EXPLORING AND DESCRIBING THE GROWTH POINTS OF LEARNERS AS THEY ENCOUNTER FUNCTIONS IN EQUATION FORM

Submitted by

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DECLARATION

I declare that this research report is my own work and no part of it has been copied from another source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list.

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Abstract

This research report confirms the value of the Framework of Growth Points in a learner's mathematical development in the area of functions in equation form. The study also shows that learners advance through the various growth points in a progressive, sequential fashion, which mirrors the results of Ronda's study, on which a part of this study was based. The study was carried out in a high school in Johannesburg. Learners in Grades 9, 10 and 11 were required to do an assessment which tested for their achievement in different growth points. This study also explores the discourse of learners while they talked about the tasks in the assessment. A smaller sample of learners was interviewed so that the researcher could explore the nature of their discourse. This research report shows that there are patterns in the discourse of learners which can be related to the growth points that they achieve.

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Chapter 1 **Introduction**

1.1 Introduction

Learning trajectories, especially in mathematics, is an area of research that focuses on learners' progression in their thinking about concepts. When teaching functions, mathematics teachers often know the outcomes they are expected to achieve, but do not necessarily know how to get there.

South African teachers use the National Curriculum Statement (NCS) to inform their teaching, which has been provided by the Department of Education (DOE). Soon all South African teachers will use the Curriculum and Assessment Policy Statement (CAPS), which is currently being phased into schools, year by year.

In 2013 the last cohort of matric learners wrote the NCS exams, and 2014 will see the first cohort of matric learners writing their exit exams according to the CAPS curriculum. The new CAPS curriculum does not differ vastly in terms of mathematical content, but is more prescriptive in the *timing* of what is to be taught and when.

Both the NCS and CAPS provide statements as to the outcomes that learners are expected to reach in each learning area, but seldom do these curriculum documents map a *path of development* that guides teachers in their teaching of specific concepts. There is no mention of strategies or thinking that learners use in order to achieve the required outcomes. This leaves teachers free to figure out the best way to achieve these outcomes (Daro, Mosher & Corcoran, 2011). The trade-off for autonomy in reaching these outcomes may be uncertainty on the part of the teachers, as they do not necessarily have the time or skill to devise the best paths to these outcomes.

To illustrate, in Appendix A, both the NCS and CAPS statements for "Functions" are shown for Grade 9. These curriculum statements show what outcomes are expected of a learner, but there is no clear learning path as to how learners may develop these concepts. CAPS is slightly more prescriptive than the NCS, but instead of showing progression through concepts in terms of learning trajectories, it is prescriptive on what outcomes to teach and assess, and when. Textbook designers have therefore become responsible for creating lessons and tasks that correspond with speculated or implied trajectories of learning. However, there is little evidence to suggest that textbook writers are following learning trajectories that have been explicitly researched or tested.

An increase of publications and articles about learning trajectories suggests that there is renewed interest in this area. Wilson, Mojica and Confrey (2013) suggest that learning trajectories are also very useful in helping teachers to understand how learners progress in their mathematical thinking, a process which may inform and improve teaching practices.

There are some theoretical frameworks that propose to explain the process of the learning of functions, but few of these have been developed in conjunction with research, and remain rather theoretical and general (e.g. DeMarois & Tall, 1996; Slavit, 1994; in Ronda, 2004). "In spite of the proliferation of educational studies on function, not much is known about the process of learning. Only a few researchers have tried to follow this process as it actually happens in the classroom." (Walter & Gerson, 2000; Yerushalmy, 2006; in Nachlieli & Tabach, 2012). This demonstrates that there is a need for a research-based model that can reveal how learners develop their understanding of mathematical concepts – and in particular, functions. The upshot of such research will be to provide educators with more explicit guidelines for their teaching.

Ronda (2004) has created such an empirically-based conceptual framework. It describes learners' development of their understanding of functions, and refers to this path of development as learning trajectories. This *Framework of Growth Points* maps out the 'big ideas' that learners typically experience on their path to understanding functions (Ronda, 2004; Ronda, 2009:31). These growth points are in approximately the order that they are expected to encounter these ideas. Ronda (2009) suggests that most learners follow a similar learning path or trajectory, reaching a set of defined growth points as they encounter functions. This research report uses Ronda's (2004) study into growth points and their description as its basis.

Algebra is claimed to be "... a great explainer" by Long, De Temple and Millman (2008: p. 36). Algebra is a very broad area of mathematics, and even when confined to the limitations of high school mathematics there are many perspectives as to what constitutes elementary algebra. In its simplest form, algebra can be seen as a generalised form of arithmetic, a way to solve problems, the study of relationships between numbers, and finding an unknown.

Introductory high school[1] algebra sets the foundation for higher level algebra, any aspects of calculus, and indeed tertiary study of mathematics. Some research, for example the CSMS study, suggests that the learning of algebra is not an easy task (Hart et al, 1981), and Watson (2009) has pointed out that algebra continues to be an area which causes dismay for many learners. The successful teaching of algebra at a high school level is seen to be important, because not only is it a major part of the mathematics syllabus, but the success in mathematics is also seen to be the gateway to many forms of tertiary education. Algebra and functions are important topics in school mathematics, and indeed in the critical thinking processes that are an aimed outcome of the process of schooling. Arcavi argues that:

> "*thus, a knowledge of mathematics, and particularly a knowledge of algebra, is crucial for, among other things, the inspection, understanding, and development of a critical appraisal of the large amounts of information and arguments with which we are confronted at all times"* (Arcavi, 2008: p. 37).

While there are many opinions on what algebra is, I will mainly be using Caspi and Sfard's (2012) definition of algebra as a discourse as a basis to my study. Caspi and Sfard (2012) categorise elementary algebra into two further meta-arithmetic discourses – *solving equations*, and *generalisation* – and these explain the use of symbolic representation which is at the heart of elementary algebra.

Solving equations asks questions about unknown quantities, where a calculation is used to find an unknown. *Generalisations* look at number patterns, and, with the help of symbolic representations (usually called variables), shows that these number patterns can be represented in the form of an equation. These can also be seen as functions, and can be represented in other ways too, for example by table, graph or diagram. At school level there is much foundation work in algebra which leads to these two categories, such as the ways in which variables and exponents are used. Function is seen as a mathematical object, but learners do not necessarily see it this way when they are first introduced to the concept of

[1] In South Africa, high school is from Grade 8 to Grade 12 – the formal schooling exit point. High School is the same as secondary school, and these terms are used interchangeably.

function (Ronda, 2009; Nachlieli and Tabach, 2012). Hence the paradoxical situation occurs as learners struggle with talking about a concept which is not necessarily yet well-defined for them (Nachlieli and Tabach, 2012).

This study also explores the way in which mathematics, algebra, and specifically functions can be expressed in terms of discourse. Sfard's communicational framework provides both the theory and the tools to explore discourse in the mathematics classroom (Sfard, 2008).

Mathematical discourse is seen as a human activity, unlike language, which is a set of passive tools (Caspi and Sfard, 2012; Lemke, 1990). Human activity evolves, which is mirrored by a change in a learner's discourse when encountering new algebraic discourses. The development in algebraic discourse can be seen to progress through levels that have been described by Caspi and Sfard (2012) as "canonic". This means that there is a hierarchy of levels of algebraic discourse, where each level of discourse builds on the previous level, and is hence more complex than the previous level (a meta-discourse of the previous level). Caspi and Sfard (2012) state that "transition from one level to another can be seen as developmental milestones", and hence this can possibly be seen to link with meeting Ronda's growth points on a learning trajectory.

Sfard suggests that *thinking is a form of communication* and that learning a subject such as mathematics is *modifying and extending one's discourse* (2007). Because, as many other authors have also shown (E.g. Caspi and Sfard, 2012; Ryve, 2011) discourse relates closely to the process of learning, I have investigated the link between the development of learners' discourses and the growth points that Ronda has proposed in her framework of growth points.

1.2 Aims of the study

The aim of this study was to use Ronda's Framework of Growth Points to investigate the learning of functions in equation form, in the South African context. This study also aimed to investigate learners' discourses in relation to the growth points that have already been achieved.

1.3 Research questions

1. Using Ronda's Framework of Growth Points, where do selected South African learners fit in - especially in relation to functions in equation form?
2. How do learners' discourses relate to the growth points they have achieved?

1.4 Background and rationale

Mathematics is important to learners as it not only fosters a questioning approach to life, but opens doors to many areas of tertiary study (Arcavi, 2008). The study of algebra contributes a large portion of the current mathematical syllabus, and is therefore relevant to investigate.

During my time as a mathematics teacher, I have observed that algebra is, more often than not, a struggle for many learners. This observation is backed by many large-scale international mathematics studies such as the CSMS study (Kuchemann; 1981) and the TIMMS study. National mathematical benchmarking tests, such as the Annual National Assessment (ANA), also support this claim. It was shown that Grade 9 learners achieved a pass rate of 13% in the 2012 test, the pass mark being set at 40%. A large part of this (approximately 40%) of the paper is algebra, or relies on algebraic knowledge. This points to difficulties in the learning of all areas of mathematics, including algebra. The 2013 ANA results did not differ much, with the pass rate being only slightly higher and only 2% of learners getting over 50% for this assessment.

Functions are not an easy concept to grasp, especially considering their abstract nature. To understand function is to understand it in its various representations, but one should not confuse it with the representations themselves. "A function is a relationship of dependency between variables … it is the *relationship* that is the function, not a particular representation of it." (Watson, 2009: 30, own emphasis) Watson's statement shows that functions should be seen as a relationship, but that this relationship manifests in different representations.

The Department of Education highlights the importance of being able to represent functions in different ways. “The mathematical models of situations may be represented in different ways – in words, as a table of values, as a graph, or as a computational procedure (formula or expression). (Department Of Education, 2003)

As seen in Appendix A, the first outcome stated in the NSC shows that the learner has achieved the outcome if the learner “draws graphs on the Cartesian plane for given equations (in two variables), or determines equations of formulae from given graphs using tables where necessary”. (DOE, 2003)

I have noticed that often the first encounter of functions (in textbooks) is in equation form. Equations are introduced as the relation between two variables which then dictate how ordered pairs can be found (usually represented in a table).

The equation form of a linear function, which is represented in the standard form of $y = mx + c$ has both process and object properties – which will be further discussed in the literature review. The process of substitution of values into this equation creates a set of ordered pairs, which can be plotted onto the Cartesian plane. However, the equation can also be seen as an object when learners can interpret the invariant properties of the function it is representing (Ronda 2009).

My own observations in the mathematics classroom have led me to believe that language plays an important role in the learning of mathematics. Indeed, there are many who have suggested that language is indeed important in the process of mathematics teaching (Sfard, 2008; Arcavi, 2008; Lemke, 1990; etc).

Lemke (1990, p. 12) reiterates that “classroom language is not just a list of technical terms, or even just a recital of definitions. It is the use of those terms in relation to one another, across a wide variety of contexts”. In this research report, I will use the word ‘discourse’ to refer to the human activity of using language.

The development of gaining mathematical skills and knowledge is a function of increasing one’s prowess in the discourse associated with mathematics and its subsidiaries. One conceivable reason for the low achievement in mathematics is the struggle with the use of mathematical discourse, and this discourse is needed for the successful study of high school mathematics.

When looking at the big picture of education in South Africa, one realises that we as a nation are in a somewhat dismal state of educational disrepair. South Africa has a unique history laden with difficulties, especially in education, and our history has had and continues to have an effect on education.

One of the ways in which education for the majority of the population was thwarted, was in the field of language. Under apartheid, the majority of South African learners were taught in a language which was not their first language. Social inequalities have been lessened over time, but the majority of teaching (at a secondary level) still happens in English, which is still not the mother tongue of the majority of learners in South Africa. This is another pertinent reason to be studying discourse in a mathematics classroom, especially a classroom in which not all learners many have English as their first language[2]. Good teaching practices are learnt through research. If research goes into informing good teaching practices, the country's education system stands to benefit (Taylor, personal communication: 2011).

As an educator and education researcher, it will be beneficial to understand the paths learners take when they learn about functions, and the discourses that are associated with the learning of functions.

Outline of the research report

This first chapter has given an introduction to the research report, and also explains the rationale for the research. It has given the context of the research and explains why this research will be helpful in the South African educational system.

Chapter Two gives an outline of the literature that was considered to be important to this study. The areas of algebra and functions are explored, and special consideration is given to functions in equation form. The two theoretical frameworks are also explained in this chapter. Ronda's Framework of Growth Points gives the outline to the theoretical

[2] First language here is the same as mother tongue, or what some have called "main language".

framework which will be used in answering the first research question. Sfard's communicational framework is also considered, along with additional input from Nachlieli and Tabach (2012) and Ben-Yehuda et al (2005).

The third chapter elaborates on how the research was conducted. Methodology, reliability and validity, sampling, data collection and analysis as well as ethical considerations are discussed.

Chapter Four discusses the analysis of the data, which was done in order to answer the first research question. This part of the study showed that South African learners could be compared with the learners from Ronda's study in terms of their achievement of growth points. Differences and similarities of the two sets of data are also discussed.

Chapter Five elaborates on the findings of the second part of the study, which studied the discourse used by learners when talking about functions in equation form. This part of the study uses Sfard's communicational framework to analyse learners' discourses.

Chapter Six offers a summary of the findings, recommendations for further research and then concludes the study.

Chapter 2 **Literature Review**

In this literature review, I will discuss the area of school algebra, with particular reference to functions, and the multiple representations thereof. I will subsequently discuss the dual nature of mathematical objects, which explains that algebraic objects can be seen to have properties which are different, yet compatible. I will explore learning trajectories and how they can inform teaching practice. I will then consider mathematics as a discourse, and specifically algebraic discourse. Finally, I will look at the teaching of algebra in South Africa.

2.1 Algebra

According to Watson (2009), algebra is the way we express generalisations about numbers, quantities, relations and functions. Algebra is seen as a means for the manipulation of symbols in order to solve complex problems (Kieran, 2007). The importance of seeing the link between arithmetic and algebra is highlighted by many, for example, noticing that algebra is a generalisation of arithmetic (e.g.: Banerjee & Subramaniam, 2012).

This is a reflection on Caspi and Sfard's (2012, p. 45) work, which states that secondary school algebra is a "meta-discourse of arithmetic". Algebra is seen to be a type of meta-arithmetic, through the formalisation of numeric patterns. Instruction on function can also be, and usually is, approached in a similar manner, where learners explore patterns, and hence generate rules in the form of an equation or formula.

The discourse on functions subsumes those discourses on algebraic expressions and graphs (A. Sfard, Personal communication, 5 September 2012). Caspi and Sfard (2012) categorise school algebra into two broad categories, based on earlier work by Sfard and Linchevski (1994). The first category – *constant value algebra* – concerns algebra where values, either known or unknown, are fixed and do not change. Within constant value algebra, Caspi and Sfard (2012) identify two sub-categories; *solving equations* refers to the process of finding an unknown variable in an equation, while *generalisation* refers to simplifying patterns or algebraic expressions. The second category – *variable value algebra* – describes processes of change or movement. This type of algebra may be represented by graphs or tables and hence results in a new mathematical object – functions (Caspi and Sfard, 2012).

2.2 Functions

An early definition of functions from Euler states that; "One says one quantity is a function of another whenever the first quantity depends on the other in such a way that if the latter is changed then the former undergoes changes itself." (Euler, 1755, in Sfard, personal communication, 2012) Over time the definition of a function has evolved and today's formal definition of a function is as follows.

The subset F of AxB is a function iff for every $x \in A$ and $y_1, y_2 \in B$,

if (x, y1) , (x , y2) $\in$ F then $y_1 = y_2$

This formal notion of a function, however, is not the way in which functions are introduced to the first-time learner. In South African schools, learners are typically introduced to formal algebra in Grade 8, although some groundwork is usually laid at primary school level in the form of recognising patterns and filling in a missing "space" to ensure an equation is equivalent. There are some schools which introduce the notion of a variable at Grade 7 level, but these are not the norm.

High school learners are introduced to the idea of variables, and build skills which involve expressions and solving for one unknown (equations). These ideas and skills become building blocks for the notion of a function as a relationship between more than one variable. The notion of equivalence is introduced when looking at equations with one variable, where learners typically have to solve for an unknown variable. However, functions – at high school level - show the relationship between two variables. "A function is a relationship of dependency between variables … it is the relationship that is the function, not a particular representation of it." (Watson, 2009, p. 30).

Anderson, in Greenes & Rubenstein (2008), states that since functions have many real world applications, it is important that learners learn to do more than *just* manipulate the equation. Learners must develop proficiency using an equation to represent relationships among variables described in a mathematics problem. The nature of an algebraic function is such that its relationship can be represented in many ways (Kieran, 2007).

The multiple representations characterise functions. However, learners do not always grasp the connections between each representation. Learners may also view the representation as a function itself. Van Dyke and Craine state this of learners:

> "We want them to realize that a function can be represented by a graph, by a table of values, and, in many of the instances they will consider, by an algebraic expression. We need to make learners see that a change in algebraic form does not necessarily mean a change in the relation that is represented but rather that an equivalent form may make a certain property of the relation become more apparent." (Van Dyke and Craine; in Moses, 1999, p. 215)

Seeing a function in many different ways can be helpful. As Even points out, "different representations give different insights which allow better, deeper, more powerful and more complete understanding of a concept" (Even, 1990, p. 524). Being able to select, use, move between and compare representations is a crucial mathematical skill (Even, 1998, in Kieran, 2007). The understanding of a function as a whole, with its many representations, instead of just as a process is addressed in the next section entitled process/object duality.

Wolloughby (in Moses, 1999: p. 197) suggests that "High school and College learners often have trouble with the function concept because of the abrupt and abstract way in which it is introduced." The difficulty with introducing complex concepts very quickly is that learners take some time to become familiar with the new discourse, which may be associated with functions. As shown in this study, the understanding of functions is an ongoing process and the journey to understanding functions as an object is not an easy one.

2.3 Functions in equation form

In my experience as a teacher, functions are almost always encountered in equation form first. This experience corresponds to the ordering of the introduction of concepts in the curriculum statements (both NCS and CAPS). The equation provides the rule which represents the relationship between two variables. Function, when represented as an equation, can lend itself well to both process and object conceptions. For example, a linear function, in the form $y = mx + c$ can be used to generate a set of co-ordinates, as well as show properties such as the gradient (represented by m), as well as the intercept (c).

Although function in equation form lends itself to be seen as both a process and an object, it is important to note that the equation is not the function itself – it is a representation. The importance of the ability to link the function with its other representations is highlighted by Ronda (2009), as all representations have their own strengths and weaknesses.

Functions in equation form are not without their own difficulties. Many learners first experience equations as a way to solve for one unknown value, in a "solving equations" sense. Learners are generally accustomed to seeing equations as containing a single unknown quantity. When functions are introduced, equations then become associated with the "generalisations" type of algebra, as an equation is thus a representation for a relationship between two variables (Ronda, 2009). The equation is now a representation of an object – function – instead of a statement of condition. Learners also have to navigate the equals sign, which in earlier years of their schooling denoted a "do something" signal, whereas with functions the equals sign shows a relationship of equality (Ronda, 2009).

2.4 Process-object duality

Sfard and Linchevski (1994) introduce the idea that mathematical objects have a dual nature, which explains that functions can be seen in more than only one way. *Reification* is the process whereby a learner is able to understand that the result of a mathematical process is indeed a mathematical object (Sfard, 1991; Sfard, 1992; Sfard & Linchevski, 1994).

Sfard's (cf. 1994) earlier work speaks about the *dual nature of algebraic objects* – and points to the fact that there can be two ways of looking at functions. Functions can be seen as having both process and object conceptions, and these are different, yet compatible. Even though this is a difficult notion to grasp, it can be likened to Bohr's concept of the nature of physical entities. Properties of light can be – and need to be – understood through both particle and wave theory (Sfard and Linchevski, 1994). Similarly, functions need to be understood both as both a process and an object.

Sfard (1991) points out that, when encountering and acquiring a new mathematical concept, learners will see the concept as an operational conception before they graduate to having a structural conception (object conception). A function, when seen as a process (or as an operational conception), is interpreted in such a way that the person seeing the equation would be inclined to 'do something' with the expression. Performing computations in a procedural way is what many learners will be inclined to do when seeing an equation.

In a typical classroom, this process may be seen as what teachers often tell learners to do when faced with the equation form of a function: "generate a set of co-ordinates by using your equation". This shows a one-dimensional approach to functions in equation form (Sfard and Linchevski, 1994). In the case of a standard linear equation $y = mx + c$, a learner may look at the equation purely as a procedure, in which case they might substitute values, in the position x, to generate co-ordinates. This shows that learners are approaching the problem from a point where they want to get an answer, but do not necessarily show understanding of the equivalence shown by the equation. Learners have learnt the procedure of generating a set of co-ordinates – in a point-wise manner, but have not grasped the relationship represented by the equation (Ronda, 2004).

Reification occurs when a learner compresses a series of processes into one object, and therefore is able to see a lengthy string of processes as one thing. This one new phenomenological object can subsequently be used as a basis for new procedures, which take place on a higher level (Sfard and Linchevski, 1994). An example of this happening can be seen where a learner shifts their thinking about functions in equation form. Instead of seeing the function $2y + 3 = 4x$ as something which should be operated on, the learner shifts their thinking and is able to see the equation holistically, as a *relationship of equivalence* instead as a *string of processes*. It is also important to note that the process of reification would not be possible if the learner did not understand the function as a process in the first place (Sfard and Linchevski, 1994; Sfard, 1991). Sfard's newer work calls this shift objectification (cf. Caspi & Sfard, 2012).

While Sfard and Linchevski (1994) state that reification comes about when learners discover the link between the process and the object, the 'object' in this case signifies a structural understanding. A structural understanding comes about when one understands the object as both a part and a whole – meaning that the object is seen as a whole, but is also linked to the parts which constitute the whole. Sfard and Linchevski, (2004) maintain that reification needs to take place in order to understand the full meaning of an object.

The above shows issues of learning about function, but does not indicate how learners progress as they learn, or what stages are met through the learning process. Hence there is a need to investigate trajectories of learning in mathematics, especially function.

2.5 Learning Trajectories

The idea of learning trajectories was first put forward by Simon (1995, in Simon and Tzur, 2004) when he constructed a *hypothetical learning trajectory.* This was an idea which was used to inform the planning of lessons which included the outcomes of the lesson, which tasks were to be used, and hypothesised about the process of learners learning (Simon, 1995; in Simon and Tzur, 2004). Through research and further development on the learning processes in mathematics, this idea has shifted from the hypothetical to an empirically-developed idea. The learning trajectories of some areas have been suggested, especially in primary school mathematics, as shown in a Consortium for Policy Research in Education report on learning trajectories in Mathematics (Daro et al, 2011). Actual learning trajectories may inform progression through key concepts or levels of thinking. Confrey et al give a broad explanation of learning trajectories:

> *"[Learning trajectories are] a researcher-conjectured, empirically-supported description of the ordered network of constructs a learner encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation and reflection, towards increasingly complex concepts over time."* (Confrey et al, 2009, p. 347, in Daro et al, 2011)

The area of learning trajectories in many studies is rather broad, but has also been narrowed to the area of mathematics by some researchers. This is described by Clements and Sarama (2004) as:

> *"descriptions of children's thinking and learning in a specific mathematical domain, and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain".*
> (Clements & Sarama, 2004, p. 83)

Wilson, Mojica and Confrey (2012) suggest that learning trajectories can help teachers in their teaching practice as they inform about learners' progression through key concepts.

Ronda (2004) has created a framework of growth points for the learning of functions which uses the idea of learning trajectories. The framework of growth points shows learning trajectories through meeting key growth points – which has been developed in conjunction with extensive empirical research. Her framework aims to articulate the way in which learners' progress in their understanding of functions, and the route that they take to move to the ultimate goal of an objectified understanding of functions.

2.6 Framework of growth points

Ronda's framework has been created after doing empirical research, and is concerned with the 'big ideas' that learners encounter while learning about algebraic functions, in particular linear and quadratic functions. These big ideas are called *growth points*, and serve as 'checkpoints' or 'milestones' to identify the learning paths of learners. The framework is based on research which shows the typical path many learners take when learning functions.

Figure 1 below shows the entire framework of growth points – over four domains (Ronda, 2004). I will be working specifically with the domain of *Equations,* which is shown in the far left column.

The Framework of Growth Points in Students' Developing Understanding of Function

Equations	Graphs	Linking Representations	Equivalent Relationships
GP 1: Equations as procedures for generating values	GP 1: Interpretations based on individual points	GP 1: Linking equations and tables	GP 1: Representations show the same relationship if they have the same set of values
GP 2: Interpretations based on relationships	GP 2a: Interpretations based on rates	GP 2: Point-by-point linking of graphs with other representations	
GP 3: Interpretations based on local properties	GP 2b: Interpretations based on continuous property	GP 3: Linking representations by trends/patterns or some properties	GP 2: Representations show the same relationship if they share some properties
GP 4: Manipulation and transformation of equation seen as objects	GP 3: Interpretations based on point-wise and holistic analysis of relationships	GP 4: Linking by invariant properties	GP 3: Representations show the same relationship if they have the same invariant properties
		GP 5: Linking representations seen as objects	

Figure 2-1 The Framework of Growth Points

Ronda (2009) describes four growth points which can be associated with functions in equation form. The description of these growth points has subsequently been refined from the above table, taken from Ronda (2004).

Growth Point 1: Equations as procedures for generating values
A learner is coded at growth point one when they are able to see an equation as a means to generate values. This growth point corresponds to Sfard's (1991) procedural conception of a function. In Ronda's study, most learners who were coded at this growth point showed a preference for solving problems by point-by-point analysis, even though it is time consuming.

Growth Point 2: *Equations are representations of relationships*
Learners coded at this growth point were able to start investigating the relationship between variables in an equation. Here learners are aware of that "equations are a statement of relationship between the varying quantities". (Ronda, 2009, p. 43)

Growth Point 3: *Equations describe properties of relationships*
Learners who were coded at growth point 3 were able to recognise, describe and interpret the properties of the function given in equation form. Properties include the gradient and the y-intercept.

Growth Point 4: *Functions are objects that can be manipulated and transformed*
Learners who were coded at this growth point were able to conceive an equation as a culmination of the previous growth points, as well as it being a mathematical object. This means that learners could perform an operation on the equation as a whole. This *holistic* notion of a function means that learners are able to see the function as an object and not just merely as a collection of co-ordinates. This growth point is consistent with Sfard's notion of having an object conception of a function.

The order in which these growth points are presented, are seen to be the order in which most learners are typically seen to reach the growth points. These four growth points can be used as the basis in the investigation of the trajectory of a typical learning path of learners encountering functions. Ronda's framework has been piloted and refined in studies in Australia and the Philippines, but has not yet been used extensively. This study investigates the use of the framework in a different educational context, and so can serve to confirm the potential generality of the framework in describing learners' learning trajectories.

Ronda (2004) was able to code learners at these growth points by way of an assessment. However, it may not always be practical for a busy teacher to assess learners constantly using a written assessment. Due to this, it may be possible to link up growth points with a learner's discourse. Hence my interest in investigating how learners' discourse relates to the growth points that they have achieved, and the dual focus of my study.

2.7 Mathematical discourse and the Communicational Framework[3]

As a precursor to discussing mathematical discourse, the clarification of acquisitionist and participationist theories of learning should be briefly discussed. In the past, Sfard (1998) talks about two metaphors – Acquisitionism and Participationism – and how they can be used to describe the process of learning. She highlights the mistake of using only one of these two metaphors to explain the process of learning.

Recently, however, Sfard has changed her mind, and is now more in favour of the Participationist view of learning. In her paper, Sfard (2006) stresses the movement from the Acquisitionist view of learning, towards a Participationist view of learning. She deems the Acquisitionist view of learning unsuitable to wholly define thinking, because it does not have the adequate complex theoretical structures to deal with the fine nuances of learning that we are able to see today, with the help of advanced technological tools (Sfard, 2006).

The commognitive approach is based on the view that the theory of Acquisition no longer suffices to adequately explain the process of learning (Sfard, 2007). Participationism, however, is able to adequately explain that learning take place as a result of the individualisation of patterned collective human activity (Lave 1993, Wenger, 1998, in Sfard, 2007).

"The study of discourse is the study of human communication; the most unique of this communication is language in use," writes Ryve (2011, p. 169). Sfard explains thinking as "an individualised form of (interpersonal) communication", and together, different types of

[3] Sfard's refers to her framework as the Commognitive framework or the Communicational Framework in different pieces of her writing. I will use these two terms interchangeably. The word "Commognition" comes from a combination of the words *Communication* and *Cognition*, which indicate that these two are different expressions of the same phenomenon.

communication are said to be discourses (Sfard, 2008, p. 81). Sfard (2008) goes on to say that, within the commognitive framework, learning may then be defined as individualising discourse.

A specialised discourse is needed to fully explain abstract mathematical concepts encountered during the learning of mathematics. Sfard (2007) defines mathematics as being a specific type of discourse. This specific discourse can be characterised by its specialised objects (technical register, visual representations), its rules and its mediators. The *communicational* view of learning suggests that discourses can change, and that learners need to keep up with these changes in discourse (Sfard, 2007). Mathematics learning, then, is the act of communication within the mathematical discourse (Sfard, 2006).

Becoming fluent in the discourse of mathematics is not an easy process. As Sfard (2007) suggests, learning mathematics is synonymous to becoming fluent in the discourse which is specific to mathematics. Caspi and Sfard (2012) elaborate on the subject of mathematical discourse by stating that algebra is a sub-category of mathematical discourse.

To answer my second critical question, I will be using Sfard's communicational approach to explain the process of learning mathematics. It is an interpretive framework which aims to make sense of discourse in the classroom (Sfard, 2007, 2008).

The communicational view of learning emphasises that mathematical thinking is a process whereby communication in social situations is individualised. Self-communication, which does not necessarily have to be in words, is therefore the product of internalisation of communication in a social environment (Sfard, 2007, 2008). Discourse is "recognisable by four characteristics, the first three of which are its specialised vocabulary, visual mediators and routines. All these, if applied properly, result in narratives that the [research] community endorses and regards as facts" (Sfard 2013, p.140). The four characteristics of discourse are discussed below.

Mathematical words: these are the specific technical words which are used in discourse (Sfard, 2008). Word use is very important in that the use of the word constitutes its meaning (Wittgenstein, in Sfard, 2007, p. 571). In functions, some examples of these mathematical words may be "parabola", "turning point" and "gradient".

Visual mediators are the images (visual or imagined) which pertain to the mathematical objects that are the subject of communication. Examples of visual mediators are graphs, equations, diagrams, etc (Sfard, 2008). There are multiple visual mediators associated with functions: graphs on the Cartesian plane, an equation, table of values, special symbols, etc.

Narratives are texts, either in spoken or written forms which are used to describe an object or the relationship between objects. Endorsed narratives are narratives which are seen to be true. Examples of endorsed narratives are definitions, proofs and theorems (Sfard, 2008).

Routines are repetitive patterns seen by an interlocutor in their action, words, and discourses (Sfard, 2008). Typical routines include proving, performing a calculation and so on. Routines are governed by sets of rules. Rules about objects in the discourse are object-level rules, whereas meta-level rules are less explicit, as they are rules about the discourse itself. For example, a meta-rule may say what a satisfactory proof is.

Sfard (2008) also distinguishes between the *how* and the *when* of a routine. The how of a routine consists of meta-rules which constrain "the course of the patterned discursive performance" (Sfard, 2008, p,208). The when of the routine is made up of meta-rules that constrain when it is appropriate to use a particular routine.

Sfard (2008) also categorises routines into three distinct categories; deeds, rituals and explorations. Explorations aim to further discourse by producing endorsable narratives. Examples of explorations are "routines of solving equations, of proving a mathematical result, or generating and investigating a mathematical conjecture" (Berger, 2013).

Rituals are seen to be "creating and sustaining a bond with other people" (Viirman, 2011, Sfard, 2008, p 241). The goal of a ritual is a social reward, like attention or approval. Rituals are identified by the imitation of speaker/participant with a colleague or a more experienced interlocutor. Rituals are associated with prompts, and are therefore usually highly situated in comparison to explorations (Sfard, 2008). Ben-Yahuda, Lavy, Linchevski and Sfard (2005) state that learning is often mimetic, where learning often takes place by "following the discursive patterns of more experienced interlocutors (Ben-Yehuda et al, 2005, p.182).

Deeds aim to change actual objects (either physical or discursive). This is different to explorations, where the change happens in narratives. A deed may be the act of choosing a box with a larger number, or dividing sweets equally amongst friends (Sfard, 2008; Berger 2013).

For an in-depth comparison of deeds, rituals and explorations, see Table 2-1 (Sfard, 2008, p.243)

Table 2-1 Deeds, explorations, and rituals - comparison

	Deed	Ritual	Exploration
Closing condition/Goal	A change in environment	Relationships with others (improving one's positioning with respect to others)	Description of the world (production of endorsed narrative about the world)
By whom the routine is performed	No special requirements	With (scaffolded by) others	No need for scaffolding – can be performed individually
For whom the routine is performed	No special requirements	*Others* (authoritative discourse)	*Others and oneself* (internally persuasive discourse)
Applicability (changing the *when*, keeping the *how* constant)		Restricted – the procedure is highly *situated*	Broad – the procedure is applicable in a wide range of situations
Flexibility (changing the *how*, keeping the *when* constant)		Almost no degrees of freedom in the course of action	The procedure is a whole class of equivalence of different courses of action
Correctibility	By tinkering	Cannot be locally corrected – has to be reiterated in its entirety	Parts can be locally replaced with an equivalent subroutine
Acceptability condition	The *result* – the change in environment – must count as adequate; no need for human mediation of the acceptance – it depends on the environment	The activity has to be shown to adhere strictly to the rules defining the routine procedure – the acceptance depends on other people	The narrative produced through the performance must be *substantiable* in such a way that the acceptance is independent of other people
Words' and mediators' use	Possibly no active use of keywords	Phrase-driven use of keywords – as descriptors of extradiscursive mediators	Objectified use of keywords – as signifying objects in their own right

The analysis of the discourse of learners, along these four categories, and especially the category of rituals, will be useful in investigating if there are any connections between the learners' discourses and their achieved growth points. What is not clear is exactly how the above criteria in terms of discourse are able to demonstrate or connect to Ronda's Growth Points. The answer may lie in Caspi and Sfard's (2012) work on algebraic discourses.

Caspi and Sfard (2012) use the characteristics of discourse as a basis for creating a guideline on how algebraic discourse develops through the process of learning algebra. They define algebra as a meta-discourse of arithmetic, and hence can be analysed according to the same four characteristics presented above. Caspi and Sfard (2012) also differentiate between parallel forms of algebraic discourse, formal and informal, and how these are linked.

Caspi and Sfard (2012) talk about three levels of algebraic discourse in the fixed-value algebra (constant-value algebra) field. Because I will be working with variable-value algebra, this is not quite applicable, but still holds some value in the comparison between types of discourse and the growth points. Caspi and Sfard (2012) have proposed a hierarchical development model of discourse for variable value algebra. It has only two levels – processual and objectified. At the time of this study, it was rather undeveloped and hence not much use to the study. See Table 2-2 for a complete outline of the hierarchical levels of elementary algebra as proposed by Caspi and Sfard (2012).

The three levels of fixed-value algebra are processual[4], granular and objectified. The processual level of algebraic discourse focuses on numerical calculations, which generally follow a linear order. This is similar to Ronda's (2004) description of point-by-point analysis. This level has been based on Sfard's earlier work showing the most basic level of understanding – the procedural conception of an object.

The granular level includes numerical calculations, but these are no longer seen as a "one-to-one reflection of the sequence of operations performed" in the course of calculations (Caspi & Sfard, 2012, p. 50). This shows that learners on this level are starting to see the relationship shown by the function. "Granules" are partially reified 'chunks' of information which are seen as intermediately objects.

The third objectified level of understanding is when a learner is able have both a process and object conception of a mathematical object. This level is characterised by being a full participant of the mathematical discourse about the object (Caspi and Sfard, 2012). This level seems to correspond with Growth Point 4 in Ronda's framework (2004). Caspi and Sfard's (2012) research showed that many learners were showing the beginning signs of a

[4] In Sfard's earlier work, she refers to this type of conception as procedural. In her later work, she prefers to call it a *processual conception* of a mathematical object.

formal algebraic discourse; however their participants were not yet near to the objectified levels in their discourse.

All three levels shown above can be investigated by using the four characteristics described above; mathematical words, visual mediators, routines and narratives. Routines especially point to the achievement of different levels. Formal and informal discourses are differentiated by the rigorousness of the discourse. Formal discourse aims to prevent ambiguity by using strict meta-rules (grammar). These discourses run in parallel, but are not necessarily identical (Caspi and Sfard, 2012). These three levels of discourse may follow both a formal progression as well as an informal progression.

> *"Whereas each of the columns can be seen as organized according to the developmental chronology, no claims are made about horizontal relations, that is, about how the stages in the growth of the informal discourse could be sequenced in relation to their formal counterpart. This abstention reflects out conviction that these two lines of development, the informal and formal, although mutually influential, can nevertheless be seen as quite independent. This assumption is what motivated our decision to present the two strands side by side rather than trying to create a unified linearly ordered scheme. If the corresponding levels of informal and formal discourses have been aligned with one another, it is not because the discourse in the right half of the row can be seen as the formal version of the one on the left". (Caspi and Sfard, 2012, p.47)* (These two columns are seen on Table 2-2)

Caspi and Sfard (2012) in their paper did not point exactly to what would constitute an informal objectified discourse, although they do not say that this type of discourse is impossible. However, if a highly complex form of informal discourse were conceivable, it would follow that there is not necessarily a need for formal discourse in algebra.

Ben-Yehuda et al (2005) differentiate between colloquial and literate discourses, which formed the basis for the development of the formal and informal discourses which were presented above. "Colloquial discourses are also known as every day or spontaneous as they often develop as if by themselves, as a by-product of repetitive actions" (Ben-Yahuda et al, 2005, p.181).

Table 2-2 Levels of Elementary Algebra Discourse

Level/task		Informal				Formal			
Level	Task	Signifiers of objects and their use		Routines		Signifiers and their use		Routines	
		Verbal signifiers	Ideographic (visual mediators)	Prompt (question, request)	Procedure	Verbal	Ideographic (visual mediators)	Prompt (question, request)	Procedure
"Constant value" algebra									
Level 1 processual	Find an unknown	Colloquial noun used as an unknown number	Geometric shapes used as a placeholder	"What was <the noun>?"	Undoing sequence of numerical operations	*unknown*	Letter as an unknown	"Arithmetic" equations[a]	Undoing
	Generalize relations	Colloquial noun used as a given number		How will you get a specific element of the pattern?	Sequence of operations with an equality sign used as the instruction "execute" recursive descriptions	*given*	Letter used as a given (parameter)		
Level 2 granular	Find an unknown	Complex noun clause used the in a processual way, as a description of a specific calculation				*equation*	Letter used as a given (parameter)	$P(x) = n$ where $p(x)$ is a complex literal expression (e.g. $3x + 7$) and n is a specific number	Operation-on-both-sides algorithm
							Complex literal expression used as a prescriptions for a process		
	Generalize relations				Granular descriptions of complex calculations	*formula (expression)*	Complex literal expression used as a prescriptions for a process		Complex formulas as descriptions of calculations
					Shortened recursion				
Level 3 objectified	Find an unknown	Complex noun clause used in an objectified way, as a description of the result of a specific calculation				*parameter* (replaces given)	Complex literal expression used as an object (a specific number)	$P(x) = R(x)$ where $p(x)$ and $R(x)$ are complex literal expressions	Equation-solving symbolic algorithms
	Generalize relations			Describe a rule for producing a pattern	Looking mainly for recursive descriptions		Complex formulas as descriptions of a results of calculations	Model a phenomenon Investigate the phenomenon	Complex formulas as descriptions of results of calculations

Nachelieli and Tabach (2012) introduce the idea of circularity in discourse, especially when learners talk about functions. They argue that mathematics is autopoetic or “a discourse that creates its own objects” (Nachelieli & Tabach, 2012, p.10). Learners often find themselves in the situation where they are required to talk about an object which they are not yet familiar. In their research, Nachelieli and Tabach (2012) showed that learners were able to participate in the discourse of function without necessarily having a full understanding of the object. Hence, the learners in the study showed that they were familiar enough with the discourse on functions to use it to perform some of the tasks on functions.

Nachelieli and Tabach (2012) also suggest that the instructional sequence of functions should be such that lower levels of discourses should be fully attained before moving on to the next level. This speaks to Caspi and Sfard’s levels of algebraic discourse, discussed previously. They also found that the learning of functions – the move from level to level – is a gradual process, where the abstract notion of a function is something which cannot be grasped in a hurry. These theoretical ideas link up to Ronda’s Framework of growth points in that the attainment of knowledge and skills about functions generally follow a sequence. This is shown by Ronda because her framework was devised empirically after researching how learners learnt about functions.

Ronda, in her research which was done in the Philippines and Australia, showed that most learners were achieving at Growth Point 1, and some at Growth Point 2. There were very few learners that were able to reach Growth Point 4 – a level which showed objectified thinking. Based on my experience as a mathematics teacher, I anticipated that these results would be mirrored in the South African context. There were, however, some slight differences in the results of my study. These results are discussed in Chapters 4 and 5.

It is not apparent from Ronda’s Framework of Growth Points, though, how learners might talk about functions differently at different growth points. By using the above framework, I will describe how learners’ discourses relate to the growth points which have been set out in Ronda’s framework. This is significant as there is not always time to continually test learners’ attainment of growth points formally. However, an attentive teacher working in a discourse-rich class will be able to listen to the discourse of learners to see how they are progressing, according to Ronda's framework. This relation may also have implications on how teachers inform their teaching practices, by explicitly regarding the discourse of learners.

2.8 Algebra teaching in South Africa

The Department of Education announced at the end of 2012 that learners in Grade 9 were achieving at an average of a mere 13% in mathematics, according to the results of their Annual National Assessment (ANA). It further emerged that only 2,3% of learners achieved over 50% in the ANA. These figures have left the general public in South Africa appalled, and have left many wondering where teaching has gone wrong.

Van Larden and Moore-Russo (2012), show in their research that teachers of high school mathematics have different ideas on what is important to teach in algebra. Van Larden and Moore-Russo questioned South African teachers on their beliefs and what they thought was important in the teaching of algebra in South Africa. The most important and frequent theme was "symbols and symbolic manipulation", whereas "thinking and reasoning" featured very low down on the list of frequency. Other themes which came out as very important were "operations and computations" and "quantity and number". This is disturbing as it shows that teachers in South Africa are most likely teaching in a very processual (procedural) way, and therefore do not seem to be placing much importance on the objectified properties of concepts in algebra. Van Larden and Moore-Russo state that:

> *"This particular group of teachers seemed to value the development of (procedural) skills over the development of concepts. Moreover, the teachers also seemed to consider algebra as being more about using and manipulating symbols than applying, communicating or reasoning about algebraic ideas"* (Van Larden & Moore-Russo, 2012, p. 55).

Caspi and Sfard (2012) have shown in their research in Israel that Grade 7 learners were able to participate in an informal algebraic discourse when solving an algebraic task. Some of these Grade 7 learners were also showing evidence of moving over to formal algebraic discourse, and although this move was not complete, there were the beginnings of reification in their discourse. Caspi and Sfard (2012) mention that the beginnings of reification were probably due to the Grade 7 learners having a well-developed arithmetic discourse, comparatively.

In the South African context, Van Larden and Moore-Russo (2012) have shown that many South African teachers emphasise the importance of process in the teaching of algebra, rather than the understanding of concepts. Since reification is the move from a procedural discourse to a more formalised, conceptual discourse, this move would be very difficult if only the procedural discourse was taught. Considering the research done by Van Larden and Moore-Russo, it would not be surprising to find that South African learners opted for a more processual level of discourse. Perhaps an understanding of why many Grade 9 learners are not able to complete algebraic tasks – as demonstrated by the ANA results – will come from studying their algebraic discourse.

In Ronda's (2004) work, she showed that many of the learners Grade 8 and some in Grade 9 were not able to progress beyond Growth Point 1 and 2. This was replicated to some extent in my results, but generalisations were not made due to differing sample sizes, and other factors. Results of the comparison with Ronda's study are discussed further and can be found in Chapter 4.

2.9 Framework

My research report uses two theoretical frameworks, each of which pertains to one of the two critical questions I have put forward.

For my first critical question: *"Using Ronda's Framework of Growth Points, where do South African learners fit in, especially in relation to functions in equation form?"* I will be using Ronda's (2004, 2009) framework which identifies and describes learners' understanding of functions in equation form.

My second critical question: *"How do learners' discourses relate to the growth points they have achieved?"* will be answered by using Sfard's communicational framework. This framework is further developed by Caspi and Sfard (2012) to include algebraic discourse.

Both frameworks have been extensively discussed in the literature review. In the following chapter, I discuss how the study was done.

Chapter 3 **Methodology**

The previous chapters provided an overview of the research, its aims and background, and gave a detailed account of the literature which relates to the study. This chapter will give an in-depth explanation of the methodology of the research.

This research report is a study which focuses on describing the learning trajectory of learners according to the Framework of Growth points, with a focus on functions in equation form. The main objectives of the study were to use the framework developed by Ronda, and test this framework in the South African context. The results will then be compared to the results of Ronda's study. The discourse a learner used was also explored with the intention to explore if the discourse of learners is in any way related to the growth points that they have achieved.

The study was conducted in two phases, which can loosely be categorised as quantitative and qualitative. The first phase of the study was largely quantitative, where a relatively large sample of learners completed an assessment. The second stage of the study was more qualitative. Interviews, based on the assessment, were held with a small number of selected learners. However, these categorisations will take shape in this chapter.

The study was guided by the following questions:

1. Using Ronda's Framework of Growth Points, where do selected South African learners fit in, especially in relations to functions in equation form?
2. How do learners' discourses relate to the growth points that they have achieved?

3.1 Limitations and assumptions of the study

Due to the nature of a master's research report, this study was small, with a relatively small sample. This means that the results cannot be generalised to a large population, but rather give insight, and reasonable generalisability into this specific case.

3.2 Describing the research methodology

A thought paradigm informs the beliefs we have about the world that we live in. Research is situated in these paradigms to align the expectations of the outcomes of the study, and how the study is carried out. I found it difficult to situate this study in one paradigm. However, let me explain where I positioned my research.

The study was done on a continuum of the constructivist-pragmatic-commognitive paradigm. The constructivist in me aimed to explore the reality which was presented to me in the small case that I researched. However, in order to present the results as more generalisable than a case study; I aimed to use both quantitative and qualitative methods as a part of the study. This pragmatic approach acknowledged that traditional paradigms are restrictive and hence the methods that I used were appropriate for the information that I wanted to gather.

The commognitive paradigm is an additional lens which focusses myself as a researcher onto the importance of the analysis of discourse to research on learning in mathematics. The unit of analysis for the second phase of the study is discourse, and the data is the verbatim discourse of the participant. The study switches from a largely dualist view of learning in the first phase, i.e. if an outcome is achieved or not, to a non-dualist view, which unifies thinking and behaviour (Sfard, personal communication, 2012).

3.3 Conduct of the study

As previously noted, the study was conducted in two phases. This was informed by the two research questions, with the first phase of the study pertaining to question 1, and the second phase of the study relating to question 2.

The diagram below gives an overview of the conduct of the study.

Phase 1: Assessments

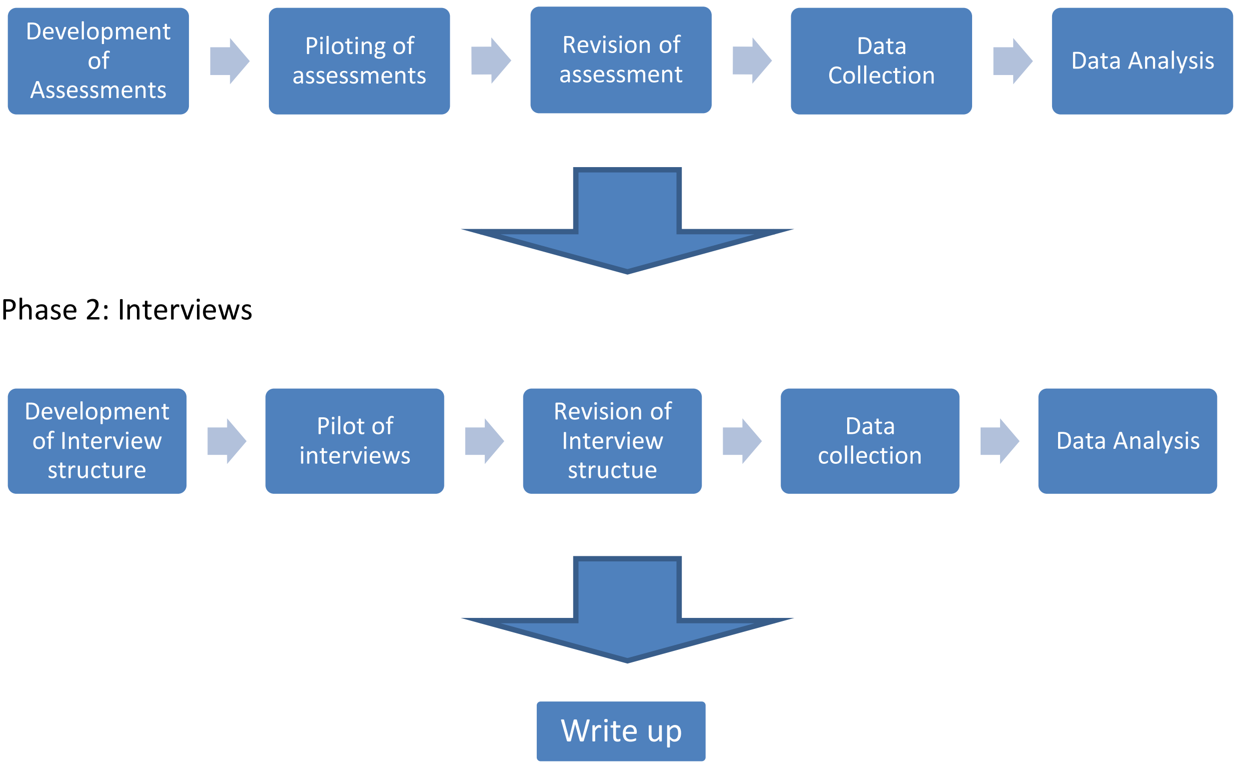


Figure 3-1 The research process.

3.4 Validity and Reliability

The validity and reliability of a study show the extent to which the results of the study are well-founded and correspond to reality, and the extent to which the results are consistent.

3.4.1 Validity

Validity ensures that the study measures what it intends to measure, and that the study can then produce results which are generalisable to some degree (Bell, in Opie, 2004).

Content validity, which is the degree to which the instrument fairly covers the topic is purports to measure, was ensured by using an instrument which had already been used successfully by Ronda (2004). Additionally, the assessment instruments, as well as the interviews were piloted and reworked to ensure appropriateness. This process was discussed extensively during supervision sessions.

3.4.2 Reliability

Reliability is defined as "the extent to which a test or procedure produces similar results under constant conditions on all occasions" (Bell, in Opie, 2004).

To ensure that my process of analysing the data from phase 1 was reliable, I requested the help of an inter-rater. Seeing that E. Ronda is a post-doctoral fellow at the University of Witwatersrand, I requested her to be the inter-rater, as she has experience with this type of data. She ensured that the coding of the data corresponded to the coding of her original data. I ensured that the rating of the data was done strictly according to the categories set out in framework. The methodology and procedures I have used are suited to my research question, which shows that my findings will be credible (Opie, 2004).

Additionally, I ensured that all interviews were carried out in a similar manner, under similar conditions. I ensured that all learners were treated similarly. Trustworthiness was ensured as I have clearly explained my methodology and procedures. All data is accounted for, and is presented in a transparent and fair manner (Lincoln and Guba, and Schife, in Opie, 2004).

3.5 Data collection

3.5.1 Assessment Task

The instrument used in phase 1 of the study was a partial adaptation of the instrument used in Ronda's study. I used all the questions which were used to investigate the understanding of functions in equation form. Some questions were adapted after the initial pilot study.

The pilot study was done approximately three months in advance of the main data collection.

Table 3-1 provides a brief explanation of each question which was used in the assessment. For a copy of the assessment as it was given to the learners, please see Appendix B. The assessment items are also discussed in more detail in Chapter 4 in the discussion as to how learners answered each item.

Table 3-1 Description of Assessment task

	Task	Description
1a	Evaluating equation	This task involves using substitution to find the level of the water in the tank.
1b	Rate	This task involves interpreting rate from a given piecewise function.
1c	Intercepts	This task involves interpreting the intercept from a given piece-wise function
2	Rate	This task involves determining the equation which gives the fastest change in *x*, when *y* takes on values from 1 to 10. The choices were all linear functions.
3	Making equations	A table of values and its corresponding linear equation is shown. A second table showing the same *x*-values, but with *y*-values which are three more than the first table. The task requires the learner to find the equation which corresponds to the second table.
4	Inverse	A table of values and its corresponding linear equation are shown. The *x* and *y* values were swapped and shown in in the second table. The task was to write the equation for the second table.
5	Relating equation/composition	Two linear equations were given. The first relates to *s* and *p*, the second relates to *p* and *n*. The task was to find an equation which relates *s* and *n*.
6	Generating values	A table of values and its corresponding quadratic equation was shown. A second table showing the same *x* values as the first table, but with *y*-values which are two units more than the *y*-values of the first table are also shown. The task was to construct the corresponding equation of the second table.
7	Relating equations	This question is similar to Question 5. The difference lies in that one of the variables was given a specific value.

The assessment questions were all taken from Ronda's study. I modified some tasks slightly, although the essence of all questions remained the same. After piloting the assessment, some small changes were made. For example, Question 3 was originally a quadratic function in Ronda's assessment task. I felt that the question could be changed to a linear function without compromising what was being tested, but also making it more accessible to Grade 9 learners who would have the understanding to answer the question but might be intimidated by a quadratic function which they had not seen before. Other minor changes included slight wording changes to accommodate the South Africa context. For example, the word "swapped" was used instead of "interchanged" in Question 4. In the assessment itself, there are 8 test items, however only 7 are used in my analysis. I added Question 8 to the assessment as a possible question for analysis. It was also taken from the test that Ronda used in her study. Upon further reflection, Question 8 did not test understanding of functions in equation form, so it was not analysed.

3.5.2 Interview

Qualitative data, in the form of interviews, were collected from some selected learners in the study. Interviews involve the collection of data from direct spoken contact between the researcher and the participants of the study (Cohen and Manion, 1994). When interviews are well structured, they are able to provide in-depth data. Interviews also increase the chance of gaining valid information from the participants, because they allow both the participant and the interviewer to ask for clarification during the interview (Cohen and Manion, 1994). This characteristic of interviews may also be a downfall, as the validity relies on how the interviewer conducts the interview. In order to overcome this potential problem, I piloted the interview and discussed it extensively with my supervisors to find ways in which to conduct the interview in the best manner possible.

The interview aimed to stimulate a conversation with selected learners in order to gather data on their discourse about functions. The interview was structured around the assessment which took place in phase 1 of the study. The questions in the interview asked learners how they completed selected tasks, and probed their thinking as to why they answered each question in a particular way. Learners were also encouraged to look at questions in a different way, to prompt them to see a different strategy for completing a question.

The interview did not require the learner to redo the entire assessment. Due to time constraints, this would have been rather difficult and also tedious for the learners partaking in the interviews. Rather, the interview was structured in a way that the learner was asked to start by answering question 3, and then tailored around how that question was answered.

In some cases, where learners answered Question 3 in a procedural way, the interviewer asked the interviewee a leading question in order to probe if the learner was capable of a more holistic understanding of the task. A similar procedure was repeated with Question 6. The interviewer also asked the interviewee some questions on other tasks in the assessment, but these were usually in conjunction with the learners' own completed assessment booklets. The learners referred to their assessments as the interviewer asked for clarification on how learners answered certain questions. This was done to get an idea of the discourse used by learners when they spoke about their answers.

3.6 Participants of the study

The study was done with Grade 9, 10 and 11 learners in a school situated in Johannesburg. The study took learners from three consecutive school years to see the progression of understanding through the years.

My study was different from Ronda's study in that I used a sample of Grade 9, 10 and 11 learners, whereas Ronda's study used Grade 8, 9 and 10 learners. I considered this necessary because I did not feel that Grade 8 learners would know enough about functions for their participation to be meaningful to the study. In Ronda's study, she mentions that learners in the Philippines start working on functions in Grade 8, which is why her study started with Grade 8 learners.

In Ronda's study, participants are assessed twice over a three-month period. I did not have the timeframe to allow for a similar assessment strategy, so I assessed each cohort only once. I was still, however, able to see the movement of understanding across grade groups. I tested my participants towards the end of the school year, so they did have the benefit of completing the curriculum for the year before being tested.

3.6.1 Participants in the pilot study

The original assessment book from Ronda's study was piloted with a group of volunteer students who were at a Maths Enrichment Camp in July 2013. This group of students were a diverse set of Grade 11 learners. I did not use any data from this set of students in my data analysis. Their answers to the assessment however, were used to inform the final structure and coding for the main data collection. Their answers also helped revise the assessment so that it was better suited to South African learners.

3.6.2 Participants in main data collection

Description of the school and learners:

The research was carried out in a secondary school in Johannesburg. The majority of learners come from middle-class backgrounds according to information from the school. Learners seem to have had a stable educational history. Permission was granted from the headmistress of the school for the research to take place. The research school was chosen by means of convenience sampling as the principal of the school in which the research was done is known to the researcher.

Convenience sampling is defined the choice of a participant population due to it being close at hand (McMillan and Schumacher, 2010). Random sampling, the most desirable type of sampling which produces generalizable results, is not always possible in educational settings. Convenience sampling does not claim that the sample is representative of the population. While convenience sampling is not always conducive to generalising results, this was deemed to be inconsequential to the study, as the aim of the study was not to produce widely generalisable results (McMillan and Schumacher, 2010). Once again, convenience sampling was used to choose the participants from the school population. For the sake of streamlining the data collection process, I used pre-arranged groups of learners in the form of school classes. I chose one class from each grade (9, 10 and 11).

The participants in the main data collection were learners in Grades 9, 10 and 11.

Table 3-2 Number of participants in the study

Number of respondents	
Grade 9	33
Grade 10	25
Grade 11	25
Total	83

3.6.3 Administration of instrument (Phase 1)

The administration of the assessment took place in November 2013 at the selected school. The assessment was written during an extended break, so that it did not encroach upon lesson time. The learners were told that the test should not take longer than 45 minutes, and most learners finished within 30 minutes, although were able to carry on if they had not finished.

3.6.4 Administration of interviews (Phase 2)

After analysing the data from the assessments, I went ahead with the interviews. Learners had been categorised into groups according to the results from their assessments and these groups showed the highest growth point they has achieved. I purposively selected six learners from each of the four groups, and depending on their own availability, interviewed three learners per growth point group.

Purposive sampling was used for the selection of the learners to take part in the interviews. “In purposive sampling, researchers handpick the cases to be included in the sample on the basis of their judgement of their typicality. In this way, they build up a sample that is satisfactory to their specific needs” (Cohen and Manion, 1994, p.89). In this research, the sample was chosen according to the answers from learners in their assessment booklet.

I interviewed 13 learners in total, four from Growth Point 1, and three each from Growth Points 2 to 4. The interviews took place in an empty classroom after school hours, so that there was no time pressure to finish the interviews. Each interview lasted approximately 10-15 minutes.

In the interview, learners were given a blank assessment book to work with. During some points in the interview, their original assessment was brought out for clarification of some questions.

3.7 Data Analysis

The data gathered from the study was gathered in two phases, as well as analysed in two phases. Data from the assessments was analysed according to the criteria set out by Ronda (2004). Ronda's study was done to devise the framework of growth points. The aim of my study is to use the already existent framework of growth points, and confirm its value.

The assessment was piloted to get an idea of the type of responses that would be given by learners. The types of responses matched the type of responses which were elicited in Ronda's study. Hence there was no need to change the record sheet which recorded the responses and types of strategies used by each learner.

Data from the interviews was analysed according to Sfard's Communicational Framework, with additional input from Caspi and Sfard's (2012) work on algebraic discourses. This part of the study is more exploratory as I have described and explained selected learners' strategies and actions found in their discourse.

3.7.1 Phase 1: Coding written responses for assessment tasks

The questions in the assessment were devised in a way such that there may be a number of different strategies to get to the correct answer to a question. Different questions were used to ascertain different growth points that learners were able to achieve. Students' answers were marked and their answers for each question were recorded on a spreadsheet.

For the answer to be considered correct, the learners had to provide an explanation. However, there were very few learners who did not provide an explanation of their answers. If learners provided more than one explanation, they were coded at the higher level of the explanation. The answers were coded according to correctness, as well as which strategy was used to answer each question. More explanation on the strategies used, as well as the specific criteria for the achievement of different growth points is shown in Chapter 4.

Figure 3-2 is an example of a record sheet which shows how learners were coded and how the data were then recorded.

A	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
	Grade	1a	1b	1c	2	3	4	5	6	7	8		GP1	GP2	GP3	GP4
		Result in Question														
Student code	Grade	1a	1b	1c	2	3	4	5	6	7	8		GP1	GP2	GP3	GP4
11.1	11	0	1	1	2	1,5	2	1	0	0	0		1	1	1	0
11.2	11	0	1	0	0	1,5	2	0	1	0	1		1	0	0	0
11.3	11	0	0	0	1	1,5	2	1	1	1	0		1	1	0	0
11.4	11	1	1	1	0	1,5	0	0	1	0	0		1	0	1	0
11.5	11	1	1	1	0	1,5	0	0	1	2	2		1	1	1	0
11.6	11	1	1	1	1	1	0	2	1	0	0		1	1	1	0
11.7	11	0	1	1	1	1,5	1	1	2	0	1		1	1	1	0
11.8	11	1	1	0	1,5	1,5	1	1	1	0	1		1	1	1	0
11.9	11	0	1	0	2	1,5	0	1	0	0	0		1	1	1	0
11.10	11	0	1	1	0	0	0	0	1	2	2		1	0	1	0
11.11	11	1	0	1	1	2	0	2	2	2	1		1	1	1	1
11.12	11	1	1	1	1	1,5	2	2	1	2	2		1	1	1	0
11.13	11	0	0	0	0	1,5	2	2	1	0	1		1	1	0	0
11.14	11	1	1	1	1,5	1,5	0	2	1	2	1		1	1	1	0
11.15	11	1	1	1	1	1,5	2	2	2	2	2		1	1	1	1

Figure 3-2 Table showing how learners' results were coded and recorded

To ensure that the coding of the data were reliable, I gave a sample of the coded assessment scripts to Erlina Ronda, who confirmed that the data had been coded correctly. The rating of the data was done strictly according to the categories set out in framework. The methodology and procedures are suited to my research question, which shows that my findings will be credible (Opie, 2004).

3.7.2 Phase 2: Coding spoken responses from interviews

In Phase 2 of the study, interviews with 13 learners were conducted. All interviews were transcribed verbatim, and their written work, which was done during the interviews, was kept as a reference. The interview was piloted with one learner in Growth Point 3. Based on this interview, the interview structure was revised. Please see Appendix B for a basic overview of the interview.

The interviews were analysed according to Sfard's Communicational Framework which looks at discourse on four planes; Word use, Visual mediators, Routines, and Endorsed Narratives. A deeper explanation of these four characteristics can be found in the previous chapter. Additional ideas on the data analysis of the interviews came from Ben-Yehuda et al (2005) and Nachlieli & Tabach (2012) which is also explained in Chapter 3. The analysis of the interviews is discussed in Chapter 5.

3.8 Ethical Considerations

"Ethics has to do with the application of moral principles to prevent harming or wronging others, to promote the good, to be respectful and to be fair." (Siever, 1993, p. 14; in Opie, 2004). Ethics is important in education, as the researcher is dealing with minors. The researcher has to ensure that the research is carried out ethically, and the resulting data is properly handled and processed. This means that the research has to be carried out while respecting the rights of all parties involved in the study.

As a researcher, it was my responsibility to ensure that all parties involved in my research were protected, and that any potentially harmful situations are avoided. Opie (2004) urges researchers to consider all possible ethical issues that may arise from research before the research process has started, which I feel that I did adequately.

In this study, ethical clearance was sought from the University of the Witwatersrand human research ethics committee (non-medical), which deals with approving research that involves human subjects. My ethical clearance was approved, and the protocol number is 203ECE114M. The ethical approval from the University of the Witwatersrand human research ethics committee (non-medical) can be found at Appendix D.

Since it is imperative for ethical conventions to be adhered to, I ensured that permission was sought from the school and the Department of Education. The school principal, teachers and all learners were all assured that all information gathered would remain confidential throughout the study. All learners have been referred to by pseudonyms in the research report to maintain their anonymity. The learners were informed of their choice to pull out of the study at any time if they felt uncomfortable, without any consequence. These conditions were clearly stated in the informed consent form. The learners taking part in the interviews were informed again at the beginning of the interview of this condition.

Parents (or guardians) of participating learners were given an information sheet which explained exactly what would happen in the study; and their permission was requested in the form of a reply letter. They were required to give permission for their children to take part in the research study, as all learners in the study were minors. These consent forms were to be signed by the learners' parents or guardians, as a measure of prevention against psychological stress and emotional injury (Frankfort-Nichmias & Nichmias cited in Cohen & Manion, 1994). Parental and learner information sheets, as well as reply letters can be found in Appendix E and F and G respectively. Permission from the school can be seen at Appendix H. Permission given from the GDE can be seen at Appendix I.

Raw data in the form of video recordings have been stored on a password-protected hard drive and will be destroyed three to five years after the study has finished. Transcriptions of the video recordings will be treated in a similar manner.

This chapter has detailed how the data was collected and analysed. The next chapters give an analysis of the data and explore findings.

Chapter 4 **Results Phase 1**

Phase 1 of the study was done in order to answer the question "Using Ronda's Framework of Growth points, where do selected South African learners fit it, especially in relation to functions in equation form?"

My focus on functions in equation form was decided because this is the way in which many learners in South Africa first encounter functions. In my experience as a teacher, and in looking at many textbooks at Grade 8 and 9 level, I have found that the most common way of introducing functions is the introduction of a string of single values, which lead to the recognition of patterns and the formulation of rules (generally being in equation form first). In general, equations are a very common, and arguably the most common representation of a function. Equations can be seen as both a process – a means for generating values – as well as an object: something which can be manipulated or transformed. The understanding of functions in equation form is therefore an important part of the overall understanding of functions.

This chapter discusses the growth points in equations, and where selected South African learners fall into the framework which was set up by Ronda in her study (2004).

The first section of this chapter shows all the tasks in the assessment, discusses these tasks, and then discusses the different strategies which were used by learners to answer the tasks. This section also shows the success rate for selected questions, and also quantifies the number of learners using different strategies used in answering these selected questions.

The second part of the chapter gives the criteria of how learners were coded at each growth point, which was done according to similar criteria in Ronda's study.

The last part of the chapter shows the overall results of where selected South African learners fall within the spectrum of growth points, and also compared this to the results of Ronda's study which was done with Grade 8, 9 and 10 learners in Australia and the Philippines.

4.1 The questions in the assessment: strategies and analysis

Seven tasks were used to assess learners to categorise them according to which growth points they were able to achieve. The tasks were designed to test learners' understanding of functions in equation form.

Each task is shown along with an explanation of the task. For a full copy of the assessment booklet; please see Appendix B. This section also shows how each question was answered typically, as some questions were answerable in more than one way. Some strategies showed a higher level of thinking in that the strategies were more holistic. For some questions, there is an analysis of the different strategies that are used by the learners to answer the question.

4.1.1 Question 1

Question 1 consisted of three parts, and was based on a real life situation of a container being filled with water. This task was used to assess Growth Point 3, as it tested learners on whether they were able to interpret properties of a function such as rate (Question 1b) and intercept (Question 1c). Question 1a didn't have an explicit purpose in ascertaining the Growth Point of a learner, but was included in order to scaffold the learners understanding for Questions 1b and 1c.

1. Imagine water flowing through a pipe into a container for 10 minutes. The following equations show how the height of the water (w) in the container, and how the height is related to the number of minutes (t) when the pipe was opened.

   $w = t + 8$ for the first four minutes

   $w = 3t$ for the remaining 6 minutes

Please use the above information to answer the following questions

   a. What was the height of the water in the container 3 minutes after the pipe was opened?

b. From the given information, do you think the height of the water in the container is increasing at the same rate throughout the 10 minutes? Circle the letter corresponding to your answer.
   a) Yes, the water level increases at the same rate throughout the 10 minutes.
   b) No, the water level is not increasing at the same rate throughout the 10 minutes.

Please show or explain how you obtained your answer.

c. From the given information, do you think the container already contains water before the pipe was opened? Circle the letter corresponding to your answer.
   a) Yes, the container already does contain some water before the pipe was opened.
   b) No, the container did not contain water before the pipe was opened.

Please show or explain how you obtained your answer.

Figure 4-1 Task 1a, 1b and 1c

Although there was some variation in the strategies used to answer question 1b and 1c, the following analysis shows the most typical correct answers. In the responses of the learners in my study, I found there to be no distinct categories of answers for this question, although in Ronda's study, there were learners who used point-by-point reasoning in their answers. In my study, there were a small number learners who attempted to use point-by-point reasoning for their answers, but none of these resulted in correct answers, and hence were not counted. There were also some common misconceptions in some Grade 9s' answers, which will be discussed later in the chapter.

A typical solution to Question 1b is shown in the figure below.

Please show or explain how you obtained your answer.

1st 4 minutes $w = t + 8$

next 6 minutes $w = 3t$

$\therefore t \neq 3t$

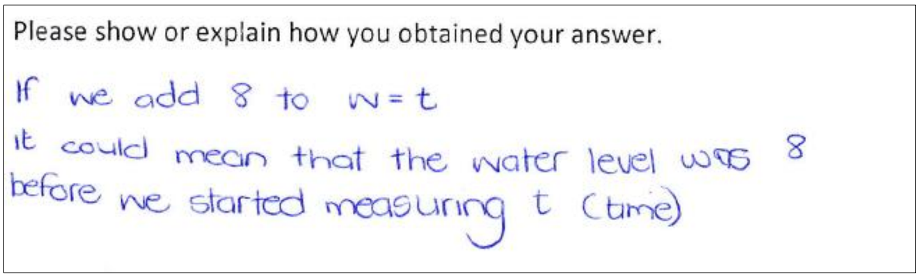
$\therefore$ not same rate

Figure 4-2 Typical Solution to Task 1b

Question 1b was difficult because it entailed answering a question about gradient involving a piece-wise function. There was another question later on in the assessment (Task 2) which provided a more straight-forward question involving gradient where the learner could demonstrate their knowledge about the gradient property of equations.

A small number of learners, when answering Task 1b gave the correct answer, but incorrectly reasoned that the rates would be different because the equations were different. While this is partially correct, it was not accepted as a correct answer because the learners were expected to show their understanding of the gradient property of a function, which entailed reasoning that the co-efficients of *x* were different.

Question 1c


Please show or explain how you obtained your answer.

If we add 8 to w = t
it could mean that the water level was 8
before we started measuring t (time)

Figure 4-3 Typical Solution to Task 1c

There were two correct methods for answering Task 1c. Evaluating the equation for t=0 gave an answer of 8, or identifying the meaning of a constant in the equation. Both were correct.

The table (Table 4.1) shows the number and percentages of learners who got Task 1a, 1b and 1c correct, as well as the number and percentages that got the all three questions in the task correct.

There were very few Grade 9 learners who were able to answer Task 1 correctly in its entirety. In fact, only one learner from the Grade 9 cohort of participants was able to correctly answer all three sub-questions in this task. In Grade 11, 53% of learners answered this question correct in its entirety, and only two learners (8%) were not able to answer the question at all, meaning that most learners, if they weren't able to answer all sub-questions correctly, they were at least able to answer some correctly. The movement between growth points over grade levels is discussed further later in the chapter.

Table 4-1 Numbers and percentages of learners getting Question 1 correct

	Grade 9		Grade 10		Grade 11	
	n=33	%	n=25	%	n=25	%
Task 1a	4	12	13	52	16	64
Task 1b	12	36	13	52	20	80
Task 1c	3	9	11	44	18	72
Entire task correct	1	3	8	32	13	53

Some very interesting misconceptions arose from Task 1. These are discussed later on in the chapter under the heading “Other Discussion”.

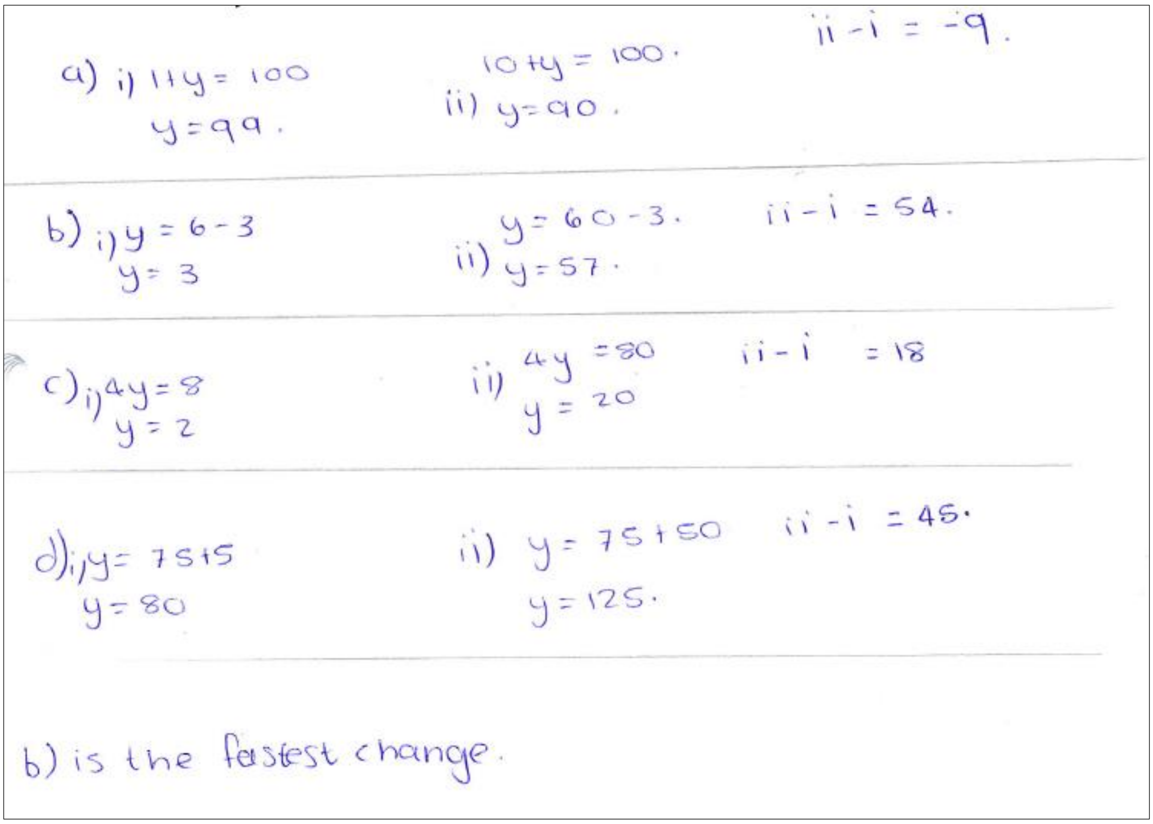
4.1.2 Question 2

Question 2 involves the interpretation of a set of linear equations. The learner is required to identify the equation which shows the fastest change in *y*. The question was seemingly straightforward and did not involve a context or piecewise functions like Question 1.

2. Which equation shows the fastest change in y when x moves from 1 to 10? Please show or explain how you got your answer.
   a. $x + y = 100$
   b. $y = 6x - 3$
   c. $4y = 8x$
   d. $y = 75 + 5x$

Figure 4-4 Question 2

Many learners struggled with the question. In Ronda’s study, she identified two strategies which were used in learners’ solutions. These two strategies were reflected in my study too. The first strategy was a point-by-point interpretation of the question (Strategy 1), and the second was a holistic solution where the learner identified the equation with the largest coefficient of *x* (Strategy 2). I identified a third category which I called Strategy 1.5. This category was where learners approached the question at first in a point-wise manner, and then saw that there was a constant difference in the change of each value of *x*, which then pointed them to working in a somewhat holistic manner.



a) i) $1+y=100$
$y=99$.

ii) $10+y=100$.
$y=90$.

$ii-i=-9$.

b) i) $y=6-3$
$y=3$

ii) $y=60-3$.
$y=57$.

$ii-i=54$.

c) i) $4y=8$
$y=2$

ii) $4y=80$
$y=20$

$ii-i=18$

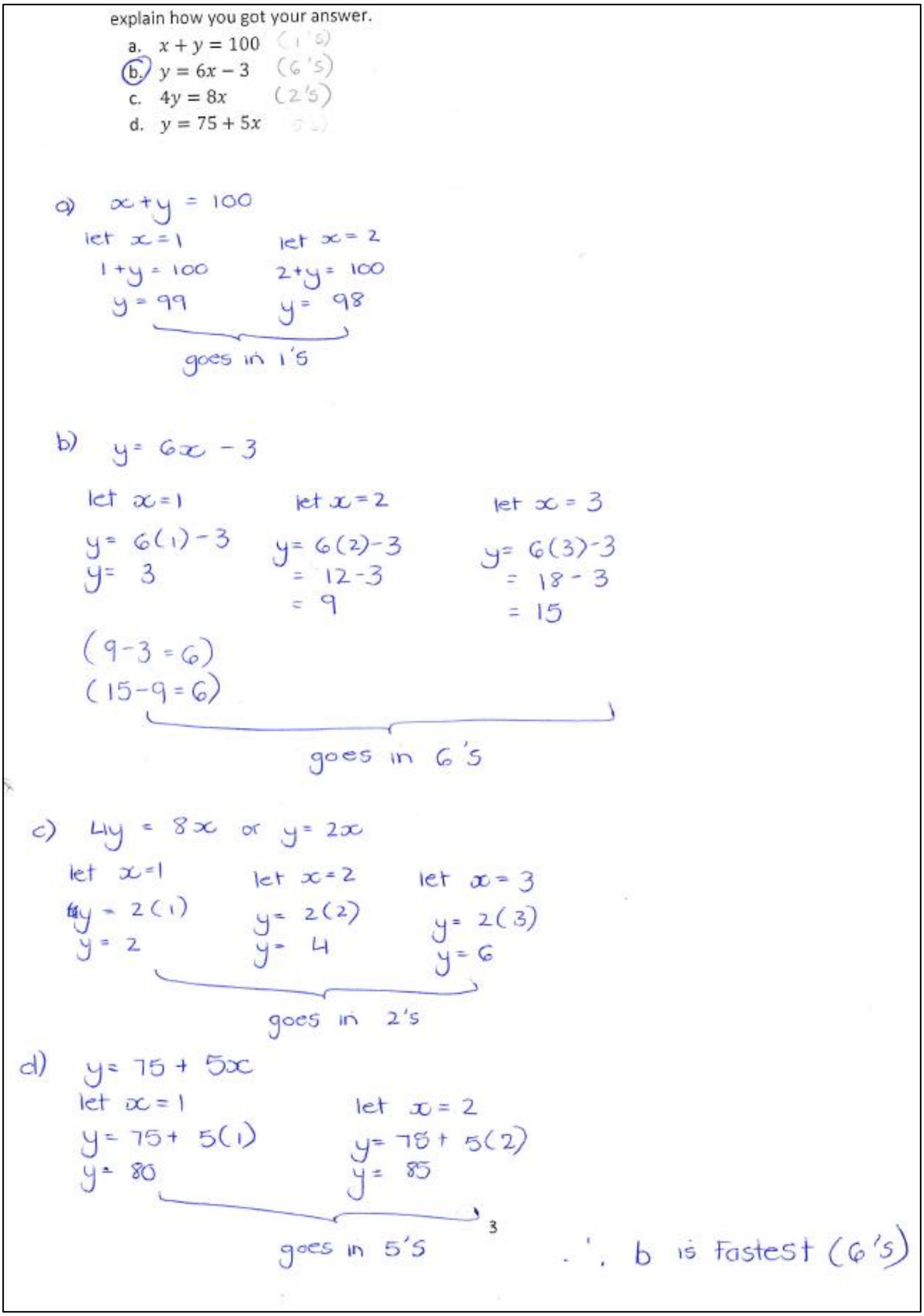
d) i) $y=75+5$
$y=80$

ii) $y=75+50$
$y=125$.

$ii-i=45$.

b) is the fastest change.

Figure 4-5 Solution to task 2 – Strategy 1 (point wise analysis)


explain how you got your answer.

a. $x + y = 100$ (1's)
b. $y = 6x - 3$ (6's)
c. $4y = 8x$ (2's)
d. $y = 75 + 5x$ (5's)

a) $x + y = 100$

let $x = 1$ $\quad$ let $x = 2$

$1 + y = 100$ $\quad$ $2 + y = 100$

$y = 99$ $\quad$ $y = 98$

goes in 1's

b) $y = 6x - 3$

let $x = 1$ $\quad$ let $x = 2$ $\quad$ let $x = 3$

$y = 6(1) - 3$ $\quad$ $y = 6(2) - 3$ $\quad$ $y = 6(3) - 3$

$y = 3$ $\quad$ $= 12 - 3$ $\quad$ $= 18 - 3$

$= 9$ $\quad$ $= 15$

$(9 - 3 = 6)$

$(15 - 9 = 6)$

goes in 6's

c) $4y = 8x$ or $y = 2x$

let $x = 1$ $\quad$ let $x = 2$ $\quad$ let $x = 3$

$y = 2(1)$ $\quad$ $y = 2(2)$ $\quad$ $y = 2(3)$

$y = 2$ $\quad$ $y = 4$ $\quad$ $y = 6$

goes in 2's

d) $y = 75 + 5x$

let $x = 1$ $\quad$ let $x = 2$

$y = 75 + 5(1)$ $\quad$ $y = 75 + 5(2)$

$y = 80$ $\quad$ $y = 85$

goes in 5's $\quad$ 3 $\quad$ $\therefore$ b is Fastest (6's)

Figure 4-6 Solution to task 2 – Strategy 1.5 (Point Wise moving to holistic)

a. $y = 100 - x.$ $\rightarrow$ $y = -x + 100.$

b. $y = 6x - 3.$

c. $y = 2x.$

d. $y = 75 + 5x.$ $\rightarrow$ $y = 5x + 75.$

I think equation b shows the faster change in y because the gradient is the greatest, which means that the change in x ~~as~~ opposed to the change in y is greater, resulting in more difference

Figure 4-7 Solution to task 2 – Strategy 2 (Holistic Analysis)

Question 2, along with Question 1b, was the question that tested for the achievement of Growth Point 3. This question required learners to use the property of rate to interpret which equation would show the fastest change over a set of *y*-values. Strategy 2 was the preferred strategy for this question, as it showed understanding of the gradient and its use.

Despite thinking that this question would yield better results than Question 1b, this was not the case. This shows that learners still struggle with the concept of gradient, and how it relates to questions of this nature.

Table 4-2 below shows a comparison of the different strategies used by learners in Question 2. The table is organised according to grade groups.

Table 4-2 Learners using different strategies for Question 2

	Grade 9 (n=33)	Grade 10 (n=25)	Grade 11 (n=25)
Achieved GP 3	2	6	19
None	31	19	6
Strategy 1	2	3	10
Strategy 1.5	0	2	5
Strategy 2	0	1	4

The table shows that Grade 9 learners had difficulty completing this task successfully. Even though two learners were able to complete the task, they still used a piece-wise method, which shows a more procedural way of mathematical thinking. The number of learners who were able to complete the task successfully increased through the grade cohorts.

Even though not all the Grade 11 learners achieved Growth Point 3, they still made a good effort at the question. Most were able to get it right using one strategy or another, but the results still show the preference for point-wise analyses (Strategy 1).

4.1.3 Question 3

Question 3 required learners to find the equation of a linear function, given a table of *x* and *y* values. This question from Ronda's assessment was originally adapted from Moschkovich, Schoenfeld and Arcavi's study (1993). The question was originally posed using a quadratic function, but was changed to a linear function to make it more accessible to Grade 9 learners.

Because Question 3 it could be answered by way of more than one strategy[5], I found it to point to the attainment of different Growth Points.

3. Examine the two tables shown below. The set of values in the table on the left shows specific values of $y = 3x + 3$.
Please give the equation which will result in values shown in the table on the right. Please show or explain how you obtained your answer.

x	y
-1	0
0	3
1	6
2	9
3	12

$y = 3x + 3$

x	y
-1	-2
0	1
1	4
2	7
3	10

$y =$ ____________

Explanation:

Figure 4-8 Question 3

[5] Many questions could be answered with more than one strategy, however Question 3 and later Question 6 were particularly telling questions in terms of the differentiation between procedural vs objectified thinking. This is also discussed in Chapter 5.

Strategy 1 shows a "guess and check" method where the learner, by trial and error, finds the correct intercept and coefficient for *x* to satisfy the output values of *y*.

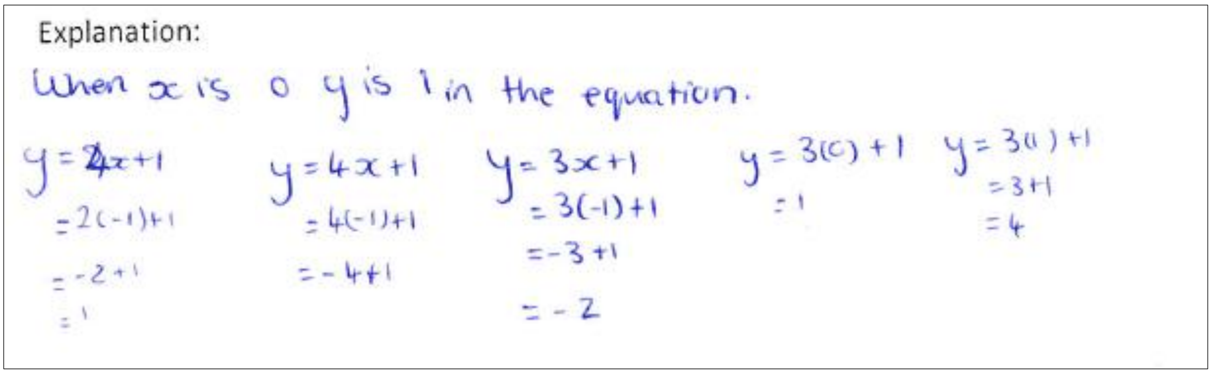


Figure 4-9 Solution to Question 3 - Strategy 1

Strategy 1.5 shows a little more understanding on the part of the learners. They are able to use some properties such as the definition of the intercept, and the formula to find gradient to get their answer. A typical answer for a learner coded at strategy 1.5 is shown in Figure 4-10.

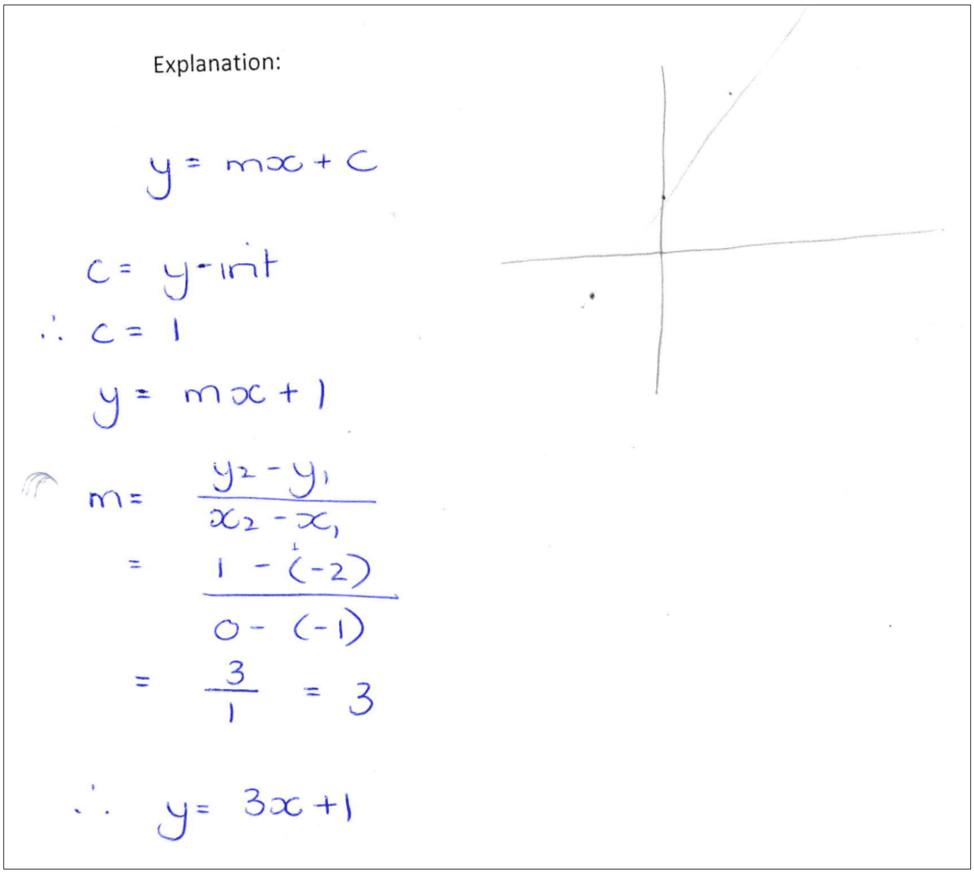
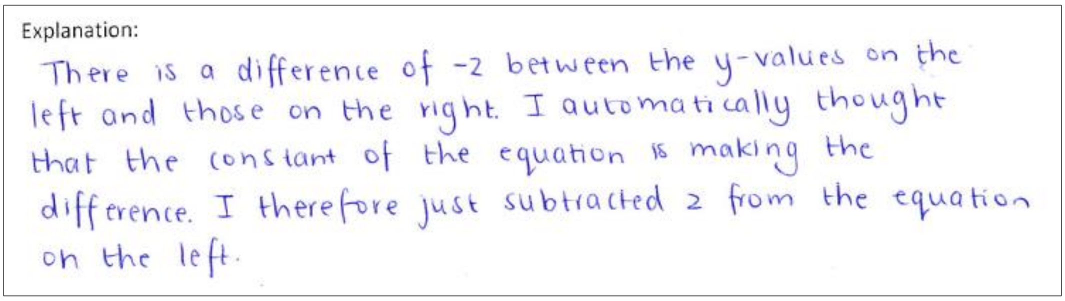


Figure 4-10 Solution to Question 3 - Strategy 1.5

Learners who were coded at Growth Point 2 showed a holistic understanding of equations, as they were able to perform an operation on the equation. The explanation in Figure 4-11 shows the answer which was typical of a learner coded at Strategy 2 for Question 3.


Explanation:

There is a difference of -2 between the y-values on the left and those on the right. I automatically thought that the constant of the equation is making the difference. I therefore just subtracted 2 from the equation on the left.

Figure 4-11 Solution to Question 3 - Strategy 2

Question 3 required learners to find an equation which represented a table of values. There was more than one way of finding this answer. The three distinct strategies identified in Ronda's study, were discussed above, and were also found in the answers of the participants of my study. Shown in the table below are the numbers of learners using each strategy in their answers.

Table 4-3 Learners using different strategies for Question 3

	Grade 9 (n=33)	Grade 10 (n=25)	Grade 11 (n=25)
None/Incorrect	16	1	1
Strategy 1	9	9	2
Strategy 1.5	5	13	17
Strategy 2	3	2	5

The table shows that the learners' advancement through grades is related to their ability to use a higher strategy. There are still some learners at a Grade 11 level who are most comfortable thinking in a procedural manner about functions in equation form.

4.1.4 Question 4

This task required learners to find the inverse of a function. This question was included as it tested for the achievement of Growth Point 4.

4. The relationship between x and y in Table 1 is $y = 2x + 1$.

In Table 2, the values of x and y in Table 1 were swapped. Please write the equation which shows the new relationship between x and y in Table 2. Please show or explain how you obtained your answer.

Table 1

x	*y*
0	1
1	3
2	5
3	7
4	9

Table 2

x	*y*
1	0
3	1
5	2
7	3
9	4

$y = 2x + 1$ ____________________

Explanation:

Figure 4-12 Question 4

Question 4 could be answered in a similar fashion to strategy 1 or 1.5 in Question 3, which involves finding the gradient and intercept of the equation in a procedural manner, and then equating it. Hence I have included only a typical response of a learner who has been coded at Strategy 2 for Question 4. This shows that the learner is able to conceive of the function as an object which can be manipulated.

Explanation:

$y = 2x+1$ gives the original results
If the x value of ~~the~~ Table 1 is the y of Table 2, and the x of Table 2 is the y of Table 1. then the variables should be swopped around

$$x = 2y + 1$$

$$x - 1 = 2y$$

$$y = \frac{x-1}{2}$$

Figure 4-13 Solution to Question 4 - Strategy 2

Many learners found it very difficult to complete this question, as it required learners to have an understanding of functions as an object that can be manipulated or transformed. Many learners ignored the Table 1, and used strategy 1 or 1.5 to find the equation that would represent the values in the table. Very few learners were able to identify the object as an object and hence use Strategy 2 to complete the task. This is demonstrated by the small number of learners who were coded at Growth Point 4. The requirements for each growth point and the numbers of learners who achieved each growth point are discussed later in the chapter.

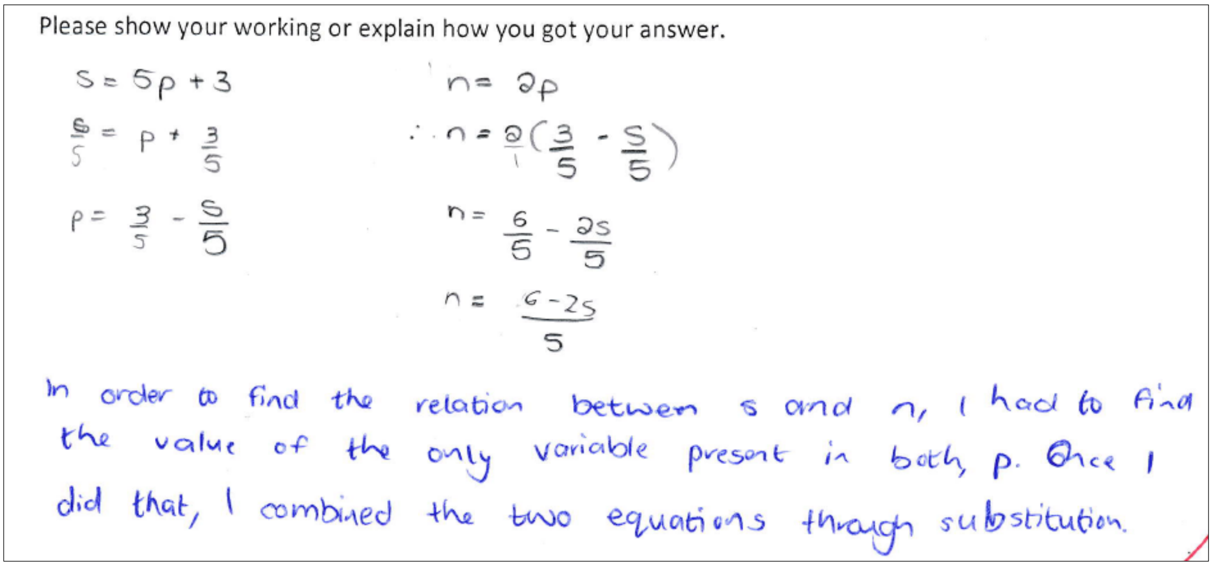
4.1.5 Question 5

This task was designed to assist testing Growth Point 2. It involved the analysis of the relationship between two equations, which did not have the same variables respectively.

5. The relation of s with p is shown in the equation: $s = 5p + 3$. The relation of p with n is shown in the equation: $2p = 6n$. From this information, please write the equation that will show the relation of s with n.
   Please show your working or explain how you got your answer.

Figure 4-14 Question 5

Learners typically answered this question in two ways. The first strategy involved partitioning; however, this strategy was not common. The second strategy used by learners was composition, which was the preferred strategy.


Please show your working or explain how you got your answer.

$S = 5p + 3$

$\frac{S}{5} = p + \frac{3}{5}$

$p = \frac{3}{5} - \frac{S}{5}$

$n = 2p$

$\therefore n = \frac{2}{1}\left(\frac{3}{5} - \frac{S}{5}\right)$

$n = \frac{6}{5} - \frac{2s}{5}$

$n = \frac{6-2s}{5}$

In order to find the relation between s and n, I had to find the value of the only variable present in both, p. Once I did that, I combined the two equations through substitution.

Figure 4-15 Solution to Question 5 - Strategy 1 (Partitioning)

Please show your working or explain how you got your answer.

$s = 5p + 3. \longrightarrow 5p = s - 3. \therefore p = \frac{s-3}{5}$ ①

$2p = 6n.$ ②

Subs ① into ②

$2\left(\frac{s-3}{5}\right) = 6n.$

$\frac{2s-6}{5} = 6n$

$2s - 6 = 30n.$

$2s = 30n + 6.$

$s = 15n + 3.$

I saw that I should use simultaneous equations to combine the two variables into one equation. I also thought that the p-variable should be eliminated so that it doesn't stand in the way.

Figure 4-16 Solution to Question 5 - Strategy 2 (Composition)

4.1.6 Question 6

Question 6 required the generation of *y*-values for a given equation. This question was also originally adapted from Moschkovich, Schoenfeld and Arcavi's study (1993). Two strategies were identified in the answer booklets of learners.

The first strategy, shown in Figure 4-18, involved the substitution of *x* values into the equation to generate *y* values. This shows a point-wise interpretation of the equation.

Strategy 2, shown in Figure 4-19, showed a more holistic interpretation of the equation, where the learner was able to show the relationship between the two equations, and then relate this to the values shown in the table.

6. Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ is shown on the table on the left.
Fill in the table on the right with values for of $y = x^2 + 3x$
Please show or explain how you obtained the y values.

$y = x^2 + 3x + 3$

x	y
0	3
1	7
2	13
3	21
4	31

$y = x^2 + 3x$

x	y
0	
1	
2	
3	
4	

Figure 4-17 Question 6

$y = x^2 + 3x$

let x be 0)
$y = (0)^2 + 3(0)$
$= 0.$

let x be 1
$y = 1^2 + 3(1)$
$1 + 3$
$= 4$

let x be 2
$y = (2)^2 + 3(2)$
$= 4 + 6$
$= 10$

let x be 3
$y = (3)^2 + 3(3)$
$9 + 9$
$= 18$

let x be 4
$y = (4)^2 + 3(4)$
$= 16 + 12$
$= 28.$

6

Figure 4-18 Solution to Question 6: Strategy 1 (Point wise analysis)

The difference between the two equations is that the second does not contain the constant +3. Therefore I subtracted 3 from all the results from the left.

$3 - 3 = 0$.
$7 - 3 = 4$
$13 - 3 = 10$
$21 - 3 = 18$
$31 - 3 = 28$

Figure 4-19 Solution to Question 6: Strategy 2 (Holistic interpretation)

Question 6 was similar to Question 3, but was slightly simpler, as it required the learner to generate a set of values instead of finding the equation which represented a set of values. There were two possible ways in which to answer this question, as explained previously.

Table 4-4 Learners using different strategies for Question 6

	Grade 9 (n=33)	Grade 10 (n=25)	Grade 11 (n=25)
None	8	1	2
Strategy 1	24	23	15
Strategy 2	1	1	8

As seen on the table, this was generally a rather well-answered task. Despite not being formally taught quadratic equations, there were many Grade 9 learners who were able to complete the task successfully. This reveals that value generation, when given an equation, is an easy skill to acquire. This table also shows that there are some Grade 9 learners who still struggle with this very basic task, which – according to CAPS – should be an acquired skill by the end of Grade 9 (earlier even, with linear equations).

Grade 10 and 11 learners are able to generate values, but there are still very few who are able to do so using Strategy 2. A discussion about the preference for procedural working came up with one learner in her interview, and this will be discussed further in Chapter 5 (See interview with learner 4.11.15 in Chapter 5).

Question 3 and 6 provided the basis of my interviews, as these two questions were able to demonstrate the movement between procedural and objective understanding of functions. This will also be discussed further in Chapter 5.

4.1.7 Question 7

Question 7 was very similar to Question 5, and the strategies present in answering Question 5 were consistent with the strategies found in answering Question 7. Answers to Question 7 were coded in the same way as Question 5. Question 7 was seen as easier than Question 5 as there was an actual value assigned to *t*.

7. The relation of r with q is shown in the equation of $r = 5q + 3$. The relation of q with t is shown in the equation $2q = 6t$. If $t = 5$, what is the value of r? Please show or explain your solution.

Figure 4-20 Question 7

After marking each task in the assessment, I went on to code each learner in term of which Growth Points they had reached. The coding of learners is discussed further. It was not necessary for a learner to be coded at a lower growth point in order to achieve a higher one. This is discussed further later in the chapter.

4.2 Coding learners at different growth points

In Phase 1 of the study, the questions in the assessments were coded according to the criteria shown above. All the questions in the assessment were marked and coded according to which strategies were used. Each Growth Point was derived according to the criteria set out in Ronda's study, but I made some slight modifications in the criteria in some Growth Points. This will be explained in each growth point. Learners who did not meet the criteria of any growth points were coded as Growth Point 0.

Growth Point 1: Equations as procedures for generating values.

To be coded at Growth Point 1, learners needed to show that they were able to see an equation as a means to generate values. In the assessment, Questions 3 and 6 were used to test this growth point, as well as other growth points. These questions asked for the equation, given a set of values (Question 3), or asked learners to generate a set of values, given an equation (Question 6). Learners had to answer either Question 3 or 6 correctly to be coded at Growth Point 1. However, Question 3 and 6 could have been answered using different strategies, which would indicate different growth points. If learners used 'higher' strategies in Questions 3 and 6, I made the assumption that they were also capable of using 'lower' strategies and were therefore coded at Growth Point 1 (with the probability of being coded at a higher growth point too).

Growth Point 2: Equations are representations of relationships

Growth Point 2 is reached when a learner is able to see that equations are not only a means for generating values, but also show a connection between two variables. Learners may start to see the equation holistically, but are not yet at the level of seeing an equation as an object.

To be coded at Growth Point 2, learners had to have at least one of the following combinations of questions correct.

- Question 5 and 7 correct
- Question 5 correct and Question 3 or 6 correct.
- Question 7 correct and Question 3 or 6 correct using strategy 1, 5 or 2

Growth Point 3: Equations describe properties of relationships

Growth Point 3 is the point at which learners are able to start seeing that functions in equation form have distinct properties (such as gradient and intercepts) and use these properties in interpreting the given task.

To be coded at Growth Point 3, learners had to answer Question 1c correctly, as well as either question 1b, or 2. Alternately if learners answered Question 2 using Strategy 2, they were coded at growth point 3. This is a slight change from Rhonda's criteria, which required that learners answer Question 1c correctly. When marking the assessments, I found that many learners had difficulty in answering all of Question 1 correctly. This may be due to the nature of the question, which is unusual in the South African context. Some learners even commented that they expected this question to be from a science paper, rather than a mathematics paper.

I decided that if learners answered Question 2 with Strategy 2, this showed adequately that they were able to interpret the equation based on the property of gradient.

The presence of Growth Points 2 and 3 show that learners do not immediately move from an introduction of functions (or procedural conception of functions), to immediately understanding functions as an object. There is a course of learning which takes time.

Growth Point 4: Equations are objects that can be manipulated and transformed.

Growth Point 4 shows the highest level of understanding a function. To be coded at Growth Point 4 learners had to show an objective understanding of functions in equation form. An objective understanding of functions was characterised by: being able to perform an operation on an equation (Strategy 2 in Tasks 3 and 6), working with the composition of equations (Strategy 2 in Task 5), and finding the inverse of a function (Strategy 3 in Task 4).

To be coded at Growth Point 4, learners had to meet at least three of the following four criteria in the assessment.

- Question 3: Strategy 2
- Question 4: Strategy 3
- Question 5: Strategy 2
- Question 6: Strategy 2

The data from the assessments was recorded on a spreadsheet (as shown in Figure 3-2 in Chapter 3) and the results were used to code learners at different growth points according to the criteria set out.

4.3 Learning trajectories

To get an idea of how learners progressed year on year, a table was created with the results of the study (See Table 4-5 below). Looking at these data, it is clear that learners progress through the big ideas in a manner which is typical of learning about functions in equations form.

Learners coded at Growth Point 0 were not coded at any other growth point, whereas learners coded at Growth Points 1 to 4 could be coded at more than one growth point. Growth points are not mutually exclusive, hence the cumulative totals do not necessarily sum to 100%. While I observed that learners do progress through a typical trajectory of learning, it does not necessarily follow that learners always progress through the growth points in a set order. This is discussed later in the chapter under the heading "Overall achievement of growth points".

Table 4-5 Number (and percentages) of Learners coded at Growth Points for equations

Growth Points	Grade 9 (n=33)	%	Grade 10 (n=25)	%	Grade 11 (n=25)	%
GP 0	6	18	0	0	0	0
GP 1	27	82	25	100	25	100
GP 2	8	24	18	72	22	88
GP 3	1	3	8	32	21	84
GP 4	0	0	1	4	7	28

The data show that at a Grade 9 level, many learners have a basic understanding of functions. However there are some learners who have not even been able to grasp the most basic understanding of functions. This is shown by 18% of learners at Growth Point 0. Eighty-two percent of Grade 9 learners achieved Growth Point 1, hence they had a basic understanding of functions in equation form.

In both Grade 10 and 11, all learners have managed to reach Growth Point 1; hence all had a basic procedural understanding of functions in equation form. The achievement of Growth Point 2 was also very high in Grade 10 and 11 with 72% and 88% of learners reaching it respectively.

The difference between Grade 10 and 11 learners came in at the achievement levels of Growth Points 3 and 4. While the majority of Grade 10 learners were able to reach Growth Point 2 (72%), there were fewer that reached Growth Point 3 (32%) and very few that reached Growth Point 4 (4%). Grade 11s achieved well in Growth Point 3, with 84% of learners reaching this level.

This table shows that many learners, even at Grade 11 level, are still not able to see an equation of a function as a holistic concept. Only seven learners (28%) in Grade 11 reached Growth Point 4.

Below, the data is shown in visual form as a graph (Figure 4-21) comparing the achievement of growth points over the cohorts of learners. Growth Point 0 was not included in this graph.

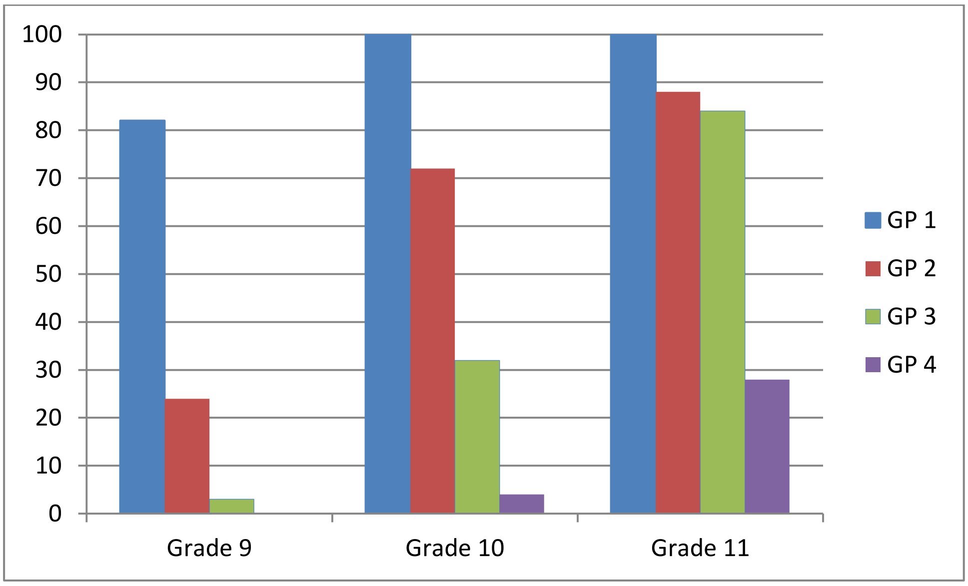


Figure 4-21 Percentages of learners coded at Growth Points

The following Table (Table 4-6) shows the results of Ronda's study. It is important to note that the participants in Ronda's study underwent two tests, approximately 5 months apart. The two tests happened during the first half of the school year in the Philippines. In all comparisons of my data to Ronda's data, I have used the results from her second data collection (D2) as I thought it would be more comparable to the data collected in my collection.

Table 4-6 Results of Ronda's study

Percentages of Students Coded at the Growth Points under Equations

Growth Points	Yr 8 (n = 149)		Yr 9 (n = 152)		Yr 10 (n = 143)	
	D1	D2	D1	D2	D1	D2
GP 0	33.6	7.4	2.0	0.0	0.0	0.0
GP 1	66.4	92.6	98.0	100.0	100.0	100.0
GP 2	10.1	28.9	23.0	52.6	44.8	68.5
GP 3	5.4	20.1	14.5	27.6	24.5	55.2
GP 4	0.0	0.7	1.3	0.7	3.5	12.6

Comparing the results of my study with the results in Ronda's study showed some interesting findings. Overall, the trend of growth is similar, with learners moving from growth point to growth point in an ordered manner, building on previous growth points. Each year there is a larger proportion of learners who are able to achieve successive growth points. However, there is a difference with the rate of development in the growth points in the two studies.

Rendering the data from my and Ronda's study into a graph (see Figure 4-22), the following shows a comparison of results between the two studies. I have taken data from the second data collection in Ronda's study, as indicated by D2 in Table 4-6 above.

In this following graph, I compare my Grade 9 group with Ronda's Grade 8 group, and so on. This was done because the South African Grade 9 curriculum was seen to be somewhat comparable with the outcomes that Grade 8 learners in Ronda's study had achieved (at least according to the curriculums on functions).

In Ronda's study, she mentions that Grade 8 learners in Science secondary schools are introduced to functions and learn about linear functions in Grade 8. In Grade 9, they further their study of functions with the introduction of quadratic functions. In Grade 10, exponential, polynomial and circle functions are taught (Ronda, 2004).

In South Africa, a basic notion of function is introduced in Grade 8. This is a very informal introduction where learners are required to find an algebraic expression which can explain a number pattern (usually linear). In Grade 9, learners are introduced to linear functions and their different representations. In Grade 10, quadratic functions are introduced, along with basic instruction on exponential and hyperbolic functions. In Grade 11, the instruction on linear, quadratic, exponential and hyperbolic functions is continued (DOE, 2003).

Without going into an in-depth analysis and comparison of the curriculums in South Africa and the Philippines, the above shows that there can be a rough mapping between Grade 9 in South Africa with Grade 8 in the Philippines, and so forth. This, however, is by no means exhaustive, hence the comparison is tentative.

It should also be stated that the scale of the two studies was very different. Ronda's study was a large scale study over many aspects of functions (as pointed out in Chapter 3) with a large number of participants (444 learners in total). My study was narrower in that it only focussed on functions in equation form, and had a smaller number of participants (83 in total).

Figure 4-22 below shows a visual comparison of the two sets of data.

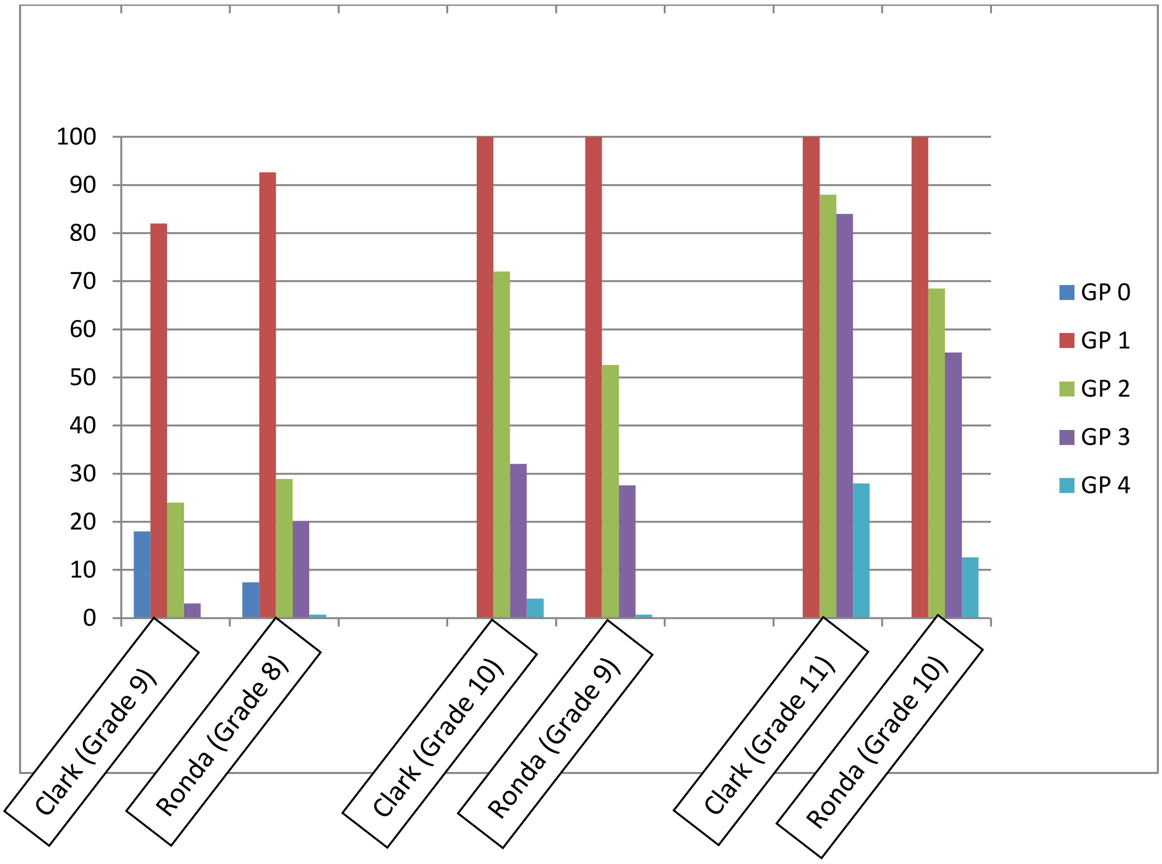


Figure 4-22 Comparison between Ronda and Clark

This table shows the relationship among the Growth Points from the data from my study and Ronda's study. Visually this shows the same basic trend – learners follow a trajectory of learning.

4.4 Comparison of growth of learners

Ronda started her study with Grade 8 learners, who at the end of the year (D2) fared better than the Grade 9 learners in my study. There is a larger proportion of Grade 8 learners (in Ronda's study) in each growth point, than the Grade 9 learners in my study. This changes when comparing my Grade 11 learners with Ronda's Grade 10 learners. In each growth point a larger proportion of my learners are achieving the respective growth points than the Grade 10 learners in Ronda's study.

The following series of graphs show the increase in the numbers of learners achieving each growth point over the Grades, and illustrates the point above.

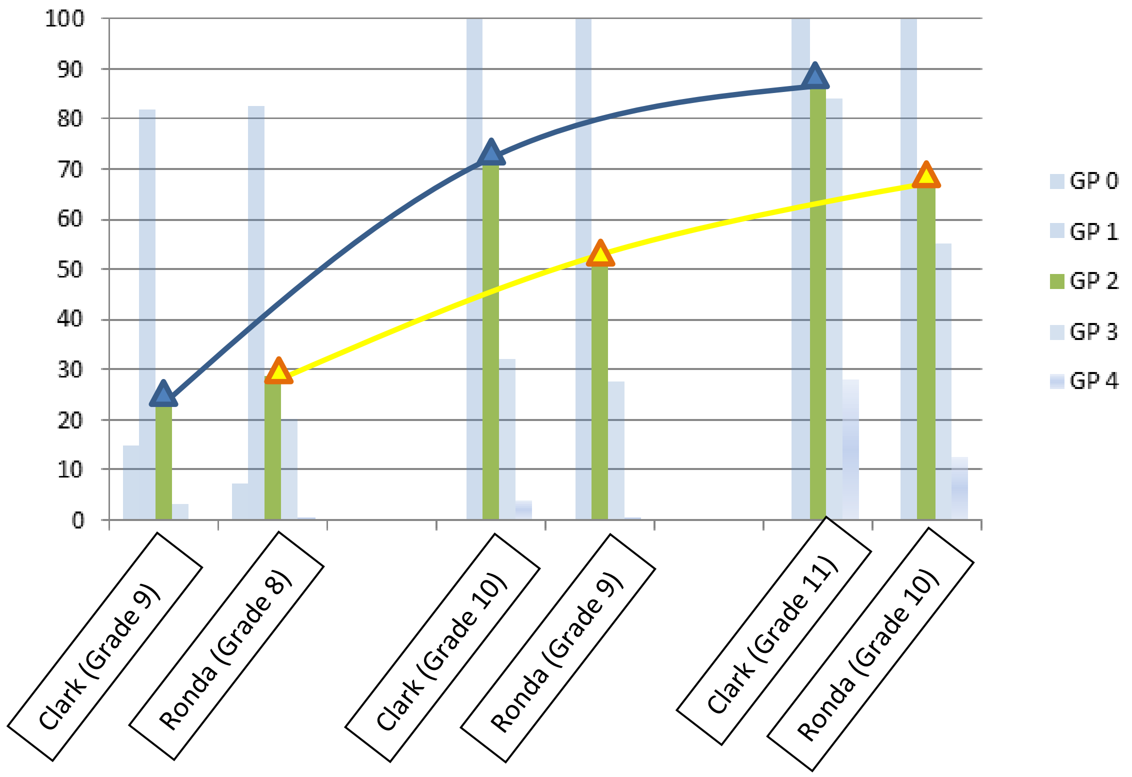


Figure 4-23 Comparison of movement through Growth Point 2

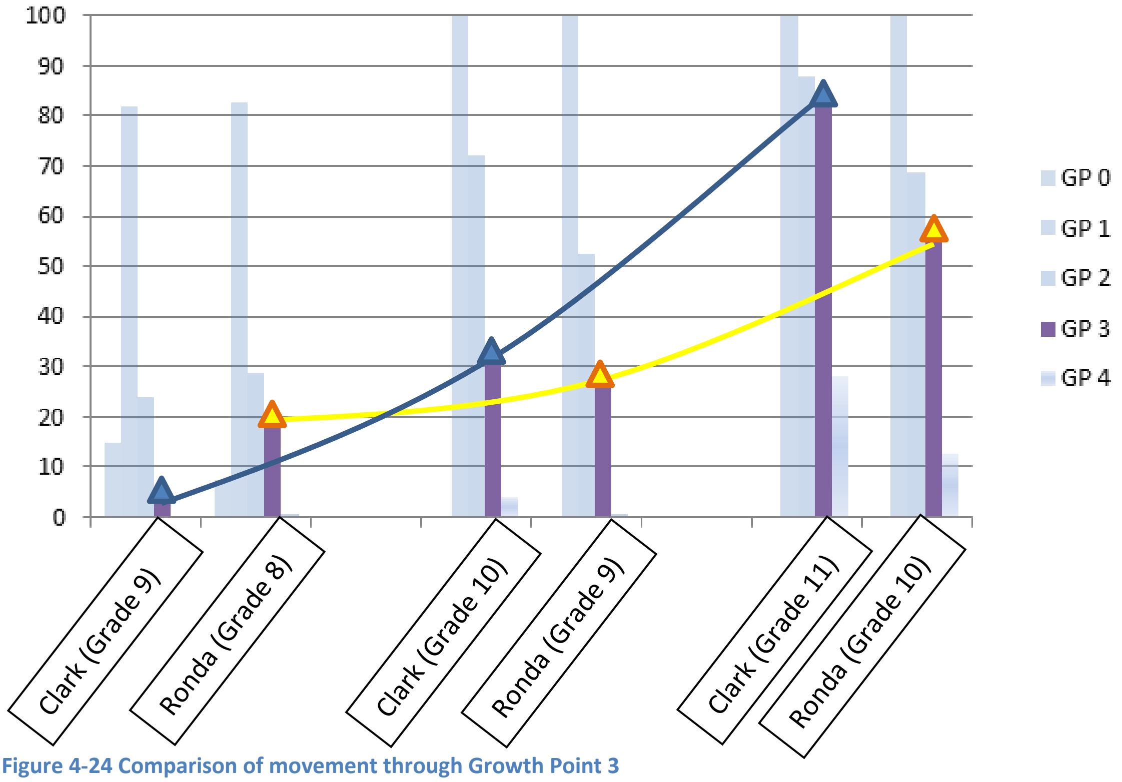


Figure 4-24 Comparison of movement through Growth Point 3

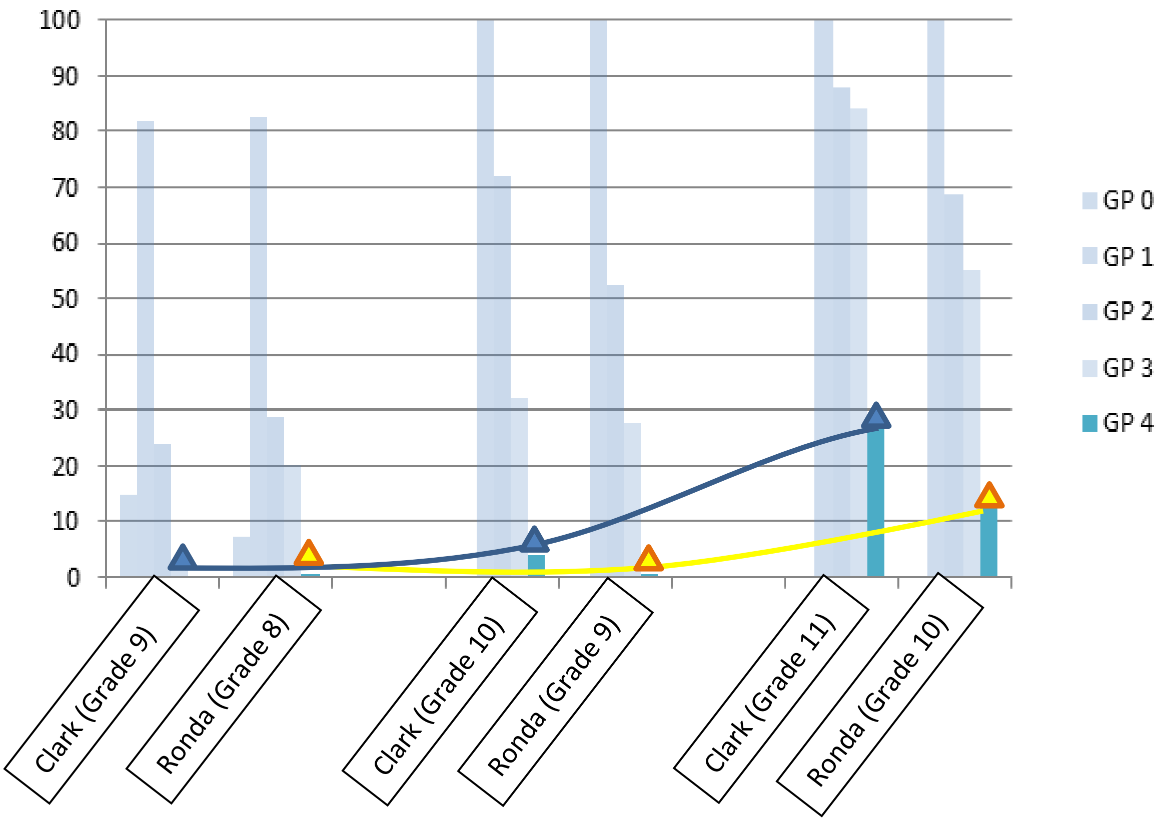


Figure 4-25 Comparison of movement through Growth Point 4

In each of these three cases, the progression of learners through the growth points happens at a faster rate in my study than in Ronda's study. A reason for this difference in rate of growth was not an aim for this study. In both cases, the study was done comparing learners in three consecutive year-cohorts. The oldest cohort in Ronda's study was Grade 10, whereas in mine, it was Grade 11.

These results seem to indicate that the rate at which learners in my study gain an understanding about functions is faster than the learners in Ronda's study. A reason for this was not speculated as this was not in the scope of my study. This may be a topic for further investigation.

4.5 Overall achievement of growth points

In the previous section of results, achievement of individual growth points was shown. The following table was included to show the trend of the order in which growth points are achieved.

Table 4-7 Frequencies of Learners at the Growth Points under equations

	Grade 9 (n=33)	Grade 10 (n=25)	Grade 11 (n=25)
GP 0	6	0	0
GP 1	19	5	1
GP 1,2	7	12	3
GP 1,2,3	1	6	12
GP 1,2,3,4	0	1	7
GP 1,3	0	1	2

This table shows that learners do typically follow a learning trajectory, where growth points are reached in consecutive order. This table served to answer whether or not it was assumed that the achievement of a higher growth point implied the achievement of a lower one.

There were some cases where learners did not seem to follow the typical path of learning as they were coded at Growth Points 1 and 3, but did not reach Growth Point 2. Upon further investigation into these three learners it emerged that they showed some signs of reaching Growth Point 2, but did not completely satisfy the conditions set out for achieving Growth Point 2.

Although most learners follow the typical path of progression through the growth points in a consecutive manner, this is not always the case as a learner may reach a higher growth point before having full understanding of a lower growth point. Interestingly, there were no instances where a learner had reached a growth point higher than GP 1, without first reaching Growth Point 1 itself. In all instances, learners who reached Growth Point 4 also reached Growth Points 1, 2 and 3.

My data shows that learners progress through the four growth points in a manner where growth points are achieved progressively. The results of my study mirror the results of Ronda's study.

4.6 Other discussion

There were some interesting misconceptions which came to light in the answers of the learners in the assessment. The most noteworthy was a misconception which was found in question 1b (See figure 4-26). Learners were asked to calculate the height of water in a tank three minutes after a pipe was opened. Although this task was included in the assessment to familiarise the learners with the context, it was still a task which proved to be problematic for some of them.

1. Imagine water flowing through a pipe into a container for 10 minutes. The following equations show how the height of the water (w) in the container, and how the height is related to the number of minutes (t) when the pipe was opened.

   $w = t + 8$ for the first four minutes

   $w = 3t$ for the remaining 6 minutes

Please use the above information to answer the following questions

   a. What was the height of the water in the container 3 minutes after the pipe was opened?

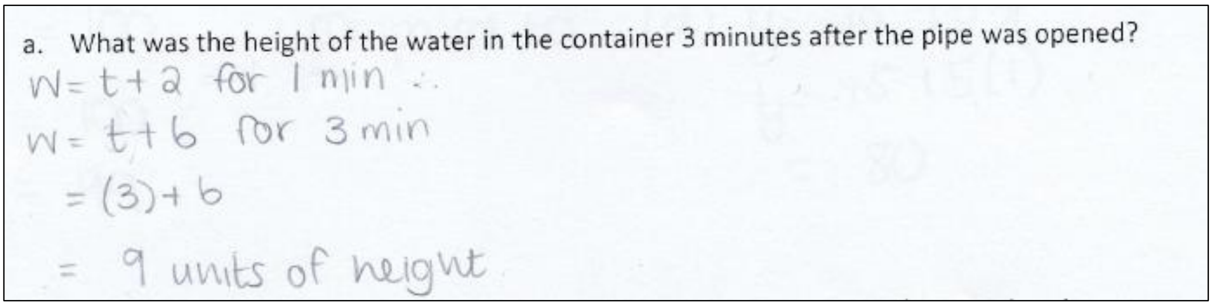
Figure 4-26 Question 1a

A table showing the number and percentages of learners who got this question right was provided in the analysis for Question 1 earlier in the chapter (See Table 4-1).

A common mistake in the question was learners not knowing which equation to use, as this was a piece-wise equation. The correct equation would be the first equation as it shows the level of the water within the first three minutes. Some learners combined both equations to get their answers, as they were unsure of which equation to use (See Figure 4-28).

A more interesting misconception was the misinterpretation of the first equation in Question 1. Many learners seemed to "divide" the equation $w = t + 8$ by 4 (owing to the instruction "for the first four minutes"). They then assumed the equation to be $w = t + 2$ for each minute, and multiplied that by 3, to get an equation of $w = t + 6$, because the question asked what the height of the water was after three minutes (See Figure 4-27 and 4-29). This misconception shows that these learners do not fully understand the meaning of a constant in a linear equation.

The figures below show three examples of learners work where answers which were incorrect. These examples further highlight that learners struggled with this task which tested the basic understanding of a linear function.


a. What was the height of the water in the container 3 minutes after the pipe was opened?

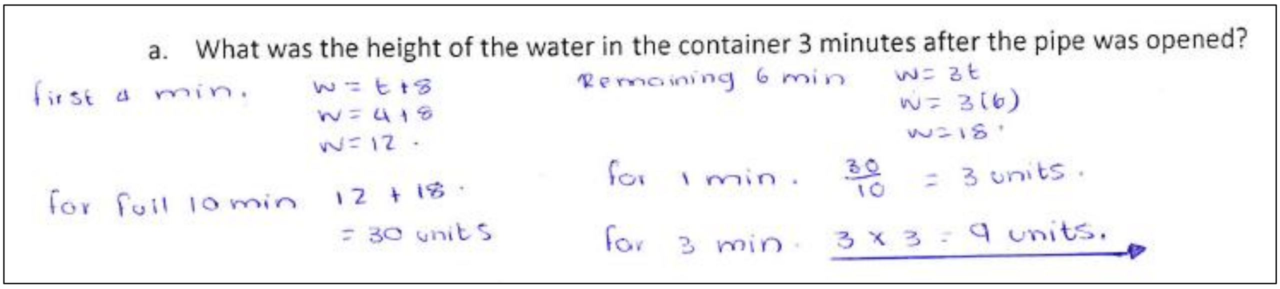
W = t + 2 for 1 min ∴

W = t + 6 for 3 min

= (3) + 6

= 9 units of height

Figure 4-27 Equation which has been 'divided'


a. What was the height of the water in the container 3 minutes after the pipe was opened?

first 4 min. w = t + 8, w = 4 + 8, w = 12.

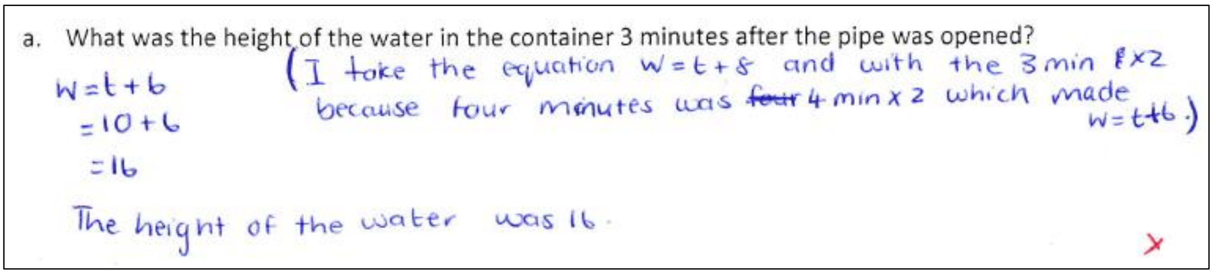
Remaining 6 min w = 3t, w = 3(6), w = 18.

for full 10 min 12 + 18 = 30 units

for 1 min. 30/10 = 3 units.

for 3 min. 3 x 3 = 9 units.

Figure 4-28 Combination of equations to find answer


a. What was the height of the water in the container 3 minutes after the pipe was opened?

$W = t + 6$
$= 10 + 6$
$= 16$

(I take the equation $W = t + 8$ and with the 3 min $t \times 2$ because four minutes was ~~four~~ 4 min x 2 which made $W = t + 6$.)

The height of the water was 16.

Figure 4-29 ‘Division’ of equation and incorrect substitution

During Phase 2 of the study (in the interviews), some learners also mentioned that the context in this question confused them.

4.7 Conclusion

This phase of the study aimed to replicate and confirm a part of Ronda’s study. The domain “Functions in Equation form” was chosen and researched. The results from my study showed an overall similarity to the results in Ronda’s study, in that the learners moved from growth point to growth point in a similar fashion.

In Ronda’s study, the order of the growth points was established empirically by the frequency of learners who achieved growth points. This was mirrored in my study, as the frequency of learners who achieved growth points decreased from Growth Points 1 to 4. Both studies showed that learners who achieved higher growth points also achieved the lower growth points (with very few exceptions). The pattern was true to all grades in my study.

The similarities between the two studies were in the way that learners progressed through the growth points. Ronda found that learners follow a typical learning trajectory when proceeding from Growth Point 1 to Growth Point 4. My findings were consistent with Ronda’s in that the participants in my study also progressed in the same typical learning trajectory from Growth Points 1 to 4.

The differences became apparent in the rate at which learners progressed through growth points over the years. The learners in my study seemed to progress from Growth Points 1 to 4 in a quicker manner than the learners in Ronda’s study. While there is no conclusive evidence to suggest a reason for this finding, it may perhaps be that the learners in the oldest cohort in my study were older than the oldest cohort in Ronda’s study. That the learners in my study progressed faster may be a function of their age; however this is a topic for further study.

The next chapter presents the findings of Phase 2 of the study.

Chapter 5 **Results Phase 2**

The second phase of the study aimed to answer my second research question: “How do learners’ discourses relate to the growth points they have achieved?” In this chapter, I will be discussing the data which was collected in the second phase of the study.

The aim of the interviews was to explore the discourse used by learners about functions in equation form. The interviews were based on the assessments which were given to all the learners, although interviews were conducted with selected learners. The interviews were conducted in such a way that learners were able to speak freely about the tasks that were in the assessment.

My focus on discourse is important for several reasons. Discourse is the first way in which a teacher or teaching assistant can identify if a learner is not learning what they should at a certain level (or in the case of this study, where a learner is not achieving growth points appropriate to their school level). In my own experience as a teacher, I have realised that the way in which a learner speaks to a teacher or fellow classmate is one of the earliest ways in establishing if a learner has understood a concept or not.

The first section of the chapter describes the interview process and how the interviews were analysed. The second section gives an in-depth analysis of the interviews that pertained to Questions 3 and 6 in the assessment. This section especially focusses on the routines that learners used in their discourses. I then discuss the extent to which the learners’ discourses were objectified or not. The last part of the chapter provided insight into some other interesting observations which were gleaned from other questions in the interviews.

5.1 Summary of interviews

The interviews were held two to three weeks after the assessments were done. Thirteen learners were interviewed in total; however one of these learners was interviewed first to pilot the process. Subsequently 12 learners were interviewed for the main data collection. However, I still used the data from the learner who took part in the pilot study, as the pilot study resulted in only minor revisions to the final interview structure.

Table 5-1 Table showing participants of the interviews

Growth Point 1	Growth Point 2	Growth Point 3	Growth point 4
Learner 9.3	Learner 10.5	Learner 10.7	Learner 11.15
Learner 9.8	Learner 10.14	Learner 10.12 (Pilot)	Learner 11.20
Learner 9.22	Learner 10.23	Learner 11.9	Learner 11.25
Learner 9.33			

The interviews took place with the researcher in a one-on-one situation. The interviews were audio recorded as well as videotaped in order to see the nuances of the learners beyond just their spoken words. The video also ensured that I was able to track the order of their written work, as well as references made to the written work by pointing and gesturing. The filming of the learners proved to be a good decision in this regard. Ethical clearance was given for the filming of learners. This is discussed in Chapter 3.

The interviews were created to elicit discourse by the learners about selected questions in the assessment. The interviews also made provision for the interviewer to examine and explore the learners thinking, and even scaffold the learners' thinking by asking leading questions. This was done to see if a learner was able to move to a different growth point with the help of a more experienced interlocutor. An outline of the structure of the interview can be found in Appendix C.

In the transcripts, learners are referred to by a coding number which I used to preserve anonymity. The coding number begins with their growth point, their Grade level, and an identifying number.

5.2 Analysis of Questions 3 and 6

After listening to and transcribing the interviews, I decided to focus my analysis on the learners' explanations of Question 3 and 6. The interviews with the learners, in all cases, began by asking learners' for their reasoning on Questions 3 and 6. This regularity provided a good starting point to the analysis, as all learners were asked the same questions at the beginning of the interview. In some cases towards the end of the interviews, the interview took a slightly different path. However, the consistency in questioning provided a good basis for the analysis of Questions 3 and 6. An overall analysis of the discourse used by each learner in Questions 3 and 6 can be seen in Table 5-1.

5.2.1 The task

I first asked learners to complete Question 3 in a blank assessment booklet. Many of the learners in the interviews were able to do this easily. Learners had different reactions when asked to do Question 3 in the interview, as this is the question that I started each interview with. Most learners were confident with this task, except for the learners who achieved only Growth Point 1. Most learners in Growth Point 1 were unsure or tentative in their explanations.

Question 3 requires a learner to find the equation given a table of values. Question 3 can be seen at Figure 4-8 in Chapter 4. Figures 4-9, 4-10 and 4-11 show the strategies used to answer Question 3.

Question 6 requires learners to find the *y*-values in a table of values given an equation. Question 6 can be seen in Figure 4-17. Figures 4-18 and 4-19 show the strategies used to answer Question 6.

That more than one strategy could be used in answering Questions 3 and 6, revealed achievement in different growth points. Some learners showed a movement between growth points during the interview.

I structured the interview in such a way that I asked learners to explain how they did Question 3 first, with no other instructions. In many cases, especially with learners in the lower growth points, they used a procedural strategy to complete the question (Strategies 1 and 1.5 in Chapter 3). If they answered using Strategies 1 or 1.5, I asked them a leading question to elicit the comparison of the *y*-values between the tables. This would often lead to the realisation that the question could be answered holistically. (Please refer to Appendix C to see the basic structure of the interview). For this reason, some learners answered Question 6 spontaneously, and hence used a different strategy to their original assessment.

5.2.2 Words

If the task were looked at in a holistic manner, the learners would have seen that the two equations/tables in each problem were the same, save for the constant value. This meant that in each case, the equation had *shifted* (In Question 3, the equation shifted down two units, in Question 6, the equation shifted down three units.) This terminology comes about in Grade 11, when the transformation of functions is explicitly taught. The use of the word “shift” emerged only twice in the interviews, and only from Grade 11 learners; one from Growth Point 3 and one from Growth Point 4. This process of using one word for a group of objects (i.e.: every *y*-value which has changed by the same value) is called *saming.*

Interviewer:	Can you see any relationship between the *y*-values?
Learner 4.11.15	They have been…. Well, they are two smaller. So that would be…. Yeah.
Interviewer	Ok, so is there another way that you could have worked out the equation [for table 2]?
Learner 4.11.15	Would it just be a **shift**? So you just adjust accordingly by minusing 2.

Interviewer	Yes, just explain to me as you go on.
Learner 3.11.9	I’m not sure if I am right, but I am assuming that the *x* values are the same. And the *y* values have changed by 2 each time. So I am assuming, that because it’s going less each time, it’s… you’re minusing 2. So the graph is **shifting** down by 2. Must I write anything?

The above excerpts show that the learners have used one word “shift” to refer to many different mathematical objects (all the *y*-values in the second table), and hence have been able to construct a new mathematical object – a function which had shifted vertically. “Saming, if applied to discursive objects that are all realisations of the same signifier, is part of the process of the learners’ construction of a new mathematical object.” (Berger, 2013:3)

5.2.3 Visual Mediators

Visual mediators did not play a large role in this study, as the focus of the study was functions in equation form. Indeed the written function itself can be seen as a visual mediator; however the equations were the same for all learners so this was not a

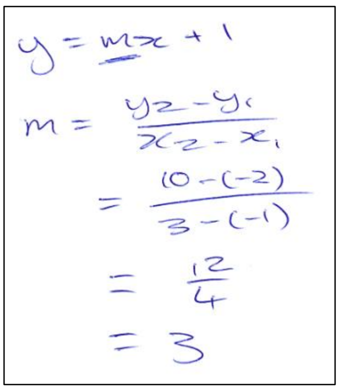


Figure 5-1 Use of equation as visual mediator

differentiating factor in the discourse. There were some cases where learners used visual mediators to help them to complete Question 3. Two examples of visual mediators were found which were common to many students.

The first was found in the assessments and in the interviews of learners who used a more procedural discourse. Figure 5-1 below shows the formula ($m = \frac{y_1 - y_2}{x_1 - x_2}$) which was used by learners to find the gradient of an equation. This can be seen as a symbolic visual mediator and it is used by learners as a part of their mathematical discourse (Sfard 2008). This formula is discussed below in the "rituals" section, as it not only is a visual mediator, it can also be seen as a ritual.

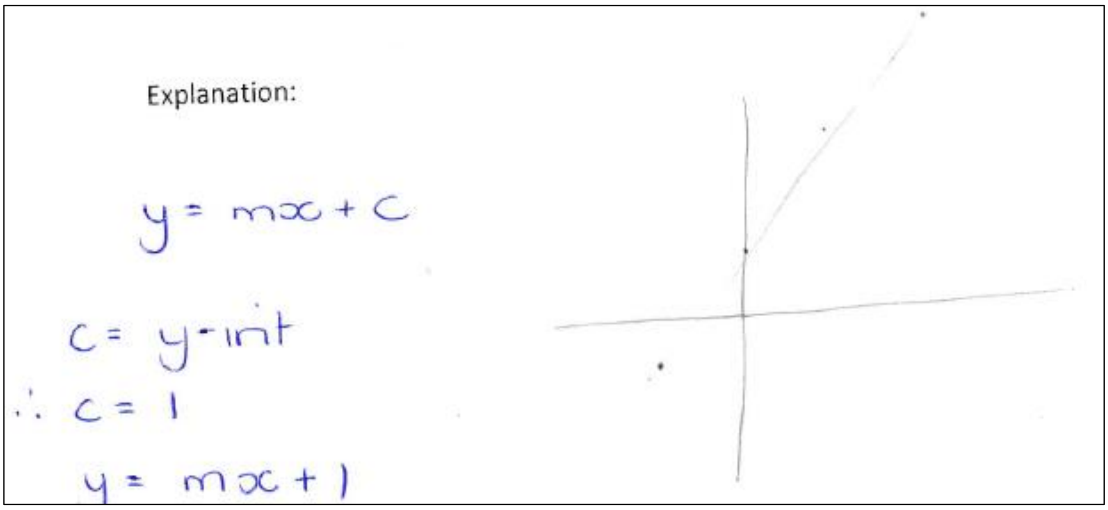


Figure 5-2 Use of rough sketch graph as visual mediator (Question 3)

Figure 5-2 shows that some learners attempted to use a rough sketch of a graph in order to help them solve question 3. No learners used a graph in the interviews, and hence no definitive conclusion can be made from this. It seems that the learner intended to use the sketches in order to check their answer, or help them find an equation in the assessment.

5.2.4 Routines

Routines are the patterned discursive activities and produce narratives about mathematical objects. There are three types of routines which exist; deeds, rituals and explorations, all three of which were present in the interviews with the participants of my study. According to Sfard "not every routine is explicably describable" however, I will endeavour to try to describe the routines found in these interviews to an extent which produces an adequate analysis of the discourse (Sfard, 2012, personal communication).

Deeds

There were two learners who used deeds in their discourse about Question 3. These learners were not very comfortable in their understanding or explanation of the question. Both learners chose an equation, almost at random, to represent the table of values in Question 3. A deed constitutes a choice, without necessarily knowing what is being chosen or why (Sfard 2008). While these learners may have wanted to use a more ritualised discourse (as it does seem that they were anxious for social approval from the interviewer too), they were not always able to produce a mathematically correct routine. Furthermore this finding is supported by the fact that these learners did not imitate the rituals which would commonly be associated with this task (e.g. by using the formula for finding the gradient of a straight line).

	Transcript	Routines
Learner 1.9.22	So the y-intercept is going to be where $x = 0$ so y equals 1. It's a positive. Then to find the x, the gradient and the x, I think. I'm not sure about this. When y is equal…. [silence] I'm not sure. I remember doing this but I am not sure.	Ritual to find y-intercept
Interviewer	Ok, don't worry. Let's go back to your test so you have a bit of a hint. The first thing you said is "when x is 0, y is 1 in the equation," and that's what you got. And then it looks like you just tried a whole lot of different gradients until you found the right one. Am I right?	Offering learner to look at assessment
Learner 1.9.22	Oh yes. Ok so… [mumbling] Ok, so… mhmmm I said y = … cos I took… I tried different gradients every time, then I substituted x which was either of these, so I put, maybe 0, and then found out thingy, y would equal to.	Deed: choosing different gradients at random
Interviewer	So I see here you tried $2x$ first, and then you substituted in -1, and you got 1, so you thought that's wrong?	Clarifying
Learner 1.9.22	Yes.	Confirming
Interviewer	Then you tried $4x$, you substituted in -1 and you got 5, and you	Clarifying

	thought that's also wrong. And then you tried *3x*, and you substitute in -1 and you got -2, and you thought great!	
Learner 1.9.22	Yes	Confirming

The excerpt above shows that the learner uses one deed in her discourse about Question 3. This deed was the apparent random choice of a co-efficient for *x* in finding the gradient for the function. The learner then fell back on a previous, very procedural ritual, which aimed to test if the chosen function was correct by using substitution.

Rituals

Learners who used rituals in Question 3 all started out by finding the gradient and the intercept to calculate the equation. This shows a procedural understanding of equations as learners use a 'recipe' to complete the task. These learners seemed to want to answer the questions correctly, and were in search of social approval from the interviewer.

The following excerpt shows a typical interview of Question 3.

	Transcript	Routine
Interviewer	So the first question that I want you to do is question number 3. Please explain to me how you would go about completing it.	
Learner 3.10.7	Um. The standard form of this is $y = mx + c$. And then. So that C is where the *x*-axis … where it is zero. The y-axis, the y intercept. So I see here where the *x* is zero, that's 1. So immediately I would say, $y = mx + c$, then $y = mx + 1$. Then I would work out the *m*. a lot of people use rise over run, but I can't do that so I use $\frac{y_2-y_1}{x_2-x_1}$. Then I would pick two points. Let's go with these two because they are easy [points at two points in the table] y_2 is 1, minus 4, over x_2, 0-1. And 1-4 is negative 3. 0-1 is negative 1. So that is 3.	Ritual: giving the standard form of the equation Procedural discourse

Interviewer	Great.	
Learner 3.10.7	So then *y* equals 3*x*+1.	Giving the answer
Interviewer	Perfect.	

To find the gradient, learners who used ritualised discourse used the formula $m = \frac{y_1-y_2}{x_1-x_2}$ in their calculation which can be seen as the imitation of their teacher's routines. That many learners used this equation shows that this is typical routine; a repetitive and well-defined discursive pattern. Their use of these routines can be seen as the individualisation of the discourse, and hence these learners are learning to participate in the discourse.

Many of the learners, during the interviews, said the equation $m = \frac{y_1-y_2}{x_1-x_2}$ out aloud, as if it were a mantra. This is hard to convey in the transcripts, but as the interviewer, I took notes on this phenomenon during the interviews. This shows that there is mimetic nature to learning. Ben-Yehuda et al (2005) say that "more often than not, such learning is mimetic (cf. Diamonstone 2002; Seeger 1998), that is, it results from following discursive patterns of more experienced interlocutors".

Another common ritual which was seen was in Question 6 where learners used substitution to find the values for y in Table B. This, however, was not often seen in the interviews as many learners spontaneously used the more holistic strategy for answering Question 6 (Strategy 2).

Rituals have a narrow scope – this is shown in the ritualistic discourses where learners are able to perform calculations on a specific set of values only – unlike explorations, which can be applicable to a broad number of calculations.

Explorations

Explorative routines were seen in two different ways in the interviews. Firstly, there were learners who used explorative routines without any prompting. Explorations talk about objects as a whole, and this was seen in the discourses of learners in Growth Point 4. Secondly, explorative routines were seen in the discourses of learners who did not use these routines in their assessments, but due to the prompts of the interviewer, were able to spontaneously use these discourses in Question 6.

Learners in Growth Point 1 who were prompted to look for a pattern in the *y*-values of the two tables really struggled to see the pattern. Some learners did see the pattern but were not able to relate this back to the equations at all.

The following excerpt from an interview shows that the learner is using an exploratory discourse which will result in an endorsed narrative – a narrative which is seen to be true.

	Transcript	Routine
Interviewer	If you could look at Question 3 for a little bit and then explain to me how you did it.	
Learner 4.11.20	Ok [silence] Ok well. I see that the *x*'s are all the same. The y's are the ones that change. So I compare these two. [pointing to the *y* columns on each table] So, it's kind of like number patterns. So, like, 0-2 will give you negative 2. Then I start to see the difference. And then because, the constant is what is affecting the *y*-value, moving up or down, that's why I will take away 2 from this function.	Exploratory discourse showing an operation on an object.

The following two excerpts show learners who both used a ritual when answering Question 3, but after the prompts from the interviewer, were able to use an exploratory discourse.

The first learner explores the holistic view of question 3. This learner was able to see the holistic view of the equation when scaffolded by the interviewer.

	Transcript	Routine
Interviewer:	Now I want you to look at these two [points to tables]. And I want you to look, and see if you notice anything – we can see that all the *x* values are the same, but the *y* values are a little bit different. So do you notice a pattern or anything interesting?	
Learner 2.10.5	Every single value on this side [Table B] is minus 2 of this [Table A].	Beginnings of exploratory routine
Interviewer	OK, so do you think you would be able to find the equation in an easier way? Do you remember what it was?	
Learner 2.10.5	It was 3*x*+1. You can just minus 2 from the *y* values!	

This learner has shown that she is now able to see the function as an object which can be transformed or manipulated. In seeing that one can subtract 2 from all the *y*-values, she has shown that it is not necessary to perform a procedural. This can be seen as the start of reification, where the learner has moved from seeing the task as a series of processes, to a holistic object. The shift in this case is the start of objectification.

This learner has spontaneously used an exploratory discourse while answering Question 6.

	Transcript	Routines
Interviewer	Now the next question I want you to look at is this question here. Number 6.	
Learner 3.10.7	Ok, well I would actually do it the long way, but from the previous table, where that, like, link thingy with the plus 3 and all that, in the other one, I think you could do it from this table without any calculation, because that [table A] is exactly the same as that [table B], with no plus 3.	Spontaneously using an exploratory routine
Interviewer	Alright	
Learner	So, I would think that is this one [Pointing to Table B] has	Justifying

3.10.7	got no 3, so you would minus the three from it [pointing to table A], so you would do the same on this side. So that would be 0,0. And then 7-3 is 4. And 10 , 18, and then 28.	
Interviewer	That's excellent. Maybe you want to substitute in one number to check that you have got the right answer.	
Learner 3.10.7	[writes] And yes that is! That's pretty cool.	Checking that answer is correct

Learners who have been labelled as having exploratory rituals were all able to immerse themselves (some spontaneously) in the new discourse, and at the same time were trying to figure out the rules of the new discourse (meta-level learning). Each time the learner provided a rationale as to why they used the new objectified discourse, instead of a procedural discourse.

5.2.5 Endorsed Narratives

Narratives are the sequences of spoken or written texts about an object. These show a description of the object, associations between objects or processes with or by objects (Sfard, 2008). Narratives are subject to endorsement or rejection. Within school algebra, narratives are endorsed if they conform to the confines of school maths[6]. Endorsed narratives are the result of explorative routines which are used to verify or produce this endorsed narrative, and are also discussed in the following section.

In the above analysis of routines, I have shown that endorsed narratives result from exploratory routines.

5.2.6 Overall analysis of Questions 3 and 6

The table below (Table 5-1) shows an overview of the discourse used by each learner in Questions 3 and 6. This overview shows that learners in the lower growth points mainly use rituals in their discourse, and rely on procedural discourse to complete tasks. Learners in higher growth points are more and more able to use exploratory routines in their discourses, and hence their discourse is mainly objectified.

[6] For example, in school mathematics, the solution to a quadratic function may not exist if there are none real roots, whereas within a more formal mathematics discourse the solution of quadratic function may be found using complex numbers.

The table also shows that those learners' discourses start off by being very procedural in nature, but as they progress to higher growth points, their discourse becomes more and more objectified.

There was one learner (Learner 2.10.23) in Growth Point 2 who started using exploratory rituals in the interview. This did not surprise me, as this learner was in Grade 10. I would have thought it to be an anomaly if a Grade 9 learner achieving Growth Point 2 started using exploratory rituals spontaneously.

There was one learner (Learner 3.10.12) who did not start using an exploratory discourse in the interview, where one would have expected this. The reason for this may be that she was the learner with whom the interview was piloted. For this reason, the interviewer did not ask the leading question "What did you notice about the *y*-values in the two tables", and hence the learner did not get the opportunity to notice the holistic nature in which the question could have been answered.

Table 5-2 Figure showing the overall comparison in learners for Questions 3 and 6.

Learner code	Main type of language use	Routine (Question 3)	Routines (Question 6)	Type of discourse
Growth Point 1				
Learner 1.9.3	Informal	Ritual	Ritual	Procedural
Learner 1.9.22	Informal	Deed	Ritual	Somewhat Procedural
Learner 1.9.8	Informal	Ritual	Ritual	Procedural
Learner 1.9.33	Informal	Deed	Ritual	Somewhat Procedural
Growth Point 2				
Learner 2.10.23	Informal	Ritual then Exploratory	Exploratory (Spontaneous)	Becoming Objectified
Learner 2.10.5	Informal	Ritual	Ritual	Procedural
Learner 2.10.14	Informal	Ritual	Ritual	Procedural
Growth Point 3				
Learner 3.11.9	Informal	Ritual then Exploratory	Exploratory (Spontaneous)	Becoming Objectified
Learner 3.10.7	Informal	Ritual then Exploratory	Exploratory (Spontaneous)	Becoming Objectified
Learner 3.10.12 **(Pilot)**	Informal	Ritual (Was not prompted – pilot interview)	Ritual	Procedural
Growth Point 4				
Learner 4.11.25	Informal	Exploratory	Exploratory	Objectified
Learner 4.11.15	Informal	Ritual then Exploratory (Used Exploratory in assessment)	Exploratory	Objectified
Learner 4.11.20	Informal	Exploratory	Exploratory	Objectified

5.3 Other observations

During the interview, there were a few additional interesting observations which I shall discuss here.

5.3.1 The informal nature of the discourse

The excerpts of the interviews show that the word use of the learners in the study is often very informal. This informal use of words can create ambiguity but because the interviews were situated, the interviewer was able to gather the meaning of the discourse in its entirety by referring back to the videos of the interviews. Even though there were some learners who had objectified discourses, and some who were well on their way to having an objectified discourse, I found it interesting that many learners still used a very informal discourse in the interviews.

The following two examples show the very informal nature of the discourse that was used in the interviews.

Learner 2.10.5	: Every single value on *this side* [pointing at Table B] is minus 2 *of this.* [pointing at Table A]

Learner 3.10.7	Ok, well I would actually do it the long way, but from the previous table, where that, *like, link thingy* with the plus 3 *and all that,* in the other one, you could do it from this table without any calculation, because that [points to table A] is exactly the same as that [points to table B], with no plus 3.

Both the excerpts above show that even though the learners are able to talk about a function in a holistic way, their discourse is still very informal. Because the interviews were in the company of the interviewer, the learners perhaps thought that they did not necessarily have to formally explain their solutions to the task. This type of ambiguous talk also confirms that the use of a video recorder for the interviews was useful, as not only was rich data collected, but the videos also serve as references to the body language (Learners pointing to tables, etc).

Ben-Yehuda et al (2005) explain that a literate discourse is one where there are no ambiguities. In the excerpts above, the discourse is clearly ambiguous, but the meaning which was conveyed to the interviewer showed that the learner had made sense of the problem and was able to provide an accurate explanation of how to complete the problem in a holistic manner. However, Caspi and Sfard (2012) explained that the achievement of algebraic milestones is common to both the formal and informal strands of discourses, and that these “find their expression in different modalities – In the purely verbal form in the case of informal discourse, and in the form of symbolic expressions combined with additional visual constructs, such as graphs, otherwise.” (Caspi & Sfard, 2012, p.49)

5.3.2 Flexibility and corrigibility in discourses

Flexibility is the manner in which a student is able to produce more than one response on how to complete a calculation. Corrigibility is the ability to self-correct one’s discourse when an error is made (Ben-Yehuda, et al, 2005). These are both aspects of routines in mathematical discourse.

Both flexibility and corrigibility were seen in the routines of learners in the interviews. Flexibility was seen in some interviews and assessments where learners provided more than one way in which they answered a task, however this was more apparent in the written answers from the assessment booklets, and not in the interviews – unless prompted by the interviewer (This was discussed earlier).

Corrigibility – or the ability to self-correct – was seen in the interviews. There were many instances during the interviews when a learner would make a mistake and then self-correct after realising that there was a mistake. Some learners took longer than others, and some were prompted by the interviewer.

See the following excerpt for an instance where a learner has self-corrected of her own volition.

Interviewer	So the next question I want you to look at is 6.
Learner 4.11.25	I just said these answers [*y* values in Table A], **plus 3**, and then put them into the new table [Table B]. [silence] Oh wait, it’s **minus 3**! Ok, 0, 4, 10, 18, 28.

This excerpt shows that the learner immediately was able to correct her own discourse without the interviewer pointing out her mistake (that she was adding 3 instead of subtracting 3). She has retraced to an earlier point in her discourse, found her mistake and corrected it (Ben-Yehuda et al, 2005).

Interviewer	Now I want us to go back to question 2. I want you to explain what you did over there.
Learner 2.10.5	Ok so the first thing I did was I made up some values for *x*, from like 1 to 4, and so I used, I substituted *x*, as 1 into both of the equations I thought would end up giving the most, and then I compared the two differences. The answers between the two – after I substituted.
Interviewer	So I noticed you used *a*, and you also used *d*. Over here [for option *a*] it went 99, 98, 97, 96… and here [for option *d*] it went 80, 85, 90. But you still told me that [referring to option 1] gave you the fastest change?
Learner 2.10.5	Oh! The fastest CHANGE…. I see now where…. I think I just complicated it a bit to which one is going to give you the **most**.
Interviewer	Ah, I see. So you were looking for the highest answer instead of the biggest change.

The learner then redid the task and successfully corrected her error. This excerpt shows that the learner retraced to find the source of confusion, and also switched mediation by recalculating her answers by writing down her calculations (Ben-Yehuda et al, 2005).

5.3.3 Revoicing by the interviewer as a means for clarification

There were instances in the interviews where I asked learners to explain their thinking for a question, and they weren't able to do so. They had not been able to complete the task in the assessment so I was curious to ask them why they were not able to do the task. In these cases, I, as the interviewer, decided to revoice the question in order to see if the actual question in the task, and its wording had been the cause of the learners' inability to complete the question. In the following case, I asked Learner 10.14 about Question 2 (See Figure 5-2).

2. Which equation shows the fastest change in y when x moves from 1 to 10? Please show or explain how you got your answer.
   a. $x + y = 100$
   b. $y = 6x - 3$
   c. $4y = 8x$
   d. $y = 75 + 5x$

Figure 5-3 Question 2

The learner was coded at Growth Point 2 after the assessment, but the following excerpt shows that the learner does have a conception of gradient (one of the indicators of Growth Point 3), however this conception only came about after the interviewer revoiced the question.

Interviewer	That is your test. I want you to first look at Question 2, which you didn't do at all on the test. I want to see if you can now do it. Because you have seen it before you may have thought about it, and can do it now? [silence] Or if you didn't understand the question, maybe you can tell me why you didn't understand it?
Learner 10.14	I think I just looked at the question and I was like "what is this?" [silence]
Interviewer	Maybe if I reworded the question, and I said, which equation has the steepest gradient, would that help you?
Learner 10.14	Yes it would.
Interviewer	Let see if that makes more sense to you now.
Learner 10.12	So I put all of them in standard form. So this one would be {writes} And I arrange this one {writes}.
	[The learner rewrote the equations so that they were all in standard form of $y = mx + c$]
Interviewer	So what would the answer be?
Learner 10.14	I think the one with the highest "m" value.

In this interview, the learner was not able to make sense of the question in the assessment to start with, but after the interviewer reworded the question, the learner quickly understood the question and was able to complete without much difficulty.

5.3.4 The difficulty of context/no-context

A learner spoke of the difficulty of understanding some tasks which were abstract in nature and had no real life context. In the following excerpt, Learner 10.12 explains how she found Question 2 to be very difficult until she adapted it to her knowledge of a scientific context. She then was able to complete the question successfully.

Learner 3.10.12	So in Question 2 I went and I put all the same constants on the same side, because you can't work out that the gradient is if they are all in a different order. So I went and I got *y* on the same side, and I worked out what each of the gradients was, for each of the graphs that were there. I thought about it in terms of Science, because it makes more sense in a practical way to think about it in Science, because Maths is very one-dimensional. Besides you can think about it actually happening. And it meant that I could understand the whole idea of somebody moving along, from point 1 to 10, instead of just a graph going nowhere.
Interviewer	So what did the gradients tell you?
Learner 3.10.12	The gradient told me how fast they … like …. The gradient is the slope, and the slope is telling me how fast they are moving. So if the gradient is a positive slope, it is getting faster. And if it is a negative slope, they are getting slower.

The excerpt shows that some learners have difficulty in understanding an abstract context, and that a real-life example may provide assistance in the understanding of the question.

This however is an incongruous observation as many learners also struggled with Question 1 in the assessment which *was* a context based question. In fact, the same learner who complained about the abstractness of Question 2 struggled with the context in Question 1. This is discussed below in relation to visual mediators.

5.3.5 Additional use of visual mediators

This shows an excerpt from the interview with Learner 10.12. This particular interview was the pilot interview, and took a bit longer than some of the other interviews. While it was ambitious to ask so many questions in the pilot interview, I managed to get some very rich data from this interview.

In this particular section of the interview, we discussed Question 1. As shown in Chapter 4, Question 1 was a difficult question for many learners. In the assessment, Learner 10.12 did not get any of the tasks right in Question 1. I decided to use this question as a part of the interview anyway.

1. Imagine water flowing through a pipe into a container for 10 minutes. The following equations show how the height of the water (w) in the container, and how the height is related to the number of minutes (t) when the pipe was opened.

$w = t + 8$ for the first four minutes

$w = 3t$ for the remaining 6 minutes

Please use the above information to answer the following questions

Figure 5-4 Question 1

Because the learner struggled quite a bit in the interview, I drew a small graph to represent the relationship between the height of the water and time. It was a piece-wise function as shown in Figure 5-5.

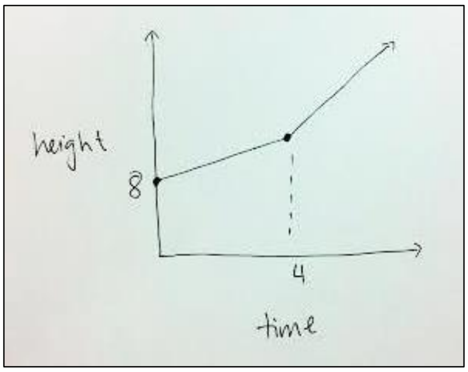


Figure 5-5 Graph which represents relationship between the height of the water and time.

Interviewer	[Draws graph as illustrated in figure 5-6]
Learner 3.10.12	[.....] and then I can work out what the height was?
Interviewer	Absolutely.
Learner 3.10.12	I don't know why I didn't think of that sooner, it's very simple. So then it would have been 11.

Interviewer	I didn't give you a unit, so it could be anything. Centimetres, meters whatever. We don't know how big the tank is.
Learner 3.10.12	So much simpler than I thought it was. I don't know why I didn't think of drawing a graph. It would have made it so much easier. Can I use the graph to show you my answers, right?
Interviewer	If you want to, sure
Learner 3.10.12	(Learner 10.12 replying to question 1b) OK. No, again. Because if you look at the graph that you drew, it shows that it starts off with one gradient, which is obviously how fast it is going then it gets a steeper gradient, which shows that it's going faster. So, in order for it to increase at the same rate, it would have to be moving at a constant slope, but it's not, so the answer would have to be no.

The resulting realisation from the learner as seen in the excerpt below showed that the visual mediator helped the learner to understand the question.

The learner uses the graph to explain to help justify her reasoning for Task 1b. This indeed shows that visual mediators form a part of the learners' mathematical discourse. This occurrence can also indicate a well-developed sense of "linking representations" which is another domain in Ronda's Framework of Growth Points (See Figure 2-1 in Chapter 2).

5.3.6 Preference for familiarity of procedural thought.

The issue of familiarity of procedure seems to snag many learners as they move from procedural to objectified thinking. The learner in the excerpt below was graded at Growth Point 4 after the assessment. In the interview she used the gradient formula to work out the equation in Question 3 almost as if doing so automatically, even though she showed evidence of objectified thinking in the actual assessment by using Strategy 2 to answer Question3. I asked her about this in the interview.

Learner 4.11.15	I just see the first thing that comes to mind, and do it even if it's the longest way around.
Interviewer	Is it because you feel more comfortable with the traditional way?
Learner 4.11.15	Yeah it's just because we've done it so often. That we have been taught that when you need to work out an equation, you work out the gradient, and then substitute a point in, and then find the c value

This excerpt has shown that even though learners may be able to partake in an objectified discourse, that this does not necessarily always take place.

5.3.7 The Difficulty of Maths and Identity

In many interviews, learners' discourses about their explanations of the solution of a task were interspersed with talk about their identity, or utterances which were about themselves.

This example below is an excerpt from a learner who began to talk about what they found difficult in the subject of mathematics. Although this was not in the scope of the study, I found this to be interesting.

Learner 2.10.5	I just think that sometimes teachers make it seem so difficult, that you always think there's something extra in there … so you start working out everything else except what you're supposed to be doing, and you'll have a page of working out but you can't get anything cos everything else didn't relate back, and in the meantime, it's so easy.
Interviewer	So what you're saying is, you learn a procedure – a recipe – of what to do, but you don't actually understand?
Learner 2.10.5	Yeah, something like that. And then you think that you do understand, then you confuse it with something else, cos then they put extra stuff on top of that, and then … In the meantime, you just have to look at it to see ... Oh, wait, you can do it like that.

Additionally, there were many other instances where the talk about mathematics was interspersed with identifying utterances. Here are some more examples.

Learner 2.10.23	**I wouldn't know** it's a straight line.
Interviewer	How did you do it last time, do you remember?
Learner 2.10.23	**I think I** did a table.

Interviewer	I'm going to ask you to go to Question 3 please.
Learner 2.10.5	I had a bit of difficulty with finding the equations, because I get confused and **then I confuse** all the different concepts that I have together, and it all comes into one.

There were cases of learners in Growth Point 4, who showed that they were able to work with functions, who still interspersed their explanations with identifying utterances of a somewhat emotional nature. These are bolded in the following excerpts.

Interviewer	And let's look at Question 1. I want you to talk about b and c, what your reasoning was. I know you wrote it down, but I just want to understand your reasoning.
Learner 4.11.20	Oh ok. So for b) **I thought** that because, even though both of them [the equations] are talking about *w* it's *a (Option a in the question)* it's an equation concerning *w*. But *t*+8 and 3*t*, they are… the one is an addition, and the other one is a multiplication, **that's why I think that**… you will always get different answers for it.

Interviewer	What does that show?
Learner 4.11.15	Agh, **I don't know. I really have no idea. It's right there in my mind but I don't know how to put it.**
Interviewer	That's ok.
Learner 4.11.15	Yeah, **I don't know**.

These excerpts from interviews confirm the point which Heyd-Metzuyamin and Sfard (2012) have made about mathematical discourse – that it is somehow inextricably linked with the discourse of identity. Heyd-Metzuyamin and Sfard (2012) see affect and social meters as aspects of the discourse

5.3.8 Other considerations of the study

A shortcoming of this study may be that learners were interviewed one-on-one with the researcher, which may have enticed the learner to take on the discourse of the researcher in some situations. However, the practice of one-on-one interviews was chosen so that learners would not be swayed by a classmates discourse. I also felt that one-on-one interviews were necessary because I wanted to ask the learner about the assessments which had been competed by them previously. Group interviews are more suited to situations where learners are completing rich tasks, generally unseen or novel.

As shown in the interview with Learner 10.14 in the section entitled “Revoicing by the interviewer as a means for clarification,” it emerged that the learner was able to do the question after the interviewer clarified the question by rewording it. This shows that Question 2 may have been answered better if it were worded differently in the assessment. This again points to the difficulties learners may experience if they are not entirely fluent in the language that they are being educated in.

5.4 Discussion and conclusion

Overall, most of the learners were comfortable to be interviewed, and hence good data were collected on their discourses. The interviews were successful in that they did show that the discourse of learners is related to the Growth Points that they have achieved.

Additionally, the interviews also brought other issues to light. For example, even though some learners seemed to have a holistic conception of functions in equation form, they still reverted back to a point-by-point analysis of the equations. Some learners said that this was due to familiarity. Another issue was that of context. Learners found the context on Question 1 to be confusing rather than helpful.

“I also hypothesize that as long as school teaching focusses on *how* routines should be performed to the almost total neglect of *when* this performance would be most appropriate, it is more likely to result in the discourse of rituals than of explorations.” (Sfard, 2008: p223).

Chapter 6 **Conclusions**

6.1 Summary of the research process

The study was done on a continuum of the constructivist-pragmatic-commognitive paradigm in order to fulfil the expectations of the study – which were to firstly affirm the framework of Growth Points, and secondly to explore the mathematical discourse of learners within the overall structure of the framework of growth Points. The research process can be summarised as a mixed methods study in which both qualitative and quantitative methods were used. These were chosen to allow me to gather the data that I required in order to answer the questions that I set out to. In collecting both quantitative and qualitative data, my data set was rich, which meant I was able to draw good data from the research process.

6.2 Summary of findings

6.2.1 Findings: Phase 1

The research question which guided the first phase of research was “Using Ronda’s Framework of Growth points, where do selected South African learners fit, especially in relation to functions in equation form?”

The finding of the study showed that learners generally followed the same learning trajectories which were described in Ronda’s study. Learners in my study moved from growth point to growth point in a similar manner to Ronda’s study, and followed the growth points mainly consecutively with the exception of a few learners.

The strategies used by learners in answering selected questions reflected the growth in learning of functions. The strategies that learners used in questions showed growth over grades. Progression through grades showed a higher number of learners used more objectified strategies.

Overall, it seemed like the learners in my study seemed to progress faster along the learning path over the duration of 3 years, than the learners in Ronda’s study. This however could be attributed to the fact that the learners in my study, at the highest grade-level for the study were a year older than the learners in Ronda's study. This opens up an opportunity for further research.

6.2.2 Findings: Phase 2.

The research question that guided this second phase of my research was "How do learners' discourses relate to the growth points they have achieved?" I explored this question by conducting interviews with selected learners from each different growth point.

Looking at the discourse of the learners, it is clear that their discourse somewhat mirrors their growth point levels. Their discourse cannot pinpoint their growth point exactly, but will indicate somewhat the degree to which their discourse is becoming objectified. I looked at the learners discourse with the end goal that they would be able to see a function holistically, instead of something which can be operated on in a point-by-point manner.

All learners in Growth Point 4 demonstrated exploratory routines, and their discourse was objectified. This corresponded with the characteristics of Growth Point 4.

The interesting finding of the study was the learners in Growth Points 2 and 3 who were able to spontaneously use an exploratory routine in the interview. This happened due to the prompting by the interviewer – who was, in this situation, an experienced interlocutor. This phenomenon verifies Sfard's (2008) view that learning takes place through participation. "The participationist vision of human development implies that any substantial change in individual discourse, one that involves a modification in meta-rules or introduction of whole new mathematical object, must be mediated by experienced interlocutor" (Sfard, 2012, p.254).

The fact that both Grade 9 and 10 learners were comfortable in using ritualistic routines to begin with shows that rituals are perhaps a precursor to exploratory discourses. Sfard (2008) explains that the transition from ritualistic to exploratory routines:

> "[The] transformation can happen quite abruptly, so that the stage of ritualization is hardly noticeable, or it can last for a long time, perhaps even forever. The transitory phase of ritualization corresponds to the period of individualising – the period during which the learner can participate in the collective implementation of the routine but is not yet capable of independent performance" (Sfard, 2012, p. 253).

Different types of exploratory routines were seen in the interviews.

> "All the exploratory routines can be divided into three types: *construction*, which is a discursive process resulting in new endorsable narrative; *substantiation*, the action that helps mathematics decide whether to endorse previously constructed narratives; and *recall*, the process one performs to be able to summon a narrative that was endorsed in the past." (Sfard, 2008, p. 225)

Learners in Growth Point 4 who were able to use exploratory routines without prompting used either *recall* or *substantiation,* whereas the learners who spontaneously used exploratory routines in the interview for the first time used the *construction* type of exploration (Sfard, 2008).

Sfard mentioned that there are conditions on the *how* and the *when* of the routine. Important to this study is the *when* of a routine. Schools seem to focus on the *how* of a routine, whereas the *when* of the routine isn't given attention. The *when* of the routine is important as one needs to know the best time to teach a concept. This links in with learning trajectories. It was seen by the learners in Growth Point 1 that they were not able to spontaneously partake in exploratory routines, even with the mediation of an experienced interlocutor. This is perhaps a sign that learners are not yet ready to see the function as an object. The idea of function as an object, "as a static 'thing', when introduced too early is doomed to remain beyond the comprehension of many students" (Sfard, 1992, p.77). This further shows that learners should be comfortable with seeing a function as a process before seeing it as an object.

6.3 Other findings of the study

Preference for point-wise analysis

In Ronda's study, she found that many learners were not able to progress further than a point wise understanding of equations in function form. In my study, some learners, even though they seemed to understand functions as an object, opted to operate on the problem in a point-wise manner. This shows that even though it is more tedious, some learners prefer to use a more familiar method, instead of an easier, objectified discourse.

This is highlighted by learner 4.11.15

Learner: *"I just see the first thing that comes to mind, and do it even if it's the longest way around….*

Interviewer: "Is it because you feel more comfortable with the "traditional way"?"

Learner: *"Yeah, it's because we have done it so often. We have been taught that when you need to work out an equation, you work out a gradient, and then substitute a point in, and then find the c-value."*

This is highlighted in Ronda's study too where she confirms the preference for point-wise thinking.

> "Although the percentage of students at the growth points was increasing from Year 8 to Year 10, which was to be expected, the majority of students only achieved Growth Points 1 and 2. Both these growth points involved point-wise thinking. This finding seems to suggest that advanced students continue to operate using point-by-point interpretations, despite their experiences with other functions" (Ronda, 2004, p.167)

Difficulty with Rate

Many learners showed difficulty in understanding rate, which was shown by the results of Questions 1 and 2 in the assessment. Although most learners in Grade 11 (84%) and some learners in Grade 10 (32%) reached Growth Point 3, this is still not as high as I would have expected, especially at Grade 10. The notion of rate is taught from Primary School, and continues in high school from Grade 8 level, even if not formally associated with linear functions. This again corroborates Nachlieli and Tabach's (2012) perception that functions are a topic which is not easily nor quickly comprehended.

6.4 Contribution to Knowledge

This research report has contributed to the field of mathematics education in the following ways.

Firstly, it has provided confirmation and clarification of the recently developed research-based framework on learning trajectories which has been proposed by Ronda (2004). Results in my study mirrored those in Rondas study to an acceptable extent and hence confirmed its validity.

Secondly, this study has explored learners' discourse while completing selected tasks on functions. Hence, I have related the learners' discourse strategies to the growth points shown in Ronda's framework. I used Sfard's communicational framework as a lens to look at the discourse of learners relating to their achievements in the framework of growth points. I found that Sfard's communicational framework provided a comprehensive way in which to analyse discourse.

6.5 Implications for teaching and learning

This study has started to find the building blocks that will build the bridge to close the gap between theoretical and practical aspects of the learning and teaching of functions. This study has replicated an empirical study on learning trajectories, and found that the Framework of Growth Points does indeed match up with practical aspects of learning of functions in equation form.

This study was done in a school which is well resourced, had good teachers, and had good leadership. Even though the results of this study cannot be widely generalised as the sample was small, I think that the findings will be able to inform some aspects of teaching.

The learning of functions is not an easy process. By the results of this study which was done in a functioning school, it can only follow that the learning of functions will be an even more difficult process in schools that are not as well run as my research school.

Because of the good teaching environment in the research school, I would argue that the results from this study showed a progression through the growth points which is standard to what should be happening in all schools. It may be a model of learning which schools could aim to replicate.

Additionally, this study has shown that there are types of tasks which can accurately test the level of understanding of a learner. The assessment task used in the study can be replicated or used as a template for other assessments which can point to the growth points of learners in everyday teaching.

This study has shown that learners follow a trajectory of learning. Ronda's framework of growth points can therefore be used in the design of the curriculum and the design of learning materials in the area of functions. This will ensure that learners are taught about functions in a sequence which is most likely to make sense to learners.

Regarding the preference of the majority of learners to use point-by-point analysis in their solutions, I would recommend that teachers design and use tasks which require a holistic analysis; however this should not be done too soon in the learning process as this may alienate learners.

6.6 Recommendations for further research

Research has been done by Ronda into the learning trajectories of learners as they encounter functions. My study only covered one area of Ronda's Framework of Growth Points and that was *functions in equation form*. Because there has been a renewed interest in the field of learning trajectories this area of research is topical and relevant. This study may provide the motivation for further research into learning trajectories in other areas of school mathematics. This study may also further provide a motivation for further research into the nature of learners' discourses in other areas of algebra.

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Appendix A – CAPS and NCS

Grade 9

Assessment Standards

We know this when the learner:

- Draws graphs on the Cartesian plane for given equations (in two variables), or determines equations or formulae from given graphs using tables where necessary.
- Determines, analyses and interprets the equivalence of different descriptions of the same relationship or rule presented:
  - verbally;
  - in flow diagrams;
  - in tables;
  - by equations or expressions;
  - by graphs on the Cartesian plane

  in order to select the most useful representation for a given situation.

Figure A-1 NCS curriculum for Grade 9 Functions

GRADE 9

Input and output values

- Determine input values, output values or rules for patterns and relationships using:
  - flow diagrams
  - tables
  - formulae
  - equations

Equivalent forms

- Determine, interpret and justify equivalence of different descriptions of the same relationship or rule presented:
  - verbally
  - in flow diagrams
  - in tables
  - by formulae
  - by equations
  - by graphs on a Cartesian plane

Interpreting graphs

- Revise the following done in Grade 8:
  - analyse and interpret global graphs of problem situations, with a special focus on the following trends and features:
    - linear or non-linear
    - constant, increasing or decreasing
    - maximum or minimum
    - discrete or continuous
- Extend the above with special focus on the following features of **linear graphs**:
  - *x*-intercept and *y*-intercept
  - gradient

Drawing graphs

- Revise the following done in Grade 8:
  - draw global graphs from given descriptions of a problem situation, identifying features listed above.
  - use tables of ordered pairs to plot points and draw graphs on the Cartesian plane
- Extend the above with special focus on:
  - drawing **linear graphs** from given equations
  - determining equations from given **linear graphs**

Figure A-2 CAPS curriculum for Grade 9 Functions

Appendix B – Assessment

Question Booklet

***NAME:*___________________________________**

***GRADE:* __________________________________**

***AGE:* ____________________________________**

Instructions

- Please attempt to answer all questions in the booklet
- For each question, please give an explanation as to how you answered each question
    - If you cannot answer a question fully – do not worry. Please do try to explain why you cannot do it, if possible.

I hope you find the questions interesting.

Thank you

1. Imagine water flowing through a pipe into a container for 10 minutes. The following equations show how the height of the water (w) in the container, and how the height is related to the number of minutes (t) when the pipe was opened.

$w = t + 8$	for the first four minutes
$w = 3t$	for the remaining 6 minutes

Please use the above information to answer the following questions

a. What was the height of the water in the container 3 minutes after the pipe was opened?

b. From the given information, do you think the height of the water in the container is increasing at the same rate throughout the 10 minutes? Circle the letter corresponding to your answer.
   a) Yes, the water level increases at the same rate throughout the 10 minutes.
   b) No, the water level is not increasing at the same rate throughout the 10 minutes.

Please show or explain how you obtained your answer.

c. From the given information, do you think the container already contains water before the pipe was opened? Circle the letter corresponding to your answer.
   a) Yes, the container already does contain some water before the pipe was opened.
   b) No, the container did not contain water before the pipe was opened.

Please show or explain how you obtained your answer.

2. Which equation shows the fastest change in y when x moves from 1 to 10? Please show or explain how you got your answer.
   a. $x + y = 100$
   b. $y = 6x - 3$
   c. $4y = 8x$
   d. $y = 75 + 5x$

3. Examine the two tables shown below. The set of values in the table on the left shows specific values of $y = 3x + 3$.
Please give the equation which will result in values shown in the table on the right. Please show or explain how you obtained your answer.

x	y
-1	0
0	3
1	6
2	9
3	12

x	y
-1	-2
0	1
1	4
2	7
3	10

$y = 3x + 3$

$y =$ ____________________

Explanation:

4. The relationship between x and y in Table 1 is $y = 2x + 1$.

In Table 2, the values of x and y in Table 1 were swapped. Please write the equation which shows the new relationship between x and y in Table 2. Please show or explain how you obtained your answer.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

Table 2

x	y
1	0
3	1
5	2
7	3
9	4

$y = 2x + 1$ ______________________

Explanation:

5. The relation of s with p is shown in the equation: $s = 5p + 3$. The relation of p with n is shown in the equation: $2p = 6n$. From this information, please write the equation that will show the relation of s with n.
   Please show your working or explain how you got your answer.

6. Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ is shown on the table on the left.
Fill in the table on the right with values for of $y = x^2 + 3x$
Please show or explain how you obtained the y values.

$y = x^2 + 3x + 3$

x	*y*
0	3
1	7
2	13
3	21
4	31

$y = x^2 + 3x$

x	*y*
0	
1	
2	
3	
4	

7. The relation of r with q is shown in the equation of $r = 5q + 3$. The relation of q with t is shown in the equation $2q = 6t$. If $t = 5$, what is the value of r? Please show or explain your solution.

8. Circle the equations that show the same function or relationship.
   a. $y = 4x + 8$

   b. $p = 4(s + 2)$

   c. $y = \frac{x-8}{4}$

   d. $p = 4s + 8$

Explanation:

THE END THANK YOU!

Appendix C – Interview Structure

Interview Schedule: The following proved a rough outline of the interviews

Opening

Hi, my name is Robyn and I am here to ask you a few questions about the assessment on functions you did 2 weeks ago. I am doing my Masters in Maths Education at Wits, and as a part of this course; I am doing a research report about how students learn about functions.

I'm going to ask you these questions because I am interested in finding out about what you were thinking as you answering the questions in the assessment.

The interview should take about 15 minutes, it that ok with you?

Like I said in the letter which was sent to you and your parents, I will be video-taping this interview. Is this still fine with you? Remember that if you feel uncomfortable at any point, you can ask me to stop video-taping.

Let me start by asking you these questions:

1. What is your name? (for clarification)
2. What grade are you in?
3. Do you remember writing this test 2 weeks ago?

Don't worry if you thought it was hard. The aim of the assessment was to get you to think! I have chosen to interview you because you wrote down some very interesting answers, and I would like you to talk to me about them.

I am going to give you some of the same questions that were in the assessment. What I want you to do is explain to me how you got to the answer, or how you would start to get to the answer.

I have given you a pen and paper, so you are more than welcome to write, if you need to do some working out, but please remember to explain to me what you are doing at the same time.

Body

LEARNER IS GIVEN UNCOMPLETED ASSESSMENT SHEET

4. Please look at this task 3. I would like you to do it for me, and as you do it, please explain your thinking and reasoning.

I let the learners complete the question and then asked the following:

If the learner used point-point reasoning for Question 3

5. Please look at the question and see that I have given you two tables. Do you notice any similarities between the tables?
6. (Another prompt if needed: Look at the x-values in the table; do you agree that they are all the same? Now look at the x-values. Do you see any patterns or similarities?)
7. (Another prompt if needed: Look at the x-values. Do you notice a common difference between them?)

If the learner used holistic reasoning for Question 3

8. How can you relate the two equations to each other?
9. How does it work?

Learners are given similar prompts and questions for Question 6.

LEARNER IS GIVEN THEIR COMPLETED ASSESSMENT

10. Please explain your thinking when you were doing this question.
    [A variety of the assessment questions will be clarified. Possible questions to ask about: 1, 2, 4, 5,7]
    Interviewer will use their discretion about which questions to ask according to Growth Point of the learner, and according to amount of time of the interview

Closing

Thank you so much for your time. It really has been interesting for me to find out how you think about functions!

Appendix D – Letter from Wits

Wits School of Education

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa
Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

Student Number: 0410732R
Protocol Number: 203ECE114M

Date: 19 July 2013

Dear Robyn Clark

Application for Ethics Clearance: Master of Education

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Exploring and Describing the Growth Points of learners as they encounter functions in equation form.

The committee recently met and I am pleased to inform you that clearance was granted.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely

MMabeta
Matsie Mabeta
Wits School of Education

011 717 3416
CC Supervisor: Prof J Adler

Appendix E – Parent information letter

Dear Parent or Guardian

My name is Robyn Clark, and I am currently engaged in Master's research in Mathematics Education at the University of the Witwatersrand. The study's focus is on how students learn about functions, and the typical paths they follow when learning.

I am interested in exploring the trajectories of learning that learners follow, especially when learning about functions. I am also interested in exploring the language used when learning about functions. The study involves myself as a Masters level student, along with my supervisor, Professor Jill Adler. We have a strong interest in issues related to mathematics teaching and learning. We believe that this research can make a meaningful contribution to current debates around learning trajectories and how this may impact the teaching of mathematics.

To this end I would like to give learners a small assessment, which will take place during class. I would also like to interview (with videotaping) some learners to talk about how they have approached tasks in the assessment. I would like to videotape the interviews so that I can hear the explanation, as well as see what is being written at the same time.

All data collected will be shared amongst the research team only. In discussions about the data and in all of our reporting of it, the anonymity of schools, teachers and learners will be upheld. Lessons will continue as scheduled throughout the process of our research.

I would like to invite your child to take part in my research study. Your child will not be advantaged or disadvantaged in any way. There are no foreseeable risks in participating and your child will not be paid for this study. I stress that participation in this study is voluntary. Your child is under no obligation to participate and there will be no consequences should she choose not to partake in the study. Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study. All participants also have the right to withdraw from the study at any point in the study. All research data will be destroyed between 3-5 years after completion of the study.

We would be very grateful for this opportunity, and if you are agreeable to this process please read and complete the attached consent form and return it to school. If you have any questions or concerns or would like to discuss the aims of my research in more detail, please do not hesitate to contact myself on 082 432 8258 (Robyn Clark).

In addition to complying with the University of Witwatersrand's ethical policies, permission has also been granted by the Principal of [redacted] for this research to take place. This permission is on condition that the research takes place in accordance with the School's Educational Surveys & Research Policy. Results from this study may be published as a journal article, or at a conference.

Yours sincerely

____

Robyn Clark

082 432 8258 or robzclark@gmail.com

Supervisor

____

Dr. Jill Adler

Appendix F – Learner information letter

Dear Learner

My name is Robyn Clark and I am a Masters Student in the School of Education at the University of the Witwatersrand. I am doing research on how learners learn about functions.

My investigation involves exploring how functions are learnt. I want to find out how, and in what order, learners progress through the concepts that are learnt in functions. I would also like to find out about the language that is used when talking about, and solving problems about functions.

I would like to invite you to be a part of my study. You will have to complete an assessment on functions. You do not have to study for this assessment. This assessment is designed to get you to think, so do not worry if you find it difficult. The assessment will not count for marks, although I would like you to take it seriously. I would also like to interview a few learners, and will be videotaping the interview. I will be videotaping the interview to hear the explanations, as well as see what is being written at the same time.

Remember, this is not for marks and it is voluntary. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

In the write up of my research, I will be using pseudonyms, so that no one can identify you. All information about you will be kept confidential in all my writing about the study. All collected information will be stored safely and destroyed between 3-5 years after I have completed my project. Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you! Please feel free to contact me if you have any questions.

Thank you

Robyn Clark

0824328258

robzclark@gmail.com

Appendix G – Parent and learner consent forms

Learner Consent form for participation in a research project (Phase 1)

(Please circle your response)

I have read the information sheet and **give consent / do not give consent** to participate in the Mathematics research project subject to the conditions laid out in the accompanying letter.

(*Please circle your response)*

I agree that data from a written assessment can be used in the study only. **YES NO**

The conditions also include the use of the data for research purposes and in articles for publication in academic journals, or presentation at conferences on condition that the school is anonymous and all participants are referred to by pseudonyms.

Name of learner: …………………………………………………………………………….

Signature of learner:…………………………………………………………………………

Date: …………………………………………………………………………………………….

Parent Consent form for participation in a research project (Phase 1)

(Please circle your response)

I have read the information sheet and **give consent / do not give consent** for my child to participate in the Mathematics research project subject to the conditions laid out in the accompanying letter – This includes a small assessment written in test-like conditions, as well as video-taped interviews with the researcher. The conditions also include the use of the data for research purposes and in articles for publication in academic journals, or presentation at conferences on condition that the school is anonymous and all participants are referred to by pseudonyms.

(Please circle your response)

I agree that data from my child's written assessment can be used in the study only.

YES NO

Name of parent or guardian: ………………………………………………………………

Signature of parent or guardian: …………………………………………………………..

Date: ……………………………………………………………………………………….

Learner Consent form for participation in a research project (Phase 2)

(Please circle your response)

I have read the information sheet and **give consent / do not give consent** to participate in the Mathematics research project subject to the conditions laid out in the accompanying letter.

(Please circle your response)

I agree to be interviewed in this study. **YES NO**

I agree to be video-taped in the interview. **YES NO**

The conditions also include the use of the data for research purposes and in articles for publication in academic journals, or presentation at conferences on condition that the school is anonymous and all participants are referred to by pseudonyms.

Name of learner: …………………………………………………………………………….

Signature of learner:…………………………………………………………………………

Date: ……………………………………………………………………………………….

Parent Consent form for participation in a research project (Phase 2)

(Please circle your response)

I have read the information sheet and **give consent / do not give consent** for my child to participate in the Mathematics research project subject to the conditions laid out in the accompanying letter – This includes a small assessment written in test-like conditions, as well as video-taped interviews with the researcher. The conditions also include the use of the data for research purposes and in articles for publication in academic journals, or presentation at conferences on condition that the school is anonymous and all participants are referred to by pseudonyms.

(Please circle your response)

I agree for my child to be interviewed in this study. **YES NO**

I agree for my child to be video-taped in the interview. **YES NO**

Name of parent or guardian: ……………………………………………………………

Signature of parent or guardian: …………………………………………………………..

Date: ……………………………………………………………………………………….

Appendix H – Permission from Research School

School Logo (removed)

13 JUNE 2013

PROPOSED RESEARCH AT [redacted]

EXPLORING AND DESCRIBING THE GROWTH POINTS OF LEARNERS AS THEY

ENCOUNTER FUNCTIONS IN EQUATION FORM

RESEARCHER: ROBYN CLARK

TO WHOM IT MAY CONCERN

Ms Robyn Clark has requested permission to use [redacted] - a research site in a research project that she is conducting.

This project serves to complete the Master's Program that she is currently involved in at the University of the Witwatersrand.

I hereby give consent for Ms Robyn Clark to conduct the research at this school and look forward to the findings and recommendations that such a study will deliver.

Yours sincerely

[redacted]

Headmistress

Appendix I – Permission from the GDE

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

For administrative use:
Reference no. D2014/119

GDE RESEARCH APPROVAL LETTER

Date:	**19 August 2013**
Validity of Research Approval:	**19 August 2013 to 20 September 2013**
Name of Researcher:	**Clark R.L.**
Address of Researcher:	**75 Greenside Road**
	Greenside
	2193
Telephone Number:	**011 646 3762 / 082 432 8258**
Email address:	**robzclark@gmail.com**
Research Topic:	**Exploring and describing the growth points of learners as they encounter functions in equation form**
Number and type of schools:	**ONE Secondary School**
District/s/HO	**Johannesburg East**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

2013/08/20

1

Making education a societal priority

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

1. *The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.*
2. *The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.*
3. *A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.*
4. *A letter / document that outlines the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.*
5. *The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.*
6. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
7. *Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
8. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
9. *It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.*
10. *The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.*
11. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.*
12. *On completion of the study the researcher/s must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.*
13. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.*
14. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

Dr David Makhado
Director: Education Research and Knowledge Management

DATE: 2013/08/20

2

Making education a societal priority

Office of the Director: Knowledge Management and Research
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