Tensor Products of C^*-algebras

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Abstract

This report deals with the tensor product of two C^*-algebras and with the norms that can be defined on the tensor product which make it into a C^*-algebra. In particular, it discusses the two most important C^*-norms, the projective and injective C^*-norms, and it shows that they are respectively the largest and smallest C^*-norms that can be placed on the tensor product of two C^*-algebras. The state spaces of the tensor product under these norms are examined and identifications between the state spaces and certain spaces of functions are demonstrated.

Declaration

I declare that this research report is my own unaided work. It is being submitted in partial fulfilment of the requirements for the degree of Master of Science at the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted for any other degree or examination at any other university.

Mark Alan Brodie
August 1989.

Preface

Given two vector spaces E and F, we may consider the algebraic tensor product $E \otimes F$. It is natural to examine what type of space $E \otimes F$ is, depending on what type of spaces E and F are. Thus, for example, if E and F are normed spaces, $E \otimes F$ can be made into a normed space in various ways.

This research report considers a similiar question for C^*-algebras. Specifically, if E and F are C^*-algebras, how is it possible to make $E \otimes F$ into a C^*-algebra? The immediate problem is that $E \otimes F$ is, in general, not complete. It is therefore necessary to examine what norms can be defined on $E \otimes F$ which satisfy the properties of a C^*-norm, and to complete $E \otimes F$ with respect to such a norm, in order to obtain a C^*-algebra.

Chapter 1 introduces the fundamental definitions, ideas and notation which are required, and lists some of the important results which will be used in the course of the report. Certain elementary facts about C^*-algebras and tensor products are assumed.

Chapter 2 begins our investigation of the two most important C^*-norms which can be defined on the tensor product of two C^*-algebras, the *min* and *max* norms. These norms are defined, and it is shown that not only are they indeed C^*-norms, but they are also cross-norms in the usual sense. We prove easily that the max norm is the greatest possible C^*-norm which can be defined on the tensor product of two C^*-algebras. We then establish, via a somewhat lengthy sequence of results ending with Theorem 5.15, that the min norm is the smallest possible C^*-norm. We also show that all C^*-norms are cross-norms.

Chapter 3 is devoted to a consideration of the state spaces. We examine how the various norms on the tensor product of two C^*-algebras affect the resulting state spaces. Two different approaches to this question are presented. The first approach makes use of results concerning completely positive maps, while the second approach is based on the concept of separating sets. Along the way we discuss the important idea of the enveloping C^*-algebra of an involutive Banach algebra, and we also prove an apparently new result, Theorem 3.4.

The notation throughout the report is quite standard. Definitions, lemmas and theorems are numbered consecutively in each section. Unless the chapter number is given explicitly, quotations of any numbered definition, lemma or theorem always refer to the same chapter in which the reference is made.

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Chapter 1

PRELIMINARIES

§1. INTRODUCTION

This brief opening chapter introduces the basic concepts, definitions, and notation concerning tensor products, C^*-algebras, and representations that we will use. It also lists some of the fundamental results which we will need. Section 3 proves a useful extension result for C^*-algebras without identity elements.

§2. THE BASIC IDEAS

Let A_1 and A_2 be C^*-algebras. We will use the usual notation $A_1 \otimes A_2$ to denote the algebraic tensor product of A_1 and A_2. Then $A_1 \otimes A_2$ is an involutive algebra under the natural definitions:

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 y_1 \otimes x_2 y_2,$$
$$(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*, \qquad x_1, y_1 \in A_1,\ x_2, y_2 \in A_2.$$

We will consider what norms we can place on $A_1 \otimes A_2$ which will enable us to turn $A_1 \otimes A_2$ into a C^*-algebra.

As usual, if A is a C^*-algebra, then A' denotes the space of all bounded (i.e. continuous) linear functionals on A. If α is a norm on $A_1 \otimes A_2$, we say that α is a **cross-norm** if:

(1) $\|x_1 \otimes x_2\|_\alpha = \|x_1\|\|x_2\|$ for all $x_1 \in A_1$, $x_2 \in A_2$;

(2) if $x_1' \in A_1'$, $x_2' \in A_2'$, then $x_1' \otimes x_2'$ is continous on $A_1 \otimes A_2$ with respect to the norm α and the functional norm satisfies $\|x_1' \otimes x_2'\| \leq \|x_1'\|\|x_2'\|$.

We know that if x is any element of a C^*-algebra A, then x can be uniquely expressed in the form $x = x_1 + ix_2$, where x_1 and x_2 are hermitian elements of A such that $\|x_1\| \leq \|x\|$, $\|x_2\| \leq \|x\|$ [Dixmier, [2], 1.1.4]. If x is any hermitian element of A, then x can be written in the form $x = x_1 - x_2$, where x_1 and x_2 are positive elements of A such that $\|x_1\| \leq \|x\|$, $\|x_2\| \leq \|x\|$, and $\|x\| = \|x_1\| + \|x_2\|$ [Dixmier, [2], 1.5.7]. A linear functional ω on A is **positive** if $\omega(x^* x) \geq 0$ for all $x \in A$. All positive linear functionals on a C^*-algebra are continuous and satisfy $\|\omega\| = \omega(1)$ if the C^*-algebra has an identity element [Dixmier, [2], 2.1.4]. If f is a hermitian functional on A, then f can be uniquely expressed in the form $f = f_1 - f_2$, where f_1 and f_2 are positive linear functionals on A such that $\|f\| = \|f_1\| + \|f_2\|$ [Takesaki, [6], III.2.1 and III.4.2].

If H is a Hilbert space, then $\mathcal{L}(H)$ will denote the space of all bounded linear maps from H into itself. A **representation** of $A_1 \otimes A_2$ will always mean a

*-representation; namely, π is a representation of $A_1 \otimes A_2$ if π maps $A_1 \otimes A_2$ into $\mathcal{L}(H)$ for some Hilbert space H and satisfies, for all $x, y \in A_1 \otimes A_2$, $\lambda \in \mathbf{C}$:

$$\pi(\lambda x) = \lambda \pi(x)$$

$$\pi(x + y) = \pi(x) + \pi(y)$$

$$\pi(xy) = \pi(x)\pi(y) \quad \text{(where the RHS multiplication is composition in } \mathcal{L}(H))$$

$$\text{and } \pi(x^*) = (\pi(x))^*.$$

If H is a Hilbert space and M is a subset of H, then $[M]$ denotes the closed subspace of H spanned by M. If π is a representation of a C^*-algebra A on a Hilbert space H, then $\pi(A)H$ is defined as $\pi(A)H = \{\pi(x)\xi : x \in A, \ \xi \in H\}$, and π is called **nondegenerate** if $[\pi(A)H]$ coincides with the whole of H.

Suppose A is a C^*-algebra. If π_1 is a representation of A on a Hilbert space H_1 and π_2 is a representation of A on a Hilbert space H_2, then π_1 and π_2 are said to be **unitarily equivalent** if there exists an isometry U from H_1 onto H_2 such that $U\pi_1(x)U^* = \pi_2(x)$ for all $x \in A$. We will repeatedly make use of the correspondence between positive linear functionals on A and representations of A: If ω is a positive linear functional on A, there corresponds uniquely, within unitary equivalence, a representation π_ω of A on a Hilbert space H_ω, and a vector $\xi_\omega \in H_\omega$, such that: (i) $[\pi_\omega(A)\xi_\omega] = H_\omega$; (ii) $\omega(x) = (\pi_\omega(x)\xi_\omega, \xi_\omega)$ for all $x \in A$. We call π_ω the **cyclic representation** of A induced by ω, and ξ_ω is called a cyclic vector for π_ω [Takesaki, [6], I.9.14]. In this case $\|\omega\| = (\xi_\omega, \xi_\omega)$ [Dixmier, [2], 2.4.3]. Thus if $\|\omega\| = 1$, then ξ_ω is a unit vector. If A is not a C^*-algebra but just an involutive algebra, this association between positive linear functionals on A and representations of A can still be made [Effros and Lance, [3], pg 8].

Let A be a C^*-algebra and, for each $i \in I$, let π_i be a representation of A on a Hilbert space H_i. Let H be the direct sum Hilbert space $\sum_{i \in I}^{\oplus} H_i$. For each vector $\xi = \sum_{i \in I}^{\oplus} \xi_i \in H$ and $x \in A$, put $\pi(x)\xi = \sum_{i \in I}^{\oplus} \pi_i(x)\xi_i$. Then it is easy to see that π is a representation of A on H, called the direct sum of the family $(\pi_i : i \in I)$, and denoted by $\sum_{i \in I}^{\oplus} \pi_i$ [Takesaki, [6], I.9.15]. Every representation of A can be uniquely expressed as the direct sum of a trivial representation and a nondegenerate representation, [Dixmier, [2], 2.2.6], so we will often restrict our attention to nondegenerate representations. Notice that by (i) above, all cyclic representations are nondegenerate. Cyclic representations are particularly important because every nondegenerate representation of a C^*-algebra is a direct sum of cyclic representations [Takesaki, [6], I.9.17].

A representation π is **faithful** if $\pi(x) \neq 0$ for every nonzero x. We know that every C^*-algebra has a faithful representation; indeed, every C^*-algebra is isometrically isomorphic to a closed subalgebra of $\mathcal{L}(H)$ for some Hilbert space H [Takesaki, [6], I.9.18]. Given a representation π of A on a Hilbert space H, a closed subspace M of H is called an invariant subspace of π if $\pi(x)M \subset M$ for every $x \in A$. A representation π is irreducible if H and $\{0\}$ are the only invariant subspaces of π.

§3. Adjunction of Identities

If A is a C^*-algebra without an identity element, then we can embed A in a C^*-algebra with identity in the usual way [Takesaki, [6], I.1.5]. We obviously need to examine how this affects the tensor product of two C^*-algebras. We know that if π is a representation of $A_1 \otimes A_2$, then there exist unique representations π_1 of A_1 and π_2 of A_2 such that $\pi(x_1 \otimes x_2) = \pi_1(x_1)\pi_2(x_2) = \pi_2(x_2)\pi_1(x_1)$ for all $x_1 \in A_1$, $x_2 \in A_2$. We call π_i the restriction of π to A_i, $i = 1,2$ [Takesaki, [6], IV.4.1].
Hence we get:

Theorem 3.1. *Let A_1 and A_2 be C^*-algebras, and $(A_1)_I$ and $(A_2)_I$ the C^*-algebras obtained by adjunction of identities to A_1 and A_2 respectively, if necessary. Then any C^*-norm β of $A_1 \otimes A_2$ can be extended to a C^*-norm of $(A_1)_I \otimes (A_2)_I$.*

Proof: Let $A_1\tilde{\otimes}_\beta A_2$ denote the completion of $A_1 \otimes A_2$ with respect to the β-norm. Since β is a C^*-norm, $A_1\tilde{\otimes}_\beta A_2$ is a C^*-algebra. Let π be an isometric representation of $A_1\tilde{\otimes}_\beta A_2$, so that $\|x\|_\beta = \|\pi(x)\|$ for all $x \in A_1 \otimes A_2$.
Let π_i be the restriction of π to A_i, $i = 1,2$. Then $\pi_1(x_1)\pi_2(x_2) = \pi_2(x_2)\pi_1(x_1)$ for all $x_1 \in A_1$, $x_2 \in A_2$.
Define π_i^0 on $(A_i)_I$ by $\pi_i^0(x,\lambda) = \pi_i(x) + \lambda$, $x \in A_i$, $\lambda \in \mathbf{C}$.
Then π_i^0 is easily seen to be a representation of $(A_i)_I$ which coincides with π_i on A_i, $i = 1,2$.
We want to define a representation on $(A_1)_I \otimes (A_2)_I$ which extends π. The natural way to do this is by defining $\pi^0(z_1 \otimes z_2) = \pi_1^0(z_1)\pi_2^0(z_2)$, $z_1 \in (A_1)_I$, $z_2 \in (A_2)_I$, and extend by linearity to the whole of $(A_1)_I \otimes (A_2)_I$.
But this is only well defined if $\pi_1^0(z_1)\pi_2^0(z_2) = \pi_2^0(z_2)\pi_1^0(z_1)$ for all $z_1 \in (A_1)_I$, $z_2 \in (A_2)_I$; i.e. the ranges of π_1^0 and π_2^0 must commute.
But this is indeed true, for, if $z_i = (x_i, \lambda_i)$, $x_i \in A_i$, $\lambda_i \in \mathbf{C}$, $i = 1,2$, then

$$\begin{aligned}\pi_1^0(z_1)\pi_2^0(z_2) &= (\pi_1(x_1) + \lambda_1)(\pi_2(x_2) + \lambda_2)\\ &= \pi_1(x_1)\pi_2(x_2) + \lambda_1\pi_2(x_2) + \lambda_2\pi_1(x_1) + \lambda_1\lambda_2,\end{aligned}$$

while

$$\begin{aligned}\pi_2^0(z_2)\pi_1^0(z_1) &= (\pi_2(x_2) + \lambda_2)(\pi_1(x_1) + \lambda_1)\\ &= \pi_2(x_2)\pi_1(x_1) + \lambda_2\pi_1(x_1) + \lambda_1\pi_2(x_2) + \lambda_2\lambda_1.\end{aligned}$$

But

$$\pi_1(x_1)\pi_2(x_2) = \pi_2(x_2)\pi_1(x_1), \quad \text{and } \lambda_1\lambda_2 = \lambda_2\lambda_1,$$

so

$$\pi_1^0(x_1)\pi_2^0(x_2) = \pi_2^0(x_2)\pi_1^0(x_1) \quad \text{for all } x_1 \in (A_1)_I,\ x_2 \in (A_2)_I.$$

Thus π^0, defined by $\pi^0(x_1 \otimes x_2) = \pi_1^0(x_1)\pi_2^0(x_2)$, $x_1 \in (A_1)_I$, $x_2 \in (A_2)_I$, is a well defined representation of $(A_1)_I \otimes (A_2)_I$.
Clearly π^0 extends π, for if $x \in A_1 \otimes A_2$, $x = \sum_{i=1}^n x_{1i} \otimes x_{2i}$, then
$\pi^0(x) = \sum_{i=1}^n \pi_1^0(x_{1i})\pi_2^0(x_{2i}) = \sum_{i=1}^n \pi_1(x_{1i})\pi_2(x_{2i}) = \sum_{i=1}^n \pi(x_{1i} \otimes x_{2i}) = \pi(x)$.
Define a norm β^0 on $(A_1)_I \otimes (A_2)_I$ by $\|x\|_{\beta^0} = \|\pi^0(x)\|$ for all $x \in (A_1)_I \otimes (A_2)_I$.
Then β^0 is clearly a C^*-norm on $(A_1)_I \otimes (A_2)_I$, and $\|x\|_{\beta^0} = \|\pi^0(x)\| = \|\pi(x)\| = \|x\|_\beta$ for all $x \in A_1 \otimes A_2$.
Hence the new norm β^0 on $(A_1)_I \otimes (A_2)_I$ extends the original norm β on $A_1 \otimes A_2$.

Thus, so far as the norm problem in tensor products of C^*-algebras is concerned, we may, without loss of generality, restrict our attention to C^*-algebras which contain identity elements. By identifying x_1 and $x_1 \otimes 1$, we may regard A_1 as a C^*-subalgebra of $A_1 \otimes A_2$, and, by identifying x_2 and $1 \otimes x_2$, we may do the same for A_2.

Chapter 2

THE MIN AND MAX NORMS

§1. INTRODUCTION

This chapter introduces the *min* and *max* norms. After giving the definitions, we show that they satisfy all the properties of a C^*-norm, and then we prove that they are cross-norms. After noting that *max* is the largest C^*-norm, we present a sequence of results which establishes that *min* is the smallest C^*-norm. We end the chapter by proving that all C^*-norms are cross-norms.

§2. THE MAX NORM

Definition 2.1: If A_1 and A_2 are C^*-algebras, and $x \in A_1 \otimes A_2$, then $\|x\|_{max} = sup\{\|\pi(x)\| : \pi$ runs through all representations of $A_1 \otimes A_2\}$.

To prove that this is indeed a C^*-algebra norm, we begin by noting that $\pi(0) = 0$ for all representations π of $A_1 \otimes A_2$, so $\|0\|_{max} = 0$.
Then, for all $\lambda \in \mathbf{C}$ and $x, y \in A_1 \otimes A_2$,

$$\begin{aligned}
\|\lambda x\|_{max} &= sup_\pi\|\pi(\lambda x)\| = sup_\pi\|\lambda\pi(x)\| = sup_\pi|\lambda|\|\pi(x)\| = |\lambda| sup_\pi\|\pi(x)\| \\
&= |\lambda|\|x\|_{max}, \\
\|x+y\|_{max} &= sup_\pi\|\pi(x+y)\| = sup_\pi\|\pi(x)+\pi(y)\| \\
&\le sup_\pi(\|\pi(x)\| + \|\pi(y)\|) = sup_\pi(\|\pi(x)\|) + sup_\pi(\|\pi(y)\|) \\
&= \|x\|_{max} + \|y\|_{max}, \\
\|xy\|_{max} &= sup_\pi\|\pi(xy)\| = sup_\pi\|\pi(x)\pi(y)\| \\
&\le sup_\pi(\|\pi(x)\|\|\pi(y)\|) \quad \text{(since } \mathcal{L}(H) \text{ is a normed algebra)} \\
&\le sup_\pi\|\pi(x)\| sup_\pi\|\pi(y)\| = \|x\|_{max}\|y\|_{max}, \\
\text{and } \|x^*x\|_{max} &= sup_\pi\|\pi(x^*x)\| = sup_\pi\|\pi(x^*)\pi(x)\| \\
&= sup_\pi\|(\pi(x))^*\pi(x)\| = sup_\pi\|\pi(x)\|^2 \quad \text{(since } \mathcal{L}(H) \text{ is a } C^*\text{-algebra)} \\
&= (sup_\pi\|\pi(x)\|)^2 = \|x\|^2_{max}.
\end{aligned}$$

Once we have shown that $x \neq 0 \Rightarrow \|x\|_{max} \neq 0$, then we will have demonstrated that $\|\cdot\|_{max}$ is a C^*-norm, called the **projective** C^*-norm, and the completion of $A_1 \otimes A_2$ under $\|\cdot\|_{max}$ will be a C^*-algebra, denoted by $A_1 \tilde{\otimes}_{max} A_2$ and called the projective C^*-tensor product of A_1 and A_2.

§3. The Min Norm

Let π_1, π_2 be representations of A_1, A_2 on the Hilbert spaces H_1, H_2 respectively. Then we can define a representation π of $A_1 \otimes A_2$ on the Hilbert space tensor product $H_1 \otimes H_2$ by $\pi(x) = \sum_{i=1}^{n} \pi_1(x_{1i}) \otimes \pi_2(x_{2i})$, where x is expressed as $x = \sum_{i=1}^{n} x_{1i} \otimes x_{2i}$, $x_{1i} \in A_1$, $x_{2i} \in A_2$, for $i = 1, ..., n$.
We prove below that this actually is a representation, and we denote it by $\pi_1 \otimes \pi_2$.

Definition 3.1: If A_1 and A_2 are C^*-algebras, and $x \in A_1 \otimes A_2$, then
$\|x\|_{min} = sup\{\|(\pi_1 \otimes \pi_2)(x)\| : \pi_i$ runs through all representations of A_i, $i = 1,2\}$.

In the course of showing that $\|.\|_{min}$ satisfies the properties of a norm, we will show that $\pi_1 \otimes \pi_2$ is indeed a representation. It follows that $\|x\|_{min} \leq \|x\|_{max}$ for all $x \in A_1 \otimes A_2$.
Clearly $\|0\|_{min} = 0$.
To show that $\|\lambda x\|_{min} = |\lambda|\|x\|_{min}$ for all $x \in A_1 \otimes A_2$, $\lambda \in \mathbf{C}$, we first show that $(\pi_1 \otimes \pi_2)(\lambda x) = \lambda(\pi_1 \otimes \pi_2)(x)$.
That is clearly true if $\lambda = 0$. If $\lambda \neq 0$, then $\lambda x = \sum_{i=1}^{n} x_{1i} \otimes x_{2i}$ if and only if $x = \sum_{i=1}^{n} \frac{1}{\lambda} x_{1i} \otimes x_{2i}$.
So,

$$\begin{aligned}
(\pi_1 \otimes \pi_2)(\lambda x) &= \sum_{i=1}^{n} \pi_1(x_{1i}) \otimes \pi_2(x_{2i}) \\
&= \lambda \sum_{i=1}^{n} \pi_1(\frac{1}{\lambda} x_{1i}) \otimes \pi_2(x_{2i}) \\
&= \lambda(\pi_1 \otimes \pi_2)(x).
\end{aligned}$$

Thus

$$\|\lambda x\|_{min} = sup\|(\pi_1 \otimes \pi_2)(\lambda x)\| = sup\|\lambda(\pi_1 \otimes \pi_2)(x)\| = |\lambda| sup\|(\pi_1 \otimes \pi_2)(x)\| = |\lambda|\|x\|_{min}.$$

Now let $x, y \in A_1 \otimes A_2$, $x = \sum_{i=1}^{n} x_{1i} \otimes x_{2i}$, $y = \sum_{j=1}^{m} y_{1j} \otimes y_{2j}$.
Put $x'_{1i} = x_{1i}$, $i = 1, ..., n$, and $x'_{1i} = y_{1i-n}$, $i = n+1, ..., n+m$, and define x'_{2i}, $i = 1, ..., n+m$, similarly.
Then

$$\begin{aligned}
(\pi_1 \otimes \pi_2)(x) + (\pi_1 \otimes \pi_2)(y) &= \sum_{i=1}^{n} \pi_1(x_{1i}) \otimes \pi_2(x_{2i}) + \sum_{j=1}^{m} \pi_1(y_{1j}) \otimes \pi_2(y_{2j}) \\
&= \sum_{i=1}^{n+m} \pi_1(x'_{1i}) \otimes \pi_2(x'_{2i}) \\
&= (\pi_1 \otimes \pi_2)(x+y) \quad (\text{since } x + y = \sum_{i=1}^{n+m} x'_{1i} \otimes x'_{2i}).
\end{aligned}$$

Thus

$$\begin{aligned}\|x+y\|_{min} &= sup\|(\pi_1\otimes\pi_2)(x+y)\| = sup\|(\pi_1\otimes\pi_2)(x)+(\pi_1\otimes\pi_2)(y)\|\\ &\le sup\{\|(\pi_1\otimes\pi_2)(x)\|+\|(\pi_1\otimes\pi_2)(y)\|\}\\ &= sup\|(\pi_1\otimes\pi_2)(x)\| + sup\|(\pi_1\otimes\pi_2)(y)\|,\end{aligned}$$

so

$$\|x+y\|_{min} \le \|x\|_{min} + \|y\|_{min}.$$

Next, we show that $\|xy\|_{min} \le \|x\|_{min}\|y\|_{min}$.
We note that

$$\begin{aligned}(\pi_1\otimes\pi_2)(x)(\pi_1\otimes\pi_2)(y) &= \left[\sum_{i=1}^{n}\pi_1(x_{1i})\otimes\pi_2(x_{2i})\right]\left[\sum_{j=1}^{m}\pi_1(y_{1j})\otimes\pi_2(y_{2j})\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\pi_1(x_{1i})\pi_1(y_{1j})\otimes\pi_2(x_{2i})\pi_2(y_{2j})\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\pi_1(x_{1i}y_{1j})\otimes\pi_2(x_{2i}y_{2j})\\ &= (\pi_1\otimes\pi_2)(xy) \quad \left(\text{because } xy=\sum_{i=1}^{n}\sum_{j=1}^{m}x_{1i}y_{1j}\otimes x_{2i}y_{2j}\right).\end{aligned}$$

So

$$\begin{aligned}\|xy\|_{min} &= sup\|(\pi_1\otimes\pi_2)(xy)\| = sup\|(\pi_1\otimes\pi_2)(x)(\pi_1\otimes\pi_2)(y)\|\\ &\le sup\|(\pi_1\otimes\pi_2)(x)\|\|(\pi_1\otimes\pi_2)(y)\|\\ &\le sup\|(\pi_1\otimes\pi_2)(x)\| sup\|(\pi_1\otimes\pi_2)(y)\|,\end{aligned}$$

which proves that

$$\|xy\|_{min} \le \|x\|_{min}\|y\|_{min}.$$

Finally, we show that $\|x^*x\|_{min} = \|x\|^2_{min}$. (Strictly speaking, this is not necessary, since any algebra norm dominated by a C^*-norm is itself a C^*-norm [Blecher, [1], Corollary 3].)
If $x^* = \sum_{i=1}^n z_{1i}\otimes z_{2i}$, then $x=\sum_{i=1}^n x_{1i}\otimes x_{2i}$, where $x_{ij}=z_{ij}^*$ for $i=1,2,\ j=1,...,n$.
Thus

$$\begin{aligned}(\pi_1\otimes\pi_2)(x^*) &= \sum_{i=1}^{n}\pi_1(z_{1i})\otimes\pi_2(z_{2i})\\ &= \sum_{i=1}^{n}\pi_1(x_{1i}^*)\otimes\pi_2(x_{2i}^*)\\ &= \sum_{i=1}^{n}(\pi_1(x_{1i}))^*\otimes(\pi_2(x_{2i}))^* = \left(\sum_{i=1}^{n}\pi_1(x_{1i})\otimes\pi_2(x_{2i})\right)^*\\ &= ((\pi_1\otimes\pi_2)(x))^*.\end{aligned}$$

So

$$\begin{aligned}
\|x^*x\|_{min} &= sup\|(\pi_1 \otimes \pi_2)(x^*x)\| = sup\|(\pi_1 \otimes \pi_2)(x^*)(\pi_1 \otimes \pi_2)(x)\| \\
&= sup\|((\pi_1 \otimes \pi_2)(x))^*(\pi_1 \otimes \pi_2)(x)\| = sup\|(\pi_1 \otimes \pi_2)(x)\|^2 \\
&= (sup\|(\pi_1 \otimes \pi_2)(x)\|)^2 \\
&= \|x\|^2_{min}.
\end{aligned}$$

It remains to show that $x \neq 0 \Rightarrow \|x\|_{min} \neq 0$.
Once this is done, $\|x\|_{min} \le \|x\|_{max}$ shows that $x \neq 0 \Rightarrow \|x\|_{max} \neq 0$, which will complete the proof that both $\|.\|_{min}$ and $\|.\|_{max}$ are C^*-norms.

To do this, we investigate the relationship between $\|.\|_{min}$ and the usual injective norm

$$\|x\|_\epsilon = sup\{|(x_1' \otimes x_2')(x)| : x_1' \in A_1', \|x_1'\| \le 1, x_2' \in A_2', \|x_2'\| \le 1\}.$$

We also need the following definitions: If $a \in A$ and $x' \in A^*$, ax' will denote the functional defined on A by $ax'(x) = x'(xa)$. If T is a linear map on a Hilbert space H, then T is a **partial isometry** if T is an isometry when restricted to the subspace (Ker T)$^\perp = \{\xi \in H : (\xi, \eta) = 0$ for all $\eta \in$ Ker T$\}$. If T is a partial isometry, then so is T^* [Reed and Simon, [5], pg 197].

Let $x_1' \in A_1'$, $\|x_1'\| \le 1$, $x_2' \in A_2'$, $\|x_2'\| \le 1$.
In what follows, let i be either 1 or 2. We apply the polar decomposition to x_i' to obtain $x_i' = v_i\omega_i$, where $v_i \in A_i$, ω_i is a positive linear functional on A_i with cyclic representation π_i, so that $\omega_i(x) = (\pi_i(x)\delta_i, \delta_i)$ for all $x \in A_i$, and $\pi_i(v_i)$ is a partial isometry [Takesaki, [6], III.4.2]. We know that $\|x_i'\| = \|\omega_i\|$ [Takesaki, [6], III.4.6] $= (\delta_i, \delta_i)$ [Dixmier, [2], 2.4.3] $= \|\delta_i\|^2$.
It follows that, for all $x \in A_i$,

$$\begin{aligned}
x_i'(x) &= v_i\omega_i(x) = \omega_i(xv_i) = (\pi_i(xv_i)\delta_i, \delta_i) \\
&= (\pi_i(x)\pi_i(v_i)\delta_i, \delta_i) = (\pi_i(x)\delta_i, \pi_i(v_i)^*\delta_i).
\end{aligned}$$

Put $\xi_i = \delta_i$, and $\eta_i = \pi_i(v_i)^*\delta_i$.
Then we have: $x_i'(x) = (\pi_i(x)\xi_i, \eta_i)$ for all $x \in A_i$, $\|\xi_i\| = \|\delta_i\|$, and $\|\eta_i\| = \|\pi_i(v_i)^*\delta_i\| = \|\delta_i\|$ because $\pi_i(v_i)^*$ is a partial isometry.
Thus $\|x_i'\| = \|\delta_i\|^2 = \|\xi_i\|\|\eta_i\|$.
Then, for $x = \sum_{i=1}^n x_{1i} \otimes x_{2i}$, we have

$$\begin{aligned}
|(x_1' \otimes x_2')(x)| &= |\sum_{i=1}^n x_1'(x_{1i})x_2'(x_{2i})| \\
&= |\sum_{i=1}^n (\pi_1(x_{1i})\xi_1, \eta_1)(\pi_2(x_{2i})\xi_2, \eta_2)|.
\end{aligned}$$

Now $\pi_1(x_{1i}) \in \mathcal{L}(H_1)$, $\pi_2(x_{2i}) \in \mathcal{L}(H_2)$, so $\pi_1(x_{1i}) \otimes \pi_2(x_{2i}) \in \mathcal{L}(H_1 \otimes H_2)$ is defined by

$[\pi_1(x_{1i}) \otimes \pi_2(x_{2i})](\xi \otimes \eta) = [\pi_1(x_{1i})\xi] \otimes [\pi_2(x_{2i})\eta], \quad \xi \in H_1, \eta \in H_2.$

Also, the inner product on $H_1 \otimes H_2$ is defined by

$(\rho_1 \otimes \rho_2, \delta_1 \otimes \delta_2) = (\rho_1, \delta_1)(\rho_2, \delta_2), \quad \rho_1, \delta_1 \in H_1, \rho_2, \delta_2 \in H_2.$

Thus

$$\begin{aligned}(\pi_1(x_{1i})\xi_1, \eta_1)(\pi_2(x_{2i})\xi_2, \eta_2) &= ([\pi_1(x_{1i})\xi_1] \otimes [\pi_2(x_{2i})\xi_2], \eta_1 \otimes \eta_2) \\ &= ([\pi_1(x_{1i}) \otimes \pi_2(x_{2i})](\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2).\end{aligned}$$

So

$$\begin{aligned}|(x_1' \otimes x_2')(x)| &= \left|\sum_{i=1}^{n}([\pi_1(x_{1i}) \otimes \pi_2(x_{2i})](\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2)\right| \\ &= \left|\left(\sum_{i=1}^{n}([\pi_1(x_{1i}) \otimes \pi_2(x_{2i})](\xi_1 \otimes \xi_2)), \eta_1 \otimes \eta_2\right)\right| \\ &\quad \text{(by linearity of the inner product)} \\ &= \left|\left(\left[\sum_{i=1}^{n}\pi_1(x_{1i}) \otimes \pi_2(x_{2i})\right](\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2\right)\right| \\ &\quad \text{(by definition of addition of maps)} \\ &= |([(\pi_1 \otimes \pi_2)(x)](\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2)| \quad \text{(by definition of } \pi_1 \otimes \pi_2) \\ &\leq \|[(\pi_1 \otimes \pi_2)(x)](\xi_1 \otimes \xi_2)\|\|\eta_1 \otimes \eta_2\| \\ &\quad \text{(since } |(x, y)| \leq \|x\|\|y\| \text{ in any Hilbert space)} \\ &\leq \|(\pi_1 \otimes \pi_2)(x)\|\|\xi_1 \otimes \xi_2\|\|\eta_1 \otimes \eta_2\| \\ &\leq \|x\|_{min}\|\xi_1\|\|\eta_1\|\|\xi_2\|\|\eta_2\| \\ &= \|x\|_{min}\|x_1'\|\|x_2'\| \\ &\leq \|x\|_{min} \quad \text{(because } \|x_1'\| \leq 1,\ \|x_2'\| \leq 1).\end{aligned}$$

Thus $|(x_1' \otimes x_2')(x)| \leq \|x\|_{min}$ for all $x_1' \in A_1'$, $\|x_1'\| \leq 1$, $x_2' \in A_2'$, $\|x_2'\| \leq 1$.

Hence $\|x\|_\epsilon \leq \|x\|_{min}$ for all $x \in A_1 \otimes A_2$.

This establishes that $x \neq 0 \Rightarrow \|x\|_{min} \neq 0$, thus showing that $\|.\|_{min}$, and hence $\|.\|_{max}$, are C^*-norms.

The norm $\|.\|_{min}$ is known as the **injective** C^*-norm.

§4. Cross-norms

In this section we investigate $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ further. We establish that they are dominated by the ordinary projective tensor product norm, and we show that they are both cross-norms.

The ordinary projective tensor product norm is defined by
$\|x\|_\pi = inf\{\sum_{i=1}^n \|a_{1i}\|\|a_{2i}\| : x = \sum_{i=1}^n a_{1i} \otimes a_{2i}\}$, $x \in A_1 \otimes A_2$.
Then, for all $x, y \in A_1 \otimes A_2$,

$$\begin{aligned}
\|xy\|_\pi &= inf\left\{\sum_{i=1}^m \|a'_{1i}\|\|a'_{2i}\| : xy = \sum_{i=1}^m a'_{1i} \otimes a'_{2i}\right\} \\
&\leq inf\left\{\sum_{i=1}^p \sum_{j=1}^q \|a_{1i}b_{1j}\|\|a_{2i}b_{2j}\| : x = \sum_{i=1}^p a_{1i} \otimes a_{2i},\ y = \sum_{j=1}^q b_{1j} \otimes b_{2j}\right\} \\
&\leq inf\left\{\left(\sum_{i=1}^p \|a_{1i}\|\|a_{2i}\|\right)\left(\sum_{j=1}^q \|b_{1j}\|\|b_{2j}\|\right) : x = \sum_{i=1}^p a_{1i} \otimes a_{2i},\ y = \sum_{j=1}^q b_{1j} \otimes b_{2j}\right\} \\
&\leq (inf\left\{\sum_{i=1}^p \|a_{1i}\|\|a_{2i}\| : x = \sum_{i=1}^p a_{1i} \otimes a_{2i}\right\})(inf\left\{\sum_{j=1}^q \|b_{1j}\|\|b_{2j}\| : y = \sum_{j=1}^q b_{1j} \otimes b_{2j}\right\}) \\
&= \|x\|_\pi \|y\|_\pi,
\end{aligned}$$

and

$$\begin{aligned}
\|x^*\|_\pi &= inf\left\{\sum_{i=1}^n \|a_{1i}\|\|a_{2i}\| : x^* = \sum_{i=1}^n a_{1i} \otimes a_{2i}\right\} \\
&= inf\left\{\sum_{i=1}^n \|a^*_{1i}\|\|a^*_{2i}\| : x = \sum_{i=1}^n a^*_{1i} \otimes a^*_{2i}\right\} \quad (\text{since } \|a\| = \|a^*\| \text{ for } a \in A_1,\ A_2) \\
&= inf\left\{\sum_{i=1}^n \|a_{1i}\|\|a_{2i}\| : x = \sum_{i=1}^n a_{1i} \otimes a_{2i}\right\} \\
&= \|x\|_\pi .
\end{aligned}$$

Thus the completion $A_1 \tilde{\otimes}_\pi A_2$ is an involutive Banach algebra. However, $\|.\|_\pi$ is not in general a C^*-norm.

We will now show that $\|x\|_{max} \leq \|x\|_\pi$ for all $x \in A_1 \otimes A_2$.
We will make use of the following results, which will be useful again later on.

Let $A_{1,h}$, $A_{2,h}$ denote the set of hermitian elements of A_1, A_2 respectively. Then $A_{1,h} \otimes A_{2,h}$ may be regarded as a subset of $A_1 \otimes A_2$. We have:

Lemma 4.1. $A_{1,h} \otimes A_{2,h} = (A_1 \otimes A_2)_h$.

Proof:

$$x \in A_{1,h} \otimes A_{2,h} \Rightarrow x = \sum_{j=1}^{n} a_{1j} \otimes a_{2j}, \quad a_{ij} \in A_{i,h},\ i = 1,2,\ j = 1,\dots,n,$$

$$\text{so } x^* = \sum_{j=1}^{n} a_{1j}^* \otimes a_{2j}^* = \sum_{j=1}^{n} a_{1j} \otimes a_{2j} = x,$$

$$\text{so } x \in (A_1 \otimes A_2)_h.$$

Conversely, let $x \in (A_1 \otimes A_2)_h$, i.e. $x = \sum_{j=1}^{n} a_{1j} \otimes a_{2j}, \quad a_{ij} \in A_i,\ i = 1,2,\ j = 1,\dots,n$, and $x = x^*$.
Then

$$\begin{aligned} x &= \frac{1}{2}(x + x^*) \\ &= \frac{1}{2}\Big(\sum_{j=1}^{n} a_{1j} \otimes a_{2j} + \sum_{j=1}^{n} a_{1j}^* \otimes a_{2j}^*\Big) \\ &= \frac{1}{4}\sum_{j=1}^{n}\big((a_{1j} + a_{1j}^*) \otimes (a_{2j} + a_{2j}^*) - i(a_{1j} - a_{1j}^*) \otimes i(a_{2j} - a_{2j}^*)\big) \end{aligned}$$

because

$$\begin{aligned} &\sum_{j=1}^{n}\big((a_{1j} + a_{1j}^*) \otimes (a_{2j} + a_{2j}^*) - i(a_{1j} - a_{1j}^*) \otimes i(a_{2j} - a_{2j}^*)\big) \\ =&\sum_{j=1}^{n}\big(a_{1j} \otimes a_{2j} + a_{1j}^* \otimes a_{2j} + a_{1j} \otimes a_{2j}^* + a_{1j}^* \otimes a_{2j}^* - (-a_{1j} \otimes a_{2j} + a_{1j}^* \otimes a_{2j} + a_{1j} \otimes a_{2j}^* - a_{1j}^* \otimes a_{2j}^*)\big) \\ =&2\sum_{j=1}^{n}(a_{1j} \otimes a_{2j} + a_{1j}^* \otimes a_{2j}^*). \end{aligned}$$

Thus $x \in (A_1 \otimes A_2)_h \Rightarrow x = \frac{1}{4}\sum_{j=1}^{n}((a_{1j} + a_{1j}^*) \otimes (a_{2j} + a_{2j}^*) - i(a_{1j} - a_{1j}^*) \otimes i(a_{2j} - a_{2j}^*))$.
Clearly $a_{ij} + a_{ij}^* \in A_{i,h}$, $i = 1,2$, while $[i(a_{1j} - a_{1j}^*)]^* = \bar{i}(a_{1j} - a_{1j}^*)^* = -i(a_{1j}^* - a_{1j}) = i(a_{1j} - a_{1j}^*)$, so $i(a_{1j} - a_{1j}^*) \in A_{1,h}$, and similarly $i(a_{2j} - a_{2j}^*) \in A_{2,h}$.
Thus $x \in A_{1,h} \otimes A_{2,h}$.
Hence $A_{1,h} \otimes A_{2,h} = (A_1 \otimes A_2)_h$.

Lemma 4.2. *Given $x \in (A_1 \otimes A_2)_h$, there exists $\alpha \geq 0$ such that $x \leq \alpha(1 \otimes 1)$.*

Proof: Let $(A_1 \otimes A_2)_+$ be the set of positive elements of $A_1 \otimes A_2$, i.e. the cone generated by elements of the form $x^* x$, $x \in A_1 \otimes A_2$. If $A_{i,+}$ denotes the positive cone of A_i, $i = 1,2$, then $A_{1,+} \otimes A_{2,+} = \left\{\sum_{j=1}^{n} a_{1j} \otimes a_{2j},\ a_{ij} \in A_{i,+},\ i = 1,2\right\}$ is clearly contained in $(A_1 \otimes A_2)_h$, since if $a_{1j} \in A_{1,+}$, $a_{2j} \in A_{2,+}$, $a_{1j} = x_j^* x_j$, $a_{2j} = y_j^* y_j$, then $a_{1j} \otimes a_{2j}$ can be written as

$a_{1j} \otimes a_{2j} = x_j^* x_j \otimes y_j^* y_j = (x_j \otimes y_j)^*(x_j \otimes y_j) \in (A_1 \otimes A_2)_+$.

In general, $(A_1 \otimes A_2)_+$ does not coincide with $A_{1,+} \otimes A_{2,+}$, but we do not need this fact.

We know that if x is a positive element of a C^*-algebra with identity 1, then $x \le \|x\|1$ [Dixmier, [2], 1.6.9].

So, if $a_1 \in A_{1,+}$, $a_2 \in A_{2,+}$, then, since $a_1 \otimes a_2 \in (A_1 \otimes A_2)_+$,

we have $a_1 \otimes a_2 \le \|a_1 \otimes a_2\|1 \le \|a_1\|\|a_2\|1$, where 1 naturally denotes the identity $1 \otimes 1$ of $A_1 \otimes A_2$.

Next, let $a_1 \in A_{1,h}$, $a_2 \in A_{2,h}$.

Then we can write $a_1 = a_{11} - a_{12}$, $a_2 = a_{21} - a_{22}$, where $a_{ij} \in A_{i,+}$, $\|a_{ij}\| \le \|a_i\|$, $i, j = 1, 2$, [Dixmier, [2], 1.5.7].

Then

$$\begin{aligned}
a_1 \otimes a_2 &= (a_{11} - a_{12}) \otimes (a_{21} - a_{22}) \\
&= a_{11} \otimes a_{21} + a_{12} \otimes a_{22} - a_{11} \otimes a_{22} - a_{12} \otimes a_{21} \\
&\le a_{11} \otimes a_{21} + a_{12} \otimes a_{22} \quad (\text{since } a_{11} \otimes a_{22},\ a_{12} \otimes a_{21} \in (A_1 \otimes A_2)_+) \\
&\le \|a_{11}\|\|a_{21}\|1 + \|a_{12}\|\|a_{22}\|1 \\
&\le \|a_1\|\|a_2\|1 + \|a_1\|\|a_2\|1,
\end{aligned}$$

so $a_1 \otimes a_2 \le 2\|a_1\|\|a_2\|1$.

Finally, let $x \in (A_1 \otimes A_2)_h$.

Then, by Lemma 4.1, $x \in A_{1,h} \otimes A_{2,h}$, so $x = \sum_{j=1}^n a_{1j} \otimes a_{2j}$, with $a_{ij} \in A_{i,h}$, $i = 1, 2$, $j = 1, ..., n$.

Hence $x \le 2 \sum_{j=1}^n \|a_{1j}\|\|a_{2j}\|1$, which completes the proof.

Now let ω be a positive linear functional on $A_1 \otimes A_2$. We will show that ω is continuous with respect to the π norm.

Let $a_1 \in A_1$, $a_2 \in A_2$. Then a_1 and a_2 can be written in the form $a_1 = a_{11} + ia_{12}$, $a_2 = a_{21} + ia_{22}$, where the a_{ij} are hermitian elements of A_i which satisfy $\|a_{ij}\| \le \|a_i\|$, $i, j = 1, 2$ [Dixmier, [2], 1.1.4].

We know from the proof of Lemma 4.2 that if $a_1 \in A_{1,h}$, $a_2 \in A_{2,h}$, then $a_1 \otimes a_2 \le 2\|a_1\|\|a_2\|1$.

Therefore $a_{11} \otimes a_{21} \le 2\|a_{11}\|\|a_{21}\|1$, and similarly for $a_{12} \otimes a_{21}$, etc.

Now $a_1 \otimes a_2 = a_{11} \otimes a_{21} + a_{11} \otimes ia_{22} + ia_{12} \otimes a_{21} - a_{12} \otimes a_{22}$,
so

$$
\begin{aligned}
|\omega(a_1 \otimes a_2)| &\leq |\omega(a_{11} \otimes a_{21})| + |\omega(a_{11} \otimes ia_{22})| + |\omega(ia_{12} \otimes a_{21}) + |\omega(a_{12} \otimes a_{22})| \\
&= |\omega(a_{11} \otimes a_{21})| + |\omega(a_{11} \otimes a_{22})| + |\omega(a_{12} \otimes a_{21}) + |\omega(a_{12} \otimes a_{22})| \\
&\text{(because } \omega \text{ is linear and } |i| = 1) \\
&\leq \omega(2\|a_{11}\|\|a_{21}\|1) + \omega(2\|a_{11}\|\|a_{22}\|1) + \omega(2\|a_{12}\|\|a_{21}\|1) + \omega(2\|a_{12}\|\|a_{22}\|1) \\
&\text{(because } \omega \text{ is positive)} \\
&= 2\|a_{11}\|\|a_{21}\|\omega(1) + 2\|a_{11}\|\|a_{22}\|\omega(1) + 2\|a_{12}\|\|a_{21}\|\omega(1) + 2\|a_{12}\|\|a_{22}\|\omega(1) \\
&\leq 2\|a_1\|\|a_2\|\omega(1) + 2\|a_1\|\|a_2\|\omega(1) + 2\|a_1\|\|a_2\|\omega(1) + 2\|a_1\|\|a_2\|\omega(1) \\
&= 8\omega(1)\|a_1\|\|a_2\| \\
&= 8\omega(1)\|a_1 \otimes a_2\|_\pi .
\end{aligned}
$$

Hence ω is continuous with respect to the π norm. Therefore it can be extended to a positive linear functional on the completion $A_1\tilde{\otimes}_\pi A_2$. Because of the correspondence between positive linear functionals and cyclic representations and the fact that *all* (non-degenerate) representations are direct sums of cyclic representations, we can conclude that any representation of $A_1 \otimes A_2$ can be extended to a representation of $A_1\tilde{\otimes}_\pi A_2$. Since $\|p(x)\| \leq \|x\|_\pi$ for all representations p of $A_1\tilde{\otimes}_\pi A_2$, [Takesaki, [6], I.5.2], we get $\|x\|_{max} \leq \|x\|_\pi$ for all $x \in A_1 \otimes A_2$.
So we have

$$\|x\|_\epsilon \leq \|x\|_{min} \leq \|x\|_{max} \leq \|x\|_\pi \quad \text{for all } x \in A_1 \otimes A_2.$$

This shows that $\|.\|_{min}$ and $\|.\|_{max}$ are cross-norms, as follows:
(1) $\|x_1\|\|x_2\| = \|x_1 \otimes x_2\|_\epsilon \leq \|x_1 \otimes x_2\|_{min} \leq \|x_1 \otimes x_2\|_\pi = \|x_1\|\|x_2\|$,
so $\|x_1 \otimes x_2\|_{min} = \|x_1\|\|x_2\|$;
(2) if $x_1' \in A_1'$, $x_2' \in A_2'$, then

$$
\begin{aligned}
|(x_1' \otimes x_2')(x_1 \otimes x_2)| &\leq \|x_1'\|\|x_2'\|\|x_1 \otimes x_2\|_\epsilon \quad \text{(because } \|.\|_\epsilon \text{ is a cross-norm)} \\
&\leq \|x_1'\|\|x_2'\|\|x_1 \otimes x_2\|_{min}.
\end{aligned}
$$

Thus $x_1' \otimes x_2' \in (A_1\tilde{\otimes}_{min} A_2)'$ and $\|x_1' \otimes x_2'\| \leq \|x_1'\|\|x_2'\|$.
This proves that $\|.\|_{min}$ is a cross-norm and the same argument clearly holds for $\|.\|_{max}$.

It turns out, as the names indicate, that $\|.\|_{min}$ is the smallest possible C^*-norm on the tensor product of two C^*-algebras, while $\|.\|_{max}$ is the largest. The first claim will take some effort to prove, but the second can be easily demonstrated:
Let α be any C^*-norm on the tensor product $A_1 \otimes A_2$ of two C^*-algebras A_1 and A_2. We know that there is an representation π of the C^*-algebra $A_1\tilde{\otimes}_\alpha A_2$ which is an isometry; i.e. $\|x\|_\alpha = \|\pi(x)\|$ for all $x \in A_1\tilde{\otimes}_\alpha A_2$. The representation π restricts to a representation of $A_1 \otimes A_2$, and so, since $\|x\|_{max} = sup\{\|\pi(x)\| : \pi$ runs through all representations of $A_1 \otimes A_2\}$, we conclude that $\|x\|_{max} \geq \|\pi(x)\| = \|x\|_\alpha$ for all $x \in A_1 \otimes A_2$.

§5. The Smallest C^*-norm

We now prove that the injective C^*-norm *min* is the smallest C^*-norm. Our approach is based on that in Takesaki, [6] and [7]. We end the section by proving that all C^*-norms are cross-norms.

Definition 5.1: If ϕ is a positive linear functional on a C*-algebra A, then ϕ is **pure** if every positive linear functional ψ on A which satisfies $\psi(x^*x) \leq \phi(x^*x)$ for all $x \in A$ is of the form $\psi = \lambda\phi$ for some scalar λ with $0 \leq \lambda \leq 1$. A **state** is a positive linear functional of norm 1, and we denote the set of all pure states on A by $P(A)$.

Lemma 5.2. *If A is a C^*-algebra and S is the set of all positive linear functionals on A of norm ≤ 1, then the pure states on A, together with the zero map, are precisely the extreme points of S.*

Proof: See Dixmier, [2], 2.5.5.

Lemma 5.3. *If A is a C^*-subalgebra of a C^*-algebra B and if the restriction of a pure state ω of B to A is a pure state, then*

$$\omega(xy) = \omega(x)\omega(y) \quad \textit{for all } x \in A,\ y \in A' \cap B,$$

where $A' \cap B$ is the C^*-subalgebra of A consisting of all elements of B which commute with A.

Proof: We begin by assuming that $0 \leq y \leq 1$, $y \in A' \cap B$. Since ω is a state, $\omega(1) = \|\omega\| = 1$, so we get $0 \leq \omega(y) \leq 1$.
If $\omega(y) = 0$, the Cauchy-Schwarz inequality $|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$ yields, for any $x \in A$,

$$\begin{aligned} |\omega(xy)|^2 &= |\omega(xy^{\frac{1}{2}}y^{\frac{1}{2}})|^2 = |\omega([(xy^{\frac{1}{2}})^*]^*y^{\frac{1}{2}})|^2 \\ &\leq \omega([(xy^{\frac{1}{2}})^*]^*(xy^{\frac{1}{2}})^*)\omega(y^{\frac{1}{2}*}y^{\frac{1}{2}}) \\ &= \omega((xy^{\frac{1}{2}})(xy^{\frac{1}{2}})^*)\omega(y^{\frac{1}{2}}y^{\frac{1}{2}}) \quad \text{(since } y \text{ is hermitian)} \\ &= \omega(xyx^*)\omega(y) \\ &= 0. \end{aligned}$$

Hence $\omega(xy) = 0$ and $\omega(xy) = \omega(x)\omega(y)$ holds in this case.
If $\omega(y) = 1$, then $\omega(1 - y) = 0$ because ω is a state, and the same argument as in the above case yields $\omega(x(1 - y)) = \omega(x)\omega(1 - y)$, whence we get

$$\omega(x) - \omega(xy) = \omega(x(1 - y)) = \omega(x)[1 - \omega(y)] = \omega(x) - \omega(x)\omega(y), \quad \text{so } \omega(xy) = \omega(x)\omega(y).$$

Now suppose $0 < \omega(y) < 1$.

Then the restriction ω_A of ω to A can be written in the form

$$\omega_A(x) = \omega(y)\frac{1}{\omega(y)}\omega(xy) + (1-\omega(y))\frac{1}{1-\omega(y)}\omega(x(1-y)) \quad \text{for all } x \in A.$$

Consider the functionals ω_1 and ω_2 defined on A by

$$\omega_1(x) = \frac{1}{\omega(y)}\omega(xy) \text{ and } \omega_2(x) = \frac{1}{1-\omega(y)}\omega(x(1-y)).$$

We claim that ω_1 and ω_2 are positive functionals. For we have

$$\begin{aligned}\omega_1(x^*x) &= \frac{1}{\omega(y)}\omega(x^*xy) = \frac{1}{\omega(y)}\omega(x^*yx) \quad \text{(because } y \text{ commutes with } A)\\ &= \frac{1}{\omega(y)}\omega(x^*z^*zx) \quad \text{(because } 0 \le y),\end{aligned}$$

$$\text{and } \frac{1}{\omega(y)}\omega(x^*z^*zx) = \frac{1}{\omega(y)}\omega((zx)^*zx) \ge 0 \quad \text{(because } \omega \text{ is positive)}.$$

Similarly ω_2 is positive.

Thus ω_1 and ω_2 are both states. (Assuming, as we may, that A contains an identity element, we get $\|\omega_1\| = \omega_1(1) = \frac{1}{\omega(y)}\omega(y) = 1$, and similarly for ω_2.)

Now we have $\omega_A(x) = \omega(y)\omega_1(x) + (1-\omega(y))\omega_2(x)$, i.e. we have written ω_A as a convex combination of states.

But ω_A is a pure state, by assumption, and is thus, by Lemma 5.2, an extreme point, so we may conclude that $\omega_A = \omega_1 = \omega_2$.

Thus $\omega_A(x) = \omega_1(x)$ for all $x \in A$, which gives $\omega(xy) = \omega(x)\omega(y)$ for all $x \in A$, $y \in A' \cap B$, $0 \le y \le 1$.

Now let y be any nonzero positive element of $A' \cap B$. Since $z \le \|z\|1$ for any positive element z of a C*-algebra, [Dixmier, [2], 1.6.9], we have $\frac{y}{\|y\|} \le \left\|\frac{y}{\|y\|}\right\| 1$, i.e. $\frac{y}{\|y\|} \le 1$.

Since $\frac{y}{\|y\|} \in A' \cap B$ because y is, we know, by the first part of the proof, that $\omega\left(x\frac{y}{\|y\|}\right) = \omega(x)\omega\left(\frac{y}{\|y\|}\right)$ for all $x \in A$.

So we get $\omega(xy) = \omega\left(x\|y\|\frac{y}{\|y\|}\right) = \|y\|\omega\left(x\frac{y}{\|y\|}\right) = \|y\|\omega(x)\omega\left(\frac{y}{\|y\|}\right) = \omega(x)\omega(y)$ for all $x \in A$, $y \in A' \cap B$, $0 \le y$.

Finally, let y be any element of $A' \cap B$. Then we can write y as $y = y_{11} - y_{12} + i(y_{21} - y_{22})$, where y_{11}, y_{12}, y_{21} and y_{22} are positive elements of B. The proof of the existence of these y_{ij} [Dixmier, [2], 1.5.7] shows that if y commutes with A, then so do

all the y_{ij}. Thus y_{11}, y_{12}, y_{21} and y_{22} all lie in $A' \cap B$.
So, for all $x \in A$,

$$\begin{aligned}\omega(xy) &= \omega(xy_{11} - xy_{12} + ixy_{21} - ixy_{22}) = \omega(xy_{11}) - \omega(xy_{12}) + i\omega(xy_{21}) - i\omega(xy_{22}) \\ &= \omega(x)\omega(y_{11}) - \omega(x)\omega(y_{12}) + i\omega(x)\omega(y_{21}) - i\omega(x)\omega(y_{22}) \\ &= \omega(x)[\omega(y_{11}) - \omega(y_{12}) + i\omega(y_{21}) - i\omega(y_{22})] \\ &= \omega(x)[\omega(y_{11} - y_{12} + i(y_{21} - y_{22}))] \\ &= \omega(x)\omega(y).\end{aligned}$$

Corollary 5.4. *If A_1 is an abelian C^*-algebra, A_2 is any C^*-algebra, and β is any C^*-norm on $A_1 \otimes A_2$, then every pure state $\omega \in P(A_1 \tilde{\otimes}_\beta A_2)$ is of the form $\omega = \omega_1 \otimes \omega_2$ for pure states $\omega_i \in P(A_i)$, $i = 1, 2$.*

Proof: Regard A_1 as a C^*-subalgebra of $A_1 \tilde{\otimes}_\beta A_2$.
Let $a_1 \in A_1$ and pick any element in $A_1 \tilde{\otimes}_\beta A_2$ of the form $x_1 \otimes x_2$, where $x_i \in A_i$, $i = 1, 2$.
Then

$$a(x_1 \otimes x_2) = (a_1 \otimes 1)(x_1 \otimes x_2) = a_1 x_1 \otimes x_2,$$

$$\text{while } (x_1 \otimes x_2)a_1 = (x_1 \otimes x_2)(a_1 \otimes 1) = x_1 a_1 \otimes x_2 = a_1 x_1 \otimes x_2 \quad (\text{since } A_1 \text{ is abelian}).$$

Thus $a_1(x_1 \otimes x_2) = (x_1 \otimes x_2)a_1$.
So A_1 commutes with elements of the form $x_1 \otimes x_2$, $x_i \in A_i$, whence A_1 commutes with $A_1 \otimes A_2$, and so A_1 commutes with $A_1 \tilde{\otimes}_\beta A_2$.
Now let $\omega \in P(A_1 \tilde{\otimes}_\beta A_2)$.
Define ω_1 on A_1 and ω_2 on A_2 by $\omega_1(a_1) = \omega(a_1 \otimes 1)$, and $\omega_2(a_2) = \omega(1 \otimes a_2)$. We show that ω_1 and ω_2 are pure.
Suppose ϕ is a positive linear functional on A_1 such that $\phi(a^*a) \le \omega_1(a^*a)$ for all $a \in A_1$.
Put $\phi'(a_1 \otimes 1) = \phi(a_1)$. Then ϕ' is a positive linear functional on $A_1 \otimes 1$ such that $\phi'(a_1^* a_1 \otimes 1) = \phi(a_1^* a_1) \le \omega_1(a_1^* a_1) = \omega(a_1^* a_1 \otimes 1)$ for all $a_1 \in A_1$.
Since ω is pure, $\phi' = \lambda\omega$ for some $0 \le \lambda \le 1$,
from which we get $\phi(a) = \phi'(a \otimes 1) = \lambda\omega(a \otimes 1) = \lambda\omega_1(a)$, so $\phi = \lambda\omega_1$, and ω_1 is pure.
Similarly ω_2 is pure.
Thus the restriction ω_1 of ω to A_1 is pure. By Lemma 5.3,

$$\omega(xy) = \omega(x)\omega(y) \quad \text{for all } x \in A_1,\ y \in A_1' \cap (A_1 \tilde{\otimes}_\beta A_2).$$

But $A_1' = A_1 \tilde{\otimes}_\beta A_2$,

$$\text{so } \omega(xy) = \omega(x)\omega(y) \quad \text{for all } x \in A_1,\ y \in A_1 \tilde{\otimes}_\beta A_2. \ (*)$$

Hence

$$\begin{aligned}(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) &= \omega_1(a_1)\omega_2(a_2) = \omega(a_1 \otimes 1)\omega(1 \otimes a_2) \\ &= \omega((a_1 \otimes 1)(1 \otimes a_2)) \quad \text{by } (*) \\ &= \omega(a_1 \otimes a_2).\end{aligned}$$

Therefore $\omega = \omega_1 \otimes \omega_2$.

Notation: Let A_1 and A_2 be C^*-algebras. Let β be any C^*-norm on $A_1 \otimes A_2$ and denote the completion $A_1 \tilde{\otimes}_\beta A_2$ of $A_1 \otimes A_2$ under the β norm by A_β. Let $P(A_i)$ be the set of pure states on A_i. Then, given $\omega_i \in P(A_i)$, $i = 1, 2$, $\omega_1 \otimes \omega_2$ may or may not be continous on $A_1 \otimes A_2$ with respect to the β norm. We define a subset S_β of $P(A_1) \times P(A_2)$ by

$$S_\beta = \{(\omega_1, \omega_2) \in P(A_1) \times P(A_2) : \omega_1 \otimes \omega_2 \text{ is continuous on } (A_1 \otimes A_2, \beta)\}.$$

Result: If $(\omega_1, \omega_2) \in S_\beta$, then $\omega_1 \otimes \omega_2$ is continuous under the β norm, and so, since $\omega_1 \otimes \omega_2$ is a positive linear functional, the functional norm is given by $\|\omega_1 \otimes \omega_2\| = (\omega_1 \otimes \omega_2)(1 \otimes 1) = \omega_1(1)\omega_2(1) = 1$ because ω_1 and ω_2 are states.

Lemma 5.5. *If $(\omega_1, \omega_2) \in S_\beta$, then $(u^*\omega_1 u, v^*\omega_2 v) \in S_\beta$ for any unitary elements $u \in A_1$, $v \in A_2$, where $u^*\omega_1 u$ is the functional defined by $u^*\omega_1 u(x) = \omega_1(uxu^*)$.*

Proof: First we must show that $u^*\omega_1 u \in P(A_1)$ and $v^*\omega_2 v \in P(A_2)$.
Clearly $u^*\omega_1 u$ is positive, linear and is a state. ($\|u^*\omega_1 u\| = u^*\omega_1 u(1) = \omega_1(uu^*) = \omega_1(1) = 1$).
To show that it is pure, let ϕ be a positive linear functional on A_1 such that $\phi(x^*x) \le u^*\omega_1 u(x^*x)$ for all $x \in A_1$, i.e. $\phi(x^*x) \le \omega_1(ux^*xu^*)$ for all $x \in A_1$.
Then $u\phi u^*(x^*x) = \phi(u^*x^*xu) = \phi((xu)^*xu) \le \omega_1(u(xu)^*xuu^*) = \omega_1(x^*x)$ for all $x \in A_1$.
Since ω_1 is pure, $u\phi u^*$ is of the form $u\phi u^* = \lambda\omega_1$, $0 \le \lambda \le 1$.
But then $\phi = \lambda u^*\omega_1 u$, so $u^*\omega_1 u$ is pure. Thus $u^*\omega_1 u \in P(A_1)$, and similarly $v^*\omega_2 v \in P(A_2)$.
Now we must show that $u^*\omega_1 u \otimes v^*\omega_2 v$ is continuous with respect to the β norm.
Let $x = a_1 \otimes a_2 \in A_1 \otimes A_2$.
Then

$$\begin{aligned}|(u^*\omega_1 u \otimes v^*\omega_2 v)(x)| &= |(u^*\omega_1 u)(a_1)(v^*\omega_2 v)(a_2)| = |\omega_1(ua_1u^*)\omega_2(va_2v^*)| \\ &= |(\omega_1 \otimes \omega_2)(ua_1u^* \otimes va_2v^*)| \\ &\le \|ua_1u^* \otimes va_2v^*\|_\beta \quad (\text{because } \|\omega_1 \otimes \omega_2\| = 1).\end{aligned}$$

Now $ua_1u^* \otimes va_2v^* = (u \otimes v)(a_1u^* \otimes a_2v^*) = (u \otimes v)(a_1 \otimes a_2)(u^* \otimes v^*)$.
Hence

$$\begin{aligned}\|ua_1u^* \otimes va_2v^*\|_\beta &= \|(u \otimes v)(a_1 \otimes a_2)(u^* \otimes v^*)\|_\beta \\ &\leq \|(u \otimes v)\|_\beta \|(a_1 \otimes a_2)\|_\beta \|(u \otimes v)^*\|_\beta \\ &= \|a_1 \otimes a_2\|_\beta \text{ (because } u \otimes v \text{ is unitary).}\end{aligned}$$

Thus $|(u^*\omega_1 u \otimes v^*\omega_2 v)(x)| \leq \|a_1 \otimes a_2\|_\beta = \|x\|_\beta$.
The result with $x = \sum_{i=1}^n a_{1i} \otimes a_{2i}$ follows by linearity.
Hence $u^*\omega_1 u \otimes v^*\omega_2 v$ is continuous with respect to the β norm.

Lemma 5.6. *If A_i' has the weak*-topology, i.e. the $\sigma(A_i', A_i)$ topology, $i = 1, 2$, then S_β is closed in $P(A_1) \times P(A_2)$.*

Proof. Let $(\omega_{1\alpha}, \omega_{2\alpha})$ be a net in S_β converging to $(\omega_1, \omega_2) \in P(A_1) \times P(A_2)$.
Let $x = \sum_{i=1}^n a_{1i} \otimes a_{2i} \in A_1 \otimes A_2$.
Then,

$$\begin{aligned}(\omega_1 \otimes \omega_2)(x) &= \sum_{i=1}^n \omega_1(a_{1i})\omega_2(a_{2i}) = \lim_\alpha \sum_{i=1}^n \omega_{1\alpha}(a_{1i})\omega_{2\alpha}(a_{2i}) \\ &= \lim_\alpha (\omega_{1\alpha} \otimes \omega_{2\alpha})(x).\end{aligned}$$

Hence

$$\begin{aligned}|(\omega_1 \otimes \omega_2)(x)| &= |\lim_\alpha (\omega_{1\alpha} \otimes \omega_{2\alpha})(x)| = \lim_\alpha |(\omega_{1\alpha} \otimes \omega_{2\alpha})(x)| \\ &\leq \lim_\alpha \|\omega_{1\alpha} \otimes \omega_{2\alpha}\|_\beta \|x\|_\beta = \|x\|_\beta,\end{aligned}$$

since $\|\omega_{1\alpha} \otimes \omega_{2\alpha}\|_\beta = 1$ for all α because $(\omega_{1\alpha}, \omega_{2\alpha}) \in S_\beta$. Thus $\omega_1 \otimes \omega_2$ is continuous with respect to the β norm and so $(\omega_1, \omega_2) \in S_\beta$.

Lemma 5.7. *Let E be a C^*-algebra and $P(E)$ the set of all pure states on E. If K is a $\sigma(E', E)$-closed subset of $P(E)$ such that $uKu^* = K$ for all unitary elements $u \in E$, then $K^\perp = \{x \in E : \omega(x) = 0 \text{ for all } \omega \in K\}$ is a closed ideal of E. If I is a closed ideal of E, then $I^\perp = \{\omega \in P(E) : |\langle x, \omega\rangle| \leq 1 \text{ for all } x \in I\}$ is a $\sigma(E', E)$-closed subset of $P(E)$ such that $uI^\perp u^* = I^\perp$ for all unitary elements $u \in E$. The correspondences $K \mapsto K^\perp$ and $I \mapsto I^\perp$ are each other's inverse.*

Proof. See Takesaki, [6], IV.4.15.

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Lemma 5.8. *If $S_\beta \neq P(A_1) \times P(A_2)$, then there exist nonzero elements $a_1 \in A_1, a_2 \in A_2$, such that $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all $(\omega_1, \omega_2) \in S_\beta$.*

Proof: By Lemma 5.6, S_β is closed in $P(A_1) \times P(A_2)$. Hence, if $S_\beta \neq P(A_1) \times P(A_2)$, there exist nonempty open sets $U_1 \subset P(A_1)$, $U_2 \subset P(A_2)$ such that $(U_1 \times U_2) \cap S_\beta = \phi$.
Let $V_1 = \cup\{u^* U_1 u : u \text{ is a unitary element of } A_1\}$.
If $\omega \in V_1$, $\omega = u^*\omega' u$ for some $\omega' \in P(A_1)$, so $\omega \in P(A_1)$. Thus $V_1 \subset P(A_1)$.
Similarly, define $V_2 = \cup\{v^* U_2 v : v \text{ is a unitary element of } A_2\}$. Then $V_2 \subset P(A_2)$.
Also, $(V_1 \times V_2) \cap S_\beta = \phi$. For suppose $(\omega_1', \omega_2') \in (V_1 \times V_2) \cap S_\beta$.
Then $(u^*\omega_1 u, v^*\omega_2 v) \in S_\beta$ for some $\omega_1 \in U_1$, $\omega_2 \in U_2$, u unitary in A_1, v unitary in A_2.
But then, by Lemma 5.5, $(uu^*\omega_1 uu^*, vv^*\omega_2 vv^*) \in S_\beta$, i.e. $(\omega_1, \omega_2) \in S_\beta$, contradicting the fact that $(U_1 \times U_2) \cap S_\beta = \phi$.
Thus we have open subsets V_1 of $P(A_1)$, V_2 of $P(A_2)$, such that $(V_1 \times V_2) \cap S_\beta = \phi$ and $V_1 = u^* V_1 u$ for all unitary elements u of A_1, $V_2 = v^* V_2 v$ for all unitary elements v of A_2.
Let $K_1 = V_1^c = P(A_1) - V_1$, $K_2 = V_2^c$. Since V_i is nonempty, $K_i \neq P(A_i)$.
We now show, using the fact that $V_1 = u^* V_1 u$, that $K_1 = u^* K_1 u$ for all unitary elements u of A_1.
If $x \in u^* K_1 u$, then $x = u^* k u$, $k \in K_1$. If $u^* k u = u^* v_1 u$ for some $v_1 \in V_1$, then $k = v_1$, contradicting the fact that $k \in K_1 = V_1^c$. Hence $x = u^* k u \notin u^* V_1 u$. Similarly, if $x \notin u^* V_1 u$, then $x \in u^* K_1 u$.
Thus $u^* K_1 u = (u^* V_1 u)^c = V_1^c = K_1$. Similarly $K_2 = v^* K_2 v$ for all unitary elements v of A_2.
Thus K_1 and K_2 are closed subsets of $P(A_1)$ and $P(A_2)$ respectively satisfying the conditions of Lemma 5.7 above.
Also,

$$
\begin{aligned}
(V_1 \times V_2) \cap S_\beta = \phi &\Longrightarrow S_\beta \subset [P(A_1) \times P(A_2)] - [V_1 \times V_2] \\
&\Longrightarrow S_\beta \subset [P(A_1) \times (P(A_2) - V_2)] \cup [(P(A_1) - V_1) \times P(A_2)] \\
&\Longrightarrow S_\beta \subset (P(A_1) \times K_2) \cup (K_1 \times P(A_2)).
\end{aligned}
$$

Thus, if $(\omega_1, \omega_2) \in S_\beta$, either $\omega_1 \in K_1$ or $\omega_2 \in K_2$.
Let $I_i = K_i^\perp = \{x \in A_i : \omega(x) = 0 \text{ for all } \omega \in K_i\}$, $i = 1, 2$.
Then, if $I_i^\perp$ is defined as in Lemma 5.7, we know that $I_i^\perp = K_i$, $i = 1, 2$.
Now $I_i = \{0\} \Longrightarrow I_i^\perp = \{\omega \in P(A_i) : |\langle 0, \omega\rangle| \leq 1\} = P(A_i) = K_i$, $i = 1, 2$.
Since $K_i \neq P(A_i)$, we must have $I_i \neq \{0\}$, $i = 1, 2$.
Choose nonzero $a_1 \in I_1, a_2 \in I_2$, and let (ω_1, ω_2) be any element of S_β. Then either $\omega_1 \in K_1$ or $\omega_2 \in K_2$. But if $\omega_1 \in K_1$, then $\omega_1(a_1) = 0$ because $a_1 \in I_1$, and if $\omega_2 \in K_2$, then $\omega_2(a_2) = 0$.
Thus $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = \omega_1(a_1)\omega_2(a_2) = 0$, as required.

Theorem 5.9. *If A is a C^*-algebra and S is the set of all positive linear functionals on A of norm ≤ 1, then S is the $\sigma(A', A)$ convex closure of zero and the pure states.*

Proof: We know that the extreme points of S are zero and the pure states. Using the fact that S is a $\sigma(A', A)$-compact convex subset of A', the result follows immediately from the Krein-Milman theorem.

Corollary 5.10. *If x is a nonzero element of a C^*-algebra A, then there exists a pure state p on A such that $p(x) \neq 0$.*

Proof: Since $x \neq 0$, the Hahn-Banach theorem yields a continuous linear functional ω on A such that $\omega(x) \neq 0$. By dividing by $\|\omega\|$ if necessary, we may assume $\|\omega\| \leq 1$. Then, by Theorem 5.9, we may write ω as the limit of a convex combination of pure states,
i.e. $\omega = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i p_i$, where $0 \leq \alpha_i \leq 1$, $p_i \in P(A)$ for $i = 1$ to n, $\sum_{i=1}^{n} \alpha_i = 1$.
If $p_i(x) = 0$ for all i, then $\omega(x) = 0$, a contradiction. Hence there exists at least one pure state p_i on A such that $p_i(x) \neq 0$.

Theorem 5.11. *If ϕ is a positive linear functional on a C^*-algebra A, then the following are equivalent: (i) ϕ is pure; (ii) The cyclic representation π_ϕ of A induced by ϕ is irreducible.*

Proof: See Takesaki, [6], I.9.22.

Lemma 5.12. *If A is a C^*-algebra and $x \in A$, then $\|x\| = sup\{\|\pi_\omega(x)\| : \omega \in P(A)\}$.*

Proof: We have that if π is a representation of a C^*-algebra A, then $\|\pi(x)\| \leq \|x\|$ for all $x \in A$ [Takesaki, [6], I.5.2].
But we know that at least one representation is an isometry, so $\|x\| = sup\{\|\pi(x)\| : \pi$ is a representation of $A\}$.
Now let π be an arbitrary (nondegenerate) representation of A. Then π can be written as the direct sum of cyclic representations on A [Takesaki, [6], I.9.17].
We now prove that if $\pi = \sum_{i \in I}^{\oplus} \pi_i$, then $\|\pi(x)\| = sup_{i \in I}\|\pi_i(x)\|$ for all $x \in A$.
If $\xi = \sum_{i \in I}^{\oplus} \xi_i$, then $\pi(x)\xi = \sum_{i \in I}^{\oplus} \pi_i(x)\xi_i$, and $\|\pi(x)\xi\| = sup_{i \in I}\|\pi_i(x)\xi_i\|$.
Thus

$$\begin{aligned}
\|\pi(x)\| &= sup_{\|\xi\|=1}\|\pi(x)\xi\| = sup_{\|\xi\|=1} sup_{i \in I}\|\pi_i(x)\xi_i\| \\
&= sup_{i \in I} sup_{\|\xi_i\|=1}\|\pi_i(x)\xi_i\| \quad \text{(easily justified)} \\
&= sup_{i \in I}\|\pi_i(x)\|.
\end{aligned}$$

We have just proved that if x is any element of A and π is any (nondegenerate) representation of A, then $\|\pi(x)\| = sup_{i \in I}\|\pi_i(x)\|$, where the π_i, $i \in I$, are cyclic representations

of A.
It follows that $sup\{\|\pi(x)\| : \pi$ is a cyclic representation of $A\} = sup\{\|\pi(x)\| : \pi$ is a representation of $A\}$.
Thus

$$\|x\| = sup\{\|\pi(x)\| : \pi \text{ is a cyclic representation of } A\}$$
$$= sup\{\|\pi_\omega(x)\| : \omega \text{ is a positive linear functional on } A\}.$$

Now, if ω is a positive linear functional on A, we can, dividing by $\|\omega\|$, if necessary, to apply Theorem 5.9, express ω as the limit of a convex combination of pure states.
Thus $\pi_\omega = \lim_{n\to\infty} \sum_{i=1}^n \alpha_i \pi_{\omega_i}$, $0 \leq \alpha_i \leq 1$, $\omega_i \in P(A)$ for all $i = 1$ to n, $\sum_{i=1}^n \alpha_i = 1$.
Hence,

$$\|\pi_\omega\| = \lim_{n\to\infty} \|\sum_{i=1}^n \alpha_i \pi_{\omega_i}\| \leq \lim_{n\to\infty} \sum_{i=1}^n \|\alpha_i \pi_{\omega_i}\| = \lim_{n\to\infty} \sum_{i=1}^n \alpha_i \|\pi_{\omega_i}\|$$
$$\leq \lim_{n\to\infty} \sum_{i=1}^n \alpha_i \|\pi_{\omega_r}\|, \quad (\text{where } \|\pi_{\omega_r}\| = sup_{1\leq i\leq n}\|\pi_{\omega_i}\|)$$
$$= \|\pi_{\omega_r}\| \lim_{n\to\infty} \sum_{i=1}^n \alpha_i = \|\pi_{\omega_r}\|.$$

So $\|\pi_\omega\| \leq \|\pi_{\omega_r}\|$, $\omega_r \in P(A)$,
which implies

$$\|x\| = sup\{\|\pi_\omega(x)\| : \omega \in P(A)\}$$
$$= sup\{\|\pi(x)\| : \pi \text{ is an irreducible representation of } A\}$$

by the correspondence between irreducible representations and pure states.

Lemma 5.13. *If A_1 and A_2 are any C^* algebras and β is a C^*-norm on $A_1 \otimes A_2$ such that $S_\beta = P(A_1) \times P(A_2)$, then $\|x\|_{min} \leq \|x\|_\beta$ for all $x \in A_1 \otimes A_2$.*

Proof: We recall the definition of the min norm: $\|x\|_{min} = sup\{\|(\pi_1 \otimes \pi_2)(x)\| : \pi_i$ is an representation of A_i, $i = 1,2\}$, $x \in A_1 \otimes A_2$.
As in the proof of Lemma 5.12, we can restrict our attention to irreducible representations only; i.e. we have $\|x\|_{min} = sup\{\|(\pi_1 \otimes \pi_2)(x)\| : \pi_i$ is an irreducible representation of A_i, $i = 1,2\}$.
If $S_\beta = P(A_1) \times P(A_2)$, then $\omega_1 \otimes \omega_2$ is continuous with respect to the β-norm for any pure states ω_1 and ω_2. It follows, via the correspondence between pure states and irreducible representations, that $\pi_1 \otimes \pi_2$ is continuous with respect to the β-norm for any irreducible representations π_1 of A_1 and π_2 of A_2. Thus $\pi_1 \otimes \pi_2$ can be extended to a representation of the C^*-algebra $A_\beta = A_1 \tilde{\otimes}_\beta A_2$, whence, [Takesaki, [6], I.5.2], $\|(\pi_1 \otimes \pi_2)(x)\| \leq \|x\|_\beta$ for all $x \in A_1 \otimes A_2$.

This holds for all irreducible representations π_1, π_2 of A_1, A_2 respectively, so $\|x\|_{min} \leq \|x\|_\beta$ for all $x \in A_1 \otimes A_2$.

We need one more fact concerning pure states. If B is a C^*-subalgebra of a C^*-algebra A and p is a pure state on B, then, by the Hahn-Banach theorem, p can be extended to a pure state on A.

Lemma 5.14. *If either A_1 or A_2 is abelian, and β is a C^*-norm on $A_1 \otimes A_2$, then the β-norm and the min norm coincide on $A_1 \otimes A_2$.*

Proof: Suppose A_1 is abelian. By Corollary 5.4, every $\omega \in P(A_\beta) = P(A_1 \tilde{\otimes}_\beta A_2)$ is of the form $\omega = \omega_1 \otimes \omega_2$ for $\omega_i \in P(A_i)$, $i = 1, 2$. Hence, if π_{ω_i} is the irreducible representation of A_i induced by ω_i, $i = 1, 2$, the irreducible representation π_ω of A_β induced by ω is given by $\pi_\omega = \pi_{\omega_1} \otimes \pi_{\omega_2}$ [Takesaki, [6], III.4.9].
Let $x \in A_1 \otimes A_2$. By Lemma 5.12, $\|x\|_\beta = sup\{\|\pi_\omega(x)\| : \omega \in P(A_\beta)\}$.
Thus we have

$$\|x\|_\beta = sup\{\|(\pi_{\omega_1} \otimes \pi_{\omega_2})(x)\| : \pi_{\omega_i} \text{ is an irreducible representation of } A_i\} = \|x\|_{min}.$$

Theorem 5.15. *Let A_1 and A_2 be C^*-algebras. The injective C^* crossnorm min is the smallest possible C^*-norm on $A_1 \otimes A_2$.*

Proof: Let β be any C^*-norm on $A_1 \otimes A_2$.
By Lemma 5.13, it suffices to prove that $S_\beta = P(A_1) \times P(A_2)$.
Suppose $S_\beta \neq P(A_1) \times P(A_2)$.
By Lemma 5.8, there exist nonzero elements $a_1 \in A_1, a_2 \in A_2$ such that $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all $(\omega_1, \omega_2) \in S_\beta$.
Let B be the abelian C^*-subalgebra of A_1 generated by a_1 and 1. By Lemma 5.14, the restriction of the β-norm to $B \otimes A_2$ agrees with the injective C^*-norm min on $B \otimes A_2$. Hence we may naturally embed $B\tilde{\otimes}_{min}A_2$ in $A_\beta = A_1\tilde{\otimes}_\beta A_2$.
Since $a_1 \neq 0, a_2 \neq 0$, there exist, by Corollary 5.10, $\rho \in P(B)$ and $\omega_2 \in P(A_2)$ such that $\rho(a_1) \neq 0$ and $\omega_2(a_2) \neq 0$. A straightforward calculation shows that $\rho \otimes \omega_2$ is a pure state of $B\tilde{\otimes}_{min}A_2$. Let ω be the pure state extension of $\rho \otimes \omega_2$ to A_β. Then the restriction of ω to A_2 is ω_2, which is pure. By Lemma 5.3, ω is of the form $\omega = \omega_1 \otimes \omega_2$, and ω_1 is also pure.
Hence $(\omega_1, \omega_2) \in S_\beta$, but $(\omega_1, \omega_2)(a_1 \otimes a_2) = \rho(a_1)\omega_2(a_2) \neq 0$, a contradiction.
Thus $S_\beta = P(A_1) \times P(A_2)$.

It follows that all C^*-norms on $A_1 \otimes A_2$ are cross-norms. To see this, let β be any C^*-norm on $A_1 \otimes A_2$.
Then (1) $\|x_1\|\|x_2\| = \|x_1 \otimes x_2\|_{min} \leq \|x_1 \otimes x_2\|_\beta \leq \|x_1 \otimes x_2\|_{max} = \|x_1\|\|x_2\|$,

so $\|x_1 \otimes x_2\|_{beta} = \|x_1\|\|x_2\|$ for all $x_1 \in A_1$, $x_2 \in A_2$,
and (2) if $x_1' \in A_1'$, $x_2' \in A_2'$, then

$$|(x_1' \otimes x_2')(x_1 \otimes x_2) \leq \|x_1'\|\|x_2'\|\|x_1 \otimes x_2\|_{min} \quad \text{(because} \|.\|_{min} \text{ is a cross-norm)}$$
$$\leq \|x_1'\|\|x_2'\|\|x_1 \otimes x_2\|_{\beta}.$$

Thus $x_1' \otimes x_2' \in (A_1 \tilde{\otimes}_\beta A_2)'$ and $\|x_1' \otimes x_2'\| \leq \|x_1'\|\|x_2'\|$.

Chapter 3

STATE SPACES

§1. INTRODUCTION

This chapter considers how the different norms on the tensor product affect the state spaces, i.e. the spaces of positive linear maps of norm 1. In section 2 we define the concept of a completely positive map, and prove the main result concerning these maps which we will need in our discussion of the state spaces. We determine the state spaces for the *pi* norm and then, using the idea of the enveloping C^*-algebra of an involutive Banach algebra, we find the state space for the *max* norm. The final section considers a different approach to the state spaces, based on the idea of separating subsets, which confirms our previous work and allows us to calculate the state space for the *min* norm.

§2. COMPLETELY POSITIVE MAPS

Let A be a C^*-algebra. For $n \in \mathbf{N}$, let $M_n(A)$ denote the space of all nxn-matrices $a = [a_{ij}]$ with entries $a_{ij} \in A$, $i, j = 1, ..., n$. We make $M_n(A)$ into an involutive algebra in the obvious way: For all $a = [a_{ij}]$, $b = [b_{ij}] \in A$, $\lambda, \mu \in \mathbf{C}$,

$$(\lambda a + \mu b)_{ij} = \lambda a_{ij} + \mu b_{ij}$$

$$(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(a^*)_{ij} = a_{ji}^*.$$

We let M_n denote the algebra of all nxn complex matrices. Suppose H is an n-dimensional Hilbert space with orthogonal basis $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. We may regard M_n as a space of linear operators on H in the usual way:

If $h \in H$ is written uniquely as $h = \sum_{i=1}^{n} \alpha_i \epsilon_i$ and $\gamma = [\gamma_{ij}] \in M_n$, then $\gamma(h) = \sum_{i=1}^{n} (\sum_{j=1}^{n} \gamma_{ij} \alpha_j) \epsilon_i \in H$. Then we can give γ the operator norm of $\mathcal{L}(H)$, and, under this norm, M_n becomes a C^*-algebra. Thus we may form the C^*-algebra tensor product $A \otimes M_n$.

We now identify $M_n(A)$ with $A \otimes M_n$, as follows:

Let e_{ij} be the matrix with a 1 in the (i-j) position and zeroes elsewhere.

Then the collection $\{e_{ij}\}_{i,j=1}^{n}$ generate M_n as a linear space.

If $a = [a_{ij}] \in M_n(A)$, then a corresponds to $\sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \in A \otimes M_n$.

Conversely, any element $x \in A \otimes M_n$ may be written uniquely as $x = \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}$, $a_{ij} \in A$, and so we have a correspondence identifying $M_n(A)$ and $A \otimes M_n$ as involutive algebras.

Lemma 2.1. *An element of $M_n(A)$ is positive if and only if it is a sum of matrices of the form $[a_i^* a_j]$, where $a_1, ..., a_n \in A$.*

Proof: Suppose $b = [b_{ij}] = [a_i^* a_j]$, $a_1, ..., a_n \in A$.
Let $a_{1j} = a_j$, $j = 1, ..., n$, and $a_{ij} = 0$, $i = 2, ..., n$, $j = 1, ..., n$.
Then $a = [a_{ij}]$ looks like

$$a = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \text{so } a^* = \begin{pmatrix} a_1^* & 0 & \dots & 0 \\ a_2^* & 0 & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_n^* & 0 & \dots & 0 \end{pmatrix},$$

and

$$a^* a = \begin{pmatrix} a_1^* a_1 & a_1^* a_2 & \dots & a_1^* a_n \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_n^* a_1 & a_n^* a_2 & \dots & a_n^* a_n \end{pmatrix} = b.$$

Thus b is positive.
So sums of elements of the form $[a_i^* a_j]$ are positive.
Conversely, suppose $c = [c_{ij}] \in M_n(A)$ is positive. Then there exists $d = [d_{ij}] \in M_n(A)$ such that $c = d^* d$.
Hence, by definition of matrix multiplication in $M_n(A)$, $c_{ij} = \sum_{k=1}^n (d^*)_{ik} d_{kj}$, $i, j = 1, ..., n$,
and then, by definition of the involution on $M_n(A)$, $c_{ij} = \sum_{k=1}^n d_{ki}^* d_{kj}$, $i, j = 1, ..., n$.
For each $k = 1, ..., n$, define $a_k = [d_{ki}^* d_{kj}] \in M_n(A)$.
Then each a_k is of the desired form $[a_i^* a_j]$, and $c = \sum_{k=1}^n a_k$.

Next we consider dual spaces. If $a \in M_n$, then define f_a on M_n by $f_a(x) = trace(ax)$. Then it is easy to see that f_a is in M_n' for every $a \in M_n$. So we have a map from M_n to M_n', given by $a \mapsto f_a$. This map is clearly linear. We now show that it is 1-1.
Suppose $f_a = f_b$, $a, b \in M_n$. Then $trace(ax) = trace(bx)$ for all $x \in M_n$. Let $x = e_{ij}$, $i, j = 1, ..., n$.
Then

$$ae_{ij} = \begin{pmatrix} 0 & \dots & 0 & a_{1j} & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{2j} & 0 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & a_{ij} & 0 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & a_{nj} & 0 & \dots & 0 \end{pmatrix},$$

so $trace(ae_{ij}) = a_{ij}$. But $trace(ae_{ij}) = trace(be_{ij}) = b_{ij}$, so $a_{ij} = b_{ij}$ for all $i, j = 1, ..., n$; i.e. $a = b$.

The correspondence $a \mapsto f_a$ is also onto. For let $g \in M_n'$. Put $a_{ij} = g(e_{ji})$, $i, j = 1, ..., n$. Then, if $a = [a_{ij}] \in M_n$, we have, for all $x \in M_n$,

$$(ax)_{ii} = \sum_{k=1}^{n} a_{ik}x_{ki}, \text{ so } trace(ax) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}x_{ki},$$

$$\text{while } g(x) = g\left(\sum_{i,j=1}^{n} x_{ij}e_{ij}\right) = \sum_{i,j=1}^{n} x_{ij}g(e_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}a_{ji} = trace(ax) = f_a(x),$$

so $g = f_a$.

The map $a \mapsto f_a$ leads to a correspondence between $(A \otimes M_n)'$ and $A' \otimes M_n$, as follows:

Put $f_{ij} = f_{e_{ji}}$, $i, j = 1, ..., n$. Then $f_{ij}(e_{ij}) = trace(e_{ji}e_{ij}) = 1$ for all $i, j = 1, ..., n$.
Let A be a C^*-algebra and let $g \in (A \otimes M_n)'$.
Define $g_{ij} \in A'$ by $g_{ij}(a) = g(a \otimes e_{ij})$ for all $a \in A$, $i, j = 1, ..., n$.
Then

$$\left(\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij}\right)(a \otimes e_{kl}) = \sum_{i,j=1}^{n} g_{ij}(a)f_{ij}(e_{kl})$$
$$= \sum_{i,j=1}^{n} g(a \otimes e_{ij})trace(e_{ji}e_{kl})$$
$$= g(a \otimes e_{kl})$$

because $trace(e_{ji}e_{kl}) = 1$ if $i = k$ and $j = l$ and is zero otherwise.

Thus $(\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij})(a \otimes e_{kl}) = g(a \otimes e_{kl})$ for all $a \in A$, $k, l = 1, ..., n$,
so $g = \sum_{i,j=1}^{n} g_{ij} \otimes f_{ij} \in A' \otimes M_n$.
Conversely, if $\sum_{i,j=1}^{n} h_{ij} \otimes e_{ij} \in A' \otimes M_n$, define h on $A \otimes M_n$ by
$h(\sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}) = \sum_{i,j=1}^{n} h_{ij}(a_{ij})$.
Then $h \in (A \otimes M_n)'$ and $h = \sum_{i,j=1}^{n} h_{ij} \otimes f_{ij}$.
Thus we may identify $(A \otimes M_n)'$ with $A' \otimes M_n$, and hence with $M_n(A')$.

Lemma 2.2. *Let A be a C^*-algebra and let $g = \sum_{i,j=1}^{n} g_{ij} \otimes f_{ij} \in (A \otimes M_n)'$. Then g is positive if and only if $\sum_{i,j=1}^{n} g_{ij}(a_i^*a_j) \geq 0$ for all $a_1, ..., a_n \in A$.*

Proof. By definition, g is positive if and only if $g(x) \geq 0$ for all positive $x \in A \otimes M_n = M_n(A)$. By Lemma 2.1, we may take $x = [a_i^*a_j]$, $a_1, ..., a_n \in A$. Thus g is positive if and only if $g(x) = (\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij})(a_i^*a_j \otimes e_{ij}) = \sum_{i,j=1}^{n} g_{ij}(a_i^*a_j) \geq 0$.

Definition 2.3: Let each of A and B be either a C^*-algebra or the dual of a C^*-algebra. Let $T \in \mathcal{L}(A, B)$. We define, for each $n \in \mathbf{N}$, a map $T_n : M_n(A) \to M_n(B)$ by $T_n([a_{ij}]) = [T(a_{ij})]$. We say that T is **completely positive** if T_n is positive for each n.

The correspondence $a \mapsto f_a$ is also onto. For let $g \in M_n'$. Put $a_{ij} = g(e_{ji})$, $i,j = 1,...,n$. Then, if $a = [a_{ij}] \in M_n$, we have, for all $x \in M_n$.

$$(ax)_{ii} = \sum_{k=1}^{n} a_{ik}x_{ki}, \text{ so } trace(ax) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}x_{ki},$$

$$\text{while } g(x) = g\left(\sum_{i,j=1}^{n} x_{ij}e_{ij}\right) = \sum_{i,j=1}^{n} x_{ij}g(e_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}a_{ji} = trace(ax) = f_a(x),$$

so $g = f_a$.

The map $a \mapsto f_a$ leads to a correspondence between $(A \otimes M_n)'$ and $A' \otimes M_n$, as follows:

Put $f_{ij} = f_{e_{ji}}$, $i,j = 1,...,n$. Then $f_{ij}(e_{ij}) = trace(e_{ji}e_{ij}) = 1$ for all $i,j = 1,...,n$.
Let A be a C^*-algebra and let $g \in (A \otimes M_n)'$.
Define $g_{ij} \in A'$ by $g_{ij}(a) = g(a \otimes e_{ij})$ for all $a \in A$, $i,j = 1,...,n$.
Then

$$\begin{aligned}\left(\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij}\right)(a \otimes e_{kl}) &= \sum_{i,j=1}^{n} g_{ij}(a)f_{ij}(e_{kl}) \\ &= \sum_{i,j=1}^{n} g(a \otimes e_{ij})trace(e_{ji}e_{kl}) \\ &= g(a \otimes e_{kl})\end{aligned}$$

because $trace(e_{ji}e_{kl}) = 1$ if $i = k$ and $j = l$ and is zero otherwise.

Thus $(\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij})(a \otimes e_{kl}) = g(a \otimes e_{kl})$ for all $a \in A$, $k,l = 1,...,n$,
so $g = \sum_{i,j=1}^{n} g_{ij} \otimes f_{ij} \in A' \otimes M_n$.
Conversely, if $\sum_{i,j=1}^{n} h_{ij} \otimes e_{ij} \in A' \otimes M_n$, define h on $A \otimes M_n$ by
$h(\sum_{i,j=1}^{n} a_{ij} \otimes e_{ij}) = \sum_{i,j=1}^{n} h_{ij}(a_{ij})$.
Then $h \in (A \otimes M_n)'$ and $h = \sum_{i,j=1}^{n} h_{ij} \otimes f_{ij}$.
Thus we may identify $(A \otimes M_n)'$ with $A' \otimes M_n$, and hence with $M_n(A')$.

Lemma 2.2. *Let A be a C^*-algebra and let $g = \sum_{i,j=1}^{n} g_{ij} \otimes f_{ij} \in (A \otimes M_n)'$. Then g is positive if and only if $\sum_{i,j=1}^{n} g_{ij}(a_i^*a_j) \geq 0$ for all $a_1,...,a_n \in A$.*

Proof: By definition, g is positive if and only if $g(x) \geq 0$ for all positive $x \in A \otimes M_n = M_n(A)$. By Lemma 2.1, we may take $x = [a_i^*a_j]$, $a_1,...,a_n \in A$. Thus g is positive if and only if $g(x) = (\sum_{i,j=1}^{n} g_{ij} \otimes f_{ij})(a_i^*a_j \otimes e_{ij}) = \sum_{i,j=1}^{n} g_{ij}(a_i^*a_j) \geq 0$.

Definition 2.3: Let each of A and B be either a C^*-algebra or the dual of a C^*-algebra. Let $T \in \mathcal{L}(A,B)$. We define, for each $n \in \mathbf{N}$, a map $T_n : M_n(A) \to M_n(B)$ by $T_n([a_{ij}]) = [T(a_{ij})]$. We say that T is **completely positive** if T_n is positive for each n.

Proposition 2.4. *If A and B are C^*-algebras and $T \in \mathcal{L}(A, B')$, then T is a completely positive map if and only if T satisfies $\sum_{i,j=1}^{n} T(a_i^* a_j)(b_i^* b_j) \geq 0$ for all $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$, $n \in \mathbb{N}$.*

Proof: T is completely positive if and only if T_n is positive for each n.
By definition, $T_n : M_n(A) \to M_n(B')$, and, by Lemma 2. T_n is positive if and only if $T_n([a_i^* a_j]) \geq 0$ for all $a_1, \ldots, a_n \in A$.
Now $T_n([a_i^* a_j]) = [T(a_i^* a_j)] \in M_n(B') = B' \otimes M_n$, so, by Lemma 2.2,
$g = [T(a_i^* a_j)] = \sum_{i,j=1}^{n} T(a_i^* a_j) \otimes f_{ij}$ is positive if and only if
$\sum_{i,j=1}^{n} T(a_i^* a_j)(b_i^* b_j) \geq 0$ for all $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$.

§3. Finding the State Spaces

If A and B are C^*-algebras and α is any norm on $A \otimes B$, $A\tilde{\otimes}_\alpha B$ will denote, as usual, the completion of $A \otimes B$ with respect to the norm α, and $S(A\tilde{\otimes}_\alpha B)$ will denote the space of states on $A\tilde{\otimes}_\alpha B$; i.e. the space of linear maps p on $A\tilde{\otimes}_\alpha B$ such that $p(x^* x) \geq 0$ for all $x \in A\tilde{\otimes}_\alpha B$ and $\|p\|_\alpha = sup\{|p(x)| : \|x\|_\alpha \leq 1,\ x \in A\tilde{\otimes}_\alpha B\} = 1 = p(1)$.

We will first consider the case $\alpha = \pi$.
We let $(A\tilde{\otimes}_\pi B)'$ denote the Banach dual of $(A\tilde{\otimes}_\pi B)$, i.e. the space of all bounded [i.e. continous] linear maps on $A\tilde{\otimes}_\pi B$. Then $S(A\tilde{\otimes}_\pi B)$ is a subspace of $(A\tilde{\otimes}_\pi B)'$.
Now we know [eg Effros and Lance, [3], pg 9] that there is an isometric isomorphism between $(A\tilde{\otimes}_\pi B)'$ and $\mathcal{L}(A, B')$, given by $T_f(a)(b) = f(a \otimes b)$, where $f \in (A\tilde{\otimes}_\pi B)'$ and $T_f \in \mathcal{L}(A, B')$.
We now investigate what this isometry tells us about $S(A\tilde{\otimes}_\pi B)$.
Clearly $S(A\tilde{\otimes}_\pi B)$ is isometrically isomorphic to a subspace of $\mathcal{L}(A, B')$. Precisely what this subspace is given by the next two theorems.

Theorem 3.1. *If $f \in (A\tilde{\otimes}_\pi B)'$ and $T_f \in \mathcal{L}(A, B')$ is defined by $T_f(a)(b) = f(a \otimes b)$, then f is positive if and only if T_f is completely positive.*

Proof: We note that f is positive

if and only if $f(x^*x) \geq 0$ for all $x \in A\tilde{\otimes}_\pi B$

if and only if $f(x^*x) \geq 0$ for all $x \in A \otimes B$

if and only if $f(\left(\sum_{i=1}^n a_i \otimes b_i\right)^* \left(\sum_{j=1}^n a_j \otimes b_j\right)) \geq 0$ for all $a_i \in A,\ b_i \in B,\ n \in \mathbb{N}$

if and only if $f(\left(\sum_{i=1}^n a_i^* \otimes b_i^*\right) \left(\sum_{j=1}^n a_j \otimes b_j\right)) \geq 0$ for all $a_i \in A,\ b_i \in B,\ n \in \mathbb{N}$

if and only if $f\left(\sum_{i=1}^n \sum_{j=1}^n a_i^* a_j \otimes b_i^* b_j\right) \geq 0$ for all $a_i \in A,\ b_i \in B,\ n \in \mathbb{N}$

if and only if $\sum_{i=1}^n \sum_{j=1}^n f(a_i^* a_j \otimes b_i^* b_j) \geq 0$ for all $a_i \in A,\ b_i \in B,\ n \in \mathbb{N}$

if and only if $\sum_{i=1}^n \sum_{j=1}^n T_f(a_i^* a_j)(b_i^* b_j) \geq 0$ for all $a_i \in A,\ b_i \in B,\ n \in \mathbb{N}$,

which, by Proposition 2.4, is equivalent to T_f being completely positive.

Since the correspondence $f \mapsto T_f$ is an isometry, it follows that f is a state on $A\tilde{\otimes}_\pi B$ if and only if T_f is a completely positive map of norm 1 in $\mathcal{L}(A, B')$.

Definition 3.2: By simple weak*-convergence, we mean pointwise convergence in the weak*-topology; i.e. $T_i \to T$ (simple weak*) if and only if $T_i(a) \to T(a)$ (weak*) if and only if $T_i(a)(b) \to T(a)(b)$ for all $a \in A,\ b \in B$.
By considering this topology, we obtain:

Theorem 3.3. *The isometric isomorphism between $S(A\tilde{\otimes}_\pi B)$ and the set of completely positive maps of norm 1 in $\mathcal{L}(A, B')$ is a homeomorphism if $S(A\tilde{\otimes}_\pi B)$ has the weak*-topology and $\mathcal{L}(A, B')$ the topology of simple weak*-convergence.*

Proof: We need to prove that the correspondence $f \mapsto T_f$ is continuous in both directions with respect to the stated topologies.
So suppose f_i is a net in $S(A\tilde{\otimes}_\pi B)$ such that $f_i \to f$ in the weak*-topology.
We know that $f_i \to f$ (weak*) $\Longleftrightarrow f_i(\sum_{j=1}^n a_j \otimes b_j) \to f(\sum_{j=1}^n a_j \otimes b_j)$ for all $a_1, \ldots, a_n \in A,\ b_1, \ldots, b_n \in B,\ n \in \mathbb{N}$.
Now clearly $f_i(\sum_{j=1}^n a_j \otimes b_j) \to f(\sum_{j=1}^n a_j \otimes b_j) \Rightarrow f_i(a \otimes b) \to f(a \otimes b)$ for all $a \in A,\ b \in B$.

For the converse, suppose $f_i(a \otimes b) \to f(a \otimes b)$ for all $a \in A$, $b \in B$.
Then

$$\|f_i(\sum_{j=1}^{n} a_j \otimes b_j) - f(\sum_{j=1}^{n} a_j \otimes b_j)\| = \|(f_i - f)(\sum_{j=1}^{n} a_j \otimes b_j)\|$$
$$= \|\sum_{j=1}^{n}(f_i - f)(a_j \otimes b_j)\|$$
$$\leq \sum_{j=1}^{n} \|(f_i - f)(a_j \otimes b_j)\| \to 0 \text{ as } i \to \infty.$$

Thus $f_i(\sum_{j=1}^{n} a_j \otimes b_j) \to f(\sum_{j=1}^{n} a_j \otimes b_j)$ for all $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$, $n \in \mathbf{N}$.
Hence

$$f_i \to f \text{ (weak}^*) \iff f_i(\sum_{j=1}^{n} a_j \otimes b_j) \to f(\sum_{j=1}^{n} a_j \otimes b_j) \text{ for all } a_i \in A,\ b_i \in B,\ n \in \mathbf{N}$$
$$\iff f_i(a \otimes b) \to f(a \otimes b) \text{ for all } a \in A,\ b \in B$$
$$\iff T_{f_i}(a)(b) \to T_f(a)(b) \text{ for all } a \in A,\ b \in B$$
$$\iff T_{f_i} \to T_f \text{ in the topology of simple weak}^*\text{-convergence.}$$

Having identified $S(A\tilde{\otimes}_\pi B)$ with the completely positive maps of norm 1 in $\mathcal{L}(A, B')$, we now consider $S(A\tilde{\otimes}_{max} B)$. The max norm is the greatest C^*-norm on $A \otimes B$, while the π-norm is the greatest cross-norm on $A \otimes B$. These two norms are of course generally unequal, so to see why their state spaces are equal we need to use the concept of the enveloping C^*-algebra of an involutive Banach algebra.

Let A be an involutive Banach algebra with (approximate) identity.
Let R be the set of representations of A, and define
$\|x\|' = sup_{\pi \in R}\{\|\pi(x)\|\}$ for all $x \in A$.
Then, since $\|\pi(x)\| \leq \|x\|$ for all $\pi \in R$, $x \in A$ [Takesaki, [6], I.5.2], $\|x\|' \leq \|x\|$ for all $x \in A$.
Moreover, $\|.\|'$ is clearly an algebra seminorm on A which satisfies $\|x^*\|' = \|x\|$ and $\|x^*x\|' = \|x\|'^2$ for all $x \in A$.
Let $I = \{x \in A : \|x\|' = 0\}$. Then I is a closed, self adjoint, two-sided ideal of A and so we may form the quotient A/I. Then $\|x + I\| = \|x\|'$ is a well-defined norm on A/I which is a C^*-norm.
Hence the completion B of A/I is a C^*-algebra. The canonical map of A into B is a norm-reducing *-homomorphism whose image is dense in B. We call B the enveloping C^*-algebra of A, denoted by $C^*(A)$.
Of course, if A is originally a C^*-algebra, then A may be identified with its enveloping C^*-algebra $C^*(A)$.

Even if A is not originally a C^*-algebra, the set of representations of A is in 1-1 correspondence with the set of representations of its enveloping C^*-algebra $C^*(A)$. Furthermore, there is a norm-preserving 1-1 correspondence between the continuous positive functionals on A and the positive functionals on $C^*(A)$. Hence there is a bijective correspondence between the states of A and the states of $C^*(A)$.
[For all the above details, see Dixmier, [2], 2.7]

Now let us consider $A\tilde{\otimes}_\pi B$, where A and B are C^*-algebras. We know that $A\tilde{\otimes}_\pi B$ is an involutive Banach algebra, but is not a C^*-algebra because π is not a C^*-norm. We therefore form its enveloping C^*-algebra $C^*(A\tilde{\otimes}_\pi B)$.
Let $x \in A \otimes B$. Then x is an element of $A\tilde{\otimes}_\pi B$, so $x + I \in C^*(A\tilde{\otimes}_\pi B)$.
Then $\|x + I\| = \|x\|' = \sup\{\|\pi(x)\| : \pi$ is a representation of $A \otimes B\} = \|x\|_{max}$ by definition of the max norm.
Thus $A \otimes B$, regarded as a subspace of $C^*(A\tilde{\otimes}_\pi B)$, is isometrically isomorphic to $A \otimes B$ regarded as a subspace of $A\tilde{\otimes}_{max}B$.
It follows that the completions $C^*(A\tilde{\otimes}_\pi B)$ and $A\tilde{\otimes}_{max}B$ are isometrically isomorphic.
Thus we may identify $A\tilde{\otimes}_{max}B$ with the enveloping C^*-algebra of $A\tilde{\otimes}_\pi B$.
Hence $S(A\tilde{\otimes}_{max}B) = S(C^*(A\tilde{\otimes}_\pi B)) = S(A\tilde{\otimes}_\pi B)$,
so $S(A\tilde{\otimes}_{max}B)$ is isometrically isomorphic (and, under the topologies given in Theorem 3.3 above, homeomorphic) to the set of completely positive maps of norm 1 in $\mathcal{L}(A, B')$.
We also have:

Theorem 3.4. *Let A and B be C^*-algebras. Let $\mathcal{L}^{cp}(A, B')$ denote the space of maps which can be written as differences of completely positive maps from A to B'.*
Then $Her(A\tilde{\otimes}_{max}B)'$ is isometrically isomorphic to $\mathcal{L}^{cp}(A, B')$ under a suitable norm.

Proof: Let $f \in Her(A\tilde{\otimes}_{max}B)'$. Then there exist unique positive maps p_1, p_2 on $A\tilde{\otimes}_{max}B$ such that $f = p_1 - p_2$ and $\|f\| = \|p_1\| + \|p_2\|$ [Takesaki, [6], III.2.1 and III.4.2]. Now the map $p \mapsto T_p$, where $T_p(a)(b) = p(a \otimes b)$, is an isometric isomorphism between positive maps on $A\tilde{\otimes}_{max}B$ and completely positive maps from A to B'. Under this correspondence, if $T_i = T_{p_i}$, $i = 1,2$, then f corresponds to $T = T_1 - T_2$, so $T \in \mathcal{L}^{cp}(A, B')$.
Conversely, if $T \in \mathcal{L}^{cp}(A, B')$, then $T = T_1 - T_2$, where T_1 and T_2 are completely positive maps from A to B'. Then, by the isomorphism, there exist positive maps p_1 and p_2 on $A\tilde{\otimes}_{max}B$ such that $T_i = T_{p_i}$, $i = 1,2$. Since the difference of two positive maps is a hermitian map, we have $f = p_1 - p_2 \in Her(A\tilde{\otimes}_{max}B)'$.
Since p_1 and p_2 are unique, so are T_1 and T_2, and thus $T = T_1 - T_2$ is the only way we can write T as the difference of two completely positive maps.
Hence we can define $\|T\|_{cp} = \|T_1\| + \|T_2\|$ for all $T \in \mathcal{L}^{cp}(A, B')$. This is well defined and it is clearly a norm on $\mathcal{L}^{cp}(A, B')$.

Since $\|p_i\| = \|T_i\|$, $i = 1, 2$, we have $\|T\|_{op} = \|T_1\| + \|T_2\| = \|p_1\| + \|p_2\| = \|f\|$, which gives the result.

We defined states as positive linear functionals with norm 1. However, the concept of a state can easily be defined without the necessity of the space having a norm defined on it. We define $S(A \otimes B)$ to be the set of all linear functionals p on $A \otimes B$ such that $p(x^*x) \geq 0$ for all $x \in A \otimes B$ and $p(1) = 1$.

We now show that $S(A\tilde{\otimes}_{max}B)$ may be identified with $S(A \otimes B)$.
Firstly, it is clear that if p is a state on $A\tilde{\otimes}_{max}B$, then the restriction of p to $A \otimes B$ is a state on $A \otimes B$. Conversely, if p is a state on $A \otimes B$, we must show that we can extend p to a state on $A\tilde{\otimes}_{max}B$. To do this, we will use the fact that to every state p on $A \otimes B$, we can associate a Hilbert space H_p, a representation π_p of $A \otimes B$ on H_p, and a cyclic (unit) vector ξ_p such that $p(x) = (\pi_p(x)\xi_p, \xi_p)$ for all $x \in A \otimes B$.

Now let $x \in A\tilde{\otimes}_{max}B$. Then $x = \lim_{i\to\infty} x_i$, $x_i \in A \otimes B$ for all i, and the limit is in the sense of the max norm. Thus $\|\pi(x_i - x_j)\| \to 0$ as $i, j \to \infty$ for all representations π of $A \otimes B$. We want to define p at x in the obvious way, so we need to show that $\lim_{i\to\infty} p(x_i)$ exists. Since $\mathbb{C}$ is complete, it suffices to show that $\{p(x_i)\}$ is a Cauchy sequence.
We have:

$$\begin{aligned}|p(x_i) - p(x_j)| = |p(x_i - x_j)| &= |(\pi_p(x_i - x_j)\xi_p, \xi_p)| \\ &\leq \|\pi_p(x_i - x_j)\xi_p\|\|\xi_p\| \\ &\leq \|\pi_p(x_i - x_j)\|\|\xi_p\|^2 \ \to 0 \text{ as } i, j \to \infty.\end{aligned}$$

Thus $p(x) = \lim_{i\to\infty} p(x_i)$ exists.
To prove that p is well defined, let y_i be another sequence in $A \otimes B$ such that $x = \lim_{i\to\infty} y_i$.
Then $|p(x_i) - p(y_i)| = |p(x_i - y_i)| \to |p(x - x)| = 0$, so p is well defined.
Hence p can be extended to a functional on $A\tilde{\otimes}_{max}B$ which is clearly a state on $A\tilde{\otimes}_{max}B$.
Hence $S(A\tilde{\otimes}_{max}B) = S(A \otimes B)$.

§4. An Alternative Approach to the State Spaces

Let A and B be C^*-algebras and consider $A \otimes B$ without any norm on it, for the moment. To each $f \in S(A \otimes B)$, there corresponds a Hilbert space H_f, a representation π_f of $A \otimes B$ on H_f, and a cyclic (unit) vector $\xi_f \in H_f$ such that $f(x) = (\pi_f(x)\xi_f, \xi_f)$ for all $x \in A \otimes B$. We will need to make use of the following relationship between f and π_f:

Lemma 4.1. $\|\pi_f(x)\| = sup_{y\in A\otimes B}\left(\dfrac{f(y^*x^*xy)}{f(y^*y)}\right)^{\frac{1}{2}}$.

Proof: By definition, $\|\pi_f(x)\| = sup\{\|\pi_f(x)\eta\| : \eta \in H_f,\ \|\eta\| = 1\}$.
Since π_f is cyclic, the subspace of H_f spanned by $\pi_f(A\otimes B)\xi_f$ is dense in H_f.
Hence $\|\pi_f(x)\| = sup\{\|\pi_f(x)\eta\| : \eta = \pi_f(y)\xi_f,\ y \in A\otimes B,\ \|\eta\| = 1\}$.
Now

$$\begin{aligned}
\|\pi_f(x)\eta\| &= (\pi_f(x)\eta, \pi_f(x)\eta)^{\frac{1}{2}} = (\pi_f(x)\pi_f(y)\xi_f, \pi_f(x)\pi_f(y)\xi_f)^{\frac{1}{2}}\\
&= (\pi_f(xy)\xi_f, \pi_f(xy)\xi_f)^{\frac{1}{2}}\\
&= \left(\frac{(\pi_f(xy)\xi_f, \pi_f(xy)\xi_f)}{(\pi_f(y)\xi_f, \pi_f(y)\xi_f)}\right)^{\frac{1}{2}} \quad (\text{since } (\pi_f(y)\xi_f, \pi_f(y)\xi_f)^{\frac{1}{2}} = (\eta,\eta)^{\frac{1}{2}} = 1)\\
&= \left(\frac{(\pi_f(xy)^*\pi_f(xy)\xi_f, \xi_f)}{(\pi_f(y)^*\pi_f(y)\xi_f, \xi_f)}\right)^{\frac{1}{2}} = \left(\frac{(\pi_f(y^*x^*xy)\xi_f, \xi_f)}{(\pi_f(y^*y)\xi_f, \xi_f)}\right)^{\frac{1}{2}}\\
&= \left(\frac{f(y^*x^*xy)}{f(y^*y)}\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence $\|\pi_f(x)\| = sup_{y\in A\otimes B}\left(\dfrac{f(y^*x^*xy)}{f(y^*y)}\right)^{\frac{1}{2}}$.

For $f \in S(A\otimes B)$, define $p_f(x) = \|\pi_f(x)\|$ for all $x \in A\otimes B$.
Then p_f is easily se[…] e a C^*-seminorm on $A\otimes B$.
Now let Γ be any subset of $S(A\otimes B)$. We want to show that sup{ […] $\in \Gamma\}$ is finite.

If $x \in A\otimes B$, then $x^*x \in (A\otimes B)_h$, and so, by Lemma 4.2 of Chapter 2, there exists $\alpha > 0$ such that $x^*x \le \alpha(1\otimes 1)$. Since, for all $y \in A\otimes B$, the map $w \mapsto y^*wy$ is clearly positive on $A\otimes B$, we get $y^*x^*xy \le \alpha y^*y$ for all $y \in A\otimes B$.
Thus, if f is a state on $A\otimes B$ and $x \in A\otimes B$, then $f(y^*x^*xy) \le \alpha f(y^*y)$ for all $y \in A\otimes B$.
Note that the constant α depends only on x, not on f.
So, if $f \in \Gamma$, then

$$\begin{aligned}
p_f(x) = \|\pi_f(x)\| &= sup_{y\in A\otimes B}\left(\frac{f(y^*x^*xy)}{f(y^*y)}\right)^{\frac{1}{2}}\\
&\le sup_{y\in A\otimes B}\left(\frac{\alpha f(y^*y)}{f(y^*y)}\right)^{\frac{1}{2}} = \alpha^{\frac{1}{2}},
\end{aligned}$$

so $p_f(x) \le \alpha^{\frac{1}{2}}$ for all $f \in \Gamma$.
Hence $sup\{p_f : f \in \Gamma\}$ exists, so we may define $p_\Gamma(x) = sup\{p_f(x) : f \in \Gamma\} = sup\{\|\pi_f(x)\| : f \in \Gamma\}$.
Then p_Γ is a C^*-seminorm because each p_f is. It is clear that p_Γ will be a C^*-norm if Γ is chosen such that, for each nonzero $x \in A\otimes B$, there exists $f \in \Gamma$ such that $\pi_f(x) \neq 0$.

In that case we call Γ a **separating subset** of $S(A \otimes B)$, and we write $A\tilde{\otimes}_\Gamma B$ for the C^*-algebra obtained by completing $A \otimes B$ with respect to p_Γ.

Thus separating subsets Γ of $S(A \otimes B)$ give rise to C^*-algebras $A\tilde{\otimes}_\Gamma B$.
Conversely, if α is a C^*-norm on $A \otimes B$, then it is true that $\alpha = p_\Gamma$ for some separating subset Γ of $S(A \otimes B)$. For we have seen before that if $A\tilde{\otimes}_\alpha B$ is a C^*-algebra, then $\|x\|_\alpha = sup\{\|\pi_\omega(x)\| : \omega \text{ is a state on } A\tilde{\otimes}_\alpha B\}$.
Since each such state ω on $A\tilde{\otimes}_\alpha B$ restricts to a state ω on $A \otimes B$, we may take Γ as the set of such restrictions of states on $A\tilde{\otimes}_\alpha B$. Then Γ is a separating subset of $S(A \otimes B)$ by Corollary 5.10 of Chapter 2, and $p_\Gamma(x) = sup\{\|\pi_f(x)\| : f \in \Gamma\} = sup\{\|\pi_\omega(x)\| : \omega \in S(A\tilde{\otimes}_\alpha B)\} = \|x\|_\alpha$.

Let Γ be a separating subset of $S(A \otimes B)$. If ω is a state on $A\tilde{\otimes}_\Gamma B$, its restriction to $A \otimes B$ is in $S(A \otimes B)$, so we obtain a bijective mapping from $S(A\tilde{\otimes}_\Gamma B)$ onto a subset of $S(A \otimes B)$ which we will denote by $S_\Gamma(A \otimes B)$. We will sometimes identify $S(A\tilde{\otimes}_\Gamma B)$ with $S_\Gamma(A \otimes B)$.
Let $f \in \Gamma$. Can we extend f to $A\tilde{\otimes}_\Gamma B$ so that the extension will be an element of $S(A\tilde{\otimes}_\Gamma B)$? If so, then by restricting the extension back to $A \otimes B$ we will obtain f again, thus showing that $f \in S_\Gamma(A \otimes B)$, i.e. that $\Gamma \subseteq S_\Gamma(A \otimes B)$.
So let $x \in A\tilde{\otimes}_\Gamma B$. Then $x = \lim_{i\to\infty} x_i$, $x_i \in A \otimes B$, the limit being with respect to p_Γ.
Thus $\|\pi_g(x_i - x_j)\| \to 0$ as $i, j \to \infty$ for all $g \in \Gamma$.
Hence $|f(x_i) - f(x_j)| = |(\pi_f(x_i - x_j)\xi_f, \xi_f)| \le \|\pi_f(x_i - x_j)\|\|\xi_f\|^2 \to 0$ as $i, j \to \infty$ because $f \in \Gamma$.
So $f(x) = \lim_{i\to\infty} f(x_i)$ exists. We have in fact repeated the argument used earlier to prove $S(A\tilde{\otimes}_{max} B) = S(A \otimes B)$, and, as in that case, it is easy to prove that the limit is unique and so f is well defined.
Thus we have extended f to a state on $A\tilde{\otimes}_\Gamma B$, so we have proved that $\Gamma \subseteq S_\Gamma(A \otimes \;\;)$

Now let $\Gamma = S(A \otimes B)$. Then Γ is clearly a separating subset of $S(A \otimes B)$.
Further, $p_\Gamma(x) = sup\{p_f(x) : f \in S(A \otimes B)\} = sup\{\|\pi_f(x)\| : f \in S(A \otimes B)\} = \|x\|_{max}$ for all $x \in A \otimes B$, so p_Γ is just the max norm. Since $\Gamma \subseteq S_\Gamma(A \otimes B)$, we have $S(A \otimes B) \subseteq S_{max}(A \otimes B)$. But $S_\Gamma(A \otimes B)$ is always a subset of $S(A \otimes B)$, so we have $S_{max}(A \otimes B) \subseteq S(A \otimes B)$, which yields $S(A \otimes B) = S_{max}(A \otimes B) = S(A\tilde{\otimes}_{max} B)$, as expected.

We continue by considering the case $\Gamma = (A^* \otimes B^*) \cap S(A \otimes B)$. Then Γ is a separating subset of $S(A \otimes B)$ since, if x is a nonzero element of $A \otimes B$, there exists $f \in (A^* \otimes B^*) \cap S(A \otimes B)$ such that $f(x) \neq 0$.
We have $p_\Gamma(x) = sup\{\|\pi_f(x)\| : f \in (A^* \otimes B^*) \cap S(A \otimes B)\}$, $x \in A \otimes B$.
Let $f \in (A^* \otimes B^*) \cap S(A \otimes B)$, $f = f_1 \otimes f_2$, where $f_1 \in A^*$, $f_2 \in B^*$. Then f_1 and f_2

are both positive, and $\pi_f = \pi_{f_1} \otimes \pi_{f_2}$ [Takesaki, [6], III.4.9].
So we have, for $x \in A \otimes B$,

$$\begin{aligned} p_\Gamma(x) &= \sup\{\|\pi_{f_1} \otimes \pi_{f_2}(x)\| : f_1 \in A^*,\ f_2 \in B^*,\ f_1 \text{ and } f_2 \text{ positive}\} \\ &= \sup\{\|\pi_1 \otimes \pi_2(x)\| : \pi_1,\ \pi_2 \text{ cyclic representations of } A,\ B \text{ respectively}\} \\ &= \|x\|_{min}, \end{aligned}$$

so p_Γ is precisely the min norm.

We will use the following result to determine $S_{min}(A \otimes B) = S(A \tilde{\otimes}_{min} B)$.

Lemma 4.2. *Let E be a C^*-algebra with identity, E_h the set of hermitian elements of E, E_+ the positive cone of E, $S(E)$ the set of states of E, and Q a subset of $S(E)$. Suppose that if $x \in E_h$ satisfies $f(x) \geq 0$ for each $f \in Q$, then $x \in E_+$. Then the weak*-closed convex hull of Q is $S(E)$.*

Proof: See Dixmier, [2], 3.4.1

We will apply this result with $E = (A \tilde{\otimes}_{min} B)$ and $Q = \Gamma = (A^* \otimes B^*) \cap S(A \otimes B)$. We first prove:

Lemma 4.3. *If $f \in \Gamma$ and $y \in A \otimes B$, then there is some $g \in \Gamma$ such that $f(y^*xy) = f(y^*y)g(x)$ for all $x \in A \otimes B$.*

Proof: Let $f \in \Gamma$ and $y \in A \otimes B$. If $f(y^*y) \neq 0$, then define g by

$$g(x) = \frac{f(y^*xy)}{f(y^*y)} \quad \text{for all } x \in A \otimes B.$$

Then $g(1) = 1$, so $g \in (A^* \otimes B^*) \cap S(A \otimes B)$, and $f(y^*xy) = f(y^*y)g(x)$ for all $x \in A \otimes B$.
If $f(y^*y) = 0$, the Cauchy-Scharwz inequality yields, for any $x \in A \otimes B$,
$|f(y^*xy)|^2 = |f((x^*y)^*y)|^2 \leq f((x^*y)^*x^*y)f(y^*y) = 0$, so $f(y^*xy) = 0$,
and $f(y^*xy) = f(y^*y)g(x)$ is satisfied by any $g \in \Gamma$.

Theorem 4.4. *The weak*-closed convex hull of Γ is $S(A \tilde{\otimes}_{min} B)$.*

Proof: Suppose that x is a hermitian element of $(A \tilde{\otimes}_{min} B)$ such that $g(x) \geq 0$ for all $g \in \Gamma = (A^* \otimes B^*) \cap S(A \otimes B)$. We shall prove that x is positive, thus allowing us to apply Lemma 4.2.
Let $f \in \Gamma$, and let π_f be the usual cyclic representation of $A \tilde{\otimes}_{min} B$ on a Hilbert space H_f with cyclic vector ξ_f. We know that $\pi_f(x) \geq 0$ if and only if $(\pi_f(x)\xi, \xi) \geq 0$ for all $\xi \in H_f$ [Dixmier, [2], 1.6.7]. Since $\pi_f(A \otimes B)\xi_f$ is dense in H_f, we get $\pi_f(x) \geq 0$ if and

only if $(\pi_f(x)\pi_f(y)\xi_f, \pi_f(y)\xi_f) \geq 0$ for all $y \in A \otimes B$.
But for all $y \in A \otimes B$ we get, using Lemma 4.3,

$$(\pi_f(x)\pi_f(y)\xi_f, \pi_f(y)\xi_f) = (\pi_f(y)^*\pi_f(x)\pi_f(y)\xi_f, \xi_f) = (\pi_f(y^*xy)\xi_f, \xi_f)$$
$$= f(y^*xy) = f(y^*y)g(x) \geq 0 \quad \text{because } y \in \Gamma, \text{ so } g(x) \geq 0.$$

Thus $\pi_f(x) \geq 0$ for all $f \in \Gamma$. The direct sum of all representations π_f, $f \in \Gamma$, is an isometric isomorphism of $A\tilde{\otimes}_{min}B$ [Takesaki, [6], I.9.18]. But if π is an isometric representation of a C^*-algebra A, then $\pi(a) \geq 0 \Rightarrow a \geq 0$ [Dixmier, [2], 2.6.2]. Thus we must have $x \geq 0$.
Hence, by Lemma 4.2, the weak*-closed convex hull of Γ is $S(A\tilde{\otimes}_{min}B)$.

Corollary 4.5. *$S(A\tilde{\otimes}_{min}B)$ is the weak*-closure of $(A^* \otimes B^*) \cap S(A \otimes B)$.*

Proof: An elementary calculation shows that $(A^* \otimes B^*) \cap S(A \otimes B)$ is convex. The result then follows from the previous theorem.

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