ADAPTIVE CONTROL AND PARAMETER IDENTIFICATION

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A THESIS SUBMITTED TO THE FACULTY OF ENGINEERING, UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN ENGINEERING

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בס"ד

Roll on, thou deep and dark blue Ocean - roll!
Ten thousand fleets sweep over thee in vain;
Man marks the earth with ruin - his *control*
Stops with the shore.

Lord Byron

Childe Harold's Pilgrimage clxxix

ABSTRACT

The broad theory of adaptive control is introduced, with motivation for using such techniques. The two most popular techniques, the Model Reference Adaptive Controllers (MRAC) and the Self Tuning Controllers (STC) are studied in more detail.

The MRAC and the STC often lead to identical solutions. The conditions for which these two techniques are equivalent are discussed.

Parameter Adaptation Algorithms (PAA) are required by both the MRAC and the STC. For this reason the PAA is examined in some detail. This is initiated by deriving an off-line least-squares PAA. This is then converted into a recursive on-line estimator. Using intuitive arguments, the various choices of gain parameter as well as the variations of the basic form of the algorithm are discussed. This includes a warning as to where the pitfalls of such algorithms may lie.

In order to examine the stability of these algorithms, the Hyperstability theorem is introduced. This requires knowledge of the Popov inequality and Strictly Positive Real (SPR) functions. This is introduced initially using intuitive

energy concepts after which the rigorous mathematical representation is derived.

The Hyperstability Theorem is then used to examine the stability condition for various forms of the PAA.

DECLARATION

I declare that this dissertation is my own unaided work. It is being submitted for the degree of Master of Science in Electrical Engineering at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

27 day of May 1983

This dissertation is dedicated to my wife Suri and to my parents, Nathan and Lorna Rabinowitz with grateful appreciation for their continued support and encouragement. This dissertation is also dedicated to my parents-in-law, Dr and Mrs Arnold Wolf, with grateful thanks for Suri.

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PREFACE

Wh e doing post-doctoral research in the United States of America, the author attended a workshop in adaptive processing at Yale University in May 1981. Having been involved in research in adaptive filters at the CSIR in Pretoria, it was thought that this workshop would be most beneficial.

While attending this workshop the author was fortunate to make the acquaintance of Prof Yoan Landau from France, who whet the author's appetite for adaptive control.

The author was privileged to attend a short course given by Prof Landau at the University of California in June 1982. This dissertation was undertaken in an effort to investigate more fully the concepts presented at this course. More specifically, the problem of proving stability in non-linear time varying feedback loops was to be investigated. This same problem occurs in many areas, including that of adaptive filters.

The author expressed sincere thanks to Prof Landau for this inspiration and the introduction to this fascinating field. Most of the concepts expressed herein were introduced to the author by Prof Landau either at the course or during personal discussions. It was only [illegible], while [illegible] the literature, that the author began to see the mathematical beauty of this field.

[illegible] on is written along a single theme. It begins [illegible] adaptive control and a motivation [illegible]. This is followed by a discussion of the Model Reference Adaptive Controllers (MRAC) and the Self Tuning Controllers (STC) which are the most commonly used in [illegible]. The similarities and conditions of equivalence [illegible] are also given.

[illegible] Parameter Adaptation Algorithm (IAA) this [illegible] detail, including various permutations [illegible] appeal is made to intuitive thinking [illegible] while the mathematical proofs are rigorous.

To analyse the stability of these algorithms the Hyperstability Theorem is introduced. All the necessary background mathematics including the Popov inequality and the Strictly Positive Real (SPR) condition are developed from the basic concepts. Once again, an intuitive approach allows for ease of comprehension.

The Hyperstability theorem is then used to [illegible] for different PAA scheems. Thus the [illegible] the PAA from an off-line algorithm to a rec[illegible] on-line algorithm, and then shows under what cond[illegible] guarantee convergence.

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CHAPTER 1

INTRODUCTION TO ADAPTIVE CONTROL

1.1 Reasons for Adaptive Control

High performance control systems require precise tuning of the controller. However, in most practical situations, the plant parameters may be unknown, or may vary with time. This may occur either because environmental conditions change or are unknown, or because we have considered a simplified linear model for a non-linear system.

An adaptive controller automatically adjusts its parameters on-line in such a manner so as to achieve and maintain an acceptable level of performance under the above conditions.

The concept of adaptive control seems to be old, however, interest in these systems has arisen only as recently as the fifties with significant development starting in the late sixties [1], [11].

The "Model Reference Adaptive Systems" (MRAS) approach will be considered in detail. This technique may be used for 1) adaptive model following control, 2) on-line and real-time parameter identification and 3) adaptive state observation. In the last two methods, the plant being identified or observed forms the reference model.

Definition (Landau [3]) "An adaptive control system measures a certain index of performance (IP) of the control system using the inputs, the states and the outputs of the adjustable system. From the comparison of the measured index of performance and a set of given ones, the adaptation mechanism modifies the parameters of the adjustable system or generates an auxiliary input, in order to maintain the IP close to the set of given ones (i.e. within the set of acceptable ones)".

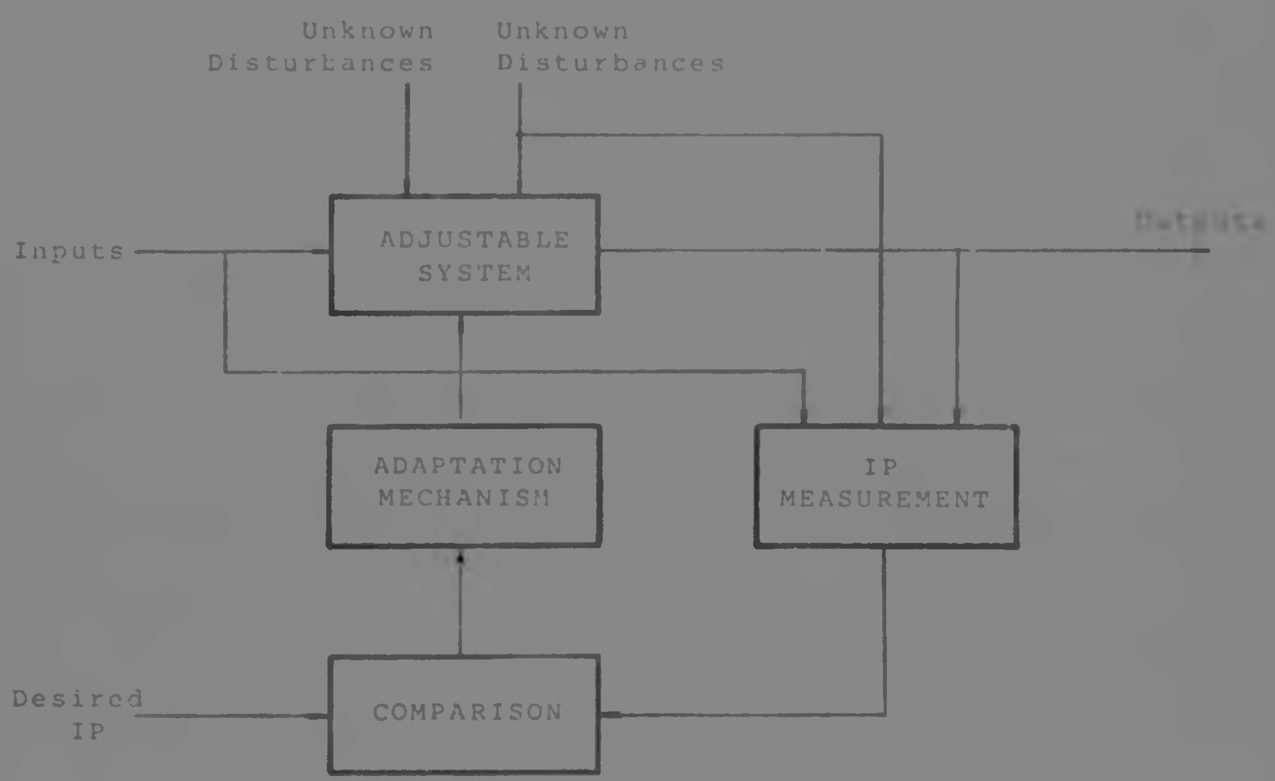


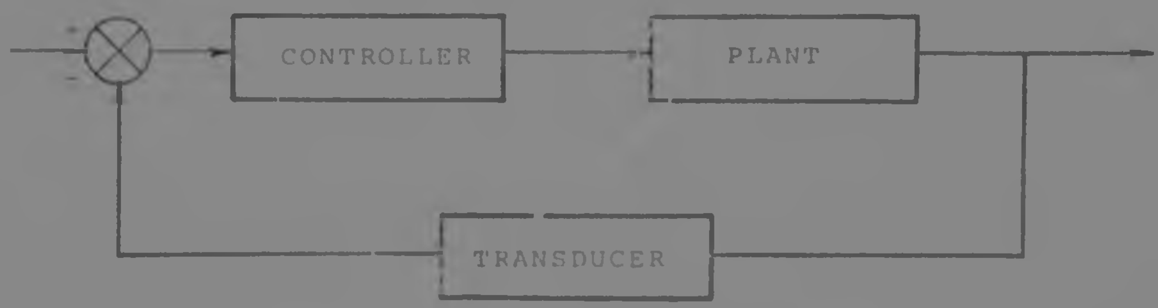
FIGURE 1.1 : Generalized Adaptive Control Mechanism

1.2 Comparison of Conventional Control and Adaptive Control

A conventional controller monitors the controlled variables under the effect of disturbances acting on them. Since it is designed assuming constant process parameters, its performance will vary under parameter disturbances.

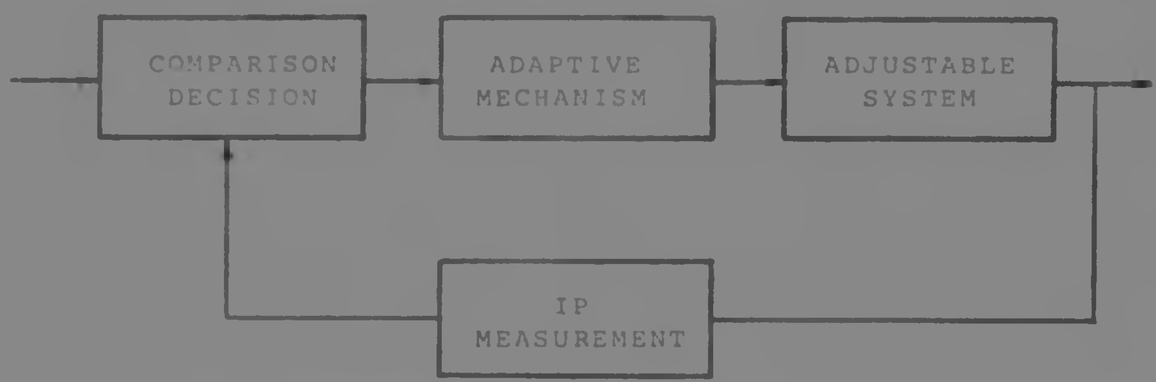
In adaptive control, the system contains a feedback control with adjustable parameters. A supplementary loop monitors the system performance and adjust the controller parameters in the presence of parameter disturbances so as to maintain acceptable performance (e.g. to maintain a specific damping ratio.

Figure 1.2 illustrates the two systems. From this figure we can make one-to-one correspondence between the system, as shown in Table 1.1.



OBJECTIVE: Control of Physical Variables

(a) CONVENTIONAL CONTROL



OBJECTIVE: Control of Performance

(b) ADAPTIVE CONTROL

FIGURE 1. : Conventional versus Adaptive Control

TABLE 1.1 : Conventional Control compared to Adaptive Control

CONVENTIONAL CONTROL	ADAPTIVE CONTROL
PLANT	ADJUSTABLE SYSTEM
TRANSDUCER	IP MEASUREMENT
REFERENCE INPUT	DESIRED IP
COMPARATOR	COMPARISON - DECISION
CONTROLLER	ADAPTATION MECHANISM

TABLE 1.1 : Conventional Control compared to Adaptive Control

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1.3 Basic Adaptive Control Techniques

1.3.1 Open-loop adaptive control

This technique is also known as "gain-scheduling" and is often used in aircraft autopilots. It assumes there is a "rigid relationship" between the environment and the system parameters. The system controller then adapts according to the environment <u>without measuring</u> the actual system performance.

This technique will fail if the "environment-system" relationship changes. The system is illustrated in Figure [illegible]. The system is not truly adaptive in terms of our definition.

It should be noted that this method is not necessarily cheaper to implement than a closed loop system, as transducers may be very costly. Closed loop adaptive systems merely require additional computer capabilities.

The designer should also be careful not to use adaptive techniques in a situation where a conventional feedback controller would suffice (where the controller is designed such that system performance is not too sensitive to parameter variations).

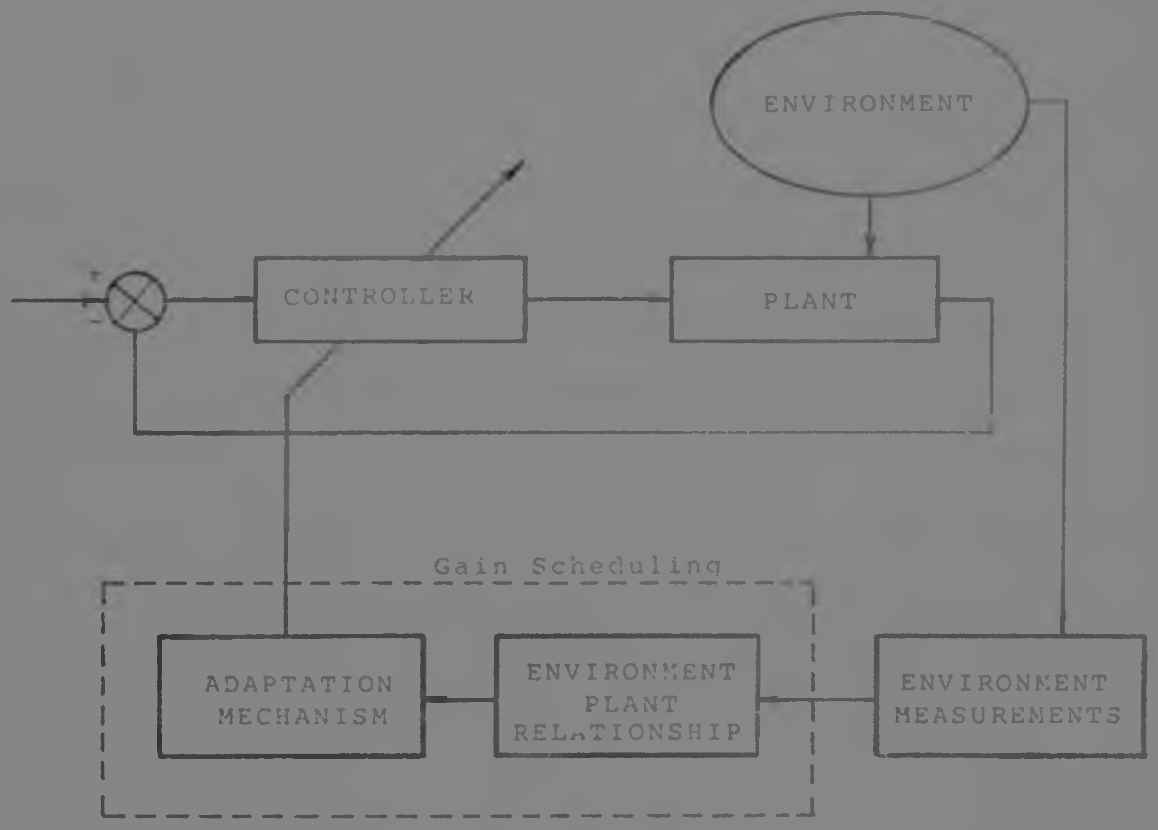
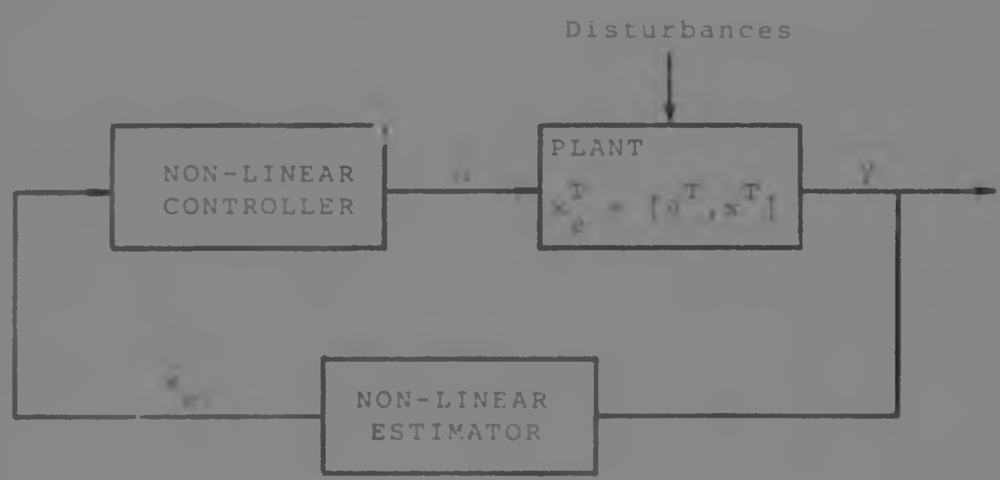


FIGURE 1.3 : An Open Loop Adaptive Control

1.3.2 Closed-loop adaptive control

(a) Dual Stochastic Control

In dual stochastic control [4], [5], the unknown parameters are considered as additional states to be estimated. This technique simultaneously tries to reduce both the control and the estimation error. This is illustrated in Figure 1.4.



x_e^T = extended state

θ^T = parameter vector

x^T = state vector

$x_e^T = [\theta^T, x^T]$

FIGURE 1.4 : Dual Stochastic Control

Due mainly to computation requirements, even the simplified approximations to this technique are extremely complicated and difficult to implement. A simple linear control problem with one unknown parameter becomes a stochastic non-linear control problem. Consider the following example:

$$\dot{x} = ax + u \qquad (1.1)$$

Suppose a is unknown

Let $x_1 = x$

$x_2 = a$

Then the system is characterized by

$$\dot{x}_1 = x_2 x_1 + u \qquad (1.2)$$

$$\dot{x}_2 = f(x_1, x_2) \qquad (1.3)$$

Equation (1.2) is non-linear and the form of $f(.,.)$ in equation (1.3) may also be unknown.

The dual approach is of theoretical interest for obtaining performance bounds for the simpler and more feasible sub-optimal techniques.

(b) Self-Tuning Control (STC) and Model Reference Adaptive Control (MRAC)

Self-tuning controllers, proposed by Kalman (1958) were originally developed for the stochastic discrete time regulation problem.

The MRAC techniques were initially developed for deterministic tracking problems by Whitaker (1958).

Both techniques were developed independently and both have been successfully implemented. The two are strongly connected, and for a variety of IP and process models the two techniques can lead to identical solutions if the desired response is specified in terms of a transfer function in a deterministic environment, or an ARMA model in a stochastic environment.

CHAPTER 2

MODEL REFERENCE ADAPTIVE CONTROLLERS AND SELF TUNING REGULATORS

2.1 Basic Principles

Both the MRAC and the STR techniques give approximations for the solution of the non-linear control problem. They are based on the hypothesis that "for any possible value of the process parameters there exists a linear controller with a fixed complexity such that the closed loop control system (process and controller) can achieve pre-specified (desired) performances". [1]

Thus one assumes that for varying plant parameters, only the controller parameters (not the controller structure) need be changed to achieve the desired performance.

The configuration of an MRAC with explicit reference model is given in Figure 2.1. The reference model characterizes the desired plant structure. The controller is adjusted by the adaptation mechanism so as to give a closed loop response that is as close as possible to that of the reference model. The adaptation mechanism uses the error signal as well as the plant inputs and outputs in its adaptation algorithm.

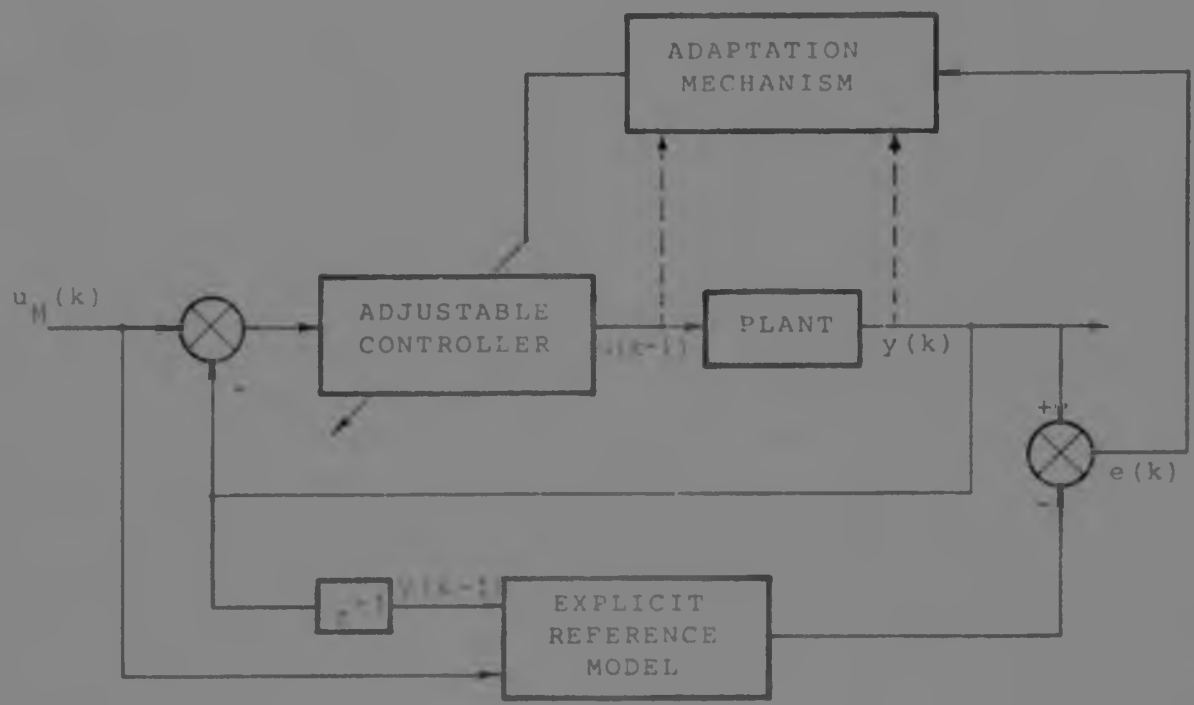


FIGURE 2.1 : Explicit MRAC

Figure 2.2 illustrates the self tuning control (STC) scheme. A model of the plant is estimated on-line using the available input and output data of the plant. The model is then used for the design of a suitable controller. The model, and therefore the controller, is continuously updated as more information becomes available. The technique of on-line estimation of the plant model will now be investigated in detail.

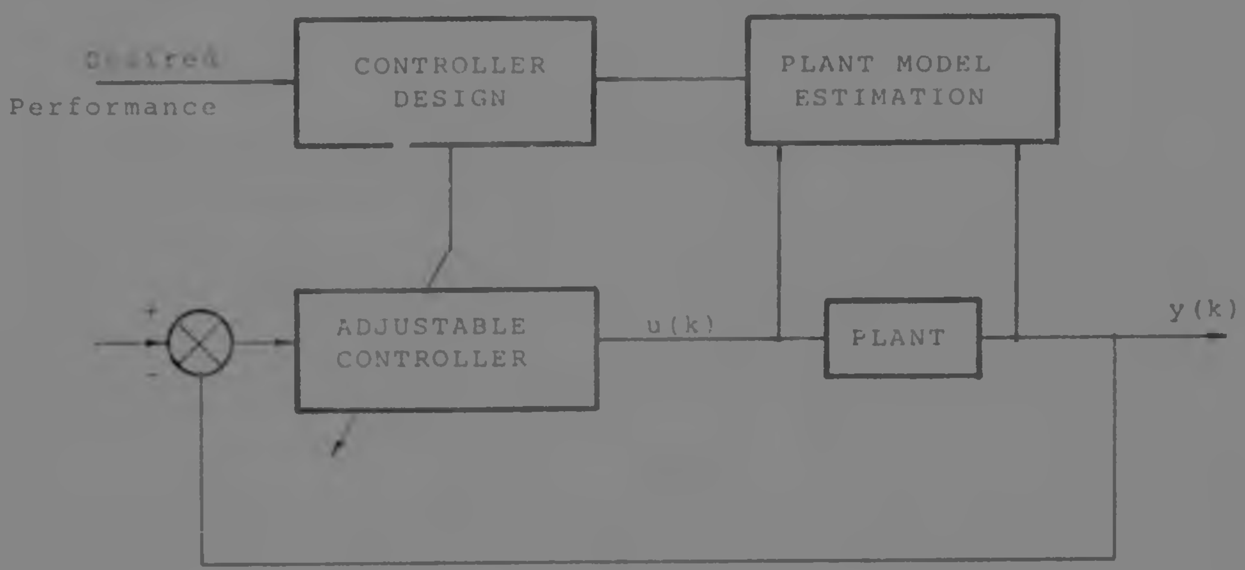


FIGURE 2.2 : Self Tuning Control Principle

2.1.1 On-line plant estimation

The basic principle for on-line parameter estimation is to build up an adjustable predictor for the plant output. This scheme is illustrated in Figure 2.3.

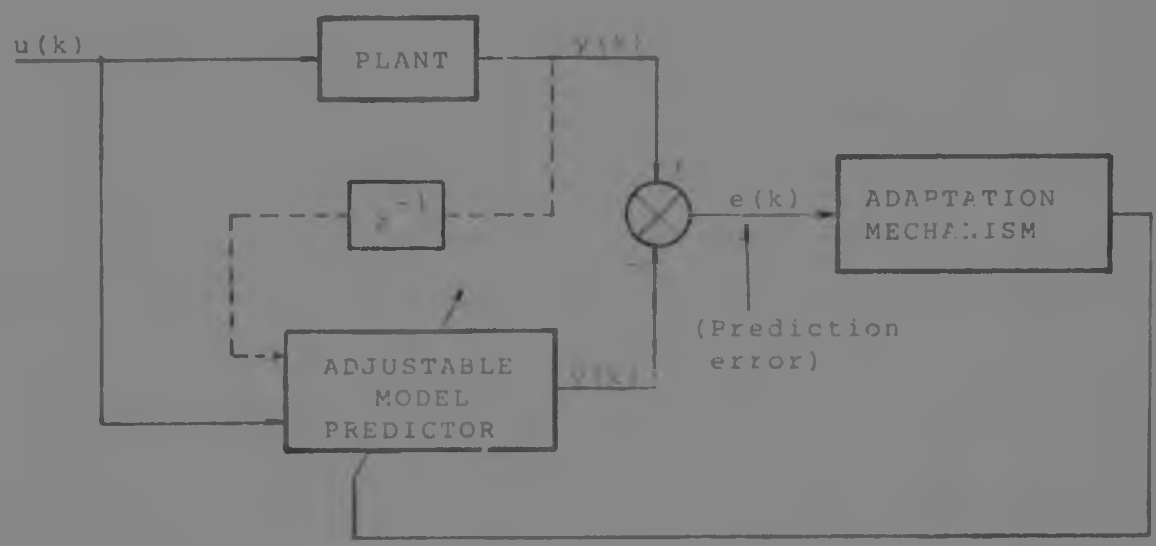
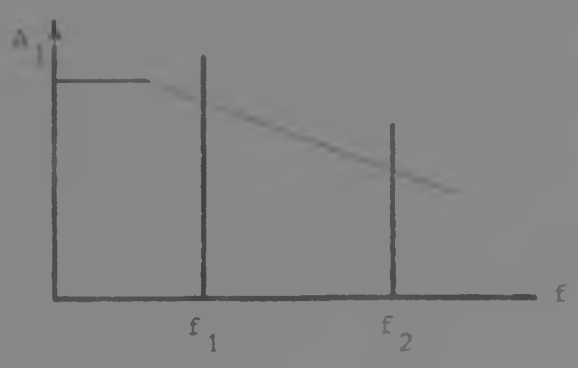


FIGURE 2.3 : On Line Parameter Estimation

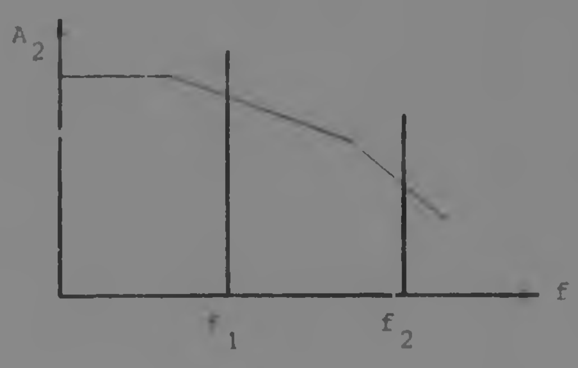
The prediction error (e(k)) is used by a recursive estimation algorithm to adjust the parameter of the model predictor. The object in a deterministic environment is to force e(k) asymptotically to zero. The stochastic environment is discussed in Section 2.2.

This scheme consists of an adaptive predictor that asymptotically gives an estimated model whose output agrees with the plant output for the given input. This is not an identification of the plant model, which would give the correct input output relationship for all possible input sequences.

In order to identify the plant, we would need a "sufficiently rich" input (one that has a rich enough spectrum) so as to excite all the modes of the plant. Figure 2.4 illustrates the Bode plot of two systems that are indistinguishable if one has an input with a single frequency f_1. However, when the input contains f_2 as well, the difference in the systems become apparent.



(a) First order system



(b) Second order system

FIGURE 2.4 : A sufficiently rich Input Spectrum is required to distinguish between systems

The controller is designed on the parameters of the predictor which, as we have seen, need not be the same as the true plant parameter. This complicates the analysis of these schemes.

The estimation scheme of Figure 2.3 is the dual of the MRAC shown in Figure 2.1. Th basic configuration is the same if we interchange the blocks as shown in Table 2.1.

<u>TABLE 2.1</u> : Duality of MRAC and Adaptive Estimation

EXPLICIT MRAC		ADAPTIVE ESTIMATOR
REFERENCE MODEL	⟺	PLANT
ADJUSTABLE SYSTEM (PLANT AND CONTROLLER)	⟺	ADJUSTABLE PREDICTOR (PLANT MODEL)

The Parameter Estimation shown in Figure 2.3 is inserted into Figure 2.2 to obtain the general configuration of the S shown in Figure 2.5.

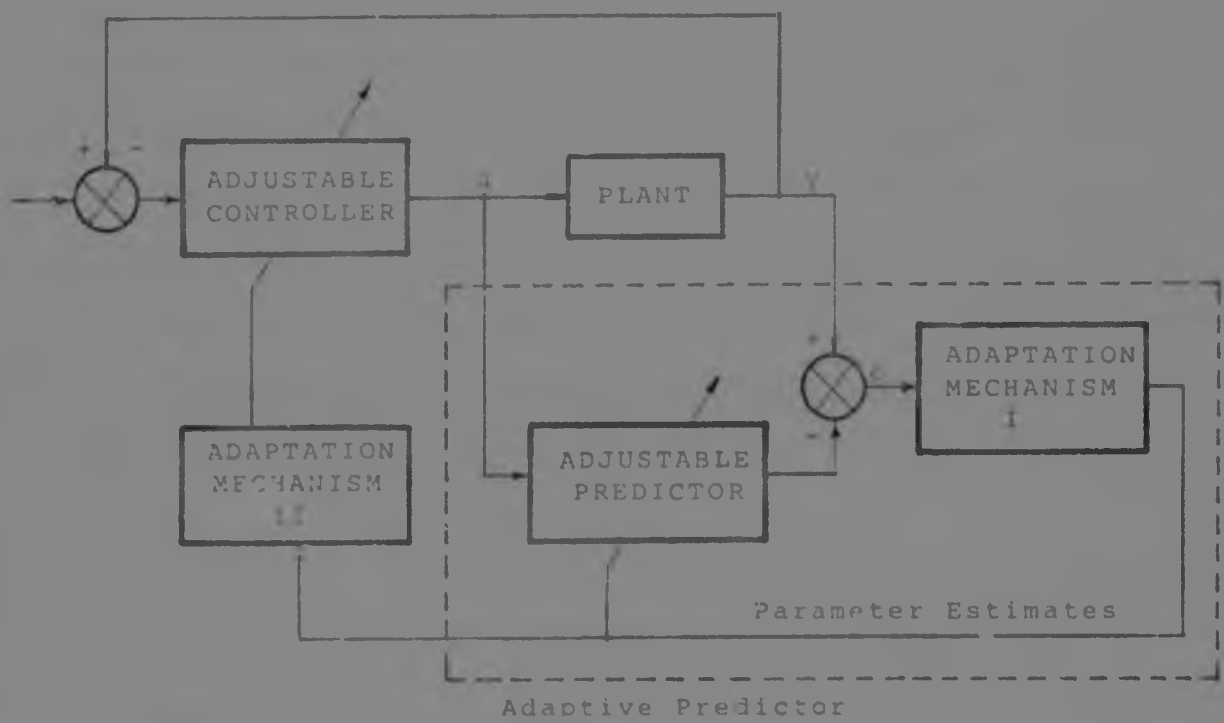


FIGURE 2.5 : General Configuration of the STC

A large variety of schemes can be obtained by combining various recursive parameter estimation schemes (Adaptation Mechanism I) with various controller design strategies based on the plant model parameters (Adaptation Mechanism II).

As we have seen, the estimates of the plant parameters are not necessarily the same as the actual plant parameters. Thus one needs to do careful analysis to determine if a specific scheme will work. "Analytical results describing the behaviour of such adaptive control schemes are available only for very limited choice of parameter estimation algorithms and control strategies". [1]

When the desired performance is given in terms of a specified transfer function and the plant is minimum phase, the resulting STC class is equivalent with the explicit MRAC shown in Figure 2.1 and theoretical results for this class is available.

2.1.2 Direct and indirect adaptive control

In the explicit MRAC shown in Figure 2.1, the controller parameters are directly updated by the adaptation mechanism. This is called "Direct Adaptive Control". The STC shown in Figure 1.5 on the other hand uses adaptation mechanism I to adapt the adjustable predictor parameters. These parameters are then used by the adaptive mechanism II to compute the controller parameters. This is known as indirect adaptive control.

In many instances, by re-parametrization, one can directly estimate the controller parameter in the adaptive mechanism I. The adaptive mechanism II then falls away and the connection of STC and explicit MRAC is then even more obvious.

The following is an example of re-parameterization [1].

Let the plant model be

$$y(k + 1) = - a\, y(k) + u(k) \qquad (2.1)$$

where y is the output, u is the input and 'a' is an unknown parameter.

The objective is to find u, such that

$$y(k + 1) = - c\, y(k) \qquad |c| < 1 \qquad (2.2)$$

when a is known, the appropriate control is

$$u(k) = - r\, y(k) \qquad (2.3)$$

with $$r = c - a \qquad (2.4)$$

Equation (2.1) can now be rewritten as

$$y(k + 1) = - c\, y(k) + r\, y(k) + u(k) \qquad (2.5)$$

and r is the only unknown parameter. By using (2.5) as the model, the parameter r is estimated by the adaptive mechanism r, and used directly in equation (2.3) to get the control u(k). No secondary adaptive mechanism is needed.

For a STC scheme where the desired performance is expressed in terms of a desired dynamic system, the controller is adjusted such that at each instant, the output of the adaptive predictor is equal to the desired system (i.e. the reference model) output. The controller and the predictor thus form an "implicit" reference mode. The error in such a scheme has the same meaning as the error in the explicit MRAC of Figure 2.1. The implicit MRAC is shown in Figure 2.6. It should be noted that the explicit reference model is <u>not</u> part of the scheme, but is merely inserted for illustrative purposes.

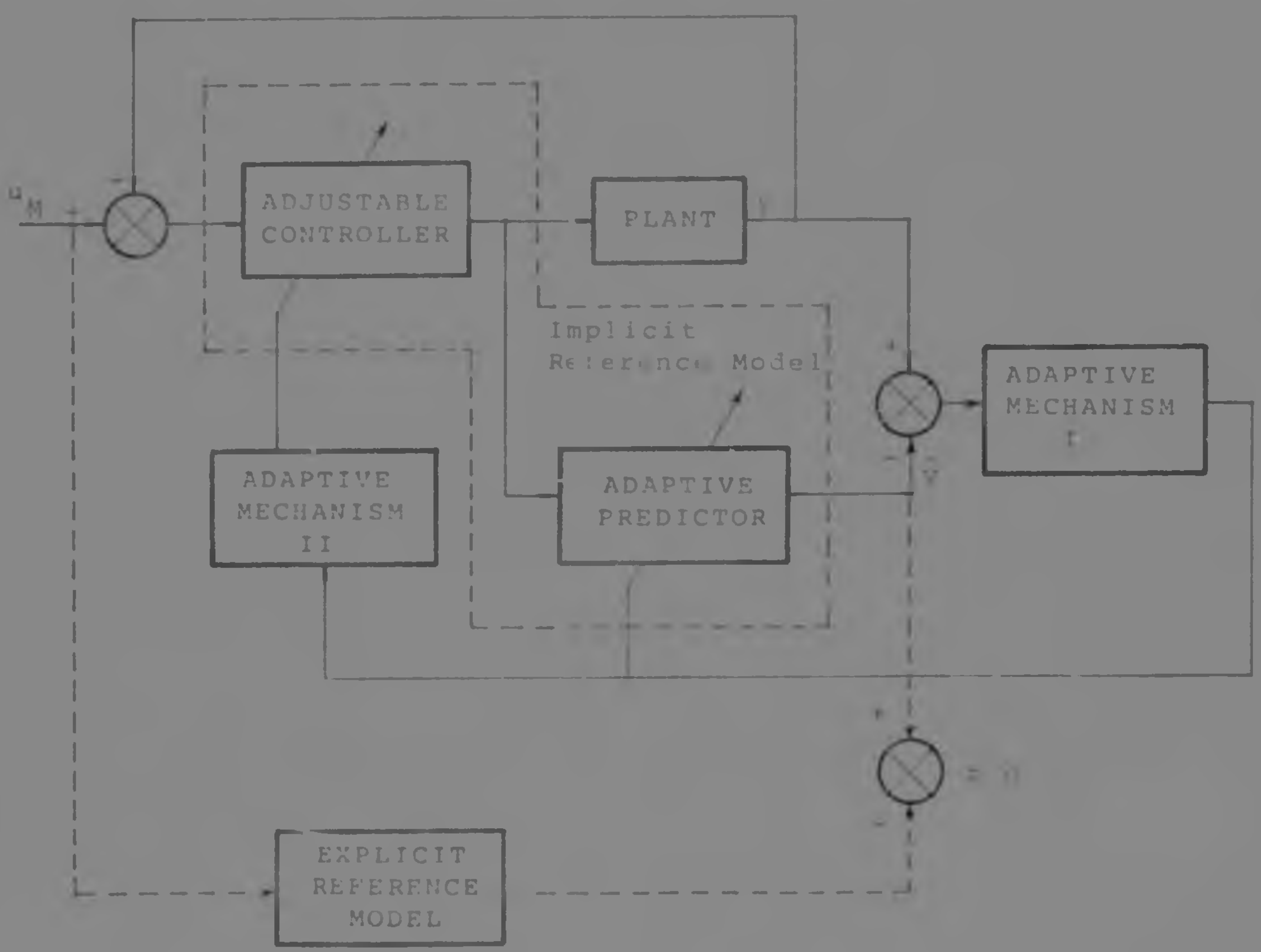


FIGURE 2.6 : Implicit MRAC

2.2 Stochastic Environments

In a stochastic environment, in addition to the plant model one needs to consider a disturbance model. We will assume that the disturbance can be modelled as an ARMA process. Consider the general structure shown in Figure 2.7. The output process y(k), is called an ARMAX process.

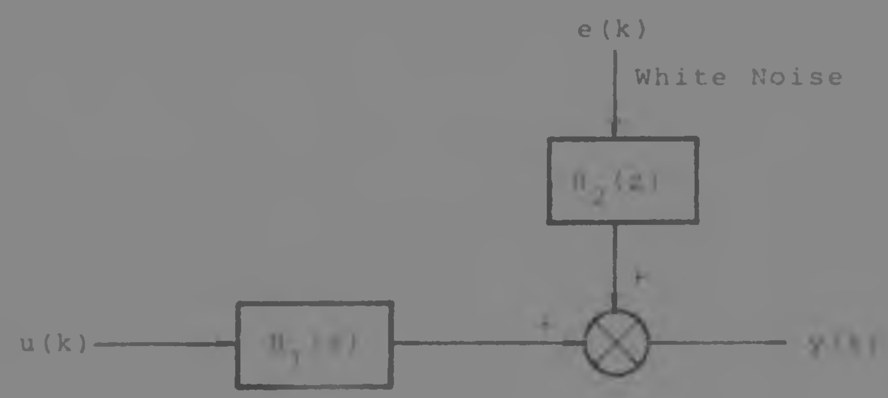


FIGURE 2.7 : The ARMAX Process

If the control law u(k) is a linear feedback of the output y(k) then y(k) is also an ARMA process, and we can specify the desired performance in terms of an ARMA process. Assuming the plant and disturbance models are known, we formulate the control strategy as shown in Figure 2.8. As the only unpredictable process is e(k), the output error $(y_k - \hat{y}_k)$ should be only in terms of e(k). The terms e(k - 1), e(k - 2),.... have been taken into account by the previous output errors which were fed back. Thus the output error should be a white noise process. This process is called an innovation sequence and its "whiteness" is a good measure of the performance of the controller.

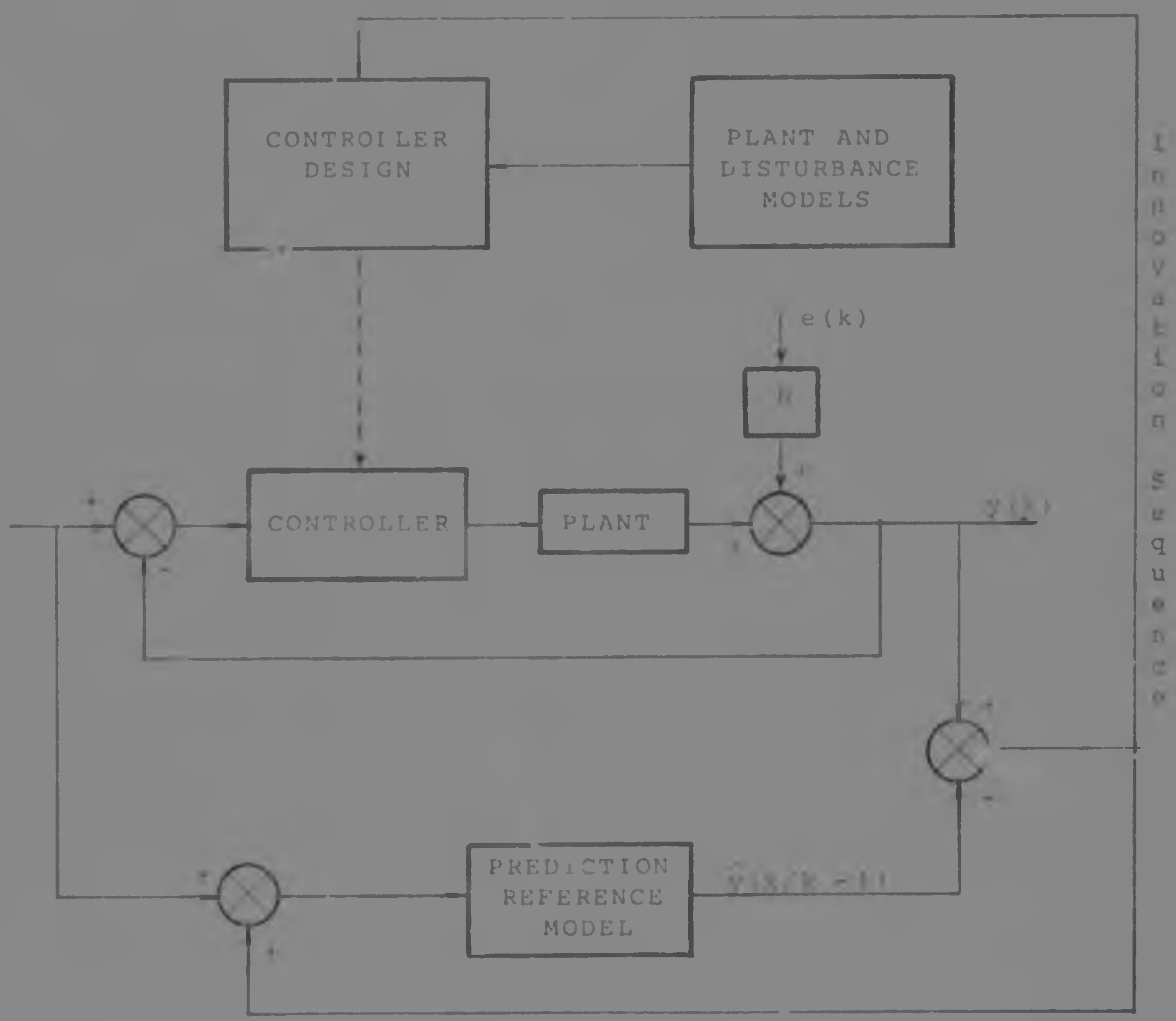


FIGURE 2.8 : Linear Controller Design in a Stochastic Environment using an Explicit Prediction Reference Model

As an example consider the following simple ARMAX process.

$$y(k + 1) = - a\ y(k) + u(k) + c\ e(k) + e(k + 1) \qquad (2.6)$$

where y(k) is the output, u(k) the input and e(k) is a sequence of identically distributed Gaussian random variables.

Assume the desired output it

$$y(k + 1) = - d\ y(k) + f\ e(k) + e(k + 1) \qquad (2.7)$$

As we have no knowledge at time k of e(k + 1) we can formulate the optimum predictor as

$$\hat{y}(k + 1/k) = - d\ y(k) + f\ e(k) \qquad (2.8)$$

and the required control would be

$$y(k + 1) - \hat{y}(k + 1/k) = e(k + 1) \qquad (2.9)$$

(This would give minimum output error).

The optimum control is then

$$u(k) = - (d - a)\ y(k) - (c - f)\ e(k) \qquad (2.10)$$

and the output error will be a white noise sequence

When the plant parameters and the disturbance model are unknown or vary these schemes can be transformed into the type shown in Figure 2.1 or Figure 2.5 with additional parameters. The deterministic reference model in Figure 2.1 is replaced by an explicit stochastic prediction reference model.

In the explicit prediction reference model, the algorithm will adapt so as to obtain (asymptotically) a prediction error that is an innovation sequence.

For the STC structure where the design object is given in terms of an ARMA model, the predictor and controller will form an implicit prediction reference model and the objective will be to achieve a prediction error that is an innovation process.

Under these circumstances, the same similarities exist between the stochastic MRAC and the stochastic STC as were indicated in the deterministic case in Section 2.1.

2.3 Analysis and Design of Adaptive Control Schemes

Since the adaptive control scheme is non-linear the analysis of these systems is non-trivial. A basic property required in the deterministic case is global stability, while global convergence is required in the stochastic case. In both cases one may reformulate the problem in terms of a stability analysis for a system disturbed from equilibrium. This approach works for analysis and design of MRAC and STC with direct adaptation. For indirect adaptation STC the problem is more complex [1].

The problem of direct adaptation of the controller parameters can be approached as a recursive estimation problem. This suggests the use of recursive parameter estimation techniques. In Chapter 3, parameter adaptation algorithms will be discussed in detail.

2.4 Conclusions

The MRAC and STC structures have been discussed both in deterministic and in stochastic environments. The similarities of both methods have been pointed out and the difference between implicit and xplicit as well as direct and indirect adaptive control have been discussed. It has also been indicated that the analysis and design of these systems can be analysed in the framework of a stability problem.

CHAPTER 3

PARAMETER ADAPTATION ALGORITHMS

3.1 The Off-line Least Squares Estimation Algorithm

Consider a system characterized by the transfer function

$$H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} \tag{3.1}$$

The letter q^{-1} is used to denote a unit delay instead of z^{-1}, since z is used to denote a complex number.

We will consider the case of a unit delay (d = 1).

It is assumed that A and B are monic polynomials with the first term of A normalized to 1.

$$A = 1 + a_1 q^{-1} + \ldots + a_n q^{-n} \tag{3.2}$$

$$= 1 + q^{-1} A^* \tag{3.3}$$

where $$A^* = a_1 + a_2 q^{-1} + \ldots + a_n q^{-n+1} \qquad (3.4)$$

and $$B = b_o + b_1 q^{-1} + \ldots + b_m q^{-m} \qquad (3.5)$$

$$= b_o + q^{-1} B^* \qquad (3.6)$$

where $$B^* = b_1 + b_2 q^{-1} + \ldots + b_m q^{-m+1} \qquad (3.7)$$

For a given input $u(k)$, the output $y(k+1)$ is given by the difference equation

$$A(q^{-1}) y(k+1) = B(q^{-1}) u(k) \qquad (3.8)$$

Note that the output has index $k+1$ instead of k, due to the unit delay. Equation (3.8) can now be reformulated using equations (3.3) and (3.6)

$$[1 + q^{-1} A^*(q^{-1})] y(k+1) = B(q^{-1}) u(k) \qquad (3.9)$$

i.e. $$y(k+1) = -A^*(q^{-1}) y(k) + B(q^{-1}) u(k) \qquad (3.10)$$

Using equations (3.4) and (3.6) in equation (3.10) we get

$$y(k+1) = -\sum_{i=1}^{n} a_i y(k+1-i) + \sum_{i=0}^{m} b_i u(k-i) \qquad (3.11)$$

The parameter vector θ is defined by

$$\theta^T = [a_1 \; \ldots \; a_n, \; b_o \; \ldots \; b_m] \qquad (3.12)$$

and the measurement vector $\phi(k)$ by

$$\phi(k)^T = [-y(k) \; \ldots \; -y(k-n+1), \; u(k) \; \ldots \; u(k-m)] \qquad (3.13)$$

Equation (3.11) can now be written as

$$y(k+1) = \theta^T \phi(k) \qquad (3.14)$$

The problem is to find the best estimate $\bar{\theta}(k)$ for θ (in the least squares sense) given the k sets of measurements

$$y(i) = \theta^T \phi(i-1) \qquad i = 1, \ldots, k \qquad (3.15)$$

i.e. find $\bar{\theta}(k)$ that minimizes the IP

$$J(\theta) = \sum_{i=1}^{k} [y(i) - \theta(k)^T \phi(i-1)]^2 \qquad (3.16)$$

Since $k \geq i$

$$\hat{y}(i/\hat{\theta}(k)) = \hat{\theta}(k)^T \phi(i-1) \qquad (3.17)$$

is called the a posteriori prediction

$$E(i/j) = y(i) - y[i/\theta(j)] \qquad (3.18)$$

is the a posteriori prediction error

We can thus rewrite (3.16) as

$$J(k) = \sum_{i=1}^{k} E^2(i/k) \qquad (3.19)$$

To find the optimum $\hat{\theta}(k)$, we set

$$\frac{\partial J(k)}{\partial \hat{\theta}(k)} = 0 \qquad (3.20)$$

This yields

$$-2 \sum_{i=1}^{k} [y(i) - \hat{\theta}^T(k)\phi(i-1)]\phi(i-1) = 0 \qquad (3.21)$$

Since $\hat{\theta}^T(k)\phi(i-1)$ is a scalar,

$$\sum_{i=1}^{k} y(i)\phi(i-1) = \sum_{i=1}^{k} \phi(i-1)\hat{\theta}^T(k)\phi(i-1) \quad (3.22)$$

$$= \sum_{i=1}^{k} \phi(i-1)\phi(i-1)^T\hat{\theta}(k) \quad (3.23)$$

since a scalar is its own transpose.

Now let

$$F(k)^{-1} = \sum_{i=1}^{k} \phi(i-1)\phi(i-1)^T \quad (3.24)$$

Then

$$\sum_{i=1}^{k} y(i)\phi(i-1) = F(k)^{-1}\hat{\theta}(k) \quad (3.25)$$

which gives

$$\hat{\theta}(k) = F(k)\sum_{i=1}^{k} y(i)\phi(i-1) \quad (3.26)$$

The off-line solution is given by equations (3.24) and (3.26). One first computes $F(k)^{-1}$ using (3.24). The inverse F(k) is then computed and used to solve for $\hat{\theta}(k)$ in (3.26). Due to the matrix inversion at each step, this is extremely time consuming. A recursive, on-line method is presented in the next section.

3.2 Recursive Least Squares (RLS) Parameter Estimation

3.2.1 The RLS algorithm

From equation (3.24), we have

$$F(k+1)^{-1} = \sum_{i=1}^{k+1} \phi(i-1)\phi(i-1)^T$$

$$= \sum_{i=1}^{k} \phi(i-1)\phi(i-1)^T + \phi(k)\phi(k)^T$$

$$= F(k)^{-1} + \phi(k)\phi(k)^T \tag{3.27}$$

Also

$$\sum_{i=1}^{k+1} y(i)\phi(i-1) = \sum_{i=1}^{k} y(i)\phi(i-1) + y(k+1)\phi(k) \tag{3.28}$$

i.e.

$$F(k+1)^{-1}\theta(k+1) = F(k)^{-1}\theta(k) + y(k+1)\phi(k) \tag{3.29}$$

Adding and subtracting $\phi(k)\phi(k)^T\theta(k)$ to the left hand side of (3.29) yields

$$F(k+1)^{-1}\hat{\theta}(k+1) = F(k)^{-1}\hat{\theta}(k) + y(k+1)\phi(k) + \phi(k)\phi(k)^T\hat{\theta}(k)$$

$$- \phi(k)\phi(k)^T\hat{\theta}(k)$$

$$= [F(k)^{-1} + \phi(k)\phi(k)^T]\hat{\theta}(k) + \phi(k)[y(k+1)$$

$$- \hat{\theta}(k)^T\phi(k)] \quad (3.30)$$

Define the a priori prediction error, $\varepsilon^o(k+1)$ by

$$\varepsilon^o(k+1) = y(k+1) - \hat{\theta}(k)^T\phi(k) \qquad (3.31)$$

Substituting equations (3.27) and (3.31) into equation (3.30) gives

$$F(k+1)^{-1}\hat{\theta}(k+1) = F(k+1)^{-1}\hat{\theta}(k) + \phi(k)\varepsilon^o(k+1) \qquad (3.32)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\varepsilon^o(k+1) \qquad (3.33)$$

$F(k)$ is called the adaptation gain. The estimate $\hat{\theta}(k)$ is corrected in the direction $\phi(k)$ modified by the matrix $F(k)$ according to the error term $\varepsilon^o(k+1)$.

3.2.2 A recursion formula for the adaptation gain

To find a recursive formula for F(k), we make use of the Matrix Inversion Lemma [6]. Given three matrices, F(nxn), R(mxm) and H(mxn) and assuming all the necessary inverses exist, then

$$(F^{-1} + HR^{-1}H^{T}) = F - FH(R + H^{T}FH)^{-1}H^{T}F \qquad (3.34)$$

Now let R = 1 (which implies that H^TFH is also a scalar)

$$F = F(k)$$

and

$$H = \phi(k)$$

Then (3.34) yields

$$[F(k)^{-1} + \phi(k)\phi(k)^{T}]^{-1} = F(k) - \frac{F(k)\phi(k)\phi(k)^{T}F(k)}{1 + \phi(k)^{T}F(k)\phi(k)}$$

i.e.

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi(k)^{T}F(k)}{1 + \phi(k)^{T}F(k)\phi(k)} \qquad (3.35)$$

This gives the required recursive algorithm for the adaptive gain F(k).

3.2.3 Reformulation of the RLS algorithm in terms of a posteriori error

The system of equations developed thus far is summarized as follows:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\varepsilon^{o}(k+1) \qquad (a)$$

$$F(k+1)^{-1} = F(k)^{-1} + \phi(k)\phi(k)^{T} \qquad (b)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi(k)^{T}F(k)}{1 + \phi(k)^{T}F(k)\phi(k)} \qquad (c)$$

$$\varepsilon^{o}(k+1) = y(k+1) - \hat{y}(k+1/k) \qquad (d)$$

$$= y(k+1) - \hat{\theta}^{T}(k)\phi(k) \qquad (e) \qquad (3.36)$$

If we multiply equation (3.36c) by $\phi(k)$ on both sides, and place the left hand side over a common denominator we get the following

$$F(k+1)\phi(k) = \frac{F(k)\phi(k)}{1 + \phi(k)^{T}F(k)\phi(k)} \qquad (3.37)$$

This is then substituted into equation (3.36a) to get

$$\bar{\theta}(k+1) = \bar{\theta}(k) + F(k)\phi(k)\frac{\varepsilon^{o}(k+1)}{1+\phi(k)^{T}F(k)\phi(k)} \qquad (3.38)$$

We now define the a posteriori prediction error $\varepsilon(k+1)$ as follows

$$\varepsilon(k+1) \triangleq y(k+1) - \bar{\theta}(k+1)^{T}\phi(k) \qquad (3.39)$$

Using equation (3.36e) in equation (3.39) yields

$$\varepsilon(k+1) = \varepsilon^{o}(k+1) - [\bar{\theta}(k+1) - \bar{\theta}(k)]^{T}\phi(k) \qquad (3.40)$$

Noting that this is a scalar we can take the transpose of the last term without affecting the equation

$$\varepsilon(k+1) = \varepsilon^{o}(k+1) - \phi(k)^{T}[\bar{\theta}(k+1) - \bar{\theta}(k)] \qquad (3.41)$$

Substituting from equation (3.36a) we get

$$\varepsilon(k+1) = \varepsilon^{o}(k+1) + \phi(k)^{T}\left[\frac{F(k)\phi(k)\varepsilon^{o}(k+1)}{1+\phi(k)F(k)\phi(k)}\right] \qquad (3.42)$$

$$= \frac{\varepsilon^{o}(k+1)}{1+\phi(k)^{T}F(k)\phi(k)} \qquad (3.43)$$

We see from equation (3.43) that the a posteriori error is always smaller than the a priori error. The adaptation mechanism acts so as to reduce the error.

We can now reformulate the system of equations given in (3.36) in terms of the a posteriori error

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1) \qquad \text{(a)}$$

$$F(k+1)^{-1} = F(k)^{-1} + \phi(k)\phi(k)^T \qquad \text{(b)}$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi(k)^T F(k)}{1 + \phi(k)^T F(k)\phi(k)} \qquad \text{(c)}$$

$$\varepsilon(k+1) = y(k+1) - \hat{\theta}(k+1)^T \phi(k) \qquad \text{(d)}$$

$$= \frac{y(k+1) - \hat{\theta}(k)^T \phi(k)}{1 + \phi(k)^T F(k)\phi(k)} \qquad \text{(e)}$$

$$= \frac{\varepsilon^0(k+1)}{1 + \phi(k)^T F(k)\phi(k)} \qquad \text{(f)} \qquad (3.44)$$

This system of equations is far more convenient for analysis purposes. It should be noted that this is a structure for recursive parameter adaptation algorithms. The LS algorithm shown is not the only possibility. The differences in the various algorithms will occur in the parameters that appear in the ϕ vector and in the form of the prediction error. Differences may also occur if an error criterion other than the LS is used. This change will be manifested in the update equation for F(k).

3.3 The Adaptation Gain F(k)

There are a number of different choices for the update algorithm for F(k). Each of these will correspond to a different off-line criterion. We will now briefly discuss the merits and failings of some of these.

1 $$F(k+1)^{-1} = F(k)^{-1} + \phi(k)\phi(k)^T \quad (3.45)$$

where $F(0) > 0$

This corresponds to a quadratic off-line criterion

$$F(k) = \sum_{i=1}^{k} [y(i) - \hat{\theta}(k)^T \phi(i-1)]^2 \quad (3.46)$$

In this case F(k) is a positive definite matrix for all k. Since $F^{-1}(k)$ is always increasing, F(k) will be decreasing (if it is not a scalar, then we refer to the norm of the matrix). This means that the new information gets less and less weight. However, if we wish to track a varying parameter this is undesirable. In fact as time goes on, F(k) will tend to zero. To overcome this we can introduce a forgetting factor, which leads to the following algorithm.

2 $$F(k+1)^{-1} = \lambda F(k)^{-1} + \phi(k)\phi(k)^T \qquad (3.47)$$

where $0 < \lambda \leq 1$

This corresponds to a quadratic off-line criterion, with a forgetting factor.

$$F(k) = \sum_{i=1}^{k} \lambda^{k-i} [y(i) - \hat{\theta}(k)^T \phi(i-1)]^2 \qquad (3.48)$$

Typically λ is between 0,95 and 0,99. One difficulty with this, is that if $\phi(k)\phi(k)^T$ is equal to zero for some time, then F(k) will tend to blow up. This will not happen in the following algorithm.

3 $$F(k+1) = F(k) = F(0) \qquad (3.49)$$

This corresponds to the simple gradient off line criterion

$$J(k) = [y(k+1) - \theta(k)\phi(k)]^2 \qquad (3.50)$$

undesirable. In fact as time goes on, F(k) will tend to zero. To overcome this we can introduce a forgetting factor, which leads to the following algorithm.

2 $$F(k+1)^{-1} = \lambda F(k)^{-1} + \phi(k)\phi(k)^T \qquad (3.47)$$

where $0 < \lambda \leq 1$

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$$F(k) = \sum_{i=1}^{k} \lambda^{k-1}[y(i) - \theta(k)^T\phi(i-1)]^2 \qquad (3.48)$$

Typically λ is between 0,95 and 0,99. One difficulty with this, is that if $\phi(k)\phi(k)^T$ is equal to zero for some time, then F(k) will tend to blow up. This will not happen in the following algorithm.

3 $$F(k+1) = F(k) = F(0) \qquad (3.49)$$

This corresponds to the simple gradient off line criterion

$$J(k) = [y(k+1) - \theta(k)\phi(k)] \qquad (3.50)$$

This constant gain algorithm allows tracking of time varying parameters. However, the convergence depends on the present state and this is poor, since it is not necessarily moving in the optimum direction. The following criterion is further generalised by introducing a second λ parameter.

$$P(k+1)^{-1} = \lambda_1(k)P(k)^{-1} + \lambda_2(k)\phi(k)\phi(k)^T \qquad (3.51)$$

where $P(0) > 0$

$0 < \lambda_1(k) \leq 1$

$0 \leq \lambda_2(k) < 2$

The corresponding off-line criterion is of little importance, and is given here mainly for completeness.

$$J(k) = \sum_{i=1}^{k} \left(\prod_{j=i}^{k} \lambda_1(j) \right) \mu(i-1)\left[y(i) - \theta(k)^T\phi(i-1)\right]^2$$

(3.52)

$$\text{Where} \quad \mu(k) = \frac{1 + \lambda_2(k)\phi(k)^T P(k)\phi(k)}{1 + \phi(k)^T P(k)\phi(k)} \qquad (3.53)$$

This will allow tracking of time varying parameters. The parameters λ_1 and λ_2 may vary with k. The condition

$\lambda_1(k) > 0$ implies $F(k+1)$ decreases. Thus one has a fair amount of control over $F(k)$. Obviously all the previous algorithms are special cases of this one with specific values assigned to the λ parameter.

The update algorithm may be manipulated using the matrix inversion lemma to give

$$F(k+1) = \frac{1}{\lambda_1(k)} \left[F(k) - \frac{F(k)\phi(k)\phi(k)^T F(k)}{\alpha(k) + \phi(k)^T F(k)\phi(k)} \right] \qquad (3.54)$$

where

$$\alpha(k) = \frac{\lambda_1(k)}{\lambda_2(k)} \qquad (3.55)$$

The trace of $F(k+1)$ is given by

$$\mathrm{tr}\, F(k+1) = \frac{1}{\lambda_1(k)} \mathrm{tr} \left[F(k) - \frac{F(k)\phi(k)\phi(k)^T F(k)}{\alpha(k) + \phi(k)^T F(k)\phi(k)} \right] \qquad (3.56)$$

By fixing a value for α (typically $0,5 < \alpha \leq 1$) we can choose $\lambda_1(k)$ so as to keep the trace of $F(k)$ constant for all k. Since α is fixed, we can now calculate $\lambda_2(k)$.

The fixed trace algorithm performs far better than the constant gain algorithm. At each step both algorithms move in the direction of least squares minimization. However, the step size in the constant trace algorithm is constant, while that of the constant gain algorithm decreases.

$$F(k) = \frac{1}{p(k)} I \tag{3.57}$$

This is known as a scalar adaptation gain. Depending on the form of p(k) we can get different algorithms.

i) $p(k) = \text{constant}$ (3.58)

This means that F(k) will be constant, giving a gradient type algorithm.

ii) $p(k) = k$ (3.59)

This gives

$$F(k) = \frac{1}{k} I$$

iii) $p(k+1) = p(k) + \phi(k)^T \phi(k)$ (3.60)

where $p(0) > 0$

The algorithms arising from (ii) and (iii) fall into the category of stochastic approximation algorithms [7], [8]. The convergence analysis of these algorithms is simpler but their performance is lower. However, techniques do exist for increasing convergence rates quite dramatically [9].

A word of caution is appropriate at this point. In going from the off-line algorithm to an on-line recursive algorithm certain problems arise.

Firstly, in the initialization of the algorithm we need to wait n steps to calculate off-line, an initial $F(n)$. Alternately we may use an arbitrary initialization for $F(k)$. For example we may choose

$$F(k) = \frac{1}{\delta} I \tag{3.61}$$

where $0 < \delta < 1$

instead of

$$F(n) = \sum_{i=1}^{n} \phi(i-1)\phi(i-1)^T \tag{3.62}$$

where n is the number of parameters.

This gives the following form for F^{-1}

$$F(k+1)^{-1} = \delta I + \sum_{i=0}^{k} \phi(i-1)\phi(i-1)^T \tag{3.63}$$

As k increases, the first term becomes negligible compared to the summation. However, strictly speaking it is not the same as the off-line procedure

Furthermore, we listed a number of possible alternatives for updating F(k) and these will need to be examined analytically to determine the effects on the overall algorithm.

One final point of consideration is that in the on-line algorithm we are using a large number of samples ($k \to \infty$).

In the light of the above, one needs to show that in spite of these changes, the prediction error (ε) will tend to zero as $k \to \infty$.

A note of interest, is that the F matrix is related to the covariance matrix of the input, and is also related to the Kalman gain matrix [6]. If the eigen-values of the F matrix are small then the input may be insufficiently rich.

3.4 The Equivalent Feedback System for Representing Parameter Adaptation Algorithms

By reformulating the PAA in terms of a feedback system, the analysis is greatly simplified. If the feedback system can be shown to be asymptotically stable, then the corresponding PAA will be algebraically stable. This allows us to use control techniques for the design and analysis of stable PAA's.

This method will be demonstrated for the RLS algorithm. The appropriate modifications for the other algorithms will be indicated.

The parameter update vector is given by

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)\varepsilon(k+1) \qquad (3.64)$$

The parameter error vector is defined as

$$\tilde{\theta}(k) \triangleq \hat{\theta}(k) - \theta \qquad (3.65)$$

Substituting (3.63) into (3.62) for $\hat{\theta}(k)$ and $\hat{\theta}(k+1)$ yields

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1) \qquad (3.66)$$

The a posteriori prediction error is given by

$$\varepsilon(k+1) = y(k+1) - \hat{\theta}(k+1)^T \phi(k) \tag{3.67}$$

where

$$y(k+1) = \theta^T \phi(k) \tag{3.68}$$

Substituting (3.68) into (3.67) yields

$$\varepsilon(k+1) = -[\hat{\theta}(k+1) - \theta]^T \phi(k) \tag{3.69}$$

Using equation (3.65) in (3.69) gives

$$\varepsilon(k+1) = -\tilde{\theta}(k+1)^T \phi(k) \tag{3.70}$$

The equivalent feedback system can now be drawn using equations (3.66) and (3.70). This is given in Figure 3.1.

The feedforward part of the loop is merely a straight connection. However, it has been represented as a block of transfer function one, since this will change if algorithms other than the RLS algorithm is used. The feedforward path is a linear time invariant (LTI) system, while that of the feedback path (indicated by broken lines in Figure 3.1) is a nonlinear time varying (NLTV) system.

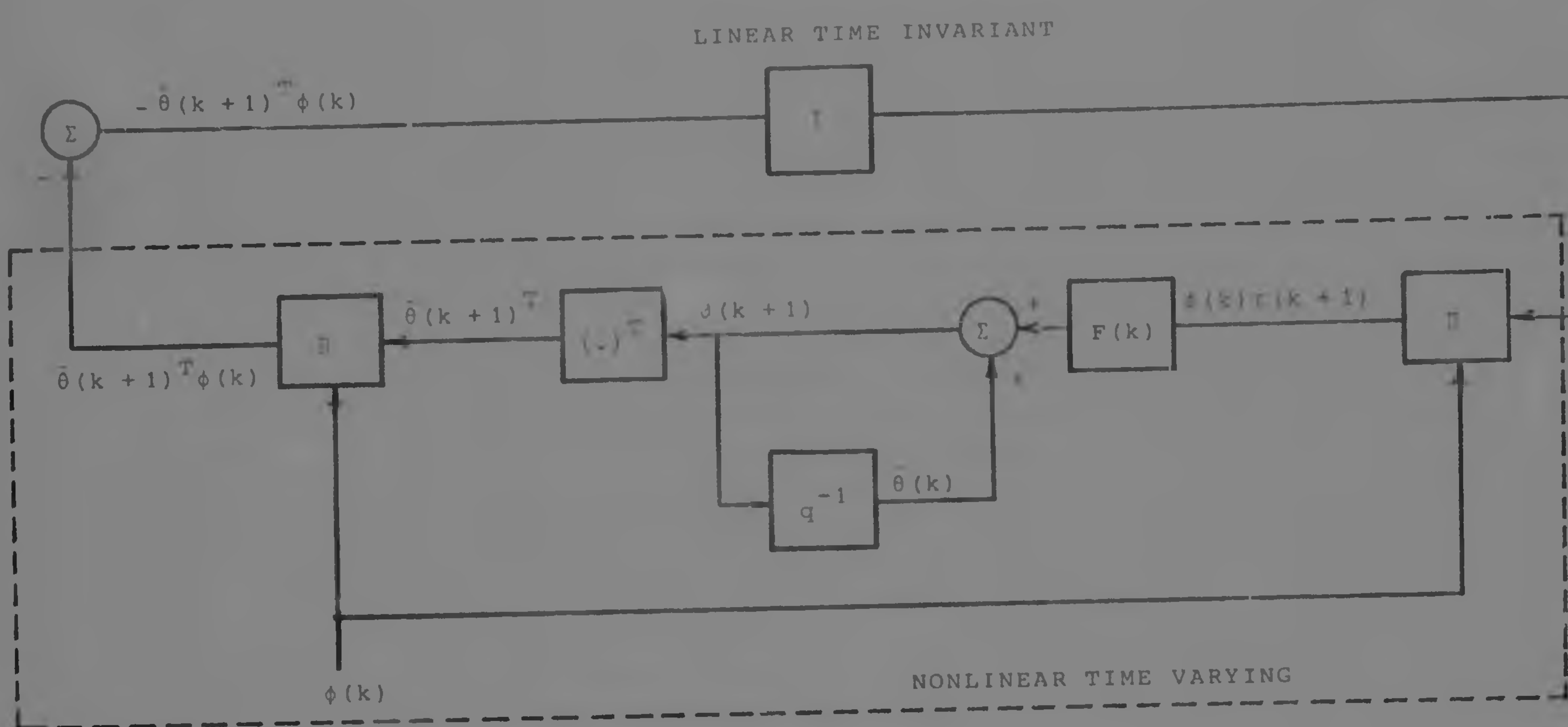

FIGURE 3.1 : Equivalent Feedback Representation for the PAA

This system may be represented by the simplified diagram shown in Figure 3.2.

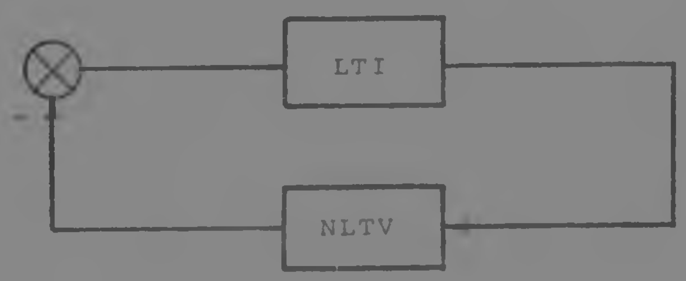

FIGURE 3.2 : Simplified Feedback Block Diagram

We thus need to examine the stability of such a system. If this system is stable then the prediction error will go to zero as $k \to \infty$ which is what we require. This analysis is non-trivial and will be dealt with in the following chapter.

3.5 Conclusion

The PAA was introduced and the RLS algorithm was developed for on line recursive estimation.

Various changes to the algorithms for updating the F matrix were suggested giving intuitive reasoning for these. However, possible problems with these changes were noted and one should proceed cautiously.

The equivalent feedback representation for PAA was introduced for the purpose of stability analysis. This is discussed in the following chapter.

CHAPTER 4

STABILITY ANALYSIS

4.1 Positive Systems

We begin the study of stability properties of the system discussed in Chapter 3, by looking at positive systems. A positive dynamic system is just the mathematical term used to describe a passive dynamic system. That is, a system which dissipates energy.

Consider the system shown in Figure 4.1.

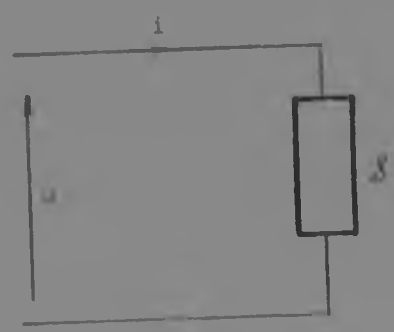

FIGURE 4.1 : Input and Output Definition for the Passive System $\mathcal{S}$

The system will be strictly passive if the energy of the system at time t is less than the initial energy plus the energy supplied.

$$E(t) < E(0) + E_s(0,t] \tag{4.1}$$

where $E_s(0,t]$ is the energy supplied from time zero up to time t.

$$E_s(0,t] = \int_0^t i(t)u(t)dt \tag{4.2}$$

If the system is discrete

$$E_s(0,kT] = \sum_{j=0}^{k} i_j u_j \tag{4.3}$$

Note in Figure 4.1 that if $\mathcal{S}$ were an active system, i would be in the opposite direction, so that E_s would be negative and not positive.

Now consider the system defined in Figure 4.2.

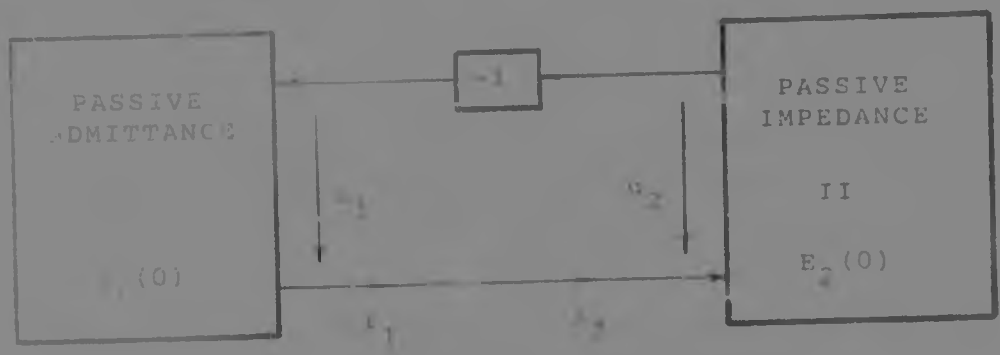

FIGURE 4.2 : Interconnection of Two Passive Systems

[illegible] it is obvious that

$$[illegible] \quad (4.4)$$

[illegible]

$$[illegible] \quad (4.5)$$

The system can be drawn as a feedback system as illustrated in Figure 4.3.

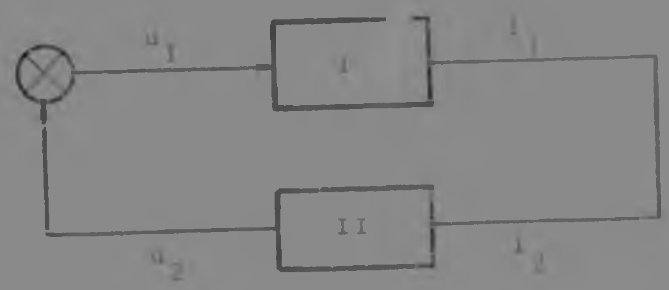

FIGURE 4.3 : Negative Feedback Representation of the Two Connected Passive Systems

The question again arises: Will this system be stable?

To answer this we consider the energy equations.

The energy supplied to each system is given by

$$E_{s,1} = \int_0^t u_1(t) i_1(t) dt \tag{4.6}$$

and

$$E_{s,2} = \int_0^t u_2(t) i_2(t) dt \tag{4.7}$$

However, from equations (4.4) and (4.5) we see that (4.6) and (4.7) imply that

$$E_{s,1} = - E_{s,2} \tag{4.8}$$

Since both systems are passive we know from (4.1) that

$$E_{s,1} > E_1(t) - E_1(0) \tag{4.9}$$

and

$$E_{s,2} > E_2(t) - E_1(0) \tag{4.10}$$

Adding these two equations and using equation (4.8) gives

$$0 > [E_1(t) + E_2(t)] - [E_1(0) + E_2(0)] \tag{4.11}$$

We define the total energy of the system at time t as

$$E(t) = E_1(t) + E_2(t) \tag{4.12}$$

Similarly for the initial energy

$$E(0) = E_1(0) + E_2(0) \tag{4.13}$$

Thus

$$0 > E(t) - E(0) \tag{4.14}$$

or $E(t) < E(0)$ $t > 0$ (4.15)

Since the origin is arbitrarily defined, we can keep moving it forward in time, and the energy at any future time will be less than the energy at this new origin. Thus E(t) is monotonically decreasing. Since this is strictly monotonically decreasing

$$E(t) \to 0$$

as $t \to \infty$. Therefore the system is asymptotically stable.

Thus the "strictly positive" condition on our systems is a sufficient condition (although not a necessary one) to ensure asymptotic stability.

We define any system H such as that shown in Figure 4.4 to be a positive system if for $\underline{x}_0 = 0$ we have

$$\int_0^t \underline{y}^T \underline{u} \, dt \geq 0 \qquad (4.16)$$

where $\underline{x}_0$ is the initial state vector.

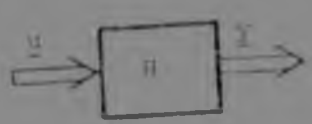

FIGURE 4.4 : Input and Output for the Generalized System H

If H is such that

$$\int_0^t \underline{y}^T \underline{u} \, dt > 0 \qquad (4.17)$$

then H is said to be a strictly positive system. It should be noted that if the strictly passive systems in the previous discussion were replaced by passive systems (i.e. the strict condition is removed) then we could not conclude that it would be asymptotically stable. However, the system would still be stable in the sense that the output is bounded and will not grow.

4.2 Positivity In Terms of Transfer Function

Consider the system defined by the equation

$$\underline{y} = P \, \underline{u} \qquad (4.18)$$

where P is a matrix

P is said to be positive definite (written P > 0) if

$$\underline{u}^T P \, \underline{u} > 0 \qquad (4.19)$$

$\forall \, \underline{u} \neq 0$

Now $\underline{u}^T P \underline{u} = \underline{u}^T \underline{y}$ (4.20)

Thus if $P > 0$

then $\underline{u}^T \underline{y} > 0$

and $\int_0^t \underline{u}^T \underline{y} \, dt > 0$

and the system is therefore strictly positive. Note that $P >$ gives $\underline{u}^T \underline{y} > 0$ at all times, not just the average value, which is more than required.

Thus a positive definite matrix is a sufficient condition to give a strictly positive system.

The class of transfer function which satisfies the inequality

$$\int_0^t \underline{u}^T \underline{y} \, dt \geq 0 \qquad (4.21)$$

is known as the class of positive real transfer function. If the inequality is strict (i.e. >) then this class is called the strict positive real transfer function (SPR).

Intuitively, if we multiply two sinusoids of similar frequency, then the product of their average is dependent on the phase lag

e.g. $u = \sin(\omega t)$

$y = \sin(\omega t + \phi)$

$$\int_0^{2\pi} uy \, d\omega t = f(\phi) \qquad (4.22)$$

$f(\phi) = 0 \qquad \phi = 0^\circ$

$f(\phi) < 0 \qquad \phi > 90^\circ$

$f(\phi) > 0 \qquad \phi < 90^\circ$

Thus if $0 < \phi < 90^\circ$ then the system is SPR. In terms of a Nyquist diagram, this requires the plot to lie in the fourth quadrant.

In Figure 4.5 H_1 is a first order system and satisfies the SPR condition. However, H_2 which is a third order system does not fulfil this condition.

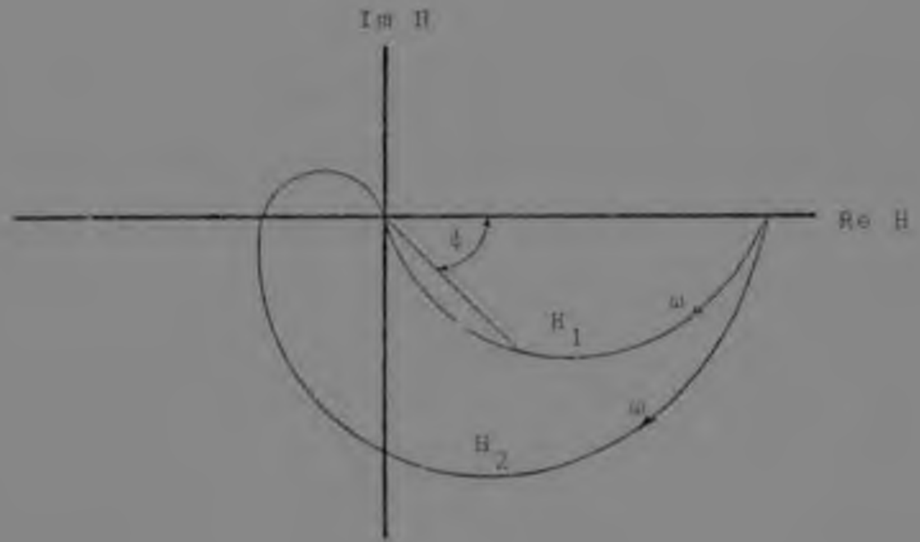

FIGURE 4.5 : Nyquist Diagram
H_1 is SPR while H_2 is not

In some non-linear cases the SPR condition is very easy to prove. Consider the input output relationship shown in Figure 4.6

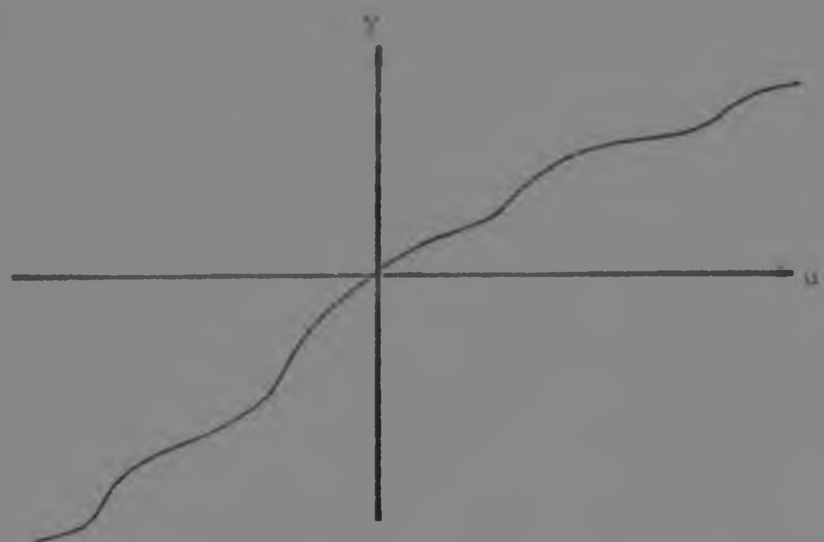

FIGURE 4.6 : SPR Input/Output relationship

Since the curve is confined to the first and third quadrants it is trivial to show that the SPR condition is satisfied.

4.3 Discrete Time Positive Systems

4.3.1 Definition

Consider the discrete system shown in Figure 4.7 where $\dim u_k = \dim y_k$

FIGURE 4.7 : Input and Output for the Discrete System H

The "energy" equation used in the previous sections now becomes

$$E_{SUPPLIED}(k_1) = E(k_1 + 1) - E(0) + E_{LOSSES}[0,k_1] \tag{4.20}$$

where k_1 is the discrete time index.

If we let $\underline{x}_k$ be the vector defining the system states at time k, then equation (4.20) can be expressed mathematically as

$$\sum_{k=0}^{k_1} \underline{y}_k^T \underline{u}_k = \alpha(k_1+1, \underline{x}_{k_1+1}) - \alpha(0, \underline{x}_0) + \sum_{k=0}^{k_1} \beta(\underline{x}_k, \underline{u}_k) \qquad (4.21)$$

where α is the system energy function and β is the energy loss function.

The system H is called a strictly positive dynamic system if

$$\exists\, \alpha(k, x_k) > 0 \qquad \forall\, k \qquad (4.22)$$

and $$\exists\, \beta(x_k, u_k) > 0 \qquad \forall\, k \qquad (4.23)$$

4.3.2 Discrete linear time invarient systems

Consider the following system

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \qquad (4.24)$$

$$\underline{y}_k = C\underline{x}_k + D\underline{u}_k \qquad (4.25)$$

We assume that (A,B) is completely controllable and that (C,A) is completely observable. Under such circumstances there is a one-to-one correspondence between equations (4.24) and (4.25), and the discrete square transfer matrix

$$H(z) = D + C(z I - A)^{-1} B \qquad (4.26)$$

The transfer matrix H(z) is said to be SPR if one of the following three conditions can be proved [2]

1 All the elements of H(z) are analytic on and outside the unit circle (i.e. all poles lie within the unit circle).

An equivalent definition is

2 H(z) is SPR if

$$\exists\, P > 0 \quad \text{and} \quad M > 0$$

where M is of the form

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

such that

$$\sum_{k=0}^{k_1} y_k^T u_k = \frac{1}{2} x_{k_1+1}^T P x_{k_1+1} - \frac{1}{2} x_0^T P x_0$$

$$+ \sum_{k=0}^{k_1} [x_k^T, u_k^T] M \begin{bmatrix} x_k \\ u_k \end{bmatrix} \qquad (4.27)$$

Thus the P matrix is used to construct the α function, and the M matrix is used for the β function.

Since the only negative term is the constant $\frac{1}{2} x_0^T P x_0$, then for all bounded x_0 and P, a SPR system will satisfy the inequality

$$\eta(0,k_1) = \sum_{k=0}^{k_1} y_k^T u_k > -\gamma_0^2 \qquad (4.28)$$

where $\gamma_0^2 < \infty$.

This is the Popov inequality.

For continuous systems, the Popov inequality is given by

$$\int_0^t y^T \underline{u}\, dt > -\gamma_0^2 \qquad (4.29)$$

In the initial development, we assumed x_0 to be 0. This would give an upper bound on γ_0 of 0 and equation (4.29) would reduce to the SPR condition given in equation (4.17). For any arbitrary x_0, the SPR condition will be given by equation (4.28) for discrete system and (4.29) for continuous systems, with

$$\gamma_0 = \underline{x}_0^T P \underline{x}_0 \qquad (4.30)$$

Thus the SPR condition is a sufficient condition for a system to satisfy the Popov inequality, but it is not a necessary condition.

The most commonly used definition is

F(z) is SPR if

$\exists\ P > 0$

$\exists\ Q, R > 0$

and $\exists\ S$ such that

$$A^T PA - P = Q \quad (4.31)$$

$$B^T PA + S^T = C \quad (4.32)$$

$$D + D^T - B^T PB = R$$

and $$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$$

Equation 4.31 is the Lyapunov equation. [illegible] systems that are not discrete LTI systems are simil[illegible] and can be found in reference [2].

4.4 Combining SPR Systems

There are three general ways of combining two SPR systems. However, a combination of SPR systems will not necessarily yield an SPR system. Each combination will now be examined.

4.4.1 Parallel systems

The general configuration is shown in Figure 4.8.

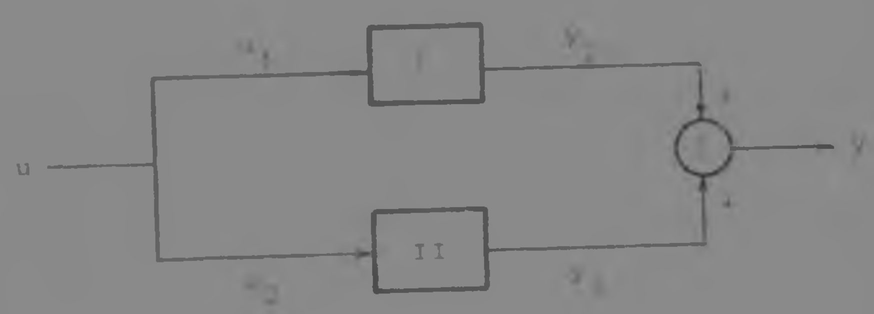

FIGURE 4.8 : Parallel Configuration of Two Passive Systems

We need to demonstrate that the combination is also SPR.

Let the initial energy in system I be γ_1^2 and the energy in system II be γ_2^2.

Then the total system energy is

$$\gamma^2 = \gamma_1^2 + \gamma_2^2 \tag{4.35}$$

$$\int_0^t uy\,dt = \int_0^t u(y_1 + y_2)dt \qquad (4.36)$$

$$= \int_0^t uy_1 dt + \int_0^t uy_2 dt$$

Since both individual systems are SPR, they each satisfy equation (4.2[illegible]). Therefore

$$\int_0^t uy\,dt > (-\gamma_1^2) + (-\gamma_2^2) \qquad (4.37)$$

Thus

$$\int_0^t uy\,dt > -\gamma^2 \qquad (4.38)$$

and the combination is SPR.

4.4.2 Feedback systems

Consider the feedback configuration shown in Figure 4.9, where both individual systems are SPR with initial energies γ_1^2 and γ_2^2 as above. The combination will still have an initial energy as defined by equation (4.35) above.

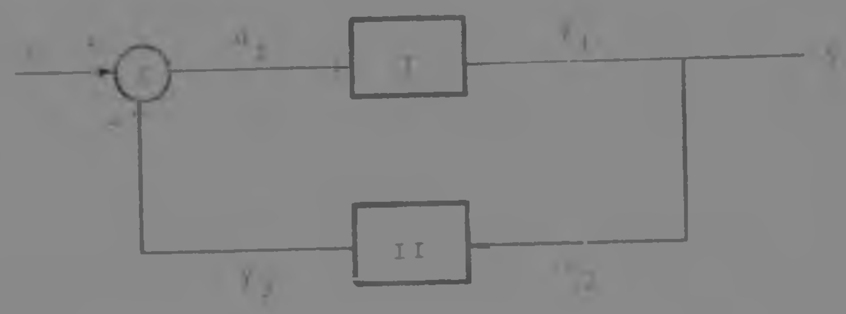

FIGURE 4.9 : Feedback Configuration of Two Passive Systems

From the figure

$$u_1 = u - y_2 \quad (4.39)$$

Thus

$$u = u_1 + y_2 \quad (4.40)$$

Also

$$y_1 = y \quad (4.41)$$

and

$$u_2 = y \quad (4.42)$$

Using these equations one gets

$$\int_0^t uy \, dt \;=\; \int_0^t (u_1 + y_2) y \, dt \tag{4.43}$$

$$= \int_0^t u_1 y_1 dt + \int_0^t u_2 y_2 dt \tag{4.44}$$

Since both systems are SPR, they both satisfy equation (4.29) giving

$$\int_0^t uy \, dt \;>\; \text{[illegible]} \tag{4.45}$$

Therefore

$$\text{[illegible]} \tag{4.46}$$

and the combination is [illegible]

4.4.3 Cascade systems

Consider the cascade system shown in Figure 4.10.

FIGURE 4.10 : Cascade Configuration of Two Passive Systems

We shall show that the combined sytem is not necessarily a SPR system by a counterexample.

Let

$$H_1 = \frac{1}{1 + sT_1} \tag{4.47}$$

and

$$H_2 = \frac{1}{1 + sT_2} \tag{4.48}$$

Both systems are first order, and have Nyquist plots confined to the fourth quadrant, and are therefore SPR.

The combined system however, has a transfer function given by

$$H = \frac{1}{(1 + sT_1)(1 + sT_2)} \tag{4.49}$$

This is the second order and has a Nyquist plot of the form shown in Figure 4.11.

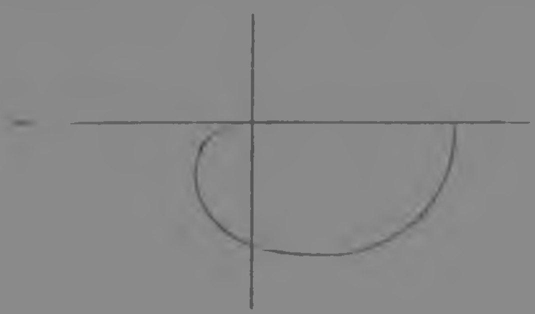

FIGURE 4.11 : Second Order Nyquist Plot

The Nyquist plot crosses the 90° line and is therefore not SPR. Thus in general cascading of SPR systems does not necessarily yield a SPR system.

4.5 The Hyperstability Theorem of Popov

Having developed the necessary mathematical background in the preceding sections, we are now able to examine the stability problem posed at the end of Chapter 3.

That is, under what conditions will the system shown in Figure 4.12 be globally asymptotically stable?

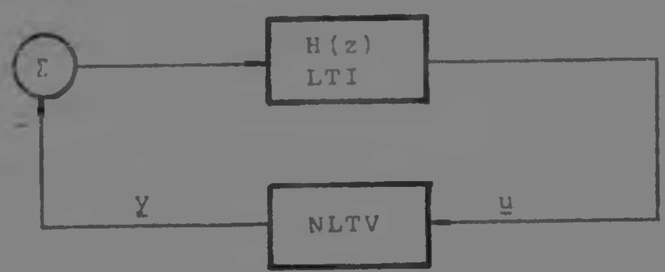

FIGURE 4.12 : Generalized Feedback System

The answer to this question is given by the hyperstability theorem (which will not be proven here) which states that if the Popov inequality is satisfied by the NLTV feedback path, i.e.

$$\eta(0,k_1) = \sum_{k=0}^{k_1} v_k^T y_k \geq -\gamma_0^2 \tag{4.50}$$

and if H(z) is SPR, then the system will be globally asymptotically stable.

This will be applied to the Parameter Adaptation Algorithm to prove convergence, in the following chapter.

4.6 Conclusions

In this chapter the concepts of SPR, Popov inequality and Hyperstability were developed. Emphasis was given to the intuitive approach of these concepts. The ideas developed were used to state the Hyperstability Theorem. This will be used in the following chapter to prove the stability of the PAA.

CHAPTER 5

STABILITY OF THE PAA

5.1 PAA with F(k) = I

We begin our analysis with the simple case of F(k) = I. The PAA has the feedback representation as shown in Figure 5.1.

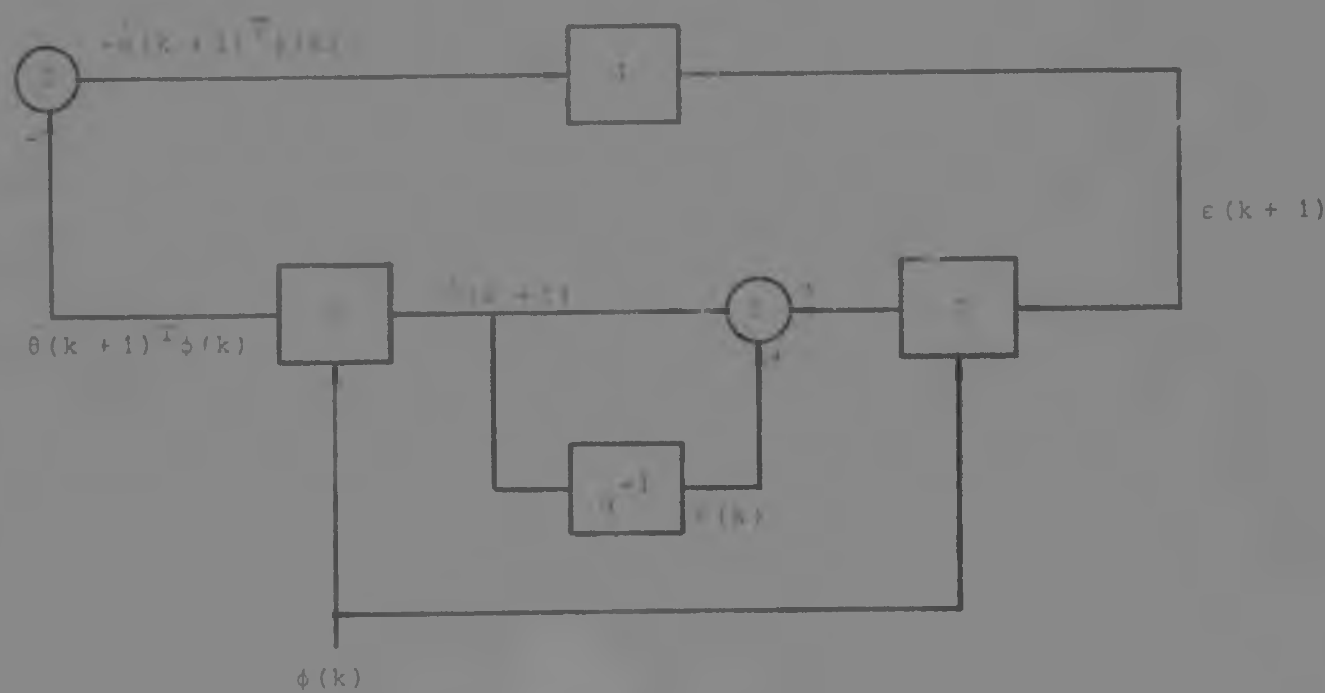

FIGURE 5.1 : PAA with F(k) = I

The LTI feedforward path is just unity, i.e.

$$H(z^{-1}) = 1 \qquad (5.1)$$

and thus obviously satisfies the SPR condition.

To verify the Popov inequality we will require the following Lemma [1]

Lemma:

Given a sequence of real vectors x(k) and a constant vector c, the following relationship holds

$$\sum_{k=0}^{k_1} x(k)^T \left[\sum_{i=0}^{k} x(i) + c \right] = \frac{1}{2} \left[\sum_{k=0}^{k_1} x(k) + c \right]^T \left[\sum_{k=0}^{k_1} x(k) + c \right]$$

$$+ \frac{1}{2} \sum_{k=0}^{k_1} x(k)^T x(k) - \frac{1}{2} c^T c \qquad (5.2)$$

Obviously this expression is $\geq - \frac{c^T c}{2}$ since all the other terms are non-negative.

Proof:

This will be done by induction.

Assume the relationship holds for $k_1 - 1$

$$\sum_{k=0}^{k_1} x(k)^T \left[\sum_{i=0}^{k} x(i) + c \right] = \sum_{k=0}^{k_1-1} x(k)^T \left[\sum_{i=0}^{k} x(i) + c \right] + x(k_1)^T \left[\sum_{k=0}^{k_1} x(k) + c \right] \qquad (5.4)$$

Since the relationship is assumed to be true for $k_1 - 1$ we get

$$\sum_{k=0}^{k_1} x(k)^T \left[\sum_{i=0}^{k} x(i) + c \right] = \frac{1}{2} \left[\sum_{k=0}^{k_1-1} x(k) + c \right]^T \left[\sum_{k=0}^{k_1-1} x(k) + c \right] + \frac{1}{2} \sum_{k=0}^{k_1-1} x(k)^T x(k) - \frac{1}{2} c^T c + x(k_1)^T \left[\sum_{k=0}^{k_1} x(k) + c \right] \qquad (5.5)$$

However

$$\frac{1}{2}\Big[\sum_{k=0}^{k_1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big] = \frac{1}{2}\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]$$

$$+ \frac{1}{2}\, x(k_1)^T x(k_1) + x(k_1)^T\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]$$

$$(5.6)$$

Also

$$\frac{1}{2}\, x(k_1)^T x(k_1) + x(k_1)^T\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]$$

$$= x(k_1)^T\Big[\frac{1}{2}\, x(k_1) + \sum_{k=0}^{k_1-1} x(k) + c\Big]$$

$$= x(k_1)^T\Big[\sum_{k=0}^{k_1} x(k) - \frac{1}{2}\, x(k_1) + c\Big]$$

$$= x(k_1)^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big] - \frac{1}{2}\, x(k_1)^T x(k_1) \qquad (5.7)$$

Substituting (5.7) into (5.6) gives

$$\frac{1}{2}\Big[\sum_{k=0}^{k_1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big] = \frac{1}{2}\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1-1} x(k) + c\Big]$$

$$+ x(k_1)^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big] - \frac{1}{2} x(k_1)^T x(k_1) \qquad (5.8)$$

Substituting (5.8) into (5.5) yields

$$\sum_{k=0}^{k_1} x(k)^T\Big[\sum_{i=0}^{k} x(i) + c\Big] = \frac{1}{2}\Big[\sum_{k=0}^{k_1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big] + \frac{1}{2} x(k_1)^T x(k_1)$$

$$+ \frac{1}{2}\sum_{k=0}^{k_1-1} x(k)^T x(k) - \frac{1}{2} c^T c$$

$$= \frac{1}{2}\Big[\sum_{k=0}^{k_1} x(k) + c\Big]^T\Big[\sum_{k=0}^{k_1} x(k) + c\Big]$$

$$+ \frac{1}{2}\sum_{k=0}^{k_1} x(k)^T x(k) - \frac{1}{2} c^T c \qquad (5.9)$$

Thus the lemma is proven to be true for k_1 if we assume it true for k_1-1. To complete our proof by induction, we must still show that it holds for the first value which is $k_1 = 0$. For this value (5.2) becomes

$$x(0)^T[x(0) + c] = \frac{1}{2}[x(0) + c]^T[x(0) + c] + \frac{1}{2}x(0)^Tx(0) - \frac{1}{2}c^Tc \qquad (5.10)$$

Expanding the right hand side of (5.10) yields

$$\frac{1}{2}x(0)^Tx(0) + \frac{1}{2}c^Tc + x(0)^Tc + \frac{1}{2}x(0)^Tx(0) - \frac{1}{2}c^Tc$$

$$= x(0)^Tx(0) + x(0)^Tc$$

$$= x(0)^T[x(0) + c] \qquad (5.11)$$

which is the left hand side of (5.10). Therefore the expression is true for $k_1 = 0$ and thus by induction, it holds for all k_1.

This will now be used to verify that the feedback path satisfies the Popov inequality.

From the block diagram we have:

$$\tilde{\theta}(k + 1) = \tilde{\theta}(k) + \phi(k)\varepsilon(k + 1) \qquad (5.12)$$

Iterating back gives

$$\tilde{\theta}(k + 1) = \sum_{i=0}^{k} \phi(i)\varepsilon(i + 1) + \tilde{\theta}(0) \qquad (5.13)$$

The input and output variables u and y are given by

$$u_k = \varepsilon(k + 1) \qquad (5.14)$$

and $$y_k = \phi(k)^T \tilde{\theta}(k + 1) \qquad (5.15)$$

(since y is a scalar, it equals its transpose).

Thus the left hand side of the Popov inequality is given by

$$\eta(0, k_1) = \sum_{k=0}^{k_1} \varepsilon(k + 1)\phi(k)^T \tilde{\theta}(k + 1) \qquad (5.16)$$

$$= \sum_{k=0}^{k_1} \varepsilon(k + 1)\phi(k)^T \left[\sum_{i=0}^{k} \phi(i)\varepsilon(i + 1) + \tilde{\theta}(0) \right]$$

(5.17)

Now let $\qquad x(k) = \phi(k)\varepsilon(k+1)$ (5.18)

and $\qquad \bar{\theta}(0) = c$ (5.19)

Thus (5.17) becomes

$$\eta(0,k_1) = \sum_{k=0}^{k_1} x(k)^T [\sum_{i=0}^{k} x(i) + c] \tag{5.20}$$

By our Lemma

$$\eta(0,k_1) \geq -\frac{\gamma_0^2}{2} \tag{5.21}$$

i.e. $$\eta(0,k_1) \geq -\frac{\theta(0)^T\theta(0)}{2} \tag{5.22}$$

Thus the Popov inequality is satisfied with

$$\gamma_0^2 = \frac{\theta(0)^T\theta(0)}{2} \tag{5.23}$$

and the system is stable and will converge.

5.2 PAA with F(k) = F

The feedback representation is given in Figure 5.2.

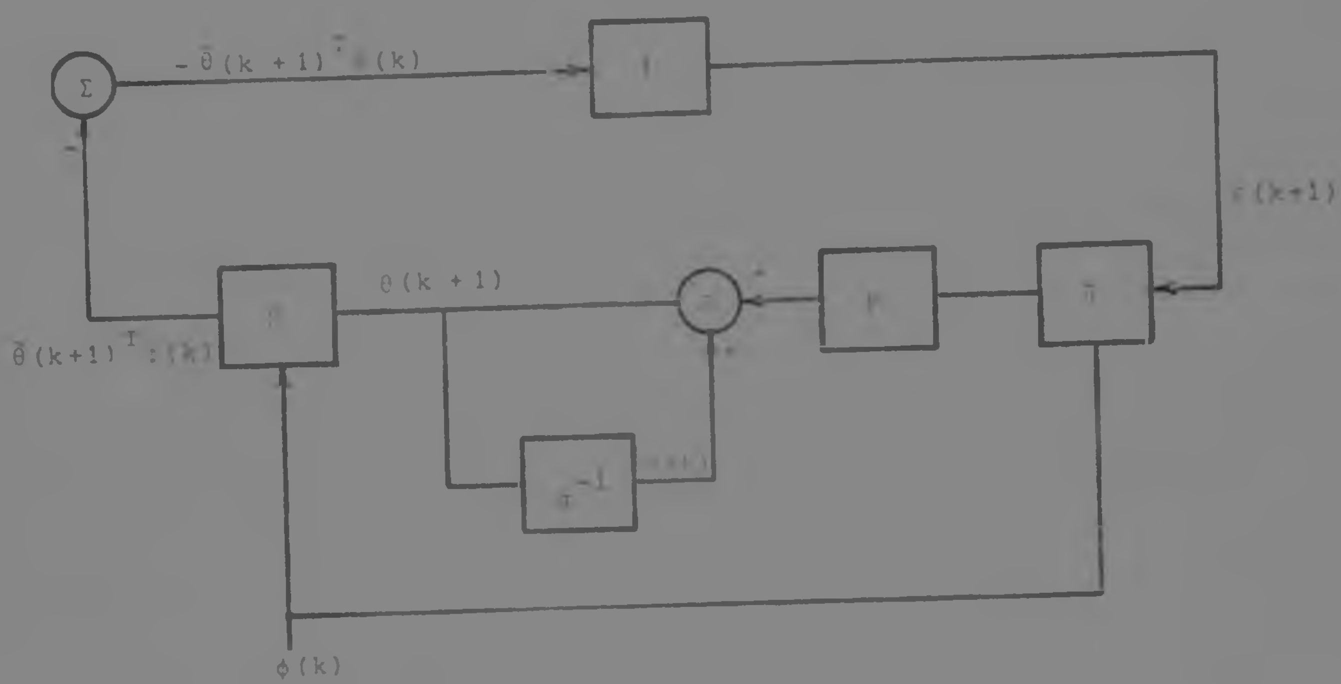

FIGURE ? : PAA with F(k) = F

From the diagram we have

$$\theta(k+1) = \theta(k) + F\phi(k)\varepsilon(k+1) \qquad (5.24)$$

Iterating back gives

$$\theta(k+1) = \sum_{i=0}^{k} F\phi(i)\varepsilon(i+1) + \theta(0)$$

The equation (5.17) of the previous case now becomes:

$$\eta(0,k_1) = \sum_{k=0}^{k_1} \varepsilon(k+1)\phi(k)^T \left[\sum_{i=0}^{k} F\phi(i)\varepsilon(i+1) + \theta(0) \right] \tag{5.25}$$

Since F is a positive definite matrix, it can be factorized as

$$F = L^T L \tag{5.26}$$

where L is a regular square matrix (L^{-1} exists)[illegible].

Define

$$\tilde{\phi}(k) = L\phi(k) \tag{5.27}$$

Substituting this into (5.25) one gets

$$\eta(0,k_1) = \sum_{k=0}^{k_1} \varepsilon(k+1)\bar{\phi}(k)^T (L^{-1})^T \left[\sum_{i=0}^{k} L^T L (L^{-1}\bar{\phi}(k))\varepsilon(i+1) + \bar{\theta}(0) \right] \tag{5.28}$$

$$= \sum_{k=0}^{k_1} \varepsilon(k+1)\tilde{\phi}(k)^T \left[\sum_{i=0}^{k} \tilde{\phi}(k)\varepsilon(i+1) + (L^T)^{-1}\tilde{\theta}(0) \right] \tag{5.29}$$

If we now define

$$x(k) = \tilde{\phi}(k)\varepsilon(k+1) \tag{5.30}$$

and

$$c = (L^T)^{-1}\tilde{\theta}(0) \tag{5.31}$$

we get

$$\eta(0,k_1) = \sum_{k=0}^{k_1} x(k)^T \left[\sum_{i=0}^{k} x(k) + c \right] \tag{5.32}$$

as before, we apply the Lemma to yield

$$\eta(0,k_1) \geq -\frac{c^T c}{2} \tag{5.33}$$

with

$$\gamma_j^2 = \frac{c^T c}{2} \tag{5.34}$$

$$= \frac{\theta(0)^T L^{-1} (L^T)^{-1} \theta(0)}{2} \tag{5.35}$$

Now if A and B are non-singular matrices, then [13]

$$(AB)^{-1} = B^{-1}A^{-1} \tag{5.36}$$

This yields

$$\gamma_0^2 = \frac{\tilde{\theta}(0)^T (L^T L)^{-1} \tilde{\theta}(0)}{2} \tag{5.37}$$

$$= \frac{\tilde{\theta}(0)^T F^{-1} \tilde{\theta}(0)}{2} \tag{5.38}$$

Thus this system is stable and will converge.

5.3 Stability of the Generalized System

We will now consider a system with

$$F(k+1)^{-1} = \lambda_1(k)F(k)^{-1} + \lambda_2(k)\phi(k)\phi(k)^T \tag{5.39}$$

We will aslo generalize the feedforward path to be a transfer function $H(q^{-1})$ instead of unity. We assume that $H(q^{-1})$ is a ratio of two monic polynomials.

$$H(q^{-1}) = \frac{H_1(q^{-1})}{H_2(q^{-1})} \tag{5.40}$$

This system is shown in Figure 5.3

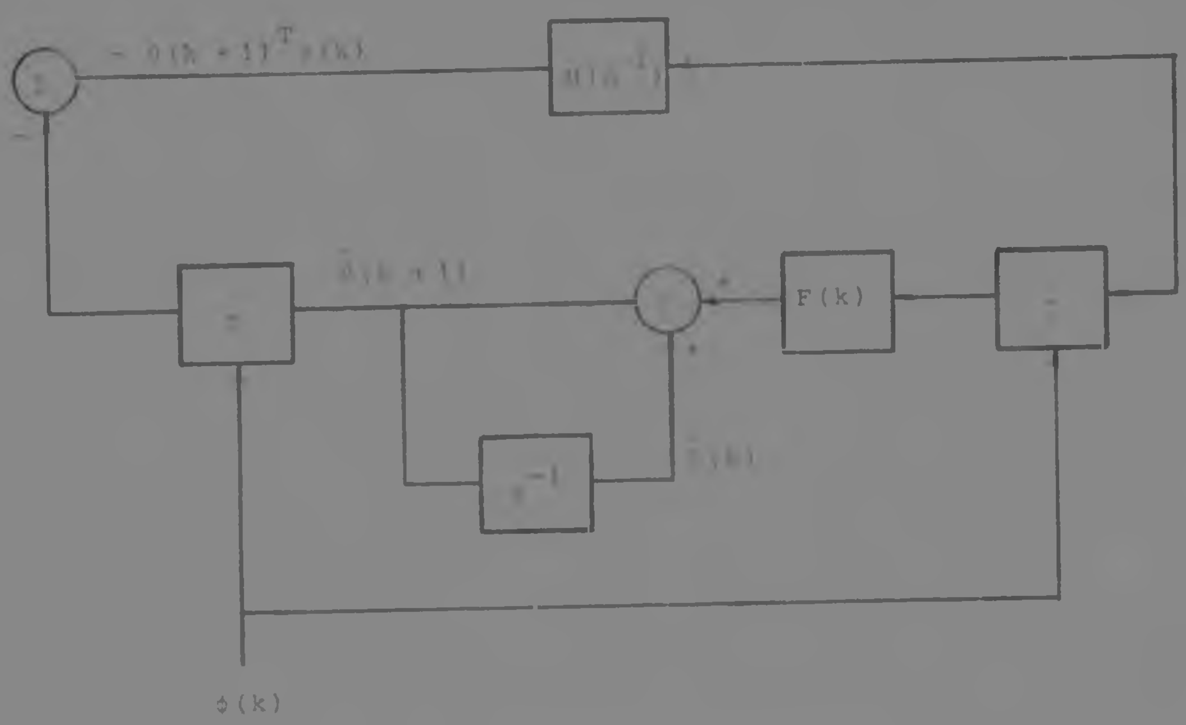

FIGURE 5.3 : Generalised Feedback System

For $\lambda_2(k) \neq 0$, the NLTV feedback path does not satisfy the Popov inequality. However, if one introduces a local feedback bath of $\frac{\lambda_2(k)}{z}$ around this loop, then the Popov inequality is satisfied [2]. To leave the system unchanged one needs a local feedforward loop of $-\frac{\lambda_2(k)}{z}$. For the case $\lambda_2(k) = \lambda_2$ (i.e. constant), this loop is obviously time invariant and can be incorporated into the LTI feedforward path. This system is shown in Figure 5.4.

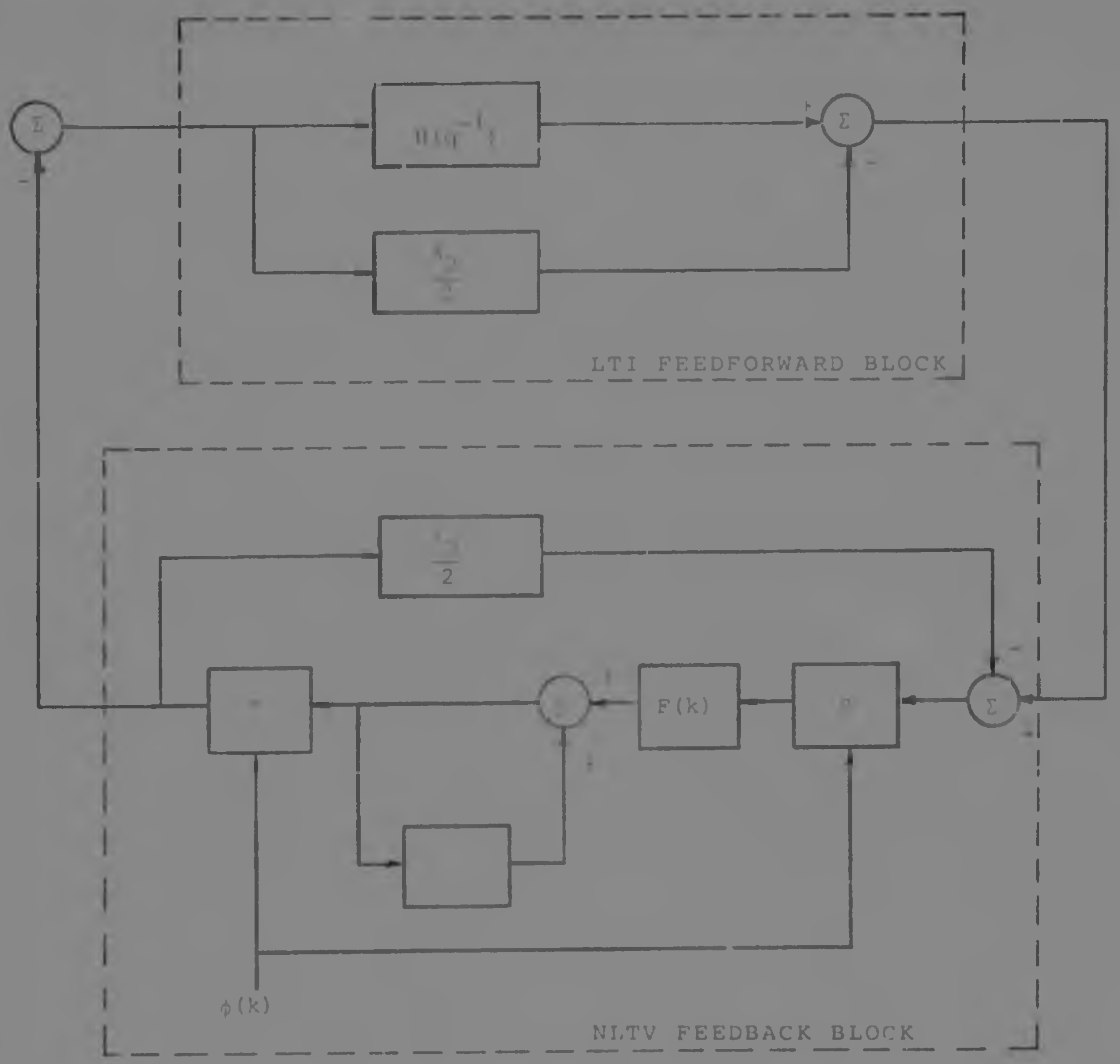

FIGURE 5.4 : Generalized system with $\lambda_{\nu}(k) = \lambda$

This system will be stable if $H(q^{-1}) - \frac{\lambda_2}{2}$ is SPR.

For a time varying $\lambda_2(k)$ we cannot incorporate the $\lambda_2(k)$ into the feedforward loop since it is time varying.

Define $\qquad \lambda = \sup_k \lambda_2(k)$ $\qquad$ (5.41)

Then $\qquad \lambda - \lambda_2(k)$ is non negative.

Now, if $\qquad \lambda < 2$ $\qquad$ (5.42)

(this is necessary since $H(q^{-1})$ is a ratio of monic polynomials [2])

and $\quad H'(z^{-1}) = H(z^{-1}) - \frac{\lambda}{2}$ $\qquad$ (5.43)

is SPR, then the system will converge since we can introduce an additional feedback path of

$$\frac{\lambda - \lambda_2(k)}{2} \qquad (5.44)$$

as shown in Figure 5.5. The feedback of $\frac{\lambda_2(k)}{2}$ causes the feedback path to satisfy the Popov inequality and this is unaffected by adding the feedback path of equation (5.44). The feedforward path now has the additional feedforward path of $\frac{\lambda}{2}$ which is time invariant. This system is shown in Figure 5.5.

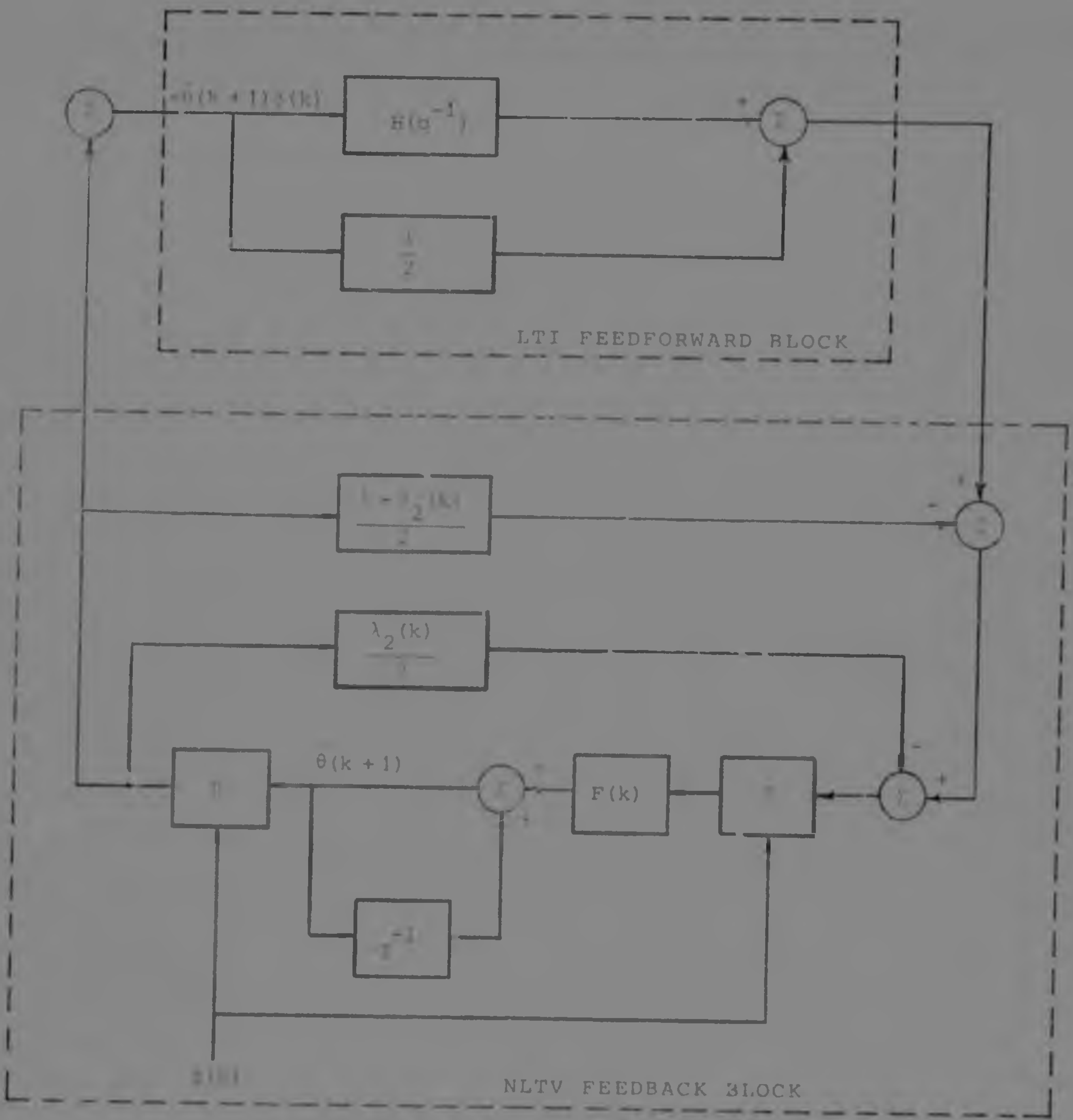

FIGURE 5.5 : Generalized system for $\lambda_2(k)$ time varying

Under such conditions, the system will converge, since the feedforward block is SPR, and the feedback path satisfies the Popov inequality.

CHAPTER 6

CONCLUSION

This thesis began with the motivation for using adaptive control techniques. Different techniques were presented and their differences and similarities were discussed superficially.

The MRAC and the STC were then discussed in some detail and the PAA was introduced. To analyse the stability of this, the theory of the hyperstability theorem was developed and subsequently applied to the PAA.

The theory behind these techniques is thus well developed and the techniques are certainly applicable to real time adaptive control. The increasing speed of microcomptuers is steadily broading the areas of applicability of these techniques.

It is the author's opinion that these controllers will soon be incorporated in most fields of control.

REFERENCES

[1] Landau, I D : Adaptive Control - Theory and Application Course Notes from Berkeley, June 1982.

[2] Landau, Y D : Adaptive Control - The Model Reference Approach, Marcel Dekker, 1979.

[3] Landau, I D : A Survey of Model Reference Adaptive Techniques - Theory and Applications, Automatica, Vol 10, No 3, pp 353-379, July 1975.

[4] Fitzgerald, R : A Gradient Method for Optimizing Stochastic Systems, PhD Thesis, MIT, May 1964.

[5] Feldbaum, A : Optimal Control System, Academic Press, New York, 1965.

[6] Anderson, B D O and Moore, J B : Optimal Filtering, Prentice Hall, New Jersey, 1979.

[7] Albert, A E and Gardner, L A : Stochastic Approximations and Non-Linear Regression, MIT Press, Mass, 1967.

[8] Wasan, M : Stochastic Approximation, University Press, Cambridge, 1969.

[9] Rabinowitz, B P : Partition Statistics in Image Processing, PhD Dissertation, Polytechnic Institute of New York, 19 1.

[10] Monopo , V : Model Reference Adaptive Control - From Theory to Practice, Proceedings of the Workshop on Applications of Adaptive Systems Theory, Yale University, May 1981.

[11] Åstrom, K J and Wittenmark, B : On Self-Tuning Regulators, Automatica, Vol 9, pp 1 5-199, 1973.

[12] Gantmacher, F R : The Theory of Matrices, Chelsea, New York, 1977.

[13] Nering, E D : Linear Algebra and Matrix Theory, Wiley, 1963.