

## Analogies between Mass-Flux and Reynolds-Averaged Equations

STEPHAN R. DE ROODE AND PETER G. DUYNKERKE

*Institute for Marine and Atmospheric Research Utrecht, Utrecht University, Utrecht, Netherlands*

A. PIER SIEBESMA

*Royal Netherlands Meteorological Institute, De Bilt, Netherlands*

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### ABSTRACT

In many large-scale models mass-flux parameterizations are applied to prognose the effect of cumulus cloud convection on the large-scale environment. Key parameters in the mass-flux equations are the lateral entrainment and detrainment rates. The physical meaning of these parameters is that they quantify the mixing rate of mass across the thermal boundaries between the cloud and its environment.

The prognostic equations for the updraft and downdraft value of a conserved variable are used to derive a prognostic variance equation in the mass-flux approach. The analogy between this equation and the Reynolds-averaged variance equation is discussed. It is demonstrated that the prognostic variance equation formulated in mass-flux variables contains a gradient-production, transport, and dissipative term. In the latter term, the sum of the lateral entrainment and detrainment rates represents an inverse timescale of the dissipation.

Steady-state solutions of the variance equations are discussed. Expressions for the fractional entrainment and detrainment coefficients are derived. Also, solutions for the vertical flux of an arbitrary conserved variable are presented. For convection in which the updraft fraction equals the downdraft fraction, the vertical flux of the scalar flows down the local mean gradient. The turbulent mixing coefficient is given by the ratio of the vertical mass flux and the sum of the fractional entrainment and detrainment coefficients. For an arbitrary updraft fraction, however, flux correction terms are part of the solution. It is shown that for a convective boundary layer these correction terms can account for countergradient transport, which is illustrated from large eddy simulation results. In the cumulus convection limit the vertical flux flows down the "cloud" gradient. It is concluded that in the mass-flux approach the turbulent mixing coefficients, and the correction terms that arise from the transport term, are very similar to closures applied to the Reynolds-averaged equations.

### 1. Introduction

Boundary layer clouds have radiative properties that influence the earth's energy balance. Also, they exchange heat, momentum, and moisture between the boundary layer and the free troposphere by means of turbulent transport. For these reasons, an accurate representation of cloud-topped boundary layers is important for general circulation models. Therefore, it is necessary to develop parameterizations for these types of boundary layers, which are for computational reasons, relatively simple but, on the other hand, sufficiently accurate. There are a variety of cloud parameterization schemes that range from the very simple to rather complex. For instance, to prognose the evolution of a well-

mixed stratocumulus-topped boundary layer, the mixed-layer model is the simplest one to use. In such a model, the vertical fluxes of temperature and moisture are rather simple functions of the surface fluxes, entrainment velocity, and thermodynamic jumps across the inversion, cloud-base height, radiation, and precipitation (Nicholls 1984; Wang 1993; Bretherton and Wyant 1997). However, when the stratocumulus deck is broken and inhomogeneous, the assumption that the boundary layer is well mixed is no longer valid, and the mixed-layer scheme therefore cannot be used. For the same reason cumulus cloud fields, which develop in conditionally unstable layers, cannot be represented by such a simple mixed-layer model, either.

In a first-order closure model the Reynolds-averaged prognostic equations for the mean state variables are solved, whereas in a higher-order closure model prognostic equations for the variances and fluxes are included. These types of equations are frequently used in many general circulation models. In a first-order closure model, it is often assumed that the vertical turbulent

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*Corresponding author address:* Stephan de Roode, Department of Atmospheric Sciences, University of Washington, Box 351640, Seattle, WA 98195-1640.  
E-mail: [roode@atmos.washington.edu](mailto:roode@atmos.washington.edu)

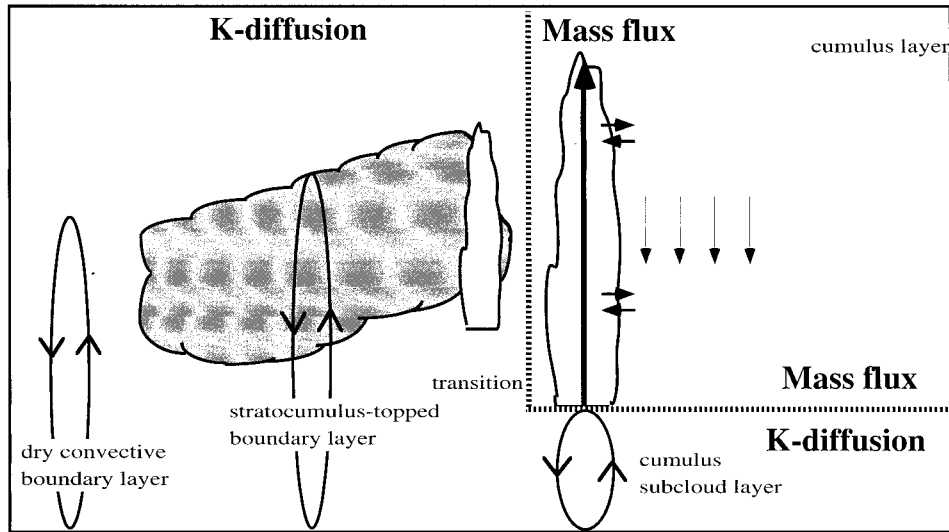


FIG. 1. Schematic of various types of atmospheric boundary layers and its representation in general circulation models.

flux  $\overline{w'\psi'}$  flows down the local mean gradient of  $\overline{\psi}$ . The vertical flux is parameterized as the product of a turbulent-mixing coefficient  $K_\psi$  and the local gradient of the mean:

$$\overline{w'\psi'} = -K_\psi \frac{\partial \overline{\psi}}{\partial z}. \quad (1.1)$$

Unless the height-dependent coefficient  $K_\psi$  is prescribed, it must be predicted and is dependent on the flow variables. For instance, the turbulent-mixing coefficient can be parameterized as a function of a velocity scale and a length scale. In a one-and-a-half-order closure model, a prognostic equation for the turbulent kinetic energy (TKE) equation is included (Duykerke and Driedonks 1987) and the square root of the TKE is used as a typical velocity scale. Because in a convective boundary layer (CBL) the flux can flow counter to the vertical mean gradient, a correction term can be included (Holtslag and Moeng 1991) that is referred to as nonlocal closure.

The mass-flux approach is typically in use for cumulus parameterizations (Arakawa 1969; Siebesma 1998; Tiedtke 1989). The basic assumption made is that the dynamical and thermodynamical variables can be described by a top-hat profile. In the mass-flux approach, the vertical turbulent flux  $F_{\psi,M}$  can be expressed as

$$F_{\psi,M} = M_c(\psi_u - \psi_d) = \kappa_{w\psi} \overline{\rho} \overline{w'\psi'}, \quad (1.2)$$

with  $M_c$  the convective mass flux,  $\psi_u$  and  $\psi_d$  the values of an arbitrary variable  $\psi$  in the cumulus updraft and in the environment, respectively;  $\kappa_{w\psi}$  a proportionality factor,  $\overline{\rho}$  the mean density; and  $\overline{w'\psi'}$  the Reynolds-averaged flux. Because the proportionality factor is approximately  $\kappa_{w\psi} \approx 1$  for the cumulus-environment de-

composition (Siebesma and Cuijpers 1995), the top-hat approximation is frequently used in large-scale models for the parameterization of cumulus clouds. However, the distinction between cloud and environment cannot be made for the CBL or the stratocumulus-topped boundary layer (STBL). Therefore another indicator function has to be chosen in order to select convective thermals that can be based on the fluctuations of humidity or temperature, or the sign of the vertical velocity (see Manton 1977; Coulman 1978; Lamb 1978; Greenhut and Khalsa 1982; Lenschow and Stephens 1980; Young 1988a; Nicholls 1989; Schumann and Moeng 1991a; De Laat and Duykerke 1998). In studies where the updraft-downdraft decomposition was applied, it was shown that the proportionality factor  $\kappa_{w\psi}$  has a typical value of about 0.6 for the CBL and STBL. This number was found from both aircraft observations and large eddy simulation (LES) results. Wyngaard and Moeng (1992) reported on the basis of theoretical arguments that  $\kappa_{w\psi}$  should be 0.627, provided that the joint probability density function of vertical velocity and scalar fluctuations is a Gaussian function. The remaining fraction of the vertical flux is transported by subplume motions that are lost when averaging over updrafts and downdrafts separately.

In many general circulation models different parameterization schemes are in use, as is illustrated in Fig. 1. Typically, the large-scale thermodynamic tendencies due to cumulus convection are computed from a mass-flux scheme, whereas for dry convection and stratocumulus, a  $K$ -diffusion parameterization is applied. This can lead to model inconsistencies when stratocumulus is gradually replaced by cumulus, a type of boundary layer that is frequently observed in the subtropics (Bretherton et al. 1995; Martin et al. 1995; de

Roode and Duynkerke 1996; de Roode and Duynkerke 1997; Wang and Lenschow 1995). To overcome the problem of convection-dependent schemes, Randall et al. (1992) formulated a “second-order bulk boundary layer” model as a compromise between a higher-order closure model and a mass-flux model. They developed a theoretical framework for a single mass-flux scheme that could deal with the simulation of all types of convective atmospheric boundary layers, ranging from the convective boundary layer to a cumulus-topped boundary layer.

In this paper, a prognostic equation for the variance using the mass-flux approach will be derived by mathematically manipulating the updraft and downdraft mass-flux equations for conserved variables (Arakawa and Schubert 1974; Tiedtke 1989; Siebesma and Cuijpers 1995). The resulting variance equation is nearly identical to the one presented by Randall et al. (1992). However, they obtained the variance equation by a direct substitution of relationships between Reynolds-averaged higher-order moments and mass-flux variables in the Reynolds-averaged variance equation. The primary difference between the variance equation of Randall et al. (1992) and the one presented in this paper lies in a slightly different formulation of the dissipation term. Here it will be shown that the parameterization of the dissipation of variance is a simple, yet unique, function of the lateral entrainment and detrainment rates. In this process, the lateral mixing rates represent typical inverse dissipative timescales. As a physical explanation, lateral mixing tends to decrease the difference between the updraft and downdraft properties. Eventually, we will discuss analogies in the closure problems encountered in mass-flux and Reynolds-averaged equations. Some illustrative examples are presented from LES results and aircraft observations made in a cumulus field.

## 2. Basic equations

In this section the prognostic equation for the variance in the mass-flux notation will be presented. Basically, the conservation equations are used for the derivation. For mass and an incompressible fluid the continuity equation reads (Stull 1988)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2.1)$$

where  $u$  and  $v$  are the mean horizontal velocity com-

ponents in the  $x$  and  $y$  direction, and  $w$  is the vertical velocity component in the  $z$  direction. The conservation equation for a scalar quantity  $\psi$  is given by

$$\frac{\partial \psi}{\partial t} + u_j \frac{\partial \psi}{\partial x_j} = \nu_\psi \frac{\partial^2 \psi}{\partial x_j^2} - \frac{S_\psi}{\rho}, \quad (2.2)$$

where  $\nu_\psi$  is the molecular diffusivity of  $\psi$  and  $S_\psi$  contains all the possible sink and source terms.

### a. Reynolds decomposition

In the Reynolds decomposition, any variable  $\psi$  is split into a mean  $\bar{\psi}$  and a turbulent part  $\psi'$ . Upon applying Reynolds-averaging rules, the conservation equation for a variable  $\psi$  can be written as

$$\frac{\partial \bar{\psi}}{\partial t} + \bar{u}_j \frac{\partial \bar{\psi}}{\partial x_j} = - \frac{\partial \bar{u}'_j \bar{\psi}'}{\partial x_j} - \frac{\bar{S}_\psi}{\bar{\rho}}. \quad (2.3)$$

Because the molecular diffusivity term is very small in comparison with the other terms it is neglected in (2.3). To derive a prognostic equation for the variance  $\overline{\psi'^2}$  one needs to expand Eq. (2.2) into mean and turbulent parts after which (2.3) is subtracted. If the remaining equation is multiplied by  $2\psi'$  then, after Reynolds averaging, one can write (Stull 1988):

$$\begin{aligned} \frac{\partial \overline{\psi'^2}}{\partial t} &= \underbrace{-2\bar{w}'\bar{\psi}'}_{(S)} \frac{\partial \bar{\psi}}{\partial z} - \underbrace{\frac{\partial (\bar{w}'\bar{\psi}'\bar{\psi}')}{\partial z}}_{(T)} - \underbrace{2\varepsilon_\psi}_{(D)} - \underbrace{2\frac{\bar{S}'_\psi \bar{\psi}'}{\bar{\rho}}}_{(Source)}, \end{aligned} \quad (2.4)$$

where the terms (S) represent storage, (P) production, (T) transport, and (D) dissipation of variance. It is assumed that the boundary layer is horizontally homogeneous. Because in the boundary layer the large-scale subsidence is typically of the order ( $\sim 1 \text{ cm s}^{-1}$ ), it is assumed that its contribution to the variance budget is negligibly small. Thus, we are only considering the effect of turbulence on the variance budget. The dissipation term  $2\varepsilon_\psi$  arises from the molecular diffusivity term in (2.1),  $\varepsilon_\psi = \nu_\psi \overline{(\partial \psi' / \partial x_j)^2}$ . The last term (Source) on the rhs of (2.4) is a covariance term of perturbations of the source function and  $\psi$ .

### b. Mass-flux decomposition

The updraft mean and the downdraft mean of any variable  $\psi$  are defined as

$$\psi_u = \frac{\int_A \chi^+ \psi \, dA}{\int_A \chi^+ \, dA}, \quad \psi_d = \frac{\int_A \chi^- \psi \, dA}{\int_A \chi^- \, dA}, \quad \begin{cases} \chi^+ = 1 & \text{and} & \chi^- = 0 & \text{if } w > 0 \\ \chi^+ = 0 & \text{and} & \chi^- = 1 & \text{if } w \leq 0, \end{cases}$$

where the integration is performed over a horizontal plane at height  $z$  and  $\chi$  is an indicator function that discriminates between updrafts and downdrafts. The parameter  $\sigma$  defines the updraft fraction,

$$\sigma = \frac{A_u}{A_u + A_d}, \quad (2.5)$$

where  $A_u$  and  $A_d$  are the updraft and downdraft areas, respectively. If the presence of liquid water is used as an indicator function then  $\sigma$  represents the cloud fraction. The vertical mass flux  $M_c$  is defined as (Randall et al. 1992)

$$M_c = \bar{\rho}\sigma(1 - \sigma)(w_u - w_d). \quad (2.6)$$

The prognostic equations for the updraft and downdraft mean of  $\psi$  can be written as (De Laet and Duynkerke 1998)

$$\begin{aligned} \bar{\rho} \frac{\partial \sigma \psi_u}{\partial t} &= -\frac{\partial M_c \psi_u}{\partial z} + L_{\text{ex}} - \sigma S_u \\ \bar{\rho} \frac{\partial (1 - \sigma) \psi_d}{\partial t} &= +\frac{\partial M_c \psi_d}{\partial z} - L_{\text{ex}} - (1 - \sigma) S_d, \end{aligned} \quad (2.7)$$

where  $L_{\text{ex}}$  represents the net lateral exchange of the variable  $\psi$  between the updrafts and downdrafts. The updraft fraction  $\sigma$  is assumed to be constant as a function of time. Note that we assumed a perfect top-hat distribution in (2.7) by using  $\kappa_{w\psi} = 1$  in (1.2). Thus, we implicitly assumed that the effect of the subplume contribution to the flux is proportional to that of the top-

hat contribution  $M_c(\psi_u - \psi_d)$  (Randall et al. 1992; De Laet and Duynkerke 1998; Petersen et al. 1999).

Often, the net lateral exchange term  $L_{\text{ex}}$  is parameterized according to a top-hat formulation (Arakawa and Schubert 1974; Siebesma and Cuijpers 1995; Tiedtke 1989):

$$L_{\text{ex}} = E_s \psi_d - D_s \psi_u, \quad (2.8)$$

where  $E_s$  and  $D_s$  represent the lateral entrainment and detrainment rate, which is the horizontal mass flow per unit of time from a downdraft into an updraft, and from an updraft into a downdraft, respectively. Since the number of unknowns is increased by one extra term in (2.8), the conditionally sampled continuity equation is used, which relates  $E_s$  and  $D_s$  to the vertical mass-flux gradient:

$$\frac{\partial M_c}{\partial z} = E_s - D_s = M_c(\varepsilon - \delta), \quad (2.9)$$

where the parameters  $\varepsilon$  and  $\delta$  represent the normalized fractional entrainment and detrainment coefficients, respectively. According to Randall et al. (1992) the variance in the mass-flux approach is given by

$$\overline{(\psi'^2)}_M = \sigma(1 - \sigma)(\psi_u - \psi_d)^2, \quad (2.10)$$

and the vertical flux of variance reads

$$\bar{\rho} \overline{(w' \psi' \psi')} _M = (1 - 2\sigma) M_c (\psi_u - \psi_d)^2. \quad (2.11)$$

After some manipulation (see appendix A) it can be shown from (2.7) that the prognostic equation for the variance in mass-flux variables can be written as

$$\begin{aligned} \bar{\rho} \frac{\partial \sigma(1 - \sigma)(\psi_u - \psi_d)^2}{\partial t} &= \underbrace{-2M_c(\psi_u - \psi_d)}_{\text{(P)}} \frac{\partial \bar{\psi}}{\partial z} - \underbrace{\frac{\partial(1 - 2\sigma)M_c(\psi_u - \psi_d)^2}{\partial z}}_{\text{(T)}} - \underbrace{(E_s + D_s)(\psi_u - \psi_d)^2}_{\text{(D)}} \\ &\quad - \underbrace{2\sigma(1 - \sigma)(\psi_u - \psi_d)(S_u - S_d)}_{\text{(Source)}}. \end{aligned} \quad (2.12)$$

The mass-flux variance equation consists of terms that have a similar physical interpretation as (2.4). Note that (S), (P), and (T) in (2.12) can be derived directly by substitution of (1.2), (2.10), and (2.11) into (2.4), as was demonstrated by Randall et al. (1992).

### 3. Lateral entrainment and detrainment

#### a. Comparison of Reynolds-averaged and mass-flux variance equations

In the previous section we identified a gradient production and transport term in the mass-flux variance equation (2.12). Because the lateral mixing rates  $E_s$  and  $D_s$  are both positive by definition it is clear that the term

(D) in (2.12) is always negative and for this reason can be interpreted as a term that acts to decrease the variance in the mass-flux representation. In a large-scale model, which includes a prognostic variance equation, the dissipation of the variance  $\varepsilon_\psi$  is generally not explicitly computed. Usually the dissipation is parameterized as a function of large-scale variables by assuming that it is proportional to the ratio of the variance and a typical turbulence timescale  $\tau_{\text{dis}}$  (Randall et al. 1992):

$$\varepsilon_\psi = \frac{\overline{\psi'^2}}{\tau_{\text{dis}}}. \quad (3.1)$$

If we neglect the source term in (2.12) and compare (2.4) and (2.12) we can conclude that the dissipation

term (D) in (2.12) is very similar to the closure assumption (3.1), and accordingly the dissipation of mass-flux variance  $2\varepsilon_{\psi,M}$  can be defined as

$$2\varepsilon_{\psi,M} = \frac{(E_s + D_s)}{\bar{\rho}} (\psi_u - \psi_d)^2. \quad (3.2)$$

If we substitute (2.10) into (3.1) we find that (3.1) and (3.2) are equivalent if

$$\tau_{\text{dis}} = \frac{2\bar{\rho}\sigma(1 - \sigma)}{(E_s + D_s)}. \quad (3.3)$$

There is thus a direct link between  $\tau_{\text{dis}}$  and  $(E_s + D_s)/\bar{\rho}$ : the sum of the lateral entrainment and detrainment rate is thus proportional to the inverse turbulence timescale in the parameterization of dissipation. If we substitute (3.3) into (2.12) we find exactly the same prognostic variance equation as presented by Randall et al. (1992). Note that in some higher-order closure models a formulation slightly different from (3.1) is used (André et al. 1978; Canuto et al. 1994; Mellor and Yamada 1982):

$$\varepsilon_{\psi} = c \frac{\sqrt{\bar{\varepsilon}}}{\Lambda} \psi'^2, \quad (3.4)$$

with  $c$  a constant,  $\Lambda$  a length scale, and  $\bar{\varepsilon}$  the turbulent kinetic energy. However, according to the definition of  $\varepsilon$  and  $\delta$  by (2.7) we recognize that (3.2) can be expressed similarly to (3.4), with the typical velocity scale given by  $M_c/\bar{\rho}$  and  $(\varepsilon + \delta)^{-1}$  representing the characteristic turbulence length scale. Therefore it can be concluded that the calculation of the characteristic lateral mixing rates in mass-flux schemes is analogous to the closure of the dissipation in the Reynolds-averaged equations.

It should be stressed that the mass-flux variance dissipation term does not directly arise from the molecular diffusivity term in (2.2), but rather is a result of the conditionally sampled horizontal flux divergence term  $L_{\text{ex}}$  in (2.7) and its parameterization by (2.8). The analogies between the mass-flux and Reynolds-averaged equations can be explained by the similar way they are derived. To obtain the prognostic Reynolds-averaged variance equation, the prognostic equation for  $\psi'$  is multiplied by  $\psi'$ , whereas in deriving (2.12), the prognostic equation (A.8) for  $(\psi_u - \psi_d)$  is multiplied by  $(\psi_u - \psi_d)$ . In the Reynolds-averaged prognostic variance equation the dissipation term acts to decrease the variance, while the lateral entrainment–detrainment term in (2.12) tends to decrease the square of the difference between the updrafts and downdrafts, which is also the variance in the mass-flux representation.

A simulation of the CBL (Table 1) was performed with the Institute for Marine and Atmospheric Utrecht/Royal Meteorological Institute of the Netherlands (IMAU/KNMI) LES model (Cuijpers and Duynkerke 1993) using a central difference scheme. To test the similarities between the mass-flux and Reynolds-averaged variance equations the LES results were used to compute the variance budgets for the potential temper-

ature. For the Reynolds-averaged variance budget we only considered the contributions due to the resolved motions because the subgrid fluxes are very small with an exception for the lowest model levels. Conditional sampling was performed by applying the updraft–downdraft decomposition. The lateral entrainment and detrainment rates were calculated following the method described in Siebesma and Cuijpers (1995). Basically, the net lateral exchange is determined as a residual from the budget equations, and we are using (2.8)–(2.9) to compute the lateral entrainment and detrainment rate. In the boundary layer the subsidence velocity is typically one order of magnitude smaller than the convective mass flux, and therefore, for simplicity, we did not prescribe any large-scale advection. As is clear from Fig. 2, the production term, which is given by the product of the vertical flux and the mean gradient, becomes negative in the bulk of the boundary layer. This means that the flux and the gradient have the same sign and therefore the downgradient formulation (1.1) cannot give a correct flux. Since the dissipation term is negative by definition, variance in the bulk of the boundary layer needs to be produced by the vertical transport term. Indeed, in these regions the transport terms in (2.4) and (2.12) both produce variance. Generally, both variance budgets exhibit the same features; a maximum dissipation and production near the surface and the top of the boundary layer, and production by the transport terms in the bulk. Because the triple moment  $(\overline{w'\theta'\theta'})_M$  becomes very small near the top of the mixed layer the mass-flux transport of variance does not consume as much variance as the Reynolds-averaged transport term (Young 1988a).

#### b. Expressions for the fractional entrainment and detrainment coefficients

In a mass-flux model it is important to use accurate values for the lateral mixing parameters. From (2.12) we can derive expressions for the fractional entrainment ( $\varepsilon$ ) and detrainment ( $\delta$ ) coefficients. If we assume a steady state and no source function, then a general formula for  $\varepsilon$  can be derived from (2.9) and (2.12):

$$\begin{aligned} \varepsilon = & -\frac{1}{\psi_u - \psi_d} \frac{\partial \bar{\psi}}{\partial z} \\ & - \frac{1}{2M_c(\psi_u - \psi_d)^2} \frac{\partial(1 - 2\sigma)M_c(\psi_u - \psi_d)^2}{\partial z} \\ & + \frac{1}{2} \frac{\partial \ln M_c}{\partial z}. \end{aligned} \quad (3.5)$$

The terms on the rhs of (3.5) can be further simplified by systematically eliminating the mass flux  $M_c$  by (2.9). If we substitute (A.5) to eliminate  $\bar{\psi}$ , write  $\partial_z(ab) = b\partial_z a + a\partial_z b$  for the first and second term, and use the continuity equation (2.9), we can rewrite (3.5) into a symmetric form:

TABLE 1. Characteristics of the large eddy simulation of a clear convective boundary layer. The boundary layer height  $h$  is defined as the level where the buoyancy flux has a minimum value.

$\overline{w'\theta'_{v,s}}$	0.05	[K m s <sup>-1</sup> ]
$u_{*c}$	0.095	[m s <sup>-1</sup> ]
$L$	-1.3	[m]
$h$	600	[m]
$\Delta\theta_v$	5	[K]

$$(1 - \sigma)\varepsilon + \sigma\delta = -\frac{(1 - \sigma)\frac{\partial\psi_u}{\partial z} + \sigma\frac{\partial\psi_d}{\partial z}}{(\psi_u - \psi_d)}. \quad (3.6)$$

### 1) CUMULUS CONVECTION ( $\sigma \rightarrow 0$ )

Let us suppose  $\sigma \rightarrow 0$ , which is the limit for cumulus convection. Then we can obtain expressions for the fractional lateral entrainment and detrainment coefficients by (2.9) and (3.6):

$$\begin{aligned} \varepsilon &= -\frac{1}{\psi_u - \psi_d} \frac{\partial\psi_u}{\partial z}, \\ \delta &= -\frac{1}{\psi_u - \psi_d} \frac{\partial\psi_u}{\partial z} - \frac{\partial \ln M_c}{\partial z}. \end{aligned} \quad (3.7)$$

This relationship for the fractional entrainment coefficient was used by Raga et al. (1990) to compute  $\varepsilon$  in cumulus clouds. By taking the limit  $\sigma \rightarrow 0$ , it is implicitly assumed that there is a balance between gradient production, vertical transport, and dissipation of variance. Currently, in some models  $\varepsilon$  is taken to be constant throughout the cumulus layer (Siebesma and Holtslag 1996; Tiedtke 1989). This can be interpreted as taking the typical length scale of cumulus convection to be constant.

### 2) CONVECTION WITH ZERO VERTICAL VELOCITY SKEWNESS ( $\sigma = 0.5$ )

If we substitute  $\sigma = 0.5$  into (3.6), and using (2.9) we can write

$$\begin{aligned} \varepsilon &= -\frac{1}{\psi_u - \psi_d} \frac{\partial\bar{\psi}}{\partial z} + \frac{1}{2} \frac{\partial \ln M_c}{\partial z}, \\ \delta &= -\frac{1}{\psi_u - \psi_d} \frac{\partial\bar{\psi}}{\partial z} - \frac{1}{2} \frac{\partial \ln M_c}{\partial z}. \end{aligned} \quad (3.8)$$

Typically, the updraft fraction is very close to  $\sigma = 0.5$  in stratocumulus clouds (De Laat and Duynkerke 1998). The expression for the fractional entrainment (3.8) differs from (3.7); the gradient of the updraft value is replaced by the gradient of the mean and an additional mass-flux gradient term is included. By taking this limit we effectively neglect the effect of the transport term in (2.12), and the production and dissipation of variance balance each other.

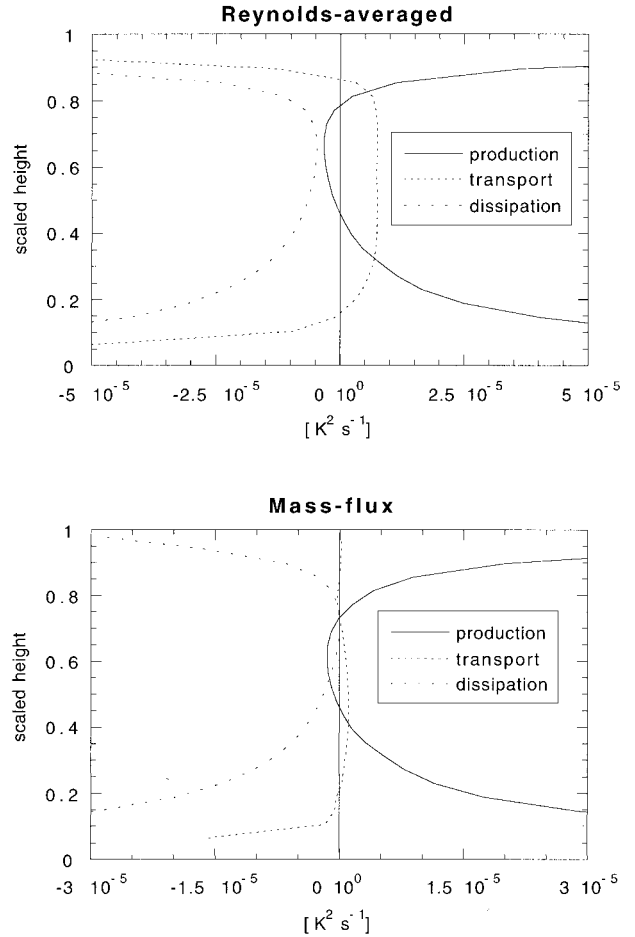


FIG. 2. Variance budget for (a) the Reynolds-averaged equations according to Eq. (2.4) and (b) the mass-flux equations according to Eq. (2.12). Line styles as indicated in the legend.

### 3) SURFACE LAYER SCALING

In many models similarity relationships are used to calculate variances and fluxes at the lowest model levels. These similarity relationships are needed to get realistic vertical gradients of the mean state variables in the surface layer (Wang and Albrecht 1990). Because vertical gradients in the surface layer determine the magnitude of the surface fluxes, Randall et al. (1992) discussed the importance of parameterizing the effect of lateral mixing, and they introduced a surface ventilation coefficient in their mass-flux model. We will suggest a parameterization for the fractional entrainment coefficient that can be applied to the surface layer. We can rewrite (3.5) if we apply the chain rule to the second term on the rhs of (3.5):

$$\begin{aligned} \varepsilon &= -\frac{1}{(\psi_u - \psi_d)} \frac{\partial\bar{\psi}}{\partial z} - \frac{(1 - 2\sigma)}{(\psi_u - \psi_d)} \frac{\partial(\psi_u - \psi_d)}{\partial z} \\ &\quad + \sigma \frac{\partial \ln M_c}{\partial z} + \frac{\partial\sigma}{\partial z}. \end{aligned} \quad (3.9)$$

According to a parameterization for the updraft fraction in a CBL proposed by Young (1988a), the updraft fraction varies between  $\sigma = 0.5$  ( $z = 0$ ) and  $\sigma = 0.48$  at  $z = 0.1z_i$ , with  $z_i$  the boundary layer height. Therefore it is reasonable to assume that the second term is much smaller than the first term on the rhs of (3.9). To reduce (3.9) further we will neglect, for simplicity, updraft fraction variations with height by using the approximation  $\sigma = 0.5$ . By (2.12) this means that we effectively neglect the effect of vertical transport of variance.

After substitution of (2.10) into (3.9) we can express  $\varepsilon$  (with  $\psi = \theta$ ) as a function of the vertical velocity variance, potential temperature variance, and the dimensionless potential temperature gradient  $\phi_h \equiv (kz/\theta_*) (\partial\theta/\partial z)$ :

$$\varepsilon = -\frac{1}{2\sqrt{\kappa_{\theta\theta}}} \frac{\theta_*}{(\theta'^2)^{1/2}} \frac{\phi_h}{kz} + \frac{1}{2} \frac{\partial \ln[(\overline{w'^2})^{1/2}/u_*]}{\partial z}. \quad (3.10)$$

Because the friction velocity  $u_*$  is independent of height  $z$  it can be included in the second term on the rhs of (3.10). The factor  $\kappa_{\theta\theta}$  relates the variance in the mass-flux approach (2.10) to the Reynolds-averaged variance,  $(\overline{\theta'^2})_M = \kappa_{\theta\theta} \overline{\theta'^2}$ . Using the similarity relationships summarized in appendix B, we find the following expressions for  $\varepsilon$  as a function of height:

neutral stratification:

$$\varepsilon = \frac{1}{2k\alpha_{\theta_1}(\kappa_{\theta\theta})^{1/2}} \frac{1}{z}, \quad (3.11a)$$

free convection limit:

$$\varepsilon = \left[ \frac{\gamma_1}{2k\alpha_{\theta_2}(\kappa_{\theta\theta})^{1/2}} + \frac{1}{6} \right] \frac{1}{z}, \quad (3.11b)$$

unstable stratification with wind shear:

$$\varepsilon = \left[ \frac{1}{2k\alpha_{\theta_2}(\gamma_2\kappa_{\theta\theta})^{1/2}(-L)^{1/6}} \right] \frac{1}{z^{6/5}} + \frac{1}{6} \frac{1}{z}, \quad (3.11c)$$

where we assumed  $-\gamma_2 z/L \gg 1$  in (B.10). Note that the fractional entrainment coefficient is dependent on the wind shear according to the Monin–Obukhov length ( $L$ ) dependency in (3.11c). For a neutral surface stratification and in the free convection limit the surface similarity relationships predict that  $\varepsilon$  is inversely proportional to the height  $z$ , where  $z$  is also a typical length scale of the dominant eddy sizes. Lenschow and Stephens (1980) presented scaling relationships for thermal velocities and temperature, and for the free convection layer they suggested similar power laws as (B.7) and (B.8). Therefore, substitution of their similarity relationships into (3.10) would give comparable expressions as (3.11b, c), even though they used humidity as an indicator of thermals.

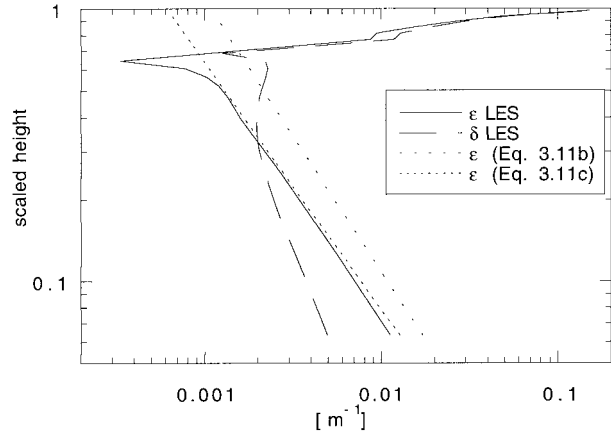


FIG. 3. Fractional lateral entrainment and detrainment coefficients for a clear convective boundary layer. Line styles as indicated in the legend.

### c. Examples of lateral entrainment and detrainment

The fractional entrainment and detrainment coefficients in a clear convective boundary layer (see Table 1 for details) are shown in Fig. 3. To plot the fractional lateral coefficients  $\varepsilon$  and  $\delta$  in a plot with logarithmic axes, we have omitted two points where values for  $\varepsilon$  and  $\delta$  were found to be negative. These negative values were due to a statistical error near the levels where the potential temperature flux changes sign and were not found from the total water content budget equations. The fractional lateral entrainment coefficient has maximum values near the surface and the top of the boundary layer. This can be attributed to the increasingly smaller eddy sizes near these interfaces. Also shown in the same figure are the fractional entrainment parameterizations according to Eqs. (3.11b,c), where we used  $\kappa_{\theta\theta} = 0.3$  (Young 1988a). In the lower half of the boundary layer (BL) the LES-derived fractional entrainment coefficient has a nearly constant lapse rate when plotted logarithmically and follows a power law,

$$\varepsilon = cz^{-\alpha}, \quad (3.12)$$

with  $\alpha$  close to 1. Although the parameterizations are obtained with surface similarity relationships and are therefore not valid in the mixed layer, Eq. (3.11c) gives a good agreement with the LES results mainly because it predicts approximately the same power law as (3.12). Also, the free convection limit Eq. (3.11b) gives the correct gradient. The results in the surface layer are difficult to compare because in this region the LES does not resolve the turbulent motions very well. However, the LES results in the mixed layer and the surface similarity relationships show that the fractional entrainment has a distinct height dependence which can be approximately parameterized by (3.12).

As another illustration, in-cloud and environmental values of the total water content and liquid water potential temperature in a cumulus cloud field were con-

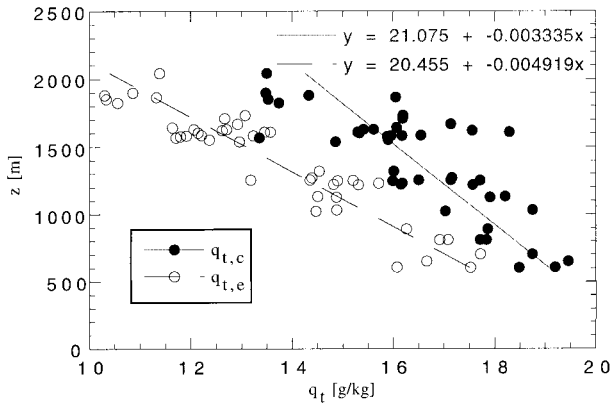


FIG. 4. Conditionally sampled total water content in cumulus clouds ( $q_{t,c}$ ) and environment ( $q_{t,e}$ ). Also indicated are vertical gradients calculated with a linear fit.

ditionally sampled. The observations were made from the National Center for Atmospheric Research (NCAR) C-130 aircraft during the Small Cumulus Microphysics Study near Florida (Gerber 1999). During flight RF12, which took place from 1300 to 1615 LT (local time) on 5 August, the observed cumuli had a base at 400 m and cloud top at 3000 m. The aircraft observations were made between cloud base and about 2000 m. The estimated cloud fraction was about 15%. We selected clouds that had a horizontal extent larger than 500 m. From the results shown in Fig. 4 we calculated the total water content gradient and the difference between the in-cloud value and the environmental value. After substituting these numbers into (3.7) we find  $\varepsilon \approx 1.5 \times 10^{-3} \text{ m}^{-1}$ . The same approach for the liquid water potential temperature leads to a similar result, namely,  $\varepsilon \approx 1.3 \times 10^{-3} \text{ m}^{-1}$ . This result agrees well with the observations of Raga et al. (1990). The horizontally averaged dissipation  $\varepsilon_{qt}$  was calculated from the Fourier spectra of the total water content in and outside the cloud. The fractional detrainment  $\delta$  needed in the mass-flux variance dissipation was calculated from the vertical mass-flux gradient and the continuity equation (2.9), and (3.7). The results from legs at approximately the same height were averaged and are shown in Fig. 5. The dissipation obtained with the two different methods shows that the dissipation in the cumulus field tends to increase with height. According to the mass-flux expression of the variance, this is due to a larger difference with increasing height between the cloud and environmental value of the total water content.

#### 4. Flux parameterizations

After multiplying (3.6) by  $M_c(\psi_u - \psi_d)/[(1 - \sigma)\varepsilon + \sigma\delta]$  and a substitution of the mean (A.5) a general expression for the vertical flux  $F_{\psi,M}$  can be written

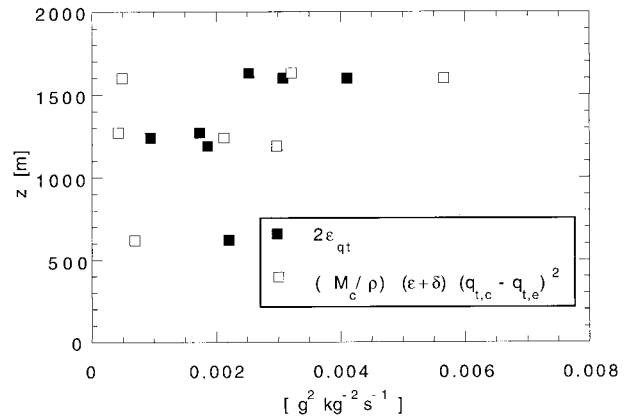


FIG. 5. Horizontal mean dissipation calculated from inertial subrange theory and according to the mass-flux variance equation (3.2) in a cumulus cloud field.

$$F_{\psi,M} = -\frac{M_c}{(1 - \sigma)\varepsilon + \sigma\delta} \times \left[ \frac{\partial \bar{\psi}}{\partial z} + (1 - 2\sigma) \frac{\partial(\psi_u - \psi_d)}{\partial z} - (\psi_u - \psi_d) \frac{\partial \sigma}{\partial z} \right]. \quad (4.1)$$

##### a. Convection with zero vertical velocity skewness ( $\sigma = 0.5$ )

If we suppose  $\sigma = 0.5$ , then (4.1) gives

$$F_{\psi,M} = -\frac{2M_c}{\varepsilon + \delta} \frac{\partial \bar{\psi}}{\partial z}. \quad (4.2)$$

The numerator in (4.2) has dimension  $[\text{m}^{-1}]$  and thus can be considered as a typical inverse length scale. Hence, for a nonskewed vertical velocity distribution, the mass-flux approach reduces to the classic  $K$ -diffusion closure (1.1), with  $K_\psi$  a function of a velocity scale and a length scale, the former given by the mass flux and the latter by the sum of the fractional lateral mixing coefficients  $\varepsilon$  and  $\delta$ ,

$$K_\psi = 2M_c(\varepsilon + \delta)^{-1}. \quad (4.3)$$

Note that the form of (4.2) is similar to the downgradient diffusion formula presented in Randall et al. (1992), and their equation can be directly obtained by a substitution of (3.3) into (4.2).

##### b. Cumulus convection ( $\sigma \rightarrow 0$ )

From (4.1), and for  $\sigma \rightarrow 0$  the vertical flux of  $\psi$  can be expressed as

$$F_{\psi,M} = -\frac{M_c}{\varepsilon} \frac{\partial \psi_u}{\partial z}, \quad (4.4)$$

where we used  $\psi_d \approx \bar{\psi}$  by (A.5). Note that this expression can be obtained directly by dividing the equa-



tion for the fractional entrainment coefficient for cumulus (3.7) by the mass flux  $M_c$ . In (4.4) the flux is proportional to the in-cloud gradient, or in other words, the flux is down the ‘‘cloud gradient.’’

For  $\sigma \rightarrow 0$ , Randall et al. (1992) obtained the cumulus-induced compensating subsidence formula:

$$-M_c \frac{\partial \bar{\psi}}{\partial z} = \frac{\partial M_c(\psi_u - \psi_d)}{\partial z}, \quad (4.5)$$

where they assumed the convective mass flux to be constant with height. Moreover, because they parameterized the dissipation by (3.1), with the variance according to (2.10), their dissipation term becomes zero in the limit  $\sigma \rightarrow 0$  and therefore does not enter the solution (4.5). The assumption of constant mass flux with height, and  $\psi_d \approx \bar{\psi}$  by (A.5), enables us to rewrite (4.5) as  $(\partial/\partial z)\psi_u = 0$ . In fact, (4.5) states that the vertical gradient in the cloud is zero, which is not very realistic, as is clear from the results shown in Fig. 4. Randall et al. noticed, however, that a detrainment term could be included by allowing the mass flux to vary with height. Indeed, if we substitute (2.9) into (4.1), and take the limit  $\sigma \rightarrow 0$  we can write

$$-M_c \frac{\partial \bar{\psi}}{\partial z} = \frac{\partial M_c(\psi_u - \psi_d)}{\partial z} + D_s(\psi_u - \psi_d). \quad (4.6)$$

Note that only if  $D_s = 0$ , we can obtain (4.5) from (4.6). The physical interpretation of the detrainment term in (4.8) is that it modifies the vertical mean gradient in the cloud due to lateral mixing (Arakawa and Schubert 1974).

### c. Flux expressions for arbitrary $\sigma$ : Flux correction terms

For the CBL a downgradient diffusion approach for the vertical flux can give an incorrect result for the potential temperature, for instance, since this flux can be countergradient (Holtslag and Moeng 1991). In many models a correction term is introduced to overcome this problem; this term is referred to as a countergradient or nonlocal transport. The  $K$ -diffusion expression (4.2) will not give countergradient transport for  $\sigma = 0.5$ , either. In general, the updraft fractions can vary with height. If we express the flux  $F_{\psi,M}$  as

$$F_{\psi,M} = -\frac{2M_c^2}{(E_s + D_s)} \frac{\partial \bar{\psi}}{\partial z} - \frac{M_c}{(E_s + D_s)(\psi_u - \psi_d)} \frac{\partial(1 - 2\sigma)M_c(\psi_u - \psi_d)^2}{\partial z}, \quad (4.7)$$

by (1.2), (2.9), and (2.12), and apply the chain rule to the second term on the rhs of (4.7), we can express the vertical flux as

$$F_{\psi,M} = -\frac{2M_c^2}{(E_s + D_s)} \frac{\partial \bar{\psi}}{\partial z} - \frac{2(1 - 2\sigma)M_c}{(E_s + D_s)} \frac{\partial M_c(\psi_u - \psi_d)}{\partial z} - \frac{M_c^2 M_c(\psi_u - \psi_d)}{(E_s + D_s)} \frac{\partial(1 - 2\sigma)}{\partial z} \frac{1}{M_c}. \quad (4.8)$$

The second term on the rhs of (4.8) is the mass-flux analogy of the countergradient term as derived by Wyngaard and Weil (1991), who concluded that the roots of the transport asymmetry lie in the interaction between the skewness of the transporting turbulence and the gradient flux of the transported scalar. These authors suggested a scalar flux parameterization according to

$$\overline{w'\psi'} = -\overline{w'^2} T_L \frac{\partial \bar{\psi}}{\partial z} - \frac{S_w \sqrt{\overline{w'^2}} T_L}{2} \frac{\partial \overline{w'\psi'}}{\partial z}, \quad (4.9)$$

with  $S_w$  the vertical velocity skewness and  $T_L$  a Lagrangian integral timescale. By  $F_{\psi,M} = \bar{\rho}(\overline{w'\psi'})_M$ , and if we neglect the third term on the rhs of (4.8), we can rewrite (4.8) after substitution of (1.2) and (2.10):

$$\frac{(\overline{w'\psi'})_M}{\bar{\rho}} = -\frac{(\overline{w'^2})_M}{2(E_s + D_s)} \frac{\partial \bar{\psi}}{\partial z} - \frac{S_{wm} \sqrt{1 - S_{wm}^2/4} \sqrt{(\overline{w'^2})_M}}{2(E_s + D_s)} \frac{\partial (\overline{w'\psi'})_M}{\partial z}, \quad (4.10)$$

where we used (Randall et al. 1992)

$$\sigma = \frac{1}{2} - \frac{S_{wm}}{2\sqrt{4 + S_{wm}^2}} \quad (4.11)$$

with the vertical velocity skewness  $S_{wm}$  in the mass-flux approach defined as (De Laat and Duynkerke 1998)

$$S_{wm} = \frac{\sigma w_u^3 + (1 - \sigma)w_d^3}{(\sigma w_u^2 + (1 - \sigma)w_d^2)^{3/2}}. \quad (4.12)$$

If we assume  $0 < S_{wm} \ll 2$  we can write (4.10) in a formulation equivalent to (4.9), with  $2(E_s + D_s)/\bar{\rho} \sim T_L^{-1}$ :

$$\frac{(\overline{w'\psi'})_M}{\bar{\rho}} = -\frac{(\overline{w'^2})_M}{2(E_s + D_s)} \frac{\partial \bar{\psi}}{\partial z} - \frac{S_{wm} \sqrt{(\overline{w'^2})_M}}{2(E_s + D_s)} \frac{\partial (\overline{w'\psi'})_M}{\partial z}. \quad (4.13)$$

Remarkably, the sum of the lateral entrainment and detrainment rate can also be interpreted as an inverse timescale that is relevant to the nonlocal part of the vertical turbulent flux. Thus the mass-flux equations contain a countergradient solution for the vertical flux. However, the flux expression (4.8) contains a third term that has to be evaluated. We have expressed this term as a function of the flux  $M_c(\psi_u - \psi_d)$ , such that by (1.2) we can rewrite (4.8) as

$$F_{\psi,M} = - \frac{\frac{2M_c^2}{(E_s + D_s)} \frac{\partial \bar{\psi}}{\partial z}}{\left[ 1 + \frac{M_c^2}{(E_s + D_s)} \frac{\partial (1 - 2\sigma)}{\partial z} \frac{1}{M_c} \right]} - \frac{\frac{2(1 - 2\sigma)M_c}{(E_s + D_s)} \frac{\partial M_c(\psi_u - \psi_d)}{\partial z}}{\left[ 1 + \frac{M_c^2}{(E_s + D_s)} \frac{\partial (1 - 2\sigma)}{\partial z} \frac{1}{M_c} \right]} \quad (4.14)$$

The numerator in the two terms on the rhs of (4.14) is a function of dynamical parameters only and has values that typically lie in the range between 1 and 1.5, with minima near the bottom and top of the boundary layer and a maximum at about  $0.6z/z_i$ . Thus, the numerator effectively tends to reduce the eddy mixing coefficient and the multiplication factor of the flux gradient.

## 5. Summary and discussion

In this paper we have derived a prognostic variance equation in the mass-flux approach for an arbitrary conserved variable (Arakawa and Schubert 1974; Siebesma and Cuijpers 1995; Tiedtke 1989) and compared it with the variance equation in the Reynolds-averaged approach. Mass-flux equations include a parameterization for the net lateral exchange of mass per unit of time between the updrafts and downdrafts in terms of lateral entrainment ( $E_s$ ) and detrainment ( $D_s$ ) rates. The major findings of this research are as follows.

- The prognostic variance equation in the Reynolds-averaged and the mass-flux approach do have striking similarities; they both contain a gradient-production, a transport, and a dissipative term. In the mass-flux approach and for a top-hat distribution, the prognostic variance equation derived in this paper is nearly identical to the one presented by Randall et al. (1992).
- The sum of the lateral entrainment and detrainment rate is equivalent to the inverse of the characteristic timescale applied in the Reynolds-averaged closure of the molecular dissipation (André et al. 1978; Canuto et al. 1994; Mellor and Yamada 1982).
- In the lower half of the convective boundary layer the fractional entrainment coefficient  $\varepsilon$  follows a power law  $z^{-\alpha}$ , with  $\alpha$  a number close to 1.
- For a skewed vertical velocity distribution,  $\sigma \neq 0.5$ , the mass-flux equations bear a solution that accounts for countergradient flux transport, similar to the formulation presented by Wyngaard and Weil (1991).

In the mass-flux approach, a variance destruction term appears in the variance equation that arises from the lateral entrainment and detrainment terms. Except for this term, the prognostic mass-flux variance equation

derived in this paper resembles the one presented by Randall et al. (1992). The analogies between the mass-flux and Reynolds-averaged equations can be explained by the similarities in their derivation. To obtain the Reynolds-averaged variance equation, the prognostic equation for  $\psi'$  is multiplied by  $\psi'$ , whereas in deriving the mass-flux variance equation, Eq. (A.8) for  $(\psi_u - \psi_d)$  is multiplied by  $(\psi_u - \psi_d)$ . In the Reynolds-averaged prognostic variance equation the dissipation term acts to decrease the variance, while the lateral entrainment/detrainment term in (2.12) tends to decrease (the square of) the difference between the updraft and downdraft properties, which is also the variance in the mass-flux representation.

From LES results of a convective boundary layer it is shown that the fractional entrainment  $\varepsilon$  and fractional detrainment  $\delta$  vary with height. Both parameters have maximum values near the surface and top of the boundary layer. In the lower half of the convective boundary layer  $\varepsilon$  follows a power law  $z^{-\alpha}$ , where  $\alpha$  is a number close to 1. In the surface layer such a height dependency is predicted by similarity relationships. Currently, some cumulus mass-flux models use a constant value for the fractional entrainment coefficient  $\varepsilon$  (Siebesma and Holtlag 1996; Tiedtke 1989). To refine these models, it might be necessary to develop a parameterization for  $\varepsilon$  that is dependent on the typical length scale or velocity scale of convection. For example, in the case of deep cumulonimbus convection a smaller value for  $\varepsilon$  and  $\delta$  might be used than for shallow cumuli, because the horizontal and vertical length scales are larger (Siebesma 1996). This can be seen after substitution of (2.6) into (3.3), which gives an expression for the sum of the fractional mixing coefficients:

$$\varepsilon + \delta = \frac{2}{\tau_{\text{dis}}(w_u - w_d)}. \quad (5.1)$$

Like Siebesma, if it is assumed that the dissipation time-scale is of the same order of magnitude as the eddy turnover time of the most active eddies we can write

$$\tau_{\text{dis}} = \frac{H}{w_u}, \quad (5.2)$$

with  $H$  the typical cumulus depth. For the limit  $\sigma \rightarrow 0$  the downdraft velocity is approximately  $w_d \approx 0$  and substitution of (5.2) into (5.1) gives

$$\varepsilon + \delta = \frac{2}{H}. \quad (5.3)$$

Such a hypothesis can be tested with an LES model by performing simulations with different vertical thermodynamical mean gradients and surface fluxes.

Several solutions for the vertical turbulent flux of a scalar have been presented. In the updraft–downdraft decomposition,  $\sigma$  is closely related to the vertical velocity skewness and is an important parameter for flux parameterizations. As is summarized in Table 2, the ver-

TABLE 2. Summary of vertical flux solutions derived from the mass-flux variance Eq. (2.12).

$\sigma$	Flux	Diffusivity coefficient
$\sigma = \sigma(z)$	$F_{\psi,M} = -K_\psi \left[ \frac{\partial \bar{\psi}}{\partial z} + (1 - 2\sigma) \frac{\partial(\psi_u - \psi_d)}{\partial z} - (\psi_u - \psi_d) \frac{\partial \sigma}{\partial z} \right]$	$K_\psi = \frac{M_c}{(1 - \sigma)\varepsilon + \sigma\delta}$
$\sigma = 0$	$F_{\psi,M} = -K_\psi \frac{\partial \psi_u}{\partial z}$	$K_\psi = \frac{M_c}{\varepsilon}$
$\sigma = 0.5$	$F_{\psi,M} = -K_\psi \frac{\partial \bar{\psi}}{\partial z}$	$K_\psi = \frac{2M_c}{\varepsilon + \delta}$
$\sigma = 1$	$F_{\psi,M} = -K_\psi \frac{\partial \psi_d}{\partial z}$	$K_\psi = \frac{M_c}{\delta}$

tical flux of a scalar is downgradient when the updraft fraction equals  $\sigma = 0.5$ , which is equivalent to a vertical velocity skewness  $S_{wm} = 0$ . When  $\sigma \rightarrow 0$ , which is the limit for convection with a large vertical velocity skewness, the relevant gradient is given by the in-cloud gradient. Also we conclude that it is necessary to include the effect of lateral detrainment in the compensating subsidence formulation (Arakawa and Schubert 1974; Randall et al. 1992) since it represents the effect of lateral mixing on the mean in-cloud vertical gradient. For an arbitrary  $\sigma$  the general solution for the flux contains a (nonlocal) correction term that is similar to that presented by Wyngaard and Weil (1991), who derived the same correction term from a Taylor expansion. These results imply that the updraft fraction needs to be parameterized, which is similar to parameterizing the third-order moment of the vertical velocity in a Reynolds-averaged closure, as is clear in Eq. (4.11). However, the general mass-flux solution gives an additional term for the nonlocal part of the vertical flux, which acts to modify the eddy diffusivity coefficient.

Typically, in the mass-flux equations the diffusivity parameter  $K_\psi$  depends on the mass flux and the fractional entrainment and detrainment coefficients. Computing the mass flux as a (vertical) velocity scale is in fact similar to introducing a TKE or  $w'^2$  equation into a Reynolds-averaged closure scheme. Analyses of the budgets of the conditionally sampled vertical velocity were discussed from both observations (Lenschow and Stephens 1980; Young 1988b) and model simulations (Schumann and Moeng 1991b; Krueger et al. 1995). As is summarized in Tables 2 and 3, the mass-flux and Reynolds-averaged equations are consistent and require closures based on similar physical concepts. This suggests that using  $K$  diffusion for convective boundary

layers and a mass-flux scheme for cumulus clouds might be an unnecessary complication; a single scheme might be sufficient. The greatest difficulty in mass-flux schemes however is that for convective boundary layers the conditional sampling approach gives the most satisfactory results for the updraft–downdraft decomposition, whereas for cumulus the cloud core–environment works better. Practically this means that the area fraction parameter  $\sigma$  can represent an updraft fraction that is closely related to the *dynamical* vertical velocity skewness parameter, whereas for cumulus it represents a cloud fraction, which can be considered as an important *optical* (radiation) parameter. If one applies the updraft–downdraft decomposition to a cumulus cloud field this leads to a weak mass-flux correlation (Siebesma and Holtslag 1996), indicating that the subplume motions give a significant contribution to the vertical flux that effect then needs to be parameterized in a mass-flux model. For the CBL and STBL it was found that the subplume contribution is a nearly constant fraction (about 40%) of the total vertical flux. For these types of boundary layers the subplume contribution can therefore be simply parameterized by assuming that it is proportional to the vertical transport due to the top-hat contribution (Petersen et al. 1999).

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TABLE 3. Summary of mass-flux variables and the analogous Reynolds-averaged closure variables.

Scale	Mass flux	Reynolds averaged
Velocity	$M_c \bar{\rho}$	$(w'^2)^{1/2}, \sqrt{\text{TKE}}$
Length	$\varepsilon^{-1}, \delta^{-1}$	$l$
Time	$(E_c \bar{\rho})^{-1}, (D_c \bar{\rho})^{-1}$	$\tau$
Asymmetry	$\sigma$	$S_w$

## APPENDIX A

**Derivation of the Prognostic Variance Equation in Mass-Flux Variables**

To derive (2.12) we will omit the source term in our calculation for the sake of readability. Then our starting equations are as follows (see 2.7):

$$\bar{\rho} \frac{\partial \sigma \psi_u}{\partial t} = -\frac{\partial M_c \psi_u}{\partial z} + E_s \psi_d - D_s \psi_u \quad (\text{A.1})$$

$$\bar{\rho} \frac{\partial (1 - \sigma) \psi_d}{\partial t} = +\frac{\partial M_c \psi_d}{\partial z} - E_s \psi_d + D_s \psi_u. \quad (\text{A.2})$$

If we multiply (A.1) by  $(1 - \sigma)$  and (A.2) by  $\sigma$  and subtract (A.2) from (A.1) we can write

$$\begin{aligned} \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)}{\partial t} = & -(1 - \sigma) \frac{\partial M_c \psi_u}{\partial z} - \sigma \frac{\partial M_c \psi_d}{\partial z} \\ & + E_s \psi_d - D_s \psi_u, \end{aligned} \quad (\text{A.3})$$

where we assume that the updraft fraction  $\sigma$  is constant with time. Our aim is to find an expression for the prognostic variance in the mass-flux notation. We know that we seek an expression of the following form:

$$\frac{\partial \overline{\psi'^2}}{\partial t} = -2\overline{w'\psi'} \frac{\partial \overline{\psi}}{\partial z} - \frac{\partial (\overline{w'\psi'\psi'})}{\partial z} - 2\varepsilon_\psi. \quad (\text{A.4})$$

Using the definition of the mean,

$$\overline{\psi} = \sigma \psi_u + (1 - \sigma) \psi_d, \quad (\text{A.5})$$

we substitute the following equality into (A.3),

$$M_c \frac{\partial \overline{\psi}}{\partial z} = M_c \frac{\partial (\sigma \psi_u + (1 - \sigma) \psi_d)}{\partial z} \quad (\text{A.6})$$

such that

$$\begin{aligned} \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)}{\partial t} = & -M_c \frac{\partial \overline{\psi}}{\partial z} - (1 - \sigma) \frac{\partial M_c \psi_u}{\partial z} - \sigma \frac{\partial M_c \psi_d}{\partial z} + E_s \psi_d \\ & - D_s \psi_u + M_c \frac{\partial (\sigma \psi_u + (1 - \sigma) \psi_d)}{\partial z}. \end{aligned} \quad (\text{A.7})$$

After substitution of the continuity equation (2.9) and some manipulation of (A.7) we can write

$$\begin{aligned} \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)}{\partial t} = & -M_c \frac{\partial \overline{\psi}}{\partial z} - E_s (\psi_u - \psi_d) + \sigma (\psi_u - \psi_d) \frac{\partial M_c}{\partial z} \\ & + M_c (\psi_u - \psi_d) \frac{\partial \sigma}{\partial z} - (1 - 2\sigma) M_c \frac{\partial (\psi_u - \psi_d)}{\partial z}. \end{aligned} \quad (\text{A.8})$$

To obtain an equation for the variance we multiply (A.8) by  $2(\psi_u - \psi_d)$ :

$$\begin{aligned} & \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)^2}{\partial t} \\ = & -2M_c (\psi_u - \psi_d) \frac{\partial \overline{\psi}}{\partial z} - 2E_s (\psi_u - \psi_d)^2 \\ & + 2\sigma (\psi_u - \psi_d)^2 \frac{\partial M_c}{\partial z} + 2M_c (\psi_u - \psi_d)^2 \frac{\partial \sigma}{\partial z} \\ & - (1 - 2\sigma) M_c \frac{\partial (\psi_u - \psi_d)^2}{\partial z}. \end{aligned} \quad (\text{A.9})$$

Now we will use

$$\begin{aligned} 2\sigma (\psi_u - \psi_d)^2 \frac{\partial M_c}{\partial z} = & (2\sigma - 1) (\psi_u - \psi_d)^2 \frac{\partial M_c}{\partial z} \\ & + (\psi_u - \psi_d)^2 \frac{\partial M_c}{\partial z}, \end{aligned}$$

$$2M_c (\psi_u - \psi_d)^2 \frac{\partial \sigma}{\partial z} = M_c (\psi_u - \psi_d)^2 \frac{\partial (2\sigma - 1)}{\partial z}, \quad \text{and}$$

$$\frac{\partial abc}{\partial z} = bc \frac{\partial a}{\partial z} + ac \frac{\partial b}{\partial z} + ab \frac{\partial c}{\partial z},$$

such that (A.9) can be rewritten as

$$\begin{aligned} & \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)^2}{\partial t} \\ = & -2M_c (\psi_u - \psi_d) \frac{\partial \overline{\psi}}{\partial z} - 2E_s (\psi_u - \psi_d)^2 \\ & + (\psi_u - \psi_d)^2 \frac{\partial M_c}{\partial z} - \frac{\partial (1 - 2\sigma) M_c (\psi_u - \psi_d)^2}{\partial z}. \end{aligned} \quad (\text{A.10})$$

Substituting the continuity equation (2.9) in (A.10) eventually gives

$$\begin{aligned} & \frac{\partial \sigma (1 - \sigma) (\psi_u - \psi_d)^2}{\partial t} \\ = & -2M_c (\psi_u - \psi_d) \frac{\partial \overline{\psi}}{\partial z} - \frac{\partial (1 - 2\sigma) M_c (\psi_u - \psi_d)^2}{\partial z} \\ & - (E_s + D_s) (\psi_u - \psi_d)^2. \end{aligned} \quad (\text{A.11})$$

The primary manipulations which were taken to obtain (A.11) were a multiplication of (A.1) by  $(1 - \sigma)$  and (A.2) by  $\sigma$ , a subtraction of (A.2) from (A.1) after which the remaining equation was multiplied by  $2(\psi_u - \psi_d)$ . It is therefore easy to show that the contribution due to the source term is given by

$$(\text{Source}) = -2\sigma (1 - \sigma) (\psi_u - \psi_d) (S_u - S_d). \quad (\text{A.12})$$

## APPENDIX B

**Surface Layer Similarity Relationship**

From similarity theory, we have the following scaling relationships that are valid for a neutral surface stratification (Garratt 1994; Stull 1988):

$$\phi_h \equiv \frac{kz}{\theta_*} \frac{\partial \bar{\theta}}{\partial z} = 1, \quad (\text{B.1})$$

$$(\overline{w'^2})^{1/2} = \alpha_{w1} u_*, \quad (\text{B.2})$$

$$(\overline{\theta'^2})^{1/2} = \alpha_{\theta1} \theta_*, \quad (\text{B.3})$$

with the friction velocity

$$u_* = [\overline{u'w_s'^2} + \overline{v'w_s'^2}]^{1/4}, \quad (\text{B.4})$$

and a temperature scale

$$\theta_* = -\frac{\overline{w'\theta'_{vs}}}{u_*}, \quad (\text{B.5})$$

and  $k = 0.4$  the von Karman constant,  $\alpha_{w1} = 1.25$  and  $\alpha_{\theta1} = 2$ . For convective boundary layers the height  $z$  is typically scaled with the Monin–Obukhov length  $L$ ,

$$L = -\frac{u_*^3}{k \frac{g}{\theta_0} \overline{w'\theta'_{vs}}} \quad (\text{B.6})$$

and the vertical velocity and potential temperature variances are given by

$$(\overline{w'^2})^{1/2} = \alpha_{w2} u_* \left(-\frac{z}{L}\right)^{1/3} \quad (\text{B.7})$$

$$(\overline{\theta'^2})^{1/2} = -\alpha_{\theta2} \theta_* \left(-\frac{z}{L}\right)^{-1/3} \quad (\text{B.8})$$

with  $\alpha_{w2} = 1.9$  and  $\alpha_{\theta2} = 0.95$ . Free convective scaling arguments suggest that  $\phi_h$  should scale as a  $-1/3$  power law of  $-z/L$ :

$$\phi_h = \gamma_1 \left(-\frac{z}{L}\right)^{-1/3}, \quad (\text{B.9})$$

although for  $-5 < -z/L < 0$  observations suggest that

$$\phi_h = \left(1 - \gamma_2 \frac{z}{L}\right)^{-1/2} \quad (\text{B.10})$$

with  $\gamma_1 \approx 0.4$  and  $\gamma_2 \approx 16$ .

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