SPIN-1999/26, ITFA-99-30, RUNHETC-2000-37, CU-TP-957
HEP-TH/0010103
OCTOBER 2000

COHOMOLOGICAL YANG-MILLS THEORIES ON KÄHLER 3-FOLDS

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ABSTRACT

We study topological gauge theories with $N_c = (2, 0)$ supersymmetry based on stable bundles on general Kähler 3-folds. In order to have a theory that is well defined and well behaved, we consider a model based on an extension of the usual holomorphic bundle by including a holomorphic 3-form. The correlation functions of the model describe complex 3-dimensional generalizations of Donaldson-Witten type invariants. We show that the path integral can be written as a sum of contributions from stable bundles and a complex 3-dimensional version of Seiberg-Witten monopoles. We study certain deformations of the theory, which allow us to consider the situation of reducible connections. We shortly discuss situations of reduced holonomy. After dimensional reduction to a Kähler 2-fold, the theory reduces to Vafa-Witten theory. On a Calabi-Yau 3-fold, the supersymmetry is enhanced to $N_c = (2, 2)$. This model may be used to describe classical limits of certain compactifications of (matrix) string theory.

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1. Introduction

The quantum field theoretic approach [1] to Donaldson's invariants of 4-manifolds [2][3] opened up new horizons in mathematics [4] through the quantum properties of the underlying physical theory uncovered by Seiberg and Witten [5]. The purpose of this paper is to examine a natural generalization of Donaldson-Witten theory to a complex Kähler 3-fold.

In [6] we considered, among others, a natural generalization of the Donaldson-Witten theory on a complex Kähler surface to a complex $d > 2$ dimensional Kähler manifold M. The path integral of the resulting model was localized to the moduli space of Einstein-Hermitian connections, or equivalently the moduli space of stable bundles. However this model had a serious problem due to the uncontrollable abundance of anti-ghosisare only a finite dimensional space of anti-ghost zero-modes. For higher dimensional Kähler manifolds however, we find an infinite dimensional solution space. Another problem arises due to zero-modes of the ghosts. These are related to the appearance of reducible connections (or strictly semi-stable bundles). These can also be found for a Kähler surface, i.e. $d = 2$, but in that case one can always get rid of these zero-modes by changing the metric. These zero-modes are responsible for the jumps in the observables as a function of the metric. In the case of higher dimensional Kähler manifold the appearance of ghost zero-modes is however much more generic and rigid; one can in general not get rid of them by a change in the metric.

In this paper we resolve these problems by starting off where we have failed in [6]. A simple observation is that one has to introduce additional degrees of freedom to control the anti-ghost zero-modes. This inductive procedure naturally leads us to a natural extension of the moduli space of Einstein-Hermitian connections or, equivalently, stable bundles. This extension is very close to the one considered by [7] in the context of homological mirror symmetry. It turns out that we have a well-defined model only for the $d \leq 3$ case. By a deformation of the model we are also able to deal with reducible connections. In fact, the important ingredient that makes this possible is the equivariant treatment, which we adopt from the start. We already anticipated this in [6][8]. Closely related models have been considered in various papers [9][10][11][12][13][14][15], largely motivated by a program of Donaldson and Thomas [16][17][18] as well as certain world-volume theories of D-branes [19].

We will follow the general approach of defining a cohomological field theory with a Kähler structure, as discussed in [20]. We begin by constructing a well-defined $N_c = (2, 0)$ model on a Kähler 3-fold. This model gives a concrete formula for Donaldson-Witten type polynomials, which is valid regardless of what the properties of the extended moduli space are. We also argue, using a S^1-symmetry and the DH integration formula, that Donaldson-Witten type invariants may be equivalent to Seiberg-Witten type invariants on Kähler 3-folds. For manifolds of special holonomy, the model reduces to various known models. The dimensional reduction of the model on a 2-torus, gives rise to the $N_c = (2, 2)$ Vafa-Witten model on a Kähler 2-fold. Finally we briefly specialize to the Calabi-Yau case. On a Calabi-Yau 3-fold the $N_c = (2, 0)$ supersymmetry is automatically enhanced to $N_c = (2, 2)$ supersymmetry. This $N_c = (2, 2)$ model can be obtained by dimensional reduction of the $N_{ws} = (2, 2)$ gauged linear sigma model in $(1+1)$ dimensions introduced in [8]. The partition function of the theory can be identified with the holomorphic Casson invariant defined in [17][18].

2. Preliminaries

In this section we give some preliminary description of supersymmetric models and moduli spaces of stable bundles on Kähler manifolds.

2.1. General $N_c = (2,0)$ Models

First we briefly summarize the general structure and some properties of cohomological field theories with a Kähler structure. A more detailed discussion can be found in [20]. As discussed in this reference, a cohomological field theory on a Kähler manifold can always be identified with a $N_c = (2,0)$ supersymmetric gauged sigma model in zero dimensions. Such a sigma model is classified by data $((\mathcal{X}, \varpi), \mathcal{G}, (\mathcal{E}, h, \mathfrak{S}, \mathfrak{J}))$, where

- $\mathcal{X}$ is a complex Kähler manifold with Kähler form ϖ. $\mathcal{X}$ is the target space of the sigma model.
- $\mathcal{G}$ is a group acting on $\mathcal{X}$ with isometries. This will be the gauge group of the sigma model.
- $\mathcal{E}$ is a $\mathcal{G}$-equivariant holomorphic Hermitian vector bundle over $\mathcal{X}$ with a Hermitian structure h and two mutually orthogonal holomorphic sections $\mathfrak{S}$ and $\mathfrak{J}$.[1]

The $N_c = (2,0)$ supersymmetry is generated by supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ satisfying the following anti-commutation relations:

$$\boldsymbol{s}^2 = 0, \qquad \{\boldsymbol{s}, \overline{\boldsymbol{s}}\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{\boldsymbol{s}}^2 = 0, \tag{2.1}$$

where $\phi = \phi^a T^a$ is a $Lie(\mathcal{G})$-valued scalar and $\mathcal{L}_a$ denotes the Lie derivative with respect to the vector field V_a on $\mathcal{X}$ generating the $\mathcal{G}$-action. The supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ can be identified with the holomorphic and anti-holomorphic differentials of the $\mathcal{G}$-equivariant cohomology of $\mathcal{X}$ after parity change. The theory has two additive quantum numbers (p,q) called ghost numbers, such that $\boldsymbol{s}_+$ has ghost numbers $(1,0)$ and $\overline{\boldsymbol{s}}_+$ has ghost numbers $(0,1)$. They define a grading for the fields and observables in the theory.

Let us now introduce the various "fields" of the model. First, we have local holomorphic coordinate fields X^ion $\mathcal{X}$, and their complex conjugate fields $X^{\bar{i}}$. They are part of

[1] More precisely, $\mathfrak{S}$ is a section of the dual holomorphic bundle, so that the notion of orthogonality is canonically defined.

holomorphic multiplets (X^i, ψ_+^i) (i.e. $\overline{s}_+ X^i = 0$), and anti-holomorphic multiplets $(X^{\bar\imath}, \psi_+^{\bar\imath})$ respectively. Their transformation laws are

$$\begin{aligned} s_+ X^i &= i\psi_+^i, & s_+\psi_+^i &= 0, \\ \overline{s}_+ X^i &= 0, & \overline{s}_+\psi_+^i &= \phi_{++}^a \mathcal{L}_a X^i, \\ s_+ X^{\bar\imath} &= 0, & s_+\psi_+^{\bar\imath} &= \phi_{++}^a \mathcal{L}_a X^{\bar\imath}, \\ \overline{s}_+ X^{\bar\imath} &= i\psi_+^{\bar\imath}, & \overline{s}_+\psi_+^{\bar\imath} &= 0. \end{aligned} \tag{2.2}$$

Associated with the group $\mathcal{G}$ we have the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$ and the invariant field ϕ_{++} taking values in $Lie(\mathcal{G})$. Their transformation laws are

$$\begin{aligned} & & s_+\eta_- &= 0, & & \\ s_+\phi_{--} &= i\eta_-, & \overline{s}_+\eta_- &= +iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & s_+\phi_{++} &= 0, \\ \overline{s}_+\phi_{--} &= i\overline{\eta}_-, & s_+\overline{\eta}_- &= -iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & \overline{s}_+\phi_{++} &= 0. \\ & & \overline{s}_+\overline{\eta}_- &= 0, & & \end{aligned} \tag{2.3}$$

Associated with the holomorphic sections $\mathfrak{S}_\alpha(X^i)$ and $\mathfrak{J}^\alpha(X^i)$, satisfying

$$\overline{s}_+\mathfrak{S}_\alpha = 0, \qquad \overline{s}_+\mathfrak{J}^\alpha = 0, \tag{2.4}$$

we have Fermi multiplets $(\chi_-^\alpha, H^\alpha)$ and their conjugate multiplets $(\chi_-^{\overline{\alpha}}, H^{\overline{\alpha}})$, with the following transformation laws

$$\begin{aligned} s_+\chi_-^\alpha &= -H^\alpha, & s_+ H^\alpha &= 0, \\ \overline{s}_+\chi_-^\alpha &= \mathfrak{J}^\alpha(X^i), & \overline{s}_+ H^\alpha &= -i\phi_{++}^a\mathcal{L}_a\chi_-^\alpha + i\psi_+^i\partial_i\mathfrak{J}^\alpha(X^j), \\ s_+\chi_-^{\overline{\alpha}} &= \mathfrak{J}^{\overline{\alpha}}(X^{\bar\imath}), & s_+ H^{\overline{\alpha}} &= -i\phi_{++}^a\mathcal{L}_a\chi_-^{\overline{\alpha}} + i\psi_+^{\bar\imath}\partial_{\bar\imath}\mathfrak{J}^{\overline{\alpha}}(X^{\bar\jmath}), \\ \overline{s}_+\chi_-^{\overline{\alpha}} &= -H^{\overline{a}}, & \overline{s}_+ H^{\overline{\alpha}} &= 0. \end{aligned} \tag{2.5}$$

Note that the section $\mathfrak{J}^\alpha(X^i)$ deforms the usual transformation $\overline{s}_+\chi_-^\alpha = 0$. The holomorphicity of $\mathfrak{J}^\alpha(X^i)$ guarantees the consistency of the above transformation laws with the commutation relations (2.1), since $\overline{s}_+^2\chi_-^\alpha = \overline{s}_+\mathfrak{J}^\alpha(X^i) = 0$. The Fermi fields χ_-^α and $\chi_-^{\overline{\alpha}}$ will be called anti-ghosts (they will have negative ghost numbers).

The action functional of the $N_c = (2,0)$ supersymmetric model can be given by the following form

$$\begin{aligned} S(\zeta) = &- s_+\overline{s}_+\Big(\langle\phi_{--}, \mu - \zeta\rangle - \langle\eta_-, \overline{\eta}_-\rangle + \Big\langle h_{\alpha\overline{\alpha}}(X^i, X^{\bar\imath})\chi_-^\alpha, \chi_-^{\overline{\alpha}}\Big\rangle\Big) \\ &+ i s_+\langle\chi_-^\alpha, \mathfrak{S}_\alpha(X^i)\rangle + i\overline{s}_+\langle\chi_-^{\overline{\alpha}}, \mathfrak{S}_{\overline{\alpha}}(X^{\bar\imath})\rangle, \end{aligned} \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ denotes a bi-invariant inner product on the Lie algebra $Lie(\mathcal{G})$, μ denotes the $\mathcal{G}$-equivariant momentum map[2] $\mu : X \to Lie(\mathcal{G})^*$, and ζ is a constant taking values in the

[2]The Hamiltonian of the $\mathcal{G}$-action on $\mathcal{X}$. Note that the Kähler manifold $\mathcal{X}$ is automatically symplectic.

central elements of $Lie(\mathcal{G})$. The condition that the action functional $S(\zeta)$ has $N_c = (2,0)$ supersymmetry is

$$\overline{\boldsymbol{s}}_+\langle \chi_-^\alpha, \mathfrak{S}_\alpha(X^i)\rangle = \langle \mathfrak{J}^\alpha(X^i), \mathfrak{S}_\alpha(X^i)\rangle = 0, \tag{2.7}$$

which motivates the orthogonality of the two sections.

Expanding the action functional S one finds that the auxiliary fields D, H^α and $H^{\overline{\alpha}}$ can be integrated out by setting

$$D = \frac{1}{2}(\mu - \zeta), \qquad H^\alpha = i h^{\alpha\overline{\beta}}\mathfrak{S}_{\overline{\beta}}. \tag{2.8}$$

Then the fixed point theorem of Witten implies that the path integral reduces to an integral over the space of solutions of the following equations,

$$\begin{aligned} \mathfrak{J}^\alpha(X^i) &= 0, \\ \mathfrak{S}_\alpha(X^i) &= 0, \\ \mu - \zeta &= 0, \end{aligned} \tag{2.9}$$

and

$$\phi^a_{++}\mathcal{L}_a X^i = 0, \qquad [\phi_{++}, \phi_{--}] = 0, \tag{2.10}$$

modulo $\mathcal{G}$-symmetry. If $\mathcal{G}$ acts freely on the solution space of (2.9), the equations (2.10) can only be solved by setting $\phi_{++} = 0$, and the path integral reduces to an integral over the symplectic quotient $\mathfrak{M}_\zeta$ of $(\mathfrak{S}_\alpha^{-1}(0)\cap\mathfrak{J}_\alpha^{-1}(0)) \subset \mathcal{X}$ by $\mathcal{G}$,

$$\mathfrak{M}_\zeta = \left(\mu^{-1}(\zeta)\cap\mathfrak{S}_\alpha^{-1}(0)\cap\mathfrak{J}_\alpha^{-1}(0)\right)/\mathcal{G}. \tag{2.11}$$

An observable of the theory $\widehat{\mathcal{O}}^{r,s}$ is induced by an element $\mathcal{O}^{r,s}$ of $\mathcal{G}$-equivariant Dolbeault cohomology of $\mathcal{X}$ after parity change. The superscript (r,s) denote the ghost numbers and the degrees respectively. A correlation function $\left\langle \prod_{m=1}^k \widehat{\mathcal{O}}^{r_m,s_m}\right\rangle$ can be non-vanishing only if

$$\sum_{m=1}^k (r_m, s_m) = (\triangle, \triangle), \tag{2.12}$$

where $\triangle$ is the net ghost number anomaly due to zero-modes of the fermions $(\eta_-, \psi^i_+, \chi^\alpha_-)$. We call $\triangle$ the formal complex dimension of $\mathfrak{M}_\zeta$.

If $\mathcal{G}$ acts freely on $(\mathfrak{S}_\alpha^{-1}(0)\cap\mathfrak{J}_\alpha^{-1}(0)) \subset \mathbb{C}$, we do not have zero-modes for the ghosts η_-. If the holomorphic sections $\mathfrak{S}$ and $\mathfrak{J}$ are generic there are no zero-modes of the anti-ghosts χ^α_-. In this situation $\mathfrak{M}_\zeta$ is a smooth complex $\triangle$-dimensional non-linear Kähler manifold. For non-generic $\mathfrak{S}$ and $\mathfrak{J}$ the zero-modes of χ^α_- span the fibre of a Hermitian holomorphic bundle $\mathcal{V} \to \mathfrak{M}_\zeta$. We call $\mathcal{V}$ the anti-ghost bundle. The correlation function $\left\langle \prod_{m=1}^k \widehat{\mathcal{O}}^{r_m,s_m}\right\rangle$ becomes

$$\left\langle \prod_{m=1}^k \widehat{\mathcal{O}}^{r_m,s_m}\right\rangle = \int_{\mathfrak{M}_\zeta} \mathrm{e}(\mathcal{V}) \wedge \widetilde{\mathcal{O}}^{r_1,s_1}\wedge\ldots\wedge\widetilde{\mathcal{O}}^{r_k,s_k}, \tag{2.13}$$

where $\mathrm{e}(\mathcal{V})$ denotes the Euler class of $\mathcal{V}$ and $\widetilde{\mathcal{O}}^{r_m,s_m}$ denotes a closed differential form on $\mathfrak{M}_\zeta$, obtained by $\mathcal{O}^{r_m,s_m}$ after the restriction and reduction. It can be non-vanishing if the condition (2.12) holds. This ghost number anomaly is reflected geometrically by the fact that $\mathfrak{M}_\zeta$ has complex dimension $\triangle + \frac{1}{2}\,\mathrm{rank}(\mathcal{V})$, while $\mathrm{e}(\mathcal{V})$ is a form of degree $(\frac{1}{2}\,\mathrm{rank}(\mathcal{V}), \frac{1}{2}\,\mathrm{rank}(\mathcal{V}))$. So the integrand of the RHS of (2.13) is a top form exactly if (2.12) holds.

2.2. A Target Space from Bundles On Kähler Manifolds

We now describe a $N_c = (2,0)$ model related to stable bundles on a Kähler manifold. For general references on these structures see [3][21]. We consider a compact complex Kähler d-fold M with Kähler form ω. The complex structure on M determines a decomposition of the space $\Omega^r(M)$ of r-form on M as $\Omega^r(M) = \oplus_{p+q=r}\Omega^{p,q}(M)$. On M any two-form $\alpha \in \Omega^2(M)$ can be decomposed into $\alpha = \alpha^+ + \alpha^-$ such that

$$\begin{aligned} \alpha^+ &= \alpha^{2,0} + \alpha_0\omega + \alpha^{0,2}, \\ \alpha^- &= \alpha^{1,1}_\perp, \end{aligned} \tag{2.14}$$

where $\alpha_0 \in \Omega^0(M)$ is a scalar function and $\alpha^{1,1}_\perp$ is a $(1,1)$-form orthogonal to ω. Corresponding to this decomposition we define the following projections

$$P^\pm : \Omega^2(M) \to \Omega^{2\pm}(M), \qquad P^{0,2} : \Omega^2(M) \to \Omega^{0,2}(M). \tag{2.15}$$

For a complex Kähler 2-fold the above decomposition coincides with the decomposition in self-dual and anti-self dual two-forms. We denote by $\Omega^p(M,E)$ the space of real p-forms on M taking values in E. Let E be a rank r vector bundle over M endowed with a Hermitian metric. The choice of E fixes the topological type for the connections on E. We denote by $\mathcal{A}$ the space of all connections and by $\mathcal{G}$ the group of all gauge transformations. The gauge group $\mathcal{G}$ is equivalent to the group of all unitary automorphisms of E (and it has structure group $\mathrm{U}(r)$). The Lie algebra $Lie(\mathcal{G})$ of $\mathcal{G}$ can be identified with $\Omega^0(M,\mathrm{End}(E))$ and we use integration over M to identify $Lie(\mathcal{G})^*$ with $\Omega^{2d}(M,\mathrm{End}(E))$. Thus the bi-invariant inner product on $Lie(\mathcal{G})$ is the integral over M combined with the trace of $\mathrm{U}(r)$

$$\langle a, a\rangle = -\int_M \mathrm{Tr}(a \wedge *a). \tag{2.16}$$

We take the infinite dimensional space $\mathcal{A}$ as our initial target space $\mathcal{X}$. (Later in this paper we shall extend this space).

To define an equivariant $N_c = (2,0)$ model we need to introduce complex and Kähler structures on our target space $\mathcal{A}$. Let A denote a connection one-form, which is decomposed into $A = A^{1,0} + A^{0,1}$. We denote by $d_A = \partial_A + \overline{\partial}_A$ the corresponding covariant derivative,

$$d_A = \partial_A + \overline{\partial}_A : \Omega^0(M,E) \longrightarrow \Omega^{1,0}(M,E) \oplus \Omega^{0,1}(M,E). \tag{2.17}$$

The space $\mathcal{A}$ is an infinite dimensional affine space. A tangent vector is represented by $\delta A \in \Omega^1(M, \mathrm{End}(E))$. Note that there is no natural complex structure on $\mathcal{A}$. Any complex structure should be induced from the complex structure on M. One introduces a complex structure on $\mathcal{A}$ by declaring $\delta A^{0,1} \in \Omega^{0,1}(M, \mathrm{End}(E))$ to be the holomorphic tangent vectors. Then $\mathcal{A}$ becomes an infinite dimensional flat Kähler manifold with Kähler form ϖ given by

$$\varpi(\delta A, \delta A') = \frac{1}{4\ d!\pi^2} \int_M \mathrm{Tr}(\delta A \wedge \delta A') \wedge \omega^{d-1}, \tag{2.18}$$

on which $\mathcal{G}$ acts with isometries preserving the Kähler structure. The Kähler potential for the Kähler form (2.18) of $\mathcal{A}$ is given by

$$\mathcal{K}(A^{1,0}, A^{0,1}) = \frac{1}{4\ d!\pi^2} \int_M \kappa\, \mathrm{Tr}(F \wedge F) \wedge \omega^{d-2}, \tag{2.19}$$

where κ is a Kähler potential for ω, i.e., $\omega = i\partial\overline{\partial}\kappa$.

Now we introduce our $N_c = (2,0)$ supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_-$ with the familiar commutation relations

$$\boldsymbol{s}_+^2 = 0, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{\boldsymbol{s}}_+^2 = 0. \tag{2.20}$$

The supercharges are identified with the differentials of $\mathcal{G}$-equivariant cohomology of our target space $\mathcal{A}$. Thus $\phi^a_{++}\mathcal{L}_a$ is the infinitesimal gauge transformation generated by the adjoint scalar $\phi_{++} \in Lie(\mathcal{G}) = \Omega^0(M, \mathrm{End}(E))$. The $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$ takes values in $\Omega^0(M, \mathrm{End}(E))$. Their transformation laws for are given by the general formula (2.3).

With the complex structure on $\mathcal{A}$ introduced above we have holomorphic multiplets $(A^{0,1}, \psi_+^{0,1})$ and conjugate anti-holomorphic multiplets $(A^{1,0}, \overline{\psi}_+^{1,0})$, respectively, where $\psi_+^{0,1} \in \Pi\Omega^{0,1}(M, \mathrm{End}(E))$ represents a holomorphic cotangent vector in $\mathcal{A}$. These are the multiplets associated to the coordinates X^i and $X^{\bar{i}}$. The transformation laws are as given as in (2.2) (or in more details in Appendix A). Note that

$$\{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\}A = -id_A\phi_{++}, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\}\psi_+^{0,1} = i[\phi_{++}, \psi_+^{0,1}], \tag{2.21}$$

which are the infinitesimal gauge transformations generated by ϕ_{++}, in accordance with (2.20).

From the transformation laws and the Kähler form (2.19) we obtain the following equivariant Kähler form

$$\begin{aligned}\widehat{\varpi}^{\mathcal{G}} &= i\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\mathcal{K} \\ &= \frac{i}{2\ d!\pi^2}\int_M \mathrm{Tr}(\phi_{++}F) \wedge \omega^{d-1} + \frac{1}{2\ d!\pi^2}\int_M \mathrm{Tr}(\psi_+^{0,1} \wedge \overline{\psi}_+^{1,0}) \wedge \omega^{d-1},\end{aligned} \tag{2.22}$$

where we used the Bianchi identity $d_A F = 0$, which implies $\overline{\partial}_A F^{0,2} = \partial_A F^{0,2} + \overline{\partial}_A F^{1,1} = 0$, and integration by parts. The second term of the equivariant Kähler form can be identified with the Kähler form ϖ (after parity change) and the first term is the $\mathcal{G}$-momentum map $\phi^a_{++}\mu_a$, $\mu : \mathcal{A} \rightarrow Lie(\mathcal{G})^* = \Omega^{2n}(M, \mathrm{End}(E))$,

$$\mu(A) = \frac{1}{2\ d!\pi^2} F \wedge \omega^{d-1} = \frac{1}{2d\ d!\pi^2}(\Lambda F)\omega^d, \tag{2.23}$$

where Λ denotes the adjoint of wedge multiplication with ω.

With the construction described until now, we have a $N_c = (2,0)$ model based on our infinite dimensional target space $\mathcal{A}$. The path integral of the resulting model will localize to the symplectic quotient $\mu^{-1}(\zeta)/\mathcal{G}$. For $d \geq 2$ the quotient space is still infinite dimensional. Thus we should supply some additional localization. According to our general discussion in the last section we may still consider a certain infinite dimension Hermitian holomorphic vector bundle $\mathcal{E} \rightarrow \mathcal{A}$ over $\mathcal{A}$ with a certain holomorphic section $\mathfrak{S}$ (we will put $\mathfrak{J} = 0$ for the moment), which determines anti-ghost multiplets accordingly. Then the path integral will be further localized to $(\mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}$, which might be a finite dimensional Kähler manifold. We will now consider such an extension of the model.

2.3. The Holomorphic Section

The remaining task is to determine an infinite dimensional vector bundle over our target space $\mathcal{A}$ with an appropriate $\mathcal{G}$-equivariant holomorphic section $\mathfrak{S}(A^{0,1})$, i.e. $\overline{\boldsymbol{s}}_+\mathfrak{S} = 0$. From our general discussion we can see that a choice of section $\mathfrak{S}$ should be compatible with the Kähler quotient such that the effective target space $\mathcal{M} = (\mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}$ inherits a Kähler structure when $\mathcal{G}$ acts freely. We introduce a bundle $\mathcal{E}$ over our target space $\mathcal{A}$ for which a holomorphic section $\mathfrak{S}(A^{0,1})$ is given by

$$\mathfrak{S} : A^{0,1} \rightarrow F^{0,2} \in \Omega^{0,2}(M, \mathrm{End}(E)). \tag{2.24}$$

We note that the above is the most natural choice on *generic* Kähler manifolds, since any holomorphic function of $A^{0,1}$ which is gauge covariant must be a function of $F^{0,2}$. A further obvious requirement is that the resulting action functional should be invariant under the Lorentz symmetry – more precisely the holonomy of a Kähler manifold M.[3] Then our effective target space will be the moduli space $\mathcal{M}_{EH}$ of Einstein-Hermitian bundles defined by

$$\mathcal{M}_{EH} = (\mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}. \tag{2.25}$$

Since our section takes values in $\Omega^{0,2}(M, \mathrm{End}(E))$ we have corresponding Fermi multiplets $(\chi^{2,0}_-, H^{2,0})$, taking values in $\Omega^{2,0}(M, \mathrm{End}(E))$. They transform according to the general transformation laws (2.5), with $\mathfrak{J} = 0$.

Now we have all the ingredientsnecessary to define a $N_c = (2,0)$ model. For example, the action can be found from the general form (2.6).

[3] There are two special cases. On a Calabi-Yau 4-fold or an arbitrary hyper-Kähler manifold one can take a certain projection of $F^{0,2}$ for the holomorphic section of $\mathcal{E} \rightarrow \mathcal{A}$. We will return to this in another paper[22].

3. Motivating The Extended Moduli Space Of Stable Bundles

In this section we motivate the notion of extended moduli space of stable bundles [23], in the context of resolving the problems of anti-ghost zero-modes.

First we set up our notation. Consider a d complex dimensional compact Kähler manifold (M,ω) with Kähler form ω, and a rank r Hermitian vector bundle $E \to M$. The curvature two-form decomposes as $F = F^+ + F^-$ according to (2.14). A connection on E is called Einstein-Hermitian (EH) with factor ζ if

$$\begin{aligned} &F^{0,2} = 0, \\ &i\Lambda F = \zeta I_E. \end{aligned} \tag{3.1}$$

The model as it now stands has a problem with the anti-ghost zero-modes. Let A be an EH connection. We consider a nearby connection $A + \delta A$, $\delta A \in \Omega^1(M, \mathrm{End}(E))$, which also is EH. After linearization we have $d_A^+\delta := P^+ d_A \delta A = 0$, with P^+ the projection operator defined in (2.15). There is still a gauge freedom $d_A\lambda$. Supplying the Coulomb gauge condition $d_A^*\delta A = 0$, local deformations δA around a point A in $\mathcal{M}_{EH}$ are represented by the kernel of the operator $d_A^+ \oplus d_A^*$ acting on $\Omega^1(M, \mathrm{End}(E))$. This structure can be summarized by the associated elliptic complex of Atiyah-Hitchin-Singer [24];

$$0 \longrightarrow \Omega^0(M, \mathrm{End}(E)) \xrightarrow{d_A} \Omega^1(M, \mathrm{End}(E)) \xrightarrow{d_A^+} \Omega^{2+}(M, \mathrm{End}(E)). \tag{3.2}$$

We compare this complex with the fermionic zero-modes of the fermions $(\overline{\eta}_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2})$ in the model introduced in Sect. 2.2, which are governed by the equations

$$\overline{\partial}_A \overline{\eta}_- = 0, \qquad \begin{aligned} \overline{\partial}_A^* \psi_+^{0,1} = 0, \\ \overline{\partial}_A \psi_+^{0,1} = 0, \end{aligned} \qquad \overline{\partial}_A^* \overline{\chi}_-^{0,2} = 0. \tag{3.3}$$

After decomposing $\overline{\eta}_- = \boldsymbol{\eta}_- + i\chi_-^0$ into its real and imaginary part, we can form real fermions $(\boldsymbol{\eta}_-, \boldsymbol{\psi}_+, \boldsymbol{\chi}_-)$ which we define as

$$\boldsymbol{\psi}_+ = \overline{\psi}_+^{1,0} + \psi_+^{0,1}, \qquad \boldsymbol{\chi}_- = \overline{\chi}_-^{2,0} + \chi_-^0\omega + \chi_-^{0,2}, \tag{3.4}$$

so that $\boldsymbol{\eta}_- \in \Omega^0(M, \mathrm{End}(E))$, $\boldsymbol{\psi}_+ \in \Omega^1(M, \mathrm{End}(E))$ and $\boldsymbol{\chi}_- \in \Omega^{2+}(M, \mathrm{End}(E))$. The zero-mode equations (3.3) are then translated into

$$d_A \boldsymbol{\eta}_- = 0, \qquad \begin{aligned} d_A^* \boldsymbol{\psi}_+ = 0, \\ d_A^+ \boldsymbol{\psi}_+ = 0, \end{aligned} \qquad d_A^{+*} \boldsymbol{\chi}_- = 0. \tag{3.5}$$

Thus the zero-modes of the fermions $(\boldsymbol{\eta}_-, \boldsymbol{\psi}_+, \boldsymbol{\chi}_-)$ are elements of the AHS complex (3.2). The above correspondence is one of the crucial ingredients of Witten's approach to Donaldson theory in four real dimensions [1]. The path integral measure contains such fermionic zero-modes and the net ghost number anomaly is precisely the index of the above complex, which is the formal dimension of the moduli space of instantons on a four manifold.

Let us undo the combination (3.4), and return to the initial equations (3.3) for the complex fermions $(\overline{\eta}_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2})$. The equations (3.3) imply that the fermionic zero-modes are in one to one correspondence with the following Dolbeault complex [25]

$$0 \longrightarrow \Omega^{0,0}(M, \mathrm{End}(E)) \xrightarrow{\overline{\partial}_A} \Omega^{0,1}(M, \mathrm{End}(E)) \xrightarrow{\overline{\partial}_A} \Omega^{0,2}(M, \mathrm{End}(E)). \tag{3.6}$$

Note that $\overline{\partial}_A^2 = 0$ at the fixed point locus. Our problem for $d \geq 3$ is that a fermionic zero-mode of $\overline{\chi}_-^{0,2}$ only needs to satisfy the condition $\overline{\partial}_A^* \overline{\chi}_-^{0,2} = 0$. As a result we always have an infinite dimensional anti-ghost bundle. Therefore the path integral would hardly make any sense. But this is exactly what the EH condition gives us via local deformations. For $d = 2$ the desired condition $\overline{\partial}_A \overline{\chi}_-^{0,2} = 0$ is void due to the dimensional reason. For $d \geq 3$ the only way of imposing the desired condition $\overline{\partial}_A \overline{\chi}_-^{0,2} = 0$ is to introduce another fermionic field $\lambda_+^{3,0}$ with ghost numbers $(1,0)$ such that the action functional contains the following term

$$S \sim \int_M \mathrm{Tr}(\lambda_+^{3,0} \wedge * \overline{\partial}_A \overline{\chi}_-^{0,2}) + \cdots. \tag{3.7}$$

Then we obtain in addition to (3.3) the two equations

$$\overline{\partial}_A \overline{\chi}_-^{0,2} = 0, \qquad \partial_A^* \lambda_+^{3,0} = 0. \tag{3.8}$$

Thus we have to generalize the $N_c = (2,0)$ model by introducing a new holomorphic multiplet $(C^{3,0}, \lambda_+^{3,0}) \in \Omega^{3,0}(M, \mathrm{End}(E))$.[4] For $d = 3$ the above additional conditions are sufficient. For $d \geq 4$ we should supply yet another additional condition $\partial_A \lambda_+^{3,0} = 0$, otherwise we have too many zero-modes for $\lambda_+^{3,0}$. Thus we should introduce another fermionic field $\overline{\xi}_-^{0,4}$ with ghost numbers $(-1,0)$ such that now the action contains

$$S \sim \int_M \mathrm{Tr}(\lambda_+^{3,0} \wedge * \overline{\partial}_A \overline{\chi}_-^{0,2} + \partial_A \lambda_+^{3,0} \wedge * \overline{\xi}_-^{0,4}) + \cdots, \tag{3.9}$$

and so on.

Thus a natural resolution of our problem is to extend the complex (3.6) all the way to the end

$$0 \longrightarrow \boldsymbol{C}^{0,0} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,1} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,2} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,3} \xrightarrow{\overline{\partial}_A} \ldots \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,d} \longrightarrow 0, \tag{3.10}$$

where $\boldsymbol{C}^{0,\ell} := \Omega^{0,\ell}(M, \mathrm{End}(E))$. To give any meaning to the above Dolbeault complex, we have to introduce the following set of fermionic fields

$$\overline{\eta}_-^{0,0}, \psi_+^{0,1}, \overline{\chi}_-^{0,2}, \overline{\lambda}_+^{0,odd}, \overline{\xi}_-^{0,even}, \tag{3.11}$$

where $2 < odd, even \leq d$. It can be seen, from the basic structure of our $N_c = (2,0)$ model, that $\overline{\lambda}_+^{0,odd}$ are superpartners of anti-holomorphic bosonic fields $C^{0,odd}$, forming anti-holomorphic multiplets;

$$C^{0,odd} \xrightarrow{\overline{s}_+} \overline{\lambda}_+^{0,odd}. \tag{3.12}$$

[4]When $\lambda_+^{3,0}$ is in a Fermi multiplet it is impossible to get the term (3.7) without breaking the $N_c = (2,0)$ supersymmetry.

Furthermore, the fields $\overline{\xi}_{-}^{0,even}$ should be in Fermi multiplets

$$\overline{\xi}_{-}^{0,even} \xrightarrow{s_+} H^{0,even}, \tag{3.13}$$

where $H^{0,even}$ are auxiliary fields. Then we may try to design an action functional which gives the following equations, in addition to (3.3), for fermionic zero-modes

$$\begin{aligned} \overline{\partial}_A \overline{\lambda}_{+}^{0,odd} &= 0, \qquad & \overline{\partial}_A \overline{\xi}_{-}^{0,even} &= 0, \\ \overline{\partial}_A^{*} \overline{\lambda}_{+}^{0,odd} &= 0, \qquad & \overline{\partial}_A^{*} \overline{\xi}_{-}^{0,even} &= 0. \end{aligned} \tag{3.14}$$

Thus the $(0,q)$-form fermionic zero-modes become the elements of the q-th cohomology group $\boldsymbol{H}^{0,q} := H^{0,q}_{\overline{\partial}_A}(M, \mathrm{End}(E))$ of the complex (3.10). Then the net ghost number violation due to the fermionic zero-modes is precisely the index $\sum_{q=0}^{d}(-1)^{q+1}\dim_{\mathbb{C}} \boldsymbol{H}^{0,q}$ of the complex (3.10). Now we are in the same situation as the Donaldson-Witten theory in the $d=2$ case.

Finally let's consider how the above extension fits into the framework of EH connections. Kim [26] introduced the followingcomplex (see also [21]), generalizing the complex given in (3.2),

$$0 \longrightarrow \boldsymbol{B}^0 \xrightarrow{d_A} \boldsymbol{B}^1 \xrightarrow{d_A^+} \boldsymbol{B}^{2+} \xrightarrow{d_A^{0,2}} \boldsymbol{B}^{0,3} \xrightarrow{\overline{\partial}_A} \ldots \xrightarrow{\overline{\partial}_A} \boldsymbol{B}^{0,d} \longrightarrow 0, \tag{3.15}$$

where $d^{0,2} = \overline{\partial}_A \circ P^{0,2}$, $\boldsymbol{B}^p = \Omega^p(M, \mathrm{End}(E))$ and $\boldsymbol{B}^{p,q} = \Omega^{p,q}(M, \mathrm{End}(E))$. It is shown that the above is a complex if the connection A is EH and elliptic. We denote the associated q-th cohomology group by $\boldsymbol{H}^q$. It is not difficult to show that

$$\sum_{q=0}^{d}(-1)^{q+1}\dim_{\mathbb{R}} \boldsymbol{H}^q = 2\sum_{q=0}^{d}(-1)^{q+1}\dim_{\mathbb{C}} \boldsymbol{H}^{0,q}. \tag{3.16}$$

It should also be obvious that the two extended complexes (3.15) and (3.10) are related in the same way as the unextended complexes (3.2) and (3.6).

We remark that Kim's complex is not the genuine deformation complex of EH connections, but rather a natural extension of it. As in the $d=2$ case we require that the index is the formal complex dimension of a certain extended moduli space of stable bundles. We define the extended moduli space $\mathfrak{M}$ of EH connections or stable bundles by extending the EH condition as the space of solutions of the following equations

$$\begin{aligned} \overline{\mathfrak{D}} \circ \overline{\mathfrak{D}} &= 0, \\ \exp(\omega) \wedge \left(\mathfrak{D} \circ \overline{\mathfrak{D}} + \overline{\mathfrak{D}} \circ \mathfrak{D}\right)|_{\text{top form}} + id\zeta\omega^d I_E &= 0, \end{aligned} \tag{3.17}$$

where $\overline{\mathfrak{D}}$ is the extended holomorphic connection

$$\overline{\mathfrak{D}} = \overline{\partial}_A + \sum_{k\geq 1} C^{0,2k+1}. \tag{3.18}$$

The versal deformation complex of the above equations is then precisely equivalent to Kim's complex (3.15). This can be checked using the Kähler identities

$$\overline{\partial}_A^* = -i[\Lambda, \partial_A], \qquad \partial_A^* = i[\Lambda, \overline{\partial}_A]. \tag{3.19}$$

In the above scheme the infinitesimal deformations of the extended moduli space always lie in $\boldsymbol{H}^{0,odd}$, while the obstructions, by Kuranishi's method, lie in $\boldsymbol{H}^{0,even}$. Thus the local model of the extended moduli space is $f^{-1}(0)$ [7], where

$$f : \boldsymbol{H}^{0,odd} \to \boldsymbol{H}^{0,even}, \quad f(A, C) = \overline{\mathfrak{D}} \circ \overline{\mathfrak{D}}. \tag{3.20}$$

The formal complex dimension of the extended moduli space $\mathfrak{M}$ can be computed using the Riemann-Roch formula

$$\sum_{q=0}^{d} (-1)^{q+1} \dim_{\mathbb{C}} \boldsymbol{H}^{0,q} = -\int_M \mathrm{Td}(M) \wedge \mathrm{ch}(E) \wedge \mathrm{ch}(E^*), \tag{3.21}$$

where $\mathrm{Td}(M)$ denotes the Todd class of M and $\mathrm{ch}(E)$ denotes the Chern character of E.

It seems that we have all the ingredients to construct a well-defined $N_c = (2,0)$ model. Unfortunately it turns out to be impossible to implant the above ideas, except for the case of at most three complex dimensions. It is not possible to maintain $N_c = (2,0)$ supersymmetry and impose the desired equations (3.14) for all fermions unless $d \leq 3$. This follows from the fact that the zero-mode equations for the fermions in the holomorphic multiplet should be completely holomorphic equations (they arise from the supersymmetry transformation of the first two equations in (2.9)). This is inconsistent with the two equations for $\lambda_+^{odd,0}$ in (3.14), therefore we can impose at most one of them. This is sufficient only for $d \leq 3$. The reason why we did not have this problem in lower dimensions was that for $\psi_+^{1,0}$ we also had the non-holomorphic supersymmetric partner of the D-term equation at our disposal. This equation is related to the gauge symmetry. So in order to extend the above ideas to higher dimensions, we are led to associate the even degree terms in the complex (3.15) with new gauge symmetries rather than obstructions. We do however not see how these can be related to gauge symmetries, except for $\boldsymbol{B}^0$. Therefore in the rest of this paper we will only consider $d \leq 3$.

4. $N_c = (2,0)$ Model On Kähler 3-Folds

We consider the $N_c = (2,0)$ model studied in Sect. 2 specializing to the case that M is a Kähler 3-fold. According to the discussion in the previous section we introduce one more bosonic field $C^{0,3} \in \Omega^{0,3}(M, \mathrm{End}(E))$ and its Hermitian conjugate $C^{3,0}$. Our goal is to construct a $\mathcal{G}$-equivariant $N_c = (2,0)$ model whose target space is the space $\mathcal{A}$ of all connections together with the space of all $C^{0,3}$ fields. Furthermore the fermionic zero-modes should be elements of the Dolbeault cohomology of the complex (3.10). It turns out there is only one way of achieving this goal.

4.1. Basic Properties Of The Model

The $N_c = (2,0)$ model here will be an example of the construction in Sect. 2.1 with $\mathfrak{J} \neq 0$ in (2.5). We first recall that the path integral of a general $N_c = (2,0)$ model is localized to the solution space of (2.9), modulo $\mathcal{G}$ symmetry. The momentum map μ is determined from the Kähler potential on the space of all X^i and from the action of $\mathcal{G}$ on it. The sections $\mathfrak{J}^\alpha$ and $\mathfrak{S}_\alpha$ should satisfy the following equations to have $N_c = (2,0)$ supersymmetry,

$$\begin{aligned} \overline{\boldsymbol{s}}_+ \mathfrak{J}^\alpha &= 0, \\ \overline{\boldsymbol{s}}_+ \mathfrak{S}_\alpha &= 0, \\ \overline{\boldsymbol{s}}_+ \mathfrak{S}_\alpha &= 0, \\ \langle \mathfrak{J}^\alpha, \mathfrak{S}_\alpha \rangle &= 0. \end{aligned} \tag{4.1}$$

In the present case our infinite dimensional target space is

$$\mathcal{X} = \mathcal{A} \oplus \Omega^{3,0}(M, \mathrm{End}(E)) \oplus \Omega^{0,3}(M, \mathrm{End}(E)), \tag{4.2}$$

and the infinite dimensional group $\mathcal{G}$ acts on the above space as the group of all local gauge transformation on M. The Lie algebra $Lie(\mathcal{G})$ of $\mathcal{G}$ is $\Omega^0(M, \mathrm{End}(E))$ and the bi-invariant inner product on $Lie(\mathcal{G})$ is (2.16). We already gave a complex structure on $\mathcal{A}$ in Sect. 2.2 by demanding that $A^{0,1}$ is a holomorphic field, i.e., $\overline{\boldsymbol{s}}_+ A^{0,1} = 0$. We have a unique holomorphic section $F^{0,2}$ from the subspace $\mathcal{A}$ and the corresponding Fermi multiplet $(\overline{\chi}_-^{0,2}, H^{0,2}) \in \Omega^{0,2}(M, \mathrm{End}(E))$ with the transformation laws (2.5). Let us see what the complex structure on the additional field space should be. We need to put a constraint on the additional fields $C^{3,0}$. From the discussion in the last section this condition is $\partial_A^* C^{3,0} = 0$. This constraint has to come from either one of the first two equations in (2.9) (or their conjugates), and therefore must be (anti-)holomorphic. Note that $\partial_A^* = - * \overline{\partial}_A *$, which is holomorphic, since $\overline{\boldsymbol{s}}_+ A^{0,1} = 0$. Therefore, for the equation to be holomorphic, we need also $\overline{\boldsymbol{s}}_+ C^{3,0} = 0$. Thus the additional holomorphic multiplet is $(C^{3,0}, \lambda_+^{3,0})$. The additional equation could be added to $\mathfrak{S}$, as it has the same form-degree $(0,2)$ (after conjugation), so that we get that the combination $F^{0,2} - \overline{\partial}_A^* C^{0,3}$ has to vanish.[5] This is however not possible in our setting, because $F^{0,2}$ is holomorphic, while the second part is anti-holomorphic, as we just argued. Therefore, the total combination is neither holomorphic nor anti-holomorphic, as is required for $\mathfrak{S}$ (respectively $\overline{\mathfrak{S}}$). Therefore, our only choice is to use the first equation in (2.9), that is we should set $\mathfrak{J} = \partial_A^* C^{3,0}$. We see that $\overline{\boldsymbol{s}}_+ \mathfrak{J} = 0$. We conclude

$$\begin{aligned} \mathfrak{J} &= \partial_A^* C^{3,0}, \\ \mathfrak{S} &= F^{0,2}. \end{aligned} \tag{4.3}$$

With this choice also the last condition in (4.1) is satisfied, as

$$\langle \mathfrak{J}^\alpha, \mathfrak{S}_\alpha \rangle = \int_M \mathrm{Tr}(\partial_A^* C^{3,0} \wedge * F^{0,2}) = \int_M \mathrm{Tr}(C^{3,0} \wedge * \overline{\partial}_A F^{0,2}) = 0, \tag{4.4}$$

[5] This implies the equation $\overline{\partial}_A^* C^{0,3} = 0$ due to the Bianchi identity. This in fact is the combination that is often used in the literature [7][13].

where we used the Bianchi identity $d_A F = 0$, which implies $\overline{\partial}_A F^{0,2} = 0$.

The above considerations determine an equivariant $N_c = (2,0)$ model, following the description in Sect. 2.1.

4.2. Fields And Action Functional

Here we recall again the fields and their supersymmetry transformation laws, to summarize what we have learned. Associated with the $\mathcal{G}$ symmetry we have the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$, all transforming as adjoint valued scalars on M. The transformation laws are given by (2.3). We have two sets of holomorphic multiplets and their anti-holomorphic partners. One set of holomorphic multiplets is $(A^{0,1}, \psi_+^{0,1})$ with anti-holomorphic partners $(A^{1,0}, \overline{\psi}_+^{1,0})$. The other holomorphic multiplet is $(C^{3,0}, \lambda_+^{3,0})$ with anti-holomorphic partner $(C^{0,3}, \overline{\lambda}_+^{0,3})$. Finally we have Fermi multiplets $(\chi_-^{2,0}, H^{2,0})$ and anti-Fermi multiplets $(\overline{\chi}_-^{0,2}, H^{0,2})$. The explicit transformation rules are written down in Appendix A. The fields and their transfoormation rules can be summarized by the following diagrams,

$$
\begin{array}{ccccc}
C^{3,0} & \xrightarrow{s_+} & \lambda_+^{3,0} & & \phi_{++} \\
 & & & & \\
\overline{\eta}_- & \xrightarrow{s_+} & D & & \overline{\lambda}_+^{0,3} \\
\uparrow{\scriptstyle \overline{s}_+} & & \uparrow{\scriptstyle \overline{s}_+} & & \uparrow{\scriptstyle \overline{s}_+} \\
\phi_{--} & \xrightarrow{s_+} & \eta_- & & C^{0,3}
\end{array}
\quad , \qquad
\begin{array}{ccc}
A^{0,1} & \xrightarrow{s_+} & \psi_+^{0,1} \\
 & & \\
\overline{\chi}_-^{0,2} & \xrightarrow{s_+} & H^{0,1}
\end{array}
\quad . \tag{4.5}
$$

The resulting $N_c = (2,0)$ model in general can not be embedded into a $N_c = (2,2)$ theory since $\boldsymbol{s}_+ \overline{\chi}_-^{0,2} \neq 0$. Such an embedding is only possible if M is a Calabi-Yau 3-fold, where our $N_c = (2,0)$ supersymmetry will automatically enhance to $N_c = (2,2)$ even without adding additional fields.

The final ingredient for the action functional is the $\mathcal{G}$-momentum map on the total space (4.2). The total space has a natural $\mathcal{G}$-invariant Kähler potential

$$
\mathcal{K}_T = \frac{1}{24\pi^2} \int_M \Big(\kappa \operatorname{Tr}(F \wedge F) \wedge \omega^2 - i \operatorname{Tr}\left(C^{3,0} \wedge C^{0,3} \right) \Big). \tag{4.6}
$$

Using the transformation laws in Appendix A we obtain from this the following equivariant Kähler form,

$$
\begin{aligned}
\widehat{\varpi}_T^{\mathcal{G}} :=& i \boldsymbol{s}_+ \overline{\boldsymbol{s}}_+ \mathcal{K}_T \\
=& \frac{1}{12\pi^2} \int_M \operatorname{Tr}\Big(i\phi_{++} \Big(F \wedge \omega^2 + \frac{1}{2} [C^{3,0}, C^{0,3}] \Big) \Big) \\
&+ \frac{1}{12\pi^2} \int_M \operatorname{Tr}\Big(\psi_+^{0,1} \wedge \overline{\psi}_+^{1,0} \wedge \omega^2 - \frac{i}{2} \lambda_+^{3,0} \wedge \overline{\lambda}_+^{0,3} \Big).
\end{aligned} \tag{4.7}
$$

The last line is the Kähler form $\widehat{\varpi}_T$, after parity change, and the term in the second line is proportional to the $\mathcal{G}$-momentum map μ_T on the total space (4.2),

$$\mu_T = \frac{1}{12\pi^2}\Big(F\wedge\omega^2 + \frac{1}{2}[C^{3,0},C^{0,3}]\Big). \tag{4.8}$$

Thus the $N_c = (2,0)$ action functional is given by, following (2.6),

$$\begin{aligned} S =&\frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{12\pi^2}\int_M \mathrm{Tr}\Big(\phi_{--}\Big(F\wedge\omega^2 + \frac{1}{2}[C^{3,0},C^{0,3}] + \frac{i}{3}\zeta\omega^3 I_E\Big)\Big)\\ &+\frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\chi_-^{2,0}\wedge *\overline{\chi}_-^{0,2}\Big) + \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{6\pi^2}\int_M \mathrm{Tr}\Big(\eta_- * \overline{\eta}_-\Big)\\ &+\frac{i\boldsymbol{s}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\chi_-^{2,0}\wedge *F^{0,2}\Big) + \frac{i\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\overline{\chi}_-^{0,2}\wedge *F^{2,0}\Big). \end{aligned} \tag{4.9}$$

This action functional indeed gives the desired equations for the fermionic zero-modes. After expanding the action functional S we have the following terms relevant for fermionic zero-modes,

$$\begin{aligned} S = -\frac{1}{6\pi^2}\int_M \mathrm{Tr}\Big(& i\overline{\eta}_- * \overline{\partial}_A^*\psi_+^{0,1} + i\eta_- * \partial_A^*\overline{\psi}_+^{1,0} + \frac{3}{2}\chi_-^{2,0}\wedge *\overline{\partial}_A\psi_+^{0,1} + \frac{3}{2}\overline{\chi}_-^{0,2}\wedge *\partial_A\overline{\psi}_+^{1,0}\\ &+\frac{3i}{2}\chi_-^{2,0}\wedge *\overline{\partial}_A^*\overline{\lambda}_+^{0,3} + \frac{3i}{2}\overline{\chi}_-^{0,2}\wedge *\partial_A^*\lambda_+^{3,0}\Big) + \cdots. \end{aligned} \tag{4.10}$$

From this we obtain the following fermionic equations of motion,

$$\begin{aligned} \overline{\partial}_A^*\psi_+^{0,1} &= 0,\\ i\overline{\partial}_A\eta_- + \frac{3}{2}\overline{\partial}_A^*\overline{\chi}_-^{0,2} &= 0,\\ \overline{\partial}_A\psi_+^{0,1} + i\overline{\partial}_A^*\overline{\lambda}_+^{0,3} &= 0,\\ \overline{\partial}_A\overline{\chi}_-^{0,2} &= 0. \end{aligned} \tag{4.11}$$

We will see below that these give rise to exactly the required equations (3.3) and (3.8).

4.3. The Path Integral

The path integral of our model is localized to the locus of the following equations, modulo $\mathcal{G}$ symmetry, see (2.9) and (2.10),

$$\begin{aligned} \overline{\partial}_A^* C^{0,3} &= 0,\\ F^{0,2} &= 0,\\ iF\wedge\omega\wedge\omega + \frac{i}{2}[C^{3,0},C^{0,3}] - \frac{\zeta}{3}\omega^3 I_E &= 0, \end{aligned} \tag{4.12}$$

and

$$
\begin{aligned}
d_A\phi_{++} &= 0, \\
[\phi_{++}, C^{0,3}] &= 0, \\
[\phi_{++}, \phi_{--}] &= 0.
\end{aligned}
\tag{4.13}
$$

We call the moduli space defined by the eq. (4.12) the *extended* moduli space $\mathfrak{M}$ of EH connections (with factor ζ) or stable bundles.

Since the path integral is localized to integrable connections $\bar{\partial}_A^2 = 0$, the fermionic equations of motion in (4.11) become

$$
\bar{\partial}_A \eta_- = 0, \qquad
\begin{aligned}
\bar{\partial}_A^* \psi_+^{0,1} &= 0, \\
\bar{\partial}_A \psi_+^{0,1} &= 0,
\end{aligned}
\qquad
\begin{aligned}
\bar{\partial}_A^* \bar{\chi}_-^{0,2} &= 0, \\
\bar{\partial}_A \bar{\chi}_-^{0,2} &= 0,
\end{aligned}
\qquad
\bar{\partial}_A^* \bar{\lambda}_+^{0,3} = 0.
\tag{4.14}
$$

Thus the zero-modes of fermions

$$
\bar{\eta}_-,\ \psi_+^{0,1},\ \bar{\chi}_-^{0,2},\ \bar{\lambda}_+^{0,3}
\tag{4.15}
$$

are elements of the cohomology group $\boldsymbol{H}^{0,p}$ of the following Dolbeault complex (3.10),

$$
0 \longrightarrow \boldsymbol{C}^{0,0} \xrightarrow{\bar{\partial}_A} \boldsymbol{C}^{0,1} \xrightarrow{\bar{\partial}_A} \boldsymbol{C}^{0,2} \xrightarrow{\bar{\partial}_A} \boldsymbol{C}^{0,3} \longrightarrow 0,
\tag{4.16}
$$

where $\boldsymbol{C}^{0,\ell} := \Omega^{0,\ell}(M, \mathrm{End}(E))$. It is also easy to check that the above is isomorphic to the versal deformation complex of the extended moduli space $\mathfrak{M}$ of stable bundles. Thus minus the index of the above Dolbeault cohomology group corresponds to the net ghost number violations in the path integral measure due to the zero-modes of fermions in (4.15). We have

$$
\begin{aligned}
\triangle &= -\#(\bar{\eta}_-)_0 + \#(\psi_+^{0,1})_0 - \#(\bar{\chi}_-^{0,2})_0 + \#(\bar{\lambda}_+^{0,3})_0 \\
&= \sum_{q=0}^{3} (-1)^{q+1} \dim \boldsymbol{H}^{0,q}.
\end{aligned}
\tag{4.17}
$$

The net ghost number violation of the path integral due to all the fermions – the fermions in (4.15) and their conjugates – is $(\triangle, \triangle)$. The above index can be computed by applying the standard Riemann-Roch formula. We find

$$
\triangle = \int_M \mathrm{c}_1(M) \wedge \left(r\, \mathrm{c}_2(E) - \frac{r-1}{2} \mathrm{c}_1(E)^2 \right) - r^2(1 - h^{0,1} + h^{0,2} - h^{0,3}),
\tag{4.18}
$$

where $h^{p,q}$ denote the Hodge numbers of M. We also note that a Hermitian vector bundle E admits an EH connection only if

$$
\int_M \omega \wedge \left(r\, \mathrm{c}_2(E) - \frac{r-1}{2} \mathrm{c}_1(E)^2 \right) \geq 0
\tag{4.19}
$$

and the equality holds if and only if E is projectively flat.

Now we take a closer look at the path integral. We note that the zero-modes of $\psi_+^{0,1}$ and $\overline{\lambda}_+^{0,3}$, thus $\boldsymbol{H}^{0,1}$ and $\boldsymbol{H}^{0,3}$, correspond to local deformations of the extended moduli space $\mathfrak{M}$. The other fermionic zero-modes $\overline{\eta}_- \in \boldsymbol{H}^{0,0}$ and $\overline{\chi}_-^{0,2} \in \boldsymbol{H}^{0,2}$ will cause some trouble. Note that we have a decomposition into trace and trace-free parts

$$\boldsymbol{H}^{0,q} = H^{0,q}(M) \oplus \widetilde{\boldsymbol{H}}^{0,q}. \tag{4.20}$$

We call $\widetilde{\triangle} = \triangle - (-1 + h^{0,1} - h^{0,2} + h^{0,3})$ the complex formal dimension of $\mathfrak{M}$. If we assume a situation that $\mathcal{G}$ acts freely on the locus of solutions of (4.12), i.e., the connection is irreducible, the extend moduli space $\mathfrak{M}$ is an analytic space with the Kähler structure induced from the $\mathcal{G}$-equivariant Kähler form (4.7). The moduli space will not have the right complex dimension $\widetilde{\triangle}$ unless $\widetilde{\boldsymbol{H}}^{0,2} = 0$ as well. In the ideal situation $\widetilde{\boldsymbol{H}}^{0,0} = \widetilde{\boldsymbol{H}}^{0,2} = 0$, the extended moduli space $\mathfrak{M}$ is smooth and the zero-modes of $\psi_+^{0,1}, \lambda_+^{3,0}$ span the holomorphic tangent space.[6] Thus the formal complex dimension is the actual dimension.

However the assumption made above, in particular $\widetilde{\boldsymbol{H}}^{0,2} = 0$, is too naive. We note that the obstruction to deformation of the extended moduli space $\mathfrak{M}$ lies in $\widetilde{\boldsymbol{H}}^{0,2}$. In two complex dimensions Donaldson proved that one can always achieve $\widetilde{\boldsymbol{H}}^{0,2} = 0$ after a suitable perturbation of the metric. In three complex dimensions one can hardly expect such a result to continue to hold. The assumption $\widetilde{\boldsymbol{H}}^{0,0} = 0$ is valid for a bundle E with degree and rank coprime.

Let us see how the path integral deals with the above problems. We assume, for simplicity, that our gauge group is $\mathrm{SU}(r)$, so that $\mathrm{End}(E)$ is always trace-free (so we also should replace $\boldsymbol{H}$ by $\widetilde{\boldsymbol{H}}$). Then the formal complex dimension $\triangle$ in (4.17) is given by

$$\triangle = r\int_M \mathrm{c}_1(M) \wedge \mathrm{c}_2(E) - (r^2-1)(1 - h^{0,1} + h^{0,2} - h^{0,3}), \tag{4.21}$$

instead of (4.18). A typical observable of the theory is the total $\mathcal{G}$-equivariant Kähler form, after parity change, $\widehat{\varpi}_T^{\mathcal{G}}$ is given by (4.7). First we consider an idealistic case that $\boldsymbol{H}^{0,0} = \boldsymbol{H}^{0,2} = 0$. Then the correlation function $\langle \exp \widehat{\varpi}_T^{\mathcal{G}} \rangle$ can be identified with the symplectic volume of $\mathfrak{M}$,

$$\langle \exp \widehat{\varpi}_T^{\mathcal{G}} \rangle = \int_{\mathfrak{M}} \exp \widetilde{\varpi}_T = \mathrm{vol}(\mathfrak{M}). \tag{4.22}$$

If there are zero-modes for the anti-ghosts $\overline{\chi}_-^{0,2}$, i.e. $\boldsymbol{H}^{0,2} \neq 0$, the above correlation function is modified,

$$\langle \exp \widehat{\varpi}_T^{\mathcal{G}} \rangle = \int_{\mathfrak{M}} \mathrm{e}(\mathcal{V}) \wedge \exp \widetilde{\varpi}_T, \tag{4.23}$$

where $\mathrm{e}(\mathcal{V})$ denotes the Euler class of the anti-ghost bundle $\mathcal{V}$. One may consider correlation functions of other observables $\widehat{\mathcal{O}}^{r,s}$ with ghost numbers (r,s) given by $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed $\mathcal{G}$

[6] We will establish this later. We remark that the case with $\boldsymbol{H}^{0,3} \neq 0$ causes no problem as this is associated with deformations of $\mathfrak{M} \supset \mathcal{M}_{EH}$ along the direction of $C^{0,3}$. It would be a problem if we work only with $\mathcal{M}_{EH}$.

equivariant differential forms $\mathcal{O}^{r,s}$. We have

$$\left\langle \prod_{i=1}^{\ell} \widehat{\mathcal{O}}^{r_i,s_i} \right\rangle = \int_{\mathfrak{M}} \mathrm{e}(\mathcal{V}) \wedge \widetilde{\mathcal{O}}^{r_1,s_1} \wedge \ldots \wedge \widetilde{\mathcal{O}}^{r_\ell,s_\ell} \tag{4.24}$$

where $\widetilde{\mathcal{O}}^{r,s}$ denotes the equivariant differential form $\mathcal{O}^{r,s}$ after the restriction and reduction to $\mathfrak{M}$. The above correlation function can be non-vanishing if

$$\sum_{i=1}^{\ell}(r_i, s_i) = (\triangle, \triangle), \tag{4.25}$$

due to the ghost number anomaly. What is remarkable is that the path integral is well-defined even if the moduli space $\mathfrak{M}$ does not satisfy conditions of unobstructedness like $\boldsymbol{H}^{0,2} = 0$.

To understand this important point in more detail, let us look up some details about how the Euler class of the anti-ghost bundle emerges. The action function S (4.9) contains the following Yukawa coupling involving the anti-ghost,

$$S = \frac{i}{4\pi^2} \int \mathrm{Tr}\left(\chi_-^{2,0} \wedge *[\phi_{++}, \overline{\chi}_-^{0,2}]\right) + \cdots. \tag{4.26}$$

It also contains the following terms, solely from the first line of the expression (4.9), depending on ϕ_{--},

$$\begin{aligned} S = -\frac{1}{6\pi^2} \int \mathrm{Tr}\Big[\phi_{--}\Big(* d_A^* d_A \phi_{++} - \frac{1}{4}[C^{3,0}, [\phi_{++}, C^{0,3}]] \\ - *\Lambda[\psi_+^{0,1}, \overline{\psi}^{1,0}] - \frac{1}{4}[\lambda_+^{3,0}, \overline{\lambda}_+^{0,3}]\Big)\Big] + \cdots. \end{aligned} \tag{4.27}$$

Assuming, for simplicity, that there $\overline{\eta}_-$ has no zero-modes ($\boldsymbol{H}^{0,0} = 0$) one can evaluate the correlation functions by solving the ϕ_{--} equations of motion and replacing all the other fields, including ϕ_{++}, by their zero-modes. Then the only non-vanishing term in the action functional S in the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ invariant neighborhood $\mathcal{C}$ of the fixed point locus comes from the expression (4.26), which can be written as

$$S|_{\mathcal{C}} = -\mathcal{F}_{\alpha\overline{\beta}i\overline{j}} \widetilde{\psi}_+^i \widetilde{\overline{\psi}}_+^{\overline{j}} \widetilde{\chi}_-^{\alpha} \widetilde{\overline{\chi}}_-^{\overline{\beta}}, \tag{4.28}$$

where $\widetilde{\psi}_+^i$ and $\widetilde{\chi}_-^\alpha$ denote the zero-modes of $(\psi_+^{0,1}, \lambda_+^{3,0})$ and $\chi_-^{2,0}$, respectively, and similarly for the conjugate fields. In the above the indices i and α run over $i = 1, \ldots, \mathbf{h}^{0,1} + \mathbf{h}^{0,3}$ and $\alpha = 1, \ldots, \mathbf{h}^{0,2}$, where $\mathbf{h}^{0,*} = \dim_{\mathbb{C}} \boldsymbol{H}^{0,*}$. The expression $\mathcal{F}^{\overline{\alpha}}{}_{\overline{\beta}i\overline{j}} \widetilde{\psi}_+^i \widetilde{\overline{\psi}}_+^{\overline{j}}$ denotes the curvature two form of the anti-ghost bundle $\mathcal{V}$ over $\mathfrak{M}$ – the space of the zero-modes a^i of $A^{0,1}$ and

$C^{3,0}$ modulo $\mathcal{G}$. Consequently the expectation value, for example $\langle \exp \widehat{\varpi}_T^{\mathcal{G}} \rangle$, becomes[7]

$$
\begin{aligned}
\langle \exp \widehat{\varpi}_T^{\mathcal{G}} \rangle = \int_{\mathfrak{M}} \prod_{\ell=1}^{\triangle + \mathbf{h}^{0,2}} da^\ell da^{\overline{\ell}} d\psi_+^\ell d\psi^{\overline{\ell}} \prod_{\gamma=1}^{\mathbf{h}^{0,2}} d\chi_-^\gamma d\chi_-^{\overline{\gamma}} \left(\det h_{\alpha\overline{\beta}}(a^\ell, a^{\overline{\ell}}) \right)^{-1} \\
\times \exp \left(\mathcal{F}_{\alpha\overline{\beta} i \overline{\jmath}}(a^\ell, a^{\overline{\ell}}) \psi_+^i \psi_+^{\overline{\jmath}} \chi_-^\alpha \chi_-^{\overline{\beta}} + \widetilde{\omega}_{i\overline{\jmath}}(a^\ell, a^{\overline{\ell}}) \psi_+^i \psi^{\overline{\jmath}} \right),
\end{aligned}
\tag{4.29}
$$

which leads exactly to (4.23).

5. Deformations Of The Model

In this section we study certain deformations of our $N_c = (2,0)$ model. The main purpose will be to be able to handle situations where non-stable bundles can occur, that is $\boldsymbol{H}^{0,0} \neq 0$ in the language of the last section, so there may be zero-modes for $\overline{\eta}_-$. In the moduli space of the theory considered until now, this situation introduces singularities. In our equivariant approach, which we implemented from the start, these type of singularities are however easily handled. The second deformation we consider might help to relate our extended model again to the unextended model based on EH bundles.

5.1. Deformation To A "Holomorphic" $N_c = (2,0)$ Model

In this subsection we consider a deformation of the original $N_c = (2,0)$ model. The resulting deformed model will have much better behavior than the original model when the effective target space $\mathcal{M}_\zeta$ has singularities. This kind of deformation is originally due to Witten [27] and applied in a similar situation to the present case in [6]. We will find the deformed and the original model as two special limits of a one-parameter family of models. In comparison with the discussion in [27], we added the extra localization Fermi multiplets χ_-; they will however be purely spectators, and the specialization to the Kähler case will simplify the procedure.

The original action was given by (2.6). We saw that the path integral of the $N_c = (2,0)$ model is localized to the symplectic quotient $\mathcal{M}_\zeta = (\mathcal{X} \cap \mu^{-1}(\zeta))/\mathcal{G}$ of $\mathcal{X}$ by $\mathcal{G}$. Now we consider the following one-parameter family of $N_c = (2,0)$ theories, given by the action functional

$$
\begin{aligned}
S(\zeta)_\lambda =& S(\zeta) + \frac{\lambda}{2} \boldsymbol{s}_+ \overline{\boldsymbol{s}}_+ \langle \phi_{--}, \phi_{--} \rangle \\
=& - \boldsymbol{s}_+ \overline{\boldsymbol{s}}_+ \Big(\langle \phi_{--}, \mu - \zeta - \frac{\lambda}{2} \phi_{--} \rangle - \langle \eta_-, \overline{\eta}_- \rangle + \langle h_{\alpha\overline{\beta}} \chi_-^\alpha, \chi_-^{\overline{\alpha}} \rangle \Big) \\
& + i \boldsymbol{s}_+ \langle \chi_-^\alpha, \mathfrak{S}_\alpha \rangle + \overline{\boldsymbol{s}}_+ \langle \chi_-^{\overline{\alpha}}, \mathfrak{S}_{\overline{\alpha}} \rangle.
\end{aligned}
\tag{5.1}
$$

[7] The determinant of the metric comes from integrating out the auxiliary fields.

If we set $\lambda = 0$ we retain the original action. Since the λ-dependent term by which we deform is $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed, the theory does not depend on λ, as long as $\lambda \neq 0$. The models with $\lambda = 0$, and $\lambda \neq 0$ can be different since new fixed points can flow in from infinity in the field space [27]. For $\lambda \neq 0$ the path integral localixes to the critical points of $I = \langle \mu - \zeta, \mu - \zeta \rangle$, while the original theory, at $\lambda = 0$, is localized to the zeros (trivial critical points) of I.

Next, we add local $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed observables $-\widehat{\varpi}^{\mathcal{G}}$, $i\langle \phi_{++}, \zeta \rangle$ and $-\frac{\varepsilon}{2}\langle \phi_{++}, \phi_{++} \rangle$ to this action functional, basically for regularization. We get

$$
\begin{aligned}
S_h(\zeta,\varepsilon)_\lambda =& S(\zeta)_\lambda - i\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\mathcal{K} + i\langle \phi_{++}, \zeta \rangle + \frac{\varepsilon}{2}\langle \phi_{++}, \phi_{++} \rangle \\
=& - i\langle \phi_{++}, \mu - \zeta \rangle - \widehat{\varpi}(\psi_+, \overline{\psi}_+) + \frac{\varepsilon}{2}\langle \phi_{++}, \phi_{++} \rangle \\
& - \boldsymbol{s}_+\overline{\boldsymbol{s}}_+ \Big(\langle \phi_{--}, \mu - \zeta - \frac{\lambda}{2}\phi_{--} \rangle - \langle \eta_-, \overline{\eta}_- \rangle + \langle h_{\alpha\overline{\beta}}\chi^\alpha_-, \chi^{\overline{\alpha}}_- \rangle \Big) \\
& + i\boldsymbol{s}_+\langle \chi^\alpha_-, \mathfrak{S}_\alpha \rangle + \overline{\boldsymbol{s}}_+\langle \chi^{\overline{\alpha}}_-, \mathfrak{S}_{\overline{\alpha}} \rangle.
\end{aligned}
\tag{5.2}
$$

The partition sum of this model computes the expectation value $\langle e^{-\mathcal{O}(\zeta,\varepsilon)} \rangle$ in the deformed theory (5.1), where $\mathcal{O}(\zeta,\varepsilon)$ denotes the extra contributions in the action above.

For $\lambda \neq 0$, we can integrate out the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$. We are then left with

$$
\begin{aligned}
S_h(\zeta,\varepsilon)_\lambda =& - i\,\langle \phi_{++}, \mu - \zeta \rangle - \widehat{\varpi}(\psi_+, \overline{\psi}_+) + \frac{\varepsilon}{2}\,\langle \phi_{++}, \phi_{++} \rangle \\
& - \boldsymbol{s}_+\overline{\boldsymbol{s}}_+\langle h_{\alpha\overline{\beta}}\chi^\alpha_-, \chi^{\overline{\alpha}}_- \rangle + i\boldsymbol{s}_+\langle \chi^\alpha_-, \mathfrak{S}_\alpha \rangle + \overline{\boldsymbol{s}}_+\langle \chi^{\overline{\alpha}}_-, \mathfrak{S}_{\overline{\alpha}} \rangle. \\
& + \frac{1}{2\lambda}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\,\langle \mu - \zeta, \mu - \zeta \rangle + \mathcal{O}(1/\lambda^2).
\end{aligned}
\tag{5.3}
$$

If we take the limit $\lambda \to 0$, while $\lambda \neq 0$, we see that the dominant contributions to the path integral come from the critical points of $I = \langle \mu - \zeta, \mu - \zeta \rangle$. Note that this includes the trivial critical points $\mu = \zeta$, whose contributions give the path integral of the original model, defined by (2.6), with the insertions of the observables added above. However, in general we also get contributions from higher critical points. So we do not get back the original model. The contributions of the higher critical points, for which $I \neq 0$, are proportional to $e^{-I/2\varepsilon}$, for $\varepsilon \to 0$ (this can be seen by integrating out ϕ_{++}). Therefore, we can easily extract the contribution from the original model. On the other hand, as the theory is independent of $\lambda \neq 0$, this limit is the same as the theory for *any* value $\lambda \neq 0$.

Now consider the limit $\lambda \to \infty$, to remove all the λ-dependent terms from (5.3). We call this model a holomorphic $N_c = (2,0)$ model.[8] The path integral of this theory is localized to critical points of $I = \langle \mu - \zeta, \mu - \zeta \rangle$, which shows that indeed this limit is the same as the deformed model given by (5.3) for finite λ.

[8]This name is inspired by the holomorphic Yang-Mills theory [6], which arises from Donaldson-Witten theory in this way.

5.2. A Use Of S^1 Symmetry

The extended equations (4.12) we have may be very useful. On the extended moduli space $\mathfrak{M}$ of EH connections we have the natural S^1-action

$$S^1 : C^{0,3} \to e^{i\theta} C^{0,3}, \tag{5.4}$$

which preserves the complex and the Kähler structure. Thus any cohomological computation can be further localized to the fixed point locus of this S^1-action. For the SU(2) case we are concentrating on it is easy to determine the fixed points. We have two branches.

- Branch (i)

  $\phi_{++} = 0$ and the SU(2) symmetry is unbroken. Then we have a trivial fixed point where simply $C^{0,3} = 0$ where we have EH connections.

- Branch (ii)

  ϕ_{++} is a constant diagonal trace-free matrix. The nontrivial fixed points occur if the gauge symmetry can undo the S^1-action. For this the SU(2) symmetry should be broken to U(1). Thus the gauge bundle splits, $E_A = L \oplus L^{-1}$ where $A \in \mathcal{A}^{1,1}$. Furthermore $C^{0,3}$ and $C^{3,0}$ become

  $$C^{0,3} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}, \qquad C^{3,0} = \begin{pmatrix} 0 & 0 \\ \overline{\gamma} & 0 \end{pmatrix}, \tag{5.5}$$

  where γ is a section of $K^{-1} \otimes L^2$, with K denoting the canonical line bundle of our Kähler 3-fold. Then we have the following fixed point equations

  $$\begin{aligned} F_L^{0,2} &= 0, \\ iF_L \wedge \omega \wedge \omega - \frac{1}{2}\gamma \wedge \overline{\gamma} &= 0. \end{aligned} \qquad \partial_L^* \gamma = 0, \tag{5.6}$$

  where F_L denotes the curvature of the line bundle L. Obviously we have a nontrivial solution if $\deg(L) > 0$. If $\gamma = 0$ we can have abelian EH connections, and also if $\deg(L) = 0$.

The equations (5.6) are analogous to the abelian Seiberg-Vafa-Witten equations [4][28]; they may be equally powerful. Thus we expect that the above equations may contain all the nontrivial information about the Donaldson-Witten type theory on Kähler 3-folds. It should be possible to establish our conjecture quite rigorously. Here we will only sketch the idea.

As a first step we map the $N_c = (2,0)$ model defined by the action functional S (4.9) to its deformed version, following the discussions in Sect. 5.1. The action functional is then defined by

$$
\begin{aligned}
S_h(\varepsilon) =& \frac{1}{4\pi^2}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\left(\chi_-^{2,0}\wedge *\overline{\chi}_-^{0,2}\right) \\
&+\frac{i}{4\pi^2}\boldsymbol{s}_+\int_M \mathrm{Tr}\Big(\chi_-^{2,0}\wedge *F^{0,2}\Big)+\frac{i}{4\pi^2}\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\Big(\overline{\chi}_-^{0,2}\wedge *F^{2,0}\Big) \\
&- i\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\mathcal{K}_T+\frac{\varepsilon}{4\pi^2}\int_M\frac{\omega^3}{3!}\mathrm{Tr}(\phi_{++}^2),
\end{aligned}
\tag{5.7}
$$

where $\mathcal{K}_T$ is given by (4.6). As we established earlier the partition function of this theory for $\varepsilon=0$ is the correlation function (4.23) with the same conditions. If the reducible connections are unavoidable we turn on ε to regularize and utilize the non-abelian localization.

Examining the supersymmetry transformation laws of the holomorphic $C^{3,0}$ and the Fermi $\chi^{2,0}$ multiplets, we can see that the S^1-action (5.4) should be extended as follows

$$
\begin{aligned}
&S^1:(C^{0,3},\overline{\lambda}_+^{0,3},\overline{\chi}_-^{0,2},H^{0,2})\to\xi(C^{0,3},\overline{\lambda}_+^{0,3},\overline{\chi}_-^{0,2},H^{0,2}),\\
&S^1:(C^{3,0},\lambda_+^{3,0},\chi_-^{2,0},H^{2,0})\to\overline{\xi}(C^{3,0},\lambda_+^{3,0},\chi_-^{2,0},H^{2,0}),
\end{aligned}
\tag{5.8}
$$

where $\xi\overline{\xi}=1$. Thus the above fields are now charged under S^1. A problem might be that the above S^1-action is not a symmetry of the action functional.[9] However the S^1-action preserves the supersymmetry transformation laws as well as the localization equations. Thus we can use it anyway. Now we modify the transformation laws of the charged fields under the S^1 by extending the $\mathcal{G}$-equivariant cohomology to $\mathcal{G}\times S^1$;

$$
\boldsymbol{s}_+^2=0,\qquad \{\boldsymbol{s}_+,\overline{\boldsymbol{s}}_+\}=-i\phi_{++}^a\mathcal{L}_a-im\mathcal{L}_{S^1},\qquad \overline{\boldsymbol{s}}_+^2=0.
\tag{5.9}
$$

We use the same form of the deformed action functional (5.7) but with the new transformation laws for supercharges according to (5.9). We obtain a new $N_c=(2,0)$ supersymmetric action functional[10]

$$
\begin{aligned}
S_h(m,\varepsilon) =& \frac{1}{4\pi^2}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\left(\chi_-^{2,0}\wedge *\overline{\chi}_-^{0,2}\right) \\
&+\frac{i}{4\pi^2}\boldsymbol{s}_+\int_M \mathrm{Tr}\Big(\chi_-^{2,0}\wedge *F^{0,2}\Big)+\frac{i}{4\pi^2}\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\Big(\overline{\chi}_-^{0,2}\wedge *F^{2,0}\Big) \\
&-\widehat{\varpi}_T^{\mathcal{G}}-imH_{S^1},
\end{aligned}
\tag{5.10}
$$

where H_{S^1} is the bosonic Hamiltonian of the S^1-action,

$$
H_{S^1}=\frac{1}{24\pi^2}\int_M\mathrm{Tr}\left(C^{3,0}\wedge C^{0,3}\right).
\tag{5.11}
$$

[9]This is due to a term like $\mathrm{Tr}(\chi_-^{2,0}\wedge *\overline{\partial}_A\overline{\psi}_+^{0,1})$.

[10]We put ε to zero. We can turn on ε whenever necessary.

The first and second lines in the action functional localize the path integral to the locus $\overline{\partial}^*_A C^{0,3} = F^{0,2} = 0$. The first term in the third line further localize the path integral to the locus $\mu_T = 0$. For simplicity we assume that there are no zero-modes of $\chi^{2,0}_-$. Then the partition function of the model reduces to[11]

$$Z = \int_{\mathfrak{M}} e^{imH_{S^1} + \widetilde{\varpi}_T}, \tag{5.12}$$

where $\widetilde{\varpi}_T$ is the Kähler form of $\mathfrak{M}$, obtained by the restriction and reduction from our equivariant Kähler form $\widehat{\varpi}^{\mathcal{G}}_T$ (4.7). Thus the partition function is given by the familiar DH integral formula over a finite dimensional Kähler manifold $\mathfrak{M}$ [29][30]. It is therefore an integral over the set of critical points of H_{S^1}, which is the same as the fixed point locus of the S^1-action on $\mathfrak{M}$. Thus we have the same two branches.

The following is a formal argument since we do not understand the compactification of $\mathfrak{M}$. However it will be sufficient to serve our purpose. We will just apply the exactness of the stationary phase integral. By setting $m \to \infty$ we may have

- Branch (i)

  Note that the value of the Hamiltonian H_{S^1} is zero at Branch (i). So its contribution to the integral is simply the volume of $\mathcal{M}_{EH}$ weighted by the one loop determinant of due to the normal bundle $N(\mathcal{M}_{EH})$ in $\mathfrak{M}$. Note that such one loop determinant contains weight m^{-s} where s denotes codimension of $\mathcal{M}_{EH})$ in $\mathfrak{M}$. Thus

$$Z(i) \sim \frac{1}{m^s}\,\mathrm{vol}(\mathcal{M}_{EH}) \times \cdots. \tag{5.13}$$

  The unwritten part is due to contribution from the normal bundle $N(\mathcal{M}_{EH})$, while we extracted its dependence on m.

- Branch (ii)

  Note that the value of the Hamiltonian at Branch (ii) is

$$H_{S^1} = \frac{1}{12\pi}\deg(L) := \frac{1}{24\pi^2}\int \mathrm{c}_1(L)\wedge\omega\wedge\omega,$$

  where L is a line bundle defined in (5.6). Thus

$$Z(ii) \sim \sum_L \frac{1}{m^{s'}}\int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi}\deg(L) + \widetilde{\omega}\,|_{\mathcal{F}(L)}\right) \times \cdots \tag{5.14}$$

[11] We remark that the action functional contains the mass term for the anti-ghosts $\chi^{2,0}_-$ and $\overline{\chi}^{0,2}_+$. If there are no zero modes for anti-ghost such the term plays no roles. If there are zero-modes of anti-ghosts we have to include contribution from the anti-ghost bundles and the mass term. Then the partition function Z becomes

$$Z = \int_{\mathfrak{M}} det(\mathcal{F} - imI)\exp\left(imH_{S^1} + \widetilde{\varpi}_T\right),$$

where $\mathcal{F} - imI$ is the S^1-equivariant curvature two form of the anti-ghost bundle $\mathcal{V}$ over $\mathfrak{M}$.

where $\mathcal{F}(L)$ denotes the fixed point locus, s' denotes its codimension and $\widetilde{\omega}|_{\mathcal{F}(L)}$ denote the Kähler form on $\mathcal{F}(L)$. The unwritten part is due to contributions from the normal bundle over the fixed point locus, while we extracted its dependence on m.

We assume that $s < s'$, otherwise the above formal formula does not make sense. Then one can take $m = 0$. Since the original formula was smooth in the limit of the reduction to the symplectic volume of $\mathfrak{M}$ the poles in $Z(i)$ and $Z(ii)$ should cancel order by order. Thus we have

$$\begin{aligned}\mathrm{vol}(\mathcal{M}_{EH}) \sim \sum_L \frac{1}{(s'-s)!}\left(\frac{im}{12\pi}\deg(L)\right)^{s'-s} \\ \times \int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi}\deg(L) + \widetilde{\varpi}|_{\mathcal{F}(L)}\right) \times \cdots\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\mathrm{vol}(\mathfrak{M}) \sim \sum_L \frac{1}{s'!}\left(\frac{im}{12\pi}\deg(L)\right)^{s'} \\ \times \int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi}\deg(L) + \widetilde{\varpi}|_{\mathcal{F}(L)}\right) \times \cdots .\end{aligned} \tag{5.16}$$

We conclude that the above formal evaluation gives evidence for our conjecture that Seiberg-Vafa-Witten type invariants defined by the equation (5.6) should be equivalent to the Donaldson-Witten type invariants on a Kähler 3-fold. It is possible to perform a similar analysis for the case with anti-ghost zero-modes, which makes life more complicated but does not alter the essential points advocated above.

6. Specialized Models

We will now shortly comment on properties of the model in some special situations when the Kähler 3-fold has additional symmetries, that is more reduced holonomy.

6.1. Reduction To A Kähler Surface

In this subsection we perform a dimensional reduction of our models on a Kähler 3-fold M to a complex Kähler surface M_2. We first assume that M is a product manifold $M_3 = M_2 \times \mathbb{C}$ and then remove dependence of our fields on $\mathbb{C}$. We have the following correspondence

$$\begin{aligned}&A^{0,1} \to A^{0,1},\ \sigma,\\ &\psi_+^{0,1} \to \psi_+^{0,1},\ \overline{\eta}_+,\\ &\chi_-^{2,0} \to \psi_-^{1,0},\ \chi_-^{2,0},\\ &H^{0,2} \to H^{0,1},\ H^{0,2},\\ &C^{0,3} \to B^{0,2},\end{aligned} \tag{6.1}$$

as well as the corresponding decomposition for their Hermitian conjugates. The other fields $(\phi_{\pm\pm}, \eta_-, \overline{\eta}_-, D)$ remain as they were. Thus we obtain a $N_c = (2,2)$ model. Similarly the equation (4.12) for the extended EH connection reduces to the Vafa-Witten equations. Furthermore our equation (5.6) for branch (ii) fixed point become the Abelian Seiberg-Witten equations. Thus our conjecture on Donaldson-Witten type invariants on a Kähler 3-fold becomes a fact [4]. The model we obtain is exactly the Vafa-Witten theory of a twisted $\mathcal{N} = 4$ super-Yang-Mills theory on the Kähler surface [28][31][32].

Now instead of the above trivial reduction we consider a product manifold $M = M_2 \times \Sigma$, where Σ is a 2-torus. Then we can follow the same steps with the same sort of assumption as [33] to conclude that the models discussed in the previous subsection are equivalent to the topological sigma model of Vafa and Witten [28]. Thus the stringy Donaldson-Witten invariants on a Kähler surface may be obtained from formulas like (5.15) and (5.16) on the product 3-fold. This supports an earlier suspicion of one of the authors that a stringy generalization of Donaldson-Witten theory as discussed in [31] does not give information beyond Seiberg-Witten, since the Seiberg-Vafa-Witten type invariants on a manifold $M_2 \times \Sigma$ most likely are just the Seiberg-Witten invariants on M_2.

6.2. The Model On Calabi-Yau 3-Folds

We now shortly comment on the case that the Kähler 3-fold M is Calabi-Yau with holomorphic 3-form $\omega^{0,3}$. For the Calabi-Yau case the $N_c = (2,0)$ supersymmetry enhances to $N_c = (2,2)$ supersymmetry. We will come back to this model in more detail in a forthcoming paper [22].

We argued in [8] that our model is the world-volume theory of parallel type IIB (Euclidean) $D5$-branes wrapped on the CY_3. We show that the $\mathcal{G}$-equivariant degrees of freedom correspond to the bulk degrees of freedom transverse to the (Euclidean) $D5$-branes. We use such a correspondence as supporting evidence that our path integral should be well-defined in any situation.

We consider the $N_c = (2,0)$ theory with supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ defined in the previous section specializing to a Calabi-Yau 3-fold M with a holomorphic 3-form $\omega^{0,3}$. Using the non-degeneracy of $\omega^{0,3}$ we may redefine the fields $(\overline{\chi}^{0,2}_-, H^{0,2}, \lambda^{0,3}_+, C^{0,3})$ as

$$\psi^{0,1}_-,\ H^{0,1},\ \eta_+, \sigma, \tag{6.2}$$

where[12]

$$\begin{aligned} \overline{\chi}^{0,2}_- &= \overline{*}(\omega^{3,0} \wedge \psi^{0,1}_-), \qquad & \lambda^{0,3}_+ &= \eta_+ \omega^{0,3}, \\ H^{0,2} &= \overline{*}(\omega^{3,0} \wedge H^{0,1}), \qquad & C^{0,3} &= \sigma \omega^{0,3}. \end{aligned} \tag{6.3}$$

[12]The anti-holomorphic Hodge star operator $\overline{*}$ is defined by $\overline{*}\alpha = *\overline{\alpha}$. Acting on a (p,q)-form on a complex d-fold gives a $(d-p, d-q)$,

$$\overline{*} : \Omega^{p,q}(M) \to \Omega^{d-p,d-q}(M).$$

It is not difficult to show that the action functional S has additional global supersymmetries generated by $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$. We have the following diagrams to be compared with (4.5);

$$
\begin{tikzcd}
\overline{\sigma} \arrow[r, "\boldsymbol{s}_+"] \arrow[d, "\overline{\boldsymbol{s}}_-"] & \eta_+ \arrow[d, "\overline{\boldsymbol{s}}_-"] & \phi_{++} \arrow[l, "\boldsymbol{s}_-"'] \arrow[d, "\overline{\boldsymbol{s}}_-"] \\
\overline{\eta}_- \arrow[r, "\boldsymbol{s}_+"] & D & \overline{\eta}_+ \arrow[l, "\boldsymbol{s}_-"'] \\
\phi_{--} \arrow[r, "\boldsymbol{s}_+"] \arrow[u, "\overline{\boldsymbol{s}}_+"] & \eta_- \arrow[u, "\overline{\boldsymbol{s}}_+"] & \sigma \arrow[l, "\boldsymbol{s}_-"'] \arrow[u, "\overline{\boldsymbol{s}}_+"]
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
\psi_-^{0,1} \arrow[dr, "\boldsymbol{s}_+"'] & A^{0,1} \arrow[l, "\boldsymbol{s}_-"'] \arrow[r, "\boldsymbol{s}_+"] & \psi_+^{0,1} \arrow[dl, "\boldsymbol{s}_-"] \\
& H^{0,1} &
\end{tikzcd}
\quad . \tag{6.4}
$$

The four supercharges satisfy the following anti-commutation relations

$$
\begin{array}{lll}
 & \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi_{++}^a \mathcal{L}_a, & \\
\{\boldsymbol{s}_\pm, \boldsymbol{s}_\pm\} = 0, & \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_-\} = -i\sigma^a \mathcal{L}_a, & \{\boldsymbol{s}_+, \boldsymbol{s}_-\} = 0, \\
\{\overline{\boldsymbol{s}}_\pm, \overline{\boldsymbol{s}}_\pm\} = 0, & \{\boldsymbol{s}_-, \overline{\boldsymbol{s}}_+\} = -i\overline{\sigma}^a \mathcal{L}_a, & \{\overline{\boldsymbol{s}}_+, \overline{\boldsymbol{s}}_-\} = 0. \\
 & \{\boldsymbol{s}_-, \overline{\boldsymbol{s}}_-\} = -i\phi_{--}^a \mathcal{L}_a, &
\end{array}
\tag{6.5}
$$

The above anti-commutation relations define a balanced $\mathcal{G}$-equivariant Dolbeault cohomology on the space $\mathcal{A}$ of all connections [31]. Thus our model becomes a $N_c = (2,2)$ model.

The action functional S in (4.9) can be rewritten in a form showing manifest $N_c = (2,2)$ symmetry,

$$
S = \boldsymbol{s}_+ \overline{\boldsymbol{s}}_+ \boldsymbol{s}_- \overline{\boldsymbol{s}}_- \Big(\mathcal{K} - \frac{1}{6\pi^2} \int_M \mathrm{Tr}(\sigma * \overline{\sigma}) \Big) + \boldsymbol{s}_+ \boldsymbol{s}_- \mathcal{W}(A^{0,1}) + \overline{\boldsymbol{s}}_+ \overline{\boldsymbol{s}}_- \overline{\mathcal{W}}(A^{1,0}), \tag{6.6}
$$

where $\mathcal{K}$ is the Kähler potential on the space $\mathcal{A}$ of all connections,

$$
\mathcal{K} = \frac{1}{24\pi^2} \int_M \kappa \, \mathrm{Tr}\,(F \wedge F) \wedge \omega, \tag{6.7}
$$

and $\mathcal{W}(A^{0,1})$ is the holomorphic Chern-Simons form,

$$
\mathcal{W}(A^{0,1}) = \frac{1}{8\pi^2} \int_M \omega^{3,0} \wedge \mathrm{Tr}\Big(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \Big). \tag{6.8}
$$

We remark that the above action functional can be obtained by the dimensional reduction of the $(1+1)$-dimensional $N_{ws} = (2,2)$ spacetime supersymmetric linear gauged sigma model in two real dimensions, whose target space is the space $\mathcal{A}$ of all connections on a Calabi-Yau 3-fold M [8]. In [8] we interpreted the model as the matrix string theory [34][35][36] compactified on a Calabi-Yau by regarding $\mathcal{A}$ as the configuration space of all D-branes wrapped on the Calabi-Yau.

7. Discussion And Conclusion

In this paper we studied an extended moduli problem of stable bundles on Kähler 3-folds, using topological field theory. The partition function of the topological field theory gives a concrete formula to calculate natural generalizations of Donaldson-Witten type invariants for higher dimensional Kähler manifolds. The bare problem of stable bundles in 3 complex dimensions generically is obstructed, which is reflected in the infinite number of anti-ghost zero-modes in the corresponding model. We argued that in order to reduce this to a finite number, we had to extend the model by adding a $(3,0)$-form field. This extended moduli problem indeed gives rise to a finite number of zero-modes, and therefore also a finite dimensional moduli space. However, the model may still have anti-ghost zero-modes, which would make the moduli space non-smooth. However, the partition function and the correlation functions can still be well defined, by using the Euler class of the corresponding anti-ghost bundle.

Another potential problem was the appearance of zero-modes for the ghosts, corresponding to the possible appearance of strictly semi-stable bundles. We saw that we could deform the model such that we are able to deal with thissituation. This deformation is similar to the one proposed in Donaldson theory in [27].

Stable bundles also appear as the BPS sector of string theory, interpreted as BPS configurations of D-branes wrapped around the Kähler manifold. It would be interesting to see if the extended moduli problem also has a string interpretation, though at firs sight this does not seem the case, as we have no natural candidate for the additional $(3,0)$-form.

The general mathematical cohomological problem has a generalization to higher dimensional Kähler manifolds. However, we could not implement these ideas into a topological field theory setting. The only solution could lie in the interpretation of the higher even forms as gauge parameters rather than field strengths (obstructions). However, we do not know anyway in which this could happen in the nonabelian case. It is interesting to compare to string theory, where there are strong hints towards "nonabelian" higher form gauge transformations.

Acknowledgements

We are grateful to Erik and Herman Verlinde for encouragement and discussions. CH was supported by the FOM foundation and DOE grant # DE-FG02-96ER40559. JSP was supported by a pioneer fund of NWO and DOE grant # DE-FG02-92ER40699.

Appendix A. Supersymmetry Transformation Laws

In this Appendix we give the explicit $N_c = (2,0)$ supersymmetry transformation rules of the fields of our model discussed in Sect. 4. The transformation laws of the *gauge multiplet* $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$ and of ϕ_{++} are given by

$$
\begin{aligned}
& & \boldsymbol{s}_+\eta_- &= 0, & & \\
\boldsymbol{s}_+\phi_{--} &= i\eta_-, & \overline{\boldsymbol{s}}_+\eta_- &= +iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & \boldsymbol{s}_+\phi_{++} &= 0, \\
\overline{\boldsymbol{s}}_+\phi_{--} &= i\overline{\eta}_-, & \boldsymbol{s}_+\overline{\eta}_- &= -iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & \overline{\boldsymbol{s}}_+\phi_{++} &= 0. \\
& & \overline{\boldsymbol{s}}_+\overline{\eta}_- &= 0, & &
\end{aligned}
\tag{8.1}
$$

We had two sets of holomorphic multiplets and their anti-holomorphic partners. The transformations follow those given in (2.2). One set of holomorphic multiplets is $(A^{0,1}, \psi_+^{0,1})$ with anti-holomorphic partners $(A^{1,0}, \overline{\psi}_+^{1,0})$,

$$
\begin{aligned}
\boldsymbol{s}_+ A^{0,1} &= i\psi_+^{0,1}, & \boldsymbol{s}_+\psi_+^{0,1} &= 0, \\
\overline{\boldsymbol{s}}_+ A^{0,1} &= 0, & \overline{\boldsymbol{s}}_+\psi_+^{0,1} &= -\overline{\partial}_A\phi_{++}, \\
\boldsymbol{s}_+ A^{1,0} &= 0, & \boldsymbol{s}_+\overline{\psi}_+^{1,0} &= -\partial_A\phi_{++}, \\
\overline{\boldsymbol{s}}_+ A^{1,0} &= i\overline{\psi}_+^{1,0}, & \overline{\boldsymbol{s}}_+\overline{\psi}_+^{1,0} &= 0.
\end{aligned}
\tag{8.2}
$$

The other holomorphic multiplet is $(C^{3,0}, \lambda_+^{3,0})$ with anti-holomorphic partner $(C^{0,3}, \overline{\lambda}_+^{0,3})$,

$$
\begin{aligned}
\boldsymbol{s}_+ C^{3,0} &= i\lambda_+^{3,0}, & \boldsymbol{s}_+\lambda_+^{3,0} &= 0, \\
\overline{\boldsymbol{s}}_+ C^{3,0} &= 0, & \overline{\boldsymbol{s}}_+\lambda_+^{3,0} &= -i[\phi_{++}, C^{3,0}], \\
\boldsymbol{s}_+ C^{0,3} &= 0, & \boldsymbol{s}_+\overline{\lambda}_+^{0,3} &= -i[\phi_{++}, C^{0,3}], \\
\overline{\boldsymbol{s}}_+ C^{0,3} &= i\overline{\lambda}_+^{0,3}, & \overline{\boldsymbol{s}}_+\overline{\lambda}_+^{0,3} &= 0.
\end{aligned}
\tag{8.3}
$$

Finally we have Fermi multiplets $(\chi_-^{2,0}, H^{2,0})$ and anti-Fermi multiplets $(\overline{\chi}_-^{0,2}, H^{0,2})$, with transformation rules as in (2.5), using the holomorphic section $\mathfrak{J}$ given in (4.4), we get

$$
\begin{aligned}
\boldsymbol{s}_+\chi_-^{2,0} &= -H^{2,0}, & \boldsymbol{s}_+ H^{2,0} &= 0, \\
\overline{\boldsymbol{s}}_+\chi_-^{2,0} &= -\partial_A^* C^{3,0}, & \overline{\boldsymbol{s}}_+ H^{2,0} &= -i[\phi_{++}, \chi^{2,0}] + i[*\psi_+^{0,1}*, C^{3,0}] + i\partial_A^*\lambda_+^{3,0}, \\
\boldsymbol{s}_+\overline{\chi}_-^{0,2} &= -\overline{\partial}_A^* C^{0,3}, & \boldsymbol{s}_+ H^{0,2} &= -i[\phi_{++}, \overline{\chi}^{0,2}] + i[*\overline{\psi}_+^{1,0}*, C^{0,3}] + i\overline{\partial}_A^*\overline{\lambda}_+^{0,3}, \\
\overline{\boldsymbol{s}}_+\overline{\chi}_-^{0,2} &= -H^{0,2}, & \overline{\boldsymbol{s}}_+ H^{0,2} &= 0.
\end{aligned}
\tag{8.4}
$$

Appendix B. Some Properties Of $\mathfrak{M}$

This is a mathematical digression to establish a property of the extended moduli space. First we recall a theorem [21][26] on the moduli space $\mathcal{M}_{EH}$ of EH connections – if $\widetilde{\boldsymbol{H}}^{0,0} = 0$ the moduli space $\mathcal{M}_{EH}$ is a complex analytic space. It is nonsingular at a neighborhood of a connection if $\widetilde{\boldsymbol{H}}^{0,2} = 0$ and its tangent space is naturally isomorphic to the space of $\boldsymbol{H}^{0,1}$. Here $\widetilde{\boldsymbol{H}}^{0,*}$ denotes the cohomology group defined by tracefree endomorphisms. We refer to [21]Ch. VII.3 for details on the notations.

Now we state an analogous theorem about the extended moduli space $\mathfrak{M}$ of EH connections on a complex Kähler 3-fold - if $\widetilde{\boldsymbol{H}}^{0,0} = 0$ the moduli space $\mathfrak{M}$ is a complex analytic space. It is nonsingular at a neighborhood of an extended connection if $\widetilde{\boldsymbol{H}}^{0,2} = 0$ and its tangent space is naturally isomorphic to the space $\boldsymbol{H}^{0,1} \oplus \boldsymbol{H}^{3,0}$. The extended moduli space $\mathfrak{M}$ is a smooth Kähler manifold with the formal dimension equal to the actual dimension if $\widetilde{\boldsymbol{H}}^{0,0} = \widetilde{\boldsymbol{H}}^{0,2} = 0$.

The proof of the above theorem is similar to that of the Einstein-Hermitian case [21]. Given an extended EH connection $\overline{\mathfrak{D}}$, a nearby deformation $\overline{\partial}_A + \alpha$, $C^{3,0} + \beta$ is governed by the equations

$$\begin{aligned} \overline{\partial}_A \alpha + \alpha \wedge \alpha &= 0, \\ \overline{\partial}_A^* \alpha &= 0, \\ \Lambda(\overline{\partial}_A \beta + \alpha \wedge \beta) &= 0. \end{aligned} \tag{9.1}$$

We only need to consider the last equation since the theorem quoted above already dealt with the first two equations. The last equation has the following orthogonal decomposition

$$\overline{\partial}_A \beta + \alpha \wedge \beta = 0 \leftrightarrow \begin{cases} \overline{\partial}_A \left(\beta + \overline{\partial}_A^* \circ G(\alpha \wedge \beta) \right) = 0, \\ \overline{\partial}_A^* \oplus \overline{\partial}_A \circ G(\alpha \wedge \beta) = 0, \\ H(\alpha \wedge \beta) = 0, \end{cases} \tag{9.2}$$

where G is Green's operator and H is the harmonic projection. We define Kuranishi map k'

$$k' : \boldsymbol{C}^{3,0} \to \boldsymbol{C}^{3,0}, \qquad k'(\beta) = \beta + \overline{\partial}_A^* \circ G(\alpha \wedge \beta). \tag{9.3}$$

Then, from the first equation on the right of (9.2) we have $\overline{\partial}_A(k'(\beta)) = 0$, while $\overline{\partial}_A^*(k'(\beta)) = 0$ by the dimensional reason. Thus we obtain $\Lambda \overline{\partial}_A(k'(\beta)) = 0 \to \overline{\partial}_A^*(k'(\beta)) = 0$. Consequently we have

$$k'(\beta) \subset \boldsymbol{H}^{3,0}. \tag{9.4}$$

Now we examine if the Kuranishi map is invertible for a given $\rho \in \boldsymbol{H}^{3,0}$, i.e., $\beta = k'^{-1}(\rho)$ and $\Lambda(\overline{\partial}_A \beta + \alpha \wedge \beta) = 0$. Taking the orthogonal decomposition of $\alpha \wedge \beta$ one finds that

$$\Lambda(\overline{\partial}_A \beta + \alpha \wedge \beta) = \Lambda \overline{\partial}_A^* \circ \overline{\partial}_A \circ G(\alpha \wedge \beta) + \Lambda(H(\alpha \wedge \beta)). \tag{9.5}$$

Note that $\Lambda(H(\alpha \wedge \beta))$ is in $\widetilde{\boldsymbol{H}}^{2,0}$, which is isomorphic to $\widetilde{\boldsymbol{H}}^{0,2}$. By our assumption we have $H(\alpha \wedge \beta)) = 0$. Denoting $\gamma = \overline{\partial}_A \alpha + \alpha \wedge \alpha$ and $\delta = \overline{\partial}_A \beta + \alpha \wedge \beta$ we have

$$\begin{aligned} \delta &= \overline{\partial}_A^* \circ G(\overline{\partial}_A \alpha \wedge \beta - \alpha \wedge \overline{\partial}_A \beta) \\ &= \overline{\partial}_A^* \circ G(\gamma \wedge \beta + \alpha \wedge \delta) \\ &= \overline{\partial}_A^* \circ G(\alpha \wedge \delta), \end{aligned} \tag{9.6}$$

where we used the fact that $\gamma = 0$ for $\widetilde{\boldsymbol{H}}^{0,2} = 0$. Applying the following estimate

$$\|\overline{\partial}_A^* \circ Gv\|_{2,k+1} \leq c\|v\|_{2,k}, \tag{9.7}$$

we have

$$\begin{aligned} \|\delta\|_{2,k} \leq \|\delta\|_{2,k+1} &= \|\overline{\partial}_A^* \circ G(\alpha \wedge \delta)\|_{2,k+1} \\ &\leq c\|\delta\|_{2,k} \cdot \|\alpha\|_{2,k}. \end{aligned} \tag{9.8}$$

Taking α sufficiently close to 0 so that $\|\alpha\|_{2,k} < 1/c$, we conclude $\delta = 0$. Thus the Kuranishi map k' is invertible if $\widetilde{\boldsymbol{H}}^{0,2} = 0$. Consequently the local model of the extended moduli space $\mathfrak{M}$ is given by $f^{-1}(0)$ where

$$\begin{aligned} f : \boldsymbol{H}^{0,1} \oplus \boldsymbol{H}^{3,0} &\to \widetilde{\boldsymbol{H}}^{2,0}, \\ (\alpha, \beta) &\to \Lambda(H(\alpha \wedge \beta)). \end{aligned} \tag{9.9}$$

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