The computation of k-defect polynomials, suspended Y-trees and its applications.

by

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

____ (Signature of candidate)

__________ day of _______________ 20 _____ in ____________________

Abstract

We start by defining a class of graphs called the suspended Y-trees and give some of its properties. We then classify all the closed sets of a general suspended Y-tree. This will lead us to counting the graph compositions of the suspended Y-tree. We then contract these closed sets one by one to obtain a set of minors for the suspended Y-trees. We will use this information to compute some of the general expression of the k-defect polynomial of a suspended Y-tree. Finally we compute the explicit Tutte polynomial of the suspended Y-trees.

— Simon Werner

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— University of the Witwatersrand School of Mathematics

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Chapter 1

Introduction

1.1 Brief History of Graph Theory

Before we discuss k-Defect Polynomials and the improper colourings of graphs, we need to understand how this theory of colourings came about. Augustus De Morgan (born 27 June 1806, died 18 March 1871) was a British mathematician and logician. One of his students, in 1852, noted that when colouring a map of England, only four colours were needed in order for each county to be a different colour to any adjacent county. De Morgan then sent a letter to Sir William Rowan Hamilton (born 4 August 1805, died 2 September 1865) who was an Irish physicist, astronomer, and mathematician. This letter contained details of the students observations and may be considered the origin of graph colouring. The problem extended to a more formal version in the letter: ”What is the least possible number of colours needed to fill in any map (real or invented) on the plane?”. Arthur Cayley (born 16 August 1821, died 26 January 1895) was a British mathematician who published the first problem in the form of a puzzle in 1878. Sir Alfred Bray Kempe (born 6 July 1849, died 21 April 1922, London) was a mathematician best known for his work on linkages and the four colour theorem (F.C.T.). In 1879 Kempe wrote his famous proof of the four colour theorem. Percy John Heawood (born 8 September 1861, died 24 January 1955) was a British mathematician who discovered an error in Kempe's proof. An error that could not be repaired. In contrast though, he did use Kempe's methods of proof to show that every map is 5-colourable. In other words, a maximum of five colours is needed in order to colour any map. George David Birkhoff (born 21 March 1884, died 12 November 1944) was an American mathematician who, in 1913, introduced the concept of reducibility, which proved to be quite essential in the ensuing proof of the F.C.T. He showed how a four colouring portion of a map can be expanded into a colouring of the entire map. Based on the technique of reducible configurations, Kenneth Ira Appel (born October 8, 1932) and Wolfgang Haken (born June 21, 1928) proclaimed a complete proof in 1976. The radical inclusion of computation in combinatorics was frowned upon by skeptics. An alternative, less computer reliant, proof was later developed by G. Neil Robertson (born 1938) and Paul D. Seymour (born July 26, 1950). This research was activated by Read in 1968, in his paper An introduction to chromatic polynomials. This research has expanded and instead of looking at the proper colouring only, researchers are also looking at improper colourings, see [13].

1.2 Overview of Dissertation

In Chapter 1, we introduce basic definitions, concepts, notations, classes and operations in graph theory that are relevant to this dissertation. We give examples of some of the cases and introduce closed sets which lead into Chapter 2. In Chapter 2, we introduce the class of Y-trees and the concept of suspended trees. We combine these to yield a suspended Y-tree, which is the main graph focus of this dissertation. We discuss some properties of suspended Y-trees. We continue in Chapter 2 with the introduction of closed sets applied to suspended Y-trees. Our main theorem in Chapter 2 is one which essentially counts the number of closed sets of a certain size in a suspended Y-tree graph of a certain order. We then see how these concepts are applicable to graph compositions. A paper was submitted for publication on this very topic - using methods of counting closed sets in suspended Y-trees to counting graph compositions of suspended Y-trees [10]. In Chapter 3, we introduce the concepts of graph colourings, chromatic polynomials, characteristic polynomials and operations to find chromatic polynomials. We then introduce the coboundary polynomial and discuss how to obtain the k-defect polynomial. We illustrate this with some examples before applying it to suspended Y-trees. We then discuss and explore explicit expressions for k-defect polynomials and verify our results. In Chapter 4, we introduce the Tutte polynomial. We discuss properties and operations before applying it to suspended Y-trees. We discover an explicit expression for the Tutte polynomial of suspended Y-trees and verify our results. We then mention how to derive the chromatic polynomial from the Tutte polynomial and verify this result using examples.

1.3 Basic Definitions

In this section, we discuss basic concepts, definitions and notions in graph theory which are relevant to this dissertation. We refer the reader to [1] and [6] unless otherwise stated.

Definition 1.3.1. A *graph*, denoted G, is made up of a set of vertices, $V(G)$, and a set of edges, $E(G)$. An edge connects two vertices. If two vertices u and v are endpoints of an edge, they are referred to as *adjacent*. If a vertex u and edge e are incident, it means that u is an endpoint of e.

Definition 1.3.2. A *loop* is an edge which connects a vertex to itself. In other words, it is an edge whose endpoints are equal.

Definition 1.3.3. A *parallel edge* or a *multiple edge* is a set of one or more edges which connect the same two vertices. In other words, edges whose endpoints are equal. A graph with no loops or parallel edges is known as a *simple graph*.

Definition 1.3.4. The *degree* of a vertex is the number of edges incident to it.

Definition 1.3.5. A graph is *connected* if there is a path from each vertex to any other vertex. In other words, if there is a path from u to v for $u, v \in V(G)$.

Definition 1.3.6. An edge $e \in E(G)$ of a graph G is called an *isthmus* if its deletion renders the graph G as disconnected, that is $k(G \backslash e) > k(G)$.

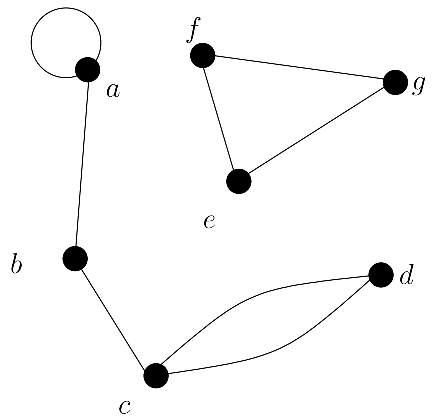

Figure 1.1: An example of a graph.

In the diagram in Figure 1.1 is a graph, G with vertex set $V(G) = \{a, b, c, d, e, f, g\}$. We can also clearly see that this graph has two components, *abcd* and efg. The edge connecting a to itself would be a *loop* while the edges connecting c to d and vice versa is a *parallel edge*.

Definition 1.3.7. A *subgraph*, H, of a graph G, is a graph with the following properties:

$$V(H) \subseteq V(G)$$

$$E(H) \subseteq E(G).$$

We say H is contained within G and we display this fact as $H \subseteq G$.

Definition 1.3.8. A *walk* is a graph or a subgraph with vertices that are listed as

$$v_0, e_1, v_1, e_2, v_2, ..., e_i, v_i, ..., e_{k-1}, v_k,$$

for $1 \leq i \leq k$. Each edge, e_i, has endpoints v_{i-1}, v_i. The length of a walk or a path is the counted number of edges between v_0 and v_k. We describe a walk as *closed* if it's endpoints are equal.

Definition 1.3.9. A *spanning subgraph* is a subgraph whose vertex set is the entirety of the vertex set of the graph for which it is a subgraph. In other words, if $H \subseteq G$, and $V(H) = V(G)$, then H is a spanning subgraph of G.

If this spanning subgraph is a tree, then it is known as a *spanning tree.*

Definition 1.3.10. The *Rank*, r or $r(G)$, of a graph G is defined to be the number of components subtracted from the number of vertices.

In Figure 1.5 we see a forest, $f_{(8,3)}$. It has a rank of $8 - 3 = 5$.

Definition 1.3.11. A *closed set* X of size k, is the largest rank-r subgraph of $E(G)$ containing X.

We denote the set of all closed sets of size k by δ_k. Thus the number of all closed sets of size k is represented by $|\delta_k|$.

In the diagram in Figure 1.2 is the cyclic graph C_4. The vertex set is labelled a, b, c, d while the edge set is labelled $1, 2, 3, 4$. This graph has $|\delta_1| = 4$ i.e they are just the edges $1, 2, 3, 4$. This graph has 6 closed sets of size 2. They are just the edges $(1, 2), (2, 3), (3, 4), (4, 1), (1, 3), (2, 4)$. This graph has 0 closed sets of size 3. Any combination of 3 edges will result in a subgraph which is not a maximum. Adding another edge to such a subgraph will not alter the rank and therefore it is not a closed set. This graph has 1 closed set of size 4. It is the only combination of 4 edges in this graph, ie: $(1, 2, 3, 4)$.

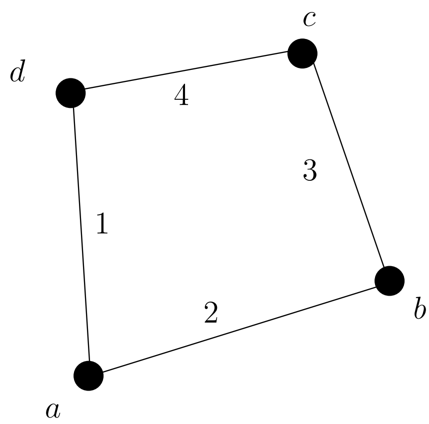

Figure 1.2: C_4.

Definition 1.3.12. A graph is not always connected. In other words, there is not always a path from one vertex to every other vertex. Such graphs are called *Disjoint* or *Disconnected graphs.*

Let G be a graph. We define $k(G)$ to be the number of components of G.

In diagram in Figure 1.1, we see an example of a graph with two components. We can clearly see the separate vertex sets, $V_1(G) = \{a, b, c, d\}$ and $V_2(G) = \{e, f, g\}$. In this case, $k(G) = 2$.

Definition 1.3.13. A *bridge* or a *coloop* is an edge, that if deleted, would separate a graph G into two components, G_1 and G_2.

Definition 1.3.14. Two graphs, G and H are said to be isomorphic if there is a bijection $\sigma : V(G) \mapsto V(H)$ which preserves adjacency. If $u, v \in G$ are adjacent, then $\sigma(u,), \sigma(v) \in H$ are adjacent.

1.4 Classes of Graphs

The are many classes of graphs which have been studied in graph theory, the following are examples of classes of graphs and are relevant to this work.

Definition 1.4.1. A *Tree*, denoted t_n, is a graph with n vertices and contains no cycles. In other words it is acyclic.

A *leaf*, usually associated with trees, is a vertex of degree 1. Some of the well known properties of trees are as follows

i) A tree has $n-1$ edges.

ii) A tree is connected.

iii) A tree is a simple graph.

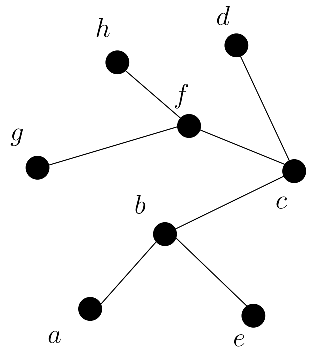

Figure 1.3: An example of a tree

In the diagram in Figure 1.3 is a tree graph, t_8, with 8 vertices and 7 edges. The vertices a, e, d, g, h are leaves as they are of degree 1.

Definition 1.4.2. A *path*, denoted P_n, is a graph or a subgraph with n vertices that are adjacent if and only if they are successive.

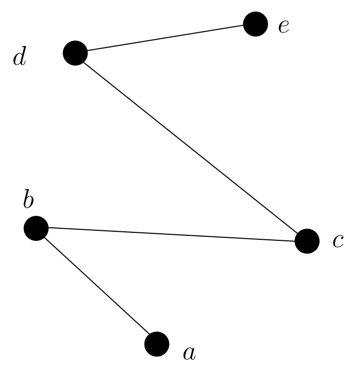

Figure 1.4: An example of a path.

In the diagram in Figure 1.4 is a path, P_5. It is a simplified tree graph as it holds all of the characteristics of a tree.

Definition 1.4.3. A *Forest*, denoted $f_{(n,m)}$, is a graph with n vertices and m components. Each of the components is a Tree.

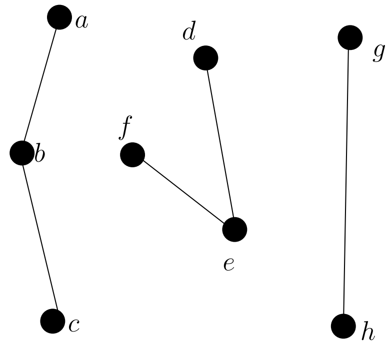

Figure 1.5: An example of a forest.

In the diagram in Figure 1.5 is a forest, $f_{(8,3)}$. It has 8 vertices and 3 components, each of which is a tree.

Definition 1.4.4. A *Cyclic Graph*, denoted C_n, is a graph or subgraph which has n vertices and n edges connected in a closed chain. Each vertex has a degree of 2.

A graph that does not contain a cycle is known as *acyclic*.

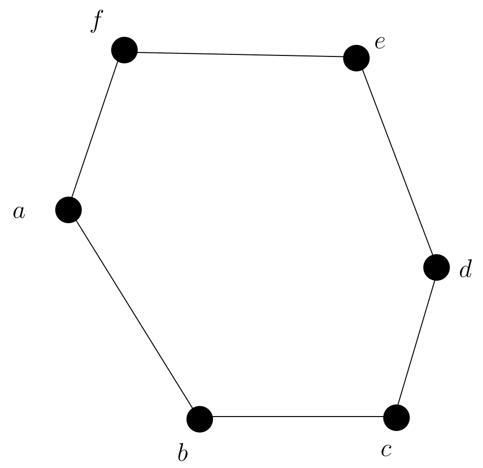

Figure 1.6: An example of a cyclic graph.

In the diagram in Figure 1.6 is a cyclic graph, C_6. It has 6 edges and 6 vertices, each with a degree of 2.

Definition 1.4.5. A *Complete Graph*, denoted K_n, is a graph or subgraph which has n vertices, each of which is connected to every other vertex.

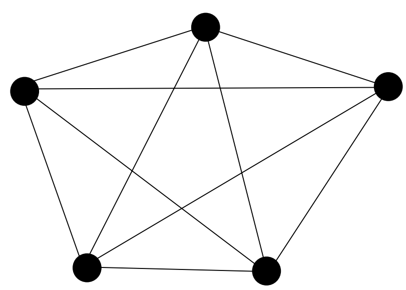

Figure 1.7: An example of a complete graph.

In the diagram in Figure 1.7 is a complete graph, K_5. It has 5 vertices, each of which is connected to the other 4.

Definition 1.4.6. Let F_{n_m} denote an $n \times m$ - complete flower graph with tensor product $C_n \otimes C_m$ for $n, m \geq 2$. F_{n_m} has $n(m-1)$ vertices and nm edges. The cyclic graphs of order m are called the petals and the cyclic graph of order n is called the center of F_{n_m}.

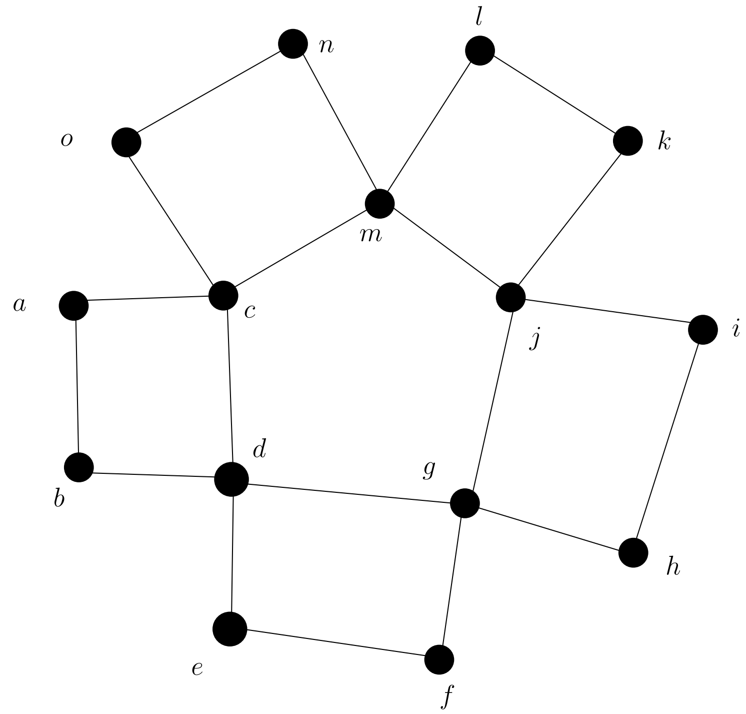

Figure 1.8: An example of a flower graph.

In the diagram in Figure 1.8 is a flower graph, F_{5_4}, with 5 petals that are each C_4, and a center C_5.

1.5 Graph Operations

In this section we will define some graph operations which are relevant to this dissertation.

Definition 1.5.1. Deletion is an operation which removes an edge from a graph completely and keeps the vertices connected by it unchanged. The deletion of an edge e from graph G is denoted as $G \setminus e$.

Definition 1.5.2. Contraction is an operation which removes an edge from a graph and combines the vertices connected by it. This reduces the total number of vertices in the graph by 1. The contraction of an edge f in a graph G is denoted as G/f.

Definition 1.5.3. A *Minor* of a graph is the result once a closed set has been contracted.

If we consider the diagram in Figure 1.2 and contract the closed set of $(3, 4)$, we obtain the result seen in the diagram in Figure 1.9. Here we see that vertices b and d have merged, edges 3 and 4 have disappeared and vertex c has been removed.

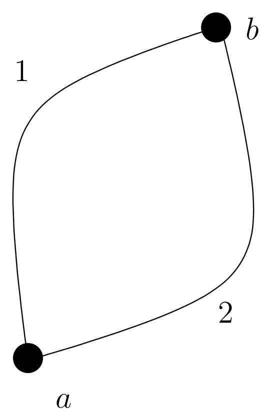

Figure 1.9: The minor of C_4 once the closed set $(3, 4)$ has been contracted.

Chapter 2

Closed sets and graph compositions of suspended Y-trees

In this chapter we introduce a class of graphs called suspended Y-trees. We give some properties of this class and we study the number of closed sets of these graphs. Finally we apply techniques to counting the number of graph compositions and find a relationship between closed sets and graph compositions.

2.1 Suspended Y-trees

In this section, we introduce a class of graphs called *suspended trees*. In particular we study Y-trees and we give some properties of this class of graphs. There is not much literature on suspended trees. We got the idea of suspended trees from a research paper with applications to *Knot Theory*, see [11].

Recall the definition of a *tree* and a *leaf* from Chapter 1, Definition 1.4.1. A tree, denoted t_n, is a graph with n vertices and $n-1$ edges which contains no cycles; and a leaf is a vertex of degree 1.

Definition 2.1.1. A *suspended tree* is a tree graph in which an additional vertex is added and subsequently connected to each leaf of the tree.

We denote a suspended tree of t_n by $\widetilde{t_n}$.

Example 2.1.2. The diagram in Figure 2.1 is a tree t_8 with vertex set $\{a, b, c, d, e, f, g, h\}$ and leaves $\{a, g, h, f\}$. The diagram in Figure 2.2 is the suspended tree of t_8, $\widetilde{t_8}$. The vertex i is added and then joined to the leaves of t_8.

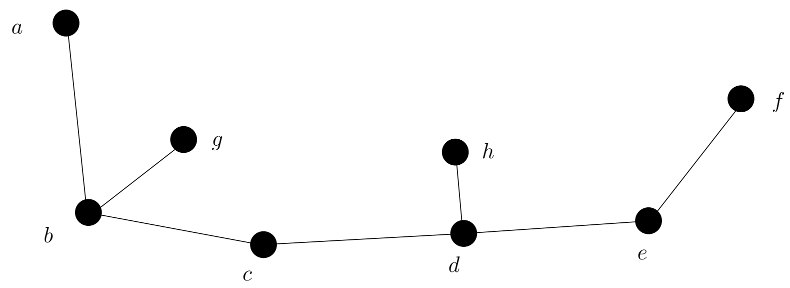

Figure 2.1: An example of a tree, t_8, not yet suspended.

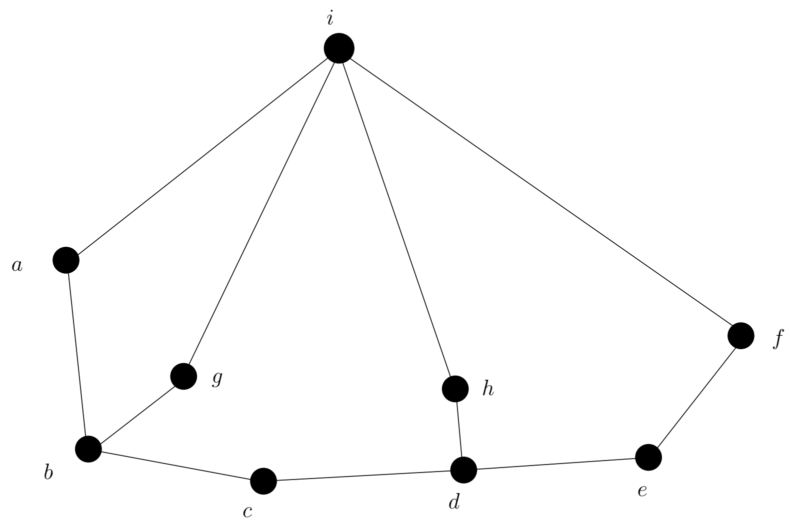

Figure 2.2: An example of a suspended tree, $\widetilde{t_8}$.

Definition 2.1.3. A *Y-Tree* is a graph with n vertices $\{a_1, a_2, ..., a_n\}$ with edge set $\{\{a_1a_2\}, \{a_2a_3\}, ..., \{a_{n-2}a_{n-1}\}, \{a_{n-2}a_n\}\}$.

We denote a Y-tree by Y_n and the suspended tree of Y_n by $\widetilde{Y_n}$.

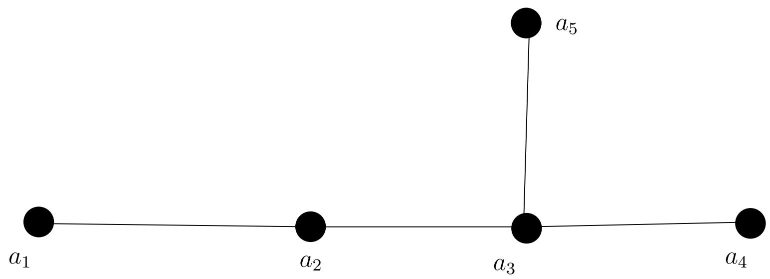

Figure 2.3: An example of a Y-Tree, Y_5

The diagram in Figure 2.3 is a Y-tree with vertex set $\{a_1, a_2, a_3, a_4, a_5\}$ and edge set $\{\{a_1, a_2\}\{a_2, a_3\}\{a_3, a_4\}\{a_3, a_5\}\}$.

Proposition 2.1.4. *Let Y_n be a Y-tree. Then Y_n has three leaves.*

Proof. By definition, vertex a_1 is adjacent to one vertex, vertex a_{n-1} is adjacent to one vertex and vertex a_n is adjacent to one vertex. These are the three leaves. $\square$

Proposition 2.1.5. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then $\widetilde{Y_n}$ has $n + 2$ edges.*

Proof. Let Y_n be a Y-tree. Then Y_n is a tree with n vertices and has $n - 1$ edges. Hence suspending this Y-tree will add a vertex and an edge to each of the 3 leaves. Therefore the number of edges will be $n - 1 + 3 = n + 2$ as required. $\square$

Proposition 2.1.6. *A suspended Y-tree, $\widetilde{Y_n}$, has one component.*

Proof. Let Y_n be a Y-tree. Then Y_n is a tree and therefore connected. Hence all pairs of vertices in Y_n are connected by a path. Assume that vertex u is a leaf. Then suspending this tree will add a vertex, say v. There is now an edge $\{uv\}$ in $\widetilde{Y_n}$. Therefore, through u, there exists a path from v to every other vertex. This implies that $\widetilde{Y_n}$ is connected and therefore has one component.

$\square$

Proposition 2.1.7. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then $\widetilde{Y_n}$ is a simple graph. It has no loops and no parallel edges.*

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. By Definition 1.4.1, a tree has no parallel edges nor loops. By construction of a suspended tree we are connecting the vertices of the tree's leaves to a new vertex. Therefore $\widetilde{Y_n}$ cannot have parallel edges or loops. □

Recall from Chapter 1, Definition 1.4.2, that P_n is a path on n vertices.

Proposition 2.1.8. *Let $\widetilde{Y_n}$ be a suspended Y-tree. There exists three cyclic subgraphs. There are two C_n subgraphs and one C_4 subgraph. Each of the C_n subgraphs has an intersection of P_3 with the C_4 subgraph.*

Proof. Let Y_n represent a Y-tree. Then Y_n has edge set

$$\{\{a_1a_2\},\{a_2a_3\},...,\{a_{n-2}a_{n-1}\},\{a_{n-2}a_n\}\}.$$

Suspending this Y-tree will result in an additional vertex (a_{n+1}) to create $\widetilde{Y_n}$, having the edge set

$$\{\{a_1a_2\},\{a_2a_3\},...,\{a_{n-2}a_{n-1}\},\{a_{n-2}a_n\}\}\cup\{\{a_1a_{n+1}\},\{a_{n-1}a_{n+1}\},\{a_na_{n+1}\}\}.$$

From this edge set we can extract the edge set

$$\{a_{n-2}a_{n-1}\},\{a_{n-2}a_n\}\},\{a_{n-1}a_{n+1}\},\{a_na_{n+1}\}.$$

This creates a cyclic graph, specifically C_4. We can also extract the edge sets

$$\{\{a_1a_2\},\{a_2a_3\},...,\{a_{n-2}a_n\}\}\cup\{a_na_{n+1}\},\{a_1a_{n+1}\}$$

$$\{\{a_1a_2\},\{a_2a_3\},...,\{a_{n-2}a_{n-1}\}\}\cup\{a_{n-1}a_{n+1}\},\{a_1a_{n+1}\}.$$

These are cyclic graphs of size n. The intersecting edge set $\{a_{n-2}a_n\}\cup\{a_na_{n+1}\}$ is a path, P_3. The other intersecting edge set $\{a_{n-2}a_{n-1}\}\cup\{a_{n-1}a_{n+1}\}$ is also a path, P_3. □

Example 2.1.9. The diagram in Figure 2.4 is $\widetilde{Y_4}$, the smallest suspended Y-tree. There are 5 vertices and 6 edges. We can clearly see C_4 on the left and right sides of the graph made up of vertices a, b, d, e and b, c, e, d respectively, with the path P_3 (vertices b, d, e) in common.

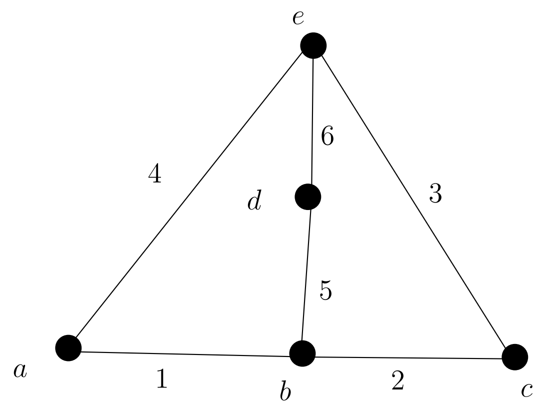

Figure 2.4: An example of a suspended Y-tree

2.2 Closed Sets of Suspended Y-trees

In this section we study the number of closed sets of $\widetilde{Y_n}$. Recall from Definition 1.3.11 that a closed set of size k, is the maximum subgraph containing k edges which does not change the rank. We will consider a few examples of closed sets of suspended Y-trees, $\widetilde{Y_n}$, before stating the theorem.

Note: If we consider all the closed sets within $\widetilde{Y_n}$, then there are potential closed sets from size 0 to size $n+2$ as there cannot be a closed set containing more edges than the graph itself.

Recall That δ_i is the set of closed sets of size i, while $|\delta_i|$ is the number of closed sets of size i.

Example 2.2.1. The diagram in Figure 2.4 we have $\widetilde{Y_4}$. Let us consider the closed sets from size 1 to size 6 in terms of the edges labeled, 1 - 6.

$$\begin{aligned}
\delta_0 &= \{\emptyset\} \\
|\delta_0| &= 1 \\
\delta_1 &= \{1,2,3,4,5,6\} \\
|\delta_1| &= 6 \\
\delta_2 &= \{(1,2),(1,3),(1,4),(1,5),(1.6),(2,3),(2,4), \\
&\quad (2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\} \\
|\delta_2| &= 15 \\
\delta_3 &= \{(1,2,5),(3,4,6),(1,2,6),(3,4,5),(1,3,6),(2,4,6),(1,3,6),(2,4,5)\} \\
|\delta_3| &= 8 \\
\delta_4 &= \{(1,2,3,4),(1,4,5,6),(2,3,5,6)\} \\
|\delta_4| &= 3 \\
\delta_5 &= \textit{there are no closed sets of size } k=5. \\
|\delta_5| &= 0 \\
\delta_6 &= \{(1,2,3,4,5,6)\} \\
|\delta_6| &= 1
\end{aligned}$$

Example 2.2.2. The diagram in Figure 2.5 is the suspended Y-tree $\widetilde{Y_5}$, with 6 vertices and edge set $\{1,2,3,4,5,6,7\}$. The edge sets $\{1,2,4,5,7\}$ and $\{3,4,5,6\}$ are the cyclic subgraphs C_5 and C_4, respectively. The intersection $\{4,5\}$ is clearly a path, P_3.

Let us consider all the closed sets of $\widetilde{Y_5}$

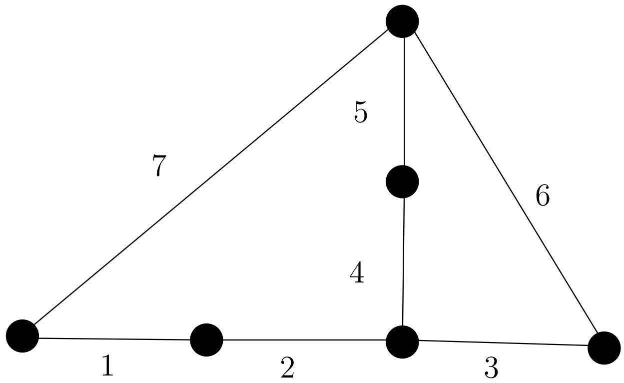

Figure 2.5: An example of $\widetilde{Y}_5$

$$
\begin{aligned}
\delta_0 &= \{\emptyset\} \\
|\delta_0| &= 1 \\
\delta_1 &= \{1,2,3,4,5,6,7\} \\
|\delta_1| &= 7 \\
\delta_2 &= \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3),(2,4),(2,5),(2,6), \\
&\quad (2,7),(3,4),(3,5),(3,6),(3,7),(4,5),(4,6),(4,7),(5,6),(5,7),(6,7)\} \\
|\delta_2| &= 21 \\
\delta_3 &= \{(1,2,3,),(1,2,4),(1,2,5),(1,2,6),(1,2,7),(1,3,4),(1,3,5),(1,3,6), \\
&\quad (1,3,7),(1,4,5),(1,4,6),(1,4,7),(1,5,6),(1,5,7),(1,6,7), \\
&\quad (2,3,4),(2,3,5),(2,3,6),(2,3,7),(2,4,5),(2,4,6),(2,4,7), \\
&\quad (2,5,6),(2,5,7),(2,6,7),(3,4,7),(3,5,7),(3,6,7),(4,5,7), \\
&\quad (4,6,7),(5,6,7),\} \\
|\delta_3| &= 31
\end{aligned}
$$

$$\begin{aligned}
\delta_4 &= \{(1,4,6,7),(1,3,4,7),(1,5,6,7),(1,3,4,7),\\
&\quad (2,4,6,7),(2,3,4,7),(2,5,6,7),(2,3,5,7),\\
&\quad (1,2,4,6),(1,2,3,4),(1,2,5,6),(1,2,3,5),(3,4,5,6)\}\\
|\delta_4| &= 13\\
\delta_5 &= \{(1,2,4,5,7),(1,3,4,5,6),(2,3,4,5,6),(3,4,5,6,7),(1,2,3,6,7)\}\\
|\delta_5| &= 5\\
\delta_6 &= \text{there are no closed sets of size } k=6.\\
|\delta_6| &= 0\\
\delta_7 &= \{(1,2,3,4,5,6,7)\}\\
|\delta_7| &= 1.
\end{aligned}$$

Example 2.2.3. The diagram in Figure 2.6 is the suspended Y-tree $\widetilde{Y}_6$, with 7 vertices and edge set $\{1,2,3,4,5,6,7,8\}$. The cyclic subgraph C_4 has edge set $\{4,5,6,7\}$. The cyclic subgraphs of C_6 have edge sets $\{1,2,3,5,6,8\}$ and $\{1,2,3,4,7,8\}$. Let us consider the number of closed sets of size k within $\widetilde{Y}_6$.

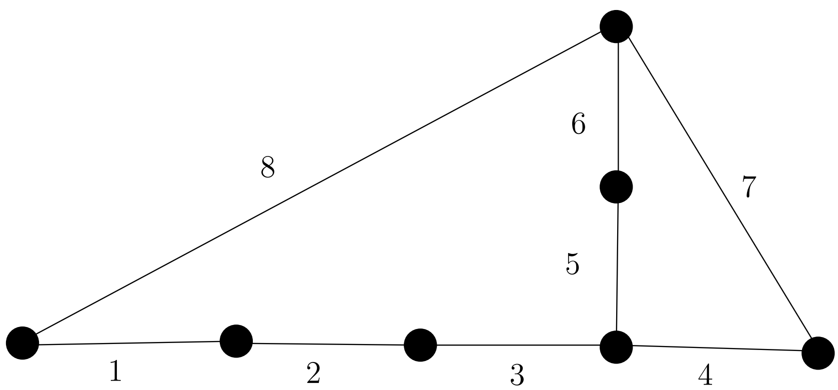

Figure 2.6: An example of $\widetilde{Y}_6$

$$
\begin{aligned}
|\delta_0| &= 1\\
|\delta_1| &= 8\\
|\delta_2| &= 28\\
|\delta_3| &= 52\\
|\delta_4| &= 54\\
|\delta_5| &= 20\\
|\delta_6| &= 8\\
|\delta_7| &= 0\\
|\delta_8| &= 1
\end{aligned}
$$

Proposition 2.2.4. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then $\widetilde{Y_n}$ cannot have a closed set of size $k > (n+2)$.*

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. Assume that this graph can have a closed set of size $k > n+2$. In other words assume that this graph, $\widetilde{Y_n}$, has at least one closed set of size $k = n+3$. By Proposition 2.1.5, $\widetilde{Y_n}$ has only $(n+2)$ edges and, by Proposition 2.1.6 one component. So to find a closed set of this size, $(n+3)$, we need an extra edge. This is a contradiction. □

Recall that we denote the set of all closed sets of size k by δ_k. Thus the number of all closed sets of size k is represented by $|\delta_k|$. In the theorems below, different values of k give different formulas. Hence we split our main result into several cases.

Theorem 2.2.5. *Let $\widetilde{Y_n}$ be a suspended Y-tree graph, with $n > 4$. Let δ_k represent the set of all closed sets of size k in $\widetilde{Y_n}$. Then for $k \leq 3$ we have*

$$|\delta_k| = \begin{cases} 1, & k = 0 \\ n+2, & k = 1 \\ \binom{n+2}{2}, & k = 2 \\ \binom{n+2}{3} - \binom{4}{3}, & k = 3. \end{cases}$$

We will prove each of these four cases separately.

Proof. Case $k = 0$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. It is clear that there is only one closed set of size $k = 0$, that is the empty set.

Case $k = 1$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. If we consider all the closed sets of size $k = 1$, we are considering each edge individually. In other words, each separate edge is, itself, a subgraph with the properties of a closed set.

By Proposition 2.1.5, there are $(n + 2)$ edges and so there are $(n + 2)$ closed sets of size $k = 1$.

Case $k = 2$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. If we consider all the closed sets of size $k = 2$, we are considering every possible pair of edges, since there are no parallel edges. Hence there are

$$\binom{n+2}{2}.$$

Case $k = 3$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. Let us consider all the closed sets of size $k = 3$. It is clear that there are $\binom{n+2}{3}$ such combinations. However, if we are to consider a closed set with 3 edges on the C_4 subgraph of $\widetilde{Y_n}$, we find that this set does not fulfill the definition of a closed set. There are 4 such possibilities which leads to the function:

$$\binom{n+2}{3} - \binom{4}{3}$$

$\square$

Theorem 2.2.6. *Let $\widetilde{Y_n}$ represent a suspended Y-tree graph, with $n > 4$. Let δ_k represent the set of all closed sets of size k in $\widetilde{Y_n}$. Then for $4 \leq k < (n-1)$,*

$$|\delta_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}.$$

Proof. Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. By Proposition 2.1.8, there exists subgraphs C_4 and C_n with intersection P_3. Selecting an edge set of size $4 \leq k < (n-1)$ allows for varied combinations, not all of which will be closed sets. We use the principle of "Inclusion/Exclusion".

There are $\binom{n+2}{k}$ possible combinations of edge sets of size k, both closed and non-closed sets. If only three of these edges are contained in the C_4 subgraph, then we do not have a closed set. But there are $\binom{4}{3}$ ways that this could happen and the other $(k-3)$ edges are chosen from the $(n-2)$ edges. Hence we have $\binom{n+2}{k}$ total combinations but we subtract the four $\binom{n-2}{k-3}$ situations. This results in

$$|\delta_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}.$$

$\square$

Lemma 2.2.7. *Let C_n be a cyclic graph of order n. There is no closed set of size $n-1$.*

Proof. Let C_n have edge set $a_1, a_2, \cdots, a_{n-1}, a_n$. Extracting a subgraph with an edge set of size $n-1$ will not be a closed set because the missing edge can still be included without changing the rank of the extracted subgraph. $\square$

The case of $k = n - 1$ gives a special formula. Considering that there are two cyclic graphs, C_n. By Lemma 2.2.7, C_n has no closed set of seize $n - 1$.

Theorem 2.2.8. *Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. Let δ_k represent the set of all closed sets of size k in $\widetilde{Y_n}$. Then for $k = (n-1)$,*

$$|\delta_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - 2\binom{n}{k}.$$

Proof. Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. By Proposition 2.1.8, there exists subgraphs C_4 and C_n. Selecting an edge set of size $k = (n-1)$ allows for varied combinations, not all of which will be closed sets. We use the principle of "Inclusion/Exclusion".

There are $\binom{n+2}{n-1}$ possible combinations of edge sets of size $(n-1)$, both closed and non-closed sets. If only three of these edges are contained in the subgraph C_4, then we do not have a closed set. We need to exclude these sets. There are $\binom{4}{3}$ ways that this could happen. In each case, of the remaining $(n-2)$ edges, $k-3$ are now unavailable for our potential closed set.

Furthermore, if all of these $(n-1)$ edges appear on either of the C_n subgraphs, by Lemma 2.2.7, there is no closed set of set $n-1$ in C_n. Hence there are $\binom{n}{k}$ such ways in both C_n subgraphs that this could happen. Hence the result

$$|\delta_k| = \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - 2\binom{n}{k}.$$

$\square$

Theorem 2.2.9. *Let $\widetilde{Y_n}$ be a suspended Y-tree graph, with $n > 4$. Let δ_k represent the set of all closed sets of size k in $\widetilde{Y_n}$. Then for $n \leq k \leq (n+2)$*

$$|\delta_k| = \begin{cases} \binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - \binom{4}{1}, & k = n \\ 0, & k = (n+1) \\ 1, & k = (n+2). \end{cases}$$

Proof. We will prove each of these three cases separately.

Case $k = n$

Let $\widetilde{Y_n}$ be a suspended Y-tree graph. Selecting an edge set of size $k = n$ allows for varied combinations, not all of which will be closed sets. We use the principle of "Inclusion/Exclusion". There are $\binom{n+2}{n}$ possible combinations of edge sets of size $k = n$ which include both closed and non closed sets. If three of these edges are contained in the C_4 subgraph, then we do not have a closed set. There are $\binom{4}{3}$ ways that this could happen. In each case, of the remaining $(n-2)$ edges, $k-3$ are now unavailable for our potential closed set, hence we exclude these. Furthermore, there are $\binom{4}{1}$, two for each C_n subgraph, where the closed set would have $(k-1)$ of its edges contained in a cyclic graph. Since, in this case, $(k-1) = (n-1)$ we cannot have such closed sets as they are not maximums.

Case $k = (n+1)$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. Any subgraph, made from a combination of $(n+1)$ edges, will not be a closed set as the final edge can be added without the rank of this subgraph being changed.

Case $k = (n+2)$

Let $\widetilde{Y_n}$ represent a suspended Y-tree graph. There is only one subgraph with $(n+2)$ edges and that is the entire graph itself, which happens to be a closed set - the only one of size $k = (n+2)$. □

Note that we excluded the case of $n = 4$. It is a special case since Proposition 2.1.8 will imply all the cyclic subgraphs are C_4 as well as the case $k = (n-1)$ and $k = 3$ being the same. See the diagram in Figure 2.7.

Theorem 2.2.10. *Let $\widetilde{Y_4}$ represent a suspended Y-tree graph. The following is true,*

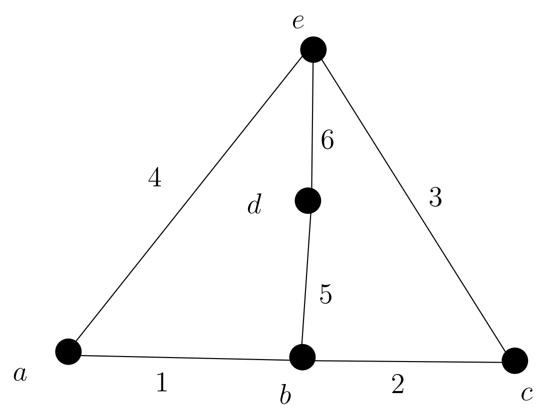

Figure 2.7: An example of a suspended Y-tree

for the specified values of k.

$$\begin{aligned} |\delta_0| &= 1 \\ |\delta_1| &= 6 \\ |\delta_2| &= 15 \\ |\delta_3| &= 8 \\ |\delta_4| &= 3 \\ |\delta_5| &= 0 \\ |\delta_6| &= 1 \end{aligned}$$

Proof. For $\widetilde{Y_4}$, the cases $k = 0, 1, 2, 5, 6$ are similar to the suspended trees $\widetilde{Y_n}$, for $n > 4$. Hence we will only show the cases $k = 3$ and $k = 4$.

Case $k = 3$

In $\widetilde{Y_4}$ there are at least three subgraphs, each being C_4. The only three-edge combinations which are not closed sets are the three-edge sets from one of the C_4 subgraphs. There are $3 \times \binom{4}{3}$ of these sets. But the number of possible three-edge sets in $\widetilde{Y_4}$ is $\binom{6}{3}$. Therefore the number of closed sets is

$$\binom{6}{3} - 3\binom{4}{3} = 20 - 12 = 8$$

as required.

Case $k = 4$

In $\widetilde{Y}_4$ there are at least three subgraphs, each being C_4. There are $\binom{6}{4}$ possible combinations, both closed and non-closed sets of edges. There can be three-edge set from any of the cyclic subgraphs and there are three of them, each with four edges. Hence our result

$$\binom{6}{4} - 3\binom{4}{1} = 15 - 12 = 3$$

as required. Note that these three closed sets are the cyclic subgraphs themselves, with edge sets $\{1,2,3,4\}$, $\{1,5,6,4\}$ and $\{2,3,6,5\}$. $\square$

2.3 Applications to Graph Compositions

In this section we will discuss the graph compositions of suspended trees $\widetilde{Y}_4$. It is interesting that all non-isomorphic trees have the same number of graph compositions but when we apply the operation of suspending a tree, the number of graph compositions are completely different. We will show that even if the trees have the same number of leaves, as long as the trees are not isomorphic, the number of graph compositions are not equal.

Definition 2.3.1. Let G be a graph with edge set $E(G)$ and vertex set $V(G)$. A composition of G, according to [7], is a partition of $V(G)$ into subsets of connected graphs $\{G_1, G_2, ..., G_m\}$ with the following properties.

$$\begin{aligned} \bigcup_{i=1}^{m} V(G_i) &= V(G) \\ V(G_i) \bigcap V(G_j) &= \emptyset \end{aligned}$$

These vertex sets, $V(G_i)$, will be called *components* of a given composition.

Definition 2.3.2. Let $C(G)$ denote the number of compositions for a graph, G.

We begin by giving all the graph compositions of the suspended Y-tree $\widetilde{Y}_5$, the diagram in Figure 2.5.

The diagram in Figure 2.8 is the graph composition of size 0. There is only one.

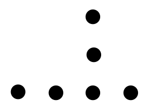

Figure 2.8:

The diagrams in Figure 2.9 are the graph compositions of size 1. There are 7 of them.

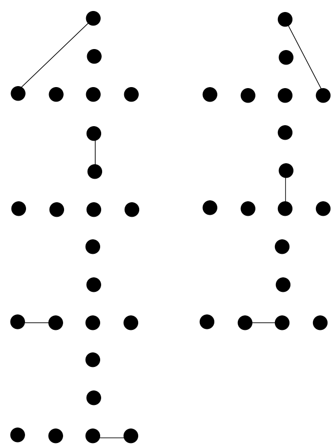

Figure 2.9:

The diagrams in Figure 2.10 are the graph compositions of size 2. There are 21 of them.

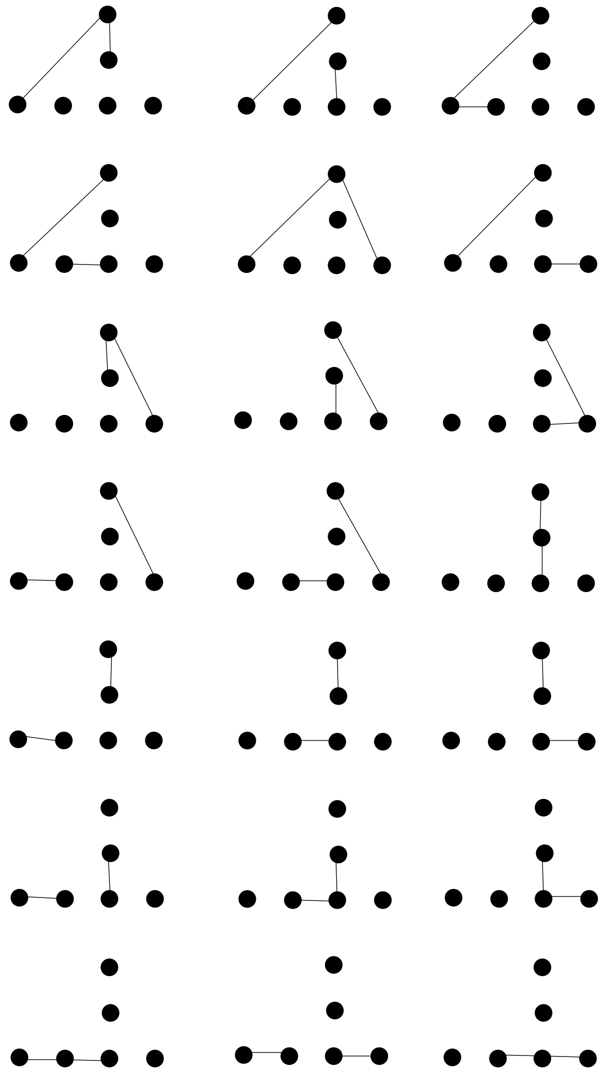

Figure 2.10:

The diagrams in Figure 2.11 are the graph compositions of size 3. There are 31 of them.

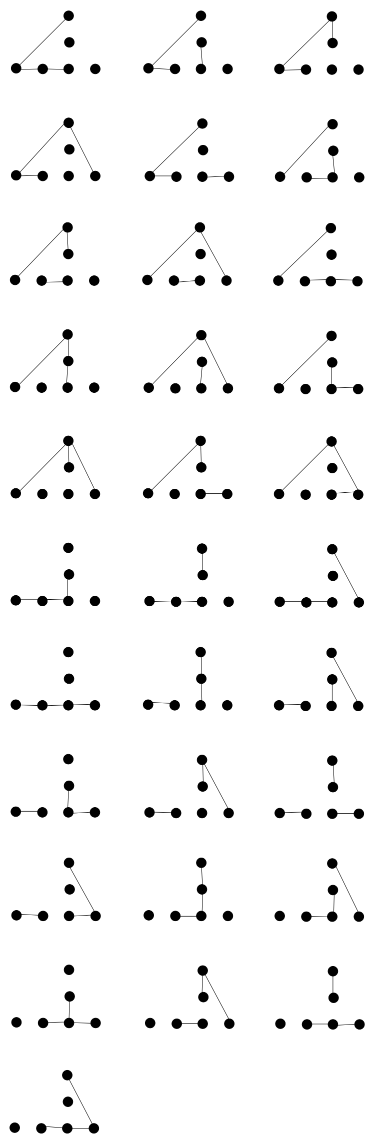

Figure 2.11:

The diagrams in Figure 2.12 are the graph compositions of size 4. There are 13 of them.

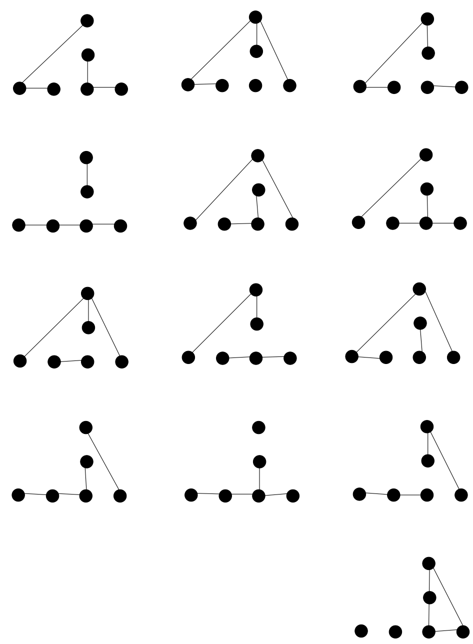

Figure 2.12:

The diagrams in Figure 2.13 are the graph compositions of size 5. There are 5 of them.

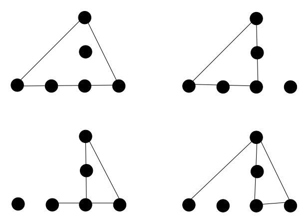

Figure 2.13:

The diagram in Figure 2.14 is the graph composition of size 7. There is only one.

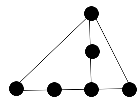

Figure 2.14:

We refer the reader to [7] for more information on the following theorems.

Theorem 2.3.3. *Let t_n be a tree graph with n vertices. Then the number of graph compositions $C(t_n) = 2^{n-1}$.*

Proof. Let us consider this proof by methods of induction.
Base case: $n = 1$.
When $n = 1$, the tree is merely one vertex. The number of compositions, $C(t_1)$ is $2^0 = 1$.
Assume true for $n = k$.
Consider $n = k + 1$. We expect to have the result of 2^k compositions for t_{k+1}.
If we have t_{k+1} and remove an edge, we now have disconnected t_{k+1} into two subgraphs, both of which are trees. Let l and $k + 1 - l$ denote the number of vertices of these subtrees, with $1 \leq l$. By Theorem 2.2.9, $2 \cdot 2^{l-1} \cdot 2^{k-l} = 2^k$ compositions for t_{k+1}, as expected. $\square$

We will state the following theorems without proofs. We refer the reader to [7] for more information.

Theorem 2.3.4. *Let K_n be a complete graph with n vertices. Then the number of graph compositions $C(K_n) = B(n)$.*

Recall from Definition 1.3.12, $k(G)$ is the number of components of a graph. Recall from Definition 2.3.2, $C(G)$ is the number of compositions of graph G.

Theorem 2.3.5. *Let G_1 and G_2 be disconnected subgraphs of G. In other words, $k(G) = 2$. We have the result that $C(G) = C(G_1) \cdot C(G_2)$. This result also holds if there is exactly one vertex in common between G_1 and G_2.*

Recall that the total number of closed sets of $\widetilde{Y}_6$ is 172. In Figure 2.6 has the following closed sets

$$\begin{aligned}
|\delta_0| &= 1 \\
|\delta_1| &= 8 \\
|\delta_2| &= 28 \\
|\delta_3| &= 52 \\
|\delta_4| &= 54 \\
|\delta_5| &= 20 \\
|\delta_6| &= 8 \\
|\delta_7| &= 0 \\
|\delta_8| &= 1.
\end{aligned}$$

The number of graph compositions of $\widetilde{Y}_6$, $C(\widetilde{Y}_6) = 172$.

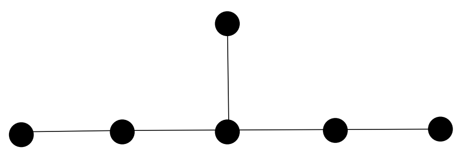

Figure 2.15:

Recall Definition 1.3.14, which states that two graphs, G and H are said to be isomorphic if there is a bijection $\sigma : V(G) \mapsto V(H)$ which preserves adjacency.

If we consider the tree τ_6 in the diagram in Figure 2.15, it is clear that it is not isomorphic to $\widetilde{Y}_6$. For instance, the degree sequence of τ_6 and Y_6 are the same but

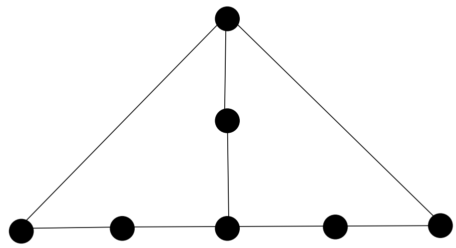

Figure 2.16:

the diagrams in Figure 2.17 can be used to prove that they are not isomorphic. The vertex set and edge set of Y_6 are the following

$$V(Y_6) = \{a, b, c, d, e, f\}, E(Y_6) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{d, f\}\}.$$

The vertex set and edge set of τ_6 are the following

$$V(\tau_6) = \{A, B, C, D, E, F\}, E(\tau_6) = \{\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{C, F\}\}.$$

Let σ be a mapping such that $\sigma(a) \mapsto A$, $\sigma(b) \mapsto B$, $\sigma(c) \mapsto C$, $\sigma(d) \mapsto D$, $\sigma(e) \mapsto E$, $\sigma(f) \mapsto F$. There is no mapping of the edge $\{d, f\}$ from Y_6 to τ_6. Therefore Y_6 and τ_6 are not isomorphic.

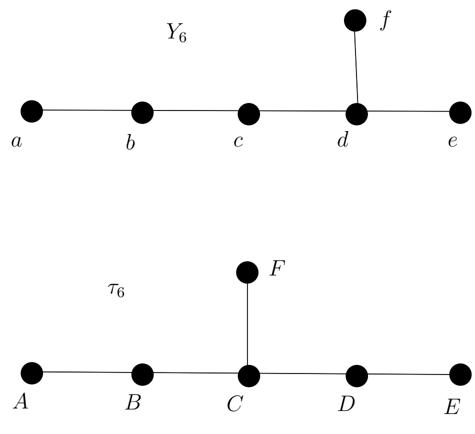

Figure 2.17:

By Theorem 2.3.3, $C(\tau_6) = C(Y_6) = 2^{6-1} = 32$. For $\widetilde{\tau}_6$ in Figure 2.16 we have the following number of closed sets

$$\begin{aligned} |\delta_0| &= 1 \\ |\delta_1| &= 8 \\ |\delta_2| &= 28 \\ |\delta_3| &= 56 \\ |\delta_4| &= 40 \\ |\delta_5| &= 30 \\ |\delta_6| &= 24 \\ |\delta_7| &= 0 \\ |\delta_8| &= 1. \end{aligned}$$

The number of graph compositions of $\widetilde{\tau_6}$, $C(\widetilde{\tau_6}) = 188$.

From the above examples, we have shown that the number of graph compositions of non isomorphic trees are the same but the number of graph compositions of these trees after suspension is different.

Let it be stated that $\sum_{k=1}^{n} \binom{n}{k} = 2^n$ for the following. For more information, we refer the reader to [2].

Theorem 2.3.6. *Let G be a labeled graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{C}_o(G)$ be the set of all distinct compositions of G such that $C(G) = |\mathcal{C}_o(G)|$ and let $\mathcal{F}(G)$ be the set of all distinct closed sets of $M(G)$. Then $C(G) = |\mathcal{F}(G)|$.*

Lemma 2.3.7.

$$\sum_{k=4}^{n-2}[\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}] = 3(2^n) + 4 - \left[\frac{70n + 2n^3}{6}\right].$$

Proof. Let $A = \sum_{k=4}^{n-2} \binom{n+2}{k}$ and let $B = \sum_{k=4}^{n-2} \binom{4}{3}\binom{n-2}{k-3}$. Then

$$\begin{aligned}
A &= \sum_{k=4}^{n-2} \binom{n+2}{k} \\
&= \sum_{k=1}^{n+2} \binom{n+2}{k} - \left[\binom{n+2}{0} + \binom{n+2}{1} + \binom{n+2}{2} + \binom{n+2}{3}\right] \\
&- \left[\binom{n+2}{n-1} + \binom{n+2}{n} + \binom{n+2}{n+1} + \binom{n+2}{n+2}\right] \\
&= 2^{n+2} - \left[2\binom{n+2}{n-1} + 2\binom{n+2}{n} + 2\binom{n+2}{n+1} + 2\binom{n+2}{n+2}\right] \\
&= 2^{n+2} - \left[2\frac{(n+2)(n+1)n}{3!} + 2\frac{(n+2)(n+1)}{2} + 2(n+2) + 2\right] \\
&= 2^{n+2} - \left[\frac{2n^3 + 12n^2 + 34n}{6} + 8\right].
\end{aligned}$$

$$\begin{aligned}
B &= \sum_{k=4}^{n-2} \binom{4}{3}\binom{n-2}{k-3} \\
&= 4\left(\sum_{k=0}^{n-2} \binom{n-2}{k} - \left[\binom{n-2}{0} + \binom{n-2}{n-4} + \binom{n-2}{n-3} + \binom{n-2}{n-2}\right]\right) \\
&= 4\left(2^{n-2} - \left[\frac{n^2 - 3n + 6}{2}\right]\right) \\
&= 2^n - \left[2n^2 - 6n + 12\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=4}^{n-2} [\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}] \\
&= A - B \\
&= 2^{n+2} - \left[\frac{2n^3 + 12n^2 + 34n}{6} + 8\right] - \left(2^n - \left[2n^2 - 6n + 12\right]\right) \\
&= 3(2^n) + 4 - \left[\frac{70n + 2n^3}{6}\right].
\end{aligned}$$

□

Lemma 2.3.8.

$$\binom{n+2}{n-1} - \binom{4}{3}\binom{n-2}{n-4} - 2\binom{n}{n-1} = \frac{1}{6}(n^3 - 9n^2 + 50n) - 12.$$

Proof.

$$\begin{aligned}
& \binom{n+2}{n-1} - \binom{4}{3}\binom{n-2}{n-4} - 2\binom{n}{n-1} \\
= \quad & \frac{(n+2)(n+1)(n)(n-1)!}{(n-1)!(3)(2)} - 4\frac{(n-2)(n-3)(n-4)!}{(n-4)!(2)} - 2\frac{n(n-1)!}{(n-1)!} \\
= \quad & \frac{(n+2)(n+1)(n)}{(6)} - 4\frac{(n-2)(n-3)}{2} - 2n \\
= \quad & \frac{n^3+3n^2+2n}{6} - \frac{4n^2-20n+24}{2} - 2n \\
= \quad & \frac{n^3+3n^2+2n}{6} - \frac{12n^2-60n+72}{6} - \frac{12n}{6} \\
= \quad & \frac{n^3-9n^2+50n-72}{6} \\
= \quad & \frac{1}{6}(n^3 - 9n^2 + 50n) - 12.
\end{aligned}$$

□

Lemma 2.3.9.

$$\binom{n+2}{n} - \binom{4}{3}\binom{n-2}{n-3} - \binom{4}{1} = \frac{1}{6}(3n^2 + 9n) - 4n + 5.$$

Proof.

$$\begin{aligned}
\binom{n+2}{n} - \binom{4}{3}\binom{n-2}{n-3} - \binom{4}{1} \quad &= \quad \frac{(n+2)(n+1)(n!)}{(2)(n!)} - 4\frac{(n-2)(n-3)!}{(n-3)!} - 4 \\
&= \quad \frac{n^2+3n+2}{(2)} - 4(n-2) - 4 \\
&= \quad \frac{n^2+3n+2}{2} - 4n + 4 \\
&= \quad \frac{3n^2+9n+6}{6} - 4n + 4 \\
&= \quad \frac{1}{6}(3n^2 + 9n) - 4n + 5.
\end{aligned}$$

□

Now we are in a position to state and prove the main theorem. Recall that we denote the set of all closed sets of size k by δ_k. Thus the number of all closed sets of size k is represented by $|\delta_k|$. We denote the set of all distinct compositions of G by $\mathcal{C}_o(G)$. Recall that $C(G)$ denotes the number of compositions of the graph G. Thus $C(G) = |\mathcal{C}_o(G)|$.

Theorem 2.3.10. *Let $\widetilde{Y_n}$ be a suspended Y tree on n vertices. The number of graph compositions of $\widetilde{Y_n}$,*

$$C(\widetilde{Y_n}) = 3(2^n - n) - 2.$$

Proof. By Theorem 2.3.6 we know that $C(\widetilde{Y_n}) = |\mathcal{F}(\widetilde{Y_n})|$ where $\mathcal{F}(\widetilde{Y_n})$ is the set of all distinct closed sets of $\widetilde{Y_n}$. Thus

$$\begin{aligned} C(\widetilde{Y_n}) &= |\mathcal{F}(\widetilde{Y_n})| \\ &= \sum_{k=0}^{n+2} |\delta_k|. \end{aligned}$$

By Theorems 2.2.5, 2.2.6, 2.2.8 and 2.2.9 we get

$$\begin{aligned} \sum_{k=0}^{n+2} |\delta_k| &= 1 + [n+2] + [\binom{n+2}{2}] + [\binom{n+2}{3} - \binom{4}{3}] \\ &+ \sum_{k=4}^{n-2} [\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3}] \\ &+ [\binom{n+2}{n-1} - \binom{4}{3}\binom{n-2}{n-4} - 2\binom{n}{n-1}] \\ &+ [\binom{n+2}{n} - \binom{4}{3}\binom{n-2}{n-3} - \binom{4}{1}] + 1. \end{aligned}$$

Hence simplifying and applying ,

$$\begin{aligned}
\sum_{k=0}^{n+2} |\delta_k| &= \left(\frac{n^3+3n^2+17n}{6}\right) + \left(3(2^n)+4-\left[\frac{70n+2n^3}{6}\right]\right) \\
&+ \left(\frac{1}{6}(n^3-9n^2+50n)-12\right) + \left(\frac{1}{6}(3n^2+9n)-4n+5\right)+1 \\
&= 3(2^n)-2 \\
&+ \left(\frac{n^3+3n^2+17n-70n-2n^3+n^3-9n^2+50n+3n^2+9n-24n}{6}\right) \\
&= 3(2^n)-3n-2.
\end{aligned}$$

Therefore

$$\sum_{k=0}^{n+2} |\delta_k| = C(\widetilde{Y_n}) = 3(2^n-n)-2.$$

□

Chapter 3

k-defect polynomials of suspended Y-trees

In this chapter, we study k-defect polynomials of suspended Y-trees. For certain k values, we find the explicit expression of the k-defect polynomial of suspended Y-trees.

3.1 Introduction

In this section we give some relevant definitions and concepts for this chapter. Unless otherwise stated, we refer the reader to [1] for details on proper colourings and the Definitions and Propositions in this section.

Definition 3.1.1. A *proper colouring* of a graph, G, is when a set of colours are assigned to the vertices such that adjacent vertices do not share a common colour.

Definition 3.1.2. The *chromatic number* of a graph G is the minimum number of colours needed for a proper vertex colouring of G. We denote it as $\chi(G)$.

Definition 3.1.3. If the chromatic number of G, $\chi(G) = \lambda$ then G is said to be λ-*colourable.*

Definition 3.1.4. The *characteristic polynomial* is the chromatic polynomial divided by a common factor of λ.

We state the following Propositions without a proof. For further details the reader is referred to [1]. Recall that $\chi(G;\lambda)$ is the chromatic polynomial of a graph G, $G\setminus e$ denotes the subgraph of G with the edge e deleted and G/e denotes the subgraph of G with the edge e contracted.

Proposition 3.1.5. *Let G be a graph with edgeset $E(G)$.*

(a) If $e \in E(G)$ and e is neither a bridge nor a loop in G, then

$$\chi(G;\lambda) = \chi(G\setminus e;\lambda) - \chi(G/e;\lambda).$$

(b) If $e \in E(G)$ and e is a bridge of G, then

$$\chi(G;\lambda) = (\lambda - 1)\chi(G\setminus e;\lambda).$$

(c) If there exists a loop in G, then $\chi(G;\lambda) = 0$.

Proposition 3.1.6. *If $e_1, e_2 \in E(G)$ are edges connecting the same vertices, then*

$$\chi(G;\lambda) = \chi(G\setminus e_1;\lambda) = \chi(G\setminus e_2;\lambda).$$

3.2 Coboundary Polynomials and k-Defect Polynomials

In this section we introduce the concept of coboundary polynomials and we show the relationship with k-defect polynomials.

Definition 3.2.1. The *coboundary polynomial* of a graph $G(V,E)$, is a polynomial with two independent variables λ and S.

We denote the coboundary as $B(G; \lambda, S)$. The coboundary polynomial of a graph G is defined as

$$B(G; \lambda, S) = \sum_{A \subseteq E} (S-1)^{|A|} \lambda^{r(E)-r(A)}$$

where $r(U)$ is the rank of U. Recall from Definition 1.3.10 that the rank of a graph is defined to be the number of components subtracted from the number of vertices.

Definition 3.2.2. Let G be a graph with a certain vertex colouring. An edge, $e \in E(G)$, is called a *bad* if it connects two vertices of the same colour.

Definition 3.2.3. We denote the *k-defect polynomial* as $\phi_k(G; \lambda)$. It is the polynomial of a graph G that counts the number of possible ways to colour G with k bad edges.

Proposition 3.2.4. *In a graph G, the 0-defect polynomial is the chromatic polynomial.*

$$\phi_0(G; \lambda) = \chi(G; \lambda).$$

Proof. The 0-defect polynomial is the polynomial for 0 bad edges. This is a proper colouring which is defined by $\chi(G; \lambda)$. □

The coboundary polynomial can be written as a generation function in S. We refer the reader to [4] for further details.

$$B(G; \lambda, S) = \sum_{k=0}^{|E|} S^k \phi_k(G; \lambda). \tag{3.1}$$

We state the following proposition without proof. For further details we refer the reader to [9].

Proposition 3.2.5. *Let G be a graph.*

i) If $e \in E(G)$ and e is neither a bridge nor a loop in G, then

$$B(G; \lambda, S) = B(G \setminus e; \lambda, S) + (S-1)B(G/e; \lambda, S).$$

ii) If $e \in E(G)$ *and* e *is a loop, then*

$$B(G;\lambda,S) = SB(G \setminus e;\lambda,S).$$

iii) If $e \in E(G)$ *and* e *is a bridge, then*

$$B(G;\lambda,S) = (S+\lambda-1)B(G/e;\lambda,S).$$

This formula is referred to as the deletion and contraction formula for the coboundary polynomial.

Proposition 3.2.6. *The coboundary polynomial of a single vertex is* 1*. In other word* $B(K_1;\lambda,S) = 1$.

Lemma 3.2.7. *Let* T_n *be a tree graph. Then the coboundary polynomial* $B(T_n;\lambda,S) = (S+\lambda-1)^{n-1}$.

Proof. Let T_n be a tree graph. Each edge of T_n is a bridge. By Proposition 3.2.5 part c, we get the result. □

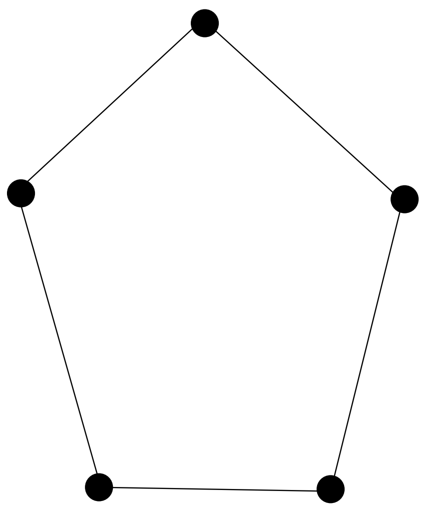

Figure 3.1: C_5

Example 3.2.8. Let G be the cyclic graph, C_5, shown in Figure 3.1. We use the deletion and contraction formula for the coboundary polynomial

$$\begin{aligned}
B(C_5;\lambda,S) &= B(C_5\setminus e;\lambda,S)+(S-1)B(C_5/e;\lambda,S)\\
&= B(T_5;\lambda,S)+(S-1)B(C_4;\lambda,S)\\
&= (S+\lambda-1)^4+(S-1)B(T_4;\lambda,S)+(S-1)^2B(C_3;\lambda,S)\\
&= (S+\lambda-1)^4+(S-1)(S+\lambda-1)^3\\
&+ (S-1)^2B(T_3;\lambda,S)+(S-1)^3B(C_2;\lambda,S)\\
&= (S+\lambda-1)^4+(S-1)(S+\lambda-1)^3)\\
&+ (S-1)^2(S+\lambda-1)^2+(S-1)^3(S+\lambda-1)+(S-1)^4(C_1;\lambda,S)\\
&= (S+\lambda-1)^4+(S-1)(S+\lambda-1)^3)\\
&+ (S-1)^2(S+\lambda-1)^2+(S-1)^3(S+\lambda-1)+S(S-1)^4.
\end{aligned}$$

This can be rewritten as a generating function

$$\begin{aligned}
B(C_5;\lambda,S) &= S^5+S^3(10\lambda-10)+S^2(10\lambda^2-30\lambda+20)\\
&+ S^1(5\lambda^3-20\lambda^2+30\lambda-15)\\
&+ S^0(\lambda^4-5\lambda^3+10\lambda^2-10\lambda+4).
\end{aligned}$$

Recall equation 3.1 that $B(G;\lambda,S)=\sum_{k=0}^{|E|}S^k\phi_k(G;\lambda)$. Hence we can extract the respective k-defect polynomials of C_5 as

$$\begin{aligned}
\phi_5(C_5;\lambda) &= 1\\
\phi_4(C_5;\lambda) &= 0\\
\phi_3(C_5;\lambda) &= 10\lambda-10\\
\phi_2(C_5;\lambda) &= 10\lambda^2-30\lambda+20\\
\phi_1(C_5;\lambda) &= 5\lambda^3-20\lambda^2+30\lambda-15\\
\phi_0(C_5;\lambda) &= \lambda^4-5\lambda^3+10\lambda^2-10\lambda+4.
\end{aligned}$$

Example 3.2.9. In Figure 2.4, we have a suspended Y-tree, $\widetilde{Y_4}$. Applying the deletion and contraction formula for the coboundary polynomial, we get

$$\begin{aligned}
B(Y_4;\lambda,S) &= (2S-2+\lambda)[(S+\lambda-1)^3 \\
&+ (S-1)(S+\lambda-1)^2 \\
&+ (S-1)^2(S+\lambda-1)+S(S-1)^3] \\
&+ (S-1)^2(S+\lambda-1)^2+2S(S-1)^3(S+\lambda-1) \\
&+ S^2(S-1)^4.
\end{aligned}$$

After expanding and collecting like terms, we get

$$\begin{aligned}
B(Y_4;\lambda,S) &= S^6 \\
&+ S^4(3\lambda-3) \\
&+ S^3(8\lambda-8) \\
&+ S^2(15\lambda^2-42\lambda+27) \\
&+ S^1(6\lambda^3-30\lambda^2+48\lambda-24) \\
&+ S^0(\lambda^4-6\lambda^3+15\lambda^2-17\lambda+7).
\end{aligned}$$

Hence, we extract the respective k-defect polynomials of $\widetilde{Y_4}$ as

$$\begin{aligned}
\phi_6(\widetilde{Y_4};\lambda) &= 1 \\
\phi_5(\widetilde{Y_4};\lambda) &= 0 \\
\phi_4(\widetilde{Y_4};\lambda) &= 3\lambda-3 \\
\phi_3(\widetilde{Y_4};\lambda) &= 8\lambda-8 \\
\phi_2(\widetilde{Y_4};\lambda) &= 15\lambda^2-42\lambda+27 \\
\phi_1(\widetilde{Y_4};\lambda) &= 6\lambda^3-30\lambda^2+48\lambda-24 \\
\phi_0(\widetilde{Y_4};\lambda) &= \lambda^4-6\lambda^3+15\lambda^2-17\lambda+7.
\end{aligned}$$

Example 3.2.10. In Figure 2.5, we have a suspended Y-tree, $\widetilde{Y}_5$. Applying the deletion and contraction formula for the coboundary polynomial and extracting the respective k-defect polynomials, we get

$$\begin{aligned}
\phi_7(\widetilde{Y}_5;\lambda) &= 1\\
\phi_6(\widetilde{Y}_5;\lambda) &= 0\\
\phi_5(\widetilde{Y}_5;\lambda) &= 5\lambda-5\\
\phi_4(\widetilde{Y}_5;\lambda) &= \lambda^2+9\lambda-10\\
\phi_3(\widetilde{Y}_5;\lambda) &= 31\lambda^2-86\lambda+55\\
\phi_2(\widetilde{Y}_5;\lambda) &= 21\lambda^3-99\lambda^2+154\lambda-76\\
\phi_1(\widetilde{Y}_5;\lambda) &= 7\lambda^4-42\lambda^3+101\lambda^2-111\lambda\\
\phi_0(\widetilde{Y}_5;\lambda) &= \lambda^5-7\lambda^3+21\lambda^3-34\lambda^2+29\lambda-10.
\end{aligned}$$

3.3 Properties of k-Defect Polynomials

In this section we list some useful properties of k-Defect polynomials for the purposes of this chapter.

The following Propositions will be stated without proofs. For more information, we refer the reader to [9].

Proposition 3.3.1. *Let G be a graph. Then*

$$\phi_k(G;\lambda) = \sum_{X\in L(G),|X|=k} \chi(G/X;\lambda)$$

if G has one or more closed sets of size k. If not, then

$$\phi_k(G;\lambda) = 0.$$

Proposition 3.3.2. *Let $\widetilde{Y_n}$ be a suspended Y-tree and let δ_{k_i} be the ith closed set of size k. Then*

$$\phi_k(\widetilde{Y_n};\lambda) = \sum_{i=0}^{|\delta_k|} \chi(\widetilde{Y_n}/\delta_{k_i};\lambda)$$

where $|\delta_k|$ is the number of closed sets of size k.

Proposition 3.3.3. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Let $0 \leq k \leq n+2$. If all minors obtained by contracting closed sets of size k, δ_k, are isomorphic, then*

$$\phi_k(\widetilde{Y_n};\lambda) = |\delta_k|\chi(\widetilde{Y_n}/\delta_k;\lambda).$$

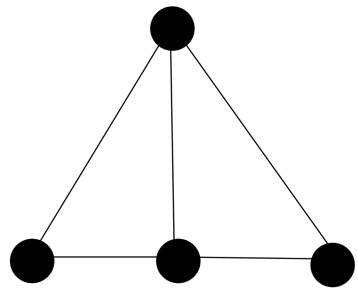

Figure 3.2: After the contraction of an edge in $\widetilde{Y_4}$, all minors are isomorphic to this graph, $\widetilde{Y_4}/\delta_1$.

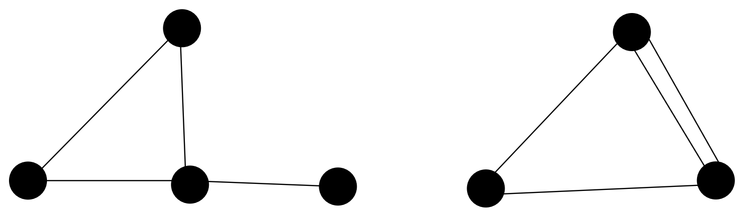

Figure 3.3: $\widetilde{Y_4}/\delta_1$, after an edge has been deleted (left) and contracted (right).

Example 3.3.4. We shall now apply these Propositions to $\widetilde{Y_4}$. Recall from Example 3.2.9 that

$$\phi_1(\widetilde{Y_4};\lambda) = 6\lambda^3 - 30\lambda^2 + 48\lambda - 24.$$

From Proposition 3.3.2, and by the properties in Proposition 3.1.5, we have that

$$\begin{aligned}
\phi_1(\widetilde{Y_4};\lambda) &= \sum_{i=0}^{6} \chi(\widetilde{Y_4}/\delta_{1_i};\lambda) \\
&= 6(\lambda-1)\chi(C_3;\lambda) - \chi(C_3;\lambda) \\
&\quad \text{from the diagrams in Figure 3.3} \\
&= 6(\lambda-2)(\lambda)(\lambda-1)(\lambda-2) \\
&= 6(\lambda-2)(\lambda-1)(\lambda-2) \\
&\quad \text{the } \lambda \text{ is dropped because we're dealing with the characteristic polynomial} \\
&= 6(\lambda^2-4\lambda+4)(\lambda-1) \\
&= 6(\lambda^3-5\lambda^2+8\lambda-4) \\
&= 6\lambda^3 - 30\lambda^2 + 48\lambda - 24.
\end{aligned}$$

3.4 Explicit Expressions for k-Defect Polynomials of Suspended Y-trees

In this section, we give the main results of this chapter. We will give some explicit expressions of some k-defect polynomials.

Proposition 3.4.1. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then the $(n+2)$-defect polynomial*

$$\phi_{n+2}(\widetilde{Y_n};\lambda) = 1.$$

Proof. From Proposition 2.1.5 we know that $\widetilde{Y_n}$ has $n+2$ edges. The minor obtained after the contraction of the closed set of size $n+2$ is K_1. There is one flat of size $n+2$. This is just a single vertex, and by Proposition 3.3.2, $\phi_{n+2}(\widetilde{Y_n};\lambda) = 1$. □

Proposition 3.4.2. *Let $\widetilde{Y_n}$ be a suspended Y-tree. The $(n+1)$-defect polynomial $\phi_{n+1}(\widetilde{Y_n};\lambda) = 0$.*

Proof. By Theorem 2.2.9, we know that there are no closed sets of size $n+1$. Therefore there are no minors and no k-defect polynomials of size $n+1$. □

Lemma 3.4.3. *Let C_n be a cyclic graph. The minor obtained by contracting m edges of C_n will be the cyclic graph C_{n-m}.*

Proof. Without loss of generality, consider C_8, the diagram in Figure 3.4. We can contract the edge set $\{e_1, e_2\}$ and the result will be the minor cyclic graph C_6. □

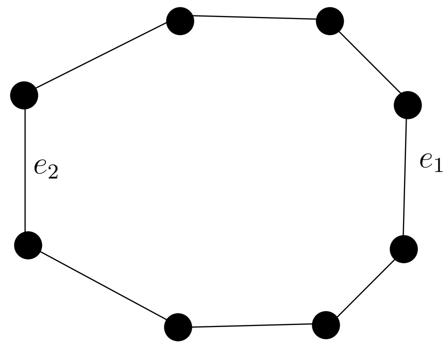

Figure 3.4: C_8

Lemma 3.4.4. *Let G be a graph without a vertex of degree 1. The minor obtained by contracting any set of edges in G will not have a vertex of degree 1.*

Proof. Let G be a graph. If there is no vertex of degree 1 then each and every edge of G is contained within a certain cyclic subgraph of G. By Lemma 3.4.3, if we contract an edge of one of those cyclic graphs, it will give us a minor which is a cyclic graph. Hence we cannot have a vertex of degree 1. □

Lemma 3.4.5. *The n-defect polynomial $\phi_n(\widetilde{Y_n}; \lambda)$ has a factor $(\lambda - 1)$.*

Proof. By Lemma 3.4.4, if we contract n of the $n+2$ edges in $\widetilde{Y_n}$, we end up with a pair of parallel edges. Therefore the characteristic polynomial of this will be $(\lambda - 1)$. Hence $(\lambda - 1)$ is a factor. □

Proposition 3.4.6. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then the n-defect polynomial*

$$\phi_n(\widetilde{Y_n};\lambda) = \frac{n^2-5n+10}{2}(\lambda-1).$$

Proof. By Theorem 2.2.9 we know that the number of closed sets of size n in $\widetilde{Y_n}$ is

$$\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - \binom{4}{1}.$$

This can be simplified using the fact that

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}.$$

$$\begin{aligned}
\binom{n+2}{k} - \binom{4}{3}\binom{n-2}{k-3} - \binom{4}{1} &= \binom{n+2}{n} - 4\binom{n-2}{n-3} - 4 \\
&= \frac{(n+2)!}{n!(2)!} - 4\frac{(n-2)!}{(n-3)!(1)!} - 4 \\
&= \frac{(n+2)(n+1)}{2} - 4(n-2) - 4 \\
&= \frac{n^2+3n+2}{2} - 4n + 8 - 4 \\
&= \frac{n^2-5n+10}{2}(\lambda-1).
\end{aligned}$$

□

We shall now investigate the explicit expression of the 0-defect polynomial of $\widetilde{Y_n}$.

Proposition 3.4.7. *Let $\widetilde{Y_n}$ be a suspended Y-tree. The 0-defect polynomial of $\widetilde{Y_n}$,*

$$\phi_0(\widetilde{Y_n};\lambda) = (\lambda-1)^{n-1}(\lambda^2-3\lambda+3) + (-1)^n(2\lambda^2-5\lambda+3).$$

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. The 0-defect polynomial is just the chromatic polynomial since it would be a proper colouring. Using the deletion and contraction methods in Proposition 3.1.5 we get

$$
\begin{aligned}
\chi(\widetilde{Y_n};\lambda) &= (\lambda-1)(\chi(C_n;\lambda)) - C_{n-1}(\lambda-2)\\
&= (\lambda-1)((\lambda-1)^n + (-1)^n(\lambda-1))\\
&- (\lambda-2)((\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1))\\
&= (\lambda-1)^{n+1} + (-1)^n(\lambda^2-2\lambda+1)\\
&- (\lambda-2)(\lambda-1)^{n-1} + (-1)^n(\lambda^2-3\lambda+2)\\
&= (\lambda-1)^{n-1}(\lambda^2-3\lambda+3)\\
&+ (-1)^n(2\lambda^2-5\lambda+3).
\end{aligned}
$$

□

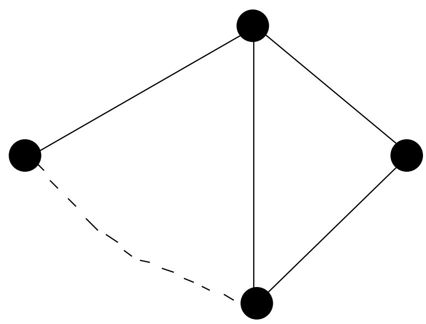

Figure 3.5: Z_n, a minor of $\widetilde{Y_n}$.

Recall that the 0-defect polynomial, $\phi_0(\widetilde{Y_n};\lambda) = \chi(\widetilde{Y_n};\lambda)$.

Lemma 3.4.8. *Let Z_n be the minor of $\widetilde{Y_n}$ shown in the diagram in Figure 3.5. Then $\chi(Z_n;\lambda) = (\lambda-2)((\lambda-1)^n + (-1)^n(\lambda-1))$.*

Proof. Let Z_n be the minor of $\widetilde{Y_n}$ shown in the diagram in Figure 3.5. Then

$$\begin{aligned}
\chi(Z_n;\lambda) &= \frac{\chi(C_n;\lambda)\chi(C_3;\lambda)}{\chi(K_2;\lambda)} \\
&= \frac{\chi(C_n;\lambda)(\lambda(\lambda-1)(\lambda-2))}{\lambda(\lambda-1)} \\
&= (\lambda-2)\chi(C_n;\lambda) \\
&= (\lambda-2)((\lambda-1)^n+(-1)^n(\lambda-1)).
\end{aligned}$$

□

We shall now investigate the explicit expression of the 1-defect polynomial of $\widetilde{Y_n}$. If we contract one edge from $\widetilde{Y_n}$, we can group all the minors into two groups. The one group of minors is Z_n, the diagram in Figure 3.5. The other group of minors is $\widetilde{Y_{n-1}}$.

Before stating the main result, the following lemma is obtained from Proposition 3.3.1.

Lemma 3.4.9. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then the 1-defect polynomial*

$$\phi_1(\widetilde{Y_n};\lambda) = \sum_{i=1}^{n+2} \chi(\widetilde{Y_n}/e_{1_i};\lambda)$$

where e_{1_i} is a closed set of $\widetilde{Y_n}$ of size 1 and $1 \leq i \leq n+2$.

Recall that the minors obtained by contracting a closed set of size 1 are either $\widetilde{Y_n}/e_{1_i} \cong Z_n$ or $\widetilde{Y_n}/e_{1_i} \cong \widetilde{Y_{n-1}}$.

Proposition 3.4.10. *Let $\widetilde{Y_n}$ be a suspended Y-tree. The 1-defect polynomial of $\widetilde{Y_n}$,*

$$\begin{aligned}
\phi_1(\widetilde{Y_n};\lambda) &= (n-2)\left[(-1)^n(2\lambda^2-5\lambda+3)+(\lambda-1)^n(\frac{\lambda^3-4\lambda^2+5\lambda-3}{\lambda^2-2\lambda+1})\right] \\
&+ 4(\lambda-2)(\lambda-1)^n(-1)^n(4\lambda^2-12\lambda+8).
\end{aligned}$$

Proof. Let $\widetilde{Y_n}$ be a suspended Y-tree. By Lemma 3.4.9 we have

$$
\begin{aligned}
\phi_1(\widetilde{Y_n};\lambda) &= \sum_{i=1}^{n+2} \chi(\widetilde{Y_n}/e_{1_i};\lambda) \\
&= 4\chi(Z_n;\lambda) + (n-2)\chi(\widetilde{Y_{n-1}}/;\lambda) \\
&= 4(\lambda-2)((\lambda-1)^n + (-1)^n(\lambda-1)) \\
&+ (n-2)(\lambda-1)^{n-2}(\lambda^2-3\lambda+3) \\
&+ (-1)^{n-1}(2\lambda^2-5\lambda+3)\text{by Proposition 3.4.7 and Lemma 3.4.8} \\
&= (n-2)\left[(-1)^n(2\lambda^2-5\lambda+3) + (\lambda-1)^n(\frac{\lambda^3-4\lambda^2+5\lambda-3}{\lambda^2-2\lambda+1})\right] \\
&+ 4(\lambda-2)(\lambda-1)^n(-1)^n(4\lambda^2-12\lambda+8).
\end{aligned}
$$

□

We have calculated the k-defect polynomials of $\widetilde{Y_4}$ in Example 3.2.9.

$$
\begin{aligned}
\phi_6(\widetilde{Y_4};\lambda) &= 1 \\
\phi_5(\widetilde{Y_4};\lambda) &= 0 \\
\phi_4(\widetilde{Y_4};\lambda) &= 3\lambda-3 \\
\phi_1(\widetilde{Y_4};\lambda) &= 6\lambda^3-30\lambda^2+48\lambda-24 \\
\phi_0(\widetilde{Y_4};\lambda) &= \lambda^4-6\lambda^3+15\lambda^2-17\lambda+7.
\end{aligned}
$$

From Propositon 3.4.7, the 0-defect polynomial of $\widetilde{Y_4}$,

$$
\begin{aligned}
\phi_0(\widetilde{Y_4};\lambda) &= (\lambda-1)^{4-1}(\lambda^2-3\lambda+3) + (-1)^4(2\lambda^2-5\lambda+3) \\
&= \lambda(\lambda^4-6\lambda^3+15\lambda^2-17\lambda+7).
\end{aligned}
$$

From Proposition 3.4.10, the 1-defect polynomial of $\widetilde{Y_4}$,

$$
\begin{aligned}
\phi_1(\widetilde{Y_4};\lambda) &= (4-2)\left[(-1)^n(2\lambda^2-5\lambda+3) + (\lambda-1)^4(\frac{\lambda^3-4\lambda^2+5\lambda-3}{\lambda^2-2\lambda+1})\right] \\
&+ 4(\lambda-2)(\lambda-1)^4(-1)^n(4\lambda^2-12\lambda+8) \\
&= \lambda(6\lambda^3-30\lambda^2+48\lambda-24).
\end{aligned}
$$

From Proposition 3.4.1, then the $(n+2)$-defect polynomial

$$\phi_{n+2}(\widetilde{Y_4};\lambda) = 1.$$

From Proposition 3.4.2, the $(n+1)$-defect polynomial

$$\phi_{n+1}(\widetilde{Y_4};\lambda) = 0.$$

From Proposition 3.4.6, the n-defect polynomial

$$\begin{aligned} \phi_n(\widetilde{Y_4};\lambda) &= \frac{4^2 - 54 + 10}{2}(\lambda - 1). \\ &= 3\lambda - 3. \end{aligned}$$

In conclusion, we see that our Theorem is verified.

Chapter 4

The Tutte Polynomial

4.1 Introduction

The Tutte polynomial has applications in graph theory, knot theory, statistical physics and more. A powerful property of the Tutte polynomial is that if it is evaluated at certain values, we obtain many parameters. The number of spanning trees, the number of acyclic orientations and the characteristic polynomial are just some examples of what the Tutte polynomial can yield. For more details, we refer the reader to [8]. In this Chapter we list some properties of the Tutte polynomial, we obtain explicit expressions of the Tutte polynomial for certain suspended Y-trees, we give specific examples of Tutte polynomials and finally we deduce the chromatic polynomial from the Tutte polynomial.

4.2 Properties

Definition 4.2.1. The *Tutte Polynomial* of a graph, G, is denoted $T(G; x, y)$. It has the following properties:

i) $T(I; x, y) = x$ where I is a bridge.

ii) $T(L; x, y) = y$ where L is a loop.

iii) If $e \in E(G)$ and e is neither a bridge nor a loop, then

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y).$$

iv) If $e \in E(G)$ and e is either a bridge nor a loop, then

$$T(G; x, y) = T(G(e); x, y)T(G \setminus e; x, y).$$

For these, and additional details on the Tutte Polynomials, We refer the reader to [3] for more information.

The coboundary polynomial can be written in terms of the Tutte polynomial. We refer the reader to [3] for more information.

$$B(G; \lambda, S) = (S-1)^r T(G; \frac{S+\lambda-1}{S-1}, S)$$

4.3 Explicit Expressions of the Tutte Polynomial for Suspended Y-trees

In this section we attempt to deduce and explicit expression for the Tutte polynomial for suspended Y-trees.

Proposition 4.3.1. *Let t_n be a tree graph. Then the Tutte polynomial*

$$T(t_n; x, y) = x^{n-1}.$$

Proof. Let t_n be a tree graph. Then every edge of t_n is a bridge and there are $n-1$ of them. By repeated application of Property (i) in Definition 4.2.1 we get

$$\begin{aligned} T(t_n; x, y) &= xT(t_{n-1}; x, y) \\ &= x^2 T(t_{n-2}; x, y) \\ &= ... \\ &= x^{n-2} T(t_2; x, y) \\ &= x^{n-1}. \end{aligned}$$

$\square$

Proposition 4.3.2. *Let C_n be a cyclic graph. Then the Tutte polynomial*

$$T(C_n; x, y) = \sum_{i=1}^{n-1} x^i + y.$$

Recall from Definition 4.2.1 that $T(L; x, y) = y$ where L is a loop.

Proof. Let C_n be a cyclic graph. Applying the properties in Definition 4.2.1, we get

$$\begin{aligned}
T(C_n; x, y) &= T(C_n \setminus e; x, y) + T(C_n/e; x, y) \\
&= T(t_n; x, y) + T(C_{n-1}; x, y) \\
&= x^{n-1} + T(t_{n-1}; x, y) + T(C_{n-2}; x, y) \\
&= x^{n-1} + x^{n-2} + T(t_{n-2}; x, y) + T(C_{n-3}; x, y) \\
&= ... \\
&= x^{n-1} + x^{n-2} + x^{n-3} + ... + x^2 + x + T(L; x, y) \\
&= \sum_{i=1}^{n-1} x^i + y.
\end{aligned}$$

$\square$

Definition 4.3.3. *For the purposes of the proof of Theorem 4.3.5, we define D_n to be C_n with an extra edge, as shown in the diagram in Figure 4.1.*

Definition 4.3.4. *For the purposes of the proof of Theorem 4.3.5, we define J_n to be C_{n-2} with a parallel edge connecting to an additional vertex, as shown in the diagram in Figure 4.2.*

Theorem 4.3.5. *Let $\widetilde{Y_n}$ be a suspended Y-tree. Then The Tutte polynomial*

$$\begin{aligned}
T(\widetilde{Y_n}; x, y) &= (x+1)T(C_n; x, y) + (x+y)T(C_{n-2}; x, y) \\
&= (x+1)\sum_{i=1}^{n-1} x^i + y + (x+y)\sum_{i=1}^{n-3} x^i + y.
\end{aligned}$$

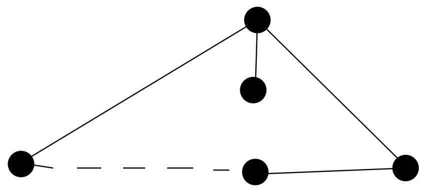

Figure 4.1: D_n

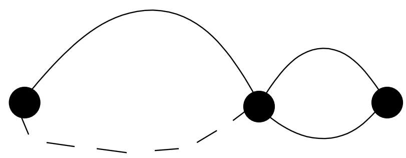

Figure 4.2: J_n

Proof. Using the properties listed in Definition 4.2.1

$$\begin{aligned}
T(\widetilde{Y_n};x,y) &= T(\widetilde{Y_n}\setminus e;x,y)+T(\widetilde{Y_n}/e;x,y)\\
&= T(D_n;x,y)+T(Z_n;x,y)\\
&= xT(C_n;x,y)+T(C_n;x,y)+T(J_n;x,y)\\
&= (x+1)T(C_n;x,y)+xT(C_{n-2};x,y)+T(C_{n-2}\cup L;x,y) \text{ where L is a loop}\\
&= (x+1)T(C_n;x,y)+xT(C_{n-2};x,y)+yT(C_{n-2};x,y)\\
&= (x+1)T(C_n;x,y)+(x+y)T(C_{n-2};x,y)\\
&= (x+1)\sum_{i=1}^{n-1}x^i+y+(x+y)\sum_{i=1}^{n-3}x^i+y \text{ by Proposition 4.3.2.}
\end{aligned}$$

□

The following Propositions will be stated without a proof. For more details, we refer the reader to [3]

Proposition 4.3.6. *Let G be a graph with vertex set $V(G)$ and $k(G)$ components. Then*

$$\chi(G;\lambda)=(-1)^{|V(G)|-k(G)}\lambda^k(G)T(G;1-\lambda,0)$$

where $\chi(G : \lambda)$ is the chromatic polynomial and $T(G; x, y)$ is the Tutte polynomial.

4.4 Specific Examples of Tutte Polynomials

In this section we are going to compute $T(\widetilde{Y}_4; x, y)$ and $T(\widetilde{Y}_5; x, y)$ using the deletion and contraction method in Definition 4.2.1. We will then compute $T(\widetilde{Y}_4; x, y)$ and $T(\widetilde{Y}_5; x, y)$ using Theorem 4.3.5. We will then compare the results.

Example 4.4.1. Let us consider $\widetilde{Y}_4$. Using the deletion and contraction method from Definition 4.2.1, we find the Tutte polynomial

$$\begin{aligned}
T(\widetilde{Y}_4; x, y) &= T(\widetilde{Y}_4 \setminus e; x, y) + T(\widetilde{Y}_4/e; x, y) \\
&= T(D_4; x, y) + T(Z_4; x, y) \\
&= xT(C_4 : x, y) + T(C_4; x, y) + T(J_4; x, y) \\
&= x^4 + x^3 + x^2 + xy + x^3 + x^2 + x + y + x(x + y) + y(x + y) \\
&= x^4 + x^3 + x^2 + xy + x^3 + x^2 + x + y + x^2 + xy + yx + y^2 \\
&= x^4 + 2x^3 + 3x^2 + 3xy + y^2 + x + y.
\end{aligned}$$

Example 4.4.2. Let us consider $\widetilde{Y}_5$. Using the deletion and contraction method from Definition 4.2.1, we find the Tutte polynomial

$$\begin{aligned}
T(\widetilde{Y}_5; x, y) &= T(\widetilde{Y}_5 \setminus e; x, y) + T(\widetilde{Y}_5/e; x, y) \\
&= T(D_5; x, y) + T(Z_5; x, y) \\
&= xT(C_5 : x, y) + T(C_5; x, y) + T(J_5; x, y) \\
&= x^5 + x^4 + x^3 + x^2 + xy + x^4 + x^3 \\
&+ x^2 + x + y + x(x^2 + x + y) + y(x^2 + x + y) \\
&= x^5 + x^4 + x^3 + x^2 + xy + x^4 + x^3 \\
&+ x^2 + x + y + x^3 + x^2 + xy + yx^2 + xy + y^2 \\
&= x^5 + 2x^4 + 3x^3 + 3x^2 + 3xy + x + y + x^2y + y^2.
\end{aligned}$$

Example 4.4.3. Let us consider $\widetilde{Y_4}$. Using Theorem 4.3.5, we can find the Tutte polynomial

$$\begin{aligned} T(\widetilde{Y_4}) &= (x+1)T(C_4; x, y) + (x+y)T(C_2; x, y) \\ &= (x+1)(x^3+x^2+x+y) + (x+y)(x+y) \text{by Proposition 4.3.2} \\ &= x^4+2x^3+3x^2+3xy+y^2+x+y. \end{aligned}$$

Example 4.4.4. Let us consider $\widetilde{Y_5}$. Using Theorem 4.3.5, we can find the Tutte polynomial

$$\begin{aligned} T(\widetilde{Y_5}) &= (x+1)T(C_5; x, y) + (x+y)T(C_3; x, y) \\ &= (x+1)(x^4+x^3+x^2+x+y) + (x+y)(x^2+x+y) \text{by Proposition 4.3.2} \\ &= x^5+2x^4+3x^3+3x^2+3xy+x+y+x^2y+y^2. \end{aligned}$$

Notice how, in the above examples, our theorem is verified in both the $\widetilde{Y_4}$ and the $\widetilde{Y_5}$ cases.

Example 4.4.5. *From Proposition 4.3.6, we have that*

$$\chi(G;\lambda) = (-1)^{|V(G)|-k(G)}\lambda^k(G)T(G; 1-\lambda, 0).$$

Let us verify this Proposition for $\widetilde{Y_4}$ and $\widetilde{Y_5}$ by comparing the results to Examples 3.2.9 and 3.2.10.

$$\begin{aligned} \chi(\widetilde{Y_4};\lambda) &= (-1)^{|V(\widetilde{Y_4})|-k(\widetilde{Y_4})}\lambda^k(\widetilde{Y_4})T(\widetilde{Y_4}; 1-\lambda, 0) \\ &= (-1)^{5-1}\lambda^1((1-\lambda)^4+2(1-\lambda)^3+3(1-\lambda)^2+0+0+(1-\lambda)+0) \\ &\quad \text{From Example 4.4.1} \\ &= \lambda((1-\lambda)^4+2(1-\lambda)^3+3(1-\lambda)^2+(1-\lambda)) \\ &= \lambda(\lambda^4-6\lambda^3+15\lambda^2-17\lambda+7) \\ &\quad \text{As in Example 3.2.9.} \end{aligned}$$

$$
\begin{aligned}
\chi(\widetilde{Y_5};\lambda) &= (-1)^{|V(\widetilde{Y_5})|-k(\widetilde{Y_5})}\lambda^k(\widetilde{Y_5})T(\widetilde{Y_5};1-\lambda,0) \\
&= (-1)^{6-1}\lambda^1((1-\lambda)^5+2(1-\lambda)^4+3(1-\lambda)^3+3(1-\lambda)^2+0+(1-\lambda)+0+0+0 \\
&\quad \text{From Example 4.4.2} \\
&= \lambda(\lambda^5-7\lambda^3+21\lambda^3-34\lambda^2+29\lambda-10 \\
&\quad \text{As in Example 3.2.10.}
\end{aligned}
$$

Proposition 4.3.6 is therefore verified.

Bibliography

[1] Balakrishnan, R, *A Textbook of Graph Theory*, Springer-Verlag, New York, 2000.

[2] Bryant, V, *Aspects of Combinatorics*, Cambridge University Press, Great Britain, 1993.

[3] T.H. Brylawski, and J.G. Oxley, The Tutte polynomial and its applications, In White, N., editor, *Matroid Applications*, Encyclopedia of Mathematics and its Applications,123-215, Cambridge University Press,Cambridge, 1992.

[4] H.H. Crapo, The Tutte polynomial, Aequationes Math.,3,(1969) 211-229.

[5] Endo, T., The link component number of suspended trees,*Graphs and Combinatorics*, 26, (2010), 483-490.

[6] Prof. J. L. Gross, Coursenotes for Rosen 6th Edition, Chapter 9, 13/11/2009.

[7] Knopfmacher, A. and Mays, M.E., Graph Compositions 1: Basic Enumeration, *Integers:The Electronic Journal of Combinatorial Number Theory*, **1**, (A04), (2001), 1-11.

[8] Mier, A. and Noy, M., On Graphs Determined by Their Tutte Polynomials, *Graphs and Combinatorics*, 2004.

[9] E.G. Mphako, Tutte polynomials, chromatic polynomials and matroids, Ph.D. thesis, Victoria University of Wellington, 2001.

[10] E.G. Mphako and S.C. Werner, Graph Compositions of Suspended Y-Trees, 2014, (submitted for publication)

[11] Graphs and Combinatorics (2010) 26:483-490 DOI 10.1007/s00373-010-0936-7

[12] R.C. Read, An introduction to chromatic polynomials, J. Combin. Theory, 4 (1968)52-71.

[13] V. I. Voloshin, Graph Coloring: History, Results and Open Problems, Troy University, Troy, AL, 2009.