Symmetric Colorings of Finite Groups

by

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Let G be a finite group and let $r \in \mathbb{N}$. A coloring of G is any mapping $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$. Colorings of G, χ and ψ are equivalent if there exists an element g in G such that $\chi(xg^{-1}) = \psi(x)$ for all x in G. A coloring χ of a finite group G is called symmetric with respect to an element g in G if $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$. We derive formulae for computing the number of symmetric colorings and the number of equivalence classes of symmetric colorings for some classes of finite groups.

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Chapter 1

Introduction

1.1 Literature review

Combinatorics is a vibrant area of mathematics. One of the main questions of combinatorics is to count the size of a finite set. Simple examples of these come up in trying to figure out how many digits one needs in a telephone number to service a certain sized population or how many different 7-place license plates are possible if the first three places are to be occupied by letters and the last four places by numbers.

Many combinatorial concepts have also appeared throughout the ancient world. Around 6th BC, an Indian mathematician showed that there are 63 combinations that can be made out of 6 different tastes. During the 19th and 20th century, combinatorics enjoyed a rapid growth which led to the birth of many subfields, in particular Ramsey theory. Ramsey theory deals with finding order amongst apparent chaos. This means that given a mathematical structure of interest, we can identify conditions on this setting under which our mathematical structure of interest must appear. If one puts 5 points on a plane so that no three points are in a straight line, one can always find a convex 4-gon. A convex 4-gon is a quadrilateral and as one moves around the quadrilateral, one will always turns left and never turn right. The mathematical generalisation of this result is named the Happy End Problem after its proposer and solver, Ester Kline and George Snekeres who married shortly after solving it. Ramsey theory was pioneered by the English mathematician called Frank Ramsey in 1928 by publishing a paper in which he proved what would be known as Ramsey Theorem. The paper showed that in a group of 6 people, it is guaranteed that either three of them will be mutual friends or three of them will be mutual strangers.

This dissertation is about graph colorings or colorings. Coloring is an assignment of labels traditionally called colors to elements of a graph subject to certain constraints. In simple terms, this dissertation is about colorings of regular polygon (n-gon). It is obvious that there are r^n colorings of n-gon using r colors. Two colorings are equivalent if we can get one from another by rotating about the regular n-gon center. Clearly, the relation is an equivalence relation. Each equivalence relation is called a necklace. To compute the number of necklaces, Burnside's Lemma (see [4]) is used. Burnside's Lemma is a result in mathematics which is often useful in taking account of symmetry when counting mathematical objects. The lemma was apparently first stated by Cauchy in 1845. Hence it is also called the Cauchy-Frobenius Lemma, or the lemma that is not Burnside's. The lemma was (mistakenly) attributed to Burnside because he quoted and proved in his 1897 book Theory of groups of finite order without attribution, apparently because he thought it was sufficiently well known.

We define coloring as symmetric if it is invariant with respect to some mirror symmetry with an axis crossing the center of the regular n-gon and one of its vertices. If a coloring is symmetric, then every coloring equivalent to it will be also symmetric. The number of equivalence classes of symmetric colorings(or symmetric necklaces) cannot be computed using the Burnside Lemma since not all colorings are symmetric. The technique (see [10]) or method that is used to compute the number of equivalence classes of symmetric colorings was developed by Yuliya Zelenyuk. The technique is based on constructing the partially ordered set of so called optimal partitions.

In mathematics, especially order theory, a partially ordered set (or poset) formalizes and generalizes the intuitive concept of an ordering, sequencing, or arrangement of the elements of a set. A partition of a set is a nonempty subset such that every element is in exactly one of these subsets. Given two partitions of a particular set with identical mathematical structure of interest and that every cell of one partition is contained in some cell of the other, the one that is contained in the other is called the optimal partition of a given set.

Recall that we defined a coloring as symmetric if it is invariant with respect to some mirror symmetry with an axis crossing the center of the regular n-gon and one of its vertices. Our definition of a symmetric coloring of is very restrictive. It only gives us the reflections with respect to an axis through the center and one of the vertices of the regular n-gon. If n is even it captures only half reflections of the regular n-gon. Our goal is to captures all the reflections of the regular n-gon, that is, we want to define symmetric colorings of as colorings which are invariant with respect to some mirror symmetry. At first the redefining of our symmetric colorings posed a challenge to mathematicians who were researching on this topic. It was Yehven Zelenyuk together with Yuliya Zelenyuk who managed to solve the problem (see [13]). Their method was much simplier that the complicated approach taken by Yuliya Zelenyuk in computing symmetric colorings of necklaces..

An r-ary bracelet of length n is an equivalence class of r-colorings of vertices of a regular n-gon, taking all rotations and mirror symmetries as equivalent. In (see [13]) it was shown that the number of symmetric r-ary bracelets of length n is $\frac{1}{2}(r+1)r^{\frac{n}{2}}$ if n is even, and $r^{\frac{n+1}{2}}$ if n is odd. The formula for counting of symmetric colorings of the vertices of a regular polygon was obtained in (see [14]).

Recall that our approach in this section was based on regular polygon. In essence, the rotational symmetries of a regular polygon forms a mathematical object called a group, the cyclic group to be specific. All symmtries of a regular polygon form a group called a dihedral group. So in the coming chapters, we will take a more robust approach using sophisticated mathematical objects. This was just a more intuitive approach to the subject.

1.1.1 Applications of Colorings

Graph coloring problems arise in several computer science disciplines. One of which is register allocation during code generation in a computer programming language compiler. In case you are not a computer scientist, a compiler is a program that translates a programming language to the native low level instructions that the CPU can execute. The CPU has several layers of memory. The computer can store data in these layers, all of which have different sizes and different access times. Three types of storage are: hard drive, RAM, and CPU registers which is a sort of small but very fast RAM which is placed physically on the CPU. Access to the hard drive is a couple of orders of magnitude slower than access to RAM. Likewise the RAM is orders of magnitude slower to access than registers.

So if you need rapid access to some data in your program, you might choose to save it in the RAM instead of on the hard drive. If it will fit (remember, registers are small) you can save it in a register and speed up the access time even more. The CPU has a fixed number of registers, so the compiler may try to optimize the usage of registers to speed up the program. This translates into a graph coloring problem, where you need the graph to be k-colorable (A k-coloring of a graph is a vertex coloring that is an assignment of one of k possible colors to each vertex of a graph such that no two adjacent vertices receive the same color) for a CPU with k registers.

A fisherman has six different types of fish, namely A, B, C, D, E and F. Because of predator-prey relationships, water conditions and size, some fish can be kept in the same tank. Fish A cannot be with fish B and C, B cannot be with A, C and E. C cannot be with A, B, D and E, D cannot be with C and F. E cannot be with B, C and F, F cannot be with D and E. What is the smallest number of tanks needed to keep all the fish? So if each vertex represents one of the types of fish and each edge connects vertices that are not compatible, then each color on the vertices represent the tanks. In this problem, it turns out that the smallest number of tanks needed to keep all the fish is three. This is due to the use of chromatic polynomials (It counts the number of graph colorings as a function of the number of colors).

Assume that we have a number of radio stations, identified by x and y coordinates in the plane. We have to assign a frequency to each station, but due to interferences, stations that are close to each other have to receive different frequencies. Such problems arise in frequency assignment of base stations in cellular phone networks. At first sight, one might think that the conflict graph is planar in this problem, and the Four Color Theorem can be used, but it is not true: if there are lots of stations in small region, then they are all close to each other, therefore they form a large clique(clique is a subset of its vertices such that every two vertices in the subset are connected by an edge) in the conflict graph. Instead, the conflict graph is a unit disk graph, where each vertex corresponds to a disk in the plane with unit diameter, and two vertices are connected if and only if the corresponding disks intersect. A 3-approximation algorithm for coloring unit disk graphs is given in, yielding a 3-approximation for the frequency assignment problem.

1.1.2 Aims and objectives of the dissertation

There are two main objectives of this dissertation.

(1) We aim to compute the number of symmetric colorings of some classes of finite groups.

(2) We are also searching for a general formula for computing the number of symmetric colorings of an arbitrary dihedral group.

1.1.3 Outline of the dissertation

The outline of the dissertation is as follows.

(1) In Chapter 2, basics in group theory are presented.

(2) Background in the symmetric colorings of finite groups is presented in Chapter 3.

(3) From Chapter 4 until Chapter 8 we compute symmetric colorings of some classes finite groups.

(4) In Chapter 9 we generalize the definition of a symmetric coloring.

(5) In Chapter 10 we will summarize our finding and provide a conclusion.

Chapter 2

Group theory

2.1 Introduction

In this chapter we will present basic theory of groups and posets which will be used in our dissertation.

2.2 Homomorphism and isomorphism

Group theory can be considered the theory of symmetry. The collection of symmetries of some object preserving some of its structure forms a group. The formal definition is given below.

Definition 2.2.1 A group consists of a non-empty set G, together with a rule for combining any two elements a, b of G to form another element of G, written $a \cdot b$. This rule must satisfy the following axioms:

(1) For all $a, b, c \in G$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative).

(2) There exist an element 1 in G called an identity of G such that for all a in G, we have $a \cdot 1 = 1 \cdot a = a$.

(3) For each a in G, there is an element a^{-1} in G called an inverse of a such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

We usually omit the dot and write ab for $a \cdot b$. There is no requirement that $ab = ba$ for all a, b in G. If this property holds, we say that G is abelian, non-abelian if the property does not hold. The following proposition shows that the identity and inverse of a group are both unique (see [6]).

Proposition 2.2.2 (see [4]) If G is a group, then

(1) The identity of G is unique

(2) For each a in G, a^{-1} is unique

(3) $(a^{-1})^{-1} = a$ for all a in G

(4) $(ab)^{-1} = b^{-1}a^{-1}$ for all a, b in G.

Definition 2.2.3 A group G is finite if G is a finite set. In this case, the number of elements in G is called the order of G and is denoted by $|G|$.

Definition 2.2.4 The order of an element a in a group G is the smallest positive integer n such that $a^n = 1$. The order of an element a is denoted by $|a|$.

Definition 2.2.5 A non-empty subset H of a group G, which is itself a group under the same operation is called a subgroup of G.

We use the notation $H \leqslant G$ to mean that H is a subgroup of G. When determining whether or not a subset H of G is a subgroup of G, one does not necessary need to directly verify the group axioms. The next proposition provides the simple test suffice to show that a subset of a group is a group (see [7]).

Proposition 2.2.6 (*Subgroup criterion*)(see [4]) Let G be a group and H a non-empty subset of G. If ab^{-1} is in H whenever a and b are in H, then H is a subgroup of G.

Example 2.2.7 **(1)** There are many group examples, the *classical dihedral group* D_n, the *quartenion group* Q_8, the *alternating group* A_4 and the group of integers modulo n ($\mathbb{Z}_n$) (see [3]). The classical dihedral group is the group of symmetries of a regular polygon including both rotations and reflections. The quaternion group is used to define a number system that extends the complex number system.

(2) Let G be an abelian group with identity 1. $B(G) = \{x \in G : x^2 = 1\}$ is a subgroup of G. When $B(G) = G$, G is called a *Boolean group*. We first note that $1 \in B(G)$, so $B(G)$ is non-empty. Let $a, b \in B(G)$. This means $a^2 = 1$ and $b^2 = 1$. Since G is abelian, $(ab^{-1})^2 = ab^{-1}ab^{-1} = a^2b^{-2} = 1$, hence $ab^{-1} \in B(G)$. By the subgroup criterion, $B(G)$ is a subgroup of G.

(3) Given $\mathbb{Z}_n$, $B(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : 2x \equiv 0 \mod(n)\}$. $2x \equiv 0 \mod(n)$ has $\gcd(2, n)$ incongruent solutions. To be precise, $B(\mathbb{Z}_n) = \{0\}$ when n is odd and $B(\mathbb{Z}_n) = \{0, \frac{n}{2}\}$ when n is even.

This results from the last example will play an important role in the later chapters.

We are about to show the notion of when two groups "'look the same", that is, have the same group-theoretic structure. This notion is known as isomorphism between two groups.

Definition 2.2.8 Let G and H be groups. A map $\varphi : G \longrightarrow H$ such that

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in G$ is called a homomorphism[1].

Definition 2.2.9 The kernel of a homomorphism $\varphi : G \longrightarrow H$ is the set $\{g \in G : \varphi(g) = 1\}$. The kernel is denoted by $ker\varphi$.

[1]It is important to keep in mind that the product ab is computed in G and the product $\varphi(a)\varphi(b)$ is computed in H.

$ker\varphi$ is a subgroup of G. Clearly, $1 \in ker\varphi$. Let $a, b \in ker\varphi$, then

$$\begin{aligned}\varphi(ab^{-1}) &= \varphi(a)\varphi(b^{-1}) \\ &= \varphi(a)\varphi(b)^{-1} \\ &= 1\end{aligned}$$

hence $ab^{-1} \in ker\varphi$.

Definition 2.2.10 The map $\varphi : G \longrightarrow H$ is called isomorphism and G and H are said to be isomorphic, written $G \cong H$, if:

(1) φ is a homomorphism

(2) φ is a bijection.

For any group G, $G \cong G$. This is because we can create an identity map on G, which is obviously an isomorphism. Apart from this isomorphism, there might be other isomorphisms on G. An isomorphism from G to itself is called an *automorphism* and the set of all automorphisms of G is denoted by Aut(G). Aut(G) is a group under composition of automorphisms.

Let G be a group and define $\varphi : G \longrightarrow G$ by $\varphi(g) = g^{-1}$. φ is known as *inversion* of G. The following proposition makes the inversion of G an automorphism provided G is an abelian group.

Proposition 2.2.11 Let G be a group and define $\varphi : G \longrightarrow G$ by $\varphi(g) = g^{-1}$. The function φ is an automorphism if and only if G is abelian.

Proof: Let $g \in G$ and φ^2 satisfies

$$\begin{aligned}\varphi^2(g) &= \varphi(\varphi(g)) \\ &= \varphi(g^{-1}) \\ &= g\end{aligned}$$

thus, φ is a bijection. Let φ be a homomorphism, now $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$, hence $\varphi(gh) = g^{-1}h^{-1} = (hg)^{-1} = \varphi(hg)$, therefore $gh = hg$. Conversely, let G be an abelian, then $\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \varphi(hg)$. ■

2.3 Generators and relations

In this subsection, we present a convenient way to define groups.

Definition 2.3.1 Let S be a subset of elements of a group G such that every element of G can be written as a finite product of elements of S and their inverses, then S is called a set of generators of G and is denoted by $\langle S \rangle$.

The equations in a group G that the generators satisfy are called *relations* in G. In D_n, we have relations $a^n = 1$, $b^2 = 1$ and $ab = ba^{-1}$. These three relations have the property that any other relation between elements of the group can be derived from these three.

In general, if a group is generated by a subset S and there is some collection of relations, say R_1,R_2,...,R_m such that any relation among the elements of S can be deduced from these, we shall call these generators and relations a *presentation* of G and write

$$G = \langle S : R_1, R_2, ..., R_m \rangle$$

Presentation for D_n is given by $D_n = \langle a, b : \ a^n = 1, b^2 = 1, ab = ba^{-1} \rangle$. It is also possible to begin with an set of generators and relations and construct a group that is uniquely described by these generators and relations, subject to the stipulation that all other relations among the generators can be derived from the original ones. The following proposition (see [4]) completely determines when two presentations are isomorphic.

Proposition 2.3.2 (see [4]) If K is a group satisfying the defining relations of a finite group G and $|K| \geq |G|$, then K is isomorphic to G.

From proposition 1.1.13, we can deduce that $Q_8 \cong \langle a, b : \ a^4 = b^4 = 1, bab = a, a^2 = b^2 \rangle$.

2.4 Group actions

In this subsection, we focus on group action on a set of which is a description of symmetries of objects using groups.

Definition 2.4.1 Let G be a finite group. A $left$ group action of a group G on a non-empty set A is a map from $G \times A$ to A (written as $g \cdot a$ for all $g \in G$ and $a \in G$) satisfying the following two axioms:

(1) $g \cdot (h \cdot a) = (gh) \cdot a$ for all $g, h \in G$ and $a \in A$

(2) $1 \cdot a = a$ for all $a \in A$.

We then say that G acts on A. We can similarly define the notion of *right* action.

Definition 2.4.2 Let G be any group, let A be non-empty set and let G act on A. If $a \in A$, the orbit of a under G is defined by

$$[a]_G = \{g \cdot a : g \in G\}$$

Definition 2.4.3 Let G be any group, let A be non-empty set and let G act on A. If $a \in A$, the stabiliser of a is defined by

$$St(a) = \{g \in G : g \cdot a = a\}$$

We can easily show that $St(a) \leqslant G$. Firstly 1 is in $St(a)$, by axiom (2) of an action. If $b \in St(a)$, then by the axioms of an action

$$a = 1 \cdot a$$

$$\begin{aligned} &= (b^{-1}b) \cdot a \\ &= b^{-1} \cdot (b \cdot a) \\ &= b^{-1} \cdot a \end{aligned}$$

then $b^{-1} \in St(a)$. Clearly, if $x, y \in St(a)$, then $xy \in St(a)$. Thus $St(a)$ is a subgroup of G.

2.5 Cyclic groups

Let G be any group and let a be any element of G. One way of forming a subgroup H is by letting H be the set of all integer powers of a. This section is concerned with such groups generated by one element.

Definition 2.5.1 A group H is cyclic if H can be generated by a single element and we write $H = \langle a \rangle$, where a is the generator.

One can show from the above definition that all cyclic groups are abelian. A cyclic group may have more than one generators, for example, if $H = \langle a \rangle$, then $H = \langle a^{-1} \rangle$.

Example 2.5.2 **(1)** Let $D_n = \langle a, b : \; a^n = b^2 = 1, ab = ba^{-1} \rangle$, let H be the subgroup of all rotations of a regular poygon, thus $H = \langle a \rangle$

(2) Let $G = \mathbb{Z}_n$. This group is generated by 1 and -1, thus $\mathbb{Z}_n = \langle 1 \rangle = \langle -1 \rangle$. There are other generators of $\mathbb{Z}_n$, in particular $\mathbb{Z}_8 = \langle 1 \rangle = \langle 3 \rangle = \langle 5 \rangle = \langle 7 \rangle$.

Finite cyclic groups are completely classified up to isomorphism using the following proposition (see [4]).

Proposition 2.5.3 Let G be a finite group of order n. If G is cyclic group, then $G \cong \mathbb{Z}_n$.

The next theorem tell us how many subgroups a finite cyclic group has and how to find them (see [4]).

Theorem 2.5.4 (*Fundamental Theorem of Cyclic Groups*) Let $H = \langle a \rangle$ be a cyclic group.

(1) Every subgroup of H is cyclic.

(2) For each positive integer m dividing n there is a unique subgroup of order m. This subgroup is the cyclic group $\left\langle a^d \right\rangle$, where $d = n/m$.

Example 2.5.5 The list of subgroups of the cyclic group $\mathbb{Z}_{20}$ is

$$\begin{aligned} \langle 1 \rangle &= \mathbb{Z}_{20} \\ \langle 2 \rangle &= \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\} \cong \mathbb{Z}_{10} \\ \langle 4 \rangle &= \{0, 4, 8, 12, 16\} \cong \mathbb{Z}_5 \\ \langle 5 \rangle &= \{0, 5, 10, 15\} \cong \mathbb{Z}_4 \\ \langle 10 \rangle &= \{0, 10\} \cong \mathbb{Z}_2 \\ \langle 20 \rangle &= \{0\} \cong \mathbb{Z}_1 \end{aligned}$$

2.6 Quotient groups

In this subsection, we will prove the single most important theorem in finite group theory, Lagrange's Theorem. But first, we introduce a powerful tool for analyzing a group, the notion of a coset. Cosets are useful in perfoming counting arguments for finite groups and cosets of a subgroup sometimes enable us to construct a new group from an old.

Definition 2.6.1 Let G be a group and H a subgroup of G. Then a *right* coset of H in G is a subset of the form $Ha = \{ha : h \in H\}$ for some $a \in G$. We define a *left* coset of H in G to be a subset of the form $aH = \{ah : h \in H\}$ for some $a \in G$.

Proposition 2.6.2 Let H be a subgroup of G. Then the *right*(*left*) cosets of H in G form a partition of H in G and any pair of distinct cosets has empty intersection.

Proof: Each element of G occurs in at least one left coset. For if $g \in G$, then $g \in gH$. Let aH and bH be two letf cosets of H in G and suppose that $aH \cap bH \neq \emptyset$. Let $g \in aH \cap bH$, then $g = ah_1 = bh_2$ for some $h_1, h_2 \in H$. Hence $a = bh_2h_1^{-1}$. Therefore $aH \subseteq bH$. Similarly, $bH \subseteq aH$. Thus $aH = bH$. ■

We mentioned that cosets are useful in counting arguments. We also know from proposition 1.1.23 that If G is of finite order and H a subgroup of G, then the cosets of H in G are disjoint. Therefore, the order of G is the sum of the number of elements in each coset. This is useful in proving the Lagrange's Theorem (see [7]).

Theorem 2.6.3 (*Lagrange's Theorem*) The order of a subgroup H of a finite group G divides the order of G.

Proof: Let the distinct cosets of H in G be g_1H , g_2H, ... , g_nH. Since these form a partition of G, then

$$|G| = |g_1H| + |g_2H| + ... + |g_nH|$$

Let $g \in G$ and let $\theta : H \longrightarrow gH$ be defined by $\theta(h) = gh$. Clearly, θ is a bijection. Therefore $|H| = |gH|$. Hence $|H|$ divides $|G|$. ■

Definition 2.6.4 Let the number of $left(right)$ cosets of H in G be called the index of H in G. Denote it by $|G : H|$.

The following corollary follows from definition 1.1.25 and Lagrange's Theorem.

Corollary 2.6.5 Let H be a subgroup of G. Then $|G| = |H||G : H|$.

Another important result in this subsection is Burnside's Lemma (see [3]). But before we prove it, we need the following Lemma (see [4]).

Lemma 2.6.6 Let G be a finite group acting on a non-empty set A. The number of elements in equivalence class containing $a \in A$ is $|G : St(a)|$.

Lemma 2.6.7 (*Burnside's Lemma*) Let G be a finite group that acts on a non-empty set A. For each g in G, let A^g denote the set of all elements of A left fixed by element g in G. The number of equivalence classes(orbits) is

$$\frac{1}{|G|}\sum_{g\in G}|A^g|$$

Proof: Let us denote the number of orbits by A/G. Clearly

$$\sum_{g\in G}|A^g| = \sum_{s\in A}|St(s)|$$

which is given by

$$\sum_{[s]_G\in A/G}\sum_{x\in[s]_G}|St(x)|$$

Now we need to show that

$$\sum_{x\in[s]_G}|St(x)| = |G|$$

Let $x, y \in [s]_G$ then $g\cdot x = y$ for some $g \in G$ and now let $\varphi : St(x) \longrightarrow St(y)$ be $\varphi(a) = gag^{-1}$. Clearly $gag^{-1} \in St(y)$ since $(gag^{-1})\cdot y = (ga)\cdot(g^{-1}\cdot y)$ of which is equal to $(ga)\cdot x$ and this is leads to $g\cdot(a\cdot x) = g\cdot x = y$.
Let $\varphi(a) = \varphi(b)$, then $gag^{-1} = gbg^{-1}$ hence $a = b$, therefore φ is injective. Clearly φ surjective thus φ is bijective. All these implies that $|St(x)| = |St(y)|$, therefore

$$\sum_{x\in[s]_G}|St(x)| = |St(j)||[s]_G|$$

where $j, s \in A$. By Lemma 1.1.27, it follows that

$$\sum_{x\in[s]_G}|St(x)| = |G|$$

So

$$\sum_{[s]_G\in A/G}\sum_{x\in[s]_G}|G_x| = \sum_{[s]_G\in A/G}|G| = |G|\sum_{[s]_G\in A/G}1$$

It then follows that

$$\frac{1}{|G|}\sum_{g\in G}|A^g|$$

is number of orbits ■

Definition 2.6.8 Let G be a group and H a subgroup of G. We say that H is normal in G if $gH = Hg$ for all $g \in G$. Write $H \lhd G$ when H is normal in G.

If G is abelian, then every subroup of G is normal and we always have $1 \lhd G$ and $G \lhd G$. Let us denote by G/H the set of left(right) cosets of H in G. We can turn G/H into a group using the following thereom (see [4]).

Theorem 2.6.9 Let H be a subgroup of G. The following are equivalent:

(1) $H \lhd G$

(2) $gH = Hg$ for all $g \in G$

(3) $gHg^{-1} \subseteq H$ for all $g \in G$

(4) $(aH)(bH) = (ab)H$ for all $a, b \in G$.

Let G be a group and let H be a normal subgroup. We can make G/H a group by using cosets multiplicaton rule $(aH)(bH) = (ab)H$ from (4) of the preceding result. The group G/H is called a *quotient group*. Another thereom that relies on the notion of cosets and quotient groups is the first isomorphism theorem.

Theorem 2.6.10 $(First\ Isomorphism\ Theorem)$ If $\varphi : G \longrightarrow H$ is homomorphism of groups, then $ker\varphi \lhd G$ and $G/ker\varphi \cong \varphi(G)$.

Proof: Let $a \in ker\varphi$ and let $g \in G$, then

$$\begin{aligned}\varphi(gag^{-1}) &= \varphi(g)\varphi(a)\varphi(g)^{-1} \\ &= \varphi(g)\varphi(g)^{-1} \\ &= 1.\end{aligned}$$

So $gag^{-1} \in ker\varphi$. By theorem 1.1.30, $ker\varphi \lhd G$.
Let $\theta : G/K \longrightarrow H$ be defined by $\theta(gK) = \varphi(g)$ for all $g \in G$, where $K = ker\varphi$. Let $a, b \in G$ and $\varphi(a) = \varphi(b)$, then $\varphi(ab^{-1}) = 1$ if and only if $ab^{-1} \in K$ and $ab^{-1} \in K$ if and only if $aK = bK$. Thus we see that θ is well-defined and injective. θ is a group homomorphism, since $ker\varphi \lhd G$. Hence $G/ker\varphi \cong \varphi(G)$. ■

The full converse of the Lagrange's Theorem is not true, meaning, if G is a finite group and n divides $|G|$, then G need not have a subgroup of order n. Let A_4 be the alternating group of order 12. Let H be a subgroup of A_4 of order 6. Since $[A_4 : H] = 2$, then $H \lhd A_4$. Now, let $g \in A_4$, then $g^2 \in H$. Let $g \in A_4$ be an element of order 3, then $g \in H$. A_4 has 8 elements of order 3 and $|H| = 6$. This is a contradiction. A partial converse which holds for arbitrary finite groups is the following result (see [4]).

Theorem 2.6.11 (*Cauchy Theorem*) If G is a finite group and p is a prime dividing $|G|$, then G has element of order p.

2.7 Direct products

In this subsection, we consider two of the easier methods for constructing larger groups from smaller ones. We begin with the definition of the direct product of groups.

Definition 2.7.1 The direct product $G_1 \times G_2 \times ... \times G_n$ of the groups $G_1, G_2, ..., G_n$ is the set n-tuples for which the ith component is an element of G_i and the operation is componentwise. In symbols

$$G_1 \times G_2 \times ... \times G_n \quad = \quad \{(g_1, g_2, ..., g_n) : g_i \in G_i\}$$

where $(g_1, g_2, ..., g_n)(h_1, h_2, ..., h_n)$ is defined to be $(g_1h_1, g_2h_2, ..., g_nh_n)$. It is understood that each product g_ih_i is performed with operation of G_i.

Let $G = G_1 \times G_2 \times ... \times G_n$. The proof that the group axioms hold for G is very easy since each axiom is the consequence of the fact that the same axiom holds in each factor G_i. Clearly, G is a group of order $|G_1||G_2|...|G_n|$ and also note that the direct product of groups is abelian if and only if each of the factors is abelian. Our first proposition gives a method of computing the order of an element in a direct product in terms of the order of the component pieces (see [4]).

Proposition 2.7.2 The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element.

Proposition 2.7.3 (see [4]) Let G and H be finite cyclic groups. Then $G \times H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

As a consequence of the proposition 2.7.3 and an induction argument, we can generalise proposition 2.7.3 to the direct product of a finite number of finite cyclic groups.

Example 2.7.4 Let p be a prime and $n \in \mathbb{N}$. Now let

$$G \quad = \quad \mathbb{Z}_p \times \mathbb{Z}_p$$

Clearly G has subgroup of order p. Intersection of two subgroups is also a subgroup. By Langrange's Theorem, intersection of two distinct subgroups of order p is trivial. The $p^2 - 1$ nonidentity elements of G are partitioned into subsets of size $p - 1$. Therefore, there are $p + 1$ subgroups of order p. Therefore $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have 4 and 3 subgroups of ordere 3 and 2 respectively.

There are two groups of order four up to isomorphism, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ and they are both abelian. In fact there is a theorem which classify all finite abelian groups up isomorphism. It is known as the Fundamental Theorem of Finite Abelian Groups (see [7]).

Theorem 2.7.5 (*Fundamental Theorem of Finite Abelian Groups*) If G is a non-trivial finite abelian group, then there are unique positive integers s and $n_1, n_2, ..., n_s$ where $n_j \geq 2$, such that n_j divides n_{j+1} for $j = 1, 2, 3, ..., s-1$ and $G \cong \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_s}$.

By the Fudamental Theorem of Finite Abelian Groups, it follows that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ are the only abelian groups of order eight.

2.8 Semi-direct products

$\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are direct products of other groups. However, most groups cannot be decomposed into a direct product but may be decomposed into what we call *semidirect products*.

Definition 2.8.1 Let H and K be subgroups of a group and define

$$HK = \{hk : h \in H, k \in K\}$$

Proposition 2.8.2 Let H and K be subgroups of the group G. If $H \cap K = 1$, then each element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$.

Proof: Let k_1 and k_2 be in K. Let also h_1 and h_2 be in H, then

$$\begin{aligned} {h_1}^{-1}h_1k_1 &= {h_1}^{-1}h_2k_2 \\ k_1{k_2}^{-1} &= {h_1}^{-1}h_2 \end{aligned}$$

$H \cap K = 1$, then ${h_2}^{-1}h_1 = 1$ and $k_2k_1^{-1} = 1$. Hence $h_2 = h_1$ and $k_2 = k_1$. ■

Theorem 2.8.3 (*Recognition theorem*) Suppose G is a group with subgroups H and K. If

(1) $H \trianglelefteq G$ and $K \trianglelefteq G$

(2) $H \cap K = 1$

then $HK \cong H \times K$.

Proof: By hypothesis (1), HK is a subgroup of G (see [4]). Let $h \in H$ and $k \in K$, hence $k^{-1}hk \in H$, since $H \trianglelefteq G$. Therefore $h^{-1}(k^{-1}hk) \in H$. Similarly, $(h^{-1}k^{-1}h)k \in K$. Since $H \cap K = 1$, then $hk = kh$. By proposition 1.1.39, each element of HK can be written uniquely as a product hk, with $h \in H$ and $k \in K$. Thus the function $\varphi : HK \longrightarrow H \times K$ defined by $\varphi(hk) = (h, k)$ is well defined. φ is a homomorphism, since every element of H commute with every element of K and φ is a bijection. Hence $HK \cong H \times K$. ■

But what if K is not necessarily normal in G. How is HK related to $H \times K$? To answer the question, let us do some further analysis. Suppose we have a group G containing subgroups H and K such that $H \trianglelefteq G$ but K is not necessarily normal in G. Clearly, HK is a subgroup of G. Then each element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$. Given two elements h_1k_1 and h_2k_2, their product in G is

$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2(k_1^{-1}k_1)k_2 \\ &= h_1(k_1h_2k_1^{-1})k_1k_2\end{aligned}$$

Our goal now is to turn this construction around, namely start with two groups H and K and try to define a group containing them in such a way that all the conditions hold. This lead us to the following theorem (see [4]).

Theorem 2.8.4 Suppose H and K are groups. Let $\theta : K \longrightarrow \text{Aut}(H)$ be a homomorphism. Let $H \rtimes_\theta K$ be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the following multiplication on $H \rtimes_\theta K$:

$$(h_1, k_1)(h_2, k_2) = (h_1\theta_{k_1}(h_2), k_1k_2)$$

(1) $H \rtimes_\theta K$ is a group

(2) $H \trianglelefteq G$

(3) $H \cap K = 1$

(4) for all $h \in H$ and $k \in K$, $\theta_k(h) = khk^{-1}$.

The group $H \rtimes_\theta K$ described in theorem 1.1.41 is called a *semi − direct product* of H and K under θ. When there is no danger of confusion, we shall write $H \rtimes_\theta K$ as $H \rtimes K$.

As in the case of a direct product, we now prove a *recognition theorem* for a semidirect product.

Theorem 2.8.5 Suppose G is a group with subgroups H and K. If

(1) $H \trianglelefteq G$

(2) $H \cap K = 1$

then $HK \cong H \rtimes K$.

Proof: Recall that HK is a subgroup of G, since $H \trianglelefteq G$. By proposition 1.1.39, HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$. Let $\theta : K \longrightarrow \text{Aut}(H)$ be defined by $\theta_k(h) = khk^{-1}$. Thus the function $\varphi : HK \longrightarrow H \rtimes_\theta K$ defined by $\varphi(hk) = (h, k)$ is well defined. Clearly, φ is a homomorphism and φ is a bijection. Hence $HK \cong H \rtimes_\theta K$. ■

Example 2.8.6 Let $D_n = \langle r, s | r^n = 1, s^2 = 1, rs = sr^{-1} \rangle$ be a *classical dihedral group*. $\langle r \rangle$ and $\langle s \rangle$ are both subgroups of D_n and $D_n = \langle r \rangle \langle s \rangle = \langle s \rangle \langle r \rangle$. $\langle r \rangle \trianglelefteq G$ and $\langle r \rangle \cap \langle s \rangle = 1$, hence $D_n \cong \langle r \rangle \rtimes \langle s \rangle$.

Let H be any finite abelian group of order n and let $K \cong \langle s \rangle \cong \mathbb{Z}_2$. Define $\theta : K \longrightarrow \text{Aut}(H)$ by $\theta_s(h) = h^{-1}$ for all $h \in H$. Clearly $shs^{-1} = h^{-1}$, hence $D_n \cong H \rtimes_\theta K$ if H is cyclic. $H \rtimes_\theta K$ is known as the *generalised dihedral group* and denoted by $D(H)$.

In fact, we are interested in all groups of order less than sixteen and some of order sixteen. Their classification is then essential. The theorems belows are only for the non-abelian groups. The abelian groups can be easily classified using the Fundamental Theorem of Finite Abelian Groups.

Theorem 2.8.7 Any non-abelian group of order 6 is isomorphic to D_3.

Proof: Let G be a group of order 6. By the cauchy theorem, G has two elements of order two and three, namely a and b. Hence $1, a, a^2, b$ are four distinct elements in G. The elements a^2b and ab are also distinct elements in G. Now, there are six distinct elements in G and $|G| = 6$. $G = \langle a, b \rangle$ and $\langle a \rangle \trianglelefteq G$ (the index of $\langle a \rangle$ in G is 2). Hence $bab^{-1} \in \langle a \rangle$. Therefore $bab^{-1} = 1$ or a or a^2. Clearly, $bab^{-1} \neq 1$. Let $bab^{-1} = a$, therefore $ba = ab$. It also follows that $G \cong \mathbb{Z}_3 \times \mathbb{Z}_2$. Let $bab^{-1} = a^2$, then $D_3 \cong G$. Thus, if G is non-abelian, then $D_3 \cong G$. ■

Similarly, the argument used in theorem 1.1.44 can be used in proving theorem 1.1.46 and theorem 1.1.48.

Theorem 2.8.8 Any nonabelian group of order 8 is isomorphic to D_4 or Q_8.

Proof: Let G be a group of order 8. Assume G has no element of order eight, that is $G \ncong \mathbb{Z}_8$. Also assume that not all the elements of G are of order 2. Clearly, G has element a of order 4 and $H = \langle a \rangle \trianglelefteq G$. Let $b \in G - H$, so $b^2 \in H$ since cosets partition a group. Let $b^2 = a$ or a^3, hence $|b| = 8$. That leads to a contradiction, since G has no elements of order 8 (otherwise it was going to be cyclic). Hence $b^2 = 1$ or a^2. Since H is normal, then $bab^{-1} \in H$. Clearly, $bab^{-1} \neq 1$ (otherwise $a = 1$). If $bab^{-1} = a^2$, then $a^2 = 1$. Contradiction as $|a| = 4$. Therefore $bab^{-1} = a$ or a^3. If $b^2 = 1$ and $bab^{-1} = a$, then $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. If $b^2 = a^2$ and $bab^{-1} = a$, then $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Now, if $b^2 = 1$ and $bab^{-1} = a^3$, then $G \cong D_4$. If $b^2 = a^2$ and $bab^{-1} = a^3$, then $G \cong Q_8$. ■

Theorem 2.8.9 (see [4]) Any nonabelian group of order 10 is isomorphic to D_5.

Theorem 2.8.10 (see [4]) Any nonabelian group of order 12 is isomorphic to one of A_4, D_6 and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.

The presentation of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is given by $\langle a, b : \; a^4 = b^3 = 1, a^{-1}ba = b^{-1} \rangle$ (see [4]).

Theorem 2.8.11 (see [4]) Any nonabelian group of order 14 is isomorphic to D_7.

Our 16 order non-abelian groups of interest are the generalised quartenion group Q_{16}, the modular group of order 16, the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, semidihedral group DQ_{16} and $Q_8 \times \mathbb{Z}_2$.

2.9 Posets

The theory of posets plays an important role in combinatorics. In particular on the theory of Möbius inversion.

Definition 2.9.1 A poset is an ordered pair $(P, \leq)$ consisting of a non-empty set P and operation $\leq$ on P satisfying the following three axioms:

(1) For all a in P, $a \leq a$

(2) For all a, b in P, if $a \leq b$ and $b \leq a$, then $a = b$

(3) For all a, b, c in P, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Let the notation $a \geq b$ mean $b \leq a$ and the notation $a < b$ means $a \leq b$ but $a \neq b$. Hence $a > b$ means $b < a$.

Definition 2.9.2 A poset P is finite if P is a finite set.

If a poset is not finite then it is infinite.

Example 2.9.3 Let the set inclusion relation be a relation on a set of all subgroups of a finite group G. The set of all subgroups of G is a poset, all the axioms of a poset definition are satisfied. Since G is a finte group, then the set of all subgroups of a finite group G under set inclusion relation is a finite poset.

Before going any further. Note that any subset of a poset is also a poset under the same relation. We call such posets subposets.

Definition 2.9.4 If $a \leq b$ in the poset P, then:

(1) A closed interval is $[a, b] = \{x \in P : a \leq x \leq b\}$

(2) An open interval is $(a, b) = \{x \in P : a < x < b\}$.

Definition 2.9.5 A poset P is locally finite if every interval of P is finite.

Proposition 2.9.6 Every finite poset is locally finite.

Proof: Given that P is finite. Assume that P is not locally finite, then P has interval which is not finite. Certainly that implies that P itself has infinite elements. Hence P is locally finite. ■

From the above proposition, it is clear that the set of subgroups of a finite group G under the set inclusion operation is a locally finite poset.

Definition 2.9.7 A non-empty subset I of a poset P is an ideal, if the following conditions hold:

(1) For every a in I, $b \leq a$ implies that b is in I.

(2) For every a, b in I, there is some elements c in I such that $a \leq c$ and $b \leq c$.

The smallest ideal that contains a given element a is a principal ideal and a is said to be a principal element of the ideal in this situation. The principal ideal $\langle a \rangle_P$ for a principal a is thus given by $\langle a \rangle_P = \{x \in P \ : x \leq a\}$. Since every subset of a finite set is finite, then principal ideals of the set of subgroups of a finite group G under the set inclusion operation are finite principal ideals.

Definition 2.9.8 Let P be a poset. Let $a, b \in P$. We say a is an immediate predecessor of b if $a < b$ but no elements in P lies between a and b.

The *Hasse diagram* of a poset P is a graphical representation of P as follows. The elements of P are represented by vertices in a plane and there is a directed line segment drawn from a to b whenever a is an immediate predecessor of b. We always place b higher than a. If $a \leq b$, there is path upwards from a to b passing through a chain of intermediate elements. The Hasse diagram of the set of subgroups of a finite group under the set inclusion operation is called the *Lattice of subgroups of a group*.

2.10 The Möbius function

Let P be a locally finite poset. Let $f : P \longrightarrow \mathbb{R}$ be a real function on the poset P. The sum function F of f is defined by

$$F(x) = \sum_{x \leq y} f(y)$$

We want to determine the general inversion formula on P in such a way that it is possible to compute f from F. There exists a function μ called the Möbius function of P such that

$$F(x) = \sum_{x \leq y} f(y) \Leftrightarrow f(x) = \sum_{x \leq y} F(y)\mu(y, x)$$

for all x in P.

Definition 2.10.1 Let P be locally finite poset. We define

$$\mathbb{A}(P) = \{f : P \times P \longrightarrow \mathbb{R} : x \nleq y \Rightarrow f(x, y) = 0\}$$

The set $\mathbb{A}(P)$ is a real vector space with respect to pointwise addition and scalar multiplication, that is, for $f, g \in \mathbb{A}(P)$ and $c \in \mathbb{R}$, the sum of f and g denoted by $f + g$ and scalar multiplication of f by c, denoted as cf is defined by:

$$\begin{aligned} (f + g)(x, y) &= f(x, y) + g(x, y) \\ (cf)(x, y) &= cf(x, y). \end{aligned}$$

The vector space $\mathbb{A}(P)$ is called the *real incidence algebra* and its elements, the *incidence functions* of P. We also define a *convolution* by

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

for all $f, g \in \mathbb{A}(P)$. Notice that the right-hand is well-difined by the local finiteness of P. The convolution is associative.

Definition 2.10.2 The Kronecker function δ in $\mathbb{A}(P)$ is defined by:

$$\delta(x, y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases}$$

From definition 1.3.2, it is clear that

$$\begin{aligned} (f * \delta)(x, y) &= \sum_{x \leq z \leq y} f(x, z)\delta(z, y) \\ &= f(x, y) \end{aligned}$$

Theorem 2.10.3 Let $f \in \mathbb{A}(P)$. the following conditions are equivalent

(1) f has a left inverse

(2) f has a right inverse

(3) f has a two-sided inverse

(4) for all $x \in P$, $f(x, x) \neq 0$.

Proof: $(1 \Leftrightarrow 4)$ Suppose $g \in \mathbb{A}(P)$ is a left inverse of f, then $g * f = \delta$. For all $x \in P$

$$1 = (g * f)(x, x) = f(x, x)g(x, x)$$

by definition of convolution and then $f(x, x) \neq 0$

Conversely, let $f(x, x) \neq 0$ for all $x \in P$. We define the left inverse of f by

$$\begin{aligned} f^{-1}(x, x) &= \frac{1}{f(x, x)} \\ f^{-1}(x, y) &= \frac{1}{f(y, y)}(-\sum_{x \leq z < y} f^{-1}(x, z)f(z, y)) \end{aligned}$$

In the same way, we can show that (2) and (4) are the equivalent.

$(3 \Leftrightarrow 4)$ A two-sided inverse is necessarily a left and right inverse. Thus this result follows from the previous arguments. ■

Corollary 2.10.4 Let $f \in \mathbb{A}(P)$. If f has any inverse, then it is the unique two-sided inverse of f.

Proof: Let g and g' be two-sided inverse of f, then $g' * f = f * g' = \delta$ and $g * f = f * g = \delta$. Thus

$$\begin{aligned} g' &= g' * \delta \\ &= g' * (f * g) \\ &= (g' * f) * g \\ &= \delta * g \\ &= g \end{aligned}$$

■

Now, let us define an important incidence function which is used to define the Möbius function.

Definition 2.10.5 The Zeta-function ζ in $\mathbb{A}(P)$ is defined by:

$$\zeta(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x \nleq y \end{cases}$$

By theorem 1.3.3, the Zeta-function is invertible and denote inverse of Möbius function. It is denoted by μ. The following proposition is an immediate concequence of our definition.

Proposition 2.10.6 (see [2]) Let μ be the Möbius function of P. For all $x, y \in P$:

$$\mu(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if} \quad x < y \\ 0 & \text{otherwise} \end{cases}$$

Proof: Recall that $\mu * \zeta = \delta$, so

$$\begin{aligned} (\mu * \zeta)(x,y) &= \sum_{x \leq z \leq y} \mu(x,z)\zeta(z,y) \\ &= \sum_{x \leq z < y} \mu(x,z)\zeta(z,y) + \mu(x,y) \\ &= \delta(x,y) \end{aligned}$$

■

Now, the main result of this subsection, the Möbius Inversion Formula follows (see [2]).

Theorem 2.10.7 Let P be a locally finite poset and $f, g : P \longrightarrow \mathbb{R}$. If all the principal ideals of P are finite, then

$$g(x) = \sum_{x \leq y} f(y) \Leftrightarrow f(x) = \sum_{x \leq y} g(y)\mu(y,x)$$

Chapter 3

Symmetric Colorings

3.1 Introduction

In this chapter we will present the important theorems that will be used to compute the symmetric colorings of some classes of finite groups.

3.2 Symmetric Colorings of finite groups

Let G be a finite group and $r \in \mathbb{N}$. A r-coloring (*coloring*) of G is any mapping $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$. The next proposition shows that there are $r^{|G|}$ r-colorings of G.

Proposition 3.2.1 Let G be a finite group, then there are $r^{|G|}$ r-colorings of G.

Proof: Every mapping $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$ can be viewed as a word of length $|G|$ from r alphabets. Since there are r ways to choose each letter. Therefore there are $r^{|G|}$ r-colorings of G.

Given any coloring χ of G and $g \in G$, we define the coloring χg by $\chi g(x) = \chi(xg^{-1})$ for all $x \in G$. From this definition, clearly G acts on the set of colorings. Colorings of

G, χ and ψ are *equivalent* if there exist an element g in G such that $\chi(xg^{-1}) = \psi(x)$ for all x in G. The relation *equivalent* is an equivalence relation on the set of all colorings of G. Each equivalence class is called a necklace of G and is denoted by $N_r(G)$. Burnside's Lemma can be used to find the number of necklaces of G.

Proposition 3.2.2 Let G be a finite group that acts on the set of r-colorings of G. The number necklaces of G is:

$$N_r(G) = \frac{1}{|G|}\sum_{g\in G} r^{|G:\langle g\rangle|}$$

Proof: When g acts on $\chi(x)$, x is transformed to xg^{-1}, xg^{-1} is transformed to xg^{-2}. In fact, xg^{-i} is transformed to $xg^{-(1+i)}$. Given that the order of g is $|g|$. A permutation of elements of G with $|G|/|g|$ cycles is formed. Since we can pick any of the r colors for each cycle, then we can conclude that the number of r-colorings fixed by g is $r^{|G:\langle g\rangle|}$, then by Burnside's Lemma it follows that the number of necklaces of G is

$$N_r(G) = \frac{1}{|G|}\sum_{g\in G} r^{|G:\langle g\rangle|}$$

Definition 3.2.3 A coloring χ of a finite group G is called symmetric with respect to an element g in G if

$$\chi(gx^{-1}g) = \chi(x)$$

for all $x \in G$.

For all $h \in G$, let χh be symmetric with respect to $g \in G$, therefore

$$\begin{aligned}
\chi h(gx^{-1}g) &= \chi h(x) \\
\chi(gh^{-1}(xh^{-1})^{-1}gh^{-1}) &= \chi(xh^{-1}) \\
\chi(gh^{-1}x^{-1}gh^{-1}) &= \chi(x)
\end{aligned}$$

thus χ is symmetric to gh^{-1}. Therefore a coloring equivalent to a symmetric one is also symmetric. Let $|S_r(G)/\sim|$ denote the set of all symmetric necklaces of G and $|S_r(G)|$ denote all symmetric r-colorings of G. To compute $|S_r(G)/\sim|$ or $|S_r(G)|$ we need to construct the partially ordered set of so called *optimal partitions* of G.

Theorem 3.2.4 (see [10]) Let P be the partially ordered set of optimal partitions of G. Then

$$\begin{aligned} |S_r(G)| &= |G| \sum_{x \leqslant P} \sum_{y \leqslant x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|} \\ |S_r(G)/\sim| &= \sum_{x \leqslant P} \sum_{y \leqslant x} \frac{\mu(y,x)|St(y)|}{|Z(y)|} r^{|x|} \end{aligned}$$

For finite abelian group G, the partially ordered set of optimal partitions can be identified with the lattice of subgroups of G.

Theorem 3.2.5 (see [5]) Let G be a finite abelian group. Then

$$\begin{aligned} |S_r(G)/\sim| &= \sum_{X \leqslant G} \sum_{Y \leqslant X} \frac{\mu(Y,X)}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}} \\ |S_r(G)| &= \sum_{X \leqslant G} \sum_{Y \leqslant X} \frac{\mu(Y,X)|G/Y|}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}} \end{aligned}$$

where $\mu(Y,X)$ is the Möbius function on a lattice of subgroups of G.

Chapter 4

Symmetric colorings of all abelian groups of order ≤ 16

4.1 The group $\mathbb{Z}_2 \times \mathbb{Z}_2$

Let $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b : \; a^2 = b^2 = 1, ba = ab \rangle$ be the presentation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. All the non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ have order two and there are three of them. Thus, there are three cyclic subgroups of order two. By Lagrange's Theorem, it follows that there are five subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The lattice of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is shown in figure 4.1.

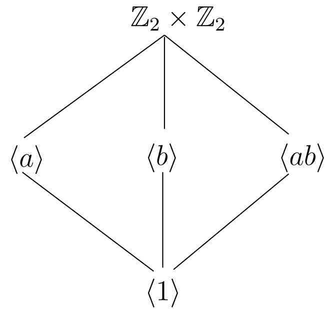

Figure 4.1: lattice of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$

And by theorem 3.2.4

$$\begin{aligned}|S_r(\mathbb{Z}_2\times\mathbb{Z}_2)| &= r^4+3(-1+1)r^2+(2-1-1-1+1)r^2\\ &= r^4\end{aligned}$$

and

$$\begin{aligned}|S_r(\mathbb{Z}_2\times\mathbb{Z}_2)/\sim| &= \frac{r^4}{4}+3\Big(-\frac{1}{4}+\frac{1}{2}\Big)r^2+\Big(\frac{2}{4}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+1\Big)r^2\\ &= \frac{r^4}{4}+\frac{3r^2}{4}\\ &= \frac{1}{4}(3r^2+r^4)\end{aligned}$$

Proposition 4.1.1 For every $r\in\mathbb{N}$, $|S_r(\mathbb{Z}_2\times\mathbb{Z}_2)|=r^4$ and $|S_r(\mathbb{Z}_2\times\mathbb{Z}_2)/\sim|=\frac{1}{4}(3r^2+r^4)$.

The number of necklaces of $\mathbb{Z}_2\times\mathbb{Z}_2$ is

$$\frac{1}{4}(3r^2+r^4)$$

This is because all the colorings of $\mathbb{Z}_2\times\mathbb{Z}_2$ are symmetric.

4.2 The group $\mathbb{Z}_4\times\mathbb{Z}_2$

Now, let $\mathbb{Z}_4\times\mathbb{Z}_2=\langle a,b:\ a^4=b^2=1, ba=ab\rangle$. There are three subgroups isomorphic to $\mathbb{Z}_2$, namely $\langle b\rangle$, $\langle a^2b\rangle$ and $\langle a^2\rangle$. Any subgroup of order four either is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_4$. $\langle a\rangle$ and $\langle ab\rangle$ are isomorphic to $\mathbb{Z}_4$. $\langle a^2,b\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$.
Let $G=\mathbb{Z}_4\times\mathbb{Z}_2$. Since $G/\langle 1\rangle\cong G$, then $B(G/\langle 1\rangle)\cong B(G)$. Recall that $B(G)$ has three subgroups of order two, thus $|B(G)|=4$. $G/\langle b\rangle\cong\mathbb{Z}_4$ or $\mathbb{Z}_2\times\mathbb{Z}_2$. $G/\langle b\rangle$ is a cyclic group generated by $a\langle b\rangle$, hence $G/\langle b\rangle\cong\mathbb{Z}_4$. Only two of the $\mathbb{Z}_4$ subroups are of order two (see example 2.2.7), then $|B(G/\langle b\rangle)|=2$. For $\langle a^2b\rangle$ and $\langle a^2\rangle$, $G/\langle a^2b\rangle\cong\mathbb{Z}_4$ and $G/\langle a^2\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$. Now, by theorem 3.2.4

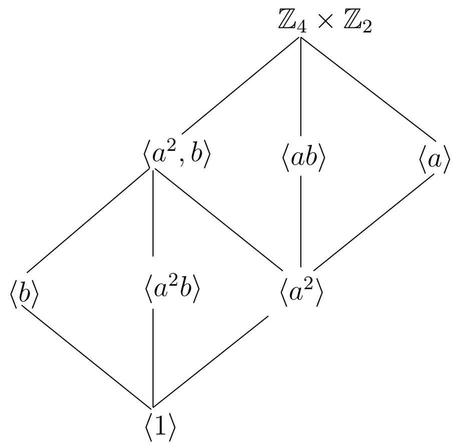

Figure 4.2: lattice of subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_2$

$$\begin{aligned} |S_r(G)| &= \frac{8}{4}r^6 + 2\Big(\frac{4}{2} - \frac{8}{4}\Big)r^3 + \Big(\frac{4}{4} - \frac{8}{4}\Big)r^4 + \Big(\frac{2}{2} - \frac{4}{2} - \frac{4}{2} - \frac{4}{4} + \frac{16}{4}\Big)r^2 \\ &= 2\Big(\frac{2}{2} - \frac{4}{4}\Big)r^2 \\ &= 2r^6 - r^4 \end{aligned}$$

and

$$\begin{aligned} |S_r(G)/\sim| &= \frac{1}{4}r^6 + 2\Big(\frac{1}{2} - \frac{1}{4}\Big)r^3 + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^4 + \Big(\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{2}{4}\Big)r^2 \\ &= 2\Big(\frac{1}{2} - \frac{1}{4}\Big)r^2 \\ &= \frac{1}{4}(r^6 + 2r^3 + r^2) \end{aligned}$$

Proposition 4.2.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_2)/\sim| = \frac{1}{4}(r^6 + 2r^3 + r^2)$ and $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_2)| = 2r^6 - r^4$.

The number of necklaces of $\mathbb{Z}_4 \times \mathbb{Z}_2$ is

$$\frac{1}{8}(4r^2 + 3r^4 + r^8)$$

4.3 The group $\mathbb{Z}_3 \times \mathbb{Z}_3$

By example 2.7.4, $\mathbb{Z}_3 \times \mathbb{Z}_3$ has four order 4 subgroups isomorphic to $\mathbb{Z}_3$. Given the presentation $\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a, b : \ a^3 = b^3 = 1, ba = ab \rangle$, the order 3 subgroups are $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$ and $\langle ab^2 \rangle$. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, then $G/\langle 1 \rangle \cong G$, $G/\langle a \rangle \cong \mathbb{Z}_3$, $G/\langle b \rangle \cong \mathbb{Z}_3$, $G/\langle ab \rangle \cong \mathbb{Z}_3$ and $G/\langle ab^2 \rangle \cong \mathbb{Z}_3$. Hence the number of symmetric colorings and necklaces is given by

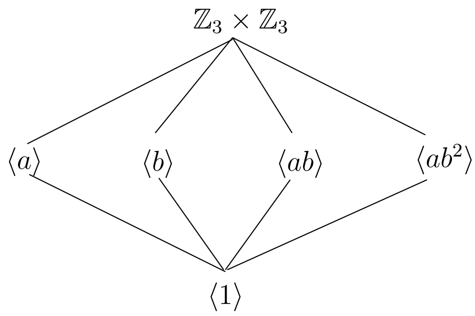

Figure 4.3: lattice of subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$

$$
\begin{aligned}
|S_r(G)| &= \frac{9r^5}{1} + 4\Big(-\frac{9}{1} + \frac{3}{1}\Big)r^2 + \Big(\frac{18}{1} - \frac{3}{1} - \frac{3}{1} - \frac{3}{1} - \frac{3}{1} + 1\Big)r \\
&= 9r^5 - 24r^2 + 16r
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(G)/\sim| &= \frac{r^5}{1} + 4\Big(-\frac{1}{1} + \frac{1}{1}\Big)r^2 + \Big(\frac{3}{1} - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} + 1\Big)r \\
&= r^5
\end{aligned}
$$

Proposition 4.3.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_3 \times \mathbb{Z}_3)/\sim| = r^5$ and $|S_r(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 9r^5 - 24r^2 + 16r$.

The number of necklaces of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is given by

$$\frac{1}{9}(8r^3 + r^9)$$

4.4 The group $\mathbb{Z}_6 \times \mathbb{Z}_2$

$\mathbb{Z}_6 \times \mathbb{Z}_2 = \langle a, b : \; a^6 = b^2 = 1, ba = ab \rangle$ has 6 order 6 elements, 3 order 2 elements and 2 order 3 elements. By the Fundamental Theorem of Finite Abelian Groups, there is only one abelian group of order six and is cyclic. Since every cyclic group of order 6 has two elements of order 6, it follows that there are 3 order 6 subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_2$. Using similar arguments, we can easily show that there is only one order 3 subgroup which is cyclic. Clearly, there are 3 subgroups of order 2 and using these subgroups, we can easily construct the only subgroup of order 4. All these subgroups are given by $\langle b \rangle \cong \langle a^3 \rangle \cong \langle a^3 b \rangle \cong \mathbb{Z}_2$, $\langle a^2 \rangle \cong \mathbb{Z}_3$, $\langle a^3, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\langle a \rangle \cong \langle a^2 b \rangle \cong \langle ab \rangle \cong \mathbb{Z}_6$. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_2$, then $G/\langle 1 \rangle \cong G$ and $G/\langle a^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. $G/\langle b \rangle \cong G/\langle a^3 b \rangle \cong G/\langle a^3 \rangle \cong \mathbb{Z}_6$. $G/\langle a^3, b \rangle \cong \mathbb{Z}_3$. Thus by theorem 3.2.4,

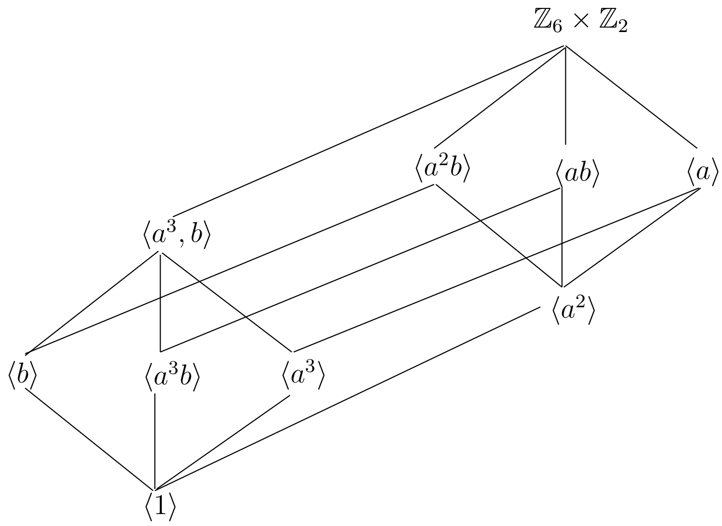

Figure 4.4: lattice of subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_2$

$$
\begin{aligned}
|S_r(G)| &= \frac{12}{4}r^8 + 3\Big(\frac{6}{2} - \frac{12}{4}\Big)r^4 + \Big(\frac{4}{4} - \frac{12}{4}\Big)r^4 + \Big(\frac{3}{1} - \frac{6}{2} - \frac{6}{2} - \frac{6}{2} + \frac{24}{4}\Big)r^2 \\
&+ 3\Big(\frac{2}{2} - \frac{4}{4} - \frac{6}{2} + \frac{12}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{2}{2} - \frac{2}{2} - \frac{2}{2} + \frac{6}{2} + \frac{6}{2} + \frac{6}{2} - \frac{3}{1} + \frac{8}{4} - \frac{24}{4}\Big)r \\
&= 3r^8 - 2r^4
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(G)/\sim| &= \frac{1}{4}r^8 + 3\Big(\frac{1}{2} - \frac{1}{4}\Big)r^4 + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^4 + \Big(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{2}{4}\Big)r^2 \\
&+ 3\Big(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} + \frac{1}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{1} + \frac{2}{4} - \frac{2}{4}\Big)r \\
&= \frac{1}{4}(r^8 + 3r^4)
\end{aligned}
$$

Proposition 4.4.1 $|S_r(\mathbb{Z}_6 \times \mathbb{Z}_2)/\sim| = \frac{1}{4}(r^8 + 3r^4)$ and $|S_r(\mathbb{Z}_6 \times \mathbb{Z}_2)| = 3r^8 - 2r^4$ for every $r \in \mathbb{N}$.

The number of necklaces of $\mathbb{Z}_6 \times \mathbb{Z}_2$ is given by

$$\frac{1}{12}(6r^2 + 2r^2 + 3r^6 + r^{12})$$

4.5 The group $\mathbb{Z}_4 \times \mathbb{Z}_4$

$\mathbb{Z}_4 \times \mathbb{Z}_4$ has 3 order 2 elements and 12 order 4 elements. There are also 3 order 2 subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$. Since every cyclic group of order 4 has 2 elements of order 4, hence there are 6 cyclic subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ of order 4. There is only one subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since there are 3 subgroups of order 2, there are 3 subgroups of order 8 and they are all isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4 = \langle a, b : \ a^4 = b^4 = 1, ba = ab \rangle$. All the non-trivial subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$

are given by $\langle a^2\rangle \cong \langle b^2\rangle \cong \langle a^2b^2\rangle \cong \mathbb{Z}_2$, $\langle a\rangle \cong \langle ab^2\rangle \cong \langle a^2b\rangle \cong \langle b\rangle \cong \langle ab\rangle \cong \langle ab^3\rangle \cong \mathbb{Z}_4$, $\langle a^2, b^2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\langle a, b^2\rangle \cong \langle a^2, b\rangle \cong \langle a^2, a^3b\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Let $H = \langle a^2\rangle$, then

$$\begin{aligned} G/\left\langle a^2\right\rangle &= \{H, aH, bH, b^2H, b^3H, ab^2H, ab^3H, abH\} \\ &= \langle aH, bH\rangle \\ &\cong \mathbb{Z}_4 \times \mathbb{Z}_2 \cong G/\left\langle a^2b^2\right\rangle \cong G/\left\langle b^2\right\rangle \end{aligned}$$

Again, let $H = \langle a\rangle$. It follows that

$$\begin{aligned} G/\langle a\rangle &= \{H, bH, b^2H, b^3H\} \\ &= \langle bH\rangle \\ &\cong \mathbb{Z}_4 \cong G/\left\langle a^2b\right\rangle \cong G/\langle b\rangle \cong G/\left\langle b^2a\right\rangle \cong G/\langle ab\rangle \cong G/\left\langle b^3a\right\rangle \end{aligned}$$

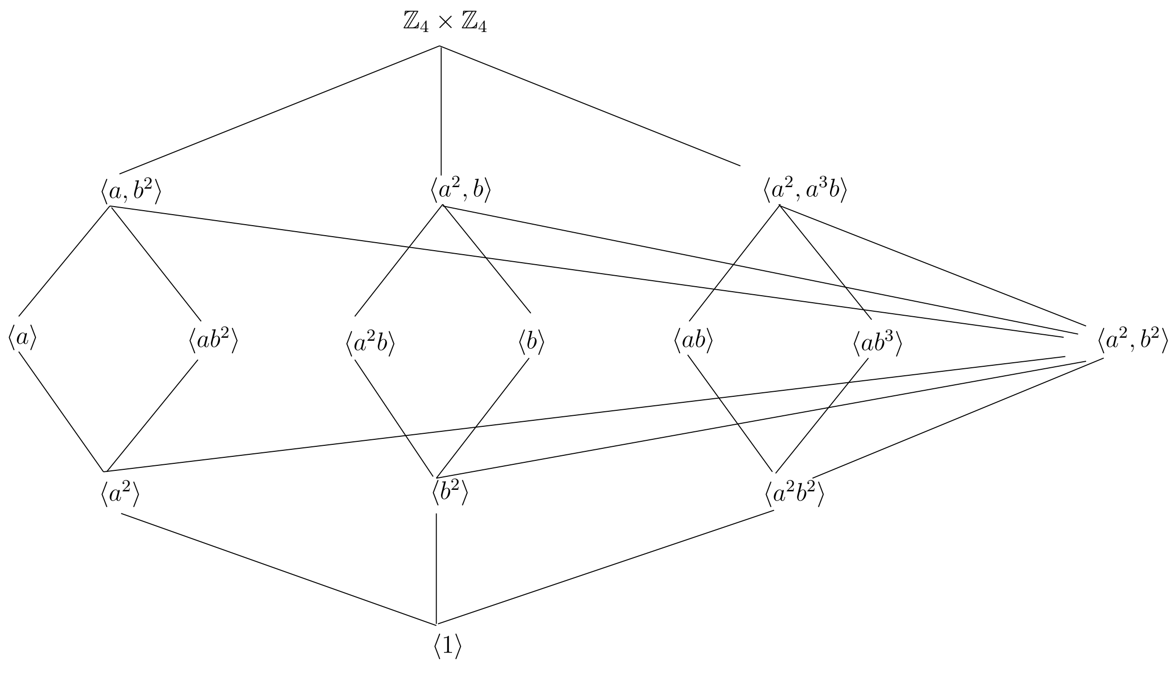

Figure 4.5: lattice of subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$

$G/\langle a^2, b^2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G/\langle a^2, b\rangle \cong G/\langle b^2, a\rangle \cong G/\langle a^3b, a^2\rangle \cong \mathbb{Z}_2$.

Using all these results together with theorem 3.2.4, it follows that

$$\begin{aligned} |S_r(G)| &= \frac{16}{4}r^{10} + 3\Big(\frac{8}{4} - \frac{16}{4}\Big)r^6 + 6\Big(\frac{4}{2} - \frac{8}{4}\Big)r^3 + \Big(\frac{4}{4} - \frac{8}{4} - \frac{8}{4} - \frac{8}{4} + \frac{32}{4}\Big)r^4 \\ &+ 3\Big(\frac{2}{2} - \frac{4}{4} - \frac{4}{2} - \frac{4}{2} + \frac{16}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{2}{2} - \frac{2}{2} - \frac{2}{2} + \frac{8}{4}\Big)r \\ &= 4r^{10} - 6r^6 + 3r^4 \end{aligned}$$

and

$$\begin{aligned} |S_r(G)/\sim| &= \frac{1}{4}r^{10} + 3\Big(\frac{1}{4} - \frac{1}{4}\Big)r^6 + 6\Big(\frac{1}{2} - \frac{1}{4}\Big)r^3 + \Big(\frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{2}{4}\Big)r^4 \\ &+ 3\Big(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + \frac{2}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{2}{4}\Big)r \\ &= \frac{1}{4}(r^{10} + 6r^3 - 3r^2) \end{aligned}$$

Proposition 4.5.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_4)| = 4r^{10} - 6r^6 + 3r^4$ and $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_4)/\sim| = \frac{1}{4}(r^{10} + 6r^3 - 3r^2)$.

The number of necklaces of $\mathbb{Z}_4 \times \mathbb{Z}_4$ is given by

$$\frac{1}{16}(12r^4 + 3r^8 + r^{16})$$

4.6 The group $\mathbb{Z}_8 \times \mathbb{Z}_2$

Another group of interest is $G = \mathbb{Z}_8 \times \mathbb{Z}_2 = \langle a, b : \ a^8 = b^2 = 1, ab = ba\rangle$. $\langle b\rangle \cong \langle a^4\rangle \cong \langle a^4b\rangle \cong \mathbb{Z}_2$, hence $\langle a^4, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. There are two cyclic subgroups of order four, namely $\langle a^2\rangle \cong \langle a^2b\rangle \cong \mathbb{Z}_4$ since only four elements of G are of order four. The subgroups of order eight are $\langle a^2, b\rangle$, $\langle ab\rangle$ and $\langle a\rangle$.

Let $H = \langle a^4 \rangle$. Therefore

$$\begin{aligned} G/\left\langle a^4 \right\rangle &= \{H, aH, a^2H, a^3H, bH, abH, a^2bH, a^3bH\} \\ &\cong \mathbb{Z}_4 \times \mathbb{Z}_2. \end{aligned}$$

Now, let $H = \langle a^4 b \rangle$,

$$\begin{aligned} G/\left\langle a^4 b \right\rangle &= \{H, aH, a^2H, a^3H, a^4H, a^5H, a^6H, a^7H\} \\ &= \langle aH \rangle \\ &\cong \mathbb{Z}_8 \cong G/\langle b \rangle \end{aligned}$$

If $H = \langle a^4, b \rangle$, then

$$\begin{aligned} G/\left\langle a^4, b \right\rangle &= \{H, aH, a^2H, a^3H, \} \\ &= \langle aH \rangle \\ &\cong \mathbb{Z}_4 \cong G/\left\langle a^2 b \right\rangle \end{aligned}$$

and lastly, if $H = \langle a^2 \rangle$

$$\begin{aligned} G/\left\langle a^2 \right\rangle &= \{H, aH, bH, abH\} \\ &= \langle aH, bH \rangle \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \end{aligned}$$

By theorem 3.2.7

$$\begin{aligned} |S_r(G)| &= \frac{16}{4}r^{10} + \Big(\frac{8}{4} - \frac{16}{4}\Big)r^6 + 2\Big(\frac{8}{2} - \frac{16}{4}\Big)r^5 + \Big(\frac{4}{2} - \frac{8}{4} - \frac{8}{2} - \frac{8}{2} + \frac{32}{4}\Big)r^3 \\ &+ \Big(\frac{4}{2} - \frac{8}{4}\Big)r^3 + \Big(\frac{4}{4} - \frac{8}{4}\Big)r^4 + 2\Big(\frac{2}{2} - \frac{4}{4}\Big)r^2 \\ &+ \Big(\frac{2}{2} - \frac{4}{2} - \frac{4}{2} - \frac{4}{4} + \frac{16}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{2}{2} - \frac{2}{2} - \frac{2}{2} + \frac{8}{4}\Big)r \\ &= 4r^{10} - 2r^6 - r^4 \end{aligned}$$

and

$$
\begin{aligned}
|S_r(G)/\sim| &= \frac{1}{4}r^{10} + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^6 + 2\Big(\frac{1}{2} - \frac{1}{4}\Big)r^5 + \Big(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + \frac{2}{4}\Big)r^3 \\
&+ \Big(\frac{1}{2} - \frac{1}{4}\Big)r^3 + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^4 + 2\Big(\frac{1}{2} - \frac{1}{4}\Big)r^2 \\
&+ \Big(\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{2}{4}\Big)r^2 + \Big(\frac{1}{1} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{2}{4}\Big)r \\
&= \frac{1}{4}(r^{10} + 2r^5 + r^2)
\end{aligned}
$$

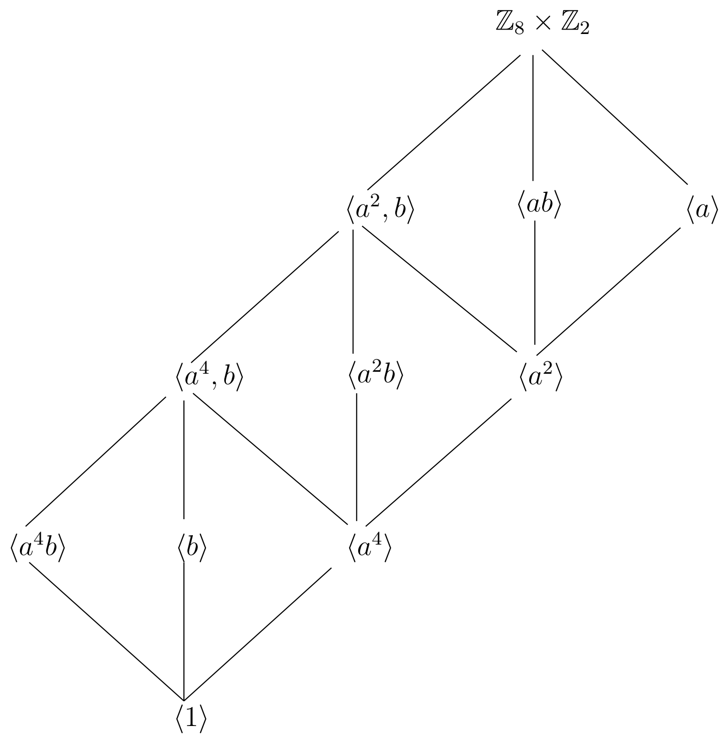

Figure 4.6: lattice of subgroups of $\mathbb{Z}_8 \times \mathbb{Z}_2$

Proposition 4.6.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_8 \times \mathbb{Z}_2)| = 4r^{10} - 2r^6 - r^4$ and $|S_r(\mathbb{Z}_8 \times \mathbb{Z}_2)/\sim| = \frac{1}{4}(r^{10} + 2r^5 + r^2)$.

The number of necklaces of $\mathbb{Z}_4 \times \mathbb{Z}_4$ is given by

$$\frac{1}{16}(8r^2 + 4r^4 + 3r^8 + r^{16})$$

4.7 Boolean Groups

Recall that all the colorings of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are symmetric and $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean group. In fact, the colorings of Boolean groups are all symmetric.

Theorem 4.7.1 (see [9]) Every r-coloring of a finite abelian group is symmetric if and only if one of the following cases holds:

(1) $r = 1$

(2) $r = 2$ and $G = \mathbb{Z}_5$ or $\mathbb{Z}_3$

(3) G is a Boolean group.

By theorem 4.7.1, the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 ... \times \mathbb{Z}_2$ (n times) has only symmetric colorings. Therefore

$$\begin{aligned} |S_r(G)/\sim| &= \frac{1}{2^n}(r^{2n} + (2^n - 1)r^{2^{n-1}}) \\ &= \frac{1}{2^n}(r^{2^n} + 2^n r^{2^{n-1}} - r^{2^{n-1}}) \end{aligned}$$

and $|S_r(G)| = r^{2^n}$. Hence $|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)| = r^8$, $|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)| = r^{16}$,

$$|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/\sim| = \frac{1}{8}(r^8 + 7r^4)$$

and

$$|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/\sim| = \frac{1}{16}(r^{16} + 15r^8)$$

Now we are going to prove the two results of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ using theorem 3.2.4. Let $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c : \ a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$. Clearly, $\langle a \rangle \cong \langle b \rangle \cong \langle c \rangle \cong \langle ab \rangle \cong \langle ac \rangle \cong \langle bc \rangle \cong \langle abc \rangle \cong \mathbb{Z}_2$ and $\langle a, b \rangle \cong \langle a, c \rangle \cong \langle b, c \rangle \cong \langle a, bc \rangle \cong \langle ac, b \rangle \cong \langle ab, c \rangle \cong \langle ab, ac \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

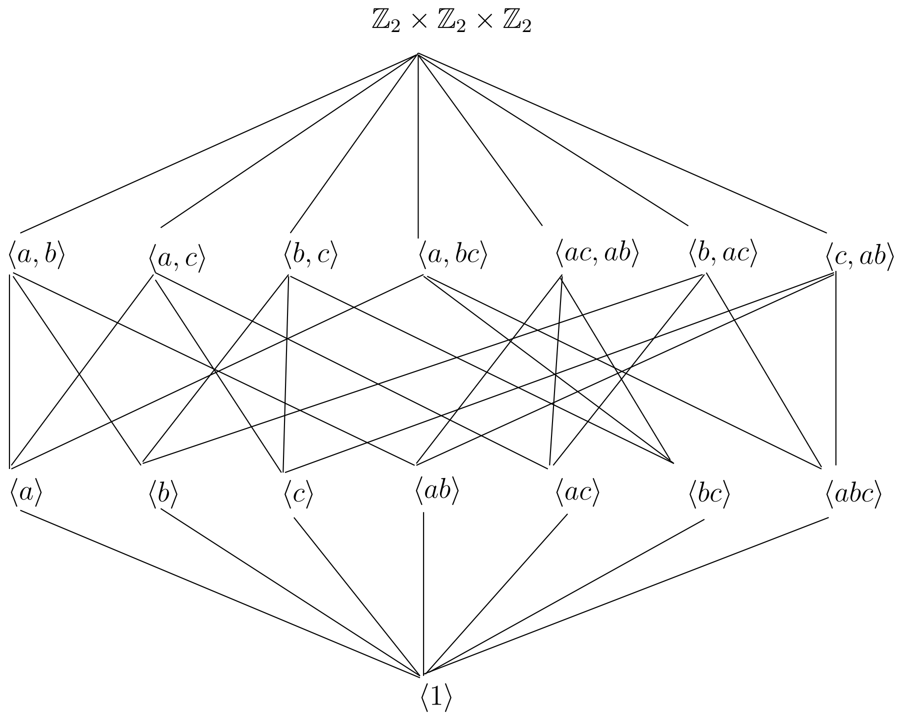

Figure 4.7: lattice of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Let $H = \langle a \rangle$, then

$$\begin{aligned} G/\langle a \rangle &= \{H, bH, cH, bcH\} \\ &= \langle bH, cH \rangle \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong G/\langle ac \rangle \cong G/\langle bc \rangle \cong G/\langle abc \rangle \cong G/\langle b \rangle \cong G/\langle c \rangle \cong G/\langle ab \rangle . \end{aligned}$$

It follows that

$$\begin{aligned} |S_r(G)| &= \frac{8r^8}{8} + 7(-\frac{8}{8} + \frac{4}{4})r^4 + (-\frac{4}{4} + \frac{2}{2})r^2 + (-\frac{64}{8} + \frac{56}{4} - \frac{14}{2} + \frac{1}{1})r \\ &= r^8 \end{aligned}$$

and

$$\begin{aligned} |S_r(G)/\sim| &= \frac{r^8}{8} + 7(-\frac{1}{8} + \frac{1}{4})r^4) + (-\frac{1}{4} + \frac{1}{2})r^2 \\ &= \frac{1}{8}(7r^4 + r^8) \end{aligned}$$

These confirm what we have proved using theorem 4.7.1.

Proposition 4.7.2 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)| = r^8$ and $|S_r(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/\sim| = \frac{1}{8}(7r^4 + r^8)$.

4.8 The group $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c |\ a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$ has seven subgroups of order two, namely $\langle a^2 \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle a^2 b \rangle$, $\langle a^2 c \rangle$, $\langle bc \rangle$ and $\langle a^2 bc \rangle$. Order 4 cyclic subgroups are $\langle a \rangle$, $\langle ab \rangle$, $\langle ac \rangle$ and $\langle abc \rangle$. Subgroups which are isomorphic to this are $\langle a^2, b \rangle$, $\langle a^2, c \rangle$, $\langle a^2, bc \rangle$, $\langle c, b \rangle$, $\langle a^2 c, b \rangle$, $\langle a^2 b, c \rangle$ and $\langle a^2 b, a^2 c \rangle$. There is only one subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. That is $\langle a^2, b, c \rangle$. $\langle a, b \rangle$, $\langle a, c \rangle$, $\langle a, bc \rangle$, $\langle ac, b \rangle$, $\langle ab, c \rangle$ and $\langle ac, bc \rangle$ are subgroups isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.
Let $H = \langle a^4 \rangle$, now

$$\begin{aligned} G/\langle a^2 \rangle &= \{H, aH, bH, cH, abH, acH, bcH, abcH\} \\ &= \langle aH, bH, cH \rangle \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned}$$

Let $H = \langle a^2 b \rangle$, then

$$\begin{aligned} G/\left\langle a^2 b \right\rangle &= \{H, aH, a^2H, a^3H, cH, acH, a^2cH, a^3cH\} \\ &= \langle aH, cH \rangle \\ &\cong \mathbb{Z}_4 \times \mathbb{Z}_2 \cong G/\left\langle a^2 c \right\rangle \cong G/\langle bc \rangle \cong G/\left\langle a^2 bc \right\rangle \cong G/\langle b \rangle \cong G/\langle c \rangle . \end{aligned}$$

Let $H = \langle a^2, b \rangle$, then

$$\begin{aligned} G/\left\langle a^2, b \right\rangle &= \{H, aH, cH, acH\} \\ &= \langle aH, cH \rangle \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong G/\left\langle a^2, c \right\rangle \cong G/\left\langle a^2, bc \right\rangle . \end{aligned}$$

Let $H = \langle c, b \rangle$, then

$$\begin{aligned} G/\langle c, b \rangle &= \{H, aH, a^2H, a^3H\} \\ &= \langle aH \rangle \\ &\cong \mathbb{Z}_4 \cong G/\left\langle a^2 c, b \right\rangle \cong G/\left\langle a^2 b, c \right\rangle \cong G/\left\langle a^2 b, a^2 c \right\rangle . \end{aligned}$$

Due to the hight number of subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and chaotic nature of the lattice of subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the number of symmetric colorings and necklaces is outside the concrete lattice at hand. So by theorem, if we substitute $\langle 1 \rangle$ we get $\frac{16}{8} r^{12}$. If we substitute $\langle a^2 \rangle$, we get $(\frac{8}{8} - \frac{16}{8}) r^8$. Now if we substitute one of the other subgroups of order two, we get $(\frac{8}{4} - \frac{16}{8}) r^6$. The cyclic subgroups of order four result in $(\frac{4}{4} - \frac{8}{8}) r^4$, and some Klein 4-subgroups, namely $\langle a^2, b \rangle$, $\langle a^2, c \rangle$ and $\langle a^2, bc \rangle$ result in $(\frac{4}{4} - \frac{8}{8} - \frac{8}{4} - \frac{8}{4} + \frac{32}{8}) r^4$ and the other Klein 4-subgroups result in $(\frac{4}{2} - \frac{8}{4} - \frac{8}{4} - \frac{8}{4} + \frac{32}{8}) r^3$. $\langle a^2, b, c \rangle$ gives us $(\frac{2}{2} - \frac{4}{4} - \frac{4}{4} - \frac{4}{4} - \frac{4}{2} - \frac{4}{2} - \frac{4}{2} - \frac{4}{2} + 2(\frac{8}{8} + \frac{8}{4} + \frac{8}{4} + \frac{8}{4} + \frac{8}{4} + \frac{8}{4} + \frac{8}{4}) - \frac{128}{8}) r^2$ and the other subgroups of order eight give us $(\frac{2}{2} - \frac{4}{4} - \frac{4}{4} - \frac{4}{4} + \frac{16}{8}) r^2$. G gives us $(\frac{1}{1} - \frac{14}{2} + \frac{54}{4} - \frac{64}{8}) r$. Combining these, we get

$$|S_r(G)| = 2r^{12} - r^8$$

and

$$|S_r(G)/\sim| \quad = \quad \frac{1}{8}(r^{12} + 6r^6 + r^4).$$

The number of necklaces of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by

$$\frac{1}{16}(8r^4 + 7r^8 + r^{16}).$$

Proposition 4.8.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2)/\sim| = \frac{1}{8}(r^{12} + 6r^6 + r^4)$ and $|S_r(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2)| = 2r^{12} - r^8$.

4.9 Cyclic groups

The number of symmetric colorings and symmetric necklaces of cyclic groups are computed using the following theorem (see [5]) where p represents a prime number.

Theorem 4.9.1 If n is odd, then

$$\begin{aligned} |S_r(\mathbb{Z}_n)| &= \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}} \\ |S_r(\mathbb{Z}_n)/\sim| &= r^{\frac{n+1}{2}}. \end{aligned}$$

If $n = 2^l m$, where $l \geq 1$ and m is odd, then

$$\begin{aligned} |S_r(\mathbb{Z}_n)| &= \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1} \\ |S_r(\mathbb{Z}_n)/\sim| &= \frac{r^{\frac{m+1}{2}} + r^{\frac{n}{2}+1}}{2}. \end{aligned}$$

Recall that we are only interested in cyclic groups of all orders ≤ 16. The table below contains all the computed number of colorings of abelian groups we are interested in.

Order	Abelian groups	No. of Symmetric Colorings	No.of Symmetric Necklaces
Order 1	$\mathbb{Z}_1$	r	r
Order 2	$\mathbb{Z}_2$	r^2	$\frac{1}{2}(r^2 + r)$
Order 3	$\mathbb{Z}_3$	$3r^2 - 2r$	r^2
Order 4	$\mathbb{Z}_4$	$2^3 - r^2$	$\frac{1}{2}(r^3 + r)$
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	r^4	$\frac{1}{4}(r^4 + 3r^2)$
Order 5	$\mathbb{Z}_5$	$-4r + 5r^3$	r^3
Order 6	$\mathbb{Z}_6$	$3r^4 - 2r^2$	$\frac{1}{2}(r^2 + r^4)$
Order 7	$\mathbb{Z}_7$	$-6r + 7r^4$	r^4
Order 8	$\mathbb{Z}_8$	$-r^2 - 2r^3 + 4r^5$	$\frac{1}{2}(r + r^5)$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	r^8	$\frac{1}{8}(r^8 + 7r^4)$
	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2r^6 - r^4$	$\frac{1}{4}(r^6 + 2r^3 + r^2)$
Order 9	$\mathbb{Z}_9$	$-2r - 6r^2 + 9r^5$	r^5
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$9r^5 - 24r^2 + 16r$	r^5
Order 10	$\mathbb{Z}_{10}$	$5r^6 - 4r^2$	$\frac{1}{2}(r^3 + r^6)$
Order 11	$\mathbb{Z}_{11}$	$-10r + 11r^6$	r^6
Order 12	$\mathbb{Z}_{12}$	$2r^2 - 4r^3 - 3r^4 + 6r^7$	$\frac{1}{2}(r^2 + r^7)$
	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$3r^8 - 2r^4$	$\frac{1}{4}(r^8 + 3r^4)$
Order 13	$\mathbb{Z}_{13}$	$-12r + 13r^7$	r^7
Order 14	$\mathbb{Z}_{14}$	$-6r^2 + 7r^8$	$\frac{1}{2}(r^4 + r^8)$
Order 15	$\mathbb{Z}_{15}$	$8r - 12r^2 - 10r^3 + 15r^8$	r^8
Order 16	$\mathbb{Z}_{16}$	$-r^2 - 2r^3 - 4r^5 + 8r^9$	$\frac{1}{2}(r + r^9)$
	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$4r^{10} - 2r^6 - r^4$	$\frac{1}{4}(r^{10} + 2r^5 + r^2)$
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$4r^{10} - 6r^6 + 3r^4$	$\frac{1}{4}(r^{10} + 6r^3 - 3r^2)$
	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$2r^{12} - r^8$	$\frac{1}{8}(r^{12} + 6r^6 + r^4)$
	$(\mathbb{Z}_2)^4$	r^{16}	$\frac{1}{16}(r^{16} + 15r^8)$

Chapter 5

Symmetric colorings of all non-abelian groups of order ≤ 10, the dihedral groups D_3, D_4, D_5 and the quaternion group Q_8

From this chapter onward, we will be dealing with non-abelian groups. In this chapter we are only focusing on the non-abelian groups of order ≤ 10. Remember that we classified all of them up isomorphism in chapter one.

5.1 Dihedral Group D_3

Let $D_3 = \langle a, b : \ a^3 = b^2 = 1, bab = a^{-1} \rangle$ be the presentation of the classical dihedral group of order six. This group has three order two subgroups, $\langle b \rangle \cong \langle ba \rangle \cong \langle ba^2 \rangle \cong \mathbb{Z}_2$ and only one order three subgroup, $\langle a \rangle \cong \mathbb{Z}_3$. To find the symmetric colorings of the group $D_3 = \langle a, b : \ a^3 = b^2 = 1, bab = a^{-1} \rangle$, we need to construct its poset of optimal partitions. The lattice of subgroups of D_3 is given in figure 5.1. The list of all optimal partions of D_3 is given below.

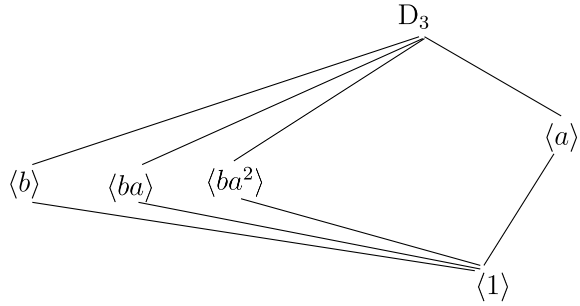

Figure 5.1: lattice of subgroups of D_3

The finest partition

$$\pi : \{1\}, \{b\}, \{ba\}, \{ba^2\}, \{a, a^2\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = 5.$$

Three partitions of the form

$$\pi : \{1\}, \{b, ba\}, \{ba^2\}, \{a, a^2\}$$
$$St(\pi) = \{1\}, Z(\pi) = \left\langle ba^2 \right\rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 4.$$

One partition

$$\pi : \{1, a, a^2\}, \{b\}, \{ba^2\}, \{ba\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 3, |\pi| = 4.$$

One partition

$$\pi : \{1\}, \{a, a^2\}, \{b, ba^2, ba\}$$

$$St(\pi) = \{1\}, Z(\pi) = \{1, b, ba, ba^2\},$$

$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 3.$$

Three partitions of the form

$$\pi : \{1, a, a^2\}, \{b\}, \{ba^2, ba\}$$

$$St(\pi) = \{1\}, Z(\pi) = \{1, a, b, a^2\},$$

$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 3.$$

Three partitions of the form

$$\pi : \{1, b\}, \{a, a^2, ba^2, ba\}$$

$$St(\pi) = \{1, b\}, Z(\pi) = \langle b \rangle,$$

$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 2.$$

One partition

$$\pi : \{1, a, a^2\}, \{b, ba^2, ba\}$$

$$St(\pi) = \{1, a, a^2\}, Z(\pi) = D_3,$$

$$|St(\pi)| = 3, |Z(\pi)| = 6, |\pi| = 2.$$

And the coarsest partition

$$\pi : D_3$$

$$St(\pi) = D_3, Z(\pi) = D_3,$$

$$|St(\pi)| = 6, |Z(\pi)| = 6, |\pi| = 1.$$

Now, we need to draw the poset of optimal partitions π together with parameters $|St(\pi)|$,$|Z(\pi)|$ and $|\pi|$. The lattice of poset of optimal partitions is shown in figure 4.2.

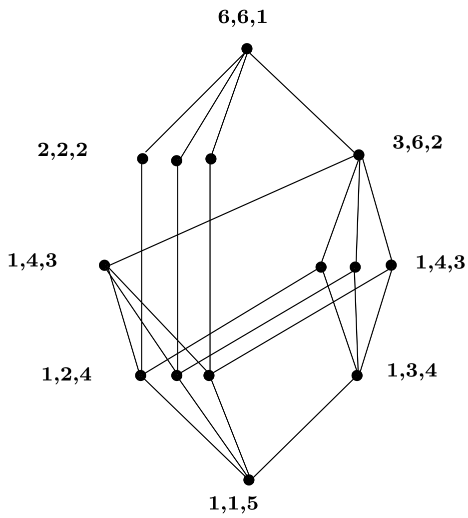

Figure 5.2: lattice of optimal partitions of D_3

Now, by theorem 3.25, it follows that

$$\begin{aligned}
|S_r(D_3)| &= 6\Big(r^5 + 3\Big(\frac{1}{2} - 1\Big)r^4 + \Big(\frac{1}{3} - 1\Big)r^4 + 3\Big(\frac{1}{4} - \frac{1}{2} - \frac{1}{3} + 1\Big)r^3 + \Big(\frac{1}{4} - \frac{3}{2} + 2\Big) \\
&+ 3\Big(\frac{2}{2} - \frac{1}{2}\Big)r^2 + \Big(\frac{1}{6} - \frac{1}{4} - \frac{3}{4} + \frac{3}{2} + \frac{2}{3} - 2\Big)r^2 + \Big(\frac{1}{6} - \frac{1}{6} - \frac{3}{2} + \frac{3}{2}\Big)r\Big) \\
&= 6\Big(r^5 - \frac{3}{2}r^4 - \frac{2}{3}r^4 + \frac{5}{4}r^3 + \frac{3}{4}r^3 - \frac{2}{3}r^2\Big) \\
&= 6r^5 - 13r^4 + 12r^3 - 4r^2
\end{aligned}$$

and

$$\begin{aligned}
|S_r(D_3)/\sim| &= r^5 + 3\Big(\frac{1}{2} - 1\Big)r^4 + \Big(\frac{1}{3} - 1\Big)r^4 + 3\Big(\frac{1}{4} - \frac{1}{2} - \frac{1}{3} + 1\Big)r^3 + \Big(\frac{1}{4} - \frac{3}{2} + 2\Big)r^3 \\
&+ 3\Big(\frac{2}{2} - \frac{1}{2}\Big)r^2 + \Big(\frac{3}{6} - \frac{1}{4} - \frac{3}{4} + \frac{3}{2} + \frac{2}{3} - 2\Big)r^2 + \Big(\frac{6}{6} - \frac{3}{6} - \frac{6}{2} + \frac{3}{2}\Big)r \\
&= r^5 - \frac{13}{6}r^4 + 2r^3 + \frac{7}{6}r^2 - r.
\end{aligned}$$

Proposition 5.1.1 (see [12]) For every $r \in \mathbb{N}$, $|S_r(D_3)| = 6r^5 - 13r^4 + 12r^3 - 4r^2$ and $|S_r(D_3)/\sim| = r^5 - \frac{13}{6}r^4 + 2r^3 + \frac{7}{6}r^2 - r$.

5.2 Quaternion group Q_8

The number of symmetric colorings and necklaces of the quartenion group of order 8 is already known (see [11]).

Proposition 5.2.1 For every $r \in \mathbb{N}$, $|S_r(Q_8)| = 4r^5 - 3r^4$ and $|S_r(Q_8)/\sim| = \frac{1}{2}r^5 - \frac{1}{4}r^4 + \frac{3}{4}r^2$.

5.3 Dihedral group D_4

Let $D_4 = \langle a, b : \; a^4 = b^2 = 1, bab = a^{-1}\rangle$ be the presentation of the classical dihedral group of order eight. The group has five order two subgroups, namely $\langle b\rangle \cong \langle ba\rangle \cong \langle ba^2\rangle \cong \langle ba^3\rangle \cong \langle a^2\rangle \cong \mathbb{Z}_2$. Again $\langle a\rangle \cong \mathbb{Z}_4$ and $\langle a^2, b\rangle \cong \langle a^2, ba\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ are the only subgroups of order four. By Lagrange's Theorem, it follows that $\langle 1\rangle$, $\langle b\rangle$, $\langle ba\rangle$, $\langle ba^2\rangle$, $\langle ba^3\rangle$, $\langle a^2\rangle$, $\langle a^2, b\rangle$, $\langle a^2, ba\rangle$, $\langle a\rangle$ and D_4 are the only subgroups of D_4. The lattice of subgroups of D_4 is shown in figure 5.3.

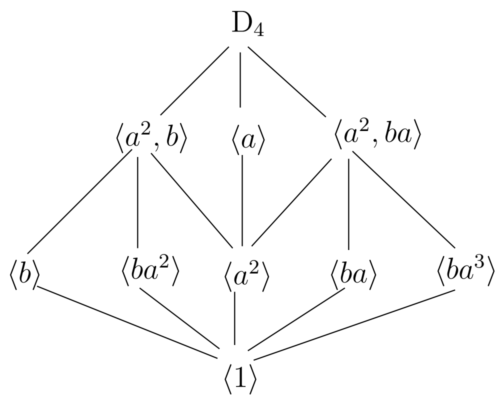

Figure 5.3: lattice of subgroups of D_4

The list of all optimal partitions π of D_4 is the following:

The finest partition

$$\pi : \{1\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{a^2\}, \{a, a^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a^2\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 7.$$

Two partitions of the form

$$\pi : \{1\}, \{b\}, \{ba^2\}, \{ba^3, ba\}, \{a^2\}, \{a, a^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a^2, b, ba^2\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 6.$$

One partition of the form

$$\pi : \{1, a^2\}, \{a, a^3\}, \{b\}, \{ba^2\}, \{ba^3\}, \{ba\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a \rangle$$
$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 6.$$

One partition of the form

$$\pi : \{1\}, \{a^2\}, \{a, a^3\}, \{b, ba^2\}, \{ba^3, ba\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a^2, b, ba, ba^2, ba^3\}$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 5.$$

Two partitions of the form

$$\pi : \{b\}, \{ba^2\}, \{ba^3, ba\}, \{1, a^2\}, \{a, a^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a, a^2, a^3, b, ba^2\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 5.$$

One partition of the form

$$\pi : \{b, ba^2\}, \{ba^3, ba\}, \{1, a^2\}, \{a, a^3\}$$
$$St(\pi) = \{1, a^2\}, Z(\pi) = D_4,$$
$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 4.$$

One partition of the form

$$\pi : \{b, ba, ba^2, ba^3\}, \{1, a, a^2, a^3\}$$
$$St(\pi) = \langle a \rangle, Z(\pi) = D_4,$$
$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 2.$$

Four partitions of the form

$$\pi : \{1, b\}, \{a^2, ba^2\}, \{a, a^3, ba, ba^3\}$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \left\langle a^2, b \right\rangle$$
$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 3.$$

Two partitions of the form

$$\pi : \{1, a^2, b, , ba^2\}, \{a, a^3, ba, ba^3\}$$
$$St(\pi) = \left\langle a^2, b \right\rangle, Z(\pi) = D_4,$$
$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 2.$$

And the coarsest partition

$$\pi : D_4$$

$$St(\pi) = D_4, Z(\pi) = D_4,$$

$$|St(\pi)| = 8, |Z(\pi)| = 8, |\pi| = 1.$$

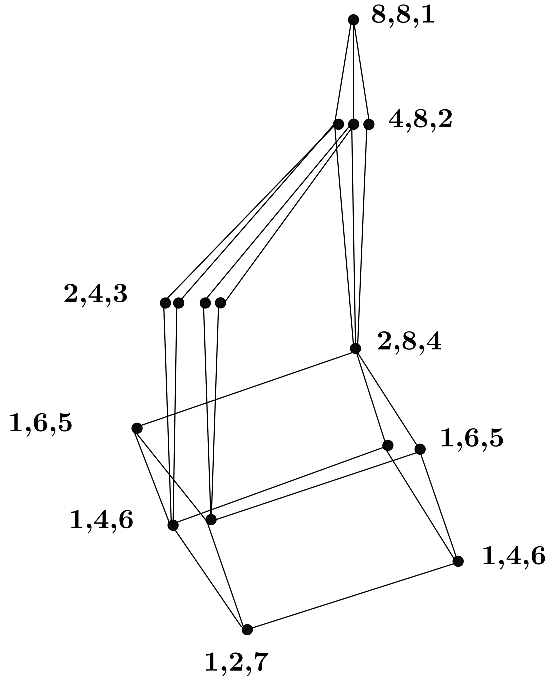

Figure 5.4: lattice of optimal partitions of D_4

Now, we obtain that

$$\begin{aligned}
|S_r(D_4)| &= 8\Big(\frac{1}{2}r^7 + 3\Big(\frac{1}{4} - \frac{1}{2}\Big)r^6 + 3\Big(\frac{1}{6} - \frac{1}{4} - \frac{1}{4} + \frac{1}{2}\Big)r^5 \\
&+ \Big(\frac{1}{8} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2}\Big)r^4 + 4\Big(\frac{1}{4} - \frac{1}{4}\Big)r^3 \\
&+ \Big(\frac{1}{8} - \frac{1}{8}\Big)r^2 + 2\Big(\frac{1}{8} - \frac{1}{8} - \frac{1}{4} - \frac{1}{4} + \frac{2}{4}\Big)r^2 + \Big(\frac{1}{8} - \frac{1}{8} - \frac{1}{8} - \frac{1}{8} + \frac{2}{8}\Big)r\Big) \\
&= 4r^7 - 6r^6 + 4r^5 - r^4
\end{aligned}$$

and

$$\begin{aligned}
|S_r(D_4)/\sim| &= \frac{1}{2}r^7 + 3\Big(\frac{1}{4}-\frac{1}{2}\Big)r^6 + 3\Big(\frac{1}{6}-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\Big)r^5 \\
&+ \Big(\frac{2}{8}-\frac{1}{6}-\frac{1}{6}-\frac{1}{6}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}\Big)r^4 + 4\Big(\frac{2}{4}-\frac{1}{4}\Big)r^3 \\
&+ \Big(\frac{4}{8}-\frac{2}{8}\Big)r^2 + 2\Big(\frac{4}{8}-\frac{2}{8}-\frac{2}{4}-\frac{2}{4}+\frac{2}{4}\Big)r^2 + \Big(\frac{8}{8}-\frac{4}{8}-\frac{4}{8}-\frac{4}{8}+\frac{4}{8}\Big)r \\
&= \frac{1}{2}r^7 - \frac{3}{4}r^6 + \frac{1}{2}r^5 + r^3 - \frac{1}{4}r^2.
\end{aligned}$$

Hence, we have

Proposition 5.3.1 For every $r \in \mathbb{N}$, $|S_r(D_4)/\sim| = \frac{1}{2}r^7 - \frac{3}{4}r^6 + \frac{1}{2}r^5 + r^3 - \frac{1}{4}r^2$ and $|S_r(D_4)| = 4r^7 - 6r^6 + 4r^5 - r^4$.

5.4 Dihedral group D_5

$D_5 = \langle a, b: \ a^4 = b^2 = 1, bab = a^{-1}\rangle$ is the presentation of dihedral group of order ten. Clearly, all the five subgroups of order two are isomorphic to $\mathbb{Z}_2$. There is one subgroup of order five, namely $\langle a\rangle \cong \mathbb{Z}_5$. By Lagrange's Theorem, $\langle 1\rangle$, $\langle b\rangle$, $\langle ba\rangle$, $\langle ba^2\rangle$, $\langle ba^3\rangle$, $\langle ba^4\rangle$, $\langle a\rangle$ and D_5 are the only subgroups of D_5. The lattices of subgroups of D_5 is given in figure 5.5 and the list of all optimal partitions π of D_5 is the following:

The finest partition

$$\pi : \{1\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}, \{a, a^4\}, \{a^2, a^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = 8.$$

Five partitions of the form

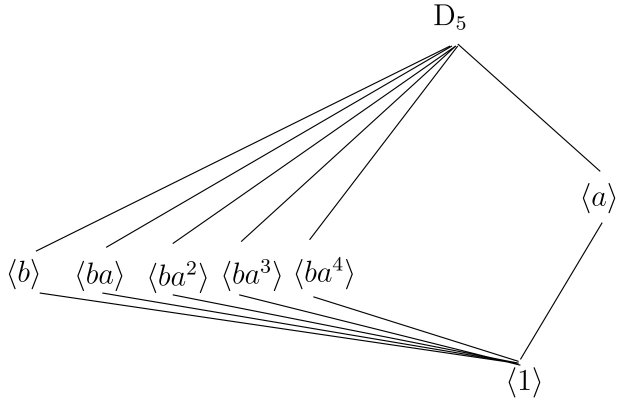

Figure 5.5: lattice of subgroups of D$_5$

$$\pi : \{1\}, \{b\}, \{ba, ba^4\}, \{ba^2, ba^3\}, \{a, a^4\}, \{a^2, a^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, b\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 6.$$

One partition

$$\pi : \{1, a, a^2, a^3, a^4\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 5, |\pi| = 6.$$

One partition

$$\pi : \{1\}, \{a, a^4\}, \{a^2, a^3\}, \{b, ba, ba^2, ba^3, ba^4\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, b, ba, ba^2, ba^3, ba^4\}$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 4.$$

Five partitions of the form

$$\pi : \{1, a, a^2, a^3, a^4\}, \{b\}, \{ba, ba^4\}, \{ba^2, ba^3\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a, a^2, a^3, a^4, b\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 4.$$

One partition

$$\pi : \{1, a, a^2, a^3, a^4\}, \{b, ba, ba^2, ba^3, ba^4\}$$
$$St(\pi) = \langle a \rangle, Z(\pi) = D_5$$
$$|St(\pi)| = 5, |Z(\pi)| = 10, |\pi| = 2.$$

Five partitions of the form

$$\pi : \{1, b\}, \{a, a^4, ba, ba^4\}, \{a^2, a^4, ba^2, ba^3\}$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \langle b \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 3.$$

And the coarsest partition

$$\pi : D_5$$
$$St(\pi) = D_5, Z(\pi) = D_5,$$
$$|St(\pi)| = 10, |Z(\pi)| = 10, |\pi| = 1.$$

Now, by theorem 3.2.5, it follows that

$$\begin{aligned} |S_r(D_5)| &= 10\Big(r^8 + 5\Big(\frac{1}{2} - 1\Big)r^6 + \Big(\frac{1}{5} - 1\Big)r^6 + 5\Big(\frac{1}{6} - \frac{1}{2} - \frac{1}{5} + 1\Big)r^4 + \Big(\frac{1}{6} - \frac{5}{2} + 4\Big)r^4 \\ &+ 5\Big(\frac{1}{2} - \frac{1}{2}\Big)r^3 + \Big(\frac{1}{10} - \frac{1}{6} - \frac{5}{6} + \frac{5}{2} + \frac{4}{5} - 4\Big)r^2 + \Big(\frac{1}{10} - \frac{1}{10} - \frac{5}{2} + \frac{5}{2}\Big)r\Big) \\ &= 10r^8 - 33r^6 + 40r^4 - 16r^2 \end{aligned}$$

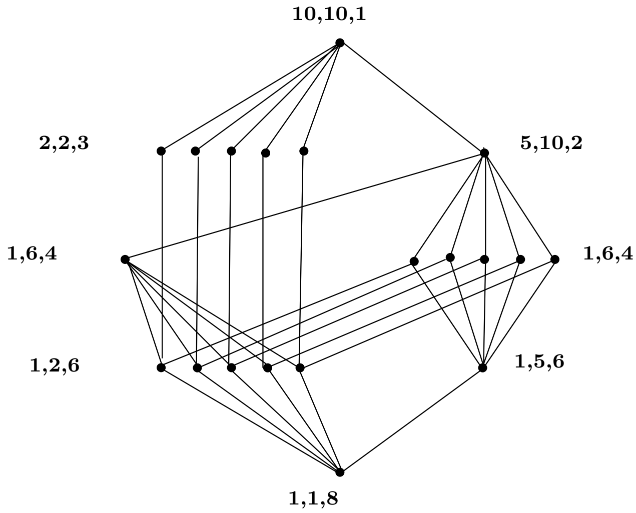

Figure 5.6: lattice of optimal partitions of D_5

and

$$\begin{aligned} |S_r(D_5)/\sim| &= r^8 + 5\Big(\frac{1}{2}-1\Big)r^6 + \Big(\frac{1}{5}-1\Big)r^6 + 5\Big(\frac{1}{6}-\frac{1}{2}-\frac{1}{5}+1\Big)r^4 + \Big(\frac{1}{6}-\frac{5}{2}+4\Big)r^4 \\ &+ 5\Big(\frac{2}{2}-\frac{1}{2}\Big)r^3 + \Big(\frac{5}{10}-\frac{1}{6}-\frac{5}{6}+\frac{5}{2}+\frac{4}{5}-4\Big)r^2 + \Big(\frac{10}{10}-\frac{5}{10}-\frac{10}{2}+\frac{5}{2}\Big)r \\ &= r^8 - \frac{33}{10}r^6 + 4r^4 + \frac{5}{2}r^3 - \frac{6}{5}r^2 - 2r. \end{aligned}$$

Proposition 5.4.1 For every $r \in \mathbb{N}$, $|S_r(D_5)| = 10r^8 - 33r^6 + 40r^4 - 16r^2$ and $|S_r(D_5)/\sim| = r^8 - \frac{33}{10}r^6 + 4r^4 + \frac{5}{2}r^3 - \frac{6}{5}r^2 - 2r$.

Chapter 6

Symmetric colorings of all non-abelian groups of order 12, the alternating group A_4, the dihedral group D_6 and the semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

6.1 Semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

Recall that $\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b : \; a^4 = b^3 = 1, ba = ab^{-1} \rangle$. The group has only one subgroup of order two and three, namely, $\langle a^2 \rangle \cong \mathbb{Z}_2$ and $\langle b \rangle \cong \mathbb{Z}_3$. $\langle a \rangle \cong \langle ba \rangle \cong \langle b^2 a \rangle \cong \mathbb{Z}_4$ are the only subgroups of order four. The order six subgroup is given by $\langle ba^2 \rangle \cong \mathbb{Z}_6$. The only other subgroups of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ are $\langle 1 \rangle$ and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. The figure 6.1 shows the lattice of subgroups of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. Now, we need to construct its poset of optimal partitions and the list of all optimal partions of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is given below.

The finest partition

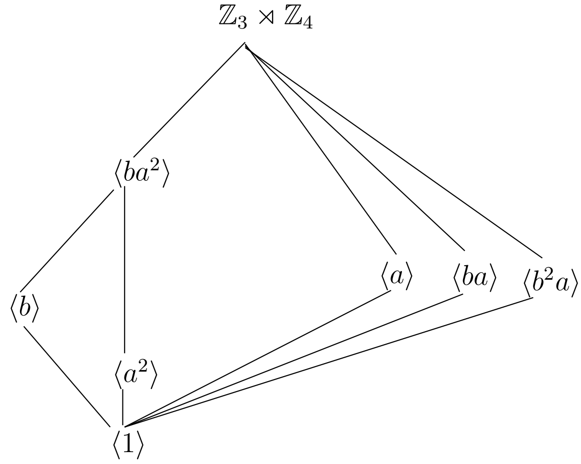

Figure 6.1: lattice of subgroups of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

$$\pi : \{1\}, \{b, b^2\}, \{ba, ba^3\}, \{ba^2, b^2a^2\}, \{b^2a^3, b^2a\}, \{a, a^3\}, \{a^2\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a^2 \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 7.$$

One partition of the form

$$\pi : \{1, a^2\}, \{a, a^3\}, \{b, b^2, ba^2, b^2a^2\}, \{ba, ba^3\}, \{b^2a^3, b^2a\},$$
$$St(\pi) = \langle a^2 \rangle, Z(\pi) = \langle a^2 \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 5.$$

Two partitions of the form

$$\pi : \{1, a^2\}, \{a, a^3, b^2a^3, b^2a\}, \{b, b^2, ba^2, b^2a^2\}, \{ba, ba^3\},$$
$$St(\pi) = \langle a^2 \rangle, Z(\pi) = \langle ba \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 4.$$

One partition of the form

$$\pi : \{1, a, a^2, a^3\}, \{b, b^2, ba^2, b^2a^2, ba, ba^3, b^2a, b^2a^3\},$$
$$St(\pi) = \langle a \rangle, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 4, |Z(\pi)| = 4, |\pi| = 2.$$

One partition of the form

$$\pi : \{a, a^3\}, \{1, b, b^2\}, \{a^2ba^2, b^2a^2\}, \{ba, ba^3\}, \{b^2a, b^2a^3\},$$
$$St(\pi) = \{1\}, Z(\pi) = \left\langle ba^2 \right\rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 5.$$

And the coarsest partition

$$\pi : \mathbb{Z}_3 \rtimes \mathbb{Z}_4$$
$$St(\pi) = \mathbb{Z}_3 \rtimes \mathbb{Z}_4, Z(\pi) = \mathbb{Z}_3 \rtimes \mathbb{Z}_4,$$
$$|St(\pi)| = 12, |Z(\pi)| = 12, |\pi| = 1.$$

By theorem 2.0.11

$$\begin{aligned} |S_r(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)| &= 12\Big(\frac{1}{2}r^7 + \Big(\frac{1}{2} - \frac{1}{2}\Big)r^5 + 2\Big(\frac{1}{4} - \frac{1}{2}\Big)r^4 + \Big(\frac{1}{6} - \frac{1}{2}\Big)r^5 \\ &+ \Big(\frac{1}{4} - \frac{1}{2}\Big)r^2 + \Big(\frac{1}{12} - \frac{1}{4} - \frac{1}{6} - \frac{1}{4} - \frac{1}{4} + \frac{2}{2} + \frac{1}{2}\Big)r\Big) \\ &= 6r^7 - 4r^5 - 6r^4 - 3r^2 + 8r \end{aligned}$$

and

$$|S_r(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)/\sim| = \frac{1}{2}r^7 + \Big(\frac{2}{2} - \frac{1}{2}\Big)r^5 + 2\Big(\frac{2}{4} - \frac{2}{2}\Big)r^4 + \Big(\frac{1}{6} - \frac{1}{2}\Big)r^5$$

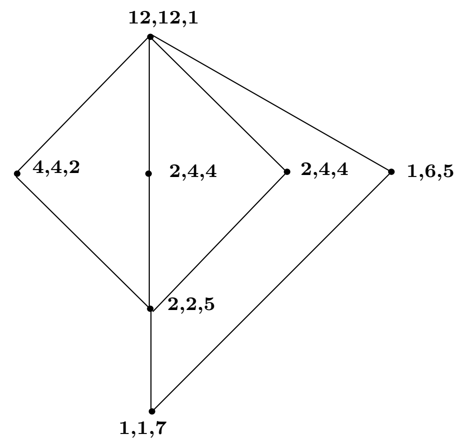

Figure 6.2: lattice of optimal partitions of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

$$
\begin{aligned}
&+ \quad \Big(\frac{4}{4}-\frac{2}{2}\Big)r^2+\Big(\frac{12}{12}-\frac{4}{4}-\frac{1}{6}-\frac{2}{4}-\frac{2}{4}+\frac{4}{2}+\frac{1}{2}\Big)r \\
&= \quad \frac{1}{2}r^7+\frac{1}{6}r^5-r^4+\frac{4}{3}r.
\end{aligned}
$$

Proposition 6.1.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)| = 6r^7-4r^5-6r^4-3r^2+8r$ and $|S_r(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)/\sim| = \frac{1}{2}r^7+\frac{1}{6}r^5-r^4+\frac{4}{3}r$.

6.2 Dihedral group D_6

Now, let $D_6 = \langle a, b : \ a^6 = b^2 = 1, bab = a^{-1}\rangle$ be the presentation of the dihedral group of order twelve. There are seven subgroups of order two, namlely $\langle b\rangle \cong \langle ba\rangle \cong \langle ba^2\rangle \cong \langle ba^3\rangle \cong \langle ba^4\rangle \cong \langle ba^5\rangle \cong \langle a^3\rangle \cong \mathbb{Z}_2$. $\langle a^2\rangle \cong \mathbb{Z}_3$. Recall that any group of order six is either isomorphic to $\mathbb{Z}_6$ or D_3. $\langle a\rangle \cong \mathbb{Z}_6$ and $\langle a^2, b\rangle \cong D_3 \cong \langle a^2, ba\rangle$ are the only subgroups of order six. Since there is no element of order four, $\langle a^3, b\rangle \cong \langle a^3, ba\rangle \cong \langle a^3, ba^2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ are the only subgroups of order four. The other subgroups are $\langle 1\rangle$ and D_6. The lattice of subgroups of D_6 is given in figure 5.3 and the list of all the optimal partions of D_6 is given below.

The finest partition

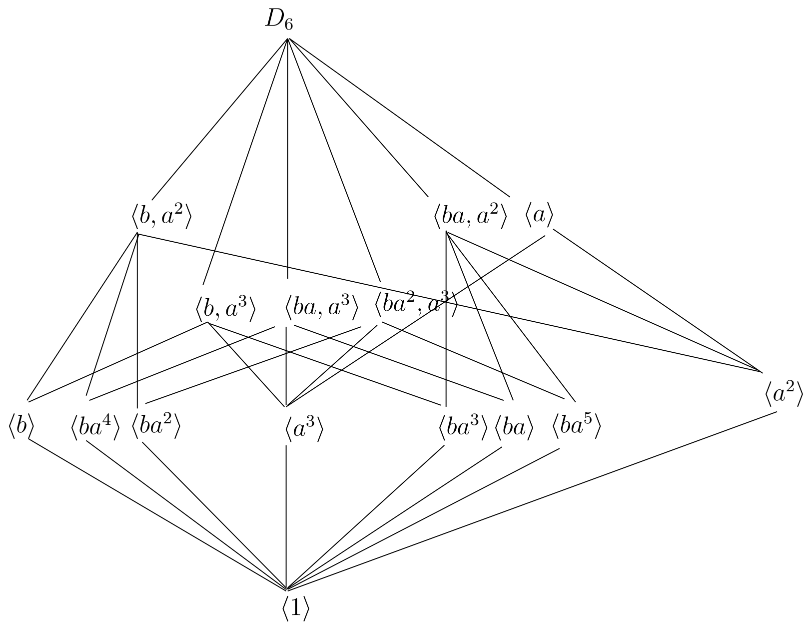

Figure 6.3: lattice of subgroups of D_6

$$\pi : \{1\}, \{a, a^5\}, \{a^2, a^4\}, \{a^3\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}, \{ba^5\}.$$
$$St(\pi) = \{1\}, Z(\pi) = \left\langle a^3 \right\rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 10.$$

One partition of the form

$$\pi : \{1, a^2, a^4\}, \{a, a^3, a^5\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}, \{ba^5\}.$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 8.$$

Three partitions of the form

$$\pi : \{1\}, \{a^2, a^4\}, \{a, a^5\}, \{a^3\}, \{b\}, \{ba, ba^5\}, \{ba^2, ba^4\}, \{ba^3\}.$$

$$St(\pi) = \{1\}, Z(\pi) = \langle a^3, b \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 8.$$

Three partitions of the form

$$\pi : \{1, a^2, a^4\}, \{a, a^3, a^5\}, \{b\}, \{ba, ba^5\}, \{ba^2, ba^4\}, \{ba^3\}.$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a, a^2, a^3, a^4, a^5, b, ba^3\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 8, |\pi| = 6.$$

One partition of the form

$$\pi : \{1\}, \{a^2, a^4\}, \{a, a^5\}, \{a^3\}, \{ba, ba^3, ba^5\}, \{b, ba^2, ba^4\}.$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a^3, b, ba, ba^2, ba^3, ba^4, ba^5\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 8, |\pi| = 6.$$

One partition of the form

$$\pi : \{1, a, a^2, a^3, a^4, a^5\}, \{b, ba^3\}, \{ba, ba^4\}, \{ba^2, ba^5\}.$$
$$St(\pi) = \langle a^3 \rangle, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 6, |\pi| = 4.$$

Three partitions of the form

$$\pi : \{1, a, a^2, a^3, a^4, a^5\}, \{b, ba^3\}, \{ba, ba^4, ba^2, ba^5\}.$$
$$St(\pi) = \langle a^3 \rangle, Z(\pi) = \{1, a, a^2, a^3, a^4, a^5, b, ba^3\},$$
$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 3.$$

One partition of the form

$$\pi : \{1, a^3\}, \{a, a^2, a^4, a^5\}, \{b, ba^3\}, \{ba, ba^4\}, \{ba^2, ba^5\}.$$

$$St(\pi) = \left\langle a^3 \right\rangle, Z(\pi) = \left\langle a^3 \right\rangle,$$

$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 5.$$

Three partitions of the form

$$\pi : \{1, a^3\}, \{a, a^2, a^4, a^5\}, \{b, ba^3\}, \{ba, ba^4, ba^2, ba^5\}.$$

$$St(\pi) = \left\langle a^3 \right\rangle, Z(\pi) = \left\langle a^3, b \right\rangle,$$

$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 4.$$

One partition of the form

$$\pi : \{1, a^3\}, \{a, a^2, a^4, a^5\}, \{b, ba^3, ba, ba^4, ba^2, ba^5\}.$$

$$St(\pi) = \left\langle a^3 \right\rangle, Z(\pi) = \{1, a^3, b, ba, ba^2, ba^3, ba^4, ba^5\}$$

$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 3.$$

One partition of the form

$$\pi : \{1, a^2, a^4\}, \{a, a^3, a^5\}, \{b, ba^2, ba^4\}, \{ba, ba^3, ba^5\}.$$

$$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = D_6,$$

$$|St(\pi)| = 3, |Z(\pi)| = 12, |\pi| = 4.$$

One partition of the form

$$\pi : \{1, a, a^2, a^3, a^4, a^5\}, \{b, ba, ba^2, ba^3, ba^4, ba^5\}.$$

$$St(\pi) = \langle a \rangle, Z(\pi) = D_6,$$

$$|St(\pi)| = 6, |Z(\pi)| = 12, |\pi| = 2.$$

Six partitions of the form

$$\pi : \{1, b\}, \{a, a^5, ba, ba^5\}, \{a^2, a^4, ba^2, ba^4\}, \{a^3, ba^3\}.$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \langle b, a^3 \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 4.$$

Three partitions of the form

$$\pi : \{1, b, a^3, ba^3\}, \{a, a^5, a^2, a^4, ba^2, ba^4, ba, ba^5\}.$$
$$St(\pi) = \langle b, a^3 \rangle, Z(\pi) = \langle b, a^3 \rangle,$$
$$|St(\pi)| = 4, |Z(\pi)| = 4, |\pi| = 2.$$

Two partitions of the form

$$\pi : \{1, a^2, a^4, b, ba^2, ba^4\}, \{a, a^3, a^5, ba, ba^3, ba^5\}.$$
$$St(\pi) = \langle b, a^2 \rangle, Z(\pi) = D_6,$$
$$|St(\pi)| = 6, |Z(\pi)| = 12, |\pi| = 2.$$

And the coarsest partition

$$\pi : D_6$$
$$St(\pi) = D_6, Z(\pi) = D_6,$$
$$|St(\pi)| = 12, |Z(\pi)| = 12, |\pi| = 1.$$

By theorem 2.0.11,

$$|S_r(D_6)| = 12\Big(\frac{1}{2}r^{10} + 3\Big(\frac{1}{4} - \frac{1}{2}\Big)r^8 + \Big(\frac{1}{6} - \frac{1}{2}\Big)r^8 + \Big(\frac{1}{8} - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} + \frac{2}{2}\Big)r^6$$

$$
\begin{aligned}
&+ \ 3\Big(\frac{1}{8}-\frac{1}{6}-\frac{1}{4}+\frac{1}{2}\Big)r^6+\Big(\frac{1}{2}-\frac{1}{2}\Big)r^5+3\Big(\frac{1}{4}-\frac{1}{2}-\frac{1}{4}+\frac{1}{2}\Big)r^4\\
&+ \ \Big(\frac{1}{6}-\frac{1}{2}-\frac{1}{6}+\frac{1}{2}\Big)r^4+\Big(\frac{1}{12}-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{2}{2}\Big)r^4\\
&+ \ 6\Big(\frac{1}{4}-\frac{1}{4}\Big)r^4+\Big(\frac{1}{8}-\frac{1}{8}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{2}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{2}{2}\Big)r^3\\
&+ \ 3\Big(\frac{1}{8}-\frac{1}{6}-\frac{1}{8}-\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}-\frac{1}{2}\Big)r^3+3\Big(\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{2}{4}\Big)r^2\\
&+ \ \Big(\frac{1}{12}-\frac{4}{8}-\frac{1}{12}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{2}{6}-\frac{2}{2}+\frac{4}{8}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{2}{6}+\frac{2}{2}\Big)r^2\\
&+ \ \Big(\frac{1}{12}-\frac{1}{12}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{2}{4}\Big)r^2\\
&+ \ \Big(\frac{1}{12}-\frac{1}{12}-\frac{1}{12}-\frac{1}{12}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}+\frac{6}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{2}{4}-\frac{2}{4}-\frac{2}{4}\Big)r\Big)\\
&= \ 6r^{10}-13r^8+12r^6-4r^4.
\end{aligned}
$$

Hence, from the lattice of optimal partitions of D_6 in figure 5.4
$|S_r(D_6)/\sim| = \frac{1}{2}r^{10} - \frac{13}{12}r^8 + r^6 + \frac{1}{2}r^5 + \frac{1}{4}r^4 + r^3 - \frac{7}{6}r^2$.
Therefore, we have

Proposition 6.2.1 For every $r \in \mathbb{N}$, $|S_r(D_6)| = 6r^{10} - 13r^8 + 12r^6 - 4r^4$ and $|S_r(D_6)/\sim| = \frac{1}{2}r^{10} - \frac{13}{12}r^8 + r^6 + \frac{1}{2}r^5 + \frac{1}{4}r^4 + r^3 - \frac{7}{6}r^2$.

6.3 The alternating group A_4

Again, the number of symmetric colorings and necklaces of the the group A_4 is already known.

Proposition 6.3.1 (see [10]) For every $r \in \mathbb{N}$, $|S_r(A_4)| = 12r^8 - 18r^6 + 9r^5 - 32r^4 + 72r^2 - 42r$ and $|S_r(A_4)/\sim| = r^8 - \frac{3}{2}r^6 + \frac{3}{4}r^5 - \frac{8}{3}r^4 + \frac{3}{4}r^3 + \frac{26}{3}r^2 - 6r$.

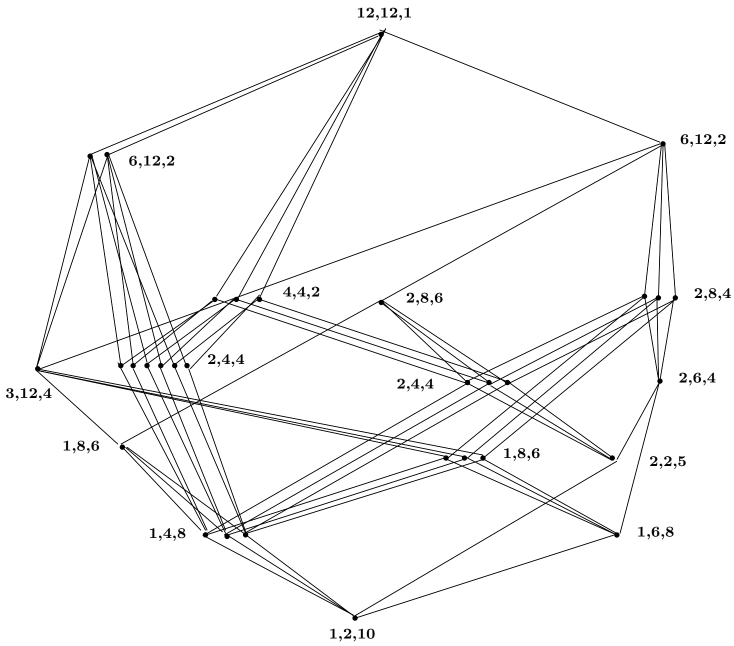

Figure 6.4: lattice of optimal partitions of D_6

Chapter 7

Symmetric colorings of all non-abelian groups of order 14

7.1 Dihedral group D_7

Let $D_7 = \langle a, b : \ a^7 = b^2 = 1, bab = a^{-1}\rangle$ be the presentation of the dihedral group of order 14. This group has seven order two subgroups, namely $\langle b\rangle \cong \langle ba\rangle \cong \langle ba^2\rangle \cong \langle ba^3\rangle \cong \langle ba^4\rangle \cong \langle ba^5\rangle \cong \langle ba^7\rangle \cong \mathbb{Z}_2$ and only one order seven subgroup, namely $\langle a\rangle \cong \mathbb{Z}_7$. Now, we need to construct its poset of optimal partitions. The subgroup lattice of D_7 is given in figure 7.1 and the list of all optimal partions of D_7 is given below.

The finest partition

$\pi : \{1\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}, \{ba^5\}, \{ba^6\}, \{a, a^2\}, \{a^2, a^5\}, \{a^3, a^4\}$

$St(\pi) = \{1\}, Z(\pi) = \{1\},$

$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = 11.$

Seven partitions of the form

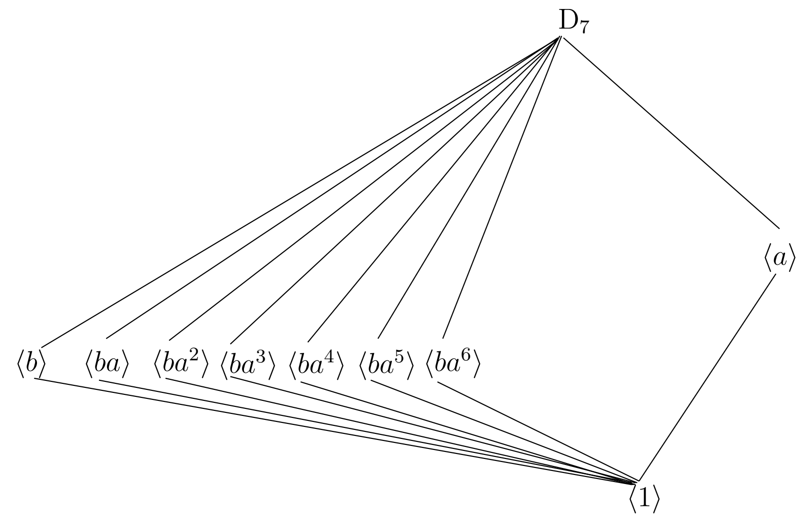

Figure 7.1: lattice of subgroups of D_7

$$\pi : \{1\}, \{ba\}, \{b, ba^2\}, \{ba^3, ba^6\}, \{ba^4, ba^5\}, \{a, a^6\}, \{a^2, a^5\}, \{a^3, a^4\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle ba \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 8.$$

One partition of the form

$$\pi : \{1\}, \{ba, b, ba^2, ba^3, ba^6, ba^4, ba^5\}, \{a, a^6\}, \{a^2, a^5\}, \{a^3, a^4\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, b, ba, ba^2, ba^3, ba^4, ba^5, ba^6\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 8, |\pi| = 5.$$

One partition of the form

$$\pi : \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\}, \{ba^5\}, \{ba^6\}, \{1, a, a^2, a^3, a^4, a^5, a^6\}$$
$$St(\pi) = \{1\}, Z(\pi) = \langle a \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 7, |\pi| = 8.$$

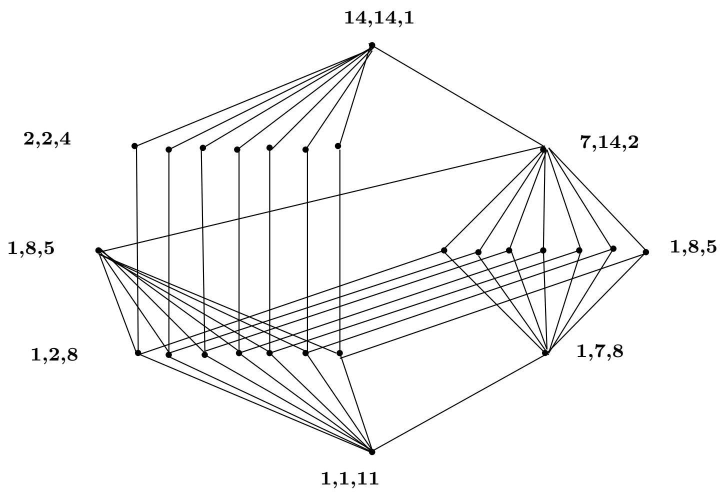

Figure 7.2: lattice of optimal partitions of D_7

Seven partitions of the form

$$\pi : \{b\}, \{ba, ba^6\}, \{ba^2, ba^5\}, \{ba^3, ba^4\}, \{1, a, a^2, a^3, a^4, a^5, a^6\}$$
$$St(\pi) = \{1\}, Z(\pi) = \{1, a, a^2, a^3, a^4, a^5, a^6, b\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 8, |\pi| = 5.$$

One partition of the form

$$\pi : \{b, ba, ba^2, ba^3, ba^4, ba^5, ba^6\}, \{1, a, a^2, a^3, a^4, a^5, a^6\}$$
$$St(\pi) = \langle a \rangle, Z(\pi) = D_7,$$
$$|St(\pi)| = 7, |Z(\pi)| = 14, |\pi| = 2.$$

Seven partitions of the form

$$\pi : \{1, b\}, \{a, a^6, ba, ba^6\}, \{a^2, a^5, ba^2, ba^5\}, \{a^3, a^4, ba^3, ba^4\}$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \langle b \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 4.$$

And the coarsest partition

$$\pi : D_7$$
$$St(\pi) = D_7, Z(\pi) = D_7,$$
$$|St(\pi)| = 14, |Z(\pi)| = 14, |\pi| = 1.$$

Now, by theorem 3.2.5

$$\begin{aligned}
|S_r(D_7)| &= 14\Big(r^{11} + 7\Big(\frac{1}{2} - 1\Big)r^8 + \Big(\frac{1}{7} - 1\Big)r^8 + 7\Big(\frac{1}{8} - \frac{1}{2} - \frac{1}{7} + 1\Big)r^5 + \Big(\frac{1}{8} - \frac{7}{2} + 6\Big)r^5 \\
&+ 7\Big(\frac{2}{2} - \frac{1}{2}\Big)r^4 + \Big(\frac{1}{14} - \frac{1}{8} - \frac{7}{8} + \frac{7}{2} + \frac{6}{7} - 6\Big)r^2 + \Big(\frac{1}{14} - \frac{1}{14} - \frac{7}{2} + \frac{7}{2}\Big)r\Big) \\
&= 14r^{11} - 61r^8 + 84r^5 - 36r^2
\end{aligned}$$

and

$$\begin{aligned}
|S_r(D_7)/\sim| &= r^{11} + 7\Big(\frac{1}{2} - 1\Big)r^8 + \Big(\frac{1}{7} - 1\Big)r^8 + 7\Big(\frac{1}{8} - \frac{1}{2} - \frac{1}{7} + 1\Big)r^5 + \Big(\frac{1}{8} - \frac{7}{2} + 6\Big)r^5 \\
&+ 7\Big(\frac{2}{2} - \frac{1}{2}\Big)r^4 + \Big(\frac{7}{14} - \frac{1}{8} - \frac{7}{8} + \frac{7}{2} + \frac{6}{7} - 6\Big)r^2 + \Big(\frac{14}{14} - \frac{7}{14} - 7 + \frac{7}{2}\Big)r \\
&= r^{11} - \frac{61}{14}r^8 + 6r^5 + \frac{7}{2}r^4 - \frac{15}{7}r^2 - 3r.
\end{aligned}$$

Proposition 7.1.1 For every $r \in \mathbb{N}$, $|S_r(D_7)| = 14r^{11} - 61r^8 + 84r^5 - 36r^2$ and $|S_r(D_7)/\sim| = r^{11} - \frac{61}{14}r^8 + 6r^5 + \frac{7}{2}r^4 - \frac{15}{7}r^2 - 3r$.

Chapter 8

Symmetric colorings of non-abelian groups of order 16, the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, generalised quaternion Q_{16}, semidihedral group DQ_{16}, modular group M_{16} and $Q_8 \times \mathbb{Z}_2$

labelchap:appl

8.1 Generalised quaternion Q_{16}

$Q_{16} = \langle a, b : a^8 = b^4 = 1, aba = b, a^4 = b^2 \rangle$ is the presentation of the generalised quaternion Q_{16} (see [3]). a^4 is the only element of order two. Hence $\langle a^4 \rangle \cong \mathbb{Z}_2$. There are ten elements of order four. Hence, there are five cyclic subgroups of order four, $\langle a^2 \rangle \cong \langle ab \rangle \cong \langle a^2 b \rangle \cong \langle a^3 b \rangle \cong \langle b \rangle \cong \mathbb{Z}_4$. The subgroups of order eight are

$\langle a^2, b\rangle$, $\langle a^2, ab\rangle$ and $\langle a\rangle$. The lattice of subgroups of Q_{16} is shown in figure 8.1 and the list of all optimal partions of Q_{16} is the following

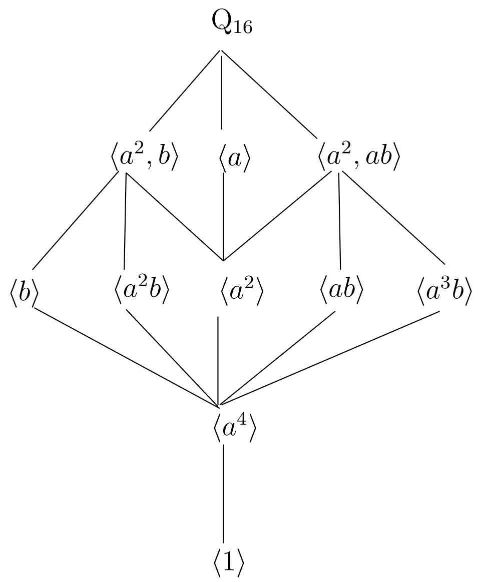

Figure 8.1: lattice of subgroups of Q_{16}

The finest partition

$\pi : \{1\}, \{a, a^7\}, \{a^2, a^6\}, \{a^3, a^5\}, \{a^4\}, \{b, b^3\}, \{ab, a^5b\}, \{a^2b, a^6b\}, \{a^3b, a^7b\}.$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^4 \right\rangle,$

$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 9.$

Two partitions of the form

$\pi : \{1, a^4\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}`, \{b, b^3, a^2b, a^6b\}, \{ab, a^5b\}, \{a^3b, a^7b\}.$

$St(\pi) = \left\langle a^4 \right\rangle, Z(\pi) = \left\langle a^2, ab \right\rangle,$

$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 6.$

One partition of the form

$$\pi : \{1, a^4\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}`, \{b, b^3\}, \{a^2b, a^6b\}, \{ab, a^5b\}, \{a^3b, a^7b\}.$$

$$St(\pi) = \left\langle a^4 \right\rangle, Z(\pi) = \left\langle a^2 \right\rangle,$$

$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 7.$$

Two partitions of the form

$$\pi : \{1, a^4, b, b^3\}, \{a, a^7, a^3, a^5, b, b^3, ab, a^5b, a^3b, a^7b\}, \{a^2, a^6, a^2b, a^6b\}.$$

$$St(\pi) = \langle b \rangle, Z(\pi) = \left\langle a^2, b \right\rangle,$$

$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 3.$$

Two partitions of the form

$$\pi : \{1, a^4, ab, a^5b\}, \{a, a^7, a^3, a^5, b, b^3, a^2b, a^6b\}, \{a^2, a^6, a^7b, a^3b\}.$$

$$St(\pi) = \langle ab \rangle, Z(\pi) = \left\langle a^2, ab \right\rangle,$$

$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 3.$$

One partition of the form

$$\pi : \{1, a^2, a^4, a^6\}, \{a, a^7, a^3, a^5\}, \{b, b^3, a^2b, a^6b\}, \{ab, a^5b, a^3b, a^7b\}.$$

$$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = Q_{16},$$

$$|St(\pi)| = 4, |Z(\pi)| = 16, |\pi| = 4.$$

One partition of the form

$$\pi : \{1, a^2, a^4, a^6, a, a^7, a^3, a^5\}, \{b, b^3, a^2b, a^6b, ab, a^5b, a^3b, a^7b\}.$$

$$St(\pi) = \langle a \rangle, Z(\pi) = Q_{16},$$

$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

Two partitions of the form

$$\pi : \{1, a^4, ab, a^5b, a^2, a^6, a^7b, a^3b\}, \{a, a^7, a^3, a^5, b, b^3, a^2b, a^6b\}.$$
$$St(\pi) = \langle a^2, ab \rangle, Z(\pi) = Q_{16},$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

And the coarsest partition

$$\pi : Q_{16}$$
$$St(\pi) = Q_{16}, Z(\pi) = Q_{16},$$
$$|St(\pi)| = 16, |Z(\pi)| = 16, |\pi| = 1.$$

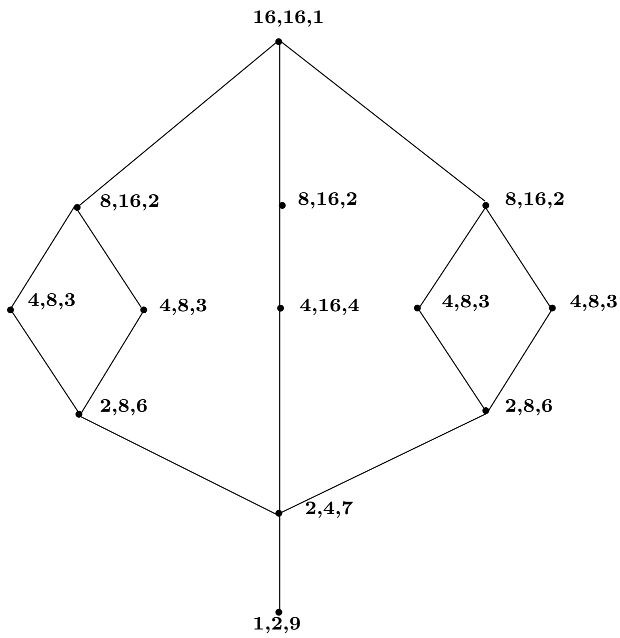

Figure 8.2: lattice of optimal partitions of Q_{16}

$$
\begin{aligned}
|S_r(Q_{16})| &= 16\Big(\frac{1}{2}r^9 + \Big(\frac{1}{4}-\frac{1}{2}\Big)r^7 + 2\Big(\frac{1}{8}-\frac{1}{4}\Big)r^6 + 4\Big(\frac{1}{8}-\frac{1}{8}\Big)r^3 + \Big(\frac{1}{16}-\frac{1}{4}\Big)r^4 \\
&+ \Big(\frac{1}{16}-\frac{1}{16}\Big)r^2 + 2\Big(\frac{1}{16}-\frac{1}{8}-\frac{1}{8}+\frac{1}{8}\Big)r^2 + (\frac{1}{16}-\frac{1}{16}-\frac{1}{16}-\frac{1}{16}+\frac{2}{4}\Big)r\Big) \\
&= 8r^9 - 4r^7 - 4r^6 - 3r^4 - 2r^2 + 6r
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(Q_{16})/\sim| &= \frac{1}{2}r^9 + \Big(\frac{2}{4}-\frac{1}{2}\Big)r^7 + 2\Big(\frac{2}{8}-\frac{2}{4}\Big)r^6 + 4\Big(\frac{4}{8}-\frac{2}{8}\Big)r^3 + \Big(\frac{4}{16}-\frac{2}{4}\Big)r^4 \\
&+ \Big(\frac{8}{16}-\frac{4}{16}\Big)r^2 + 2\Big(\frac{8}{16}-\frac{4}{8}-\frac{4}{8}+\frac{2}{8}\Big)r^2 + \Big(\frac{16}{16}-\frac{8}{16}-\frac{8}{16}-\frac{8}{16}+\frac{4}{4}\Big)r \\
&= \frac{1}{2}r^9 - \frac{1}{2}r^6 - \frac{1}{4}r^4 + r^3 - \frac{1}{4}r^2 + \frac{1}{2}r.
\end{aligned}
$$

Proposition 8.1.1 For every $r \in \mathbb{N}$, $|S_r(Q_{16})| = 8r^9 - 4r^7 - 4r^6 - 3r^4 - 2r^2 + 6r$ and $|S_r(Q_{16})/\sim| = \frac{1}{2}r^9 - \frac{1}{2}r^6 - \frac{1}{4}r^4 + r^3 - \frac{1}{4}r^2 + \frac{1}{2}r$.

8.2 Semidihedral group DQ_{16}

Another group of interest is the semidihedral group $DQ_{16} = \langle a, b : \ a^8 = b^2 = 1, bab = a^3\rangle$. $\langle a^4\rangle \cong \langle b\rangle \cong \langle a^2b\rangle \cong \langle a^4b\rangle \cong \langle a^6b\rangle \cong \mathbb{Z}_2$ are subgroups of order two. The cyclic subgroups of order four are $\langle a^2\rangle \cong \langle ab\rangle \cong \langle a^3b\rangle$. From the five order two subgroups, we can easily construct two subgroups isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$, namely $\langle a^4, b\rangle$ and $\langle a^4, a^2b\rangle$. The subgroups of order eight are $\langle a^2, b\rangle$, $\langle a^2, ab\rangle$ and $\langle a\rangle$. The lattice of subgroups of DQ_{16} is shown in figure and the list of all optimal partions of Q_{16} is given below.

The finest partition

$\pi : \{1\}, \{a, a^7\}, \{a^2, a^6\}, \{a^3, a^5\}, \{a^4\}, \{b\}, \{ab, a^5b\}, \{a^2b\}, \{a^4b\}, \{a^6b\}, \{a^3b, a^7b\}.$

$St(\pi) = \{1\}, Z(\pi) = \langle a^4\rangle,$

$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 11.$

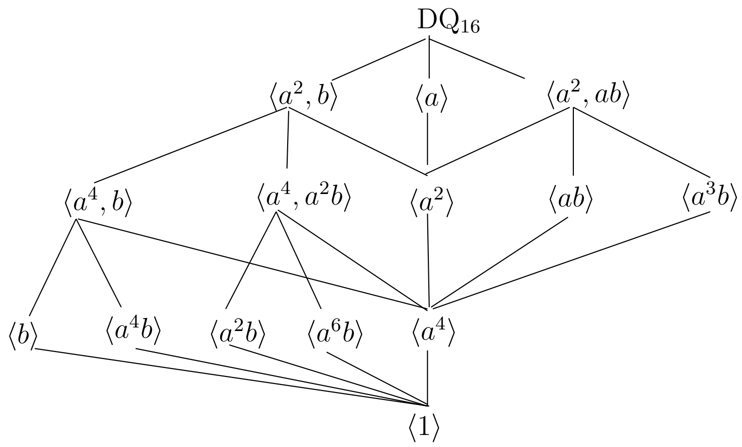

Figure 8.3: lattice of subgroups of DQ_{16}

Four partitions of the form

$\pi : \{1\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}, \{a^4\}, \{b\}, \{ab, a^5b, a^3b, a^7b\}, \{a^2b, a^6b\}, \{a^4b\}.$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^4, b \right\rangle,$

$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 8.$

One partition of the form

$\pi : \{1, a^4\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}, \{b\}, \{ab, a^5b\}, \{a^2b\}, \{a^4b\}, \{a^6b\}, \{a^3b, a^7b\}.$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^2 \right\rangle,$

$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 9.$

Four partitions of the form

$\pi : \{1, b\}, \{a, a^7, a^3, a^5, ab, a^5b, a^3b, a^7b\}, \{a^2b, a^6b, a^2, a^6\}, \{a^4b, a^4\}.$

$St(\pi) = \langle b \rangle, Z(\pi) = \left\langle a^4, b \right\rangle,$

$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 4.$

One partition of the form

$$\pi : \{1, a^4\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}, \{b, a^4b\}, \{ab, a^5b\}, \{a^2b, a^6b\}, \{a^3b, a^7b\}.$$
$$St(\pi) = \left\langle a^4 \right\rangle, Z(\pi) = \left\langle a^2 \right\rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 7.$$

Two partition of the form

$$\pi : \{1, a^4, ab, a^5b\}, \{a^2, a^6, a^3b, a^7b\}, \{a, a^3, a^5, a^7, b, a^2b, a^4b, , a^6b\}.$$
$$St(\pi) = \langle ab \rangle, Z(\pi) = \left\langle a^2, ab \right\rangle,$$
$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 3.$$

One partition of the form

$$\pi : \{1, a^2, a^4, a^6\}, \{a, a^3, a^5, a^7\}, \{, b, a^2b, a^4b, , a^6b\}, \{ab, a^3b, a^5b, a^7b\}.$$
$$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = DQ_{16},$$
$$|St(\pi)| = 4, |Z(\pi)| = 16, |\pi| = 4.$$

One partition of the form

$$\pi : \{1, a^2, a^4, a^6, a, a^3, a^5, a^7\}, \{, b, a^2b, a^4b, a^6b, ab, a^3b, a^5b, a^7b\}.$$
$$St(\pi) = \langle a \rangle, Z(\pi) = DQ_{16},$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

Two partitons of the form

$$\pi : \{1, a^2, a^4, a^6, b, a^2b, a^4b, , a^6b\}, \{a, a^3, a^5, a^7, ab, a^3b, a^5b, a^7b\}.$$
$$St(\pi) = \left\langle a^2, b \right\rangle, Z(\pi) = DQ_{16},$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

And the coarsest partition

$$
\begin{aligned}
&\pi : DQ_{16}\\
&St(\pi) = DQ_{16}, Z(\pi) = DQ_{16},\\
&|St(\pi)| = 16, |Z(\pi)| = 16, |\pi| = 1.
\end{aligned}
$$

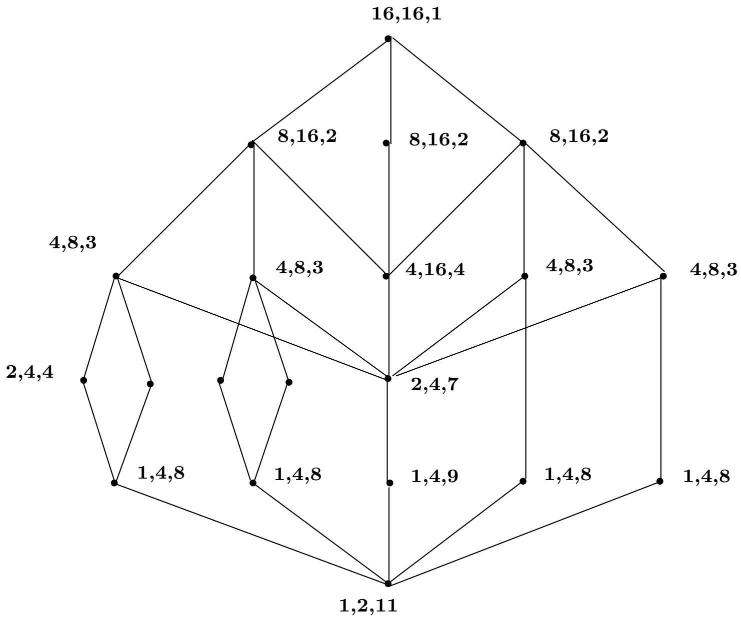

Figure 8.4: lattice of optimal partitions of DQ_{16}

$$
\begin{aligned}
|S_r(DQ_{16})| &= 16\Big(\frac{1}{2}r^{11} + 4\Big(\frac{1}{4}-\frac{1}{2}\Big)r^8 + \Big(\frac{1}{4}-\frac{1}{2}\Big)r^9 + \Big(\frac{1}{4}-\frac{1}{4}\Big)r^7 + 4\Big(\frac{1}{4}-\frac{1}{4}\Big)r^4\\
&+ \Big(\frac{1}{16}-\frac{1}{4}\Big)r^4 + 2\Big(\frac{1}{8}-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\Big)r^3 + \Big(\frac{1}{16}-\frac{1}{16}\Big)r^2 + 2\Big(\frac{1}{8}-\frac{3}{4}+\frac{1}{4}+\frac{1}{2}\Big)r^3\\
&+ \Big(\frac{1}{16}-\frac{1}{8}-\frac{1}{8}-\frac{1}{16}+\frac{2}{4}\Big)r^2 + \Big(\frac{1}{16}-\frac{1}{8}-\frac{1}{8}-\frac{1}{16}+\frac{2}{4}\Big)r^2\\
&+ \Big(\frac{1}{16}-\frac{1}{16}-\frac{1}{16}-\frac{1}{16}+\frac{2}{16}\Big)r\Big)\\
&= 8r^{11} - 4r^9 - 16r^8 - 3r^4 + 8r^3 + 8r^2
\end{aligned}
$$

and

$$\begin{aligned}|S_r(DQ_{16})/\sim| &= \frac{1}{2}r^{11}+4\Big(\frac{1}{4}-\frac{1}{2}\Big)r^8+\Big(\frac{1}{4}-\frac{1}{2}\Big)r^9+\Big(\frac{2}{4}-\frac{1}{4}\Big)r^7+4\Big(\frac{2}{4}-\frac{1}{4}\Big)r^4\\ &+ \Big(\frac{4}{16}-\frac{2}{4}\Big)r^4+2\Big(\frac{4}{8}-\frac{2}{4}-\frac{1}{4}+\frac{1}{2}\Big)r^3+2\Big(\frac{4}{8}-\frac{6}{4}+\frac{1}{4}+\frac{1}{2}\Big)r^3\\ &+ \Big(\frac{8}{16}-\frac{4}{16}\Big)r^2+\Big(\frac{8}{16}-\frac{4}{8}-\frac{4}{8}-\frac{4}{16}+\frac{4}{4}\Big)r^2\\ &+ \Big(\frac{8}{16}-\frac{4}{8}-\frac{4}{8}-\frac{4}{16}+\frac{4}{4}\Big)r^2+\Big(\frac{16}{16}-\frac{8}{16}-\frac{8}{16}-\frac{8}{16}+\frac{8}{16}\Big)r\\ &= \frac{1}{2}r^{11}-\frac{1}{4}r^9-r^8+\frac{1}{4}r^7+\frac{3}{4}r^4+\frac{3}{4}r^2.\end{aligned}$$

Proposition 8.2.1 For every $r \in \mathbb{N}$, $|S_r(DQ_{16})| = 8r^{11}-4r^9-16r^8-3r^4+8r^3+8r^2$ and $|S_r(DQ_{16})/\sim| = \frac{1}{2}r^{11}-\frac{1}{4}r^9-r^8+\frac{1}{4}r^7+\frac{3}{4}r^4+\frac{3}{4}r^2$.

8.3 The semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ has three elements of order two and twelve elements of order four. Hence, $\langle a^2\rangle$, $\langle b^2\rangle$ and $\langle a^2b^2\rangle$ are sugroups of order two. The six cyclic subgroups of order four are $\langle a\rangle$, $\langle ba^2\rangle$, $\langle b^2a\rangle$, $\langle b\rangle$, $\langle ba\rangle$ and $\langle ba^3\rangle$. From the three subgroups of order four, we can construct the Klein 4-group $\langle a^2, b^2\rangle$. $\langle a^2, b\rangle$, $\langle a^2, ba\rangle$ and $\langle a, b^2\rangle$ are the subgroups of order six. The lattice of subgroups of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ is shown in figure 8.5 and the list of all optimal partions of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ is given below.

Finest partition

$\pi : \{1\}, \{a, a^3\}, \{a^2\}, \{b, b^3\}, \{b^2\}, \{ba, b^3a\}, \{ba^2, b^3a^2\}, \{ba^3, b^3a^3\}, \{b^2a^2\}, \{b^2a, b^2a^3\}$

$St(\pi) = \{1\}, Z(\pi) = \langle a^2, b^2\rangle$

$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 10.$

One partition of the form

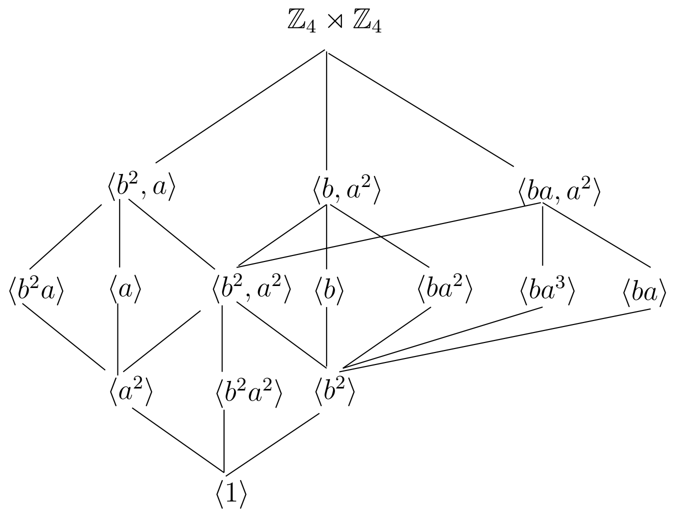

Figure 8.5: lattice of subgroups of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

$$\pi : \{1, b^2a^2\}, \{a, a^3, b^2a, b^2a^3\}, \{a^2, b^2\}, \{b, b^3, ba^2, b^3a^2\}, \{ba, b^3a, ba^3, b^3a^3\}$$

$$St(\pi) = \left\langle a^2b^2 \right\rangle, Z(\pi) = \left\langle a^2, b^2 \right\rangle$$

$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 5.$$

One partition of the form

$$\pi : \{1, b^2\}, \{a, a^3, b^2a, b^2a^3\}, \{a^2, b^2a^2\}, \{b, b^3\}, \{ba, b^3a\}, \{ba^2, b^3a^2\}, \{ba^3, b^3a^3\}$$

$$St(\pi) = \left\langle b^2 \right\rangle, Z(\pi) = \left\langle a^2, b^2 \right\rangle$$

$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 7.$$

One partition of the form

$$\pi : \{1, a^2\}, \{a, a^3\}, \{b, b^3, ba^2, b^3a^2\}, \{b^2, b^2a^2\}, \{ba, b^3a, ba^3, b^3a^3\},$$

$$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = \left\langle a, b^2 \right\rangle$$

$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 6$$

One partition of the form

$$\pi : \{1, b^2\}, \{a, a^3, b^2a, b^2a^3\}, \{a^2, b^2a^2\}, \{b, b^3\}, \{ba, b^3a, ba^3, b^3a^3\}, \{ba^2, b^3a^2\}$$

$$St(\pi) = \left\langle b^2 \right\rangle, Z(\pi) = \left\langle a^2, b \right\rangle$$

$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 6.$$

One partition of the form

$$\pi : \{1, b^2, a^2, b^2a^2\}, \{a, a^3, b^2a, b^2a^3\}, \{ba, b^3a, ba^3, b^3a^3\}, \{b, b^3, ba^2, b^3a^2\}$$

$$St(\pi) = \left\langle b^2, a^2 \right\rangle, Z(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4$$

$$|St(\pi)| = 4, |Z(\pi)| = 4, |\pi| = 16.$$

Two partitions of the form

$$\pi : \{1, a, a^2, a^3\}, \{b^2, b^2a^2, b^2a, b^2a^3\}, \{ba, b^3a, ba^3, b^3a^3, b, b^3, ba^2, b^3a^2\}$$

$$St(\pi) = \langle a \rangle, Z(\pi) = \left\langle b^2, a \right\rangle$$

$$|St(\pi)| = 3, |Z(\pi)| = 4, |\pi| = 8.$$

Two partitions of the form

$$\pi : \{1, b^2, b, b^3\}, \{a, a^3, b^2a, b^2a^3, ba, b^3a, ba^3, b^3a^3\}, \{ba^2, b^3a^2, a^2, b^2a^2\}$$

$$St(\pi) = \langle b \rangle, Z(\pi) = \left\langle a^2, b \right\rangle$$

$$|St(\pi)| = 3, |Z(\pi)| = 4, |\pi| = 8.$$

Two partitions of the form

$$\pi : \{1, b^2, ba, b^3a\}, \{a^2, b^2a^2, ba^3, b^3a^3\}, \{b, b^3, ba^2, b^3a^2, a, a^3, b^2a, b^2a^3\}$$

$$St(\pi) = \langle ba \rangle, Z(\pi) = \left\langle a^2, ba \right\rangle$$

$$|St(\pi)| = 3, |Z(\pi)| = 4, |\pi| = 8.$$

One partition of the form

$$\pi : \{1, b^2, ba, b^3a, a^2, b^2a^2, ba^3, b^3a^3\}, \{b, b^3, ba^2, b^3a^2, a, a^3, b^2a, b^2a^3\}$$
$$St(\pi) = \left\langle ba, a^2 \right\rangle, Z(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

One partition of the form

$$\pi : \{1, b^2, b, b^3, ba^2, b^3a^2, a^2, b^2a^2\}, \{a, a^3, b^2a, b^2a^3, ba, b^3a, ba^3, b^3a^3\}$$
$$St(\pi) = \left\langle a^2, b \right\rangle, Z(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

One partition of the form

$$\pi : \{1, a, a^2, a^3, b^2, b^2a^2, b^2a, b^2a^3\}, \{ba, b^3a, ba^3, b^3a^3, b, b^3, ba^2, b^3a^2\}$$
$$St(\pi) = \left\langle b^2, a \right\rangle, Z(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

And the coarsest partition

$$\pi : \mathbb{Z}_4 \rtimes \mathbb{Z}_4$$
$$St(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4, Z(\pi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_4,$$
$$|St(\pi)| = 16, |Z(\pi)| = 16, |\pi| = 1.$$

By theorem 3.2.5, it follows that

$$\begin{aligned} |S_r(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)| &= 16\Big(\frac{1}{4}r^{10} + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^7 + \Big(\frac{1}{8} - \frac{1}{4}\Big)r^6 + \Big(\frac{1}{8} - \frac{1}{4}\Big)r^6 + \Big(\frac{1}{4} - \frac{1}{4}\Big)r^5 \\ &+ \Big(\frac{1}{16} - \frac{1}{8} - \frac{1}{8} - \frac{1}{4} + \frac{2}{4}\Big)r^4 + 2\Big(\frac{1}{8} - \frac{1}{8}\Big)r^3 + 4\Big(\frac{1}{8} - \frac{1}{8}\Big)r^3 \end{aligned}$$

$$
\begin{aligned}
&+ \quad 3\Big(\frac{1}{16} - \frac{1}{16} - \frac{1}{8} - \frac{1}{8} + \frac{2}{8}\Big)r^2 + \Big(\frac{1}{16} - \frac{1}{16} - \frac{1}{16} - \frac{1}{16} + \frac{2}{16}\Big)r\Big) \\
&= \quad 4r^{10} - 4r^6 + r^4
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)/\sim| \quad &= \quad \frac{1}{4}r^{10} + \Big(\frac{2}{4} - \frac{1}{4}\Big)r^7 + \Big(\frac{2}{8} - \frac{2}{4}\Big)r^6 + \Big(\frac{2}{8} - \frac{1}{4}\Big)r^6 + \Big(\frac{2}{4} - \frac{1}{4}\Big)r^5 \\
&+ \quad \Big(\frac{4}{16} - \frac{2}{8} - \frac{2}{8} - \frac{2}{4} + \frac{2}{4}\Big)r^4 + 2\Big(\frac{4}{8} - \frac{2}{8}\Big)r^3 + 4\Big(\frac{4}{8} - \frac{2}{8}\Big)r^3 \\
&+ \quad 3\Big(\frac{8}{16} - \frac{4}{16} - \frac{4}{8} - \frac{4}{8} + \frac{4}{8}\Big)r^2 + \Big(\frac{16}{16} - \frac{8}{16} - \frac{8}{16} - \frac{8}{16} + \frac{8}{16}\Big)r \\
&= \quad \frac{1}{4}r^{10} + \frac{1}{4}r^7 - \frac{1}{4}r^6 + \frac{1}{4}r^5 - \frac{1}{4}r^4 + \frac{3}{2}r^3 - \frac{3}{4}r^2.
\end{aligned}
$$

Proposition 8.3.1 For every $r \in \mathbb{N}$, $|S_r(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)/\sim| = \frac{1}{4}r^{10} + \frac{1}{4}r^7 - \frac{1}{4}r^6 + \frac{1}{4}r^5 - \frac{1}{4}r^4 + \frac{3}{2}r^3 - \frac{3}{4}r^2$ and $|S_r(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)| = 4r^{10} - 4r^6 + r^4$.

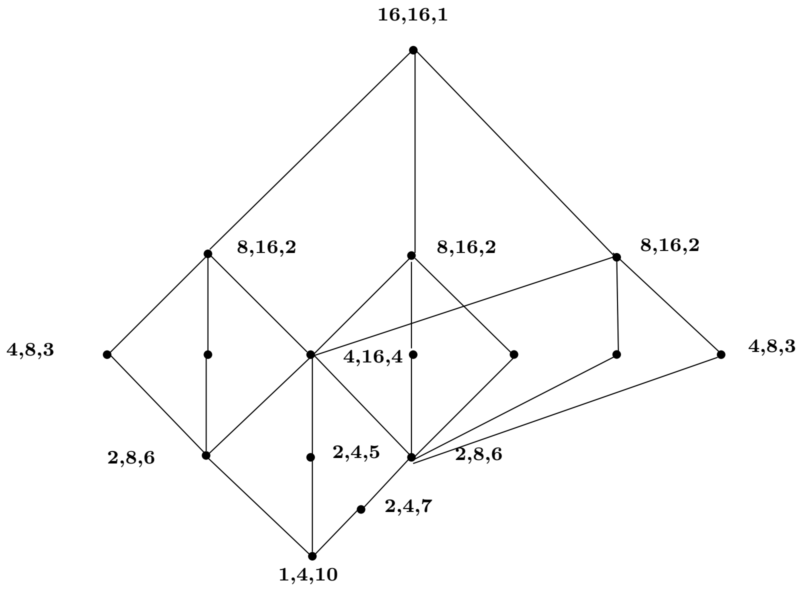

Figure 8.6: lattice of optimal partitions of $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

8.4 The group $Q_8 \times \mathbb{Z}_2$

$Q_8 \times \mathbb{Z}_2 = \langle a, b, c : \; a^4 = b^4 = 1, bab = a, a^2 = b^2, ac = ca, bc = cb \rangle$. $Q_8 \times \mathbb{Z}_2$ has three sugroups of order two, namely $\langle a^2 \rangle$, $\langle a^2 c \rangle$ and $\langle c \rangle$. Hence we can construct the subgroup $\langle a^2, c \rangle$ which is isomorphic to the Klein 4-group. There are twelve elements of order four. Thus, it follows that $\langle a \rangle$, $\langle ac \rangle$, $\langle b \rangle$, $\langle bc \rangle$, $\langle ab \rangle$ and $\langle abc \rangle$ are the only cyclic sugroups of order four. There are seven subgroups of order eight. Three of them are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ and they are $\langle a, c \rangle$, $\langle b, c \rangle$ and $\langle ab, c \rangle$. Subgroups $\langle a, b \rangle$, $\langle a, bc \rangle$, $\langle ac, b \rangle$ and $\langle ac, bc \rangle$ are isomorphic to Q_8. The figure 8.7 shows the lattice of subgroups of $Q_8 \times \mathbb{Z}_2$. Now, we need to construct the poset of optimal partitions of $Q_8 \times \mathbb{Z}_2$ and the list of all optimal partions of $Q_8 \times \mathbb{Z}_2$ is given below.

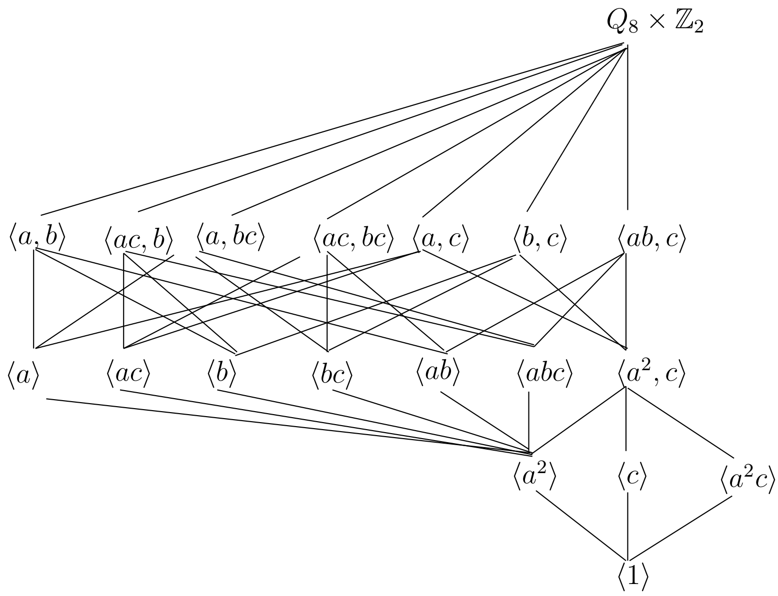

Figure 8.7: lattice of subgroups of $Q_8 \times \mathbb{Z}_2$

Finest partition

$$\pi : \{1,\}, \{a, a^3\}, \{b, b^3\}, \{a^2\}, \{ab, ab^3\}, \{c\}, \{ac, a^3c\}, \{a^2c\}, \{bc, b^3c\}, \{abc, ab^3c\}$$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^2, c \right\rangle$

$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 10.$

One partition of the form

$$\pi : \{1, a^2\}, \{a, a^3\}, \{b, b^3\}, \{ab, ab^3\}, \{c, a^2c\}, \{ac, a^3c\}, \{bc, b^3c\}, \{abc, ab^3c\}$$

$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = Q_8 \times \mathbb{Z}_2$

$|St(\pi)| = 2, |Z(\pi)| = 16, |\pi| = 8.$

Two partitions of the form

$$\pi : \{1, c\}, \{a, a^3, ac, a^3c\}, \{b, b^3, bc, b^3c\}, \{ab, ab^3, abc, ab^3c\}, \{a^2, a^2c\}$$

$St(\pi) = \langle c \rangle, Z(\pi) = \left\langle a^2, c \right\rangle$

$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 5.$

One partition of the form

$$\pi : \{1, c, a^2, a^2c\}, \{a, a^3, ac, a^3c\}, \{b, b^3, bc, b^3c\}, \{ab, ab^3, abc, ab^3c\}$$

$St(\pi) = \left\langle a^2, c \right\rangle, Z(\pi) = Q_8 \times \mathbb{Z}_2$

$|St(\pi)| = 4, |Z(\pi)| = 16, |\pi| = 4.$

Six partitions of the form

$$\pi : \{1, a, a^2, a^3\}, \{b, b^3, ab, ab^3\}, \{c, a^2c, ac, a^3c\}, \{bc, b^3c, abc, ab^3c\}$$

$St(\pi) = \langle a \rangle, Z(\pi) = Q_8 \times \mathbb{Z}_2$

$|St(\pi)| = 4, |Z(\pi)| = 16, |\pi| = 4.$

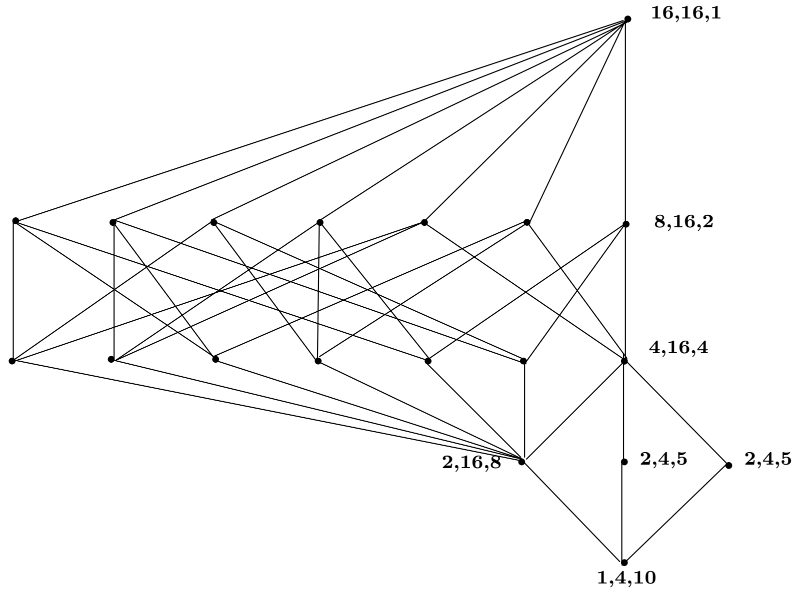

Figure 8.8: lattice of optimal partitions of $Q_8 \times \mathbb{Z}_2$

Three partitions of the form

$$\pi : \{1, a, a^2, a^3, c, a^2c, ac, a^3c\}, \{b, b^3, ab, ab^3, bc, b^3c, abc, ab^3c\}$$

$$St(\pi) = \langle a, c \rangle, Z(\pi) = Q_8 \times \mathbb{Z}_2$$

$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

Four partitions of the form

$$\pi : \{1, a, a^2, a^3, b, b^3, ab, ab^3\}, \{c, a^2c, ac, a^3c, bc, b^3c, abc, ab^3c\}$$

$$St(\pi) = \langle a, b \rangle, Z(\pi) = Q_8 \times \mathbb{Z}_2$$

$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

And the coarsest partition

$$\pi : Q_8 \times \mathbb{Z}_2$$
$$St(\pi) = Q_8 \times \mathbb{Z}_2, Z(\pi) = Q_8 \times \mathbb{Z}_2,$$
$$|St(\pi)| = 16, |Z(\pi)| = 16, |\pi| = 1.$$

Now, by theorem 3.2.5

$$\begin{aligned}
|S_r(Q_8 \times \mathbb{Z}_2)| &= 16\Big(\frac{1}{4}r^{10} + 2\Big(\frac{1}{4} - \frac{1}{4}\Big)r^5 + \Big(\frac{1}{16} - \frac{1}{4}\Big)r^8 + \Big(\frac{1}{16} - \frac{1}{16} - \frac{1}{4} - \frac{1}{4} + \frac{2}{4}\Big)r^4 \\
&+ 6\Big(\frac{1}{16} - \frac{1}{16}\Big)r^4 + 7\Big(\frac{1}{16} - \frac{1}{16} - \frac{1}{16} - \frac{1}{16} + \frac{2}{16}\Big)r^2 \\
&+ \Big(\frac{1}{16} - \frac{7}{16} + \frac{14}{16} - \frac{8}{16}\Big)r\Big) \\
&= 4r^{10} - 3r^8
\end{aligned}$$

and

$$\begin{aligned}
|S_r(Q_8 \times \mathbb{Z}_2)/\sim| &= \frac{1}{4}r^{10} + 2\Big(\frac{2}{4} - \frac{1}{4}\Big)r^5 + \Big(\frac{2}{16} - \frac{1}{4}\Big)r^8 + \Big(\frac{4}{16} - \frac{2}{16} - \frac{2}{4} - \frac{2}{4} + \frac{2}{4}\Big)r^4 \\
&+ 6\Big(\frac{4}{16} - \frac{2}{16}\Big)r^4 + 7\Big(\frac{8}{16} - \frac{4}{16} - \frac{4}{16} - \frac{4}{16} + \frac{4}{16}\Big)r^2 \\
&+ \Big(\frac{16}{16} - 7\frac{8}{16} + 7\frac{8}{16} - \frac{16}{16}\Big)r \\
&= \frac{1}{4}r^{10} - \frac{1}{8}r^8 + \frac{1}{2}r^5 + \frac{3}{8}r^4.
\end{aligned}$$

Proposition 8.4.1 For every $r \in \mathbb{N}$, $|S_r(Q_8 \times \mathbb{Z}_2)/\sim| = \frac{1}{4}r^{10} - \frac{1}{8}r^8 + \frac{1}{2}r^5 + \frac{3}{8}r^4$ and $|S_r(Q_8 \times \mathbb{Z}_2)| = 4r^{10} - 3r^8$.

8.5 Modular group of order 16

$M_{16} = \langle a, b : \ a^8 = b^2 = 1, bab = a^5\rangle$ has three subgroups of order two, namely $\langle a^4\rangle$, $\langle a^4b\rangle$ and $\langle b\rangle$. Since there are four elements of order four, $\langle a^2\rangle$ and $\langle a^2b\rangle$ are the only cyclic subgroups of order four. $\langle a^4, b\rangle$ is also another subgroup of order four isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. From these subgroups, we can construct the only subgroups of order eight, namely $\langle a^2, b\rangle$, $\langle ab\rangle$ and $\langle a\rangle$. The list of all optimal partions of M_{16} is given below.

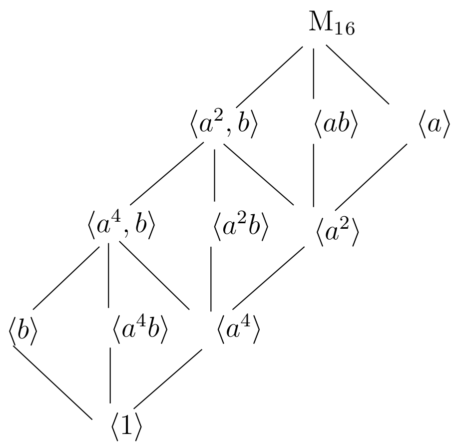

Figure 8.9: lattice of subgroups of M_{16}

Finest partition

$\pi : \{1\}, \{a, a^7\}, \{a^2, a^6\}, \{a^3, a^5\}, \{a^4\}, \{b\}, \{ab, a^3b\}, \{a^2b, a^6b\}, \{a^4b\}, \{a^5b, a^7b\}$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^4 \right\rangle$

$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 10.$

One partition of the form

$\pi : \{1, a^4\}, \{a, a^7\}, \{a^2, a^6\}, \{a^3, a^5\}, \{ab, a^3b\}, \{a^2b, a^6b\}, \{b, a^4b\}, \{a^5b, a^7b\}$

$St(\pi) = \{1\}, Z(\pi) = \left\langle a^2 b \right\rangle$

$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 8.$

One partition of the form

$$\pi : \{1\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}, \{a^4\}, \{b\}, \{ab, a^3b, a^5b, a^7b\}, \{a^2b, a^6b\}, \{a^4b\}$$
$$St(\pi) = \{1\}, Z(\pi) = \left\langle a^4, b \right\rangle$$
$$|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 8.$$

One partition of the form

$$\pi : \{1, a^4\}, \{a, a^7, a^3, a^5\}, \{a^2, a^6\}, \{ab, a^3b, a^5b, a^7b\}, \{a^2b, a^6b\}, \{b, a^4b\}$$
$$St(\pi) = \left\langle a^4 \right\rangle, Z(\pi) = \left\langle a^2, b \right\rangle$$
$$|St(\pi)| = 2, |Z(\pi)| = 8, |\pi| = 6.$$

Two partitions of the form

$$\pi : \{1, b\}, \{a, a^7, a^3, a^5, ab, a^3b, a^5b, a^7b\}, \{a^2, a^6, a^2b, a^6b\}, \{a^4, a^4b\}$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \left\langle a^4, b \right\rangle$$
$$|St(\pi)| = 2, |Z(\pi)| = 4, |\pi| = 4.$$

Two partitions of the form

$$\pi : \{1, a^4, b, a^4b\}, \{a, a^7, a^3, a^5, ab, a^3b, a^5b, a^7b\}, \{a^2b, a^6b, a^2, a^6\}$$
$$St(\pi) = \left\langle a^4, b \right\rangle, Z(\pi) = \left\langle a^2, b \right\rangle$$
$$|St(\pi)| = 4, |Z(\pi)| = 8, |\pi| = 3.$$

One partition of the form

$$\pi : \{1, a^4, a^2, a^6\}, \{a, a^7, a^3, a^5\}, \{ab, a^3b, a^5b, a^7b\}, \{a^2b, a^6b, b, a^4b\}$$
$$St(\pi) = \left\langle a^2 \right\rangle, Z(\pi) = M_{16}$$
$$|St(\pi)| = 4, |Z(\pi)| = 16, |\pi| = 4.$$

Three partitions of the form

$$\pi : \{1, a, a^2, a^3, a^4, a^5, a^6, a^7\}, \{ab, a^3b, a^5b, a^7b, a^2b, a^6b, b, a^4b\}$$
$$St(\pi) = \langle a \rangle, Z(\pi) = M_{16}$$
$$|St(\pi)| = 8, |Z(\pi)| = 16, |\pi| = 2.$$

And the coarsest partition

$$\pi : M_{16}$$
$$St(\pi) = M_{16}, Z(\pi) = M_{16}$$
$$|St(\pi)| = 16, |Z(\pi)| = 16, |\pi| = 1.$$

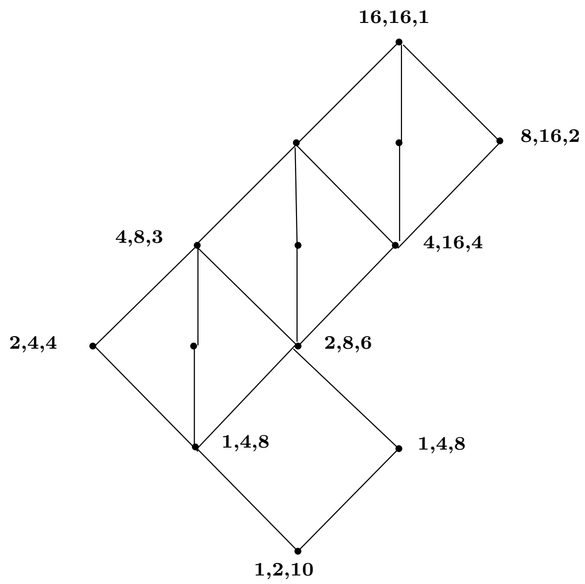

Figure 8.10: lattice of optimal partitions of M_{16}

Now, it follows that

$$
\begin{aligned}
|S_r(M_{16})| &= 16\Big(\frac{1}{2}r^{10} + 2\Big(\frac{1}{4} - \frac{1}{2}\Big)r^8 + \Big(\frac{1}{8} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3}\Big)r^6 + 2\Big(\frac{1}{4} - \frac{1}{4}\Big)r^4 \\
&+ \Big(\frac{1}{16} - \frac{1}{8}\Big)r^4 + \Big(\frac{1}{8} - \frac{1}{8}\Big)r^3 + \Big(\frac{1}{8} - \frac{1}{4} - \frac{1}{4} - \frac{1}{8} + \frac{2}{4}\Big)r^3 \\
&+ 2\Big(\frac{1}{16} - \frac{1}{16}\Big)r^2 + \Big(\frac{1}{16} - \frac{1}{8} - \frac{1}{8} - \frac{1}{16} + \frac{2}{8}\Big)r^2 \\
&+ \Big(\frac{1}{16} - \frac{1}{16} - \frac{1}{16} - \frac{1}{16} + \frac{2}{16}\Big)r\Big) \\
&= 8r^{10} - 8r^8 + 2r^6 - r^4
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(M_{16})/\sim| &= \frac{1}{2}r^{10} + 2\Big(\frac{1}{4} - \frac{1}{2}\Big)r^8 + \Big(\frac{2}{8} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3}\Big)r^6 + 2\Big(\frac{2}{4} - \frac{1}{4}\Big)r^4 \\
&+ \Big(\frac{4}{16} - \frac{2}{8}\Big)r^4 + \Big(\frac{4}{8} - \frac{2}{8}\Big)r^3 + \Big(\frac{4}{8} - \frac{2}{4} - \frac{2}{4} - \frac{2}{8} + \frac{2}{4}\Big)r^3 \\
&+ 2\Big(\frac{8}{16} - \frac{4}{16}\Big)r^2 + \Big(\frac{8}{16} - \frac{4}{8} - \frac{4}{8} - \frac{4}{16} + \frac{4}{8}\Big)r^2 \\
&+ \Big(\frac{16}{16} - \frac{8}{16} - \frac{8}{16} - \frac{8}{16} + \frac{8}{16}\Big)r \\
&= \frac{1}{2}r^{10} - \frac{1}{2}r^8 + \frac{1}{4}r^6 + \frac{1}{2}r^4 + \frac{1}{4}r^2.
\end{aligned}
$$

Proposition 8.5.1 For every $r \in \mathbb{N}$, $|S_r(M_{16})/\sim| = \frac{1}{2}r^{10} - \frac{1}{2}r^8 + \frac{1}{4}r^6 + \frac{1}{2}r^4 + \frac{1}{4}r^2$ and $|S_r(M_{16})| = 8r^{10} - 8r^8 + 2r^6 - r^4$.

Chapter 9

Bracelets

9.1 Introduction

Let $\chi : \mathbb{Z}_n \longrightarrow \{1, 2, 3, ..., r\}$ be a coloring of $\mathbb{Z}_n$. Colorings of $\mathbb{Z}_n$ have a geometric intepretation. A coloring of $\mathbb{Z}_n$ represents a regular n-gon with its vertices coloured using r distinct colours. Two colorings of $\mathbb{Z}_n$ are equivalent if we can get one from another by rotating about the regular n-gon center. A coloring of $\mathbb{Z}_n$ is symmetric if it is invariant with respect to some mirror symmetry with an axis crossing the center of the regular n-gon and one of its vertices.

Our definition of a symmetric coloring of $\mathbb{Z}_n$ is very restrictive. It only gives us the reflections with respect to an axis through the center and one of the vertices of the regular n-gon. If n is even it captures only half reflections of the regular n-gon. Our goal is to captures all the reflections of the regular n-gon, that is, we want to define symmetric colorings of $\mathbb{Z}_n$ as colorings which are invariant with respect to some mirror symmetry. We are then going to derive a general formula for counting them.

9.2 Symmetric bracelets

$D(G) = G \rtimes_\theta \langle s \rangle$, where $\theta : \langle s \rangle \longrightarrow \mathrm{Aut}(G)$ is defined by $\theta_s(g) = g^{-1}$ (*inversion*) for all $g \in G$. The group naturally act on the set of colorings of G by $g\chi(x) = \chi(xg^{-1})$ and $gs\chi(x) = \chi(gx^{-1})$. Colorings of G, χ and ψ are equivalent if there exist an element $a \in D(G)$ such that $a\chi(x) = \psi$. Clearly the relation is an equivalence relation on the set of all colorings of G and each equivalence class is called a *bracelet* of G and is denote by $B_r(G)$. The following theorem is used to compute the number of bracelets of G (see [13]).

Theorem 9.2.1 If G is an abelian group, then

$$B_r(G) = \frac{1}{2}N_r(G) + \begin{cases} \frac{1}{2|B(G)|}(r^{\frac{B(G)}{2}} + |B(G)| - 1)r^{\frac{|G|}{2}} & \text{if} \quad |G| \text{ is even} \\ \frac{1}{2}r^{\frac{|G|+1}{2}} & \text{if} \quad |G| \text{ is odd.} \end{cases}$$

In example 1.1.7, we proved that $B(\mathbb{Z}_n) \cong \mathbb{Z}_2$ for even values of n and $B(\mathbb{Z}_n) = 1$ for odd values of n. This leads to the following corollary.

Corollary 9.2.2 Let $\mathbb{Z}_n$ be an integers modulo(n) group, then

$$B_r(\mathbb{Z}_n) = \frac{1}{2}N_r(\mathbb{Z}_n) + \begin{cases} \frac{1}{4}(r+1)r^{\frac{n}{2}} & \text{if} \quad n \text{ is even} \\ \frac{1}{2}r^{\frac{n+1}{2}} & \text{if} \quad n \text{ is odd.} \end{cases}$$

Let $\mathbb{Z}_n$ be the group integers modulo(n). Let a positive integer d divide n. Hence $\langle d \rangle = \langle j \rangle$ if and only if gcd(n, d)=gcd(n, j)=d. Recall that gcd(n, j)=d if and only if gcd$(\frac{n}{d}, \frac{j}{d})$=1, it follows that there are $\varphi(\frac{n}{d})$ integers less than n such that gcd(n, j)=d. φ is an *Euler totient function* (see [8]). Hence there are

$$\frac{1}{n}\sum_{d|n}\varphi(\frac{n}{d})r^d$$

necklaces of $\mathbb{Z}_n$. Let $n = p$ a prime, then there are

$$\frac{1}{p}(\varphi(1)r^p + \varphi(p)r)) = r + \frac{1}{p}(r^p - r)$$

necklaces of $\mathbb{Z}_p$.

Example 9.2.3

(1) Let G be $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since G is a Boolean group, then $B(G) = G$. Hence

$$B_r(G) = \frac{1}{4}(3r^2 + r^4)$$

(2) The group $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ has three subgroups of order two, so $|B(G)| = 4$. Since

$$N_r(G) = \frac{1}{8}(4r^2 + 3r^4 + r^8),$$

$$B_r(G) = \frac{1}{16}(4r^2 + 9r^4 + 3r^8)$$

(3) We also know that the finite abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has seven subgroups of order two and

$$N_r(G) = \frac{1}{8}(7r^4 + r^8).$$

Therefore

$$B_r(G) = \frac{1}{8}(7r^4 + r^8).$$

(4) Recall that

$$N_r(G) = \frac{1}{9}(8r^3 + r^9)$$

for $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. Since G has odd order,

$$B_r(G) = \frac{1}{18}(8r^3 + r^9 + 9r^5).$$

(5) $G = \mathbb{Z}_6 \times \mathbb{Z}_2$ has three subgroups of order two and

$$N_r(G) = \frac{1}{12}(6r^2 + 2r^2 + 3r^6 + r^{12}).$$

Hence

$$B_r(G) = \frac{1}{24}(8r^2 + 12r^6 + 3r^{10} + r^{12}).$$

(6) The group $G = \mathbb{Z}_8 \times \mathbb{Z}_2$ has three subgroups of order three. Since G has

$$\frac{1}{16}(8r^2 + 4r^4 + 3r^8 + r^{16})$$

necklaces, then

$$B_r(G) = \frac{1}{32}(8r^2 + 4r^4 + 15r^8 + 4r^{10} + r^{16}).$$

(7) $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ has three subgroups of order three and has

$$\frac{1}{16}(12r^4 + 3r^8 + r^{16})$$

necklaces. Hence,

$$B_r(G) = \frac{1}{32}(12r^4 + 15r^8 + 4r^{10} + r^{16}).$$

(8) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has fifteen subgroups of order two and

$$N_r(G) = \frac{1}{16}(15r^8 + r^{16}).$$

Hence

$$B_r(G) = \frac{1}{32}(2r^{16} + 30r^8).$$

In the chapter 2, we defined a coloring of $\mathbb{Z}_n$ as symmetric with respect to an element g in a group $\mathbb{Z}_n$ if $\chi(gx^{-1}g) = \chi(x)$ for all $x \in \mathbb{Z}_n$. That is, a coloring is symmetric if it is invariant with respect to some mirror symmetry with an axis crossing the center of the polygon and one of its vertices. Now, we want to define symmetric colorings of $\mathbb{Z}_n$ as colorings which are invariant with respect to some mirror symmetry.

Definition 9.2.4 A coloring χ of G is called symmetric with respect to an element g in G if

$$\chi(gx^{-1}) = \chi(x)$$

for all $x \in G$.

Lemma 9.2.5 (see [13]) Let χ be a coloring of a finite abelian group G. Then the following statements are equivalent.

(1) χ is symmetric

(2) $[\chi]_G$ is symmetric

(3) $[\chi]_{D(G)}$ is symmetric

(4) $[\chi]_G = [\chi]_{D(G)}$

(5) $s\chi$ belongs to $[\chi]_G$.

Proof: (1)⇒(2) Let $g, h \in G$ such that $gs\chi(x) = \chi(x)$ and $h\chi(x) = \lambda(x)$. λ is symmetric,

$$\begin{aligned} gh^2 s\lambda(x) &= gh^2 sh\chi(x) \\ &= ghs\chi(x) \\ &= hgs\chi(x) \\ &= h\chi(x) \\ &= \lambda(x) \end{aligned}$$

(2)⇒(3) is obvious

(3)⇒(4) Clearly, $[\chi]_G \subseteq [\chi]_{D(G)}$. Let $\lambda \in [\chi]_{D(G)}$, such that $\lambda(x) = gs\lambda(x)$ for some $g \in G$. $\lambda(x) = hs\chi(x)$ for some h. Hence

$$\begin{aligned} \lambda(x) &= gshs\chi(x) \\ &= gh^{-1}\chi(x) \end{aligned}$$

therefore $\lambda \in [\chi]_G$.

(4)⇒(5) Let $s\chi(x) \in [\chi]_{D(G)}$, then $s\chi(x) \in [\chi]_G$.

(5)⇒(1) Let $g \in G$, then $s\chi(x) = g\chi(x)$, hence $g^{-1}s\chi(x) = \chi(x)$.

Lemma 9.2.6 (see [13]) Let χ be a coloring of a finite abelian group G. If $[\chi]_{D(G)}$ is not symmetric, then $[\chi]_{D(G)}$ is a disjoint union of $[\chi]_G$ and $[s\chi]_G$.

Proof: By theorem 8.0.6, $[\chi]_G$ and $[s\chi]_G$ are distinct orbits. Therefore $[\chi]_G \cup [s\chi]_G \subseteq [\chi]_{D(G)}$. The converse inclusion also holds. Therefore $[\chi]_G \cup [s\chi]_G = [\chi]_{D(G)}$.

The last result of this section uses both lemma 8.0.6 and 8.0.7 in order to find the formula for counting symmetric bracelets.

Theorem 9.2.7 If G is a finite abelian group, then

$$N_r^*(G) = \begin{cases} \frac{1}{|B(G)|}(r^{\frac{B(G)}{2}} + |B(G)| - 1)r^{\frac{|G|}{2}} & \text{if} \quad |G| \text{ is even} \\ r^{\frac{|G|+1}{2}} & \text{if} \quad |G| \text{ is odd} \end{cases}$$

is the number of symmetric bracelets on G.

Proof: By theorem 8.0.6, the symmetric necklaces and bracelets are the same. Hence

$$\begin{aligned} B_r(G) &= N_r^*(G) + \frac{1}{2}(N_r(G) - N_r^*(G)) \\ &= \frac{1}{2}(N_r(G) + N_r^*(G)). \end{aligned}$$

Therefore $N_r^*(G) = 2B_r(G) - N_r(G)$.

Example 9.2.8

(1) Recall that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has three subgroups of order two. Therefore

$$N_r^*(\mathbb{Z}_2 \times \mathbb{Z}_2) = \frac{1}{4}(r^4 + 3r^2).$$

Notice that this is equal to the number of necklaces of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) $\mathbb{Z}_4 \times \mathbb{Z}_2$ has three subgroups of order two. Thus

$$N_r^*(\mathbb{Z}_4 \times \mathbb{Z}_2) \;=\; \frac{1}{4}(r^6 + 3r^4).$$

(3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has seven subgroups of order two. Hence it has

$$\frac{1}{8}(7r^4 + r^8)$$

symmetric necklaces.

(4) The order of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is nine(odd number), so $N_r^*(\mathbb{Z}_3 \times \mathbb{Z}_3) = r^5$.

(5)

$$N_r^*(\mathbb{Z}_6 \times \mathbb{Z}_2) \;=\; \frac{1}{4}(r^8 + 3r^6)$$

since there are three subgroups of order two.

(6) The group $\mathbb{Z}_8 \times \mathbb{Z}_2$ has three subgroups of order two. Therefore

$$N_r^*(\mathbb{Z}_6 \times \mathbb{Z}_2) \;=\; \frac{1}{4}(r^{10} + 3r^8).$$

(7) All the non-trivial elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are of order two, hence there are

$$\frac{1}{16}(15r^8 + r^{16})$$

symmetric necklaces of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(8) $\mathbb{Z}_4 \times \mathbb{Z}_4$ has three subgroups of order two and has

$$\frac{1}{4}(3r^8 + r^{10})$$

symmetric bracelets.

(9) $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has eight subgroups of order two. Hence, it has

$$\frac{1}{8}(7r^8 + r^{12})$$

symmetric bracelets.

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