IDEAS AND EXPLORATIONS
Brouwer's Road to Intuitionism

IDEEËN EN VERKENNINGEN;
Brouwer's weg naar het intuïtionisme

(met een samenvatting in het Nederlands)

Proefschrift

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Voor de vijf vrouwen rondom mij:
Janny, Ingeborg, Simone, Jessica en Rita.

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Life sometimes takes an unexpected and unforseen turn. After retiring at the age of 57 from 35 years of civil aviation (part of that time having filled with studies in theoretical physics at the Leiden University and in numerous hotel rooms), I started a study in philosophy at the Utrecht University, thereby specializing in logic and the foundations of mathematics. The unexpected turn came with a telephone call from Dirk van Dalen, with the invitation to join the 'L.E.J. Brouwer-group' immediately after having completed my philosophy study. He thereby offered me the opportunity to write a doctoral thesis on the early work of Brouwer and on the beginning of his road to intuitionism.

The result of four years research, reading, writing, crossing out and rewriting again is this dissertation. Writing a doctoral thesis is often a lonely job, especially when the author is constantly threatened by a hostile computer.

It will be obvious that I have to express my gratitude in the first place to my thesis advisor Dirk van Dalen for the numerous hours that he spent on my drafts and on the long discussions and explanations of the many topics that are discussed in this dissertation. This all at the cost of the precious time he needed so badly for writing at the same time two biographies about the life and work of Brouwer (one in Dutch and a scientific one in English). It is of course a relief for me that this dissertation is now completed, but undoubtedly it must be a relief for him too (as well as for his wife Dokie, who is almost just as busy with the work on the biographies).

In the second place I want to thank Mark van Atten. He was willing to share his thorough and detailed knowledge of the philosophical foundations of mathematics with me by reading through my drafts and manuscripts, pointing to many smaller and larger imperfections; he even was willing to look at the smallest details in punctuation marks and typing errors.

I also owe many thanks to Dennis Dieks for several long discussions about Brouwer's ideas on the character of physical laws and on the nature and aim of physical practice and research.

I thank Theo Verbeek and Maarten van Houte, who were so kind to read through, and comment on, what I had to say about Kant, who was of such great influence on Brouwer's ideas about the ultimate foundations of all of mathematics.

Finally, my gratitude goes to the whole staff of the faculty of philosophy. Despite the rather exceptional combination of being the lower in scientific rank and (almost) the senior in age, I have spent wonderful years here.

It should, again, be said that the whole project was also made possible by the NWO, the Netherlands Organisation for Scientific Research, in the form of a grant, which is gratefully acknowledged.

Ten slotte het 'thuisfront':

Lieve Janny, we kunnen, na alle drukte van de afgelopen jaren, natuurlijk niet zeggen dat we ons 'mijn pensioen anders hadden voorgesteld', want het was allemaal opgenomen in onze planning, hoewel de laatste vier jaren er natuurlijk als een verrassing extra bij kwamen. Maar jij hebt gelukkig je eigen interesse met je studie kunstgeschiedenis, een onderwerp dat mij ook boeit. Jouw interesse voor wat ik doe blijkt uit het feit dat jij met veel genoegen de Brouwer-biografie hebt gelezen en nu dus over een en ander mee kunt praten.

En er was gelukkig nog tijd over voor onze kinderen en kleinkinderen en, afgezien dan misschien van het laatste half jaar, voor geregeld museum- en concertbezoek en voor de vakanties samen.

Moge dit alles, naast het verdere werk aan het Brouwer project, nog lang voortgang vinden.

Contents

Preface

Almost a century after L.E.J. Brouwer took his doctoral degree at the University of Amsterdam, the dissertation which he defended on that occasion is still of interest to us, both from a historical *and* from a philosophical-mathematical point of view. A recent republication of the dissertation,[1] as well as the publication of two biographies of Brouwer[2] bear witness to this general interest.

Despite the fact that Brouwer's intuitionistic mathematics does not play a leading part in the mainstream of the mathematical practice of today, the constructivistic founding, based on the ur-intuition alone, of his 'old' intuitionism from before, say, 1919 is still widely discussed and commented on. There certainly were constructive foundations of mathematics long before Brouwer,[3] but unique for Brouwer is the ur-intuition of successive and well-separated events, connected by a 'flowing', as the most fundamental basis possible for *every* mathematical construction.

Another unique feature of Brouwer's mature intuitionism from after 1917 (in which the ur-intuition remains the ultimate foundation) is, as the reader may know, formed by the 'two acts'.[4] We will see that the two acts are already present in the dissertation, albeit in a more or less latent form and not yet explicitly denoted as 'acts'.

Let us put first that there is not one main subject in Brouwer's dissertation, which has to be discussed; there is a number of topics in it, which ask for a reinterpretation, for a correction of a misinterpretation or even for a first interpretation; it concerns topics which are sometimes thoroughly discussed and sometimes merely touched upon by Brouwer. Just to mention a few: the exact construction of the integers and of the rationals, departing from the ur-intuition; another one is Brouwer's solution to the continuum problem or, a third one, the 'denumerably infinite unfinished' cardinality. A justified reinterpretation is often made possible by the recent discovery of a number of notebooks by Brouwer's hand, written during the time of preparation of his dissertation. The

[1] [Dalen 2001].

[2] One in Dutch, [Dalen, D. van 2001] and one scientific biography in English, of which the first volume has been published, [Dalen, D. van 1999].

[3] [Beth 1967], page 149, where for example Plotinus, Nicolaus Cusanus and Hobbes are mentioned.

[4] See page 57.

result of a reinterpretation more than once leads to the conclusion that the topic concerned is another example of *mathematics as a free creation of the human mind, solely based on the ur-intuition of the two-ity continuous and discrete*, which is the heart and the essence of Brouwer's summary on the last page of his dissertation. A second central item of this summarizing conclusion will be the very limited role that is granted to logic in the construction of mathematics; logic just plays its modest part in the language that describes this construction.

In our dissertation we will investigate the subjects concerning the most basic foundational matters from Brouwer's doctoral thesis, *On the foundations of mathematics*, which will lead to his intuitionistic mathematics of ten years later. This limited selection explains the omission of mathematical-technical matters like the thoroughly discussed group-theoretical foundation of the arithmetical operations on the measurable continuum, whereas the foundation and construction of the measurable continuum itself forms an important part of our dissertation. Also parts which we regard as today's common and commonly shared knowledge are very briefly discussed, just mentioned, or even completely omitted.

We realize that the last two phrases might give rise to some confusion about which dissertation we are sometimes referring to, and this brings us to the following introductory remark: since this work is a dissertation about a dissertation, a sharp distinction has to be made between *this* or *our dissertation* and *Brouwer's dissertation.* We will stick to this terminology unless the context makes it unambiguously clear which one we are talking about. In practice this will come down to the following: when, for instance, a reference is made to 'chapter 3' without further specification, or to 'our chapter 3', the third chapter of this dissertation is meant; in case we have the third chapter of Brouwer's dissertation is mind, this will be explicitly stated.

As said, the aim of this dissertation is to present a (re)interpretation of those topics in Brouwer's dissertation, which are of foundational interest for his future development. This concerns mainly the following subjects:

– The ur-intuition and the construction of the ω-scale, departing from that intuition,

– the status of 'signs',

– the construction of the η-scale,

– the scale of integers,

– the everywhere dense η-scale,

– the Bolzano-Weierstrass theorem,

– covering an everywhere dense scale with a continuum.

– the possible point sets, in particular the third construction rule for sets,

– the continuum problem and Brouwer's solution,

– Brouwer's view on (theoretical) physics and on natural sciences in general. We realize that this is outside the realm of pure foundational mathematics, but in view of the importance that Brouwer himself attached to this subject, we feel that we cannot get around it; it is too directly linked with his general view on mankind and on human society.

– objectivity and apriority,

– the role of logic in the construction of the mathematical building,

– the hypothetical judgement and its constructivistic interpretation,

– the possible cardinalities, in particular the 'denumerably infinite unfinished' cardinality,

– the denumerably unfinished cardinality of the set of mathematical theorems versus Gödel's first incompleteness theorem,

– the actual infinite and related problems like e.g. those of 'known/unknown' and 'finished/unfinished' of the (parts of) a mathematical construction.

For that purpose the arrangement of the chapters is as follows:

In the first chapter a general and concise survey of set theory will be presented, as known among mathematicians in the year 1907, the year of Brouwer's academic promotion. In this survey there will be an emphasis on the work of Cantor, with additional remarks on (for us relevant) parts of the work of Dedekind, Poincaré, Zermelo, Schoenflies and Bernstein.

In the subsequent chapters the content of Brouwer's dissertation will be discussed in regard to the following subjects:
Chapter 2: The ur-intuition of mathematics,
Chapter 3: The continuum,
Chapter 4: The possible point sets,
Chapter 5: The continuum problem and Brouwer's solution to it,
Chapter 6: Mathematics and Experience,
Chapter 7: The role of logic,
Chapter 8: The summary and the theses.
In an appendix Brouwer's own bibliography is presented, as pieced together from his dissertation and from the notebooks.

For the relevant research we have mainly used the republication (1981) of Brouwer's dissertation,[5] which edition also includes the Brouwer-Korteweg correspondence, the *Rejected Parts*, two reviews of the dissertation by Mannoury, the *Addenda and Corrigenda*[6] (1917) and the 1908-paper *The unreliability of the logical principles*.[7] As an additional source of information we were also able

[5] [Brouwer 1981].
[6] *Addenda en corrigenda over de grondslagen der wiskunde*, [Brouwer 1917a].
[7] *De onbetrouwbaarheid der logische principes*, [Brouwer 1908a].

to use, thanks to the mentioned recent new edition of Brouwer's dissertation,[8] the corrections by Brouwer's hand, made in his own copy of the dissertation in view of a reissue of it, in or shortly after 1917. For the English version of the dissertation, *On the foundations of mathematics*, we have mostly adopted the translation as this was published in the *Collected works*.[9] When quoting from the dissertation, we always refer to the original page numbers, which are added in the margin of both the Dutch and the English editions.

As a reference to Brouwer's published work we have adopted the codification from *A bibliography of L.E.J. Brouwer*.[10]

To gain a better insight into Brouwer's ideas on the several topics that will be discussed, but also into the process of development leading to his views, we have, apart from the dissertation itself, the following sources at our diposal:

1^{st}. *Brouwer's nine notebooks.*

About six months before the date of taking his doctoral degree Brouwer composed a synopsis of what seemed to be a rather random collection of notes about different mathematical and philosophical topics. This synopsis was known for a long time and, judging by its content and reference system, it had to refer to a set of notebooks, unknown until some five years ago.[11]

The notebooks, nine in number, which turned up later, matched perfectly the content and the references of the synopsis. A transcription of these notebooks is now completed, annotations are in preparation, and the whole will be published later.

The dating of the notebooks can be made within reasonable margins. On the first page of the first notebook a reference in the margin is made to the Heidelberger Congress, which took place in 1904. The proceedings of this congress were published in 1905, hence this year most likely marks the beginning of the notes.

On page 28 of the ninth and last notebook we find a reference to the *Revue de Métaphysique et de Morale* of the year 1906, which gives us a reliable indication of the end of the notes.

However, at the end of this last notebook, on page 32, Brouwer mentioned the second volume of *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* by A. Schoenflies, which was published in 1908. And on the last page (page 33)

[8] [Dalen 2001].

[9] *L.E.J. Brouwer. Collected works*, [Brouwer 1975].

[10] [Dalen, D. van 1997].

[11] Most likely the synopsis is not a first draft for his dissertation. A little more than a month after the letter in which he announced the composition of the synopsis (7 September 1906), Brouwer wrote another letter (16 October 1906) to his thesis supervisor Korteweg, which contained the planned arrangement of the chapters for his dissertation on the basis of the notebooks and its synopsis. The first draft for this chapter arrangement shows some similarity with the final result in the dissertation and no similarity whatsoever with the chapter arrangement of the synopsis. Moreover, the synopsis does not contain fundamentally new insights, it just presents in a systematic way concise and summarizing remarks from pages of the notebooks, thereby referring to the relevant notebook pages.

there is a reference to two papers from Brouwer's own hand, *On the structure of perfect point sets*,[12], published in 1910, and *Zur Analysis Situs*, written in May 1909 and published in the *Mathematische Annalen*, also in 1910.[13]

Apparently Brouwer continued to make some notes for at least one year after the defence of his dissertation, which took place on 19 February 1907. Most likely the last blank pages of the last notebook were made useful, since on earlier pages not a single trace can be found of any note, definitely dating from later than the date of the defence.

As remarked already, Brouwer started composing the mentioned synopsis some six months before the public defence of his dissertation; this 'period of six months' can be concluded from the letter of 7 september 1906 from Brouwer to Korteweg:

> For some time I am now in Blaricum, where I can spend more efficiently all my time on my work. I stopped reading others and I am busy arranging my notes into chapters.
>
> I am feeling so much the stronger in my conviction, now that I perceive that I still hold as my view the notes of about two years ago, after all the reading since. Except that I now can support them better with the help of mathematical developments, than at that time.[14]

The justified conclusion seems to be that the notebooks were mainly written during the years 1905 and 1906, with some additional notes on the last two pages of the last notebook from the years 1907, 1908 and 1909.

The content of the notebooks consists of short notes, varying in length from one line to one page, discussing and commenting on a large variety of topics, like the foundations of mathematics, the foundations of (projective) geometry, philosophy, mysticism (often with pessimistic overtones),[15] but there are also long discussions on the continuum and on sets, which are the subjects of our main concern. Brouwer also made elaborate notes on potential theory; the results of those notes were published separately in the *Verslagen van de Koninklijke Nederlandse Akademie van Wetenschappen*.[16]

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[12] Over de Structuur der Perfecte Puntverzamelingen; [Brouwer 1910b] and [Brouwer 1910a], published in the *Verslagen van de Koninklijke Nederlandse Akademie van Wetenschappen* and in the *Proceedings* of the same Academy, of 1910.

[13] [Brouwer 1910c].

[14] included in [Brouwer 1981] and [Dalen 2001]: Sinds enige tijd ben ik in Blaricum, waar ik beter al mijn tijd aan mijn werk kan geven. Met het lezen van anderen ben ik opgehouden, en ben nu bezig mijn aantekeningen te ordenen en onder hoofdstukken te brengen.

Ik voel mij des te sterker in mijn overtuiging, nu ik merk, mijn aantekeningen van ongeveer twee jaar geleden ook nu nog, na mijn lectuur van de tussentijd, geheel voor mijn rekening te kunnen nemen. Alleen kan ik ze nu beter met wiskundige ontwikkelingen steunen dan toen.

[15] To get an idea of Brouwer's mysticism and his pessimistic outlook on life and on humanity, see [Brouwer 1905]; it is also translated in English, see [Stigt 1996].

[16] The *Proceedings of the Royal Dutch Academy of Sciences*. In fact the Dutch version was published in the *KNAW Verslagen* and the English translation in the *KNAW Proceedings*.

To the modern reader it appears to be difficult to transform this totality of seemingly loose remarks, thought experiments and comments into a systematic overview of the developments in Brouwer's mathematical and philosophical thinking that we are about to investigate.

Especially in the last three notebooks Brouwer was very much concerned with the continuum, the admissible sets and their possible cardinalities. Whereas in the first three or four notebooks only a small number of paragraphs was devoted to sets and the continuum, from the second half of the sixth notebook onwards page-long discussions follow on these topics; apparently these pages must have been written during the year 1906.

Also we find in the notebooks many rudimentary ideas and thought experiments that return only in much later developments of his mathematical thinking.

We adopt Brouwer's convention to indicate the page references to the notebooks: III–7 refers to page 7 of the third notebook. This system of codification is Brouwer's own, as he used it in the synopsis of his notebooks. In this synopsis Brouwer composed from the seemingly random and chaotic abundance of comments and remarks a more systematic whole, by collecting the different subjects into chapters.

Finally we make the following comment on the notebooks: Brouwer frequently quoted from the writings of others (Cantor, Poincaré, Russell, Couturat and many others). It is striking that in many cases, in fact more often than not, the quotes are not exactly verbatim, despite the fact that Brouwer put them between quotation marks. He often composed the quotes himself by selecting parts from longer sentences (without using the modern convention of inserting '(...)' for deleted words or groups of words), or he added new words to make from part of a sentence a complete one. Brouwer either took that liberty or quoted by heart. Since in the dissertation the quotes are strictly verbatim, the ones in the notebooks are most likely just references and mnemonics for later use in the dissertation.

2^{nd}. *The correspondence with his thesis supervisor Korteweg.*

During the time of preparation of his dissertation and especially during the final stage of actually writing it all down, Brouwer was in frequent and close contact with his thesis supervisor Korteweg, via personal visits as well as via letters.[17] As a result of criticism from the side of Korteweg, several parts of Brouwer's draft text did not find their way into his dissertation. These 'rejected parts' have been preserved.

The earliest known correspondence dates back to a letter of 15 February 1906, concerning the extension of Brouwer's study grant. The more substantive part begins with a letter of 7 September 1906, through which a reasonable dating of the period of writing down the content of the synopsis could be made; the relevant quotation was given above.

[17] Included in [Brouwer 1981] and in [Dalen 2001].

3^{rd}. *The Rejected Parts of the manuscript.*

These consist of the parts of Brouwer's draft for his dissertation which, as remarked above, were rejected as a result of Korteweg's criticism.[18]

4^{th}. *Preserved notes concerning the public defence of the dissertation, i.e. Mannoury's and Barrau's opposition, and Brouwer's reply to it.*[19]

An academic 'promotion', i.e. the ceremony of taking one's doctoral degree, consists of a public defence of the dissertation against objections raised by the examining professors, but in the old days in Amsterdam the examination was opened with the *opposition from the floor*, that is, opponents from the audience were allowed to attack the dissertation and the theses. That the opponents concerned and the content of their opposition did not come as a complete surprise to the candidate, can be concluded from the Korteweg correspondence.[20]

Also the following lecture and paper by Brouwer, as well as a letter to De Vries, a review and an inaugural lecture, although they were given, held, sent, or read after obtaining his doctoral degree, are relevant to a proper understanding and interpretation of the subjects, especially when viewed as a transitional stage on his way to intuitionistic mathematics:

5^{th}. The Rome lecture *Die möglichen Mächtigkeiten*, held at the International Congress of Mathematicians in Rome,1908.[21]

6^{th}. *The unreliability of the logical principles*,[22] published in the *Tijdschrift voor Wijsbegeerte*.[23]

7^{th}. Brouwer's *letter to J. de Vries*, professor in mathematics at Utrecht University, 1907.[24]

8^{th}. *Mannoury's review of the dissertation and Brouwer's reply.*[25]

9^{th} *Intuitionism and Formalism*,

Brouwer's inaugural adress on the occasion of his professorship at the University of Amsterdam, 1912.[26]

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[18] Included in [Brouwer 1981] and in [Dalen 2001].

[19] Found as loose sheets in the last notebook. The Mannoury-part is included in [Brouwer 1981] and in [Dalen 2001].

[20] For a detailed description of the ceremony, see [Dalen, D. van 1999], page 118, 119.

[21] Atti IV Congr. Intern. Mat. Roma III, page 569 – 571. Also included in [Brouwer 1975], page 102 – 104.

[22] *De onbetrouwbaarheid der logische principes.* See [Brouwer 1908a]. Also included in [Brouwer 1981] and in [Dalen 2001]; English translation in [Brouwer 1975], page 107 – 111.

[23] Dutch *Journal of Philosophy.*

[24] Included in [Dalen 2001].

[25] See [Brouwer 1981] and [Dalen 2001]. In fact, Mannoury wrote two reviews; Brouwer replied to the one which was published in the *Nieuw Archief voor Wiskunde.*

[26] See [Brouwer 1981] and [Dalen 2001]; for English translation see [Brouwer 1975].

Occasionally some additional short quotes will be drawn from other writings, not mentioned in the given list and usually from a much later date.

For each subject the mentioned sources will be examined on their relevance, beginning with the dissertation and thereupon, if needed or desired as an elucidation or as an aid to a proper interpretation of the views expressed in the dissertation, the other sources. It will be evident that particularly the notebooks will offer us the opportunity to trace any possible development in fundamental notions, as there are the continuum, sets and their possible cardinalities, but also developments in future notions like choice sequences and the spread concept. For a proper understanding of these remarks and notes we often have to refer to much later work by Brouwer's hand.

A final remark about the translation of the quotes: for the English version of the quotations from the work of Brouwer we have mainly used, as point of departure, the translation as it was published in the mentioned *Collected Works* (C.W.), edited by A. Heyting. However, we have modified the translation when that was, in our opinion, considered necessary.

Translations by others (e.g. Mancosu, Dresden) of texts by Brouwer, but not included in the C.W., will be mentioned separately.[27]

Original texts in German or French are left untranslated.

[27]Many English translations of Brouwer's texts, as far as they were originally written in Dutch, can be found in [Heijenoort 1967], [Benacerraf and Putnam 1983], [Stigt 1990], [Mancosu 1998] or [Ewald 1999].

Chapter 1

Sets and the continuum before 1907

For a fruitful discussion of Brouwer's early work, a reasonable amount of knowledge of the development of set theory, as this took place during the second half of the nineteenth century, is required. We must keep in mind that Brouwer reacted to a new and very current branch of mathematics, which received a lot of attention during the period beginning just a few decades before Brouwer's active life as a mathematician and philosopher.

This first chapter contains background material, needed for a proper understanding of the discussion about the foundational parts of Brouwer's dissertation. The content will be familiar to many readers, since it consists mainly of basic set theory. For those readers a quick glance will suffice.

After 1870 set theory developed rapidly into a separate branch of mathematics, despite resistance from mathematicians like Kronecker, but also from philosophers and even from theologians. This new development took place mainly through the work of Georg Cantor, who, for that reason, must be considered as the founder of set theory.

Before Cantor, use was made of certain collections possessing certain properties, like the locus in geometry, but collections like that did not seem to have the status of a mathematical object with which and on which certain mathematical operations could be performed. Likewise collections with an infinite number of elements, although their existence had of course to be admitted, seemed to possess strange properties to such an extent (like the possibility of a one-one correspondence between a collection and a part thereof) that they too seemed to lack the status of mathematical object.[1]

[1]see [Galileo 1974], page 40, 41 where one of the participants in the discussion concludes, because of the possible one-one relation between natural numbers and their squares, that concepts like 'equal', 'greater' and 'smaller' are not applicable in case of infinite quantities.

To get an insight into the development and in the state of knowledge of set theory at the time of Brouwer's public defence of his dissertation, the following persons and subjects are of importance, and will be discussed:

1.1 *Cantor*
1.2 *Dedekind*
1.3 *Poincaré*
1.4 *Zermelo*
1.5 *Schoenflies*
1.6 *Bernstein*
1.7 *The paradoxes*

An important predecessor in the development of set theory was Bolzano (1781–1848), whose main publication on this topic, *Paradoxien des Unendlichen*, was published posthumously in 1851.[2] Bolzano was attracted by the seemingly paradoxical nature of the infinite. Without going into the details of Bolzano's work, we just mention the fact that he very clearly saw the problems concerning the infinite, that he asked the proper questions, but did not take the steps and did not draw the conclusions, that Cantor did twenty five years later.

Because Brouwer made use of Cantor's results (and disagreed in many respects with these results), it is, for a proper understanding of the references that Brouwer made, of importance to give a survey of the content of Cantor's work in a systematic way. This will be done in the first section of this chapter, despite the fact that it concerns rather basic set theory with which many readers will be familiar.

1.1 Cantor (1845–1918)

Cantor started publishing on set theory in 1870, shortly after his *Habilitation.* From this time onwards the idea that a set can have a one-to-one relation to a proper subset of itself is generally accepted as a typical property of infinite sets (and is certainly no reason to turn away from those sets).[3]

The word *set* (German: *Menge*, French: *ensemble*) did not appear right from the beginning in the work of Cantor. In his earliest letters to Dedekind during the years 1873 and 1874[4] and in his paper *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen,* published in 1874, he employed the mathematically undefined term *totality* (Inbegriff) e.g. of the natural numbers or of the rational numbers. In the 1877-correspondence to Dedekind and in the 1878-paper *Ein Beitrag zur Mannigfaltigkeitslehre,* the term *Mannigfaltigkeit* was used to indicate such totalities, but now also *set* (Menge) appeared, and in

[2] [Bolzano 1851].

[3] For Dedekind the possibility of a one-to-one mapping of a set onto a proper subset of itself became the definition of the infinity of a set, now known under the term *Dedekind infinite.* See under section 1.2.

[4] Cantor communicated his new ideas on sets and cardinalities with Dedekind in a series of letters after 1873; see [Cantor 1937].

the eighties of that century this term became the accepted one. We will from now on use only the term *set*. In the *Beiträge zur Begründung der transfiniten Mengenlehre*[5] the following definition of a set is given (§ 1):

> Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen.

To indicate the magnitude of a set M the term *power, cardinality* or *cardinal number* (Mächtigkeit, Cardinalzahl) is used: two sets have the same power or cardinality if a one-to-one mapping of one set onto the other is possible, under abstraction from the order and the nature of their elements. This cardinality with its double abstraction is indicated for a set M by $\overline{\overline{M}}$. Sets satisfying that mapping condition are then said to be *equivalent* (gleichmächtig).

Consulted literature for this section about Cantor consists of the following material:

– *Briefwechsel Cantor-Dedekind*, edited by E. Noether and J. Cavaillès, Paris, 1937.

This volume contains the Cantor-Dedekind correspondence between 1872 and 1882; the important 1899-part of this correspondence is contained in Cantor's *Gesammelte Abhandlungen.*

– Cantor, *Gesammelte Abhandlungen,* edited by E. Zermelo, Berlin, 1932.
From this volume in particular the following articles:

– *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. (Journal für die reine und angewandte Mathematik*[6] 1874.)

– *Ein Beitrag zur Mannigfaltigkeitslehre.* (idem, 1878.)

– *Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten. (Göttinger Nachrichten,* 1879.)

– *Über unendliche lineare Punktmannigfaltigkeiten. (Mathematische Annalen,* 1879, '80, '82, '83, '84; M.A. Volumes 15, 17, 20, 21 and 23 respectively.)

– *Über verschiedene Theoreme aus der Theorie der Punktmengen in einem n-fach ausgedehnten stetigen Raume G_n. (Acta Mathematica,* 1885.)

– *Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche. (Zeitschrift für Philosophie und philosophische Kritik* bd. 88, 1890.)

[5][Cantor 1897], also in [Cantor 1932], page 282 ff.

[6]Brouwer usually referred to this journal under the name of its editor *Crelle*, and we will follow this practice henceforth.

> – *Mitteilungen zur Lehre vom Transfiniten. (Zschr. f. Philos. u. philos. Kritik,* 1887.)
>
> – *Über eine elementare Frage der Mannigfaltigkeitslehre. (Jahresbericht der Deutschen Mathematiker-Vereinigung,* Bd.1, 1890–91.)
>
> –*Beiträge zur Begründung der transfiniten Mengenlehre. (Mathematische Annalen* 1895, '97, M.A. Volumes 46 and 49 respectively.)

As far as possible, the papers mentioned above will be discussed in a chronological order, which results not only in an overview of the state of knowledge on sets in the year 1907, but additionally in a survey of the origin and history of set theory. We must keep in mind that there was only one complete textbook on set theory available during the years of Brouwer's preparation for his thesis.[7] Brouwer drew most of his information directly from Cantor's papers.

In the first sections frequent use will be made of the Cantor-Dedekind correspondence; this in particular gives a good insight in the development of his ideas. It will be noticed that in the beginning Cantor often made use of terms and concepts which were only properly defined in a later stage of the development.

1.1.1 The discovery of non-equivalent infinite sets

In a letter to Dedekind of 29 November 1873,[8] Cantor considered the possibility of the construction of a one-to-one relation between the natural and the real numbers. He did not know a definite answer to this problem yet, and asked Dedekind to think it over. Intuitively the answer should be no, but there was justified doubt:

> Auf den ersten Anblick sagt man sich, nein es ist nicht möglich, denn (n) besteht aus discreten Theilen, (x) aber bildet ein Continuum;
>
> (...)
>
> Wäre man nicht auch auf den ersten Anblick geneigt zu behaupten, dass sich (n) nicht eindeutig zuordnen lasse dem Inbegriffe $(\frac{p}{q})$ aller positiven rationalen Zahlen $\frac{p}{q}$?

But this last statement is easy to prove, just as the theorem of the equivalence between the natural numbers and the algebraic numbers.

In a following letter to Dedekind of 7 December 1873, Cantor proved in an ingenious way that the assumed equivalence between the natural numbers and

[7] The first general textbook on set theory was Schoenflies' *Bericht über die Mengenlehre*, [Schoenflies 1900a], published in the *Jahresbericht der Deutschen Mathmatiker-Vereiniging*; see under section 1.5. In fact there was also Young and Young's *The theory of sets of points* ([Young and Chisholm Young 1906]), but probably Brouwer was not familiar with this book; he never mentioned the name Young, neither in his thesis, nor in his notebooks, although W.H. Young published in German in the *Mathematische Annalen* in 1905.

[8] [Cantor 1937], page 12; all letters between Cantor and Dedekind during the years 1872 – 1882 are published in this volume. The letters written in the nineties of the nineteenth century can be found in [Cantor 1932].

the reals leads to a contradiction. The proof was performed with the help of a method of nested intervals and was published a year later.[9] With this proof the existence of at least two different kinds of infinity was shown for the first time.

1.1.2 On the equivalence of R_1 and R_n

On 5 January 1874 Cantor asked to Dedekind in a letter the following question:

> Lässt sich eine Fläche (etwa ein Quadrat mit Einschluss der Begrenzung) eindeutig auf eine Linie (etwa eine gerade Strecke mit Einschluss der Endpunkte) eindeutig beziehen, so dass zu jedem Puncte der Fläche ein Punct der Linie und umgekehrt zu jedem Puncte der Linie ein Punct der Fläche gehört?

In the letter of 20 June 1874 he returned to this question and gave a proof that such a relation is possible, and he even proved that for any finite n a one-to-one relation between a line and an n-dimensional manifold is possible. On the basis of some remarks from Dedekind,[10] a correction by Cantor was given in his reply of 25 June 1877 to Dedekind and at the end of this letter Cantor added:

> Da sieht man, welch' wunderbare Kraft in den gewöhnlichen reellen rationalen und irrationalen Zahlen doch liegt, dass man durch sie im Stande ist die Elemente einer ρ fach ausgedehnten stetigen Mannigfaltigkeit eindeutig *mit einer einzigen Coordinate zu bestimmen.*

It appears that Cantor was surprised by his own results. In his next letter to Dedekind he stated *'je le vois, mais je ne le crois pas'.*

Cantor was in doubt, because it violated the principle of invariance of dimension in one-to-one mappings of one 'Punktmannigfaltigkeit' onto an other; but on 2 July 1877 Dedekind answered him that the invariance of dimension is only valid for continuous mappings and that authors, when discussing mappings, always more or less tacitly assume that the mappings concerned are continuous.

Cantor published this proof, with the mentioned remarks by Dedekind incorporated, in 1878,[11] and he returned to this subject in 1879[12] with the same conclusion: discontinuous mappings do not require invariance of dimension.

In the paper of 1878 Cantor introduced the term *equivalence* of two sets and showed that in case of an infinite set an equivalence is possible between that set and a proper subset thereof; he also proved that the system of the natural

[9] [Cantor 1874]. The in later time frequently employed diagonal method of proof of the uncountability of a set was only introduced in [Cantor 1891], also to be found in [Cantor 1932], page 278 ff., in which short article the countability of the algebraic numbers and the uncoutability of the reals is proved. Both the *interval method* and the *diagonalization method*, are called *spoiling arguments* in modern logic.

[10] letter of 22 June 1874 to Cantor, [Cantor 1937], page 27.

[11] [Cantor 1878], § 1, also included in [Cantor 1932], page 122 ff.

[12] [Cantor 1879b], see also [Cantor 1932], page 134 ff.

numbers is the 'smallest' possible infinite set, which means that any subset of the set of the natural numbers is either finite or equivalent to the original set.

1.1.3 On linearly ordered point sets

In the *Mathematische Annalen* (volumes 15, 17, 20, 21, and 23, appearing between 1879 and 1884) the first of two series of papers was published, under the titel *Über unendliche lineare Punktmannigfaltigkeiten.* In the next sections up to and including § 1.1.7 a selection from the content of this series will be presented and occasionally commented on. The beginning of the first article reads:

> In einer, im Crelleschen Journale, Bd. 84, herausgegebenen Abhandlung[13] (...) habe ich für ein sehr weitreichendes Gebiet von geometrischen und arithmetischen, sowohl kontinuierlichen wie diskontinuierlichen Mannigfaltigkeiten den Nachweis geführt, daß sie eindeutig und vollständig einer geraden Strecke oder einem diskontinuierlichen Bestandteile von ihr sich zuordnen lassen.[14]

Cantor used the term *lineare Punktmannigfaltigkeiten*, or briefly, *lineare Punktmengen.*[15] In the Crelle paper (volume 84), to which he referred in the last quotation, a linearly ordered set is defined as:

> Unter einer *linearen* Mannigfaltigkeit reeller Zahlen wollen wir jede wohldefinierte Mannigfaltigkeit reeller, voneinander verschiedener, d.i. ungleicher Zahlen verstehen, so daß eine und dieselbe Zahl in einer linearen Mannigfaltigkeit nicht öfter als einmal als Element vorkommt.[16]

Thus for Cantor not much was lost by restricting himself to the infinite linearly ordered sets of natural, rational, algebraic, or real numbers.

At the beginning of the first paper of the series which we are discussing now, the concept *derivative* of a linearly ordered point set is defined: the *derivative* of a point set is the set of all limit points of the original set, where:

A *limit point* of a set is defined as a point (not necessarily an element of the set) with the property that every neighbourhood of that point contains at least one point of the set, different from the limit point itself.

The *second derivative* is the derivative of the derivative set, etc.

A set is of the *first kind (von der ersten Gattung)* if there is an n^{th} derivative that has no derivative itself.[17] This is the case if the n^{th} derivative is a set with finitely many elements in every interval of finite length.

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[13] [Cantor 1878], also in [Cantor 1932], page 119 ff.

[14] [Cantor 1932], page 139.

[15] See on page 15 for the more abstract definition of *einfach geordnete Menge* (simply ordered set).

[16] [Cantor 1932]. page 124.

[17] To be precise: some n^{th} and all subsequent derivatives are empty.

A set is of the *second kind (von der zweiten Gattung)* if every derivative of the set has a derivative.

A set is *everywhere dense* if every element of the set is a limit point.

A set, which is everywhere dense in an interval, is necessarily of the second kind; hence a set of the first kind is certainly *not* everywhere dense.

Additionally the following concepts are defined:

The *cardinality (Mächtigkeit)* of a set is a measure of its magnitude. Two sets have different cardinality if one is equivalent to a proper subset of the other, whereas the converse is *not* true.

A set is *denumerable* if it is equivalent to the natural number system. Well-known examples of denumerable sets are the set of rational numbers and the set of algebraic numbers, as was proved by Cantor.

An *equivalence class*[18] contains sets which are all equivalent, hence sets of different classes are by definition not equivalent; The denumerable sets form together one class and the set of all natural numbers, the set of all rational numbers, the set of all algebraic numbers or any infinite subset of any of these three sets, belong to this class, from which it follows that sets of the first kind (Gattung) and of the second kind can belong to the same class.

Another equivalence class is formed by the following sets: 1) open or closed continuous intervals, 2) combinations of these intervals, and also

> 3) Jede Punktmenge, welche aus einem stetigen Intervalle dadurch hervorgebt, daß man eine *endliche* oder *abzählbar unendliche* Mannigfaltigkeit von Punkten $\omega_1, \omega_2, ..., \omega_\nu, ...$ daraus entfernt.[19]

or any set which is equivalent to any of these three.

We will return to this third remarkable possibility when discussing the sets that Brouwer in 1907 admitted as possible.

1.1.4 Irrational numbers, the continuum, the continuum hypothesis, reducible and perfect sets

The fifth paper in this series is the most important and also the longest one. It bears the title *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* and it was also separately published in 1883 under the same title. Brouwer made use of the latter publication, as can be concluded from his quotations which include the relevant page references.

In § 9 of this paper the real irrational numbers are discussed and an overview is presented of the method of the Dedekind cuts to define these real numbers.[20]

[18] Cantor used here the term *class* ('Klasse'), but we will henceforth use the term *equivalence class* for this type of class, to prevent confusion with several other class concepts that Cantor used.

[19] op. cit. page 142.

[20] see under *Dedekind*, section 1.2.

In § 10 a discussion follows about the *continuum* concept, beginning with a short history of this concept, in which it ranges from a religious dogma to a mathematical-logical concept.

In connection with Brouwer's continuum concept at the time of taking his doctoral degree in 1907, the following quotations are of importance:

> Zunächst habe ich zu erklären, daß meiner Meinung nach die Heranziehung des *Zeitbegriffes* oder der *Zeitanschauung* bei Erörterung des viel ursprünglicheren und allgemeineren Begriffs des Kontinuums *nicht* in der Ordnung ist; die *Zeit* ist meines Erachtens eine Vorstellung, die zu ihrer deutlichen Erklärung den von ihr unabhängigen Kontinuitätsbegriff zur Voraussetzung hat und sogar mit Zuhilfenahme desselben weder objektiv als eine Substanz, noch subjektiv als eine notwendige apriorische Anschauungsform aufgefaßt werden kann, sondern nichts anderes als eine *Hilfs– und Beziehungsbegriff* ist, durch welchen die Relation zwischen verschiedenen in der Natur vorkommenden und von uns wahrgenommenen Bewegungen festgestellt wird. So etwas wie *objektive* oder *absolute Zeit* kommt in der Natur nirgends vor.[21]

Hence, according to Cantor, the continuum concept is prior to the concept of time. As for the space concept:

> Ebenso ist es meine Überzeugung, daß man mit der sogenannten *Anschauungsform* des *Raumes* gar nichts anfangen kann, um Aufschluß über das *Kontinuum* zu gewinnen, da auch der *Raum* und die in ihm gedachten Gebilde nur mit Hilfe eines begrifflich bereits *fertigen* Kontinuums denjenigen Gehalt erlangen, mit welchem sie Gegenstand nicht bloß ästhetischer Betrachtungen oder philosophischen Scharfsinnes oder ungenauer Vergleiche, sondern nüchtern-exakter mathematischer Untersuchungen werden können.[22]

The last two quotations tell us that Cantor's concept of the continuum neither depends on time, nor on space. In his own words:

> Somit bleibt mir nichts anderes übrig, als mit Hilfe der in § 9 definierten reellen Zahlbegriffe[23] einen möglichst allgemeinen rein arithmetischen Begriff eines Punktkontinuums zu versuchen.[24]

[21]op. cit. page 191.

[22]op. cit. page 192.

[23]In § 9 Cantor discussed three methods for the definition of irrational numbers, all three starting from the system of rational numbers: the Weierstrass method by means of infinite sequences with an upper (or lower) bound, the Dedekind method by means of cuts and (in Cantor's terms) a 'Weierstrass-like' method with the help of fundamental sequences with all terms, except a finite number, in pairs less than a certain, arbitrary small, positive number ε apart. Hence all three methods give a definition of irrational numbers on a purely arithmetical basis.

[24]op. cit. page 192.

Hence Cantor's continuum is the arithmetical continuum of all real numbers, defined by means of the method of the Dedekind cuts or by similar arithmetical methods.[25]

We will return to this when discussing Brouwer's completely opposite concept of the continuum and his ideas on the role of time in mathematics.[26]

The *continuum hypothesis* is expressed by Cantor in the following way:

> Ich habe in Crelles J. Bd 84, S. 242 bewiesen, daß alle Räume G_n, wie groß auch die sogenannte Dimensionenanzahl n sei, *gleiche* Mächtigkeit haben und folglich *ebenso mächtig* sind wie das Linearkontinuum, wie also etwa der Inbegriff aller reellen Zahlen des Intervalles (0....1). Es reduziert sich daher die Untersuchung und Feststellung der Mächtigkeit von G_n auf dieselbe Frage, spezialisiert auf das Intervall (0...1), und ich hoffe, sie schon bald durch einen strengen Beweis dahin beantworten zu können, daß die gesuchte Mächtigkeit keine andere ist als diejenige unserer *zweiten* Zahlenklasse (II).[27]

The first part of this quotation was discussed in section 1.1.2 and in the second part Cantor conjectured that, in his terminology, $\overline{\overline{C}} = \aleph_1$.

This results, according to Cantor, in the theorem that infinite *point* sets either have the cardinality of the first numberclass or that of the second; in a footnote at the end of the paper two more theorems are added: the set of all continuous and integrable functions has the cardinality of the second numberclass and the set of all real functions, continuous or discontinuous, has the cardinality of the third class.[28]

Point sets P, as subsets of R_n, can be divided in, again, two classes, according to the cardinalities of their first derivative $P^{(1)}$:[29]

1. P is *reducible* if $P^{(1)}$ has the cardinality of the first number class. In that case there is a number α of the first or second number class, such that $P^{(\alpha)}$ is empty ('verschwindet').

P is *perfect* if $P^{(\gamma)} = P$ for every γ.

2. If $P^{(1)}$ has the cardinality of the second number class (i.e. 'nicht abzählbar'), then it can be uniquely expressed as $P^{(1)} = R + S$, with R reducible and S perfect. (This is a slightly modified form of the Cantor-Bendixson theorem, which states the just given theorem for every point set, i.e. for every closed set on the real line, or, for every closed set with a denumerable basis).[30]

Other topological concepts like *connected point sets*[31] are introduced in this paragraph. In a footnote to this paragraph Cantor presented a surprising ex-

[25] see also 1.1.8, the linear continuum.

[26] see chapter 2 and 3 of this dissertation.

[27] op. cit. page 192.

[28] Number classes were also defined in this section, but for a more proper definition, see under section 1.1.5.

[29] Op. cit. page 193.

[30] See [Dalen, D. van, H.C. Doets, H.C.M. de Swart 1975], page 291.

[31] See under 1.1.6.

ample of a particular kind of set, now known as the *Cantor set*: a perfect point set which is in no interval everywhere dense.[32]

1.1.5 Generation principles (Erzeugungsprinzipien) and the Limitation principle (Hemmungsprinzip) for finite and transfinite numbers

In § 11 of the *Grundlagen* the generation principles for ordinals are defined. The series of natural numbers $1, 2, 3, ..., n, ...$ originates from the repeated expression, either spoken or in writing, of equal units and combining the result into a unity. A natural number n is the expression for a certain finite number of those units, as well as for the union into a totality (set) of the expressed units.

The *first generation principle* is the addition of one unit to the already constructed number:

> Es beruht somit die Bildung der endlichen ganzen realen Zahlen auf dem Prinzip der Hinzufügung einer Einheit zu einer vorhandenen schon gebildeten Zahl; ich nenne dieses Moment, welches, wie wir gleich sehen werden, auch bei der Erzeugung der höheren ganzen Zahlen eine wesentliche Rolle spielt, das *erste Erzeugungsprinzip*.[33]

The *second generation principle* guarantees the existence of a number, which follows immediately after *all* finite natural numbers. Although there is of course no largest finite natural number, Cantor stated:

> (...) hat es doch andrerseits nichts Anstössiges, sich eine *neue* Zahl, wir wollen sie ω nennen, zu denken, welche der Ausdruck dafür sein soll, daß der ganze Inbegriff (I) in seiner natürlichen Sukzession dem Gesetze nach gegeben sei.
>
> (...)
>
> Es ist sogar erlaubt, sich die neugeschaffene Zahl ω als *Grenze* zu denken, welcher die Zahlen ν zustreben, wenn darunter nichts anderes verstanden wird, daß ω die *erste* ganze Zahl sein soll, welche auf alle Zahlen ν folgt, d.h. größer zu nennen ist als jede der Zahlen ν.[34]

ω is the first number following all finite natural numbers and it is also the first number *not* belonging to number class I. After having obtained ω by means of the second principle we can again employ the first principle to construct the continuation of the number series $\omega + 1, \omega + 2, ..., \omega + \nu, ...$ until, again, with the help of the second principle, we reach $\omega + \omega = \omega 2$ (in modern notation; Cantor himself wrote it as 2ω).

[32]See page 95 for a more detailed description.
[33]op. cit. page 195.
[34]op. cit. page 195.

> Die logische Funktion, welche uns die beiden Zahlen ω und 2ω geliefert hat, ist offenbar verschieden von dem *ersten* Erzeugungsprinzip, ich nenne sie das *zweite Erzeugungsprinzip.*[35]

Then again with the first principle and alternating with the second $\omega 2 + 1, \omega 2 + 2, ..., \omega 2 + \nu, ...$ up to $\omega\omega = \omega^2$.

Continuing in this way, using both principles in turn, one gets $\omega^2\lambda + \omega\mu + \nu$, with λ, μ and ν natural numbers, and, as a general form, the *normal form* of the numbers of the second number class:

$$\omega^{\mu}\nu_0 + \omega^{\mu-1}\nu_1 + ... + \omega\nu_{\mu-1} + \nu_{\mu}$$

with ν_i a natural number for all i. And ultimately, with the second principle, we reach ω^{ω}; but then we still can continue, by means of the first two principles only, with $\omega^{\omega} + 1$ and so on.

Restriction to those two principles gives us the impression of an unlimited totality without any chance of some form of completion. For that reason Cantor introduced a third principle, the *limitation principle* (Hemmungs– oder Beschränkungsprinzip), which enabled him to define the second number class as a completed totality:

> Bemerken wir nun aber, daß alle bisher erhaltenen Zahlen und die zunächst auf sie folgenden eine gewisse Bedingung erfüllen, so erweist sich diese Bedingung, *wenn sie als Forderung an alle zunächst zu bildenden Zahlen gestellt wird*, als ein neues, zu jenen beiden hinzutretendes *drittes* Prinzip, welches von mir *Hemmungs– oder Beschränkungsprinzip* genannt wird und das, wie ich zeigen werde, bewirkt, daß die mit seiner Hinzuziehung definierte zweite Zahlenklasse (II) nicht nur eine höhere Mächtigkeit erhält als (I), sondern sogar genau die *nächst höhere*, also *zweite Mächtigkeit.*
>
> (...)
>
> Wir definieren daher die zweite Zahlenklasse (II) als *den Inbegriff aller mit Hilfe der beiden Erzeugungsprinzipe bildbaren, in bestimmter Sukzession fortschreitenden Zahlen* α
>
> $$\omega, \omega + 1, ..., \nu_0\omega^{\mu} + \nu_1\omega^{\mu-1} + ... + \nu_{\mu-1}\omega + \nu_{\mu}, ..., \omega^{\omega}, ..., \alpha ...,$$
>
> *welche der Bedingung unterworfen sind, daß alle der Zahl* α *voraufgehenden Zahlen, von 1 an, eine Menge von der Mächtigkeit der Zahlenklasse (I) bilden.*[36]

All transfinite numbers of the second number class are equivalent to the set of all natural numbers, hence they have the same cardinality as ω and are denumerably infinite, but in § 12 it is proved that the second number class as a totality has a cardinality which exceeds that of the natural number system. In § 13 Cantor proved that the second number class is, qua cardinality, the direct successor of the first number class.

[35] op. cit. page 196.
[36] op. cit. page 197.

1.1.6 The Cantor-Bernstein theorem and the definition of further new concepts

The same § 13 of this fifth paper (the *Grundlagen ...*) continues with just mentioning a special case of an important and well known theorem, which is now known as the *Cantor-Bernstein theorem*; it is presented in the following formulation:

> Hat man irgendeine wohldefinierte Menge M von der zweiten Mächtigkeit, eine Teilmenge M' von M und eine Teilmenge M'' von M' und weiß man, daß die letztere M'' gegenseitig eindeutig abbildbar ist auf die erste M, so ist immer auch die zweite M' gegenseitig eindeutig abbildbar auf die erste und daher auf die dritte.[37]

A proof is not given; see for a further discussion the sections on Dedekind[38] and on Bernstein.[39]

In the fourteenth section of Cantor's work, in which a host of new ideas is introduced, a number of theorems for transfinite numbers is given and proved, such as the theorem that the distributive law is only valid in the form $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$, in which $(\alpha + \beta)$, α and β are the multiplicator; it is also shown that $\alpha + \xi = \beta$ always has a unique solution for ξ, whereas $\xi + \alpha = \beta$ generally has no solution.

A similar theorem applies for $\beta = \xi\alpha$; in this form it has a unique solution, whereas $\beta = \alpha\xi$ generally has not.

The sixth and last paper of this series may be seen as marking the beginning of the Cantor-topology. It is a direct continuation of the previous paper, (the *Grundlagen*); it appeared one year later in the *Mathematisch Annalen* and was not included in the already published book version.

In this paper a *perfect set* is defined as a set which coincides with its first derivative (§ 16), so $P^{(1)} = P$. This condition differs from the one for a closed set:

A set is *closed* (abgeschlossen) if it contains its first derivative as a subset, i.e. $P^{(1)} \subseteq P$ or $P^{(1)} \cap P = P^{(1)}$.

A set is *dense in itself* (in sich dicht) if $P \subseteq P^{(1)}$ or $P^{(1)} \cap P = P$.[40]

Note that Cantor also gave a definition of a perfect set in § 10 in a different formulation but with the same content.[41] Also in section 1.1.10 definitions are

[37] op. cit. page 201.

[38] see under 1.2.

[39] see under 1.6.

[40] In Cantor's notation: a set is closed if $D(P, P^{(1)}) = P^{(1)}$ and a set is dense in itself if $D(P, P^{(1)}) = P$; for the union of two sets A and B he used the notation $M(A, B)$. The D and the M were written in Gothic typeface. Brouwer used a Gothic D and S for intersection and union ([Brouwer 1918 B], section 1); we will use the modern notation of $\cap$ and $\cup$ respectively.

Note that there is a difference between the concepts *everywhere dense* and *dense in itself*; see for an example the Cantor set, page 95.

[41] see under section 1.1.4.

given, again in different wordings, of the concepts *perfect set, closed set* and *dense in itself*, which definitions can, however, again be proved to be equivalent to the ones given above.

A *continuum in the proper sense* (Kontinuum in eigentlichen Sinne, §19) is a set which is perfect and *connected* (in sich zusammenhängend), in which 'connected' (zusammenhängend) means, according to §19 of the *Grundlagen*, that for every two elements t and t' of the set, and for every $\varepsilon > 0$ there exist a finite number of elements $t_1 = t, t_2, ..., t_n = t'$ of the set such that for every i $|t_i - t_{i+1}| < \varepsilon$.[42]

In this same § 19 the theorem, which claims that all linear perfect point sets are equivalent, is proved.[43] Cantor also noticed that the perfect set has the cardinality of the continuum, and he was able to define this set with the help of a formula for its individual elements (see 1.1.8).

There certainly was opposition against Cantor's new ideas. In the paper *Über die verschiedene Standpunkte in Bezug auf das aktuelle Unendliche* (1885) Cantor discussed the possibility of the existence of the actual infinite, and the paper *Mitteilungen zur Lehre vom Tranfiniten* is composed of a number of letters, written to mathematicians, philosophers and theologians, in which Cantor defended his standpoint.

1.1.7 Beitrage zur Begründung der transfiniten Mengenlehre

During the years 1895 and 1897, the second of the two important series of papers was published in the *Mathematische Annalen*, under the title of this section.[44] In this series the results of twenty five years of development of the Cantorian set theory are presented in a concise and precise way, this time with the emphasis on ordered and well-ordered sets, resulting in the possibility to discuss the continuum of the real numbers in a mathematically rigorous manner.

The remainder of the discussion of Cantor's work will be concerned with this second series. Several topics which are treated in this series, came up already in earlier sections and will therefore either be just mentioned or completely omitted.

The first paper begins with the earlier quoted definition of a set and of its cardinality; the latter is introduced as the result of double abstraction.[45]

Ordering of cardinal numbers in *less than* and *greater than* is defined in the usual way, just like *addition* and *multiplication*. The sum (product) of two

[42] Note that, if 'connected' is the proper translation for 'zusammenhängend', this differs from the modern topological concept of 'connectedness': a set S is connected if it cannot be split into two open non-empty subsets A and B, such that $A \cap B = \emptyset$ and $A \cup B = S$. According to Cantor's definition the rational scale is connected, according to the modern definition it is not. In non-standard analysis the two definitions can be made to be equivalent again.

[43] Compare this to the theorem in section 1.1.9.

[44] [Cantor 1897], also in [Cantor 1932], page 282 ff.

[45] See the beginning of section 1.1.

cardinal numbers is the cardinality of the sum (product) of two disjoint sets with the two given cardinal numbers.

Also *exponentiation* of cardinal numbers is specified (in § 4) by means of the concept of assignment. An *assignment* (Belegung)[46] of the set N with elements of the set M (or simply: *the assignment of* N *with* M) is a law (ein Gesetz), according to which, with *every* element n of N, a specific element $m = f(n)$ of M is associated. Repeated use of elements of M is permitted with this operation. The set of all assignments of N with M is called the *assignment set of* N *with* M, in symbols as M^N (in Cantorian notation $(N|M)$).

If now a and b are two cardinal numbers and M and N two representing sets for a and b (that is, that M is of cardinality a and N of cardinality b), then Cantor defined:

$$a^b = \overline{\overline{M^N}}$$

This is in accordance with the traditional definition of exponentiation, and the usual laws apply.

If N is an arbitrary set with cardinality b and M is the set $(0, 1)$, representing the cardinal number 2, then the *power set* of N is defined as the set of all subset of N, and can be shown to have cardinality $\overline{\overline{M^N}} = 2^b$.

A well-known theorem states that the cardinality of the power set of a set is greater than the cardinality of the original set, i.e. there exists no one-to-one mapping of the original set onto its power set. This results in the absence of a greatest cardinal number.

1.1.8 The linear continuum

In § 4 of the *Begründung* papers the linear continuum X is defined as the totality (Inbegriff) of all real numbers x on the closed interval $[0, 1]$[47] and its cardinality is written as c. If the continuum is expressed in a dual number system (only $0's$ and $1's$ in the expansion), then an arbitrary real number x on that interval can be written as the expansion:

$$x = \frac{f(1)}{2} + \frac{f(2)}{2^2} + ... + \frac{f(n)}{2^n} + ... \text{ with } f(n) = 0 \text{ or } 1,$$

Because the numerator $f(n)$ can only have the values 0 or 1, the series for x is a Cauchy sequence[48] with a denumerable ($\aleph_0$) number of terms and with the only possible values for those terms: 0 or $\frac{1}{2^n}$, that is two possible values at each dual place. Hence the total number of possible values for x is $2^{\aleph_0}$, which is then the cardinality of the set $\{x | x = \frac{f(1)}{2} + \frac{f(2)}{2^2} + ... + \frac{f(n)}{2^n} + ... \text{ with } f(n) = 0 \text{ or } 1\}$, which set is the perfect set.

Since the power of the continuum does not change when rewriting its elements in the decimal number system, the conclusion follows that the cardinality

[46] Philip Jourdain translates this as 'covering', which can be misleading.

[47] Compare this with the definition of the continuum, that Cantor used in 1.1.4, where the continuum hypothesis was discussed.

[48] A Cauchy sequence is a sequence $\{t_n\}$ with the property that for every ε there exists a n such that $|t_{n+p} - t_n| < \varepsilon$ for every p.

of the continuum, in the Cantorian meaning[49] of the totality of all real numbers on the interval $[0,1]$, is $c = 2^{\aleph_0}$.

In § 5 the finite cardinal numbers (the natural numbers) are defined, as well as the rules for addition and multiplication for the finite numbers in combination with the smallest infinite cardinal number $\aleph_0$, that is the cardinality of each number of the second number class.[50]

1.1.9 Order types

A definition of a *simply ordered* set is, in the formulation of Cantor:

> Eine Menge M nennen wir *einfach geordnet,* wenn unter ihren Elementen m eine bestimmte *Rangordnung* herrscht, in welcher von je zwei beliebigen Elementen m_1 und m_2 das eine den *niedrigeren,* das andere den *höheren* Rang einnimmt, und zwar so, dass wenn von drei Elementen m_1, m_2 und m_3 etwa m_1 dem Range nach niedriger ist als m_2, dieses niedriger als m_3, alsdann auch immer m_1 niedrigeren Rang hat als m_3.[51]

Note that the ordering of a set can be obtained in different ways.[52] Given a simply ordered set, the concept of *order type* (Ordungstypus) can be defined, by a single abstraction from the nature of the objects:

> Jeder geordneten Menge M kommt ein bestimmter *Ordnungstypus* oder kürzer ein bestimmter *Typus* zu (...); hierunter verstehen wir *den Allgemeinbegriff, welcher sich aus M ergiebt, wenn wir nur von der Beschaffenheit der Elemente m abstrahieren, die Rangordnung unter ihnen aber beibehalten.*[53]

Two simply ordered sets are of the same *order type*, if they can be mapped ono-to-one onto each other under preservation of the order.[54] In this case the sets are called *similar* (ähnlich).

If an order type is written as α, then $\overline{\alpha}$ is the corresponding cardinal number. All order types with the same cardinal number $\overline{\alpha}$ build up together the *typeclass* $[\overline{\alpha}]$. Members of a typeclass are not necessarily similar, but they are equivalent.

Rules for addition and multiplication for order types are given in § 8. For addition and multiplication associativity applies but commutativity generally not and distributivity only for the form $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$ with $(\beta+\gamma)$ the

[49] This is not Brouwer's idea of the continuum, as we will see in chapter 3.
[50] see section 1.1.5.
[51] [Cantor 1932], page 296.
[52] For instance the ordering of the rationals in their natural order or in one of the systematic orders with which their denumerability can be proved.
[53] op. cit. page 497.
[54] This is not yet the ordinal number, see under 1.1.11.

multiplicator.[55] Also the following applies: $\omega = 1 + \omega \neq \omega + 1$, where ω is the order type of the natural numbers.

Finally, it is proved in this paragraph that the cardinal number of the sum (product) of two order types equals the sum (product) of the cardinal numbers of the two order types.

Standard symbols for the most well-known order types are: ω for the order type of the natural numbers, η for that of the rationals and ϑ for that of the reals.

In § 9 a theorem of fundamental importance is stated and proved:

> Hat man eine einfach geordnete Menge M, welche die drei Bedingungen erfüllt:
>
> 1) $\overline{\overline{M}} = \aleph_0$.
>
> 2) M hat kein dem Range nach niedrigstes und kein höchstes Element.
>
> 3) M ist überall dicht,
>
> so ist der Ordnungstypus von M gleich η.[56]

Thus all densely ordered and denumerable sets without endpoints are isomorphic (similar) and have the order type η of the rational numbers.

As an example of Cantor's proof-method we give a sketch of the proof of the last theorem, which comes down to the following 'back and forth' procedure: because of condition 1) M can be written as a well-ordered set[57] $M_0 = (m_0, m_1, ..., m_n, ...)$ of order type ω. Also R, the system of rational numbers, can be written as a well-ordered set $R_0 = (r_0, r_1, ..., r_n, ...)$ of the same order type. We now map r_0 on m_0. Consider m_1, this is situated in M in a certain order relation compared to m_0, say $m_1 > m_0$. Because of condition 2) and 3) there are infinitely many elements r_i in R with the same relation $r_i > r_0$. Pick the r_i with the smallest value for i, say r_{i_1} and map m_1 on r_{i_1}. Now take r_1 and suppose $r_1 \neq r_{i_1}$. This is situated in R in a certain order relation compared to r_0 and r_{i_1}. Select in M the m_i with the lowest value for i and with the same order relation compared to m_0 and m_1, say m_{i_1} and map r_1 on m_{i_1}.

In general, at step $2n$, pick r_n; if r_n has already been treated, go to the next step; else find the first m_i, such that m_i is in the same position with respect to $m_0, m_1, ..., m_{n-1}$ as r_n with respect to $r_0, r_1, ...r_{n-1}$ and map r_n on this m_i.

Next, at step $2n+1$, pick m_n; if m_n has already been treated, go to the next step; else locate m_n with respect to the already treated elements of M and find the first untreated element r_i in the same relative position.

Continuing in this way, all elements of R are mapped on elements of M under preservation of order, and by means of a simple inductive argument it can be shown that every element of M is a mapping of exactly one element of R.

[55] Compare this to the distribution laws for cardinal numbers in section 1.1.6.
[56] op. cit. page 304.
[57] for well-ordering, see under 1.1.12.

Hence the defined relation is a one-to-one relation preserving the order, hence a similarity relation.

1.1.10 Fundamental sequences and perfect sets

A *fundamental sequence of the first order* is a subset M' of a simply ordered transfinite set M (M' therefore is also simply ordered) with order type ω or $^*\omega$ (the reverse ordering of the natural numbers), which can be represented as $a_0, a_1, a_2, ...a_n, ...$ (§ 10).[58]

In the case of an order type ω the sequence is called *increasing*, in the case of order type $^*\omega$ the sequence is called *decreasing*.

For an increasing sequence a *limit* (Grenzelement) of a fundamental sequence $\{a_n\}$ is an element m_0 such that $a_n < m_0$ for every n and for every $m < m_0$ a p can be given such that $a_q > m$ for every $q > p$. In that case m_0 is limit point (Hauptelement) of the set M in which the fundamental sequence is defined. Also m_0 is the least upperbound of the given sequence.

For the case of a decreasing sequence a similar definition is given.

Cantor introduced in this paragraph the definitions of a set which is *dense in itself*, of a *closed set* and of a *perfect set*, all in terms of fundamental sequences defined in the sets concerned, and of limit points of the fundamental sequences which are elements of the sets. These definitions can be proved to be equivalent to the definitions given in 1.1.6, and will not be presented here.

1.1.11 Well-ordering

In the second part of this *Begründung* series (from § 12 to the end), published in 1897, the *well-ordered sets* are discussed; these sets can be defined as the simply ordered sets in which every non-empty subset has a first element.[59] Cantor conjectured, as he did before in the series *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, that every set can be well ordered.

A *segment* (Abschnitt) of a well-ordered set is the set of all elements preceding a given element, distinct from the first element (hence a segment is a finite subset, which is never empty). A great number of theorems on (segments of) well-ordered sets is deduced in § 13.

An *ordinal number* is the order type of a well-ordered set, hence ω is an ordinal number and η and ϑ are not. It follows that the ordinal numbers are the finite natural numbers, closed by ω and then followed by the numbers of the second number class, or, in other words, the ordinal numbers are all numbers, generated by the the first two of Cantor's three generation principles (see 1.1.5).

Rules and theorems for ordinal numbers are deduced in § 14.

[58]Compare with Brouwer's notion of 'fundamental sequence' on page 92 in the footnote.

[59]This is the usual definition of a well-ordered set, which Schoenflies already used in 1900 in the Jahresbericht der D.M.V. (see 1.5). Cantor's definition is different and the definition given here is Cantor's theorem A of § 12.

The *second number class* is defined as the totality $\{\alpha|\alpha$ is an order type of a well-ordered set of cardinality $\aleph_0\}$ (§ 15).[60] This class has as first and smallest element $\omega = lim_n n$; it is constructed with the help of the two generation principles, it is completed by the third (the Hemmungsprinzip), and is of cardinality $\aleph_1$.[61]

Theorem A of § 14 says:

> Sind α und β zwei beliebige Ordnungszahlen, so ist entweder $\alpha = \beta$, oder $\alpha < \beta$, oder $\alpha > \beta$.

Hence two arbitrary ordinal numbers are always comparable. This, in combination with the conjecture that every set can be well-ordered which means that every set can be assigned an ordinal number, results in the fact that every set can be assigned a cardinal number, including the continuum. So the continuum can be assigned an $\aleph_i$ for some i. The questionable part of this argument is of course the, for Cantor still unproved, well-ordering theorem.

The conclusion that any two cardinal numbers are comparable will eventually turn out to be equivalent to the *axiom of choice*, on which the proof of the well-ordering theorem depends.

One final remark before closing the section on Cantor: In his letter of 28 July 1899 to Dedekind[62] he mentioned the possibility that set theory can give rise to paradoxes:

> Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines 'Zusammenseins' *aller* ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als 'ein fertiges Ding' aufzufassen. Solche Vielheiten nenne ich *absolut unendliche* oder *inkonsistente Vielheiten.*
>
> Wie man sich leicht überzeugt, ist z.B. der 'Inbegriff alles Denkbaren' eine solche Vielheit; (...)
>
> Wenn hingegen die Gesamtheit der Elemente einer Vielheit ohne Widerspruch als 'zusammenseiend' gedacht werden kann, so daß ihr Zusammengefaßtwerden zu '*einem* Ding' möglich ist, nenne ich sie eine *konsistente Vielheit* oder eine 'Menge'.
>
> (...)
>
> Zwei äquivalente Vielheiten sind entweder beide 'Mengen' oder beide inkonsistent.

[60] In Cantor's definition the elements of the first number class, i.e. the natural numbers, are not included in the second number class; Cantor's definition of classes is not cumulative, contrary to, for instance, Hilbert's definition. See for this Hilbert's *Über das Unendliche*, e.g. in [Heijenoort 1967], page 375 and 386.

[61] Cantor touched upon the concept of number classes in the first section of his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre,* but the definition given above is from the viewpoint of his further development the proper one.

[62] Not included in [Cantor 1937], but in [Cantor 1932], page 443.

We will come back to the paradoxes at the end of this first chapter.

1.2 Dedekind (1831–1916)

Although Dedekind was senior to Cantor by more than thirteen years, it was the latter who started the development of set theory. Dedekind published his *Stetigkeit und irrationale Zahlen*[63] in 1872. In this essay the real numbers are defined by means of a partitioning of the rational numbers into two classes, where *every* partitioning defines a real number, either rational or irrational. (Such a partitioning of the rationals is called a *Dedekind cut*, see below.) But there is as yet hardly any set theoretic aspect in this short essay. This aspect can, however, be recognized in his *Was sind und was sollen die Zahlen*, published in 1888. Hence it is justified to say that Cantor was the pioneer in building the new set theory.

However, the influence of Dedekind on Cantor was great, as can be concluded from the correspondence between the two. Dedekind was a valuable critic for him, several corrections and suggestions from the side of Dedekind can be identified in the papers that Cantor published.

In his 1888 paper *Was sind und was sollen die Zahlen*[64] Dedekind used the terms *System* and *Ding* for set and element of a set respectively. A *System* itself is also a *Ding*.[65]

> Ein solches System S (oder ein Inbegriff, eine Mannigfaltigkeit, eine Gesamtheit) ist als Gegenstand unseres Denkens ebenfalls ein Ding.[66]

With his own characteristic symbols and terminology Dedekind defined subsets, in which the empty set was explicitly excluded, and conjunctions (unions) and disjunctions (intersections)[67] of sets as well as a mapping of one set onto or into another. For one-to-one mappings the concept of *equivalence class* was introduced.

> Erklärung. Die Systeme R, S heißen *ähnlich,* wenn es eine derartige ähnliche Abbildung φ von S gibt, daß $\varphi(S) = R$, also auch $\overline{\varphi}(R) = S$.[68]
>
> (...)

[63] [Dedekind 1912].
[64] [Dedekind 1930b].
[65] Here we will use the modern terms *set* and *element*.
[66] op. cit. § 1, item 2.
[67] Dedekind used the notation of a Gothic M and S for the union and intersection of two sets, whereas Cantor used here a Gothic M and D, Schoenflies the notation (M, N) for the union of the sets M and N, and Gothic D for intersection and Brouwer a Gothic S and D respectively. These rather confusing notations were later replaced by the uniform notation of $\cup$ and $\cap$ for union and intersection. See also the footnote on page 12.
[68] op. cit. § 3, item 32.

> Erklärung. Mann kann daher alle Systeme in Klassen einteilen, indem man in eine bestimmte Klasse alle und nur die Systeme Q, R, S, ... aufnimmt, welche einem bestimmten System R, dem Repräsentanten der Klasse, ähnlich sind.[69]

In § 4 Dedekind introduced the key notion 'chain' (Kette), which is determined by a set and a mapping φ of the set into itself. If K is a subset of a set S, then K is a *chain* ('Kette') relative to a mappig φ if $\varphi(K)$ is a subset of K. Then $\varphi(K)$ is also a chain and the conjunction and disjunction of chains form a chain.

By means of the concept of *chain* the system $\mathbb{N}$ of the natural numbers is defined (see below).

The theorem of the principle of *complete induction* for chains is proved as 'Satz 59' in the following form:

> Satz der vollständigen Induktion. Um zu beweisen, daß die Kette $\varphi_0(A)$ Teil irgendeines Systems Σ ist – mag letzteres Teil von S sein oder nicht –, genügt es zu zeigen,
>
> ρ. daß $A \subset \Sigma$, und
>
> σ. daß das Bild jedes gemeinsamen Elementes von $\varphi_0(A)$ und Σ ebenfalls Element von Σ ist.[70]

and this is followed by the remark:

> Der vorstehende Satz bildet, wie sich später zeigen wird, die wissenschaftliche Grundlage für die unter dem Namen der vollständigen Induktion (des Schlusses vom n auf $n+1$) bekannte Beweisart, (...)[71]

Dedekind was the first mathematician who gave a purely set theoretical definition of *infinite* (which definition was contested by Brouwer):

> Erklärung. Ein System S heißt *unendlich*, wenn es einem echten Teile seiner selbst ähnlich ist; im entgegengesetzten Falle heißt S ein *endliches* System.[72]

This notion is nowadays called *Dedekind infinite*. On the basis of this definition, Dedekind proved the existence of infinite systems:

> Es gibt unendliche Systeme.[73]

[69]op. cit. § 3, item 34.

[70]op. cit. § 4, item 59. Note that this is for Dedekind a *theorem*, which has to be proved; for Poincaré this 'raisonnement par récurrence' is the 'raisonnement mathématique par excellence'; for Brouwer the principle of complete induction is a natural act of mathematical construction, not in need of any further justification (see thesis II of his list of theses).

[71]op. cit § 4 item 60.

[72]op. cit. § 5, item 64.

[73]op. cit, § 5, Satz 66.

Schematically the proof proceeds via his *realm of thoughts* S; thinking of a thought or the recognition of a thought as such is also a thought, which can be regarded as an image of that thought. We create in this way an image $\varphi(S)$ of my realm of thoughts S. The fact that there are thoughts which are not images of other thoughts shows that $\varphi(S)$ is a proper subset of S. Together with the similarity of this mapping (different thoughts give different thoughts of thoughts) proves the infinity of the set of my thoughts. Note that Dedekind proved here the existence of *denumerable* infinities only.

We notice that in the first notebook Brouwer judged negatively about the method employed by Dedekind to prove the existence of infinite systems.[74]

Dedekind was now in the position to define the system of the natural numbers:

> Erklärung. Ein System N heißt *einfach unendlich*, wenn es eine solche ähnliche Abbildung φ von N in sich selbst gibt, daß N als Kette eines Elementes erscheint, welches nicht in $\varphi(N)$ enthalten ist. Wir nennen dies Element, das wir im folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und sagen zugleich, das einfach unendliche System N sei durch diese Abbildung φ geordnet.[75]

An improved version (up to isomorphism) of this definition for the system of natural numbers is: N is the smallest system with a mapping φ of the system into itself, such that there is exactly one element, which is not in the range of φ.

Dedekind then showed that every infinite set contains a *simply infinite system* (einfach unendliches System) and stated that if we abstract from the nature of the elements of a simply infinite set, we call these elements the *natural numbers* or *ordinal numbers* or just *numbers*. The *basic element* (Grundelement) 1 is called the *base of the sequence* N and he added:

> (...) kann man die Zahlen mit Recht eine freie Schöpfung des menschlichen Geistes nennen.

and he then proved:

> N ist die einzige Zahlenkette, in welcher die Grundzahl 1 enthalten ist.[76]

He concluded this paragraph with a proof of the theorem of complete induction, which he indeed stated, on the basis of his definition of the system of natural numbers, as a theorem, which, in fact, is a direct corollary of theorem 59:

---

[74] The reader may verify this in I – 25, 30 and 38. A transcription of the notebooks will be published later.

[75] op. cit, §6, item 71.

[76] op. cit, §6, item 79.

> Satz der vollständigen Induktion. (Schluß von n auf n')
>
> Um zu beweisen, daß ein Satz für alle Zahlen n einer Kette m_0 gilt, genügt es zu zeigen.
>
> ρ. daß er für $n = m$ gilt, und
>
> σ daß aus der Gültigkeit des Satzes für eine Zahl n der Kette m_0 stets seine Gültigkeit auch für die folgende Zahl n' folgt.[77]

We will contrast this with Brouwer's completely different views on complete induction in several of the following chapters.

After having defined the system of the natural numbers, Dedekind went on with the deduction of its properties, viz. the laws of inequalities, of addition and of multiplication. Most proofs were carried out with the help of the theorem of complete induction.

In the first paragraph of the preface to the first edition of *Was sind ...* Dedekind wrote:

> Indem ich die Arithmetik (Algebra, Analysis) nur einen Teil der Logik nenne, spreche ich schon aus, daß ich den Zahlbegriff für gänzlich unabhängig von den Vorstellungen oder Anschauungen des Raumes und der Zeit, daß ich ihn vielmehr für einen unmittelbaren Ausfluß der reinen Denkgesetze halte. Meine Hauptantwort auf die im Titel dieser Schrift gestellte Frage lautet: die Zahlen sind freie Schöpfungen des menschlichen Geistes, sie dienen als ein Mittel, um die Verschiedenheit der Dinge leichter und schärfer aufzufassen.[78]

For Dedekind the number system did not depend on any form of time intuition or space intuition, contrary to Brouwer's concept of the natural number system, as we will see. Compare this also with Cantor's view on page 8.

Dedekind also played a crucial role in the theory of the continuum in his earlier mentioned *Stetigkeit und irrationale Zahlen.* When the system of rationals is represented in its natural order on a line with a freely chosen point zero, we can define *right* and *left* with respect to any given point and thus define a Left and a Right class. Every *rational number* on that line belongs to exactly one of the classes, but there remain infinitely many points on that line which are not representable by a rational number (e.g. the point that represents the number whose square is 2). If, however, we do require that every partitioning into a right and left class defines a number, then the system of numbers apparently has to be extended. Dedekind performed this with the *cut*: every partitioning of the rationals into a Right and Left class represents a real number, and every

[77] op. cit, item 80.
[78] [Dedekind 1930b], page V. See also chapter 8, page 313 of this dissertation for a further comment on this quote.

cut which does not represent a rational number, defines an *irrational number*. The result is that the continuity of the line is mirrored in the continuity of the set of the real numbers. Dedekind proved that the result really is a continuum and that there remain no gaps.

1.3 Poincaré (1854–1912)

A short discussion about some aspects of Poincaré's work is useful, mainly in view of his concept of the continuum on the straight line in relation to the set of the real numbers. Also his criterium for the existence of mathematical objects and his view on the nature of complete induction are of interest.

Of course, his main interest in regard to Brouwer lies in his view on the role of the intuition in the development mathematics. This will be discussed separately in the chapters to follow.

In *La grandeur mathématique et l'expérience*, the second chapter of his book *La Science et l'Hypothèse* (1902),[79] he claimed that the continuum can only be studied via analysis and not via a line or a plane in geometry, since in the latter the line serves merely as an aid and illustration. Hence Poincaré agreed with the Cantorian method of an arithmetical definition of the continuum. In analysis one starts from the scale of natural numbers and subsequently constructs, by means of repeated division, the system of *rational* or *commensurable* numbers. But in this proces one never reaches the *irrational* or *incommensurable* numbers.

> Le continu ainsi conçu n'est plus qu'une collection d'individus rangés dans un certain ordre, en nombre infini, il est vrai, mais extérieurs les uns aux autres. Ce n'est pas là la conception ordinaire, où l'on suppose entre les éléments du continu une sorte de lien intime qui en fait un tout, où le point ne préexiste pas à la ligne, mais la ligne au point.[80]

And on the same page 30:

> Mais c'est assez pour nous avertir que le véritable continu mathématique est tout autre chose que celui des physiciens et celui des métaphysiciens.

To construct the irrational numbers, Poincaré applied the earlier mentioned method of Dedekind cuts. In order to understand that the result of this method can be interpreted as a number, it is stated on page 32:

> Les mathématiciens n'étudient pas des objets, mais des relations entre les objets; il leur est donc indifférent de remplacer ces objets par d'autres, pourvu que les relations ne changent pas. La matière ne leur importe pas, la forme seule les intéresse.

[79] [Poincaré 1916], page 29.
[80] op. cit. page 30.

But this definition of irrational numbers is, in itself, not sufficient for Poincaré *to accept them as existing numbers.* He asked for a good and sufficient reason *why* to ascribe to them a definite existence; in fact the same question can already be asked in the case of fractions. Can we have a notion of those numbers without, beforehand, accepting the existence of an infinitely divisible matter? For if the infinite divisibility was a necessary condition, then the concept of *mathematical continuum* would result from experience. And experience teaches us the paradoxical possibility $A = B,\ B = C,\ A < C$. That paradox remains in spite of an ever increasing refinement in measuring technique, albeit on a different scale. We ourselves have to invent and construct the mathematical continuum, in which always applies $A = B, B = C \Rightarrow A = C$.

1.3.1 The first stage in the construction of the mathematical continuum

The physical idea of a limit on the distinguishability of two points must be abandoned and an infinite divisibility and countability has to be accepted. As a result of that point of departure, paradoxes like 'the whole is similar to a part thereof' and 'the number of rationals equals the number of natural numbers' cease to be paradoxes. Poincaré defined the *mathematical continuum of the first order* as:

> tout ensemble de termes formés d'après la même loi que l'échelle des nombres commensurables.[81]

and next:

> Si nous y intercalons ensuite des échelons nouveaux d'après la loi de formation des nombres incommensurables, nous obtiendrons ce que nous appellerons un continu du deuxième ordre.

Hence the rationals form the first order continuum and the reals the second order continuum, and both are constructed in this first stage.

1.3.2 The second stage

After the construction of the continuum of the first and second order, one has to understand *why* this step to the second order continuum has to be taken. The formulation of the answer to this question takes place in the second stage, hence this stage does *not* relate to some form of construction. Briefly, the answer amounts to the following:

Intuitively we know that two lines, which we assume without width, intersect in a point without measure, but that, if we only accept a continuum of the first order, this would lead to a contradiction in case of the intersection of a line with a circle. This impending contradiction shows us the necessity of a second order continuum.

[81]op. cit. page 38.

The summary in Poincaré's text, which follows on the section about this second stage, shows us the dilemma: on the one hand the continuum is intuitive, because of the intuitive existence of the intersection of two lines. On the other hand:

> (...) l'esprit a la faculté de créer des symboles, et c'est ainsi qu'il a construit le continu mathématique, qui n'est qu'un système particulier de symboles. Sa puissance n'est limitée que par la nécessité d'éviter toute contradiction; mais l'esprit n'en use que si l'expérience lui en fournit une raison.[82]

Hence, on the other hand the mathematical continuum is a construction.

1.3.3 The measurable quantity

> Les grandeurs que nous avons étudiées jusqu'ici ne sont pas *mesurables;* nous savons bien dire si telle de ces grandeurs est plus grande que telle autre, mais non si elle est deux fois ou trois fois plus grande.[83]

We can compare quantities in the sense that we can tell that one is larger or smaller than the other, but not how much or how many times larger or smaller. For that purpose we have to select a unit length and a convention of measuring out that unit length along a length of which we want to know the quantified measure, thus creating a 'measurable continuum'.

1.3.4 A continuum of the third order?

According to Poincaré the creative power of man is not exhausted with the construction of the mathematical continuum of the second order; there is the possibility of a continuum of the third order. However, he did not elaborate this idea, he merely saw it as an intellectual game without any practical meaning or any chance of a mathematical application. He was motivated in this thought experiment by Du Bois-Reymond and he seemed to accept the existence of 'infinitesimals'.

> Nous pouvons nous poser plusieurs questions importantes:
>
> 1^0 La puissance créatrice de l'esprit est-elle épuisée par la création du continu mathématique?
>
> Non, les traveaux de Du Bois-Reymond le démontrent d'une manière frappante.

Poincaré sketched the possible existence of infinitely small quantities of different order as a result of man's continued creative power.

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[82] op. cit. page 40.
[83] op. cit. page 41.

> Il serait aisé d'aller plus loin, mais ce serait un vain jeu de l'esprit; on n'imaginerait que des symboles sans application possible, et personne ne s'en avisera. Le continu du troisième ordre auquel conduit la considération des divers ordres d'infiniment petits est lui-même trop peu utile pour avoir conquis droit de cité, et les géomètres ne le regardent que comme une simple curiosité. L'esprit n'use de sa faculté créatrice que quand l'expérience lui en impose la nécessité.[84]

This may remind the reader of the development of the non-standard analysis by Robinson, who, however, found applications for it in pure analysis and outside the realm of just curiosity.[85]

This second chapter of *La Science et l'Hypothèse* is concluded with the observation that, despite the construction of the continuum, there still are apparent paradoxes, like 'the curve without tangent', thereby referring to the work of Weierstrass and others.

The final section of this chapter concerns the physical and mathematical continuum of more than one dimensions, which will not be discussed here.

1.3.5 Mathematical existence and complete induction

Poincaré's criterium for the *existence* of a mathematical object is simply 'being free of contradiction':

> En mathématiques le mot exister ne peut avoir qu'un sens, il signifie exempt de contradiction.[86]

This is completely different from Brouwer's criterium for existence, as we will see. But Brouwer and Poincaré agree about the status of the principle of mathematical induction. For both it is a most natural and fundamental method and tool for the development of mathematics. About formal logic, Poincaré remarked:

> Si, au contraire, toutes les propositions qu'elle énonce peuvent se tirer les unes des autres par les règles de la logique formelle, comment la mathématique ne se reduit-elle pas à une immense tautologie? Le syllogisme ne peut rien nous apprendre d'essentiellement nouveau et, si tout devait sortir du principe d'identité, tout devrait aussi pouvoir s'y ramener.[87]

But about mathematical induction he noted:

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[84]op. cit. page 42, 43.
[85]See [Robinson 1966].
[86][Poincaré 1920], page 162.
[87][Poincaré 1916], page 9 and 10.

> L'induction mathématique, c'est-à-dire la démonstration par récurrence, s'impose au contraire nécessairement, parce qu'elle n'est que l'affirmation d'une propriété de l'esprit lui-même.[88]

We remark that Brouwer, in his first notebook, quoted this fragment and agreed with it, and that the second thesis from the list of theses at the end of his dissertation also explicitly claims that mathematical induction is an act of mathematical construction, justified by the ur-intuition of mathematics.

1.4 Zermelo (1871–1953)

The bulk of Zermelo's foundational work was published after the time of Brouwer's defence of his thesis, and will therefore not be discussed now. The proof of the well-ordering theorem, however, was published in 1904 in the *Mathematische Annalen*.[89] Cantor mentioned the theorem in § 3 of his *Grundlagen* as a conjecture and promised to reopen the issue later.[90] Hilbert included it, in relation to his first problem, on his list of unsolved mathematical problems in 1900 at the Paris conference.[91]

Zermelo's proof hinges upon the *axiom of choice*, which states that, if A is a set of which the elements consist of non-empty sets,[92] a new set can be formed by choosing one element from each of the sets that compose the set A. The use of the axiom is exactly the part of Zermelo's proof which is most sensitive to criticism and it is indeed this part at which, in several articles by Jourdain, Borel, and Schoenflies in subsequent issues of the *Mathematische Annalen*, the criticism is aimed.[93]

Later it turned out that the axiom of choice has an impressive list of equivalent statements, which seem to be not as obvious as the axiom itself.[94] Doubt was cast on the axiom because of certain seemingly paradoxical consequences among, or following from, those equivalent statements.

Brouwer disagreed with the validity of the axiom of choice, since, as he remarked in one of the notebooks,[95] there does not, and cannot exist a law which picks an element from every set; every living individual, when asked to compose a choice set, composes one 'according to the structure of his brain'.[96] Moreover, he can only perform that for well-defined sets and sets of sets, and,

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[88] [Poincaré 1916], page 24. See also for comment on Poincaré's views [Largeault 1993], page 42 ff.

[89] [Zermelo 1904].

[90] [Cantor 1932], page 169.

[91] *Göttinger Nachrichten*, 1900, pages 253 – 297. Also included in [Hilbert 1932], vol. III, page 298.

[92] .. welche mindestens ein Element m enthalten muß, ..

[93] See M.A. 59 and 60.

[94] e.g. the well-ordering theorem itself, Zorn's lemma etc.; see for an impressive number of equivalents of the axiom of choice [Rubin and Rubin 1963].

[95] Notebook VII, page 14.

[96] 'volgend uit zijn hersenstructuur.'

when the same question is asked in a different language, the result might very well be completely different.

In the dissertation Brouwer did not discuss the axiom, but he agreed with Borel when the latter stated that the axiom of choice is just equivalent to the well-ordering theorem.[97] Either one may be taken as axiom, from which the other then can be derived as a theorem. But the main argument in Brouwer's criticism and rejection of the well-ordering theorem is that the vast majority of the elements of the continuum is unknown and hence cannot be mutually compared or selected for the composition of the choice-set. Moreover, well-ordering would turn the continuum into a denumerable set, which it is not. Therefore in the dissertation the rejection of the axiom is an indirect one.

1.5 Schoenflies (1853–1928)

In a separate volume of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* Band VIII (1900), Schoenflies published the first complete textbook on set theory.[98] Until that year all publications on sets were in the form of papers in one of the major mathematical journals.[99] Schoenflies' book consists of three parts (Abschnitte), each divided into chapters.

The grand title is *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, Bericht über die Mengenlehre*, with as subtitles for the three parts:

I *Allgemeine Theorie der unendlichen Mengen.*

II *Theorie der Punktmengen.*

III *Anwendung auf Functionen reeller Variabelen.*

In the first part a systematic survey of Cantor's set theory is presented, in which also Dedekind's influence on Cantor is shown. The second part gives a further development of set theory in the direction of what was later to be called *Cantor-Schoenflies topology.*

This work is mentioned here because Brouwer frequently referred to it in his notebooks and in his dissertation, when discussing sets.

1.6 Bernstein (1878–1956)

In 1905 Bernstein published a paper in the *Mathematisch Annalen* under the title *Untersuchungen aus der Mengenlehre*[100] and in his notebooks Brouwer discussed this paper elaborately, and disagreed with it.

The introduction of the *Untersuchungen* begins as follows:

[97] *Mathematische Annalen* 60, page 194, 195. Also in [Borel 1972], vol 3, page 1251.

[98] [Schoenflies 1900a].

[99] apart from Cantor's monograph *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (1883), and [Young and Chisholm Young 1906], the latter probably being unknown to Brouwer.

[100] [Bernstein 1905], page 117 ff.

> Gegenwärtig stehen zwei Probleme innerhalb der Mengenlehre im Vordergrund des Interesses. Das eine bezieht sich auf das *Kontinuum,* d.h. die Menge, welche aus allen reellen Zahlen besteht, das andere bezieht sich auf die *Grundlagen* der Mengenlehre.[101]

In the first section of the first chapter Bernstein proved the theorem, which Cantor mentioned but did not prove (the *Cantor-Bernstein theorem*)

> Satz 1. Ist M äquivalent einem Teile M_1 von N und N einem Teile N_1 von M, so ist M äquivalent N.[102]

Unknown to Bernstein, Dedekind had already proved the theorem in 1887.[103] Bernstein further mentioned proofs by Zermelo, by Schröder and by himself. Bernstein's proof was published in the above mentioned textbook by Schoenflies.[104]

In § 5 Bernstein proved the following theorem:

> Teilt man das Kontinuum in eine endliche Anzahl gleicher Teilmengen, so ist jede dieser Teilmengen gleich dem Kontinuum.
>
> [*with, in a footnote, the addition:*]
>
> Diese Teilmengen sind nicht als Intervalle, sondern als ganz unregelmäßig verteilte Punktmengen vorzustellen.[105]

Via the proof of a number of important theorems, the third chapter begins as follows:

> Die Behauptung von G. Cantor, das im Kontinuum nur zwei verschiedene Mächtigkeiten vorkommen, ist eine Aussage, welche sich auf *alle* Teilmengen des Kontinuums bezieht. Man kann sie in der Form aussprechen:
>
> Jede Teilmenge T des Kontinuums ist entweder abzählbar oder von der Mächtigkeit des Kontinuums.
>
> Bisher bewiesen ist dieser Satz nur für die *abgeschlossenen* Mengen A, d.h. diejenigen, welche alle ihre Grenzelemente enthalten.[106]

We will see that, with regard to the continuum, Brouwer came in part to a completely different conclusion: the continuum is no point set but it is given to us in its entirety. One can only construct points on it. But with respect to the

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[101] op. cit. page 118.

[102] op. cit. page 121.

[103] and again in 1899, probably having forgotten the 1887 proof; see for this a footnote by E. Noether with the paper *Ähnliche (deutliche) Abbildung und ähnliche Systeme*, in [Dedekind 1930a], Volume III, page 447.

[104] [Schoenflies 1900a], page 16; in a footnote on that page it is added that this proof was first published in Borel's *Leçons sur la théorie des fonctions, [Borel 1950].*

[105] [Bernstein 1905], page 133.

[106] op. cit. page 145.

theorem that every subset of the continuum is either denumerable or has the cardinality of the continuum, Brouwer's conclusion is the same, and, as we will see, he gave a proof of it in his eighth notebook and subsequently a different proof in his dissertation.

1.7 The paradoxes

In 1895 Cantor was aware of the possibility of paradoxes in the case of an uncritical use of set theory. And indeed Cantor wrote to Dedekind in a letter of 28 July 1899:

> Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines 'Zusammenseins' *aller* ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als 'ein fertiges Ding' aufzufassen. Solche Vielheiten nenne ich *absolut unendliche* oder *inkosistente Vielheiten.*[107]

If the totality of a certain infinite number of elements ('eine Gesamtheit der Elemente einer Vielheit') can be comprehended without contradiction, then we have a 'konsistente Vielheit' or a set. Cantor showed in the mentioned letter that the system Ω of all ordinal numbers is inconsistent, as is the system of all $\aleph' s$.

In 1897 *Burali-Forti* published his *una questione sui numeri transfiniti*, which is based on the same fundamental idea as the one we saw in Cantor's publications (perhaps it even originates from Cantor).[108]

Burali-Forti considered the set Ω of all ordinal numbers, which is itself an ordinal number because this set Ω is also well-ordered, and thus can be represented by $\Omega + 1$. Then we have the following contradiction:

$\Omega+1 > \Omega$ (obvious) *but also* $\Omega+1 \leq \Omega$, because $\Omega+1$ is itself also an ordinal number and hence belongs to Ω.

Russell and Zermelo independently discovered another paradox, not concerning the ordinal numbers, but about the more primitive concept of a set without order and the property of being 'member of a set'. The paradox arises if we consider the set of all sets that are not elements of themselves (Russell), or a set which contains all its subsets as elements (Zermelo). Russell published the paradox, which is now known under the name of *Russell paradox.* In §78 of his *Principles of Mathematics* he presented it in the following form:

> The predicates which are not predicable of themselves are, therefore, only a selection from among predicates, and it is natural to suppose that they form a class having a defining predicate. But if so, let

[107][Cantor 1932], page 443. This quote is the first paragraph of a more complete quotation which was presented at the end of section 1.1.11.

[108][Heijenoort 1967], page 104 in a translation by J. van Heijenoort.

> us examine whether this defining predicate belongs to the class or not.[109]

In a letter of 16 June 1902 Russell informed Frege about this paradox, which should, according to Russell, be a consequence of Frege's Begriffschrift, and Russell concluded:[110]

> From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.

The effect of this paradox on Frege was devastating;[111] Dedekind too was very unsettled by this unexpected result.[112]

A well known attempt to lift the paradox is the requirement to construct the elements of a set *before* constructing the set itself, thus creating a hierarchy in mathematical objects, and thus preventing that a set can ever be an element of itself. See for this chapter 10 of Russell's 'Principles'. Brouwer discussed the paradoxes elaborately in his notebooks; in chapter 3 of his dissertation they are briefly treated and Russell's solutions are rejected there.

1.8 Final remarks

There are, as yet, no conclusions to be drawn from this concise survey of the history of set theory before 1907 and of the content of that theory. As said in the beginning of this chapter, it just serves as an aid in the thorough analysis and in the discussion of the foundational aspects of Brouwer's dissertation.

In the following chapters the most fundamental notions in the process of Brouwer's systematic construction of the 'mathematical edifice' will be treated, and we will meet many of the concepts that were sketched in this first chapter. Even Brouwer's most revolutionary ideas of those days, that ultimately developed into his intuitionism, have to be seen and interpreted in the light of the, then very recent, history of this branch of mathematics.

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[109] [Russell 1938], page 80.

[110] See for the English translation, authorised by Russell, of a letter originally written in German, [Heijenoort 1967], page 124.

[111] see the epilogue of his *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, written in 1902 after completion of the second volume of the work (volume one appeared in 1893), but before its publication.

[112] see the introduction of the third edition (1911) of [Dedekind 1930b], page XI.

Chapter 2

Brouwer's ur-intuition of mathematics

2.1 Introduction

Brouwer's opposition against *logicism* as the view that ultimately logic alone is the sole basis for all of mathematics, and against *formalism* which views mathematics as a system of rule-following manipulations and deductions of meaningless symbols, thereby departing from a set of axioms which only has to satisfy consistency, does not stand isolated; it has its history.

Mach, in *Erkenntnis und Irrtum*, claimed that 'die Grundlage aller Erkenntnis ist also die *Intuition*'.[1] However, he was referring here to knowledge of natural sciences in general. The French semi-intuitionists Poincaré and Borel granted an important role to intuition in pure mathematics. According to them, logic plays its indispensable role in a mathematical argument, but, as Poincaré remarked in the first chapter of *La Valeur de la Science*, logic on its own only teaches us tautologies.[2] A 'mathematical intuition' is needed, but this concept should then be explained and specified. Poincaré distinguished three kinds of intuition:[3]

1. *L'appel aux sens et à l'imagination*, i.e. falling back on mental images in a mathematical argument.

2. *La généralisation par induction*, e.g. one can directly imagine a triangle, but not a thousand-angle.

3. *L'intuition du nombre pur*; this gives, for Poincaré, pure mathematical reasoning, and here he comes closest to what Brouwer had in mind. Poincaré emphasized that it is on this third kind of intuition that the 'raisonnement mathématique par excellence', that is the method of *complete induction*, is

[1] [Mach 1968], page 314, 315.

[2] See page 221 for a short discussion about the role of logic in mathematics according to Poincaré and Borel.

[3] [Poincaré 1923], page 22.

based; this is the way of reasoning that brings us from the particular case of a mathematical statement to its general case; it also produces the system of the natural numbers and, as Poincaré stressed, this method cannot be obtained from logic.

However, Poincaré (like Borel, and other semi-intuitionists) did not specify the exact nature of the intuition.[4] Brouwer was the first to be very specific on this point; he described exactly (although briefly worded) what exactly is intuited and how, from this intuition, mathematics is built up.

In this chapter we will analyze the exact character of the, in Brouwer's view, most basal intuition for the whole of mathematics, the ur-intuition of all human experience, and we will see that this concept and its corollaries require, for their proper understanding, some interpretation. We will also analyze and interpret how, departing from this ur-intuition, Brouwer succeeded in the construction of the system of the natural numbers, the integers, and the rational numbers. For the construction of the rationals, the character of the continuum as the 'matrix to construct points on' is not so obvious at first sight, but we will show after some interpreting remarks about the insertion of new elements that it can indeed be viewed as such.

We will also examine how these different kinds of numbers, as constructions of the individual, are retained in memory for later use. It will turn out that the rules of arithmetic are a natural consequence of Brouwer's way of founding the number system, and are no longer in need of an axiomatic foundation.

Right at the beginning of Brouwer's dissertation, on page 8, the fundamental concept of the *ur-intuition of mathematics* is introduced. This ur-intuition is the ultimate foundation on which, in Brouwer's terms, the mathematical edifice is constructed and, as we will see, it consists of the intuition of the flow of time in which the individual experiences perceptions of change. This intuited time essentially differs from the external 'physical time', which can be numerically expressed on the measurable time continuum, which, in its turn, is the result of a *construction* on the intuitive time continuum.

Brouwer's foundation of mathematics is thus completely different from the attempted foundation on logic alone (Frege, Russell) or from the formalist foundation (Hilbert).

The consequences of Brouwer's way of founding are far-reaching. His constructivistic attitude, as a necessary result of the ur-intuition as a sole basis, brings about strong limitations in the formation of sets and their possible cardinalities. But separate *proofs of consistency* are no longer required: the mere possibility of the continuation of a construction or its successful completion *is the guarantee for* its consistency or, rather, *is* the *proof* of its consistency.

However, the exact nature of the ur-intuition and the way in which, from the ur-intuition, the system of the natural numbers, the system of the integers,

[4]For Borel, intuition consists of the *experience* of a mathematical object, the *being familiar with it*, its commonly shared experience with other mathematicians. See *L'infini mathématique et la réalité*, [Borel 1914], included in [Borel 1950], page 175.

and the rational scale can be constructed, requires, as said, some discussion, investigation, and interpretation, since Brouwer is, in his dissertation and to a lesser extent also in his notebooks, usually rather briefly worded about these basic topics.

2.2 The dissertation about the ur-intuition

The first page of the first chapter of the dissertation comes straight to the point; the chapter is entitled *The construction of mathematics*,[5] and opens with the following phrases:

> 'One, two, three, ...', we know by heart the sequence of these sounds (spoken ordinal numbers) as an endless row, that is to say, continuing for ever according to a law, known as being fixed.
>
> In addition to this sequence of sound-images we possess other sequences proceeding according to a fixed law, for instance the sequence of written signs (written ordinal numbers) 1, 2, 3, ...
>
> These things are intuitively clear.[6]

From this 'intuitive counting' the 'main theorem' of arithmetic (the number of elements of a finite set is not depending on the order in which they are counted) can be deduced; in fact the theorem is mentioned without a proof, it is merely made plausible by means of an example, suggesting that the proof should be by complete induction. Note that the principle of complete induction is a natural corollary of the way of generation of the natural numbers. For Brouwer complete induction was neither an axiom, nor a theorem which requires a proof (cf. Dedekind), but a natural mathematical act.[7]

Addition, multiplication, and exponentiation are subsequently defined by means of continued and/or repeated counting, and after that the laws of commutativity, associativity, and distributivity for these operations are deduced. But just as in the case of the proof of the main theorem of arithmetic, these definitions and deductions are merely sketches thereof, suggesting that, again, a proper definition or proof should be inductive. The fact that these operations

[5] *De opbouw der wiskunde.*

[6] (page 3) 'Een, twee, drie ...', de rij dezer klanken (gesproken ordinaal-getallen) kennen we uit ons hoofd als een reeks zonder einde, d.w.z. die zich altijd door voortzet volgens een vast gekende wet.

Naast deze rij van klankbeelden bezitten we andere volgens een vaste wet voortschrijdende voorstellingsreeksen, zo de rij der schrifttekens (geschreven ordinaal-getallen) 1, 2, 3 ...

Deze dingen zijn intuïtief duidelijk.

(In his corrected text Brouwer even strengthened the last sentence of the quote by replacing it by 'These things are intuitive', which is stronger than the original.)

[7] See thesis II from the list of theses, added to the dissertation; see also chapter 8 of this dissertation. Compare also with Poincaré's attitude towards complete induction in section 1.3.5.

can be *proved* follows directly from the ur-intuition and the status of the principle of complete induction as a natural act, but the reader should realize that in the first pages of the dissertation Brouwer had not yet reached a discussion on foundational matters; he merely sketched what was supposed to be intuitively known and familiar to all.

As a result of the main theorem of arithmetic, an abstraction can be made from the ordered sequence of numbers to finite sets without order. Negative numbers are indicated as a continuation to the left of the ordinal number sequence, after which follows the definition of rational numbers as ordered pairs of ordinal numbers and the definitions of the operations on these rational numbers.

In agreement with the normal process of extension of the number system, the irrational numbers are subsequently introduced, 'in the first place those with fractional exponents', which can be represented by, in Brouwer's terminology, 'symbolic aggregates of previously introduced numbers'. We can imagine this in the case of, for instance, $(\frac{p}{q})^{\frac{r}{s}}$ as one of the symbolic aggregates (p, q, r, s) or $(\frac{p}{q}, \frac{r}{s})$; or, in the case of the roots of a polynomial equation, as the aggregate of the degree and the coefficients of the equation.

It is noteworthy and typical for Brouwer's approach to the construction of the number system, that he stated this in the dissertation in the following way:

> Next we can introduce successively the usual irrationals (first of all the expressions containing fractional exponents) by writing them as symbolic aggregates of previously introduced numbers and then looking upon each of these as defining a partition of the earlier introduced numbers into two classes, of which the second follows as a whole after the first and has no first element; (...)[8]

Brouwer clearly referred, without mentioning the name, to the Dedekind cut, but this cut is *not* employed here to define 'arbitrary' new irrational numbers; in other words, it is *not* the key to the definition of the non-denumerable set of the real numbers. It is introduced here as if he wanted to point to the fact that for his procedure by means of 'symbolic aggregates' one can also use the Dedekind cut to fix the same definable irrational numbers, but that, contrary to the approach via symbolic aggregates, this method of Dedekind cuts can be employed as a means to establish for such a new irrational number its natural and proper place in the sequence of the previously introduced numbers, i.e. to fit it into the natural order of the rationals. Hence the order relations between the newly defined irrational numbers and the rationals is now determined. Also the basic arithmetical operations on those new numbers can be defined on the basis of the partition, which operations may give rise to again new irrational numbers. A *rational* ('earlier introduced') number can then be made to correspond with

[8] (page 6) Vervolgens kunnen we stap voor stap de gebruikelijke irrationalen, (in de eerste plaats de vormen met gebroken exponenten) invoeren, door ze als een symbolisch agglomeraat van reeds eerder ingevoerde getallen te schrijven, en daarin verder te lezen een verdeling dier reeds ingevoerde getallen in twee klassen, de tweede waarvan geheel op de eerste volgt, en geen eerste element heeft;(...)

a partition of which the lower class contains a least upper bound as element. Brouwer just mentioned all these facts and possibilities, as yet without further foundation or elaboration.

But already on page 6 of his dissertation Brouwer stated explicitly, and this is a central theme throughout the dissertation, that the totality of the numbers in any constructed system remains denumerable during every continued stage of its development, because any 'symbolic aggregate' may contain any finite number of earlier introduced numbers, and because the cardinality of the union of a finite or a denumerable number of denumerable sets is denumerable.[9]

Also the final result of a construction of a totality of rational or irrational numbers is *everywhere dense in itself*,[10] i.e. between any two numbers another number exists, i.e. can be constructed, hence the result is of the order type η.[11]

In summary, the first six pages (page 3 – 8) of the dissertation are no more than a reminder of the well-known systems of the integers, of the rationals, and of the definable irrational numbers, as based on the intuitively known natural numbers. These pages show us the following: The system of the natural numbers is considered as being intuitively clear to us; there is no need of a reduction à la Dedekind, who had to make use of concepts like *object* (Ding) and *chain* (Kette) to define the number system. In the quoted beginning of the dissertation we read that sequences of expressed 'sounds' and sequences of written 'signs' are both identified with 'ordinal-numbers'.[12] Intuition tells us that both sequences of sounds and signs, different as they may be, refer to the same ordinal number system. All rules for the elementary operations on this system are subsequently defined by continued or repeated counting.

The key message of this brief reminder is that the various steps in the development of the number system are perfectly constructive. The difference with the formalist approach in the construction of mathematics[13] is also emphasized in this résumé; intuitive knowledge of the sequence of numbers is manifest from the first page, in contrast to the formalist's concept of natural numbers. Hence there was for Brouwer no need to found arithmetic on a set of axioms, as Peano did. Arithmetic arises naturally as a result of his concept of the number system. *Any* axiomatic foundation of mathematics was rejected by Brouwer as inade-

[9] A theorem by Cantor from the beginning of *Ein Beitrag zur Mannigfaltigkeitslehre*, see [Cantor 1878].

[10] In zich overal dicht; On page 7 of his dissertation Brouwer gave the classical definition of this concept, but in his later intuitionism he distinguished between 'everywhere dense' and 'everywhere dense in itself'. An ordered set is *everywhere dense* if between any two different elements of the set another element of that set can be indicated. A set is *dense in itself* if every element is a *main element*, that is, every element is the limit element of a monotone fundamental sequence. In classical mathematics the continuum satisfies both conditions, but in intuitionistic mathematics the continuum is everywhere dense, but not dense in itself. See e.g. the *Groningen lectures* (1933), § 13 and 14. At this stage Brouwer had of course the classical interpretation in mind.

[11] See [Cantor 1897].

[12] We will further discuss the status of *signs* on page 56.

[13] The construction of the mathematical building, in Brouwer's terms.

quate and beside the point. What he is about to show now is that the intuitive foundation goes deeper; the 'intuitively known' number system rests on a more fundamental basis: the ur-intuition of an abstracted human experience.

The truly foundational work begins in earnest on page 8 of the dissertation. Here the ur-intuition of mathematics appears, in its explicit form, for the first time:

> In the following chapters we will examine further the ur-intuition[14] of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a 'between', which is never exhausted by the insertion of new entities. Since continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid one of them as a primitive entity, trying to construe it from the other one, the latter being put forward as self-sufficient; in fact it is impossible to consider it as self-sufficient. Having recognized that the intuition of continuity, of 'fluidity', is as primitive as that of several things conceived as forming together a unit, the latter being at the basis of every mathematical construction, we are able to state properties of the continuum as a 'matrix of points to be thought of as a whole'.[15]

This quote forms another often recurring theme in Brouwer's views on the foundations of mathematics, which is further developed in subsequent papers from his hand: the abstraction from all observation is a unity of continuous and discrete. The continuous is the 'flowing' in which the discrete takes place in the form of events and *both, the flowing and the discrete, are equally fundamental*. This has to be interpreted as taking place in time.[16] The individual subject experiences events ('perceptions of change', in Brouwer's terms), which

[14]The English translation in the *Collected Works* uses the term *basic intuition*, but the Germanism *ur-intuition* has become current in this context.

[15]In de volgende hoofdstukken zullen we nader ingaan op de oer-intuïtie der wiskunde (en van alle werking van het intellect) als het van kwaliteit ontdane substraat van alle waarneming van verandering, een eenheid van continu en discreet, een mogelijkheid van samendenken van meerdere eenheden, verbonden door een 'tussen', dat door inschakeling van nieuwe eenheden, zich nooit uitput. Waar dus in die oer-intuïtie continu en discreet als onafscheidelijke componenten optreden, beide gelijkgerechtigd en even duidelijk, is het uitgesloten, zich van een van beide als oorspronkelijke entiteit vrij te houden, en dat dan uit het op zichzelf gestelde andere op te bouwen; immers het is al onmogelijk, dat andere op zichzelf te stellen. De continuüm-intuïtie, het 'vloeiende', dus als oorspronkelijk erkennende, zo goed als het samendenken van meerdere dingen in één, die aan elk wiskundig gebouw ten grondslag ligt, kunnen we van het continuüm als 'matrix van samen te denken punten' eigenschappen noemen.

[16]Note that for Brouwer the intuition of time is sufficient; no separate intuition of space in the Kantian sense is required (see page 52). The intuition of time gives us arithmetic and as a result of Descartes' 'coordinate geometry', the geometries, up to any finite number of dimensions, Euclidean as well as non-Euclidean, can be reduced to arithmetic. See for this for instance Brouwer's inaugural lecture [Brouwer 1912], translated into English by A. Dresden and published in November 1913 in the Bulletin of the American Mathematical Society (see

are seperated by a 'flowing' time lag, which makes the separation manifest. This experience is the foundation of Brouwer's concepts of the system of natural numbers and of the continuum.[17]

For a better comprehension of this basic notion, we will first, before continuing our discussion about the interpretation of the ur-intuition, investigate a few other sources of information as far as their views on the ur-intuition are concerned. The relevant items are, in a chronological order, Mannoury's 'opposition from the audience' during the public defence of Brouwer's dissertation (1907), Barrau's 'opposition' on that occasion, Brouwer's Rome lecture *Die möglichen Mächtigkeiten* (1908), and then, after a discussion of the interpretation, his inaugural lecture *Intuitionism and Formalism* (1912).

2.3 Mannoury's opposition

For a proper understanding of Brouwer's arguments in regard to the ur-intuition, it is useful to take a look at some objections against it from others, and at Brouwer's defence against that opposition.

Although Mannoury's objections were mainly aimed at the role of language and logic in the construction of the mathematical building, we clearly discern in Brouwer's reply his concept of the ur-intuition of mathematics.[18] Mannoury claimed that, in a mathematical construction, *either* one has a living representation without irreducible concepts (but here the continuum and the infinite etc. are irreducible), *or* there is the discrete 'language building', in which case *thing* and *relation* are sufficient. This latter is the case in e.g. the definition of Cantor's (discrete) continuum, where 'unit' acts as the *thing* and 'successor' as the *relation*.

Brouwer answered that neither is the case, there is only the intuitive construction in the intellect.[19] He emphasized that concepts (and not their representing words) as *unity, once again, continuum* and *and so on* are irreducible concepts in the creation of a multiplicity. The intuition of *thing – medium of cohesion – second thing*, or *thing – asymmetric relation – other thing*, that is the

[Benacerraf and Putnam 1983], page 77). Remember also that for Cantor the continuum concept neither depends on time, nor on space (see page 8 of this dissertation).

[17] There are also social and moral aspects involved in Brouwer's concept of man who is perceiving changes, but these will not be considered here; see [Brouwer 1905].

[18] Mannoury was the first of the two 'opponents from the audience'. At that time he was 'privaatdocent', and in later years professor, in Amsterdam.

[19] This is also the content of the first sentence of the summary of his dissertation on page 179:

> Mathematics is created by a free action independent of experience; it develops from a single aprioristic ur-intuition, which may be called *invariance in change* as well as *unity in multitude.*
>
> (De wiskunde is een vrije schepping, onafhankelijk van de ervaring; zij ontwikkelt zich uit een enkele aprioristische oer-intuïtie, die men zowel kan noemen *constantheid in wisseling* als *eenheid in veelheid*.)

See chapter 8 for a discussion on this summary.

unity of two different things, clearly separated by a continuous flow of time, may turn this combination into a new unit on which the same operation of adding a new thing may be applied, *and so on.*

Also the fact that two events are clearly separated makes a repeated insertion of a new thing possible:

> That intuition of cohesion (...) results immediately in the succession of three things, namely *first thing – medium of cohesion – second thing,* literally translated *primum – continuum – secundum* (..); we might as well say: *first thing – asymmetric relation – second thing;* in other sounds: *first thing – second thing – third thing;* hence we have recognized as an inseparable attribute of the possibility of holding together of *two,* the possibility of insertion, which can always be continued, without completely covering the medium of cohesion with elements.[20]

For a further discussion about the interpretation of the content of this quote, see page 46.

2.4 Barrau's objections against the ur-intuition

In his objections against Brouwer's dissertation, Barrau[21] argued that *the thinking together of two points is the only mathematical primal activity.* According to Barrau the existence of irrational points emerges only after their successful construction. Brouwer defined the arithmetical operations with the help of transformations on the continuum, but these transformations, Barrau claimed, can only take place with the earlier successfully constructed points, and cannot be used to define 'new' points. Hence one cannot speak of a 'matrix of unconstructed points'.

Judging by Brouwer's reply, Barrau was, in his argument, referring to the Cantorian continuum,[22] which was mentioned already by Mannoury in his objections. Brouwer returned to the matter, agreeing with Barrau that the discrete continuum of Cantor does not exist. But his (Brouwer's) own continuum consists of the intuitive keeping together of two points, occuring in the unity of *thing – time lag – thing*, or *thing – asymmetric relation – thing*, whereas in Barrau's argument the 'thinking together' follows from the two things, and is not a fundamental entity of its own. Then Brouwer continued:

[20][Dalen 2001], page 149, 150: Die samenhoudingsvoorstelling (...) geeft nu direct de opvolging van drie dingen, nl. *eerste ding – samenhoudingsmedium – tweede ding;* letterlijk vertaald *primum – continuum – secundum;* (...) we kunnen ook zeggen: *eerste ding – asymmetrische relatie – tweede ding;* in andere klanken: *eerste ding – tweede ding – derde ding;* we hebben dus als onafscheidelijk attribuut van de mogelijkheid van samendenken van *twee* herkend de mogelijkheid van tussenvoeging, die steeds verder kan worden voortgezet, zonder dat het samenhoudingsmedium ooit geheel zal worden overdekt met elementen.

[21]Barrau was a fellow student of Brouwer, and he was the second 'opponent from the audience', whose objections, as well as Brouwer's reply, are preserved. They were found as loose sheets in one of the notebooks.

[22]i.e. a continuum of the real numbers; see page 9.

> For instance, if you say: I think these two together, then you introduce – and your formulation clearly accompanies this – a third thing, the 'being together', in which you connect both earlier given things by an asymmetric relation.[23]

And, he argued, the intuition of the time lag between two things may indeed be called 'matrix of not yet existing points', since an order type η, or every denumerable unfinished set for that matter, *can* be constructed between the two clearly separated things.[24] Hence this matrix exists before any constructed points on it exist and it exists even independent of the fact whether or not points will be constructed on it; this independence emphasizes its existence on its own (as part of the two-ity of course). So far Brouwer's reply.

This concept of the continuum fits properly the above quoted paragraph from page 8 of the dissertation, where Brouwer stated that the ur-intuition of mathematics is formed by 'a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a *between*, which is never exhausted by the insertion of new entities'. Hence for Brouwer the *discrete* and the *continuous* form an unseparable combination which makes the ur-intuition possible: *two events, separated by a continuous flow of time, can be seen as a new unity*, to which a third, separated from that unity by again a continuous flow of time, can be added, and so on, thus creating the system of numbers (see also the previous section, the reply to Mannoury). The effect of the ur-intuition also becomes manifest in the interpolation of new elements. In Brouwer's answer to Barrau we can read that the awareness of the 'between', the continuous time span between the two events, is itself the inserted third thing. We will work out on page 46 ff. how we can combine this conclusion with the concept of the 'between' as an intuition of the continuum, acting as a matrix 'which is never exhausted by the insertion of new entities', as it was formulated on page 8 of Brouwer's dissertation.

It is interesting to note that the content of Brouwer's reply to Barrau is in agreement with the résumé at the end of the second chapter of the dissertation and with two other papers, the *Rome lecture*[25] (1908), and the *Wiener Gastvorlesungen*[26] (1927) (see below). For a short discussion of the relevant parts of these two lectures, see the next section. As for the mentioned résumé: Brouwer gave at the end of his second chapter as an example of a synthetic judgement a priori:

> 2. the possibility of interpolation (namely that one can consider as a new element not only the *totality* of two already compounded, but

[23] Als u b.v. zegt: ik denk die twee *samen*, dan voert u – en uw woorden begeleiden dit zeer duidelijk – een derde ding, het 'samen zijn' in, waaraan u beide dingen, die tevoren waren gegeven, door een asymmetrische relatie verbindt.

[24] See for this construction page 46.

[25] [Brouwer 1908b], also in [Brouwer 1975].

[26] [Brouwer 1929] and [Brouwer 1930b], both Vienna talks are also published in [Brouwer 1975].

> also that which *binds* them: that which is not the totality and not an element).[27]

2.5 The ur-intuition in the Rome and Vienna lectures

In his lecture *Die möglichen Mächtigkeiten*,[28] held at the International Conference of Mathematicians in Rome in 1908, Brouwer opened with a summary of the ur-intuition and its direct consequences:

> Wenn man untersucht, wie die mathematischen Systeme zustande kommen, findet man, dass sie aufgebaut sind aus der Ur-Intuition der Zweieinigkeit. Die Intuitionen des continuierlichen und des discreten finden sich hier zusammen, weil eben ein Zweites gedacht wird nicht für sich, sondern unter Festhaltung der Erinnerung des Ersten. Das Erste und das Zweite werden also *zusammengehalten,* und in dieser Zusammenhaltung besteht die Intuition des continuierlichen (continere = zusammenhalten). Diese mathematische Ur-Intuition ist nichts anderes als die inhaltlose Abstraction der Zeitempfindung, d.h. der Empfindung von 'fest' und 'schwindend' zusammen, oder von 'bleibend' und 'wechselnd' zusammen.
>
> Die Ur-Intuition hat in sich die Möglichkeit zu den beiden folgenden Entwickelungen:
>
> 1) Die Construction des Ordnungstypus ω; wenn man nämlich die ganze Ur-Intuitiom als ein ganzes Erstes denkt, kann man ein neues Zweites hinzudenken, das man 'drei' nennt, u. s. w.
>
> 2) Die Construction des Ordnungstypus η; wenn man die Ur-Intuition empfindet als den Uebergang zwischen dem 'Ersten für sich' und dem 'Zweiten für sich', ist die 'Zwischenfügung' zustande gekommen.[29]

This is Brouwer's definitive notion: the awareness that the two experienced events do not coincide is itself an experienced event, and therewith *is* the newly inserted element between the two given elements (the two experienced events), as can also be concluded from the following fragment of the second *Vienna lecture* (1927):

> Denn in der Urintuition ist die Möglichkeit der Zwischenfügung zwischen zwei Elemente (nämlich die Betrachtung der Bindung als neues Element), mithin auch die Konstruktion im intuitiven Kontinuum

[27](page 119, 120) 2. de mogelijkheid van tussenvoeging, (dat men n.l. als nieuw element kan zien niet alleen het *geheel* van twee reeds samengestelde, maar ook het *bindende*: dat wat niet het geheel is, en niet element is).

[28][Brouwer 1908b].

[29][Brouwer 1975], page 102.

von einer Menge von einander nicht berührenden geschlossenen Intervallen enthalten (...)[30]

We underline that Brouwer explicitly stated in point 2 of the Rome lecture that if, in the construction of order type η, the ur-intuition is experienced as the transition from the 'first on its own' to the 'second on its own', *then the interpolation is completed.* Hence the simple fact of experiencing the transition is already the interpolation.

This might give the impression that the awareness of the 'flowing' between the first and the second is not the matrix onto which new elements can be constructed in the process of building η, as was stated in the reply to Barrau, but, instead, the interpolated element itself. A plausible answer to this dilemma is, that the 'between' is awarded a double role: as matrix *and* as interpolated entity. This interpretation is indeed the most likely one, since Brouwer mentioned both roles repeatedly, e.g. on page 8 of the dissertation *'We are able to state properties of the continuum as a matrix of points to be thought of as a whole'*, and also in the quote mentioned above from the Rome lecture *'Wenn man die Ur-Intuition empfindet als den Uebergang zwischen dem 'Ersten für sich' und dem 'Zweiten für sich' ist die 'Zwischenfügung' zustande gekommen'.*

The second role, taken literally, causes difficulties in the further construction of η, since it is hard to imagine how between this transition, viewed as new element, and, e.g. the original first element further points can be constructed. The matrix, on which always more elements can be constructed, seems to be lost; the new element is put in its place. But also the other quoted fragments lead to the same conclusion: the 'Zwischenfügung' *is* the inserted element. In the next section we will present what seems to us the proper interpretation.

2.6 Interpretation of the ur-intuition

The construction of the system of the natural numbers, departing from the ur-intuition alone, and the subsequent construction of the order types ω and η on the basis of the same ur-intuition, must, in the light of the quotes given above, take place in the following way:[31]

[30] [Brouwer 1930b], page 434.

[31] Note that this interpretation is not only based on the dissertation, but also on later writings, like the 1932 lecture *Will, Knowledge and Speech* [Brouwer 1933], partial translation in [Brouwer 1975], (see below).

2.6.1 The construction of natural numbers

Man, in his ur-state, experiences an event,[32] and after that experience another event, clearly separated from the first one by a certain time span, while the first event is retained in his memory. Divested of all quality, he calls the first event *one* and the second event *two*.[33] The flow of time between one and two is called *the continuum* (or *a continuum*), and this continuum cannot be experienced without the two events, just as the two events cannot be experienced and recognized as not coinciding without the flowing continuum in between. It is important to realize that *two consecutive events are necessarily accompanied by a connecting continuum.*

The combination *event – continuum – next event* may, in its turn, be considered as one single event, retained in memory as such, and separated by a time span from a new event which, again divested of all quality, results in another two-ity; this new two-ity can, in its totality, be viewed as forming a three-ity, which may be called *three*. Iteration of this process results in the system $\mathbb{N}$ of the natural numbers and in the ordinal number ω.

As said, this combination *event – time span – event*, where the first event may be composed of several earlier perceived combinations of events and time spans, is retained in memory, and can therefore also be retrieved from memory for further mathematical use.

For a proper understanding of Brouwer's ideas, we need not go into the physiological details of our brain functions. It is not immediately clear what exactly is stored in one's memory: the process of the construction of the number itself, or a constructed sign representing that process and its resulting number (see further page 56). The important thing is that the individual can, in some way, retrieve from memory old (abstracted) experiences and abilities. When we consider the first sentence of the earlier quoted paragraph from page 8 of the dissertation,

> In the following chapters we will examine further the ur-intuition of mathematics (and of every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a 'between', which is never exhausted by the insertion of new entities.[34]

[32]That is to say, in Brouwer's view, the subject experiences a sensation. There are no independent external events.

As a kind of warning we stress that one should not view Brouwer's explanation too realistically (in the sense of too literally and lifelike), e.g. as a protocol of experimental psychology. Different individuals have different intellectual histories, and the various stages of Brouwer's 'subject' run different courses in different individual minds. Brouwer's account is of course idealized.

[33]Note that Brouwer starts counting with 'one', whereas modern mathematics usually begins counting with *zero*; we will stick for the moment to Brouwer's convention, and begin with *one*.

[34]In de volgende hoofdstukken zullen we nader ingaan op de oer-intuïtie der wiskunde (en van alle werking van het intellect) als het van kwaliteit ontdane substraat van alle waarneming

the conclusion seems justified that the actual (not yet abstracted) experience consists of events that actually happen in the mental life of the subject. However, the most fundamental *mathematical* intuition is the observation of a series of events, combined into a totality in the sketched way, and *divested of all quality.* Hence that, what is retained in memory after, for instance, experiencing two consecutive and well-separated events, must be some constructed symbolic sign, which represents the experienced combination, after abstraction from its actual content. After another experience of two well-separated events on another occasion, the individual again abstracts from all content, and he notices that that, what is retained in his memory, is on both occasions the same abstract entity and he labels this 'of all quality divested entity' with a *symbolic* sign, which he names *'two'.*

This conclusion can also be drawn from the following quote from Brouwer's lecture at the University of Amsterdam *Will, Knowledge and Speech* (1932), which shows that Brouwer, in this respect, did not change his view during the years:

> However, only at the highest levels of civilization does mathematical activity reach full maturity; this is achieved through the mathematical abstraction, which divests two-ity of all content, leaving only its empty form as the common substratum of all two-ities.
>
> This common substratum of all two-ities forms the primordial intuition of mathematics, which through self-unfolding introduces the infinite as a perceptual form and produces first of all the collection of natural numbers, then the real numbers and finally the whole of pure mathematics or simply of mathematics. We shall not concern ourselves here further with the manner of these constructions.[35]

One question remains to be answered about the construction of the number system. On page 5 of his dissertation Brouwer stated:

> Now we can continue the sequence of ordinal numbers to the left by 0, -1, -2, etc., (...)[36]

The question is: how can the system of integers, the system of the positive and negative whole numbers together, be consistent with the ur-intuition as

van verandering, een eenheid van continu en discreet, een mogelijkheid van samendenken van meerdere eenheden, verbonden door een 'tussen', dat door inschakeling van nieuwe eenheden, zich nooit uitput.

[35] *Willen, Weten, Spreken*, [Brouwer 1933]: Tot volle ontwikkeling komt echter het bedrijf der wiskundige handeling eerst op hogere cultuurtrappen, en wel door middel van de wiskundige abstractie, die de tweeheid van alle inhoud ontdoet, en daarvan slechts de ledige vorm als gemeenschappelijk substraat van alle tweeheden overlaat. Dit gemeenschappelijk substraat van alle tweeheden vormt de oer-intuïtie der wiskunde, die door haar zelfontvouwing het oneindige als gedachtenvorm invoert, en op hier verder buiten beschouwing blijvende wijze eerst de verzameling der natuurlijke getallen, vervolgens die der reële getallen, en ten slotte de gehele zuivere wiskunde, of kortweg wiskunde levert. (English translation by W. van Stigt; see [Stigt 1990]).

[36] We kunnen nu de rij der ordinaalgetallen naar links voortzetten met 0, -1, -2, enz., (...)

a foundation for the natural numbers. Where are the negative integers to be found in the result of that ur-intuition? We postpone the answer till the end of the following section on the rational numbers.

2.6.2 The rational numbers

In order to construct the order type η from the system of the natural numbers $\mathbb{N}$, we must realize that every experience of a time span between two events is identical in character, since, in the construction of η in a dual system, every interval can always be split into two parts. In that sense every two continuous intervals are, as a result of an identification process, recognized as being of the same nature. For Brouwer there was no need to *discover* their similarity in character, nor to *declare* them as similar by axiom; it is an *act,* that makes them similar, one *forces* them as such, in the same sense as one forces a constructed scale on the continuum to be everywhere dense,[37] according to the principle that mathematical activity is neither a process of discovery, nor a necessity, but a free act, resulting in the free creation of the mathematical building. This principle is repeatedly applied in the dissertation and is explicitly stated in its summary,

> Mathematics is a free creation, independent of experience; it develops from a single aprioristic basic intuition, which may be called *invariance in change* as well as *unity in multitude.*[38]

This element of 'free will' in mathematical activity is more extensively elaborated in the mentioned *Will, Knowledge and Speech.* In this lecture Brouwer argued that knowing and speaking are forms of action, through which human life is maintained and enforced. They originate in the anthropological phenomena of 1) mathematical viewing, 2) mathematical abstraction and 3) enforcement of will by means of signs. As for the first item, this comes into being in two phases, that of becoming aware of time and that of causal attention. About mathematical viewing in general (and this equally applies to all forms of mathematical viewing and not only to causal sequences), Brouwer noticed in this same lecture:

> Mathematical attention is not a necessity, but a phenomenon of life, subject to the free will, everyone can find this out for himself by internal experience: every human being can at will either dream-away time-awareness and the separation between the Self and the World-of-perception or by his own powers bring about this separation and call into being in the world-of-perception the condensation of separate things. *Equally arbitrary is the identification of different*

[37]see page 79 under item 7.

[38](page 179) De wiskunde is een vrije schepping, onafhankelijk van de ervaring; zij ontwikkelt zich uit een enkele aprioristische oer-intuïtie, die men zowel kan noemen *constantheid in wisseling* als *eenheid in veelheid.* See chapter 8 for a further discussion on mathematics as a free creation.

> *temporal sequences of phenomena which never forces itself on us as inevitable.* [*my italics*][39]

Now, for a proper understanding of the construction of the rational scale (which scale is, as we saw, not depending on the exact nature of defining events; all connecting time spans are identical, or better, are 'identified', this to emphasize the character of mathematics as an act) we proceed via an interpreting remark about the earlier mentioned double role of the connecting continuum between two events, viz. the role of matrix *and* that of interpolated entity. This gives rise to the following interpretation of the continuum in regard to the construction of the η-scale.

The awareness of the flowing between the first event and the second one is itself a sensation, a third sensation in addition to the two abstracted events that we will call now *zero* and *one*;[40] this third sensation is substantialized to an event between the two original events, and may be called *a half.* We emphasize that the newly defined '$\frac{1}{2}$' is not a point, constructed on the continuum 'somewhere halfway' between 0 and 1, since there is no 'halfway' yet; the point '$\frac{1}{2}$' is the insertion ('Zwischenfügung') between 0 and 1 itself, it is the 'experience of the flowing between 0 and 1, divested of all quality'. There simply is no other choice for the inserted event, since the awareness of the connecting continuum is the *only* experience between the original two events, and it becomes so *only after* the second event.

Having now recognized the 'event' $\frac{1}{2}$ between the two events 0 and 1 as being clearly different and therefore separated from the two, there is, as a result of that separation, a connecting continuum between 0 and $\frac{1}{2}$, as well as between $\frac{1}{2}$ and 1, because, as we saw, two not coinciding events are necessarily connected by a continuum. Hence we can iterate the process between every two adjacent points and thus construct the order type η.

In this interpretation the flowing continuum may be seen as the *matrix for points to be constructed between* 0 *and* 1.

---

[39] Dat wiskundige beschouwing geen noodzaak, doch een aan de vrije wil onderworpen levensverschijnsel is, daarvan kan ieder bij zichzelf de inwendige ervaring opdoen: ieder mens kan naar willekeur hetzij zich zonder tijdsgewaarwording en zonder scheiding tussen Ik en Aanschouwingswereld verdromen, hetzij de genoemde scheiding door eigen kracht voltrekken en in de aanschouwingswereld de condensatie van aparte dingen in het leven roepen. En even willekeurig is de zich nooit als onvermijdelijk opdringende identificering van verschillende temporele verschijnselreeksen. (For the English translation of this frament from *Will, Knowledge and Speach* see [Stigt 1990], page 418, 419.)

[40] Note that we now call the first event 'zero' (we are free to do so; we will have to change this again for the construction the negative integers and rationals). On page 9, 10 and 11 of his dissertation Brouwer is constructing the 'measurable continuum', without giving names to the constructed points yet. In fact he selects an arbitrary point as the 'zero' point only after the construction of the everywhere dense dual scale, therewith (after also selecting a unit distance) turning it into a measurable scale, but for easy reference to earlier constructed points in our present argument we name the original events and the intercalations already at this early stage, in advance of Brouwer's results.

The scale of integers

Now, we still have the unanswered question about the negative integers. There are several ways to give them a place in Brouwer's construction of the number system. With all of them, the answer lies again in viewing mathematics as a free creation of the human mind. We might, for instance, imagine the construction of the negative integers to take place in the following way. We experience an event and, well separated from it, a second event. We are free to baptize this second event 'zero'. Viewing the two events *and* their connecting continuum as one single event, then the next event, separated from the previous two-oneness, we call 'one', and so on. Now, the awareness of the flowing between the first, unnamed, event and the second event which we called 'zero', is, as we saw, substantialized to a new event, which we may call 'minus one'. We repeat this procedure between the first unnamed event and the new event 'minus one', thus creating another new event 'minus two'. Iteration of the process between the first unnamed event and the last newly added one, results in the system of the negative integers. (The first and always unnamed event we might then informally dub 'minus infinity'.)

There is of course also the standard method of constructing $\mathbb{Z}$ by means of ordered pairs of natural numbers, and this method could also have been used for the construction of the rationals; but our aim is, following Brouwer's line of reasoning, to construct the complete number system (i.e. integers and rationals) on the basis of the ur-intuition alone.

One can imagine the basic arithmetical operations on the system of the integers to be defined, as Brouwer did this on the first pages of his dissertation, by continued and/or repeated counting (see page 4 of his dissertation, see also our page 35). However, after a closer analysis of the continuum (see the next chapter) Brouwer developed a more sophisticated definition with the help of group theoretic arguments, to be applied to the measurable continuum and, again, no axiomatization is needed for these operations, since, ultimately, they are all based on the ur-intuition and its corollaries, i.e. the repeated experience of the move of time, resulting in the extension of the natural number system and the insertion of always new elements between earlier constructed natural or rational numbers.

2.7 The ur-intuition in the inaugural lecture

The conclusion, following from the interpretation of the role of the connecting continuum, can also be drawn from the following quotation from the inaugural lecture *Intuitionism and Formalism*[41] on the occasion of Brouwer's professorship in 1912, in which he said the following:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remain-

[41] [Brouwer 1913].

> ing separated by time as the fundamental phenomenon of the human intellect, passing by abstraction from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, this ur-intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this ur-intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e. of the 'between', which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. [42]

Hence in the awareness of the fact that we can view two separated events as a unity, we perceive the intuition of the continuum as the 'between', as that what makes the two events a single one, as that what binds them. And as a

[42] [Brouwer 1912]; see also [Brouwer 1919c], pages 11 and 12. For the English text see [Benacerraf and Putnam 1983], page 80. Dit neo-intuïtionisme ziet het uiteenvallen van levensmomenten in qualitatief verschillende delen, die alleen gescheiden door de tijd zich weer kunnen verenigen, als oer-gebeuren in het menselijk intellect, en het abstraheren van dit uiteenvallen van elke gevoelsinhoud tot de intuïtie van twee-enigheid zonder meer, als oergebeuren van het wiskundig denken. Deze intuïtie der twee-enigheid, deze oer-intuïtie der wiskunde schept niet alleen de getallen *één* en *twee,* doch tevens alle *eindige ordinaalgetallen,* daar één der elementen der twee-enigheid als een nieuwe twee-enigheid kan worden gedacht, en dit proces een willekeurig aantal malen kan worden herhaald. Verder wordt, door dezelfde herhaling onbepaald voortgezet te denken, het *kleinste oneindige ordinaalgetal* ω geconstrueerd. Eindelijk is in de oer-intuïtie der wiskunde, waarin het saamgehoudenen en het gescheidene, het continue en het discrete verenigd liggen, mede onmiddellijk aanwezig de intuïtie van het lineaire continuüm, d.w.z. van het 'tussen', dat door inschakeling van nieuwe eenheden zich nooit uitput, dus ook nooit als verzameling van eenheden zonder meer kan worden gedacht.

(The English translation of the originally Dutch text is by A. Dresden and was published in the *Bulletin of the American Mathematical Society* in November 1913.)

In [Brouwer 1912] Brouwer employed the term 'neo-intuitionism' to contrast it with Kant's intuitionism. According to Brouwer, we find with Kant an 'old form of intuitionism, in which time and space are taken to be forms of conception inherent in human reason'. This results for Kant in the apriority of the axioms of arithmetic and geometry. The discovery of the non-Euclidean geometry was a severe blow for Kant's theory, but intuitionism has recovered, in Brouwer's terms, 'by abandoning Kant's apriority of space, but adhering the more resolutely to the apriority of time'. This is called by Brouwer in his inaugural address *neo-intuitionism.* For Brouwer the intuition of time is sufficient, thanks to the arithmetization of geometry as a result of the work of Descartes. Already in [Brouwer 1914] the 'neo' is dropped from neo-intuitionism and the term 'intuitionism' is introduced for his own mathematics. This mathematics is elaborately worked out in [Brouwer 1918] and in [Brouwer 1919a]. Also his paper [Brouwer 1919b], which summarizes the previous two, only speaks of 'intuitionism'. To mark the difference with the work of Poincaré, Borel, Lebesgue and others (who claimed that mathematics is more than a formalistic construction; it has a *content* too), the term 'pre-intuitionism' is introduced to designate the work of those French mathematicians from around the turn of the century (see e.g. [Brouwer 1952] and [Brouwer 1954c]). In the second Vienna lecture ([Brouwer 1930b], page 2) Brouwer called this pre-intuitionism 'die altintuitionistische Methode'.

result of the fact that we can view this binding 'between' as an event (in fact as the *only possible* event), separated from the original two, the 'between' can act as the matrix for always more intercalations, since the process can be iterated.

2.8 The ur-intuition in the notebooks

In the notebooks we can roughly observe during which phase of the preparation of his dissertation Brouwer was occupied with the ur-intuition, the possible sets and the continuum. This turned out to be rather late. Whereas in the notebooks 1 to 5 Brouwer was mainly working on foundational matters, thereby criticizing or responding to the work of Cantor, Russell, Poincaré and Couturat, from halfway number 5 onwards he elaborated his own ideas on sets and their possible cardinalities. It is significant that the ur-intuition in its explicit form, as it was written down in the dissertation, only appears in the last notebook. Hence the concept of the two-ity discrete-continuous as the ultimate foundation of mathematics must have been developed shortly before taking his doctoral degree.

When considering the possibility to construct the infinite, Brouwer remarked in notebook II, page 7:

> (II–7) We only have the awareness of 'infinite', that is 'always continuing', in one dimension. Hence we can only use this one-dimensional infinity in the construction of a geometrical system.[43]

Brouwer's concept of the 'infinite' is, similar to Dedekind's, that of 'always continuing', that is, the process of growth to the ordinal number ω.[44] The only method of construction of denumerably infinite sets (which is the only possible constructible infinite set[45]), is the continued repetition of the same operation of 'adding one element', which already implies the concept of time in the composition of the system of the natural numbers.

> (VIII–24) One should always keep in mind that ω only makes sense as a living and growing induction in motion; as a stationary abstract entity it is senseless; ω may never be conceived to be finished, e.g. as a new entity to operate on; however, you may conceive it to be finished in the sense of turning away from it while it continues growing, and to think of something new.[46]

[43] De bewustheid van 'oneindig', dat is 'altijd maar door' hebben we alleen ééndimensionaal. Alleen van die ééndimensionale oneindigheid kunnen we dus gebruik maken om een meetkundig systeem op te bouwen.

[44] In regard to the *continuum*, there is quite a difference between Dedekind and Brouwer.

[45] See further in chapter 4, when discussing possible sets.

[46] Men bedenke steeds dat ω alleen zin heeft, zolang het leeft, als groeiende, bewegende inductie; als stilstaand abstract iets is het zinloos; zo mag ω nooit àf gedacht worden, om m.b.v. het *geheel* als nieuwe eenheid te werken: wel mag je het àf denken in de zin, van je er van af te keren, terwijl het doorloopt, en iets nieuws te gaan denken.

We should never see ω as finished in the sense of a stationary and completed process of counting; it persists in its growing, even without our active interference. The induction 'does the work', but the continuation of the act of counting into the second number class from ω onwards may be looked at, and *can* be interpreted as an example of 'thinking of something new'.[47] We may conceive ω as finished in the sense that its method of generation is determinately given; the sequence of natural numbers continues in its process of growing, following the same successor operation, in which nothing new happens.[48] It is finished in the *intensional* sense.[49]

The concept of time as the basic intuition of mathematics, appears in the third notebook:

> (III–7) Time acts as that, which can repair the separation.[50]

Despite the content of the paragraph in which this short quote is found, and which speaks of 'sin' and 'self-preservation', this could very well be understood to refer to a time awareness between two events. The connecting time continuum, 'which can repair the separation' between two events, is the medium onto which an everywhere dense scale of rationals can be constructed. It involves the one-dimensional continuum, which we meet in the seventh notebook; the higher-dimensional space still carries here the 'sign of hostility', the property of the external world when man faces this world and acts on it:

> (VII–4) Meanwhile I can only use the one-dimensional continuum to build on (that is the primal intuition of time); I feel that I am able to construct the multidimensional by myself (it is the space, the hostile outside of myself, no externalization of myself).[51]

Time clearly becomes the primal intuition for arithmetic, and hence for the whole of mathematics.

But there is also a different kind of intuition: that of *real space*, which Brouwer mentioned in notebook eight, but to which he already alluded in the first one:

> (I–3) As a consequence of our doom it is self-evident, that our space

[47] This continuation, however, can no longer be viewed as finished in the same interpretation of 'letting it persist in its growing and continue from the beginning of the next class'; here the result is a *denumerably unfinished* set, the maximum attainable in the set construction; see further the discussion in chapter 7.

[48] See also page 322.

[49] See Brouwer's dissertation, page 176; see also chapter 8 of this dissertation in which the actual versus the potential infinite and the finished versus the unfinished will be discussed.

[50] De tijd treedt op als dat, wat de scheiding wel weer in orde kan brengen.

[51] Intussen kan ik alleen het eendimensionaal continuüm gebruiken om op te bouwen (dat is de intuïtie van de *tijd*, die primair is), het meerdimensionale voel ik, niet zelf te kunnen bouwen (het is de ruimte, het vijandige buiten mij, geen veruiterlijking van mijzelf). (see also *Life, Art and Mysticism*, [Brouwer 1905]).

> has three dimensions. That number is within us as our externalization.[52]

And then in number eight:

> (VIII–1) Real space is intuitive, but mathematical space, which is also intuitive, is constructed out of the one-dimensional intuition of time.[53]

We observe in the notebooks, more than in the resulting dissertation, the ultimate foundation of human mathematical thinking: his mystic experience of 'just being there' and the observation of what occurs to him. Any form of interference is 'sinful'.

We also note that *real space* is distinguished here from *mathematical space*. Mathematical space is intuitive in the sense that it can be fully described and constructed departing from the intuition of time alone. Real space is also intuitive, but in a different sense. An intuition of a three-dimensional space 'as a consequence of our doom' should be understood to be the outcome (effect) of the abandoning by mankind of his natural destiny in his attempt to control the world. Man wants to rule, and for that purpose alone he 'mathematizes' the world, he invents mathematically described physical laws, projected on a three-dimensional real space. Here man is far removed from his one-dimensional mathematical space which is based on the ur-intuition.[54] One should note that this quote is from the first notebook, written at a time during which the influence of his recent publication *Life, Art and Mysticism* was still great compared to a few years later when arguments like the one in the quotation more or less slip out of the picture. But a certain mystic tendency in Brouwer's work was to remain.

Kant needed time *and* space as 'Anschauungsformen' to ground arithmetic and geometry. For him space and time are within us, in our perceptive faculty, as a necessary means to observe the objects in the external world. We cannot imagine the absence of space, but we can imagine empty space, since the a priori space presents itself to us as 'appearance' (Erscheinung).[55]

In Brouwer's words, when discussing Kant's concept of space in chapter II of the dissertation:

> Kant defends the following thesis on space:
>
> The perception of an external world by means of a three-dimensional Euclidean space is an invariable attribute of the human intellect;

[52] Het spreekt volgens onze doem vanzelf, dat onze ruimte drie afmetingen heeft. Dat getal zit daar in ons als onze veruiterlijking.

[53] De reële ruimte is intuïtief, maar de mathematische ruimte, die ook intuïtief is, is opgebouwd uit de eendimensionale tijdsintuïtie.

[54] This all will be extensively discussed in chapter 6 of this dissertation. Brouwer treats it in his second chapter.

[55] [Kant 1995]: *Transzendentale Elementarlehre*, §§ 1 – 5.

> another perception of an external world for the same human beings is a contradictory assumption.
>
> Kant proves his thesis as follows:
>
> Of empirical space we notice two things:
>
> 1^0. We obtain no external experiences barring those placed in empirical space, and cannot imagine those experiences apart from empirical space. (...)
>
> 2^0. For empirical space the three-dimensional Euclidean geometry is valid (...);
>
> from which it follows that the three-dimensional Euclidean geometry is a necessary condition for all external experiences and the only possible receptacle for the conception of an external world so that the properties of Euclidean geometry must be called *synthetic judgements a priori* for all external experience.
>
> Both premisses serve to demonstrate in a certain sense (...) the *objectivity* in the first instance of empirical space per se, without which no external experience is said to be accessible to thought, and secondly of the group of Euclidean motions constructed thereon.[56]

As a comment on this we state first that Brouwer's summary of Kant's conclusion that *three-dimensional Euclidean geometry* is a necessary condition for all external experience, should be limited to the statement that *three dimensional Euclidean space* is the required condition for such experience. The latter expresses Kant's intention better. A second comment is that Kant (Brouwer is summarizing Kant, after all) would not grant an objective per se status to empirical space. The Anschauungsform of space is the only way for us to experience objects. Space and time are the two 'Anschauungsformen' (forms of intuition), but an objectively existing space remains unknown to us on principle.

For Brouwer the intuition of time had to be sufficient as a foundation for the construction of the mathematical edifice since an intuition of space for the foundation of Euclidean geometry became untenable due to the development of

[56](dissertation, page 113 – 115) Kant verdedigt omtrent de ruimte de volgende stelling:

De voorstelling van een uitwendige wereld door middel van een driedimensionale Euclidische ruimte is van het menselijk intellect een onveranderlijk attribuut; een ándere voorstelling van een uitwendige wereld bij dezelfde mensen is een contradictoire onderstelling.

Kant bewijst zijn stelling als volgt:

Van de empirische ruimte merken we twee dingen op:

1^0. wij krijgen geen uitwendige ervaringen, dan geplaatst in de empirische ruimte, en kunnen ons die ervaringen niet los van de empirische ruimte denken (...),

2^0. voor de empirische ruimte geldt de driedimensionale Euclidische meetkunde (...),

waaruit volgt, dat de driedimensionale Euclidische meetkunde noodzakelijke voorwaarde voor alle uitwendige ervaringen en het enig mogelijke receptaculum voor de voorstelling ener uitwendige wereld is, zodat de eigenschappen der Euclidische meetkunde *synthetische oordelen a priori* voor alle uitwendige ervaringen moeten worden genoemd.

De beide premissen betogen in zekere zin (...) de *objectiviteit*, eerst van de empirische ruimte zonder meer, zonder welke geen uitwendige ervaring heet te kunnen worden gedacht, (...) en vervolgens van de daarin geconstrueerde Euclidische bewegingsgroep.

non-Euclidean geometries. The intuition of time guarantees us arithmetic (just as with Kant) but, as a result of Descartes' work, geometry becomes definable in terms of arithmetic (or rather: analysis).

We must interpret Brouwer's concept of the intuition of real space in the sense that we *experience* it, we observe it, as we experience and observe causal sequences in our daily life, as a consequence of man, operating in, and attempting to control, his surroundings. There is no objective external real space for Brouwer.

In regard to causality and causal sequences, note the difference with Kant, for whom causality is one of the categories (reine Verstandesbegriffe), which play their role in our reflection on the data, obtained by our senses and presented to the mind (Verstand). These data necessarily are observed through the Anschauungsformen space and time.

Brouwer's view on space is expressed in the following quote from the second notebook, in which again mathematics as the result of a free act can be recognized:

> (II–16) It is an act of free will to put space. It is an act of free will to put in it the relation of distance and straight line, which represents a Euclidean group.[57]

See also page 105 of the dissertation, where Brouwer strongly criticized Russell's view that a multidimensional continuum is a necessary condition for the experience of objects:

> To which we answer again that such a world of objects (*things*) is not a necessary condition for experience; that empirical space is an arbitrary creation [*added correction in handwriting:* of our imagination] to enable us, all the same, with the aid of mathematical induction to bring *different* causal sequences (of results of measurements) together *under one point of view*; (...).[58]

However, the creation of real external space seems not to be completely an act of free will, but is in its turn based on the postulate of the existence of continuous functions which man needs for an easy and successful description of the physical nature (which remains of course a consequence of the 'fall of mankind'):

---

[57] Het is willekeur om de ruimte te stellen. Het is willekeur om daarin het afstands- en rechtelijnverband te stellen, dat een Euclidische groep representeert. See also, again, [Brouwer 1933].

[58] Waarop wij weer antwoorden, dat een dergelijke wereld van *objecten (choses)* voor de ervaring niet nodig is, dat de empirische ruimte een willekeurige schepping (*toegevoegd*: onzer verbeelding) is, om *verschillende* causale volgreeksen (van meetresultaten), tòch met behulp van mathematische inductie *onder één gezichtspunt* samen te brengen; (...).

Note: In our opinion 'arbitrary' (willekeurig) has to be taken in its literal meaning of 'out of free will' and not as 'random'. In his translation Van Stigt uses 'arbitrary' but in our view the free-will-aspect has to be emphasized.

> (VIII–42) We would not have any representation of space if we were not thinking in terms of the postulate of mutual measurability of coordinates, that is, the existence of certain continuous functions.[59]

So far in the notebooks (still in the eighth) the primal intuition of time is just designated as the 'basic mathematical intuition', but not yet in the explicit form of the ur-intuition of the twofoldness discrete and continuous, the unity of *event – time span – other event*, i.e. the *move of time*, which, however, we encounter for the first time in the last notebook:

> (IX–26) The construction of the sequence one, two, three, ... out of the ur-intuition 0 proceeds as follows:
>
> (0) 1^{st}. one – two (separated by flow of time)
>
> (0) 2^{nd}. two – three (separated by flow of time)
>
> This sequence is applied when counting points with the help of the ur-intuition:
>
> (0) $\underline{1^{st}}$ one – visual perception of a (first) point
>
> (0) $\underline{2^{nd}}$ two – visual perception of a second point
>
> etc.[60]

and also, a few pages later:

> (IX–29) The sequence ω can only be constructed on the continuous intuition of time.[61]

Although in the first notebook Brouwer alluded to time as a necessary condition for the construction of a set,

> (I–27) Dedekind, in his 'example of an infinite system' actually constructs the 'chain' by a constant repetition of the successor operation. Because in fact he says: 'the whole, which I *can* construct is formed in this way' (where does he get the time for that?)[62]

---

[59] Wij zouden geen ruimtevoorstelling hebben, als we niet uitgingen van het postulaat der onderlinge meetbaarheid van coördinaten, dus van het bestaan van zekere stetige functies.

[60] Het opbouwen van de rij één, twee, drie, ... uit de oerintuïtie 0, gebeurt aldus:
(0) 1^e. één – twee (gescheiden door tijdsvloeiing)
(0) 2^e. twee – drie (gescheiden door tijdsvloeiing)
Deze rij wordt toegepast bij het tellen van punten, door middel van de oerintuïtie:
(0) $\underline{1^e}$ één – gezichtsgewaarwording van een (eerste) punt
(0) $\underline{2^e}$ twee – gezichtsgewaarwording van een (tweede) punt
enz.

[61] De reeks ω is alleen op te bouwen op de continue tijdsintuïtie.

[62] Dedekind bouwt feitelijk bij zijn 'voorbeeld van een oneindig systeem' de 'keten' op door telkens herhaling van de volgoperatie. Want feitelijk zegt hij: 'het geheel, dat ik maken *kan* (waar haalt hij de tijd daarvoor?) is aldus gevormd.

(Note that we are not yet discussing sets, their possible constructions and cardinalities. At this place we merely discuss the role of time in the construction of numbers.)

and in the third notebook time was mentioned as that, which can repair all separation (see above, quote III-7), it was only in the seventh that that Brouwer emphasized the intuition of time as the most basic ur-intuition for the construction of a sequence and in the ninth that this concept was worked out in the given quotation IX–26. Only here it is explicitly stated *how* the twofoldness of discrete and continuous arises out of the ur-intuition of the awareness of time, and how, as a result of this, the system of the natural numbers becomes feasible.

It was only about six months before the public defence of his dissertation, most probably in August or September 1906 as can be concluded from the correspondence with Korteweg, that the fundamental concept of the ur-intuition of the twofoldness became manifest in writing. Through this concept mathematical objects became free creations of the individual mind.

2.9 On the status of spoken or written signs

Returning now, after what seems to us a proper and justifiable interpretation of the ur-intuition and its corollaries, to the discussion on the status of the signs representing the numbers,[63] we note that in the construction of the order types ω and η the role of spoken or written signs is not mentioned; the numbers are a direct creation of the mind, and are retained in memory as *numbers* and not represented by symbols. On the role of spoken or written *language* in this process of creation, Brouwer stated in the earlier mentioned inaugural lecture *Intuitionism and Formalism*:

> And in the construction of these sets[64] neither ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set.[65]

Time is the only basic intuition, as we again can see in the same inaugural lecture:

> In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, (...). For since Descartes we have learned to reduce all these geometries to arithmetic by means of the calculus of coordinates.[66]

[63] See the first two paragraphs of Brouwer's dissertation; see also the beginning of section 2 of this chapter.

[64] i.e. denumerable sets.

[65] En bij de opbouw dezer verzameling kan noch de gewone, noch enige symbolische taal een andere functie vervullen, dan die van een onwiskundig hulpmiddel, om het wiskundig geheugen te ondersteunen of om door meerdere individuen dezelfde wiskundige verzameling te doen bouwen.

[66] Hiermede echter zijn op grond van de aprioriteit van de tijd niet slechts de eigenschappen der rekenkunde als synthetische oordelen a priori gekwalificeerd, doch ook die der meetkunde, (...). Immers sinds Descartes heeft men achtereenvolgens al deze geometrieën door middel van coördinatenrekening op de rekenkunde leren terugvoeren. (translation A. Dresden)

and *this* will remain one of the basic concepts for Brouwer, *together* with the concept of the experience of the move of time on which the intuition of continuous and discrete as an inseperable two-ity is based, *and also together* with that of the role of language as no more than an aid since mathematics is fundamentally a mental construction. This can be concluded from the quotes mentioned above, as well as from several published papers of later time, e.g. the paper *Historical background, principles and methods of intuitionism*, published in the *South African Journal of Science* in 1952. In this paper the development of intuitionistic mathematics is sketched, which development is in later years usually explained by Brouwer in terms of the 'two acts of intuitionism', the first of which goes as follows:

> (...) the First act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomena of language, which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind, having its origin in the perception of a *move of time,* i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the *empty form of the common substratum of all two-ities.* It is this common substratum, this empty form, which is the basic intuition of mathematics.
>
> (...)
>
> In the edifice of mathematical thought thus erected, language plays no other part than that of an efficient, but never infallable or exact, technique for memorizing mathematical constructions, and for suggesting them to others; so that mathematical language by itself can never create new mathematical systems.[67]

Hence signs *in spoken or written form, considered as words or symbols in a language*, are, when viewed on their own, no free creation of the individual mind, like the numbers are. We have a language (as a cultural phenomenon) at our disposal which makes communication possible, and the only role of this language in mathematics is *to refer to* the creations by the mind, with the purpose to support one's own memory and to communicate mental creations with others. Thus a very plausible interpretation of the first six pages of the dissertation seems to be that Brouwer presented a short overview of how we are able to know the system of the numbers, and how we are able to operate on it, this all being purely based on the intuition of time. *These things are intuitively clear* should, in this interpretation, be read as: *these matters are the intuitive awareness of the intuition of time and its two-ity, as man experiences it.* This will be described in chapter I of Brouwer's dissertation from page 8 onwards. The first six pages present the construction of the mathematical building, starting from the result of this intuition.

[67][Brouwer 1952], also in [Brouwer 1975], page 508 – 515.

However, as we said before, by the end of the first world war Brouwer was considering a revised edition of his dissertation. In view of that he made many corrections in his own copy of the dissertation, either based on later views or merely expressing his original intentions more clearly.[68] E.g. the sentence *'these things are intuitively clear'* was corrected as *'these things are intuitive'*, which certainly expressed Brouwer's intentions better, and the interpretation can now be read as *these matters are the intuition of time and its two-ity,* contrary to the original text, which has to be understood as 'these matters are immediately and intuitively understood by us'.

There is, however, more to say about written or spoken signs. We mentioned in the quote from the *South African Journal of Science* the first act of intuitionism as the separation of language from all mathematical activity, which abolishes all mathematical construction by means of linguistic applications of the principles of logic. The second act enables us to reconstruct mathematics, now as a purely *mental* building. The two acts of intuitionism were formulated explicitly for the first time in the *Berliner Gastvorlesung* (1927)[69] in similar terms as in the *South African Journal of Science.* The second act reads here:

> Es ist *die zweite Handlung* des Intuitionismus, welche hier einen kompensierenden Ausweg schafft, nämlich das Erkennen der Selbstentfaltung der mathematischen Ur-Intuition zur *Mengenkonstruktion.* Diese Mengenkonstruktion, welche leider trotz ihres einfachen gedanklichen Inhaltes eine einigermassen langatmige Beschreibung erfordert, welche aber ganz allein das ganze Gebäude der intuitionistischen Mathemathik trägt, besteht in Folgendem:
>
> Zunächst wird eine unbegrentzte Folge von Zeichen festgelegt mittels eines ersten Zeichens und eines Gesetzes, das aus jedem dieser Zeichenreihen das nächstfolgende herleitet. Wir wählen z.B. die Folge ζ der 'Nummern' $1, 2, 3, \ldots$. Sodann ist eine *Menge* ein *Gesetz*, (...)[70]

[68]See the new edition of Brouwer's dissertation, in [Dalen 2001].

[69]These lectures appeared in print only posthumously in 1992; see [Dalen, D. van (ed.) 1992].

[70][Dalen, D. van (ed.) 1992], page 23. It can be, and has been defended that this second act follows from the first: in the Cambridge lecture of November 1951 *Changes in the relation between classical logic and mathematics. (The influence of intuitionistic mathematics on logic)*, Brouwer stated: 'One of the reasons that led intuitionistic mathematics to this extension was the failure of classical mathematics to compose the continuum out of points without the help of logic'. Brouwer meant with this *extension* the creation of choice sequences as elements for spreads in order to be able to handle arbitrary elements of the continuum and to bring the continuum beyond its role of just the matrix to construct points on. But then Brouwer added in the margin of the manuscript: 'incorrect, the extension is an immediate consequence of the selfunfolding'. In other words, the second act of the recognition of the selfunfolding of the mathematical ur-intuition to the construction of spreads (sets, Mengen) follows from the first act without the need to state it explicitly; it is included in it, it is included in the recognition of intuitionistic mathematics as an essentially languageless activity of the mind, having its origin in the perception of a move of time. See [Dalen, D. van (ed.) 1981a], page 93.

The exact character of the mentioned law is not yet of importance here,[71] for now we just want to point to the fact that Brouwer used the concept 'Zeichen' (sign, symbol). Does that contradict the first act, that makes mathematics an essentially languageless activity of the mind? It seems not, because the 'unlimited sequence of signs' is here, as in the beginning of the dissertation, the written reproduction of the mental construction of, for instance, the natural number system. In view of the elucidation that Brouwer gave about the role of language in mathematics, this seems to be a reasonable explanation. See also the quoted phrase from his inaugural speech on page 56: '(...) neither ordinary language nor any symbolic *language* can have any other role than that of serving as a non-mathematical auxiliary, (...)' (my italics).

However, in 1947 Brouwer published a one-page paper, entitled *Guidelines of intuitionistic mathmatics*,[72] in which he felt the need for a further elaboration about the role of symbols in mathematics, and which ends with the phrase:

> Because mathematics is independent of language, the word *symbol (Zeichen)* and in particular the words *complex of digits (Ziffernkomplex)* must be understood in this definition[73] in the sense of *mental symbols,* consisting in previously obtained mathematical concepts.[74]

So, did Brouwer after all allow symbols an existence and a role in mathematics itself, and not just in the language that accompanies it? Or did he still intend to say: a symbol stands for a mathematical construction?

The exact text in the English translation speaks of 'symbols, consisting in previously obtained mathematical concepts'. The original Dutch text uses the words 'gedachtentekens, *bestaande in* reeds verkregen mathematische denkbaarheden'. 'Bestaande in' can be translated as 'consisting in' or 'existing in', but the term 'consisting in' has also the meaning of 'being based on' or 'being dependent on', whereas the meaning of 'existing in' is limited to the more literal meaning of just 'being in'. Hence 'consisting in' points to a dependence of the sign on the already present mental mathematical concepts, and must indeed be regarded as the proper translation.

The last quote tells us exactly how far Brouwer went in his further development of the symbol concept as a mathematical entity. If we again read the first page of his dissertation, especially the last phrase of it (in Brouwer's improved version 'these things are intuitive'), but this time in the light of this 1947 paper, we must conclude that *first* the mind creates the concept, i.e. the number or the number system, and *then* the mind creates an *abstract* symbol (i.e. not a physical symbol in ink or sound waves) for that concept, which stresses the similarity

[71] see chapter 4.

[72] Richtlijnen der Intuïtionistische Wiskunde, [Brouwer 1947], English translation in [Brouwer 1975], page 477.

[73] i.e. the definition of a spread, i.e. a 'Menge', which is a law ('ein Gesetz'); see also our fourth chapter.

[74] Wegens de taalloosheid der wiskunde behoort in de genoemde definitie [van spreiding] bij *teken (Zeichen),* in het bijzonder ook bij het woord *cijfergroep (Ziffernkomplex),* gedacht te worden aan gedachtentekens, bestaande in reeds verkregen mathematische denkbaarheden.

with other experiences of the move of time. In fact, man has no choice. He simply *has* to create a mental abstract symbol after divesting the actual experiences of all content, in order to represent the natural numbers constructed so far; and this symbol *has* to be a part of mathematics since it is needed for the extension of the natural number system: it has to be connected to new abstracted experiences by a new continuum, identical to all others. Hence the mental symbol is still no element of the language. It only becomes so if it is transformed into a linguistic symbol, which is orally expressed or written down and *used in a logical or other language game,* apart from mathematics. Only then it has become the symbol that Brouwer meant in the earlier quoted phrase from the inaugural lecture: 'neither ordinary language, nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary'.

In the second chapter of his dissertation, when discussing the use of mathematics for the intervention in nature by means of the creation of causal sequences, it is expressed as follows:

> the simplest example of this being the sound image (or written symbol) of a cardinal number obtained by the process of counting, or the sound image (or written symbol) of the measuring number obtained by the process of measurement.[75]

Here again: the act of counting is mental, the act of assigning a symbol to the result *and* the symbol itself are also mental, but the *expressed* sound or the *written* symbol becomes part of the language and, when used as such, does no longer belong to the mental building of mathematics. The mind creates a cardinal number and then it creates a symbol which stands for the created number and, in order to prevent an endless series of arbitrary signs, it creates a systematic way of constructing signs, for example the dual or the decimal notation of numbers.

Note the essential and fundamental difference with the formalist approach where every system of symbols, operating rules, and axioms, stripped of all intuition, forms a mathematical theory.

2.10 The notebooks on spoken or written signs

Little is said on this subject in the nine notebooks. It seems as if the concept was almost taken for granted in the dissertation, or was not yet completely worked out, perhaps because Brouwer was not fully aware of all its implications. The really important remarks were made in later publications, beginning with the inaugural lecture. Only a few allusions to the sign concept in the notebooks are worth to be mentioned here.

[75](page 84) het eenvoudigste voorbeeld hiervan is het door de telhandeling verkregen klankbeeld (of schriftteken) van aantal, of het door de maathandeling verkregen klankbeeld (of schriftteken) van maatgetal.

In the fifth notebook the construction of the system of the natural numbers is discussed:

> (V–14) Now the practice of counting is the construction of the fantasy system *one, two, three etc.,* to the images of which are related the objects of reality as a method.[76]

Hence the symbols *one, two, three, ...* are *images* of things in reality, they are reflecting a facet of the external world by means of a mental creation of the symbols, and are, as a mental creation, *not yet* elements of a spoken or written language.

In the seventh notebook, when discussing the creation of Cantorian numbers of the second number class, it is, in a deleted fragment, expressed as follows:

> (VII–14) I can write down all Cantorian numbers of the second number class by means of a finite number of signs. But those signs are symbols, which have to be shaped with the help of the 'and so on'.[77]

The signs are in this case part of the language, and stand for the mental creation of the symbols for the Cantorian numbers.

Finally, on the first page of the eighth notebook, as a note in the margin:

> (VIII–1) Whereas signs, that stand for something arbitrary, e.g. for 'arbitrary finite number', just belong to the system of signs, that accompany the passion for building, not the building itself.[78]

It seems as if Brouwer treated signs and symbols in the notebooks and in the dissertation intuitively, and that the concept was only fully elaborated and viewed in all its consequences *after* the public defence of the dissertation. There was more to it than he originally realized.

2.11 Summary and conclusions of this chapter

We have seen in this chapter that the exact charaterization and specification of the ur-intuition appeared very late in the notebooks, hence we may assume that the details of this concept only materialized shortly before writing it down in the draft for his dissertation, and hence shortly before the date of his academic promotion.

The other sources in which this basal concept showed up (Mannoury's and Barrau's 'oposition from the floor', the Rome lecture and the Vienna lecture)

[76]Nu is de praktijk van het tellen de opbouw van het fantasie systeem *een, twee, drie enz.,* op welke beelden de dingen der werkelijkheid als methode worden betrokken.

[77]Alle Cantorse getallen der tweede getalklasse kan ik opschrijven met een eindig aantal tekens. Maar die tekens zijn de nieuw te vormen symbolen met behulp van het 'èn dat en zovoorts'.

[78]Terwijl tekens voor iets willekeurigs, b.v. voor 'willekeurig getal', alleen horen in het tekensysteem, begeleidend iets van de bouw-passie, niet van het bouwwerk zelf.

were very clarifying, compared to the briefly worded text from the dissertation. However, an interpretation still had to be worked out. The construction of the several number systems, as corollaries of the ur-intuition, required a certain amount of explication, but the main results for arithmetic (and hence for the whole of mathematics) turned out to be that time is the only intuition required for the construction of the mathematical edifice. The intuition of space is (contrary to Kant's view) no longer needed.

Brouwer did recognize an 'intuition of real space', but this is just an *experience* of it, just as the whole of our external world and the causal sequences in it are experienced by us; but this experience does not have the character of an 'ur-intuition', and this 'experience without content' is not needed for the construction of the building of mathematics. The individual mind constructs the numbers as abstractions from sequences of experienced events that are taking place in time, and he also constructs mental signs to represent these numbers for future use in e.g. arithmetic or set theory, and this is sufficient for the ultimate foundation of his whole enterprise.

As a final remark we note that in Brouwer's later work the idea of the continuum as a 'matrix to construct points on' disappeared with the emergence of choice sequences and spreads, but that the concept of the continuum as the immediate result of the ur-intuition remained (see page 74).

Chapter 3

The continuum, intuitive and measurable

3.1 Introduction

This chapter covers Brouwer's analysis of the result of the ur-intuition, viz. the intuitive continuum, its character and its properties.[1] We ended the preceding chapter with the remark that the intuitive continuum as the immediate corollary of the ur-intuition was emphasized repeatedly in the dissertation, as well as in other publications of later date. Also after the introduction and the development of the 'perfect spread', which gives us the 'full continuum' of all unfinished choice sequences on the unit interval, the continuum still remains the immediate result of the ur-intuition. We will see that, in order to prevent confusion, the different concepts about the continuum (like the 'intuitive' continuum, the 'reduced' continuum, the 'full' continuum) have to be distinguished sharply.

We find the origin of Brouwer's continuum concept, as we saw, in the flow of time as the medium of cohesion between two well-separated events. Apart from the construction of the integers and of the rational scale, more properties of the continuum and of these scales can be deduced, still on the basis of the mathematical intuition and of the concept of mathematics as a free creation of the individual human mind.

Several properties of the continuum were discussed in the previous chapter, and new ones will come up in the present one, again requiring interpretation about what exactly Brouwer had in mind, like the idea of a 'freely constructed scale', or the property of being 'partlially unknown' of an infinite sequence approximating a point.

We will argue that 'partially unknown' can be interpreted in two ways:
1. Partially unknown on principle. This is the case with lawless choice sequences; the respective elements are not individualized.

[1] See Brouwer's dissertation, page 9 ff.

2. Partially unknown but individualized. The elements are not yet known since they are not yet computed, but they are 'intensionally known'; this is the case with e.g. the sequences defining $\sqrt{2}$ or π.

Another important item to be discussed is the notion of the 'everywhere dense scale of the rationals'. Brouwer claimed that in the process of constructing the rational scale, there might be 'points not reached'; we can even leave (or find out afterwards that we have left) a complete segment of the continuum unpenetrated by inserted points. Brouwer declared the everywhere dense characteristic of the rational scale η to be the result of an 'agreement' (which we will argue to be the result of a free act of the mind); but then the following two questions have to be examined:
1. How can we exclude a segment of the continuum from the insertion of new elements, given the fact that the awareness of the non-coincidence of two experiences *is* the inserted element? In other words, how do we create an empty segment under this interpretation?
2. Once we succeed in the construction of such a segment or we become aware of it, how do we eliminate it again?

Another point of discussion is the Bolzano-Weierstrass theorem. Brouwer stated that this theorem is a direct corollary of the measurability of the continuum, but closer inspection reveals that he employed in its proof the principle of 'reductio ad absurdum', as well as the 'tertium exclusum'.

Of special interest for this subject are the notebooks, which show a development from a possible constructibility of the continuum (albeit that this possibility was only expressed once, at the level of a thought experiment), via 'something mysterious but nevertheless intuitively known' to the continuum 'as the result of the ur-intuition'. An evolution in Brouwer's thoughts about the exact nature of the continuum can be observed in the nine notebooks.

In the first section of this chapter a concise overview will be presented of just a few different (and sometimes mutually opposing) opinions about the continuum, as well as the names of the philosophers and mathematicians attached to them. This overview serves as an introductory remark, to make a comparison with Brouwer's ideas possible, which ideas were often deviating from those of the mainstream of mathematics. Moreover, this overview shows the importance that, through the ages, mathematicians and philosophers have attached to the continuum.

3.1.1 Aristotle

Euclid held the position that a line is composed of points, as can be concluded form definition 4 in Book 1 of *The Elements*:

> 4. A *straight* line is a line which lies evenly with the points on itself.[2]

[2]A not too lucid definition; see [Heath 1956], page 153.

Aristotle, who lived some fifty years earlier, took up a different position. For him, a line (the continuum) is not composed of points; all one can say is that a line is indefinitely divisible. In his *Physics*, in Book III, under B–6, a–14, this is explained as follows:

> Now, 'to be' means either 'to be potentially' or 'to be actually', and a thing may be infinite either by addition or by division. I have argued that no actual magnitude can be infinite, but it can still be infinitely divisible (it is not hard to disprove the idea that there are indivisible lines), and we are left with things being infinite potentially.[3]

and the first part of Book VI contains the following sections:

> Proof that no continuum is made up of indivisible parts.

and

> Proof that distance, time and movement are all continua.

In the first section of book VI Aristotle argued as follows:

> For instance, a line, which is continuous, cannot consist of points, which are indivisible, first because in the case of points there are no limits to form a unity (since nothing indivisible has a limit which is distinct from any other part of it), and second because in their case there are no limits to be together (since anything which lacks parts lacks limits too, because a limit is distinct from that of which it is a limit).[4]

A similarity between Aristoteles' conclusions and Brouwer's ideas will be (or will become) clear, and the contrast with e.g. Cantor is obvious.

3.1.2 Georg Cantor

In chapter 1 of this dissertation (see page 8 and page 14) Cantor's views on the continuum were given: the continuum is composed of points. Cantor's continuum is the arithmetical continuum of the real numbers. The concepts of time and space presuppose the continuum concept.

3.1.3 Hermann Weyl

In 1918 Weyl published *Das Kontinuum.* In chapter II of this booklet, *Zahlbegriff und Kontinuum*, the system of the real numbers is defined in a way similar to Dedekind. In § 6 the 'anschauliches Kontinuum' (intuitive continuum) and the 'mathematisches Kontinuum' are discussed. In this section it is stated:

[3] [Aristotle 1999], page 71.
[4] Op. cit. page 138.

> Bleiben wir, um das Verhältnis zwischen einem anschaulich gegebenen Kontinuum und dem Zahlbegriff besser zu verstehen (..), bei der *Zeit* als dem fundamentalsten Kontinuum;
>
> (...)
>
> Um zunächst einmal überhaupt die Beziehung zur mathematischen Begriffswelt herstellen zu können, sei die ideelle Möglichkeit, in dieser Zeit ein streng punktuelles *'jetzt'* zu setzen, sei die Aufweisbarkeit von Zeitpunkten zuzgegeben.
>
> (...)
>
> Zwei Zeitpunkte A, B, von denen A der frühere ist, begrenzen eine *Zeitstrecke* AB; in sie hinein fällt jeder Zeitpunkt, der später als A, aber früher als B ist. Der Erlebnisgehalt, welcher die Zeitstrecke AB erfüllt, könnte 'an sich', ohne irgendwie ein andrer zu sein als er ist, in irgend eine andere Zeit fallen; die Zeitstrecke, die er dort erfüllen würde, ist der Strecke AB *gleich*.[5]

Note the similarity with Brouwer's concept of the combination *discrete* and *continuous* which we observed in the previous chapter. From available evidence we know that Weyl was in the possession of the English translation of Brouwer's inaugural address from 1912, in which this inseparable combination is stressed. He may also have read *Die möglichen Mächtigkeiten* (1908) in which also the 'Zeitstrecke' between two events as a matrix for 'Zeitpunkte' is underlined.[6] Weyl then posed the question whether we can use this concept of 'points in a time continuum' and their mutual relation of 'earlier' and 'later' as the foundation of the real number concept, and thus, whether this time continuum can serve as a model for the mathematical continuum of real numbers, and he claimed:

> Gewiß: das anschauliche und das mathematische Kontinuum decken sich nicht; zwischen ihnen ist eine tiefe Kluft befestigt. Aber doch sind es vernünftige Motive, die uns in unserm Bestreben, die Welt zu begreifen, aus dem einen ins andere hinübertreiben;
>
> (...)
>
> Beschränken wir uns hinsichtlich des Raumes auf die Geometrie der Geraden! Will man nun doch versuchen, eine Zeit- und Raumlehre als selbständige mathematisch-axiomatische Wissenschaft aufzurichten, so muß man immerhin folgendes beachten.
>
> 1. Die Aufweisung eines einzelnen Punktes ist unmöglich. Auch sind Punkte keine Individuen und können daher nicht durch ihre Eigenschaften charakterisiert werden. (Während das 'Kontinuum'

[5] [Weyl 1918], page 67.

[6] Note that Brouwer's thesis *On the Foundations of Mathematics* was not yet available in any other language except Dutch.

> der reellen Zahlen aus lauter Individuen besteht, ist das der Zeit- oder Raumpunkte homogen.)[7]

Note the terminology: the continuum of time- or spacepoints is 'homogeneous', which has to be understood in the sense of 'having no gaps', i.e. in the Brouwerian meaning of 'flowing'. But the conclusion is clear: the intuitive continuum of space and time is *not* composed of individual points which can be indicated one by one, contrary to the 'continuum' of real numbers. The quotation-marks around the term *continuum* serve to stress that, for Weyl, the totality of the real numbers, even though it has the properties of *denseness* and *perfectness*, is not a continuum in the intuitive sense, like the continua of time and space are. But this intuitive continuum has to be transformed into a *measurable* continuum, and therefore the following formulation is offered:

> 2. Das Stetigkeitsaxiom muß dahin formuliert werden, daß mit Bezug auf eine Einheitsstrecke OE jedem Punkt P eine reelle Zahl als Abzisse entspricht und umgekehrt.

3.1.4 Otto Hölder

Hölder published in 1924 *Die mathematische Methode*.[8] In this book he discussed, on the one hand, in Part I, vierter Abschnitt, the 'points of a line' when dealing with mathematical continuity:

> (§ 33) (...) der Begriff einer zusammenhängende Linie. Er kann dadurch definiert werden, daß eine Linie, in der Weise auf eine geradlinige Strecke bezogen wird, daß jedem Punkte der Strecke ein einziger Punkt der Linie zugeordnet erscheint; (...)

On the other hand, in Part III, *Zusammenhang mit der Erfahrung*, § 134, *Apriorisches und aposteriorisches Wissen*, he stated:

> Da ferner, was die Hauptsache ist, gewichtige innere und äussere Gründe für die Annahme des Kontinuums sprechen, so kann ich kein Hindernis sehen, die Idee des einfachen Kontinuums als eine unbedingte (a priorische) Form anzusehen, welche die Bedingung für gewisse Arten der Erkenntnis darstellt.

with, in a footnote, the addition:

> Die Gedanke, das Kontinuum einfach als gegeben anzunehmen, hat neuerdings auch unter den Mathematikern an Boden gewonnen, wie

[7] Op. cit. page 70 – 73.

[8] [Hölder 1924]; full title: *Die mathematische Methode: logisch erkenntnistheoretische Untersuchungen im Gebiet der Mathematik.*

> aus einer 1920 in Nauheim bei der Mathematikerversammlung gepflogenen Diskussion hervorzugehen scheint.[9]

In §§ 75 and 76 the irrational numbers are studied; they are defined with the help of Dedekind cuts. At the end of § 76 Hölder wrote:

> Bedenkt man, daß die Definition jedes Schnittes ein besonderes Gesetz erfordert, so bedeutet der Begriff der Gesamtheit aller Schnitte, daß wir glauben, uns die Gesamtheit aller der Gesetze, die noch einer gewissen Forderung entsprechen, denken zu können. Eine solche gänzlich unbestimmte Gesamtheit dürfte aber einen unzulässigen Begriff vorstellen; demgemäß bin ich der Ansicht, daß *das Kontinuum nicht rein arithmetisch erzeugt werden kann.*

with, again in a footnote, the addition:

> Ich habe dies bereits im Jahre 1892 ausgesprochen (vgl. Göttingische gelehrte Anzeigen, 1892, S 594). H. Weyl in seiner Schrift: Das Kontinuum, kritische Untersuchungen über die Grundlagen der Analysis, 1918, vertritt sehr entschieden eben diese Ansicht.

But there seems to be a fundamental difference between Hölder and Weyl: for Hölder the continuum is composed of points, but cannot in any way be constructed from points. It is presented to us as a totality.

3.1.5 Émile Borel

Émile Borel presented in his *Éléments de la Théorie des Ensembles* (1949) in chapter II a definition of the continuum in terms of Dedekind cuts. For him the line is composed of points. In § 11 he remarked about the continuum and its cardinality:

> Nous définissons la puissance du continu comme étant celle de l'ensemble de points d'un segment de droite, par exemple du segment $[0, 1]$

and the title of § 12 reads *Le continu n'est pas dénombrable.*

Note, on the other hand, that Borel, as a French intuitionist, required for a mathematical object a finite definition and, for that reason, did not accept infinite choice sequences as representations for arbitrary non-lawlike irrational numbers.[10]

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[9] Clearly referring to Brouwer, who, at the Nauheim conference of 1920, presented his lecture *Does every real number have a decimal expansion?* ([Brouwer 1921a], in English [Brouwer 1921b], also published in the Mathematische Annalen, [Brouwer 1921c]) Weyl was also present at this conference, but he lectured on a physical subject.

[10] [Borel 1950], page 160.

3.1.6 A modern sound

Finally, a modern dictionary like *The Collins Dictionary of Mathematics* (edition 1989) defines the continuum simply as the set of all real numbers.

This is the concept of the continuum that eventually survived till the present time: the continuum as the set of all real numbers, or the set of all Dedekind cuts. This is exactly the continuum concept which was contested by Brouwer in 1907. For him the continuum is intuitively given to us, as the flowing medium of cohesion between two events, not itself consisting of points (events) but an inexhaustible matrix for a continued insertion of points. Originally there are no points on the continuum, we can *construct* points on it, or *indicate* a place on it, which we call 'indicated point'.

As a conclusion of this introduction, we may state that, *firstly*, there is in the history of mathematics no communis opinio about the nature of the continuum, and, *secondly*, Brouwer's view reminds us of what Aristotle already had stated in the *Physics*.

Brouwer's continuum concept was explained in the previous chapter; in the next section we summarize the properties of the intuitive continuum which result from that concept, and which are presented in his dissertation. Also we try to piece together from the notebooks the development of Brouwer's ideas on the intuitive and the measurable continuum. We must, however, be aware of the fact that in the notebooks the concepts of the (measurable) continuum and of sets (which are to be discussed in chapter 4 of this dissertation) are frequently too interwoven to keep them strictly separate.

3.2 A few other publications by Brouwer

The first seven pages of Brouwer's dissertation presented us the intuitive continuum as the continuous flow of time, of which we become aware by two well-separated sensations. Divested of all quality, this combination event-connecting medium-event gives us the numbers *one* and *two*, and this two-ity can be seen as a new single thing, to which another well-separated event can be added as a third, giving the number *three*, and so on.

Before going into the details of the properties of the intuitive continuum as they were further elaborated by Brouwer, we will first briefly summarize the content of three other papers (partly of much later date) on this concept, as a first indication of where Brouwer's new ideas and their elaborations ultimately were leading to.

It will be obvious that in Brouwer's reply to Mannoury's opposition during the public defence of the dissertation, the same concept of the intuitive continuum was pronounced as in the dissertation. He was, after all, defending his dissertation. In the previous chapter about the ur-intuition of mathematics we already referred to this reply, in which Brouwer also clarified the origin of the continuum concept from the move of time. Brouwer emphasized again that the continuum intuition *is* the intuition of the medium of cohesion between the two events. There is *first thing – medium of cohesion – second thing*, or, literally translated *primum – continuum – secundum.*

In an earlier quoted (see chapter 2) fragment from his *Inaugural lecture* (*Intuitionism and Formalism*, 1912) the ur-intuition of mathematics is introduced in similar wording as in the dissertation:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness.[11]

This ur-intuition gives rise to the system of the natural numbers (finite ordinal numbers) and, as a natural corollary, also to the smallest infinite ordinal number ω. Immediately after that, on the same page, it is stated:

> Finally this basal intuition of mathematics, in which the connected and the separate, the continuum and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e. of the 'between', which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units.[12]

Also in the *Rome lecture* (1908) it is presented in the same formulation and with the same content (see above, on page 42):

> Das Erste und das Zweite werden *zusammengehalten,* und in dieser Zusammenhaltung besteht die Intuition des continuierlichen (continere = zusammenhalten).

We discussed this in chapter 2, but it is summarized here to stress the contrast with Brouwer's later views on the continuum and its decimal representation. For that purpose we present the following quotes from the *Begründung* papers (1918–1919).[13] In this paper the new set concept, that of *spread* (in

[11][Brouwer 1912]; see [Benacerraf and Putnam 1983], page 80 for the English translation by A. Dresden; or, in Dutch, [Dalen 2001], page 182. Dit neo-intuïtionisme ziet het uiteenvallen van levensmomenten in qualitatief verschillende delen, die alleen gescheiden door de tijd zich weer kunnen verenigen, als oergebeuren in het menselijk intellect, en het abstraheren van dit uiteenvallen van elke gevoelsinhoud tot de intuïtie van twee-enigheid zonder meer, als oergebeuren van het wiskundig denken.

[12]Eindelijk is in de oerintuïtie der wiskunde, waarin het samengehoudene en het gescheidene, het continue en het discrete verenigd liggen, mede onmiddellijk aanwezig de intuïtie van het lineaire continuüm, d.w.z. van het 'tussen', dat door inschakeling van nieuwe eenheden zich nooit uitput, dus ook nooit als verzameling van eenheden zonder meer kan worden gedacht.

[13]*Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*, [Brouwer 1918] and [Brouwer 1919a]. Further clarification and discussion on this foundational paper follows on page 133, in which section the quotes will be presented and discussed more extensively.

German: Menge), is defined, and one easily might get the impression that with the development of this concept, Brouwer is giving up the notion of the intuitive continuum from the dissertation. But this impression will turn out to be a false one.

> Der Mengenlehre liegt eine unbegrentzte Folge von Zeichen zu Grunde, welche bestimmt wird durch ein erstes Zeichen und das Gesetz, das aus jedem dieser Zeichen das nächstfolgende herleitet. Unter den mannigfachen hierzu brauchbaren Gesetzen erscheint dasjenige am geeignetesten, welches die Folge ζ der Ziffernkomplexe 1, 2, 3, 4, 5, ... erzeugt.
>
> Eine *Menge* ist ein *Gesetz,* auf Grund dessen, wenn immer wieder ein willkürlicher Ziffernkomplex der Folge ζ gewählt wird, jede dieser Wahlen entweder ein bestimmtes Zeichen, oder nichts erzeugt, oder aber die Hemmung des Prozesses und die definitive Vernichtung seines Resultates herbeiführt, wobei für jedes n nach jeder ungehemmten Folge von $n-1$ Wahlen wenigstens ein Ziffernkomplex angegeben werden kann, der, wenn er als n-ter Ziffernkomplex gewählt wird, *nicht* die Hemmung des Prozesses herbeiführt. Jede in dieser Weise von der Menge erzeugte Zeichenfolge (welche also im allgemeinen nicht fertig darstellbar ist) heisst ein *Element der Menge.* Die gemeinsame Entstehungsart der Elemente einer Menge M werden wir ebenfalls kurz als *die Menge M* bezeichnen.[14]

The simplest example of a spread is one, in which the basic collection is the set of the natural numbers, and in which on every node the sign is composed of the same natural number as the one chosen from the basic collection. In this representation and after removal of every finitely terminating branch, the elements of the set consist of infinite sequences of natural numbers.

If, then, we allow at every node *every* choice[15] from the system ζ of the natural numbers, we obtain the set C:

> Ein zweites Beispiel einer unendlichen Menge bildet die Menge C der unbeschränkt fortgesetzten Folgen von zu ζ gehörigen Ziffernkomplexen, deren Kardinalzahl wir mit c bezeichnen werden.[16]

This can be interpreted in more than one way as a representation of the open or half open set of the real or the irrational numbers between 0 and 1, as Brouwer explained on the same page. Also on page 17 of this paper it is expressed as follows:

[14] [Brouwer 1918], page 3; see also [Brouwer 1975], page 150.

[15] This 'every' is of course in its most basic form limited to the representation of the number system, with which we operate. In the binary system the number of choices is two, in the decimal system it is ten, and we are talking about a finite spread in case of a binary or decimal representation of the set C of the real numbers.

[16] *Begründung* paper I, [Brouwer 1918], page 9 or [Brouwer 1975], page 156.

> Ein Beispiel einer (im weiteren Sinne) *überall dichten, perfekten* Menge liefert die Menge C, geordnet auf Grund der natürlichen Rangordnung der von ihr erzeugten reellen Zahlen zwischen 0 und 1.[17]

Hence the set C is not itself the set of the reals, but can be interpreted as such. Apparently, in 1918 Brouwer was of the opinion that the 'continuum of the set of the real numbers' on the open interval $(0, 1)$ is an ordered set. In 1923 he proved this theorem to be wrong.[18]

It is important to note that it is now possible to speak of *arbitrary elements of the set* C, hence of an *arbitrary* real number, as the continuation of this quote on the same page shows:

> Seien nämlich $a_1...a_n, b_1, b_2...$ und $a_1..a_n, c_1, c_2...$ $(b_1 > c_1)$ zwei willkürliche Elemente von C ...

We draw attention to the fact that in the second *Begründung* paper[19] 'a point on a line or on a plane' is neither constructed according to some algorithm, nor is it given in the form of a sequence of signs (from ζ); it consists instead of a non-terminating choice sequence of nested intervals, defined with the help of the rational η-scale, and which sequence remains fundamentally unfinished. In fact in this paper such a set of nested intervals is defined in the form of the *ebene Punktmenge*, but the definition also applies to n dimensions or to one dimension:

> In derselben Weise, wie Punkte der Ebene und ebene Punktmengen, können *Punkte des n-dimensionalen Raumes* und *n-dimensionale Punktmengen* definiert werden.[20]

with the following clarification in a footnote on the next page:

> Die Bezeichnung 'Punkt der geraden Linie', bzw 'Punkt des n-dimensionalen Cartesischen Raumes' ist schon S. 10 des ersten Teiles einmal gebraucht worden, aber in einem von dem hier definierten verschiedenen Sinne.

the difference being, that in the first paper a 'point' is an infinite sequence of elements of ζ, whereas in the second paper 'points' are defined as infinite sequences of nested intervals.

One might indeed get the impression that Brouwer has abandoned his concept of the continuum as the result of the ur-intuition. But he never retracted his view as expressed in the dissertation and in the notebooks, viz. that the continuum is intuitively presented to us; neither did he mention the concept in

[17] Ibid, page 164.
[18] See page 105.
[19] *Theorie der Punktmengen*, [Brouwer 1919a].
[20] On the first page of [Brouwer 1919a]; see [Brouwer 1975], page 191.

its explicit form any more, except for a remark at the end of the first section of the second Vienna lecture, in which the validity of the ur-intuition of the two-ity is stressed and, in addition to that, a small but important handwritten note in that lecture, both showing that he, in fact, held on to his old view of an intuitively given continuum. This handwritten note is in the form of a reminder, that it should be added 'at the end of section I of the continuum lecture, that, nevertheless, the continuum is still the immediate result of the ur-intuition' (see page 74 for the relevant quote).

Hence Brouwer distinguished between the intuitively given continuum as the connecting medium between two separated events, and the 'full continuum' (as he named it in the second Vienna lecture) of the set of the real numbers between 0 and 1, represented in the form of unfinished choice sequences. We must keep in mind this distinction when reading the *Begründung* papers and the second Vienna lecture. In the *Begründung* paper I, page 9, the set C was defined as the 'unbeschränkt fortgesetzten Folgen von zu ζ gehörigen Ziffernkomplexen, deren Kardinalzahl wir mit c bezeichnen werden', but Brouwer was clearly referring to a representation of the real numbers on the unit line segment, and not to the intuitive continuum.[21] Also in the second Vienna lecture Brouwer spoke of the 'finished elements of the reduced continuum' and of the 'unfinished elements (*choice sequences*) of the full continuum'.

Brouwer may have been aware of a possible confusion in the whole of the continuum concept, and perhaps for that reason the intuitive continuum is explicitly restored in all its dignity in the last paragraph of the first section of this second Vienna lecture; in fact it was never abandoned, and it remained the only ur-intuition on which all of mathematics, including the 'full continuum of the unfinished elements' is founded:

> Die Einführung der Mengenkonstruktion, auf welcher also die fertige überabzählbare Vielfachheit des Kontinuums beruht, bedarf nach stattgefundener Besinnung auf die mathematische Urintuition der Zweiheit, welche dem gesamten Intuitionismus zugrundeliegt, keiner weiteren Besinnung, und impliziert auch keine petitio principii (so daß die anfangs erwähnte Betrachtung des Kontinuums als reine Anschauung a priori nach Kant und Schopenhauer[22] sich im Lichte des Intuitionismus im wesentlichen behauptet).[23]

To stress the content of this quote, but perhaps also to preclude or correct in a next edition wrong conclusions by the reader, which may very well be caused by the use of the term 'continuum' in the two presented ways, Brouwer added

[21] Note the similarity with Weyl (page 65), who explicitly stated that the space continuum is not composed of points, and has to be distinguished from the set of the real numbers of the unit line interval, which is also emphasized in Brouwer's dissertation.

[22] See e.g. [Schopenhauer 1970], *Handschriftliche Nachlass* III, *Reisebuch 1819*, § 33 and § 85. See also [Schopenhauer 1977], *Parerga und Paralipomena* II, § 29, 30 about respectively time and space as a priori Anschauungsformen.

[23] [Brouwer 1930b], page 6; see also [Brouwer 1975], page 434.

the handwritten remark in the margin of his own copy of the second Vienna lecture:

> Add at the end of section I of the continuum lecture that, nevertheless, the continuum is still the *immediate result* of the ur-intuition, just as with Kant and Schopenhauer.[24]

As a conclusion from the text of the beginning of the dissertation and of the discussed parts of the other publications, we can now state the following:
One should distinguish between the intuitively given continuum based on the ur-intuition, and the 'full continuum' of the unfinished elements of the unit segment. The latter depends for its existence on the former, since only the ur-intuition makes the existence of the general concept of the continuum possible; in other words, the continuum has to be a familiar intuition-based concept before we can 'simulate' it by defining arbitrary elements on it by means of choice sequences. This latter act is called by Brouwer the *intuitive initial construction of mathematics* (die intuitive Anfangskonstruktion der Mathematik). In a letter to Fraenkel, dated January 12 1927, Brouwer even stated that this initial construction was already present in the dissertation.[25] For a further discussion of this claim, see page 117 and page 156.

On the intuitive continuum, which is turned into a measurable continuum,[26] an arbitrary point P can be approximated by an infinite sequence of nested intervals, determined by rationals of (limited) free choice. Conversely, a freely created infinite choice sequence of nested intervals, with a width becoming smaller than any positive rational, defined on a measurable continuum, determines a point P, in the sense that not the limit of the sequence designates the point, but the always unfinished sequence itself *is* the point.[27] Apparently, Brouwer was looking for a clear presentation of these concepts by means of a lucid and concise terminology. Note that the single term 'continuum' is always referring to the intuitive continuum, which remains the most fundamental basis for all mathematics. To designate the infinite and everywhere dense sets on the intuitive continuum, an additional term is added, giving the 'reduced continuum' of the denumerable set of finished and well-defined elements (e.g. the η scale), and the 'full continuum' of the non-denumerable set of the unfinished elements of choice sequences which act as a representation of the set of the reals.

The set of all real numbers is henceforth (i.e. after 1918) defined with the help of spreads, and the spread in which on every node all choices are available, and which stands for the 'full continuum' of the real numbers, is now called the *universal spread* or the *perfect spread* (a term of Heyting).

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[24] In continuum voordracht, aan het slot van I toevoegen, dat het continuum dus toch weer uit de oerintuïtie *onmiddellijk gegeven* is, juist als bij Kant en Schopenhauer.
(This copy is kept in the Brouwer archives.)

[25] See [Dalen, D. van 2000], page 289.

[26] See the next section of this chapter.

[27] After all, if the *limit* of the sequence would define a point, then the continuum would be (or become) atomic again.

3.3 Properties of the intuitive continuum

After this preview on future developments, we will now return to the discussion in the dissertation, and we will present what seems to us, also in the light of the mentioned papers from Brouwer's hand, a proper interpretation of the construction of a scale on the intuitive continuum in order to transform it into a measurable continuum.

On page 8 and 9 of the dissertation, properties of this intuitive continuum are formulated; these properties are valid for the time continuum *and* for the straight line (as well as for multidimensional space). Several of the following properties merit a further discussion. They are not all of equal importance, but we will mention them all for completeness sake:

The construction of the everywhere dense scale

1. There is neither a first, nor a last point in a sequence of points that can be constructed on the intuitive continuum by means of the ur-intuition of the two-ity; a sequence of points can de extended to the left and to the right, forming together a sequence of ordertype of the integers.

The statement that there is no first point in a sequence of points on the continuum is not further explained by Brouwer and might seem contradictory. After all, there is a first event which is labelled as 0 or as 1. We discussed this dilemma in the previous chapter (see page 48) and we suggested the following interpretation: we can label the second event as 0 and the third as 1 etc. The interpolated point between the first and the second we may call then -1, the interpolated point between the first event and -1 we then call -2 etc. By iteration of this process, the first event in this interpretation then gets informally the status of $-\infty$. By repeated and continued interpolation between the constructed numbers on this scale of integers, the positive and negative rationals can be constructed. But this, again, is an interpretation, and more interpretations are possible of Brouwer's mere statement that the sequence of points on the intuitive continuum contains neither a first point nor a last point.

2. Every interval on the intuitive continuum is always divisible by the interpolation of a new point, resulting in the ordertype η of the rational numbers in a dual representation. This was also dealt with in our chapter 2. Brouwer added as a handwritten remark in his corrected version of the dissertation, that this procedure may be called the 'split' of the intervals.[28]

3. At every stage of this 'split' process there are always points not reached; we can select such a point in advance and this point can even be approximated arbitrarily close by some infinite dual fraction:

> We can even arrange the construction in such a way that the approximation of a point by an infinite dual fraction is given by an

[28] De 'tweedeling' der intervallen.

> arbitrarily given law of progression, (...)[29]

The expression 'arbitrarily given law of progression' comes across in this context as a 'contradictio in terminis', or at least as a confusing concept. Does it intend to say that *any* law of progression will do, or that there is just one law which prescribes an arbitrary free choice for every next term; in other words, does it mean that we have a free choice for any law of progression or that we have a free choice at every next dual (or decimal) place? It seems that both conditions satisfy Brouwer's intention; there is only one strict requirement: a given point has to be approximated arbitrarily close, which can be specified by the condition that between the point concerned and the last term of the dual fraction alway more 'dual places', i.e. extensions of the dual fraction, can be inserted without ever reaching the point. Now, one can of course imagine that always new points are inserted between the last one and the point concerned, so that one approximates the point arbitrarily close (or that one still remains an appreciable distance removed from it), but these words and the accompanying picture unavoidably involve this process to take place on an everywhere dense and measurable space- or time continuum, which can hardly be reconciled with the 'awareness of the between' as the newly inserted element between two consecutive events (two constructed elements), and therefore a closer analysis is required.

For a proper solution to these apparent dilemmas of 'arbitrarily given laws of progression' and of 'approximation of a point', see the items 4 and 7 of this section. Also the selection beforehand of a point which has to be avoided (or a segment which has not to be penetrated) asks for an interpretation. See for this page 81 and note that in that argument a beforehand assumed measurability of the continuum is avoided.

4. For the moment we assume that we have indeed constructed a scale of order type η on the intuitive continuum, on which every point, not belonging to that scale, can be approximated arbitrarily close. We noticed already that the resulting scale needs not to be everywhere dense on the continuum, that is, there may remain unintended 'open segments' after ω steps of interpolation. How this is possible, and how this can be cured, will be discussed under item 7.

But the quote given in item 3 continues as follows:

> (...), though the continuum with the scale constructed in this way differs in no respect from a continuum on which the scale is constructed in complete freedom. Conversely we see that for any scale which has been constructed on the continuum, there exists a point corresponding to any conceivable law of progression.[30]

[29](dissertation, page 9, 10) (...) we kunnen zelfs zorgen, dat de benadering van een punt door een oneindige duaalbreuk volgens een willekeurige denkbare voortschrijdingswet plaatsheeft;

[30](dissertation, page 10) (...) terwijl dan toch het continuüm met schaal, op deze wijze geconstrueerd, in niets zich onderscheidt van een continuüm met geheel vrij geconstrueerde schaal; omgekeerd leiden we hieruit af, dat voor een eenmaal op het continuüm geconstrueerde schaal voor elke denkbare voortschrijdingswet een punt bestaat.

We observe that Brouwer used the expression ‘law of progression’ and not ‘law of approximation’, but from the context it is obvious that Brouwer had in mind sequences, converging to a ‘point’.

More important is the contrast that Brouwer created between the dual scale, constructed in the given way by the interpolation of a point on every subsegment, and a ‘freely constructed’ scale, without giving any further clarification or explanation of the meaning of ‘freely’.

The construction of the dual scale of order type η seems to be a free construction in the sense that, when constructing the scale, the first two points, i.e. the first two natural numbers *one* and *two* (or in a more modern notation *zero* and *one*) determine the unit measure. The first interpolated point is by definition the point *a half* (or, on the dual scale, the point 0.1), and this point is not placed somewhere halfway, since there is no ‘halfway’ yet between zero and one at this stage of the construction; there is only the connecting medium between the two sensations, experienced as a new sensation (see page 46). This experienced new sensation becomes the interpolated point and therewith *defines* or *creates* the halfway point.

What more freedom can there be in the construction of a scale on the continuum?

There is, however, one restriction or one lawlike rule in Brouwer’s sketched scale construction that limits its ‘complete freedom’: the selection beforehand of a point P, and subsequently taking care that this point is avoided in the construction, and even (see the quote above) arrange things in such a way that the point is, by some law, approximated arbitrarily close by points of the scale. This is a restriction in the freedom of the construction, and Brouwer must have referred to this in the quoted paragraph, and he contrasted it with a scale, constructed in complete freedom.

That the resulting scale differs in no respect from a ‘freely constructed’ scale, is not yet further explained at this place, but will be clarified on the same page under *the measurable continuum*, when the resulting scale is made *everywhere dense* (see item 7 on page 81).

We remarked already that *if* the ‘awareness of the between’ *is* the interpolated point, then it is hard to see how we can avoid a certain point on a continuum. Intuitively such an ‘avoidance’ can only be imagined when operating on a completed measurable continuum (see, again, item 7).

5. A sequence approximating a point must be considered as partially unknown:

> However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partially unknown.[31]

First of all we note that Brouwer, in his corrected version of the disserta-

[31] (dissertation, page 10) We kunnen de benaderingsreeks van een *bepaald aangewezen* punt evenwel nooit *af* denken, dus moeten we haar als gedeeltelijk onbekend beschouwen.

tion, added a footnote to the expression '*given definite* point': 'Take e.g. the number π.'[32] This new footnote makes the quote even more difficult in its interpretation. On a constructed scale a lawlike sequence, i.e. a sequence which evolves according to a fixed and known law, and which is convergent and is approximating an earlier chosen point, as in the case of the expansion of π, is certainly never finished, but we may consider it as completely known, just as we may consider the sequence of natural numbers $\mathbb{N}$ as completely known but never finished. Moreover, a fundamental sequence, built up from a first element and a repeated application of the same algorithm (e.g. adding the successor element in the case of $\mathbb{N}$, or a more complicated algorithm in the case of π or $\sqrt{2}$), may be understood as 'finished' in the sense that for every natural number n the n^{th} term is known, or 'the n^{th} term is individualized', or 'the sequence is individualized'.[33]

An approximating sequence, however, which is constructed by free choice, is never finished *and* partially unknown, since only the finished part is known, but this type of sequence appears as a mathematical object only in Brouwer's later work.

In regard to lawlike sequences, there are in the dissertation clear indications as to their being fully known. On page 142 and 143, the first paragraph under *Ad* 2^0 reads as follows:

> In the first chapter we have seen that there exist no other sets than finite and denumerably infinite sets and continua; this has been shown on the basis of the intuitively clear fact that in mathematics we can create only finite sequences, further by means of the clearly conceived 'and so on' the order type ω, but only consisting of *equal elements*; (consequently we can, for instance, never imagine *arbitrary* infinite dual fractions as finished, nor as individualized, since the denumerably infinite sequence of digits cannot be considered as a denumerable sequence of *equal* objects), and finally the intuitive continuum (by means of which we have further constructed the ordinary continuum, i.e. the *measurable continuum*), but no other sets.[34]

Hence a sequence of the order type ω, constructed according to a simple algorithm, i.e. with the help of the infinite repetition of one and the same object

[32] [Dalen 2001], page 44.

[33] See above, on page 50, the quote from the notebookpage VIII–24.

[34] We hebben in het eerste hoofdstuk gezien, dat er geen andere verzamelingen bestaan, dan eindige en aftelbaar oneindige, en continua; hetgeen is aangetoond op grond van de intuïtieve waarheid, dat wij wiskundig niet anders kunnen scheppen, dan eindige rijen, verder op grond van het duidelijk gedachte 'enzovoort' het ordetype ω, doch alleen bestaande uit *gelijke elementen*, (zodat we ons b.v. de *willekeurige* oneindige duaalbreuken nooit af, dus nooit geïndividualiseerd kunnen denken, omdat het aftelbaar oneindige aantal cijfers achter de komma niet is te zien als een aftelbaar aantal *gelijke* dingen), en tenslotte het intuïtief continuüm, (met behulp waarvan we vervolgens het gewone continuüm, het *meetbaar continuüm*, hebben geconstrueerd). (The third chapter of the dissertation, from which this quote is drawn, will be further discussed in chapter 6.)

or operation,[35] is never finished, but is 'individualized', while the *arbitrary dual fraction* can neither be imagined to be finished, nor to be individualized.

Clearly, the idea of 'free choices' is already present in the quoted part in a very rudimentary form (in fact merely to underline the contrast with an algorithmic sequence), and not yet worked out as an extremely useful means in the construction of sets and as an efficient way to handle the continuum of the real numbers without the need for an underlying matrix in order to make the construction of points possible.

In trying to draw a conclusion, in which we do justice to the quoted fragment from Brouwer's dissertation, which amounts to 'never completed, hence partially unknown', we attempt to interpret the concept 'partially unknown'.

For lawless choice sequences things are obvious: the unfinished part (the unchosen decimal places) are unknown on principle.

For a lawlike expansion like π or $\sqrt{2}$, or for the value of a recursive function, the decimal places of the expansion or the values of the function can be computed for any value of n. But, in view of the fact that Brouwer explicitly mentioned π as an example, one might claim that, as long as the value for a given n is *not* computed, it must be considered as still unknown, even if the rule is known and we know it to be individualized in advance. Hence 'partially unknown' can have two meanings: 1) partially unknown on principle and not individualized; only a free choice makes it known and no law prescribes this choice, and 2) partially unknown 'for the time being', but potentially known; the term is in principle individualized, only the computation has yet to be made. Brouwer must have had this second meaning in mind when mentioning the given example of π.[36]

However, a different interpretation, in which we also do justice to other quotes from Brouwer's dissertation or from the notebooks, is still possible; see for a further and more comprehensive discussion of this topic chapter 8, page 320, when discussing the existence of the actual and potential infinite, either or not lawlike.

6. According to Cantor the totality of points on the continuum is not denumerable, but Brouwer claimed that we may not speak of the *totality of points* of the continuum. The most one can speak of is *the totality of the already constructed points on the continuum at any specific time,* and this quantity is finite, potentially denumerably infinite or, at the most, denumerably infinite unfinished (this latter to be understood metaphorically; it expresses an intention or a process; see chapter 7).

7. So far, the presented construction of a scale on the continuum does not give us the certainty that the result is everywhere dense, since, as we saw, we can in the construction of the scale easily avoid a point or even a whole segment.

[35] See the footnote on page 143 of Brouwer's dissertation.

[36] Brouwer's 'partly unknown' should not be confued with Wittgenstein's 'on following rules'. See [Wittgenstein 1984], §185 – 242. See for comment and analysis on this [Kripke 1982].

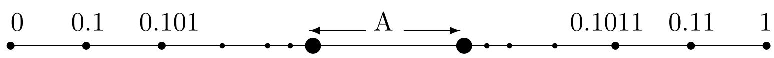
Therefore Brouwer added the following:[37]

> (...) but we agree to contract every segment not penetrated by the scale into one point, in other words, we consider two points as different only when their approximating dual fractions differ after a finite number of digits.

The details of this construction are, again, not elaborated in the dissertation, but we can imagine it to take place in the following way:

The algorithm of the 'splitting of intervals', according to which the scale η is constructed on a line segment, may, as Brouwer claimed, avoid a specific point, or we may even discover after ω steps that a certain segment is left empty. In that case one can easily verify that, in the process of repeated subdivision of an interval of which the empty segment concerned is a sub-interval, we can proceed in such a way that the dual fractions of the boundaries after ω interpolations will not differ after any finite number of digits. To obtain this result, we proceed in the following way:

Let A be the segment in the interval $(0, 1)$ which has to remain void of interpolated points; so after any step the newly inserted point will not be in A. The first interpolated point is called 0.1 and is, say, to the left of A, leaving A in the subsegment $(0.1, 1)$. (In the following argument we will only consider the newly interpolated points in the segment containing A.) We then *prescribe* the next interpolated point in this segment $(0.1, 1)$, which is called 0.11, to be to the right of A, leaving A in the subsubsegment $(0.1, 0.11)$. We note that the first dual place of the two boundaries of the segment containing A are equal and will remain so in the further development. The next interpolated point in the segment $(0.1, 0.11)$ is called 0.101, and is put again to the left of A, resulting in A to be in the sub^3-segment $(0.101, 0.11)$. The next point 0.1011 in this segment is again to the right of A, leaving A in the sub^4-segment $(0.101, 0.1011)$. The first three dual places of the boundaries of the segment containing A are now, and will remain, equal during the continued construction.



Proceeding in this way of alternating the insertion of new points left and right of the segment, thereby approximating its boundaries arbitrarily close,

[37](dissertation, page 10) (...) maar we spreken af, dat we elk segment, waarin de schaal niet doordringt, tot een enkel punt denken samengetrokken, m.a.w. we stellen twee punten dan alleen verschillend, als hun duale benaderingsbreuken na een eindig aantal cijfers gaan verschillen.

(Note that in Brouwer's later terminology the term 'difference' in this sense is replaced by the more positive term 'apartness'.)

while avoiding the segment itself including its boundaries, we obtain the required result, since at every subsequent combination of adding one point to the left and one to the right of A the number of equal dual places increases by two. Hence, in the approximation of A from the right and from the left, the dual fractions will after ω steps not differ after any finite number of digits.

A segment satisfying these conditions is then contracted into one point by identifying its two boundaries.

It will be clear that this construction does not satisfy the intuitionistic standards of later years, since the principle of the 'tertium exclusum' is applied, in the sense that the segment A is determined to such an extent that every next interpolated point can be clearly identified to be either to the left or to the right of A.

One question remains to be answered: how can we select and indicate on the continuum a point P which is to be avoided, or a segment which is not to be penetrated under the sketched interpretation in which 'the awareness of the between *is* the interpolated point', hence without having a scale to mark these points. That is, how can we indicate points on an intuitive continuum?

The most obvious way seems to be to select the first interpolated point (or any point after a finite number of interpolations) as the point P to be avoided, and e.g. the first two points (or any two consecutive points after a finite number of interpolations) as defining the interval not to be penetrated, and start (or continue) the naming of new points only *after* this 'point-or-interval-defining' selection.

The measurable continuum and the Bolzano-Weierstrass theorem

If then, the text of the dissertation continues on page 11, we select an arbitrary point as zero-point, the scale has turned the continuum into a measurable continuum. In order to be able to measure a certain distance from the zero-point to a given point, we also need a unit of distance. Brouwer does not speak explicitly of a unit of distance at this place in his dissertation, but it is obvious that the first two points, the points 'one' and 'two' (or 'zero' and 'one') are in a 'natural way' determining the unit distance, and this distance is made to be equal to the distance between any two consecutive points in the construction of $\mathbb{N}$ by another free act of the constructive mathematician.

The following theorem can now be stated; it is, according to Brouwer, directly following from the measurability of the continuum:

> From the measurability we conclude that every denumerably infinite set of points, lying in the segment determined by two points as its endpoints, has at least one limit point, i.e. at least one point such that on at least one of its sides in every segment contiguous to it there are other points of the set.[38]

[38](dissertation, page 11) Uit de meetbaarheid leiden we af, dat elk aftelbaar oneindig aantal punten, gelegen binnen een door twee punten begrensd segment, minstens één grenspunt

Brouwer's argumentation for this theorem (known as the Bolzano-Weierstrass theorem) is the following: suppose the theorem does not hold, i.e. there is no limit point, then the set of intervals defined by consecutive points must have an infimum which is unequal to zero, hence which is positive; then the bounded interval would be covered by a finite times this infimum, which is apparently false since the set of points is denumerably infinite. Hence we have a contradiction, so the theorem holds. But we see immediately that the proof is not constructive and that the method of *reductio ad absurdum* is applied, as well as the *principle of the excluded third*, and therefore the proof will, again, not satisfy later intuitionistic standards.

A closer and more formal analysis of Brouwer's argument, which also enables us to attempt to replace it by a constructive proof, shows that he in fact did the following: suppose there is no limit point P in the set A of the elements forming the measurable continuum, defined on the interval $[0, 1]$, hence suppose:

$$\neg\exists P\forall\varepsilon\exists x \in A(x \in U_\varepsilon(P))$$

which is (classically) equivalent to:

$$\forall P\exists\varepsilon\forall x \in A(x \notin U_\varepsilon(P))$$

so for every point P of A there is an ε such that there is no other point of A in an open segment $U_\varepsilon(P)$ of A, containing P. So $\bigcup U_\varepsilon(P) = [0, 1]$, and according to the Heine-Borel theorem this union has a finite subset, also covering the interval $[0, 1]$. Hence the interval is covered by a finite number of intervals, each containing only one point of A, which leads to a contradiction. Therefore there must be a limit point in the resulting set A. But now we have applied 'reductio ad absurdum' *and* the 'tertium non datur'!

If we attempt to prove it constructively, that is, without using the tertium non datur, we might proceed as follows:

Take the collection $\{|x - y| | x, y \in A \wedge x \neq y\}$. Suppose now this set has an infimum $\varepsilon > 0$, so suppose

$$\exists\varepsilon\forall x, y(x, y \in A \wedge x \neq y \longrightarrow |x - y| > \varepsilon)$$

Then there is a minimum in the set of all distances between the points, that is, only finitely many points are on the interval A, which is a contradiction. So:

$$\neg\exists\varepsilon\forall x, y(x, y \in A \wedge x \neq y \longrightarrow |x - y| > \varepsilon)$$

or:

$$\forall\varepsilon\neg\forall x, y(x, y \in A \wedge x \neq y \longrightarrow |x - y| > \varepsilon)$$

But this is as far as we can get, since the next step $\forall\varepsilon\exists x, y\neg(...)$ cannot be made intuitionistically. Hence the existence of a limit point cannot be proved, and

heeft, d.w.z. minstens één punt zó, dat naar minstens één van beide kanten binnen elk eraan grenzend segment, hoe klein ook, nog andere punten liggen.

with good reason, since in intuitionistic mathematics the Bolzano-Weierstrass theorem is in general not valid, as can be concluded from the following counterexample:
Define the denumerably infinite set $\{x_n\}, (n = 1, 2, 3...)$ on the interval $[0, 1]$ as follows:

$x_n = 2^{-n}$ if for some $m \leq n$ at the m^{th} decimal place in the decimal expansion of π for the first time the sequence 0123456789 begins.

$x_n = 1 - 2^{-n}$ if this is not the case.

Since, at least in Brouwer's days, this was an unsolved problem (but may now be replaced by any other unsolved problem in the expansion of π), we cannot say that either 0 or 1 will be a limit point.

In a footnote to the quoted Bolzano-Weierstrass theorem on page 11, Brouwer added that a bounded open segment on the continuum is equivalent to the whole continuum. Both form the 'open continuum', from which the 'closed continuum' can be built in the following somewhat notable way:

Let there be given an open continuum; placing an arbitrary point P on it results in two open continua left and right of P, e.g. from the open continuum $(0, 1)$ we can make, by 'placing the point $\frac{1}{2}$' on it, two open continua $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Conversely, from two open continua, one new open continuum can be built by coupling two endpoints by the insertion of one point between the two, e.g. by the insertion of the point $\frac{1}{2}$ we can glue together the two open continua $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ to form the single open continuum $(0, 1)$. In the same way, Brouwer continued, one can transform an open continuum into a closed one by identifying the end-points by means of one inserted point. The result is either a sort of a 'loop' which indeed is a closed continuum, or the inserted point acts as the point-at-infinity or the 'ideal point' from projective geometry. That this last option seems to be the correct one, at least in the year 1907, follows from the concluding remarks about the group definition of the operation of addition on page 19 of the dissertation. On this page Brouwer called the resulting coinciding endpoints the *point at infinity*.[39] Note, however, that the major part of this concluding remark was deleted in Brouwer's planned improved and corrected edition of the dissertation.[40]

It is a notable procedure indeed; one could of course also have added two points at both boundaries of an open continuum (that is, added the boundaries of the open continuum to it), to turn it into a closed one; this certainly is a more direct and more intuitive idea of defining a closed continuum from an open one, and this certainly is what one would expect. After all, a closed interval has two endpoints, both being an element of that interval. One can only guess why Brouwer used this method of turning an open interval into a closed one. A reason could be one of minimality: he only needed one point for the closure instead of two. However, Brouwer's method is not correct from an intuitionistic point of view: the addition of one point may be insufficient to transform an open

[39] Oneindigheidspunt.
[40] [Dalen 2001], page 49.

continuum into a closed one, since a condition for a closed set is, that every limit point is element of the set, i.e. every convergent sequence of elements has a limit element, belonging to that set. Perhaps Brouwer was eventually of the same opinion, which might then be the reason for deleting the fragment of page 19 during the correction of his copy of the dissertation; a partial deletion of the footnote on page 11 might then have been forgotten.

After the construction of the measurable continuum, the dissertation continues with the definition of addition and multiplication on this continuum by means of transformation groups. Since we are concentrating on the foundational aspects of the continuum in Brouwer's work, this group theoretical part will not be discussed here, just as, for the same reason, the subsequent treatment of projective geometry is left out.

3.4 The notebooks and the intuitive continuum

Numerous notes and remarks on the intuitive and measurable continuum were jotted down throughout the nine notebooks. In the following subsections we have attempted to present a significant selection that might shed light on the evolution of his ideas about this fundamental topic and that also might illustrate the results, conclusions, and interpretations from the previous section.

The idea, in the first notebook, of a constructible continuum (from logical principles alone) is an isolated remark.[41] Immediately after that the continuum appears as intuitively given. Brouwer repeatedly quoted and discussed the ideas of Poincaré, who felt the need to *define* a mathematical continuum, in order to overcome the paradoxes, arising from a physical continuum;[42] this paradox was denied by Borel.[43] Evidently, Poincaré had a great influence on Brouwer, but in Poincaré's definition of the continuum, as in all his definitions of mathematical entities, the use of only finitely many words was a strict requirement, whereas for Brouwer no definition at all was needed for the continuum, since it is intuitively given to us.

Brouwer distinguished between the *intuitive* continuum (intuitively given, arising from the ur-intuition alone, not constructed, not composed of points) and the *mathematical* continuum (or *definable* continuum, or *fictive* continuum), which consists of points, constructed on the intuitive continuum.[44] This process of construction never terminates, but, in the case of the construction of some lawlike sequence or set (e.g. $\mathbb{N}$ or η), its result may nevertheless from a certain point of view be understood as known, because of the application of one and the same algorithm ω times. Therefore 'known' has to be interpreted in the

[41] See the next subsection, page 85.
[42] See page 24.
[43] [Borel 1909], see also [Borel 1972], volume IV, page 2151.
[44] We saw on page 73 that after 1919 also the 'full continuum' of the real numbers was introduced.

sense of 'potentially known', and not 'actually known' since the sequence is never finished (see the preliminary remarks on page 77; see page 320 for a more comprehensive discussion).[45]

But despite his general views on the intangibility of the intuitive continuum, as expressed in the notebooks but also in the dissertation itself, Brouwer was not completely satisfied with a concept that forbade him to speak of 'arbitrary irrational numbers'. Especially in the last two notebooks we can clearly recognize attempts to get a better grip on the intuitive continuum by means of the admission of infinite sequences of free choice as objects in his mathematical realm, thus turning the intuitive continuum into the 'full continuum' of the reals on the unit interval. These objects did not find their way into the dissertation yet, but certainly were an adumbration of later developments in his work.

3.4.1 The continuum, its possible construction from logical principles alone

Brouwer was, as we will see, already at an early stage convinced of the apriority of the intuitive continuum, but not from the very beginning.

In his dissertation he had arrived at his seemingly final notion of the continuum; however, from the notebooks we can conclude that this final notion was not that final; in the very last notebook he still persevered in his attempts to grant properties to the continuum, beyond its simple role as 'matrix to construct points on'. On page 133 we will see that he indeed succeeded in this attempt.

The beginning of the first notebook still leaves open the question on the apriority of the continuum:

> (I–9) It is an interesting but philosophically unimportant question whether geometry (including projective geometry) can be built up from just logical principles and the one-dimensional continuum (and maybe even, to construct that one-dimensional continuum from logical principles).[46]

This is a completely isolated remark, in which Brouwer had considered founding the continuum on logical principles alone. It even looks like a rhetorical question with a negative answer. His final viewpoint would soon be that the notion of the continuum is intuitively given to us. In his last notebook this is, as we saw, for the first time expressed in terms of the ur-intuition of the combination of two discrete events, separated by a continuous time interval. Hence the continuum concept is not founded on logic or on logical priniples at all, but on the mathematical intuition of the connecting medium between two experiences.

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[45] The 'denumerably unfinished' set is of course a completely different story; see for this chapter 7.

[46] Een interessante, maar filosofisch onbelangrijke vraag is: is het mogelijk de geometrie (ook de projectieve) op te bouwen uit enkel logische principes en het eendimensionale continuüm? (En misschien zelfs, om ook dat eendimensionale continuüm uit logische principes op te bouwen?)

3.4.2 The continuum, the intuition

Despite the reference in the first notebook to the constructibility of the continuum, there is also, in the same notebook, the statement that the continuum is not depending on the concept of number (I–40), but that it is something which one can take for granted, since everybody has the same mental picture attached to this word; there simply is no need for a construction or definition. In the following quotes the similarity with Borel's view on the intuition of the continuum and its existence as a mathematical object is striking (see the footnote on page 34); for Borel the continuum is real because of its, among mathematicians, commonly shared notion of it.

> (II–1) We can take the continuum as a starting point, because here people understand each other.[47]

This means that there is no misunderstanding of the matter; everybody knows intuitively what he is talking about when discussing the continuum, despite the fact that it is undefinable and mysterious:

> (II–25) Until now the continuity of the line was the unknown, about which no misunderstanding was possible.[48]

and on II–27:

> In the unconscious continuity of the straight line –not yet eliminated by mathematical logic– something of the old unconscious $\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota$ lay hidden.[49]

On the one hand Brouwer still considered the process of 'building up', on the other hand there is the intuition, and these two views are not necessarily exclusive; on the contrary, both are needed in the construction of the mathematical edifice since the intuition gives us the two-ity of discrete and continuous, and thus the continuum; and departing from this continuum the act of constructing the mathematical building can begin. Therefore the one (the intuition) necessarily precedes the other (the process of building). This concept of the continuum is, in an early stage, still the result of some 'vague' intuition, apparently shared by everybody, but in the $\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota$ we already recognize his view of the continuum as the flowing and connecting, being enacted in time:

> (III–7) Time acts as that, which can repair the separation.[50]

[47]Van het continuüm kunnen we uitgaan, omdat de mensen zich daarop verstaan.
[48]De continuïteit van de lijn was tot nog toe het onbekende, waarover geen gemis aan verstandhouding mogelijk was.
[49]In de onbewuste continuïteit der rechte lijn –nog niet door mathematische logica weggenomen– lag nog wat van het oude onbewuste $\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota$.
[50]De tijd treedt op als dat, wat de scheiding wel weer in orde kan brengen.

The shifts and changes in Brouwer's opinion can often be inferred from erased lines and paragraphs. Cantorian views were often crossed out, or diverging comments were added. As an example of a deleted fragment we give the following fragment:

> (II–35) This modern mathematics starts from thing and relation, hence from discreteness; but suppose this was wrong and one had to start from continuity?
>
> And suppose the systems with simple relations can *never* be constructed directly (except axiomatically of course, and supported by the existence theorem afterwards) because they do not arise genetically from the mind? But in everybody is the *desire* for axioms, to which then the world will be adapted?[51]

The terms in which Brouwer expressed 'this modern mathematics' which is just based on the non-intuitive 'thing and relation', and which can possibly only be constructed on an axiomatic foundation and whose existence can only be determined afterwards, are somewhat belittling. He clearly referred to the axiomatic method as the foundation of mathematics:

In chapter III of his dissertation, *Mathematics and logic*,[52] Brouwer argued first that mathematics is not dependent on logic, but, instead, logic on mathematics. He subsequently applied this argument to four subjects,[53] of which the first one is entitled *The foundation of mathematics on axioms*.[54] In this first application of his view of the role of logic, the geometries of Euclid, Schur, Hilbert, Lobatchevski and others are discussed, with the conclusion (not in the original dissertation, but added by Brouwer to the improved and corrected version of it):

> Résumé: one should, in axiomatic investigations, not look for clarification of the foundations of mathematics, but only for solutions of *mathematical* problems, (...)[55]

Hence mathematics cannot be *founded* on axioms, since axioms only serve the purpose of making the mathematical building manageable, to make solutions for mathematical problems possible in a systematic way.[56]

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[51] Die moderne wiskunde gaat uit van ding en relatie, dus van discreetheid; maar als dat nu eens fout was, en men moest uitgaan van continuïteit?

En als de stelsels met eenvoudige relaties nu eens *nooit* direct zijn op te bouwen — dan axiomatisch natuurlijk, en zo achteraf gesteund door de existentiestelling — omdat ze niet genetisch uit de geest voortkomen? Maar in allen is de *wil* tot de axioma's, waaraan de wereld dan wel wordt aangepast?

[52] *Wiskunde en logica.*

[53] See page 133 of Brouwer's dissertation; see also chapter 7 of this dissertation.

[54] *De grondvesting der wiskunde op axioma's.*

[55] Résumé: men mag in de axiomatische onderzoekingen geen opheldering van de grondslagen van de wiskunde zoeken, maar alleen oplossingen van *wiskundige* opgaven, (...) [Dalen 2001], page 116.

[56] See further page 252 ff.

It must be emphasized that the quote (II–35) is not directed against *strict formalism*, in which mathematical objects have disappeared, and in which mathematics is merely the manipulation with meaningless symbols according to specific rules. Formalism was developed by Peano, Dedekind, Hilbert and others, and Brouwer's public opposition against this formalism emerged again in 1912 in his inaugural address.

The reason for deleting the quoted passage (II–35) will most likely be the following: neither discreteness *alone*, nor continuity *alone* will do for the construction of mathematics; both *continuity and discreteness* are required to build the mathematical edifice. Neither one can be based on the other, two different discrete events are necessarily connected by a continuum and a continuum cannot be experienced without two different events. But in the notebooks this explicitly expressed ur-intuition will only appear much later, in the last notebook IX (See quote IX–26 on page 55).

3.4.3 The physical continuum

In *La Science et l'Hypothèse* Poincaré discussed the physical continuum, with the conclusion that this continuum concept can lead to the following paradox: $A = B$; $B = C$; $A \neq C$[57], which can only be avoided in mathematics by the introduction of the *mathematical continuum*. As a comment on this, Brouwer asked and answered:

> (III–24) Investigate if the indefinitely refinable physical continuum of one dimension (satisfying, according to Poincaré, $A = B$; $B = C$; $A \neq C$, which is then automatically everywhere dense) is, according to Cantor, equivalent to the 'Geordnete Menge' η. Yes, of course.[58]

Poincaré needed a mathematical continuum in order to get rid of the paradoxes of the physical continuum. The required continued refinement of Poincaré's physical continuum is similar to Brouwer's construction of the order type η, but the result of Poincaré's construction will be everywhere dense, without the need for a contracting operation, which Brouwer needed for his measurable continuum. This follows from the *physical* nature of Poincaré's construction. When we observe that $A = B$, then we can, by improving our measuring equipment, find a C between A and B, such that $A = C$, $C = B$ but $A \neq B$. Poincaré used, as an example, in *La Science et l'Hypothèse* for A, B and C weights of e.g. 10, 11 and 10.5 grams (instead of indicated points on a line). If we assume that the measuring equipment (the balance in this case) can always be improved and refined, there will not remain a 'gap' between two equal weights, which has to be 'contracted into one point'. After all, we are doing physics here and not mathematics, and a gap is identical to a difference in weights. When two

[57]See the discussion on Poincaré, section 1.3. See [Poincaré 1916], page 34 ff.

[58]Ga eens na, of het onbepaald te verfijnen fysisch continuüm van één dimensie (van Poincaré volgens $A = B$; $B = C$; $A \neq C$; dat dus vanzelf überall dicht is) volgens Cantor niet gelijkwaardig is met de 'Geordnete Menge' η. Ja natuurlijk.

weights are, in case of an unlimited accuracy in the measuring apparatus, the same, there cannot be a gap between the two.[59]

In IV–21–23 Brouwer once more discussed this physical continuum, when stating that we want continuity in nature in order to be able to describe it by means of continuous and even differentiable functions; we force continuity upon nature for the benefit of our mathematical model of it.

3.4.4 The mathematical versus the intuitive continuum

In the dissertation Brouwer's conclusion for the continuum was: a line is not composed of points, one cannot speak of a line as a collection of points, they can only be *constructed on* the continuum. The justified, but not yet expressed conclusion at this place in the notebooks could already be that the continuum is a cardinality of its own. However, Brouwer limited the role of the intuitive continuum to that of a matrix to construct a different kind of continuum on: the mathematical continuum:

> (IV–30) The continuum is intuitive, a correction on the definiteness; the mathematical continuum is totally different: *that* must be constructible by us from finite numbers and inductively (that is, leaving freedom for a jump[60] without change of property, hence one-dimensional jump.[61]

Brouwer distinguished here the concept of the intuitive continuum from that of the 'mathematical continuum'; the terminology is Poincaré's, and the statement can be best understood in terms of Brouwer's 'measurable continuum' or 'definable continuum'. For the term *mathematical continuum* see also the next section.

In the following quote, in which Brouwer apparently is still searching for his final terminology, the 'building up' of the continuum is expressed; this is now in terms of the construction of the definable continuum on the intuitive one. Here the group theoretical method of the construction of points is given; in his dissertation, Brouwer used this method to define the basic operations on the constructed definable continuum.

> (IV–23) The one-dimensional continuum is built as a group of transformations of some points. We indeed have it intuitively; but that we specifically consider this intuitive aspect is because we know that

[59] Borel disagreed with this argument of Poincaré. Imperfections in measuring equipment do not lead to a contradiction in the concept of the physical continuum which would necessitate the definition or construction of a mathematical continuum. See [Borel 1909], see also [Borel 1972], page 2151.

[60] Brouwer used the term 'saltatory change', Dutch: sprongverandering, an abrupt discontinuous jump, as opposed to continuous change.

[61] Het continuüm is intuïtief, correctie op de bepaaldheid, het mathematisch 'continuüm' is heel iets anders: *dat* moeten we helemaal kunnen construeren uit eindige getallen en inductief (d.i. vrijheid laten van sprongverandering zonder eigenschap-verandering, dus eendimensionale sprongverandering.

> we will never run out of points for all the groups that we possibly will build.[62]

We know the continuum is intuitively given. The continuum that is 'built as a group of transformations of some points', is the measurable continuum, constructed on the intuitive one. See also the quote (V–19) below.

We note here that the terminology gradually transforms into that of the dissertation.

3.4.5 The mathematical continuum is the measurable continuum with rational scale

The conclusion from the last quote (IV–23) was, that 'building' means here the transformation of the intuitive continuum into the measurable continuum according to the method of transformation groups, as explained in the dissertation. In his later notebooks this method is worked out in more detail. Already in the fifth one Brouwer made the next step:

> (V–18) The continuum *has* no cardinality: I can construct on it as many points as I wish; it is intuitive (I *build* the points on it) like everything of which the moral basis can be sensed.[63]

Hence the continuum resists classification in the range of cardinal numbers à la Cantor. At this stage Brouwer did not yet grant the continuum the status of a separate cardinality in itself.

Thereupon the construction of a rational scale on the intuitive continuum is formulated in V–19: construct on the line a group (that is, a denumerable *set*; the term 'group' has to be understood here in this non-algebraic sense), and hence a scale; then every point not on that scale can be approximated by means of that scale, of course after completion of the 'contraction'-operation as described on page 79 (page 10 of Brouwer's dissertation):

> (V–19) If I have the intuitive continuum, then I can, in some arbitrary way, construct on it a number continuum, arbitrarily designating point by point for every number, but only within the proper interval.[64]

[62]Het eendimensionaal continuüm is opgebouwd als groep van transformaties van enige punten. We hebben het wel intuïtief; maar dat we juist dat intuïtieve nemen, komt, omdat we alleen zó weten, dat we voor allerlei groepen, die we eventueel zullen bouwen, nooit punten tekort zullen komen.

[63]Het continuüm *is* niet van een machtigheid: ik kan er zo veel op zetten als ik wil; het is intuïtief (de punten *bouw* ik er op), zoals alles, waarvan de morele ondergrond kan worden gevoeld.

[64]Heb ik het intuïtieve continuüm, dan kan ik daar op een of andere willekeurige manier een getallen continuüm op construeren, punt voor punt willekeurig aanwijzen bij elk getal, alleen binnen het juiste interval.

The last part of this sentence ('but only within the proper interval') points to the fact that we can insert points on every interval, but that the naming of a newly inserted point is determined by the interval in which it is put (e.g. the first inserted point in the interval $(0, 1)$ can, in the dual representation, only be named 0.1).

In V–20 it is stated that a rational scale (of order type η, 'fine without a bound'[65]) can be placed at random. If another similar scale is constructed on the same continuum, then every point of the line can be expressed in both scales and a conversion from one scale to the other is possible, after selecting a zero point for both. Clearly one scale is sufficient since a point, not lying on that rational scale, can be approximated arbitrarily close by the rational scale and is identical with its approximating sequence expressed in the scale:

> (V–20) In case of the fictitious continuum I *understand* $\sqrt{2}$ to be identical to the fundamental sequence of its decimal fraction.[66]

This is a statement of fundamental importance. In this early stage Brouwer already declared an irrational number on the continuum to be identical to the fundamental sequence itself, and not to its limit. This view, which is now communis opinio had, however, not yet been generally accepted at that time. It will also be Brouwer's view when defining 'arbitrary real numbers' by means of nested intervals.[67]

3.4.6 The continuum is no point set

In the fifth notebook we recognize Brouwer's concept of the continuum, as it was laid down in the dissertation: the continuum is intuitive, it is not composed of points, it cannot be expressed in a Cantorian cardinality, but it is an unlimited *source* for points. As a comment on § 207 of Russell's *Fondements de la Géométrie*,[68] Brouwer remarked:

> (V–30) With the continuum you never reach a point; this for those who want to see the continuum as a point set.[69]

In VI–18 it is stated that the continuum is a priori, just like arithmetic, but the fact that the theory of groups of rational numbers is applicable to the measurable continuum, is empirical.

The construction of the rational scale on the one-dimensional intuitive continuum is defined in VI–22 by means of the group of motions (translations) just as this was done in the earlier quoted fragment from IV–23, and not yet, as in the dissertation and at other places in the notebooks, by repeated splitting of intervals.

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[65]'fijn zonder perk'.

[66]Bij het fictieve continuum *versta* ik onder $\sqrt{2}$ de Fundamentalreihe van de decimaalbreuk er van.

[67]see page 126, in the section about the Review of Schoenflies.

[68][Russell 1901], page 248.

[69]Met het continuüm bereikt men nooit een punt; dit aan 't adres van hen die het continuüm als een Punktmenge willen opvatten.

3.4.7 The vast majority of the irrational numbers can only be approximated and not defined

In the sixth notebook Brouwer presented, for the first time, a more or less detailed analysis of irrational numbers: the majority cannot be directly represented on the measurable continuum:

> (VI–21) For the fictitious continuum[70] it holds that:
>
> $\underline{1^{st}}$ Every element is a main element.
>
> $\underline{2^{nd}}$ For every fundamental sequence there is a limit element.
>
> But can I imagine that?
>
> I can only specify a fundamental sequence by its limit element[71] (in general I *cannot* specify a particular infinite decimal fraction; I can only define some decimal fractions as identical with the limit element which is speciefied in a different manner). Hence I cannot speak of *every* fundamental sequence, independent of a limit element. But it *is* an intuitive axiom of the intuitive continuum that every point can be approximated by a given scale. (cf. $\underline{1^{st}}$).
>
> (...)
>
> The intuitive continuum has the property that for every arbitrary scale construction, between every two points of the scale an intuitive continuum remains.[72]

[70]or *fictive continuum*. Note the difference with the *mathematical* or *measurable continuum* of the η-scale, in which not every fundamental sequence has a limit-element belonging to that scale. The term *fictive continuum* is not used in the dissertation. At this place in the notebooks it has the meaning of the mathematical continuum of the η-scale, extended by the algorithmically defined irrational numbers. 'Every fundamental sequence' (under item 2) has then to be read as 'every well-defined fundamental sequence', since there are no other fundamental sequences. Compare this with Borel's *practical continuum* ([Borel 1908b]) and Weyl's *atomistic continuum* ([Weyl 1921]). Compare this also with Brouwer's sets of cardinality *denumerably infinite unfinished* (his dissertation chapter III).

[71]Note that the term *fundamental sequence* is used here in the meaning of *Cauchy sequence*, as, in fact, it is often used. For Cantor however (see e.g. his *Begründung*) a fundamental sequence is defined as a denumerable subset of a simply ordered transfinite set, and needs therefore not to be convergent. The same applies on most occasions for Brouwer; see e.g. the *Berliner Gastvorlesungen*, chapter 3, the end of the second paragraph ([Dalen, D. van (ed.) 1992], page 31): Mit der Folge ζ in der 'natürlichen Rangordnung' ähnliche Spezies heissen *Fundamentalreihen*. Hence these sequences need not necessarily be convergent.

[72]Voor het fictieve continuüm geldt:
$\underline{1^e}$ Elk element is hoofdelement.
$\underline{2^e}$ Bij elke fundamentaalreeks is een grenselement.
Maar kan ik me dat voorstellen?
Een fundamentaalreeks kan ik alleen aangeven [een bepaalde oneindige decimaalbreuk *kan* ik in 't algemeen niet aangeven; ik kan alleen sommige decimaalbreuken definiëren als identiek met het, op andere wijze aangegeven, grenselement] door zijn grenselement. Ik kan dus niet spreken van *elke* fundamentaalreeks, onafhankelijk van een grenselement. Maar *wel* is een intuïtief axioma van het intuïtief continuüm, dat elk punt door een gegeven schaal te benaderen is. (St $\underline{1^e}$). (.....) De eigenschap van het intuïtieve continuüm is, dat bij elke willekeurige schaalconstructie tussen elke twee schaalpunten weer een intuïtief continuüm overblijft.

At this place in the notebooks, and in contrast to the earlier quoted V–20, Brouwer again distinguished the fundamental sequence from its limit element. In his more modern view the irrational number *is* the sequence itself (the choice sequence, algorithmically defined or by free choice), and not its limit-element; in this form it also appeared in quote V–20 on page 91. But when Brouwer wrote down VI–21, in which the irrational number was defined as the limit element of the fundamental sequence, two irrational numbers a and b are equal, if their defining sequences $\{a_n\}$ and $\{b_n\}$ have the property

$$\forall k \exists n \forall m_1 m_2 > n \ \ |a_{m_1} - b_{m_2}| < 2^{-k} \qquad (k, n, m_1, m_2 \text{ natural numbers})$$

The last property in VI–21 is fundamental for the intuitive continuum. Between every two non-coinciding points indicated or constructed on it (two points satisfying the property of apartness, in Brouwer's more modern terminology), there always remains another intuitive continuum, on which other points can be indicated or constructed. This iterated possibility of intercalation of new points results in the construction of the dual rational scale.

But of course every fundamental sequence (here to be understood as a convergent fundamental sequence) stands for a number, and, later on, *is* a number.

About a possible 'totality of points' on a line, Brouwer remarked:

> (VI–23) I can say '*if* a point is on a line', but I cannot speak of '*all* points of a line' (a class of units).
>
> (VI–33, 34) I cannot speak of *all* groups[73] of a set of a certain cardinality. After creating a certain order, e.g. the order of the points of the plane, I can indicate some groups, but never *all* groups.
>
> In the same way I can indicate some real numbers (...), and as many as I wish, but never *all* (...)
>
> I can however point to *every* real number (with a finite number of gestures or sounds) by applying the continuum intuition, and then 'pointing to' a number.[74]

The last phrase of this quote carries with it a kind of haziness, in the sense that one can wonder what exactly is meant by 'pointing to a real number'. This haziness might, again, include an early anticipation of the choice aspect. Just

[73] Here, as elsewhere, the term *group* is used in the more general meaning of a set or, in this case, of a subset.

[74] (VI-23): Ik kan wel spreken van '*als* een punt ligt op een lijn', maar niet van: '*alle* punten van een lijn' (een klasse van eenheden).

(VI-33, 34): Van *alle* groepen uit een Menge van zekere machtigheid kan ik niet spreken. Eerst als ik een zekere ordening heb gegeven, b.v. de ordening der punten van het platte vlak, kan ik sommige groepen aangeven, maar toch nooit *alle.*

Zo kan ik sommige reële getallen aangeven (...), en zo veel als ik wil, maar nooit *alle* (...)

Wel kan ik *ieder* reëel getal aanwijzen (met een eindig aantal gebaren of klanken n.l.) door gebruik te maken van de continuüm-intuïtie, en daarop een punt 'aan te wijzen'.

like a choice sequence at every stage of its 'becoming' is only an approximation of its continuation, the actual 'pointing to' a real number can only be a rough approximation of what is intended with the gesture, however accurate that is.

Not all arguments in the notebooks seem to be, at first sight, of equal cogency, since in the following phrase the conclusion seems not to follow from the argument; after all, the premise also apply to the rational scale alone:

> (VI–34) The fact *that* the intuition of the continuum consists of *more* than the rational scale, follows from this, that you know you cannot jump from zero directly to a point of that scale, however large its denominator may be!
>
> Apart from this, never defend your intuition with the help of a logical suggestion like that.[75]

In order to make sense of the content of this quote, we may suppose that Brouwer most likely meant to say that on the rational scale, constructed on the intuitive continuum, when jumping from zero to any rational number, there are of course always other rationals between zero and that number, but that there is *also* necessarily a continuum between any two rationals, however close together they may be. This interpretation agrees with the last phrase of the last quote.
At the bottom of that same page Brouwer argued:

> (VI–34) I need the continuum; I cannot say: I choose ω times 1 or 2, since the intuitive induction only applies to equal things, not to varying (and ω chances are equal things).[76]

This means that the continuum can in no way be constructed or fully described as a scale of dual fractions of infinite length, for lack of a suitable algorithm; that is, the vast majority of its elements cannot be named or written down or algorithmically defined. Hence the only way to have the continuum at our disposal is, that it is presented to us in its entirety.

We also have intuitively the process of mathematical induction which is mentioned in the last quoted passage; it is not an axiom, nor a theorem, but it is the natural mathematical act of repeatedly applying the same rule or algorithm in order to reach the n^{th} term of a sequence governed by some law, or to prove a formula, not 'for all natural numbers n', but for any given finite natural number n. A theorem of the form $\forall x P(x)$ has therefore to be interpreted in its potential meaning: for every n you give me, I can prove $P(n)$ by the iterating process $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, ..., according to the rule $P(1) \wedge \forall m(P(m) \rightarrow P(m+1))$ up to $P(n)$.

[75] Dát de intuïtie van het continuüm bestaat uit *meer* dan de rationale schaal, volgt wel hieruit, dat je weet, dat je van 0 niet direct op een punt van die schaal, met hoe grote noemer ook, kunt springen!

Overigens roep nooit zo'n logische suggestie te hulp, om je intuïtie te verdedigen.

[76] Ik heb het continuüm nodig; kan niet zeggen: ik kies ω maal 1 of 2, want de intuïtieve inductie is alleen voor gelijke dingen, niet voor wisselende (en ω kansen zijn gelijke dingen).

Hence we can construct a decimal expansion of any finite length, as long as the expansion is governed by a law (or the 'chance of some outcome' is governed by a law; that chance will then always be 1 for one specific outcome and 0 for the rest); ω times a free choice is not possible, that is, it is only a potential possibility.

Note that in the last quote Brouwer is, again, far removed from the choice sequences (where an unlimited but finite number of free choices is allowed, even though the sequence never terminates, is never finished and may not be conceived as such) as a method to handle the continuum, (but which then additionally requires the continuity priciple, which states that the outcome of $P(\alpha)$, P being a function with choice sequences as argument and α being a choice sequence, is determined by an initial segment of α). But shortly after this quote, as well as in earlier ones, we can already discern ideas in that direction. And even in this last quote the *concept* of choice is present.

3.4.8 The continuum; is there a way to get more grip on it, apart from a constructed scale on it?

In notebook six Brouwer is very much occupied with the continuum and the perfect set; in VI–35 there is a reference to the 'Cantor set',[77] which is constructed on the continuum in the following way:

Trisect the closed interval $[0, 1]$ at the points $\frac{1}{3}$ and $\frac{2}{3}$ and then delete the open interval $(\frac{1}{3}, \frac{2}{3})$, called the 'middle third'. What remains is the union of two closed intervals $T_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now trisect each of the two segments of T_1 at $\frac{1}{9}$ and $\frac{2}{9}$, respectively at $\frac{7}{9}$ and $\frac{8}{9}$ and again delete the middle third from each segment, giving $T_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing in this way ω times results in a descending sequence of sets $T_1 \supset T_2 \supset T_3 \supset ...$, where T_m consists of the segments in T_{m-1} excluding their middle thirds. The Cantor set is now defined as the intersection of these sets, i.e. $T_{Cantor} = \bigcap\{T_i | i \in N\}$.

Observe that T_m consists of 2^m disjoint intervals and that we can speak, when the intervals are numbered from left to right, of the odd and the even intervals of T_m.

If we now define a function f on T as follows:

$f(x) = \langle a_1, a_2, ...\rangle$, where

$a_m = 0$ if x belongs to an odd interval of T_m, and

$a_m = 2$ if x belongs to an even interval of T_m,

then the sequence $\langle a_1, a_2, ...\rangle$ corresponds to the ternary expansion of the elements x of T, i.e. the expansion of x to the base 3:

$$x = \sum_{i=1}^{\omega} \frac{a_i}{3^i} \qquad (a_i = 0 \text{ or } 2)$$

[77]Brouwer referred, in a rather obscure paragraph about infinite dual fractions, to 'Cantor Grundlagen page 46 note 11'. This concerns a 'note by the author' (one of the 'Anmerkungen des Verfassers'), to Cantor's *Grundlagen* paper; see [Cantor 1932], page 207.

It is in this form that Cantor defined this set in note 11 of the *Anmerkungen des Verfassers* to his *Grundlagen*:

> Als ein Beispiel einer perfekten Punktmenge, die in keinem noch so kleinen Intervall überall dicht ist, führe ich den Inbegriff aller reellen Zahlen an, die in den Formel
>
> $$z = \frac{c_1}{3} + \frac{c_2}{3^2} + ... + \frac{c_\nu}{3^\nu} + ...$$
>
> enthalten sind, wo die Koeffizienten c_ν nach Belieben die beiden Werte 0 und 2 anzunehmen haben und die Reihe sowohl aus einer endlichen, wie aus einer unendlichen Anzahl von Gliedern bestehen kann.

The Cantor set T has the following properties:
– It is non-denumerable, since at every step the number of segments in which certainly a point of the set will be present, doubles; it has the cardinality $2^{\aleph_0} = c$.
– It does not contain any open set.
– It is perfect, since it coincides with its derivative.
– It is dense in itself and closed (follows from it being perfect).
– It has measure zero.

We emphasize the fact that T is in no interval of the continuum $[0, 1]$ everywhere dense, but it is *dense in itself* and even perfect. It is *not* equal to the continuum on which it is constructed, but it is equivalent to the continuum. In fact, it pictures a perfect set in dual representation. Cantor just mentioned it as an example of a perfect set; indeed it is a remarkable and extremely clever example.

Brouwer noticed, in a deleted fragment in VI-36, about Cantor's perfect set:

> (VI–36) Now it seems that, after all, I cannot speak of *all* elements of that set, hence the set is not real, since I can never say with certainty in a finite time whether a point, indicated on the continuum, belongs to it (*sometimes* I can say that it does not belong to it). But *nevertheless* I can speak of the *reality* of that set, and of all its elements;[78]

Why is this fragment deleted? Of course one cannot speak of *all* its elements, not even potentially since their number is non-denumerable. But may one nevertheless speak of the reality of the Cantor set? Brouwer answers this in the affirmative since it is defined by the clearly given rule of repeatedly trisecting every segment and deleting the middle third, but after any finite number of

[78]Nu schijnt het, dat ik toch niet kan spreken van *alle* elementen dier Menge, dus die Menge toch niet reëel is, want ik kan nooit zeker zeggen binnen eindige tijd van een op het continuüm aangegeven punt of het er toe behoort (wel *soms*, dat het er niet toe behoort). Maar *toch* kan ik spreken van de *realiteit* der Menge, en van *alle* elementen ervan.

steps no single element of the set is isolated yet, and therefore no single element can be isolated in a finite time. Therefore it is not constructed and cannot be constructed. But if I indicate a point on the continuum, then, contrary to what Brouwer wrote, I can *sometimes* say that that point *does* belong to it (or will eventually belong to it), since the boundaries of each new segment will, after ω steps, become an element of the set. But there is no law or rule that decides in a finite time whether or not an arbitrary indicated point belongs to the set. In that sense the set is not real, no more than a set of which the elements are determined by free choice from a given set is a real and well-defined set.[79] It seems as if Brouwer had in some way this choice concept in mind when rejecting the reality of the Cantor set.

Immediately after this deleted fragment the next paragraph continues with the following fragment, this time not deleted:

> (VI–36) One might say: the continuum is intuitive and the rational numbers are denumerable, hence intuitive, therefore also the continuum with the rational scale on it.
>
> Indeed, but if I indicate a point on the continuum, I cannot say whether it belongs to the scale or not.[80]

It is obvious that the continuum is intuitive. The rational scale is intuitive in a derived and indirect sense: the natural numbers come into being out of the ur-intuition of mathematics, and the rational scale from the repeated intercalation of elements on the connecting medium between any two consecutive earlier constructed elements. In that sense the combination may be seen as 'derivatively' intuitive.

The last remark of the last quote (VI–36) seems obvious, but Brouwer gave a physical illustration of it, with the help of 'barycentric coordinates':

> (VI–36) (...) first I put one mass point at each end, next 1 at one end and 3 at the other, then 3 at one end and 5 at the other, or 1 at one end and 7 at the other, and so on, thus approximating what is required, with the help of natural numbers.[81]

This means that I approximate the indicated point in a decimal expansion, each time filling in a next decimal place (supposing that, e.g. at each next

[79] Compare this with Borel, for whom a set as a totality may be well defined, without being able to define any of its elements individually. See e.g. [Borel 1914], also in [Borel 1972], vol. IV, page 2137. See also page 298 ff. where Hilbert's *Heidelberg lecture* ([Hilbert 1905]) is discussed. Rule III for the expansion of the laws of mathematical thought admits the concept of 'set' before that of its elements.

[80] Je zou zo zeggen: het continuüm is intuïtief en de rationale getallen zijn aftelbaar dus intuïtief, dus ook het continuüm met een rationale schaal er op.

Ja juist, maar als ik een punt op het continuüm aanwijs, kan ik niet zeggen of het tot de schaal behoort.

[81] (...) eerst leg ik in elk der uiteinden één massapunt, dan in 't ene 1 en 't andere 3, dan in 't ene 3, 't andere 5 of in 't ene 1 en 't andere 7 enz: zo benader ik, werkend met gehele getallen, wat moet.

addition of mass points, the masses are smaller by a factor 10). If I 'reach' the point in finitely many steps, then the point belongs to the rational scale; if not, it is not a scale point.

The sketched method is an idealized situation, and the question is of course: how can it be decided that the point will be reached in finitely many steps; this can only sometimes be decided afterwards if we accidentally 'hit' a rational number. But then another problem arises: even the rational scale cannot be seen as finished and therefore my ignorance remains complete:

> (VI–36) Only *that* continuum is intuitive, on which, when selecting a point, I am still completely ignorant of its approximation, so I am neither observing the rational scale on it as finished and individualized. Because I cannot perceive $\aleph_0$ things as being finished, I can only imagine them to be growing, and simply let that continue and turn away from it; in the same way for the rational scale.[82]

A similar remark was made on VIII-24. The problem whether or not a well-defined algorithmic sequence or set may be viewed as finished, will be discussed in chapter 8 of this dissertation, on page 322. It was already touched upon in chapter 2, page 50, and under item 5 on page 78 of this chapter.

The given quote expresses the fact that the continuum is dense to the extent that, however far we succeed in the construction of an everywhere dense scale, the approximation of the vast majority of points remains unknown.

We see that in this sixth notebook Brouwer was 'brainstorming' on the question of the perfect set, whether it can be defined as a set and how the continuum can possibly be approached and treated as more than just the 'mysterious matrix of all point sets' (VII–3). The next quotation is another experimental move in this direction:

> (VI–37) *All* algebraic numbers minus the rationals makes sense.
>
> *All* real numbers minus the rationals only makes sense in the following way: I imagine this set to be mapped on the continuum; make an arbitrary choice from the continuum and then I have, when mapping this choice, only a chance of a point of our set.
>
> (...)
>
> I *cannot* speak of the cardinal number of the continuum (that is not included in its intuition); no more can I speak of that of the infinite decimal fractions, since the *all* of it makes no sense in itself, neither via the continuum since neither the continuum possesses 'all points'.[83]

[82] Alleen dát continuüm is intuïtief, waarbij ik, een punt kiezend, nog niets van de benadering weet, dus ook de rationale schaal niet af en geïndividualiseerd erop zie staan. Want $\aleph_0$ dingen kan ik niet *af* zien staan, ik kan me alleen denken, dat ze groeien, en dan dát rustig zijn gang laten gaan en mij afwenden; zo ook met de rationale schaal.

[83] *Alle* algebraïsche getallen min de rationale heeft zin.

The first part of this extract again alludes to the choice aspect, just as we saw that in VI–36. It even alludes to the perfect spread, i.e. the set C, as a representation of the reals, as we met this on page 71. One cannot indicate a point of it, there is only the 'chance of being an element' for each indicated point. Hence 'all real numbers minus the rationals' hardly makes any sense, if one holds on to Brouwer's view that 'chance' or 'choice' cannot determine a set.

One can only, the text continues, speak of the 'reductible' sets, which are, as stated in VI–37, the sets which may be built intuitively from a finite number and induction, without the continuum principle, thus resulting in the denumerable sets.[84]

But in the first part of this quote we still perceive an attempt in the direction of the notable third construction possibility for sets, mentioned on page 66 of Brouwer's dissertation, and which will be discussed further on page 122 of our dissertation. In Brouwer's dissertation it is stated in the following terms: removing a rational constructed scale from the continuum results in a infinite possibility of approximating sequences, which are expressed in the withdrawn points. Such an approximating sequence then stands for a 'point' of the remaining part.

3.4.9 The second number class does not exist as a finished totality

In VIII–36, in which the possible denumerability of the second number class is discussed, it is said that this question cannot be answered by *yes* or *no*, but that, instead, it has to be considered as a meaningless question. In the logical sense something can be said about it, but not in the mathematical sense. This returns in Brouwer's dissertation in chapter 3: in the logical sense only negative statements can be made about this number class.

The cardinality of the second number class can (classically) be proved to be $\aleph_1$, the next higher cardinality after $\aleph_0$, and the impossibility of $\aleph_1$ (according to Brouwer) is expressed for the first time in notebook six:

> (VI–37) I shall thus have to demonstrate that Cantor's Aleph-one makes no sense;[85]

and, as a correction on this last phrase:

Alle reële getallen min de rationale heeft alleen zin als volgt: ik denk mij deze Menge op het continuüm afgebeeld; kies dan willekeurig uit het continuüm, en heb dan de afbeelding daarvan nemend alleen kans op een punt uit onze Menge.

(...)

Ik kan niet spreken van het cardinaalgetal van het continuüm, (zo iets ligt niet in de intuïtie er van); evenmin van die der oneindige decimaalbreuken, omdat op zichzelf het *alle* daarvan geen zin heeft, en via het continuüm evenmin, omdat ook het continuüm geen '*alle* punten' heeft.

[84] Brouwer referred (in VI–37) to Cantor's 'reduktibele Punktmengen', which are sets 'von der ersten Gattung' (Cantor, *Grundlagen*, page 31). See section 1.1.3 and 1.1.4. See [Cantor 1932], page 193 and [Schoenflies 1900b], page 68, 69.

[85] Ik zal dus moeten kunnen aantonen, dat Cantors Alef-eins zinloos is.

> (VI–37) No, his higher numbers certainly exist; only I just know certain defined individuals among them, and the few defined, which I can indicate, are denumerable.[86]

This is indeed Brouwer's view in the dissertation: Cantor's second number class is, as a result of the two generation principles, certainly not empty, but it does not exist as a finished totality; we can only know certain specified individuals from that class. We can always extend the quantity of known individuals, but the lack of one simple algorithm for the construction of *all* its elements limits the result to a denumerable amount, even if, instead of one simple algorithm, denumerably many different algorithms are allowed. Brouwer called the cardinality of this ever extending set *denumerably infinite unfinished* (see at the end of this section).

Nevertheless the idea of a third number class turned up, be it only once, but in fact it was already rejected in the previous quote:

> (VI–38) Does $\omega^{\omega^{\omega^{\cdot^{\cdot^{\textit{infinitely many times}}}}}}$ still belong to the second number class?
>
> Certainly: the *and so on* (in relation to a number or operation, definable with known things from the second number class, as unit) generates the second number class. I call the totality of those construction methods etc^2; and I define the 3^{rd} class as the totality, which can be obtained from earlier constructed numbers of that class with the help of etc^2.[87]

The number $\omega^{\omega^{\omega^{\cdot^{\cdot^{\textit{infinitely many times}}}}}}$ is still *in* the second number class, that is, it is an ordinal number, greater than ω, but still denumerable, but the *totality* of the second number class is of course never generated.[88] Brouwer never returned, neither in the notebooks nor in any other work, to the idea of a third number class which is, even in the last quote, already a rather vague concept. Its rejection immediately follows from the fact that the second number class cannot be finished.

In the seventh notebook the ideas are further developed in the direction of the result in the dissertation. Nevertheless old topics regularly return, like the possible constructibility of the continuum and its possible equivalence to the

[86] Neen, zijn hogere getallen bestaan zeker; alleen: ik ken niet anders, dan bepaalde individuen er uit, en de enkele bepaalde, die ik kan aanwijzen, zijn aftelbaar.

[87] Behoort $\omega^{\omega^{\omega^{\cdot^{\cdot^{\textit{oneindig maal}}}}}}$ nog tot de tweede getalklasse?

Natuurlijk: het *enzovoort* (t.o.v. als eenheid een getal of bewerking, die met reeds bekende dingen uit de tweede getalklasse zijn te definiëren) is de generering der tweede klasse. Het geheel van die vormingswijzen noem ik *enzovoort*^{2}; en definieer de 3^e klasse als het geheel wat te krijgen is uit reeds gevormde getallen dier klasse met behulp van het *enzovoort*^{2}.

[88] See [Rucker 1983], page 74 and 244.

second number class. In the next fragment such a possibility is considered, with the correct conclusion that one can commence the construction of the second number class, but that this process, by definition, cannot be conceived as being terminated:

> (VII–4) I can say that I build the continuum out of the point set, but I cannot speak of its 'cardinality', since this set, in its construction from individuals, simply is the second number class, and then 'denumerable' and 'not finished'.[89]

We indeed observe here the appearance of the term *unfinished* ('nicht fertig') as the possibly maximum attainable cardinality of a set. And also we see that at this place it is stated that one can commence the construction of the 'continuum' in the sense of the totality of points on the continuum, or the totality of all irrational numbers, but that one never succeeds in going beyond the second number class. This class remains, on principle, unfinished.[90]

3.4.10 Attempts to introduce the 'in practice unmeasurable numbers'

In the eighth notebook attempts are made to introduce the, qua definition, most difficult type of numbers, the ones that are 'in practice unmeasurable' on the continuum; these are the infinite decimal fractions without an algorithm (hence, in fact, choice sequences). Brouwer searched for alternatives for the intuitive continuum:

> (VIII–1) Denumerable ordering is my only means to individualization. *That* is the only way to specify a certain point of the continuum; (since I have there at my disposal a finite number of signs and a denumerable number of digits); and without that I only can operate with an 'arbitrary point' thereof.[91]

Hence the only points we can know and specify are the algorithmically defined ones, and their number is limited to denumerably many and can therefore be well-ordered. Arbitrary points cannot be specified.

The next two quotes clearly demonstrate that Brouwer was not satisfied with only the intuitive nature of the continuum:

[89]Ik kan uit de Punktmenge wel zeggen, dat ik het continuüm opbouw, alleen kan ik niet spreken van de 'machtigheid' er van, want deze Menge is in haar opbouw uit individuen gewoon de tweede getalklasse, en dan 'abzählbar' en 'nicht fertig'.

[90]See page 266 for this type of cardinality.

[91]Aftelbare ordening is mijn enig middel tot individualisering. Van het continuüm kan ik ook zó alleen een bepaald punt aangeven; (immers heb daar tot mijn beschikking een eindig aantal tekens en aftelbar aantal cijfers); en zonder dat kan ik alleen werken met 'een willekeurig punt' er van.

> (VIII–11) Just as we cannot imitate the construction of nature, we cannot logistically imitate the construction of the intuitive continuum: But we can – obviously – imitate from both that, which we do with it ourselves.[92]

and therefore a first attempt is made to, what will be called later, the concept of choice sequences:

> (VIII–13) We can only ground the intuition of continuous:
>
> $\underline{1^{st}}$ to view it as counterpart of discontinuity, which is our externalization.
>
> $\underline{2^{nd}}$ as a probability theorem, which *always* gives equal chances for each digit at every next decimal place. But we gaze at the system, which has that as a result, as a phenomenon of nature, we cannot construct it with our externalization of discontinuity.[93]

These last three quotes from the eighth notebook indeed show us Brouwer's attempts to analyze the continuum beyond its role of the 'mysterious matrix' on which points can be constructed.

The first quote (from VIII–1) summarizes the gist of the dissertation: the rational scale points, and the irrational points which are constructed with the help of an algorithm, and which are expressed in terms of infinite sequences of rationals, are the only definable points and the result is a denumerable but unfinished quantity.

The second quote (VIII–11) expresses and emphasizes the intuitive nature of the continuum, and the same is expressed in the first item of the last quote (in which the continuum is the inseparable counterpart of two discrete and not coinciding events). In the second item of VIII–13, however, we discern attempts to move beyond the intuitive continuum and beyond any restriction, to approximating sequences of a stochastic nature that stand for irrational numbers: at each next decimal place all digits (that is, from zero to nine in case of a decimal representation) have an equal chance, resulting in a representation of the continuum, in the form of the 'set of all real numbers on the unit interval'. But Brouwer still 'gazes at the system as a phenomenon of nature' and not yet as a useful and successful medium to have at one's disposal *arbitrary elements* of the continuum of the real numbers, thus opening the way to a real analysis.[94] For a further discussion about choice sequences see page 148.

[92] Zo min als wij de natuur kunnen nabouwen, zo min kunnen wij logistisch het intuïtieve continuüm nabouwen: alleen kunnen we –natuurlijk– van beide nabouwen dát, wat we er zelf mee doen.

[93] Te begründen is de intuïtie van continu niet dan:
$\underline{1^e}$ te bekijken als tegenhanger van discontinuïteit, die onze veruiterlijking is.
$\underline{2^e}$ als de waarschijnlijkheidsstelling, die *steeds* weer bij elke volgende decimaal voor elk cijfer gelijke kansen geeft. Maar het stelsel, dat dat geeft, staren wij aan als natuurverschijnsel, kunnen het niet opbouwen met onze discontinuïteitsveruiterlijking.

[94] Brouwer still speaks here in terms of 'chance' instead of 'choice', probably under Borel's influence.

The intuitive continuum still remains primal (which is, as we saw, still explicitly expressed in the first *Wiener Gastvorlesung* ([Brouwer 1929]):

> (VIII–20) But the unbounded open continuum is primal ([*in the margin:*] This exists before mathematics, but only in mathematics it comes into the open as the genetrix of limit points) (just like time, but not time itself). It is different from its points (viz. point matrix): but I can construct points on it, add points to it as limits. A limit point gives the possibility to split the continuum into two parts (...) I obtain the closed continuum by putting together the lower and the upper bound (as I do when glueing together a dissected continuum).[95]

This last phrase, in which an open continuum can be split into two open ones, and in which the closed continuum is formed out of an open one, shows up in Brouwer's dissertation in a footnote on page 11 (cf. page 83 of this dissertation).

In VIII–18 the ur-intuition of mathematics and the twin-concept of discrete and continuous is applied to physics:

> (VIII–18) Should matter consist of points alone? But why then remain these points separated? By the tension of 'something' between them: but that 'something' should then be continuous. And if a gas just consisted of flying points, how could they act on each other as solid bodies, if *nothing* was between them? Anyway, Maxwell's theory directly explained the theory of action at a distance.[96]

In Brouwer's opinion the physical world of discrete objects follows the pattern of the mathematical continuum: they are connected by a continuous medium. Brouwer's view on mathematics and its applications was a coherent one, his philosophy is global and not restricted to mathematics alone, and therefore our ur-intuition of mathematics applies equally well to the description of the physical world. This is of course in agreement with the dissertation, since also the observation of two separated 'flying points' can be seen as the experience of two well-separated events, connected by a 'medium of cohesion', and taking place in time.

The discussion about the continuum returns time and again. In the following passage we can observe Brouwer's attempts to 'operate on' the continuum: all

[95] Primair is het onbegrensde open continuüm. (*(met in de kantlijn)* Dit staat al vóór de wiskunde: maar in de wiskunde komt het alleen naar buiten, als genetrix van grenspunten) (zoals de tijd, maar niet de tijd zelf). Het is iets anders als zijn punten (n.l. puntenmatrix): maar ik kan er punten opzetten, er bij voegen als grenzen. Een grenspunt geeft de mogelijkheid tot verdeling van het continuüm in tweeën. (...) Het gesloten continuüm krijg ik door een begin- en eindgrens samen te koppelen (zoals ik ook doe, als ik een verbroken continuüm weer samenlijm).

[96] De materie zou alleen uit punten bestaan? Waarom blijven dan die punten gescheiden? Door de spanning van 'iets' er tussen: maar dat 'iets' is dan toch continu. En was een gas alleen vliegende punten, hoe zouden die dan als vast lichaam toch op elkaar werken, als er *niets* tussen was? Trouwens de theorie van Maxwell verklaarde direct de Fernwirkungstheorie.

indicated points on the continuum can be 'reached', that is, arbitrarily closely approximated:

> (VIII–19) Since I cannot speak of *all* points of the continuum, I do not express continuity as: *all* intermediate values are reached, but as: *if* I give an intermediate value, then it will be reached (the exact location can be found by means of consecutive approximating measurements.[97]

In the eighth notebook, on page 21 and 22, Brouwer speaks in a deleted fragment about the possible ordering of the continuum. The first half of this quote consists of a fragment in which the continuum and the second number class are compared. The reason for deleting this fragment is clear from the outcome of the comparison:

> (VIII–21, 22) The continuum can be ordered in a natural way as a sequence of all integers with finitely many or ω digits. (The next integer is approximated together with the integer itself.) Likewise T can be ordered as a sequence of ω arbitrary numbers of the first cardinality.
>
> (...)
>
> It thus appears that the cardinality of T certainly exceeds that of c. (...). [98]

The main point we want to emphasize here is, that in Brouwer's opinion the continuum of the real numbers is an ordered set, the order being the 'natural' one. This is of course the generally accepted view, but we must now be aware of the fact that Brouwer is, in this notebook, referring to the continuum of the real numbers and not to the intuitively given continuum of the connecting medium between two events, on which always more points can be constructed. Brouwer is in fact referring to, what he called in the first *Begründung* paper, the set C of the 'unbeschränkt fortgesetzten Folgen von zu ζ gehörigen Ziffernkomplexen', which can be interpreted as a representation of the real numbers between zero and one.[99]

We noted already that in 1918 Brouwer was still of the opinion that the continuum as the set of the real numbers is an ordered set (but he contested Zermelo's proof of the well-ordering of the continuum).

In 1923, however, he proved by means of a counter example that the theorem

[97]Omdat ik niet kan spreken van *alle* punten van het continuüm, zeg ik voor Stetigkeit niet: *alle* tussenwaarden worden bereikt, maar: *als* ik een tussenwaarde geef, wordt hij bereikt (de plaats waar, is door opvolgende benaderingsmeting te vinden.

[98]Het continuüm is lopend te ordenen als rij van alle gehele getallen met eindig of ω aantal cijfers. (Het eerstvolgende getal wordt benadered tegelijk met het getal zelf.) Evenzo is T te ordenen als rij van ω willekeurige getallen der eerste machtigheid. (...) Zo blijkt dan T van zeker groter machtigheid dan c. (...)

[99]See page 71.

> Die Punkte des Kontinuums bilden eine geordnete Punktspezies.[100]

is wrong, hence not only Zermelo's claim of a well-ordered continuum is incorrect, but even the generally accepted opinion that the continuum is ordered. Note again that the term 'continuum' refers to the set C from the *Begründung* paper which can represent the set of the real numbers on the open interval $(0, 1)$, and not to the intuitive continuum from the dissertation.

3.4.11 Pro and contra the pure and intangible continuum

A surprising question, asked by Brouwer in the seventh notebook (in a crossed out fragment), is the following, almost rhetorical one. After all, the answer was supposed to be known and familiar to all mathematicians:

> (VII–4) It is the question, whether the 'fertige' intuitive two-dimensional continuum is equivalent to that of one dimension; most likely not.[101]

This statement should by all means come as a surprise to the reader, as Cantor gave a proof of their equivalence thirty years earlier (in 1874, cf. page 5). Apparently Brouwer had his doubts about Cantor's theorem and, for that reason, deleted the paragraph; however, he did not follow up his view. His doubts were fully vindicated by his later intuitionistic mathematics: the equivalence is only valid if we accept a discontinuous mapping, based on a total function, whereas every total function in intuitionistic mathematics is continuous.

Despite earlier exploratory attempts in the direction of spreads and choice sequences (in VIII–13, he spoke of 'sequences of chance'), Brouwer returned to, and remained of the opinion that the continuum is an entity that exists on its own; the only way to discuss and study it, is by means of an everywhere dense 'scale', that is constructed on it (VIII–22). In this way one can approximate a point which is indicated on the continuum, and is one able to determine the cardinality of a set, defined on it (cf. Brouwer's Rome lecture *Die möglichen Mächtigkeiten* [Brouwer 1908b]; see also page 170).

But immediately following this conclusion, Brouwer distinguished between the 'known point' for which one can give an algorithmic approximating sequence, and the 'unknown point':

> (VIII–28) The unknown irrational point is rather the limit of a segment ('the intuitive continuum' or 'the *alter ego* of the point on the continuum' or 'the relation between two points of the continuum') than of a point. But the known irrational point, e.g. $\sqrt{2}$ is definitely a point.[102]

[100] [Brouwer 1923a], page 3, or [Brouwer 1975], page 270.

[101] Het is de vraag, of het 'fertige' intuïtieve continuüm van twee dimensies gelijkmachtig is met dat van één dimensie; waarschijnlijk niet.

[102] Het onbekende irrationale punt is meer de limiet van een Strecke ('het intuïtieve continuüm' of 'het andere van het punt van het continuüm' of 'de *relatie* van twee punten van het continuüm'), dan van een punt. Maar het bekende irrationale punt, bv. $\sqrt{2}$, is wel degelijk een punt.

The unknown point is the 'limit of a segment'; this has to be interpreted as the definition of an 'algorithmless' irrational number in the form of a set of nested intervals, as Brouwer presented this in the second *Begründung* paper.[103] In this definition the unknown point *is* the infinite choice sequence, and not its limit. Weyl described it in a similar way in the second part of his *Über die neue Grundlagenkrise der Mathematik*,[104] in which he distinguished between a known point in the form of a 'bestimmte Folge' which can only be defined by a law, and a 'werdende Folge', determined by free choice; similar to Brouwer, Weyl stated that in order to operate with such a 'werdende Folge', a certain property has to be determined by an initial segment of this sequence. At any stage of the development, the segment reached so far is symbolic for the 'werdende Folge' (the genetic point). In fact the segment represents the infinity of 'points' on it (i.e. it represents all sequences of nested intervals, lying in that segment), since there is an infinity of different sequences of intervals possible on that segment, standing for 'infinitely many points'; Brouwer could very well have shared the same opinion.

But in view of the fact that the last two mentioned publications are from 1919 and 1920, and Brouwer's fragment VIII–28 from 1906 or 1907, it is obvious but surprising that the concept of 'sequence of free choices of nested intervals' was in his mind at such an early stage.
See also VIII–38:

> (VIII–38) (...) One defines (on the basis of those algorithms) the *known* irrational numbers (...) as limits of known sequences. (...)[105]

In the same paragraph the question is raised whether the unknown irrational numbers can be defined as the limits of unknown sequences, but according to VIII–41 one cannot give any relation between the known and the unknown irrational numbers:

> (VIII–40/41) One constructs, completely independent of each other, the everywhere dense scale and the *unknown* irrational points. One cannot give any relation between the two groups, never determine of an element of the second group, whether it belongs to the first group.[106]

Compare this again with Weyl,[107] when he stated that 'zwischen dem Kontinuum und einer Menge diskreter Elemente eine absolute Kluft befestigt, die

[103]See [Brouwer 1919a], page 3 or [Brouwer 1975], page 191. In fact Brouwer defined here 'points on a plane' by means of λ-squares, but the definition applies to any number of dimensions.

[104][Weyl 1921], page 49 ff.

[105]Men definieert nu (op grond van die bewerkingen) de *bekende* irrationale getallen (...) als limieten van bekende reeksen. (...).

[106]Men bouwt geheel onafhankelijk van elkaar op, de überall dichte schaal en de *onbekende* irrationale punten. Men kan tussen deze beide groepen geen verband brengen, nooit van een element van de tweede groep etwa uitmaken, dat het tot de eerste groep behoort.

[107]op. cit. page 51.

jeden Vergleich ausschliesst', and therefore no comparison between the two is possible. (The reader will recognize that Brouwer's view as expressed in the last quote is rather sophisticated for the year 1906.)

In the last quote, Brouwer in fact stated that nothing can be said about unknown approximating sequences. 'Every' in e.g. 'every element of a set' can only refer to constructed elements, and unknown limit points are never like known ones.

About Couturat's definition of a perfect set,[108] Brouwer commented in this notebook:

> (VIII–41) I cannot say: 'every fundamental sequence has a limit', since I cannot consider a general fundamental sequence because it is never finished.
>
> (...)
>
> And neither can I say: 'every term A has between itself and another term B at least one term'; this applies only to well-defined sets.
>
> (...)
>
> In general I cannot say anything about sets of greater cardinality than ω; their elements are not definable; hence I can say nothing about *every* element.[109]

We again recognize the term 'finished' with its two possible interpretations: finished in its literal meaning and in its potential meaning. This discrepancy is not of too great importance here, not only because it occurs 'only' in a notebook and not in some published text, but also (and especially) because the meaning here is clearly 'never finished on principle', since it concerns a choice sequence (which of course in later time *is* a point). And since the continuation of such a sequence is unknown on principle, one cannot speak of its limit.

In modern terminology the content of VIII–41 comes down to the impossibility of quantification over unknown points. The problem of quantification over choice sequences was only much later successfully tackled by Brouwer in his continuity principle.[110]

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[108] [Couturat 1905], chapter IV, § B: '(...) on dit qu'un ensemble est *parfait*, si toutes ses suites fondamentales ont des limites, et si tous ses termes sont des limites de suites fondamentales'. Couturat obviously means here *convergent* fundamental sequences. Note the difference with Brouwer in regard to this concept. For Brouwer a fundamental sequence is (usually) any lawlike sequence of order type ω.

[109] Ik kan niet zeggen: *elke* fundamentaalreeks heeft een limiet, immers ik kan niet een algemene fundamentaalreeks beschouwen, want hij is nooit af.

(...)

En ik kan ook niet zeggen: 'elke term A heeft tussen zich en een andere term B nog minstens één term'; anders dan in de zin voor een welgedefinieerd ensemble.

(...)

In 't algemeen kan ik van ensembles groter dan ω niets zeggen; hun elementen zijn niet definieerbaar; ik kan dus niets zeggen van *elk* element.

[110] See our chapter 4.

An unexpected argument for the intuitivity of the continuum is the following one, which reminds us of the quote from II–1 from page 86 (which, in its turn, reminded us of Borel's view on the continuum).

> (VIII–38) The strongest proof that the continuum is intuitive, must be, that a child does not understand all the relevant reasoning about it, but nevertheless applies it immediately without hesitation in the proper way.[111]

The cardinality of the continuum is, for the first time, mentioned as an individual cardinality in VIII-43: here it is stated, that the cardinality c of the continuum can be understood in the following way:

> (VIII–43) The cardinality c means: that cardinality, of which the individuals can be approximated by a denumerable sequence, and can be thought as such.[112]

This quote must be understood in the sense that c is a cardinality of its own, with, as the most conspicuous property, that 'individuals' can only be approximated. Brouwer is venturing here into unknown territory. No specification is given of *how* exactly to approximate an individual, even the concept of 'individual' remains unspecified, just as it remains unspecified *how* exactly equal cardinality has to be defined for two sets with that cardinality. The method of approximation will eventually turn out to be the one of choice sequences, lawlike for the 'known' and non-lawlike for the 'unknown' elements.

We are inclined to assign to a set a cardinality of one of Cantor's aleph's as a measure for its size, but the reader must be aware of the fact that for Brouwer only $\aleph_0$ exists as an aleph: the denumerably infinite cardinality. The next higher cardinality is the 'denumerably infinite unfinished'[113] and the only higher one after that is c.

But a set with cardinality c, that is, the continuum, does not have *individuals* by itself, one can only construct individuals on it; and a constructed set of individuals on the continuum can subsequently be considered as a finite or denumerable subset of the continuum.

This characterization is not easily interpreted. Evidently, c presupposes the intuitive continuum, and refers to elements that can be approximated. The fact that Brouwer considered individuals, suggests that the elements have a description of some sort. If one takes that point of view, then c refers to an unfinished set (cf. page 266). But it is equally possible, and in harmony with other statements, that he was willing to grant individuality to non-lawlike sequences, and that thus c is the *cardinality of the choice reals avant la lettre*, or in terms of

[111] Het beste bewijs dat het continuüm intuïtief is, is wel, dat een kind over al de erop betrekking hebbende redeneringen heen hoort, maar ze toch zonder aarzelen direct zuiver toepast.

[112] De machtigheid c wil zeggen: die waarvan de individuen door een aftelbare reeks zijn te benaderen, en kan als zodanig worden gedacht.

[113] This cardinality will be discussed in chapter 7.

the dissertation and of the notebooks, the 'unknown reals'.

However, the same paragraph on VIII–43 continues, the cardinality f which is the cardinality of all functions defined on c, is contradictory, since one can make a free choice ω times[114] (that is, continue indefinitely in making free choices, thereby, again, cautiously referring to a possible choice sequence), but not c times:

> The cardinality f is contradictory. After all one can imagine that the game of chance makes a free choice ω times (that is, indefinitely continuing); but not c times. The fact, that one cannot conceive this is, on being asked, told us immediately by our intuition.[115]

Brouwer's conclusion was that c is an individual cardinality, not expressible in some sequence of aleph's. In fact Brouwer did not recognize such a sequence and he used Cantor's term $\aleph_0$ just for convenience to express the denumerably infinite cardinality. But he already attempted to express the cardinality c in terms of choice sequences: for an arbitrary element of c, one can make a free choice ω times; this has to be interpreted as an indefinitely continuing choice for the next decimal place. The concept 'c times a free choice' is absolutely meaningless, since potentially an individual can make a free choice 'only' ω times, which stands for the always continued possibility of making a choice.

We can make a comparison with Borel, who made the same observation in his *Leçons sur la théorie des fonctions*, where he wrote in an added note:

> Il me paraît ressortir clairement de ce qui précède que l'ensemble des points d'une droite qui peuvent être effectivement définis d'une manière individuelle est un ensemble *dénombrable*, mais non *effectivement énumérable*.
>
> Il n'est pas possible d'indiquer le moyen de fixer sur la droite un point unique et bien déterminé qui n'appartienne pas à cet ensemble; la proposition après laquelle *il y a* de tels points est vraie ou fausse suivant qu'on admet ou non la possibilité d'une infinité dénombrable de choix succesifs.[116]

Note the mention of successive choices in Borel's text.

[114] Note that here 'ω times a free choice' is possible, whereas in VI-34 (see section 3.5.7) this possibility is denied. This should not immediately be seen as a contradiction in Brouwer's work; the notebooks remain a collection of thought experiments, a 'laboratory' for his present and future mathematics.

[115] De machtigheid f is contradictoir. Immers men kan zich denken, dat ω maal achtereen (d.w.z. *steeds* maar door) het kansspel een vrije keus doet, maar niet c maal. Dat men dit niet kan denken, antwoordt ons, desgevraagd, direct onze intuïtie.

[116] [Borel 1950], page 164; which is 'Note IV' of the said work, *Les polémiques sur le transfini et sur la démonstration de M. Zermelo*, a collection of earlier published papers in several journals. The relevant quote is from the sixth paper of Note IV, *Les 'paradoxes' de la théorie des ensembles* from 1908, published in the *Annales de l'École Normale*.

To conclude this survey of the continuum according to the notebooks, some quotations:

> (VIII–48) The continuum is the means to preserve the everywhere dense scale of one transformation group also for another, and can be defined with the help of the continuity axiom, which is therefore inseparably linked to the continuum.[117]

That is, Dedekind's continuity axiom[118] is required as a heuristic principle to include the otherwise missing elements in the continuum. If, then, we map an everywhere dense scale A, as subset of a continuum which is defined by means of Dedekind cuts, onto another continuum of the same kind, such that the image of a cut on the first continuum which defines an element a of A, is a cut on the second continuum, defining the image of a as element of the image of A, then this mapped scale is also everywhere dense.
About the limited role of symbolic logic for mathematics:

> (VIII–55) One can probably prove by means of symbolic logic, that it is not contradictory to state that c and T are equivalent. For, under a mapping, I can only construct c individually; hence c is denumerably unfinished.[119]

This means that in symbolic logic the most one can perform is the proof of negative statements, in this case the non-contradictority of the equivalence of the continuum and the second number-class, both then of course viewed as denumerably unfinished sets. Brouwer elaborated this idea in the third chapter of his dissertation (see our chapter 7, page 266).

The next two phrases relate to the 'spread-concept' on the continuum:

> (VI–34) The cardinality of all groups from ω is of course $2^{\aleph_0}$; (...)[120]

The continuum hypothesis, according to Cantor, claims that this should be equal to $\aleph_1$, but for Brouwer, as he expressed it already in several notebooks, $\aleph_1$ is not a cardinality, since we cannot define with the help of some algorithm the elements of a set having that cardinality.

---

[117] Het continuüm is het middel, om de überall dichte schaal van één transformatiegroep ook te kunnen behouden voor een andere en is te definiëren door middel van het Stetigkeitspostulat, dat dus aan het continuüm onafscheidelijk is verbonden.

[118] This axiom was published by Dedekind in 1878 in his *Stetigkeit une irrationale Zahlen*, [Dedekind 1912], page 10: 'Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, daß jeder Punkt der ersten Klasse links von jedem Punkte der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der geraden in zwei Stücke hervorbringt.'

[119] Met symbolische logica zal waarschijnlijk wel zijn aan te tonen, dat het niet contradictoir is, om c en T gelijkmachtig te stellen. Immers bij afbeelding kan ik c niet anders, dan individueel opbouwen; c is dus aftelbaar onaf.

[120] De machtigheid van alle groepen uit ω is natuurlijk $2^{\aleph_0}$; (...)

> (IX–25) The continuum as a never terminating sequence of chances is nonsense, since with an always continuing source of chances I just get 2^ω, never $2^{\aleph_0}$.[121]

ω is the ordinal number of the (ever unfinished) set of the natural numbers, and $\aleph_0$ is the cardinality of every denumerable set; Brouwer makes here a clear distinction between 2^ω and $2^{\aleph_0}$.

Both citations (VI–34 and IX–25) seem to express in a binary representation $2^{\aleph_0}$ as the finished and completed totality of all infinite choice sequences, hence as the completed set of the reals, and such a completion is of course never reached and is unthinkable. 2^ω, then, is representing the indefinitely continuing *process* of free choices (Brouwer still used the Borelian term of 'chance'), so it represents the denumerably infinite unfinished process (see page 266). Therefore 2^ω should be interpreted as the continued process of the construction of all *finite* sequences of free choice.[122] We might put this as $2^\omega = 2^0 \cup 2^1 \cup 2^2 \cup 2^3 ...$ According to the last quote, this always growing totality of the increasing number of finite choice sequences is all we can obtain in our attempt to represent the continuum. In the last quote Brouwer was referring to the intuitive continuum, onto which a denumerably infinite unfinished number of points can be constructed.

3.5 Summary and conclusions

The continuum is intuitive, and yet ... and yet. (VIII–18)

Brouwer's continued occupation with this subject, which is of fundamental importance for him, can be concluded from his notebooks, rather than from the dissertation. The notebooks witness his 'thinking aloud' and his doubts, even his doubts about the final results in the dissertation.

Brouwer's fundamental concept is a continuum in the proper sense: a purely flowing medium of connection between well-separated events, not composed of points, but on which an everywhere dense scale without 'gaps' turned out to be constructible, as we showed in the present chapter.

On such a scale, or with the help of such a scale, non-terminating sequences could be defined, of which the terms, under a certain interpretation, had to be viewed as 'partially unknown'. This term turned out to ask for interpretation.

In later years the concept of the continuum developed further. In the *Begründung* papers,[123] Brouwer was able to speak of 'arbitrary elements' of the 'continuum of the real numbers', represented by non-terminating sequences of natural numbers or of nested intervals defined by rationals. After that he

[121] Het continuüm als oneindig voortlopende kansenrij is onzin, want als oneindig voortlopende kansenbron krijg ik alleen 2^ω, nooit $2^{\aleph_0}$.

[122] In a modern notation this totality of all finite sequences is written as $2^{<\omega}$.

[123] [Brouwer 1918] and [Brouwer 1919a].

proved that the continuum is not linearly ordered: one can define a real number, of which it is impossible to prove whether it is positive, negative or equal to zero. Also one can prove that not every definable real number has a decimal expansion, and that one cannot split the continuum into two non-empty parts.[124]

It will be clear that a dissertation should represent a concluding position, and, in case that position is one of doubt, that too has to be expressed. Apparently, during the time of his academic promotion, the doubt had not yet materialized into a fundamentally new approach to mathematics.

In this chapter we have attempted to systematize the developments of Brouwer's ideas and viewpoints on the topic of the continuum, from the notebooks as well as from the dissertation, but despite this attempt the result might look somewhat confusing, since the many quoted paragraphs in this section are rather chaotically spread over the several notebooks in the form of short notes or concise discussions. Also (seemingly) mutually conflicting opinions and statements are a frequently occurring phenomeneon. However, as a conclusion we can state that the dissertation presents a clear point of view regarding the intuitive continuum with its own specific cardinality, not comparable to any other. Points can always be constructed on the continuum, according to an algorithm given in advance in finitely many terms, resulting in sets with the cardinalities of *finite, denumerable and denumerably unfinished*. See chapters 4 and 5 for more detailed discussions on this. Every subsegment of the continuum is again a continuum, it has no atomic constituent: a line is not composed of points.

But the doubt whether the result as published in the dissertation was to be his ultimate viewpoint, remained until his last notebook: the 'and yet, ... and yet ...', his frequently deleted 'thought experiments' on free choices of nested intervals and on branching, these experiments can only be found in the notebooks and not in the resulting dissertation; they are a clear indication that the process of *studying and investigating the character of* the continuum continued. His rejection of the Cantorian continuum was one of principle and a permanent one. The concept of the ur-intuition as the ultimate foundation of mathematics was to remain. Still, Brouwer was not satisfied with his result. It is certainly worth observing that his later ideas, which turned out to be of paramount importance, were already present in the notebooks in the years 1905 through 1907, albeit often 'in statu nascendi'.

[124]See further under chapter 4.

Chapter 4

The possible point sets

4.1 Introduction

The nine notebooks that Brouwer filled with his thought experiments, his new ideas and his comments on these ideas, as well as on those of others, are for us a rich source of information in our attempts to trace the development of his views on the ur-intuition of mathematics and on the continuum. Also Brouwer's notions on possible point sets, and especially the *development* of these notions to the ones that we meet in his dissertation and even beyond, can be recognized in the notebooks from the first one onwards: the formation of a set is completely governed by the algorithm for the construction of its individual elements.[1] For that reason the totality of the elements of the second number class does not exist, since the two generation principles that Brouwer admitted enable us to construct always more elements for this set, but does not give a closure for it. The number of possible cardinalities of sets, however, is not univocally stated from the beginning; we will see in chapter 7 that this number gradually increased from two (finite and denumerably infinite) to four (finite, denumerably infinite, denumerably infinite unfinished and the continuum), which was to remain Brouwer's final number.

In the next sections of this chapter we will investigate the set concept as it appears to us from several of Brouwer's writings, in the first place from his dissertation. We will see that fundamentally new and revolutionary notions will appear in print only in 1919, so well *after* 1907, the year of his doctoral degree, but we can identify many traces, leading to these later developments, already in the notebooks. We will also meet several concepts in connection with the set concept, in the dissertation as well as in the notebooks, that require a thorough interpretation.

[1] Note that this changes with the introduction of the concepts of spreads and species in 1918.

It should not come as a surprise to the reader that there is one strict requirement which, according to Brouwer, every set has to satisfy, and that is that, ultimately, its elements have to be developed out of the ur-intuition, and are constructed with the help of some algorithm.

In the preceding chapter we entered at length into the construction of the ω- and the η scale on the intuitive continuum, and this will form the basis for the construction of all sets. In his dissertation Brouwer distinguished three modes of set construction: 1) the just mentioned basic constructions, which, in every combination, can be put together into one set; 2) an everywhere dense set, which can be completed to a continuum; 3) deleting from a continuum a constructed dense scale.

All three modes require an analysis and a discussion; especially the third mode will turn out to be problematic, and Brouwer will eventually drop it. We will argue that this mode originates from Cantor and that Brouwer in his early days simply could not get around Cantor, his influence being too great.

In 1914 a new development set in when non-terminating sequences of free choice could become elements for sets. This enabled Brouwer to handle the continuum of the real numbers by means of its representing 'perfect spread', and this allowed him to prove the non-denumerability of the reals in a surprisingly simple way with the help of his 'continuity principle'.

We will also see in this chapter that Brouwer's new development reached full maturity in 1918 with the publication of his *Begründung* papers.

The notebooks are a rich source to trace the growth of his ideas from simple and relatively primitive concepts to notions that went already beyond the results in the dissertation: We will show that the notion of choice sequence is slowly developing and is becoming demonstrably present in the eighth and ninth notebook.

In order to get a clear picture of the concept of point set in the year 1907, but also to recognize where this concept eventually was leading, the discussion of the relevant parts of the dissertation will be alternated with that of some of Brouwer's papers from later years.

4.2 Set construction

On page 62 of Brouwer's dissertation, after having completed the discussion on the foundational aspects of geometry, the thread of the treatment of the continuum and the construction of points on it is resumed. In his first chapter, Brouwer had shown the construction of two different point sequences, viz. the order types ω of the positive ordinal numbers (or the reversed order type $^*\omega$), and the in itself everywhere dense denumerable sequence of the rational numbers, i.e. the order type η. As a result of the construction of the everywhere dense sequence we were able to turn the continuum, after the selection of an arbitrary point as zero-point, into a measurable continuum, on which points can be approximated in a dual scale. The continuum as a whole, however, is not composed of points and cannot be constructed; it is intuitively given to us in its entirety.

The set construction, as presented in the dissertation, which we will analyze in this chapter, certainly did not yet meet the standards of later years. During the years 1916 – 1918, when lecturing on set theory at the University of Amsterdam, his mature ideas developed. We know this thanks to a few short notes in the margin of his lecture notes on that topic, and at this place we are witness to the birth of the intuitionistic ideas on spreads and species. On page 128 we will devote a discussion on this foundational development.[2]

A few years before this turning point in his development, in 1912, Brouwer was appointed professor at the University of Amsterdam, and his Inaugural address, *Intuitionism and Formalism*, was still written in the spirit of his 'first intuitionistic period', in which he was firmly embedded at that time. It sums up his views on sets and their construction, departing from his constructivistic position:[3]

> From the present point of view of intuitionism therefore all mathematical sets of units which are entitled to that name can be developed out of the ur-intuition, and this can only be done by combining a finite number of times the two operations: 'to create a finite ordinal number' and 'to create the infinite ordinal number ω'; here it is to be understood that for the latter purpose any previously constructed set or any previously performed constructive operation may be taken as a unit. Consequently the intuitionist recognizes only the existence of denumerable sets, i.e., sets whose elements may be brought into one-to-one correspondence either with the elements of a finite ordinal number or with those of the infinite ordinal number ω.[4]

This view is not essentially different from the one in the dissertation, as we will see now.

[2]The original notes, including the remarks in the margin, are kept in the Brouwer archives.

[3]We can roughly put Brouwer's *first intuitionistic period* between 1907 and World War I. During this period Brouwer is struggling with the continuum concept and the unknown rationals. We can date the beginning of his mature intuitionism in 1916, when he wrote the annotations in the margin of the lecture notes for his course 'set theory'; see further page 128.

[4][Benacerraf and Putnam 1983], page 81: Van het tegenwoordige standpunt van het intuïtionisme zijn dus alle wiskundige verzamelingen van eenheden, die die naam verdienen, uit de oerintuïtie op te bouwen, en kan dit uitsluitend geschieden door de beide operaties: *'schepping van een eindig ordinaalgetal'* en *'schepping van het oneindige ordinaalgetal ω',* een eindig aantal malen met elkaar te combineren, waarbij als aan het oneindige ordinaalgetal ω ten grondslag liggende eenheid natuurlijk elke tevoren opgebouwde verzameling of elke tevoren uitgevoerde constructieve operatie fungeren kan. Dientengevolge bestaan voor de intuïtionist slechts *aftelbare* verzamelingen, d.w.z. verzamelingen, wier eenheden in ééneénduidige correspondentie zijn te brengen met de eenheden ener deelverzameling van het oneindige ordinaalgetal ω.

Set construction in Brouwer's dissertation

On page 62, in the introduction to the possible point sets, it is once more emphasized that the mathematical intuition can only construct individually the elements of a denumerable quantity according to a fixed algorithm. Brouwer added in his own copy of the dissertation a handwritten remark, that this algorithm has to produce each element in finitely many steps.

Furthermore, the text continues, mathematical intuition can construct a scale of order type η and we can then imagine this to be covered by a continuum:

> But it [*i.e. the mathematical intuition*] is able, after having created a scale of order type η, to superimpose upon it a *continuum as a whole*, which afterwards can be taken conversely as a measurable continuum, which is the matrix of the points on the scale.[5]

At first sight this last phrase seems to involve a superfluous act, since we needed the intuitive continuum (between two well-separated events) for the construction of a (dual) scale of order type η. Hence we seem to be covering with a continuum something that is already built on a continuum. However, if the scale of order type η is *not* constructed by repeated splitting of an intuitively known continuum, but is generated in the way as was explained on the first pages of Brouwer's dissertation, hence if we have the set $\mathbb{N}$ of the natural numbers by intuition (Brouwer's page 3), then a proper interpretation of 'superimposing a scale η with a continuum' becomes feasible as follows: On the basis of the intuitively known natural numbers the rationals were defined as ordered pairs of natural numbers (page 5 and 6 of Brouwer's dissertation), followed by the definition of the order relations and the basic operations on those numbers. As a result we have the order type η, without an underlying continuum, defined in the 'classical' way on the basis of the intuitively given set $\mathbb{N}$.[6] This order type can then be covered by a continuum afterwards, as will be described now.

Brouwer did not explain how this 'covering by a continuum' could be executed, but we can imagine it in the following way: if we have a scale of order type η without an underlying continuum, and, in addition, we have a continuum with a scale of order type η constructed on it by ω-times splitting, then there exists a one-to-one mapping of both η-scales onto one another under preservation of order and, because of this possible mapping, we may imagine the continuumless scale as 'lying on' the scale on the continuum, thus picturing the continuum to cover the continuumless η-scale, since, because of their similarity, the two scales are 'identical up to isomorphism'.

An arbitrary point of that continuum can then be identified with the limit point of its approximating sequence (or, in modern terms, with the sequence

[5] Maar wel kan zij, eenmaal een schaal van het ordetype η opgebouwd hebbend, er *een continuüm als geheel* overheen plaatsen, welk continuüm dan achteraf weer omgekeerd als meetbaar continuüm als matrix van de punten der schaal kan worden genomen.

[6] Another way to have at one's disposal a scale of order type η without an underlying continuum, is to depart from a continuum with a scale of this order type constructed on it, and subsequently define a copy of this scale in the form of a similar set, which then needs no underlying continuum for its definition.

itself) of rational numbers on the everywhere dense scale of order type η. This interpretation fits with page 63 of Brouwer's dissertation: 'In this way we can state: the points of the second continuum are a part of those of the first'.

All this resulted in three modes for the construction of point sets on the continuum, of which the first two will be discussed in this section (the third one will follow on page 122). Brouwer put it, as usually, very briefly worded in his dissertation, almost without further comment:

> 1. we can construct on the continuum discrete, individualized sets of points which are finite, of order type ω, of order type η, or can be obtained from such sets of points by alternation or subordination. (...)[7]

This first mode refers to the construction of points on the continuum, one by one, each point in finitely many steps according to some algorithm, as this was explained in the preceding chapter (see page 75). The result is, Brouwer continued, a finite or a denumerably infinite set of points and, consequently, finitely many or denumerably many intervals:[8]

> The number of these points is always denumerable, and likewise the number of the intervals determined on the continuum by pairs of points from the set is denumerable. In each of these intervals, and also in its totality, the set may be dense or not (by *dense* we mean: of the order type η after every well-ordered or inversely well-ordered subset has been contracted to a single point).[9]

In order to be able to specify whether or not a resulting point set is dense in an arbitrary segment, Brouwer introduced on page 65 the *branching method* to characterize the points of the set by means of an arbitrary, everywhere dense dual scale on the relevant segment. Brouwer specified here 'dense' as the condition that the first derivative of the set has a perfect subset on the interval; hence dense does *not* mean 'everywhere dense on the complete interval', but 'dense in itself on parts of the interval'.[10]

This branching method proceeds as follows: since the segment to be investigated may be considered as a unit segment, we verify whether there are elements

[7](page 63): (...) kunnen we er volgens eindige getallen of de ordetypen ω of η, of ook in afwisseling of onderschikking aan elkaar van deze drie, discrete, geïndividualiseerde puntverzamelingen op bouwen; (...)

[8]This last expression ('finitely or denumerably many intervals') is of importance in Brouwer's solution to the continuum problem, especially in his proof in the notebooks; see chapter 5, page 175

[9](page 63): het aantal dezer punten is steeds aftelbaar, en evenzo het aantal der door puntenparen daaruit op het continuüm bepaalde intervallen; in elk van haar intervallen, en evenzo in haar geheel is de puntverzameling al of niet *dicht* (hieronder verstaan we: van het ordetype η, nadat alle welgeordende of omgekeerd welgeordende verzamelingen erin tot een enkel punt zijn samengetrokken.

[10]See section 1.1.6 for these notions.

of the set that begin, in their representation in dual expansion, with 0.1.[11] Next we check if both ways of branching to 0.10 and 0.11 also occur, that is, we check if there are elements that begin with 0.10 and 0.11. After that we check for 0.10 and 0.11 separately if both ways of branching occur for each of them, that is for the case 0.10 whether or not there are elements in the segment that begin with 0.100 and 0.101; likewise for the case 0.11 whether or not there are elements beginning with 0.110 and 0.111. If, at any stage, only one of the two is present, this one is then fully determined by the previous step.[12]

Continuing in this way, that is, checking at every stage if the splitting into two branches occurs in the approximation of the points of the set, we cut off every branch that does not split anymore in any later stage of this procedure. Note that the cutting off of branches that do not split anymore amounts to the same as 'contracting every well-ordered subset into a single point'. The result is either nothing or a branch which does not terminate in the proces of splitting. In the latter case the set is dense in itself in a certain interval.

However, the given procedure does not meet the standards of Brouwer's later intuitionism. First, the 'principle of the excluded third' is applied in the argument given above (as it was applied elsewhere in the dissertation). This principle will be rejected later as non-constructive.[13]

A second item of criticism is that a point, 'indicated on the continuum' can, as Brouwer stated, either be given as a dual fraction or can be approximated by an infinite sequence of dual fractions via the branching method. However, if it is not already known in advance which of the two applies and if there exists no algorithm to decide that, then, strictly speaking, this statement is in Brouwer's

[11] Since the segment concerned will not be empty, its elements will either begin with 0.0 or 0.1; there are no other possibilities. For elements beginning with 0.0 the same reasoning applies.

[12] We notice that Brouwer employed, at this place in the dissertation, the concept 'choice', albeit not yet in the sense of free choice:

> in determining each dual digit the preceding ones either determine it or they leave open the choice between two digits;
>
> (bij bepaling van elk volgend duaalcijfer is dat òf bepaald door het vorige òf laat de keus tussen twee).

[13] See *The unreliability of the logical principles*, [Brouwer 1908a], in English included in [Brouwer 1975], page 107 – 111. See also the next section for a short discussion about this principle.

Additionally, in a marginal note added to a phrase in the earlier mentioned lecture notes on set theory (1915/1916), Brouwer stated:

> From an intuitionistic point of view (even though I frequently applied the principle of the excluded third in my own work) which then probably did not give *correct* results, but merely *non-contradictory* results.
>
> (Van intuïtionistisch standpunt (ofschoon ik in mijn eigen werk ook dikwijls het principium tertii exclusi heb toegepast) wat dan waarschijnlijk geen *juiste*, doch alleen *niet-contradictoire* resultaten heeft gehad.)

On page 131 in the second chapter of his dissertation Brouwer merely judged it as empty, without yet rejecting it on principle. Also in Brouwer's own corrected version of the dissertation the principle was not yet rejected ([Dalen 2001], page 111).

views without meaning. How can it be decided that no splitting will occur in any later stage of a branch? This aspect of decidability is apparently taken for granted in the dissertation. It is, however, stated explicitly by Brouwer in the *Addenda and Corrigenda to the dissertation* from 1917. See below on page 130.

Also in a handwritten correction in his own copy of the dissertation we find, in an extended footnote on page 65, a remark, indicating that Brouwer realized the insurmountable problems in regard to this procedure and principled objections to it. The short and clarifying footnote in the original edition is extended to a rather long one in the corrected edition, ending as follows:

> According to my later views it is, though, very well possible that, in a well-defined branching conglomerate, the intended process of cutting off cannot be performed.[14]

The idea of *splitting* will play a major role in the later definition and construction of sets as spreads.[15] At first sight it seems obvious that in his dissertation Brouwer employed the branching method only as a way to characterize the elements of a point set in order to determine whether or not the set is dense, and not yet as a means to *define* elements of spreads (Mengen) by means of choice sequences. However, in a letter to Fraenkel, dated 12 January 1927, and to which we referred on page 74, Brouwer wrote the following:

> Dass das Cantorsche Haupttheorem für die vollstandig abbrechbaren Punktmengen 'selbstverständlich', für allgemeinen Punktmengen aber 'falsch' ist, hat nichts mit 'allmähliche Verschärfung' der Grundbegriffe zu tun, sondern nur damit, dass die intuitive Ausgangskonstruktion der Mathematik (welche, wo sie bei meinen Vorläufern vorkommt, nirgends über das abzählbare hinausgeht) von mir zuerst (1907) als vollständig abbrechbare, finite Menge, so dann als vollständig abbrechbare (nicht notwendig finite) Menge und schliesslich als Menge ohne weiteres erklärt wurde, aber immer im Stadium ihrer Einführung kurz als 'Menge' bezeichnet wurde.[16]

The expression 'intuitive Anfangskonstruktion der Mathematik' refers to the construction of spread elements.[17] So, according to what Brouwer wrote in 1927, he certainly had in 1907 the possibility of free choices and 'choice sequences' in mind, but, according to the text on pages 64 and 65 of his dissertation, at that time not yet sequences of free choices as a representation of the continuum, but only sequences to characterize the elements of a given set. Towards the end of the first chapter of his dissertation Brouwer is checking, by means of the technique of sequences, whether or not a given set is dense (by inspecting

[14] (dissertation, page 65, corrected edition [Dalen 2001], page 77): Naar mijn latere inzichten kan het overigens zeer goed zijn, dat van een welgedefinieerd vertakkingsconglomeraat het bedoelde afbrekingsproces onuitvoerbaar is.

[15] see page 133.

[16] See for English translation [Dalen, D. van 2000], page 289.

[17] See page 74.

if its derivative is perfect). The fact that sequences of free choice, unfinished on principle, and known only in as far as the choices are made, can be used to represent arbitrary real numbers and, in combination with the continuity principle, made intuitionistic analysis possible, must have occurred to him in 1916.[18]

Brouwer described, as we saw on page 117, in his first rule the construction of a finite or denumerably infinite set. This set can then be of order type ω or η or combinations thereof, but the result need not be everywhere dense; it may even be nowhere dense or only dense on one or more subintervals. The second mode of construction may, under conditions, result in a set which is everywhere dense:

> 2. in intervals, where the previous set is dense, we can transform it by the contractions described above into an everywhere dense set, and then apply to this set the operation of 'completion to a continuum'; the selected intervals are always clearly definable since, as their number is denumerable, they are individualized.[19]

Hence, after having contracted a set on a selected segment into a set which is everywhere dense, we can 'complete it to a continuum' by covering it with a continuum as a matrix for the existing points of the set, but also for points that can eventually be constructed on it afterwards. How this covering may be performed was sketched above. This matrix is then an inexhaustible source for more points, again and again. Every point which we are able to specify on the continuum, is either a finite dual fraction of the scale, or can be approximated arbitrarily closely by a sequence of rational dual fractions.

The reason why the operation 'completion to a continuum' has to be performed in this second mode is twofold:
1. By this operation the continuum, as the inexhaustible source, becomes itself one of the possible sets.
2. The third construction mode (see below) needs an underlying continuum for its definition.

The second possibility of set construction was elucidated by Brouwer in the Rome lecture *Die Möglichen Mächtigkeiten*[20] as follows:

> Man kann das mit dem discreten gleichberechtigten Continuum als Matrix von Punkten oder Einheiten betrachten, (...). Man bemerkt dann, dass das in dieser Weise definierte Continuum sich niemals als Matrix von Punkten erschöpfen lässt, und hat der Methode zum Aufbau mathematischer Systeme hinzugefügt die Möglichkeit, über eine

[18] See page 128.

[19] (page 65) 2. kunnen we in intervallen, waarbinnen de laatste puntverzameling dicht is, haar eerst door de boven beschreven samentrekkingen maken tot een overal in zich dichte verzameling, en dan daarop de operatie 'completering tot een continuüm' toepassen; de intervallen, die we daartoe uitkiezen, zijn steeds duidelijk te definiëren, want, daar hun aantal aftelbaar is, zijn ze geïndividualiseerd.

[20] [Brouwer 1908b], also in [Brouwer 1975], page 102 – 104.

> Skala vom Ordnungstypus η ein Continuum (im jetzt beschränkten Sinne[21]) hinzulegen.

The lecture *Die möglichen Mächtigkeiten* will be discussed more extensively in chapter 7.

4.3 The principle of the excluded middle

We know that at this stage the 'tertium non datur' was not yet rejected as not valid, but that in later work Brouwer constructed, with the help of simple algorithms, well-defined real numbers with remarkable properties, which demonstrate that this principle is not generally valid.

Brouwer's first formulation of the incorrectness of the 'principle of the excluded third' or 'principle of the excluded middle' (PEM) appeared in his paper *The unreliability of the logical principles*, where the term 'principium tertii exclusi' was used.[22] The incorrectness (or, at this stage, the unreliability) of PEM and of other classically accepted principles and theorems is usually demonstrated by means of *counterexamples*, and indeed in this paper the first counterexample is given. Since this paper is comprehensively discussed in our seventh chapter, *The role of logic*, the reader is referred to page 247, where also this counterexample is worked out. For now we stress that this one, as well as the next, is a *weak counterexample*, that is, we have no evidence for a given problem (like the possible existence of a certain number, or of a property of that number) as long as some other outstanding mathematical problem remains unsolved.[23] Whereas Brouwer's counterexample in the *unreliability* paper is still rather 'vague' in the sense that it is not likely to convince the 'hesitating' mathematician, the next one is, although still 'weak', more realistic and deserving the name 'counterexample' in its literal sense.

It can be found in Brouwer's lecture *Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie.*[24] The counterexample, presented in this lecture, shows that the generally accepted theorem that the points on the continuum of the reals form an ordered set, is incorrect. It proceeds as follows:

> Sei d_ν die ν-te Ziffer hinter dem Komma der Dezimalbruchentwickelung von π und $m = k_n$, wenn es sich in der fortschreitenden Dezimalbruchentwickelung von π bei d_m zum n-ten Male ereignet, daß der Teil $d_m d_{m+1} ... d_{m+9}$ dieser Dezimalbruchentwickelung eine Sequenz 0123456789 bildet. Sei weiter $c_\nu = (-\frac{1}{2})^{k_1}$, wenn $\nu \geq k_1$,

[21] that is, as just a source for always more points.

[22] [Brouwer 1908a].

[23] This attitude was quite a disturbance for a large part of the mathematical community, since it introduced a time element in a theorem: a theorem should be either true or false, independent of our knowledge of which is the case.

[24] [Brouwer 1923c]. The lecture was held in August 1923 in Belgium in the Dutch language and it was held in German in September of the same year. The German text appeared in Brouwer's collected works; see [Brouwer 1975], page 268.

> sonst $c_\nu = (-\frac{1}{2})^\nu$, dann definiert die unendliche Reihe $c_1, c_2, c_3, ...$ eine reelle Zahl r, für welche weder $r = 0$, noch $r > 0$, noch $r < 0$ gilt.[25]

We remind the reader that Brouwer considered already in the notebooks, albeit on different grounds, the impossibility to represent an arbitrary point on the continuum by an infinite decimal fraction. See the relevant quote in VI–21 on page 92, which was discussed in the section about the continuum in our third chapter.

4.4 The third construction rule

There is yet one more rule for set construction. The dissertation continues on page 66 with this third possibility:

> 3. we can construct a set of points by deleting from a continuum a dense scale, constructed on it on some interval.[26]

This is, from a constructivist's point of view, a remarkable rule for the construction of a set indeed. Compare this also with the quotation from page 37 of notebook VI (see page 98), where 'all real numbers minus the rationals' form a 'set of chances': if it is mapped on a continuum, then an arbitrary 'choice' on the continuum gives, relative to the set, a chance to hit one of its elements. In the dissertation Brouwer presented this third construction rule without any further comment or explanation. We have a continuum, which is not constructed but intuitively given, and which is not composed of points. A scale can be constructed on that continuum and the 'scale elements' can be considered as points (elements of a denumerable set). Now by taking away that scale there remains the third kind of set. But does this set then consist of points, is it composed of elements which are the result of a finite construction according to some algorithm? This rule seems to be in conflict with his principle to construct individually, one by one, the elements for a set, each according to a finite algorithm. A few remarks have to be made here:

On page 67, when discussing his solution to the continuum problem (see chapter 5), Brouwer noted the following:

> One can only speak about a continuum as a point set with respect to a scale of order type η.[27]

[25] The existence of the sequence 0123456789 in the decimal expansion of π is Brouwer's standard example. It was unknown to exist in Brouwer's days, but it was shown in 1997 that the 17,387,594,880th decimal is the beginning of this sequence. In the announcement of this result it was added that several of Brouwer's proofs now lost their validity, which is of course not true: we simply change the requirement of the sequence 0123456789 into the uninterrupted succession of two, three or n times the sequence for some fixed n.

[26] (page 66) 3. kunnen we een puntverzameling scheppen, door aan een continuüm in een zeker interval een er op geconstrueerde dichte schaal te onttrekken.

[27] over een continuüm als puntverzameling kan niet worden gesproken, dan in betrekking tot een schaal van het ordetype η.

Of course the meaning of this phrase is not that the continuum has, after all, turned into a point set; it means that we can always define more points on a continuum with an everywhere dense scale constructed on it, and add them to a given set; the new points then have to be defined with the help of infinite approximating sequences of elements from that scale, which then has to be of the order type η. In this interpretation a continuum might indeed be viewed as a 'point set relative to an everywhere dense scale'.

A set, constructed according to the third rule, is the remainder of a continuum after taking away a dense denumerable scale, and 'points' of this remainder can indeed be defined in relation to the removed scale of order type η. But one still cannot imagine this set, in the form of a 'remainder', to be a 'point set'. Moreover, the vast majority of the remainder of the continuum cannot be defined by approximating sequences in a lawlike way. Hence in order to let the result be a set in the proper sense, i.e. composed of individual elements, the acceptance of lawless approximating sequences as individual elements seems to be the only way, which is not very satisfactory either, since in that case the algorithmic character of the elements is lost.

Now, one important and possibly surprising peculiarity has to be noted here, viz. that the same idea of set-construction can be found with Cantor in the first part of his earlier discussed article *Über unendliche lineare Punktmannigfaltigkeiten.*[28] In this paper Cantor discussed possible classes of infinite sets; the first class contains the countable sets, the second class consists of those sets that can be represented by an arbitrary continuous interval. Cantor then remarked about this class:

> In diese Klasse gehören beispielsweise:
>
> 1) Jedes stetige Intervall $(\alpha...\beta)$.
>
> 2) Jede Punktmenge, die aus mehreren getrennten, stetigen Intervallen $(\alpha...\beta), (\alpha'...\beta'), (\alpha''...\beta'')...$ in endlicher oder unendlicher Anzahl besteht.
>
> 3) Jede Punktmenge, welche aus einem stetigen Intervalle dadurch hervorgeht, daß man eine *endliche* oder *abzählbar unendliche* Mannigfaltigkeit von Punkten $\omega_1, \omega_2, ...\omega_\nu, ..$ daraus entfernt.[29]

There is little doubt that Brouwer was familiar with Cantor's 1879-paper. But for Cantor the continuum consists of points, so for him it was all right. For Brouwer things were different.

A possible interpretation of this method of set construction could be the one we just alluded to: Brouwer recognized here as a set the collection of all possible Cauchy sequences that can be represented by means of the constructed dense scale on the continuum. The only way to express on the continuum arbitrary points not belonging to a dense scale, is by means of approximating sequences of

[28] [Cantor 1879a], cf. our page 6 ff.

[29] See [Cantor 1932], page 142.

points of the dense scale, where sequences of free choice then have to be admitted. After removal of the dense scale there remain the approximating sequences as elements of the third set. The removed elements are no longer elements of the set, but serve only for the composition or definition of a new element. They appear as elements in the defining sequences for the set-elements, but are no longer elements themselves. Brouwer thus indeed satisfied the mentioned condition, that one can only speak of the continuum as a point set in relation to a scale of order type η.

However, in this interpretation there still remains the possibility to express a removed rational number a by means of the Cauchy sequence $a, a, a, a, \ldots$. One could, of course, still define the rational a to be *no* element of that set, whereas the sequence $a, a, a, a, \ldots$ *is* an element, or, more likely, add the condition that a sequence should not converge to a point of the removed scale. But a more serious objection is the one we mentioned above, viz. the necessary admittance of sequences of free choice. By admitting this kind of sequence, there cannot be any longer the condition that every element should be given according to a known and fixed law in finitely many steps. In hindsight we know that choice sequences as elements for sets had to wait for at least another ten years.

Therefore the last objection is a very good candidate for the possible reason why in the *Addenda and Corrigenda on the Foundations of Mathematics*[30] under point 3 this third possibility was withdrawn: no algorithm can be given according to which a set, in compliance with rule 3, can be constructed (see page 131). It cannot be generated with the help of one mental act.[31]

Nevertheless, there still is a different but also possible explanation: Cantor's authority on the field of sets. Brouwer simply could not get around Cantor, and could not leave unnoticed the set-constructions that Cantor allowed. Brouwer was a newcomer in mathematics and he had to mature for some more years. The fact *that* his authority eventually had matured can be concluded from the extra text, in later time added to the footnote on page 65 of his dissertation and which we discussed in the previous section (see page 119); the addition to the footnote makes also this interpretation a good candidate. It shows his authority and independence in set theory in that period.

But one can only guess whether there is only one single reason for Brouwer to drop the rule, or that a combination of arguments led him to its rejection.

On a loose sheet, dated 1 November 1912, inserted on page 67 and added to the text of the 2001-republication of his dissertation,[32] Brouwer remarked the following about 'negative definitions' of sets by the exclusion of elements or of denumerable sets:

[30][Brouwer 1917a], also in [Dalen 2001], page 195 ff.

[31]In this third item of the *Addenda and Corrigenda*, Brouwer referred to the 'soon to be published work' (the *Begründing* papers), in which new methods for set construction will be explained. Probably he had noticed already that a set, defined by the removal of a dense scale, is not a spread.

[32][Dalen 2001], page 77.

> The best thing to do is, to recognize a set of points on the linear continuum as defined only then –we may do such a thing as long as the possibility of unsolvable problems exists– when we have constructed it by putting term by term in a well-ordered way, whether or not under the addition of the fundamental sequence of free number choices. Then every non-denumerable point set contains a *perfect* subset.
>
> We only recognize a definition by the *exclusion* of points as sufficient, if it can be translated into another definition in the form given above.
>
> For instance 'all points between 0 and 1, *except* those ending in an infinite number of consecutive digits 4' can be translated into '*free* choices of fundamental sequences of digits, not being 4, and between every two of those digits a free choice of an arbitrary finite number of digits 4'.[33]

In 1912 the concept of choice played already an important role (a role that we can in fact already observe in the notebooks), but only after 1918 this choice concept became of paramount importance in the form of choice sequences as elements of spreads. However, from this inserted sheet it becomes clear that as early as 1912 Brouwer reconsidered this third way of set construction, which was eventually completely dropped in the *Addenda* from 1917.

Another significant remark is made on an insert on page 87, on which Brouwer gave another argument in regard to the *negative definitions* of a set:

> '*All* points of the continuum *except* the set α' is no definition, since for that we should perceive the continuum as *finished* (in order to give a meaning to *all*); it only becomes a definition, if it is translated into a positive (i.e. in terms without *except*) denumerable *construction, possibly with the aid of a fundamental sequence of choices.*[34]

which clearly shows that Brouwer distanced himself from the third construction method.

---

[33] Het beste is, een puntverzameling op het lineaire continuüm eerst dàn als gedefinieerd te erkennen – en zo iets mogen we doen, zolang de mogelijkheid van onoplosbare problemen bestaat – als we haar hebben opgebouwd, door welgeordend punt voor punt te plaatsen, al of niet onder toevoeging der fundamentaalreeks van vrije cijferkeuzen. Elke niet aftelbare puntverzameling bevat dan een *perfecte* deelverzameling.

Definitie door *uitsluiting* van punten erkennen we dus alleen dàn als afdoend, als ze zich in een nieuwe definitie van bovenstaande vorm laat vertalen.

Bijvoorbeeld 'alle punten tussen 0 en 1, *behalve* die op ∞ veel opeenvolgende cijfers 4 eidigen' is te vertalen in '*vrije* keuzen van fundamentaalreeksen van cijfers, die *niet* 4 zijn, en tussen elke 2 dier cijfers *vrije* keuze van een willekeurig *eindig* aantal cijfers 4'.

[34] '*Alle* punten van het continuüm *behalve* de verzameling α' is geen definitie; immers daartoe zouden we het continuüm moeten *af* denken (om *alle* een zin te geven); het wordt eerst een definitie, als ze is omgezet in een positieve (d.w.z. zonder *behalve* geformuleerde) aftelbare *opbouw*, eventueel onder tehulpname ener fundamentaalreeks van keuzen.

In the following sections the fundamental developments in the set concept in the years 1914 – 1919 will be sketched, in order to make the search for their adumbrations in the notebooks and a comparison with the results in the dissertation feasible.

The possible cardinalities for sets will be discussed in chapter 7, when the third chapter of Brouwer's dissertation will be studied and commented on.

4.5 The review of Schoenflies' Bericht

In 1900 Schoenflies published the first volume of *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, Bericht über die Mengenlehre*, as a separate volume with the *Jahresbericht der Deutschen Mathematiker-Vereinigung* volume 8.[35] In 1908 it was followed by the second volume.[36]

In 1913 an updated and thoroughly corrected edition appeared.[37] In 1914 Brouwer published a review in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* volume 23,[38] in which he wrote down his objections against Schoenflies' new edition.

Brouwer characterized Schoenflies' work as almost encyclopedical, and written with 'something for everyone'. For the intuitionist there is a lot of surplus information in it.

> Um dies näher zu beleuchten, erinnere ich daran, daß für den Intuitionisten nur wohlkonstruierte unendliche Mengen existieren, welche sich zusammensetzen aus einem Teile erster Art, das sich als eine einzige Fundamentalreihe erzeugen läßt, und einem Teile zweiter Art, dem eine Fundamentalreihe f als Fréchetsche V-klasse zugrunde liegt, während seine Elemente in solcher Weise durch je eine Folge von Auswahlen unter den Elementen einer endlichen Menge oder einer Fundamentalreihe bestimmt werden, daß jeder Folge von Auswahlen eine Folge von einander einschließenden Teilgebieten von f mit gegen Null konvergierender Breite entspricht, und in den je zwei verschiedenen Folgen von Auswahlen entsprechenden Gebietsfolgen zwei außerhalb voneinander liegende Endsegmente existieren.[39]

And this immediately leads to some conclusion (without proof, as Brouwer explicitly added for the second conclusion which is now known as the Cantor-Bendixson theorem):

> 1. Die Summe einer endlichen Zahl oder einer Fundamentalreihe von elementefremden wohlkonstruierten Mengen ist wiederum eine wohlkonstruierte Menge.

[35] [Schoenflies 1900a].
[36] [Schoenflies 1908].
[37] For a historical account, see [Dalen, D. van 1999], page 229 ff.
[38] [Brouwer 1914], also in [Brouwer 1975], page 139 – 144.
[39] [Brouwer 1975], page 140.

> 2. Jede abgeschlossene wohlkonstruierte Punktmenge setzt sich aus einer perfekten und einer abzählbaren Punktmenge zusammen, d.h. das Cantorsche Haupttheorem bedarf für den Intuitionisten keines Beweises.
>
> 3. Jede nichtabzälbare wohlkonstruierte Punktmenge enthält eine perfekte Teilmenge, d.h. die 'total imperfekten' Punktmengen (vgl. S 361 – 364 des Schoenfliesschen Werkes) sind für den Intuitionisten illusorisch.

The exact definition of Fréchet's V-class is not relevant here; the main novelty is Brouwer's definition of 'Menge' as the union of a denumerable set and a perfect set, where the latter is composed of choice sequences. In fact the definition is quite general, but it is convenient to restrict our attention to the continuum.

In this review a choice sequence is defined as a set of nested neighbourhoods, converging to a final neighbourhood with width zero, such that the neighbourhoods of two choice sequences which become disjunct at a certain step, will remain so at all subsequent steps. Hence two such sequences stand for two *different* choice sequences and thus for two different real numbers on the continuum, where, as we stressed earlier, the real number is not the limiting point of the sequence, but the sequence itself in its totality.[40] This form of 'being different' of two choice sequences is made more specific in Brouwer's second *Begründung* paper as 'lying apart' (örtlich verschieden). This is in Brouwer's later intuitionism stronger than merely 'different'. Two choice sequences 'coincide' if every neighbourhood of one is partly covered by every neighbourhood of the other. In general two species or two elements of a species are 'different' if the assumption of their equality leads to a contradiction. They are 'apart' if the condition given above is satisfied, which is the condition of being 'demonstrably separated'.

There are two details in the quoted paragraph from the Schoenflies review which merit some extra attention.

Firstly, the paragraph begins with 'erinnere ich daran ...', followed by Brouwer's definition of 'Menge'. This opening is a bit surprising, since this is the first time that a definition of a set is given in which the elements partly consist of choice sequences. There is no known publication of an earlier date in which this definition appeared. In his inaugural address from 1912, Brouwer still spoke of well-defined sets in the sense of his dissertation; that is: constructed out of the ur-intuition, point by point, on the continuum.

Secondly, it is also for the first time that Brouwer called his mathematics 'intuitionistic'. In the inaugural address he mentioned Kant's intuitionism, which differs from his own 'neo-intuitionism' by giving up Kant's apriority of space, and maintaining only the apriority of time (see also the footnote on our page 49).

From now on Brouwer fully accepted choice sequences as mathematical objects, and spreads and species will take the place of the 'set of which the elements

[40]cf. [Brouwer 1919a], § 1.

have to be constructed individually'

'Spreads' and 'species' as the new concepts for sets are, at this stage of the development and in this review, implicitly there. We are now entitled to speak of an arbitrary element of the continuum of the real numbers. The continuum is no longer just a matrix for points, to be constructed onto it. A 'theory of the continuum' comes into being. This, however, does not imply the disappearance of the intuitve continuum, as we noticed earlier. We observed already that in the second *Wiener Gastvorlesung*, entitled *Die Struktur des Kontinuums*, Brouwer added a handwritten note that the continuum remains the immediate result of the ur-intuition:

> Add at the end of section I of the continuum lecture that, nevertheless, the continuum is still the *immediate result* of the ur-intuition, just as with Kant and Schopenhauer.[41]

How a choice sequence is defined will will be determined by decidable conditions. These conditions may prescribe the terms of a sequence to be fixed either completely by free choice, or, in the case of sequences as set elements, within the more or less strict constraints of some limiting prescription.

4.6 The lecture notes 1915/'16 on Set theory

The next important step in Brouwer's development is to be found in a small number of notes, written in pencil, in the margin of his lecture notes for his course 'set theory', lectured in 1915 – 1916 and in 1916 – 1917.[42] In 1915/1916 Brouwer still lectured set theory in the traditional constructivistic way. For instance, in 1915 he still proved the non-denumerability of the real numbers by means of Cantor's diagonal argument, but in 1916 he sketched in the margin of his own notes a proof with the help of the continuity argument:

> (...) that it is on the other hand impossible to map *all* the elements of f_1 on different elements of ρ,[43] follows from the fact that the choice of the element of ρ should take place at a certain place of the never terminating choice sequence, and in this way all extensions of such a *finite* branch, defining the element of ρ, have the same image in ρ.[44]

[41] Kept in the Brouwer archives. This was earlier quoted and discussed on page 74 of this dissertation:

In continuum voordracht, aan het slot van I toevoegen, dat het continuum dus toch weer uit de oerintuïtie *onmiddellijk gegeven* is, juist als bij Kant en Schopenhauer.

[42] These lecture notes, with the handwritten marginal notes on it, are kept in the Brouwer archives.

[43] f_1 is the set of real numbers, presented as infinite choice sequences (decimal or dual fractions) and ρ is the set of natural numbers.

[44] (...) dat het omgekeerd onmogelijk is *alle* elementen van f_1 af te beelden op verschillende elementen van ρ, volgt daaruit, dat de keuze van het element van ρ dan zou moeten plaatsvinden op zeker punt van de (immers nooit aflopende) keuzenreeks, en op deze manier krijgen alle verlengingen van zulk een *eindige* keuzentak, die het element van ρ bepaalt, hetzelfde beeld in ρ.

A very simple and ingeneous argument, based on the continuum of the reals as represented by the totality of all choice sequences. After all, from the purely constructivistic standpoint in the dissertation, the totality of all constructible irrational numbers remains denumerably unfinished.[45] If, however, one admits all choice sequences, and presupposes their denumerability, then the determination of the natural number which is paired to a specific choice sequence (thus mapping the specific choice sequence on a natural number by means of some mapping function) should take place after a finite number of choices for the sequence. In symbols, if the totality of all reals, represented as choice sequences, were denumerable, i.e. the totality can be mapped on $\mathbb{N}$ by some mapping function F, then the value of $F(\alpha)$ for some choice sequence α has to be determined after an initial segment $\overline{\alpha}$ of α. So $F(\alpha) = F(\overline{\alpha}) = n$, but then $F(\overline{\alpha}.\beta) = n$ for every extension β of $\overline{\alpha}$, which leads to a contradiction, hence no such F exists.

From a constructivistic point of view it is obvious that, in the case of a mapping $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, this mapping can only be performed if an initial segment of the argument in the form of an infinite choice sequence, is sufficient to determine the value of F for that argument. This is the *'Continuity principle for natural numbers'*,[46] symbolically written as:

$$\forall\alpha\exists x A(\alpha, x) \Longrightarrow \forall\alpha\exists m\exists x\forall\beta[\overline{\beta}m = \overline{\alpha}m \Rightarrow A(\beta, x)]$$

in which α and β are choice sequences, m and x natural numbers, $\overline{\alpha}m$ the initial segment of the sequence α with length m and $A(\alpha, x)$ is some predicate that defines an unambiguous relation between the choice sequence α and the natural number x.

The above argument does not cover the general case; it does not take into account the fact that choices of a higher order may play a role. Thus Brouwer's note in the margin is more a first idea than an exact argument.[47]

We also find in the margin of these lecture notes the important, and for intuitionistic set theory fundamental, concept of 'spread' (Menge, Brouwer usually employed the word *set* for spread).

> A mathematical entity is *either* an element of a previously constructed fundamental sequence F (governed by induction, as the sequence ρ), *or* a fundamental sequence f (which never terminates and is not governed by induction) of arbitrarily chosen elements from F. (One can very well operate with such a sequence, if one just has to operate with a *suitable* beginning segment of f for every entity d or function sequence r which has to be deduced), (neither r is ever terminated in general).
>
> A set now is a *law*, which deduces a d or r from an f; this r can contain as element e.g. relation symbols (e.g. those of ordering),

[45] See chapter 7 for the notion 'denumerably unfinished'.

[46] To be precise, the *weak* continuity principle, since it only states the existence of a suitable m for each α separately.

[47] For an analysis of the continuity principle, in which different types of sequences (lawlike and non-lawlike) are included in the argument, see [Atten, M. van and Dalen, D. van 2000].

> such that the law can result in well-ordered sets or other ordered sets or in a function (though one cannot obtain in this way the set of *all* ordered sets or of *all* well-ordered sets).[48]

The content of this quote reminds us of the passage we cited from the Schoenflies review. But the paragraphs in the margin of Brouwer's lecture notes do not speak of the union of the two composing parts of a set, as was done in the Schoenflies review (see page 126), but intend to say that only one of two possibilities is the case. A set is a law which deduces its elements in the form of either discrete entities d, or non-terminating choice sequences r governed by law, with as input a choice sequence f.

Here the most important result is, that infinite choice sequences which never terminate become legitimate mathematical objects. Infinite sequences, governed by some law, had of course been familiar objects since long, but these new sequences are of a completely different character. In the case of sequences of free choice one *cannot*, as in the case of an algorithmic sequence, turn away from it and let it grow while doing something else, as Brouwer expressed it in one of the notebooks (see page 98).

4.7 Addenda and corrigenda to the dissertation

In the year 1917 Brouwer published in the *Proceedings* of the KNAW[49] a list of 15 corrections to his dissertation.[50] Three of those (the numbers 3, 7 and 11) are relevant to our present subject.

3. Set construction

This item concerns the three modes of set construction.[51] In this correction the third construction mode is dropped as a result of the consequences of his intuitionistic point of view, as expressed in the Schoenflies reviews. The second mode becomes the most general one and the first can be seen as a particular case of the second.

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[48]Een wiskundig ding is òf een element uit een tevoren geconstrueerde fundamentaalreeks (door inductie beheerst, zoals de rij ρ) F, òf een fundamentaalreeks f (die nooit af is en niet door inductie beheerst wordt) van willekeurig gekozen elementen uit F. (Met zulk een reeks kan men zeer goed werken als men voor later uit af te leiden ding d of functiereeks r altijd maar *in elke fase* met een *passend* beginsegment van f heeft te werken), (r is dan i.h.a. óók nooit af).

Een verzameling is nu een *wet*, waarmee uit een f een d of een r wordt afgeleid; deze r kan dan b.v. als elementen ook relatie-symbolen (b.v. ordenende) bevatten, zodat de wet b.v. tot welgeordende verzameling of andere geordende verzameling of tot een functie kan voeren (overigens kan men zo niet komen tot de verzameling *der* geordende verzamelingen of *der* welgeordende verzameling).

[49]*Koninklijke Nederlandse Academie van Wetenschappen*, the Royal Dutch Academy of Science. When Brouwer started publishing in the *Proceedings*, this institute was still called *Koninklijke Academie van Wetenschappen*, or Royal Academy of Science.

[50][Brouwer 1917a], also in [Dalen 2001], page 195 ff. English translation in [Brouwer 1975], page 145 – 149.

[51]See pages 117, 120 and 122.

As for the first mode, Brouwer explicitly referred to the parts of the Schoenflies review that we discussed on page 126: choice sequences enable us to go beyond the matrix role of the continuum and to represent the continuum of the reals in a direct way. One might even be inclined to call this representation a '*construction* of the continuum of the real numbers, without the intuition', but we saw on page 74 that Brouwer never abandoned his ur-intuition of continuous and discrete, and, moreover, a construction presupposes an algorithm for individual elements, and therefore a *representation* (or simulation) of the continuum is the proper expression.

The introduction of choice sequences, in a process of growth, as legitimate objects of the mathematical universe, certainly ruled out the third construction mode. The result of the first mode is either a finite, or a denumerably infinite set, and the result of the second mode is either a denumerably infinite set or a continuum. Since the representing tree of the result of the first mode has either finitely many, or a denumerably infinite number of branches in which no further splitting occurs, the first mode can be considered as a special case of the second, and the second then becomes the general rule for set construction.

Two essential assumptions form the basis for the analysis in the dissertation:

> in the first place that the set can be constructed in such a way that it is individualized, i.e. so that the different infinitely proceeding branches of the tree produce different points, and further that the individualized point set can be *internally dissected*, i.e. that the process of breaking off the branches which do not ramify any more, which must terminate after a denumerable number of steps, really can be effected.[52]

In the *Addenda and Corrigenda* Brouwer presented an updated version of his modes of set construction, covering the first two modes and omitting the third one.

The second assumption was, in fact, already put forward when discussing the branching method: *how* can it be decided that no further splitting will occur? A decision procedure is required for this.

However, in this item of the *Addenda*, the role of the given use of choice sequences is still limited to that of 'describing set elements' or of 'simulation of the continuum', hence for the analysis afterwards of an already existing set, and not yet in the more creative sense for their *construction*. But, as Brouwer remarked in this same third item of the *Addenda*, there is also the possibility of a new construction principle, in which these assumptions are no longer needed. Brouwer certainly must have had in mind the concept of Menge (spread), as can be concluded from a reference at this place, to a work which was 'soon to

[52]ten eerste, dat de puntverzameling *geïndividualiseerd* kan worden geconstrueerd, d.w.z. zo, dat twee verschillende oneindig voortgezette takken van het vertakkingsagglomeraat tot twee verschillende punten voeren,

ten tweede, dat de geïndividualiseerd geconstrueerde puntverzameling *inwendig ontleed* kan worden, d.w.z. dat het afbrekingsproces der zich niet weer vertakkende takken, dat na een aftelbaar aantal schreden tot een eind moet voeren, werkelijk kan worden uitgevoerd.

be published'. That work can only have been his *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten,*[53] of which the first part appeared in 1918, so he certainly must have been working on it in 1917.

7. The principle of the excluded third

This correction refers to two items in two different places in the dissertation, both about the role of logic in mathematics. Firstly, on page 131 of his dissertation, Brouwer was still of the opinion that the principle of the excluded third is nothing but a useless tautology, harmless for the rest. In the corrections this claim is changed into the stronger version that this principle results in 'improper petitiones principii', as he argued this already in his 1908 paper *The unreliability of the logical principles*; he referred explicitly to this paper at this place in the *Addenda*.[54]

The second item is about the *comprehension axiom*, which asserts the existence of a set on the basis of a certain property of its elements alone. On page 135 of Brouwer's dissertation, the axiom is in its implicit application limited to entities that belong to a previously constructed mathematical system. Brouwer mentioned as an example the Euclidean axioms of geometry, of which the blame of incompleteness is only unjustified if Euclid saw the building of geometry as already finished; axioms then only serve the purpose of handy and concise summary of its basic properties.[55] But this is, according to Brouwer in his *Addenda*, in general not sufficient for the definition of a *set* or of a mathematical system within an existing system. Also the set or the system, defined by its properties within a completed system (hence characterized according to the Aussonderungs axiom), has to be the result of a construction.[56] In the *Addenda* Brouwer referred to page 177 of his dissertation, where this corrected view is already properly applied. On this page he criticized Poincaré, for whom 'existence' only means 'exempt of contradiction', contrary to Brouwer's dictum:

> but *to exist* in mathematics means: *to be constructed by intuition*; and the question whether a corresponding language is consistent, is not only unimportant in itself, it is also not a test for mathematical existence.[57]

This constructivistic point of departure remained of course in his later intuitionism in the definition of subspecies of species, even if a subspecies is by

[53][Brouwer 1918] and [Brouwer 1919a], called for short: the *Begründung*.

[54]For a more detailed discussion of this paper, cf. page 247 of our dissertation.

[55]See also chapter 7, page 253.

[56]Also the collection of elements that have a certain property among earlier constructed elements involves a constructive act. Compare this with page 229, where the concept of a 'building within a building' is discussed.

[57]diss. page 177: maar *bestaan* in wiskunde betekent: *intuïtief zijn opgebouwd*, en of een begeleidende taal vrij van contradictie is, is niet alleen op zichzelf zonder belang, maar ook geen criterium voor het wiskundig bestaan.

definition a property.

11. The non-denumerable point set

This item is about the well-ordering theorem, already conjectured by Cantor and proved by Zermelo.[58] Brouwer referred in this item to page 152 and 153 of his dissertation, where he noticed that the theorem is, according to Borel, equivalent to the axiom of choice; either one can be taken as axiom and the other one subsequently proved. In his dissertation the solution was simple for him: for denumerable sets the theorem is trivial and for the only other infinite cardinality, the continuum, the theorem does not apply because 1) the vast majority of its elements is unknown, and 2) well-ordering includes denumerability.

If we admit non-denumerable sets of points, defined by an infinite tree representing the continuum, then the impossibility of well-ordering the perfect spread can be proved. An earlier proof was given in the Schoenflies review, and Brouwer made use of the opportunity to correct his earlier proof.

4.8 The ‘Begründung’ papers, 1918/19

The breakthrough of Brouwer’s intiuitionistic mathematics came in the year 1918 with the publication of the first part of a revolutionary paper, bearing the long, but fully explanatory title *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil, Allgemeine Mengenlehre*;[59] the second part, *Begründung ... Zweiter Teil, Theorie der Punktmengen*[60] appeared in 1919. Both are usually referred to, for short, as the *Begründung* papers.

The term ‘intuitionistic’ was, except in the title, not used in these papers. We met it in its modern meaning in the Schoenflies review, and it is used again in 1919 in Brouwer’s short paper *Intuitionistische Mengenlehre*,[61] published in the *Jahresbericht der Deutschen Mathematiker Vereinigung*.[62] From now on the terms ‘intuitionistic’ and ‘intuitionism’ are used to designate Brouwer’s fundamentally new approach to mathematics and its construction.[63]

We already discussed some of the new notions and tools from the *Begründung* papers on page 71 when introducing the ‘full continuum’ of the real numbers, and a part of the following quotes was also given in that section, but will be repeated here for completeness’ sake.

The *Begründung* paper begins with the following definition:

[58]cf. page 17 and page 27 respectively.

[59][Brouwer 1918], see also [Brouwer 1975], page 151–190.

[60][Brouwer 1919a] or [Brouwer 1975], page 191–221.

[61]See page 135.

[62]This latter paper, although it appeared in print in 1920, can very well be read as an introduction to the *Begründung* papers.

[63][Brouwer 1919b]. Brouwer used terms like ‘neo-intuitionism’, ‘old intuitionism’, ‘semi intuitionism’; for some clarification of this terminology we refer again to the footnote on page 49 of this dissertation.

> Der Mengenlehre liegt eine unbegrentzte Folge von Zeichen zu Grunde, welche bestimmt wird durch ein erstes Zeichen und das Gesetz, das aus jedem dieser Zeichen das nächstfolgende herleitet. Unter den mannigfachen hierzu brauchbaren Gesetzen erscheint dasjenige am geeignetesten, welches die Folge ζ der Ziffernkomplexe 1, 2, 3, 4, 5, ... erzeugt.

and it continues with the definition of *spread:*

> Eine *Menge* ist ein *Gesetz*, auf Grund dessen, wenn immer wieder ein willkürlicher Ziffernkomplex der Folge ζ gewählt wird, jede dieser Wahlen entweder ein bestimmtes Zeichen, oder nichts erzeugt, oder aber die Hemmung des Prozesses und die definitive Vernichtung seines Resultates herbeiführt, wobei für jedes n nach jeder ungehemmte Folge von $n-1$ Wahlen wenigstens ein Ziffernkomplex angegeben werden kann, der, wenn er als n-ter Ziffernkomplex gewählt wird, *nicht* die Hemmung des Processes herbeiführt. Jede in dieser Weise von der Menge erzeugte Zeichenfolge (welche also im allgemeinen nicht fertig darstellbar ist) heisst ein *Element der Menge.* Die gemeinsame Entstehungsart der Elemente einer Menge M werden wir ebenfalls kurz als *die Menge M* bezeichnen.

Thereupon the concepts *mathematische Entität* and *Species* are defined:

> Mengen und Elemente von Mengen werden *mathematische Entitäten* genannt.
>
> Unter einer *Species erster Ordnung* verstehen wir eine Eigenschaft, welche nur eine mathematische Entität besitzen kann, in welchem Falle sie ein *Element der Species erster Ordnung* genannt wird. Die Mengen bilden besondere Fälle von Species erster Ordnung.
>
> Unter ein *Species zweiter Ordnung* verstehen wir eine Eigenschaft, welche nur eine mathematische Entität oder Species erster Ordnung besitzen kann, in welchem Falle sie ein *Element der Species zweiter Ordnung* genannt wird.

Hence choice sequences, according to the first quotation, consist of infinite sequences of elements, which are constructed from a collection of objects, given in advance. Choice sequences, in turn, are the elements of a spread and a spread is a particular case of the more general concept of a species (of the first order).

The simplest example of a spread is one, in which the basic collection is the set of the natural numbers, and where the law is such that at any stage of its development, every sequence is composed of the natural numbers in the order in which they were chosen from the set ζ 'on their way to the node'.

If every finitely terminating branch is removed, the elements of the spread consist of infinite choice sequences of 'signs' (for instance of the natural numbers or of intervals on the continuum). These choice sequences can be, and sometimes are, constructed according to a specific law, but also sequences of free choices are possible, e.g. in a spread as a representation of (a segment of) the continuum.

In the next quotation C is the 'universal spread',[64] which represents the set of the reals on the open interval $(0,1)$ and A is the set ζ of the natural numbers. In this fragment we immediately recognize the marginal notes from the lectures on set theory:

> *Die Menge C ist grösser als die Menge A.* Ein Gesetz, das jedem Elemente g von C ein Element h von A zuordnet, muss nämlich das Element h vollständig bestimmt haben nach dem Bekanntwerden eines gewissen Anfangssegmentes α der Folge von Ziffernkomplexen von g. Dann aber wird jedem Elemente von C, welches α als Anfangssegment besitzt, dasselbe Element h von A zugeordnet. Es ist mithin unmöglich, jedem Elemente von C ein verschiedenes Element von A zuzuordnen. Weil man andererseits in mannigfacher Weise jedem Elemente von A ein verschiedenes Element von C zuordnen kann, so ist hiermit der aufgestellte Satz bewiesen.[65]

Brouwer stated as a theorem that the set C is non-denumerable, and subsequently proved it, this time not with the help of Cantor's diagonal method, but in his own (intuitionistic) way; however, hidden in this proof is Brouwer's 'continuity principle', here for the first time appearing in print. This principle was discussed on page 128, and thanks to it we can calculate values of functions with as input arbitrary real numbers represented by infinite choice sequences.[66] This principle is a corollary of Brouwer's constructivism, in which every construction has to completed in a finite time by means of a finite number of acts. As a consequence of this, $F(\alpha)$, with α an infinite choice sequence and $F(\alpha)$ a natural number, only has a computable value if this value can be determined in a finite construction, that is, after a finite initial segment of α.[67]

Another corollary is that, if we can speak of an 'arbitrary α', for which a certain property A holds, we can also say that it holds 'for all α' that is, $\forall\alpha A(\alpha)$, but the sign $\forall$ then has to be understood as 'for *every* α you give me, I can prove that $A(\alpha)$ holds', rather than 'for *all* α'.

4.9 Intuitionistische Mengenlehre, 1919

In 1919 Brouwer published his *Intuitionistische Mengenlehre*, in which he again employed the term 'intuitionistic' in its new mathematical meaning, after its initial appearance in 1914 in the Schoenflies review. The *Intuitionistische Mengenlehre* paper can be seen as a summary of the *Begründung* papers, as well as a

[64] A term introduced by A. Heyting, see e.g. [Heyting 1981], page 128.

[65] [Brouwer 1918], page 13.

[66] For a discussion of the concept of choice sequence in the notebooks, see also pages 102 and the last section of this chapter.

[67] We stress that this argument is only a crude approximation to an explanation of the principle in which different types of choice sequences are included; see [Atten, M. van and Dalen, D. van 2000].

less technical-mathematical elucidation of it. It can also perfectly well be read as an introduction to the *Begründung* papers.

It opens with the presentation of two statements, which are, in a more rudimentary form, already present in the dissertation:[68]

I. The comprehension axiom, which defines a set on the basis of certain mathematical objects having certain properties, is unsuitable, since it may lead to contradictions. Only a *constructive* definition of a set can be the basis for a set theory, and this definition is the spread law.

II. The axiom, as formulated by Hilbert, that every mathematical problem has its solution or that its non-solvability can be proved, is equivalent to the logical law of the excluded third, and there is neither proof nor evidence for this rule.

These two theses form the basis for the intuitionistic concept of mathematics, but Brouwer immediately admitted that he occasionally applied the non-constructive PEM himself:

> Von der in diesen beiden Thesen kondensierten *intuitionistischen* Auffassung der Mathematik habe ich übrigens in den in Anm. 2)[69] zitierten Schriften bloss fragmentarische Konsequentzen gezogen, habe auch in meinen gleichzeitigen philosophiefreien mathematischen Arbeiten regelmässig die alten Methoden gebraucht, wobei ich allerdings bestrebt war, nur solche Resultate herzuleiten, von denen ich hoffen konnte, dass sie nach Ausführung eines systematischen Aufbaues der intuitionistischen Mengenlehre, im neuen Lehrgebäude, eventuell in modifizierter Form, einen Platz finden und einen Wert behaupten würden.
>
> Mit einem solchen systematischen Aufbau der intuitionistischen Mengenlehre habe ich erst in der eingangs erwähnten Abhandlung[70] einen Anfang gemacht. Hier möchte ich kurz hinweisen auf einige der am tiefsten einschneidenden, nicht nur formalen, sondern auch inhaltlichen Aenderungen, welche die klassische Mengenlehre dabei erfahren hat.[71]

Hence in his pure mathematics Brouwer did use the principle, but only in those cases where he expected the same positive result when using a more complicated argument without the PEM, and, in hindsight, when applying his intuitionistic set theory. In his dissertation Brouwer judged the principle of the excluded middle as a useless principle, but harmless for the rest. Apparently he was aware of the fact that his proofs, although not wrong on principle, lacked absolute confidence and strength, but that he relied on the possibility of a PEM-free stronger proof.

[68] [Brouwer 1919b]; see also [Brouwer 1975], page 230.
[69] Brouwer is referring, among other publications, to his dissertation and to [Brouwer 1908a].
[70] the *Begründung* papers.
[71] [Brouwer 1919b], page 2.

Next in this paper follow the already quoted definitions of spread and species.

As we stated already on page 126, the reason for this relatively long elaboration on future notions is to make the discovery of their early traces in the notebooks possible and useful.

4.10 The notebooks on set formation

Again, the nine notebooks contain numerous remarks about sets, their construction, limitations on their magnitude etc. The subject of 'sets' is often interwoven with that of the continuum to the extent that a sharp distinction between the two is not always possible. Therefore a repetition of quotes from our chapter 3 will occasionally occur.

We must also keep in mind the fact that the notebooks were written within a time span of only two years; in view of the different notions, we can, on the one hand, clearly observe a development in the direction of Brouwer's position in his dissertation; on the other hand we can already discern traces of his later ideas, which were sketched in the previous sections of this chapter, and which only turned up in his published work after the year 1914. From several of the following quotations we can conclude that the latter is the case for the notion of choice in the construction of elements for sets, and for his attempts to make the continuum manageable with the help of infinite sequences of free choices.

4.10.1 Sets, general

During the first few years after taking his doctoral degree, Brouwer stuck to his principle of constructibility in the definition of sets. A set is never given to us in its entirety, but it is always defined by an algorithm for the construction of its individual elements, and therefore there are strong limitations on the resulting cardinality; this in contrast to Cantor for whom no such limitations existed:

> (III–16) One cannot speak about an already existing cardinality, having certain properties; one can construct it and then e.g. conclude afterwards that it is equivalent to some other one.[72]

The phrase 'one can construct it' is limiting the cardinality of the result since a construction, which is carried out stepwise by the subject, can only take finitely many steps, or run parallel to the generation of the ordinal number ω. Therefore any ordered construction yields either a finite set, or one of cardinality $\aleph_0$, i.e. the second number class cannot be viewed as an intensionally completed set.

Brouwer held on to this view during the years before the first worldwar. Although he added the 'denumerably infinite unfinished' and the continuum to

[72] Men kan niet spreken over een machtigheid, die er al is; en dan zekere eigenschappen heeft; men kan haar opbouwen, en dan achteraf b.v. zeggen dat zij gelijkmachtig is met een zekere andere.

his list of possible cardinalities, the strict requirement for an algorithm for the construction of individual elements initially remained.

About Cantor's transfinite numbers, Brouwer remarked in a (later on deleted) paragraph:

> (II–30 and 31) Everything about transfinite numbers I must be able to see intuitively (directly or with the help of simple induction). It is meanigless to speak of other, *non-intuitable* things.
>
> (...)
>
> The only new aspect in Cantor's transfinite numbers is the construction of geometry from the theory of numbers (i.e. units and simple induction).[73]

It will be obvious that Brouwer did not restrict his constructivism to sets alone, but that every 'mathematical building' requires a proper construction, hence also Euclidean and non-Euclidean geometry and arithmetic. One of the results of his constructivism is of course a high degree of transparancy in the resulting building.

We can detect Brouwer's constructivistic attitude in most of his notes. For instance in the third notebook, the cause of the paradoxes in set theory is attributed to a lack of establishing individually the set elements; one can avoid impredicative definitions of sets if, in the construction of its elements, one only makes use of objects and concepts which were constructed earlier, and subsequently collect them in a constructive way. And by avoiding impredicative definitions one precludes the paradoxes into which set theory ran around the year 1900 by the work of Richard, Berry and Russell:

> (III–17) Be careful with the definitions of sets; they might not exist just like Russell's contradictory 'class of classes not belonging to their elements'.[74]

This is one of the earliest statements about sets in the notebooks and we will attempt to demonstrate, via a series of quotations, that there is a development in Brouwer's concept of sets to the ideas as they were laid down in his dissertation, and even beyond: there are clues and signs of later concepts of spreads and species. But from the very beginning he required a proper construction for all mathematical objects, the only question being *which* constructions were admissible.

[73] Alles van de transfiniete getallen moet ik kunnen zien aanschouwelijk (direct of met behulp der enkelvoudige inductie). Van andere dingen te spreken die ik niet *kan* aanschouwen ware zinloos. (...)

Het enige nieuwe van Cantors transfiniete getallen, is het het opbouwen van de meetkunde uit theorie van getallen (d.i. eenheden en enkelvoudige inductie).

[74] Wees met de definities van 'Menge' voorzichtig; ze zijn misschien zo min bestaanbaar als de contradictoire 'class of classes not belonging to their elements' van Russell.

IV–18 reveals that Brouwer's terminology was not always consistent with its later specific and strict meaning, when he was jotting down the several notes:

> A 'Menge', which I can enumerate cannot be 'ähnlich' to a part of itself. This gives the fundamental property of arithmetic.[75]

For this quotation to be true, 'enumerable' must have the meaning of 'finite', whereas nowadays it has normally the meaning of finite or denumerably infinite.

In IV–26 the tone becomes more gloomy, when Brouwer called set theory the 'centralizing science', which starts from counting and which classifies empirical geometry in set theory, as a hypothesis for the totality of the physical phenomena. The original and forgivable 'fall'[76] of counting changed into the consciously 'continued sin' of doing mathematics. Things go from bad to worse in this paragraph IV–26: the empirical geometry, which is closest to the intuition, is abstracted into set theory, and this abstraction is the beginning of mankind alienating from itself. This negative and pessimistic attitude, especially in relation to the application of mathematics, remained through all the notebooks (although less frequent and less pronounced towards the end); we also meet this attitude in the synopsis of the notebooks and apparently they were also present in the draft of his dissertation. From this draft they mainly found their way, as a result of Korteweg's veto, into the preserved *Rejected parts.* But a certain pessimistic overtone never completely disappeared from his writings, not even from his later ones.

4.10.2 Sets, constructibility as condition for their existence

In the notebooks Brouwer paid much attention to the actual construction of a set. Also with respect to the geometrical set of lines and planes:

> (V–13) Planes are not yet given in space; they are *built* in it, like on earth houses are built from the elements of it. Hence in practice the sinful geometry only applies to the sinful constructions of mankind.[77]

As we have seen, the move of time is the only ur-phenomenon for Brouwer, resulting in the two-ity of continuous and discrete, which, in its turn, makes the awareness and counting of numbers possible, from which arithmetic, analysis and also geometry can be built.

In the last given quote we recognize again his pessimistic outlook on the world, where the sinful act of arranging the surrounding nature into physical

[75] Een 'Menge', die ik kan aftellen, *kan* niet aan een deel van zichzelf 'ähnlich' zijn. Hieruit volgt de grondeigenschap der rekenkunde.

[76] Brouwer used here the word 'tumble', Dutch: tuimeling.

[77] In de ruimte zijn nog niet de vlakken gegeven; die worden er in *gebouwd*, zoals op de aarde uit de elementen er van de huizen worden gebouwd. En zo geldt de zondige meetkunde ook in de praktijk slechts voor de zondige bouwwerken der mensen.

laws led to 'externalization' and to the downfall of mankind;[78] in this view already the mere act of doing geometry belongs to the sinful activity of 'arranging the external world'. In his early work, 'sin' has to be understood as 'every activity which will lead man away from his ur-state of just being there'.[79]

In a marginal note, added to the above quoted fragment, the act of construction by the *individual mathematician* is emphasized. This act of individually doing mathematics in the flow of time will, in Brouwer's later years after the second world war, ultimately result in the concept of the *creating subject*:

> (V–13) The word '*every* element' of an infinite manifold does not make sense, if I have not built that manifold myself (they do not exist in nature) and how else could I do that, but by induction. And the latter is impossible without indiscernibility in the manifold, and *that* includes the fundamental theorem of arithmetic.
>
> And just because of that indiscernibility I can conceive the machine, that *'continuously'* (intuitive idea) adds points, one by one (which is impossible in case of discernibility).[80]

Hence the construction of an infinite quantity is permitted by means of the algorithm of iterating the same act of adding one element (which makes the successive acts mutually indiscernible), together with the confidence that that, which is constructed, remains. The most basic method for this is the simple act of counting (V–14). There is a beginning, a growing and a limit that gives us the cardinal number as 'reflection of desire for possession'.[81] In this fifth notebook we recognize the tone that, under the influence of Korteweg, was removed from Brouwer's concept of the dissertation.

In the last quotation we also notice the emphasis on the individual mathematician, who personally has to construct his objects and sets. This is already the concept of mathematics 'as the free creation of the individual human mind', as presented in the dissertation, and again underlined in its concluding summary at the end.

In V–14,15 Brouwer specified explicitly that, *first*, we construct a system of numbers, and only *after* that we build the relations between the numbers in that system. He elucidated this statement in a long fragment, filled with notions like *sin* and *desire*. Only after performing the construction in the prescribed order, we observe that this system has the property of indiscernibility, resulting in the applicability of the main theorem of arithmetic:

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[78] See [Brouwer 1905].

[79] See *Life, Art and Mysticism,* chapter I, *The sad world.*

[80] En het woordje '*elk* element' van een oneindige hoeveelheid heeft geen zin, als ik die hoeveelheid niet zelf heb opgebouwd (in de natuur zijn ze er niet), en hoe kan ik dat anders doen dan door inductie? En dat kan ik niet doen zonder ononderscheidenheid in de veelheid, en *die* sluit in de hoofdstelling der rekenkunde.

Ook kan ik juist uit hoofde van die ononderscheidenheid voorstellen de machine, die '*altijd maar door*' (intuïtieve voorstelling) punten een voor een bijlegt (bij onderscheidenheid gaat dat niet).

[81] afspiegeling van begeerte naar bezit.

> (V–15) But once we pose the question: is induction possible and are the units to be seen as equal (in other words, can a cardinal number be formulated), then a negative answer would collapse not only its own question, but would at the same time collapse the 'form' as useless for the desire..
>
> *Definition* A finite quantity is one, which is accurately constructed by me, without induction.
>
> (...)
>
> How could I formulate a syllogism or a theorem about something, which I cannot intuit? If I formulate it about something which is defined, then it applies only for the illustrative *examples* of the defined entities.[82]

We can see in the last part of this quotation, as well as in the next one, his argument that one can do logic only *after* the construction of a mathematical system; only *then* one has material to perform logic on; logic applies to classes of constructed objects, not to classes defined by comprehension.

A basic idea of his 1908-paper *The unreliability of the logical principles*, in which also the priority of a mathematical construction is underlined, occurs verbatim in notebook VI:

> (VI–35) Once more: it is not true that I can consider mathematics (e.g. of the transfinite numbers) to be derived from given logical relations, since logical relations only make sense if they are applied to a mathematically constructed system. Hence sometimes a mathematical system runs parallel to another mathematical system, viz. if a logical substratum of a mathematical system can be constructed independently of it as a mathematical system of its own (e.g. Hilbert in Ens. Math.), but otherwise the system of departure is often necessary as Existenzbeweis of the logical substratum, which is not itself a mathematical system.[83]

In V–16 Brouwer stated that there are, roughly, three areas of representation of the concepts of our mind:

[82]Maar stellen we eenmaal de vraag: is inductie mogelijk en zijn de eenheden als gelijk te zien (m.a.w. is er een cardinaalgetal te zeggen), dan zou een ontkennend antwoord behalve zijn eigen vraag tegelijk doen instorten de 'vorm' als onbruikbaar voor de begeerte.

Definitie Een eindige hoeveelheid is een exact door mij opgebouwd, zonder inductie.

(...)

Hoe kan ik een syllogisme of stelling opstellen omtrent iets, dat ik me niet kan voorstellen? Stel ik zo'n ding op over iets gedefinieerds, dan geldt het eigenlijk alleen over de aanschouwelijke *voorbeelden* van het gedefinieerde.

[83]Nog eens: het is nièt waar, dat ik de wiskunde (b.v. der transfiniete getallen) kan beschouwen afgeleid uit gegeven logische relaties, omdat logische relaties pas zin krijgen, als ze zijn toegepast op een wiskundig opgebouwd systeem. Soms loopt dus een wiskundig systeem, als het logisch substraat onafhankelijk er van zèlf kan worden opgebouwd als een wiskundig systeem, parallel met een ander wiskundig systeem (voorbeeld hiervan is Hilbert Ens. Math.), maar anders is het systeem van uitgang ook vaak nodig als Existenzbeweis van het logisch substraat, dat zelf geen wiskundig systeem is.

> There are three areas of representation ((...) one should not interpret them too strict (...))
>
> 1. From the world of observation as the antipoles of our sins (represented objects not exact, words not exact)
>
> 2. From mathematics: the medium of the 'Beharrung' of those representations (represented objects *exact,* since they are coming from me, words not exact (...))
>
> 3. From logic: (represented objects *and* words exact (...).[84]

The three areas show an increasing abstraction, in which objects and words become increasingly exact, that is 'lending equilibrium in the mind';[85] hence the order in which this happens, is essential: logic can never be in the first place.

One more observation about the notebooks in general: the pages V–13 through V–16 are the first ones where Brouwer devoted several pages in succession (four, in this case) to one single subject: the foundational construction of mathematics. From here on this will happen more and more often, whereas at the same time the negative and pessimistic remarks decrease in number.

4.10.3 Sets and the Russell-paradox

This paradox is discussed in VI–26 through –33. According to Brouwer the paradox is caused by the confusion between the concepts '*if* something is the case' and 'the class of all objects, *for which* this is the case'. The Russell paradox, as far as it appears in the notebooks, will be discussed on page 293.

4.10.4 Sets, limitations resulting from the method of construction

Possible sets, possible elements of those sets, or possible mathematical objects in general are restricted by the requirement of an algorithmic instruction according to which they are given. 'Arbitrary' objects only exist within a pre-given domain, as can be concluded from Brouwer's comment on a short quote by Cantor, referring to the comprehension axiom:

> (VII–23) (Cantor) 'Von jedem beliebigen Object muss man angeben können, ob es seiner Definition zufolge der Menge angehört oder nicht'. Nonsense ([*in the margin*]: Russell's basic mistake originates from this idea). Mathematics does not know 'beliebige Objecte', it

[84] Er zijn drie gebieden van voorstelling ((...) men vatte ze niet te strict op (...))
1. Uit de aanschouwingswereld als tegenpolen onzer zonden (voorgestelde dingen niet exact, woorden niet exact).
2. Uit de wiskunde: het medium der Beharrung dier voorstellingen (voorgestelde dingen exact, want uit mijzelf, woorden niet exact (...).
3. Uit de logica: (voorgestelde dingen *en* woorden exact).

[85] evenwicht gevend in het hoofd.

> only knows self-constructed objects; and the *definition* may only be a limitation on the construction, through which the *intrinsic* construction should again become possible from combinations of 1, ω and c. It is just that the construction is limited by the definition.[86]

Hence, a definition may only *limit* the construction of a set from an available stock of elements or from building blocks for those elements, but it may not give a type of 'elements' for which no clear algorithm can exist.

Nevertheless, Brouwer attempted to define 'in practice unmeasurable numbers' on the continuum, but despite this effort to 'understand' the continuum, he stated (VIII–16) that we can only create in our intellect denumerable quantities, 'according to our life time', which may suggest finitism, and emphasizes the notion of time.

Brouwer again stressed that we can neither construct, nor conceive Cantor's second cardinality T (that is the cardinality $\aleph_1$) as a completed totality, but the notion 'denumerably unfinished' is not employed here:

> (VIII–16) For everything, that we can create *mathematically,* is denumerable; if we want to create T, we observe that our creation is never finished by giving isolated *acts;* and *laws,* which are denumerable sequences of facts; but for that reason we may not postulate that there are more things *apart from* what we can create.[87]

T cannot be created as a finished entity by means of 'isolated acts and laws', that is to say, by means of clearly stated algorithms. Hence there are only two 'modes of existence' for point sets:

> (VIII–16) For pointsets in c, there are only two modes of existence:
>
> 1^{st} The mathematical free creation (1^{st} cardinality).
>
> 2^{nd} The indefinitely continuing possibility of physical approximation (2^{nd} cardinality).
>
> Hence there *exist* only 2 cardinalities for pointsets.[88]

The 'second cardinality' in this quote reminds us of any process of actually executed approximation (because of the term 'physical'), which is on prinicple never completed, e.g. the construction of Poincaré's physical continuum.

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[86](Cantor) 'Von jedem beliebigen Object muss man angeben können, ob es seiner Definition zufolge der Menge angehört oder nicht'. Larie (*met in de kantlijn:* in zo'n gedachte zit ook de grondfout van Russell). De wiskunde kent geen beliebige Objecte, dan de zelf opgebouwde; en de *definitie* mag alleen zijn een bouw-beperking, waarna de *intrinsieke* opbouw weer mogelijk moet worden uit combinaties van 1, ω en c. Alleen is door de definitie de bouw beperkt.

[87]Want alles, wat wij *wiskundig* kunnen scheppen, is aftelbaar; willen we T gaan scheppen, dan merken we, dat ons scheppen nooit klaar komt met het geven van geïsoleerde *daden;* en *wetten,* dat zijn aftelbaar oneindige feitenreeksen; maar daarom mogen we niet postuleren, dat er nog dingen zijn *buiten* hetgeen wij scheppen kunnen.

[88]Voor Punktmengen in c zijn er maar twee manieren van bestaan:

1^e De wiskundige vrije schepping (1^{ste} machtigheid).

2^e De onbepaald voortlopende fysische benaderingsmogelijkheid (2^{de} machtigheid).

Er *bestaan* dus maar 2 machtigheden van *(voor)* Punktmengen.

Because Brouwer speaks here of 'cardinality', he is most likely not referring to the never terminated process of the composition of a single choice sequence, but, instead, to the algorithmic construction of a denumerably infinite set, which may be seen as finished only potentially. He may even allude to the 'denumerably unfinished' cardinality.

Brouwer mentioned the term 'unfinished cardinality' for the first time explicitly in VII–23, as a comment on Schoenflies' *Bericht über die Mengenlehre*:

> (VII–23) [Bericht page 13, theorem IV] All definable real numbers are denumerably unfinished.[89]

But neither this concept, nor the continuum has the status of a separate cardinality yet. The number of possible cardinalities for sets is still limited to two, finite and denumerable, instead of the four in the dissertation.

In the eighth notebook, pages 14 up to 45, Brouwer was constantly and intensely searching for the limitations and bounds of set theory. He frequently discussed the work of other mathematicians (Bernstein, Klein, Cantor) and commented on it. We recognize this process of seeking e.g. on page VIII–17, where Brouwer explained that we can construct the everywhere dense rational scale R which is denumerable; we can add to this the *known* limit points, the limits of known algorithmic sequences, like the sequences for π or $\sqrt{2}$, and 'it remains the same set', that is, it remains denumerable; repeat this ω times and it still remains the same set, hence, as Brouwer concluded in VIII–17, the 'perfect set[90] cannot be constructed and therefore does not exist.[91] It only exists in the 'physics of the intuition' and we can *postulate* it, that is, we can express the words and postulate the concept of the perfect set, but that is *all* we can actually do.

Clearly, there is for Brouwer only the general idea of the perfect set, but then we cannot speak of its cardinality since there is no way of constructing it.

4.10.5 The perfect set cannot be constructed

In IV–12 the possibility of constructing the perfect set is, again, considered. This fragment is not crossed out, but, judging by the handwriting, a remark is added to it later, and these two phrases together give a good impression of Brouwer's old view, compared to his new: the 'construction' of the perfect set is replaced by the intuition of the continuum:

> (IV–12) The only way to 'construct' the perfect set (which is required) must be according to Cantor Mathematische Annalen 46,

[89] Bericht page 13 st. IV Alle aangeefbare reële getallen zijn aftelbaar onaf.

[90] This is the set that coincides with its derivative, as defined by Cantor (it is not the Cantor set as we sketched this on page 95) For Cantor's general defnition of a perfect set, see page 12.

[91] See for the relevant quote page 152.

> page 488.[92]
>
> [*with the later addition:*]
>
> If we do not want to appeal aprioristically to continuity, which is the purest thing to do.[93]

Immediately after that, on the same page IV–12, Brouwer realized that, among the numbers of a perfect set, some are special: the ones that can be given by an algorithm; but how can we be sure that the continuum is exhausted by the Cantorian points (the real numbers)? In IV–13 he stated that we cannot define every fundamental sequence, that is, not every number, given in the Cantorian way, can be named by means of a known (algorithmic) sequence. And that applies to the vast majority of the elements of a perfect set, which is the reason that in IV–17 Brouwer wrote:

> (IV–17) I cannot speak of *all* points of a straight line in a collective sense, and give properties for them; I only can constantly construct points on the continuum, but then I generate them.[94]

Brouwer realized that the majority of elements of a perfect set escapes our ability to construct them according to some rule or algorithm. This conclusion, together with the constant occupation with the subject 'continuum', *must,* one would say, sooner or later lead to the concept of choice sequence, which eventually was Brouwer's escape from the realm of the lawlikeness.

4.10.6 The second number class

Contrary to the continuum, Cantor's set of the second cardinality (the second number class) cannot be postulated intuitively:

> (VIII–17) And now T. While constructing T we notice, that we never can finish it, not even after ω operations.
>
> Hence we have to conclude that this 'finishing' does not exist, in other words that T *does not exist.* Since there is no intuitive foundation to postulate its 'Fertigkeit', like there is in the case of c.[95]

---

[92] On page 488 of M.A. 46 Cantor's general construction of the continuum (the perfect set) is presented; see page 14 ff. Again, this is not the Cantor set from page 95.

[93] De enige manier, om de perfecte Menge 'op te bouwen' (wat toch vereist wordt), zal wel zijn volgens Cantor M. Ann. 48 pg 488.

[*met de latere toevoeging:*]

Als we niet aprioristisch aan de continuïteit willen appelleren, wat het zuiverst is.

[94] Ik kan niet samenvattend spreken over *alle* punten van een rechte lijn, en daarover dingen, eigenschappen zeggen; ik kan alleen voortdurend punten vormen op een continuüm, maar dan genereer ik ze.

[95] En nu T. Bij het opbouwen van T merken we, dat we nooit klaarkomen, ook niet na ω operaties.

We moeten dus rekenen, dat het klaar komen, m.a.w. de Menge T *niet bestaat.* Want een intuïtieve grond, om de Fertigkeit er van te postuleren, zoals bij c, is er niet.

In Brouwer's view the continuum is intuitively given and may, for that reason, be postulated as a finished ('fertig') object; also ω may, from a certain viewpoint according to Brouwer, be postulated as being finished, namely as a closure of the simple algorithm of the successor operation.[96]

But, as he emphasized above, both these arguments for being finished do not apply for the second number class T: there is no closure for T, neither definable, nor intuitively given. Denumerably many elements can be added in denumerably many ways, still resulting in a denumerable set, according to a theorem by Cantor.

We have the intuition of the continuum and the knowledge that one can only speak about the continuum with the help of a constructed scale on it which is dense and of order type η. That scale, in Brouwer's words, 'expresses the whole essence of the continuum', that is to say that every *definable* subset of the continuum must be expressible in that scale. This was elaborated a few years later, in 1908, in his paper *Die möglichen Mächtigkeiten*.[97] Hence the fundamental difference between the two concepts 'continuum' and 'second number class' can be construed from the different ways in which the two are described and in which there properties are explained. The continuum is not constructed, but intuitively given instead, and the second number class will be one of the typical examples of a 'denumerably infinite unfinished' set, which stands for a continued process and not for a (potentially) completed entity.

4.10.7 A third and a fourth cardinality

The possible cardinalities are again investigated in VIII–25, and the number now increases to three:

> (VIII–25) In any case a certainly existing set is: $C^{\aleph_0}$, in which at every next decimal place, instead of a digit, appears an arbitrary point of the continuum. But its cardinality is c.
>
> On the other hand, the cardinality $F = C^c$, that of all functions, does not exist.
>
> If I want to search for all 'sets of limit points' which can be constructed from ω, then I have to construct all possible infinite groups from it, or, for this all possible groups; and this happens by approximationin the dual system, resulting in the kinds of groups with cardinality E (finite), A ($\aleph_0$) and C.[98]

[96] See for a discussion on a 'finished' or an 'actual' infinity chapter 8.

[97] [Brouwer 1908b], the Rome lecture.

[98] In elk geval is een zeker bestaande Menge: $C^{\aleph_0}$, waar op elke volgende decimaal in plaats van een cijfer, een willekeurig punt van het continuüm valt. Maar de machtigheid daarvan is c.

Daarentegen de machtigheid $F = C^c$, die van alle functies, bestaat niet.

Wil ik alle mogelijke 'Mengen van grenselementen' zoeken, die uit ω zijn te vormen, dan moet ik alle mogelijke oneindige groepen er uit vormen, of hiertoe maar alle mogelijke groepen; en dit geschiedt door de benadering in het tweetallig stelsel, die als soorten van groepen dus alleen geeft, van machtigheid E (eindig), A ($\aleph_0$) en C.

At this place the continuum is recognized as a possible cardinality,[99] which raises the number of cardinalities to three; only the denumerably infinite unfinished one is still missing.

But in VII–16 and in VIII–16 references are made to this unfinished cardinality; in the seventh notebook we read:

> (VII-16) T cannot be mapped on ω by a finite law; neither can T be completed by a finite procedure; but during the construction of T in an infinite time it remains possible to map it on ω. And that is all I can say. Of course T remains unfinished. ω is finished (by our innate mathematical induction).[100]

This paragraph contains interesting information in regard to Brouwer's thought experiment about the 'unfinished mapping'. This kind of mapping is mentioned only once in a footnote on page 149 of Brouwer's dissertation. It is hardly worked out and he never came back to this notion, but in this quote we observe its rough draft. See further page 271 for a detailed discussion.

And finally, about an unfinished cardinality, from the eighth notebook:

> (VIII-16) For everything that we can create *mathematically* is denumerable; if we want to create T, we find out that our creating is never finished by giving isolated *acts*; and *laws*, which are denumerable sequences of facts; but for that reason we may not postulate that there are more things *apart from* what we can create.[101]

with which we reach the final number of four different cardinalities.

Barring a number of conjectures (in IV–12 the 'construction of the perfect set' according to Cantor's method), Brouwer generally proceeded towards the conclusions in the dissertation: one cannot speak of *all* points of a straight line (IV–17); we have intuitively the line as the continuum (IV–23); one can only speak of 'every element of a set' in case of a self-constructed set; the continuum *has* no cardinality, but *is* a cardinality; an arbitrary point of the continuum can only be approximated with the help of constructed points (V–30, VI–21); mathematics does not know 'beliebige Objekte', it only knows self constructed objects (VII–23).

4.10.8 Later developments, suggested in the notebooks

Finally we will point out a selection of quotes and sections from the notebooks, which can be viewed as a germ of, or even as direct or indirect evidence for,

[99] And the continuum will eventually be the only *actually infinite set* in the literal sense.

[100] T is niet op ω af te beelden door een eindige wet; maar T komt ook niet klaar door een eindig werk; maar, T vormende in oneindige tijd, blijft zij onder haar vorming steeds op ω afbeeldbaar. En dat is het enige wat ik kan zeggen. T is uit dien aard der zaak onaf; ω is af (door de mathematische inductie, die in ons is).

[101] Want alles, wat wij *wiskundig* kunnen scheppen, is aftelbaar; willen we T gaan scheppen, dan merken we, dat ons scheppen nooit klaar komt met het geven van geïsoleerde *daden*; en *wetten*, dat zijn aftelbaar oneindige feitenreeksen; maar daarom mogen we niet postuleren, dat er nog dingen zijn *buiten* hetgeen wij scheppen kunnen.

later important developments in the area of set theory, which were already partly mentioned or discussed in previous sections. Again, we note in advance that in the dissertation as well as in the notebooks, a complete separation between the two subjects of *sets* and of *the continuum* is hard to make, since the two are, understandably, often too interwoven. This sometimes results in a repetition of the same quote in case its content is relevant to both subjects.

The concept of *choice* in the definition of choice sequences as elements for spreads, is in the notebooks frequently expressed in the French terms *chance* or *prendre au hasard*, which are expressions originating with Borel.

From the sixth notebook onwards, one regularly recognizes germs of new ideas that only after 1907 came to full development, and, probably for that reason, were in 1907 often crossed out. Take for instance the following:

> (VI–36) Now it seems that I cannot speak of *all* elements of that set [*Brouwer is referring to the continuum*], hence that set is not real, since I cannot say with certainty within a finite time lag whether a point, indicated on the continuum, belongs to it (*sometimes* I can say that it does not belong to it). But *nevertheless* I can speak of the *reality* of that set, and of *all* its elements;[102]

and also:

> (VI–37) I *cannot* speak of the cardinal number of the continuum, (that is not included in its intuition); neither can I speak of that of the infinite decimal fractions, since the *all* makes no sense in itself, no more than via the continuum because neither the continuum possesses the '*all* points' concept.[103]

After the appearance of choice sequences as elements for sets, and of the 'universal spread', one still cannot speak of *all* elements of the continuum of the real numbers, but one *can* speak of an *arbitrary* element of it. It is of importance to note that in the last quotation an infinite decimal fraction is mentioned as an element of the continuum, but obviously Brouwer is now referring to the continuum of the reals, and not to the intuitive continuum which resulted from the ur-intuition. Denoting both concepts with the name 'continuum' may give rise to confusion, as we noticed earlier in section 3.3, page 79.

In the seventh notebook we witness the appearance of the concept of choice in the formation of sets. The term 'prendre au hasard' in the next quotation (which is again crossed out in this notebook) can of course not be specified any further, since in that case it would give us a denumerable result. In this

[102](VI-36): Nu schijnt het, dat ik toch niet kan spreken van *alle* elementen dier Menge, dus die Menge toch niet reëel is, want ik kan nooit zeker zeggen binnen eindige tijd van een op het continuüm aangegeven punt of het er toe behoort (wel soms, dat het er niet toe behoort). Maar *toch* kan ik spreken van de *realiteit* der Menge, en van *alle* elementen ervan.

[103](VI-37): Ik *kan* niet spreken van het cardinaalgetal van het continuüm, (zoiets ligt niet in de intuïtie ervan); evenmin van die der oneindige decimaalbreuken, omdat op zichzelf het *alle* daarvan geen zin heeft, en via het continuüm evenmin, omdat ook het continuüm geen '*alle* punten' heeft.

quote Brouwer compared the continuum with the second number class. Here 'continuum' is defined as the fictive continuum in combination with the 'axiom of limit points', which axiom implies that the limit element of a convergent sequence of elements of a set also belongs to the set (the set is closed under the construction of its limit points). Hence the continuum is the full continuum of the reals.

> (VII–15) Both (the continuum and the second number class) are composed of a multiplicity of the right to 'prendre au hasard' (and every 'prendre au hasard' is slightly different. (...)
>
> But a closer specification of the 'prendre au hasard' is not possible, since otherwise it would fall under an old denumerable field of numbers.
>
> I may postulate the order of the different 'hasards' on the continuum, which I perceive empirically afterwards via the infinite decimal fraction, in the case of my simultaneous or analogous 'hasard' for the second number class arbitrarily, exactly as in the case of its partner in the continuum.
>
> [*crossed out:*] Since I know by experience that the ω-fold free choice can be extended to the 'prendre au hasard' (for the continuum).[104]

The 'arbitrary choice' (prendre au hasard) is by definition *not* governed by a rule, and is used here to denote the continuum or the second number class (which two totalities need *not* be equivalent). In case of a real number, the 'prendre au hasard' is a free choice for each decimal place of the empirically observed infinite decimal fraction as an element of the full continuum.

As for the second number class, things are different. The general term of a member of this class is also composed of an infinite sequence of choices, but the result has to be a representation of an element of this class in the form of a Cantorian normal form, and the inherent limitations in the admitted choices may (and will) make the cardinality of this class smaller than that of the continuum.

The term 'prendre au hasard', which Brouwer often employed, originates with E. Borel in his publications on set theory and the transfinite. Whereas the regular French term for arbitrary choice is 'choix arbitraire',[105] Borel used the term 'prendre au hasard', e.g. in his paper *L'antinomie du transfini*, published in 1900 in the *Revue Philosophique*:

[104] Beide (continuüm en 2^e klasse) bestaan uit de veelheid van recht tot 'prendre au hasard' (en elk 'prendre au hasard' is weer iets verschillend. (...)

Maar nadere aanduiding van het 'prendre au hasard' is niet mogelijk, anders zou het vallen in een oud aftelbaar getallenlichaam. De ordening van de verschillende 'hasards' op het continuüm, die ik achteraf empirisch merk volgens de oneindige decimaalbreuk, mag ik bij mijn gelijktijdig of analoog 'hasard' bij de tweede getalklasse willekeurig nét zo, als bij zijn partner in het continuüm postuleren.

(en in een doorhaling:) Immers ik weet bij ondervinding, dat de ω-voudige vrije keuze zich laat uitbreiden tot het 'prendre au hasard' (bij het continuüm).

[105] See e.g. *Cinq lettres sur la théorie des ensembles* in Borel's collected works, containing letters by Hadamard, Baire, Lebesgue and Borel.

> Si l'on prend au hasard un entier positif quelconque (...)

(which in this case of course refers to a single choice).

In the next phrase (also crossed out) the question about the denumerability of a system of 'prendre au hasard' is possibly raised:

> (VII–16) If the 'prendre au hasard' is projected on a denumerable quantity, then this is *only* possible via an empirical infinite decimal fraction. Finite is impossible, since then the 'hasard' would have disappeared, and we had a free creation, defined by ourselves.[106]

This paragraph might refer to a single choice sequence which, in case of arbitrary choices, necessarily has to be of infinite length, since otherwise the arbitrariness is lost, but it may also allude to the necessary non-denumerability of a set which is composed of the sequences of a denumerable number of free choices from the set of the natural numbers. In both cases such a sequence, if it really has to be arbitrary, has to be infinite and may never (potentially) terminate, since otherwise the 'hasard' (i.e. the arbitrariness of the resulting sequence) is lost and it becomes our own unique making, created for this unique occasion.

An interesting remark (not erased) about choice sequences can be read in VII–19. After having discussed his solution to the continuum problem,[107] Brouwer added in the margin the following comment about 'points on the continuum':

> (VII–19) Of course I can, apart from the continuum with its point-scale, also build in the ω-sequence of chance-decimals, which I can arrange everywhere dense. But then the question is: how many of those decimals I leave for free choice? If their number is finite, then finite cardinality. If their number is ω, then cardinality c.[108]

In VII–20 rational and real numbers are compared; Brouwer declared these two to be of a fundamentally different kind: the rational numbers are constructions whereas the real numbers are 'chances in nature'. He used this argument to argue that the separation of all rational numbers from all real numbers is not a permitted operation. However, a real number as a 'sequence of chances in nature' (a never terminating choice sequence) and therefore a never completed mathematical object, becomes a mathematical object all the same:

[106] Wordt het 'prendre au hasard' geprojecteerd op een aftelbare hoeveelheid, dan kàn het niet anders dan door een empirische oneindige decimaalbreuk. Eindige kan nooit, want dan zou het hasard weg zijn, en hadden we onze eigen gedefinieerde vrije schepping.

[107] See our next chapter.

[108] Behalve in het continuum met zijn schaal van punten kan ik natuurlijk ook bouwen in de ω-rij van kansdecimalen, die ik ook weer überall dicht kan ordenen. Maar dan is het maar de vraag: hoeveel van die decimalen laat ik over voor de vrije keus? Zijn het een eindig aantal, dan machtigheid eindig. Zijn het er ω, dan machtigheid c.

> (VII–20) In his deduction of: Cardinality of the continuum = $2^{\aleph_0}$ Cantor forgets that one is not allowed to subtract *all* rational numbers from *all* real numbers. They are objects of a different kind: the former I construct, the latter are chances in nature.[109]

The eighth notebook is the most relevant and interesting one for the present subject. In these notes the problems of sets and of the continuum are investigated in a very systematic way, and no longer via loose and randomly scattered remarks only. Together with the arguments, leading to the well known conclusions in the dissertation, we do recognize here, again in phrases which are often crossed out, the germs of later growth towards a mature intuitionistic set theory:

> (VIII–13) We can only ground the intuition of continuous:
>
> $\underline{1^{st}}$ to view it as counterpart of discontinuity, which is our externalization.
>
> $\underline{2^{nd}}$ as a probability theorem, which *always* gives equal chances for all digits at every next decimal place. But we gaze at the system, which has that as a result, as a phenomenon of nature, we cannot construct it with our externalization of discontinuity.[110]

The first item relates to Brouwer's view on the sinful activity of mankind when observing nature with the aim of intervention and control.

The second one already suggests the spread in which on every next node, in this case at every next decimal place, all possible branches are permitted. Brouwer still spoke of chances, juding by the terminology almost certainly under the influence of Borel, but he clearly was searching for a way of founding the continuum of the real numbers, instead of accepting only the intuitive continuum as a given matrix for the construction of sets and the algorithmically constructible elements as the only points on it. But a possible *construction* of the intuitive continuum remains entirely unthinkable:

> (VIII–14) Suppose I had constructed an object with all the properties of the intuitive continuum; I would gaze at this result in amazement, hence I would absolutely have no reason to suppose that the

[109] Cantor in zijn afleiding van: machtigheid continuüm = $2^{\aleph_0}$ vergeet, dat je niet *alle* rationale getallen mag aftrekken van *alle* reële getallen. Het zijn ongelijksoortige dingen: de eerste bouw ik op, de laatste zijn kansen in de natuur.

'Chance in nature' resembles the act of dice-throwing to determine the decimals of a real number, which was rejected by Brouwer during the Berlin lectures; see [Dalen, D. van (ed.) 1981a], page xi.

[110] Te begründen is de intuïtie van continu niet dan:

$\underline{1^{e}}$ te bekijken als tegenhanger van discontinuïteit, die onze veruiterlijking is.

$\underline{2^{e}}$ als de waarschijnlijkheidsstelling, die *steeds* weer bij elke volgende decimaal voor elk cijfer gelijke kansen geeft. Maar het stelsel, dat dat geeft, staren wij aan als natuurverschijnsel, kunnen het niet opbouwen met onze discontinuïteitsveruiterlijking.

> constructed continuum had anything to do with the intuitive continuum.[111]

On an attempt by Klein to construct the continuum and to establish the postulate for the possibility of an infinite degree of accuracy, Brouwer gave the following comment:

> (VIII–15) And it is nonsense when Klein wants to determine in more detail the postulate of an indefinitely continued degree of accuracy by the construction of a fictive continuum. The postulate would then be a theorem of the probability theory applied to nature: continuing with my measurements I can always establish a next decimal, and all decimals have equal chances; but a postulate of induction about nature, albeit a fictitious one, is no mathematics, but physics. And I must obtain my continuum independently of anything external to me. But where do I get that theorem? From the intuition of the continuum.[112]

The content of this quote reminds us of the content of Poincaré's argument in *La Science et l'Hypothèse*, chapter II *La grandeur mathématique et l'expérience* to such an extent, that Brouwer may erroneously have referred to Klein instead of to Poincaré.[113] The concept of the continuum as a 'spread' in which, in case of a decimal representation, all digits $(0, 1, 2, ...9)$ have equal chance at every next decimal place, is sketched here as the result of a physical process of always refined measurement, hence as a physical continuum instead of as the result of a mathematical intuition. This physical approach of the continuum forced Poincaré, because of the involved paradox, to *postulate* a mathematical continuum.[114]

We already encountered the physical approach on page 15 and 16 of the eighth notebook (see our page 143), and it is mentioned again on the next page of the same notebook:

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[111]Gesteld al, ik had een ding met al de eigenschappen van het intuïtieve continuüm geconstrueerd; dat resultaat zou ik met verwondering aanstaren, dus zou ik niet de minste reden hebben, aan te nemen, dat dat geconstrueerde continuüm iets met het intuïtieve te maken had.

[112]En als Klein het postulaat van oneindig voortgezette graad van nauwkeurigheid, nader wil gaan vastleggen door het fictieve continuüm op te bouwen, is dat onzin. Het postulaat zou dan zijn een stelling van waarschijnlijkheidsrekening over de natuur: doorgaande met meten kan ik steeds een nieuwe decimaal vinden, en alle decimalen hebben gelijke kansen; maar een inductiepostulaat over de natuur, zij het een fictieve, is geen wiskunde, maar fysica. En ik moet mijn continuüm hebben onafhankelijk van iets buiten mij. Maar waar haal ik die stelling vandaan? Uit de intuïtie van continuüm.

[113]On the last page of this eighth notebook we find a short list of consulted papers or books, apparently referring to the relevant literature for the content of this notebook. The following papers by, or about Klein are mentioned:

– *Neuere Geometrie* (a paper about Klein),
– *Das Erlanger Programm* ([Klein 1872]),
– *Mathematische Annalen* 4, 6, 7, 17. This consists of some 14 papers, neither of which has a clear reference to the given quote.

[114]See section 1.3, page 24.

> (VIII–17) R the everywhere dense denumerable set of the rational numbers (that is, the 'everywhere dense' set of the first cardinality); add to it all known limit points ($\sqrt{2}$, π etc.), and it remains the same set; apply the addition again and again, ω times; we still have the same set. Hence the 'perfect' set cannot be constructed, and therefore it does not *exist*: we perceive it only in the physics of the intuition, and we can postulate axioms of the calculus of probability of it.[115]

The term 'same set' in this quote has of course the meaning of 'set with the same cardinality'. Any possible continued construction always remains at the most denumerable.[116] The 'perfect set' is, again, not the Cantor set (see page 95), but the continuum in the Cantorian sense of the 'set of all real numbers'. Departing from the set of rational numbers, which is everywhere dense but not perfect, one can add to it new elements ω times, that is, one can add all lawlike limit points, but obviously the cardinality of the resulting set remains the same, hence the quoted conclusion.

On the same page the second cardinality $\aleph_1$, the cardinality of the totality of all numbers of the second number class T and the next higher after $\aleph_0$, is treated. Brouwer proved that T, just as the perfect set of the real numbers, does not exist as a finished entity either.

But in the following quote a certain amount of doubt concerning the character of the continuum is again present. It was deleted by Brouwer in the notebook, probably when his continuum concept became the final one of the dissertation:

> (VIII–18) And yet ... and yet ... Maybe our continuum is a paradox, in approximation usable as the result of the laws of large numbers in physics.
>
> And maybe our 'intuition' of the line is nothing but the relation of separation between two points.[117]

In this fragment we recognize a mixture of critique and doubt. Brouwer was certainly searching for a way to say more about the continuum, just as in the next, and again deleted, paragraph:[118]

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[115](VIII-17): R de überall dichte aftelbare Punktmenge der rationale getallen (d.w.z. de 'overal dichte' Menge van de eerste machtigheid); neem er alle bekende grenzen bij ($\sqrt{2}$, π enz.), dan blijft het dezelfde Menge; pas er weer en weer en ω maal die toevoeging op toe; we houden dezelfde Menge. De 'perfecte' Menge is dus niet op te bouwen, *bestaat* dus niet: alleen in de fysica der intuïtie zien we haar, en we kunnen er axioma's van stellen van waarschijnlijkheidsrekening.

[116]This is an example of *denumerably unfinished.*

[117]En toch ... en toch ... Misschien is ons continuüm een paradox, die bij benadering bruikbaar is als resultaat van wetten van grote getallen in de fysica.

En is onze 'intuïtie' van lijn niets, dan de relatie van scheiding tussen twee punten.

[118]cf. Begründung paper I, [Brouwer 1918].

> (VIII–21) The continuum can be linearly ordered as a sequence of all integers with a finite number or ω digits (the next number is approximated together with the number itself).[119]

This sentence, written in 1906 or 1907, describes in a nutshell a method of ordering the continuum of the reals on the interval $(0, 1)$, as represented by the set C; this ordering is described in Brouwer's first *Begründung* paper via the technique of continued fractions (see [Brouwer 1918], page 9).

Again we see the concept of the continuum as the set which is composed of elements of infinite sequences of digits, in which we recognize choice sequence, still indicated by Brouwer with the term 'chance sequence'.[120]

The reason for deleting this paragraph probably was that in 1906 or 1907 Brouwer rejected the idea to view the continuum as an ordered set of 'integers with ω digits'. It seems as if he felt that he was forced to *choose* between the 'continuum' as (or, rather, represented by) the set of all real numbers on the one hand, and the unrepresentable intuitive continuum on the other hand, hence that he was in two minds about the continuum concept. In the section about the *Begründung* papers[121] we saw that after 1918 Brouwer *distinguished* between the two, *both* being mathematical concepts. But the existence of the uncountable set C of the reals (the full continuum of the real numbers) ultimately remained to be based on the intuitive continuum.[122]

In his later intuitionistic period, when non-terminating choice sequences were accepted as arbitrary elements of the continuum, the same quote would again, but now on different grounds, be crossed out: he proved that the continuum is not linearly ordered.[123]

The notion of choice sequence is also present in the next quotation (this time not deleted), in which the set of chance sequences is seen as a constructed and everywhere dense scale with the power of the continuum. If a set is 'directly defined', then its cardinality is at most denumerable; if it is defined by means of non-terminating sequences of free choice ('chance'), then the resulting cardinality is that of the continuum:

> (VIII–22) Nothing can be said of the continuum, but with the help of an everywhere dense scale, constructed on it. (That scale completely expresses the character of c). Hence every subset must after all be expressible with such a scale. That is only possible in two ways:
>
> 1^{st} directly defined. Then the set is denumerable.
>
> 2^{nd} with the help of an infinite chance sequence. Then the set is of the cardinality of c.[124]

[119] Het continuüm is lopend te ordenen als rij van alle gehele getallen met eindig of ω aantal cijfers (het eerstvolgende getal wordt benaderd tegelijk met het getal zelf).

[120] kansenrij.

[121] Page 133, and also page 71.

[122] See page 74.

[123] See page 121.

[124] Er is niets te zeggen van het continuüm, dan met behulp van een er op geconstrueerde

The second item is another early reference to 'choice sequences', but Brouwer used the term only much later; at this stage the concept did not yet have the status of a mathematical object. We have the intuitive continuum with an everywhere dense scale of rationals constructed on it. If we have an algorithm by means of which we can define new elements on the continuum, not belonging to the scale, hence irrationals, then the result always remains denumerable (or denumerably infinite unfinished). But if we admit infinite choice sequences, not governed by some algorithm, then the totality is no longer denumerable (see page 128). But then, in Brouwer's view, the cardinality becomes c, since the totality of the choice sequences represents the continuum of the reals and there is no cardinality between 'denumerably unfinished' and c which is the highest one.

On the one hand, we observe in the eighth notebook that every point on the continuum can only be defined by means of a beforehand algorithmically constructed dense scale; on the other hand, we see attempts to approximate an arbitrary point of the continuum by means of a chance sequence, expressed in an everywhere dense scale and not governed by an algorithm. One can construe this as an 'element of the continuum' in the shape of a non-terminating sequence of decimals, in which every *finite* segment represents a rational number, hence an element of the constructed everywhere dense scale, but without an algorithm for its composition.

Page VIII–23 is one of the more interesting pages, since at this place Brouwer attempted to approximate subsets of the continuum of the reals with the help of the branching method. This method was developed to determine the cardinalities of sets, hence Brouwer started from an already existing set. This technique shows, in its form, a similarity with the future process of constructing a set according to a given law, the spread law, which *determines* on every node the admissible branches, and subsequently the signs to be attached to the nodes. This is the concept of the spread in its new form, not operating with a system of nested intervals, but with the addition of a new natural number to the sequence of natural numbers, which were assigned to the preceding nodes.

But here, at this place in the notebooks, the branching method is still employed to determine the cardinality of subsets of the continuum only.

A first attempt is deleted, and stops halfway a sentence, apparently when Brouwer realized a better (or the proper) method to handle it, which method later turned out to be the perfect one for the spread concept. The deleted part must be considered unsuccessful, it is not further elaborated by Brouwer, and for us it remains in its content rather vague; it stops in the middle of a sentence.

The part which is not crossed out is the text that finally appeared in the dissertation, but we must be well aware of the fact that a *construction* of a set

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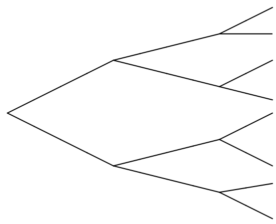
überall dichte schaal. (Die schaal drukt het hele wezen van c uit). Dus ook elke deelverzameling moet ten slotte met zo'n schaal zijn uit te drukken. En dat kan maar zijn op 2 manieren:

1[e] direct gedefinieerd. Dan is de verzameling aftelbaar.

2[e] met behulp van een oneindige kansenrij. Dan is de verzameling van de machtigheid van c.

with the cardinality of the continuum is of course completely out of the question and will remain so:[125]

> (VIII–23) *In approximating subsets of c.*
>
> Successively every decimal place is approximated in a dual system, whether or not by free choice.
>
> We then get a repeated twofold branching:
>
> 
>
> However, we cut off every branch, which comes to an end or which splits no more; there remains in the end:
>
> either *nothing*, or a *complete infinitely continuing twofold branching tree.* The latter case certainly will give the cardinality c for the limit points. For the first case, imagine that we cut off only the branches coming to a dead end, then there can remain:
>
> a) *nothing*; in this case we had finite cardinality.
>
> b) *a tree with a finite number of infinitely long branches*: in that case we had the cardinality $\aleph_0$ for the set and finite for the limit points.[126]

This method for the determination of the cardinality of a set found, as said, its way into the first chapter of the dissertation (see the discussion on page 117).

The argument of quote VIII–23 requires the use of the principle of the excluded middle (PEM) (which was already judged to be meaningless): there remains 'either nothing or a complete infinitely continuing twofold branching tree'; apparently this choice was supposed to be decidable.

[125]However, in a letter to Fraenkel, Brouwer claimed that the 'initial construction of mathematics', that is, the definition of elements for a set by means of choice sequences, was already present in the dissertation. See also page 74.

[126]Bij de benadering van Teilmengen van *c*. Achtereenvolgens wordt elke decimaal in tweetallig stelsel benadered, al of niet met vrije keuze. We krijgen dan een telkens herhaalde tweevoudige vertakking:

(see diagram in the text)

We breken nu echter elke tak die doodloopt, of zich nooit meer vertakt, af; er blijft dan ten slotte over: òf *niets*, òf *een volledige oneindig voortlopende tweevertakking.* Het laatste geval geeft zeker de machtigheid c voor de grenspunten. Denk voor het eerste geval, dat we alleen de doodloopende takken afbreken; dan kan overblijven:

a) *niets*; dan hadden we eindige machtigheid.

b) *een boom met een eindig aantal oneindig lange takken*: dan hadden we machtigheid $\aleph_0$ voor de Menge en eindig voor de grenspunten.

In the new mode of set definition (after 1918) the branching method is no longer employed to determine the cardinality of a set, but, instead, to define a set which then becomes a law. See for this the section about the *Begründung* papers on page 133.

According to VIII–23, the 'complete and indefinitely continuing twofold branching tree' (that is, with both branches on every node) certainly has the cardinality c. This is the continuum of the reals in dual form.[127]

In the following fragment Brouwer was aware of the problems related to the decidability of a branching tree:

> (IX–26) One could say: can one find out whether a point sequence on the continuum is dense or not? In other words, is the character of a branch always decidable? In any case I can say: if it is not yet decided, I certainly *cannot* apply the completion to a continuum, hence it has to be limited to a denumerable quantity.[128]

This phrase refers to the 'branching method', as described by Brouwer on pages 64 and 65 of his dissertation, and also in notebook VIII–23. See also our pages 117 and 156.

The following paragraph was quoted earlier:

> (VIII–24) One should always keep in mind that ω only makes sense as a living and growing induction in motion; as a stationary abstract entity it is meaningless; ω may never be thought as finished, as a new entity to operate on; however you may think it to be finished in the sense of turning away from it while it continues growing, and to think of something new.[129]

Here we see once more the concept of an infinite sequence: on the one hand, it is never finished, the process of growth continues and since the process takes place in time, it never terminates. On the other hand, it may be seen as finished in an idealized way. We may leave the process alone, it does not require our permanent personal intervention. In this way we may see the system of natural numbers as finished, or any other infinite lawlike system for that matter. Brouwer expressed himself metaphorically with the terms 'turning away from it, while it continues growing and think of something else', meaning that the process of growth is set into motion by applying the algorithm (inductive or recursive). We have, as it were, fed the machine with the necessary data in the

[127] Later by Heyting to be called the *universal spread*.

[128] Men zou kunnen zeggen: is het uit te maken, of een puntrij op het continuüm dicht is of niet? M.a.w. is het karakter van de boomtak altijd uit te maken? In elk geval kan ik zeggen: heb ik het nog niet uitgemaakt, dan kan ik de completering tot continuüm *zeker niet* toepassen, moet dus zeker tot een aftelbare hoeveelheid beperkt blijven.

[129] Men bedenke steeds dat ω alleen zin heeft, zolang het leeft, als groeiende, bewegende inductie; als stilstaand abstract iets is het zinloos; zo mag ω nooit àf gedacht worden, om m.b.v. het *geheel* als nieuwe eenheid te werken: wel mag je het àf denken in de zin, van je er van af te keren, terwijl het doorloopt, en iets nieuws te gaan denken.

form of the first term of the sequence and the algorithm, and set it into motion. No more intellectual activity is required by us, we can think of something else. We will return to this matter when discussing Brouwer's concept of the 'actual infinite'.[130]

About the cardinalities of sets that can eventually be constructed from a denumerably infinite ordered set with ordinal number ω, Brouwer noticed the following:

> (VIII–25) If I want to find all possible 'sets of limit elements' that can be constructed from ω, then I have to construct all possible infinite groups from it, or all possible groups for that matter; and this can be performed by the approximation in the dual system, which results in the only kinds of groups with the cardinality E (finite), A ($\aleph_0$) and C.[131]

In this paragraph Brouwer applied a 'Cantorian way of reasoning': all possible infinite subsets of an ordered set with ordinal number ω (Brouwer used the term 'group', but, again, this should not be understood in its algebraic sense) expressed in the dual system (just for convenience) results in the cardinality of the continuum, which makes the total number of possible cardinalities three: E, A and C. The argumentation here is of course the definition of the universal spread: on every node of the spread both branches occur. But Brouwer again referred to a way of constructing or describing the continuum of the reals, or, at least to a way of 'parallelling' it.

The continuity theorem is far too sophisticated at this early stage of the development of his intuitionistic mathematics, but Brouwer was already searching for a solution in a constructive sense for the definition of functions with 'unknown irrational numbers' as argument, and even for their continuity, thereby speaking of a 'continuity postulate':

> (VIII–38) One has the rational scale and some continuous operations in it (e.g. extraction of a root). Then one defines on the basis of those operations, the *known* irrational numbers (on the basis of an extension to a *postulate of continuity*) as limits of known sequences (the known order relations are assigned to those limits).
>
> One might define also the *unknown* irrational numbers as the limits of unknown sequences. One assigns to them the known order relations, and only afterwards one needs to introduce the continuity postulate, in order to be able to perform operations on those

[130]See chapter 8 of this dissertation, page 320.

[131]Wil ik alle mogelijke 'Mengen van grenselementen' zoeken, die uit ω zijn te vormen, dan moet ik alle mogelijke oneindige groepen er uit vormen, of hiertoe maar alle mogelijke groepen; en dit geschiedt door de benadering in het tweetallig stelsel, die als soorten van groepen dus alleen geeft, van machtigheid E (eindig), A ($\aleph_0$) en C.

> irrational numbers.[132]
>
> (VIII–40) Sometimes I can assign certain irregular (unstetige) values for a function to *known irrationals,* the values of the unknown irrationals, however, remain always determined by the continuity postulate.[133]

In both quotes the 'continuity postulate' is mentioned, which is needed to define 'known irrational numbers' as limits of known sequences of rationals (we have not yet arrived at the stage where the *whole* sequence, and not its limit, stands for the irrational number). The postulate then defines the existence of the limit of every convergent sequence of rationals, lawlike or not, thus extending the system of the rational numbers to include the irrationals, giving the system of the reals.[134] Alternatively, the term 'continuity postulate' may refer to Dedekind's axiom from *Stetigkeit und irrationale Zahlen.* See page 22.

In the last quotes, Brouwer was referring to the way in which Cantor, in the *Mathematische Annalen*, volume 5 from 1872, defined the irrational numbers:

> Die rationlen Zahlen bilden die Grundlage für die Feststellung des weiteren Begriffes einer Zahlengrösse; ich will sie das Gebiet A nennen (mit Einschluss der Null).
>
> Wenn ich von einer Zahlengrösse im weiteren Sinne rede, so geschieht es zunächst in dem Falle, dass eine durch ein Gesetz gegebene unendliche Reihe von rationalen Zahlen $a_1, a_2, ...a_n, ...$ (1) vorliegt, welche die Beschaffenheit hat, dass die Differenz $a_{n+m} - a_n$ mit wachsendem n unendlich klein wird, was auch die positive ganze Zahl m sei.
>
> (...)
>
> Diese Beschaffenheit der Reihe (1) drücke ich in den Worten aus: *'Die Reihe (1) hat eine bestimmte Grenze b'.*
>
> (...)
>
> Die Gesammtheit der Zahlengrössen b möge durch B bezeichnet werden.

---

[132](VIII-38): Men heeft de rationale schaal en enkele stetige bewerkingen daarin (b.v. worteltrekking). Men definieert nu op grond van die bewerkingen, de *bekende* irrationale getallen (op grond van uitbreiding tot een stetigkeits'postulaat') als limieten van bekende reeksen (aan welke limieten dan de bekende orderelatie wordt toegekend).

Of ook men definieert de *onbekende* irrationale getallen als limieten van onbekende reeksen. Men kent er de bekende orderelatie aan toe, en behoeft eerst achteraf, om *bewerkingen* met deze irrationalen te kunnen uitvoeren, het stetigkeitspostulaat in te voeren.

[133](VIII-40): Soms kan ik aan *bekende irrationalen* bepaalde irregulaire (unstetige) waarden voor een functie geven, de waarden der onbekende irrationalen blijven dan echter altijd nog bepaald door het stetigkeitspostulaat.

[134]The reader will understand that this is not yet Brouwer's *continuity principle* from his intuitionistic mathematics!

> Mittelst obiger Festsetzungen lassen sich die Elementaroperationen, welche mit rationalen Zahlen vorgenommen werden, ausdehnen auf die beiden Gebiete A und B zusammengenommen.[135]

However, after the publication of *Stetigkeit und irrationale Zahlen* (also in 1872) the most well-known definition of irrational numbers is by means of 'Dedekind cuts'[136] and Dedekind's Stetigkeits Postulat:

> Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, daß jeder Punkt der ersten Klasse links von jedem Punkte der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt.

In the second paragraph of quote (VIII–38), as well as in (VIII–40), attempts were made to define the unknown irrationals as limits of 'unknown sequences', that is, as limits of non-lawlike choice sequences. A continuity postulate is then required afterwards to make operations on the unknown irrationals possible. E.g. if we have two convergent sequences (defining two irrationals), we can construct a third sequence by adding up the corresponding terms of the two sequences. With the help of the continuity postulate the limit of the third sequence (this sequence can easily be proved to be convergent too) then defines an irrational number which is the sum of the first two irrationals.

Also the definition of the order relation needs the continuity postulate: if α and β are two irrationals and $\{a_n\}$ and $\{b_n\}$ are their defining sequences of rationals, then $\alpha < \beta$ if there exists an m such that $a_n < b_n$ for all $n > m$.

About still higher cardinalities, Brouwer wrote:

> (VIII–43) The cardinality of f is contradictory. After all, one can imagine that the game of chance makes a free choice ω times in succession (that is to say: *always* continuing); but not c times. Our intuition tells us, if requested, that this is unimaginable. One can only read Schoenflies' *Bericht*, page 24 § 4 in the following way: It is not true that:
>
> f is conceivable and can be mapped one-one on c.[137]

f is the cardinality of all functions, defined on the continuum, which is an impossible and contradictory cardinality. However, the last quote from the eighth notebook clearly tells us that ω times a free choice is thinkable, and that

[135] [Cantor 1871], § 1.

[136] See page 22 of this dissertation.

[137] De machtigheid f is contradictoir. Immers men kan zich denken, dat ω maal achtereen (d.w.z. *steeds* weer door) het kansspel een vrije keus doet; maar niet c maal. Dat men dit niet kan denken, antwoordt ons, desgevraagd, direct onze intuïtie. Men kan dus Schoenflies' *Bericht*, p. 24 § 4 alleen lezen: het is niet waar dat:
f denkbaar zou zijn en eenduidig af te beelden op c.

therefore the cardinality c is not contradictory, thereby referring to the spread in which on every node (in case of the dual representation) both choices are allowed, ω times. But c times is unthinkable, a number of choices always remaining denumerable at the most. Brouwer was referring to Schoenflies' *Bericht über die Mengenlehre*, from the *Jahresbericht der Deutschen Mathematiker-Vereinigung* volume 8.[138] In chapter 4 of this book, the most simple non-denumerable sets are discussed, and under item 4 Schoenflies claimed:

> Die einfachste uns bekannte Menge, deren Mächtigkeit grösser als c ist, ist die Menge F aller Functionen einer reellen Variabelen.

Schoenflies proved that its cardinality $f = c^c > c$.

In a note in the margin of the eighth notebook, added afterwards, Brouwer again stated that the existence of unknown infinite sequences, not determined by some law and hence being choice sequences of numbers, is very well possible and certainly not unthinkable and not contradictory (he was not alluding at this place to the 'nested interval' type of choice sequences):

> (VIII–45) I can *think* the unknown infinite sequences in case of the regular continuum, since I know a close connection with *all* finite sequences; only by that connection, independent of the formal generation, hence intuitively, the unknown infinite sequences can be thought as existing, as not absurd.[139]

But the notion of sequences of ω times a free choice is certainly not yet a definitive and permanent one.

In the ninth and last notebook remarks to the contrary are made again: no infinite chance sequences as a model for the continuum. Towards the end of this notebook, and shortly before the public defence of its result, he apparently had to make his choice for the standpoint of the dissertation, not yet fully realizing and neither able to work out the consequences of the continuously boiling new ideas. *That* these ideas were boiling may be concluded from the fact that not all quotations, alluding to choice sequences and spreads, are crossed out. And even *if* they were deleted, then he apparently *had* these ideas and thoughts beforehand.

4.11 Concluding remarks about sets

On the basis of the discussions and quotations in this chapter, we can draw the following conclusions about Brouwer's present (1907) and future notions of set:

[138] [Schoenflies 1900b].

[139] Bij het gewone continuüm kan ik de onbekende oneindige reeksen *denken,* omdat ik een nauw verband weet met *al* die eindige reeksen; alleen door dat verband, onafhankelijk van deze formele generering, dus intuïtief, kunnen die oneindige onbekende reeksen als bestaand, als niet onzinnig worden gedacht.

– Brouwer was a constructivist from the beginning. Elements have to be constructed individually; only then a set is well-defined. This also applies for 'species': the properties referred to in the definitions apply to pre-existing objects only.

Three modes for the construction of a set were given in the dissertation, but, for reasons of non-constructibility, the third mode, most likely introduced under Cantor's influence, was rejected in the *Addenda and Corrigenda* and thus disappeared from the list of construction methods for sets.

– The number of cardinalities in the notebooks gradually increased from two to four, which is the number that we find in the dissertation. The four possible cardinalities, in particular the *denumerably infinite unfinished*, will be extensively discussed in chapter 7.

– In 1914, the new development set in with the publication of the Schoenflies review, followed by the notes in the margin of the lecture notes on set theory (1915), and with the publication of the *Begründung* papers (1918). This development eventually resulted in notions like choice sequences, spreads, species, the perfect spread as a representation of the continuum of the reals, and in an intuitionistic real analysis.

With respect to all those future developments we may conclude to the following four summaries, all originating from the notebooks, in particular from the eighth one:

> -1- The concept of the choice sequence can often be indirectly sensed or directly recognized in the notebooks, but the idea of a 'free choice' is not yet explicitly mentioned; it is still expressed in terms of 'chance sequences', every next decimal has equal chances; however, this corresponds to the admission of a free choice for every next decimal place in the determination of a non-terminating sequence.
>
> -2- A function can have a value for an unknown irrational number as argument. This includes that the function has a value for a choice sequence as argument, since the unknown irrational can be expressed as such a sequence.
>
> -3- Without explicitly expressing his continuity principle, Brouwer employed it, as a result of the same claim that a function can have an unknown irrational value for an unknown irrational argument (see page 159). This latter seems to imply, in view of his constructivism, the continuity principle: if the function value has to be constructed decimal by decimal, then each next decimal has to be deduced from a finite number of consecutive decimals from the expansion of the variable.[140]

[140] This reasoning follows Charles Parsons' argument in the introduction to [Brouwer 1927] on page 446 of [Heijenoort 1967]. However, this argument is not correct; there is more to say to it. See [Atten, M. van and Dalen, D. van 2000] for an analysis of the continuity principle.

-4- The basic idea of the construction of a set according to the spread concept can already be recognized in the branching method, when this method was employed in the determination of the possible cardinalities of a set and to find out whether or not a set is dense.

At this point we are still in a very early stage of Brouwer's mathematical and philosophical development. Many ideas are present in a very rudimentary form, but it is evident that Brouwer did not hesitate to make up his own mind and to draw his own conclusions, facing the authorities of his day, when necessary.

Chapter 5

The ‘continuum problem’

5.1 Introduction

It was in 1900 at the Paris conference that Hilbert presented his list of unsolved mathematical problems; this list can be viewed as an assignment for the mathematical community for the century ahead. As number one on that list, which was entitled *Mathematische Probleme*, stands the continuum problem, already conjectured by Cantor, and expressed by Hilbert as follows:

> Jedes System von unendlich vielen reellen Zahlen, d. h. jede unendliche Zahlen- (oder Punkt)menge, ist entweder der Menge der ganzen natürlichen Zahlen $1, 2, 3, \ldots$ oder der Menge sämtlicher reellen Zahlen und mithin dem Kontinuum, d. h. etwa den Punkten einer Strecke, äquivalent; *im Sinne der äquivalentz gibt es hiernach nur zwei Zahlenmengen, die abzählbare Menge und das Kontinuum.*[1]

As a related problem, of which the solution may lead to the solution of the original problem, Hilbert mentioned the well-ordering of the continuum. In fact we have to distinguish between the *continuum problem* which asks for an answer to the question ‘what is the cardinality of the continuum’, and the *continuum hypothesis*, which conjectures that this cardinality is the next higher after $\aleph_0$, the cardinality of the set of the natural numbers. Since it was proved already by Cantor that the cardinality of c exceeds $\aleph_0$, the claim of the hypothesis is that $c = 2^{\aleph_0} = \aleph_1$.

In this form it is still an unsolved mathematical problem. Different other questions, related to the continuum hypothesis, have been solved, but the hypothesis itself still stands as such.

[1] Presented at the Paris conference of 1900, and published in the *Göttinger Nachrichten* in 1900. See [Hilbert 1900], see also [Hilbert 1932], vol. III, page 298. In our dissertation it was mentioned earlier on page 9.

For Brouwer, the continuum problem is of a completely different nature. The form $c = 2^{\aleph_0} = \aleph_1$ is a meaningless one for him since $\aleph_1$ is a non-existing cardinality. The question that he intends to answer on page 66 of his dissertation is: what are the possible cardinalities of subsets of the continuum. If we take the standard interval $(0, 1)$ (which is, in fact, the intuitive continuum between the first and the second experienced event) as point of departure, then a subset of the continuum cannot be anything else but a point set, constructed on the continuum, or a finite or denumerable set of subsegments of the continuum.

The aim of this chapter is an analysis to the solution of the continuum problem, as Brouwer presented it in his dissertation and in the notebooks. Brouwer's solution is simple and is, as we will argue, a direct corollary of the modes of set construction that he admitted. In fact his solution is almost trivial, since the limited number of possible cardinalities immediately follows from his set constructions.

The solution from the notebooks is different, but it is just as well an immediate corollary of Brouwer's constructivism.

One can wonder why Brouwer did not present the solution from the notebooks in his dissertation; it certainly is a more direct and intuitive (although possibly a less sophisticated) one. On the other hand, the proof from the dissertation is more in agreement with, and fits better, the three admitted modes of set construction that Brouwer just had given.

5.2 The solution in the dissertation

On page 66 ff. of his dissertation, Brouwer proved that every point set which is defined on the measurable continuum (hence constructed on the intuitive continuum), either has the cardinality of the continuum or is denumerable, thus solving the problem and confirming the hypothesis. The proof proceeds as follows:[2] let there be given a set, constructed according to the given rule 2, whether or not in combination with the third rule (see our pages 117, 120 and 122), *and* a continuum. The set can now be mapped on the continuum in the following way:

> (...) in both sets (the continuum and the given set) a well-defined, and consequently denumerable, point set is chosen in such a way that all the other points can be obtained by approximation from everywhere dense parts of this set. Then the undefined points are brought into one-to-one correspondence by mapping the everywhere dense parts, with respect to which the infinitely proceeding approximations must be taken, onto each other; then the points which were

[2] In Brouwer's presentation of the proof the term 'group' is used; as said earlier, Brouwer frequently used the term *group* when the term *set* clearly expresses the intention; apparently the terminology in this matter was not yet completely fixed.

> defined can still be brought into correspondence with each other, because they are denumerable in both sets.[3]

As usual, Brouwer put it briefly worded; the conclusion of the foregoing procedure is drawn without further arguments or explanation:

> It follows that every set of points on the measurable continuum (and consequently also on the simple intuitive continuum, which in fact we can only handle after having made it measurable – or built up out of individualized measurable parts) which is not denumerable, has the power of the continuum.[4]

As an elucidation of this very concise argument and its conclusion, the following discourse expresses, in our opinion, the intentions and content of Brouwer's solution. Suppose we have constructed a set according to method 2, i.e. everywhere dense and 'completed to a continuum'; or, according to rule 3, deleted from a continuum a constructed dense scale. The result of the (earlier discussed) rules 2 and 3 can be fully mapped one-to-one onto the continuum, hence has the cardinality of the continuum. This 'mapping' can easily be understood if we recall to mind the way in which an everywhere dense set was 'completed to a continuum'. We discussed this on page 116: we have an everywhere dense denumerable set which is 'covered by a continuum', and this 'covering' can now be identified with the 'completion to a continuum' from rule 2. But then, if we take a continuum C and define an everywhere dense scale (e.g. η) on it, by which every point on C, not belonging to that scale, can be approximated, then the proof of the statement that the result of rule 2 is similar to C, is a trivial one, hence the result of rule 2 has the power of the continuum.

And if we delete from C the scale η, then the result is similar to that of rule 3. Also the proof of this claim is a trivial one: map the denumerable scales onto each other and map the approximating sequences in the denumerable scale for undefined points onto each other.

Hence a set satisfying the first method of construction is denumerable in virtue of its definition, and the set A, constructed according to method 2 and/or 3, has the cardinality of the continuum, since the above sketched mapping between A and C can be completed one-to-one. No other sets can be constructed and therefore no other sets exist.

---

[3](page 66) in beide verzamelingen (het continuüm en de gegeven puntverzameling) wordt een welgedefinieerde, dus aftelbare puntgroep uitgekozen zó, dat alle andere punten als benaderingen ten opzichte van overal dichte delen van die groep kunnen worden beschouwd, en vervolgens worden de ongedefinieerde punten met elkaar één-éénduidig in correspondentie gebracht, door de overal dichte delen, ten opzichte waarvan in beide de oneindig voortlopende benaderingen moeten worden genomen, op elkaar af te beelden; de wel gedefinieerde punten kunnen dan altijd nog daarna in correspondentie met elkaar worden gebracht, daar ze in beide aftelbaar zijn.

[4](page 66, 67) Waaruit volgt, dat elke puntverzameling op het meetbaar continuüm (dus ook op het intuïtief continuüm zonder meer, waarmee we immers eerst kunnen werken nadat we het meetbaar – of uit geïndividualiseerde meetbare stukken opgebouwd – hebben gemaakt), die niet aftelbaar is, de *machtigheid van het continuüm* bezit.

Brouwer then concluded:

> Thus the continuum problem, put forward by Cantor in 1873[5] and mentioned by Hilbert as still open ('Mathematische Probleme', Problem no. 1, page 263),[6] seems to be solved, in the first place by keeping strictly to the view: a continuum as set of points must be considered with respect to a scale of order type η.[7]

Brouwer did *not* claim that the continuum problem *was* solved, but that it *seemed* so. But we immediately see that Brouwer's solution to the continuum problem is a direct consequence of 1) the three methods of set-construction that he admitted, which methods, in turn, are a consequence of the basic intuition of all mathematics which states that the continuum is *no* point set, *and* 2) of his point of departure that only constructible objects are acceptable as mathematical entities on the continuum. 'Undefined points' exist only on a continuum, or in a denumerable set which is 'completed to a continuum', in relation to an (everywhere dense) scale of constructed points. But then the conclusion that every set either is denumerable or has the cardinality of the continuum becomes obvious and almost trivial. Even if we consider (thereby anticipating on future developments) the perfect spread as a representation of the continuum of the reals, then the conclusion should remain the same under this argument, as a result of the requirement that every subset of the continuum needs an algorithm for the construction of its individual elements, and the *result* of a construction can never surpass a denumerable cardinality. There simply does not and cannot exist anything between the cardinalities $\aleph_0$ and c.[8]

Any object in mathematics has to be the result of a construction; this is also what Brouwer wrote to his thesis supervisor Korteweg in a letter of 5 November 1906, in which he announced his solution:

> Allow me to send you the enclosed copy of the Göttinger Nachrichten, in which Hilbert's Paris lecture 'Mathematische Probleme' is printed. You will see that I have completely discussed no.1 ('Cantor's Problem von der Mächtigkeit des Continuums') in the first chapter of my dissertation, by going back to the intuitive construction, that must exist for all mathematics.[9]

[5] [Cantor 1874], also in [Cantor 1932]

[6] [Hilbert 1900].

[7] (page 67) Hiermee schijnt het 'continuümprobleem', door Cantor in 1873 opgesteld en door Hilbert ('Mathematische Probleme', Problem no. 1, pag 263) als nog steeds actueel gesignaleerd, te zijn opgelost, en wel in de eerste plaats door streng vast te houden aan het inzicht: over een continuüm als puntverzameling kan niet worden gesproken, dan in betrekking tot een schaal van het ordetype η.

[8] Note that 'denumerably unfinished' does not represent a result, but a non-terminating process.

[9] Mag ik u misschien nog bijgaande jaargang der Göttinger Nachrichten sturen, waarin de voordracht van Hilbert te Parijs: 'Mathematische Probleme' staat afgedrukt. U zult dan zien, dat ik *no. 1,* (Cantors Problem von der Mächtigkeit des Continuums") in het eerste hoofdstuk van mijn dissertatie volledig heb behandeld, en dat juist door mijn teruggaan op de intuïtieve opbouw die voor alle wiskunde moet bestaan.

Nevertheless, one gets the impression that Brouwer was not completely satisfied with the proof he gave, for the same reason that he was not satisfied with the third construction rule. Indeed five years later he added a correction to his proof, written on a sheet and added to his own corrected copy of his dissertation. This sheet begins as follows:

> The best thing to do is, to recognize a set of points on the linear continuum as defined only then –we may do such a thing as long as the possibility of unsolvable problems exists– when we have constructed it by putting term by term in a well-ordered way, whether or not under the addition of the fundamental sequence of free number choices. Then every non-denumerable point set contains a *perfect* subset.
>
> We only recognize a definition by the *exclusion* of points as sufficient, if it can be translated into another definition in the form given above.[10]

A definition must be constructive and one should not speak of 'undefined points', as Brouwer actually did in the proof. We saw that he also used this negative definition of a point set in his third mode for the construction of sets, and that he rejected this possibility in the *Addenda and Corrigenda*. However, the inserted sheet only questions the quality of the proof, and not the content of the theorem itself.

And indeed one may put forward as a point of criticism the impossibility to map the undefined points of the set on the undefined points of the continuum, because of the impossibility of an algorithm for sequences that represent 'undefined points' on the continuum. One can define lawlike approximating sequences for 'undefined' points (i.e. points not belonging to the 'well-defined point set') on both the set and the continuum, but the result remains denumerable and there is no fixed algorithmic rule which allows us to consider all undefined points on both the set and on the continuum as a well-defined totality. On the basis of this proof alone one can never conclude to a cardinality of the continuum. In fact we have, in case of construction rule 2 or 3, that continuum already; the proof becomes superfluous.

Again, as can be concluded from the quoted proof and our elaboration of it, Brouwer had to appeal to sequences of free choices as the only possible argument for his solution to the continuum problem, since choice sequences of rationals on the continuum, in any phase of their development, can be mapped on similar choice sequences in the investigated set, with a denumerable result.

---

[10] Het beste is, een puntverzameling op het lineaire continuüm eerst dàn als gedefinieerd te erkennen – en zoiets mogen we doen, zolang de mogelijkheid van onoplosbare problemen bestaat – als we haar hebben opgebouwd, door welgeordend punt voor punt te plaatsen, al of niet onder toevoeging der fundamentaalreeks van vrije cijferkeuzen. Elke niet-aftelbare puntverzameling bevat dan een *perfecte* deelverzameling.

Definitie door *uitsluiting* van punten erkennen we dus alleen dàn als afdoend, als ze zich in een nieuwe definitie van bovenstaande vorm laat vertalen.

(See page 119, where the possible set constructions were discussed. The given correction was also quoted at that place. See for this correction [Dalen 2001], page 77.)

This argument clearly shows the necessity of the later developments to choice sequences as proper mathematical objects. In order to be able to handle the continuum, to discuss continuous functions and to work out an intuitionistic and constructive analysis, a development into the direction of choice sequences turned out, in hindsight, to be the only way. And in that development the continuum problem disappeared as improper and unimportant.

Brouwer came already to that conclusion in his inaugural address (see below). In the first *Begründung* paper the continuum problem is not mentioned any more, the only two examples of infinite cardinalities are, again, the familiar 'denumerable' and 'continuum'.

In *Intuitionistische Mengenlehre*[11] Brouwer articulated it as follows:

> Die klassischen Kardinalzahlen a und c bleiben bestehen, dagegen wird das in der klassischen Theorie durch die Menge aller Funktionen einer Variablen gelieferte Beispiel einer Kardinalzahl $> c$ hinfällig.

5.3 Two other publications on this problem

5.3.1 The Rome lecture

In 1908, the year after his academic promotion, Brouwer lectured at the International Conference of Mathematicians in Rome. His contribution was *Die möglichen Mächtigkeiten*,[12] in which Brouwer defined a certain procedure to decide on the cardinality of a set which is defined on the continuum. Without, at this place, going into the details of the nature of this operation,[13] Brouwer concluded:

> Wird die Operation wenigstens einmal vollführt, so ist die Mächtigkeit von M jene des Continuums; wird sie nicht vollführt, so ist M abzählbar.
>
> Es existiert also nur eine Mächtigkeit für mathematische unendliche Mengen, nämlich die abzählbare. Man kan aber hinzufügen:
>
> 1) *die abzählbar-unfertige,* aber dann wird eine *Methode,* keine Menge gemeint;
>
> 2) *die continuierliche,* dann wird freilich etwas Fertiges gemeint, aber nur als *Matrix,* nicht als Menge.
>
> Von anderen unendlichen Mächtigkeiten, als die abzählbare, die abzählbar-unfertige, und die continuierliche, kann gar keine Rede sein.[14]

This, once more, shows Brouwer's conclusion of his solution to the continuum problem: *every* set, hence also every subset of the continuum, either has the cardinality of the continuum, or is denumerable (or denumerably unfinished).

[11] [Brouwer 1919b], page 3.
[12] [Brouwer 1908b], also in [Brouwer 1975], page 102.
[13] See chapter 7
[14] [Brouwer 1908b], conclusions at the end of this paper.

5.3.2 The inaugural address

But in 1912 the dawn of his intuitionism was approaching. In his inaugural address *Intuitionism and Formalism*[15] Brouwer returned to the question as posed by the continuum hypothesis, this time with a different outcome: the question becomes meaningless. In the inaugural address, the attitude of the formalist with that of the intuitionist is compared on different mathematical topics. In the following quote the subject is *the continuum*:

> Let us consider the concept 'real number between 0 and 1'. For the formalist this concept is equivalent to 'elementary series of digits after the decimal point', for the intuitionist it means 'law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations.' And when the formalist creates the 'set of all real numbers between 0 and 1', these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of real numbers of the intuitionist, determined by finite laws of construction. Because it is possible to prove to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of real numbers between 0 and 1 can be constructed in various ways, and second that for every such set it is possible to assign a real number between 0 and 1, not belonging to the set, the formalist concludes: 'the power of the continuum, i.e. the power of the set of real numbers between 0 and 1, is greater than aleph-null', a proposition that is without meaning for the intuitionist; the formalist further raises the question, whether there exist sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null, in other words, 'whether the power of the continuum is the second smallest infinite power,' and this question, which is still waiting for an answer, he considers to be one of the most difficult and most fundamental of mathematical problems.
>
> For the intuitionist, however, the question as stated is without meaning; and as soon as it has been so interpreted as to get a meaning, it can easily be answered.[16]

[15] [Brouwer 1912], for English translation see [Benacerraf and Putnam 1983], page 77.

[16] Beschouwen we het begrip: *'reëel getal tussen 0 en 1'.* Voor de formalist is dit begrip gelijkwaardig met : *'fundamentaalreeks van cijfers achter de komma',* voor de intuïtionist met: *'door een eindig aantal operaties geconstrueerde voortbrengingswet van een fundamentaalreeks van cijfers achter de komma'.* En waar de formalist de *'verzameling van alle reële getallen tussen 0 en 1'* creëert, zijn deze woorden voor de intuïtionist van zin ontbloot, hetzij daarbij aan de door een fundamentaalreeks van vrije cijferkeuzen bepaalde formalistische reële getallen, of aan de door eindige voortbrengingswetten gedetermineerde intuïtionistische reële getallen wordt gedacht. Daar men op voor de formalist zowel als voor de intuïtionist bindende wijze kan aantonen *ten eerste,* dat op de meest verschillende manieren aftelbaar oneindige verzamelingen van reële getallen tussen 0 en 1 kunnen worden geconstrueerd, *ten tweede,* dat naast elke zodanige verzameling terstond een *niet* tot de verzameling behorend reëel getal

The 'easily answerable' questions that Brouwer was referring to in this quote, are the intuitionistic restatements of the continuum problem; these questions were put by Brouwer in one of the following forms:[17]

1. Is it impossible to construct a non-denumerable set of real numbers between 0 and 1, such that its cardinality is smaller than that of the continuum, but greater than $\aleph_0$? The answer simply is 'yes, that is impossible'.
2. Can one, while constructing on the following two sets, maintain a one-to-one correspondence between a set of real numbers between 0 and 1, and the set of denumerable infinite cardinal numbers? Again, the answer is in the affirmative, and the resulting cardinality is denumerably infinite unfinished.[18]
3. But if one asks for the possibility to construct a law, which generates a one-to-one correspondence between all real numbers between 0 and 1 (all 'fundamental sequences of digits') and the set of all denumerably infinite ordinal numbers, then the answer is no.

This conclusion is correct, but the proof that Brouwer presented in the inaugural address is a questionable one:

> for this law of correspondence must prescribe in some way a construction of certain denumerably infinite ordinal numbers at each of the successive places of the elementary series; hence there is for each place c_ν a well-defined largest denumerably infinite number α_ν, the construction of which is suggested by that particular place; there is also a well-defined denumerably infinite ordinal number α_ω, greater than all α_ν's and that can not therefore be exceeded by any of the ordinal numbers involved by the law of correspondence; hence the power of that set of ordinal numbers cannot exceed $\aleph_0$.[19]

Brouwer's argument is the following: suppose the law of correspondence exists, then in a dual representation the first dual place has to prescribe two

tussen 0 en 1 kan worden aangegeven, concludeert de formalist tot de voor de intuïtionist zinloze stelling : *'de continue machtigheid, d.w.z. de machtigheid der verzameling der reële getallen tussen 0 en 1, is groter dan aleph-nul',* stelt de vraag, of er verzamelingen van reële getallen tussen 0 en 1 *bestaan,* waarvan de machtigheid kleiner is dan de continue, doch groter dan aleph-nul, m.a.w. *'of de continue machtigheid op één na de kleinste oneindige machtigheid is',* en beschouwt deze vraag, die nog steeds geen oplossing heeft gevonden, als een der moeilijkste en fundamenteelste wiskundige problemen.

Voor de intuïtionist daarentegen is de vraag in de geciteerde vorm zinloos, en, zodra men haar door precisering een zin gegeven heeft, gewoonlijk gemakkelijk te beantwoorden.

[17] See the continuation of Brouwer's *Inaugural address.*

[18] See chapter 7.

[19] ([Dalen 2001], page 189; English text [Benacerraf and Putnam 1983], page 86): immers die correspondentiewet moet bij de opvolgende cijfers der fundamentaalreeks telkens een constructie van aftelbaar oneindige ordinaalgetallen voorschrijven; er is dan bij elke cijferplaats c_ν een welgedefinieerd *grootste* aftelbaar oneindig ordinaalgetal α_ν, waarvan ze de constructie suggereert, en er is een welgedefinieerd aftelbaar oneindig ordinaalgetal α_ω, groter dan alle α_ν's dat door geen der in de correspondentiewet betrokken ordinaalgetallen kan worden overschreden, zodat van de verzameling dezer ordinaalgetallen de machtigheid niet groter kan zijn, dan $\aleph_0$.

ordinal numbers, one of which is the largest, and, in general, the n^{th} dual place has to prescribe 2^n ordinal numbers, with also one as the largest. Hence we would get a denumerable sequence of largest ordinal numbers $a_1, a_2, ...a_n, ...$, so then there certainly should exist an ordinal number a_ω, larger than any from the denumerable sequence. But this would result in a denumerable largest ordinal number in the one-to-one mapping of the real numbers (in a dual representation) into the second number class, which is contradictory, hence there is no such mapping. But remember that Brouwer considered in his argument only the set of all *finite* dual fractions, and concluded from that, that the supposed mapping leads to a contradiction since the set of all *infinite* dual fractions in non-denumerable. Hence either the proof is not correct, or he employs in it a sort of continuity argument, which states that the result of the mapping from the reals into the second number class can be decided at a finite decimal place of the real number to be mapped.

But tools like choice sequences and the continuity principle, that Brouwer had at his disposal in his second intuitionistic period, were not yet available to him in 1912. However, in intuitionistic mathematics the continuum hypothesis completely changed its meaning and became incomparable with the one from 1907 and 1912, since in 1919 Brouwer was able to speak of arbitrary elements of the continuum of the reals, and the continuum could be represented by the perfect spread. The ur-intuition of continuous and discrete remained, but Brouwer's solution to the continuum hypothesis is no longer needed, and the problem itself is not mentioned any more. The continuum hypothesis disappeared from Brouwer's writings, since nothing could be said about it. As a result of the continuity principle, there are numerous non-denumerable sets that cannot be mapped one-to-one, e.g. R_1, R_2, R_3, ... See also Brouwer's *Addenda and Corrigenda*.[20]

5.4 The dissertation again

In the third chapter of his dissertation, in which the (limited) role of logic in mathematics is discussed, Brouwer returned to the continuum problem, this time from a logical point of view. The content of this short discussion shows a strong similarity to that of the relevant part of the inaugural address:

> Now neither the totality of numbers of the second number class, nor the continuum as a system of individualized points, exist mathematically; therefore it appears from what we said above, that the only clear idea which we can find behind this problem, is the following logical theorem, which belongs outside mathematics proper:
>
> *It is possible to introduce as logical entities the totality of numbers of the second number class and the totality of points of the continuum*

[20] [Brouwer 1917b]; see also page 130.

> *in such a way that it is non-contradictory to suppose a one-to-one mapping between them, leaving out no element of one of them.*[21]

But, Brouwer continued, both these totalities can only be defined as a denumerably infinite unfinished set, for which the equivalence applies because of their unfinished character (see chapter 7). However, this has no further consequence for their possible mathematical equivalence as a totality, since mathematically these totalities do not exist, from which it follows that it makes no mathematical sense to speak of their equivalence.

5.5 The notebooks and the continuum problem

In the notebooks only a few remarks are made on the continuum problem; in fact only one page is (almost completely) devoted to it, but several short paragraphs in earlier notebooks give already an indication, without further proof, what the outcome of Brouwer's solution will be, e.g.:

> (V–19) If I have the intuitive continuum, then I can construct in some arbitrary way a continuum of numbers on it, (...).[22]

This quote and other ones tell us that we can construct on the intuitive continuum finite sets or denumerably infinite sets, e.g. the everywhere dense scale of the rationals as the ‘continuum of numbers’. Hence the only possible ‘subset of the intuitive continuum’ that we can construct on it, seems to be a denumerable set. What remains then necessarily has the cardinality of the continuum (see below).

In VII–19 Brouwer presented a proof of the theorem

> ‘every subset of the continuum either is denumerable, or has the cardinality of the continuum’.

This implies that there is no cardinality between the continuum and denumerable (including denumerably unfinished). It follows from the formulation in which the same conclusion in the dissertation is put (with a somewhat more sophisticated proof in the dissertation) that the proof was, for Brouwer, not absolutely conclusive, but nevertheless Hilbert's first problem from the list of 1900 ‘seems to be solved’. However, the continuum hypothesis in its original

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[21] dissertation, page 150, Brouwer's italics: Uit het voorgaande blijkt nu, dat men daarmee, aangezien nòch het geheel der getallen van de tweede getalklasse, nòch het continuüm als systeem van geïndividualiseerde punten wiskundig bestaan, niets duidelijk gedachts kan zoeken, dan de volgende buiten de eigenlijke wiskunde staande logische stelling:

‘Men kan als logische entiteiten invoeren het geheel der getallen van de tweede getalklasse en het geheel der punten van het continuüm zó, dat de aanname dat daartussen een correspondentie één aan één bestaat, waarbij geen enkel element van een van beide buiten die correspondentie valt, niet-contradictoir is’.

[22] Heb ik het intuïtieve continuüm, dan kan ik daar op een of andere willekeurige manier een getallencontinuüm op construeren, (...).

form, viz. $2^{\aleph_0} = \aleph_1$, is of course not proved; such a proof would imply the recognition of the existence of $\aleph_1$, a non-existing cardinality for Brouwer.

The proof in notebook VII proceeds as follows:

> (VII–19) *Proof that every defined subset of the continuum is either denumerable, or has the power of the continuum.*
>
> What I construct is denumerable. If, now, I alternate on the continuum segments *yes* and *no*, then I have to construct one of those sequences of segments, say, A. Then the rest is B.
>
> 1^{st} *case*: A is ordered according to its construction. Then A and B have the power $\aleph_0$ or c, depending on whether they have 'content' or not.
>
> 2^{nd} *case*: A is constructed everywhere dense (or of a type that results from the splitting of elements of the everywhere dense set).[23] Then also B is everywhere dense. That, which has a content, certainly has cardinality c. But that, which has no content is like A of cardinality $\aleph_0$, but like B of cardinality c. For B the segments remain, which are only reached at the ω^{th} decimal in the construction of the rational scale; despite the fact that these segments are merely points, their cardinality remains c; whereas for A only those segments count, that are singled out at a finite decimal place.[24]

The proof amounts to the following:
All I can construct on the continuum are points, or segments (intervals) defined by two points.

First case: I construct on the continuum a denumerable number of points or segments, 'ordered according to their construction', and I call the result of this construction A.[25] If A contains one or more non-empty segments (i.e. has positive measure), then A has the cardinality c of the continuum; If A is only composed of constructed points, then it has the cardinality $\aleph_0$. If, for the

[23] The phrase between the brackets is somewhat puzzling; the condition 'A is everywhere dense' seems sufficient.

[24] *Bewijs, dat elke gedefinieerde deelverzameling van het continuüm is òf aftelbaar, òf heeft de machtigheid* c. Wat ik opbouw is aftelbaar. Ga ik nu het continuüm alterneren in segmenten van wèl en niet, dan moet ik een van die segmenten reeksen, b.v. A, opbouwen. B is dan de rest.

1^{ste} *geval.* A is lopend geordend opgebouwd. Dan hebben A en B de machtigheid $\aleph_0$ of c, naarmate ze geen of wel inhoud hebben.

2^{de} *geval.* A is überall dicht, of volgens een type, dat ontstaat door splitsing van elementen der überall dichte Menge, opgebouwd. Dan is ook B überall dicht. Wie inhoud heeft, is zeker van machtigheid c. Maar wie geen inhoud heeft, is als A van machtigheid $\aleph_0$, maar als B van machtigheid c. Immers aan B blijven de segmenten, die eerst bij de ω^{de} decimaal geraakt worden bij opbouw van de rationale schaal; al zijn dus die segmenten slechts punten, hun machtigheid blijft c; terwijl aan A alleen die segmenten komen, die bij een eindige decimaaltrap worden afgezonderd.

[25] As an example of a point set (not having content), ordered according to its construction, take e.g. $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$. The denumerable η scale is an example of a set which is not ordered according to its construction.

complement B, there also remains at least one non-empty segment, then also B has the cardinality c; if B contains no non-empty segments, then apparently B consists only of boundary points of the denumerable number of non-empty segments that compose A and 'fill' the continuum, and since the number of segments that I constructed is denumerable, the number of its boundary points is also denumerable, hence of cardinality $\aleph_0$.

The second case is, that A is a constructed everywhere dense points set and that it is therefore not ordered according to its construction, e.g. A is the η-scale. In that case also B has to be everywhere dense and does not contain any non-empty segments either; hence, if A is the η-scale, then B is composed of the irrational numbers on the continuum. Then the cardinality of A is $\aleph_0$ since it is constructed and B then has, despite the fact that all its elements are just points (i.e. its measure is zero), the cardinality c, because the elements of A are all, by virtue of their construction, reached at a finite decimal place, and this limitation of finiteness is not applicable to the remaining elements for B.

On the same page of this notebook Brouwer continued with the (not entirely clear) remark that the method of proof presented above can be read in two ways:

> (VII–19) (...) A are the segments that are reached at a finite decimal place; B are the segments that are *'not reached'* at an infinite decimal place (to be read as a positive term). After all, the addition of the infinite decimal place is the postulate for c. But the points that *'are not reached'* and that *'are reached'* at an infinite decimal place, are the same. And by postulate I can add them to A. B then has the cardinality ω or 0, depending on whether or not I add the boundary points to B.[26]

Note that the proof from the notebooks is different from the one that was presented in the dissertation, but that both hinge upon the same concept of which sets can be constructed on the continuum.

However, in the addition to the given proof (the last quote of VII–19), it is questionable how it can be decided that a point is reached at an infinite decimal place, and even how this can be decided for a finite decimal place. The criterium for the 'infinite decimal place' was rejected earlier, and again on the next page VII–20 Brouwer returned to this, when criticizing Cantor's method of deduction of the cardinality of the continuum:

> (VII–20) In his deduction of: cardinality of the continuum $= 2^{\aleph_0}$, Cantor forgets that one cannot substract *all* rational numbers from

[26] (...) A zijn de segmenten, die bij een eindige decimaal bereikt worden; B zijn de segmenten, die bij oneindige decimaal *'niet bereikt'* (als een positieve term gelezen) worden. Immers die toevoeging van de oneindige decimaal is het postulaat voor c. Maar de punten, die bij oneindige decimaal *'niet bereikt'* of *'wel bereikt'* worden zijn dezelfde. En ik kan ze bij postulaat even goed bij A voegen. En dan wordt B van machtigheid ω *of 0* (naarmate ik de grenspunten van de segmenten bij B tel of niet.

> *all* real numbers. They are things of a different kind: the first ones I construct, the last ones are chances in nature. And in the sense in which I can add, without change in cardinality, to the group of all rational numbers plus something all rational numbers without change in cardinality: in that sense I cannot conceive a cardinality of the continuum.[27]

A real number is only a 'chance in nature', another important and promising idea in the light of later intuitionistic developments.

In the solution, given in VII–19, Brouwer apparently departed from a continuum on which points and intervals are subsequently constructed, whereas in the dissertation the point of departure is a set, constructed according to rule 2 or 3, i.e. a dense point set which is completed to a continuum.

5.6 Concluding remarks

The conclusion of this chapter can be a rather short one; the fact that Brouwer rejected Cantor's cardinality $\aleph_1$ and all higher cardinalities, makes his solution to the continuum problem a trivial one. One can construct on the continuum either points or intervals as subsets, and nothing else. In the first case the cardinality of the constructed set is denumerably infinite at the most, in the second case it is that of the continuum.

In this light the solution from the notebooks is the most direct and constructive one, since a set is only properly defined if an algorithm for the construction of its elements is given. In the dissertation, on the other hand, a subset of the continuum is given and we have to figure out its cardinality. Well, we construct a denumerable subset on it, and we construct also a denumerable subset on a continuum, and compare the results: the outcome is almost self-evident.

One can be in doubt as to the reason for the relatively great difference between the two proofs, and why the proof from the dissertation is completely absent in the notebooks. We hypothesized about this already in the introduction to this chapter.

And, finally, we emphasize once more that Brouwer did not compare the cardinalities $2^{\aleph_0}$ and $\aleph_1$ (Hilbert's problem). But how could he, $\aleph_0$ and c being the only existing infinite cardinalities for him.[28]

[27] Cantor in zijn afleiding van: machtigheid continuüm = $2^{\aleph_0}$ vergeet, dat je niet *alle* rationale getallen mag aftrekken van alle reële getallen. Het zijn ongelijksoortige dingen: de eerste bouw ik op, de laatste zijn kansen in de natuur. En in de zin, waarin ik zonder verandering der machtigheid bij de groep der rationale getallen plus iets nog wel eens alle rationale getallen mag optellen zonder verandering der machtigheid: in die zin kan ik mij niet een machtigheid van continuüm denken.

[28] The denumerably infinite unfinished cardinality (see chapter 7) is recognized as a method rather than as a cardinality proper.

Chapter 6

Mathematics and experience

6.1 Introduction

In contrast to the first chapter of Brouwer's dissertation, in which the mathematical edifice is constructed departing from the ur-intuition alone, and in contrast to its third chapter, where the role of logic in the mathematical construction is investigated, the second chapter treats the part that, in Brouwer's view, mathematics plays in daily life. Man exploits mathematics in order to control his life and to gain power over the world that surrounds him. He can accomplish this, owing to his ability to 'take a mathematical view of his life'.

In Brouwer's second chapter, and especially in the first half of it, the tone sometimes reminds us of one of his earliest publications, *Life, Art and Mysticism*, which is composed of a series of lectures held for students of the Institute of Technology of Delft.[1] Especially in the 'rejected parts' (i.e. the parts that Brouwer originally wrote for his dissertation, but which fell victim to Korteweg's veto) we often recognize its rather pessimistic tone. A considerable share of the 'rejected parts' belonged to Brouwer's draft of the second chapter.

One can wonder what Brouwer had in mind with this chapter, which stands in such a sharp contrast with the other two, and which seems to be, especially the first half of it, so unrelated to the construction of pure mathematics. We can find an answer to this question in his letter to Korteweg, dated 7^{th} November 1906:

> With reference to our discussions on Sunday may I add that the purpose of chapter 2 is:
>
> a. to explain *how the mathematical experience accompanies all essentially-human acting*;

[1] The 'Technische Hogeschool', cf. [Dalen, D. van 1999], § 2.6.

> b. and with reference to the preceding: to investigate *to what extent experience-based mathematics can be a-priori*, in particular whether space and time are *both* a-priori.[2]

In answer to the question why this chapter is discussed in our dissertation, instead of completely omitting it, we claim that it contains foundationally relevant material, and that it expresses Brouwer's global philosophy. This has significant consequences for his ideas on objectivity, on natural sciences and on apriority.

Not all parts in Brouwer's second chapter are of equal importance to us. Some fall outside the scope of this dissertation (but are sometimes mentioned for completeness' sake), and therefore we will make a selection.

In the present chapter we will see that Brouwer's idea of physics, its role in our world and its importance to mankind, is different from that of the average physicist. According to Brouwer, man can take a mathematical view of his life, a view in the form of causal sequences, and he does this for the sole purpose of increasing and maintaining his power over the surrounding world and over his fellow men. Causality is imposed instead of discovered, and the same applies to continuity and differentiability of the describing functions. This latter is not an uncommon position in the philosophy of science, but Brouwer's opinion that we force laws upon nature as a means to rule and to control (and that this is judged as most negative and sinful) distinguishes him from the mainstream of physical practice.

This attitude is a consequence of his general view on mankind, which was expressed in his mentioned booklet *Life, Art and Mysticism* and we will, for a better understanding of his ideas, pay some attention to this work.

Brouwer is also of the opinion that the laws, which describe phenomena in nature, are in fact just laws describing our measuring instruments. This, too, is not a common view on the nature of physical laws and this view asks for an analysis and a comment. In this comment we will try to do justice to Brouwer's standpoint.

On the basis of Poincaré's *La Science et l'Hypothèse*, Brouwer discussed topics like 'the meaning of a physical theory' and 'the value of an explanation', and, on the basis of Russell's *An Essay on the Foundations of Geometry*, concepts like 'objectivity' and 'apriority of space and time'. We will argue that Brouwer's interpretation of these terms should be understood from the angle of his solipsism.

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[2] [Stigt 1990], page 492; Dutch text [Dalen 2001], page 15: Mag ik u, naar aanleiding van het zondag besprokene, nog opmerken, dat de bedoeling van het tweede hoofdstuk is a) toelichting, *hoe de wiskundige ervaring het essentieel menselijke handelen begeleidt*, en b) naar aanleiding van het vorige: onderzoek, *in hoeverre ervaringswiskunde a priori kan zijn*, in het bijzonder, of ruimte en tijd *beide* a priori zijn.

6.2 Man's desire for knowledge and control

The second chapter opens with the following significant paragraph, which sets the tone for the next eight pages:

> Proper to man is a faculty which accompanies all his interactions with nature, namely the faculty of taking a *mathematical view* of his life, of observing in the world repetitions of sequences of events, i.e. of causal systems in time. The basic phenomenon therein is the simple intuition of time, in which repetition is possible in the form: 'thing in time and again thing', as a consequence of which moments of life break up into sequences of things which differ qualitatively. These sequences thereupon concentrate in the intellect into mathematical sequences, not *sensed* but *observed*.[3]

Man first of all observes recurring sequences of events; after that, the sequences of successive events are interpreted in his imagination to be '*causal* sequences', especially when such sequences always lead to the same result when departing from the same initial state. Because of the recurrence of such sequences man *believes* that the events taking place later in time are *caused* by the earlier ones. This, in fact, is a normal procedure in the practice of physical research, and certainly not typical for Brouwer. A physical theory is based on sequences of observed events, in which regularity and a pattern is noticed which lends itself to a mathematical description. In general, the content of the second chapter reflects the methods of mathematical physics of Brouwer's days in a faithful way. Brouwer's deviating and mysticism-based views can be found in the dissertation 'between the lines', but elsewhere, in the rejected parts and in

[3]dissertation, page 81: De mensen is een vermogen eigen dat al hun wisselwerkingen met de natuur begeleidt, het vermogen n.l. tot *wiskundig bekijken* van hun leven, tot het zien in de wereld van herhalingen van volgreeksen, van causale systemen in de tijd. Het oer-fenomeen is daarbij de tijdsintuïtie zonder meer, waarin herhaling als 'ding in de tijd en nog eens ding' mogelijk is, en op grond waarvan levensmomenten uiteenvallen als volgreeksen van kwalitatief verschillende dingen; die vervolgens zich in het intellect concentreren tot niet *gevoelde*, doch *waargenomen* wiskundige volgreeksen.

A handwritten correction to this text was made, in which Brouwer changed the phrase

> (..) of observing in the world repetitions of sequences of events, i.e. of causal systems in time.

into:

> of observing in the world sequences of events, repeating themselves as causal systems in time.

(in Dutch: 'tot het zien in de wereld van volgreeksen, zich herhalend als causale systemen in de tijd'.)

By this rewording Brouwer emphasized that the definition of a causal sequence is not 'a sequence of events, repeating itself more than once', but that, thanks to the repetition, *it forces itself on us as being causal.* Rightly or wrongly we *suppose* that, as a rule, real *causal* sequences, when departing from the same initial state, will proceed in the same way and will lead to the same final result. For that reason every such sequence gives us the impression of being causal.

Note that these events now have content, and are no longer abstractions of experiences.

some other papers from a later date, more openly, albeit in later time not in the extreme pessimistic tone larded with terms like 'sin' and 'sinful desire' from his early days.

For instance, we recognize his interpretation of a mathematical viewing of life in the first of the two *Wiener Gastvorlesungen* from 1928.[4] See also (and especially) the lecture *Will, Knowledge and Speech*, held in 1932:

> Mathematical viewing is an attitude which man has adopted in his struggle for existence. It comes into being in two phases, the phase of becoming-aware-of-time and the phase of causal attention.
>
> The *becoming-aware-of-time* is the fundamental event of the intellect: a moment of life falls apart into two qualitatively different things of which the one gives way to the other but is retained by memory.
>
> (...)
>
> *Causal attention* is an act of human imagination linking and identifying different sequences of phenomena; such a phantasy is called a *causal sequence*.[5]

In this paper Brouwer once more emphasized the fact that both, the mathematical attention *and* the causal attention, are not a necessity, but a phenomenon of life, subject to the free will. The separation between the self, and the world of perception which comes to us in the form of sequences of successive events, is brought about by man's own power; hence on the basis of this argument there need not be at all any form of causal coherence in the world, independent of man.[6]

Immediately following the quoted opening paragraph of this chapter, the text of the dissertation now continues with a short discussion on man's ability to influence and to control his surrounding world. He notices that, by interfering in an early stage of the development of a sequence, the final result changes too. If a specific final result is desired, but an intermediate goal, being a necessary station to be passed on the way to that desired result, is easier attainable via a slightly diverging route, then an intervention in an early stage of a causal sequence of events will be directed towards this intermediate goal, as a means to the final desired situation.

[4] [Brouwer 1929].

[5] [Brouwer 1933]. For the English text see [Stigt 1990] appendix 5, page 418. For the original Dutch text: [Brouwer 1933], page 177, 178: Wiskundige beschouwing is een in de strijd om het bestaan aangenomen houding, die in twee fasen tot stand komt, de fase der *tijdsgewaarwording* en de fase der *causale aandacht*.

De *tijdsgewaarwording* is het grondgebeuren van het intellect: uiteenvalling van een levensmoment in twee qualitatief verschillende dingen, waarvan het ene voor het andere terugtreedt en niettemin door herinnering wordt vastgehouden.

(...)

Causale aandacht vervolgens is *fantasering der identificering van verschillende verschijnselreeksen*, en een zodanige fantasie wordt een *causale reeks* genoemd.

[6] [Stigt 1990], page 418, 419.

In order to make an early intervention successful, man observes as many as possible of those causal sequences. In this way he obtains a large stock of sequences which makes, in case of some desired goal, a proper choice for the best one easier:

> And human behaviour includes attempts to observe as many of these mathematical sequences as possible, in order, whenever in the real world intervention at an earlier member of such a sequence seems more successful than at a later member, to choose the earlier one as a guide for his actions, even when his instinct is only affected by the later one. (Substitution of the *means* for the *end*.) Nevertheless the non-instinctive nature of his intellectual action renders the certainty that the parts of a sequence really belong together, anything but perfect. Consequently it can always be falsified, which is observed in the discovery that 'the rule no longer applies'.[7]

Mankind thus obtains his power over the world, and this is an ability which was, on moral grounds, judged as most negative in *Life, Art and Mysticism* since it removes him from his natural state and from his natural destiny. The pessimistic tone which we meet in this early publication and in the rejected parts (but not in the dissertation, or at the most in a latent form) appears again in the mentioned lecture *Will, Knowledge and Speech*, which shows that despite Korteweg's intervention the 'old' ideas are still there, though in a slightly milder form. In this lecture Brouwer made the following remark about causality:

> (...); there can therefore be no question of a causal coherence of the world independent of man. On the contrary, the so-called causal coherence of the world is a dark force of human thought serving a dark function of the will of mankind, which it uses like a cloud of stupefying gas, in an attempt to make the world defenseless and ready to be assaulted by its desires.[8]

In order to maintain power over his environment, man needs certainty of regularities; the dissertation continues as follows about this desired certainty:

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[7] dissertation, page 81, 82: En het levensgedrag der mensen zoekt zoveel mogelijk van die wiskundige volgreeksen te kunnen waarnemen, om telkens, waar in de werkelijkheid bij een vroeger element van zulk een reeks met meer succes schijnt te kunnen worden ingegrepen, dan bij een later, ook dan, wanneer alleen bij dat latere het instinct wordt aangedaan, het eerste te kiezen als richting voor hun daden. (Vervanging van het *doel* door het *middel*.) Het oninstinctieve van deze intellectuele handeling maakt echter de zekerheid dat werkelijk de delen ener volgreeks bijeen behoren alles behalve volkomen, zodat ze steeds kan worden gelogenstraft, wat waargenomen wordt als de ontdekking 'dat de regel niet langer doorgaat'.

[8] [Stigt 1990], page 419; Dutch text [Brouwer 1933], page 179: (...) kan er derhalve van het bestaan van een causale samenhang der wereld onafhankelijk van de mens geen sprake zijn. Integendeel, de zogenaamde causale samenhang der wereld is een donkere kracht der gedachte in dienst ener donkere wilsfunctie der mensheid, die daardoor, als door afwolking van een bedwelmend gas, de aanschouwingswereld tegenover haar weerloos en voor haar verlangen stormrijp tracht te maken.

> In order to maintain as long as possible the certainty of an observed regularity, one tries to *isolate* systems, i.e. to exclude those observations which disturb the regularity; in such a way man *makes* far more regularity in nature than originally occurred spontaneously; he *desires* this regularity, because it strengthens him in the struggle for life, rendering him capable of predicting, and taking action.[9]

But, as we saw, man goes further in his quest for knowledge and power: independent of any direct or even possible applicability, he has already created his own stock of pure theoretical and for the time being non-materialized causal sequences, just waiting for an opportunity to apply them in his daily practice. One may ask if Brouwer is now referring to pure mathematics, or to some form of theoretical physics based on earlier and successful causal sequences. In view of the subsequent argument, the former seems to be the case:

> Here it must be remembered that in those mathematical systems in which no time-coordinate appears, it is found that in practical applications all their relations still become relations in time. Thus, for instance, Euclidean geometry when applied to reality gives the causal relations between the results of different measurements executed with the aid of the group of rigid bodies.[10]

Therefore only a small number of sequences from that stock (or, before viewing these causal relations in time, a small number of these mathematical substructures) represent causal sequences in reality.

But man's power and his ability to knowledge (and therewith to control) still increase by the further abstraction from direct experience to a more general viewpoint, by the creation of theories and laws:

> Mathematical natural science does not, however, derive its great power solely from the observation of sequences which are approximately equivalent for the instinct, but from combining a very large number of such sequences from one point of view by means of a mathematical system built up with the aid of mathematical induction. Such a system is called a *law*.[11]

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[9]dissertation, page 82: Om de zekerheid van een waargenomen regelmaat zo lang mogelijk te handhaven, tracht men daarbij systemen te *isoleren* d.w.z. het als regelmaat storend waargenomene, verwijderd te houden; zo *maakt* de mens in de natuur veel meer regelmatigheid dan er oorspronkelijk spontaan in voorkwam; hij *wenst* die regelmatigheid, omdat ze hem sterkt in de strijd om het bestaan, doordat ze hem in staat stelt te voorspellen, en zijn maatregelen te nemen.

[10]dissertation, page 83: Men bedenke hierbij dat de wiskundige systemen, waarin geen tijdcoördinaat voorkomt, bij practische toepassing toch al hun relaties tot causale relaties in de tijd zien worden. Zo b.v. de Euclidische meetkunde geeft, op de werkelijkheid toegepast, het causaal verband tussen de resultaten van verschillende metingen, met behulp van de groep der rigide lichamen uitgevoerd.

[11]dissertation, page 84: Haar grote macht krijgt echter de wiskundige natuurwetenschap nog niet door het opmerken van voor het instinct ongeveer gelijkwaardige volgreeksen, maar

In that context individual causal sequences only differ in the values of the relevant parameters, and the ones that occur in nature are the *coincidental occurrences.*

And man still goes on: Brouwer described on page 85 how discrete observations, by the interpolation between these observations, are completed to continuous functions, not only with respect to the time coordinate, but with respect to every functional variable. This completion to a continuous function finds its justification in the fact that 'it works'. By the act of interpolation between the observed values, the resulting functions become analytic; this type of functions has a strong preference since they are differentiable, which is a desired property in the 'anthropomorphisation' of nature; that is, man decides for that type of functions and is fully satisfied with them, because only gradual changes in a desired direction apply at a time, hence a similar behaviour of the describing function in closely adjacent argument values is a requirement. Analytic functions meet this requirement, and apparently nature obeys laws which can be appropriately described by this type of functions. This, again, is common practice in physical research, although continuous or analytic functions are not always the proper ones to describe and to predict. Nature sometimes resists or refuses a 'smooth' continuous description.

At first sight it may seem to the unsuspected reader that Brouwer mentioned the term *analytic* just in passing, but on closer examination it becomes obvious that the term is used in its strict mathematical sense.[12] For instance, when Brouwer stated on page 87 of his dissertation that for functions in nature which can be measured in practice, thereby starting from the postulate of 'nearly equal behaviour in closely adjacent argument values', one can prove the existence of all higher derivatives. In a footnote he added that this, by itself, does not give the certainty that the resulting function is analytic, thereby referring to Pringsheim in the *Mathematische Annalen* 44 (1894).[13]

Beyond this, Brouwer did not refer to analytic functions any further in his dissertation, but in the notebooks there are several references to this type of function, clearly in the strict mathematical sense, as for instance in notebook seven:

> (VII–7) We only postulate the functions in nature to be *continuous*, and that is merely human externalization.
>
> But that imitation from our own creation, the 'analytic' function, has in Faraday's physics no value.[14]

door het samenvatten van een zeer groot aantal van zulke volgreeksen onder één gezichtspunt door middel van een met behulp van mathematische inductie opgebouwd wiskundig systeem, dat *wet* wordt genoemd.

[12] A function is *analytic* in a point a if it can be locally (i.e. in an interval around a) expanded in a Taylor series.

[13] Pringsheim's paper *Ueber Funktionen, welche in gewissen Punkten endliche Differentialquotienten jeder endlichen Ordnung, aber keine Taylor'sche Reihenentwickelung besitzen*, M.A. 44 (1894), page 41 - 56.

[14] De functies der natuur postuleren we alleen *continu* en dat is zuivere mensenveruiterlijking.

In the notebooks Brouwer strongly approved of Faraday's physics as being purely descriptive, contrary to the theoretical approach by, for instance, Lorentz or Maxwell.[15]

Also in the eighth notebook, again with the 'old' negative undertone and with the same purpose of 'ruling' and 'controlling':

> (VIII–11) Function theory is only concerned with very special functions: the analytic functions; but that does not matter: it is strict, that is, it is a free and self constructed building, which intends to imitate a part of nature, in order to control it, (...)[16]

And in the ninth, with the same goal:

> (IX–17) We construct, within certain constraints, analytic functions, which of course can approximate *every* empirical function, (...) since we postulate the functions of nature such that they can be interpolated indefinitely.[17]

6.2.1 Comments on Brouwer's views on physics

Brouwer claimed that man has a faculty of 'taking a mathematical view of his life', that is, to view the sequences of events that allow a mathematical description as causal sequences. And as a result of the possibility to intervene in such causal sequences, mankind derives his power over the surrounding world. Brouwer's description in the dissertation of the praxis of physical research can be seen as a lifelike one, but his view on the *aim* of the natural sciences seems to be that they just serve as a tool to obtain power over our environment and to control it. The more positive vision of a physical practice as one of fundamental research and creation of models, guided by and born out of pure curiosity or admiration, is not the one that, at first sight, appears from his writings. True, in its most primitive state the human race in the low countries, constantly threatened by floods, tried in a combined effort to ensure itself against this threat, but after this early period the history of modern science is filled with examples of curiosity as the only motivation for fundamental research. Of course man employs a great number of his discoveries for his own benefit, but usually technical applications come only after fundamental discoveries, and moreover, this occurs in a limited number of cases only. Certainly in his early years, Brouwer identified fundamental discoveries with their possible but relatively rare applications. This was a direct result of his philosophy, which appeared in

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Maar dat na-bouwen uit onze eigen schepping: de 'analytische' functies, is Faraday-fysisch zonder waarde.

[15]See the next section for a comment on this rather diverging view.

[16]De functieleer houdt zich slechts met zeer bijzondere functies bezig, de analytische; maar dat hindert niet: het is streng, d.w.z. een vrij zelfgeschapen bouw-werk, dat een gedeelte van de natuur wil nabouwen, om er vat op te hebben, (...)

[17]Wij bouwen binnen bepaalde grenzen analytische functies, die natuurlijk elke empirische functie kunnen benaderen, (...), immers we postuleren de functies der natuur onbegrensd interpoleerbaar.

its strongest form during the years in which he was maturing intellectually. The attitude, expressed in *Life, Art and Mysticism*, reappeared in his draft for the second chapter of his dissertation that we are discussing now; about this draft Korteweg commented in a letter, dated 11^{th} November 1906 as follows:

> (...)
>
> Therefore only chapter two remains.
>
> After receiving your letter I have again considered whether I could accept chapter two as it stands. But honestly, Brouwer, I cannot. I find it all interwoven with a kind of pessimism and mystical attitude to life which is not mathematics and has nothing to do with the foundations of mathematics. In your mind it may well have grown together with mathematics, but that is wholly subjective. (...) I am convinced that any promotor, young or old, sharing this philosophy of life or not, would object to it being included in a mathematical dissertation.[18]

Indeed if we compare Brouwer's original draft for the beginning of the second chapter with the earlier quoted final result, the difference is striking and the text of the draft indeed reminds us of the content of his early writings. We easily can imagine Korteweg's objections and his resistance to its inclusion in a mathematical dissertation when he was reading the following lines:

> All human life originated in a one-sided construction of nature and has protracted its existence in an 'externalization', man impregnating nature with the human self and repressing other one-sided developments. ([*footnote:*] This externalization of life and the holding off of death, from the point of view of religion, reflects a lack of wisdom and the absence of a bond with the universe. Moreover, this externalization, the will to destroy and rule, immediately obstructs any nourishing of the heart by nature. Those who rule are already damned and damned are those qualities that promote man's rule.)
>
> This externalization by man, making his environment subservient to the full development of his humanity, appears to us ([*footnote:*] If we view the world intellectually, i.e. with a mathematical causal eye.) as a process whereby nature itself becomes linear and regular and all other life repressed or adapted to mankind. ([*footnote:*] Since the adaption of the environment leads human life further away

[18] See e.g. [Stigt 1990], page 90 and for the original Dutch version [Dalen 2001], page 17: (...) Blijft dus alleen het tweede hoofdstuk. Na ontvangst van uw brief heb ik nogmaals overwogen of ik dit zoals het daar ligt accepteren kan. Maar waarlijk Brouwer het gaat niet. Daarin ligt een soort pessimistische en mystieke levensbeschouwing ingevlochten die geen wiskunde meer is en ook met de grondslagen der wiskunde niet te maken heeft. Zij moge in uw geest hier en daar met wiskunde zijn samengegroeid; maar dat is dan geheel subjectief. (...) Ik ben overtuigd dat iedere promotor, jong of oud, die levensbeschouwing delende of niet, tegen de opname daarvan in een wiskundige dissertatie bezwaar zou maken.

> from the natural state which originally supported him, this conquered and adapted environment will ultimately become intolerable to mankind.)
>
> What then is the nature of this human externalization which evidently is so much more powerful than the brute assimilation and destruction practised by other creatures? We feel linearity and regularity, for example, also in bees; there it does not result in any sort of special power. But man has the faculty, accompanying all his interactions with nature, of objectifying the world, of seeing in the world causal systems in time. ([*footnote:*] This 'seeing', however, is a human act of externalization: there is no real *existence* of objective natural phenomena as can be ascribed to nature itself: the seeing originates in man, is an expression of man's will alone, independent of nature which itself exists independent of man's will.)[19])

At the end of this long passage we recognize the text of the beginning of chapter two of the dissertation. We quoted this rejected part at length to emphasize its gloomy tone and the negative role that Brouwer granted to physics in human life. Pure physics, that is the search for the building blocks of nature, was in Brouwer's (early) opinion identified with its worst imaginable application: to rule and to submit our environment and its inhabitants. Man is no longer a discovering subject in awe, trying to understand nature and its building blocks *for the sake of understanding only*, but he is a ruling subject, and for that purpose mathematics is employed to force a world into mathematical equations, thus enabling man to manipulate.

[19][Stigt 1979], or [Stigt 1990] page 405 or, for the original Dutch text [Dalen 2001], page 29: Alle leven, na ontstaan te zijn als vereenzijdiging van de natuur, rekt zijn bestaan in een 'veruiterlijking', een doordringing van de natuur met zichzelf, in terugdringing van andere vereenzijdigingen. ([footnote] Dát het zich veruiterlijkt, en niet te sterven legt, wordt als gemis van wijsheid, als gemis van verband met het Àl, door religie gevoeld. – Overigens snijdt het zich veruiterlijken, het willen vernietigen of willen heersen, meteen af van alle voeding van het hart uit de natuur. – Wie heerst, is reeds gevloekt, en het zijn gevloekte eigenschappen, die helpen tot heersen.)

De veruiterlijking der mensheid, het dienstbaar maken van de omgeving aan de ontplooiing der menselijkheid, verschijnt ons ([footnote] n.l. als wij intellectueel, met wiskundig-causale blik de wereld bekijken.) als een rechtlijnig en regelmatig worden van de natuur; waardoor al het andere leven wordt teruggedrongen of aan de mensheid aangepast. ([footnote] Daar het aanpassen van het milieu het steeds verder afvoert van de natuurtoestand, zoals die oorspronkelijk de mensheid droeg, wordt elk overwonnen en aangepast milieu ten slotte voor de mensheid zelf onhoudbaar.)

Wat is nu het essentiële van de menselijke veruiterlijking, dat zoveel machtiger blijkt, dan de brutale assimilatie of vernieling, die van andere schepselen uitgaat? De rechtlijnigheid en regelmatigheid komt b.v. ook bij de bijen voor, maar daar brengt ze generlei bijzondere macht. Maar de mens heeft een vermogen, dat al zijn wisselwerking met de natuur begeleidt, het vermogen n.l. tot objectivering der wereld, het zien in de wereld van herhalingen van volgreeksen, het zien in de wereld van causale systemen in de tijd. ([footnote] Dat zien is intussen niets, dan een *daad* van veruiterlijking; van een *bestaan* der objectieve verschijnselen der natuur, in dezelfde zin als van een betaan der natuur zonder meer, is geen sprake: het zien gaat uit van de bekijker, is een *wilsuiting* van de bekijker alleen, buiten de natuur om, die zelf bestaat voor het subject buiten zijn *wil* om.

The attitude towards nature and towards the world, which appeared from his early writings, is a direct consequence of Brouwer's mystic bias with its solipsistic character, which we find in its most pronounced form in the mantioned *Life, Art and Mysticism.* It will be useful to pay some attention to Brouwer's mystical monograph, as it is the background for many of his early convictions and reflections. In the first chapter of this essay the human predicament is sketched, which was caused by the abandonment of man's natural destiny: a complete introspection, an unconditional 'turning into oneself'. Not surprisingly the only remedy for mankind is a return to this natural state:

> (*from chapter 1*) The Netherlands came into existence and was preserved by the deposit of silt of the rivers; a balance between the dunes, the delta, the tides and the discharge of water was established – a balance in which temporary floodings of parts of the delta were incorporated. And in that land a strong human race could live and endure.
>
> (...)
>
> The people originally lived separated, and each tried to preserve for himself his balance in the supporting environment of nature, amidst sinful seductions; thàt filled their lives, no interest in each other, no worry about the morrow. Hence, also, no work and no grief; no hatred, no fear; also no pleasure. Meanwhile, one was not content; one sought power over each other, and certainty about the future. Thus the equilibrium was destroyed, ever more sore labour for the suppressed, ever more infernal conspiracies for the rulers, and all are the suppressed and the rulers at the same time; and the old instinct of separation lingers on as pale envy and jealousy.
>
> (...)
>
> (*from chapter 2, about the possible remedy*) If it is nevertheless given to you to overcome all inertia, and to continue, then the passions become silent; you feel yourself pass away from the old exterior world, from time and space and all other manifold things. And the eyes of a joyous silence, which are no longer tied, open up.
>
> (...)
>
> You recognize your 'Free Will', in sofar as it was free to withdraw itself from the world, in which there was causality, and then remains free, and yet only then has a really determined Direction, which it reversibly follows in freedom.[20]

[20](*from capter 1*) Nederland ontstond en werd in stand gehouden door het afslibsel der rivieren; er vormde zich een evenwicht van duinen, delta, getijden en afwatering, een evenwicht, waarin mee waren opgenomen tijdelijke overstromingen van gedeelten der delta. En in dat land kon leven en voortleven een krachtig mensengeslacht.

(...)

De mensen leefden uit oorsprong gescheiden, en ieder voor zich zocht te houden zijn even-

The picture we are left with from these quotes is one of pessimism and solipsism. Man's destiny is one of introspection, of being completely on his own. He has to accept life and the world as it happens to him. Every act to improve his environment, especially when this is performed in cooperation with others, is a sinful act. In that light we have to interpret Brouwer's early attitude towards the natural sciences.

True, under Korteweg's influence the tone became more moderate and true-to-life, but the pessimistic undertone remained. Whereas most physicists will rather agree with Galileo's famous dictum:

> Philosophy is written in that great book which continually lies open before us (I mean the Universe). But one cannot understand this book until one has learned to understand the language and to know the letters in which it is written. It is written in the language of mathematics, and the letters are triangles, circles and other geometric figures. Without these means it is impossible for mankind to understand a single word; without these means there is only vain stumbling in a dark labyrinth.[21]

which is usually remembered in its brief form *'the book of nature is written in the language of mathematics'*, for Brouwer the surrounding world is forced into a mathematical form; this on itself is of course normal physical practice, but for Brouwer this was not done as a means to learn and to discover it, but to let it be ruled and manipulated by a mankind who has forgotten his natural destiny. This attitude, though not present in the dissertation and becoming much milder in later years in comparison with 1905, never completely disappeared; it remained one of the foundational aspects of his philosophy.

When reading the relevant pages in the dissertation or in *Will, Knowledge and Speech* there seems to be, however, at first sight a certain similarity between

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wicht in de dragende natuur tussen zondige verleidingen; dát vulde hun leven, geen belangstelling in elkander, geen zorg om de dag van morgen. Dus ook geen werk, en geen verdriet; geen haat, geen vrees; ook geen genot. Intussen, men was niet tevreden; macht zocht men over elkaar, en zekerheid over de toekomst. Zo werd het evenwicht verbroken, steeds pijnlijker arbeid voor de onderdrukten, steeds helser samenspanning voor de machthebbers, en allen zijn onderdrukten en machthebbers tegelijk; en het oude instinct van scheiding leeft voort als bleke nijd en jalouzie.

(...)

(*from chapter 2*) Wordt het u niettemin gegeven, alle traagheid te overwinnen, en voort te gaan, dan gaan de hartstochten zwijgen, ge voelt u afsterven van de oude aanschouwingswereld, van tijd en ruimte en alle andere veelheid, en die niet langer gebonden ogen ener blijde stilte gaan open.

(...)

Ge erkent uw 'Vrije Wil', in zoverre hij vrij was, zich te onttrekken aan de wereld, waarin causaliteit was, en dan vrij blijft, en toch eerst dán een recht bepaalde Richting heeft; die hij in vrijheid, omkeerbaar volgt.

The English translation is by D. van Dalen, see [Dalen, D. van 1999], page 66 – 68. The content of chapter 2, section 2.6 of this book gives a better and a more complete picture of Brouwer's attitude. For an integral English translation of this early work see [Stigt 1996].

[21] From his *Il saggiatore nel quale con bilancia esquisita ...*, translation by J. Summers. See also [Galileo 1957] or [Galileo 1960].

Brouwer's view and by Galileo's dictum just quoted, in particular in regard to the mathematical techniques. For instance the last two paragraphs of section two of *Will, Knowledge and Speech* read as follows:

> Some scientific theories are coined the 'theories of exact science'. They are those theories which first of all refer to causal sequences that are particularly stable, either because they are perceived as laws of nature or because they are artificially called into being as technical facts; secondly, their hypothesis achieves a very considerable simplification; thirdly, the causal sequences in them that are to be governed correspond to special values of numerical parameters whose whole domain is present in the more extended mathematical system of the hypothesis.
>
> It is in these exact-scientific theories in particular that the phenomenon of the heuristic character of scientific hypotheses becomes evident; this consists in discovering behind sequences –which were originally added as hypotheses– corresponding real causal sequences in the perceptional world.[22]

But, following this quote, Brouwer expressed a different intention. At the end of the first section of *Will, Knowledge and Speech* Brouwer spoke of mankind who, even at the lowest levels of civilization, attempts 'to stabilize his range of causal influence and to create an ordered domain under his power'. And two paragraphs earlier he referred to the mathematical attention of which the only justification 'lies in the expediency of the 'mathematical act', which is based on it and which is within the grasp of man because of his causal attention.' Man notices an element in a causal sequence of events, of which the last event is some desired goal that cannot be achieved directly, and for that reason man shifts his attention to an earlier element in the sequence and selects that one as a provisional goal, as a means to the originally desired end. One can of course interpret the phrase 'to create an ordered domain under his power' in the sense of isolating sequences of events for a more directed research, which is, again, a common way of doing physical research, but the quoted paragraph on our page 183 clearly speaks of a defenseless world, ready to be assaulted by man's transformation of a sequence of events into a causal one.

For Galileo the use of the mathematical language as a successful means to describe nature is an astonishing discovery, the 'book of nature' appears to be

[22] [Stigt 1990], page 420; in Dutch [Brouwer 1933], page 180: Speciaal tot *exactwetenschappelijke theorieën* worden zodanige wetenschappelijke theorieën gestempeld, die ten eerste op zeer bijzonder stabiele, hetzij als natuurwetten *waargenomen*, hetzij als technische feiten *kunstmatig in het leven geroepen*, causale reeksen betrekking hebben; door welker hypothesen ten tweede een zeer aanzienlijke vereenvoudiging wordt bereikt; en waarin ten derde de te beheersen causale reeksen aan speciale waarden van *numerieke parameters* beantwoorden, waarvan het volledige waardengebied in het hypothetische uitgebreidere wiskundige systeem aanwezig is. Het is in het bijzonder bij de exactwetenschappelijke theorieën, dat het verschijnsel van het *heuristische karakter van wetenschappelijke hypothesen* aan de dag treedt, hierin bestaande dat bij oorspronkelijk als hypothetisch ingevoegde reeksen achteraf daaraan beantwoordende werkelijke causale reeksen der aanschouwingswereld ontdekt worden.

written in that language, whereas for Brouwer it is an *act* by which man *forces* nature into a mathematical model, with the sole aim to control. And this aim is man's sinful attitude, by which he strays from his natural state and destiny.

Well, of course the creation of a mathematical model for the description of nature is a human act, and to that extent physicists indeed force nature into such a model. As long as it works, we can make progress in the development of the natural sciences, and in case of failure, which is of course very likely to occur in a continued development, we do not accept the conclusion that nature cannot be forced into our mathematical model, but we attempt to create an improved or a new model instead, in which the successful part of the old model is included as a particular case. Hence the main difference between Galileo and Brouwer is that, although they both successfully apply the same mathematical technique (apart from 300 years development in those techniques), the discovery of a fruitful application of mathematics to causal sequences was for the former an astonishing discovery which filled him with joy, whereas for the latter it was the result of human sin and sinful desire, filling him with regret and sorrow for violating nature and man's destiny in it.[23]

Another example of Brouwer's diverging view on physics can be read in the correspondence with his thesis supervisor Korteweg, in the same letter from which we quoted earlier, dated 11^{th} November 1906. Korteweg wrote to Brouwer:

> You think that I considered as absurd the view 'that astronomy is no more than an easy way of lumping together causal sequences of the readings of our measuring instruments'. No, not that view! I recognize that it is possible to look at it in this way, although in my opinion the general law of gravity has little to do with the instruments that led to its discovery except in as far as they make measuring at all possible. But the assertion that the uniformity of laws which apply in very different areas of physics would find its origin in the uniformity of the instruments used, that assertion to me seems absurd.[24]

Brouwer wrote an extensive reply to Korteweg's letter, which, in its turn, apparently was a reaction to a discussion on that topic that took place on the 4^{th} of November. In a long letter, dated 13^{th} November 1906, Brouwer

[23] It is of course a different philosophical question whether nature can be exactly described in mathematical equations, or that every theory is ultimately and fundamentally an approximation.

[24] [Stigt 1990], page 495; in Dutch [Dalen 2001], page 18: (...) ge dacht dat ik de mening 'dat astronomie niets is dan een gemakkelijke samenvatting van causale volgreeksen in aflezingen op onze meetinstrumenten' absurd vond. Neen niet *die* mening; ik erken dat men de zaak zo voorstellen kan, hoewel m.i. de algemene aantrekkingswet al heel weinig meer te maken heeft met de instrumenten die tot haar ontdekking hebben geleid dan alleen in zoverre deze het meten 'überhaupt' mogelijk maken; maar dat de gelijksoortigheid der wetten die op zeer verschillend natuurkundig gebied heersen haar oorsprong zou vinden in de gelijksoortigheid der gebruikte instrumenten die bewering was het die mij ongerijmd voorkwam.

defended his view that the laws of nature in the different parts of physics show a strong mutual similarity because the laws merely express relations between measurements, taken from the rigid group. The similarity is then a result of the similarity in the different measuring instruments used; physical aspects are never directly measured, only angles of torsion of wires are measured.

To stress his point, Brouwer gave the following example which is, although it makes his point clear, not convincing from a physical point of view:

> When the electromagnetic field of a Daniell element and that of a Leclancher element are projected onto our measuring instruments no difference is shown up, and yet looking at this problem for the first time one would expect there is to be as great a difference as there is between copper sulphate and ammonium chloride. Only our counting and measuring instinct, working with certain instruments, is affected by them in the same way and it then appears that the same mathematical system can be applied. It is only the lack of suitable instruments that has prevented us so far from finding other mathematical systems which can be applied to one but not the other.[25]

Brouwer's claim is, in other words, the following: in the days of preparing his thesis, there was no difference in instrument readings between two electrostatic or electromagnetic fields of, say, equal strength but originating from two qualitatively different sources. They have the same effect on our counting and measuring instinct and *therefore* we can describe them by means of the same mathematical system. By refining and improving our measuring equipment there might show up some difference in the measurements, in which case also the describing mathematical system would have to be altered. Brouwer's conclusion is then that we describe in our mathematical model the instruments used, rather than the measured fields themselves.

The following comment on this view is of relevance:

1. Two fields originating from two qualitatively different sources need not, with sufficiently sophisticated instruments, necessarily show a difference in instrument readings. Both mentioned examples are galvanic elements, producing a potential difference or a direct current via different chemical reactions. In an equilibrium situation of equal magnitude of the resulting fields, the measurements will produce identical results, since there is fundamentally no difference between electrons and other charged elementary particles in the different molecules.

[25] [Stigt 1990], page 496. In Dutch [Dalen 2001], page 20: Geprojecteerd op onze meetinstrumenten is er geen onderscheid tussen het electromagnetisch veld van een Daniell-element en dat van een Leclancher-element; maar als we het onbevangen bekijken, moeten we toch verwachten, dat tussen beide velden een even groot verschil moet heersen, als tussen kopersulfaat en chloorammonium bestaat; alleen op ons tel- en meetinstinct, werkend met zekere bepaalde instrumenten, werken ze gelijk; daar blijkt zich een zelfde wiskundig systeem op beide te laten toepassen, maar het is alleen gebrek aan geschikte instrumenten, dat ons tot nog toe belet heeft andere wiskundige systemen te vinden, die zich op het ene veld wel, op het andere niet, laten toepassen.

We admit that one can claim that this comment and its underlying theory is based exactly on the inherent limitations in measuring instruments. But already in Brouwer's days the electron as the carrier of the negative elementary charge was known (though not yet the magnitude of that charge); there was also a theory describing the electrostatic (and the electromagnetic) field, and this theory predicted no difference in the measured fields of the two different galvanic elements. Modern theories predict the same result and extremely refined measuring aparatus confirm this. In regard to this it is significant for Brouwer that in the notebooks he valued so negatively the unifying theory of Maxwell and the work of Newton and Van der Waals, in comparison with the (for the rest of the greatest importance) method of just mathematically describing the observed phenomena of Faraday.[26]

2. In the dissertation Brouwer expressed himself in a different and more balanced way (e.g. when stating that sometimes different theories can be brought together into one), but still physical practice differs from the way as it is sketched by him, which is caused by the difference in its purpose. Ideally, physics is an interplay between theory and experiment, and is guided by curiosity alone, without thinking of practical applications. Sometimes a theory asks for certain specific experiments, but often experimental physicists play the leading part and are ahead of existing theories. One starts with pure and unbiased observation, which is of course the most fundamental level of all physics. Then follows a theory, based on those observations and on the basis of that theory further, and this time directed, observations are made or experiments are performed for its confirmation, and even special instruments are designed for those observations, thereby in the first place *not* thinking of practical applications in order to rule or control our environment.

If, for instance, on the basis of a theory about electromagnetic fields originating from different sources, a difference in observable values is expected when measuring the different fields, instruments are selected, refined, or designed to confirm that prediction. If no difference is observed then one either adapts the theory or designs better instruments. If, however, a difference is observed where no one was expected, then one searches for an explanation for this difference, whether or not in the form of an improved theory. This ideal 'model of interplay' is of course common to all physical practice, including the one as described by Brouwer, but in his view the *aim* of the physicist is different, viz. that of gaining power (which is judged negatively by Brouwer), and if some physicist with such a view on his field of research is doing physics, then that cannot but have an impact on that practice and its results.

To summarize, Brouwer's view on physics can be sketched as follows: Based on observations, man notices 'causal sequences', which can be expressed mathematically. These causal sequences are employed to reach a certain desired goal. On top of that, new causal sequences are observed (or constructed), having no practical application yet but just waiting for one, to reach new goals, if necessary via intermediate targets ('means to an end'). Also attempts are made to

[26]See also the quotes at pages 185 and 199.

combine different groups of sequences into one type of sequence (hence, also with Brouwer we find attempts to create grand theories by means of the interplay between theory and experiment). But physical practice, as sketched by Brouwer, though from a theoretical, experimental and technical point of view similar to the above sketched 'ideal' practice, has a different purpose, viz. to steer and force nature for our own benefit, and, as we just remarked, this difference in purpose must influence the result. But let us emphasize once more that this is not, according to Brouwer, the ideal situation; man wants to control and this is an extremely sinful desire. Brouwer presented a very negative picture of the aim of physical research, which picture is born out of his general outlook on mankind. His view on physics is not from within the physical community, it is a meta-view, and he passes a meta-judgement on it. This meta-view about aim and purpose as the main impetus to do physics and the negative judgement on this practice is in fact the main difference with our view on what physical research should be.

Two remarks have to be made here to put things into perspective:
Firstly, as said, Brouwer acknowledged the possibility that two different groups of phenomena eventually can be brought together into 'one continuous region of possibilities', hence can be subsumed under one theory; but the aim remains the same: to forecast results and consequently, to intervene successfully. (cf. page 207).
Secondly, it has to be admitted that besides the 'idealized' picture of physics as fundamental research for its own sake, also an 'instrumentalism' always has existed (and still exists), which more or less resembles Brouwer's view on physical practice. Instrumentalism claims that every theory is just a tool for controlling and changing the observable reality for one's own benefit. However, our point is that Brouwer's view is not all there is in physics, that it even is a 'minority opinion' (also in Brouwer's days), and that there is a different and more 'common' approach.

In all, one could say that the methods of modern physics and their accompanying (or guiding) philosophies are 'extensionally similar' (or, in terms of modern logic: 'elementary equivalent') to Brouwer's methods and philosophies, the main difference being the aim of doing physics and, for Brouwer, the moral aspect involved in physical practice.

In his subsequent letters Korteweg did not come back to this topic but one can hardly imagine that he agreed with Brouwer.[27]

[27] In the third chapter of his dissertation, when discussing the role of logic, Brouwer sketched in a footnote on page 135 a seemingly different, more generally accepted and 'objective' picture of the physical practice, when stating:

> In addition to this delusion on the *freedom* of logic stands as an analogous overrating the idea of Aristotle and the scholastics (...) namely that logic would be able to uncover secrets of nature which are not clear a priori. In reality the conclusions reached by this method do not hold for nature itself, but only for the mathematical system which has been arbitrarily projected on nature (and only part of which covers what has been directly perceived, while the rest has

6.2.2 The notebooks on causality and man's desire to rule

Causality and causal sequences

Most topics in regard to 'mathematics and experience' are discussed through all nine notebooks, from the beginning of the first one onwards:

> (I–3) The scientific system of means and ends attempts to include in itself as many dimensions (formerly: disturbances) as possible, but there always remains an infinite number of disturbances.[28]

This quote from the first notebook reveals that the idea of sequences of events in which one can interfere and which are employed in order to reach a certain end with the help of choosing a provisional end as a means to the final one, is an old idea which one can trace as early as in the third chapter of *Life, Art and Mysticism.* The (almost) impossibility of this effort is expressed in terms of the infinite number of disturbances (that is, the number of different

> been added by induction). One should verify for every conclusion anew (and every verification ought to be completed by mathematical induction) that it is also true for nature (i.e. that it is efficient as a guide for humen action). Such a verification is necessary even if the premisses are completely true, in the same way as every new consequence of a physical hypothesis ought to be expressly checked, no matter how useful the hypothesis has proved so far.
>
> (Naast deze waan van de *vrijheid* der logica staat als een analoge overschatting ervan het idee van Aristoteles en de scholastici (...) dat men door logica niet a priori duidelijke geheimen der natuur zou kunnen ontdekken, terwijl in werkelijkheid de conclusie waartoe men zo geraakt, niet voor de natuur zelf, maar alleen voor het in willekeur daarop geprojecteerde wiskundige systeem (waarvan dan slechts een deel het direct doorleefde dekt, terwijl het overige een uitbreiding door inductie daarvan is) geldig zijn; dat die conclusies ook voor de natuur *juist* zijn (d.w.z. als leidraad voor het menselijk handelen doel treffen), dient voor elke conclusie opnieuw geverifieerd (en elke verifiëring door wiskundige inductie aangevuld). Zulk een verifiëring is nodig, hoe juist de gebruikte premissen ook waren, zo goed als van een fysische hypothese, hoe bruikbaar ook tot nog toe gebleken, elke nieuwe consequentie uitdrukkelijk dient gecontroleerd te worden.)

But subjectivity returns in the rest of this footnote, (thereby at least *giving room* again to solipsism):

> Moreover this verification can lead to different results for different persons, because they check the words of the conclusion with different mathematical systems which they mentally connect with these words. It is also possible that, lacking such mathematical systems, verification is impossible for the time being and that it must wait for further experience, i.e. for the building of new ([*handwritten correction:*] effective) mathematical systems.
>
> (Die verifiëring kan verder door verschillende personen tot verschillend resultaat leiden, omdat zij de woorden der conclusie toetsen aan verschillende voor die woorden in hun geest bestaande wiskundige systemen, of ook zij kan bij gebrek aan zulke wiskundige systemen in afwachting van latere ondervinding, dat is vorming van nieuwe ([*toegevoegde correctie:*] doeltreffende) wiskundige systemen voorlopig onmogelijk zijn.)

[28] Het wetenschappelijk doel-middelstelsel tracht zoveel mogelijk dimensies (vroeger storingen) in zich op te nemen; toch blijft er altijd nog een oneindig aantal storingen over.

variables) that influence our actions. Man attempts to simplify this effort and, for that reason, tries to use syllogisms as simple causal schemes in his reasoning:

> (I–28) The desire towards syllogism is the desire towards the equality of means and ends.[29]

At an early stage the concept of causality and its related difficulties are brought up for discussion:

> (III–20) Causality is only observed in a viewpoint which is restricted by a freakish arbitrariness (in which is neglected 1. that there are unobserved channels of supply and drain through the sides of the vessel, and 2. that the supposed constancy of the laws inside the vessel is depending on all kinds of situations of the environment, which are unjustifiably supposed to be constant; and above all the value of the elements of the content is precarious, and suddenly can have disappeared). It is always connected with a passion of fear and desire: it notices limitations and also untrue constancy of that what is limited.[30]

The meaning of the first part of the quote being: in all cases of a supposed and supposedly understood causality there are too many unknown factors and influences, which makes causality at least very doubtful. The fact that man ascribes causality to nature, despite this inherent uncertainty, *proves* that he wants causality to serve his purpose. This inherent uncertainty *and* desire can also be read in the following:

> (VII–6) Part of your own externalization by straightforward acts is the frivolous postulation of 'habit' in nature (and the subsequent limitation of your attention to these things in nature (which *are* there, since you *want* to see them) and granting the term 'laws' to those habits according to your own externalization) (conditions for equilibrium, mechanical explanations) in order to be able to contest them or to handle them on the basis of these laws.[31]

Whereas in the dissertation man's effort to control nature is emphasized, with the aim to increase his knowledge for his own benefit, in all the notebooks

[29] De wil tot syllogisme is de wil tot gelijkstelling van doel en middel.

[30] Causaliteit wordt alleen gezien bij een in grillige willekeurigheid beschränkt gezichtspunt. (Daarbij is dan verwaarloosd 1. dat er door de wanden van het vat verborgen kanalen van toe- en afvoer leiden, 2. dat de vooronderstelde constantheid der wetten binnen het vat van allerlei niet-standvastige, maar als standvastig gerekende milieutoestanden, afhankelijk is; en vooral de waarde der elementen van de inhoud precair is en plotseling kan zijn verdwenen.) Het hangt steeds samen met een passie van vrees en begeerte: die ziet beperkingen en tevens onware constantheid van dat beperkte.

[31] Tot de veruiterlijking van jezelf door rechtlijnige daden, hoort het lichtzinnig postuleren van 'gewoonten' in de natuur (en zich dan in zijn attentie tot dié (die, daar je ze wilt zien er natuurlijk zijn) dingen van de natuur te beperken, en toekennen aan die gewoonten van wetten volgens je eigen veruiterlijking) (evenwichtsvoorwaarden, mechanische verklaring), om ze op grond van die wetten te kunnen bestrijden of behandelen.

Brouwer's doubts as to the possibility and the success of this effort are expressed. But despite this uncertainty, of which man is aware, his 'sinful desire' forces him to continue. Brouwer's opinion that this all is a sinful act of mankind and a consequence of abanoning his natural destiny can be read time and again in the notebooks, as well as in the rejected parts of his second chapter.

In order to control, man isolates parts of nature from their natural unity, in order to perform his sinful interference successfully:

> (VIII–60) Causality in human life is the sinful splitting of a unity into two parts, in order to let, by the intellect, the desire act on one of the parts.
>
> Causality in science is a juxtaposition of constructed systems into a new system (or the splitting of a system into two parts); the term here means nothing but the mathematical term 'mere relation'.[32]

Mathematization of observation and experience

The 'sinful act' of viewing the world mathematically is emphasized in the beginning of the first notebook, as if setting the tone for what will follow:

> (I–2) The mathematical viewing of a phenomenon in mathematical physics (...) is the haughty impregnation of the humble view with the human intellectual burden.[33]

In the synopsis of the nine notebooks this is in a most gloomy way expressed in the introduction of its fourth chapter, *Mathematics and society*, as follows:

> (How society is deformed by it, and is getting always more miserable and complicated, and how mathematics is just mutual understanding, and how, on the other hand, mathematics is practised in society). Instead of being a foundation of society, mathematics is just a commodity.[34]

On our page 181 we noted that the act of viewing the world mathematically comes in two phases: the becoming-aware-of-time and thereafter the causal attention. Only after these two phases man attempts to express a causal sequence itself in a mathematical formula. This too can be performed in two ways: purely

[32]Causaliteit in het leven is de zondige splitsing in tweeën van een eenheid, opdat op een der delen de begeerte door het intellect kunne werken.

Causaliteit in de wetenschap is een juxtapositie van opgebouwde systemen, of splitsing van een systeem in tweeën, tot een nieuw systeem; het woord betekent hier niets, dan het wiskundig 'relatie zonder meer'.

[33]Het wiskundig zien van een verschijnsel in mathematische physica (...) is het doortrekken van de nederige visie hovaardig met de menselijke intellectuele belasting.

[34](Hoe de samenleving er door vervormd wordt en steeds ellendiger gecompliceerd, en hoe de wiskunde verstandhouding is, en hoe aan de andere kant de wiskunde in de samenleving wordt bedreven). In plaats van de grondslag der samenleving te zijn, is wiskunde een gewoon handelsartikel.

descriptive and on the basis of an abstract theory. In view of Brouwer's attitude towards 'man doing physics', it will be clear that he strongly rejected the abstract-theory based method. He preferred Faraday's descriptive method, rather than Newton's and Van der Waals' theory-based results (one may wonder of course if a theory-free description is possible at all):

> (I–4) The Faraday theory is just a direct *description* and therefore pure of its kind. But not so the *hypotheses* of Newton and Van der Waals.[35]

The act of any physical observation arises from *desire*, which has a negative undertone: It is the *desire* to see regularity, which opens the way to *ruling*; the only reason for this desire is, to exert power. For that purpose man *needs* physical laws and therefore he creates them; a law is for that reason no discovery. A natural phenomena is an object introduced by *us*:

> (II–10) How about the *explanation* of phenomena, like Korteweg's explanation of the phenomenon of Huygens? Well, all those phenomena are objects introduced by our discretion, willing to externalize its own mathematical laws, which mathematical discretion is always partially thoughtless.[36]

This second notebook contains the most outspoken ideas of how Brouwer saw the role of physics in our life: to force nature into a mathematical form in order to rule. This is another opinion which was hardly subject to change during the composition of the nine notebooks: physical laws are our *imposed* rules:

> (II–19) Physical nature does not act according to scientific laws, but people deform nature according to *their own* scientific laws (in experiment and in technology), of which little is to be found in nature, but which small amount was soon to be noticed by science.[37]

And physical laws can only be imposed after expressing the phenomena of nature in numerical values:

> (II–27) Science is the desire to compute, to give everything numerical values and derivations of those values according to laws, which

[35] De Faraday theorie is niets dan direct beschrijving, is daarom in zijn soort zuiver. Maar niet de *hypothesen*, als van Newton en Van der Waals.

[36] Wat te denken van het *verklaren* van verschijnselen, zoals Kortewegs verklaring van het verschijnsel van Huygens? Och, vooreerst zijn ál die verschijnselen dingen, [die] door onze willekeur, die zijn eigen mathematische wetten tracht te veruiterlijken, zijn ingesteld; evenwel is die mathematische willekeur altijd voor een deel ondoordacht.

[37] De natuur beweegt zich niet volgens de wetenschappelijke wetten, maar de mensen vervormen de natuur volgens hún wetenschappelijke wetten (in experiment en techniek), waarvan oorspronkelijk in de natuur zeer weinig zat; maar dat weinige wordt door de wetenschap snel opgemerkt.

> originate in onesidedness. The world is flowing; if one grips a part, then one distances oneself all the more from the rest.
>
> When introducing hypotheses in physics by calculation, one even gets more confused and one sacrifices an even greater part.
>
> In this all, mathematics is the instrument, which greedily swallows down all onesidedness of the invariabilities, but which has no grip on the totality.[38]

The physical world, described by mathematics (i.e. forced by us into a mathematical model), is not a discovery, but the result of our wickedness:

> (IV–15) The 'wisdom by experience': *'such is the world'* (including the mathematical) is said in a complacent and acquiescent way, as if the world was not *such* by our own wickedness.[39]

One easily recognizes in this small anthology of quotes the tone of *Life, Art and Mysticism* and of the rejected parts, rather than the tone of the dissertation. The reason will be obvious: the rejected parts are, in content, rather mild and moderate compared to the notebooks, and even those rejected parts fell already victim to Korteweg's strong objections.

Continuity and differentiability of functions in nature

Also the continuity of functions which describe nature is a chimera, just introduced to make interference easier:

> (VII–7) We just postulate the functions of nature to be *continuous* and that is pure externalization of man.
>
> (...)
>
> It does not matter that we *observe* that continuity is not preserved for the infinitesimally small. It is after all our externalization that prevents our ability to think sudden jumps, but instead continuous segments of change and dimensionless points without change. We again give the infinitesimally small particles small and continuously changing sizes.[40]

[38]Wetenschap is de wil tot berekening, geven van waarden van alles, en afgeleiden van die waarden volgens wetten, die komen uit eenzijdigheid. De wereld vloeit; houdt men nu een deel vast, dan verwijdert men zich des te meer van de rest.

Gaat men nu hypothesen in physica invoeren door de berekening, dan raakt men nog verder van de wijs; offert nóg groter deel op.

De wiskunde is bij dat alles het instrument, die alle eenzijdigheden der vastigheden gretig opslokt, maar op het volle geen vat heeft.

[39]Het 'door ondervinding wijs gewordene': *'zo is de wereld'* (ook het wiskundige) wordt zelfgenoegzaam berustend gezegd, alsof de wereld niet zó was door eigen slechtheid.

[40]De functies der natuur postuleren we alleen *continu* en dat is zuivere mensenveruiterlijking.

(...)

Dat wij *merken*, dat in de natuur in 't oneindig kleine de continuïteit niet blijft, doet er niet toe. Onze veruiterlijking is nu eenmaal, ons geen plotselinge sprongen te kunnen denken, maar continue segmenten van verandering en dimensieloze punten *zonder* verandering. De oneindig kleine deeltjes geven wij toch weer opnieuw kleine afmetingen, die continu verlopen.

Much is said in the notebooks about the continuous, differentiable, or analytic character of functions that describe nature, and, again, only in the negative and pessimistic sense, which is standing in such a sharp contrast with his creative and constructive attitude in mathematics. But we are concerned here with the young Brouwer; eventually his standpoint became more moderate.

> (III–30) Where do we get the axiom of differentiability of physical functions? Well, because one chooses, out of free will, the measures of distance on the different continua as 'belonging together', (the axiom of time of the uniform motion in the limit), one measures time in such a way that the axiom applies.[41]

This concept of differentiability of physical functions was introduced, as we saw, to increase man's control over his surrounding world. Finitely many observations are extended to infinitely many points between the observed values, thus creating continuous functions:

> (IV–3) The function concept is just the presentation of the mapping of infinitely many points in a finite form, to be performed with the help of a finite number of values under the application of a finite number of *certain* mathematical inductions.[42]

Another paragraph about forcing nature into a set of differential equations, and about our natural and innate abhorrence of discontinuities since they thwart our ruling capabilities, goes as follows:

> (VIII–18) We postulate in nature the accuracy and regularity of the functions and differential quotients; however, our observation is inaccurate, and our mathematical simulation suffers from singularities, hence becomes inaccurate with respect to nature. Why don't we want them in nature? Because we can 'hardly think them', and because they don't fit in our externalization.
>
> Does nature give infinite $\frac{dy}{dx}$? Yes, but y and x are coordinates, introduced by *us*, hence are part of the simulation.[43]

---

[41] Hoe komt men aan het axioma der differentieerbaarheid van fysische functies? Wel, doordat men de maatafstanden op de verschillende continua vrijwillig 'zusammengehörig' kiest (het tijdsaxioma van eenparige beweging bij de limiet), zo meet men de tijd, dat het axioma uitkomt.

[42] Wat is het functiebegrip anders, dan het geven van een afbeelding van een oneindig aantal punten in eindige vorm, uit te voeren met behulp van een eindig aantal waarden onder toepassing van een eindig aantal *bepaalde* mathematische inducties.

[43] Wij postuleren in de natuur exactheid en regulariteit van de functies en differentiaalquotienten; onze waarneming echter is inexact, en onze wiskundige nabootsing lijdt aan singulariteiten, wordt dus t.o.v. de natuur inexact. Waarom willen we die niet in de natuur? Omdat we het 'moeilijk kunnen denken', slecht vinden passen in onze veruiterlijking. Komt in de natuur dan geen oneindige $\frac{dy}{dx}$ voor? Jawel, maar die y en x zijn door óns ingevoerde coördinaten, behoren dus bij de nabootsing.

It is of course common physical practice to attempt to describe series of numerical observations in a continuous (or even differentiable) function, which is then preferably part of a more comprehensive theory, and in sofar Brouwer's quote reflects, again, everyday physical reality. But the frequent use of terms like 'sinful' and 'externalization' indicates that Brouwer certainly was referring to the use of physics to intervene in nature for man's own benefit.

The next quote again speaks of *forced* continuity, just to make 'ruling' possible:

> (VIII–26) In order to operate on nature, we venture the generalization of continuity and of differentiability, indeed the possibility of interpolation. Discontinuity is out of the question, since for that we should be able to observe 'points' in nature, and we notice very well that they are our own externalization.[44]

There are many more relevant paragraphs in this eighth and ninth notebook, often referring to Poincaré. The following reminds us of Mach, to whom Brouwer seems to react. Nature is forced by us into a mathematical description, and therefore we introduce the principle of inertia. 'Mach's principle' states that inertia is the expression of the dynamical interaction of an object with the 'cosmic surrounding'. Without a reference system of objects (which may even consist of remote galaxies) there is no observable inertia. But in the following fragment Brouwer most likely refers to Mach's *Erkenntniss und Irrtum*[45] from 1905, in which it is claimed that all external phenomena are observed by our own physical (muscular) experience.:

> (VIII–70) In a representation of nature, a system of ordering (i.e. of arithmetic and algebra) is not sufficient; we introduce in nature the principle of inertia (maybe as a consequence of our own muscular sensation) and then we can represent all kinds of problems by some sort of *function*, deduced from mechanics. Such a function can only vary in a differentiable way, with a finite differential quotient with respect to place and time.[46]

[44]Om op de natuur te kunnen werken wagen wij de generalisatie van continuïteit, en van differentieerbaarheid, immers interpoleerbaarheid. Discontinuïteit is geen sprake van, daartoe zouden wij in de natuur 'punten' moeten kunnen zien, terwijl we heel goed merken, dat die alleen onze veruiterlijking zijn.

[45][Mach 1968], page 8. This conjecture is based on the list of references at the end of the eighth notebook, and fits with the content of the quoted paragraph from this notebook.

[46]Bij het nabouwen van de natuur is het geordend systeem (dat is dat van aritmetiek en algebra) alleen niet genoeg; we voeren het traagheidsbeginsel in de natuur in (misschien overigens is het het gevolg van ons eigen spiergevoel) en kunnen dan voor allerlei problemen een of andere soort van *grootheid* uit mechanica afgeleid, laten optreden; zo'n grootheid kan alleen differentieerbaar (met eindig differentiaalquotient tijd en plaats) veranderen.

Mach's name and his work *Erkenntniss und Irrtum* were not mentioned in regard to this quote, but it was on the list of consulted literature on the last page of this notebook. Also Poincaré introduced muscular movement when describing a 'third dimension', but the 'inertia concept' refers to Mach, rather than to Poincaré.

6.3 Poincaré's *La Science et l'Hypothèse*

The analysis of the role of science in human life and of its value is continued on page 88 of Brouwer's dissertation, with a comment on the tenth chapter of *La Science et l'Hypothèse* by Poincaré, entitled *Les théories de la Physique moderne*.[47] Because of the weight that Brouwer, in his dissertation as well as in the notebooks, lent to Poincaré's views on physical theories, on mechanical explanations for phenomena of nature, and on the actual state of affairs in the field of science, we will present a brief sketch of the content of this chapter, thereby emphasizing the items Brouwer commented on. It is divided into the following three sections:

The meaning of physical theories

The purpose and the meaning of a physical theory is not the truth about its fundamental building blocks (e.g. *ether* or *atoms*), but its ability to predict correctly the values of the observables, thus indicating that their mutual relations are properly described despite the fact that the character of the variables may change over time as a result of further research :

> Mais ces appellations [*i.e. names we give to the phenomena*] n'étaient que des images substituées aux objets réels que la nature nous cachera éternellement. Les rapports véritables entre ces objets réels sont la seule réalité que nous puissons atteindre, et la seule condition, c'est qu'il y ait les mêmes rapports entre ces objets qu'entre les images que nous sommes forcés de mettre à leur place.[48]

The fact that there is such a close relationship between the different models which are so successfully employed by us to represent the unknown reality, suggests that those models are all based on, and are, according to Poincaré, a consequence of, more general and more fundamental principles: the *principle of conservation of energy* to which all processes must obey, and the *principle of least action* that prescribes which one of all the possible processes satisfying the first principle, will actually occur. An apparent contradiction between two theories, both correct in their predictions, frequently reduces to mere contradictions between our metaphoric images, e.g. the comparison of colliding molecules with colliding billiard balls or with interacting charged particles. In different theories we often recognize the conservation of a certain *scalar* quantity. What is conserved, we then call *energy*:

> Est-ce à dire que le principe n'a aucun sens et s'évanouit en une tautologie? Nullement, il signifie que les différents choses auxquelles nous donnons le nom d'*énergie* sont liées par une parenté véritable; il affirme entre elles un rapport réel.[49]

[47] [Poincaré 1916], page 189 ff.
[48] [Poincaré 1916], page 190.
[49] l.c. page 195.

The introduced principles remain valid as long as they are useful to us.

Physics and the mechanism

There is always a strong preference among physicists for an explanation of phenomena in terms of mechanical processes, i.e. an explanation in terms of colliding molecules, whether or not in combination with attractive or repulsive forces. This is, according to Poincaré, always possible provided that the two principles mentioned above are satisfied.[50]

In the case of electromagnetic phenomena there was the ether, often considered as the *only* existing matter; atoms were then considered to be points of condensation or vortex points in the ether (Kelvin). With the help of the ether theory, physicists hoped to *demonstrate* the existence of a measurable *absolute* motion.[51] Poincaré opposed the idea of absolute motion (and turned out to be right at this point).

The present state of affairs in science

This is of course the situation in science around 1900, the time of publishing *La Science et l'Hypothèse*.

There are two different developments: 1) In the direction of a greater unity by the unification of different theories into one unified theory. In Poincaré's days the most well-known example was Maxwell's theory, which united electricity and magnetism into one single theory, governed by four differential equations. 2) In the direction of a greater complexity by the discovery of new and inexplicable phenomena, the most well-known example in those days probably being Planck's theory of black body radiation. Because of the second development the optimism of around 1850 had diminished by 1900, despite progress made by e.g. Lorentz, who, however, still needed the ether for his electron theory. The two principles (least action and conservation of energy) remained fundamental in modern theories, but new ones were added like Carnot's principle and the law of increasing entropy.[52] New analogies were recognized, such as that between electric resistance and viscosity of fluids.

> Néamoins les cadres ne sont pas rompus; les rapports que nous avions reconnus entre des objets que nous croyions simples, subsistent en-

[50] As a comment on this section about Poincaré's ideas, the following remark can be made. During the nineteenth century there was indeed a strong preference for explanations in terms of mechanics for natural phenomena. However, at the end of that century there was a transition towards a different approach: not mechanics, but electromagnetism was seen as fundamental, and one tried to interpret for instance inertia electromagnetically. Note that the first edition of *La Science et l'Hypothèse* was published in 1902.

[51] The concept of 'absolute motion' was an older one, already present in Newton's work, and not depending on ether theory.

[52] Carnot's principle from thermodynamics: All Carnot engines (an engine operating according to a Carnot cycle), operating between two given temperatures, have the same efficiency, and this efficiency is the maximum attainable for any cyclic heat engine.

> core entre ces mêmes objets quand nous connaissons leur complexité, et c'est cela seul qui importe.[53]

As a final remark, not directly related to our subject, but interesting in itself, Poincaré observed that the fundamental laws of celestial mechanics could be discovered thanks to the *imperfection* of the measuring instruments and the unfamiliarity with the complexity of nature. Familiarity with relativistic effects in astronomy could have prevented Newton's laws.

> C'est un malheur pour une science de prendre naissance trop tard, quand les moyens d'observation sont devenus trop parfaits.[54]

6.3.1 Brouwer's comment on Poincaré

Poincaré certainly was one of the greatest minds of his time in the area of mathematics and physics, and, as the direct precursor of French pre-intuitionism,[55] was of great influence on Brouwer. Brouwer discovered in *La Science et l'Hyothèse* certain aspects, which strengthened his views or which were useful to him as arguments in his pessimistic and solipsistic outlook on the role of science. According to Brouwer,[56] man concludes a posteriori from his observations of the surrounding nature that many physical phenomena that come to us in the form of causal sequences, can be mathematically expressed in second order differential equations, e.g. in its simplest form Newton's equation $F = m\frac{d^2x}{dt^2}$. The mass m is then to be understood as a convenient coefficient in this equation.[57] Observation teaches us that a system is 'approximately isolated' if its centre of gravity moves rectilinearly and uniformly, from which one gets by abstraction the principle of (scalar) equality of action and reaction as well as the concept of an 'isolated system' in absolute terms. By means of the equations of motion of Lagrange, simple systems of rigid bodies can be described, but also theoretical astronomy becomes subject to mathematical description, as well as most parts of physics that are governed by reversible processes. The preference for mechanical explanations of natural phenomena, which was also emphasized by Poincaré without giving reasons for this preference, is made plausible by Brouwer in the following terms:

> [The preference for an explanation in terms of a rigid mechanism] is probably the result of the fact that man is more familiar with the setting up of rigid constructions and mechanisms, and that he can control rigid bodies in their behaviour more easily; and that therefore the idea that nature only builds rigid mechanisms removes its mystery in so far, as if nature were to build things which man

[53] l.c. page 210, 211.

[54] l.c. page 211.

[55] See [Bockstaele 1949], page 23 e.v. and page 79.

[56] Dissertation, page 90 ff.

[57] Compare [Poincaré 1916], page 127: 'les masses sont des coefficients qu'il est commode d'introduire dans les calculs'.

> could not imitate by building from matter; and also of the fact that in this way the considerable reliance on the invariability of the laws governing rigid bodies strenghtens the illusion that 'nature can be controlled'.[58]

In this passage we once more recognize Brouwer's view on what man's aim in life is. This is not *to know and to understand* nature for its own sake, but *to control* it, and for that control a certain technical knowledge is needed. In achieving this knowledge Brouwer showed, just like Poincaré and many others, a natural preference for an explanation of physical phenomena in terms of rigid body dynamics.[59] But, as we saw, the main difference with Poincaré and others was the *aim* of doing physics. No curiosity, but a desire to rule, which desire was indeed recognized by Brouwer, but also rejected on moral grounds.

Due to his pessimistic view on mankind, Brouwer saw the application of mathematics to the physical world only as an attempt to increase man's power over it; and the most effective way to increase power, so it seemed to him, is by means of a demystification of nature and therefore by treating it like familiar rigid dynamics. The 'familiarity' with this method of explaining and controlling can be interpreted as referring to a personal familiarity, which reminds us again of his solipsism. Doing physics out of pure curiosity escapes his outlook on mankind, at least in his early years. Brouwer is still observing mankind in the context of his own *Life, Art and Mysticism*.

This all makes the objections from the side of the mathematical physicist Korteweg understandable, which resulted in the *Rejected Parts*.[60]

In a footnote Brouwer proposed another reason for the said preference of explanations of physical phenomena in terms of rigid body dynamics: a rigid body is the familiar example of an object, fixed by a finite and very limited number of coordinates. Indeed, a physical phenomenon is normally determined by a finite number of objects, which number, however, is usually very large, for instance the number of molecules in a certain volume of a gas. Therefore:

> the remaining coordinates could only show discontinuous variations and could therefore be determined by means of integers, so that nature would only possess the order of freedom of a permutation

[58]dissertation, page 90: [De voorkeur voor een verklaring in termen van een rigide mechanisme] heeft waarschijnlijk zijn oorsprong daarin, dat het bouwen van rigide constructies en mechanismen de mensen het meest vertrouwd is, en dat men de rigide lichamen het gemakkelijkst in hun gedrag beheerst; dat dus het idee, dat de natuur alleen rigide mechanismen bouwt, haar mysterie, in zoverre zij dingen zou bouwen, die de mensen niet materieel zouden kunnen nabouwen, wegneemt; en ook hierin, dat zó het zeer grote vertrouwen op de onveranderlijkheid der wetten, die de vaste lichamen beheersen, de illusie, 'de natuur te kunnen beheersen', versterkt.

[59]We remarked already that the preference to explain all physical phenomena by means of mechanical model, was dominant during the second half of the nineteenth century, and was followed by a preference for electromagnetism as fundamental in explanations. See the footnote on page 204.

[60][Stigt 1979].

> group. (...) That would bring nature still closer to the material structures of man, and to the limited freedom experienced in their creation.[61]

Hence a finite number of rigid bodies, each determined by a finite number of coordinates, makes the remaining coordinates only show discontinuous jumps, but which can be described to a very good approximation by continuous functions owing to the large number of rigid bodies involved.

Note that, just like Poincaré, Brouwer did not follow the tendency of his days in the physical community, i.e. the switch to an electromagnetic worldview.

The value of an explanation of phenomena

If it is man's desire to control and to rule, then physical phenomena which can serve that purpose stand in need of an explanation, but the concept 'explanation' here has to be interpreted with care. To compare different explanations and to select one of them as the most valuable, one has to attach a certain value to an explanation.

This value is, in Brouwer's terms, the possibility of a clear separation of the *essential* component from the *accidental* component of an observed phenomenon. The essential part can subsequently be 'extended to a larger field of possibilities', i.e. brought together into one model with essential components of seemingly different and independent phenomena, such that the prediction of new phenomena becomes possible.

The most reasonable interpretation of Brouwer's concept of 'explanation' and of its 'value' seems to be as follows: For Brouwer, an explanation is not intended for a more profound understanding of phenomena, it is not an objective for its own sake. An explanation is just a device that makes predictions possible, and those predictions are not meant to confirm the explanation as being true, but merely to intervene in nature efficiently:

> because man always wish, and be able, to extend the totality of inductively summarized phenomena, in the realm of which he can *forecast results* and consequently intervene successfully.[62]

When observing a physical phenomenon and when searching for an explanation of that phenomenon in the form of a model that can make predictions possible, which model is then preferably but not necessarily in terms of rigid body dynamics, one devises it such that it accounts for the relevant observed

[61] dissertation, page 90: dat nu ook de nog overblijvende coördinaten slechts discontinue sprongen zouden vertonen, dus door gehele getallen zouden zijn te bepalen, en de natuur zou slechts de orde van vrijheid van een permutatiegroep behouden. (...) Dat zou de natuur nóg dichter brengen bij de materiële gebouwen der mensen, en de beperkte vrijheid in het scheppen daarvan gevoeld.

[62] (dissertation, page 93) want het geheel der inductief samengevatte verschijnselen, waarin de mensen vermogen *gevolgen te voorspellen*, en uit hoofde daarvan met succes in te grijpen, zullen zij steeds willen en kunnen uitbreiden.

data. These relevant data are the ones that are included as parameters in the theory; they belong to the *essential components* of the theory or of the explanation. The non-relevant numerical data belong to the *accidental components.*

As a simple example take the basic Newtonian mechanics of billiard balls, in which mass, force and acceleration form the essential variables, and in which, say, the colour of the balls is accidental. Or take the theory of radioactivity (a topical subject in Brouwer's early years), in which the temperature is an accidental variable (in absolute terms to every known degree of approximation).

However, we must also keep in mind what might be *Brouwer's* criteria for being an essential component. If a physical law is not discovered, but created by us with the sole purpose of gaining power, then the degree of either or not being essential might be determined by the end we set ourselves, by that 'sinful desire' of ours. We will return to that at the end of this section.

It will be obvious that, in the strict physical sense, the role of accidental and essential component may change in time. For instance when applying the Newtonian laws to the kinetic theory of gases, temperature becomes a relevant parameter in the equations. In fact, temperature is *defined* in terms of the basic mechanical quantities.

Brouwer also expressed it as follows: 'a successful "explanation" opens a field of induction', that is, all consequences or all possibilities following from that explanation can be verified and if that is successfully completed, the essential part again has to be separated from its previously unnoticed accidental subparts, thus widening the field of induction. The result will be the unification of two hitherto separated groups of phenomena.[63] In regard to this, Brouwer speaks in a footnote of 'He who believes in the reality of hypotheses will speak of "going deeper into the nature of the phenomena"',[64] thereby emphasizing his anti-realism.

About an explanation of a physical phenomenon, which, after a further development of the theory (and its verification by experiment), turned out to be not correct, Brouwer noticed:

> Let us remark further that it can never be said afterwards that an explanation, which served its purpose in extending the region of known sequences by means of induction, was shown to be incorrect. For, in that case, a clash with experience proves no more than that on the strength of the explanation a field of induction was openend which was *too large.* In such a case the explanation can always be saved by re-extending the essential parts in the mathematical image of the phenomena on which it was based, at the expense of the part which had been assumed as essential.[65]

[63]Compare Maxwell's unification into one theory of electromagnetism of the previously separated theories of electricity and magnetism, and the subsequent unification of the different manifestations of electromagnetism.

[64]dissertation, page 93, footnote 3.

[65]dissertation, page 93: Merken we nog op, dat nooit een verklaring, die haar diensten bij

This is correct in the case of the Newtonian mechanics, which was improved by Einstein to relativistic mechanics. The first one can still be considered as a very good approximation for the second in case of relatively low speeds, as we experience them in our daily life. However this claim is not so conclusive in the case of Zeeman's own explanation of the effect called after him, i.e. the splitting of spectral lines in a strong and inhomogeneous stationary magnetic field.

The last citation and the discussion on it might give the impression, that Brouwer is speaking here in a strict physical sense, but we must constantly keep in mind Brouwer's early ideas of the role of physics in human life. The reason why Brouwer attached so much importance to the concept of 'value of an *explanation* of phenomena' is a corollary of his concepts of nature, of physics, and of man's role in it. In order to control and to manipulate his environment, man has to create *laws* and *theories* that make predictions possible in his system of causal sequences. And in a relatively simple set of laws, explanations, and techniques of operating on the world, man can live and control in his own small world, in himself. In this world his fellow-men are merely a special kind of active and dynamic ('living') objects.

Let it again be said that this attitude shows itself in its most pronounced form in his early years, the tone becoming more moderate as time went on. But his tendency towards mysticism and his pessimistic philosophy was to stay.

The notebooks on Poincaré

Brouwer read and studied Poincaré, and often agreed with him, for instance when discussing *force* and *mass*, which are no 'natural phenomena', but are introduced by us for convenience:

> (II–19) Force, to be distinguished in static and kinetic (i.e. hypothetical), and mass are proper fossilizations for the summary of some phenomena.[66]

This is clearly influenced by Poincaré. For the latter, *mass* is just a convenient coefficient. In his notebooks Brouwer expressed this informally as 'proper fossilizations for the summary of phenomena'.
The following fragment is written as a reaction to page 212 of Poincaré's *La Science et l'Hypothèse*. This is the last page of the tenth chapter, entitled *Les Théories de la Physique moderne*, in which Poincaré remarked: 'A mesure qu'on connait mieux les propriétés de la matière, on y voit régner la continuité':

de uitbreiding door inductie van het gebied der bekende volgreeksen heeft gedaan, later kan worden gezegd, onjuist te zijn gebleken. Immers dan bewijst een démenti der ervaring alleen, dat men op grond der ervaring een *te groot* veld van inductie had geopend. En in zulk een geval kan men de verklaring steeds redden, door in het erop gegronde wiskundig beeld der verschijnselen weer het essentiële gedeelte uit te breiden ten koste van het reeds als toevallig gestelde.

[66] Kracht (te onderscheiden in statische en kinetische d.i. hypothetische) en massa zijn geschikte verstarringen om sommige verschijnselen samen te vatten.

> (V–4) The continuous divisibility with respect to time of the phenomena (in physics and geometry) is the basic axiom. In other words, one can write down differential relations as the basic idea.[67]

Brouwer also read and commented on Poincaré's *La Valeur de la Science*. Chapter 9 of this book is entitled *La science et la réalité*, and in § 5 of this chapter, *Contingence et déterminisme*, the postulate of the indefinite possibility of interpolation of functions in nature is discussed:[68]

> (IX–17) We construct, within certain limits, analytical functions that can, of course, approximate *every* empirical function.[69]

We *even force* nature into analytic functions:

> (IX–20) We would like to catch nature and, because analytical functions are the main tool for that, we have the tendency to view equilibrium situations expressed in such functions.[70]

Finally, in the following we note Poincaré's idea of a third dimension in the perceptible world, as a consequence of our own physical movements, which was also discussed in chapter 4 of *La science et l'hypothèse* in the section *L'espace visuel*:

> (IX–29) The fact that space is a *living thing* for us, means that our muscular movements are living sensations.[71]

6.4 Space, time, objectivity and apriority

The question about objectivity and apriority of space and time was actualized again by Russell in his *An essay on the foundations of geometry*.[72]

Brouwer raised this question in order to investigate and decide in how far the concepts of *objectivity* and *apriority* can be ascribed to mathematical systems in general, and what exactly, from his point of view, these terms mean.

[67] De continue splitsbaarheid der verschijnselen naar de tijd (voor fysica en geometrie) is het grondaxioma. M.a.w. men kan differentiaalbetrekkingen opschrijven als grondidee.

[68] [Poincaré 1923], page 248 ff.

[69] Wij bouwen binnen bepaalde grenzen analytische functies, die natuurlijk *elke* empirische functie kunnen benaderen.

[70] Wij zouden de natuur willen vangen, en omdat analytische functies daartoe het hoofdzakelijke middel zijn, hebben we altijd neiging, om ook de evenwichtsfiguren zo te willen denken.

[71] Dat de ruimte voor ons *leeft*, wil zeggen, dat onze spierbewegingen zo levend zijn.

[72] [Russell 1897]; Brouwer used in his discussion the French translation [Russell 1901], probably as a consequence of the subsequent polemic in French in the journal *Revue de Métaphysique et de Morale* between Couturat, Poincaré, Lechalas and Russell. Another reason might be the many corrections, made by Russell in this four years later French edition. In the sixth notebook, on the pages 24 – 33, the polemic in the *Revue de Métaphysique et de Morale* between Russell, Poincaré, Couturat and Lechalais, which followed on the publication of Russell's foundational work, is commented on.

When discussing objectivity we must keep in mind Brouwer's solipsistic viewpoint, which remained in his later work. This point of view also applies to mathematics in the sense that this is a free creation of the *individual* mind. One can never be *certain* that other individuals will build the same mental constructions when communicating their mathematical arguments and results. In its strict sense one can even never be certain about other minds at all.

Brouwer on objectivity

A consequence of a solipsistic and constructivistic standpoint is that *objectivity* cannot have its common meaning of 'a real substantial existence, external to, and independent of the subject'. Therefore Brouwer's possible definitions of that concept differ from the usual and traditional ones. He is seeking his solution in 'invariance under mathematical transformations representing natural phenomena'. Take for instance *mass*; according to Brouwer (after Poincaré),[73] this is merely 'a coefficient whose introduction simplifies the mathematical image of nature and which remains invariant under mathematical transformations representing natural phenomena'; mass thereby becomes a mathematical concept and is described in a mathematical system.[74] However, this concept of mass may change with the development of physics in such a way that explanations of newly discovered natural phenomena become simpler when mass becomes a variable quantity (which in fact happened in relativistic mechanics). Then mass only remains invariable, and therewith an 'objective' concept, under transformations in a representation of a *very important group* of natural phenomena, viz. the phenomena of non-relativistic physics.

Hence 'objectivity' of some specific quantity or law (in the given meaning of invariance under mathematical transformations) is always relative to a certain mathematical model of nature, which model is chosen on the basis of simplicity and utility of its description. But then the definition of *objectivity per se*, not relative to some specific mathematical model, has to be adapted.

The reader may still wonder why 'objectivity' is so closely linked to the notion of 'invariability under a mathematical transformation', whereas the traditional idea is that of 'existence, independent of an observer'. Brouwer even mentioned 'indestructibility' as a first thought in regard to the objectivity of mass. But we emphasize once more Brouwer's solipsism, where a world, external to and independent of an observer, remains a hypothetical reality, unknown on principle. Indestructibility as the characteristic for the objectivity of mass

[73] [Poincaré 1916], page 127.

[74] dissertation page 95. As a simple and obvious example we might take the case of the mass of billiard balls and a Galileo transformation from one coordinate system (the billiard table) to another one which is in motion relative to the first one; or, as another more complicated example of a 'mathematical transformation representing natural phenomena' we might again take the billiard balls. Between an initial state and a final state, both given by the position coordinates and the momenta of the balls, the transition is given by a mathematical transformation, governed by Newtonian laws of motion, under which the masses remain invariant.

is an untenable one, since mass is just 'a convenient mathematical coefficient'.

Objectivity of mass as a mathematical notion from Brouwer's point of view, where at least the 'old' idea of 'external to and independent of an observer' can be recognized, can then be expressed as 'invariance under mathematical transformations in the time and space coordinates of the equations', where mass then remains invariant in time or space.

Brouwer distinguished three possible definitions for the general concept of *objectivity* of a mathematial system or notion:

> *either* invariance for a *certain given* interpretation of *all* phenomena which are so far known; (...)
>
> *or* invariance for the simplest or the most common interpretation of *all* phenomena so far known; (...)
>
> *or* invariance for the simplest or the most common interpretation of a *very important* group of phenomena;[75]

In *the first case* objectivity of a quantity would be, according to Brouwer, an *arbitrary* property, depending on the chosen interpretation, consequently resulting in the construction of rather artificial systems. This may be the case if e.g. one sticks too long to an outdated and improper interpretation. Artificiality would then indeed be the result, since there is, apparently, in this case no interpretation which offers itself as a natural one; we just have selected one (or stuck to an old one), in which all known phenomena have to fit (are forced to fit) in order to let the quantity concerned be an objective one.

In *the second case* we clearly have chosen, according to sound physical practice, the simplest interpretation for all known phenomena (an ideal situation from a physical point of view), but then objectivity would become a *transient* property which may lose its value and therefore may disappear upon the appearance of a new phenomenon. A new 'simplest and most common' theory then has to be invented, since, if we hold on to the old theory, we are back in the first case. For a given set of observed phenomena, there is, ideally, one simplest interpretation, in which a certain concept, (e.g. mass) remains invariant under a mathematical transformation. But new observations will be added to the existing ones, which might make the interpretation no longer tenable. As an example we again mention the concept of mass and the appearance of relativity theory.

The third definition is 'the greatest mainstay', despite (or precisely because of) the subjective aspect in it. Mass, as a simplifying coefficient, is considered to be objective, until a variable mass is asked for to improve (i.e. to make simpler or

[75]dissertation, page 95: òf invariabiliteit bij *een zekere* verklaring van *alle* tot nog toe bekende verschijnselen. (...)
òf invariabiliteit bij de eenvoudigste of de meest gebruikelijke interpretatie van *alle* tot nog toe bekende verschijnselen. (...)
òf invariabiliteit bij de eenvoudigste of de meest gebruikelijke interpretatie van *een zeer belangrijke* groep van verschijnselen.

to extend the range of) the explanation of phenomena. Under the old condition ('the very important group of phenomena') mass remains objective.

The fact that Brouwer's choice is the third alternative, can again be accounted for by looking at it from his standpoint: the arbitrariness, resulting in a forced system, of the first possibility and the transient characteristic of the second one are far greater disadvantages compared to the subjectivity of the third possible definition. This subjectivity is in fact no disadvantage at all in the light of his solipsistic standpoint; subjectivity becomes a most natural and obvious property of the selected definition: a quantity need not be invariable under transformations in the simplest and most common interpretation of all phenomena, but of *a very important group* of phenomena. What is 'a very important group' and wat is 'the simplest interpretation'? That is decided by the individual who asks that question, since there is no absolute certainty about other individuals to agree with. According to Brouwer, mathematics and a mathematical model for nature cannot be anything else but an individual creation.

In accordance with the last definition, *space* and *time* with the physical space and scientific time[76] constructed on it, are objective. Space and time are, as Brouwer observed, a necessary tool in the description of the physical phenomena, and *for a very important* group of phenomena they are invariable magnitudes, i.e. length and duration are invariable quantities under a transformation from one coordinate system to another, with constant (and relativistically low) relative speed between the two systems, hence they are invariable in the mechanics of all our daily experiences, which man needs to control and to rule.[77] In relativity theory they are no longer invariable magnitudes, but this theory has its own invariants under transformations.[78]

Apriority

Brouwer's concept is of course influenced by Kant, who discussed apriority and a priori knowledge thoroughly in his *Kritik der reinen Vernunft*.

Very briefly Kant's concept amounts to the following:

All our knowledge *begins* with sense experience, but does not necessarily *originate* from sense experience, since it can have its source in our own faculty of knowledge (*Erkenntnisvermogen*) (page B 1 of the *Kritik der reinen Vernunft*). Kant raised the question (B 2) whether or not there is knowledge independent of all experience and of all sense perception. This type of knowledge he called *a priori knowledge* or *pure knowledge* (B 3).

A property of an *a priori statement* is, that its content is *necessary* and at the same time of *general validity*, that is, it is valid without exceptions.

On page B 38 (about space) and B 46 (about time) Kant stated, among other

[76] This concept has to be distinguished from the intuitive time, as we saw earlier.

[77] Even today in the century of space travels.

[78] We could of course also have included an arument about Kant's influence on Brouwer in regard to objectivity, as we have done this in regard to apriority. See Van Atten's comment on intersubjectivity in [Atten 2004], chapter 6.

properties, that 1. space and time are not empirical and 2. they are necessary a priori notions, underlying all external experience ('eine notwendige Vorstellung, a priori, die allen äußeren Anschauungen zum Grunde liegt'). Hence in this latter sense the a priori notions of space and time are a necessary condition for the possibility of experience, they are 'a priori Anschauungsformen' (a priori forms of intuition). Neither space, nor time as a priori Anschauungsformen are objects of knowledge, but in geometry one can obtain a priori knowledge about spatial properties of objects of experience, like one can obtain a priori knowledge about temporal properties of the objects of experience in arithmetic.

Brouwer on apriority

According to Brouwer, the concept of apriority of a mathematical system can be interpreted either as[79]

1. existence of that mathematical system, independent of experience,

or, more generally and fundamentally, as

2. necessary condition for the possibility of science.

Ad. 1: In the first alternative the whole of mathematics is a priori knowledge (e.g. both Euclidean and non-Euclidean geometry), since it is constructed on the basis of the ur-intuition alone, independent of the content of any external experience. Of course the ur-intuition is an abstraction from the content of internal experiences of different external (or internal) events, separated by a continuum,[80] hence a sense experience marks the beginning of mathematical knowledge, but it is not the *source* of this knowledge since the source is the awareness of the abstracted two-ity after a second sense experience. This can perfectly well be understood in the Kantian sense (see above).

This interpretation, in which mathematics in its totality becomes a priori knowledge, is not to Brouwer's satisfaction; a reduction to a more primitive concept is desirable. The obvious option is precisely the ur-intuition itself, the individual experience of the two-ity *event–separation* or *discrete–continuous*, which makes mathematics a *free* creation of the *individual* mind. The next alternative leads to this option.

Ad. 2: This option presents apriority as the 'necessary condition for the possibility of science'. Brouwer's subsequent argument amounts to the following:

[79]Note that the English translation of Brouwer's dissertation, as published in [Brouwer 1975] states:

'Next the *apriority*; this can mean *one of two things*' (my emphasis),

whereas the original Dutch text states:

'Nu de aprioriteit; men kan hiermee twee begrippen bedoelen, (...)'

i.e. it can have two interpretations. The English text as it were forces one to a choice, whereas the Dutch text seems to leave both options open, which apparently was meant by Brouwer, as can be concluded from the subsequent two paragraphs: one is not forced to a choice but one is led automatically to the second alternative.

[80]dissertation, page 8:

> the substratum, divested of all quality, of any perception of change (...)

science can be distinguished in 1) experimental science, originating from the application of intuitive mathematics to the observed reality, and 2) the properties of intuitive mathematics which was applied in 1). Since there is no aprioristic element in our observation of reality, the only aprioristic element in science consists of that one thing which is common to all mathematics, i.e. the ur-intuition of mathematics.

Brouwer's conclusion then is:

> And since in this intuition we become conscious of time as change per se, we can say:
>
> *The only aprioristic element in science is time.*[81]

In a footnote to this conclusion Brouwer added the following interesting remark, which casts light on his view about what *science* is:

> Strictly speaking the construction of intuitive mathematics in itself is an *action* and not a *science*; it only becomes a science, i.e. a totality of causal sequences, repeatable in time, in a mathematics of the second order, which consists of the *mathematical consideration of mathematics* or *of the language of mathematics*; only there one meets with causal connections in the way in which mathematical systems on the one hand, mathematical symbols, words or ideas on the other hand, succeed one another. But there, as in the case of theoretical logic, we are concerned with an *application of mathematics*, that is with an *experimental science*.[82]

Brouwer subsequently discussed a number of topics from Russell's mentioned book *Essai sur les Fondements de la Géométrie*, and he more or less strongly disagreed with the content and conclusions of the treated items. We will not examine this part of Brouwer's dissertation.

6.4.1 Kant's point of view, compared to that of Russell and Brouwer

The last pages of Brouwer's second chapter consist of a comparison between the concepts of space of Kant, Russell, and Brouwer himself.

[81] dissertation, page 98: En daar deze samenvalt met de bewustwording van de tijd als verandering zonder meer, kunnen we ook zeggen:

Het enige aprioristische element in de wetenschap is de tijd.

[82] dissertation, page 98, footnote: Eigenlijk is het gebouw der intuïtieve wiskunde zonder meer een *daad*, en geen *wetenschap*; een wetenschap, d.w.z. een samenvatting van in de tijd herhaalbare causale volgreeksen, wordt zij eerst in de wiskunde der tweede orde, die het *wiskundig bekijken van de wiskunde* of *van de taal der wiskunde* is: eerst daar bestaat causaal verband in de wijze van opvolging der wiskundige systemen enerzijds, en der wiskundige tekens, woorden of begrippen anderzijds; maar daar, evenals bij de theoretische logica, hebben we ook weer te doen met een *toepassing der wiskunde*, met een *ervaringswetenschap*.

Kant

Kant's view concerning space is briefly summarized by Brouwer in his dissertation on pages 113 – 115. The relevant passage was quoted on page 52. Kant's conclusion, as summarized by Brouwer, is 'that three-dimensional Euclidean geometry is a necessary condition for external experiences and the only possible receptacle for the conception of an external world', and therefore the properties of Euclidean geometry are *synthetic judgements a priori* for all external experience.[83] Both premises (see page 52) more or less claim objectivity of empirical space per se (supposed to be necessary for external experience) and of the Euclidean group of motion (translation) defined in that space.

However, for Brouwer the experiences of an external world can occur without mathematics:

> But it can immediately be objected that we *obtain* our experiences apart from all mathematics, hence apart from any space conception; mathematical classifications of groups of experiences, hence also the creation of a space conception, are free actions of the intellect, and we can arbitrarily refer our experiences to this catalogization, or undergo them unmathematically.[84]

Nevertheless Brouwer claimed that, even *if* Kant's premises are correct, it could just as well be that the human intellect is composed in such a way that it can place the conception of an external world in other receptacles, which is not experienced due to a lack of practice or lack of effect. Hence apart from our experience of an external world by placing it in empirical space only, one could also have that experience by placing it in time, but possibly also in some other unnoticed framework for experience, e.g. in non-Euclidean space.

For Brouwer, however, mathematics is for its existence not depending on any external experience. The ur-intuition as described in our second chapter is the only a priori element for the development of mathematics, and this ur-intuition is not depending on the *content* of any experience. Hence, also experience is independent of any mathematical system. The only synthetic judgements a priori generally (that is, whether or not for external experience) are those, 'obtained as possibilities of mathematical constructions by virtue of the basic intuition of time'.

Therefore possible judgements are:

> 1. the very possibility of mathematical synthesis, of thinking many-one-ness, and of the repetition thereof in a new many-one-ness.
>
> 2. the possibility of intercalation (namely that one can consider as a new element not only the *totality* of the two already compounded,

[83] For a short comment of this summary, see page 52.

[84] dissertation, page 115: Maar er kan direct tegen worden ingebracht, dat wij onze ervaringen *krijgen* los van alle wiskunde, dus ook van alle ruimtevoorstelling; wiskundige classificatieën van groepen van ervaringen, dus ook de schepping der ruimtevoorstelling, zijn vrije daden van het intellect, en wij kunnen naar verkiezing onze ervaringen op die catalogisering betrekken, of onwiskundig ondergaan.

> but also that which *binds* them: that which is not the totality and not an element).
>
> 3. the possibility of infinite continuation (axiom of complete induction).[85]

And, as a concluding remark of this chapter,

> Experience a posteriori can teach us nothing about the necessity of the occurrence of definite mathematical systems in experimental *science.*[86]

The notebooks on Kant

In the first quote Poincaré's concept of space is compared with that of Kant:

> (III–22) All Poincaré's talking does not take away anything from Kant's apriority. The relevant questions can always be tackled from two sides, but can only be solved morally.[87]

Brouwer is referring to Poincaré's *La Valeur de la Science*, chapter IV, *'L'espace et ses trois dimensions'*, in which a physical three-dimensional continuum is the result of *observation* and which is called by Poincaré 'le continu ou le groupe de déplacements':

> (III–23) With his apriority of space, Kant only wants to say that, in case of self-reflection of your consciousness, one has to depart from something, but then one cannot do without the act of counting and without space.[88]

The following paragraph is a direct reaction to Kant's *Kritik der reinen Vernunft*, page A-22 and A-24:

> (V–16) If somebody uses the word *space* and elaborates on it, then he immediately has to be silenced: the word *space* should not be admitted. Neither the word *not* to speak about.

---

[85] dissertation, page 119: 1. de mogelijkheid zelf van wiskundige synthese, van het denken van veelenigheid, en van de herhaling daarvan in een nieuwe veelenigheid.
2. de mogelijkheid van tussenvoeging, (dat men n.l. als nieuw element kan zien niet alleen het *geheel* van de twee reeds samengestelde, maar ook het *bindende*: dat wat niet geheel is en niet element is).
3. de oneindige voortzetbaarheid (axioma van volledige inductie).

[86] dissertation, page 120, the last paragraph: De ervaring a posteriori kan omtrent het noodzakelijk optreden van bepaalde wiskundige systemen in de ervarings*wetenschap* niets leren.

[87] Al het gepraat van Poincaré doet niets af aan Kant's aprioriteit. De kwestie daarvan is steeds van twee kanten aan te pakken, alleen moreel is ze op te lossen.

[88] Kant wil met zijn aprioriteit van de ruimte eigenlijk niets zeggen, dan dat je bij zelfbekijking van het bewustzijn van iets moet uitgaan, en dan wel niet kunt buiten het tellen en de ruimte.

> (Kant, page 24) [*this should be: A-24*] 'Think', yes, but I ought not to think anything, but for the sake of the struggle; the rest is foolishness.[89]

The first paragraph of this fragment most likely refers to Kant's *Transzendentale Ästhetik*, erster Abschnitt *Von dem Raume*, § 2 *Metaphysische Erörterung dieses Begriffes* (A–24 ff.), to which Russell reacted in *Fondements de la Géometrie*. But why then should one not speak about space? Most likely, Brouwer's argument will be that the discussion by Russell gets bogged down in a formalist treatment of the concept; it becomes an object, void of life. Space has to be experienced intuitively.

The second paragraph of this last quote is also about page A–24 of Kant's *Transzendentale Ästhetik*. Brouwer is reacting to the following fragment:

> Der Raum ist eine notwendige Vorstellung a priori (...) Man kann sich niemals eine Vorstellung davon machen, daß kein Raum sei, ob man sich gleich ganz wohl *denken* kann, daß keine Gegenstände darin angetroffen werden.

Russell

1. Russell too claimed the possibility of external experience apart from Euclidean three-dimensional space, but, according to him, the properties of projective geometry and the axiom of free mobility are necessary properties of the receptacle of external experience. The different geometries of constant curvature remain as alternatives for experience.
2. Experience teaches us that only Euclidean geometry can serve to describe the external world as we observe it, hence that this geometry is 'true' to a high degree of approximation. Russell attempted to prove these two points in his *Fondements de la Géométrie* in a manner that was unacceptable for Brouwer. His criticism of Russell's proofs amount to the fact that Russell's assumptions are arbitrary and superfluous since the only necessary prerequisite in the mathematical receptacle of experience is the ur-intuition of mathematics or the intuition of time.

Brouwer was, in his dissertation as well as in the notebooks, sharp in his criticism of Russell, but the relevant parts of Brouwer's dissertation and his notebooks contain no new viewpoints, and will therefore not be discussed here.

The notebooks on Brouwer's own point of view about space and time

Brouwer's own remarks, especially in the earliest notebooks, often contain terms like 'sin', 'desire', 'doom' and the like. The concept of space is a sinful human

[89] Wanneer iemand het woord *ruimte* gebruikt, en daarover gaat spreken, moet hem direct de mond toegesnoerd worden: het woord *ruimte* mag niet toegelaten worden. Evenmin het woord *niet* om óver te spreken.

(Kant page 24) denken, ja, maar ik heb mij niets te denken, dan ter wille van de strijd; de rest is dwaasheid.

desire. However, in pure mathematics we can do without space as a basic concept in the Kantian sense, the ur-intuition being sufficient as a foundation of mathematics (which does not alter its sinfulness).

As a comment on Poincaré's (non-verbatim!) quote from *La Science et l'Hypothèse*, Brouwer remarked:

> (II–12) (Poincaré page 72) La troisième dimension vient du phénomène, que si deux sensations de convergence A et B sont indiscernables, les deux sensations d'accomodations A' et B' qui les accompagneront respectivement seront également indiscernables.
>
> That is not true, the three dimensions exist only in our description of nature, when observing the motions of rigid bodies with our senses, but we are not aware of the senses; we may not and we cannot observe the senses themselves. In retrospect we can conclude that it fits, but not in advance; we just experience the dimensionless change in our body, but we observe the external world in dimensions.[90]

There is an objectively existing space, *not* in the traditional sense of 'external to, and independent of the observer', but in the sense that we experience it. But the belief in it on the basis of that experience is a sinful belief:

> (II–29) The belief in an objectively existing (i.e. which you have to be afraid of) space is at the same time a punishment for the desire, and that desire itself.[91]

Brouwer's concept of objectivity was discussed on page 211: invariance under mathematical transformations representing natural phenomena for the simplest interpretation of a very important group of phenomena, which concept could be understood and interpreted as following from his solipsism. However, in the previous and in the following quote the term 'objective' has indeed to be understood in its 'popular' sense as 'existing external to us', his other interpretation apparently being from a later date. But both interpretations fit in his solipsistic attitude: there is only the self, and one has to be afraid of everything that forces itself upon the self, including an objective space, since it leads man away from his natural destiny: a return to his ur-state. Note that we are now quoting only from the early notebooks two and three. In this 'old' interpretation space has to be feared; it is threatening our existence:

> (III–10) Man's externalization has the illusion of constancy (animals don't have that), therefore they want to count and to measure, and

[90] [*Na de Franse tekst van Poincaré volgt Brouwer's commentaar:*] Niet waar, de drie afmetingen zijn er slechts, voorzover wij de natuur beschrijven, en daarbij bekijken we de beweging van rigida met onze zintuigen, maar aan die zintuigen denken we niet; die mogen we niet waarnemen, en kunnen we ook niet waarnemen. Achteraf kunnen we wel kijken, dat het klopt, maar het is niet waar vooraf; wij voelen alleen de dimensieloze 'wisseling' in ons lichaam, maar de buitenwereld bekijken we met dimensies.

[91] Het geloof in een objectief bestaande (d.i. waarvoor je bang moet zijn) ruimte is tegelijk de straf voor de begeerte, en die begeerte zelf.

> successfully, as it seems; but in the meantime time goes its own way, not only the measurable (for which there is consolation and which can be assimilated), but also the unmeasurable, that is the ageing of the self.[92]

6.5 Conclusions

We have seen in this chapter that Brouwer's view on physics in general, and the application of mathematics for the description of physical processes in particular, is different from the usual opinions about physical research and the status of physical theories.

This is not in the first place caused by Brouwer's view on the day-to-day practice of the physical community, but by his view on man's aim in applying mathematics in our daily life. This aim is, according to Brouwer, man's sinful desire to rule, which is the reason that man is not searching for fundamental theories, but for collections of causal sequences to make intervention in nature possible. Hence, for Brouwer physics must be descriptive instead of foundationally explaining; he prefers the experimental work and its elaborations by Faraday, rather than the unifying theories of Lorentz and Maxwell. Causality is imposed on nature instead of discovered, which is, as we remarked, not an uncommon view; only man's aim is different in Brouwer's option. Brouwer adduced uncommon, and sometimes even untenable ideas in support of his general view on physics (like the idea that theories about nature are in fact theories about the instruments used, although this idea appeared only in his correspondence with Korteweg).

We have seen that Brouwer's opinions on physics in general can be made comprehensible by taking into consideration 1) Brouwer's rather gloomy view on man, his aims and his motives, and 2) Brouwer's solipsistic standpoint.

Other concepts which could be made understood by taking into account in particular the second of these two important ingredients of Brouwer's thinking, are his notions of objectivity, apriority, and value of an explanation.

We have also seen that his most pessimistic views became more moderate during the years, but that they never disappeared completely.

[92] De veruiterlijking der mensen heeft de waan der constantheid (de dieren hebben die niet), zo willen ze tellen en meten, en dat schijnt zo ook goed te gaan; maar intussen gaat de tijd zijn gang niet alleen de meetbare, waarvoor nog troost is, en die is te assimileren, maar ook de onmeetbare, d.w.z. de zelfveroudering.

Chapter 7

The role of logic

7.1 Introduction

For centuries Aristotelian logic was believed to be the final word in this branch of knowledge; some improvements and extensions were introduced by his immediate and later successors, but the main body of this discipline was assumed to be settled by Aristotle.

Only in the nineteenth century substantial progress was made by the introduction of mathematical methods in logical reasoning, and *symbolic logic* came into being. The pioneers in that area were Peano, Boole, Frege and Russell. Peano for instance had devised a notation of logical symbols in order to discuss logic in a mathematical way. Boole was the first to study the algebraic properties of propositional logic in a systematic way, and he created the 'Boolean algebra'.

But the most prominent representative of the *logistic school* was Gottlob Frege.[1] He can be considered as the founder of this school, whose aim it was to build the whole of mathematics on a foundation of logical principles alone. Apart from Frege, Russell in his *Principles of Mathematics*[2] and Couturat in *Les Principes des Mathématiques* and in *Pour la Logistique*[3] made attempts in this direction.

Among the several opponents of logicism, Poincaré and Borel played an important role, and had a great influence on the development of Brouwer's view on the role of logic in mathematics. Poincaré was of the opinion that:

> La logique tout pure ne nous mènerait jamais qu'à des tautologies; elle ne pourrait créer du nouveau; ce n'est pas d'elle tout seule

[1] See his *Begriffschrift* from 1879, *Grundlagen der Arithmetik* from 1884 and *Grundgesetze der Arithmetik, begriffschriftlich abgeleitet* from 1893 and 1903; respectively [Frege 1879], [Frege 1884] and [Frege 1893].

[2] [Russell 1938].

[3] [Couturat 1905] and [Couturat 1906] respectively.

> qu'aucune science peut sortir.[4]

Logic, according to Poincaré, plays its important and even indispensable role in mathematical reasoning, but more is needed for the construction of arithmetic and geometry or of any other science, apart from pure logic. That 'more' consists of an intuition, which is not based on the senses and which gives us some fundamental principles and relations. This kind of intuition is not denied by Russell and Couturat either, but Poincaré is hereby referring to specific and intuitively given *mathematical* objects and principles like the system of the natural numbers and the basic rules of arithmetic.[5] Mathematics is more than a game of manipulation of symbols, it has an extra-logical content too. Also the choice of a set of axioms from the many possible, given by logic alone, is led by this intuition.

> Ainsi, la logique et l'intuition ont chacune leur rôle nécessaire. Toutes deux sont indispensables. La logique qui peut seule donner la certitude est l'instrument de la démonstration: l'intuition est l'instrument de l'invention.[6]

Poincaré's typical example of the intuitively given mathematical reasoning pur sang is the *principle of mathematical induction*, which is not a logical principle, and can neither be derived from logical principles, nor proved in any other way. Poincaré's opinion about the status of the principle of mathematical induction is exactly Brouwer's view,[7] but Poincaré's criterium for *mathematical existence* is fundamentally different from Brouwer's: For Poincaré *existence* is identical to *consistency*, for Brouwer it is *to be the result of a construction.*

Borel agreed with Poincaré about the role that intuition (Borel also used the term *invention* to designate it) plays in the selection of the proper axioms from the multitude available:

> Je voudrais montrer que la logique fournit seulement aux mathématiques leur matière, c'est-à-dire un ensemble innombrable de formules possibles. La science commence lorsque l'on choisit parmi ces formules, (...)[8]

According to Borel, logic is a reliable instrument when studying the mathematical reality, but it cannot be the source of that reality; mathematics has a content of its own. Borel also has a typical and non-Brouwerian criterium for the existence of mathematical objects: such an object is real if it is a commonly shared and familiar concept among mathematicians, about which no misunderstanding or ambiguity exists.[9]

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[4] [Poincaré 1923], page 20.
[5] Op. cit. page 22.
[6] [Poincaré 1923], page 29.
[7] Compare Brouwer's second thesis from the list of theses at the end of his dissertation.
[8] From *La logique et l'intuition en mathématiques* (1907); see [Borel 1972], page 2084.
[9] See [Borel 1950], page 175.

Brouwer's view on logic and its role in the founding and construction of mathematics was, during his early active years as mathematician, one of violent opposition. This attitude became one of acceptance after the formalization of intuitionistic logic by Heyting and Kolmogorov.[10] Logic was merely a formalization of the language accompanying mathematical reasoning (or of any other language), and had nothing to do with mathematics proper and its construction. Brouwer's main claim is that, in contrast to the view of the logicists, logic is depending on mathematics instead of the other way round; it may play its limited and modest role of the accompanying language, suitable for memory and communication, but only *after* the construction of mathematics proper. In his criticism and argumentation, Brouwer limited himself to the role of traditional logic, despite the availability of modern developments in the work of Frege, Russell and others. Therefore his rejection of the role of logic in the deduction of mathematics was not based on some specific kind of formalization of logic; it was a matter of principle and not of form. For that reason Aristotelian logic served him well enough to argue his rejection, and modern developments in logic would not have added any new elements to this argumentation.

It is striking that in his dissertation not a single reference is made to Frege, the founder of modern logic after two millennia of Aristotelean logic. But one should keep in mind that the work of Frege was rather neglected by mathematicians of that time.[11] There are several possible reasons for this omission: one could be that the formalism of his logic was rather forbidding and therefore not too attractive to be studied; another one could be that Russell's influence was far greater and that he was often read instead. Indeed Brouwer read, studied and criticized (in the second and third chapter of his dissertation) Russell's *Principles of Mathematics*;[12] he also he read Whitehead's *A Treatise on Universal Algebra*, judging by a footnote on page 159 of his dissertation.[13]

Just to illustrate the contrast with Brouwer's opinion, we will devote a few words to Frege's view in passing. For Frege, logic does not *describe* the psychological process of how man thinks, but *prescribes* how man *must* think if his thought is to remain within the bounds of reason; logic is the study of truth-preserving inference:

> (...) its focus is truth in general, it strives not to define the predicate 'true', but to characterize the conditions under which truth is transferred from proposition to proposition. (...) The laws of logic are

[10] See page 226.

[11] It is worth noting, however, that [Frege 1884] and [Frege 1893] were available in the library of the University of Amsterdam in the year 1907, according to information gained from the University Library,

[12] [Russell 1938].

[13] That Brouwer was at least familiar with some of Frege's work can be concluded from the following: in the first place, he attended the lectures by Mannoury who discussed Frege's work on the foundations of mathematics; (cf. [Mannoury 1909], Vorwort and page 78 ff.); secondly, Brouwer referred on one occasion in notebook 8 to an article by Frege in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* number 12, *Über die Grundlagen der Geometrie II*, in which Frege reacted on Hilbert's book with the same title. In his Synopsis of the notebooks Brouwer again referred to the relevant paragraph in notebook 8.

> what make it possible to infer one sentence from another, to justify one claim by appeal to another.[14]

According to Frege, nothing is needed to justify our arithmetical knowledge beyond the axioms of logic and its rules of inference.[15]

Mathematicians around 1900 were indeed of the opinion that logic codifies truth and that logic tells mathematicians how to think during the process of doing mathematics. According to them (but in more modern terminology) mathematics is about structures and relations, obeying the laws of logic.

Brouwer's diverging view on the role of logic is already cleary expressed in his own introduction to his third chapter:[16] Mathematics is the result of a construction, ultimately based on the ur-intuition alone. The criterium of a successful construction is not that the laws of logic are obeyed, but merely the simple fact that the construction *can* be completed. Brouwer usually expressed himself in the metaphor of a *building*. A proof of a theorem in a certain well-constructed mathematical theory is then a *sub-building in a building*. This idea is elaborated on page 229 ff.

Now, a natural corollary of this concept of 'building-in-a-building' is a constructive interpretation of the *hypothetical judgement* $A \to B$, in which the antecedent A actually has to be constructed before granting B the status of a *constructed* sub-building. We will discuss this at length on page 230 ff. Brouwer worked out an example of a hypothetical jugement in one of his last letters to Korteweg before the date of taking his doctoral degree, clearly to get Korteweg on his side in this matter. See for this page 241.

Brouwer's contrasting views in regard to logic can also be read in many of his publications and papers, from the dissertation onwards to his later work: mathematics is a free creation of the mind, and only the free creative mind tells us how to do mathematics. In *Consciousness, Philosophy and Mathematics*,[17] Brouwer stated that 'there is a system of general rules called *logic* enabling the subject to deduce from systems of word complexes conveying truths, other word complexes conveying truths as well', but '*truth* is only in *reality*'. *'There are no non-experienced truths'*. 'Logic is not a reliable instrument to discover truths and cannot deduce truths which would not be accessible in another way as well'. This attitude and these arguments make it clear that Brouwer did not need any form of *modern* logic to reject logic as a foundation of mathematics.

In subsequent papers about this issue he is becoming more and more specific on this matter. In the dissertation it is emphasized that logical principles only apply to the language of logical reasonings that accompany mathematically constructed systems.[18] *Only in that limited area* their applications are

[14][George and Velleman 2002], page 18, 19.
[15]ibid, page 16, 17.
[16]See page 228.
[17][Brouwer 1948], see also [Brouwer 1975], page 480.
[18]dissertation, page 132.

reliable. They are never reliable if used as the sole means in the attempt to lay the foundations of mathematics. In his paper *The Unreliability of the Logical Principles*[19] Brouwer became more specific: of the four classical logical principles, that of identity, of contradiction, of syllogism and of tertium exclusum, the first three are correct in mathematics, whereas the last one is valid for finite sets only; for infinite sets this principle is 'unreliable' as yet, without, however, being faced with a contradiction in case of an unjustified application. But non-contradictority is not sufficient for the correctness of a mathematical theory.[20] In 1917 in the *Addenda and Corrigenda*[21] its applications became 'unjustified petitiones principii', and in 1919 in the *Intuitionistische Mengenlehre*[22] Brouwer arrived at a complete rejection:

> Meiner Ueberzeugung nach sind das Lösbarkeitsaxiom und der Satz vom ausgeschlossenen Dritten beide *falsch* und ist der Glaube an diese Dogmen historisch dadurch verursacht worden, dass man zunächst aus der Mathematik der Teilmengen einer bestimmten endlichen Menge die klassische Logik abstrahiert, sodann dieser Logik eine von der Mathematik unabhängige Existenz a priori zugeschrieben und sie schliesslich auf Grund dieser vermeintlichen Apriorität unberechtigterweise auf die Mathematik der unendlichen Mengen angewandt hat.[23]

It may therefore be obvious that Brouwer's rejection of logic as a basis for mathematics and his subsequent development of intuitionistic mathematics, is not a result of the paradoxes that appeared in mathematics which was based on logic alone.[24] Brouwer's rejection is more fundamental, although he frequently mentioned the paradoxes as an example of what can happen in case of too great a trust in logic.

Moreover, Brouwer was certainly not a logician; in fact he never published on logic and took little interest in it.[25] Intuitionistic logic was later formalized by Kolmogorov and Heyting,[26] and because this logic basically contains Brouwer's foundational ideas, its proof interpretation which was published a few years later is now generally referred to as the Brouwer-Heyting-Kolmogorov interpretation.[27] Brouwer of course knew this formalization and he agreed with

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[19] [Brouwer 1908a]; for a more detailed discussion of this foundational paper see page 247.

[20] [Brouwer 1912], see also [Brouwer 1919c].

[21] [Brouwer 1917b], item 7, see also [Dalen 2001], page 197.

[22] [Brouwer 1919b].

[23] ibid, footnote 4; see also [Brouwer 1975], page 231.

[24] E.g. Russell's paradox as a result of Frege's improper use of the axiom of extension.

[25] The reader may check [Dalen, D. van 1997], the annotated bibliography of Brouwer. Note, however, that in his paper *Intuitionistische Zerlegung mathematischer Grundbegriffe* ([Brouwer 1925]) he proved in plain language that $\neg\neg\neg p \leftrightarrow \neg p$.

[26] See [Kolmogorov 1925] and [Heyting 1930]; see for comment [Troelstra 1978].

[27] Or the BHK proof-interpretation for short. This form of logic is in agreement with, and implicit present in Brouwer's constructivism, e.g. in his treatment of negation in [Brouwer 1923b]; see also [Dalen, D. van, A.S. Troelstra 1988], page 31. See also [Brouwer 1929], the first Vienna lecture.

it. There is, however, an important difference between the proof-interpretations of Heyting and of Kolgomorov, as they were originally published in Heyting's[28] and in Kolmogorov's[29] papers. For Heyting the principle of *ex falso sequitur quodlibet* is a valid one, though not explicitly mentioned as such:

> Aus diesen Gründen sind die Formeln[30]
>
> 2.14 $\vdash b \to (a \to b)$ *und* 4.1 $\vdash \neg a \to (a \to b)$
>
> aufgenommen.

The latter includes the ex falso principle.[31]
In contrast to this, Kolmogorov explicitly stated several years earlier:

> Hilbert's first axiom of negation, 'Anything follows from the false', made its appearance only with the rise of symbolic logic, as did also, incidentally, the first axiom of implication. But, while the first axiom of implication follows with intuitive obviousness from a correct interpretation of the idea of logical implication, the axiom now considered does not have and cannot have any intuitive foundation since it asserts something about the consequences of something impossible: we have to accept B if the true judgement A is regarded as false.
>
> Thus, Hilbert's first axiom of negation cannot be an axiom of the intuitionistic logic of judgement, no matter which interpretation of negation we take as a point of departure.[32]

The 'BHK-interpretation' in its present form[33] *does* accept the ex-falso principle. We will see in the introduction to his third chapter that Brouwer's view

[28] See [Heyting 1980], page 191.

[29] [Kolmogorov 1925], see [Heijenoort 1967], page 414.

[30] The grounds being that, after having proved $a \to b$, one discovers that b is always true. The symbol $\to$ has then to be interpreted in such a way that the formula $a \to b$ remains valid. Heyting then applies the same argument for the case that a is always false.

[31] The justification in this paper of the ex-falso principle is not convincing, but in a personal communication Heyting told Van Dalen that in 1927 he was in the possession of a proof interpretation of this principle. See also [Bertin and Grootendorst 1978], in which [Heyting 1930] in included, with a comment by A. Troelstra.

[32] [Heijenoort 1967], page 421. However, in a later publication by Kolmogorov (*Zur Deutung der intuitionistischen Logik*, [Kolmogorov 1932]), he accepted Heyting's axiom $\vdash \neg a \to (a \to b)$ on the following grounds:

> [page 62] Was insbesondere die Aufgabe 4.1 betrifft, [*i.e. the above given axiom from Heyting*] so ist, sobald $\neg a$ gelöst ist, die Lösung von a unmöglich und die Aufgabe $a \to b$ inhaltlos.

and he argued a few pages earlier in the same paper:

> [page 59] Der Beweis, daß eine Aufgabe inhaltlos ist, wird weiter immer als ihre Lösung betrachtet werden.

[33] e.g. as published in Van Dalen's *Logic and Structure* [Dalen, D. van 1994], chapter 5.

on logic is rather different, in the sense that it only forshadows the BHK proof-interpretation and that the ex falso-principle is implicitly rejected in Brouwer's dissertation.

In *Points and Spaces* from 1954 we find one of the rare occasions (possibly the only one) where Brouwer himself employed the term *intuitionistic mathematical logic*:

> However, notwithstanding this rejection of classical logic as an instrument to discover mathematical truths, intuitionist mathematics has its general introspective theory of mathematical assertions, a theory which with some right may be called *intuitionistic mathematical logic*, and to which belongs a theory of the *principle of the excluded middle*. In intuitionism this principle is also called the *principle of judgeability*.[34]

Considering the year of publication of this paper, we may assume that Brouwer was referring to Heyting's formalisation of intuitionistic mathematical logic, but in none of his later papers, including the *Addenda and Corrigenda* to his dissertation[35] and the two short papers *Addenda and Corrigenda* and *Further Addenda and Corrigenda* (both dating from 1954) to the paper *On the significance of the principle of the excluded middle in mathematics, especially in function theory* from 1923,[36] did Brouwer withdraw his implicitly different interpretation, which will be elaborated in the next section.

Our aim with this chapter

Brouwer begins his third chapter with a general introduction to logic and its role in a mathematical construction. Mathematics is, for its construction, not depending on any laws of logic, not even when the opposite seems to be the case, viz. in a mathematical construction on the basis of a hypothesis.

We will argue that 1) in the most literal interpretation of his argument, this implies the rejection of the *ex falso* principle, and 2) this literal interpretation must be the proper one. Brouwer mentioned three examples which are not elaborated, but we will present a few more.

Four different subjects are worked out by Brouwer, all four lending themselves to illustrate his arguments against the foundational role that is often granted to logic in the construction of the mathematical building.

The first one is about the axiomatic foundation of mathematics. The interesting aspects in this subject can be found towards the end, mainly in a footnote, about Brouwer's claim that consistency is not a sufficient condition for the existence of a mathematical sub-building. Only a successful construction guarantees us this. A consistent set of axioms does not give automatically an existent mathematical system.

[34] [Brouwer 1954c], page 3; see also [Brouwer 1975], page 524.
[35] [Brouwer 1917b].
[36] Respectively [Brouwer 1923c], [Brouwer 1954a] and [Brouwer 1954b].

The second one is about Cantor's theory of transfinite numbers, and it is this one that deserves most of our attention (also Brouwer spent most space to this subject). In particular the notion of the 'denumerably infinite unfinished set' is worth a long treatment. We will argue that:

1) A close analysis leads to the conclusion that Brouwer's definition of this type of set and its cardinality is at least incomplete because of the theorem that he claims as fundamental for the cardinality of this set.

2) A slightly modified definition can solve this problem of incompleteness.

The notion of 'unfinished mapping', which is introduced without definition, will turn out to be even more problematic. It can have two interpretations, both giving rise to problems.

For *the third subject*, the logic of Peano–Russell, a shorter discussion will be sufficient. The main point of interest for us is Brouwer's counter-argument against the well-known Russell paradox and against Russell's proposed solution to it.

The fourth subject is about the logical foundation of mathematics after Hilbert. The main topic to be discussed is Hilbert's formalistic approach (see e.g. his definition of the system of the real numbers on page 296) versus Brouwer's constructivistic modus operandi.

7.2 Brouwer's introduction to chapter III

The third chapter of Brouwer's dissertation bears the title *Mathematics and logic*; it intends to investigate the mutual influence of logic and mathematics; in fact Brouwer's intention is to show that this influence is one-sided, in the sense that logic is depending on mathematics and not the converse.

> We want to show, that mathematics is independent of the so called *logical laws,* (laws of reasoning or of human thought). This seems paradoxical, for usually mathematics is expressed, orally or in writing, in the form of argumentation, deduction of properties, by means of a chain of syllogisms.[37]

For Brouwer the *mental construction* in mathematics comes first and the written or otherwise expressed proof or reasoning, in which the rules of logic are followed, belongs to the accompanying language and is just an aid to one's own memory or a way of communicating its content to others. Intuitionistic logic *preserves* constructibility, but does not create new constructions.[38] Logic turns out to be an unreliable instrument in the creation of a mathematical building.[39]

[37]Dissertation, page 125: We willen tonen, dat de wiskunde onafhankelijk is van de zogenaamde *logische wetten*, (wetten van redenering of van menselijk denken). Dit schijnt paradox, want wiskunde wordt gewoonlijk gesproken en geschreven als bewijsvoering, afleiding van eigenschappen, en in de vorm van een aaneenschakeling van syllogismen.

[38]See the 'Einleitung' of [Heyting 1930].

[39]See [Brouwer 1908a], [Brouwer 1954a] or [Brouwer 1954b]; see also the previous and the next section of this chapter.

We have remarked earlier that, when discussing a mathematical theory or theorem, Brouwer often employed the metaphor of the construction of a building from its constituent parts.[40] It is more or less his standard figure of speech to describe the act and the result of doing mathematics. This metaphor serves very well the purpose of stressing the difference with the realist concept of mathematics.[41] Of course in the end every comparison falls short, but clearly a building is not merely composed of its bricks and its other construction materials alone, but includes the way in which, according to the architect's design, the constituent parts are laid and connected to form the resulting structure. Likewise a 'mathematical building' does not, e.g. in the case of geometry, consist of points and lines alone, but includes the *defined* relations between those elements. The deduced relations (geometrical theorems) then are the properties of the relevant mathematical building. Such a property and its proof is a construction of its own, a sub-building within the original building. Hence in general a theory may be seen as constituting a building, and a mathematical theorem is then a relation between the parts of that building and may be considered to become a separate part itself in the form of a sub-building. Also the logical structure in the accompanying language which describes a mathematical building, can itself be viewed as a building: a *linguistic building.* A *construction element* of a mathematical building may be composed of more elementary parts and the relations between these elementary parts may very well consist of simple and basic tautologies.

After this small digression on the building metaphor (we will come back to this comparison on page 231) we will now resume the thread of Brouwer's argument about the role of logic.

The relation between the main parts of the resulting building may turn out to be too complicated for an immediate insight, but may reveal itself as the result of a chain of simple tautologies:

> The proofs which we gave in Chapter I for the very first theorems of mathematics, taught us to read these theorems as tautologies. The fact that in more complicated cases a theorem is not immediately clear, but is only understood after a chain of tautologies, proves merely that we construct our buildings too complicated to be comprehended in one view.[42]

Brouwer then pointed out that there are chains of syllogisms giving the impression of a purely logical reasoning instead of a mathematical construction. We will quote the first part of the relevant paragraph:

---

[40] In the collected works, volume 1 [Brouwer 1975] the Dutch word *gebouw* is translated as *structure*, but we prefer the more literal translation *building*.

[41] See e.g.[Maddy 1990], page 20 ff., and [Penrose 1989], page 96.

[42] diss. page 125, 126, see also [Brouwer 1975], page 72: De bewijzen, die we in het eerste hoofdstuk van de allereerste stellingen der wiskunde gaven, bestonden in het leren lezen van die stellingen als tautologieën. Dat in meer gecompliceerde gevallen een stelling niet direct duidelijk is, maar eerst na een reeks van tautologieën wordt ingezien, bewijst alleen, dat wij onze gebouwen ingewikkelder bouwen, dan we ineens kunnen overzien.

> In one particular case the chain of syllogisms is of a somewhat different kind, which seems to come nearer to the usual logical figures and which actually seems to presuppose the hypothetical judgement from logic. This occurs when a building is defined by some relation in another building, while it is not immediately clear how to effect its construction. Here it seems that the construction is *supposed* to be effected, and that starting from this hypothesis a chain of hypothetical judgements is deduced.[43]

In the first paragraph of his third chapter,[44] Brouwer emphasized the fact that the chains of syllogisms form only the linguistic accompaniment of a mathematical construction and not the construction itself. Mathematics really *consists of* objects, given by their relations with simple or composed construction parts of a mathematical building. This building, in turn, has of course to be the result of a proper mathematical construction, with the ur-intuition of mathematics as the fundamental construction step. The given relations with the construction parts are then, by subsequent reasoning in the form of chains of tautologies, transformed into relations with other parts of the building. This is what is usually called a 'logical proof' of a new theorem.

Now, the important observation in the last quoted fragment is that after a proper interpretation of its terms A and B in the mathematical building under consideration, and a correct logical derivation of B from the hypothesis A (i.e. a logical proof of $A \to B$), the logical figure of the hypothetical judgement *'assume A, then B'* is not automatically recognized by Brouwer as a mathematical theorem. Close reading of the quote results for us in the following interpretation: A building is defined in another building by some relation (i.e. a relation between the newly defined building and the original one), but it is not immediately clear (i.e. from that definition) how to construct it (i.e. the 'building in the building'). One assumes it to be constructed (i.e. still the sub-building; the consequent of the hypothetical judgement is still not under discussion), and on the basis of that assumption one deduces a chain of hypothetical judgements (i.e. a theorem about the sub-building in the building; only now the conclusion of the hypothetical judgement appears!). Hence in the hypothetical judgement $A \to B$, A is the newly defined, and as yet unconstructed sub-building, and B is some theorem about that sub-building, in the form of the chain of hypothetical judgements.

[43]diss. page 126, 127, see also [Brouwer 1975], page 72: Er is een bijzonder geval, waar de aaneenschakeling van syllogismen een enigszins ander karakter heeft, dat aan de gewone logische figuren meer nabij schijnt te komen, en werkelijk het hypothetische oordeel der logica schijnt te vooronderstellen. Dat is, waar een gebouw in een gebouw door enige relatie wordt gedefinieerd zonder dat men daarin direct het middel ziet het te construeren. Het schijnt, dat men daar *onderstelt* dat het gezochte geconstrueerd was, en uit die onderstelling een keten van hypothetische oordelen afleidt.

[44]The first quote of this section.

This seems to us the proper interpretation of Brouwer's text,[45] which implies the implicit rejection of the *ex falso* principle.[46]

We will devote the remainder of this section to Brouwer's building metaphor in general, and its use in this chapter in particular. In a subsequent section we will work out several examples as an elucidation.

The reader of the third chapter of Brouwer's dissertation will find that 'building' is a somewhat overburdened term. In the opening paragraph 'building' means exactly what it should: a structured conglomerate of mathematical objects, relations and operations (which may also be considered as objects), satisfying certain definitions, axioms and rules of operation (which form the 'design' of the building, thus being an essential part of it). Thus e.g. the computation of the greatest common divisor of two integers is a mathematical building, consisting of certain integers and operations on those integers in some specific order.

Also the ring of integers is a mathematical building, consisting of the integers and the basic arithmetical operations.

The proof of the theorem that the sum of two odd numbers is an even number is a building, consisting of certain numbers and operations, resulting in certain other numbers.[47]

A few pages later the concept of the 'luinguistic building' is introduced.[48] So here the *descriptions* in a suitable language of the erection of mathematical buildings become themselves building blocks for another kind of building. In this sense one could say that the language in which, say, group theory is explained, is a building. We will meet this concept again at the end of this section.

[45]On the basis of our interpretation of Brouwer's text we cannot agree with Van Atten's more 'liberal' interpretation of the hypothetical judgement, as published in [Atten 2004], page 22, where the actual construction of the antecedent, in which all given data are included, is not strictly required.

> (page 22) An answer in general terms [*to Griss' arguments for a negationless mathematics*] would be that, to see that one construction is possible on the assumption of another, there is no need actually to carry out the assumed one. It suffices to consider its intentional content or meaning, which can be done in abstraction of any evidence or counterevidence for the construction thus intended.

Van Atten gives the following example:

> 'If something is a square circle, then it is a square'. If we did not see this truth, it would be inexplicable how we could come to see that 'square circle' is a contradictory concept.

For the logician this is a correct reasoning (see the elaborated examples on page 241 ff.), but not for Brouwer, at least not in 1907. In an attempt to construct the antecedent A of some theorem concerning a property B of a square circle (viz. that it is square), the constructive mathematician concludes to the impossibility to construct A, and therefore the hypothetical judgement $A \rightarrow B$ is mathematically an invalid theorem. The defined sub-building in which the theorem applies, cannot be constructed.

[46]See also the discussion on page 234.

[47]All these three examples can also be viewed as sub-buildings in the mathematical building of arithmetic.

[48]'Taalgebouwen' in Dutch. See dissertation, page 132; see also below.

A crucial role is played in Brouwer's considerations by the 'fitting of a building into another building'. This is something of such a great generality, that it will be rewarding to give some more detailed examples after the following remarks about the character of the defintion of a sub-building.

A building in another building can be given either *extensionally* or *intensionally*. As a simple example, take the mathematical building of the ring of the integers and the sub-building of the squares of the integers.[49] If a is the construction of a specific integer α and f the operation to be performed on a to get the square β of α then we can represent this in symbols as: $a : \alpha \rightarrow f(a) : \beta$. Now the *extensional definition* of the sub-building of all squares consists of the set of all pairs $\{< x, x^2 > | integer(x)\}$. The *intensional definition* consists of the operator f, operating on the construction of the integers. This second definition results in a 'dynamical building', with open places in it, to be filled by the operator f when operating on an integer as argument, instead of in the 'fixed' result from the extensional definition.

In the intensional case the operators are an essential part of the sub-building, in fact they are defining the sub-building (or even, they *are* the sub-building). This fits best the fragmentary comments of Brouwer on the building concept, since it is in agreement with his constructive approach: for every construction a of a part of the premise A, the (part of the) sub-building $f(a)$ is subsequently constructed. Note that we can consider the operator f as the sub-building, but that we can equally well view each individual case $f(a)$ as a sub-building in the building of, for instance, the squares, which, in turn, is then a sub-building in the larger building of the integers. This small excursion emphasizes the generality of the concept under discussion now.

On page 240 ff. we will work out several examples in order to illustrate and clarify Brouwer's constructive concepts of the 'hypothetical judgement' and 'building-in-a-building' for the 'particular case'. For the moment we will restrict our attention to the general case, where the construction of A gives no extra problem. Let us therefore take as an introductory example the following elementary case:

'If n is an even number, then n is the sum of two odd numbers.

Hence in the hypothetical judgement $A \rightarrow B$, A stands for 'n is even' and B for 'n is the sum of two odd numbers'. The logician is satisfied with the following proof: if n is even, then $n-1$ is odd; also 1 is odd and $(n-1)+1 = n$, which completes the proof for the logician. However, the constructive mathematician asks for a construction, which results in an even number n, or a proof that a given number is even. Departing from that construction or proof, two odd numbers have to be constructed whose sum is again n. So take an arbitrary number. A number is even if its last digit is even, in other words, if the last digit is 0, 2, 4, 6 or 8. If one of those conditions is satisfied, so if n is for instance 326, then we may construct the two numbers $n-1$ and 1 as the two odd ones,

[49] In fact, we see here the main building of arithmetic, in it the sub-building of the ring of integers, and in that sub-building the sub-sub-building of the squares of the integers.

and the constructive proof is completed.

Brouwer is only satisfied with a construction of (the proof for) B, hence this construction has to be carried out, and this requires first of all a construction of (the proof for) A. Brouwer's compelling argument for this interpretation must be that if f is a construction of the sub-building $A \to B$ (symbolically written as $f : A \to B$) and if a proves A (in symbols $a : A$, a is the construction of the sub-building A), then $f(a) : B$ that is, $f(a)$, the proof of B, is B's construction. Now, if we accept f as a construction which actually can be carried out, then also $f(a)$ must be actually and successfully performable and this is only possible if we take a seriously as a performable construction. Hence for the successful construction of $A \to B$ and of B, A has to be constructed first.

Note that in this view the *modus ponens* (that is, from $A \to B$ and A follows B), is an accepted technique; however, the derivation $[x : A] \vdash f(x) : B$, with an open assumption x, is not acceptable, since it does not provide a concrete proof-building for B.

Compare this (and note the diffenence) with the proof interpretation of intuitionistic mathematics as formulated by Kolmogorov and by Heyting.[50] In this proof interpretation, *'if A, then B'* has the character of a 'promise': 'I promise to give a proof of B for any proof of A you give me'. This was at the time of the dissertation unacceptable to Brouwer: neither with a promise, nor with a 'for the time being missing part A' one can successfully construct the sub-building B.[51]

Kolmogorov expressed his interpretation as follows:

> The meaning of the symbol $A \to B$ is exhausted by the fact that, once convinced of the truth of A, we have to accept the truth of B too.
>
> (...)
>
> Thus the relation of implication between two judgements does not establish any connection between their contents.[52]

The first part of this quote forms the core of Kolmogorov's argument, the second part is a sort of concluding afterthought.

Heyting explicitly stated in the earlier mentioned 'Einleitung' to his 1930-paper:

50 See the earlier mentioned [Kolmogorov 1925] and [Heyting 1930]; see also for a concise overview [Dalen, D. van 1994], chapter 5.

51 Note the difference between, firstly, the *naive logicist*, whose reasoning simply takes for granted the actual existence of the building and of the building-in-the-building, secondly, the *traditionalist*, whose reasoning consists of strings of logical arguments without asking for a possible construction of the premise-building, and, thirdly, Brouwer's requirement of a successful construction of the premise-building.

52 [Kolmogorov 1925], II, § 2.

> Die Formel $a \rightarrow b$ bedeutet im allgemeinen: Wenn a richtig ist, so ist auch b richtig.

In the modern formulation of the BHK proof-interpretation the hypothetical judgement runs as follows:

> a proves $\varphi \rightarrow \psi := a$ is a construction that converts any proof p of φ into a proof $a(p)$ of ψ.[53]

Brouwer's condition for a hypothetical judgement to be a theorem is therefore more strict than the one from the (later) proof interpretation for intuitionistic mathematics: an actual construction, instead of a 'promise', is demanded.

For this reason the *'Ex Falso'* principle ($\perp \rightarrow A$ for any A) is implicitly rejected by Brouwer since the premise $\perp$ of this hypothetical judgement cannot be the result of a proper mathematical construction. Brouwer insisted on a construction that in the end produces a proof for B (or in the case of a negative statement, a concrete blocking of every intended construction) and such a construction cannot depart from an inconsistent statement A. We noted earlier in the introduction to this chapter that the principle was adopted in [Heyting 1930], but that Kolmogorov argued for its rejection in his formalization of Intuitionistic Logic.[54]

So Brouwer not only asked for a construction to get B from an assumed A, but also the construction of A itself, followed by the result of the construction of B when the operation to construct B is fed with the one for A. Brouwer missed, so to speak, the higher order aspect of implication. If the construction of the implication is viewed as a manipulation on constructions, but not as constructions itself, one gets into trouble in Brouwer's view.

The nature of the conditions for the validity for $A \rightarrow B$ to be a theorem becomes clear in the remainder of the quoted paragraph:

> But this is no more than apparent; what actually happens is the following: one starts by setting up a structure which fulfills part of the required relation, thereupon one tries to deduce from these relations, by means of tautologies, other relations, in such a way that these new relations, combined with those that have not yet been used, yield a system of conditions, suitable as a starting point for the construction of the required structure. Only by this construction will it be proved that the original condition can be fulfilled.[55]

[53] See e.g. [Dalen, D. van 1994], chapter 5.

[54] His later acceptance was for reasons of epistemic closure; see the footnote on our page 226.

[55] Maar meer dan schijn is dat niet; wat men hier eigenlijk doet, bestaat in het volgende: men begint met een systeem te construeren, dat aan een deel der geëiste relaties voldoet, en tracht uit die relaties door tautologieën andere af te leiden zó, dat ten slotte de afgeleide zich met de nog achteraf gehoudenen laten combineren tot een stelsel voorwaarden, dat als uitgangspunt voor de constructie van het gezochte systeem kan dienen. Met die constructie is dan eerst bewezen, dat werkelijk aan de voorwaarden kan worden voldaan.

(We note that Brouwer speaks of the 'initial use' of part of the required relations, later on

The *particular case* which was mentioned in the quote on page 229, amounts to the following (Brouwer's style is always very briefly worded):

Suppose we have an existing mathematical structure M and some relation a between parts of the structure M which *defines* a substructure A of M. M might be for instance a theory like Euclidean solid geometry with as building blocks lines, triangles, cubes etc. and its definitions and theorems; A can be the premise of some complicated theorem (which premise itself can be a theorem) or even a complete subtheory (e.g. Euclidean plane geometry as a substructure of the Euclidean solid geometry).

In our 'particular case' the *construction* of A from its defining relations a is, according to the quoted text, not immediately clear; one *may suppose* A to be constructed and on the basis of that supposition one deduces a hypothetical judgement $A \to B$.

But the actual construction of the premise A is required before accepting the form $A \to B$ as a mathematical theorem, this in contrast with the sketched and quoted BHK proof interpretation from later years. The difference between the interpretation of the traditional logician and Brouwer's concept of the hypothetical judgement can be shown in the following summary of a proof of $A \to B$.

The 'traditional' logician's argument runs as follows:
– Assume A and, departing from this assumption, logically deduct B.
– In case of success we have proved $A \to B$, or, symbolically $\vdash A \to B$.
– In case of failure $\nvdash A \to B$.
– If the validity of A cannot be assumed because of an internal contradiction, then we succeed in the proof $\vdash A \to B$ on the basis of Ex falso.

Brouwer's argument goes as follows:
– Construct A.
– If the construction of A is not immediately clear we try to get more information, which is needed for its construction, from derived properties and relations for A. At first sight we seem to have not enough direct data to compose the srtucture A.
– In case of success, then, departing from this successful construction, construct B.
– If this is successful, then $\vdash A \to B$,
– If every construction of a proof of B, departing from A fails, then $\nvdash A \to B$.
– If we cannot construct (the proof of) A, then the proof of $A \to B$ cannot be performed either, it simply becomes impossible.

What actually happens, in Brouwer's view, can best be explained by means of the examples, presented below, in which also the idea behind the 'new relations, combined with those that have not yet been used' will become clear. In

followed by the application of 'those that have not yet been used'. Apparently, in the end *all* given relations are used in the construction of the antecedent, which, again, rules out the *ex falso* principle.)

addition, Brouwer's concept of *a building* and *a building-in-a-building* can be further clarified together with these examples.

Brouwer himself mentioned three examples of what he exactly meant,[56] and we will now give a rough sketch of the third example about a theorem in plane geometry:

The problems of Apollonius concern the construction of a circle, tangent to three given circles (or, in fact, tangent to any given combination of three from point, lines, or circles, but the former two are special cases of circles). The building is that of plane geometry, the 'building in the building' is the circle A, *defined* to be tangent to the three given circles, which forms then the antecedent of the hypothetical judgement $A \to B$. The consequent B is then a theorem, expressing some property of the tangent circle. The content of B is of no importance in the argument. The relevant thing is that A is only defined by the given relations of tangency to the three circles, and not yet constructed. B, as the consequent of the hypothetical judgement $A \to B$, does not yet have the status of a mathematical theorem, despite its logical construction from the assumed construction of A.

But 'one starts by setting up a structure which fulfills part of the required relations (...)'. Which required relations? Clearly the relations that define A, being the only relations coming up in the discussion so far. The result is a set of new relations, of relevance to the possible construction of A. Of course, properties of the consequent B may give clues for the desired construction of A. Only after the completion of the construction of A, B becomes the result of a proper construction, and by that a mathematical theorem.

The following quote is an additional remark by Brouwer, as if to cut the ground from under the logicians feet and to deprive him of the argument that the *logical* figure of the 'principium contradictionis' proves the impossibility of a construction, hence that *logic* teaches us the failure of a construction:

> 'But', the logician will retort, 'it might have happened that in the course of these reasonings a contradiction turned up between the newly deduced relations and those that had been kept in store. This contradiction, to be sure, will be observed as a logical figure, and this observation will be based upon the principium contradictionis.' To this we can reply: 'The words of your mathematical demonstration merely accompany a mathematical *construction* that is effected without words. At the point where you enounce the contradiction, I simply perceive that the construction no longer *goes*, that the required structure cannot be embedded in the basic structure. And when I make this observation, I do not think of a principium contradictionis.[57]

[56]See page 240.

[57]Dissertation, page 127: 'Maar', zal de logicus zeggen, 'het had ook kunnen zijn, dat bij de redeneringen een strijdigheid tussen de afgeleide en de nog wachtende voorwaarden was voor de dag gekomen, en die strijdigheid wordt toch waargenomen als logische figuur en bij

Brouwer obviously intended to stick to his constructivistic point of departure, but his argument in this quote is rather vague and not very persuasive. Its only (and probably unintended) merit is that it sketches the logician's way of working, together with that of the constructivist, which comparison certainly does not end up in favour of the constructivist. Whereas the logician observes a contradiction between two relations, and therefore stops his proof procedure, the constructivist gets stuck in his attempt. The logician may even claim to have gone the more 'elegant' road, but both will encounter a block.

But in favour of Brouwer we can state that at least he was consequent in his constructivism, and that he did not need any logical principles. In this single case his consistency required some extra labour. Notice that he did not mention the *ex falso* principle; he either rejected it here implicitly, or he simply did not think of it. If A cannot be constructed because of an internal inconsistency, then the construction of $A \to B$ cannot even be attempted because any chance of success is completely out of the question. In this sense the quote argues *against* the ex falso principle. This implicit rejection later on changed into an implicit acceptance, since Heyting clearly accepted this principle in his published formalization of intuitionistic logic, with which Brouwer agreed.

7.2.1 The notebooks on the building metaphor and on the hypothetical judgement; the 'building in a building'

The metaphor of mathematics as a 'building' or a 'construction' occurs frequently in the nine notebooks. Here are some examples, just as an illustration:

> (II–31) Hardly ever (just occasionally, cf. Hilbert) one can know whether the undefinables and their axioms are independent, except from the buildings shown.[58]

This quote, just like the next one, illustrates Brouwer's constructive and anti-formalist approach to mathematics. In fact the whole concept of the building metaphor includes this attitude:

> (V–32) A relation, found in a mathematical building, is itself a new building, which could find a place in the old one.[59]

Here we see the concept of the 'building-in-a-building', which we meet frequently in the quotes. It has to be interpreted, as we will see, as a sub-theory in a larger theory, or as one specific problem in a theory. Note, however, that the

het inzicht van die strijdigheid steunt men op het principium contradictionis.' Waarop kan worden geantwoord: 'De woorden van uw wiskundig betoog zijn slechts de begeleiding van een woordloos wiskundig *bouwen*, en waar gij de strijdigheid uitspreekt, merk ik eenvoudig, dat het bouwen niet verder *gaat*, dat er geen plaats is te vinden in het gegeven grondgebouw voor het opgegeven gebouw. En waar ik dat merk, denk ik aan geen principium contradictionis'.

[58] En ook zal men bijna *nooit* (een enkele maal, cf. Hilbert) kunnen weten of de undefinables en hun axioma's onafhankelijk zijn, anders dan uit de getoonde gebouwen.

[59] Een in een wiskundig gebouw gevonden betrekking is zelf een nieuw bouwwerk, dat in het oude een plaats kon vinden.

accompanying linguistic building has to be distinguished from the mathematical one:

> (VI–30) *Because* the mathematical language talks about exact objects, that language can be made exact itself (by logistics for a limited existing mathematical totality). However, when mathematics is constantly extending, then the system of signs has to be extended too, albeit in just a limited building in which repeatedly new small buildings are constructed.[60]

It is clear that only a *consistent* stock of building blocks makes a construction possible. In case of inconsistency, the constructive mathematician simply concludes to the impossibility of the construction, because his constructing activity is blocked.

In the eighth notebook it is emphasized that also the *completeness* of a set of axioms can only be concluded from the resulting construction; if two different constructions, satisfying the same set of axioms, are possible, then clearly the axiom set is not complete, that is, it can be extended without inconsistency or redundancy. This reflects the old idea that a set of axioms can determine a structure uniquely. If that is the case, then the axiom set is called 'categorical'; if no structure satisfies the set of axioms, it is called 'vacuous', and if more structures satisfy, the set is called 'ambiguous'.[61] The Skolem-Löwenheim theorem teaches us that for every model, satisfying a set of axioms, always another and larger model can be given which satisfies the same set, hence that no categorical set of axioms exists.

> (VIII–31) The 'major terms' which are used in mathematical syllogisms may be nothing else, but tautologies.
>
> (...)
>
> Likewise the axioms. Mathematical theorems then are constructions within the large building, of which the remotely separated parts cannot be surveyed at once intuitively. They serve as self-constructed road signs within that building.
>
> (...)
>
> Now, the axioms can be either complete or not, i.e. it can either or not be the case that other constructions are possible, satisfying the same axioms. The latter is the case [*i.e. no other construction possible*] if I had faithfully followed the building process itself in its totality.
>
> (...)

[60] *Omdat* de taal der wiskunde handelt over exacte dingen, daarom kan die taal zelf ook exact gemaakt worden (door de logistiek, voor een bestaand beperkt wiskundig geheel). Maar breidt de wiskunde zich steeds uit, dan moet ook het tekensysteem steeds worden uitgebreid, zij het ook in het beperkte gebouw, waarin alleen telkens nieuwe gebouwen worden gemaakt.

[61] See [Kleene 1952], chapter II, § 8.

> I sometimes can observe that it [*i.e. the set of axioms*] is not complete, because I indicate another building, clearly different from the given one (...) and still satisfying the axioms.[62]

In V–17 Brouwer discussed a typical and specific example of the *empirical judgement* which is not yet classified under a mathematical hypothesis: 'if it's freezing, then just planted trees will die'. About this type of judgement (which is, because of its empirical character, in fact a causal sequence), Brouwer commented as follows, thereby clearly referring to empirical hypothetical judgements in general which are excluded from this type of purely *logical* judgements:

> (V–17) This is not the logical implication ìf ..., thèn ..., but merely a simple coordination, albeit that it is often used as logic.[63]

But, Brouwer said, in case of an unjustified application of logical reasoning to empirical facts, mathematics would be degraded to an empirical science:

> (V–17) However, the wise man reads them [*this type of judgement in mathematics*] as tautologies, as two different approaches constructed about the same topic. Therefore its proof involves both predicates in one and the same building.[64]

In the sixth notebook Brouwer discussed again the hypothetical judgment on a few occasions and, just as in the last quote, usually on the basis of its 'application to the world', that is, the *empirical* hypothetical judgement, which is in fact an empirical causal sequence.

7.2.2 Conclusions about Brouwer's concept of the hypothetical judgement

Several options suggest themselves as an explanation or a reason for his different standpoint in regard to the hypothetical judgement, compared to the later proof

[62] De 'maioren', waarvan bij de wiskundige syllogismen wordt gebruik gemaakt, mogen niet anders zijn dan tautologieën.

(...)

Zo ook de axioma's. De wiskundige stellingen zijn dan samenbouwsels uit het grote gebouw, waarvan de ver van elkaar verwijderde delen niet zo direct intuïtief zouden zijn te overzien; zijn dus zelfgebouwde wegwijzers in dat gebouw.

(...)

Nu kunnen die axioma's volledig zijn of niet, d.w.z. het kan zijn, dat er nog andere gebouwen mogelijk zijn, die aan dezelfde axioma's voldoen, of niet. Het laatste is het geval, als ik met de axioma's trouw het bouwen zelf geheel heb gevolgd.

(...)

wel kan ik soms merken, dat het niet volledig is, doordat ik een ander gebouw aanwijs, d.w.z. een gebouw, dat duidelijke verschillen heeft met het gegevene, en toch aan de axioma's voldoet.

[63] Dit is niet het logisch implicerende àls ..., dàn ..., maar een eenvoudige coördinatie, al wordt het vaak als logica gebruikt.

[64] De wijze leest ze echter als tautologieën, twee verschillende benaderingen over hetzelfde opgebouwd. Het bewijs vergt dan ook beide leden in eenzelfde gebouw.

interpretation by Kolmogorov and Heyting. There is, however, not an obvious or a unique interpretation to be found.

First of all we can observe that neither in the published *Addenda and Corrigenda*[65] nor in any other paper did Brouwer correct this, in comparison with the BHK proof-interpretation, very strict form of constructivism, in which a premise actually has to be constructed before applying it in the derivation of other theorems.

Since Brouwer certainly was one of the first consistent constructivists, and probably also because this type of judgement is in the notebooks usually experience-based and drawn from daily life (hence a causal sequence), which is then given a mathematical expression only afterwards, it becomes understandable that he went, in hindsight, too far in his constructivism. In addition to that (or as a separate clarification) it could very well be that this type of judgement was not of fundamental importance to Brouwer, and was added just for completeness' sake. The unprejudiced reader of the first pages of Brouwer's third chapter will agree that Brouwer's comment on hypothetical reasonings is more or less meant as a side remark, inserted to correct the rather short-sighted view on the application of the hypothetical judgement in mathematical reasoning. It certainly cannot be taken as illustrative for his later and more mature views. This 'not carrying too much weight', this lack of *fundamental* importance makes it at least comprehensible why he did not correct it, or return to the topic on some later occasion.

On the other hand, Brouwer did accept hypothetical judgements in general. They are, after all, closely related to causal sequences. Also Brouwer highly appreciated Heyting's work in logic, in which this type of judgement holds a prominent position. Apparently, if only because of refraining from criticism, Brouwer consented with Heyting's interpretation of hypothetical judgements.[66]

7.2.3 Elaboration and examples of the 'particular case' of the concept of 'building-in-a-building'

The examples below are meant to clarify the meaning of the building metaphor. From the main text it is not immediately clear on what scale Brouwer was thinking. On the one hand we can imagine the macro version of the Euclidean plane geometry as a substructure, defined in the building of the Euclidean solid geometry. On the other hand there is the sub-building of the premise of an arithmetical theorem defined in the building of arithmetic. But the reader should note that in a footnote Brouwer mentioned three examples of a seemingly defined building *in* a mathematically constructed building, which examples suggest that, at least in this section, he was rather thinking in terms of the smaller scale of theorems and their proofs as the sub-building, defined in the larger building of a theory. The mentioned examples are:

[65] [Brouwer 1917b]; see [Dalen 2001], page 195.
[66] [Heyting 1930].

1. The proof of unicity by Hilbert and Lie for transformation groups with given properties.

2. The elementary construction problems in projective geometry, like the construction of a common harmonic pair to two given pairs of points.[67]

3. The problems of Apollonius.[68]

Hence, in Brouwer's third example, we should indeed view as the original building that of plane geometry. The sub-building then is the tangent circle to the three given circles, as earlier sketched on page 236.

We will not further discuss the other two rather complicated examples, but we will present, after some further clarification of the building-concept, a few simple and small-scale examples instead, which may equally well illustrate Brouwer's intentions (possibly even better in view of their simplicity).

As a first example we will have a look at one of the last letters from the Brouwer-Korteweg correspondence, in which Brouwer discussed the purpose of the third chapter: to clarify the fundamental difference between *logical* reasoning and *mathematical* reasoning and to show that 'mathematics, for the lack of a pure language, had to make do with the language of logical reasoning, whereas its thoughts do not proceed as a logical reasoning but as mathematical reasoning, which is something quite different'.[69] Improper use of the logical language in mathematics had already led to false notions in set theory.

Brouwer presented the following example:

> The theorem: 'if a triangle is isosceles it is an acute triangle'[70] is expressed as a logical theorem: the predicate 'isosceles' in the case of triangles is considered to imply the predicate 'acute', i.e. one imagines all the triangles of a given plane represented by the points of an R_6 and one then sees that the domain of R_6 representing isosceles triangles is contained in the domain representing all acute triangles. This is in fact true, and logical formulation and logical language can therefore safely be applied.
>
> But the thoughts of the mathematician, who because of the poverty of his language formulated this theorem as a logical theorem, proceed

[67] See for this example [Kindt 1993] for an introduction on involutions and [Prüfer 1939], page 122 ff. for the required construction.

[68] 'Problems' is the term used in the English translation by W.P. van Stigt; in Brouwer's original Dutch text it says the 'werkstukken van Apollonius', which could be translated indeed as the 'problems' or as the 'projects' of Apollonius. In the English literature on this topic the term 'tangency problems' is frequently used. This problem concerns the construction of a circle which is tangent to any combination of three from lines, points or circles; see e.g. [Molenbroek 1948], page 506 ff.

[69] Letter from Brouwer to Korteweg, dated 23rd January 1907. See [Stigt 1990], page 503; for the Dutch text see [Dalen 2001], pge 25.

[70] The reader has to overlook the defective formulation: 'isosceles' may have been a slip of the pen for 'equilateral', or, in case he *did* have an isosceles triangle in mind, he was referring to the equal base angles which are of course acute. None of this matters for the argument, keeping in mind the pressure of time and the informal occasion. Korteweg apparently understood what Brouwer meant to say.

> in a way quite different from the above interpretation. He imagines that he is going to construct an isosceles triangle, and then finds that either at the end of the construction all angles appear to be acute or that on the postulation of a right or obtuse angle the construction cannot be executed. In other words, he thinks the construction mathematically, not in its logical interpretation.[71]

The intention and the content of this quotation will be clear. The theorem is presented as a hypothetical judgement in the first sentence of the quote. The subsequent reasoning in the first paragraph is a logical one and can be viewed as a syllogism: the predicate 'isosceles' implies the predicate 'acute' and therefore any isosceles triangle will also be acute. The fact that this logical reasoning leads to the correct result is no guarantee for the correctness of logic as a basis for mathematics; a correct mathematical result can only be given by a proper construction.

The second paragraph gives the mathematician's construction, which can be interpreted in the building metaphor: the construction within the building of all triangles (which can itself be seen as a sub-building in the building of the plane geometry) results in the sub-building of the isosceles triangles. He begins with the execution of a construction and either observes the correct result or concludes to the impossibility of the construction when departing from two right or obtuse angles at the basis.

The 'mathematician's reasoning' again gives us the feeling that in case of a failure of the construction, the logical reasoning might be judged to be more elegant, compared to the blocked construction of the mathematician (compare page 237). That this is not a common conclusion for all hypothetical judgements, but rather an exception instead, can be seen in the examples below. Also notice that the last part of the argument seems to use the 'tertium non datur' argument: suppose the angles are not acute; this blocks our construction, hence the angles are acute.

For a further clarification of Brouwer's ideas, especially in regard to the 'particular case', we will present some examples of a simple arithmetical character, which may even look trivial but which, however, do require some basic and easily demonstrable constructions, and for that reason serve the purpose of an

[71] De stelling: *Als een driehoek gelijkbenig is, is ze scherphoekig* wordt gebruikt als een logische stelling – het predikaat *gelijkbenig* wordt voor driehoeken beschouwd, het predikaat *scherphoekig* te *impliceren*, d.w.z. men denkt zich alle driehoeken (van een plat vlak b.v.) afgebeeld door de punten van een R_6, en ziet dan, dat het gebied van R_6 dat de gelijkbenige driehoeken representeert besloten is in dat wat de scherphoekige driehoeken representeert. Dit is in casu werkelijk waar, de logische formulering en de logische taal kan hier dus veilig worden gebruikt.

Maar de wiskundige, die de genoemde stelling door armoede van taal met een logische stelling formuleert, denkt zich iets anders, dan de genoemde logische interpretatie. Hij denkt zich, dat hij een gelijkbenige driehoek gaat construeren, en dan hetzij dat na afloop der constructie de hoeken als scherp voor de dag komen, hetzij dat het blijkt dat bij postulering van een rechte of stompe hoek de constructie niet *gaat.*

elucidation.

First we will take the building M of arithmetic with as constituent parts the natural numbers, and as defined objects addition, multiplication, and exponentiation. Next we define the sub-building A as the, for the time being hypothetical, premise of the *theorem: If a is a power of 4 ànd of 6, then a is even.* Hence the sub-building A consists of the (construction of) all natural numbers, which are at the same time power of 4 and of 6. The conclusion B in the hypothetical judgement $A \to B$ is then 'a is even'. Note that the 'chain of syllogisms' (as Brouwer called the consecutive links in his chain of reasonings in the quoted paragraph on page 229) in this and in the following case has only one 'link', viz. the immediate conclusion 'a is even'.

The correctness of this hypothetical judgement can, in its logical form, easily be seen: if a is a power of 4 *and* a power of 6, then it certainly is a power of 4; every power of 4 is even, hence all such a are even, regardless of the actual existence of a; the logical form of the implication is satisfied but we do not know whether or not there is such an a. *The logician is satisfied but not so Brouwer.* The latter might argue as follows: It is not immediately clear whether or not the building A, the antecedent of the hypothetical judgement $A \to B$, can be constructed. *If* a number a, being at the same time power of 4 and of 6, exists, then it certainly will be even; but this does not guarantee us the existence of such an a. We have to deduce other properties, not immediately evident, which have to be satisfied by the elements of A, and which should make the actual construction of A possible, thereby raising it above the level of merely 'hypothesis'.

Well, we know that a has to be a power of 4, hence $a = 4^m$, which can be written as $a = 2^{2m}$. We also know that a has to be a power of 6, or $a = 6^n = (2 \cdot 3)^n = 2^n \cdot 3^n$. Since *both* properties have to be satisfied in A, $2^{2m} = 2^n \cdot 3^n$, or $2^{2m-n} = 3^n$. One can easily verify that $2m - n$ is a positive natural number, and now we apply the 'unused relation' that every power of 2 is even and every power of 3 odd, and we conclude that this equation has no solutions and that therefore A is empty. Hence the sub-building A cannot be the result of a construction in M. Therefore, in Brouwer's present interpretation, the theorem is neither true nor false, despite its *logical* correctness; all we can say is that Brouwer's proof procedure breaks down. Neither the theorem, nor its negation can be proved.

In view of Brouwer's remark about the 'logician's retort' on page 127 of his dissertation, which was briefly discussed on page 237 (Brouwer's introduction to chapter III), we can say that the logician, in his view, might encounter in subsequent deductions a contradiction between the deduced relations, seen by him as a logical figure based on the *principium contradictionis*, but that the mathematician has already at an early stage concluded to the impossibility of the construction of the premise, without the need of a principium contradictionis or of any other logical figure.

As a second simple example, looking even more trivial, we depart from the same building as in the previous example, but now we define the sub-building A as the hypothetical premise of the *theorem: If a is a power of 4 and at the same time a power of 8, then a is even.*

The logician immediately concludes to the correctness of this implication for the same reason as in the previous example, viz. all powers of 4 are even, hence a is even.

The constructivist again argues as follows: To fix the required deduced relations for the construction of A we note that all powers of 4 are even, hence the condition is already satisfied, and it remains valid after application of the second condition, namely that it also has to be a power of 8. But again we have the 'unused relation' (the relation between the powers of 4 and of 8) which can give us information about the content of the building. We know that all powers of 4, say 4^m, can be written as 2^{2m}. The second condition that a also is a power of 8 gives the following information: all powers of 8, say 8^n, can be expressed as 2^{3n}. Now in order to satisfy $4^m = 8^n$, also $2^{2m} = 2^{3n}$ has to be satisfied, which results in a non-empty sub-building satisfying both conditions. The constructivist has shown that he can construct numbers of the required type and that the result of any such construction is an even number.[72]

7.2.4 Concluding remarks about the introduction to chapter III

Brouwer's conclusion is that logic is dependent on, and subordinate to mathematics, and not the other way round. First mathematics as a mental construction comes into being and only *after* that its accompanying language which describes the construction process; the language of logical reasoning is a special case of the latter. The choice of the mathematical language and its accompanying signs is a rational choice, directed by economy, purpose and use, and is determined by culture and environment. So it is easily conceivable

> that, given the same organisation of the human intellect, and consequently the same mathematics, a different language would have been formed, into which the language of logical reasoning, well known to us, would not fit. Probably there are still peoples, living isolated from our culture, for which this is actually the case. And no more it is excluded that in a later stage of development the logical reasonings will lose their present position in the language of the cultural peoples.[73]

[72]The reader may contemplate for himself the arguments of the traditional logician versus those of Brouwer in the case of the numbers 3 and 5 in the above given examples; if a is at the same time power of 3 and of 5, then a is even. This is only seemingly worse. Another example is formed by the same hypothetical judgement for the numbers 4 and 7.

[73]Dissertation, page 129: dat bij dezelfde organisatie van het menselijk intellect, dus bij dezelfde wiskunde, een andere taal van verstandhouding ware ontstaan, waarin voor de ons bekende taal der logische redeneringen geen plaats zou zijn. En waarschijnlijk zijn er nog wel buiten het cultuurverband levende volken, waarbij dat werkelijk het geval is. En evenmin is voor de taal der cultuurvolken uitgesloten, dat in een ander ontwikkelingsstadium de logische

Hence mathematics as a constructed science is universal as far as the human intellect is such, but the accompanying language is certainly not universal. The mistake that, according to Brouwer, was made by the logicians was caused by the fact that they were looking at the language of logical reasoning in a mathematical way, as if that language itself was mathematics. As a result *theoretical logic* came into being. After that, mathematical language in particular was studied in a mathematical way which resulted in *logistic*, but both, theoretical logic and logistic, are empirical sciences and teach us nothing about the human intellect. The most they can teach us about is our own culture. Brouwer makes the comparison:

> And the language of logical reasoning is no more an *application of theoretical logic* (...) than the human body is an application of anatomy.[74]

As one of the examples of logical forms, Brouwer discussed the syllogism and compared it with the principle of the excluded middle:

> While in the syllogism a mathematical element could be discerned, the proposition:
>
> A function is *either* differentiable *or* not differentiable
>
> *says nothing;* it expresses the same as the following:
>
> If a function is not differentiable, then it is not differentiable.[75]

In classical logic the two statements are equivalent qua truth value, both being always true as can be concluded from their truth tables. Whereas the first one provides a useful insight, the second equivalent statement is completely devoid of interest.

We should, however, keep in mind that Brouwer must have had a constructive interpretation for the logical forms, hence there must have been another reason or a certain influence to state the given quote in this form. Van Dalen offered the following explanation of Brouwer's second reading of the principle:[76] Bellaar-Spruyt, the resident philosopher in Amsterdam, had incorporated in his philosophy course a treatment of traditional logic. It is likely that Brouwer attended the course, or obtained notes of it. Bellaar-Spruyt illustrated in his course the principle of the excluded middle by an instructive example:

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redeneringen er hun plaats zullen verliezen.

[74]Dissertation, page 130: En de taal der logische redeneringen is zomin een *toepassing van de theoretische logica* (...) als het menselijk lichaam een toepassing der anatomie is.

[75]Dissertation, page 131: Was in het syllogisme nog een wiskundig element te onderkennen, de stelling:

Een functie is òf differentieerbaar òf niet differentieerbaar

zegt *niets;* drukt hetzelfde uit, als het volgende:

Als een functie niet differentieerbaar is, is ze niet differentieerbaar.

[76]See for this interpretation [Dalen, D. van 1999], page 106 or [Dalen, D. van 2001], page 96.

> 'If you deny that Alexander was a great man, well, then you have to acknowledge that he was not a great man. Both opposite judgements, A. was a great man and A. was not a great man, cannot both be false.'[77]

From here it is only one step to Brouwer's second formulation.

Its content is empty, Brouwer claimed, but the logician turns it into the 'principle of *tertium non datur*'. Note that Brouwer did not yet reject the 'tertium non datur' for reasons of principle, but that he merely brushed it aside as useless. The rejection will only follow later (see the next section). As for syllogisms, he continued, they are a valid method of reasoning in the language of logical arguments, but they produce only a reliable outcome when used within a mathematically constructed system, just as the other logical principles are only reliable within such a structure. At this stage Brouwer introduced the earlier mentioned 'linguistic building':

> On the basis of linguistic images which accompany basic mathematical truths in actual mathematical structures, it is somtimes possible to build up *linguistic* structures, sequences of sentences, proceeding according to the logical laws. If it turns out that such a structure can never produce the linguistic form of a contradiction, then all the same it belongs to mathematics only in its quality of a linguistic structure, and it has nothing to do with mathematics outside of it, such as ordinary arithmetic or geometry ([*handwritten addition*]: except as an accompaniment, of which one can never be absolutely certain).[78]

Hence mathematical buildings are constructed and the *descriptions* of these constructions become building blocks for a different kind of building: the linguistic one. For instance, if we take the logical derivation in the form of natural deduction of B from a number of assumptions, which include A, then this construction is a building of the *linguistic* type. We have seen that Brouwer emphasized that this only describes a mathematical building *after* a proper mathematical construction of the premise A, which includes first of all that A has to have a 'mathematical content', i.e. that its most fundamental 'atomic' building blocks are formed by the ur-intuition of mathematics.

---

[77][Bellaar-Spruyt 1903], page 18: Ontkent gij, dat Alexander een groot man was, welnu, dan moet gij erkennen, dat hij geen groot man was. Beide tegengestelde oordelen: A. was een groot man, A. was niet een groot man, kunnen niet beide vals zijn.

[78]Dissertation, p 132: En wanneer het gelukt *taal*gebouwen op te trekken, reeksen van volzinnen, die volgens de wetten der logica op elkaar volgen, uitgaande van taalbeelden, die voor werkelijke wiskundige gebouwen, wiskundige grondwaarheden zouden kunnen accompagneren, en het blijkt dat die taalgebouwen nooit het taalbeeld van een contradictie zullen kunnen vertonen, dan zijn ze toch alleen wiskunde als taalgebouw en hebben met wiskunde buiten dat gebouw, bijv. met de gewone rekenkunde of meetkunde niets te maken [dan als accompagnement, dat nimmer geheel zeker is.]

Logic cannot provide this and therefore it is impossible to establish a general foundation of mathematics with the help of logic alone.

Brouwer's concept of logic is now applied to the following four subjects:
1. The foundations of mathematics on axioms,
2. Cantor's theory of transfinite numbers,
3. The logic of Peano–Russell,
4. The logical foundations of mathematics after Hilbert.[79]

Before extensively elaborating on some of these subjects (the others will be more or less briefly discussed), we will have a short look at, and spend a few remarks on, one other paper about the role of logic, viz. *the Unreliability of the Logical Principles* from 1908.

7.3 The unreliability of the logical principles

Immediately after obtaining his doctorate, Brouwer continued to work out his ideas on logic, especially on its use and applications in the area of the foundations of mathematics. He rapidly developed a more critical attitude, ending in a complete rejection on principle grounds of the principle of the excluded middle.

Already in the year 1908 he published a paper in the Dutch journal *Tijdschrift voor Wijsbegeerte*, entitled *De Onbetrouwbaarheid der logische principes.*[80] In 1919 it appeared in reprint in the collection *Wiskunde, Waarheid en Werkelijkheid*,[81] where it opens with the following added line:

> This essay could have been written today in the same form. The defended views have not found many supporters so far.[82]

In this essay, in which his mystic attitude again can be recognized, he dissociated himself from his view on the principle of the excluded middle, as laid down in his dissertation (see page 245). According to his new views in the paper under discussion now, a mathematical system which arises independent of any observation, but, departing from scientifically accepted premises, develops by logical reasoning alone, may very well lead to unreliable conclusions. Moreover, one should be very cautious when applying logical principles to mathematical systems which are 'void of life content',[83] (that is, which are not the result of a successful mental construction, based on the human ur-intuition), since in that case paradoxes may very well arise.

[79]See Brouwer's dissertation, page 133.
[80]*The unreliability of the logical principles*, [Brouwer 1908a], also included (in English) in [Brouwer 1975], page 107 – 111.
[81]*Mathematics, Truth and Reality*, [Brouwer 1919c].
[82]Dit opstel zou ook thans nog in dezelfde vorm geschreven kunnen zijn. Medestanders hebben de er verdedigde opvattingen nog weinig gevonden.
[83]van levensinhoud vrije systemen.

> [It can be shown that paradoxes] originate where regularities in the language which accompanies mathematics are extended to a language of mathematical words which is not connected with mathematics. Further we see that logistics is also concerned with the language of mathematics instead of with mathematics itself, consequently it cannot throw light on mathematics. Finally all the paradoxes vanish when we confine ourselves to speaking about systems which can be built up explicitly from the basic intuition, in other words, when we consider mathematics as presupposed in logic, instead of logic in mathematics.[84]

In this paper the following question is raised:

> Is it allowed, in purely mathematical constructions and transformations, to neglect for some time the idea of the mathematical system under construction, and to operate in the corresponding linguistic structure, following the principles of *syllogism,* of *contradiction* and of *tertium exclusum,* and can we then have confidence that each part of the argument can be justified by recalling to the mind the corresponding mathematical construction?[85]

The syllogism is considered to be a valid mode of reasoning; in fact it amounts to $b \subset c, a \subset b \Longrightarrow a \subset c$, which is merely a tautology, as Brouwer put it.

Also the principle of contradiction is incontestable. However, the principle of the excluded middle requires that every assumption is *either* true *or* false and this is equivalent to the statement:

> that for every supposed embedding of a system into another, satisfying certain given conditions, we can either accomplish such an embedding by a construction, or we can arrive by a construction at the unfeasibility of the process which would lead to the embedding.[86]

The validity of this principle is claimed to be equivalent to the validity of

[84] [Brouwer 1975], page 108, also [Brouwer 1908a], page 8. [Aangetoond kan worden dat paradoxen] ontstaan waar regelmatigheid in de taal, die wiskunde begeleidt, wordt uitgebreid over een taal van wiskundige woorden, die geen wiskunde begeleidt; dat verder de logistiek eveneens zich bezighoudt met de wiskundige taal in plaats van met de wiskunde zelf, dus de wiskunde zelf niet verheldert; dat ten slotte alle paradoxen verdwijnen, als men zich beperkt, slechts te spreken over expliciet uit de oer-intuïtie opbouwbare systemen, m.a.w. in plaats van logica door wiskunde, wiskunde door logica laat vooronderstellen.

[85] [Brouwer 1975], page 109, also [Brouwer 1908a], page 8. Kan men bij zuiver wiskundige constructies en transformaties de voorstelling van het opgetrokken wiskundig systeem tijdelijk verwaarlozen, en zich bewegen in het accompagnerend taalgebouw, geleid door de principes van *syllogisme*. van *contradictie* en van *tertium exclusum*, in vertrouwen dat door tijdelijke oproeping van de voorstelling der beredeneerde wiskundige constructies telkens elk deel van het betoog zou kunnen worden gewettigd?

[86] [Brouwer 1975], page 109, also [Brouwer 1908a], page 9: dat van iedere onderstelde inpassing van systemen op bepaalde wijze in elkaar hetzij de beëindiging, hetzij de stuiting op onmogelijkheid kan worden geconstrueerd.

the statement that every mathematical problem has a solution, and, as Brouwer put it, there is not a shred of evidence for this statement.[87]

To understand this claim we should again depart from a constructivistic point of view and from the BHK proof-interpretation. If someone advances the correctness of $A \vee \neg A$ for every statement A, then the BHK interpretation asks for a decision procedure to know which one, A or $\neg A$, is the case, and for a proof of the valid case. Hence if A is some mathematical theorem, then $A \vee \neg A$ expresses in the BHK proof interpretation Brouwer's representation of Hilbert's belief.

The fact that for finite discrete systems the principle is a valid and reliable one (we can check in finitely many steps the possibility or impossibility of an embedding), does in no way guarantee its reliability in case of a generalization of its use to arbitrary non-finite sets. However, an unjustified application of the principle to infinite systems will, according to Brouwer's conviction in 1907, never lead to a contradiction.

Brouwer mentioned as an example the generally accepted and trusted theorem that in the theory of transfinite numbers every number is either finite or infinite; but this theorem was never proved, and neither did its application ever lead to a contradiction.

Another example is the following problem:

> Do there occur in the decimal expansion of π infinitely many pairs of consecutive equal digits?[88]

Brouwer explicitly stated that, as long as this problem remains unsolved, we cannot claim that one of the two possibilities certainly must be the case.

In a way, this is a prototype of the so-called *Brouwerian counterexamples*,[89] although the decimal expansion of π is not yet applied to disprove a hitherto accepted mathematical theorem.[90] Brouwer will eventually *prove* the incorrectness of the statement that the principle of the excluded middle is a generally valid principle. This proof proceeds by means of counterexamples of classically valid theorems: it is false to state that every real number is either rational or

[87] In a footnote in his dissertation, Brouwer claimed Hilbert's conviction to be that unsolvable mathematical problems do not exist. However, Hilbert's claim is that for every mathematical problem either a solution can be given, or that a proof can be presented that no such solution exists. This is almost, but not exactly what Brouwer said, unless one considers the proof that no solution exists also as a solution. This, in fact, is Kolmogorov's position in 1932; see the footnote on page 226.

[88] Komen bij de decimale ontwikkeling van π oneindig veel paren van gelijke opeenvolgende cijfers voor?

[89] It is sometimes claimed that Borel was the first to apply the technique of counterexamples. See e.g. Von Plato's review of [Dalen, D. van 1999] in the *Bulletin of Symbolic Logic*, vol. 7, nr. 1 from March 2001. Von Plato refers to *Les Paradoxes de la Théorie des Ensembles* ([Borel 1908a]; see also [Borel 1972], page 1274; observe, however, that Borel's use of it is rather vague, compared to that by Brouwer. Moreover, Borel mentioned his example just in passing, without making any use of it in the strong way Brouwer did, viz. to *disprove* familiar and classically accepted theorems.

[90] We may refer to this type as a 'weak couterexample'.

irrational, or that every real number, which is unequal to zero, is either positive or negative, or that every natural number is either finite or infinite etc.[91]

7.4 The notebooks on the role of logic

The references in the notebooks are numerous, but in regard to the present subject there is hardly any interesting development in Brouwer's ideas, like there is in the case of sets and their cardinalities. Already in the first notebook the great distance between formal logic and 'living mathematics' is emphasized:

> (I–7) Mathematical logic deprived mathematics of all illusions of 'truth, which is in contact with life', and one notices to have worked only with a chimera; a chimerical extract, applied to reality, but not in direct contact with it.[92]

The expression in this fragment 'truth, which is in contact with life' shows a strong similarity with the phrase we met earlier: 'there are no non-experienced truths'. The two clearly express the same intention.

In this first notebook the questions about foundations of mathematics mostly refer to geometry; the possibility of a logical foundation of arithmetic shows up in later notebooks, when discussing the arithmetical aspects of Russell's *Principles of mathematics*. Meanwhile the logical-linguistic accompaniment of mathematics can play its useful role in communicating the result of a mathematical construction:

> (II–30) The logical construction is only needed to prevent harm to the mutual understanding between two persons, caused by a possible different intuition. That is why axioms are needed in the old instinctive Euclidean geometry.
>
> And that was caused by the fact that persons, who have a mutual understanding, did not reconstruct the whole system, but only some parts of it, which parts were held together by vague impressions; these parts then have to be made precise by axioms.[93]

But logic plays its limited role, outside the realm of pure mathematical thought in which the mental constructions are made on the basis of the ur-intuition alone and independent of any language or linguistic accompaniment:

[91]These are the 'strong counterexamples; see e.g. [Brouwer 1921b], [Brouwer 1923c] and [Brouwer 1930b].

[92]De mathematische logica ontnam aan de wiskunde alle illusie van 'waarheid die het leven raakt', en men merkt, met niets anders dan met een *hersen*schim te hebben gewerkt; een hersenschimmig extract, dat op de werkelijkheid is 'toegepast', maar haar niet raakt.

[93]Die logische opbouw is alleen nodig, om te voorkomen, dat eventuele verschillende aanschouwing bij twee personen schade zou doen aan de verstandhouding. Zo waren in de oude ondoordachte Euclidische meetkunde axioma's nodig.

En dát komt omdat de verstandhoudende personen vaak niet het hele systeem zelf hebben nagebouwd, maar alleen enkele delen, welke delen door vage indrukken worden samengehouden; die dienen dan wel gepreciseerd door axioma's.

> (V–2) Mathematical logic might trace and show some mistakes (although it is itself also a logical 'construction' in which mistakes can be made). But it stands outside the great intuitive logical recapitulations (which, on their own, might be difficult to analyze logically), with which science operates; hence it would throw away the essential in mathematical thought.[94]

The dependence of logic on mathematics, and the secondary role that logic plays in relation to mathematics, is expressed in the following paragraph, in which only a blocked construction is a foreboding of a contradiction in its linguistic accompaniment:

> (VII–24) It is certainly not possible to deduce the symbolic 'contradiction' for a symbol that accompanies a mathematically existing object; since in that case there would have been a contradictory construction-force during the construction, and the object could not have existed. Conversely, if we can show for an object which has to be constructed mathematically, that no symbolic contradiction can be deduced in the symbolic system, then the object can exist.[95]

The last sentence of this quote illustrates that, occasionally, Brouwer is still very far from his view in the dissertation. Popularly reformulated it says 'if an object figures consistently in a symbolic system, then it can exist'. Here he is still rather close to Poincaré, whose only criterium for existence of a mathematical object is consistency. Although Brouwer in this quote only claimed that such an object *can* exist, this is still more or less in contrast with page 132 of his dissertation where Brouwer stressed that a logical and logically consistent structure has nothing to do with a mathematical structure.

But other quotes from earlier and later notebooks show us that a mathematical construction remains primal to all logic:

> (VIII–19) The 'rule of contradiction' should only be applied to what is 'self-constructed', and likewise all laws of logic.[96]

As for the relation between language and logic, Brouwer is very brief:

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[94]De mathematische logica kan misschien wel hier of daar fouten opsporen en aantonen, ofschoon ze zelf even goed een logisch 'bouwen' is, waarin je fouten kunt maken. Maar de grote intuïtieve logische samenvattingen die alleen zeer lastig logisch geheel te ontleden zouden zijn, waarmee de wetenschap werkt, daar staat ze buiten; de hoofdzaak in het wiskundig denken zou ze dus wegwerpen.

[95]Voor een symbool, dat een wiskundig bestaand ding begeleidt is dan zeker niet mogelijk, de symbolische 'contradictie' af te leiden; immers dan zou contradictoire bouw-dwang bij het bouwen zijn geweest, en het ding had niet kunnen bestaan. Aan de andere kant, kúnnen we voor een wiskundig te bouwen ding aantonen, dat geen symbolische contradictie is af te leiden in het symbolisch systeem, dan is het ding bestaanbaar.

[96]De 'Satz vom Widerspruch' laat men alleen gelden omtrent het 'zelf opgebouwde', en evenzo alle logische wetten.

> (IX–2,3) Language *is* not logical, but it is an understanding by means of sounds in coarse-material things, shaped by habit.
>
> (...)
>
> But it is nonsense to view your own language mathematically.[97]

The synopsis

In the summarizing synopsis of the notebooks, there are a few remarks about the role of logic in the construction of mathematics:

> (from chapter 1) In the deductions of mathematical logic one lacks all guiding stimulus if one does not keep in mind the *meaning*. It has always to be viewed as abstracted from something living, and that is only successful for something mathematical.[98]

In this, as well as in the following two quotes, a comparison with literature can be made. Writing a novel without any 'contact with life' results in an empty collection of words and phrases, in which no reader will perceive any recognizable situation, despite the possible grammatical and syntactical correctness of the composing sentences.

'Contact with life' in mathematics then means 'born out of the intuition'.

> (from chapter 6) Logic never can (...) be of any help in explaining life, since it is abstracted from life, and only from life which is viewed mathematically.[99]

> (from chapter 6) One should not compare mathematics with ordinary logic, since logic is itself second order mathematics.[100]

7.5 The founding of mathematics on axioms

At the end of the introduction of his third chapter Brouwer mentioned four subjects, to be discussed in more detail in explanation of his claim that mathematics in independent of logical laws as we sketched this in the introduction to his third chapter.[101] We will limit the discussion of the beginning of the first subject to

[97] De taal *is* niet logisch, maar een verstandhouding door klanken in grof-materiële dingen, door gewoonte gevormd.
(...)
Maar het is onzin, je eigen taal wiskundig te bekijken;

[98] Bij het afleiden in de mathematische logica zonder aan de betekenis te denken, mist men alle leidende stimulans. We moeten haar telkens weer geabstraheerd denken uit iets levends, en dat gaat alleen voor iets wiskundigs.

[99] Logica kan (...) nooit het leven helpen verklaren, want is geabstraheerd uit het leven, en nog wel alleen uit het door een wiskundige bril geziene leven.

[100] Met gewone logica moeten we wiskunde niet vergelijken, want die logica is zelf wiskunde (der tweede orde).

[101] See page 247.

a concise summary of its content. However, towards the ending Brouwer made some interesting remarks, which deserve a more detailed treatment.

Understandably, Brouwer's main claim and conclusion will be that a successful completion of a construction of a mathematical building *is* a proof of its consistency and that no axiomatic foundation is needed for this. A consistent set of axioms on itself is not a sufficient condition for the existence of the mathematical system described by that axiom set.[102] Hilbert, as formalist, proved the consistency of geometry, thereby taking consistency of arithmetic for granted. A proof of the latter then has to be completed with the help of logic, which for that reason requires an independent development, parallel to arithmetic.

Brouwer agreed with the geometers of his day that the five axioms of Euclid are insufficient as a foundation for the *logical* building of geometry that was named after him. He is explicitly referring here to the linguistic accompaniment of the mathematical structure of geometry. Brouwer proposed two possible explanations for Euclid's omission in the axiomatization, which shows that the axiom-concept is in itself a sensible one for him, albeit that axioms only serve a practical purpose in the solution of problems and in communication to others.[103] In this limited sense they can be useful, as long as we are aware of the fact that they can never be the ultimate foundation of the mathematical structure of geometry, but that they only play their role in the structure of the linguistic building that *describes* the mental construction of geometry.

The place of the axioms in the mathematical hierarchy is expressed in the synopsis of his notebooks as follows:

> It is just as foolish to view a tree merely as a weight of planks, as it is one-sided to view mathematics as an axiomatic system.[104]

Hence, *first* comes the mathematical structure and only thereafter, if needed, the axiomatic foundation for its accompanying linguistic structure.

It did not cost the mathematicians too much trouble to improve the linguistic building. After all, they were not operating in mathematics proper, but in its linguisitic and logical description. Especially Pasch and Hilbert added many, hitherto tacitly assumed, axioms to the original set of five.[105] However, Hilbert's building became, like the buildings of other mathematicians of those days, in Brouwer's terms a 'linguistic building of pathological geometries'. But, Brouwer stated, the reproach at Euclid about the incompleteness of his axiom set is dropped in the following cases (of which especially the first one fits Brouwer's ideas about the role of axioms):

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[102] Compare this to Gödel's completeness theorem for predicate logic; see further page 255.

[103] See page 261, the quote II–35.

[104] Even dwaas als het is in een boom slechts een gewicht aan planken te zien, even eenzijdig in de wiskunde een axiomasysteem.

[105] For instance for Hilbert, see *Die Grundlagen der Geometrie*, [Hilbert 1899a]. The third edition of this work, with the addition of seven appendices, appeared as his *Festschrift* in 1909 under the official title *Grundlagen der Geometrie*, [Hilbert 1909].

– Euclid conceived his mathematical structure of geometry as a *mental* construction which was finished beforehand, his reasonings about it merely serving as an elucidation of his already constructed building.

– Another possibility, this time not fitting Brouwer's concept of the purpose of axioms, is that Euclid made the mistake that so many others have made, who thought that they could reason logically about other subjects than mathematical structures built by themselves.

In Brouwer's view, the geometries of Lobatchevski and Bolyai were just logical-linguistic buildings, and only Riemann showed the proper way in founding geometry by viewing space as a 'Zahlenmannigfaltigkeit'. Of course Brouwer saw this as the proper way because it allowed him to return to the ur-intuition as the ultimate foundation; but, as Brouwer regretfully established, this way was not followed consequentially to the end by Riemann.

Intuitive mathematics was introduced by Hilbert (and by others) only for a proof of consistency and for a proof of independence.[106] Both proofs proceed, in modern terminology, by presenting a model, or, in Brouwer's terminology, for the case of a proof of consistency by:

> (...) indicating a system with the property that a certain set of axioms, and consequently all the theorems deduced from it, can be held to express properties of that system.[107]

and for the proof of independence:

> In order to prove that some axioms cannot be logically deduced from some other axioms, a mathematical system is indicated such that the latter axioms can, but the former cannot be held to express properties of that system.[108]

This short summary covers the essentials of the main text of the first subject, but some noteworthy remarks follow in a long footnote, as well as in the remainder of the main text, which deserve extra attention, although they are no longer only concerned with the foundation of mathematics on axioms.

A first remark which merits a closer analysis is made by Brouwer in the footnote with the phrase in the main text about the requirement to present a model in order to prove consistency. In this footnote Brouwer gave an important elaboration and clarification of this idea, which takes on an interesting character in the light of much later revolutionary developments, in particular in

[106](dissertation, page 85) In his hand-corrected copy of the dissertation, Brouwer added the remark that the investigations by Schur, Hilbert and Pasch did not have, as an addition to Euclid, any *mathematical* value. At the most (some) logical value may be ascribed to their investigations.

[107](dissertation, page 137) (...) aangeven van een systeem, waarvan een zeker stelsel logische axioma's en dus ook alle eruit afgeleide stellingen kunnen worden beschouwd, eigenschappen uit te drukken.

[108]dat men, om aan te tonen, dat een zeker axioma uit zekere andere niet logisch is af te leiden, een wiskundig systeem aangeeft, waarvan de laatste wel, het eerste niet, beschouwd kunnen worden eigenschappen uit te drukken.

Gödel's work. Brouwer stressed that 'indicating a mathematical system' (that is, defining a sub-building in a mathematical building), such that the axioms correspond to certain properties of that system, suffices to prove the consistency of that set of axioms and their corollaries.

> It is clear that indicating a mathematical system such that the axioms *might* accompany properties of that system, suffices to prove that never two contradicting theorems can be deduced from the axioms, for contradictory theorems cannot hold for a mathematical building.[109]

Although the presentation of a model, according to Brouwer, only proves the existence of the *logical* structure (so one sensed intuitively that a mathematical structure needs no proof for its existence), this method was general practice in nineteenth-century geometry. However, Hilbert had to give up these modeltheoretic constructions, when he started to consider formal arithmetic. In 1904 Hilbert lectured about 'die Grundlagen der Logik und der Arithmetik' at the Heidelberg mathematical conference. We will not go into te details of this lecture now,[110] but Hilbert sketched there his 'program': When founding geometry, certain difficulties of arithmetical nature could be left aside for the moment, but for a solid foundation of both, geometry and arithmetic, the latter had to be axiomatized. And for a proof of consistency of arithmetic with the help of logic, the laws of logic and arithmetic had to be developed simultaneously, but independently from one another, to avoid unintentional use of arithmetical concepts in logic. This consistency proof was initiated in the Heidelberg lecture, but the axiomatization of arithmetic was introduced already five years earlier in Hilbert's paper *Über den Zahlbegriff*,[111] and Brouwer explicitly mentioned and referred to this paper in the long footnote under discussion now. In the said paper Hilbert argued that the concept of number and the basic operations on numbers usually are introduced by a *genetic method*, whereas in the construction of geometry an *axiomatic method* is the familiar way of introduction. Hilbert then raised the question whether these methods are the proper ones for the two different branches of mathematics, at the same time answering it by stating that both branches need a complete *axiomatic* foundation. Therefore a set of axioms for arithmetic is presented.

For Brouwer the successful *construction* of a mathematical structure (a sub-building in a building) is sufficient to prove the consistency or the existence of that structure, and no axioms for its foundation are needed. He has, still in the same footnote, the following comment on Hilbert's procedure: The presentation of a mathematical model (arithmetic) to prove the consistency or the existence

[109](dissertation, p 137, first sentence of the long footnote) Het is duidelijk, dat door het aangeven van een wiskundig systeem, waarvan de axioma's eigenschappen *zouden kunnen* accompagneren, bewezen is, dat nooit twee strijdige stellingen uit die axioma's kunnen worden afgeleid, want twee strijdige stellingen kunnen niet van een wiskundig gebouw gelden.

[110]These details will be discussed when treating the last subject from Brouwer's list of four from his third chapter; see page 296.

[111]See [Hilbert 1899b].

of a logical system (geometry) might give the impression that the mathematical model needs no existence proof. But not so for Hilbert: also arithmetic has to be captured by an axiomatic method, and he devised a set of axioms for that mathematical system. The theory, derived from that axiom set, required in its turn a consistency proof, since Hilbert did not recognize 'intuitive mathematics' in the Brouwerian sense, that is, for him the successful construction of the building of arithmetic is in itself not sufficient as a consistency proof.
Brouwer's conclusion was the following:

> 1^0 he [*i.e. Hilbert*] has not made any progress, compared to the preceding stage, 2^0 the consistency of the axioms does not involve the existence of the corresponding mathematical system, 3^0 even if the mathematical system of reasonings exists, this does not entail that it is *alive*, i.e. that it accompanies a sequence of thoughts, and even if the latter is the case, this sequence of thoughts need not be a *mathematical* development, so it need not be convincing.[112]

Brouwer ended his comment on Hilbert's work with the following phrase (still in the same footnote):

> We will see later on how Hilbert tried to escape from this difficulty and in how far he succeeded.[113]

This, again, refers to the fourth subject, 'the logical foundations after Hilbert' (page 296) from Brouwer's third chapter. The main difference between Brouwer and Hilbert in regard to this topic is obvious: for Brouwer, a properly and successfully built construction *is* a consistency proof, whereas for Hilbert the model of arithmetic, on which the foundation of geometry is based, needs to be axiomatized and this axiomatized system needs a consistency proof too. So, according to Brouwer, 'no progress is made by Hilbert'; another construction is needed which has to prove the consistency of the axiom set of arithmetic and its corollaries. A sort of regress threatens to set in. Of course Hilbert was aware of that, but he took a different route; he realized that, in founding arithmetic, no further appeal to a more basal discipline is possible. Therefore consistency had to be proved from 'outside', with the help of *logic*; but to avoid difficulties, the laws of logic and of arithmetic had to be developed simultaneously and indepently from one-another. Hence for Hilbert consistency of arithmetic had to be proved with the help of logic (his Heidelberg lecture was an initial impetus to this program), whereas for Brouwer logic can never be a basis for any branch of mathematics, and consistency alone never guarantees us the existence of an

[112](dissertation, p 138) 1^0 hij is dan nog even ver, als zo-even, 2^0 volgt uit de niet-strijdigheid der axioma's nog *niet* het *bestaan* van het bijbehorend wiskundig systeem, 3^0 volgt uit het *bestaan* van dat wiskundig redeneersysteem nog niet, dat dat taalsysteem *leeft*, m.a.w. een aaneenschakeling van gedachten begeleidt, en dán nog niet, dat die aaneenschaleling van gedachten een *wiskundige* ontwikkeling is, dus overtuigingskracht bezit.

[113](dissertation, page 138) We zullen beneden zien, hoe Hilbert zich hieruit heeft trachten te redden, en in hoeverre hij daarin geslaagd is.

accompanying mathematical structure.

So far the discussion of the first part of the long footnote (altogether stretching over four pages of the dissertation), and some direct or indirect conclusions from it.

In the second part of this footnote Brouwer focused his attention on Dedekind's *Was sind und was sollen die Zahlen*,[114] where an attempt is made to construct the arithmetic of the integers by means of logic, thereby departing from the primitive concepts of a certain logical system (a 'mathematical building of words'), which has as axioms linguisitic representations of relations between the primitive concepts, and which is constructed *finitely* (that is, without using complete induction).

But, Brouwer stressed, in order to give a mathematical meaning to such a logical construction, we cannot do without a mathematical proof of existence (i.e. a *construction*), and such a proof needs complete induction as a tool. This proof then results in a direct way in a much simpler and more natural system of arithmetic, compared to Dedekind's artificial system which has no mathematical meaning in itself.

In Brouwer's view, complete induction is a necessary tool in the construction of the system of the natural numbers, of the system of the integers and of arithmetic, and this construction is based on the ur-intuition alone, as it should. Remember that the principle of complete induction is neither a theorem, nor an axiom; it is a natural act of intuition-based construction instead.[115]

To conclude the discussion about this long, but important footnote, we emphasize once more that, given a set of axioms and their corollaries (a logico-linguistic building), the consistency of that totality of axioms and corollaries is not a sufficient basis for its existence. If, however, we can build an arithmetical structure, of which the basic properties can be *described* by that axiom set, then the success of that construction *is* the consistency proof of that set and of the complete theory following from it. And for that construction the principle of complete induction is a necessary and natural tool. The role of the axiom set is merely that of a handy device for a concise description of that system and for problem solving.

It is interesting to compare the conclusion about the need for a mathematical construction in order to prove the consistency of an axiom set that describes it with Gödel's second incompleteness theorem:

The formal system of Peano arithmetic cannot prove its own consistency.

This claim of unprovability of the consistency of a formal system within that formal system itself is not proved in the strict sense by Brouwer, but it is made

[114] [Dedekind 1930b].

[115] Thesis II from the list of 21 theses at the end of Brouwer's dissertation; see also page 316. Note that in 1902 Poincaré commented on this principle in a similar way in *La Science et l'Hypothèse*, judging it to be 'une propriété de l'esprit lui-même; see page 26. Brouwer certainly knew this comment, and agreed with it.

a plausible conclusion from his reasoning which was based on a constructive argument.

A second remark by Brouwer, which is worth extra attention, is made in the main text, at the end of Brouwer's discussion of this first subject, where he once more underlined that the origin of the 'pathological geometries' lies in the juggling with axioms in the linguistic building of geometry. But the axiomaticians still want mathematical application for their logical system:

> Now the following question arises: suppose we have proved by some method, without thinking of mathematical interpretation, that the logical system, built up out of certain linguistic axioms, is consistent, i.e. that two contradictory theorems can occur at no stage of development of the system; suppose further that afterwards we find a mathematical interpretation of the axioms (which of course will require the construction of a mathematical system whose elements satisfy certain given mathematical relations); does it follow from the consistency of the logical system that such a mathematical system *exists*?[116]

Brouwer accepted for a moment the possibility to prove the consistency of a linguistic building *without* appeal to a mathematical structure. But then a mathematical interpretation of the *logical* consistent set of linguistic axioms and its resulting structure still does *not* guarantee the existence of the corresponding *mathematical* building. So the existence of a model is not guaranteed by a mathematical interpretation of a consistent accompanying linguistic building. As an example Brouwer mentioned that there is no proof for the fact that a non-contradictory set of logical conditions for a finite number is sufficient for the existence of a 'model' for that number. This example is not further elaborated in the dissertation, but a specific case can be found in the technique of the 'Brouwerian counterexamples': Is there a number a (i.e. can we construct that number) such that $a = 0$ if the Goldbach conjecture is correct and $a = 1$ if it is false?

This condition is logically consistent since the Goldbach conjecture cannot be true and false at the same time, hence from a purely logical point of view a has to be either 0 or 1. But mathematically, a cannot be constructed as long as the

[116](Dissertation page 141) Nu rijst de vraag: gesteld we hebben op een of andere manier, zonder aan wiskundige interpretaties te denken, bewezen dat het uit enige taalaxioma's opgebouwde logische systeem niet strijdig is, d.w.z. dat op geen moment der ontwikkeling van het systeem twee strijdige stellingen komen; vinden we vervolgens een wiskundige interpretatie voor de axioma's, (die dan natuurlijk bestaat uit de eis, een wiskundig gebouw te construeren met aan gegeven wiskundige relaties voldoende elementen), volgt dan uit de niet-strijdigheid van het logische systeem, dat zulk een wiskundig gebouw *bestaat*?

(N.b. As a handwritten correction Brouwer replaced the beginning of this quote *'Now the following question arises'* by *'Now at least these investigations would be of some importance to a logical foundation of mathematics, if the following question had to be answered in the affirmative'*, which makes the question more imperative and directed.)

Goldbach conjecture has not been settled. So the existence of a cannot (yet) be confirmed.

Also the second number class does not exist as a totality, despite the fact that the defining terms of this class form a logical consistent set of conditions. For the construction of a mathematical building, logical consistency of its defining terms is of course a necessary condition, but not a sufficient one.

A third and last remark of importance can again be found in a footnote, this time on page 142 of the dissertation; it is referring to the just given example that a consistent set of conditions in the definition of a certain number does not automatically include its existence. It shows again an interesting analogy with Gödel's later work:

> A fortiori it is not certain that any mathematical problem can either be solved or proved to be unsolvable, though Hilbert, in 'Mathematische Probleme', believes that every mathematician is deeply convinced of it.[117]
>
> But for this question as well, it is of course uncertain whether it will ever be possible to settle it, i.e. either to solve it or to prove that it is unsolvable (a logical question is nothing else than a mathematical problem).[118]

The first paragraph of the quoted footnote claims that the truth of Hilbert's dogma is uncertain. In regard to any properly and consistently given mathematical problem, Hilbert is convinced that eventually either this problem can be solved, or that its unsolvability can be proved. But Brouwer claimed that neither of the two might be the case and that therefore theorems (or 'problems') *may*

[117] Hilbert's conviction (the first half of the footnote) can be read in [Hilbert 1900] (See for this also [Hilbert 1932], volume III, page 297) in the following terms. He stated as a comment on the solutions of some long standing mathematical problems which were finally found:

> (...) welche in uns eine Überzeugung entstehen läßt, die jeder Mathematiker gewiß teilt, die aber bis jetzt wenigstens niemand durch Beweise gestützt hat – ich meine die Überzeugung, daß ein jedes bestimmte mathematische Problem einer strengen Erledigung notwendig fähig sein müsse, sei es, daß es gelingt, die Beantwortung der gestellten Frage zu geben, sei es, daß die Unmöglichkeit seiner Lösung und damit die Notwendigkeit des Mißlingens aller Versuche dargetan wird.

And on the next page, at the end of the introduction and just before the presentation of the 23 open mathematical problems in the year 1900:

> Diese Überzeugung von der Lösbarkeit eines jeden mathematischen Problems ist uns ein kräftiger Ansporn während der Arbeit; wir hören in uns den stetigen Zuruf: *Da ist das Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik gibt es kein Ignorabimus!*

[118] Het is dus a fortiori niet zeker, dat van elk wiskundig probleem òf de oplossing kan worden gegeven òf logisch kan worden aangetoond dat het onoplosbaar is; iets, waarvan intussen Hilbert in 'Mathematische Probleme' meent, dat iedere wiskundige ten innigste is overtuigd.

Maar van deze kwestie zelf is het natuurlijk ook weer niet zeker, dat ze ooit zal kunnen worden afgedaan, d.w.z. òf opgelost, òf als onoplosbaar aangetoond (een logische kwestie is ook niets dan een wiskundig probleem).

exist, of which neither the solution can be constructed, nor the non-existence of such a solution be proved. Note that Brouwer did not *prove* anything here, but merely expressed his doubts as to the possibility of a construction on the basis of logical consistency of its conditions alone.[119]

If we indeed interpret the first paragraph of the last quote as expressing that, according to Brouwer and contrary to Hilbert's conviction, there might be a problem, of which neither the solution can be constructed, nor the impossibility of such a construction can be proved, then the second paragraph expresses the doubt that the first paragraph will eventually be decided. In other words, the first paragraph claims that there may be problems of which we do not know whether they are solvable or provably unsolvable, whereas the second paragraph leaves open the chance that the possible existence of such undecidable problems is itself undecidable, so that we can neither indicate such an undecidable problem, nor show that there is none.

But, with reference to this last sentence, there appears another interesting remark, which shows that Brouwer had an argument for the 'ignorabimus'. It was written on one of the loose sheets, found in the ninth notebook, in the form of the following note:

> Will one ever be able to prove about a problem, that it can never be decided? No, since such a proof should proceed via reductio ad absurdum. One should have to say: Suppose the question is decided for sentence a, and deduce from that a contradiction. But then one would have proved that *not* a is true, and the question remained decided.[120]

According to Brouwer one can never *prove* a concrete problem to be undecidable, i.e. we can never be positive that a certain problem neither can be solved nor that its unsolvability can be proved. Hence Brouwer did not agree with Hilbert's dogma, but he did not go as far as a full acceptance of an 'ignorabimus' either; in other words, Brouwer is not supporting the view that every mathematical problem will be decidable (i.e. that there is no ignorabimus), but he is neither defending the other extreme that there *is* ignorabimus, i.e. that there are problems that are provably undecidable. He just leaves open its possibility, he accepts it as an option.

[119]Later, Brouwer will give counterexamples of numbers of which one cannot decide whether they are positive or negative (or equal to zero), hence that neither a claim about this number, nor the negation of this claim can be proved (at least as long as some outstanding mathematical problem remains unsolved; in case of a solution, however, such a problem can always be replaced by another unsolved one). None of the mentioned possibilities can, in that case, claimed to be true!

[120]Zal men nu ooit van een vraag kunnen bewijzen, dat ze nooit uitgemaakt kan worden? Neen, want dat zou moeten uit het ongerijmde. Men zou dus moeten zeggen: Gesteld dat het was uitgemaakt in zin a en daaruit afleiden, tot een contradictie kwam. Dan zou echter bewezen zijn, dat *niet* a waar was; en de vraag bleef uitgemaakt.

(Heyting was the first to put this in print in [Heyting 1934], page 53 ff. A more detailed treatment can be found in the paper by Martin-Löf *Verificationism then and now* in [Pauli-Schimanovich and Stadler 1995], pp. 187–196.

It cannot be stated with certainty that he held this view already in 1907 when taking his doctoral degree so that at least the *content* of the loose sheet from the ninth notebook was familiar to him at that time. Perhaps this view materialized only after February 1907, and perhaps before that time Brouwer was just attacking Hilbert's belief as being unfounded, without, however, resorting to the opposite position of an unconditional ignorabimus.

Anyway, Brouwer was not certain whether or not unsolvable mathematical problems exist. Only Gödel's first imcompleteness theorem can be viewed as an answer to this doubt: There exist in Peano Arithmetic, or in any decidable ω-consistent extension of P.A., true sentences that cannot be proved in P.A.

In addition to the notes and arguments on the 'loose sheet', we can add the following comment: In the first place, it is written on a loose sheet, found in a notebook. Since the last pages of the last notebook contain notes, definitely written after the date of the defence of the dissertation, it is not certain when exactly these notes were made. In the second place, and even more important, the fragment just quoted is crossed out on the relevant page, it never appeared in print and Brouwer never returned to it. One can wonder why Brouwer crossed it out and why he did not come back to it on some later occasion; it certainly proves something. One can hardly imagine that Brouwer would not have seen its importance. As Van Dalen noted in his comment, it explains why Brouwer never tried to find some absolutely undecidable problem.[121]

Note that Brouwer's argument does not disprove Hilbert's conviction; it just argues that there might be *ignorabimus* in mathematics. It does not disprove Gödel's incompleteness theorem either, since Gödel only referred to true sentences in a system, which sentences can be neither proved nor disproved within the system itself.

7.5.1 The notebooks on Axiomatic foundations

There is, again, hardly any development in Brouwer's point of view in regard to the foundation of mathematics on axioms. Already in the second notebook he wrote that the independence of a set of axioms can only be shown by the result of a construction, i.e. by a model:

> (II–31) Almost never can one be sure of the independence of the undefinables and their axioms, except for the demonstrated constructions.[122]

In the same notebook the *cause* of man's need for axiomatization is said to be due to the need to communicate the result of a construction to others, with, additionally, the certainty that the hearer obtains the same mental picture of the construction:

[121] Extract from a footnote from [Dalen 2001], page 174.

[122] En ook zal men bijna *nooit* kunnen weten of de indefiniabelen en hun axioma's onafhankelijk zijn, anders dan uit getoonde gebouwen.

> (II–35) It is the *words* that cause axiomatization, to hold together the will of the individuals. Because it turns out that those words cannot be fixed securely, and in order to keep them secure, one axiomatizes them.[123]

But then, still in the second notebook, one should be warned that axiomatization is fundamentally different from a mathematical construction. Only the latter gives us a mathematical building, which, by virtue of its existence, guarantees consistency. Axiomatization is, in fact, leading us away from the living mathematics which is essentially languageless:

> (II–38) In mathematics, just as in the arts, it is dangerous to depart from the 'Schaffe Künstler, rede nicht', since also here the basic principles *cannot be expressed*, but can only be read between the lines.[124]

Brouwer's general attitude towards axiomatization can also be read in the following paragraph, which is about arithmetic that needs no axiomatization since it is a direct result of the ur-intuition. Brouwer is referring to Vahlen's *Abstrakte Geometrie*,[125], page 14 ff. *Zahlensysteme*:

> (IV–16) The investigation of the possible independence of the axioms of arithmetic makes no sense, because arithmetic is an aprioristic system of operation.[126]

And to stress once more the subservient role of logic:

> (IV–20) The proof of existence of arithmetic is the reality in the partialization of the barter trade.
>
> The proof of existence of mathematical logic is the arithmetic.
>
> Therefore mathematical logic can only serve as a centralization of arithmetic. It derives its life from arithmetic.[127]

[123]De axiomatisering komt vooral door de *woorden*; om de wil der individuen samen te houden. Want die woorden blijken dan toch niet zeker vast te houden, en om ze toch zeker te houden, gaat men ze axiomatiseren.

[124]Het is bij wiskunde, evenals bij kunst, hoogst gevaarlijk af te wijken van het 'schaffe Künstler, rede nicht', want de grondprincipes zijn ook hier niet te zeggen, alleen tussen de regels te lezen. (According to the 'Levensbericht Lidy van Eijsselstijn' in the 'Jaarboek van de Maatschappij der Nederlandse Letterkunde te Leiden', 1986-1987, page 77–83, the German quote originates from Goethe).

[125][Vahlen 1940] of which the first edition was published in 1905.

[126]Het onderzoek naar eventuele onafhankelijkheid van de axioma's der rekenkunde is onzin, want rekenen is een aprioristisch operatiesysteem.

[127]Het Exstenzbeweis voor de aritmetik is de werkelijkheid in de partiëring van de ruilhandel.

Het Existenzbeweis voor de mathematische logica is de aritmetik.

Zo kan die mathematische logica alleen als een centralisering gelden van de arithmetik, ontleent haar leven aan de aritmetik.

Again: the completeness of a set of axioms can only be concluded from the resulting building if this building is unique, that is, if there is only one building satisfying the axioms.[128] The next quote from notebook eight shows that Brouwer's ideas on this topic were not liable to much change during the years of preparation of his dissertation. Here he compares the role and the status of axioms for a building with that of the 'major terms' in syllogisms, which are just tautologies. Likewise the status of an axiom is merely a concise description afterwards, rather than something new.

> (VIII–31) The 'major terms' which are used in mathematical syllogisms may be nothing else, but tautologies.
>
> (...)
>
> Likewise the axioms. Mathematical theorems then are constructions within the large building, of which the parts that are remotely separated, cannot be surveyed at once intuitively. They serve as self-constructed road signs within that building.
>
> (...)
>
> Now the axioms can either or not be complete, i.e. it can either or not be the case that other constructions are possible, satisfying the same axioms. The latter is the case [*i.e. no other construction possible*] if the construction was completely governed by the axioms.
>
> (...)
>
> I sometimes can observe that it [*i.e. the axioms system*] is not complete, because I can point to another building, clearly different from the given one (...) and still satisfying the axioms.[129]

In the last quote we can point to another example of the principle of the excluded middle: 'it can either or not be the case (...)'. Brouwer recognized and admitted this, realizing that old habits often have a long life.

As a concluding remark we once more underline that the most interesting aspect of the discussion of this first subject from Brouwer's list of four hardly lies in the development of Brouwer's ideas in this field, but for the greater part in the content and corollaries of some of the footnotes.

[128] This a *categorical* axiom set; see page 238.

[129] De 'maioren', waarvan bij de wiskundige syllogismen wordt gebruik gemaakt, mogen niet anders zijn dan tautologieën.

(...)

Zo ook de axioma's. De wiskundige stellingen zijn dan samenbouwsels uit het grote gebouw, waarvan de ver van elkaar verwijderde delen niet zo direct intuïtief zouden zijn te overzien; zijn dus zelfgebouwde wegwijzers in dat gebouw.

(...)

Nu kunnen die axioma's volledig zijn of niet, d.w.z. het kan zijn, dat er nog andere gebouwen mogelijk zijn, die aan dezelfde axioma's voldoen, of niet. Het laatste is het geval, als ik met de axioma's trouw het bouwen zelf geheel heb gevolgd.

(...)

wel kan ik soms merken, dat het niet volledig is, doordat ik een ander gebouw aanwijs, d.w.z. een gebouw, dat duidelijke verschillen heeft met het gegevene, en toch aan de axioma's voldoet.

7.6 Cantor's transfinite numbers

7.6.1 The second number class

The second item from the list of four, discussed by Brouwer to clarify his view on logic, is Cantor's theory of transfinite numbers and the (in Brouwer's eyes often unjustified) role that logic plays in it. The first part of this item, a discussion of the second number class, can be treated briefly since it contains hardly any new ideas which were not worked out by Brouwer in his first chapter, where he already reached the conclusion that there are no other sets than those with cardinalities *finite, denumerably infinite*, and that of the *continuum*, which latter is a separate cardinality.[130] The limitation to only three cardinalities is the result of the restrictions, dictated by the admissible modes of constructing a set. The definition of a set must be in the form of an algorithm for the construction of its individual elements and the term 'and so on' may only refer to the repeated application of the same rule of construction.

The main point of discussion in this section will be Brouwer's fourth and most problematic cardinality, the 'denumerably infinite unfinished' one, which, however, becomes an acceptable notion when viewed, as Brouwer does, not as the cardinality of an intensionally finished set, but as the cardinality of a 'process of continued and never (not even intensionally) finished growth'. But it will turn out that Brouwer's definition for this concept is incomplete.

Even more difficulties are caused by the notion of 'unfinished mapping'. This is introduced in a footnote, not completely worked out and perhaps not well-thought up either.

It can be shown that sets, constructed according to Brouwer's modes from 1907, do not yield any form of *decidability*, which is the property that for every arbitrary entity it can be decided whether or not it belongs to a given set.[131]

[130] See chapter 4 of this dissertation.

[131] P. Maddy's, in *Realism in Mathematics* ([Maddy 1990], page 16), claims that Brouwer, because of his disbelief in an objective reality of mathematical entities, turns to verificationism, that is the view that for every statement a verification condition exists to decide on its truth. This implies that for every statement of the form 'an arbitrary mathematical entitiy a belongs to a set A', a verification on its truth is possible, and this again includes the principle of the excluded middle, which is, in Brouwer's eyes, a non-valid principle.

Brouwer himself objected in 1930 in a review on Fraenkel's *Zehn Vorlesungen über die Grundlegung der Mengenlehre* against the way Fraenkel described Brouwer's concept of a set; ([Fraenkel 1927]; for Brouwer's review see [Brouwer 1930a], also in [Brouwer 1975], page 441) he specifically objected in this review against the aspect of decidability for sets, which Fraenkel ascribed to him:

> (S.44) daß der Mengenbegriff von Brouwer eingeengt worden sei auf *entscheidungsdefinite* Gesamtheiten, für die die Frage, ob es darin Elemente von vorgegebener Eigenschaft gibt, stets, und zwar auf konstruktiven Wege, entscheidbar sei;
>
> (S.58) daß die Intuitionisten mittels einer ganz engen Mengendefinition kurzerhand einen großen Teil der Analysis vom mathematischen Gesamtkörper amputieren;

This property would require the principle of 'tertium non datur'. For Brouwer the elements for sets may only be determined by their defining algorithm.[132]

Brouwer accepted the existence of well-ordered sets according to Cantor's definition, first of all the order type ω of the sequence of finite order types, which may be viewed as finished in the intensional sense. On the basis of the repeated application of Cantor's first two generation principles (the successor operation and the closure by ω of all finite natural numbers), Brouwer was also able to construct numbers of the second number class, but by restricting himself to those two prinicples only, he was not able to view this class as a finished totality, not even in the intensional sense.

Cantor's third principle (Hemmungsprinzip) enabled the Cantorians to view the second number class (as well as e.g. the power set of a denumerable set) as a finished totality (Inbegriff); therefore it also allowed them to express the continuum hypothesis as a concrete and possibly solvable problem. This third principle was rejected by Brouwer: Cantor speaks about something without giving a concrete construction rule for it, because the *and so on* is no longer limited to the continued same operation and its closure by ω. For Cantor this was no limitation but then he should at least, according to Brouwer, *first* prove that the 'Inbegriff' is a logically sound concept; but he proved nothing of that kind. In Brouwer's words: *'So Cantor loses here mathematical ground'.*[133] The fact that paradoxes, like those of Russell, Burali-Forti and Richard, were the result of the unrestricted use of those princples, was certainly not the reason for Brouwer's restriction to only the first two, but the paradoxes were brought up by him afterwards to illustrate what can happen in case of an ill-founded (that is, non-constructive) set theory.[134]

From a constructivist's point of view it is not possible to consider the second number class as a completed totality, and neither is it possible to speak of it, since every sequence, which is definable in the second number class by means of the two generating principles, has a limit within that class,[135] hence always larger numbers can be constructed without reaching a closure for this class, let alone entering a third class. The only thing one can say about it is, in Brouwer's view, the following triviality:

> If the logical entity T (power of the second number class) is introduced, then the axiom $T = A$ (A is the power of ω) will lead to a contradiction in the logical structure; likewise the introduction of a logical entity I, playing the part of a cardinal number, which would be supposed to satisfy the axioms $A < I < T$. This is the logical result, without any value for mathematics, of these proofs by Cantor.

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[132] Around 1907 the set definition was purely constructive. Elements are *constructed* and not *selected* from an available stock, hence Zermelo's Aussonderungsaxiom can play no role. In his *Begründungspapers* [Brouwer 1918] and [Brouwer 1919a], the species concept appears, and then a set becomes a property. See page 134.

[133] Dissertation, page 46: Cantor verliest hier dus de wiskundige bodem.

[134] See page 144 of Brouwer's dissertation.

[135] [Schoenflies 1900b], page 46.

> If one wishes to look at it in the light of mathematics, then one finds no more than the following : The next two statements are *false:*
>
> 1^0 The second number class is conceivable and denumerable.
>
> 2^0 The second number class is conceivable and there is a cardinal number between its power and that of the first number class.
>
> But that these two statements are false, we knew already, for we knew that the first part of both (the second number class is conceivable) is false.[136]

Brouwer gave no *proof* for the non-existence of the second number class. There was no need for such a proof either, since the two principles only result in denumerable ordinals, and Cantor proved already that the second number class, which he was able to view as a totality, is non-denumerable.

So the only result consists of negative statements. A set with a cardinality greater than $\aleph_0$ does not exist; the maximum cardinality of a constructible set is denumerably infinite. But nevertheless Brouwer felt the need for an extra status for 'sets' like the second number class, that is, he wanted a special characterization for a class of objects that can grow unlimited without ever reaching some form of 'being completed', not even intensionally. We may describe them as a *process* of the construction of increasing ordinal numbers, all with the same cardinality and without a closure. It is, as it were, a 'pseudo set', a set 'by way of speaking'. This will be the subject for the next subsection.

7.6.2 The denumerably unfinished sets

Despite the restriction of the cardinalities for sets to the mentioned three (finite, denumerably infinite and the continuum), Brouwer made one further step. He intended to include as a possible cardinality the concept of an always repeated process of extending a set, thereby departing from a denumerably infinite set and one specific algorithm. Clearly, it is introduced by him as the only way to speak in some way of the 'totality of the well-ordered numbers'. Its cardinality is said to be *denumerably infinite unfinished*, but in order to speak of this totality, one has to reconsider the concept of 'totality' or of 'set':

> *The power of the totality of well-ordered numbers is denumerably unfinished*; here we call a set *denumerably unfinished* if it has the

[136](dissertation page 147, see [Brouwer 1975], page 82.) Wordt de logische entiteit T (machtigheid der tweede getalklasse) ingevoerd, dan zou het axioma $T = A$ (A is de machtigheid van ω) in het logisch gebouw tot een contradictie voeren; evenzo de invoering van een logische entiteit I, die de logische functie van een machtigheid zou moeten vervullen, en aan de axioma's $A < I < T$ zou moeten voldoen. Dat is het logische, voor de wiskunde waardeloze resultaat dezer bewijzen van Cantor. Wil men het in wiskundig licht bezien, dan kan men niet anders vinden dan de volgende uitspraak: *Onwaar* zijn de beide stellingen:

1^0 De tweede getalklasse is denkbaar en aftelbaar.

2^0 De tweede getalklasse is denkbaar, en er ligt een machtigheid tussen de hare, en die der eerste getalklasse.

Maar dat deze twee stellingen onwaar zijn, wisten we al, want we wisten al dat het eerste deel van beide (de denkbaarheid der tweede getalklasse) onwaar is.

> following properties: we can never construct in a well-defined way more than a denumerable subset of it, but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted to the original set. But from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention.[137]

The examples given for this type of set are, in addition to the *totality* of the well-ordered numbers, also the *totality* of definable points on the continuum and the *totality* of all possible mathematical systems.[138]

Now, the term *totality* should *not* be interpreted as if, e.g., *all* well-ordered numbers form a completely defined set, since such a totality is just what Brouwer objected to and contested. It should rather be viewed as a process of growth, starting, for instance, from the well-ordered denumerably infinite set of the natural numbers, and extending this by means of an algorithm, in this case the first two generation principles, into the second number class. Since the third principle (the 'Hemmungsprinzip') is not accepted by Brouwer, the resulting set keeps its unfinished character; we might label it as a 'pseudo-set'.

One initial remark has to be made: We note that the only places where the concept of the denumerably infinite unfinished set is mentioned and more or less briefly discussed, are the dissertation and the Rome lecture *Die möglichen Mächtigkeiten.*[139] On a few later occasions it was only mentioned, e.g. in a footnote in the inaugural address of 1912, *Intuitionism and Formalism*, and in the second Vienna lecture *Die Struktur des Kontinuums*,[140] on page 3, where Brouwer was speaking of 'abzählbar unfertigviele Elemente', and on a few other places. The concept was not withdrawn in the *Addenda end corrigenda* of 1917,[141] but in Brouwer's major papers dealing with the set concept, the *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlosse-*

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[137] page 148 of the dissertation, see also [Brouwer 1975], page 82. *De machtigheid van het geheel der welgeordende getallen is aftelbaar onaf;* we verstaan dan onder een *aftelbaar onaffe* verzameling een, waarvan niet anders dan een aftelbare groep welgedefinieerd is aan te geven, maar waar dan tevens dadelijk volgens een of ander vooraf gedefinieerd wiskundig proces uit elke zodanige aftelbare groep nieuwe elementen zijn af te leiden, die gerekend worden eveneens tot de verzameling in kwestie te behoren. Maar streng wiskundig bestaat die verzameling als geheel niet; evenmin haar machtigheid; we kunnen deze woorden echter invoeren als willekeurige uitdrukkingswijzen voor een bekende bedoeling.

[138] 'Het *geheel* der welgeordende getallen, het *geheel* der definieerbare punten op het continuüm en a fortiori het *geheel* van alle mogelijke wiskundige systemen'.

[139] [Brouwer 1908b]. In this lecture, held by Brouwer at the International Mathematical Congress in 1908 in Rome, the possible cardinalities of sets are analyzed in a similar way as in the third chapter of his dissertation, viz. by approximating the points of the set to be investigated, via the branching method. The presentation in this lecture, however, is more precise and also more abstract.

[140] [Brouwer 1930b].

[141] *Addenda and corrigenda over de grondslagen der wiskunde*, [Brouwer 1917b], see also [Dalen 2001], page 195.

nen Dritten[142] it was not named any more as a possible cardinality. Heyting did not speak of the concept either in his survey *De telbaarheidspraedicaten van Prof. Brouwer.*[143] The early papers which *did* deal with the concept, emphasized the fact that this concept refers to a method and not to a (potentially or intensionally) finished set.

We also note that for this type of 'set' the starting point is always a denumerable set, constructed in a well-defined way, together with a previously defined mathematical algorithm to create new elements. Hence the original well-defined denumerable basic set becomes, during the process of growth, a subset of the growing denumerably infinite unfinished set.

From the text of the dissertation it is not clear whether this *whole* denumerable basic set has to be employed for the construction of the first new element, and whether the basic set plus the so far constructed new elements has to be used for the construction of the next new element, or that any denumerable subset of the basic set or the so far constructed unfinished set will do as well. The latter seems more likely and the Rome lecture, *Die möglichen Mächitgkeiten*, confirms this in the following terms:

> Man kann eine Methode zur Bildung eines mathematischen Systems angeben, die aus *jeder gegebenen zum Systeme gehörigen abzählbaren Menge ein neues gleichfalls zum Systeme gehöriges Element erzeugt.* Mit einer solchen Methode sind aber wie überall in der Mathematik nur abzählbare Mengen zu construieren, das ganze System ist niemals zu construieren, weil es eben nicht abzählbar sein kann. Es ist unrichtig, dieses ganze System eine mathematische Menge zu nennen, denn es ist nicht möglich, es aus der mathematischen Ur-Intuition fertig aufzubauen. Beispiele sind: das Ganze der Zahlen der zweiten Zahlenklasse, das ganze der definierbaren Punkte auf dem Continuum, das Ganze der mathematischen Systeme.[144]

Hence at any stage during the construction of the unfinished set, any denumerable subset of it (which subset may then of course contain earlier constructed new elements) may be employed for the process, and will result, after the application of the algorithm, in a *new* element for the system.

But now another question emerges:

If we have a denumerably infinite unfinished set, e.g. the set of the definable points on the continuum, which has as its basic set the well-defined denumerable set η of the rationals (see below for an elaboration of this example), together with an algorithm to construct new elements from η or every well-defined denumerable subset D thereof, is it then a requirement that the newly constructed element, which, by definition, is not an element of D but does belong to the set of definable points, is *not* an element of η, that is, is not a rational point?

[142] the *Begründung* papers, for short. See [Brouwer 1918] and [Brouwer 1919a].
[143] [Heyting 1929].
[144] See [Brouwer 1975], page 102; my emphasis.

Judging by the German text above the answer *seems* to be affirmative: a mathematical system is constructed, which creates from every denumerable *subset of the system a new element for the system.* For instance, in case of 'the system of the definable points on the continuum', the well-defined part of it is the η-scale of the rationals and the application of the algorithm to every well-defined subset of the η-scale results in a *new element for the system*, hence in an irrational number. This conclusion *seems* justified, but the *Inaugural Address* (1912) gives a clear, but different answer:

> Calling *denumerably unfinished* all sets of which the elements can be individually realized, and in which for every denumerably infinite subset there exists an element not belonging to this subset, we can say in general, in accordance with the definitions of the text: *All denumerably unfinished sets have the same power.*[145]

Hence, according to this phrase, not every denumerably infinite subset of the η-scale needs to result in an *irrational* number, and, similarly, not every denumerably infinite subset of $\mathbb{N}$ needs to result in a member of the second number class. We will adopt this interpretation as being the most 'common sense' and practical one for the following analysis. But, clear as the answer may seem, exactly this interpretation will create difficulties, as we will see.

The conclusion so far is that a set A is denumerably infinite unfinished if there exists an algorithm F, such that for every denumerable and well-defined subset B of A, F assigns an element of A to B, not being an element of B. In symbols:

$$\exists F \forall B \{B \subseteq A \wedge B \text{ denumerable} \Rightarrow F(B) \in A - B\}$$

Obviously, it is not forbidden that different subsets may lead to the same new element and thus that for a definable point on the continuum or for an arbitrary element of the second number class the subset leading to that point or to that element can not always be uniquely determined. See for this the examples that Brouwer gave and which are elaborated below.

Then Brouwer presented arguments for what we will call on this occasion and in this chapter *Brouwer's lemma*:

All denumerable unfinished sets have the same power.

This conclusion was already mentioned at the end of the quoted footnote from the inaugural address (see above). Brouwer's arguments in his dissertation for this lemma are simple:

[145] [Benacerraf and Putnam 1983], page 86 or, in Dutch, [Dalen 2001], page 188: Verzamelingen van construeerbare elementen, waarin naast elke aftelbaar oneindige deelverzameling een daartoe niet behorend element kan worden aangegeven, als *'aftelbaar onaf'* kwalificerend, kan men in de zin van de tekst algemeen formuleren: *Alle aftelbaar onaffe verzamelingen zijn gelijkmachtig.*

> While constructing, without ever coming to an end, a denumerably unfinished set, we can map the elements in succession on the sequence of the well-ordered sets, which likewise is never exhausted. Extending the notion of equality in power, so that it may be applied to this case, we can say: *All denumerably unfinished sets have the same power.*[146]

So, if there exists an algorithm to construct the irrationals in a systematic way, one by one, from the set of the rationals (on the interval $(0, 1)$), then we can map the successively constructed irrationals on the sequence of the well-ordered sets, i.e. on the successive elements of the second number class $\omega + 1, \omega + 2, \ldots$. This construction is possible, but it fulfils only partially Brouwer's definition for the denumerably infinite unfinished set (see for this page 278).

But there are also arguments which, in the light of more recent developments, make the claim 'all denumerably unfinished sets have the same power' an implausible one. To see this, let us consider the *recursive reals*, which is a subset of the definable reals, since not every definable real needs to be recursively definable.[147] Let a real number α be given as the limit of a convergent sequence of nested intervals, each interval being defined by a pair of rationals. If f is a function such that $f(0), f(1), f(2), \ldots$ gives a convergent sequence of nested intervals, defining the real number α, we say that f represents α. (If we have a fixed effective coding of pairs of rationals onto $\mathbb{N}$, $f(n)$ may have as output code numbers for the rational intervals). The real number α is called a *recursive real* if it is represented by a recursive function; z is the index for the recursive real α if φ_z represents α. But there is no effective procedure by means of a decision function, given any x and y, to decide whether $\varphi_x = \varphi_y$,[148] hence equality of recursive reals is undecidable. However, equality of elements of the second number class is decidable. Therefore the two unfinished sets, the recursive reals and the second number class, must have a fundamentally different structure.

Now there are many non-recursive algorithms to define irrational numbers as new elements for the unfinished set, but the consideration just given makes it at least doubtful that between any two denumerably infinite unfinished sets always a one-to-one mapping can be given.

It will turn out that the actual execution of a 'one-to-one mapping onto' for the two given examples (viz. the well-ordered numbers and the definable points on the continuum, in combination with a specified algorithm for each of the two sets) creates difficulties, and may even lead to the conclusion that Brouwer's definition for this type of set (which has to satisfy the lemma mentioned above) is not adequate.

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[146] Bij het nooit klaarkomend opbouwen van een aftelbaar onaffe verzameling kunnen we al voortbouwende naar opvolging afbeelden op de rij der welgeordende verzamelingen, die eveneens nooit uitgeput raakt; het begrip van gelijkmachtigheid uitbreidend, om het hier toepasbaar te houden, kunnen we zeggen: *Alle aftelbaar onaffe verzamelingen zijn gelijkmachtig.*

[147] See [Rogers 1987], page 371.

[148] ibid, page 33.

Before discussing these examples, we first draw the reader's attention to the following remark that Brouwer added in a footnote to his lemma:

> Still, in a certain sense, one can say that denumerably unfinished sets have the same power as denumerable sets, for every denumerably unfinished set can be mapped on ω^2 (every part which I add in the course of the construction of the denumerably unfinished set, can be mapped on ω, for it is denumerable; constructing such a mapping for each constructed part, I map the unfinished set on $\omega + \omega + \omega + \ldots = \omega^2$); only this mapping remains *always unfinished*; the proof that a mapping of a denumerably unfinished set on a denumerable set is impossible, holds only for a *finished* mapping.[149]

Brouwer explained in this argument that a denumerably infinite unfinished set is denumerable 'in a certain sense', but that the relevant mapping is then necessarily an unfinished one. However, the concept 'unfinished mapping' is neither defined, nor explained. Two interpretations present themselves, both involving questionable implications:

On the one hand we can define it as a mapping of which the domain is a denumerably unfinished set. But then, if this mapping is one-to-one, the range necessarily has to be a denumerably unfinished set too. To see this, suppose that F is a one-to-one mapping from a denumerably unfinished set A onto a denumerable but (intensionally) finished set B, hence $F^{-1} : B \rightarrow A$ is defined too. But B is denumerable, hence $F^{-1}B$ is denumerable, and is assumed to be composed of all elements of A since the mapping is one-to-one onto. But now the algorithm for the creation of new elements for the denumerably unfinished set A may be applied to $F^{-1}B$, being a well-defined set of elements of A. This results in a new element $a \in A$ with $a \notin F^{-1}B$ (according to the definition of a denumerably infinite unfinished set), hence $Fa \notin B$. This is a contradiction, based on the assumption that B is denumerable. Therefore B must be denumerably unfinished too.

So an unfinished mapping (according to this first interpretation) of a denumerable unfinished set onto ω^2 is impossible, and in this case Brouwer's argument fails.

On the other hand we can imagine a one-to-one mapping of A onto B (both infinite sets, but either both (intensionally) finished or both unfinished) to be a *finished* mapping if for every $a \in A$ its image $Fa \in B$ can be uniquely determined and, conversely, for every $b \in B$ its image $F^{-1}b \in A$. This definition also applies

[149] dissertation, page 149: Intussen kan men in zekere zin ook zeggen, dat aftelbaar onaffe en aftelbare verzamelingen gelijkmachtig zijn, daar elke aftelbaar onaffe verzameling is af te beelden op ω^2 (immers elk gedeelte, dat ik telkens weer toevoeg, als ik de aftelbaar onaffe verzameling opbouw, is af te beelden op ω, immers aftelbaar; construeer ik zulk een afbeelding voor elk toegevoegd gedeelte, dan beeld ik de onaffe verzameling af op $\omega + \omega + \omega + \ldots = \omega^2$); alleen is deze afbeelding *steeds onaf*; het bewijs, dat een afbeelding ener aftelbaar onaffe verzameling op een aftelbare onmogelijk is, geldt dan ook alleen voor een *affe* afbeelding.

if A and B are both denumerably unfinished, viz. if the original denumerable parts are mapped one-to-one and if extensions of A and B, each by a new element according to its own algorithm, are mapped onto each other (this is, contrary to Brouwer's theorem, only *sometimes* possible, under conditions as specified on page 278; see also the remark about McCarty's approach on page 277).

In this light, then, we have to find an interpretation for the concept 'unfinished mapping', an interpretation that does (at least partly) justice to the quoted paragraph. When building up the denumerably infinite unfinished set of the second number class, departing from $\mathbb{N}$, we easily can, at any stage, prove the denumerability of the result so far. For instance, after the construction of $n+1$ new elements $\omega, \omega+1, ..., \omega+n$ of the unfinished set of the second number class (which is then composed of the elements $1, 2, 3, ..., \omega, \omega+1, ..., \omega+n$, since the second number class is cumulative to the first class), we prove its denumerability by ordering them as $\omega, \omega+1, ..., \omega+n, 1, 2, 3, ...$ and by mapping these elements respectively on $1, 2, ..., n+1, n+2, n+3, n+4, ...$[150]

Hence, to prove its denumerability for different values of n, no fixed (or 'finished') mapping is available. For each next value of n, that is, for each new element of the second number class, a different mapping onto $\mathbb{N}$ has to be devised, so under this interpretation an unfinished mapping necessarily refers to a mapping between an (intensionally) finished set and an unfinished one (both denumerably infinite).

It seems reasonable to suppose that Brouwer had such an interpretation in mind for the concept 'unfinished mapping', but it still remains a misleading term, since it simply is not a mapping. In the text he claims that 'every denumerably infinite unfinished set can be mapped on ω^2 (and thus on ω). This cannot be true since an unfinished mapping is not a mapping of two sets onto each other. A mapping of one set onto another either has to proceed via a finished mapping, or is impossible, And in the case of an 'unfinished mapping' of a denumerably unfinished set onto ω^2, this is merely a per element changing finished mapping of the part of the denumerably infinite unfinished set so far constructed, onto ω^2. So there cannot exist a mapping of the second number class (or any denumerably unfinished set) onto ω^2.

A positive outcome of this analysis is, that it clearly demonstrates the fundamental difference in cardinality between a denumerable and a denumerably unfinished set.

Brouwer made some remarks about unfinished mappings in his eighth notebook, thereby referring to work of Bernstein and Hardy. But neither of them used the expression 'unfinished mapping' as such; the term is Brouwer's (see page 284).

Brouwer's concluding remark on the subject of the denumerably infinite unfinished sets is the following:

[150] Different techniques of mapping apply after any denumerable infinite number of constructed elements.

> Thus we distinguish for sets the following cardinal numbers, in order of magnitude:
>
> 1. the various finite numbers.
> 2. the denumerably infinite.
> 3. the denumerably unfinished.
> 4. the continuous.[151]

In a handwritten remark in his own copy of the dissertation, Brouwer noted that the highest possible cardinality seems to be the continuum of a 'denumerably unfinished number of dimensions'. The remark continues somewhat enigmatically:

> But the continuum of two and more dimensions can only be conceived as a continuous cardinality, if an unknown point can be approximated in a denumerable sequence (in all coordinates together, for if I want to handle one coordinate first, that would never terminate, and the other coordinates never got their turn). But that approach is only possible if the decimal sequence ω is ordered as a certain ordinal number, so that all coordinates are treated in turn. Hence the sequence of coordinates is a part of that denumerable number, so it is also denumerable. And the denumerably unfinished number of dimensions is a part of it.[152]

Hence the continuum of a denumerably unfinished number of dimensions also has the cardinality of the continuum.

We will now analyze the first two of the three examples that were mentioned on page 148 of Brouwer's dissertation: the second number class and the definable points on the continuum. In both examples we will encounter some difficulties.

Example 1: the second number class

For all denumerably unfinished sets the requirement applies that only a well-defined denumerable subset of it, together with a previously defined algorithm,

[151]Dissertation, p. 149: We onderscheiden dus dan voor verzamelingen naar volgorde van grootte de volgende machtigheden:
1. de verschillende eindige.
2. de aftelbaar oneindige.
3. de aftelbaar oneindig onaffe.
4. de continue.

[152]Maar het continuüm van 2 en meer dimensies kan als continue machtigheid alleen worden gezien, als een onbekend punt in een aftelbare reeks, kan worden benaderd (in alle coördinaten tezamen; want als ik eerst een wou doen, kwam ik daarmee nooit klaar, en kwamen de andere niet aan de beurt). Maar die benadering kan alleen als de decimaalreeks ω is gerangschikt als een bepaald ordinaalgetal, om zo alle coördinaten op hun beurt een beurt te geven. De reeks der coördinaten is dus een deel van dat aftelbaar getal, is dus ook aftelbaar. En het aftelbaar onaffe aantal dimensies is een onderdeel daarvan.

is the point of departure for the construction of a new element. In the case of the second number class the choice of $\mathbb{N}$, the set of the natural numbers, or of any denumerably infinite well-defined subset B of $\mathbb{N}$ as a well-defined subset, seems to be the obvious one, together with the successor operation as the algorithm F, to be applied to the supremum of the original set.

There is of course no difference in the resulting element if we take as the original set any denumerably infinite subset of $\mathbb{N}$ and if we define the *first* new element ω to be the supremum of that set. Every subsequent element after the first one is then defined to be the successor of the supremum of the denumerable subset.[153]

Symbolically, for the first new element:

$\omega = sup\{\alpha|\alpha \in \mathbb{N}\}$ or $\omega = sup\{\alpha|\alpha \in B, B \subset \mathbb{N},\ B\ \mathit{infinite}\}$.

And for subsequent new elements we have:

$\alpha_{new} = sup\{\alpha|\alpha \in \mathbf{S}\}+1$, where $\mathbf{S}$ is the denumerable result of the construction so far, or an infinite subset thereof which contains as one of its elements the last constructed element.

Note that with this algorithm every denumerably infinite subset with the same supremum results in the same new element.

The obvious alternative is to consider as algorithm the application of *both* generating principles of Cantor, which indeed results in every desired number of the second number class. Rejection of Cantor's third principle (the *Hemmungsprinzip*) then guarantees us the unfinished character of this class. But this algorithm involves the idea of the 'transfinite application' of a rule or a combination of rules, a problem which Borel encountered in his attempt to define the second number class.[154] Nevertheless this is the only possible way if one maintains that every number of the second number class has to be a possible candidate for a new element of the unfinished set. At any moment during the process of growth, the set reached at that stage can be viewed as a new basic well-defined denumerable set. So every ordinal number $m_1\omega^{p_1} + m_2\omega^{p_2} + ...(p_r > p_{r+1})$ is, for every m_i and p_i, point of departure. Conversely every number of the second number class can now be viewed as a result of the algorithm, thus preventing the problem of running into the barrier of $\omega 2$ after ω times the application.

Example 2: the definable points on the continuum

We now consider the set of the *definable continuum*, that is, the totality of all points that can be defined with the help of some fixed algorithm, to be applied to a well-defined subset of that set. The rational scale η on the continuum is the most natural and obvious point of departure for its construction.

[153] If we take for the subsequent element only the supremum of the defining subset, we would not create any new element since the supremum of, say, $\{1, 2, ...\omega, \omega + 1\}$ is $\omega + 1$, which is already in the defining set.

The definition for the *first* new element ω has to be just the supremum; otherwise we would never get ω, since $sup\{n|n \in \mathbb{N}\} + 1 = \omega + 1$.

[154] See [Borel 1950], page 112 ff, *La croissance des fonctions et les nombres de la deuxième classe.*

Suppose we have a well-defined denumerable subset D of the rational scale. We now seek an algorithm G, such that $G(D)$ results in a new element, i.e. an element not belonging to D. This algorithm G will turn out to have the character of a 'diagonalization' procedure or of a 'spoiling argument'.[155]

By a simple transformation we can, without loss of generality, restrict our analysis to the open scale $(0, 1)$, as the most general interval, since, if D is the unbounded rational scale, we can transform it into an equivalent bounded subset D' by the transformation $x' = \frac{1}{\pi} \arctan x$, with $x' \in D'$ and $x \in D$. D' is then bounded on the open interval $(0, 1)$.

Since D is denumerable, we can write its elements as an infinite sequence $(a_0, a_1, a_2, ...)$, all on the interval of rationals $(0, 1)$. A suitable algorithm now consists of the following procedure: we split the interval of rationals into three parts of equal magnitude (the boundaries being rational numbers), resulting in three half-open or open segments (or possibly one closed segment). At least one of the three segments will not contain a_0 as an inner point (possibly exactly one, if a_0 is on the boundary between two segments).[156] Select a segment not containing a_0 as inner element, in the preferred order: 1) the middle segment of the three, 2) the right one, 3) the left one. The preference for the middle segment is needed to preclude the possibility that the resulting new point will either be 0 or 1, which numbers fall outside the interval. Again split the selected segment into three equal parts; at least one of the three segments will not contain a_1 as inner element (possibly neither of the three will contain a_1 as such a point), and again select such a segment in the same preferred order. Proceeding in this way ω times results in a set of nested intervals with an ultimate width smaller that any positive rational number, thus defining a new point, in virtue of its algorithmic construction belonging to the definable continuum, but not belonging to D. Note that the resulting point may very well be a rational point and, in fact *will* be a rational point if we end up, after finitely many steps, in a segment which is nowhere dense, i.e. which contains only a finite number of elements, as the reader may verify.

Now it is of course a condition to be an element of this type of denumerably unfinished set that it is defined by means of the given algorithm G, but we can prove that every element of the definable continuum (every point which is well-defined in *some* way) can also be defined by applying our algorithm G on some well-defined denumerable subset of the definable continuum. The question now is to establish such a subset. Observe that the Cauchy-sequence which defines the point concerned, does not automatically satisfy the requirements of the defining denumerable subset that we are looking for, since the application of the algorithm G to the Cauchy-sequence most likely will result in a different

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[155]See page 5.

[156]For the sake of simplicity we will avoid in this way the question in which segment exactly the point a_0 is. Since the boundary certainly belongs to one of the segments (one of the two segments being open, the other being half-open or closed), a_0 belongs to that segment too, but now we need not define which of the two neighbouring segments is the open one and which the closed one. The boundary simply belongs to one of the segments, without specifying to which one.

point, as the reader may verify. Therefore, for some defined point A on the bounded open continuum $(0,1)$ we proceed as follows:

Split the interval $(0,1)$ into three equal (half open or open) segments C_1, C_2 and C_3,

1. If A is in C_1, select a_0 on the boundary of the segments C_2 and C_3, (i.e. the rational $\frac{2}{3}$, this boundary of course does belong either to C_2 or to C_3; note that this also applies, without explicitly mentioning it, to the following options under 2, 4 and 5),

2. If A is on the boundary of the segments C_1 and C_2, select a_0 in the middle of the segment C_3 (i.e. the rational $\frac{5}{6}$),

3. If A is in C_2, select a_0 in the middle of the segment C_1 ($\frac{1}{6}$),

4. If A is on the boundary of C_2 and C_3, select a_0 in the middle of C_1 ($\frac{1}{6}$),

5. If A is in C_3, select a_0 on the boundary of C_1 and C_2 ($\frac{1}{3}$).

Now split the segment containing A into three equal parts and repeat the same procedure to select the point a_1, etc. After ω repetitions we obtain the denumerably infinite subset of rationals $(a_0, a_1, a_2, ...)$, which, after application of the algorithm G, results in a set of nested intervals, which will define the same point A again, as can again be verified by the reader

On the basis of the first part of this analysis of the 'definable continuum' one might get the impression that the intuitive continuum itself (or, anachronistically, the full continuum of the reals) is also denumerably unfinished,[157] since by means of the everywhere dense scale η any irrational point on the continuum should be definable by means of a denumerable subset of the rationals according to the just given procedure. Applying the algorithm G to this subset will result again in the given 'arbitrary point'.

But this analysis does not hold for the (intuitive) continuum, since, at this stage of Brouwer's development, there is no 'arbitrary element' on the continuum; the vast majority of points on it cannot be 'named' by some defining algorithm, or by some well-defined set.

In one of Brouwer's notebooks this is stated as follows:

> (VII-3) I cannot name the points of the linear continuum, but just point at them (or rather imagine myself that I point at them).[158]

On the equivalence of denumerably infinite unfinished sets

Now, on the basis of Brouwer's argument it seems as if we can map the elements of the denumerably infinite unfinished sets of the second number class and of the definable points of the continuum one-to-one onto each other in the following way:

[157]See also below, under Mannoury's comment on this concept.

[158]Ik kan de punten van het lineair continuum niet noemen, maar aanwijzen (of althans mij denken dat ik ze aanwijs).

Denumerable subsets of both sets can be mapped one-to-one onto each other because of their denumerability (in the examples: $B \sim D$), and the newly defined elements $F(B)$ and $G(D)$ are subsequently mapped onto each other.

This mapping seems to be complete, since every element of the second number class *and* of the definable continuum can be determined by their respective algorithms, to be applied on a suitable denumerable subset.

Now, it is indeed the case that, in the given examples both algorithms create from a denumerably infinite well-defined subset of the second number class, or of the definable continuum respectively, a new element, different from the elements of the well-defined subset. But we must realize that *every* well-defined denumerably infinite subset of $\mathbb{N}$, including $\mathbb{N}$ itself, will, with the given algorithm, result in the *same* new element ω, and never in another *finite* number different from the elements of the defining subset, whereas *not* every well-defined subset of η will result in an irrational number, let alone the same irrational number. Many subsets of the rationals will have another rational as a result; in many cases even the *same* rational will result from an infinite number of different subsets of η (of course this new rational is always different from the rationals of the defining subset).

In fact one can easily define two different denumerable sets of rational numbers, giving rise to two different rationals or two different irrationals or one rational and one irrational, whereas the corresponding denumerable subsets of $\mathbb{N}$, which result from the mapping of η onto $\mathbb{N}$, both give rise to the same new element ω.

However, we must call to mind Brouwer's argument leading to his lemma (as we baptized it just for this chapter), which was the following: During the construction of the elements of any denumerably infinite unfinished set we can map every next element on the next element of the, always denumerable but always unfinished, sequence of the well-ordered sets, i.e. the second number class. Then the two sets certainly remain of the same power, without the need to alter the mapping after every next step. But in this case, it is a strict condition for a complete one-to-one mapping that the definable numbers on the continuum are constructed in a systematic way, hence that there exists a system according to which the definable infinite subsets of the rationals for the construction of the successive definable numbers are also given in a systematic (algorithmic) way, or, which amounts to the same, that all definable subsets of the rationals on the unit interval can be coded in order to give them a sequence number. This coding and its corresponding sequence number is required to identify the subset of the rationals of which the resulting irrational is mapped on a certain given number of the second number class. McCarty, in his dissertation *Realizability and Recursive Mathematics*, gave construction rules for this coding. In its Introduction he wrote:

> If recursive mathematics is a field of mathematics all of whose objects and all of whose basic morphisms are computable, then the early undecidability results, for arithmetic and logical validity, can

> be viewed as setting limits to that field, as long as computability is understood as recursivity.
>
> (...)
>
> It was natural to ask how much mathematics could be accomplished within the limits and to wonder what its character might be.
>
> (...)
>
> More significantly, the restriction to coded domains is not as severe as might first appear.
>
> (...)
>
> Any subset of ω, the set of natural numbers, is, in this sense, automatically coded and set-theoretic operations on $P\omega$, the powerset of ω, affords prime candidates for 'recursive' investigation.[159]

But this only refers to subsets of the scale of the rationals, thus excluding the newly constructed irrationals, hence this mode does not fully satisfy Brouwer's definition of the unfinished set. Of course, when McCarty states 'any subset of ω ...', he is referring to 'any definable subset of ω ', since we cannot speak of other subsets; but his subject is *recursive mathematics*, and therefore he speaks about *recursively definable* subsets. And we emphasized earlier that equality of elements on ω_1 (the second number class) is decidable, in contrast to equality of recursively defined elements on the definable continuum, which is not decidable.

On top of that, we still have the additional problem that not every definable subset of the η-scale will result in an irrational number, whereas every definable subset of $\mathbb{N}$ produces an element of the second number class, as we explained above.

So, despite the fact that the point of departure is a well-defined denumerable set and an algorithm (a 'generating function') to construct new elements, there still seems to be a missing condition in Brouwer's definition of the denumerably infinite unfinished set: only *irrational* results have to be mapped on the successive members of the second number class, and an irrational number as a result of the application of the algorithm can only occur if after *every* next step the selected segment contains at least one limit point, not being a rational itself.

A possible way out of this dilemma

But there remains one *positive* property of the denumerably infinite unfinished set to be discussed: the second number class can be embedded *one-to-one into* the definable continuum (or in every denumerably infinite unfinished set for that matter). This may allow us to straighten out the problems we encountered at the previous pages.

Suppose we have the same algorithm G which enables us to construct from every denumerably infinite subset of the η-scale a new element, not belonging

[159] [McCarty 1984], page 7, 8.

to that subset. We now begin the construction of a new definable number by the application of G to the whole η-scale; this results in a new number, not belonging to the η-scale, hence an irrational number i_1, which we map on $\omega+1$. Next we put i_1 as the first element in the denumerable arrangement of the η-scale, hence in front of the enumeration of the rationals. Applying G again to this scale, i.e. on the 'η-scale-plus-one', gives us a second irrational number, different from the first one, and we map this second irrational number on $\omega+2$. Proceeding in this way, including a closure after ω times the application of the algorithm and then followed by the $\omega+1^{st}$ application, and so on, will result in a one-to-one mapping of the second number class onto a subset of the definable continuum.

If Brouwer's definition of the denumerably infinite unfinished set is altered in such a way that the first application of G should be on the whole denumerably infinite set (and not on a well-defined proper subset of it), and subsequent applications of G then are on the 'result so far', with the newly constructed elements put in front of the original defining denumerably infinite set, then a one-to-one mapping between the second number class and the definable continuum (defined according to this revised algorithm) becomes feasible.

The stated condition of page 148 of the dissertation:

> '(...) we can never construct in a well-defined way more than a denumerable subset of it'

should then be interpreted as follows: the 'subset constructed in a well-defined way' is the original denumerable set, which is now viewed as a subset of the always growing denumerably infinite unfinished set under the successive addition of the newly constructed elements in front of its enumeration.

But note that the interpretation which we sketched earlier, viz. that the newly constructed elements in their order of construction (the choice of the subset on which G is applied then being necessarily a random one), are mapped on $\omega+1$, $\omega+2$, $\omega+3$, ..., also remains a valid interpretation, be it that the result is not uniquely determined, and that only the irrational results of the algorithm G count (which irrational result will always be the case if the subset to which G is applied, is everywhere dense).

But it is highly unlikely that Brouwer had this interpretation in mind, because, in view of the random character of the result, the same objections apply now as the ones that Brouwer raised against the axiom of choice.

7.6.3 The Mannoury review on the denumerably unfinished sets and Brouwer's reply to it

It is in one of the two reviews that Mannoury wrote about Brouwer's dissertation,[160] that the *denumerably infinite unfinished cardinality* showed up once more in a published paper.

[160] Both reviews are included in [Dalen, D. van (ed.) 1981b] *and* in [Dalen 2001]. The first review was published in 1909 in the *Nieuw archief voor Wiskunde*. This review is the one of

He raised objections to more topics than the one under discussion now, but we will confine ourselves to his comment on this type of cardinality.

For Mannoury the concept of 'denumerably unfinished' implies a certain vagueness and indefiniteness. After quoting verbatim the definition (or characterization) of the denumerably infinite unfinished set from Brouwer's dissertation (page 148) Manoury commented:

> If we understand it correctly, this definition (?) is satisfied by all point sets with a higher cardinality than that of denumerably infinite (Cantor's first cardinality), but the author gives as an example the totality of the definable points on the continuum, i.e. of those points or numbers, each of which can be defined by a finite number of symbols, either digits, signs or words.[161]

According to Mannoury, this last phrase has to be interpreted in the sense that Brouwer's 'unfinished sets' are just denumerable ones, 'arranged or defined in a special way'. Mannoury's concluding remark about the denumerably unfinished cardinality is then:

> Anyway, the expatiation of the author about the unconceivability of Cantor's second number class can *only* be made to correspond with *this* interpretation (which interpretation is, however, in direct contradiction with the inclusion on the list of possible cardinalities of the 'unfinished' sets).[162]

About the second number class Mannoury agreed with Brouwer that Cantor's definition of this class is at least incomplete. The 'Inbegriff aller'[163] is too vague a concept in Mannoury's eyes for a proper mathematical definition. But for him the insufficiency of the definition by Cantor does not imply the impossibility of a better and more precise one:

> The much more strict introduction of higher cardinalities with the help of the 'assignment' or transfinite exponentiation is discussed in passing by the author, but not in the least refuted.[164]

interest to us now. The other one is a more popular and less mathematical-technical review, and was published in 1907 in *De Beweging*.

[161] Aan deze definitie (?) nu voldoen, als wij haar goed begrijpen, alle puntverzamelingen van hogere machtigheid dan die van het aftelbaar oneindige (Cantors eerste machtigheid), doch de schrijver noemt als voorbeeld het geheel der definieerbare punten op het continuüm, d.i. van die punten of getallen welke ieder afzonderlijk door een eindig aantal symbolen, hetzij cijfers, tekens of woorden, kunnen worden gedefinieerd.

[162] op. cit. page 162: Trouwens alleen met deze bedoeling (welke dan evenwel in lijnrechte strijd is met het opnemen van de 'onaffe' verzameling in de reeks der mogelijke machtigheden) kan in overeenstemming worden gebracht schrijvers uitvoerig betoog, dat Cantors tweede getalklasse ondenkbaar is.

[163] See [Cantor 1932], page 197; see also section 1.1.5. of this dissertation.

[164] op. cit. page 162: De veel strengere invoering der hogere machtigheden door middel der 'belegging' of transfiniete machtsverheffing, wordt door schrijver dan ook wel terloops besproken, doch allerminst weerlegd.

Brouwer's reply, in which Mannoury's review is discussed step by step, is included in the mentioned two books.[165] Item 3^e of Brouwer's reply is about the denumerably infinite unfinished cardinality and he only examined the contradiction that was observed by Mannoury: on the one hand, one of Brouwer's examples of a denumerably infinite unfinished set is the totality of the definable points on the continuum; on the other hand, (and Mannoury agreed with this claim) this set is just denumerably infinite.

For Brouwer there exists no contradiction here. For a denumerably infinite system, use is made of the possible combinations of a *finite* number of earlier introduced symbols. He thereby referred to page 170 of his dissertaton, apparently to the first footnote on that page:

> For the set of all combinations of a finite number of signs (in which the sign = is included and which are finite in number for each mathematical theory) remains denumerable, a fortiori this holds for the set of those special combinations of signs which may be read as *true* equations.[166]

However, for the totality of definable points on the continuum new symbols may be introduced, time and again, indefinitely many times, replacing an infinite number of earlier introduced symbols, if so desired.

So far Brouwer's comment on the relevant part of Mannoury's review. But Brouwer could have said more in his reply to Mannoury, so it seems.

On the basis of Brouwer's definition alone (although we concluded to its possible incompleteness), Mannoury inferred that all point sets of higher cardinalities than the first one, i.e. higher cardinalities than denumerably infinite, have the denumerably infinite unfinished cardinality. Only as a result of the *examples* that Brouwer gave for sets of this cardinality, Mannoury changed his conclusion and declared the denumerably unfinished sets to be just denumerable sets, arranged in a special way.

But both conclusions are wrong, and they are wrong precisely on the basis of Brouwer's definition and characterization.

Let us consider first Mannoury's conclusion that all sets of higher cardinality than the first belong to the class of the denumerably unfinished sets. Hence according to him the continuum is denumerably unfinished. 'Continuum' can, and (in 1907) should, be read as 'intuitive continuum'. In that case the answer is clear: It is a separate cardinality, not comparable to any other.

In case we read it (again anachronistically) as the 'full continuum of the reals', the following summary of our earlier argument about this point applies:

[165] [Dalen, D. van (ed.) 1981b] and [Dalen 2001]. This reply was published in 1908 in the *Nieuw Tijdschrift voor Wiskunde*.

[166] Immers het geheel van alle combinaties van een eindig aantal der ingevoerde tekens (waartoe het teken = behoort, en die eindig in aantal zijn voor elke wiskundige theorie) blijft aftelbaar, a fortiori dus het geheel van die bijzondere der tekencombinaties, die als *ware* vergelijkingen zijn te lezen.

Every well-defined denumerable subset of the continuum results, when the prescribed algorithm is applied to it (including our proposed 'improvement' on it; see page 278), in an irrational number, not belonging to the denumerably unfinished set constructed so far, but certainly belonging to the continuum. However, the converse does not apply. Indeed, for every definable irrational number on the continuum a denumerable subset can be constructed which, when the algorithm is applied to it, again results in that definable irrational number. But *the vast majority of points on the continuum is not definable*, i.e. cannot be specified by a denumerably infinite well-defined sequence of rationals and therefore the continuum is *not* denumerably infinite unfinished. But the totality of definable points on the continuum certainly has that property, since for the definition of its elements (i.e. for the definition of the *definable* irrationals) always new symbols may be introduced, as Brouwer explained in his reply.

But also Mannoury's revised conclusion that denumerably infinite unfinished sets are merely denumerably infinite sets, defined or arranged in a special way, is contestable. This interpretation does not do justice to the concept of this type of set or cardinality either. Brouwer stated explicitly that

> from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention.[167]

Hence this type of set is an expression for a *process* of extending a well-defined denumerably infinite set with always more new elements, not belonging to the denumerable set, but which are definable in terms of denumerably many elements of that set. And this is certainly not one of the two interpretations, given by Mannoury.

Brouwer did not go into these details of Mannoury's misinterpretation of this type of set; he limited his comment to the given short argument, although, in our view, he had better ones to counter Mannoury's objections. A reason could be that Brouwer already started to distance himself from the whole concept. He made clear that there was little use for the concept in his intuitionistic mathematics; it disappeared for all practical purposes. It only turned up a few times in passing after the year of publication of this reply (although the concept was not retracted in the *'Addenda and Corrigenda* to the dissertation' from 1917).

One of the last publications in which the concept returned is Brouwer's Inaugural Address *Intuitionism and Formalism* from 1912 on the occasion of his nomination to professor in mathematics at the Amsterdam University. There are three published versions of this lecture:[168] the academic version, the commercial

[167] dissertation, page 148: Maar streng wiskundig bestaat die verzameling als geheel niet; evenmin haar machtighied; we kunnen deze woorden echter invoeren als willekeurige uitdrukkingswijzen voor een bekende bedoeling.

[168] See [Dalen 2001], page 177.

edition and the English translation by A. Dresden, which latter was published in the *Bulletin of the American Mathematical Society*.[169]

In this lecture the question is asked:

> The formalist further raises the question, whether there exist sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null, in other words, 'whether the power of the continuum is the second smallest infinite power', and this question, which is still waiting for an answer, he considers to be one of the most difficult and most fundamental of mathematical problems.[170]

One of the possible, and for the intuitionist obvious, answers goes as follows:

> If we restate the question in the form: 'Is it possible to establish a one-to-one correspondence between the elements of a set of denumerably infinite ordinal numbers on the one hand, and a set of real numbers between 0 and 1 on the other hand, both sets being indefinitely extended by the construction of new elements, of such a character that the correspondence shall not be disturbed by any continuation of the construction of both sets?' then the answer must also be in the affirmative, for the extension of both sets can be divided into phases in such a way as to add a denumerably infinite number of elements during each phase.
>
> [*with, in a footnote, the addition:*]
>
> Calling *denumerably unfinished* all sets of which the elements can be individually realized, and in which for every denumerably infinite subset there exists an element not belonging to this subset, we can say in general, in accordance with the definitions of the text: *'All denumerably unfinished sets have the same power'*.[171]

[169] For the academic version see [Brouwer 1912], the commercial version is included in [Brouwer 1919c]. Both Dutch editions are brought together in a very surveyable way, by means of insertions and footnotes, and is included in [Dalen 2001]. For the English translation see [Brouwer 1913], also included in [Benacerraf and Putnam 1983], page 77 et. seq.

[170] [Benacerraf and Putnam 1983], page 85, 86; Dutch text [Dalen 2001], page 187: [De formalist] stelt de vraag, of er verzamelingen van reële getallen tussen 0 en 1 *bestaan*, waarvan de machtigheid kleiner is dan de continue, doch groter dan aleph-nul, m.a.w. *'of de continue machtigheid op één na de kleinste machtigheid is'*, en beschouwt deze vraag, die nog steeds geen oplossing heeft gevonden, als een der moeilijkste en fundamenteelste wiskundige problemen.

[171] See [Benacerraf and Putnam 1983], page 86; in Dutch [Dalen 2001], page 188: Preciseert men de vraag in de vorm: 'Kan men, onbepaald voortbouwend enerzijds aan een verzameling van aftelbaar oneindige ordinaalgetallen, anderzijds aan een verzameling van reële getallen tussen 0 en 1, een door de voortzetting der constructie niet gestoorde één-éénduidige correspondentie tussen de elementen der beide verzamelingen tot stand brengen?' dan moet het antwoord eveneens bevestigend luiden; immers voor beide verzamelingen kan men de constructie in zodanige fasen verdelen, dat gedurende elke fase een aftelbaar oneindig aantal elementen aan de verzameling wordt toegevoegd.

[*Met in een voetnoot de toevoeging:*]

In intuitionistic set theory the concept *denumerably infinite unfinished set* subsequently disappeared; it is not mentioned anymore in Heyting's survey [Heyting 1929]. The reason for this is the different approach towards the set concept with the help of choice sequences, in which there is no place any more for this type of cardinality. A set becomes a *law*. However, it was mentioned occasionally, e.g. in the second of the Vienna lectures, as we noted earlier on page 267.

In 1917 the concept was also mentioned in one of the theses at the end of the doctoral dissertation of B.P. Haalmeyer, a student of Brouwer. He claimed that *either* this type of cardinality is not basically a new one and therefore has to be removed from the list of four possible powers, *or*, if one wants to view it as a new cardinality, then its place in the order of magnitudes is unjustified. There is, of course, no further explanation given (typical for this type of statements), but undoubtedly Haalmeyer did not consider it as a *separate* cardinality, and simply placed it under the denumerably infinite one since the result is always denumerable (which is very well defensible).

But an interesting coincidence is that a copy of Haalmeyer's list of theses is kept in the Brouwer archives, and that this copy is filled with a number of handwritten comments by Brouwer on some of the theses. Interesting for the present discussion is the fact that one of the comments is about thesis 8, the one under consideration now. In his comment Brouwer still defended this type of cardinality. He claimed that the denumerable unfinished set A_1 'überdeckt' the denumerable set A.[172] Moreover, the elements of A and A_1 are 'finished', contrary to the arbitrary elements of the continuum C. Hence $C > A_1$ if at least, according to Brouwer, the 'unfinished mappings' of Bernstein and Hardy are admitted (see page 271). This last remark by Brouwer is a comment on Bernstein's *Untersuchungen aus der Mengenlehre*[173] and most likely on Hardy's paper *A theorem concerning the infinite cardinal numbers*, published in the *Quarterly Journal of Mathematics 1903*, page 87.[174] But note that, despite references to this type of cardinality as late as 1930 (in the second Vienna lecture and even later) it did no longer play any role in intuitionistic mathematics. In the first *Begründung* paper, page 7, the denumerable cardinality is subdivided in *abzählbar, zählbar, auszählbar, durchzählbar and aufzählbar.*

Verzamelingen van construeerbare elementen, waarin naast elke aftelbaar oneindige deelverzameling een daartoe niet behorend element kan worden aangegeven, als *'aftelbaar onaf'* kwalificerend, kan men in de zin van de tekst algemeen formuleren: *'Alle aftelbaar onaffe verzamelingen zijn gelijkmachtig'.*

[172] The notion 'überdeckt' is defined in [Brouwer 1918], page 11.

[173] [Bernstein 1905].

[174] This conjecture is based on notebook 8, pages 47, 48 and 80. On these pages of this notebook Brouwer discussed Bernstein's proof of the theorem *Das Kontinuum ist äquivalent der Gesamtheit O aller Ordnungstypen einfach geordneter Mengen erster Mächtigkeit* (see [Bernstein 1905], §8 and 9). But, Brouwer commented and criticized in his notebook, in his comparison of cardinalities Bernstein only spoke of (and *could* only speak of) unfinished mappings.

Also Hardy, in his proof of $2^{\aleph_0} \geq \aleph_1$ in the mentioned paper (cf. [Hardy 1979], page 427 ff.) defines a correspondence between numbers of the second number class and sequences of numbers (from the first or the second number class). This mapping also is unfinished.

7.6.4 The notebooks on Cantor's transfinite numbers

Transfinite numbers also have to be based on the mathematical ur-intuition, which excludes, as we saw, higher cardinalities than $\aleph_0$:

> (II-29) Transfinite numbers have to be built intuitively in full givenness; after that one can reason about them logically; but they differ in nature from a logical system.[175]

Brouwer's view on Cantorian numbers follows from the several discussions we presented earlier, and this view is not basically different from what is said about these numbers in the notebooks. Towards the end of the sixth notebook several pages are devoted to the discussion of the Cantorian cardinalities. In the following quote the existence of elements of the second number class is admitted, but not the cardinality of their totality:

> (VI-37) I shall thus have to demonstrate that Cantor's Aleph-eins makes no sense. No, his higher numbers certainly exist; only I just know certain defined individuals among them, and the few defined, which I can indicate, are denumerable.[176]

That is: numbers larger than ω exist, hence elements of the second number class exist, but, lacking an algorithm to collect them in one, they do not exist as a finished totality, and all elements of the second number class are of the same cardinality $\aleph_0$. There is no such thing as the completed second number class. The idea of an 'unfinished cardinality' turned up for the first time in the seventh notebook. This paragraph was quoted earlier in a different context, but it will be repeated here:

> (VII–4) I can say, from the point set, that I build the continuum, but I cannot speak of its 'cardinality', since this set, in its construction from individuals, simply is the second number class, and then 'denumerable' and 'not finished'.[177]

The existence of unfinished sets is also emphasized in the following paragraph (also quoted earlier); note that ω is here assumed to be finished:

> (VII-16) T [*the second number class*] cannot be mapped on ω by a finite law; neither can T be completed by a finite procedure; but during the construction of T in an infinite time it remains possible

[175] De transfiniete getallen moeten aanschouwelijk intuïtief opgebouwd worden; redeneren er dan achteraf, maar logisch over: maar ze zijn iets anders dan een logisch systeem.

[176] Ik zal dus moeten kunnen aantonen, dat Cantor's Alef-eins zinloos is.

Neen, zijn hogere getallen bestaan zeker; alleen, ik ken niet andere, dan bepaalde individuen er uit, en de enkele bepaalde, die ik kan aanwijzen, zijn aftelbaar.

[177] Ik kan uit de Punktmenge wel zeggen, dat ik het continuüm opbouw, alleen kan ik niet spreken van de 'machtigheid' er van, want deze Menge is in haar opbouw uit individuen gewoon de tweede getalklasse, en dan 'abzählbar' en 'nicht fertig'.

> to map it on ω. And that is all I can say. Of course T remains unfinished. ω is finished (by our innate mathematical induction).[178]

This seems to refer again to the unfinished mapping, which we discussed on page 271. ω may be viewed to be finished, but not so T and the mapping of T onto ω is always an unfinished mappinng in a Brouwerian sense (hence, is in fact no mapping at all).

And finally a fragment from the eigth notebook (quoted earlier in chapter 4), which is also stressing the unfinished character of the second number class T:

> (VIII-16) For everything that we can create *mathematically*, is denumerable; if we want to create T, we find out that our creating is never finished by giving isolated *acts*; and *laws*, which are denumerable sequences of facts; but for that reason we may not postulate that there are more things *apart from* what we can create.[179]

7.6.5 The notebooks on the denumerably unfinished set

There is one very interesting paragraph in the eighth notebook, concerning the unfinished set of the mathematical theorems.
In VIII–44 the following theorem is claimed:

> (VIII–44) The totality of mathematical theorems also, among other things, constitutes a set, which is denumerable but never finished.[180]

One can interpret this statement as a rudimentary form of Gödel's theorem, (despite obvious differences between the two). Gödel's theorem was mentioned earlier (see page 261); it states that there exist in Peano Arithmetic, or in any decidable ω-consistent extension of P.A., theorems which are true, but not provable from the set of axioms for that system. In other words: Peano arithmetic (and any axiomatizable extension) is incomplete, and the set of theorems is effectively denumerable, but remains unfinished due to the incompleteness of the system (in technical terms: it is 'productive'). Compare this conclusion with the content of the quote above: the totality of mathematical theorems constitutes a denumerable but unfinished set.

[178]T is niet op ω af te beelden door een eindige wet; maar T komt ook niet klaar door een eindig werk; maar, T vormende in oneindige tijd, blijft zij onder haar vorming steeds op ω afbeeldbaar. En dat is het enige wat ik kan zeggen. T is uit dien aard der zaak onaf; ω is af (door de mathematische inductie, die in ons is).

[179]Want alles, wat wij *wiskundig* kunnen scheppen, is aftelbaar; willen we T gaan scheppen, dan merken we, dat ons scheppen nooit klaar komt met het geven van geïsoleerde *daden*; en *wetten*, dat zijn aftelbaar oneindige feitenreeksen; maar daarom mogen we niet postuleren, dat er nog dingen zijn *buiten* hetgeen wij scheppen kunnen.

[180]Het aantal wiskundige stellingen is o.a. ook een Menge, die aftelbaar is, maar nooit af.

Gödel's own proof of his theorem is modeled on the reasoning involved in the paradoxes of Richard and of the 'Liar'.[181]

A proof of Brouwer's statement turns out to be problematic. If we specialize Brouwer's claim to arithmetic, then the number of arithmetical theorems is said to be denumerably infinite unfinished. Since Brouwer knew Richard's paradox and the liar paradox, it is not impossible that, if pressed, he would have argued along those lines. In fact, when he was informed about Gödel's incompleteness theorem, he said that it did not surprise him at all, and that he had, on conceptual grounds, come to the same conclusion.[182] But speculation of this sort belongs to retrograde science fiction, as long as no substantial support is obtained.

In the dissertation the 'totality of all possible mathematical systems' is mentioned as one of the examples of a denumerably unfinished set, and this is expressed in similar terms in the two foundational publications from 1908, his lecture *Die möglichen Mächtigkeiten* and the paper *The unreliability of the logical principles*. In the dissertation Brouwer argues that a proof has to proceed by referring to the second number class.

Now, an arithmetical *theorem* can be identified with its *proof*, which is the construction of a sub-building in the building of arithmetic, and can therefore be viewed as a 'mathematical system'. Of course every arithmetical identity yields an arithmetical theorem. We can also consider the extension of the building of arithmetic to include operations on the elements of the second number class, which makes the total number of elements of the extended building denumerably unfinished. If we now consider the infinitary language of the second number class (as Brouwer woud have allowed), then there are at least as many theorems as there are elements of the second number class. Hence this would show that the collection of *all* mathematical theorems is denumerably unfinished.

Of course, this falls short of Gödel's precise and technical theorem (and proof). We must realize the difference between the argument we just presented and Gödels way of reasoning: Gödel made use of the fixed and well-defined language of arithmetic of the *Principia Mathematica*, whereas with Brouwer the mathematical universe as well as its language are open ended. We usually em-

[181] See Gödel's paper *On formally undecidable propositions of Principia Mathematica and related systems*, [Gödel 1931]. See for this [Heijenoort 1967], page 598:

> The analogy of this argument with the Richard antinomy leaps to the eye. It is closely related to the 'Liar' too.

With in a footnote the addition:

> Any epistemological antinomy could be used for a similar proof of the existence of undecidable proposistions.

[182] As a historical note we may add that Gödel was influenced by Brouwer's 'inexhaustibility of mathematics' (and that thus, from a constructivistic viewpoint, the principle of the excluded middle becomes untenable). See [Wang 1987], page 50, where the author is quoting from Carnap's diary. On page 57 the author remarks that Brouwer informed him during a visit at his [Brouwer's] house that 'the conclusions [of Gödel's incompleteness results] had been evident to him [Brouwer] for a long time before 1931'. An interesting remark in the light of the notebook quote VIII–44.

ploy theories with a recursively enumerable set of axioms, or a language with a recursively enumerable alphabet. But the second number class requires a denumerably unfinished alphabet, since otherwise this class would be denumerable, which is not the case. We might attempt to extend the language of the second number class, together with the class itself, but this solution does not apply for standard languages. Hence the theorem is not so obvious as it seems.

See also Brouwer's criticism on Bernstein,[183] when the latter gave, on the basis of the properties of the 'ordering functions' (Ordnungsfunktionen), a proof of Cantor's theorem:[184] the continuum is equivalent to the set ('Gesamtheit') O of all ordertypes of simply ordered sets of the first cardinality.[185] In a reaction to this Brouwer claimed:

> (VIII–45) If I speak of e.g. the set of all simply ordered types of cardinality $\aleph_0$, then I first should ask: 'can I imagine that?' and if the answer was 'yes', then it also appeared to be like a constructible type according to a number T or c. Hence in this case one should say: order the cardinality like ω; put down the first one; the second in front or behind (two times); the third gives for its place three choices, etc. In this way I gradually approximate the cardinality 1.2.3.4.... $= c$.
>
> In this way Bernstein construes it in Math. Ann. 61, page 140 ff. But it is not true that one can see in this way the growth of all order types; no matter how far one continues this process, different ways of progress *never* give rise to different order types. No matter how far I will have continued, still I know *nothing* about the increasing order type. That is only the case if all laws are formulated in advance. But then the objection arises that one cannot speak of *all* laws.[186]

One cannot speak of the set of all laws of progression that define a point on the continuum, its cardinality being denumerably unfinished. In VIII–46 Brouwer continued that, if one wants to define the continuum as a set of laws of progression, (...) then one can no longer speak of its cardinality. It results

[183]Discussed in the dissertation on pages 156 and 157.

[184][Bernstein 1905], page 135.

[185]Das Kontinuum ist äquivalent der Gesamtheit O aller Ordnungstypen einfach geordneter Mengen erster Mächtigkeit.

[186]Als ik spreek van b.v. de Menge aller enkelvoudig geordende typen van machtigheid $\aleph_0$; moet ik mij eerst vragen: 'kan ik mij dat denken?' en is het antwoord 'ja' geweest, dan is het ook gebleken te zijn als een opbouwbaar type volgens een getal T of c. Zo in dit geval zou men zeggen: orden de machtigheid als ω; zet de eerste neer; de tweede er voor of er achter (2 keren); de derde geeft drie keuzen voor plaatsing enz. Zo benader ik langzamerhand tot een machtigheid 1.2.3.4.... $= c$.

Zo vat het Bernstein op in Math. Ann. 61 p 140 vlgg. Maar het is niet waar, dat men zo alle ordetypen naast elkaar ziet groeien; hoe ver men ook voortgaat met het proces, *nooit* doet verschillende voortgang een in bepaalde zin verschillend ordetype ontstaan. Hoe ver ik ook ben voortgegaan, nog *niets* weet ik omtrent het groeiend ordetype. Dat komt eerst doordat *wetten* vooraf worden geformuleerd. Maar dan komt weer het bezwaar, dat men van *alle wetten* niet kan spreken.

in something between $\aleph_0$ and c, no more than partly finished, and as such a mathematical entity, which Brouwer called c'. That applies to all 'denumerable but unfinished' sets, resulting in their equivalence. We can find this conclusion in his dissertation. Bernstein's proof of equivalence is false because he saw unfinished sets as finished and was unable to fix their equivalence (VIII–47).

7.6.6 The remainder of Brouwer's second example

Brouwer ended his discussion of the second application of the role of logic in mathematics with the treatment of some related topics of Cantor's theory of transfinite numbers.

About the *continuum problem*: this does not contain any new viewpoints worth a renewed discussion; the conclusion will suffice: from a logical point of view the continuum hypothesis does not involve a contradiction, since the sets concerned (the set of the real numbers and the second number class) are both denumerably unfinished. Mathematically, the question was discussed in chapter 5 of this dissertation with the conclusion that every subset of the continuum is either denumerable or of the power of the continuum.

Also the other examples of unjustified conclusions in regard to Cantor's theory, viz. the *Burali-Forti paradox*, the proof by Zermelo of the *well-ordering theorem* and the *Bernstein-Schröder theorem* (also known as the Cantor-Bernstein theorem) do not contain any new aspects asking for a further discussion.

7.7 The Peano-Russell logistics

7.7.1 Introduction

The third example of his arguments about the role of logic bears the title *The Peano-Russell logistics*; in this example Brouwer mainly commented on the first part of Russell's *Principles of Mathematics*. Since this discussion does not contain any new viewpoints and because the content of this part is regarded as common knowledge today, we just briefly mention some interesting fragments from it. Brouwer stated that according to Frege and Russell classical logic is not sufficient as a foundation to build the mathematical edifice on, and for that reason the foundational basis was expanded by them to include modern propositional logic. But contrary to, among others, Frege, Russell and Couturat, the role of logic for Brouwer remained limited to that of the linguistic accompaniment of a mathematical structure and just for that reason logic never led to a contradiction as long as it was properly applied within those constraints. But that was not what the logicians did; the well-known logical principles were used as a starting point and were applied to a, in Brouwer's eyes, chimerical 'everything' instead of to a mathematical system.

7.7.2 The paradoxes

Exactly this non-mathematical application of the logical principles led to the paradox, now generally known as the Russell paradox,[187] which is described in chapter X of Russell's *Principles*:

> In terms of classes the contradiction appears even more extraordinary. A class as one may be a term of itself as many. Thus the class of all classes is a class; the class of all the terms that are not men is not a man, and so on. Do all the classes that have this property form a class? If so, is it as one a member of itself as many or not? If it is, then it is one of the classes which, as ones, are not members of themselves as many, and *vice versa*. Thus we must conclude again that the classes which as ones are not members of themselves as many do not form a class – or rather, that they do not form a class as one, for the argument cannot show that they do not form a class as many.[188]

Russell proposed several solutions to neutralize the paradoxes, but these solutions could not pass the standards of rigour in Brouwer's eyes: in case of the examples of 'predicates which are not predicable of themselves', this property led to a contradiction, reason why Russell proposed not to grant them the status of predicates, which is for Brouwer simply evading the question. The reason for us to discuss this well-known topic is an analysis of Brouwer's counter-arguments.

As a possible general solution Russell proposed:

> Perhaps the best way to state the suggested solution is to say that, if a collection of terms can only be defined by a variable propositional function,[189] then, though a class as many may be admitted, a class as one must be denied. (...) We took it as axiomatic that the class as one is to be found wherever there is a class as many; (...) A natural suggestion for escaping from the contradiction would be to demur the notion of *all* terms or of *all* classes. Thus the correct statement of formal truths requires the notion of *any* term or *every* term, but not the collective notion of *all* terms.[190]

The reader can imagine Brouwer's comment on this solution: of course one cannot speak of *any element* since 'logical principles hold exclusively for words with a mathematical meaning'.[191] But Brouwer could have expressed himself even stronger (which he probably also meant to say): logical principles only apply to words that refer to objects which are the result of a mathematical construction, the result of a free creation, based on the experience of the move

[187] Note that this paradox was earlier described by Zermelo; see [Rang and Thomas 1981].

[188] [Russell 1938], page 102.

[189] A propositional function is an assertion with one or more real variables, and for all values of the variables the expression involved is a proposition. See [Russell 1938], page 13.

[190] l.c. page 104, 105.

[191] dissertation, page 163.

of time, the ur-intuition of constancy in change or of unity in multiplicity.[192] If logical reasonings are limited to those applications, then paradoxes simply cannot occur.

Brouwer also employed, on the same page 163, a different argument to let the 'common sense' repudiate the argument of Russell (we number the different elements in Brouwer's reasoning for easy reference in the subsequent discussion of their validity and strength):

1) Suppose there is an 'everything' and a totality of relations between all the objects and a totality of propositions, and suppose I know this 'everything'.

2) Then it is possible for a propositional function to decide for any arbitrary object, on the basis of its given relations, whether or not the function is satisfied by this object, resulting into two classes of objects for each function.

3) Now, if I want to investigate whether or not the class concerned satisfies the propositional function, then this investigation requires its completion, hence the investigation cannot be performed, which solves the paradox.

ad. 1) Of course, Brouwer placed himself in Russell's position to 'fight him with his own weapons', but Brouwer must have found it hard to put it in these terms. There simply is no 'everything'; every such indefinable and algorithmless totality is unthinkable. Nevertheless he made this shift in position in this first argument, whereas in the third he does not, but used his own standpoint and arguments instead.

ad. 2) Therefore there is no arbitrary object, arbitrary relation or arbitrary propositional function.

ad. 3) Now Brouwer views the class not as a unit, not as a single variable to which the propositional function can be applied, but as a collection of individuals (which, as we saw, is undefinable). For him the application of the function to the class requires its application to all of its composing objects separately. Hence Brouwer did not place himself in Russell's position in a consistent and tactically successful way. Therefore his argument on the basis of the ur-intuition followed by a proper mathematical construction, is better founded and is better defensible than this 'common-sense' reasoning.

An older (dating from the sixth century B.C. probably being the oldest, and ascribed to the Greek poet Epimenides) and for the general public more famous paradox is not mentioned in the dissertation; Brouwer referred to it in *The unreliability of the logical principles*[193] as a warning of what can happen if one performs logic on a language, independently of any mathematical system:

> Moreover, the function of the logical principles is not to guide arguments concerning experience subtended by mathematical systems, but to describe regularities which are subsequently observed in the language of the arguments. To follow such regularities in speech,

[192] See the summary of the dissertation on its last page 179.

[193] [Brouwer 1908a], also included in [Brouwer 1919c] as the first article.

> independently of any mathematical system, is to run the risk of paradoxes like that of Epimenides.[194]

Brouwer clearly is referring to the liar-paradox.[195] Not only is this paradox (one of) the oldest known, but it is still famous and under discussion today, as can be concluded from rather recent publications on this topic.[196]

The claim, as expressed by several mathematicians and stated in the beginning of this section, was that, if mathematics is to be based on logic, the latter has to be extended beyond classical logic to the logic of relations. Russell's *Principles of Mathematics*, chapter I, defines *pure mathematics* as the class of all propositions 'p implies q'. Mathematics is, according to Russell, based on some fundamental notion of logic and a number of fundamental principles of logic.[197] The most elementary relation is the relation between an element and its successor.

> The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age.[198]

For Brouwer, however, building mathematics on a logical foundation only is building a construction on quicksand:

> It is self-evident that in the language which accompanies mathematics, the succession of words obeys certain laws, but to consider these laws as directing the building up of mathematics, it is therein that the mistake lies.[199]

The remainder of this section in Brouwer's dissertation is devoted to the foundation of arithmetic, as this was developed by the logicians, especially by Peano and Russell. Since no new foundational arguments are involved here, a discussion of this section is omitted.

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[194] [Brouwer 1919c], page 7: Bovendien zijn bij betogen betreffende op wiskundige systemen gespannen ervaringswerkelijkheden de logische principes niet het richtende, maar in de begeleidende taal achteraf opgemerkte regelmatigheid, en zo men los van wiskundige systemen spreekt volgens die regelmatigheid, is er altijd gevaar voor paradoxen als die van Epimenides.

[195] In the New Testament of the Bible, in the letter of Saint Paul to Titus, the paradox is mentioned, perhaps without realizing that a paradox was expressed. See Titus 1:12–13, from the King James Bible:

> One of themselves, even a prophet of their own, said, The Cretans are always liars, evil beasts, slow bellies. This witness is true. Therefore rebuke them sharply, that they may be sound in the faith.

Saint Paul is generally regarded as a wise man and he is still read and studied today, but he obviously was not a mathematician, nor a logician.

[196] See e.g. [Martin 1984] and [Barwise and Etchemendy 1989]. For a thorough analysis of this paradox see [Visser 2002].

[197] l.c. page 4.

[198] l.c. page 5.

[199] see dissertation, page 165: Dat in de taal, die de wiskunde begeleidt, de opvolging der woorden aan wetten gehoorzaamt, spreekt vanzelf; maar dié wetten als het leidende bij de opbouw der wiskunde te beschouwen, daarin ligt de fout.

7.7.3 The notebooks on the Russell paradox

In the notebooks Brouwer mainly commented on Russell's *Fondements de la Géométrie*[200] and on the *Principles of Mathematics*.[201] The latter is of our main concern now. The paradox was rather extensively discussed by Brouwer in the sixth notebook, mainly in the form of thought experiments.

We mentioned earlier about the paradoxes, that the one of Epimenides was not discussed in the dissertation, but that it was referred to in [Brouwer 1908a]. There is also a reference to this paradox in one of the notebooks, in which Brouwer expressed how dangerously close Dedekind came to a paradox:

> (III-18) If Pete says: A Cretan said: 'I always lie', then Pete cannot have an impression of true or false about the Cretan. Keep that in mind if Dedekind wants to comprehend the totality of my impressions. That is impossible since that totality is included in itself.
>
> This leads to paradoxes like Russell's, or crocodile-'schluss', 'cannot be repeated' etc.[202]

From halfway the sixth notebook a discussion started on Russell's *Principles of Mathematics* and on the subsequent polemic in the *Revue de Métaphysique et de Morale* between Russell, Poincaré and Couturat.

The Russell paradox is analyzed and explained, not always completely clearly and transparantly, but the idea is there; remember that in the notebooks it concerns just thought experiments about this paradox. To give an idea of how Brouwer analyzed the problem and how he thought about the way to prevent all paradoxes by a proper construction of sets, we present the following quotation from the sixth notebook:

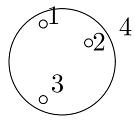
> (VI-26, 27) Russell's contradiction is based on the confusion between *if* something is the case and the *class of all things* for which that is the case. Imagine a finite number of things and compose from them all possible classes; there are among them that are not one of their own elements.
>
> But this *is* the case for the class consisting of four elements, of which

[200][Russell 1901], which is the French translation of [Russell 1897]. Brouwer preferred the French edition, undoubtly because many corrections were made by Russell in this later edition.

[201][Russell 1938].

[202]Als Piet zegt: Een Cretenzer zei: 'Ik lieg altijd', dan heeft Piet van die Cretenzer geen indruk van waarheid of leugen kunnen krijgen. Bedenk dat bij Dedekind, als die de gezamenlijkheid van al mijn voorstellingen wil omvatten. Dat kan niet, want dan was die verzameling er eerst al bij.

(hier uit voortvloeiende contradicties als van Russell, krokodilschluss, 'is niet herhaalbaar' enz.)



> one is the class of the others.
>
> That is what Russell means. Now compose from a finite number of dots *all* groups and groups of groups. And the paradox concerns the question whether or not the critical group has *one* element which is the class of all others.
>
> In case of the continued construction of groups of always higher order from those of lower order, the critical group is never reached; I can only speak about groups which can be indicated a priori, which don't have to wait until *everything* is constructed that cannot be constructed.
>
> The criterium for the constructed classes is of course independent of their union into a new class based on that criterium. The union to the newest class is a new concept, to be deduced from the old one; then the *existence* of that union can be sensed intuitively, in the other case not. Therefore the criterium of a class of classes should not be: it either or not belongs to its elements; except in the case of a completed totality of classes.[203]

There are more of those 'thought experiments' in the same sixth notebook. On one occasion Brouwer returned to this paradox in terms of propositions about given propositions, followed by a repetition of the previous argument:

> (VI-33) Put Russell's paradox as follows: (*the proper terminology*). The proposition (about given propositions):
>
> *If a proposition about given propositions does not satisfy itself.* Now ask that question *about* that proposition itself (that is: add it to the

[203] De 'contradictie' van Russell berust op de verwarring van *als* iets het geval is en de *klasse van al de dingen*, waarbij dat het geval is. Stel je maar eens een eindig getal dingen voor en vorm daaruit alle klassen; er zijn er daaronder, die *niet* zelf een van hun elementen zijn. Wel b.v. de klasse bestaande uit 4 elementen, waarvan een de klasse is van de andere.

[*See picture in the quoted text*]

Die bedoelt Russell. Vorm nu uit een eindig getal stippen alle groepen en groepen van groepen. En de paradox loopt er over, of de kritieke groep, al of niet één element heeft, dat de klasse is van alle andere.

Bij het opbouwen van telkens groepen van hogere orde uit die van lagere, vormt zich de kritieke groep nooit, en ik kan alleen spreken van groepen, die a priori zijn aan te wijzen, die niet hoeven te wachten, tot alles is opgebouwd, wat niet op te bouwen is.

Het criterium der gevormde klassen is natuurlijk onafhankelijk van hun vereniging tot een nieuwe klasse op dat criterium gegrond. Vereniging tot de nieuwste klasse is een nieuw concept, dat uit de oude moet worden afgeleid, dan kan ik het bestaan van die vereniging intuïtief voelen, anders niet. Daarom mag het criterium van een klasse van klassen niet zijn: ze hoort al of niet tot haar elementen, dan bij een bepaald reeds afgerond geheel van klassen.

> given set of propositions), then the paradox appears in the case *that I would answer that question* (which is absurd).
>
> Because a proposition about given objects means a separation of those objects with respect to their truth and falsity during the construction of the system and its relations; hence the 'either or not belonging to itself' is supposed to be convertible into symbols, that express something completely different from the words 'belonging to itself'. But for our critical proposition the argumentation to the solution should not depart from something previous, but from itself, which is absurd, and therefore it does not come as a surprise that it leads to its own opposite if it departs from itself.[204]

There are some additional smaller remarks made on 'class of classes', in this notebook, in particular about a possible definition of 'number' as a 'class of classes'. To give just one example:

> (VI-31) A number as class of classes makes no sense, I cannot survey that infinity of classes; numbers as *condition* for classes would do better.[205]

In notebook VI there is also quite an amount of general discussion on the role of logic, about what logic can do for the mathematicians, and what it can*not* do. The discussion in this notebook is mainly based on the earlier mentioned polemic in the *Revue de Métaphysique et de Morale.* This leads e.g. on page 29 to Brouwer's conclusion:

> (VI-29) Logistics, performed in its pure form, should consist of a finite number of rules composed of symbols in a row, without further text. It should not contain 'etc.', not even in the case that the principle of induction was proved, since one should not apply this rule on the *act of* writing symbolically, but only on the represented signs.
>
> Or shall we help the logicians by saying: Just as your ordinary human desire and calculation leads you to 'doing mathematics', your

[204]Stel de Russellse contradictie als volgt (*de juiste formulering*) De propositie (over gegeven proposities):
àls een propositie over gegeven proposities niet aan zichzelf voldoet. Stel nu die vraag *over* die propositie zelf. (Voeg haar dus aan het gegeven stel proposities toe.), dan komt een paradox *voor het geval, dat ik een antwoord op die vraag zou geven* (wat absurd is).

Want een propositie over gegeven dingen wil zeggen een scheiding van die dingen, ten opzichte van het *waar* of *vals zijn* van bij het opbouwen van het systeem mede opgebouwde relaties; het 'àl of niet tot zichzelf behoren' wordt dus ondersteld te zijn om te zetten in symbolen, die heel iets anders uitdrukken, dan de woorden 'tot zichzelf behoren'. Terwijl voor onze kritieke propositie de redenering tot de oplossing niet van iets vroegers kan uitgaan, maar van zichzelf zou moeten uitgaan; wat widersinnig is, en niets te verwonderen, dat hij, van zichzelf uitgaande, tot het tegendeel van zichzelf voert.

[205]Getal als klas van klassen is onzin (die oneindigheid van klassen kan ik niet overzien); als voorwaarde voor klassen zou beter gaan.

> mathematical viewing of things leads you in building your chimera; hence that chimera presupposes life *and* mathematics. (Therefore, in my proofs of existence, I may only use all systems built up in pure mathematics, that is from number, continuum and mathematical induction, but I may not use examples from life, since logic presupposes mathematics (and not the converse).)[206]

7.8 The logical foundations à la Hilbert

7.8.1 Introduction

Brouwer kept his most fundamental objection against the unjustified role that logic often played in the construction of the mathematical building for the last item. It opens as follows:

> *Consistency proofs for formal systems, independent of their interpretation.*
>
> The most uncompromising conclusion of the methods we attack, which illustrates most lucidly their inadequacy, has been drawn by Hilbert (...)[207]

Brouwer is referring to Hilbert's Heidelberg lecture *Über die Grundlagen der Logik und der Arithmetik.*[208] In an earlier paper *Über den Zahlbegriff*,[209] to which Brouwer also referred in the same paragraph, Hilbert stated that usually arithmetic is defined *genetically*, (i.e. via extensions from the system of the natural numbers) and geometry is defined *axiomatically* but that he now intended to present an axiomatization for the system of the real numbers and their basic operations. For this purpose Hilbert presented a list of 18 axioms, divided into 4 groups,[210] and at the end of this paper he stated:

[206]Praktisch zuiver uitgevoerde logistiek zou moeten zijn een eindig aantal regels van symbolen onder elkander, zonder tekst. Enz. mag er niet in voorkomen; want zelfs al had men het principe van inductie aangetoond: men mag het niet toepassen op de *handeling van* het symbolisch schrijven, alleen op de tekens, die er worden voorgesteld.

Of zullen we de logistici helpen en nu zeggen: Zo goed als uw gewone menselijke begeerte en berekening u leidt bij uw 'wiskunde doen', zo leidt uw gewoon wiskundig de dingen bekijken u bij het opbouwen van uw 'chimère'; dat chimère vóóronderstelt dus het leven èn de wiskunde. (Ik mag dan bij mijn Existenzbeweise alle in zuivere wiskunde, d.i. uit getal, continuüm en mathematische inductie opgebouwde systeem gebruiken, maar geen voorbeelden uit het leven, want *logica vooronderstelt wiskunde*, (niet omgekeerd).)

[207]dissertation page 169. *Niet-strijdigheidsbewijzen van tekensystemen, onafhankelijk van hun betekenis.*

De zuiverste consequentie van de hier bestreden methoden, waaraan tegelijk het eenvoudigst en helderst de ontoereikendheid er van blijkt, is getrokken door Hilbert (...)

[208][Hilbert 1905], included as *Anhang VII* in [Hilbert 1909]. An English translation can be found in [Heijenoort 1967], page 129.

[209][Hilbert 1899b], included as *Anhang VI* in [Hilbert 1909].

[210]The 'Axiome der Verknüpfung, der Rechnung, der Anordnung, der Stetigkeit'.

> unter der Menge der reellen Zahlen haben wir uns hiernach (...) zu denken (...) ein System von Dingen, deren gegenseitige Beziehungen durch das obige *endliche und abgeschlossene* System von Axiomen I – IV gegeben sind, (...)

This immediately gives us a clear picture of what Brouwer is going to dispute: the system of the real numbers is *defined formally* instead of *constructed out of the ur-intuition*. This contrast between Hilbert's and Brouwer's approach will be the main topic of this section. Brouwer's analysis of Hilbert's method (especially the method used in the Heidelberg lecture) will be that this turns out to be an accumulation of identifications of, on the one hand, mathematical techniques and, on the other, their descriptions in the accompanying languages.

But, as one may expect from someone of the calibre of Hilbert, he also had his point and argument: In the paragraph preceding the last given quote from *Über den Zahlbegriff*, Hilbert raised the question of *how to prove* the consistency of his totality of axioms and all their corollaries. Because of the abstract character of the axioms, this proof had to be independent of any mathematical intuition, and this question formed the second problem from the Paris list (see the next subsection). According to Brouwer, this independence can only be guaranteed

> by considering the very signs which express the axioms as a mathematical system, by formulating the principles of logic, in the manner of the algebra of logic, as rules allowing to extend this system, and by proving mathematically that these rules taken from the algebra of logic can never lead to an equation together with its negation.[211]

7.8.2 The content of Hilbert's Heidelberg lecture

In another paper by Hilbert, *Über die Grundlagen der Geometrie*,[212] the geometrical axioms as stated there are proved to be independent, and the consistency of the system is established on the basis of the consistency of aritmetic. This arithmetical consistency was not investigated in the *Grundlagen*; it was the second problem on the list of 23 problems that Hilbert presented at the Paris conference in 1900. These 23 problems were intended to be solved in the century ahead.

In the presentation of this second problem the concept of completeness was also touched upon, which for Hilbert was *syntactical completeness*, that is:

> und jede Aussage innerhalb des Bereiches der Wissenschaft, deren Grundlage wir prüfen, gilt uns nur dann als richtig, falls sie mit-

[211] dissertation, page 170: door de tekens, die de axioma's uitdrukken, zelf als een wiskundig systeem te beschouwen, de principes van de logica volgens de algebra der logica te formuleren als regels om dat systeem verder uit te bouwen, en dan wiskundig te bewijzen, dat die uit de algebra der logica afgelezen bouwregels nooit tegelijk een vergelijking en haar ontkenning zullen kunnen afleiden.

[212] [Hilbert 1902].

> tels einer endlichen Anzahl logischer Schlüsse aus den aufgestellten Axiomen ableiten läßt.[213]

In the Heidelberg lecture[214] Hilbert commenced his *Programm*, which was the proof of consistency of an axiomatically founded arithmetic, but in this lecture the proof is still limited to a simple system, the axiomatically founded system of the natural numbers extended with the concept of a set and its elements and with only one very basic operation defined in it, viz. the successor operation; the operations of addition and multiplication are not yet axiomatically founded, but the *logical* connectives $\wedge$ (u in Hilbert's notation), $\vee$ (o with Hilbert) and the implication-arrow $\rightarrow$ ($|$ with Hilbert) are given and used. Rules are stipulated to extend the system and to prove its consistency after each extension.[215] The 1904 lecture is just the initial impetus to this program, but Hilbert had shown that the consistency of a system can be recognized without the construction of a model.[216]

Before commenting on Brouwer's discussion of this subject, we will briefly sketch the content of Hilbert's lecture.

At the beginning of this lecture Hilbert stated that, in order to prevent ending up in a vicious circle, arithmetic should not be considered as part of logic, since in deducing the laws of logic certain arithmetical principles are applied. Therefore the laws of logic and those of arithmetic have to be developed together and 'teilweise gleichzeitig',[217] so that proofs can be viewed as *finite mathematical* objects; as a result of that it can be shown that such formal proofs cannot lead to a contradiction.[218]

For his foundation of the number system Hilbert defined so called basic 'Gedankendinge' (thought-objects), viz. 1 (*one*),[219] $=$ (*equal*), u (*infinite* or *infinite set*), f (*successor*) and f' (*successor operation*).

Any combination of 1 and $=$ is also a thought-object and belongs either to the class of the *entities* (Klasse der Seienden) or to the class of the *non-entities*

213[Hilbert 1900], see also in [Hilbert 1932], Vol III, page 299, 300.

214[Hilbert 1905].

215For a critical survey of the elaboration of the so called *Hilbert Program* see [Smorynski 1988] and [Sieg 1999], the latter especially for the progress of the program during the years 1917 – 1922.

216See e.g. [Smorynski 1988], page 11.

217Hilbert was, at least at that time, convinced that mathematics cannot be deduced from logic alone, but that both systems have to be developed simultaneously, whereas e.g. for Russell logic precedes mathematics. This difference in priority between Hilbert and Russell is partly caused by the fact that, what the one calls arithmetic, is a part of logic for the other, as Poincaré remarked in [Poincaré 1906b]. Brouwer, as we know, went much further in stating that logic only comes into being *after* the completion of a mathematical building. However, years later Hilbert, in the continued development of his program, was again of the opinion that logic has to be axiomatized, giving the most fundamental basis for mathematics, and that number theory and set theory, being the foundations for other branches of mathematics, are just parts of logic ('Teile der Logik'). See e.g. [Hilbert 1918], page 153. See also [Hilbert 1922] and [Hilbert 1923].

218see [Sieg 1999], page 7.

219as a symbol for a thought-object, not as a number (finite numbers will be defined later).

(Klasse der Nichtseienden). The class of the entities is defined by the axioms

1. $x = x$, and
2. $\{x = y \text{ u } w(x)\}|w(y)$.

Simple objects a, i.e. assertions without suppositions, constructed inductively from the two axioms, belong to the entities, i.e. are proper assertions. All others that 'differ' (which concept is also defined) from this form belong to the non-entities. From the axioms 1 and 2 only statements of the form $a = a$ follow, belonging to the class of the entities, therefore the defined object '=' is a consistent concept.

We write a if the proposition a belongs to the entities, and $\overline{a}$ if a belongs to the non-entities.

The last three thought-objects u, f and f' are defined by the axioms

3. $f(ux) = u(f'x)$, that is, the successor of an element of an infinite set is also an element of that set.
4. $f(ux) = f(uy)|ux = uy$, that is, two equal elements have two equal predecessors.
5. $\overline{f(ux) = u1}$, that is, the element 1 is no successor of any element.

The important question is now: do the axioms $1-5$ together with all their inductively constructed consequences form a consistent system, or can any contradiction be derived from them?

Any statement forming a contradiction with the axioms should be of the form

6. $f(ux^{(o)}) = u1$ (which is Hilbert's notation for $\exists x(f(ux) = 1$, $x^{(o)}$ standing for $x_1 \vee x_2 \vee x_3 \vee ...$), since axiom 5 is the only one containing a proposition a belonging to the non-entities. The proof of the impossibility of a contradiction follows for Hilbert from the fact that axioms $1-4$ *and* all their consequences are homogeneous, whereas 6 is not ($a = b$ is homogeneous if a and b contain the same number of basic thought-objects), but for a rigorous completion of this proof the notion of *finite ordinal number* is required, as well as that of *equinumerousness*. Note that the proof here employs the principle of complete induction.

Again, all inductively constructed assertions, built up from the axioms $1-4$, form the set of the entities; *all* others form the non-entities. The objects 1, =, u, f and f' are consistent concepts, just as all their consequences are consistent propositions, owing to the fact that the axioms and their consequences are homogeneous expressions, whereas the axiom 5, the only one giving an assertion a belonging to the class of the non-entities, is not homogeneous.

Hilbert then remarked:

> Die eben skizzierte Betrachtung bildet den ersten Fall, in dem es gelingt, den direkten Nachweis für die Widerspruchslosigkeit von Axiomen zu führen, während die sonst – ins besondere in der Geometrie – für solche Nachweise übliche Methode der geeigneten Spezialisie-

> rung oder Bildung von Beispielen hier notwendig versagt.[220]

The constructed consequences certainly are homogeneous and not in contradiction with the axioms, but the completeness of the defined system is tacitly assumed. Hilbert's proof of consistency is completely founded on the homogeneity of all true expressions, inductively deducible from the four axioms, and on the assumption that all inhomogeneous ones are non-entities, hence false.

Adoption of the well-known axioms of complete induction (which in fact were used already) and transforming them into the language selected by Hilbert, gives us the set ω as the consistent set of the smallest infinite. Subsequent addition of the axiom:

> daß jede Menge, die das erste Element der Ordnungszahl und, falls ihr irgend eines angehört, auch das diesem folgende enthält, gewiß stets das letzte Element enthalten muß,

provides us with the foundation of the *finite ordinal numbers.* The proof of the consistency of the axioms 1 – 4 and all their deducible results, is then given with the help of an example (which method of proof is justifiably criticized by Brouwer), after the addition of a new 'Gedankending' $<$, defined by the axiom:

> $(x < y \ \mathrm{u}\ y < z)|x < z$, in which x, y and z are arbitrary finite ordinals.

Finally, Hilbert presented in his Heidelberg lecture a number of principles to be used for a further expansion of the laws of mathematical thought, which then of course have to end up in a complete arithmetic:

I. Assertions, apparently not deducible from the axioms, are permitted and are true if they, when viewed as axioms, will not give rise to contradictions with the already existing axioms. This is the *creative principle.*

II. 'Arbitrary' elements in the axioms only refer to thought objects and their properly deduced combinations.

III. A set is a thought-object m, and combinations mx are elements of that set, hence the set concept precedes that of its elements. Other thought-objects are *mapping, transformation, relation* and *function.* By means of suitable axioms, their consequences in the form of combinations of the thought-objects can be distributed to the class of the entities or to the class of the non-entities. II and III are the *paradox-preventing principles.*

IV. A condition for a proper investigation of a given system of axioms consists of the possibility of a partition of all possible combinations into the two classes. The question for the possibility of this partition is equivalent to the question whether the consequences which can be obtained from the axioms, are consistent, under the addition of the familiar modes of logical inference.[221]

[220] [Hilbert 1905], page 181.

[221] We recognize here the 'side by side' development of mathematics and logic, which, according to Hilbert, is required for a successful proof of consistency.

Consistency can then be recognized in one of the following ways: 1. by showing that a possible contradiction should occur at an early stage of the development, or 2. by assuming that a contradiction can be deduced from the axioms and then showing that such a proof would itself contain a contradiction.

We again observe (more or less between the lines) that Hilbert tacitly assumed that, despite possible difficulties in its effectuation, this partition can be performed. This is in the spirit of the introduction of his Paris lecture in 1900 about the unsolved mathematical problems: every mathematical problem can be solved or its unsolvability proved. Hilbert assumed that every true proposition can be deduced from the axioms, and that every false proposition can be proved to be so by means of a resulting contradiction.

V. When speaking of *several* thought-objects or *several* combinations thereof, a limited number of them has to be understood.[222] Since we have established the concept of finite number, the exact meaning of the 'arbitrary' and of the 'differing' between propositions can be exactly described, based on that concept. Also the above sketched proof that the proposition $f(ux^{(0)}) = u1$ differs from every consequence of the axioms $1 - 4$, can be obtained in a finite number of steps:

> man hat eben den Beweis selbst als ein mathematisches Gebilde, nämlich eine endliche Menge zu betrachten, deren Elemente durch Aussagen verbunden sind, die zum Ausdruck bringen, daß der Beweis aus 1.– 4. auf 6. führt, und man hat dann zu zeigen, daß ein solcher Beweis einen Widerspruch enthält und also nicht in unserem definierten Sinne widerspruchsfrei existiert.[223]

Also the existence of the totality of the real numbers can be demonstrated in a similar way as the existence of ω was shown, including its completeness axiom:

> so bringt dasselbe zum Ausdruck, daß der Inbegriff der reellen Zahlen im Sinne der umkehrbar eindeutigen elementweisen Beziehbarkeit jede andere Menge enthält, deren Elemente ebenfalls die vorangehenden Axiome erfüllen.[224]

And the axioms for the totality of the real numbers do not differ basically from the axioms needed for the definition of the integers.

We discussed this fifth rule of Hilbert rather extensively since Brouwer criticism is mainly directed against this one.

[222] Note that Hilbert speaks of a *limited* number, not of a *finite* number!

[223] [Hilbert 1905], page 185. Note that also Hilbert used the metaphor of the 'building', but not in the strict constructive sense as Brouwer did.

[224] op. cit, page 185.

7.8.3 Brouwer's comment on Hilbert's Heidelberg lecture

Hilbert thus sketched in his Heidelberg lecture a system of a simple arithmetic without addition and multiplication, but with the successor operation as the only defined operation, and for this simple system a proof of non-contradictority was given. About this proof, Brouwer commented that Hilbert intuitively employed terms like *one*, *two*, *some*. In regard to the latter term, Hilbert explicitly stated in item V that this is used in the meaning of 'finite number', which concept was defined earlier. Brouwer also pointed out that Hilbert intuitively applied all the laws of logic as well as complete induction (which was also one of the points of criticism from Poincaré towards Hilbert; see below). Hence, Brouwer meant to say, intuition plays a major role in Hilbert's argument, contrary to his aim. This aim shows itself in item V from the given list of principles for the further extension of the laws of mathematical thought. In this fifth item Hilbert attempted, in the eyes of Brouwer, to make the consistency proofs independent of the intuition, by reviewing what he wrote so far and considering this as a mathematical building with its own rules.

In his comment Brouwer put the following paragraph between quotation marks, as if quoting from Hilbert's Heidelberg lecture, which in fact was not the case; it is not even a partial or a composed quote. One should view it as Brouwer's concise summary of this fifth item:

> 'I [*i.e. Hilbert*] have proved just now that the rules from which I have seen the linguistic structure develop, are consistent and therefore correct. In other words, the reasonings which I have made in that language justify at the same time the intuitive element in the act by which they were made.'[225]

Hilbert did not employ the term *intuition* or *intuitive*, but he indeed intuitively employed terms like *one, two, some* etc!

Brouwer's comment on Hilbert's axiomatic method amounts to the following:

Hilbert's true foundation remains the intuition, and intuition cannot *logically* be proved to be correct. If the mathematical intuition is correct, and the mathematical building based on that intuition is properly constructed, then automatically the accompanying words develop logically correct; and moreover, the consistency of a linguistic system, developed on the basis of a mathematical intuition, does not prove the correctness of that intuition.

Brouwer's main point is: There is mathematics and there is the accompanying language in which the logicians discover a separate structure; on this linguistic structure the rules of logic can be applied. As long as the two areas intentionally remain separated, problems in the form of paradoxes can be avoided. But, Brouwer claimed, Hilbert made the transition from the level of a mathematical structure to the one of the linguistic structure repeatedly, and after each transistion he remained at the next level, developing and extending it

[225]dissertation, page 171: 'De wetten volgens welke ik dat taalgebouw zich zie ontwikkelen, heb ik zo-even bewezen, dat niet-strijdig, dus juist zijn. M.a.w. de daar in die taal van mij gehouden redeneringen bewijzen meteen het intuïtieve in hun eigen daad als gerechtvaardigd.'

as if it were a mathematical structure itself, instead of a linguistic one, thereby using the previous levels only to give a meaning to the new ones.

In order to make his objections against Hilbert's methods clear and to pinpoint the places in Hilbert's arguments where he went astray in Brouwer's eyes, Brouwer presented a summary of the several stages of the interplay between mathematics and its accompanying language, which stages are frequently confused by Hilbert by viewing a linguistic level as a mathematical one. Compared to Hilbert, Brouwer is far more specific in the details of these several stages, and he observed eight different levels of reasoning:

1. The pure construction of the intuitive mathematics,
2. Its linguistic description,
3. The mathematical study of that language,
4. Forgetting the meanings of the linguistic elements and symbols used, he treated that language like a second order mathematics. This is the system which is worked out and developed by the logicians.
5. The accompanying language of the system of item 4.

This is, in Brouwer's view, the scheme followed by Hilbert in the Heidelberg lecture when presenting the several principles for the further expansion of mathematical thought, up to and including rule IV. There certainly is a form of intuition present in Hilbert's scheme, as Brouwer showed, but Hilbert neglected this and his treatment is axiomatic. Therefore he passed the levels 2 (in which he used linguistic terms like *one, two, several, for some* etc., the concepts of those terms apparently already being there from the intuition), 3 (in which these linguistic terms got an arithmetical meaning within the language used, and which arithmetical meaning was subsequently studied), 4 (in which these terms and their relations were axiomatized, now dissociated from their original and intuition-based meanings), ending up in 5, the language of the axiomatized system.

6. The language of logic is, in its turn, again studied mathematically. This is, according to Brouwer, what happened in rule V. But even more happened there:
7. Also the meaning of the elements used in 6 is forgotten, thus creating a third order mathematical system.

This is a schematic survey of what Brouwer described in the section *attempts to make these proofs independent of intuition* (page 171 of the dissertation). Hilbert looked upon all what he wrote down (up to and including his rule IV) time and again as a mathematical edifice. He observed in the first paragraph of rule V the consistency of this, by now third order, mathematical system (the first one being the intuition-based level) and claimed: it is consistent, hence it exists and *therefore* my intuition is correct, which is exactly the converse of Brouwer's way of reasoning.

8. Brouwer mentioned in his comment that Hilbert still went one step further: the linguistic accompaniment of this third order system. This happened in the same first paragraph of V when viewing its *linguistic* terminology. Brouwer ended the discussion with the statement that only the results of the *first* stage gives us mathematics and that the second stage is a necessary accompaniment for communication to others and for one's own memory.

Returning now to what Brouwer said in the beginning of the third chapter of his dissertation, in the footnote on page 142: it is not certain that every mathematical problem can be solved, though Hilbert was deeply convinced that any mathematical problem either can be solved or proved to be unsolvable.[226] Also Hilbert was convinced, as can be concluded from the Heidelberg lecture, that his system was consistent, thereby tacitly assuming its completeness (see our page 299).

As we have remarked on an earlier occasion, Brouwer expressed on a loose sheet in the ninth notebook his doubt about this conviction, thereby anticipating on Gödel's later work.[227] He also presented an extensive analysis of Hilbert's attempts to prove the consistency of arithmetic (which is the basis for a more general consistency proof of the different branches of mathematics: Hilbert's program). He thereby showed that Hilbert made the well-known mistake of the logicians concerning the different levels, not one time, but several times in succession.

On the last pages Brouwer compared his criticism on the role of logic with Poincaré's comment, as this was expressed in a polemic in the journal *Revue de Métaphysique et de Morale* by Poincaré and others in the years 1905 and 1906.[228]

7.8.4 Poincaré's criticism

In view of some of Brouwer's theses which he intended to add at the end of his dissertation,[229] we will discuss briefly the last page of the dissertation, before its summary. Especially the second item, the existence of the actual infinite, is of interest to us.

In 1902 Russell published the first edition of *Principles of Mathematics* and in 1904 and 1905 Couturat's *Les Principes des Mathématiques* appeared as a series of articles in the *Revue de Métaphysique et de Morale*, followed in 1905 by the publication of this series in a book version.[230]

Russell's and Couturat's publications triggered a polemic in the journal *Revue de Métaphysique et de Morale*, mainly during the years 1905 and 1906 between Poincaré, Couturat and Russell and others, on the relation between mathematics and logic. A number of the published papers written by the semi-

[226] See the footnote on our page 259 for the relevant quote.
[227] See page 260.
[228] [Poincaré 1905] and [Poincaré 1906b].
[229] There are several drafts for the list of theses; see [Dalen 2001], page 144 ff.
[230] [Couturat 1905].

intuitionist Poincaré appeared under the combined title *Les mathématiques et la logique* and commented in a critical way on the work of Russell, Couturat, Peano, Burali-Forti and Hilbert (the latter's *Heidelberg lecture*).

We discussed earlier Brouwer's criticism which he passed on Hilbert and we note now that the reaction of Poincaré to Hilbert's Heidelberg lecture is completely different from Brouwer's comments on it. As we saw, Brouwer limited his criticism to the fact that Hilbert did not keep properly separated the different levels of his metamathematical linguistic construction from mathematics itself, whereas Poincaré took more notice of the details of Hilbert's attempted consistency proofs.

In his dissertation Brouwer did not comment on Poincaré's criticism towards Hilbert, which is remarkable since in the notebooks some specific critical comment towards Poincaré's Revue papers is given. In the dissertation Brouwer restricted his reaction to Poincaré to the items of the *petitio principii* in logistics and of his rejection of the *actual infinite*; According to Brouwer, Poincaré thereby omitted the main point, i.e. Hilbert's confusion between the act of the mathematical construction and its accompanying language.

The petitio principii

1. *Poincaré's analysis*

In the mentioned *Revue* papers, Poincaré in the first place opposed Couturat's opinion that there is no synthetic a priori judgement in mathematics, and that mathematics is completely reducible to logic. Intuition should play no part in it. For the refutation of this claim, Poincaré made the comparison with a game of chess: just knowing the rules of how to move the pieces is not sufficient to play a good game; for that a good deal of chess-intuition is a strict condition. And whereas Poincaré is of the opinion that the principle of complete induction is a necessary principle for the development of mathematics, which is not reducible to, or provable from logic (the proof of this principle requires the principle itself), to the logician it is merely a 'definition in disguise' for the system of the natural numbers.

Poincaré discussed in the *Revue* papers the difficulties that arise when axioms and definitions are stated: what is the fundamental difference between the two in regard to their purpose and to their character? Is, for instance, complete induction an axiom, a definition, or simply a convention? Also in the case of the definition of numbers, one can already conclude it to be a problematic concept from the great number of different definitions and approaches that exist for this concept.

If one takes logic for granted then it becomes impossible, or at least extremely difficult, to avoid the *petitio principii* when defining the number concept. The use of a symbolic language, like Burali-Forti did in *Una Questione sui Numeri Transfiniti*[231] does not solve anything, as Poincaré emphasized. Intuition plays its part, possibly unconsciously, in the development of the number concept or

[231] See [Heijenoort 1967], page 104.

any other mathematical concept for that matter.

2. *Brouwer's comment.*

According to Brouwer, Poincaré's criticism is mainly aimed at the petitio principii and at the acceptation of the actual infinite, thereby missing the main point of criticism, viz. the fundamental difference between the mental construction of mathematics and its accompanying language.

Surprisingly, Brouwer then continued as follows:

> In a sense the petitio principii is allowable, for where it occurs in the act of construction of the linguistic system, it does not as such affect the perfection of that linguistic system;[232]

Brouwer clearly meant to say that a linguistic structure is perfect if and only if it is based on a successfully constructed mathematical edifice, of which it is the proper and correct linguistic accompaniment. As long as the mathematical building is properly constructed, i.e. *exists* since it is based on the ur-intuition alone, then the structure of the accompanying language cannot be anything else but perfect and no linguistic petitio principii can harm it. However, in the case that a primal mathematical intuition reappears as a mathematical *result* further down the road of the constructional development, only then we recognize an inadmissible petitio principii.

The frequently made mistake by the logicians is that they view the linguistic structure as primal and continue the construction of that structure, which is then no longer a mathematical building but a linguistic one.

The actual infinite

Another item about which Brouwer disagreed with Poincaré's critique is the latter's rejection of the actual infinite in Cantorism. This is an interesting point and Brouwer's opinion on it seems rather puzzling. We will come back to this item in the next chapter of this dissertation.

1. *Poincaré's view*

In *Les mathématiques et la Logique II*[233] this rejection is touched upon by Poincaré, but not yet explicitly expressed. However, in two other papers by Poincaré, dating from the years 1906, also published in the *Revue de Métaphysique et de Morale*, viz. *A propos de la logistique* and *Les mathématiques et la logique*,[234] Poincaré explicitly declined the actual infinite.

In *Les Mathématiques et la Logique*, in which again the content of the other two papers bearing the same title (to which it also refers) is discussed, but now

[232] dissertation, page 176: De petitio principii is in zekere zin geoorloofd, want waar die in de daad van de opbouw van het taalsysteem wordt uitgevoerd, raakt zij aan de volkomenheid van dat taalgebouw als zodanig niet.

[233] [Poincaré 1906b].

[234] [Poincaré 1906c] and [Poincaré 1906a] respectively. Note that the latter bears the same title as [Poincaré 1905] and [Poincaré 1906b], but it certainly is a different paper. In fact in [Poincaré 1906a] reference is made to the former two, which form together one large paper.

in a more concise way, the same conclusion is reached at the end: there is no actual infinite.

> *Il n'y pas d'infini actuel*; les Cantoriens l'ont oublié, et ils sont tombés dans la contradiction. Il est vrai que le Cantorisme a rendu des services, mais c'était quand on l'appliquait à un vrai problème, dont les termes étaient nettement définis, et alors on pouvait marcher sans crainte.[235]

2. *Brouwer's comment*
   In the dissertation this is limited to just one short remark:

> And as to the actual infinite of the Cantorians, it does exist, provided we confine it to that which can be intuitively constructed, and refrain from extending it by logical combinations that cannot be realized.[236]

This looks puzzling, since on the one hand Brouwer views infinite sequences, or algorithmically constructed sets as always unfinished, and on the other hand 'the actual infinite exists'. We will return to this topic in the next chapter.

7.8.5 The notebooks on the Foundations after Hilbert

The foundational subject, as it was discussed in the dissertation on the basis of Hilbert's Heidelberg lecture, is treated in the notebooks in a different way. Here the discussion is mainly a comment on Hilbert's foundations of geometry, in which (for Hilbert) the point of departure was a consistent arithmetic which formed the basis for the construction of geometry. Hilbert's main publications to which Brouwer referred in the first five notebooks are the *Festschrift*, officially entitled the *Grundlagen der Geometrie*, and the paper *Über die Grundlagen der Geometrie*.[237]

Only in the sixth notebook Hilbert's Heidelberg lecture comes up for discussion, albeit on the basis of Poincaré's comment on it from *Les mathématiques et la logique*. Later on, towards the end of notebook VIII, a short argument about the Heidelberg lecture, this time not based on the comment of others, concludes Brouwer's concern in the notebooks on this historical and foundational lecture by Hilbert.

First Brouwer remarked on the basis of [Poincaré 1906b]:

> (VI-28) (Poincaré about Hilbert, appropriately) 'Les indéterminées qui figurent dans les axiomes (en place du quelconque ou du tous de la logique ordinaire) représentent exclusivement l'ensemble des objets et des combinaisons *qui nous sont déjà aquis* en l'état actuel de la théorie.

[235] [Poincaré 1906a], page 316.
[236] (dissertation, page 176) En het actueel oneindige der Cantorianen, dit bestaat wel degelijk, als we het maar beperken tot het intuïtief opbouwbare, en dat niet door niet te verwezenlijken logische combinaties te willen uitbreiden.
[237] [Hilbert 1899a] and [Hilbert 1902] respectively.

The 'undeterminates' x, to which Poincaré referred, are undetermined in absolute sense in Russell's *Principles*, but for Hilbert they consist of one or both of the defined 'Gedankendinge' or any combination thereof, hence still have a certain structure. Of course Brouwer agreed with Poincaré: the constructivist only operates with building blocks that are constructed earlier.

In § XXV of *Les mathématiques et la logique*, Poincaré discussed, what he called, Hilbert's attempted *replâtrage* in his fifth rule from the set of rules for the extension of a consistent system. Poincaré commented that in this fifth rule Hilbert analyzed and tried to solve the difficulties he was well aware of in his Heidelberg lecture. Hilbert's analysis of these difficulties were dubbed by Poincaré as 'la tentative de replâtrage'.

In Brouwer's eyes, again in agreement with Poincaré, Hilbert is evading the real problem by limiting himself to a 'bounded number' (nombre limité) instead of to an arbitrary finite number. Brouwer's comment goes as follows:

> (VI-31) (Poincaré ib. 126). Hilbert justifies vicious circles by defining a 'proof' only by postulates, thus turning it into a new dead mathematical element. But should not an existence proof or the absence of possible contradictions be given for this new symbol? And is this not just moving the difficulty?
>
> (...)
>
> From our point of view Hilbert's 'replâtrage' is superfluous.[238]

As said, in the eighth notebook Brouwer spent a couple of paragraphs on a comment on Hilbert's Heidelberg lecture, this time not on the basis of comment from others:

> (VIII-65) Hilbert's logic is a hollow structure, built up from differently coloured types of bricks, in which the arithmetic of the natural numbers is tacitly assumed, including induction; but it cannot prove *anything* which is in some vague way connected to our known mathematical systems.[239]
>
> (VIII-66) The way in which Hilbert escapes the Russell paradox, completely without his logic, amounts to that he only speaks of a class of *earlier constructed objects*.[240]

[238](Poincaré ib. page 26) Hilbert rechtvaardigt vicieuze cirkels, door een 'Bewijs' zelf alleen door postulaten te definiëren; en er een nieuw dood mathematisch element van te maken. Maar geldt dan voor dit nieuwe symbool niet, dat er een existenzbeweis of afwezigheid van mogelijke contradictie voor moet worden gegeven? En krijgen we zo niet slechts verplaatsing van de moeilijkheid?
(...)
Van ons standpunt is de 'replâtrage' van Hilbert onnodig.

[239]De Hillbertse logica is een hol gebouw van verschillend gekleurde steensoorten, waarbij hij de aritmetica der gehele getallen (inductie incluis) stilzwijgend gebruikt ; maar dat niets kan bewijzen, dat ook maar enigszins verband houdt met onze reeds bekende wiskundige systemen.

[240]De manier, waarop Hilbert aan de Russellse paradox ontsnapt, komt, geheel buiten zijn logica om, hierop neer, dat hij alleen spreekt over een klasse van *reeds opgebouwde dingen.*

The construction of building blocks comes first, but the most basic is the ur-intuition:

> (VIII-67) Hilbert's replâtrage, *by means of* what is already constructed; but why not by means of the intuition? The result of the construction as a mathematical building has nothing to do with the intuitive construction *of* the building.[241]

And finally, Hilbert again confused, in Brouwer's eyes, arithmetic, its language and the signs thereof:

> (VIII-67) He [*i.e. Hilbert*] does not build on the foundation of logic and arithmetic, but on the system of signs thereof; in the construction he uses the logic (syllogism from a general theorem for x) and the arithmetic (mathematical induction) as something meaningless and independent.
>
> Moreover he presupposes as known the whole of mathematics as a guideline in the introduction of new symbols; and he employs subtle logical reasonings to convince us that he is on the right track.[242]

7.9 Conclusions

In his own introduction to the third chapter of his dissertation, Brouwer summarized in a few lines the way in which the construction of the mathematical edifice takes place, thereby stressing that this construction is performed without an assumed logic. He intended to elaborate this in his third chapter. Logic only comes afterwards, but we have seen that his (concise, as usual) explanation of a seemingly opposite case, where a mathematical construction was apparently based on a hypothetical judgement, asked for a long elaboration and interpretation.

Our interpretation, in which we closely followed Brouwer's views from 1907, ended in a definite, though implicit, rejection by Brouwer of the ex falso principle (although some of his examples elsewhere may sometimes give a different impression).

An intuitionistic logic, worked out by Heyting around 1930, however, accepted this principle. Brouwer knew this formalization and approved of it. He actually accepted it for publication in the Mathematische Annalen, before the great crisis. We may view this as a form of approval. Hence, in regard to this,

[241]De replâtrage van Hilbert, *met* het opgebouwde; maar waarom dan niet met de intuïtie? Dat opgebouwde als wiskundig gebouw heeft met het intuïtief bouwen ván het gebouw toch niets te maken.

[242]Hij bouwt niet op de logica en rekenkunde, maar het tekensysteem daarvan, als iets betekenisloos' en onafhankelijk, gebruikt bij die opbouw de logica (syllogisme uit algemene stelling voor x), en de rekenkunde (mathematische inductie).

Verder onderstelt hij als richting bij de invoering van nieuwe symbolen de hele wiskunde al bekend; en gebruikt fijn-strenge logische redeneringen, om ons te overtuigen, dat hij op de goede weg is.

he must have changed his position.

We have seen that from the four examples that Brouwer gave in order to substantiate his claim about the role of logic in the construction of the mathematical edifice, the second one received most of our (and his) attention. This example just fitted best into the foundational aspects of the construction of mathematics as it was extensively discussed in our previous chapters. But this example also raised most obstacles and problems, asking for explanation and interpretation. The concept *denumerably infinite unfinished set* and its cardinality needed an extensive analysis, partly with the help of the given examples for those sets. The main problem here turned out to be the theorem that 'all denumerably infinite unfinished sets are equivalent'. Another great (in a way unsurmountable) difficulty we encountered, was the notion of 'unfinished mappings'.

But also the first example, the founding of mathematics on axioms, gave an unexpected interesting outcome, which appeared in the long footnote (covering several pages). Here (but on other places as well) Brouwer showed to have an insight that certainly can be seen as a forboding of Gödel's later work.

Chapter 8

The Summary, the Theses and Conclusions

8.1 Introduction

In our last chapter three items will come up:

– *First*, Brouwer's own summary of his dissertation on page 179 and 180. It only makes one important assertion: *Mathematics is a free creation of the individual mind.* Since the expression 'free creation' is used by more mathematicians, it is interesting and important to make a comparison, and see in what respect Brouwer was different. The term 'free creation' as an act also reminds us of the 'two acts of intuitionism'.

– *Second*, the theses, added at the end of the dissertation. There are several known drafts for this list. Of special interest in regard to this is a letter to De Vries, in which some of the theses are clarified. Especially the thesis about the existence of the actual infinite (one that appeared only in one of the drafts for the list) and its accompanying elucidation, is of major importance. Seemingly conflicting statements about the infinite can be read in the dissertation, as well as in the notebooks. An interpretation has to be searched for, and can be found in connection with the interpretation for a set or a sequence to be 'actually finished' or 'potentially finished'.

– *Third*, a summary of our conclusions from the different chapters.

8.2 The summary of Brouwer's dissertation

After having finished the discussion of the four examples that substantiated his view on the role of logic in mathematics, the last two pages of Brouwer's dissertation are devoted to some concluding remarks about the three chapters that compose his dissertation.

On page 179 and 180 the central ideas of the three chapters are presented in the form of a very compact summary about what mathematics is, and what it is not. It opens as follows:

> Mathematics is a free creation, independent of experience; it develops from a single aprioristic ur-intuition, which may be called *invariance in change* as well as *unity in multitude*.[1]

But more mathematicians have characterized mathematics as a 'free creation', and therefore we compare Brouwer's opinion with the views of two others. The difference between Brouwer on the one hand, and Cantor and Dedekind on the other, is striking.

Cantor's 'freie Entwickelung'

In § 8 of the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* from 1883, Cantor dealt with two interpretations of the concept of 'existence of integers',[2] and according to Cantor these two interpretations always occur together, which is caused by the fact that they both find their origin

> in der *Einheit* des *Alls, zu welchem wir selbst mitgehören.* – Der Hinweis auf diesen Zusammenhang hat nun hier den Zweck, eine mir sehr wichtig scheinende Konsequentz für die Mathematik daraus herzuleiten, daß nämlich letztere bei der Ausbildung ihres Ideenmaterials *einzig* und *allein* auf die *immanente* Realität ihrer Begriffe Rücksicht zu nehmen und daher *keinerlei* Verbindlichkeit hat, sie auch nach ihrer *transienten* Realität zu prüfen.[3]

which distinguishes mathematics from all other sciences; a consequence of this is the following:

---

[1] dissertation, page 179: De wiskunde is een vrije schepping, onafhankelijk van de ervaring; zij ontwikkelt zich uit een enkele aprioristische oer-intuïtie, die men zowel kan noemen *constantheid in wisseling* als *eenheid in veelheid.*

[2] These two interpretations are:

> 1. 'Einmal dürfen wir die ganzen Zahlen insofern für wirklich ansehen, als sie auf Grund von Definitionen in unserm Verstande einen ganz bestimmten Platz einnehmen', von allen übrigen Bestandteilen unseres Denkens aufs beste unterschieden werden, zu ihnen in bestimmten Beziehungen stehen und somit die Substanz uneres Geistes in bestimmter Weise modifizieren.

Cantor called this form of reality of the numbers the *intrasubjective* or *immanent* reality. The second mode of ascribing reality to numbers is:

> 2. 'als sie für einen Ausdruck oder ein Abbild von Vorgängen und Beziehungen in der dem Intellekt gegenüberstehenden Aussenwelt gehalten werden müssen.

Cantor called this the *transsubjective* or *transient* reality.

[3] [Cantor 1883], page 19.

> Die Mathematik ist in ihrer Entwickelung völlig frei und nur an die selbstredende Rücksicht gebunden, daß ihre Begriffe sowohl in sich widerspruchslos sind, als auch in festen durch Definitionen geordneten Beziehungen zu den vorher gebildeten, bereits vorhandenen und bewährten Begriffen stehen. Im besondern ist sie bei der Einführung neuer Zahlen nur verpflichtet, Definitionen von ihnen zu geben, durch welche ihnen eine solche Bestimmtheit und unter Umständen eine solche Beziehung zu den älteren Zahlen verliehen wird, daß sie sich in gegebenen Fällen unter einander bestimmt unterscheiden lassen. Sobald eine Zahl allen diesen Bedingungen genügt, kann und muß sie als existent und real in der Mathematik betrachtet werden.[4]

According to Cantor, this freedom is in no way a threat to science, thanks to the small margin for arbitrariness, together with its selfcorrecting structure. Any further limitation by rules is not required:

> denn das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*.

Hence also for Cantor the system of the natural numbers is the result of a 'freie Entwickelung', but, different from Brouwer, this free development is not purely based on the ur-intuition alone, and is not the result of a construction. Cantor's freedom is a total freedom within the constraints of consistency of its concepts, internally *and* in their mutual relations. For Brouwer, freedom is expressed by the admissibility of *any* mathematical construction which is solely based on the ur-intuition.

Dedekind and 'freie Schöpfung'

Also Dedekind, in the preface to the first edition of *Was sind und was sollen die Zahlen*, claimed that the system of the natural numbers is the result of a free creation of the human mind:

> Indem ich die Arithmetik (Algebra, Analysis) nur einen Teil der Logik nenne, spreche ich schon aus, daß ich den Zahlbegriff für gänzlich unabhängig von den Vorstellungen oder Anschauungen des Raumes und der Zeit, daß ich ihn vielmehr für einen unmittelbaren Ausfluß der reinen Denkgesetze halte. Meine Hauptantwort auf die im Titel dieser Schrift gestellte Frage lautet: die Zahlen sind freie Schöpfungen des menschlichen Geistes, sie dienen als ein Mittel, um die Verschiedenheit der Dinge leichter und schärfer aufzufassen.[5]

In this quote Dedekind appears as a logicist: Arithmetic is just a part of logic. So neither with Dedekind the system of the natural numbers is the result of a construction, based on the ur-intuition of mathematics; for Dedekind the

[4] loc. cit. page 19.

[5] [Dedekind 1930b], page V. This paragraph was quoted earlier; see chapter 1, page 22.

system $\mathbb{N}$ is based on the concepts of 'mapping' and 'chain'.[6] The existence of infinite systems is proved by means of the theorem 'es gibt unendliche Systeme', in the proof of which Brouwer noticed a looming paradox, which he mentioned in the third notebook, and which we quoted earlier on page 293.

Brouwer's free creation

Brouwer brought back the basis and foundation of all mathematics to its most primitive form, the ur-intuition of 'invariance in change' or 'unity in multitude'.[7] The interpretation of Brouwer's *free creation* should be the following: in this creation the ur-intuition is the most fundamental element, and, departing from this the construction of the mathematical building is a free creative act, as this was sketched in the previous chapters of this dissertation.

See for instance our page 44, the *construction of the natural numbers*, in which the abstraction from the content of an event or a sequence of events is an *act*, thus making a number independent of the nature and content of such a sequence.

Another example is the construction of the rational numbers, in which the time span (the continuous interval) between any two consecutive events is identical in character to the time interval between any other two successive events. This is neither the result of a discovery, nor a corollary of an axiom, but it is the result of an *act* of our mind. We *force* them as it were to be similar (see page 46).

And again we perceive this free creation in the construction of the *measurable continuum* (page 79, item 7), which is, after ω times the splitting of every interval into two parts, *made to be everywhere dense by our own act* of contracting every unpenetrated segment into one point:

> But we agree to contract every segment not penetrated by the scale into one point, in other words, we consider two points as different only when their approximating dual fractions differ after a finite number of digits.[8]

A fourth example of mathematics as a free creation is thesis II, in which the principle of complete induction is declared to be neither a theorem, nor an axiom, but simply an *act* in the mathematical construction (see the next section).

[6]Compare also page 19. Clearly, for Dedekind the number concept is not depending on space or time, as this is also the case with Cantor; see page 8.

[7]In a footnote to the quoted beginning of the summary, Brouwer mentioned F. Meyer, who stated in the *Verhandlungen des Heidelberger Kongresses* that one thing will be sufficient, since the act of thinking that thing automatically includes a second thing, viz. the act of thinking itself. This is contested by Brouwer, since Meyer presupposed in his argument the intuition of *two*.

[8]dissertation, page 10: Maar we spreken af, dat we elk segment, waarin de schaal niet doordringt, tot een enkel punt denken samengetrokken, m.a.w. we stellen twee punten alleen dán verschillend, als hun duale benaderingsbreuken na een eindig aantal cijfers gaan verschillen.

These examples show best Brouwer's view of mathematics as the result of the free creative act.[9] In the continuation of the summary of his dissertation, Brouwer claimed that also the projection of mathematical systems on the experience of our environment is a free act. In the discussion of the second chapter of Brouwer's dissertation (chapter 6 of this dissertation) we dealt with man's faculty of taking a mathematical view of his life. Man observes regularity in the world, he observes recurring sequences of events and he discovers the possibility of expressing these sequences in a mathematical way. He also discovers methods of early interference in these sequences, thus changing its course into a desired direction. The *free act* consists of the creation of a mathematical model of the physical world, we *force* nature into such a model, in order to rule the surrounding world for our own well-being.

> In this respect one mathematical system can appear more practical, more economical than another, at least relative to a definite kind of purpose which one wishes to attain: none of them is absolutely efficient.[10]

The last two paragraphs of Brouwer's dissertation once more emphasize the difference between mathematics, its accompanying language and logic.

The *second last* clearly states Brouwer's (by now well-known) dictum about the employed language when expressing mathematical statements:

> In mathematics, mathematical definitions and properties ought not to be studied again by mathematical methods; they ought to be no more than a means of conducting as economically as possible one's own memory and communication with other people.[11]

In the definitions (and generally in all mathematical language) there are primitive and irreducible concepts like *continuous, entity, once more, and so on*. These concepts are elements of construction, immediately perceived in the ur-intuition of the continuum. This paragraph clearly rejects any form of meta-mathematics, i.e. mathematics about the structures of mathematical language instead of mathematics itself. This form of meta-mathematics has as objects mathematical *words*, it has as relations the *rules*, according to which these words

[9] Compare this with a quote from E.W. Beth's *Modern Logic*, who, when discussing the status of mathematical knowledge in the several philosophical movements, noted about intuitionism:

> The intuitionist conceives it as a form of self-knowledge. ([Beth 1967], page 102: De intuïtionist vat haar op als een vorm van zelfkennis.)

[10] Dissertation, page 180: Het ene wiskundige systeem kan daarbij praktischer, economischer blijken, dan het andere, althans voorzover betreft een bepaalde categorie van doeleinden, die men door middel van die systemen tracht te bereiken: absoluut doeltreffend zijn ze geen van alle.

[11] dissertation, page 180: In de wiskunde behoren wiskundige definities en eigenschappen niet zelf weer wiskundig te worden bekeken, maar alleen een middel te zijn, om eigen herinnering of mededeling aan anderen van een wiskundig gebouw zo economisch mogelijk te leiden.

can be grouped into meaningful sentences, and it has as results *statements about the language of mathematics*, about logistics, but not about mathematics itself.

The *last* paragraph deals with the impossibility of a construction of the mathematical building on the foundation of logic alone, without any mathematical intuition. Again, one is doing theoretical logic, or, at the best, logistics, but certainly not mathematics. One is just constructing a language-building:

> A logical construction of mathematics, independent of the mathematical intuition, is impossible – for by this method no more is obtained than a linguistic structure, which irrevocably remains separated from mathematics – and moreover it is a contradictio in terminis – because a logical system needs the basic intuition of mathematics as much as mathematics itself needs it.[12]

As a final conclusion we can say that we know that metamathematics and mathematical logic are not themselves methods of constructing mathematics, but merely the observation and the study of the accompanying language of a mathematical construction.

8.3 The theses

Until recent time a compulsory list of theses formed an integral part of the dissertation and had to be defended together with it. Brouwer's theses, 21 in number, all have a philosophical and/or mathematical content and consist for a great part of the conclusions from the several topics that were discussed in the dissertation. Here are some examples:

> (II) It is not only impossible to prove the admissibility of complete induction, but it ought neither to be considered as a special axiom nor as a special intuitive truth. Complete induction is an act of mathematical construction, already justified by the basic intuition of mathematics.[13]
>
> *[referring to conclusions from chapter I:]*

[12]dissertation, page 180, the last paragraph: Een logische opbouw der wiskunde, onafhankelijk van de wiskundige intuïtie, is onmogelijk – daar op die manier slechts een taalgebouw wordt verkregen, dat van de eigenlijke wiskunde onherroepelijk gescheiden blijft – en bovendien een contradictio in terminis – daar een logisch systeem, zo goed als de wiskunde zelf, de wiskundige oer-intuïtie nodig heeft.

[13]De geoorloofdheid der volledige inductie kan niet alleen niet worden bewezen, maar behoort ook geen plaats als afzonderlijk axioma of afzonderlijk ingeziene intuïve waarheid in te nemen. Volledige inductie is een daad van wiskundig bouwen, die in de oer-intuïtie der wiskunde reeds haar rechtvaardiging heeft.

(Compare this to Poincaré in [Poincaré 1916], chapter I, section V: consistency of complete induction cannot be proved; it is a 'propriété de l'esprit lui-même'.)

> (V) The arithmetical operations on the measurable continuum ought to be defined by means of group theory.[14]
>
> *[referring to conclusions from chapter II:]*
>
> (VII) Attributing 'objectivity' to physical notions like *mass* and *number* is based upon their invariability with respect to an important group of phenomena in the mathematical image of nature.[15]
>
> (VIII) Human understanding is based upon the construction of common mathematical systems, in such a way that for each individual an element of life is connected with the same element of such a system.[16]
>
> *[referring to conclusions from chapter III:]*
>
> (IX) Mathematics is independent of logic; practical logic and theoretical logic are applications of different parts of mathematics.[17]
>
> (XII) Besides the finite there are no other cardinalities than: denumerably infinite, denumerably infinite unfinished, continuous.[18]
>
> (XIII) Cantor's second number class does not exist.[19]

Several other theses are about potential theory, a subject which was not discussed in the dissertation but to which Brouwer devoted several of his earliest papers.

An interesting aspect is formed by the different drafts for the theses to which often clarifying notes were added; many of the draft-theses did not find their way into the dissertation.[20] Some interesting (and sometimes puzzling) observations can be made in the different drafts, especially in combination with a draft-letter to J. de Vries, in which Brouwer elucidated the main aspects of his dissertation.[21] This elucidation is done in four sections, and each section ends with a reference to relevant theses from the list of 21, followed by the page numbers from the dissertation to which these theses refer; however, often one or more 'theses in plain language' (that is, not specifically denominated as a thesis)

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[14] De hoofdbewerkingen op het meetbaar continuüm behoren door groepentheorie te worden gedefinieerd.

[15] Het toekennen van 'objectiviteit' aan fysische grootheden als *massa* en *aantal* berust op de invariabiliteit daarvan bij een belangrijke groep van verschijnselen in het wiskundig natuurbeeld.

[16] De verstandhouding der mensen berust op het bouwen van gemeenschappelijke wiskundige systemen, en het verbinden aan eenzelfde element van zulk een systeem van een levenselement voor elk der individuen.

[17] Wiskunde is onafhankelijk van logica; practische logica en theoretische logica zijn toepassingen van verschillende gedeelten der wiskunde.

[18] Behalve de eindige, bestaan er geen andere machtigheden dan: aftelbaar oneindig, aftelbaar oneindig onaf, continu.

[19] De tweede getalklasse van Cantor bestaat niet.

[20] These theses and their clarifying notes can be found in the new edition of Brouwer's dissertation, [Dalen 2001].

[21] Professor J. de Vries (1858-1938), Utrecht University. The letter is undated, but most likely it is from shortly after his public defence in February 1907.

are added, and these latter are always specimen from the several drafts which did not end up in the dissertation. They apparently were removed from the final list, possibly under Korteweg's influence, either because of their content or simply because the list became too long and a choice had to be made. But they were certainly *not* removed because of a change in Brouwer's opinion, since they were explicitly mentioned in the letter to J. de Vries, including relevant page references.
The four different sections in the letter to De Vries are:

A. *About the classification of mathematics as a special branch of logic.* It is impossible to classify mathematics under logic, since in case of an attempted proof of a mathematical truth from logic, that mathematical truth is tacitly and intuitively presupposed in the deduction. One of the theses added to this conclusion is the unpublished one from the second draft:

> In a logical treatment of mathematics there is nothing against the petitio principii, provided it is read from the intuition. (see page 176) [*of Brouwer's dissertation*][22]

This item was discussed on page 305, in Poincaré's criticism towards Hilbert.

B. *About the actual execution of the intuitive construction.* This was mainly treated in the first chapter, and partly in chapter 3. Among the relevant theses there is an unpublished one, from draft 2:

> A strict separation should be made between the intuitive time and the scientific time.[23]

This thesis is added as a footnote to Brouwer's main conclusion on mathematical intuition (*'The only a priori element in science is time'*) on page 99 of his dissertation.

C. *About the general character of science and the relation between mathematics and other sciences.* This mainly refers to the second chapter of Brouwer's dissertation. Science consists of the projection of mathematics on our world of experience, which seems peculiar, since mathematics is not depending on any daily experience; we force a mathematical description on nature instead. The added thesis, which is not from the dissertation-list, is thesis 26 (not verbatim) from an extra list with additional clarifications:

> Mathematics is not a science like other sciences, but it is a moral act consisting of doing science.[24]

A similar thesis is XXIV from draft 2:

[22] Bij een logische behandeling der wiskunde is niets tegen een petitio principii, mits die uit de intuïtie wordt afgelezen.

[23] Men behoort streng te onderscheiden tussen de intuïtieve en de wetenschappelijke tijd.

[24] Wiskunde is niet een wetenschap als een andere, maar een morele daad die het bedrijven van de wetenschap is.

> Mathematics should not be considered as a science as any other, but as a medium to the different sciences.[25]

D. *About the question whether or not actual infinite sets exist.*[26] This question was also dealt with on our page 78, as well as in the discussion of Poincaré's critique on Hilbert (see page 306); see also the quote from Aristotle in the beginning of chapter 3. Poincaré, on the one hand, completely rejected the actual infinite; Cantor, on the other hand, admitted infinities of always higher cardinalities. In his letter to De Vries, the *actual* infinite is restricted by Brouwer to the following sets:

> I acknowledge denumerably infinite sets, and with a restriction, the continuous cardinality, and finally, with another restriction, a new cardinality, which I call denumerably infinite unfinished. I expose however, all the higher cardinalities of Cantor as a logical chimera. At the same time I try to strip transfinite set theory of its parasite parts, such as transfinite exponentiation, the theorem of Bernstein with its applications, and more; all of which result from the false logical foundations of set theory. In this connection I can formulate:
>
> 1. Actual infinite sets can be created mathematically, even though in the practical applications of mathematics in the world only finite sets occur.[27]

This last claim is also the first thesis from the first draft of unpublished theses, and we notice that, on this point, Brouwer seems not always to be clear, consistent and unambiguous in his texts. Brouwer knew of course that actual infinities exist, for instance the system of the natural numbers or that of the rational numbers; the problem for him was to subsume them in a mathematical construction.

In the dissertation (page 176, in the discussion of Poincaré's comment) also the 'actual infinite of the Cantorians' is said to exist, but here it is explicitly restricted to that 'which can be intuitively constructed'. This can only refer to the denumerably infinite and possibly also the denumerably infinite unfinished cardinality (under the proper interpretation of the latter). The continuum is

[25]Wiskunde behoort niet te worden beschouwd als een wetenschap als een andere, maar als het medium tot de verschillende wetenschappen.

[26]Note that in 1885 Cantor published in the *Zeitschrift für philosophie und philosophische Kritik* a paper, *Über die verschiedene Standpunkte in bezug auf das aktuelle Unendliche*, which is of historical interest. See [Cantor 1932], page 371.

[27](English translation by D. van Dalen in [Dalen, D. van 1999], page 118. [Ik] erken aftelbaar oneindige verzamelingen, en met een restrictie de continue machtigheid, en ten slotte met een andere restrictie een nieuwe machtigheid, die ik noem aftelbaar oneindig onaf. Alle hogere machtigheden van Cantor echter toon ik aan als logische hersenschimmen. Tegelijk tracht ik de transfiniete Mengenlehre van haar parasitaire gedeelten als transfiniete machtsverheffing, theorema van Bernstein met zijn toepassingen, en meer, die alle uit de valse logische grondslagen ervan voorkomen, te ontdoen. Ik kan in dit verband formuleren: 1. 'Actueel oneindige verzamelingen zijn wiskundig te scheppen, ook al treden bij de practische toepassing der wiskunde in de wereld slechts eindige verzamelingen op' (Zie page 120, 142-143).

not intuitively constructed since this is given to us in its entirety. Nevertheless it is subsumed under one of Brouwer's four possible cardinalities and it certainly cannot be regarded as finite. Also in the letter to De Vries the continuum is implicitly included as one of the 'actual infinite' sets. But from the dissertation it follows that the continuum is no point set, it is only the matrix, *onto* which denumerably infinite many points can be constructed; it is an *infinite source*. Apparently, this goes very well together for Brouwer.

In spite of Brouwer's claim quoted above that 'in the practical applications in the world only finite sets occur, one should of course suppose an *actual* infinite set to be completed, but on page 9 and 10 of his dissertation, Brouwer pointed out that a denumerable set, which is by definition given by some algorithm, may *not* be considered as an example of a finished totality. On these pages, where the construction of a scale on the intuitive continuum is discussed, we also find the method of approximation of some arbitrary point. This method is of relevance to the concept of the actual infinite, since, when selecting a point P, we can approximate this point, without ever reaching it, by an infinite dual fraction (which can be viewed as an infinite sequence of dual fractions),

> (...) given by an arbitrary given law of progression, (...) However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partly unknown.[28]

In a handwritten correction to his own copy of the dissertation, Brouwer even added as an example: 'take for instance the number π'.[29] Thus a *lawlike* sequence of progression for the number π has to be regarded as partially unknown, whereas every element of this sequence can be computed directly and unambiguously.

Also the notebooks contain (more or less) conflicting remarks on the actual infinite; e.g. on the one hand:

> (VIII–20) I can think a fundamental sequence as *finished*, just as (the value of) a convergent sequence (the first one gives certainty of the equality of the terms, the second of the limiting value).[30]

So, considering Brouwer's concept of a fundamental sequence, viz. any sequence of ordertype ω,[31] together with the fact that every well-defined, i.e. algorithmically given, denumerable set can be given in the form of a fundamental sequence, results in such a set as an example of an 'actual infinity'.

But, on the other hand, in the same notebook we find the following paragraph:

[28][gegeven] volgens een willekeurige denkbare voortschrijdingswet (...) We kunnen de benaderingsreeks van een *bepaald aangewezen* punt evenwel nooit *af* denken, dus moeten haar als gedeeltelijk onbekend beschouwen.

[29]See also the discussion on page 77.

[30]Een fundamentaalreeks kan ik 'af'denken; eveneens de waarde van een convergente reeks (de eerste geeft de vastigheid van de gelijkheid der termen, de tweede die van de limiet waarde).

[31]See for instance the *Berliner Gastvorlesung*, [Dalen, D. van (ed.) 1992], page 31.

> (VIII–24) One should always keep in mind that ω only makes sense as a living and growing induction in motion; as a stationary abstract entity it is senseless; ω may never be conceived to be finished, as a new entity to operate on; however you may conceive it to be finished in the sense of turning away from it while it continues growing, and to think of something new.[32]

This, again, is clear, but it seems to be in conflict with the preceding quote: even the set of the natural numbers should in this option be considered as a 'for ever unfinished and always growing' sequence.

The conclusion from the letter to J. de Vries is unambiguous: the actual infinite does exist and the conclusion from page 176 of the dissertation is the same; moreover there are other places in his dissertation where Brouwer presented direct or indirect arguments for the existence of the actual infinite. Whereas on page 9 the phrase 'it is easy to construct on the continuum a sequence of points having the order type of the positive and negative whole numbers' still can be interpreted as expressing a process of never terminating growth, the sentence on page 62:

> The mathematical intuition is unable to create other than denumerable sets of individuals. But it is able, *after having created* a scale of order type η (...)[33]

The expression '*after* having created' *seems* to refer to a finished, actually denumerable set. On Brouwer's page 142 it is expressed as follows (we quoted and discussed this earlier in a different context; see our page 78):

> In the first chapter we have seen that there exist no other sets than finite and denumerably infinite sets and continua; this has been shown on the basis of the intuitively clear fact that in mathematics we can create only finite sequences, further by means of the clearly conceived 'and so on' the order type ω, but only consisting of equal elements; (consequently we can, for instance, never imagine *arbitrary* infinite dual fractions as finished, nor as individualized, since the denumerably infinite sequence of digits cannot be considered as a denumerable sequence of *equal* objects), and finally the intuitive continuum, (...)[34]

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[32] Men bedenke steeds dat ω alleen zin heeft, zolang het leeft, als groeiende, bewegende inductie; als stilstaand abstract iets is het zinloos; zo mag ω nooit àf gedacht worden, om m.b.v. het *geheel* als nieuwe eenheid te werken: wel mag je het àf denken in de zin, van je er van af te keren, terwijl het doorloopt, en iets nieuws te gaan denken.

[33] De mathematische intuïtie is niet in staat andes dan aftelbare hoeveelheden geïndividualiseerd te scheppen. Maar wel kan zij, eenmaal een schaal van het orde type η opgebouwd hebbend, (...)[*my emphasis in the main text*].

[34] We hebben in het eerste hoofdstuk gezien, dat er geen andere verzamelingen bestaan, dan eindige en aftelbaar oneindige, en continua; hetgeen is aangetoond op grond van de intuïtieve waarheid, dat wij wiskundig niet anders kunnen scheppen, dan eindige rijen, verder op grond van het duidelijk gedachte 'en zovoort' het orde type ω, doch alleen bestaande uit

Hence *only* arbitrary infinite dual fractions (a choice sequence in dual representation) cannot be imagined as finished and *thus*, one should say, a lawlike sequence may be regarded as finished, giving an actual infinity.

Brouwer's view is unambiguously expressed in the letter to De Vries: an actually infinite set exists if this notion is limited to algorithmically constructed denumerable sets. Also the continuum exists as an intuitively given actuality, but no sets of higher cardinality than denumerable can be constructed and therefore do not exist. Hence this also applies to other 'Cantorian' sets like the set of all subsets of a denumerably infinite set since it cannot be defined by an algorithm.

But then, how should we construe the conflicting quotes from page 9 and 10 of the dissertation and from the paragraph from notebook VIII, page 20? How is it possible that, on the one hand, the lawlike sequence of π must be considered to be partially unknown, and, on the other hand, (VIII–20) 'I can think a fundamental sequence as finished'? When is an infinite set finished and when does the actual infinite exist; and when are the terms of an infinite expansion known or unknown. Of course, arbitrary infinite dual fractions' (hence nonlawlike choice sequences) are only known as far as the choices are actually made and therefore they are in their totality unknown on principle.

Different interpretations seem to be possible for 'finished' and 'unfinished', as well as for 'known' and 'unknown'. In an attempt to create order in these seemingly conflicting remarks by Brouwer, we propose the following interpretations for these concepts. They do, we are convinced, justice to Brouwer's views; we are even inclined to imagine that he would have offered the same mathematical exegesis.

In regard to the concept 'finished': When constructing for instance the system ω (or η) in a systematic algorithmic manner, then the result of the actual construction of the elements, one by one, forms of course a never terminating and always finite sequence. The process is never actually finished. But, because of the repeated application of the same algorithm, because of the always equal steps, we may *declare* the set ω to be finished. This is another example of mathematics as an act, as a free creation of the mind: we may jump as it were over the whole procedure of the successor operation. Just as we may consider a first and a second event and their connecting continuum together as one single event, retained in memory as such and separated by a time span from a new event, thus constructing the system $\mathbb{N}$, we may consider ω as one single unit, as one 'experienced event', and add a new element, called $\omega + 1$. We *consider* or *idealize* the actual infinite set to exist and we have arguments for this *act* since we can, without hesitation, mention every member of the set (but of course not *all* members!), exactly because of the simplicity and the constancy of the algorithm.

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gelijke elementen, zodat we ons b.v. de *willekeurige* oneindige duaalbreuken nooit af, dus nooit geïndividualiseerd kunnen denken, omdat het aftelbaar oneindig aantal cijfers achter de komma niet is te zien als een aftelbaar aantal *gelijke* dingen, en tenslotte het intuïtief continuüm, (...)

We can put this in the following terms: *extensionally* speaking the actual infinite never exists, since an extensionally given set is supposed to be the result of a proper mathematical construction of its individual terms, including its termination, which is impossible. But we may also conceive the actual infinite to be an *intensional* mathematical object, e.g. the system of the natural numbers defined by the successor operation. This makes the quotes given above from VIII–24 comprehensible: as a stationary abstract entity it is senseless, but you may turn away from it and let it grow while doing something else. That 'doing something else' may then, for instance, consist of the continuation of the act of counting from $\omega + 1$ onwards. In that sense you may conceive it to be finished. The intensional definition of the algorithm makes the free act of declaring it to be finished defensible, and this makes the claim that the actual infinite exists equally defensible. Intensionally it is there; see e.g. also the following quote from the seventh notebook:

> (VII–16) ω is finished by our innate mathematical induction.[35]

Now the only remaining quote that does not fit in this picture, is the one from page 10 of the dissertation, so it seems:

> However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partly unknown.

and this is directly connected with the question of 'when is something known or not known'. Clearly, judging by the quotes from the dissertation and from the notebooks, several interpretations are possible. Of course the unchosen terms of a non-lawlike choice sequence are unknown on principle; they remain so as long as no choice has been made. But a different form of 'unknown' must be meant by Brouwer in the quote given above from the dissertation. Even for a lawlike sequence like the one for π, the uncomputed terms are unknown (for the time being), even if they are known in principle, even if a computer can fix its value in an instant; as long as the computation is not actually performed we do not yet know its outcome; this has to be understood in its most basic and primitive sense. Clearly this form of 'unknown' is closely linked with the extensional definition for the sequence of π, since these two concepts are used in one and the same sentence.

It may be confusing to the reader that Brouwer employed two interpretations for the 'actual infinite' and two interpretations for 'unknown', in a seemingly random way, but close reading of the several quotes reveals which interpretation Brouwer exactly had in mind on that specific occasion.

When the actual infinite is declared to exist (which we interpreted to be the result of a free creating act of the mind), i.e. in case of an intensional definition, the elements of the set or the terms of the sequence may be declared to be known. However, if an infinite quantity is extensionally defined and therefore never finished, then only the finite finished part of the elements is known.

[35] ω is af door de mathematische inductie, die in ons is.

8.4 Summary of our conclusions

Many interpretations or re-interpretations were made in the course of the discussion of the several topics from Brouwer's dissertation. Many conclusions were drawn at the end of each of the previous chapters or their sections or subsections. In the following we will present a summary of the most relevant items in regard to the foundational aspects of mathematics.

– *The ur-intuition and the construction of the ω-scale* (page 44).

Two well-separated and actually experienced events, combined into the unity *event – connectiong medium (time span) – next event*, and divested of all quality, form the beginning of the ω-scale (i.e. the numbers 'zero' and 'one') to which unity can be added a new event, well-separated from that unity and which forms, after abstraction, the number 'two', etc. Hence, what is retained in memory is a *sign*, which stands for the result of that abstraction.

– *The status of this sign* (page 56).

This representing sign is also a mental construction in the form of an abstract symbol, and does not belong to the accompanying language yet. Only its oral or written expression belongs to that language.

– *The construction of the η-scale* (page 46).

The experience of the flowing, of the connecting medium between a first event ('zero') and a next ('one') can itself be seen as an event and therewith it *is* the first intercalated element of the η-scale (the element 'half'), and, since this 'flowing' is experienced as well-separated form the events 'zero' and 'one', it is in its turn again connected by a flowing with both, zero and one. Hence the procedure of intercalation can be repeated indefinitely.

– *The scale of integers* (page 48).

If we call the second event 'zero' and the third 'one' (we are free to do so), then we may call the intercalation between the first and the second event 'minus one'; the second intercalation between the first event and 'minus one' we call 'minus two', etc. The first event then (informally) becomes 'minus infinity'.

– *The everywhere dense η-scale* (page 79).

This scale on the 'intuitive continuum in a graphic (i.e. geometrical) representation' is the result of the free act of contracting every not-penetrated segment of this continuum into one point; that is, identifying two points which do not differ after any finite number of decimal places, as one and the same point.

– *The Bolzano-Weierstrass theorem* (page 81).

In Brouwer's argumentation for this theorem, the principle of the excluded middle is employed. Attempts to prove it in a strictly constructive way turn out to be unsuccessful.

– *Covering by, or completion to a continuum of an everywhere dense scale* (page 120).

This can take place by mapping the set of the rationals on a continuum on which an everywhere dense scale is constructed.

– *The third construction rule for sets* (page 122).

Brouwer's choice for this rule as one of the possibilities to construct a set, must be the result of the great influence that Cantor still had on the young Brouwer, and was, with good reasons, later on rejected by him in the *Addenda and Corrigenda* (page 130).

– *Brouwer's solution to the continuum problem* (chapter 5).

His solution was the only possible and almost trivial one in view of his constructive approach of sets of points on the continuum, but it did not answer Cantor's conjecture that $2^{\aleph_0} = \aleph_1$, which is also impossible because Brouwer did not recognize any aleph's, apart from $\aleph_0$.

– *Brouwer's view on physics* (page 181).

His stern view on the moral aspects of physical practice was the result of his pessimistic outlook on mankind. Man's only desire is, according to Brouwer, to rule and to increase his power. The result is an approach towards physics, which is certainly not common among physicists.

– *Objectivity and apriority* (page 210)

Brouwer's concept of objectivity turned out to be a direct and natural corollary of his solipsism. Also his view on apriority is a result of the ur-intuition, which makes, in contrast with Kant, space superfluous as an apriorisitc element in the construction of mathematics.

– *The role of logic* (page 228)

This role is reduced to that of a set of rules for the accompanying linguistic reasoning in the construction of the mathematical building. The principium tertii exclusi is ultimately rejected for infinite sets.

– *The hypothetical judgement in mathematics* (page 230).

This judgement is subject to stricter rules than the ones from the later BHK proof interpretation. The premise of the judgement has to be the result of a properly performed mathematical construction. This was clarified with the help of several examples.

– *The denumerably infinite unfinished cardinality* (page 266).

This is the third in the list of possible cardinalities for sets. This cardinality and in particular 'Brouwer's lemma' can only be properly interpreted and understood under a stricter and more limiting set of conditions than the ones given in Brouwer's dissertation.

– *Brouwer and Gödel's first incompleteness theorem* (page 286).

A single remark, made by Brouwer in one of the notebooks:

> (VIII–44) The totality of mathematical theorems also, among other things, constitutes a set, which is denumerable but never finished.

can be interpreted and defended as a foreshadowing of Gödel's first incompleteness theorem.

– *The actual infinite* (page 320).

The concepts 'finished/unfinished' and 'known/unknown' as well as the existence of the 'actual infinite' were discussed in this chapter, with the conclusion that, for a proper interpretation, a distinction has to be made between an extensional and an intensional definition of a set.

We also mentioned the *two acts of intuitionism* (page 57), explicitly expressed in Brouwer's later work, viz.
1) the strict separation between mathematics and its accompanying language. Mathematics is fundamentally languageless,
and
2) the actual construction of mathematics, strictly separated from any language, but solely based on the ur-intuition. In this construction a set becomes a *law*.

We clearly recognized both of Brouwer's acts already in his dissertation, without, however, denoting them as such. See for this for instance our page 57 and chapter 4, which stipulate the construction of the elements for a set.

The *spread concept* could be identified in chapter 1 of Brouwer's dissertation (page 117 of this dissertation), in which its role is still limited to decide whether or not a set is dense in some specific interval, instead of using it for the construction of set elements in the form of choice sequences.

Choice sequences did not yet appear in the dissertation, but we met them several times in the notebooks, often in the form of thought experiments. We can observe similar experiments with other mathematicians (see for instance [Borel 1908b], page 16, or [Borel 1950], page 160), but, whereas with others it did not lead to anything revolutionary new notions, the choice sequence concept in the notebooks clearly were a foreboding of developments ahead.

This and similar forebodings remain one of the fascinating aspects of reading in the dissertation in combination with the notebooks. Intuitionistic mathematics had not yet matured in those early days, but the signs were there; all was waiting for a breakthrough, which was to come in 1917, with far-reaching consequences.

Probably a further and more detailed study of the nine notebooks, in combination with Brouwer's later work, will reveal more seeds of later developments. An annotated publication of the notebooks is in preparation.

Appendix
Brouwer's own bibliography

It is a happy coincidence that the notebooks allow us, in addition to the consulted literature which is mentioned by Brouwer in his dissertation, to give a more complete overview of all the material that he read and studied when preparing his doctoral thesis. However, his list of references is often very fragmentary and sketchy, and it frequently required some investigation to find out what book or paper Brouwer exactly had in mind; sometimes only a short reference is made to a specific journal, without mentioning the exact relevant article. From the resulting list it becomes clear that Brouwer was very well up-to-date in the modern literature, especially in the field of the foundations of mathematics.

Brouwer often referred to journals or other official publications under a shortened name. The following list gives these 'nicknames', followed by the full names of the journal during the years they were read and studied by Brouwer:

– Abhandlungen Göttingen: *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen; Mathematisch-Physikalische Klasse*,

– Archiv der Mathematik und Physik: *Archiv der Mathematik und Physik; mit besonderer Rücksicht auf die Bedürfnisse der Lehrer an höhern Unterrichtsanstalten*,

– Bulletin of the American Mathematical Society: *Bulletin of the American Mathematical Society: a historical and critical review of mathematical science*,

– Crelle: *Journal für die reine und angewandte Mathematik*,

– L'Enseignement Mathématique: *L'Enseignement Mathématique; Revue Internationale*,

– Göttinger Nachrichten: *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen; Mathematisch-Physikalische Klasse*,

– Jahresbericht der D.M.V.: *Jahresbericht der Deutschen Mathematiker-Vereinigung*,

– Math. Ann.: *Mathematische Annalen*,

– Phil. Magazine: *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*,

– Rendiconti Palermo: *Rendiconti del Circolo Matematico di Palermo*,

– Sächsische Berichte: *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaft zu Leipzig, Mathematisch-Physische Klasse*,
– The Monist: *The Monist; International Quarterly Journal of General Philosophical Inquiry.*

The booktitles which are (also or only) mentioned in the dissertation, are marked with an asterisk with the author's name.

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[36]First published in the *Revue de Métaphysique et de Morale*, nrs 12 (1904) and 13 (1905).

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[37] Festschrift zur Feier des 150-jährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

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[39] First print 1783.

[40] Written and published together with S. Lie.

[41] Brouwer is referring in one sentence to four articles by Klein, published in the Mathematische Annalen 4, 6, 7 and 17. From the many articles from the hand of Klein during those years we selected the following four items as the ones most likely meant by Brouwer, all about the same subject, viz. (non-Euclidean) geometry.

[42] Full title: *Nicht-Euklidische Geometrie; Vorlesung gehalten während des Wintersemesters 1889 – 1890*, ausgearbeitet von Fr. Schilling.

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[43] Brouwer often referred to this work under the short title 'Prinzipien'.
[44] See also Russell, Couturat and Poincaré.
[45] Written and published together with F. Klein.
[46] Originally published in Kazan, 1856.

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[47] In his notebooks Brouwer just mentions *Annales de la Société Scientifique de Bruxelles* 29, without further specification. Mansion published some five papers in this volume of the 'Annales'. We selected the ones most likely meant by Brouwer.

[48] Meyer gave two lectures in Heidelberg: the mentioned one and also 'Über Grundzüge einer Theorie des Tetraeders'. Brouwer most likely had the first lecture in mind.

[49] This remark is not quite clear but it may refer to Math. Ann. 57, 1903 'Preisaufgabe der Fürstlich Jablonowskischen Gesellschaft für 1906'. However, Brouwer only writes 'Neumann', without 'J. von'. Therefore it also might refer to the winning Preisschrift of R. Neumann, *Die Arengen der Urkunden Ottos des Grossen*, but what has this to do with Brouwer's work?.

[50] This journal was not found.

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Samenvatting

Als inleiding op de samenvatting van dit proefschrift is het nuttig om eerst een kort overzicht te geven van de stand van zaken in de wiskunde ten tijde van Brouwers promotie in 1907.

Brouwer studeerde en promoveerde in een periode van grote veranderingen in deze meest abstracte tak der wetenschap. De verzamelingenleer van Cantor was slechts enkele decennia oud en ook het werk van Dedekind was nog steeds zeer actueel. Van nog recenter datum waren de publicaties van Frege, Russell, Poincaré, Borel en Hilbert. Van deze laatste stamt de lijst van 23 nog op te lossen 'Mathematische Probleme', gepresenteerd in 1900 te Parijs tijdens de grote internationale wiskunde conferentie aldaar. In diezelfde jaren rond de eeuwwisseling ontstond grote beroering toen ontdekt werd dat paradoxen mogelijk bleken te zijn in de recent ontwikkelde verzamelingenleer. Met name de paradox van Russell bracht iemand als Frege in grote verwarring.

Maar men was reeds daarvoor, gedurende de tweede helft van de negentiende eeuw, in verschillende richtingen op zoek naar een solide fundering voor de wiskunde:

– Het *logicisme* trachtte alle wiskunde te funderen op, en af te leiden uit, de logica alleen. Protagonisten hierin waren Frege in Duitsland, Russell in Engeland en Couturat in Frankrijk.

– Het *formalisme* stelde dat wiskunde slechts bestaat uit het manipuleren met symbolen volgens tevoren vastgelegde regels. Hierbij moet niet aan inhoud of interpretatie gedacht worden. Hilbert speelde in deze stroming een grote rol.

– Een tegenstroming ontstond door het werk van Poincaré en Borel, die stelden dat er meer is dan vorm, logica en taal. Volgens deze Franse 'pre-intuïtionisten' is logica slechts geschikt ter controle achteraf van de juistheid van een redenering, maar levert wiskundig gezien niets nieuws op. Ook taal is een noodzakelijk hulpmiddel om wiskundige redeneringen met anderen te kunnen communiceren, maar is zelf niet een bron voor wiskunde. Er is meer nodig dan logica en taal: er is wiskundige *intuïtie* vereist, die ons uit de veelheid van definities en axioma's de juiste doet kiezen en die ons de richting doet aanvoelen waarin de oplossing van een probleem gezocht moet worden.

De Aristotelische logica, en in het bijzonder het syllogisme, brengt ons, zo stelt Poincaré, nooit van het particuliere naar het algemene. Het wiskundig middel dat ons daar wel toe in staat stelt is het onbewijsbare en intuïtief aangevoelde werktuig van de volledige inductie. Echter, als criterium voor de existentie van een wiskundig object hanteerden de Franse pre-intuïtionisten nog niet de strenge eis van construeerbaarheid die Brouwer stelde. Voor Poincaré bestaat een mathematisch object als het vrij is van interne contradictie in het relevante wiskundige systeem; voor Borel is het criterium van existentie van een wiskundig object een met andere wiskundigen gedeelde vertrouwdheid ermee.

Dit proefschrift behandelt de beginperiode van de grote rol die de nog jonge Brouwer zou gaan spelen in de tegenbeweging tegen formalisme en logicisme. Zijn dissertatie *'Over de Grondslagen der Wiskunde'*, verdedigd in 1907, kan gezien worden als een aanloop naar zijn Intuïtionisme. Hoewel dit Intuïtionisme pas tot volle ontwikkeling kwam vanaf 1918, zien we in zijn proefschrift, en in nog grotere mate in een aantal aantekenschriften voor dit proefschrift, al duidelijk een grote hoeveelheid, in de richting van zijn latere ontwikkeling wijzende, ideeën.

De volgende van deze ideeën, die ook in zijn latere werk kenmerkend voor Brouwer zullen blijven, zijn in de voorgaande hoofdstukken uitvoering behandeld:

–1– *De essentiële taalloosheid van de wiskunde.* Wiskunde is een schepping van de individuele menselijke geest, en de rol van taal hierbij is slechts die van een gebrekkig hulpmiddel voor communicatie met anderen en voor het oproepen uit het eigen geheugen van die schepping. Een probleem hierbij is dan dat van de intersubjectiviteit.

–2– *De oer-intuïtie* als de meest fundamentele basis van alle wiskunde. Deze oer-intuïtie is de 'tijdsbeweging', de gewaarwording van het niet-samenvallen van ervaren gebeurtenissen, van de ervaring van een 'tijdscontinuüm' tussen die twee gebeurtenissen. In tegenstelling tot Kant, die ruimte *en* tijd beide als 'Anschauungsformen' poneerde ter fundering van de aprioriteit van respectievelijk meetkunde en rekenkunde, had Brouwer aan de oer-intuïtie van de tijdsbeweging genoeg.

–3– Een *strict constructivisme.* Alleen dat wat vanuit de oer-intuïtie volgens een tevoren vastgesteld algoritme of constructiemethode door de geest kan worden opgebouwd, telt als wiskundig object of als stelling (relatie tussen objecten); in Brouwers terminologie: als wiskundig gebouw.[52]

–4– *De rol van de logica* is slechts een begeleidende, alleen nuttig voor een effectieve en beknopte beschrijving van een mentale wiskundige constructie. Logica komt *na* wiskunde, in plaats van er een basis en uitgangspunt voor te zijn.

–5– Een *stelsel axioma's* als fundament voor een wiskundige constructie wordt afgewezen. Er is slechts één basis en fundament en dat is de eerder genoemde oer-intuïtie. Axioma's hebben slechts de dienende functie van het beknopt weergeven van de voornaamste kenmerken van een gebouw.

Deze uitgangspunten hebben voor de op te bouwen wiskunde verstrekkende gevolgen, die door Brouwer worden beschreven in zijn dissertatie. De beschrij-

[52]Dit constructivisme zal echter in zijn intuïtionistische wiskunde een geheel andere vorm krijgen.

vingen en uitwerkingen van die gevolgen zijn echter vaak dermate beknopt weergegeven, dat in vele gevallen een uitvoerige interpretatie vereist is. Deze interpretaties zijn in de voorafgaande hoofdstukken uitgebreid uitgewerkt en met voorbeelden toegelicht. We noemen er enkele:

– Slechts eindige of aftelbaar oneindige verzamelingen bestaan. ‘Aftelbaar oneindig’ wil zeggen dat de elementen op systematische wijze één voor één geteld kunnen worden zonder een enkele over te slaan. Elk aanwijsbaar element van een dergelijke verzameling kan van een uniek natuurlijk getal voorzien worden en, omgekeerd, elk natuurlijk getal van een element. Dit houdt in dat oneindige verzamelingen bestaan in intensionele zin, echter niet extensioneel. Brouwer verzette zich hier tegen Poincaré, voor wie het oneindige geheel ondenkbaar was. Maar uit de aantekenschriften en uit de dissertatie blijkt dat Brouwer soms het bestaan van het oneindige stelt, maar dit op andere plaatsen juist weer ontkent. Dit vereist steeds een interpretatie, die echter telkens met goede argumenten mogelijk blijkt.

– Het continuüm (een tijdsverloop of een rechte lijn) bestaat niet uit punten, maar is ons als geheel direct gegeven uit de oer-intuïtie. Hierbij is het tijdscontinuüm primair. Dit niet-atomair zijn van het continuüm is overigens een opvatting die reeds door Aristoteles verdedigd werd. Het continuüm kan tot een ‘meetbaar’ continuüm gemaakt worden door er een ‘overal dichte rationale schaal’ van punten op te *construeren*, waarbij het ‘dicht’ maken van deze schaal een van de voorbeelden is van een vrije creatieve act van de menselijke geest, zoals heel de opbouw van de wiskunde het resultaat is van een vrije menselijke schepping. Volgens Brouwer zijn er echter daarna *altijd* weer opnieuw meer punten te construeren op een continuüm met een reeds overal dichte schaal erop, hetgeen voor hem aanleiding was om zijn concept ‘aftelbaar oneindig onaf’ te introduceren.

– Cantors tweede en hogere getalklassen bestaan voor Brouwer niet, omdat er geen algoritme bestaat dat alle elementen van die klassen doet ontstaan zodanig dat deze als één af geheel kan worden gezien. Er is geen afsluiting van die hogere klassen denkbaar, zoals dit wel denkbaar is voor de eerste getalklasse (de natuurlijke getallen) in de vorm van ω, het eerste element van de tweede getalklasse.

– Het door Cantor gestelde ‘continuüm probleem’ (i.e. op welke plaats in de hiërarchie van de kardinaalgetallen staat het systeem van de reële getallen, en de daarbij uitgesproken ‘continuüm hypothese’ dat de verzameling van alle reële getallen gelijkmachtig is met de tweede getalklasse) heeft, als gevolg van Brouwers constructivisme, een bijzonder eenvoudige oplossing.

Dit proefschrift behandelt uitdrukkelijk niet in expliciete zin Brouwers latere intuïtionistische wiskunde, die van keuzerijen, van verzamelingen als ‘spreiding’ en ‘species’, van de continuïteitsstelling en van het ‘bar-theorema’, wat allemaal pas na 1918 tot volle ontwikkeling kwam. Maar sommige aspecten hiervan komen al min of meer duidelijk ter sprake in de eerder genoemde aantekenschriften die een aantal jaren geleden teruggevonden werden, maar waarvan het bestaan al veel eerder bekend was. In deze schriften zijn vele fragmenten te vinden die er duidelijk op wijzen dat Brouwer zoekende was in de richting van de later ingeslagen weg. Dankzij de inhoud van de schriften zijn ook veel fragmenten in de dissertatie te interpreteren in intuïtionistische zin.

Ten slotte is ook in de voorgaande hoofdstukken Brouwers kijk op de samenleving ter sprake gekomen. Dit is een facet in zijn denken waar niet omheen gegaan kan worden omdat dit een steeds terugkerend thema is in zijn werk, zij het dat in latere tijd de toon aanmerkelijk milder wordt. In een van zijn eerste publikaties, *Leven, Kunst en Mystiek* uit 1905, ontstaan uit een serie lezingen die Brouwer in Delft hield, komt een uiterst pessimistisch mensbeeld naar voren. We vinden dit terug op vele plaatsten in de aantekenschriften: elk ingrijpen van de mens in de natuur met behulp van wiskundige 'causale reeksen' is een zondige activiteit; het leidt tot 'externalisatie' terwijl de mens volgens Brouwer moet inkeren tot zichzelf; deze opvatting is terug te vinden in het tweede hoofdstuk van zijn dissertatie. Hoewel de door hem geschetste werkwijze van het natuurkundig onderzoek een reële weergave van die praxis is, is het doel van natuurkundig onderzoek volgens Brouwer slechts het beheersen van natuur en medemens. Daarom velt hij er op morele gronden een negatief oordeel over: de mens onderzoekt de natuur slechts om zijn macht te vergroten over die natuur en daarmee over zijn medemens. Het lijkt ons dat een fysicus, die zo denkt over zijn terrein van onderzoek, daarvan wel invloed *moet* ondervinden in zijn werkwijze en resultaten.

Maar laat ons deze samenvatting besluiten met de constatering dat de werkelijke betekenis van het werk van Brouwer ligt in de schoonheid en de originaliteit van zijn totaal nieuwe opbouw van de wiskunde en de filosofische onderbouwing daarvan. Deze opbouw geschiedt op zeer consequente en strenge wijze, geheel op basis van de menselijke oer-intuïtie van de van alle inhoud geabstraheerde tijdsbeweging. Deze vergaande abstractie kan men hanteren als één van de argumenten bij de benadering van het probleem der intersubjectiviteit.

Curriculum Vitae

– John Kuiper werd geboren op 4 december 1935 te Arnhem. Na de middelbareschoolopleiding (diploma HBS-B) volgde een opleiding tot vlieger (1954 – 1957), gevolgd door een carrière bij de KLM. Na 35 dienstjaren, waarvan 21 jaar als gezagvoerder, ging hij met pensioen in 1993.

– Tijdens deze 35-jarige periode werd in 1968 aan de Rijksuniversiteit Leiden Candidaatsexamen afgelegd in Natuurkunde en Wiskunde.

– Van 1993 tot 1999 volgde een studie Wijsbegeerte aan de Universiteit Utrecht. In 1999 werd doctoraalexamen afgelegd (cum laude) in de Theoretische Filosofie.

– Aansluitend aan dit doctoraalexamen werd hij aangesteld als Onderzoeker in Opleiding bij de vakgroep Theoretische Filosofie van de Faculteit Wijsbegeerte, als medewerker aan het Brouwer Project onder leiding van Prof.Dr. D. van Dalen. Het voor u liggende proefschrift werd in deze periode (december 1999 – november 2003) geschreven.

– Op 27 februari 2004 wordt dit proefschrift verdedigd.

– De werkzaamheden bij het Brouwer Project worden na de promotie nog enige tijd gecontinueerd ter afronding van dit Project.