambda_n]$ since it is a P-algebra map extending Δ on P and it satisfies $\Delta \circ \lambda = (\sigma_1, \sigma_2) \circ \Delta^{\otimes n}$. Thus it suffices to show that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$ as maps of $\mathsf{P}[\lambda_n]$-algebras. We compute

$$\begin{aligned}((\varepsilon \otimes \mathrm{id}) \circ \Delta) \circ \lambda &= (\varepsilon \otimes \mathrm{id}) \circ (\sigma_1 \otimes \lambda_n + \lambda_n \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes n} \\ &= (\varepsilon \circ \sigma_1 \otimes \lambda + \varepsilon \circ \lambda \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes n} \\ &= (\varepsilon^{\otimes n} \otimes \lambda + 0) \circ \tau \circ \Delta^{\otimes n} \\ &= \lambda.\end{aligned}$$

Similarly, $(\mathrm{id}\otimes\varepsilon)\Delta=\lambda$.

Regarding coassociativity we compute

$$\begin{aligned}((\Delta\otimes\mathrm{id})\circ\Delta)\circ\lambda &= ((\Delta\circ\sigma_1)\otimes\lambda+(\Delta\circ\lambda)\otimes\sigma_2)\circ\tau\circ\Delta^{\otimes n}\\ &= (\sigma_1^{\otimes 2}\otimes\lambda+\sigma_1\otimes\lambda\otimes\sigma_2+\lambda\otimes\sigma_2^{\otimes 2})\circ((\Delta\otimes\mathrm{id})\circ\Delta)^{\otimes n}\end{aligned}$$

Similarly,

$$((\mathrm{id}\otimes\Delta)\circ\Delta)\circ\lambda=(\sigma_1^{\otimes 2}\otimes\lambda+\sigma_1\otimes\lambda\otimes\sigma_2+\lambda\otimes\sigma_2^{\otimes 2})\circ((\Delta\otimes\mathrm{id})\circ\Delta)^{\otimes n}.$$

This proves the theorem. QED

We now have the following generalization of Moerdijk [12]:

4.2 COROLLARY *Under the conditions of theorem 4.1, $P_n=\mathsf{P}[\lambda_n](0)$ is a Hopf P-algebra.*

PROOF The inclusion of $i:\mathsf{P}\to\mathsf{P}[\lambda_n]$ as a Hopf operad makes the initial Hopf $\mathsf{P}[\lambda_n]$-algebra P_n a Hopf P-algebra, since the functor $\bar{i}^*$ is the forgetful functor from Hopf $\mathsf{P}[\lambda_n]$-algebras to Hopf P-algebras. QED

In the following examples $|t|$ denotes the number of applications of λ_n in an element $t\in\mathsf{P}[\lambda_n](m)$, for any m. In the case $\mathsf{P}=\mathsf{Ass}$, or $\mathsf{P}=\mathsf{Com}$, or $\mathsf{P}=\mathsf{k}$ this corresponds to counting the number of vertices in a tree.

4.3 EXAMPLE Let P be a Hopf operad. We consider the case $n=1$.

i) For any $q_1,q_2\in k$, the endomorphisms $\sigma_i(t)=q_i^{|t|}t$ define a coassociative Hopf operad structure on $\mathsf{P}[\lambda_1]$ (cf. Moerdijk [12] for the initial algebra).

ii) Suppose that P is a Hopf operad with multiplication. For any $q_1,q_2\in k$, the endomorphisms $\sigma_i(t)=q_i^{|t|}r^{|t|}$ define a Hopf operad structure on $\mathsf{P}[\lambda_1]$. (Here r is $\lambda(1,\ldots,1)$, and powers are with respect to the multiplication in P.)

iii) Moreover, if P is a Hopf operad with multiplication, then for any $q_1,q_2\in k$ the endomorphisms $\sigma_1(t)=q_1^{|t|}r^{|t|}$ and $\sigma_2(t)=q_2^{|t|}t$ define a coassociative Hopf operad structure on $\mathsf{P}[\lambda_1]$.

4.4 EXAMPLE Let P be a Hopf operad with multiplication. For any choice of $q_{i\,j}\in k$ for all $j\le n$, the maps

$$\sigma_i(t_1,\ldots,t_n)=q_{i\,1}^{|t_1|}\cdot\ldots\cdot q_{i\,n}^{|t_n|}t_1\cdot\ldots\cdot t_n$$

(where $i=1,2$) give a Hopf operad structure on $\mathsf{P}[\lambda_n]$.

5 Cohomology

In the previous section we constructed Hopf operad structures on $\mathsf{P}[\lambda_n]$. This section aims to describe the relation of these Hopf operads and Hochschid cohomology for coalgebras.

Let A be a Hopf Ass-algebra, and let $C^*_{(p)}$ be the graded vector space, which in degree q is $\mathrm{Hom}_k(A^{\otimes p}, A^{\otimes q})$. For $\varphi \in \mathrm{Hom}_k(A^{\otimes p}, A^{\otimes q})$ we define a differential by the formula

$$d\varphi = (\mu^{(p)} \otimes \varphi) \circ \tau \circ \Delta^{\otimes p} + \sum_{i=1}^{q} (-1)^i \Delta_{(i)} \circ \varphi + (-1)^{q+1} (\varphi \otimes \mu^{(p)}) \circ \tau \circ \Delta^{\otimes p},$$

where $\Delta_{(i)}(x_1 \otimes \ldots \otimes x_q) = (x_1 \otimes \ldots \otimes \Delta(x_i) \otimes \ldots \otimes x_q)$. This is the coalgebra-Hochschild complex with respect to the left and right coaction of A on $A^{\otimes p}$ given by

$$(\mu^{(p)} \otimes \mathrm{id}) \circ \tau \circ \Delta : A^{\otimes p} \longrightarrow A \otimes A^{\otimes p}$$

(or with $(\mathrm{id} \otimes \mu^{(p)})$, where $\mu^{(p)}$ is the unique p-fold application of μ. This complex (for $p, q > 0$) is the p-th column in the bicomplex used by Lazarev and Movshev [8] to compute what they call the cohomology of a Hop algebra.

Let P be a Hopf operad, and let A be a Hopf P-algebra, let $p \geq 1$ and $\sigma_1, \sigma_2 : A^{\otimes p} \to A$ be coalgebra morphisms. Then

$$\begin{aligned} (\sigma_1 \otimes \mathrm{id}) \circ \tau \circ \Delta^{\otimes p} &: A^p \to A \otimes A^{\otimes p} \qquad \text{and} \\ (\mathrm{id} \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes p} &: A^p \to A^{\otimes p} \otimes A \end{aligned}$$

define a left and a right coaction of A on A^p and we can define the complex $C^q_{\sigma_1\sigma_2}$ with the boundary $d_{\sigma_1\sigma_2}$. We can write the differential explicitly as

$$d\varphi = (\sigma_1 \otimes \varphi) \circ \tau \circ \Delta^p + \sum_{i=1}^{q} (-1)^i \Delta_{(i)} \circ \varphi + (-1)^{q+1} (\varphi \otimes \sigma_2) \circ \tau \circ \Delta^p.$$

This is the Hochschild boundary with respect to these coactions. The cohomology of this complex will be denoted $H^*_{\sigma_1\sigma_2}(A)$.

Let P be a Hopf operad. A **natural n-twisting function** φ is a map from Hopf P-algebras B to coalgebra maps $\varphi^{(B)} : B^{\otimes n} \to B$, such that the map φ commutes with augmented P-algebra morphisms $f : A \to B$ (i.e. $f \circ \sigma_i^{(A)} = \sigma_i^{(B)} \circ f^{\otimes n}$ for $i = 1, 2$).

If σ_1 and σ_2 are natural p-twisting functions such that $\sigma_i^{(B)}$ satisfies the conditions from 6.1 for any Hopf P algebra B; then $H^*_{\sigma_1\,\sigma_2}(B)$ is defined for any Hopf-P algebra B, and natural in B.

Let (B, β) be a Hopf P-algebra. Then β is a 1-cocycle in the (σ_1, σ_2)-complex iff

$$(\sigma_1 \otimes \beta)\Delta - \Delta\beta + (\beta \otimes \sigma_1)\Delta = 0.$$

5.1 Example Let P be a Hopf operad. Then $\sigma = p \in \mathsf{P}(n)$ defines a natural n-twisting function. More generally, if $\sigma = (\varepsilon^{\otimes k} \otimes p)\tau$ for some $p \in \mathsf{P}(n-k)$ and some $\tau \in S_n$, then σ defines a natural n-twisting function.

5.2 Theorem *Let $\sigma_i : \mathsf{P}[\lambda_n]^{\otimes n} \to \mathsf{P}[\lambda_n]$ satisfy the conditions of theorem 4.1. If σ_i for $i = 1, 2$ extends to a natural twisting function of P-algebras, then Hopf $\mathsf{P}[\lambda_n]$-algebras are Hopf P-algebras together with a cocycle of degree 1 in the σ_1, σ_2-twisted complex.*

Proof According to theorem 4.1, $\mathsf{P}[\lambda_n]$ is a Hopf operad. Algebras for $\mathsf{P}[\lambda_n]$ are P-algebras A together with a linear map $\alpha : A^{\otimes n} \to A$. For (A, α) to be a Hopf $\mathsf{P}[\lambda_n]$-algebra means that A is a Hopf P-algebra and that α satisfies

$$(\sigma_1 \otimes \alpha)\Delta + (\alpha \otimes \sigma_2)\Delta = \Delta\alpha,$$

which is the cocycle condition. QED

The following universal property of λ_n in Hochschild cohomology is the analogue of the universal property of Connes and Kreimer [3] and Moerdijk [12] in this setting.

5.3 Corollary *Under the conditions of theorems 4.1 and 5.2, let B be a Hopf P algebra. If $\beta : B^{\otimes n} \to B$ is a cocycle in the σ_1, σ_2-twisted complex; Then there is a unique Hopf P-algebra morphism $c : P_n \to B$ such that the following diagram commutes:*

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow{\scriptstyle c^{\otimes n}} & & \downarrow{\scriptstyle c} \\ B^{\otimes n} & \xrightarrow[\beta]{} & B. \end{array}$$

Proof The result follows since $P_n = \mathsf{P}[\lambda_n](0)$ is the initial Hopf $\mathsf{P}[\lambda_n]$-algebra. QED

5.4 Example If $\sigma = (\varepsilon^{\otimes k} \otimes p)\tau$ for some $p \in \mathsf{P}(n-k)$ and some $\tau \in S_n$, then σ satisfies the conditions of theorems 4.1 and 5.2. In this case maps of operads $\psi : \mathsf{Q} \to \mathsf{P}$ and $\varphi : \mathsf{P} \to \mathsf{Q}$ induce functors P-HopfAlg $\to$ Q-HopfAlg that map the natural twisting function σ to a natural twisting functions $\bar{\psi}^*\sigma$ and $\bar{\varphi}_!\sigma$ of Hopf Q-algebras.

6 Hopf algebras

This section lists some results for linear maps σ_i that define a Hopf algebra structure on P_n but do not extend to $\mathsf{P}[\lambda_n]$. The results in this section are obtained along the same lines as those in Moerdijk [12].

Let (P_n, λ_n) be the initial $\mathsf{P}[\lambda_n]$-algebra, then any pair of linear maps σ_1, σ_2 defines a unique P-algebra morphism Δ which makes the diagram

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow{\scriptstyle \Delta^n} & & \downarrow{\scriptstyle \Delta} \\ (P_n \otimes P_n)^{\otimes n} & \xrightarrow{(\sigma_1, \sigma_2)} & P_n \otimes P_n \end{array}$$

commute. We can use this universal property to deduce a result similar to corollary 4.2 if the σ_i are only defined on the initial algebra P_n and do not extend to $\mathsf{P}[\lambda_n]$.

6.1 PROPOSITION *Let $n \in \mathbb{N}$, λ_n and P_n be defined as above. Let $\sigma_i : P_n^{\otimes n} \to P_n$ for $i = 1, 2$ be linear maps. If σ_i satisfies*

$$\epsilon \circ \sigma_i = \epsilon^{\otimes n}, \quad \textit{and}$$
$$(\sigma_i \otimes \sigma_i) \circ \tau \circ \Delta^{\otimes n} = \Delta \circ \sigma_i;$$

then there exists a Hopf-P algebra structure on P_n such that $\Delta \circ \lambda = (\sigma_1, \sigma_2) \circ \tau \circ \Delta^{\otimes n}$ and $\varepsilon \circ \lambda = \varepsilon$.

There is also a result in cohomology if the the σ_i on P_n do not extend to $\mathsf{P}[\lambda_n]$.

6.2 PROPOSITION *Let $\sigma_i : P_n^{\otimes n} \to P_n$ satisfy the conditions of theorem 6.1. If σ_1, σ_2 extend to natural n-twisting functions, and $\beta : B^n \to B$ is a cocycle in the σ_1, σ_2-twisted complex; then there is a unique Hopf P-algebra morphism $c : P_n \to B$ such that the following diagram commutes:*

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow{\scriptstyle c^n} & & \downarrow{\scriptstyle c} \\ B^{\otimes n} & \xrightarrow[\beta]{} & B. \end{array}$$

This section concludes with some functorial properties of the construction. Parts *(ii)* and *(iii)* are trivial if the Hopf P-algebra structure on P_n extends to a Hopf operad $\mathsf{P}[\lambda_n]$.

6.3 PROPOSITION *Let $\varphi : \mathsf{Q} \to \mathsf{P}$ be a map of Hopf operads, and let σ_1, σ_2 satisfy the conditions of 6.1 on Q_n such that $\sigma_1\, \sigma_2$ extend to natural n-twisting functions of Q-algebras; then*

(i). The σ_i induce natural twisting functions σ_i of P-algebras, satisfying the conditions of 6.1 on H.

(ii). The unique map of $\mathsf{Q}[\lambda_n]$-algebras $j_0 : Q_n \to \varphi^*(P_n)$ is a map of Hopf Q-algebras.

(iii). The unique map of $\mathsf{P}[\lambda_n]$-algebras $j : \varphi_!(Q_n) \to P_n$ is a map of Hopf P-algebras.

7 A Deformation

Moerdijk [12] gives an inductive formula for a 2-parameter deformation of Δ on P_1 for any Hopf operad P. Regarding P_n, we have seen in example 4.4 that there is a $2n$-parameter deformation of the coproduct given by $\sigma_1 = \sigma_2 = u \circ \varepsilon^{\otimes n}$ (provided that P is an Hopf operad with multiplication). This section obtains an explicit formula for the $2n$-parameter deformation of the coproduct in C_n and the Lie algebra structure underlying the dual.

7.1 CONVENTION In this section we fix $\mathsf{P} = \mathsf{Com}$, and $n \in \mathbb{N}$. We use notation $C_n = \mathsf{P} = \mathsf{Com}[\lambda_n](0)$. For $i = 1, 2$ and $1 \leq j \leq n$, let $q_{ij} \in k$, and define

$$\sigma_i(t_1, \ldots t_n) = t_1 \cdot \ldots \cdot t_n \cdot \prod_j q_{ij}^{|t_j|}.$$

According to example 4.4, these σ_i make C_n a commutative Hopf algebra.

A subforest s of a rooted tree t is a subset of the partially ordered set t with the induced partial ordering. For colored trees, the colour of the edge connecting $v > w$ in s is the colour of edge connecting w to its direct predecessor in the unique path from v to w in t. For $v \in s$ we denote by $p_k(v, s, t)$ the number of edges of colour k in the path in t from v to the root of t that have their lower vertex in s^c. For forests t we define $p_k(v, s, t)$ as $p_k(v, s \cap t', t')$, where t' is the connected component of t containing v. There is an easy but useful lemma on the calculus of the p_k.

7.2 LEMMA *Let t and s be subforests of a forest u. Let $v \in s$ and set $t' = t \cup v$, $s' = s \cap t'$, $t'' = t^c \cup v$ and $s'' = s \cap t''$. Then*

$$p_k(v, s, u) = p_k(v, s', t') + p_k(v, s'', t''),$$

where t', t'', s' and s'' are interpreted as subforests of u.

PROOF The lemma follows at once when we observe that a vertex in the path from v to the root in u that is not in s is either in t' or in t''. QED

Define

$$q(s,t) := \prod_j q_{1j}^{\sum_{v\in s} p_j(v,s,t)} \cdot \prod_j q_{2j}^{\sum_{v\in s^c} p_j(v,s^c,t)}.$$

Intuitively, $q(s,t)$ counts for $v \in s$ the number of edges of colour j are in the path from v to the root that have their lower vertex in s^c and adds a factor q_{1j} for each of these, and $q(s,t)$ counts for $v \in s^c$ the number of edges of colour j are in the path from v to the root that have their lower vertex in s and adds a factor $q_{2,j}$ for each of these.

7.3 THEOREM *Under the assumptions of 7.1, we have for a forest $t \in C_n$ the formula*

$$\Delta(t) = \sum_{s\subset t} q(s,t)\, s \otimes s^c,$$

where the sum is over all subforests s of t.

PROOF We use induction with respect to the number of applications of λ. The formula is trivial for the empty tree.

Let $t = \lambda(x_1, \ldots, x_n)$ be a tree and suppose that the formula holds for all forests with less then $|t|$ vertices. Subforests of t are either of the form $s = \cup_i s_i$, a (disjoint) union of subforests of the x_i, or of the form $s = r \cup (\cup_i s_i)$, a (disjoint) union of subforests of the x_i together with the root. By definition,

$$\begin{aligned}\Delta(t) = &\sum_{s_i\subset x_i} s_1 \cdot \ldots \cdot s_n \otimes \lambda(s_1^c, \ldots, s_n^c) \prod_i q_{1i}^{|s_i|} q(s_i, x_i)\\ &+ \sum_{s_i\subset x_i} \lambda(s_1, \ldots, s_n) \otimes s_1^c \cdot \ldots \cdot s_n^c \prod_i q_{2i}^{|s_i^c|} q(s_i, x_i).\end{aligned}$$

But by the lemma above,

$$\prod q_{1i}^{|s_i|} q(s_i, x_i) = \prod_j q_{1j}^{\sum_{v\in s} p_j(v,s,t)} \cdot \prod_j q_{2j}^{\sum_{v\in s^c} p_j(v,s^c,t)}$$

for $s = \cup_i s_i = s_1 \cdot \ldots \cdot s_n$ and $s^c = r \cup (\cup_i s_i^c) = \lambda(s_1^c, \ldots, s_n^c)$; and

$$\prod q_{2i}^{|s_i^c|} q(s_i, x_i) = \prod_j q_{1j}^{\sum_{v\in s} p_j(v,s,t)} \cdot \prod_j q_{2j}^{\sum_{v\in s^c} p_j(v,s^c,t)}$$

for $s = r \cup (\cup_i s_i) = \lambda(s_1, \ldots, s_n)$ and $s^c = \cup_i s_i = s_1^c \cdot \ldots \cdot s_n^c$. Putting these together yields the formula. QED

An **antipode** for a Hopf Ass-algebra A is an inverse for the identity map id with respect to the convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

on the k-linear maps $\mathrm{Hom}_k(A, A)$.

In the situation of theorem 7.3 it makes sense to ask if there exists an antipode for arbitrary choice of the coefficients q_{ij}. An explicit description of S follows by a general argument. A Hopf Ass-algebra A is called **connected** if it is $\mathbb{Z}$-graded, concentrated in non-negative degree, and satisfies $A_0 = k \cdot 1$. The **augmentation ideal** of a connected Hopf algebra A is the ideal $\bigoplus_{n\geq 1} A_n$.

7.4 LEMMA (MILNOR AND MOORE [11]) *Let A be a connected Hopf Ass-algebra. Then there exists an antipode on A which for x in the augmentation ideal is given by*

$$S(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\mu^{(k)} \circ \bar{\Delta}^{(k)}(x),$$

where $\bar{\Delta} = \Delta - (id \otimes 1 + 1 \otimes id)$, and $\mu^{(0)} = id = \bar{\Delta}^{(0)}$, and $\mu^{(k)} : A^{\otimes k+1} \to A$ and $\bar{\Delta}^{(k)} : A \to A^{\otimes k+1}$ are defined using (co)associativity for $k > 0$ and $\mu^{(0)} = id = \bar{\Delta}^{(0)}$.

7.5 COROLLARY *Under the assumptions of convention 7.1, there exists an antipode on the Hopf algebra of theorem 7.3, given by the formula*

$$S(t) = \sum_{k=1}^{|t|} \sum_{\cup_i s_i = t} (-1)^k s_1 \cdot \ldots \cdot s_k \prod_{1\leq j<k} q(s_j, s_{j+1} \cup \ldots \cup s_k, s_j \cup \ldots \cup s_k),$$

where we only sum over partitions $t = s_1 \cup \ldots \cup s_k$ of the forest t with all forests s_i non-empty.

PROOF The grading with respect to $|t|$ gives a grading that makes the Lie algebra connected, since $\mathsf{Com}(0) = k$. QED

7.6 EXAMPLE In the case $n = 1$ with $q_1 = 1$ and $q_2 = 0$ we get the formula of Connes and Kreimer [3] for the antipode on the Hopf algebra H_R of section 2.

We now state a corollary to the formula for the coproduct in theorem 7.3, that gives an explicit formula for the Lie bracket on the primitive elements of the cocommutative Hopf algebra C_n^*. We know by the Milnor-Moore theorem [11] that C_n^* is the universal enveloping algebra of the Lie algebra of its primitive elements.

7.7 COROLLARY *Let $\mathsf{P} = \mathsf{Com}$, and $t \in C_n$, and $\Delta = \Delta_{q_{1\,1}\ldots q_{2\,n}}$; then the graded dual C_n^* is the universal enveloping algebra of the Lie algebra which* *as a vector space is spanned by elements D_t, where t is a rooted tree in C_n. The bracket is given by $[D_s, D_t] = D_s * D_t - D_t * D_s$, where*

$$D_s * D_t = \sum_{w=s\cup t} q(s,w)\, D_w,$$

the sum ranges over all rooted trees in C_n that have s and t as complementary colored subforests.

PROOF For any cocommutative Hopf algebra we can define an operation $*$ on the primitive elements, such that its commutator is the Lie bracket on primitive elements: Simply define $*$ as the truncation of the product at degree > 1, with respect to the primitive filtration F. In this case, $F_m C_n^*$ is spanned by the elements D_u dual to forests u consisting of at most m trees. The product in C_n^* is determined by the coproduct in C_n. Explicitly, for forest u, we can write the multiplication in C_n^* as

$$\begin{aligned} D_s D_t(u) &= (D_s \otimes D_t)\Delta(u) \\ &= \sum_{u=w\cup w^c} q(w,u) D_s(w) D_t(w^c). \end{aligned}$$

When we then restrict to the primitive part, we conclude that $D_s * D_t$ is given by the desired formula. QED

7.8 REMARK With some minor changes, there is an analogue of the above for the case $\mathsf{P} = \mathsf{Ass}$. The only change is that we have to remember the ordering of upgoing edges at each vertex. To be very explicit, for the coproduct on $A_n = \mathsf{Ass}[\lambda_n](0)$ we have the formula

$$\Delta(t) = \sum_{s\subset t} \prod_j q_{1\,j}^{\sum_{v\in s} p_j(v,s,t)} \cdot \prod_j q_{2\,j}^{\sum_{v\in t} p_j(v,s^c,t)} s \otimes s^c,$$

where the product of trees is associative, non-commutative. The order of multiplication is given by the linear order on the roots of the trees defined by the linear on the incoming edges at each vertex and the partial order on vertices. Dually, the primitive elements of the dual are again spanned by elements dual to single rooted trees (with linear ordering on incoming edges in each vertex). The truncated product $*$ is given by

$$D_s * D_t = \sum_{w=s\cup t} \prod_j q_{1\,j}^{\sum_{v\in s} p_j(v,s,w)} \cdot \prod_j q_{2\,j}^{\sum_{v\in t} p_j(v,t,w)} D_w,$$

where w, s and t are trees with a linear ordering on the incoming edges of the same colour at each vertex and the inclusions of s and t in w have to respect these orderings.

7.9 REMARK For certain values of the parameters q_{ij}, the formula of theorem 7.3 defines a coproduct on $\mathsf{k}[\lambda_n](0)$. If for fixed $i = 1, 2$ there is at most one j such that $q_{ij} \neq 0$, then the subspace spanned by single rooted trees which do not involve multiplication is preserved under Δ. Since this subspace is isomorphic to $\mathsf{k}[\lambda_n](0)$, this implies that in this case the formulas above define a coproduct on $\mathsf{k}[\lambda_n](0)$. Consequently, this defines an operation $*$ on the dual, since each element in the dual is primitive. A formula for this is given by the formula for Com, modulo trees involving multiplication (their span is denoted μ): the quotient $\mathsf{Com}[\lambda_n](0)^*/(\mu)$ is canonically isomorphic to $\mathsf{k}[\lambda_n](0)^*$.

8 Pre-Lie algebras

A **(right) pre-Lie algebra** (cf. Gerstenhaber [4], and Chapoton and Livernet [2]) is a k-vector space L together with a bilinear map $- * - : L \times L \to L$ which satisfies

$$(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y),$$

for each x, y and z in L. Note that the bracket $[x, y] = x * y - y * x$ defines a Lie algebra structure on L. This Lie algebra is the **associated Lie Algebra** of the pre-Lie algebra $(L, *)$.

Remember from [2] that the free pre-Lie algebra L_p on p generators is given by the vector space spanned by rooted trees with vertices labeled with elements of $\mathbf{p} = \{1, 2, \dots, p\}$. The pre-Lie algebra product is given by grafting trees. That is,

$$t * s = \sum_{v \in t} s \circ_v t$$

for trees s and t, where the colour of the vertices is preserved. For any tree $t \in L_p$ define $r_i(t)$ to be the tree t with the colour of the root changed to i. Thus r_i defines a linear endomorphism of the vector space L_p.

8.1 CONVENTION Let $\mathsf{P} = \mathsf{Com}$, choose Δ on $C_n = \mathsf{Com}[\lambda_n](0)$ defined by σ_i as in the deformation used in convention 7.1 and use the notation χ_S to denote the characteristic function of a subset $S \subset X$ which has value 1 on S and value 0 on $X - S$.

8.2 PROPOSITION *Let $\mathbf{p} \subset \{1, \dots, n\}$ and define $q_{1j} = \chi_{\mathbf{p}}(j)$ and $q_{2j} = 0$. Then*

(i). The product $*$ of corollary 7.7 defines a pre-Lie algebra structure on the primitive elements of C_n^*.

(ii). If $\mathbf{p} = \{1, \ldots, n\}$*, then there is a natural inclusion of this pre-Lie algebra into the free pre-Lie algebra on* n *generators. The image in the free pre-Lie algebra is spanned by all sums* $\sum_{i \in \mathbf{p}} t_i$*, of trees that only differ in that the colour of the root of* t_i *is* i*.*

Proof First note that with this choice of σ_i, the formula for the coproduct simplifies to

$$\Delta(t) = \sum_{c \in C(t)} P^c(t) \otimes R^c(t),$$

where we sum over admissible cuts as described in section 2, provided that the cuts are at edges labeled by an element of $\mathbf{p}$ and connected to the root by edges labeled by elements of $\mathbf{p}$. To see this, consider the formula of corollary 7.7 and note that the coeffient of $s \otimes s^c$ is 0 iff s^c has a connected component with respect to edges of colours in $\mathbf{p}$ which does not contain the root.

The operation $*$ of theorem 7.7 is then given by

$$D_s * D_t = \sum_{c \in \mathbf{p}} \sum_{v \in \bar{s}} D_{t \circ_v^c s},$$

where $\mathbf{p}$ is the set of colours and $t \circ_v^c s$ is the tree obtained by grafting t on $v \in s$ by an edge of colour c and $\bar{s}$ is the subtree of s of vertices connected to the root by edges labeled by elements of $\mathbf{p}$. To check the pre-Lie algebra identities, note that in $D_u * (D_t * D_s)$ we can graft the root of u on a vertex in either t or s. The result then follows directly, since $(D_u * D_t) * D_s$ consists of the terms obtained by grafting u on t.

Let $\mathbf{p} = \{1, \ldots, n)$ and consider the pre-Lie algebra of *(i)*. A map from this pre-Lie algebra into the free pre-Lie algebra on n generators (trees with n-colored vertices) is given by assigning the colour of an edge to the vertex on top and summing over all colours in $\mathbf{p}$ for the root. That this is an inclusion of pre-Lie algebras and that the image can be described as above is a direct consequence of the definitions. QED

8.3 Example In the case $n = 1$, $\mathsf{P} = \{1\}$, this yields the identification of L_1 with the primitive elements of H_R^* with the pre-Lie product $*$ as in Chapoton and Livernet [2].

9 The Pruning coproduct

Brouder and Frabetti [1] use a pruning operator on planar binary trees to define a coproduct on the k-vector space spanned by planar binary trees.

We will see that this coproduct is related to the coproducts described in section 7. We first give a family of coproducts on planar n-ary trees. In this context the pruning Hopf algebra of Brouder and Frabetti is a direct corollary.

A **planar binary tree** is a non-empty oriented planar graph in which each vertex which is not a leaf or the root has exactly two direct predecessors and one direct successor. Let T be a planar binary tree. We call each vertex which is not a leaf a **internal vertex** of T. If T has $n+1$ leaves, it has n internal vertices. Following Loday and Ronco, we denote the k-linear span of the set of planar binary trees $k[Y_\infty]$. On planar binary trees we have the binary operation λ adding a new root (smallest element) to the union of planar binary trees T_1 and T_2.

Likewise, we define a **planar n-ary tree** as a non-empty oriented graph in which each vertex which is not a leaf has exactly n direct predecessors and one direct successor. The n-ary operation λ adds a new root to an n-tuple of planar n-ary trees. Denote the vectorspace spanned by the set of n-ary planar trees by $k[Y_\infty^{(n)}]$. The basepoint in $k[Y_\infty^{(n)}]$ is the tree with one leaf.

9.1 PROPOSITION *Let $n \geq 0$.*

(i). The pointed vector space $k[Y_\infty^{(n)}]$ is isomorphic to $\mathsf{k}[\lambda_n](0)$.

(ii). The unitary free associative algebra on the pointed vector space spanned by planar n-ary trees has a family of natural Hopf algebra structures parametrized by $i \in \{1, \dots, n\}$.

PROOF Define a bijection $\xi : k[Y_\infty^{(n)}] \to \mathsf{k}[\lambda_n](0)$ by mapping the planar n-ary tree with one leaf to the empty tree and

$$\xi(\lambda(T_1, \dots, T_n)) = \lambda_n(\xi(T_1), \dots, \xi(T_n)),$$

for planar n-ary trees T_i. Each vertex in a tree in $\mathsf{k}[\lambda_n](0)$ corresponds to an internal vertex in the corresponding n-ary tree. It is easy to check that this is a basepoint preserving bijection of bases. This proves *(i)*.

Let $i \in \{1, \dots, n\}$. The coproducts induces by $q_{1j} = \chi_{\{i\}}(j)$ and $q_{2j} = 0$ are well defined on $\mathsf{k}[\lambda_n](0)$ by remark 7.9. This shows that we have a family of coproducts on $\mathsf{k}[\lambda_n](0)$. The result *(ii)* is now a direct consequence of the deformation in remark 7.8 and functorial properties. The functor $\bar{i}_! : \mathsf{k}[\lambda_n]$-HopfAlg $\to \mathsf{Ass}[\lambda_n]$-HopfAlg induced by the inclusion of Hopf operads $i : \mathsf{k} \to \mathsf{Ass}$ is the unitary free algebra functor and the result follows since the σ_i are a special case of example 5.4. QED

9.2 COROLLARY (BROUDER AND FRABBETTI [1]) *Let the pruning operator $P : k[Y_\infty] \to k[Y_\infty] \otimes k[Y_\infty]$ be defined inductively as*

$$P(\lambda(T,1)) = 0$$
$$P(\lambda(T,S) = \sum_i \lambda(T, S_i') \otimes S_i'' + \lambda(T,1) \otimes S,$$

if $P(S) = \sum_i S_i' \otimes S_i''$. Then the formulas

$$\Delta^P(1) = 1 \otimes 1 \quad and$$
$$\Delta^P(T) = 1 \otimes t + P(T) + T \otimes 1,$$

define a coassociative coproduct on $k[Y_\infty]$. Thus the unitary free associative algebra $T(k[Y_\infty])$ is a Hopf algebra with the coproduct described on generators by Δ^P.

PROOF Let $n = 2$, and $i = 2$. Apply proposition 9.1. QED

10 Associative and dendriform algebras

A **dendriform algebra** (cf. Loday [9], and Loday and Ronco [10]) is a vector space together with two bilinear (non-associative) products $\prec$ and $\succ$, satisfying the identities

$$(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$$
$$a \succ (b \prec c) = (a \succ b) \prec c$$
$$a \succ (b \succ c) = (a \prec b) \succ c) + (a \succ b) \succ c.$$

A dendriform algebra D defines an associative algebra on the same vector space with associative product $*$ defined by

$$a * b = a \prec b + a \succ b.$$

A **dendriform Hopf algebra** (Ronco [13]) is a dendriform algebra together with a coproduct that satisfies

$$\Delta(x \prec y) = \sum x' * y' \otimes x'' \prec y'' + x' * y \otimes x'' + y' \otimes x \prec y'' + y \otimes x$$
$$\Delta(x \succ y) = \sum x' * y' \otimes x'' \succ y'' + x * y' \otimes y'' + y' \otimes x \succ y'' + x \otimes y.$$

10.1 REMARK One motivation for studying dendriform algebras is the classification of Hopf algebras analogous to the Milnor-Moore theorem (cf. Milnor and Moore [11]). The Milnor-Moore theorem gives an equivalence of categories between the category of cocommutative Hopf algebras and the category of Lie algebras (in characteristic 0). A Milnor-Moore theorem for dendriform Hopf algebras can be found in Ronco [13]. This theorem states the equivalence of dendriform Hopf algebras and brace algebras.

10.2 CONVENTION Let $k, l \in \mathbb{N}$. Use $\mathsf{P} = \mathsf{k}$, and take $q_{1j} = \delta_{kj}$ and $q_{2j} = \delta_{lj}$ to define σ_1 and σ_2 in the deformation of 7.3. According to remark 7.9 this choice of parameters indeed defines a coproduct on $\mathsf{k}[\lambda_p](0)$.

We use the augmentation of $u^* : \mathsf{k}[\lambda](0)^* \to k$.

10.3 PROPOSITION *Consider the graded vector space $(\mathsf{k}[\lambda_p](0))^*$, graded by the number of applications of λ_p in trees. Then the augmentation ideal of $(\mathsf{k}[\lambda_p](0))^*$ is a graded dendriform algebra with associated multiplication $*$.*

PROOF The most convenient way to check the statement is to give an inductive characterization of the operations $\prec$ and $\succ$. To avoid awesome notation, we identify trees t with their dual D_t.
The product is characterized by its unit $1 = \emptyset$ $(= D_\emptyset)$, and the identity

$$s * t = \lambda(s_1, \ldots, s_k * t, \ldots, s_n) + \lambda(t_1, \ldots, s * t_l, \ldots, t_n),$$

as is readily checked. Now define operations

$$\begin{aligned} s \prec t &= \lambda(s_1, \ldots, s_k * t, \ldots, s_n) \quad \text{and} \\ s \succ t &= \lambda(t_1, \ldots, s * t_l, \ldots, t_n). \end{aligned}$$

Let $t = \lambda(t_1, \ldots, t_n)$, $u = \lambda(u_1, \ldots, u_n)$ and $s = \lambda(s_1, \ldots, s_n)$ and suppose that $\prec$ and $\succ$ satisfy the dendriform identities on all trees with in total $< |t| + |s| + |u|$ vertices. Then we have

$$\begin{aligned} (s \prec t) \prec u &= \lambda(s_1, \ldots, (s_k * t) * u, \ldots, s_n) \\ &= \lambda(s_1, \ldots, s_k * (t * u), \ldots, s_n) \\ &= s \prec (t * u) \\ (s \succ t) \prec u &= \lambda(t_1, \ldots, s * t_l, \ldots, t_k * u, \ldots, t_n) \\ &= s \succ (t \prec u). \end{aligned}$$

The first identity follows by the definition of $*$ and associativity on smaller trees. The second is clear, but uses associativity on smaller trees in case $k = l$. The proof of the third dendriform identity can be copied from the first almost verbatim. It is now left to show that the dendriform operations preserve the grading in the sense that for s of degree m and t of degree n, the products $s \prec t$ and $s \succ t$ are of degree $m + n$. If the dendriform operations are graded in total degree smaller then $m + n$, assume s and t of degree m and n respectively. We now know that $*$ is graded in total degree smaller then $m + n$. Since the definition of $\prec$ and $\succ$ only uses multiplication of smaller trees, one checks that the grading is preserved. QED

11 The Loday-Ronco construction

We still assume convention 10.2. There is a isomorphism of graded vector spaces between $\mathsf{k}[\lambda_n](0)$ and $(\mathsf{k}[\lambda_n](0))^*$ that maps t to D_t. Proposition 10.3 shows that we have an associative multiplication $*$ on $(\mathsf{k}[\lambda_n](0))^*$. The isomorphism give an associative multiplication on $\mathsf{k}[\lambda_n](0)$, which we will also denote by $*$.

Denote by $*_{(n)} : \mathsf{k}[\lambda_n](0)^{\otimes n} \to \mathsf{k}[\lambda_n](0)$ the map induced by the associative multiplication, and by $\varepsilon_{(n)} : \mathsf{k}[\lambda_n](0)^{\otimes n} \to \mathsf{k}[\lambda_n](0)$ the map $u \circ \varepsilon^{\otimes n}$.

11.1 PROPOSITION *Use convention 10.2 and let the product $*$ of proposition 10.3. Then $\sigma_1 = *_{(n)}$ and $\sigma_2 = \varepsilon_{(n)}$ define a Hopf algebra structure on $\mathsf{k}[\lambda_n](0)$ iff $k = n$ and $l = 1$.*

Moreover, the augmentation ideal of $\mathsf{k}[\lambda_n](0)$ is a dendriform Hopf algebra.

PROOF We check the requirements of proposition 6.1. The requirement for the counit is fulfilled for any element of $(\mathsf{k}[\lambda_p](0))^*$. We prove the claim for coassociativity inductively. Let σ_i as above and assume that the requirement for Δ holds upto degree n. Then coassociativity of Δ holds up to degree n. Compute for $k = n$ and $l = 1$ (using Sweedler notation)

$$\begin{aligned}\Delta(a * b) &= \Delta(\lambda(a_1, \dots, a_n) * \lambda(b_1, \dots, b_n)) \\ &= \Delta(\lambda(a_1, \dots, a_n * b) + \Delta(\lambda(a * b_1, \dots, b_n)) \\ &= \sum a' * \dots * (a_n * b)' \otimes \lambda(a_1'', \dots, (a_n * b)'') \\ &+ \sum (a * b_1)' * \dots * b_n' \otimes \lambda((a * b_1)'', \dots, b_n'') \\ &+ (a * b) \otimes 1\end{aligned}$$

for $a = \lambda(a_1, \dots, a_n)$ and $b = \lambda(b_1, \dots, b_n)$ such that the sum of their degrees is m. Then use the induction hypothesis for degrees smaller then m (in particular $\Delta(a_p * n) = \Delta(a_n) * \Delta(b)$ and $\Delta(a * b_1) = \Delta(a) * \Delta(b_1)$). Deduce

$$\begin{aligned}\Delta(a \prec b) &= \sum a' * b' \otimes a'' \prec b'' \qquad \text{and} \\ \Delta(a \succ b) &= a' * b' \otimes a'' \succ b'',\end{aligned}$$

and conclude that $(\mathsf{k}[\lambda_p](0))^*$ is a Hopf algebra if $k = n$ and $l = 1$. The formulas also show that for other choices of k and l the induction fails and coassociativity does not hold since $*$ is not commutative.

The dendriform Hopf identities for $\bar{\Delta} = \Delta - \mathrm{id} \otimes 1 - 1 \otimes \mathrm{id}$ follow from the above. QED

Loday and Ronco [10] construct a Hopf algebra of planar binary trees. Remember the definition of the vector space $k[Y_\infty]$ as the vector space spanned by binary trees.

11.2 COROLLARY (LODAY AND RONCO [10]) *Let* $T, T' \in k[Y_\infty]$. *If* $T = \lambda(T_1, T_2)$ *and* $S = \lambda(S_1, S_2)$, *then*

$$T * S = \lambda(T_1, T_2 * S) + \lambda(T * S_1 S_2)$$

defines an associative product on $k[Y_\infty]$ *with as unit the tree with one leaf. Moreover there is a coproduct making* $k[Y_\infty]$ *a Hopf algebra. This coproduct is defined inductively by*

$$\Delta(T) = \sum (T_1' * T_2') \otimes \lambda(T_1'', T_2'') + T \otimes 1,$$

where $\Delta(T) = \sum_{(T)} T' \otimes T''$ *is the Sweedler notation of the coproduct. The counit* ε *is the obvious augmentation of the product by projection on the span of the unit.*
Finally, the augmentation ideal is a dendriform Hopf algebra.

PROOF Recognize the star product on $\mathsf{k}[\lambda_2](0)$ given by $q_{12} = 1 = q_{21}$ and $q_{11} = 0 = q_{22}$, and the coproduct associated to $*$. Apply proposition 11.1. QED

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