JUMP NUMBERS, HYPERRECTANGLES AND CARLITZ COMPOSITIONS

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Abstract

Let $A = (a_{ij})$ be an $m \times n$ matrix. There is a natural way to associate a poset P_A with A. A jump in a linear extension of P_A is a pair of consecutive elements which are incomparable in P_A. The jump number of A is the minimum number of jumps in any linear extension of P_A. The maximum jump number over a class of $n \times n$ matrices of zeros and ones with constant row and column sum k, $M(n,k)$, has been investigated in Chapter 2 and 3. Chapter 2 deals with extremization problems concerning $M(n,k)$. In Chapter 3, we obtain the exact values for $M(11,k)$, $M(n,6)$, $M(n,n-3)$ and $M(n,n-4)$.

The concept of frequency hyperrectangle generalizes the concept of latin square. In Chapter 4 we derive a bound for the maximum number of mutually orthogonal frequency hyperrectangles. Chapter 5 gives two algorithms to construct mutually orthogonal frequency hyperrectangles.

Chapter 6 is devoted to some enumerative results about Carlitz compositions (compositions with different adjacent parts).

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Cheng

Bo Cheng

23rd day of December, 1998

to my parents and my wife

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Contents

Chapter 1

A General Introduction

Combinatorics is concerned with arrangements of the objects of a set into patterns. Three general types of problems occur repeatedly:

(i) Existence of the arrangement,

(ii) Enumeration or classification of the arrangements, and

(iii) Study of a known arrangement.

In this thesis, three combinatorical problems are discussed:

(i) Jump number of a matrix,

(ii)Frequency hyperrectangles, and

(iii)Enumeration of Carlitz compositions.

1.1 Jump numbers

In real life, the following question occurs: a single machine performs a set of jobs one at a time; precedence constraints prohibit the start of certain jobs until some others are already completed; a job which is performed immediately after a job which is not constrained to precede it requires a "setup" or "jump"—entailing some fixed additional cost. The schedule problem is to construct a schedule to minimize the number of jumps. The jump number problem is the precedence constrained scheduling problem written in the language of ordered sets.

Let P be a finite poset, and $|P|$ be the number of vertices in P. A *chain* C in P is a subset of P which is a linear order. The length of the chain C is $|C| - 1$. A *linear extension* L of a poset P is a linear ordering $x_1, x_2, ..., x_n$ of the elements of P such that $x_i < x_j$ in P implies $i < j$.

Let $\ell(P)$ be the set of all linear extention of P. E. Szpilrajn [Sz1930] showed that $\ell(P)$ is not empty. Algorithmically, a linear extention L of P can be defined as follows:

1. Choose a minimal element x_1 in P.

2. Given $x_1, x_2, ..., x_i$, choose a minimal element from $P \setminus \{x_1, x_2, ..., x_i\}$ and call this element x_{i+1}.

A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* (or *setup*) of P in L if x_i is not comparable to x_{i+1} in P. If $x_i < x_{i+1}$ in P, then (x_i, x_{i+1}) is called a *stair* (or *bump*) of P in L. Let s(L,P) [b(L,P)] be the number of jumps [stairs] of P in L, and let s(P) [b(P)] be the minimum [maximum] of s(L,P) [b(L,P)] over all linear extensions L of P. The number s(P) is called the *jump number* of P, and the number b(P) is the *stair number* of P.

We have $\quad s(L,P)+b(L,P)=|P|-1$

and $\quad s(P)+b(P)=|P|-1 \qquad$ for every poset P.

Each of jump number and stair number determines the other. If s(L,P)=s(P) [equivalently, b(L,P)=b(P)], then L is called an *optimal linear extention* of P. The *jump number problem* is to compute s(P) and to find an optimal linear extension of P.

In Figure 1 there is given a poset and three of its linear extention; the first linear extention is not optimal, but the second and third are optimal.

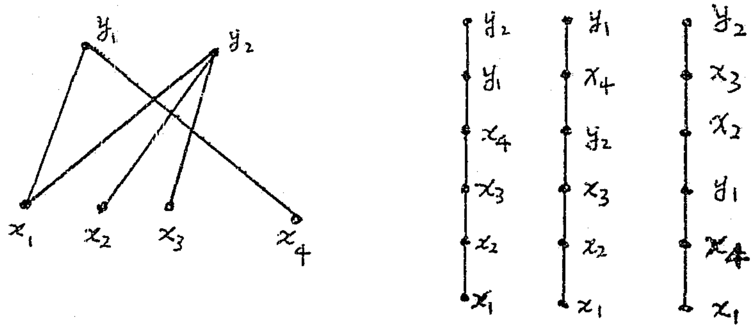

Fig. 1

The jump number problem was introduced by Chein and Martin [ChMa1972]. This problem has been shown to be NP-hard (see [BoHa1987]). In the past twenty years, many papers have been devoted to the study of this problem (Cf. [BrJuTr1994], [ChHa1980], [DuRiWi1982], [Mi1991] and [ShZa1992]).

Now we define the jump number of a matrix. Let A=(a_{ij}) be an $m \times n$ matrix. There is a natural way to associate a poset P_A with A. Let X=$\{x_1, ..., x_m\}$ and Y=$\{y_1, ..., y_n\}$ be disjoint sets of m and n elements, respectively, and define $x_i < y_j$ if and only if $a_{ij} \neq 0$. In P_A $x_1, ..., x_m$ are minimal elements and $y_1, ..., y_n$ are maximal elements. The poset P_A is the usual bipartite poset [the Hasse diagram of P_A is BG(A) drawn with the y's above the x's]. We use s(A) [b(A)] for the jump [stair] number of P_A instead of s(P_A) [b(P_A)].

Two matrices A_1 and A_2 are said to be *permutation equivalent*, denoted by $A_1 \cong A_2$, if one can be obtained from the other by independent row and column permutations.

It is clear that b(A_1) =b(A_2) and s(A_1) =s(A_2) if $A_1 \cong A_2$. Therefore the jump number of a matrix does not change under arbitrary row and column permutations.

The following formulas have been noticed in [BrJu1992].

$\max\{m,n\}-1 \leq s(A) \leq m+n-1$,

$0 \leq b(A) \leq \min\{m,n\}$,

$s(A)=s(A^T)$, where A^T is the transpose of A,

$s(A \oplus B) = s(A)+s(B)+1$, $b(A \oplus B) = b(A)+b(B)$, where $\oplus$ denotes the direct sum of matrices.

In the following we will concentrate on (0,1) matrices.

Let $R=\{r_1,...,r_m\}$ and $S=\{s_1,...,s_n\}$ be nonnegative integer vectors with $r_1+...+r_m = s_1+...+s_n$. We denote by Λ(R,S) the set of all $m \times n$ matrices $A=(a_{ij})$ of 0's and 1's such that $\sum_{j=1}^{n} a_{ij} = r_i$ $(i = 1,...,m)$ and $\sum_{i=1}^{m} a_{ij} = s_j$ $(j = 1,...,n)$. Suppose $M(R,S)=\max\{s(A):A\in \Lambda(R,S)\}$,

$$m(R,S)=\min\{s(A):A\in \Lambda(R,S)\}.$$

The necessary and sufficient conditions for the set Λ(R,S) to be nonempty were given in [Ga1957] and [Ry1957]. In this thesis, we assume that Λ(R,S) is nonempty and R (or S) is not a zero vector.

In section 2 of chapter 2, we derive an upper bound and a lower bound for s(A), where $A\in \Lambda(R,S)$.

Let $\Lambda(n,k)$ be the set of all (0,1) matrices of order n with k 1's in each row and column. Suppose $m(n,k)=\min\{s(A):A\in \Lambda(n,k)\}$, $M(n,k)=\max\{s(A):A\in \Lambda(n,k)\}$. $\lceil a \rceil$ ($\lfloor a \rfloor$) is used for the smallest (greatest) integer no less (more) than a.

The minimum jump number is easy to compute.

Theorem 1.1.1 ([BrJu1992]). If $1 \leq k \leq n$, then $\mathrm{m}(n,k) = n+k-2$.

The determination of the maximum jump number is much more difficult, and Brualdi and Jung ([BrJu1992]) offered some partial results:

Theorem 1.1.2 ([BrJu1992]). $\mathrm{M}(n,k) \leq 2n-1-\lceil n/k \rceil$ for $1 \leq k \leq n$.

The following question is immediate.

Question 1.1.3. Characterize the extremal regular matrices classes attaining the upper bound $2n$-1-$\lceil n/k \rceil$.

In order to answer this question, Brualdi and Jung stated a conjecture.

Conjecture 1.1.4. If $k \nmid n$ and $(n \bmod k) \nmid k$ for $1 \leq k \leq n$, then $\mathrm{M}(n,k) < 2n-1-\lceil n/k \rceil$.

In Chapter 2, we prove this conjecture.

From Theorems 1.1 and 1.2, we have

Theorem 1.1.5. If $\mathrm{A} \in \Lambda(n,k)$, where $1 \leq k \leq n$, then $n+k-2 \leq \mathrm{s(A)} \leq 2n$-1-$\lceil n/k \rceil$.

Two natural questions are as follows.

Question 1.1.6. Characterize the extremal regular matrices attaining the upper bound $2n$-1-$\lceil n/k \rceil$.

Question 1.1.7. Characterize the extremal regular matrices attaining the lower bound $n+k-2$.

We answer Questions 1.1.6 and 1.1.7 in Chapter 2.

Brualdi and Jung have obtained the exact values of M(n,k) for $1 \le k \le n \le 10$ in [BrJu1992]. In chapter 3, we continue to get M(11,k).

M(n,k)

n	$k=1$	2	3	4	5	6	7	8	9	10	11
1	0	–	–	–	–	–	–	–	–	–	–
2	1	2	–	–	–	–	–	–	–	–	–
3	2	3	4	–	–	–	–	–	–	–	–
4	3	5	5	6	–	–	–	–	–	–	–
5	4	6	6	7	8	–	–	–	–	–	–
6	5	8	9	9	9	10	–	–	–	–	–
7	6	9	10	10	10	11	12	–	–	–	–
8	7	11	11	13	12	13	13	14	–	–	–
9	8	12	14	14	14	15	14	15	16	–	–
10	9	14	15	16	17	16	16	17	17	18	–
11	10	15	16	17	18	17	18	18	18	19	20

Fig.2

The following formulas are derived in [BrJu1992].

$$M(n,1) = n-1, \tag{1.1}$$

$$M(n,2) = 2n-1-\lceil n/2 \rceil , \tag{1.2}$$

$$M(n,3) = 2n-1-\lfloor n/3 \rfloor - a$$

where $a \equiv n \pmod 3$ and $0 \leq a \leq 2$, (1.3)

$$M(n,4) = 2n-1-\lceil n/4 \rceil - a$$

where $a = 1$ if $4 |\ n-3$ and 0 otherwise, (1.4)

$$M(n,n) = 2n-2, \tag{1.5}$$

$$M(n,n-1) = 2n-3, \tag{1.6}$$

$$M(n,n-2) = \begin{cases} 2n-3 & \text{if } n \text{ is even} \\ 2n-4 & \text{if } n \text{ is odd} \end{cases}, \tag{1.7}$$

and $M(qk+1,k) = 2qk-q$ for $q \geq 1$. (1.8)

In Chapter 3 we deduce

$$M(n,6) = 2n-1-\lceil n/6 \rceil - a$$

$$\text{where } a = \begin{cases} 0 & \text{if } 6 \mid n \text{ or } (n \bmod 6) \mid 6 \\ 1 & \text{if } (n \bmod 6) = 4 \text{ or } 5, \text{ and } n \neq 11 \\ 2 & \text{if } n = 11 \end{cases}$$

$$M(n,n-3) = \begin{cases} 2n-3 & \text{if } 3 \mid n \\ 2n-4 & \text{if } 3 \nmid n \end{cases} \quad \text{for } n > 4,$$

$$M(n,n-4)=\begin{cases}2n-3 & \text{if } 4\mid n\\ 2n-4 & \text{if } 4\nmid n\end{cases}\quad \text{for } n>5,$$

$$M(qk+2,k)=\begin{cases}2qk-q+2 & \text{if } k \text{ is even}\\ 2qk-q+1 & \text{if } k \text{ is odd}\end{cases}, \text{ where } q\geq 1 \text{ and } k>1.$$

Moreover, two other conjectures in [BrJu1992] are also discussed in Chapter 3.

1.2 Frequency hyperrectangles

A *latin square* of order n is an $n \times n$ matrix L whose entries are taken from a set S of n distinct symbols and which has the property that each symbol from S occurs exactly once in each row and exactly once in each column of L.

Two latin square $L_1 = (a_{ij})$ and $L_2 = (b_{ij})$ on n symbols, say $0, 1, ..., n-1$, are said to be *orthogonal* if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}), $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$.

The question of existence of orthogonal latin squares was discussed by

Euler ([Eu1779]) in the following problem of the 36 officers:

Is it possible to arrange 36 officers of 6 different ranks and from 6 different regiments in a square formation of size 6 by 6 such that each row and each column of this formation contains exactly one officer of each rank and exactly one officer from each regiment?

This problem asks for two orthogonal latin squares of order 6. Euler was unable to find such a pair of latin squares and conjectured that no pair existed. Euler had been able to construct pairs of orthogonal latin squares of every odd order and of every order divisible by 4 but not of any order congruent to 2 module 4. He conjectured that pairs of orthogonal latin squares do not exist for such orders.

Tarry ([Ta1901]) verified that the conjecture was true for the order six. Much later, Bose, Shrikhande and Parker ([BoShPa1960]) proved that the Euler Conjecture was false for all orders n of the form $4k + 2$ except $n = 2$ or 6 by providing a constructive method of obtaining pairs of orthogonal latin squares of all of these orders.

The applications of latin squares to statistical designs, projective geometry and information theory are discussed in [DéKe1974] and [DéKe1991].

The idea of generalizing the concept of latin square to that of frequency square was introduced by P. A. MacMahon ([Ma1898], pages 276-280). The same idea was discussed anew by Finney ([Fi1945], [Fi1946a], [Fi1946b]), Addelman([Ad1967]) and Freeman ([Fr1966]). The formal definition of frequency square was first given in A. Hedayat's Ph. D. thesis of 1969 and the properties of such squares were developed by A. Hedayat, D. Raghavaro and E. Seiden ([HeSe1970], [HeRaSe1975]).

Definition 1.2.1. Let A=(a_{ij}) be an $n \times n$ matrix and let $\sum = \{c_1, c_2, ..., c_m\}$, $m \leq n$, be the set of distinct elements of A. Suppose further that, for each i, where $i = 1, 2, ..., m$, the elements c_i appears precisely λ_i times ($\lambda_i \geq 1$) in each row and column of A. Then A is called a *frequency square* (F-Square) of order n on the set $\sum$ with *frequency vector* $(\lambda_1, \lambda_2, ..., \lambda_m)$.

Note that, by virtue of the definition, $n = \lambda_1 + \lambda_2 + ... + \lambda_m$.

Definition 1.2.2. The two F-squares $F_1(n, n; \lambda_1, \lambda_2, ..., \lambda_g)$ defined on the set $\sum_1 = \{a_1, a_2, ..., a_g\}$ and $F_2(n, n; \mu_1, \mu_2, ..., \mu_m)$ defined on the set $\sum_2 = \{b_1, b_2, ..., b_m\}$ are *orthogonal* if each ordered pair $a_i b_j$ appear $\lambda_i \mu_j$ times when the square F_1 and F_2 are placed in juxtaposition.

The generalized concepts of frequency rectangles and frequency hyperrectangles have crept into the literature quite naturally. The following definition was given in [Ch1980] and [MaFe1984].

Definition 1.2.3. Coordinatize the $\prod_{i=1}^{d} n_i$ cells of a d-dimensional hyperrectangle of size $n_1 \times ... \times n_d$ by the d-tuples of integers $(j_1, ..., j_d)$ where $1 \le j_i \le n_i$. A frequency hyperrectangle (F-hyperrectangle) is an arrangement of m symbols into the $\prod_{i=1}^{d} n_i$ cells, if $m \mid \prod_{j \neq i} n_j$ for $i=1,...,d$, such that each of the m symbols appears $m^{-1}(\prod_{j \neq i} n_j)$ times in each of the n_i sets $H_1^i, ..., H_{n_i}^i$, where H_j^i is the set of all cells with j as the i-th coordinate, $= 1, ..., n_i$.

Suchower [Su1989] and Cheng [Ch2] gave the definition for a frequency hyperrectangle which allows different frequencies for different symbols, generalizing the above definitions.

Definition 1.2.4. An F-hyperrectangle of size $n_1 \times ... \times n_d$, denoted by $F(n_1, ..., n_d; \lambda_{1,1}, ..., \lambda_{1,m}; ...; \lambda_{d,1}, ..., \lambda_{d,m})$ where for each i, $1 \le i \le d$, $\prod_{j \neq i} n_j = \lambda_{i,1} + ... + \lambda_{i,m}$, is an $n_1 \times ... \times n_d$ array consisting of $m \ge 2$ symbols, say $\{1, ..., m\}$, with the property that for each i and j, $1 \le i \le d$, $1 \le j \le m$, the symbol j occurs exactly $\lambda_{i,j}$ times in each of the n_i subarrays $H_1^i, ... H_{n_i}^i$, where H_k^i is the subarray of all cells with k as the i-th coordinate, $k = 1, ..., n_i$.

Laywine, Mullen and Whittle [LaMuWh1995] introduced the definition for a hypercube with a prescribed type.

Definition 1.2.5. For $d \ge 2$, a d-dimensional *hypercube* of order n is an $n \times ... \times n$ array with n^d points based upon n distinct symbols. Such a hypercube has type j with $0 \leq j \leq d-1$ if, whenever any j of the coordinates are fixed, each of the n symbols appears n^{d-j-1} times in that subarray.

In this thesis, we provide a versatile definition for a frequency hyperrectangle that allows different frequencies for different symbols and has a prescribed type.

For a natural number n, we use $\underline{n}$ for the set $\{1, 2, ..., n\}$. $P_k(S)$ denotes the set consisting of all k-subsets of the set S. We define $P_0(S)$ as $\{\phi\}$ where ϕ is the empty set.

Definition 1.2.6. A frequency hyperrectangles (F-hyperrectangles) of size $n_1 \times ... \times n_d$, and type t, $0 \leq t \leq d-1$, denoted by $\mathrm{HR}(n_1, ..., n_d; t; \lambda = \Phi((i_1, ..., i_t), j))$, where $\lambda = \Phi((i_1, ..., i_t), j)$ is a function with the domain $P_t(\underline{d}) \times \underline{m}$ and $\prod_{k \neq i_1, ..., i_t} n_k = \sum_{j=1}^{m} \Phi((i_1, ..., i_t), j)$ for each $(i_1, ..., i_t) \in P_t(\underline{d})$, is an $n_1 \times ... \times n_d$ array consisting of $m \geq 2$ symbols with the property that whenever any t of the coordinates, say $i_1, ..., i_t$, are fixed, the symbol j occurs exactly $\Phi((i_1, ..., i_t), j)$ times in that subarray.

Counting the number of times that the symbol j, $1 \leq j \leq m$, appears in the array, we obtain

$$n_{i_1} \times ... \times n_{i_t} \times \Phi((i_1, ..., i_t), j) = n_{i'_1} \times ... \times n_{i'_t} \times \Phi((i'_1, ..., i'_t), j)). \tag{1.9}$$

This means that all $\Phi((i_1,...,i_t),j)$'s are determined by $\Phi((1,...,t),1)$,..., $\Phi((1,...,t),m)$ and $n_1,...,n_d$. We can use this fact to simplify the notation to

$$\mathrm{HR}(n_1,...,n_d;t;\lambda_1,...,\lambda_m), \text{ where } \lambda_j = \begin{cases} \Phi((1,...,t),j) & \text{if } t>0 \\ \Phi(\phi,j) & \text{if } t=0 \end{cases}.$$

Let us look at some examples.

$$F_1 = \mathrm{HR}(6,3,3;2;1,2): \begin{array}{ccccccccccc} 0&1&1&&1&1&0&&1&0&1 \\ 0&1&1&&1&1&0&&1&0&1 \\ 1&1&0&&1&0&1&&0&1&1 \\ 1&1&0&&1&0&1&&0&1&1 \\ 1&0&1&&0&1&1&&1&1&0 \\ 1&0&1&&0&1&1&&1&1&0 \end{array}.$$

$$F_2 = \mathrm{HR}(6,3,3;2;1,1,1): \begin{array}{ccccccccccc} 0&1&2&&2&0&1&&1&2&0 \\ 0&1&2&&2&0&1&&1&2&0 \\ 1&2&0&&0&1&2&&2&0&1 \\ 1&2&0&&0&1&2&&2&0&1 \\ 2&0&1&&1&2&0&&0&1&2 \\ 2&0&1&&1&2&0&&0&1&2 \end{array}$$

Frequency squares and hyperrectangles have numerous statistical properties and as a result, there has been considerable interest in various aspects of the theory and construction of such objects (See [DéKe1974] and [DéKe1991]).

Now we give the definition for mutually orthogonal F-hyperrectangles.

Definition 1.2.7. Two F-hyperrectangles $\mathrm{HR}(n_1, ..., n_d; t; \lambda_1, ..., \lambda_{m_1})$ and $\mathrm{HR}(n_1, ..., n_d; t; \mu_1, ..., \mu_{m_2})$ are *orthogonal* if upon superposition, each ordered pair (i, j), $1 \leq i \leq m_1$, $1 \leq j \leq m_2$, appears exactly

$$(\prod_{i=1}^{d} n_i)\lambda_i\mu_j/(\prod_{k=t+1}^{d} n_k)^2 \text{ times.}$$

A set of F-hyperrectangles is called *mutually orthogonal* if every pair of F-hyperrectangles are orthogonal.

It is easy to verify that the above examples F_1 and F_2 are orthogonal.

In Chapter 4 of this thesis, we derive a bound for the maximum number of such mutually orthogonal frequency hyperrectangles. This result simultaneously generalizes Suchower's bound in [Su1989] and LMW's bound in [LaMuWh1995], as well as some results in [Ch1980], [PeMa1986], [MaLeFe1981], [HeRaSe1975].

In Chapter 5, we give two algorithms to construct mutually orthogonal frequency hyperrectangles, which generalize Bose [Bo1938] and Mullen [Mu1988] construction, and Laywine, Mullen and Whittle's Algorithm [LaMuWh1995], respectively.

1.3 Carlitz compositions

A *composition* of an integer n is a sequence $(x_1, x_2, ..., x_k)$ of integers such that

$$n = x_1 + x_2 + ... + x_k, \text{ where } x_i \geq 1.$$

A *partition* of an integer n is a sequence $(x_1, x_2, ..., x_k)$ of integers such that

$$n = x_1 + x_2 + ... + x_k \text{ and } 1 \leq x_1 \leq x_2 \leq ... \leq x_k.$$

In both cases, the x_i's are called the summands or the parts. We refer to n as the size and to k as the number of summands of the composition (partition). It is convenient to extend these definitions by regarding 0 as obtained by an empty sequence of summands.

Partitions and compositions of integers are, besides their intrinsic interests, usually used as theoretical models for evolutionary processes in different contexts: statistical mechanics, theory of quantum strings, population biology, nonparametric statistics, etc.. Therefore properties (Statistical, algebraic, analytic,...) of these objects have received constant attention in the literature ([An1976], [HwYe1997], [KnMa], [OdRi1979], [RiKn1995]).

We know that there are 2^{n-1} compositions of the integer n with generating function $\frac{1}{1-\frac{z}{1-z}}$. The average number of summands in a random composition of size n is $\frac{n+1}{2}$.

Carlitz [Ca1976] first discussed the following restricted composition, which is called a Carlitz composition in [KnPr1998].

Definition 1.3.1. A *Carlitz composition* of an integer n is a sequence $(x_1, x_2, ..., x_k)$ of integers such that

$$n = x_1 + x_2 + ... + x_k, \text{ where } x_i \geq 1 \text{ and } x_j \neq x_{j+1}$$

$$\text{for } i = 1, ..., k \text{ and } j = 1, ..., k-1.$$

Let $c(n)$ denote the number of Carlitz compositions of size n. Carlitz [Ca1976] found the generating function

$C(z) := \sum_{n \geq 0} c(n) z^n = \frac{1}{1-\sigma(z)}$, where $\sigma(z) = \sum_{j \geq 1} \frac{z^j(-1)^{j-1}}{1-z^j}$.

Here are the first few values of $c(n)$:

$1 + z + z^2 + 3z^3 + 4z^4 + 7z^5 + 14z^6 + 23z^7 + 39z^8 + 71z^9 + 124z^{10} + ...$; it is sequence A003242 in [SlPl1995].

Carlitz [Ca1976] noted only that the radius of convergence of $C(z)$ is at least 1/2. Knopfmacher and Prodinger [KnPr] showed that there is a dominant pole ρ, where $\rho = 0.571349....$

Therefore $c(n) \sim 0.456387 \cdot (1.750243)^n$.

Knopfmacher and Prodinger [KnPr1998] also proved that the average number of summands in a random Carlitz composition of size n is asymptotic to $0.350571 \cdot n$.

For ordinary compositions, the statistic "largest summand of a composition" has obtained a lot of attention.

It is noted in [OdRi1979] that the average value of the largest summand in a random composition of size n is asymptotic to

$$\log_2 n + \gamma/\ln 2 - 3/2 + \lambda(\log_2 n) + o(1)$$

where γ is Euler's constant (numerically $\gamma = 0.577216$), and $\lambda(x)$ is non-constant continuous functions periodic with period 1 and mean 0.

Knopfmacher and Prodinger [KnPr1998] carried out the analogous analysis for the case of Carlitz compositions. The average number of the largest summand in a random Carlitz composition of size n is asymptotic to

$$\log_{1/\rho} n - \log_{1/\rho} \sigma'(\rho) - \log_{1/\rho}(1-\rho) - \gamma/\ln\rho + 1/2 + \overline{\lambda}(\log_{1/\rho} n)$$ where $\overline{\lambda}(x)$ is a periodic function that has period 1, mean 0 and small amplitude.

For ordinary compositions it is meaningless to allow the x_is to be zero, since then there would be infinitely many compositions for each n. However, in the context of Carlitz compositions, it makes sense, so the number $d(n)$ of Carlitz compositions with zeros allowed is meaningful. Such compositions have been discussed in [Ca1976] and [KnPr1998].

Knopfmacher and Prodinger [KnPr1998] obtained the generating function

$D(z) := \sum_{n \geq 0} d(n) z^n = \frac{1+2\sigma(z)}{1-2\sigma(z)}$.

Therefore $d(n) \sim 1.337604 \cdot (2.584243)^n$.

In Chapter 6 of this thesis, we derive:

the average number of summands in a random Carlitz composition with zeros allowed of size n, and show that this is asymptotic to $0.871626n$, and

the average size of the largest summand in a random Carlitz composition with zeros allowed of size n, and show this to be asymptotic to

$\log_{1/\tau} n - \log_{1/\tau} \sigma'(\tau) - \log_{1/\tau}(1-\tau) - \gamma / \ln \tau + 1/2 + \overline{\delta}(\log_{1/\tau} n)$ where $\tau = 0.386960$ and $\overline{\delta}(x)$ is a periodic function that has period 1, mean 0 and small amplitude.

In the last section of Chapter 6, we introduce another related object: the Carlitz word, which is a generalization of Carlitz composition.

Chapter 2

Extremization Problems

In this chapter, we mainly consider extremal matrices concerning Questions 1.1.6 and 1.1.7 and extremal matrix classes concerning Question 1.1.3. In section 2.1, the symbiotic relationship of bipartite posets and (0,1) matrices is discussed. We derive an upper bound and a lower bound for s(A) in section 2.2. A special class of matrices is constructed in section 2.3. This class of matrices is useful in answering Question 1.1.6. In section 2.4, we give the main results of this chapter. a proof of Conjecture 1.1.4 and answers to Questions 1.1.3, 1.1.6 and 1.1.7.

2.1 Bipartite posets and (0,1) matrices

In this section, we will introduce the symbiotic relationship between bipartite posets and (0,1) matrices.

A poset P is *bipartite* if its vertices may be partitioned into two subsets X and Y such that each ordered pair of P is of the form $x < y$ where $x \in$X and $y \in$Y. Such a partition (X,Y) is called a *bipartition* of the poset. A bipartite poset P with a partition (X,Y) is denoted by P(X,Y).

Let P(X,Y) be a given bipartite poset, where X=$\{x_1, ..., x_m\}$ and Y=$\{y_1, ..., y_n\}$. We define $a_{ij} = 1$ if $x_i < y_j$, otherwise $a_{ij} = 0$. The resulting m-by-n matrix A=(a_{ij}) is called the *reduced adjacency matrix* of P(X,Y). The matrix A characterizes P(X,Y).

For example, consider the bipartite poset P(X,Y) is in Figure 1, where X=$\{x_1, ..., x_4\}$ and Y=$\{y_1, y_2\}$. The reduced adjacency matrix of P(X,Y) is

$$\text{A}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

On the other hand, let A=(a_{ij}) be a (0,1) m-by-n matrix. There is a natural way to associate a bipartite poset P(X,Y) with A. Let X=$\{x_1, ..., x_m\}$

and Y=$\{y_1, ..., y_n\}$ be disjoint sets of m and n elements, respectively, and define $x_i < y_j$ if and only if $a_{ij} \neq 0$. In P(X,Y) $x_1, ..., x_m$ are minimal elements and $y_1, ..., y_n$ are maximal elements.

Two bipartite posets $P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ are *isomorphic* if and only if there are two bijections $\theta : X_1 \to X_2$ and $\phi : Y_1 \to Y_2$ such that $x < y$ in P_1 if and only if $\theta(x) < \phi(y)$ in P_2.

We see that two bipartite posets $P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ are isomorphic if and only if their resulting reduced adjacency matrices are permutation equivalent.

Two bipartite posets $P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ are *disjoint* if they have no vertex in common. The *union* of $P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ is the poset with the vertex set $X_1 \cup Y_1 \cup X_2 \cup Y_2$ and the ordered pair set consisting of all ordered pairs in P_1 and P_2. If $P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ are disjoint, we denote their union by $P_1(X_1,Y_1)+P_2(X_2,Y_2)$.

We always designate a zero matrix by O, a matrix with every entry equal to 1 by J. In order to emphasize the size of these matrices we sometimes include subscripts. Thus $J_{m,n}$ denotes the all 1's matrix of size m by n, and this is abbreviated to J_n if $m = n$. The notations $O_{m,n}$ and O_n have similar meanings. In displaying a matrix we often use $*$ to designate a submatrix of no particular concern.

We define the direct sum of matrices A_1 and A_2 to be the matrix

$$A_1 \oplus A_2 = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix},$$

and the oblique direct sum of A_1 and A_2 to be the matrix

$$A_1 \overline{\oplus} A_2 = \begin{bmatrix} J & A_2 \\ A_1 & J \end{bmatrix}.$$

Suppose A_i is the reduced adjacency matrix of $P_i(X_i,Y_i)$, where $1 \le i \le m$, then $A_1 \oplus ... \oplus A_m$ is the reduced adjacency matrix of the poset $P_1(X_1,Y_1)+...+P_m(X_m,Y_m)$.

Since the Hasse diagram of P(X,Y) is the usual bipartite graph drawn with the y's above the x's, terms related to bipartite graphs, such as vertex degree, isolated vertex, etc., carry over directly to bipartite posets. Their definitions can be found in [BoMu1975, Chapter 1-10].

A completed bipartite poset is a simple bipartite graph with bipartion (X,Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a poset is denoted by $P_{m,n}$.

The bipartite complement of a bipartite poset $P(X,Y)$, denoted by $\overline{P(X,Y)}$, is a bipartite poset with partition (X,Y) in which $x < y$ in $\overline{P(X,Y)}$ if and only if x is not comparable with y in $P(X,Y)$, where $x \in X$ and $y \in Y$. For example, Figure 3 shows a given poset P(X,Y) and its bipartite complement.

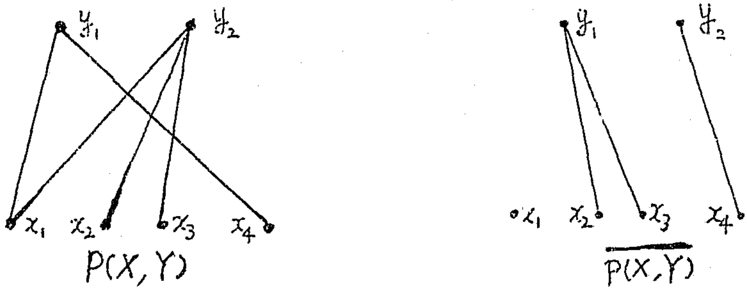

Fig. 3

The reduced adjacency matrix of $P_{m,n}$ is $J_{m,n}$. Let A and B be the reduced adjacency matrices of $P(X,Y)$ and $\overline{P(X,Y)}$, respectively, then

$$B = J_{|X|,|Y|} - A.$$

In this section, we will write some results in poset language as well as in matrix language.

Matrices (bipartite posets) whose stair number is 1 or 2 have been characterized in [BrJu1992]. The assumption in the following theorems that A has no rows or columns consisting of all 0's or all 1's involves no essential loss of generalization.

Theorem 2.1.1 ([BrJu1992]). Let A be a (0,1) matrix having no rows or columns consisting of all 0's. Then b(A)=1 if and only if A$\cong$ J.

Theorem 2.1.1$'$. Let P(X,Y) be a bipartite poset with no isolated vertex. Then b(P(X,Y))=1 if and only if P(X,Y)$\cong$ $P_{|X|,|Y|}$.

Theorem 2.1.2 ([BrJu1992]). Let A be a (0,1) matrix having no rows or columns consisting of all 0's or all 1's. Then b(A)=2 if and only if the rows and columns of A can be permuted to give an oblique direct sum

$O\overline{\oplus}...\overline{\oplus}O$ of zero matrices.

Theorem 2.1.2$'$. Let P(X,Y) be a bipartite poset with no isolated vertex, and each vertex degree satisfies d(x)<|Y| where x $\in$X,

and d(y)<|X| where y $\in$Y.

Then b(P(X,Y))=2 if and only if P(X,Y)$\cong$ $\overline{P_{m_1,n_1} + ... + P_{m_k,n_k}}$, where $m_1 + ... + m_k = |X|$ and $n_1 + ... + n_k = |Y|$.

2.2 Normal forms

The following theorems show the relations between matrix structure and jump number.

Theorem 2.2.1 ([BrJu1992]). Let A be a (0,1) matrix with no zero row or column. Let b(A)=p. Then there exist permutation matrices P and Q and integers $m_1, ..., m_p$ and $n_1, ..., n_p$ such that PAQ equals

$$\begin{bmatrix} J_{m_1,n_1} & A_{12} & \dots & A_{1p} \\ O & J_{m_2,n_2} & \dots & A_{2p} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & J_{m_p,n_p} \end{bmatrix}. \tag{2.1}$$

A matrix (2.1) satisfying the conclusion of Theorem 2.2.1 is called a normal form of A.

The next result is straightforward from the above theorem.

Theorem 2.2.2. Let A be a (0,1) matrix with exact h_1 zero rows and h_2 zero columns. Let b(A)=p. Then there exist permutation matrices P and Q and integers $m_1, ..., m_p$ and $n_1, ..., n_p$ such that

$$\mathrm{PAQ}=\mathrm{A}'=\begin{bmatrix} J_{m_1,n_1} & A_{12} & \dots & A_{1p} & O_{m_1,h_2} \\ O & J_{m_2,n_2} & \dots & A_{2p} & O_{m_2,h_2} \\ \dots & \dots & \dots & \dots & \dots \\ O & O & \dots & J_{m_p,n_p} & O_{m_p,h_2} \\ O_{h_1,n_1} & O_{h_1,n_2} & \dots & O_{h_1,n_p} & O_{h_1,h_2} \end{bmatrix} \tag{2.2}$$

Now we prove the main theorem of this section.

Theorem 2.2.3. Let R=$(r_1, ..., r_m)$ and S=$(s_1, ..., s_n)$.

Define $r_{\min} = \min\{r_i \mid 1 \le i \le m, r_i \ne 0\}$, $h_1 = |\{i \mid r_i = 0,\ 1 \le i \le m\}|$,

$s_{\min} = \min\{s_i \mid 1 \le i \le n,\ s_i \ne 0\}$, $h_2 = |\{i \mid s_i = 0,\ 1 \le i \le n\}|$,

$r_{\max}$ =max$\{r_i\}$ and $s_{\max} = \max\{s_i\}$.

Suppose A∈ Λ(R,S), then

s(A)$\ge m + n - 1 - \min\{m - h_1 - s_{\min} + 1,\ n - h_2 - r_{\min} + 1\}$

($i.e.$, $b(A) \le \min\{m - h_1 - s_{\min} + 1,\ n - h_2 - r_{\min} + 1\}$),

and s(A)$\le m + n - 1 - \max\{\lceil (m - h_1)/s_{\max} \rceil,\ \lceil (n - h_2)/r_{\max} \rceil\}$

($i.e.$, $b(A) \ge \max\{\lceil (m - h_1)/s_{\max} \rceil,\ \lceil (n - h_2)/r_{\max} \rceil\}$).

Proof. Let b(A)=p. Then A≅ A' of the form (2.2).

Since A has at least $s_{\min}$ 1's in each nonzero column, the first stair occurs in row $s_{\min}$ or later. Hence there are at most $m - h_1 - (s_{\min} - 1)$ stairs, and hence s(A)$\ge m + n -$ 1-$[m - h_1 - (s_{\min} - 1)]$

$= m + n - 1 - (m - h_1 - s_{\min} + 1)$.

Since s(A)=s(A^T), we may get s(A)$\ge m + n - 1$–$(n - h_2 - r_{\min} + 1)$.

So s(A)$\ge m + n - 1 - \min\{m - h_1 - s_{\min} + 1,\ n - h_2 - r_{\min} + 1\}$.

In the form (2.2), we have $m_1 + ... + m_p + h_1 = m$, where each m_i satisfies $m_i \le s_{\max}$, and hence $p \times s_{\max} + h_1 \ge m$, i.e., $p \ge \lceil (m - h_1)/s_{\max} \rceil$.

Then s(A)$\leq m+n-1-\lceil (m-h_1)/s_{\max} \rceil$.

Similarly s(A)$\leq m+n-1-\lceil (n-h_2)/r_{\max} \rceil$.

Hence s(A)$\leq m+n-1-\max\{\lceil (m-h_1)/s_{\max} \rceil, \lceil (n-h_2)/r_{\max} \rceil\}$. $\square$

The following corollaries are clear.

Corollary 2.2.4. Let R=$(r_1, ..., r_m)$ and S=$(s_1, ..., s_n)$.

Define $r_{\min} = \min\{r_i \mid 1 \leq i \leq m, r_i \neq 0\}$, $h_1 = |\{i \mid r_i = 0,\ 1 \leq i \leq m\}|$,

$s_{\min} = \min\{s_i \mid 1 \leq i \leq n, s_i \neq 0\}$, $h_2 = |\{i \mid s_i = 0,\ 1 \leq i \leq n\}|$,

$r_{\max}$ =max$\{r_i\}$, and $s_{\max} = \max\{s_i\}$.

Then

$$m(R,S) \geq m+n-1-\min\{m-h_1-s_{\min}+1,\ n-h_2-r_{\min}+1\},$$

$$M(R,S) \leq m+n-1-\max\{\lceil (m-h_1)/s_{\max} \rceil, \lceil (n-h_2)/r_{\max} \rceil\}.$$

Corollary 2.2.5. Let A be an $m \times n$ (0,1) matrix without any zero row, if there are at most k 1's in each column, then b(A)$\geq \lceil m/k \rceil$.

Corollary 2.2.6. Let A be an $m \times n$ (0,1) matrix without any zero column, if there are at most k 1's in each row, then b(A)$\geq \lceil n/k \rceil$.

In $\Lambda(n,k)$, $m=n$, $h_1=h_2=0$, $s_{\max}=r_{\max}=k$, then we have

Corollary 2.2.7 ([BrJu1992]). $M(n,k) \leq 2n-1-\lceil n/k \rceil$.

2.3 A special class of matrices

We say matrix A *contains* matrix B if there exists distinct integers $i_1, ..., i_s$ and distinct integers $j_1, ..., j_t$ such that the new matrix consisting of the i_1-th, ..., i_s-th rows and the j_1-th, ..., j_t-th columns of A is B. For example, matrix $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ contains matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, since the new matrix consisting of the third, second rows and the first, second columns of A is B.

First we prove some lemmas.

Lemma 2.3.1. Let A be a (0,1) matrix.

(i) If A has no zero column, then b($\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}$)=b(A).

(ii) If A has a zero column, then b($\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}$)=b(A)+1.

Proof. Let b($\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}$) $= p$, then $\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}$ is permutation equivalent to the normal form

$$
B=\begin{bmatrix} J_{m_1,n_1} & A_{12} & \dots & A_{1p} \\ O & J_{m_2,n_2} & \dots & A_{2p} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & J_{m_p,n_p} \\ O_{h_1,n_1} & O_{h_1,n_2} & \dots & O_{h_1,n_p} \end{bmatrix}.
$$

It is clear that the first m_1 rows of the normal form B must contain s all 1's rows. Hence $m_1 \geq s$.

In (i), since A has no zero column, it implies $m_1 > s$. Therefore after we delete these s all 1's rows from B, the stair number does not change.

Then $b\left(\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}\right) = b(A)$.

In (ii), we have $m_1 \geq s$.

Since A has a zero column, it follows that $m_1 = s$.

Hence after we delete these s all 1's rows from B, the stair number decreases by 1.

There $b\left(\begin{bmatrix} J_{s,t} \\ A \end{bmatrix}\right) = b(A)+1$. □

Similarly, we have

Lemma 2.3.2. Let A be a (0,1) matrix.

(i) If A has no zero row, then b($\begin{bmatrix} J_{s,t} & A \end{bmatrix}$)=b(A).

(ii) If A has a zero row, then b($\begin{bmatrix} J_{s,t'} & A \end{bmatrix}$)=b(A)+1.

Lemma 2.3.3. Let A and B be (0,1) matrices, and assume that either (i) b(A)$\geq$b(B)+2, and A has no zero column or row, or (ii) b(A)$\geq$b(B)=2, both A and B have no zero column or row. Then b(A$\overline{\oplus}B$) =b(A).

Proof. (i) Let b(A$\overline{\oplus}B$) $= p$, then A$\overline{\oplus}B$ $= \begin{bmatrix} J_{s_1,t_1} & B \\ A & J_{s_2,t_2} \end{bmatrix}$ is permutation equivalent to the normal form

$$C=\begin{bmatrix} J_{m_1,n_1} & A_{12} & \dots & A_{1p} \\ O & J_{m_2,n_2} & \dots & A_{2p} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & J_{m_p,n_p} \end{bmatrix}.$$

The following claim is clear.

Claim 2.3.4. Any two-row submatrix of A$\overline{\oplus}B$ consisting of one row from $\begin{bmatrix} J_{s_1,t_1} & B \end{bmatrix}$ and one row from $\begin{bmatrix} A & J_{s_2,t_2} \end{bmatrix}$ has no zero column.

Due to Claim 2.3.4, we see that the first m_1 rows of C contain $\begin{bmatrix} J_{s_1,t_1} & B \end{bmatrix}$ or $\begin{bmatrix} A & J_{s_2,t_2} \end{bmatrix}$.

If the first m_1 rows of C contain $\begin{bmatrix} J_{s_1,t_1} & B \end{bmatrix}$, then $m_1 > s_1$. (Because $m_1 = s_1$ implies A has a zero column, a contradiction.)

Hence after we delete $\begin{bmatrix} J_{s_1,t_1} & B \end{bmatrix}$ from C, the stair number does not change.

Therefore b(A$\overline{\oplus}B$) =b($\begin{bmatrix} A & J_{s_2,t_2} \end{bmatrix}$).

Since A has no zero row, we have b($\begin{bmatrix} A & J_{s_2,t_2} \end{bmatrix}$) =b($A$) by Lemma 2.3.2.

Hence b(A$\overline{\oplus}B$) =b(A).

If the first m_1 rows of C contain $\begin{bmatrix} A & J_{s_2,t_2} \end{bmatrix}$, then $m_1 \geq s_2$.

Therefore b(A$\overline{\oplus}B$) $\leq$b($\begin{bmatrix} J_{s_1,t_1} & B \end{bmatrix}$) $+1$

$\leq$b(B) $+1+1$ (by Lemma 2.3.2)

$\leq$b(B) $+2$ $\leq$b(A).

Since b(A$\overline{\oplus}B$) $\geq$b(A), it implies b(A$\overline{\oplus}B$) =b(A).

(ii) can be proved in a similar way. □

We now recursively construct a particular class of matrices $\widetilde{A_{n,k}}$ which have occured in [BrMaRo1986].

Set $n_1 = n$ and $k_1 = k$. We now apply steps 1 and 2 below, beginning with step 1 and alternating between steps 1 and 2.

Step 1. Suppose n_i and k_i have been defined where $n_i \geq k_i > 0$ and i is odd.

Let $n_i = q_i k_i + r_i$, where $0 \leq r_i < k_i$ and $q_i > 0$. If $r_i = 0$, we set $A_i = \oplus_{q_i} J_{k_i}$, a direct sum of q_i copies of J_{k_i}, and stop. Otherwise A_i=$(\oplus_{q_i-1}J_{k_i}) \oplus A_{i+1}$, and $A_{i+1} \in \Lambda(n_{i+1}, k_{i+1})$ is defined in step 2, and where $n_{i+1} = k_i + r_i$ and $k_{i+1} = k_i$.(when q_i=1, the direct sum $\oplus_{q_i-1}J_{k_i}$ is an empty matrix.)

Step 2. Suppose n_{i+1} and k_{i+1} have been defined where $n_{i+1} \geq k_{i+1} > 0$ and $i+1$ is even. Let $n_{i+1} = q_{i+1}(n_{i+1} - k_{i+1}) + r_{i+1}$, where $0 \leq r_{i+1} < n_{i+1} - k_{i+1}$ and $q_{i+1}>0$. If r_{i+1}=0, we set A_{i+1}=$\overline{\oplus}_{q_{i+1}}O_{n_{i+1}-k_{i+1}}$, an oblique direct sum of q_{i+1} copies of $O_{n_{i+1}-k_{i+1}}$ and stop. Otherwise , A_{i+1}=$[\overline{\oplus}_{q_{i+1}-1}O_{n_{i+1}-k_{i+1}}]\overline{\oplus}A_{i+2}$, where $A_{i+2} \in \Lambda(n_{i+2}, k_{i+2})$ with $n_{i+2} = (n_{i+1} - k_{i+1}) + r_{i+1}$ and $k_{i+2} = r_{i+1}$.

Note that for odd i, $q_{i+1} \geq 2$ and $q_{i+2} \geq 2$ whenever they are defined. That is so because $n_{i+1} = k_i + r_i > 2r_i$ and hence $n_{i+1} > 2(n_{i+1} - k_{i+1})$. Also $n_{i+2} = (n_{i+1} - k_{i+1}) + r_{i+1} > 2r_{i+1} = 2k_{i+2}$. It follows that this construction terminates after a finite number of steps. Let t be the smallest positive integer such that $r_t = 0$.

We set $\widetilde{A_{n,k}}$ equal to A_1.

Now we evaluate the stair number of $\widetilde{A_{n,k}}$.

Lemma 2.3.5. If $t \geq 3$, then b($\widetilde{A_{n,k}}$) =b(A_3) + $q_1 - 1$.

Proof. $\widetilde{A_{n,k}} = (\oplus_{q_1-1}J_{k_1}) \oplus A_2$ where $A_2 = O_{n_2-k_2}\overline{\oplus}...\overline{\oplus}O_{n_2-k_2}\overline{\oplus}A_3$.

b($\widetilde{A_{n,k}}$) =b(A_2) + $q_1 - 1$.

According to the way of constructing $\widetilde{A_{n,k}}$, A_3 has no zero column or row. Due to lemma 2.3.3, b(A_2) =b(A_3).

Then b($\widetilde{A_{n,k}}$) =b(A_3) + $q_1 - 1$. □

Theorem 2.3.6.

$$b(\widetilde{A_{n,k}}) = \begin{cases} \sum_{i=1}^{t/2} q_{2i-1} - (t-4)/2 & \text{if } 2 \mid t \\ \sum_{i=1}^{(t+1)/2} q_{2i-1} - (t-1)/2 & \text{if } 2 \nmid t \end{cases}.$$

Proof. The proof is by induction on t.

If t=1, then $\widetilde{A_{n,k}} = \oplus_{q_1} J_k$, b($\widetilde{A_{n,k}}$)=$q_1$.

If t=2, then $\widetilde{A_{n,k}}$=($\oplus_{q_1-1} J_{k_1}$)$\oplus A_2$, where $A_2 = \overline{\oplus}_{q_2} O_{n_2-k_2}$, hence b($\widetilde{A_{n,k}}$) = $q_1 - 1 + 2 = q_1 + 1$.

Then when t=1 or 2, we are done.

Assume that we have proved the result holds for $t \leq m$, $m \geq 2$. Then when $t = m + 1$, by Lemma 2.3.5,

$$b(\widetilde{A_{n,k}}) = b(A_3) + q_1 - 1. \tag{2.3}$$

By the induction assumption,

$$b(A_3) = \begin{cases} \sum_{i=2}^{t/2} q_{2i-1} - (t'-4)/2 & \text{if } 2 \mid t' \\ \sum_{i=2}^{(t+1)/2} q_{2i-1} - (t'-1)/2 & \text{if } 2 \nmid t' \end{cases} \tag{2.4}$$

where $t' = t - 2$.

Combining (2.3) and (2.4), we prove the statement. □

2.4 Main results

For a matrix A in block form, we use $A[i_1, i_2, ..., i_s \mid j_1, j_2, ..., j_t]$ for the submatrix consisting of the i_1-th, i_2-th, ..., i_s-th block rows and j_1-th, j_2-th,..., j_t-th block columns of A. $A(i_1, i_2, ..., i_s \mid j_1, j_2, ..., j_t)$ is used for the submatrix of A obtained by deleting the i_1-th, i_2-th, ..., i_s-th block rows and j_1-th, j_2-th,..., j_t-th block columns from A. Instead of $A[i_1, i_2, ..., i_s \mid i_1, i_2, ..., i_s]$ and $A(i_1, i_2, ..., i_s \mid i_1, i_2, ..., i_s)$, we write $A[i_1, i_2, ..., i_s]$ and $A(i_1, i_2, ..., i_s)$ respectively.

A square matrix is called *upper triangular* if all the elements below the main diagonal are zero.

Firstly we consider Question 1.1.6.

Theorem 2.4.1. Suppose $A \in \Lambda(n, k)$ where $1 \leq k \leq n$. Then $s(A) \leq n + k - 2$, and the equality holds if and only if A has an order $n - k + 1$ nonsingular submatrix which is permutation equivalent to an upper triangular matrix.

Proof. Brualdi and Jung have proved $s(A) \leq n + k - 2$ in [BrJu1992].

s(A)$= n+k-2$ ($i.e.$, b(A) $= n-k+1$) if and only if A contains an order $n-k+1$ submatrix of the form $\begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$, if and only if A has an order $n-k+1$ non-singular submatrix which is permutation equivalent to an upper triangular matrix. □

Now we discuss Question 1.1.7.

If A$\in \Lambda(n,k)$ where k$|$ n and b(A)$=\frac{n}{k}$, then from the form (2.1), we see that $m_1 = \dots = m_p = n_1 = \dots = n_p = k$.

Thus we obtain the following assertion.

Theorem 2.4.2. Let A$\in \Lambda(n,k)$ where $k \mid n$, then s(A)$=2n-1-\frac{n}{k}$ iff A$\cong \widetilde{A_{n,k}}$.

In order to investigate the case $k \nmid n$, we first prove the following lemmas.

Lemma 2.4.3. Suppose that A$\in \Lambda(n,k)$ and $k < n < 2k$, then s(A)$=2n-3$ iff $(n-k) \mid k$, and A$\cong \widetilde{A_{n,k}}$.

Proof . If $(n-k) \mid k$ and A$\cong \widetilde{A_{n,k}}$, then A$\cong O_{n-k}\overline{\oplus}\dots\overline{\oplus}O_{n-k}$ and thus s(A)$=2n-3$.

If s(A)=$2n-3$, then b(A)=2.

By Theorem 2.1.2, we see that A$\cong O_{m_1,n_1}\overline{\oplus}...\overline{\oplus}O_{m_t,n_t}$.

Since A$\in \Lambda(n,k)$, it implies that $m_1 = n_1 = ... = m_t = n_t = n-k$.

Hence $(n-k) \mid n$, A$\cong \widetilde{A_{n,k}}$.

Since $(n-k) \mid (n-k)$, we have $(n-k) \mid k$. □

Lemma 2.4.4. If A is permutation equivalent to the block triangular form

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1t} \\ O & A_{22} & \dots & A_{2t} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & A_{tt} \end{bmatrix} \text{ or } \begin{bmatrix} A_{11} & O & \dots & O \\ A_{21} & A_{22} & \dots & O \\ \dots & \dots & \dots & \dots \\ A_{t1} & A_{t2} & \dots & A_{tt} \end{bmatrix},$$

then b(A)$\geq \sum_{i=1}^{t}$b(A_{ii}).

Proof. Since A can achieve at least b(A_{11})+b(A_{22})+...+b(A_{tt}) stairs, it implies that b(A)$\geq \sum_{i=1}^{t}$b(A_{ii}). □

Lemma 2.4.5. Let A be an $m \times n$ (0,1) matrix having no rows or columns consisting of all 0's or 1's and b(A)=2. Then the number of 0's in any two unequal columns is not more than m.

Proof. Due to Theorem 2.1.2, A=$O_{m_1,n_1}\overline{\oplus}...\overline{\oplus}O_{m_t,n_t}$, where $t \geq 2$ and $m_1 + ... + m_t = m$.

The number of 0's in any two unequal columns equals $m_{i_1} + m_{i_2} \leq m$, where $1 \leq i_1, i_2 \leq t$ and $i_1 \neq i_2$. □

For the case $k \nmid n$, we have the following result.

Theorem 2.4.6. Suppose that $pk < n < (p+1)k$, A$\in \Lambda(n,k)$, then s(A)$=2n-1-(p+1)$ iff $(n \bmod k) \mid k$ and A$\cong \widetilde{A_{n,k}}$.

Proof. The sufficiency is clear.

To prove the necessity, when p=1, we are done.

Suppose when $p < h$, $h \geq 2$, the statement holds.

Then when $p = h$, b(A)$=h+1$.

According to Theorem 2.2.1, we may assume A is in the normal form

$$\begin{bmatrix} J_{m_1,n_1} & A_{12} & \dots & A_{1,h+1} \\ O & J_{m_2,n_2} & \dots & A_{2,h+1} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & J_{m_{h+1},n_{h+1}} \end{bmatrix} \tag{2.5}$$

where $m_i \leq k$, $n_i \leq k$, $i = 1, \dots, h+1$.

In the form (2.5), $hk < n < (h+1)k$ and A$\in \Lambda(n,k)$, the following assertion is clear.

Claim 2.4.7. For any two distinct integer i, j, $1 \leq i, j \leq h+1$, we have $k < m_i + m_j \leq 2k$, $k < n_i + n_j \leq 2k$.

In the form (2.5), suppose $D_i = (A_{i,i+1} \ldots A_{i,h+1})$, $i = 1, \ldots, h$.

Let h_i be the smallest integer of set $\{i+1, \ldots, h+1\}$ such that $A_{i,h_i} \neq O$. If $D_i = O$, then let $h_i = i$.

Suppose $F_j = \begin{bmatrix} A_{1,j} \\ \ldots \\ A_{j-1,j} \end{bmatrix}$, $j = 2, \ldots, h+1$.

Let t_j be the largest integer of set $\{1, \ldots, j-1\}$ such that $A_{t_j,j} \neq O$. If $F_j = O$, then let $t_j = j$.

Claim 2.4.8. If $h_i > i$, then $b(A_{i,h_i}) = 2$.

Proof of claim. If $b(A_{i,h_i})>2$, then it is clear that $b(A(h_i \mid i)) > h+1$, hence $b(A)>h+1$, which contradicts the assumption that $b(A)=h+1$.

If $b(A_{i,h_i}) = 1$, by Theorem 2.1.1, $A_{i,h_i} \cong \begin{bmatrix} J_{a,b} & O \\ O & O \end{bmatrix}$.

Assume $a=m_i$, then $m_i + m_{h_i} \leq k$, where $i \neq h_i$, which contradicts claim 2.4.7.

Then $a<m_i$. Similarly, $b<n_{h_i}$.

Since $A[i, h_i] \cong O_{m_{h_i},n_i} \overline{\oplus} A_{i,h_i}$, $A[i, h_i]$ has at least three stairs.

Since A$\cong \begin{bmatrix} A(i,h_i,h_i+1,...,h+1) & * & * \\ O & A[i,h_i] & * \\ O & O & A[h_i+1,...,h+1] \end{bmatrix}$,

it implies that

b(A)$\geq$b($A(i,h_i,h_i+1,...,h+1)$)+b($A[i,h_i]$)+b($A[h_i+1,...,h+1]$)

$\geq (h_i-1-1)+3+[h+1-(h_i+1)+1]$

$= h+2 > h+1$, a contradiction.

Hence b(A_{i,h_i}) $= 2$. □

Here we can get the following statements, whose proofs are given in the next section.

Claim 2.4.9. If $h_1 > 1$, then $A_{i,h_1} = O$, $i = 2,...,h_1-1$.

Claim 2.4.10. If $1<h_1 < h+1$, then $A_{1i} = O$, $i = h_1+1,...,h+1$.

Applying the above assertions, we now finish the proof of Theorem 2.4.6.

In the form (2.5), if $D_1 = O$, then $m_1 = n_1 = k$, A=$J_k \oplus A(1)$, A(1)$\in$ $\Lambda(n-k,k)$.

Let $n' = n-k$, then $(h-1)k < n' < hk$,

and s(A(1))=s(A)-s(J_k) -1

$= [2n-1-(h+1)]-(2k-1-1)-1$

$$= 2(n-k) - 1 - h$$

$$= 2n' - 1 - h.$$

Due to the induction assumption, A(1)$\cong \widetilde{A_{n-k,k}}$,

$$((n-k) \bmod k) \mid k.$$

Then $(n \bmod k) \mid k$ and A$\cong \widetilde{A_{n,k}}$.

If $D_1 \neq O$, i.e., $h_1 > 1$, then we distinguish the following two cases.

Case 1. $h_1 = h+1$. Then $A_{12} = A_{13} = \ldots = A_{1h} = O$.

Due to claim 2.4.9, $A_{2,h+1}=A_{3,h+1}=\ldots=A_{h,h+1}=O$.

Therefore A$\cong A[1,h+1] \oplus A(1,h+1)$.

Then we have $A[1,h+1] \in \Lambda(n',k)$, $A(1,h+1) \in \Lambda(n'',k)$ where $k < n' < 2k$, $(h-2)k < n'' < hk$, $n' + n'' = n$.

We get $b(A[1,h+1]) \geq 2$, $b(A(1,h+1)) \geq h-1$ by corollary 2.2.5.

Since $b(A[1,h+1])+b(A(1,h+1)) = h+1$, 't implies that

$$b(A[1,h+1]) = 2, \text{ and}$$

$$b(A(1,h+1)) = h-1.$$

Due to corollary 2.2.5, $n'' \leq (h-1)k$, then $(h-2)k< n'' \leq (h-1)k$.

Suppose that $n'' < (h-1)k$.

By the induction assumption, $(n' \bmod k) \mid k$ and $(n'' \bmod k) \mid k$.

Then $k < n' \leq k + k/2$ and $(h-2)k < n'' \leq (h-2)k + k/2$.

Hence $(h-1)k < n \leq hk$, which contradicts $hk < n < (h+1)k$.

So $n'' = (h-1)k$.

It implies that $m_2 = n_2 = ... = m_h = n_h = k$,

therefore A(1,h+1)$\cong \oplus_{h-1} J_k$, A[1,h+1]$\cong \widetilde{A_{n-(h-1)k,k}}$.

Then A$\cong \widetilde{A_{n,k}}$, $(n \bmod k) | k$.

Case 2. $1 < h_1 < h+1$. Then $A_{12} = A_{13} = ... = A_{1,h_1-1} = O$.

Due to claim 2.4.9 and 2.4.10, $A_{1,h_1+1} = ... = A_{1,h+1} = O$ and

$$A_{2,h_1} = ... = A_{h_1-1,h_1} = O.$$

Next, we will prove $D_{h_1} = O$.

Suppose not, by claim 2.4.8, we have $b(A_{1,h_1}) = 2$ and $b(A_{h_1,h_{h_1}}) = 2$.

It implies that $b(\begin{bmatrix} A_{1h_1} & O \\ J_{m_{h_1},n_{h_1}} & A_{h_1,h_{h_1}} \end{bmatrix}) \geq 4$. Then $b(A(h_{h_1} \mid 1) > h+1$, hence b(A)$>h+1$, a contradiction.

Then $D_{h_1} = O$, A$\cong A[1,h_1] \oplus A(1,h_1)$.

Similar discussion as case 1, we have A$\cong \widetilde{A_{n,k}}$ and $(n \bmod k) | k$. □

The next result follows from Theorems 2.4.2 and 2.4.6.

Theorem 2.4.11. Suppose that A$\in \Lambda(n,k)$ where $1 \leq k \leq n$, then s(A)$=2n-1-\lceil n/k \rceil$ iff $k \mid n$ or $(n \bmod k) \mid k$, and A$\cong \widetilde{A_{n,k}}$.

Corollary 2.4.12. Suppose that A$\in \Lambda(n,k)$ where $1\leq k \leq n$, then s(A)$=2n-1-\lceil n/k\rceil$ iff $k \mid n$ and A$\cong \oplus_{\frac{n}{k}} J_k$, or $(n \bmod k) \mid k$ and A$\cong \oplus_{\lceil n/k\rceil-2} J_k \oplus (\overline{\oplus}_{\frac{k}{n \bmod k}+1} O_{n \bmod k})$.

The following corollary shows Conjecture 1.1.4 is true.

Corollary 2.4.13. If $k \nmid n$ and $(n \bmod k) \nmid k$ for $1 \leq k \leq n$, then $M(n,k) < 2n-1-\lceil n/k\rceil$.

Corollary 2.4.14. Suppose that $n \geq 2k$. Let A$\in \Lambda(n,k)$ satisfy

$$s(A)=M(n,k) = 2n-1-\lceil n/k\rceil .$$

Then there exits B$\in \Lambda(n-k,k)$ such that A is permutation equivalent to $J_k \oplus B$ where s(B)$=M(n-k,k)$.

The next corollary give an answer to Question 1.1.3.

Corollary 2.4.15. $M(n,k)=2n-1-\lceil n/k\rceil$ iff $k \mid n$ or $(n \bmod k) \mid k$.

2.5 Completion of proofs

In this section, we will prove claim 2.4.9 and 2.4.10 in order to complete the proof of Theorem 2.4.6.

From claim 2.4.7, we see that

Claim 2.5.1. If $h_i > i$, then A_{i,h_i} has no column or row consisting of all 1's.

Claim 2.5.2. If $h_i > i$, then A_{i,h_i} has no column or row consisting of all 0's.

Proof of claim. Suppose A_{i,h_i} has a column or row consisting of all 0's. Due to claim 2.4.8, $A[i, h_i]$ has at least three stairs, then $b(A) > h + 1$, a contradiction. □

Since $b(A)=b(A^T)$, we can obtain the following claims.

Claim 2.5.3. If $t_j < j$, then $b(A_{t_j,j}) = 2$.

Claim 2.5.4. If $t_j < j$, then $A_{t_j,j}$ has no column or row consisting of all $0's$ or $1's$.

Now we prove claim 2.4.9.

Proof of claim 2.4.9. Suppose not, then there exists t_{h_1}, such that $2 \leq t_{h_1} \leq h_1 - 1$ and $b(A_{t_{h_1},h_1}) = 2$.

By claim 2.4.8, 2.5.1, 2.5.2 and Theorem 2.1.2, we have

$$A_{1,h_1} \cong O_{p_1,q_1} \overline{\oplus} \ldots \overline{\oplus} O_{p_w,q_w},\ w \geq 2.$$

There exist two unequal columns in A_{1,h_1} opposite two unequal columns in $A_{t_{h_1},h_1}$, otherwise we see that all columns of $A_{t_{h_1},h_1}$ are identical, which contradicts $b(A_{t_{h_1},h_1}) = 2$.

Without loss of generality we assume the numbers of 0's in these two unequal columns of A_{1,h_1} are p_1, p_2 respectively, there are a_1, a_2 0's in their respectively opposite columns of $A_{t_{h_1},h_1}$.

Due to lemma 2.4.5, $a_1 + a_2 \leq m_{t_{h_1}}$, $p_1 + p_2 \leq m_1$.

Now consider the numbers of 1's in these two columns.

Since each column sum of A is k, and $m_1 = k$, we have

$m_{h_1} + (m_{t_{h_1}} - a_1) + (m_1 - p_1) \leq k$,

$m_{h_1} + (m_{t_{h_1}} - a_2) + (m_1 - p_2) \leq k$,

equivalently $m_{h_1} + m_{t_{h_1}} - a_1 \leq p_1$,

$m_{h_1} + m_{t_{h_1}} - a_2 \leq p_2$.

Then $m_{h_1} + m_{t_{h_1}} < 2m_{h_1} + m_{t_{h_1}}$

$\leq 2m_{h_1} + 2m_{t_{h_1}} - a_1 - a_2$

$\leq p_1 + p_2$

$\leq k$, contradicting lemma 2.4.7.

This completes the proof. □

From claim 2.4.9, we have the following assertion.

Claim 2.5.5. If $h_1 > 1$, then $A_{1,h_1} \cong O_{q,q_1} \overline{\oplus} \ldots \overline{\oplus} O_{q,q_w}$, $w \geq 2$, $wq=k$,

$m_{h_1} = q \leq k/2$.

Finally we will consider claim 2.4.10.

In the form (2.5), if $1 < h_1 < h+1$, we suppose that $(A_{1,h_1+1} \ldots A_{1,h+1}) \neq O$. Let s be the smallest number of the set $\{h_1+1, \ldots, h+1\}$ such that $A_{1s} \neq O$.

By claim 2.4.7, $m_{h_1} + m_s = q + m_s > k$.

Due to claim 2.5.5, $m_{h_1} \leq k/2$, hence $m_s > k - m_{h_1} \geq k/2$.

We see that $b(A_{1s}) \leq 3$, otherwise we contradict $b(A)=h+1$.

The following statements can be deduced.

Claim 2.5.6. $b(A_{1s}) \neq 1$.

Proof of claim. Suppose that $b(A_{1s}) = 1$. Then $A_{1s} \cong \begin{bmatrix} J_{a,b} & O \\ O & O \end{bmatrix}$.

Since each column sum is k, and $m_s > k/2$, it implies that $a<k/2$.

Due to claim 2.5.5, we have $b([A_{1,h_1}\ A_{1s}]) \geq 3$.

By claim 2.4.7, $b < n_s$, then $b(\begin{bmatrix} A_{1h_1} & A_{1s} \\ O & J_{m_s,n_s} \end{bmatrix}) \geq 4$, hence $b(A)>h+1$, a contradiction.

So $b(A_{1s}) \neq 1$. $\square$

Claim 2.5.7. If A_{1s} has no zero row, then $b(A_{1s}) = 3$.

If A_{1s} has a zero row, then $b(A_{1s}) = 2$.

Proof of claim. If A_{1s} has no zero row, due to the fact that column sum of A_{1s} is at most $k - m_s < k/2$, then we have

$$b(A_{1s}) \geq \lceil k/(k-m_s) \rceil \geq k/(k-m_s) > 2 \text{ by corollary 2.2.5.}$$

So $b(A_{1s}) = 3$.

If A_{1s} has a zero row, then $b(A_{1s}) < 3$, otherwise we contradict $b(A)=h+1$.

Due to claim 2.5.6, we have $b(A_{1s}) = 2$. □

Due to claim 2.5.3, 2.5.4 and 2.5.7, we have the following assertion.

Claim 2.5.8. $1<t_s < s$, and $b(A_{t_s,s}) = 2$.

Claim 2.5.9. A_{1s} has no zero column.

Proof of claim. Assume that A_{1s} has a zero column.

Now consider whether A_{ts} has a zero row or not.

Case 1. A_{1s} has a zero row, then $b(A_{1s}) = 2$ by claim 2.5.7.

So $b(\begin{bmatrix} J_{m_t,n_t} & A_{1s} \\ O & J_{m_s,n_s} \end{bmatrix}) \geq 4$, therefore $b(A)>h+1$, a contradiction.

Case 2. A_{1s} has no zero row, then $b(A_{1s}) = 3$ by claim 2.5.7.

So $b(\begin{bmatrix} A_{1s} \\ J_{m_s,n_s} \end{bmatrix}) \geq 4$, therefore $b(A)>h+1$, a contradiction.

Hence A_{1s} has no zero column. □

Claim 2.5.10. Let the numbers of 1's in any two unequal columns of A_{1s} be a_i and a_j, respectively, then $a_i + a_j \geq k/2$.

Proof of claim. Suppose not, then there exist two unequal columns of A_{1s} such that $a_1 + a_2 < k/2$, where a_1 and a_2 are the numbers of 1's in these two unequal columns of A_{1s}, respectively. We denote by B the submatrix of A_{1s} consisting of these two columns, then b(B)=2 by claim 2.5.9.

Since $a_1+a_2 < k/2$, it implies that the number of zero rows of B is greater than $k/2$, hence $b(\begin{bmatrix} A_{1,h_1} & A_{1s} \end{bmatrix}) \geq 4$ by claim 2.5.5.

Therefore $b(A) > h+1$, a contradiction. □

Next we will complete the proof of claim 2.4.10.

We see that there exist two unequal columns in $A_{1,s}$ opposite two unequal columns in $A_{t_s,s}$, otherwise we see that all columns of A_{1s} are identical, which contradicts $b(A_{1s}) \neq 1$.

Without loss of generality we assume the numbers of 1's in these two unequal columns of A_{1s} are c_1, c_2 respectively, and there are p_1, p_2 0's in their respectively opposite columns of $A_{t_s,s}$.

Due to lemma 2.5.10, $c_1 + c_2 \geq k/2$.

Due to claim 2.5.8, $b(A_{t_s,s}) = 2$, and we get $p_1 + p_2 \leq m_{t_s}$ by lemma 2.4.5.

In matrix A we consider the numbers of 1's in these two columns, then

$$m_s + (m_{t_s} - p_1) + c_1 \le k, \quad m_s + (m_{t_s} - p_2) + c_2 \le k.$$

Hence $2m_s + m_{t_s} + (m_{t_s} - p_1 - p_2) + c_1 + c_2 \le 2k$.

So $2m_s + m_{t_s} + k/2 \le 2k$.

Due to $m_s > k/2$, then $m_s + m_{t_s} < k$, a contradiction.

Hence $A_{1i} = O$, $i = h_1 + 1, ..., h + 1$. $\square$

Chapter 3

Some exact values concerning maximum jump numbers

In the first section of this chapter, we get the exact values for $M(11,k)$ where $1 \leq k \leq 11$. Two types of recursive constructions are discussed in section 2. Consequently we derive some inequalities for $M(n,k)$. In the last section, we deduce exact values for $M(n,6)$, $M(n,n-3)$, $M(n,n-4)$ and $M(qk+2,k)$.

3.1 M(11,k)

Theorem 3.1.1. The following hold:

(a) $M(11,1)=10$, $M(11,2)=15$, $M(11,3)=16$, $M(11,4)=17$, $M(11,5)=18$, $M(11,9)=18$, $M(11,10)=19$, $M(11,11)=20$.

(b) $M(11,7)=18$, $M(11,8)=18$.

(c) $M(11,6)=17$.

Proof. (a) can be obtained from formulas (1.1)–(1.8) in Chapter 1.

(b) By Corollary 2.4.13, $M(11,7)\leq 18$ and $M(11,8)\leq 18$.

Let $A_1=(J_3\oplus(O_1\overline{\oplus}O_1\overline{\oplus}O_1\overline{\oplus}O_1))\overline{\oplus}O_4$, then we see $s(A_1)=18$. Thus M(11,7)=18.

Let $A_2=((J_2\oplus(O_1\overline{\oplus}O_1\overline{\oplus}O_1))\overline{\oplus}O_3)\overline{\oplus}O_3$, then we see $s(A_2)=18$. Thus M(11,8)=18.

(c) Now we determine the value of M(11,6).

The following lemma is quoted from [BrJu1992].

Lemma 3.1.2 ([BrJu1992]). $M(2k+1,k+1)\geq 4k-\left\lceil\sqrt{k}\right\rceil$.

By lemma 3.1.2 and Corollary 2.4.13, $17\leq M(11,6)<19$. We only need to prove $M(11,6)\neq 18$.

For any matrix $A\in\Lambda(11,6)$, if $b(A)=3$, then from Theorem 2.2.1, we have

$$A\cong\begin{bmatrix} J_{6,5-q} & A_{12} & A_{13} \\ O & J_{p,q} & A_{23} \\ O & O & J_{5-p,6} \end{bmatrix}, \qquad (3.1)$$

where $1\leq p\leq 4$, $1\leq q\leq 4$.

Since b(A)=b(A^T), we may assume b(A_{12})$\leq$b(A_{23}) in the form (3.1).

In the form (3.1), b(A_{12})≥ 1, otherwise b(A_{12})=0, then $A_{12} = O$, hence p=6, a contradiction.

Moreover b(A_{23})≤ 2 and b(A_{13})≤ 3, otherwise b(A)>3, a contradiction.

Then 1$\leq$b(A_{12}) $\leq$b(A_{23}) ≤ 2 and 0$\leq$b(A_{13}) ≤ 3.

Lemma 3.1.3. In $\Lambda(11,6)$, there does not exist a matrix of the form (3.1) whose stair number is 3, and b(A_{12})=1.

Proof. Suppose there exists a matrix A of the form (3.1) and A$\in \Lambda(11,6)$, b(A)=3 and b(A_{12})=1.

Since A$\in \Lambda(11,6)$ and b(A_{12})=1, we see that $A_{12} \cong \begin{bmatrix} J_{6-p,q} \\ O \end{bmatrix}$.

Therefore $A \cong B_1 = \begin{bmatrix} J_{6-p,5-q} & J_{6-p,q} & C_1 \\ J_{p,5-q} & O & C_2 \\ O & J_{p,q} & A_{23} \\ O & O & J_{5-p,6} \end{bmatrix}$.

Claim 3.1.4. C_1 has a zero column.

Proof of the claim. Suppose C_1 has no zero column. By the fact that C_1 has 6 columns, then the number of 1's in C_1 is not less than 6.

Since each row of C_1 only have one 1's, we see that the number of 1's in

C_1 is $6-p<6$, a contradiction. □

Thus, we have

Claim 3.1.5. $b(C_1)=1$ or 2.

We distinguish the following two cases.

Case 1. $b(C_1)=2$. Then $C_1 \cong \begin{bmatrix} J_{s_1,1} \oplus J_{s_2,1} & O_{6-p,4} \end{bmatrix}$, where $s_1+s_2 = 6-p$.

Therefore $A \cong B_2 = \begin{bmatrix} J_{6-p,5-q} & J_{6-p,q} & J_{s_1,1} \oplus J_{s_2,1} & O_{6-p,4} \\ J_{p,5-q} & O & D_1 & D_2 \\ O & J_{p,q} & D_3 & D_4 \\ O & O & J_{5-p,2} & J_{5-p,4} \end{bmatrix}$.

Since $b(A)=3$, we see that $b(\begin{bmatrix} D_2 \\ D_4 \end{bmatrix}) \leq 1$, then $b(\begin{bmatrix} D_2 \\ D_4 \end{bmatrix}) = 1$.

Due to the fact that the column sum of B_2 is 6, we have

Claim 3.1.6. D_2 has no zero column. D_4 has no zero column.

Claim 3.1.7. $\begin{bmatrix} D_2 \\ D_4 \end{bmatrix}$ has no zero row.

Proof of the claim. Suppose D_2 has a zero row, then $q=1$. Hence D_2 can not have an all 1's row. Therefore $D_2 = O$. Then the last column sum of B_2 is at most 5, a contradiction.

Therefore D_2 has no zero row. Similarly, we can prove that D_4 has no zero row. $\square$

Since $b\left(\begin{bmatrix} D_2 \\ D_4 \end{bmatrix}\right) = 1$, by claim 3.1.6, 3.1.7 and Theorem 2.1.1, we have

$$\begin{bmatrix} D_2 \\ D_4 \end{bmatrix} = J_{2p,4}.$$

We choose any one row from the second and the third row blocks of B_2, respectively. Those two rows sum $\geq (5-q)+4+q+4 =$ [illegible], a contradiction.

Case 2. $b(C_1)=1$. Then $A \cong B_3 = \begin{bmatrix} J_{6-p,5-q} & J_{6-p,q} & J_{6-p,1} & O_{6-p,5} \\ J_{p,5-q} & O & E_1 & E_2 \\ O & J_{p,q} & E_3 & E_4 \\ O & O & J_{5-p,1} & J_{5-p,5} \end{bmatrix}$.

Due to the fact that the column sum of B_3 is 6, we have

Claim 3.1.8. E_2 has no zero column. E_4 has no zero column.

Due to the fact that the row sum of B_3 is 6, we have

Claim 3.1.9. $\begin{bmatrix} E_2 \\ E_4 \end{bmatrix}$ has no zero row.

Claim 3.1.10. E_2 has no all 1's row.

Proof of the claim. Suppose E_2 has an all 1's row, then $5-q=1$, i.e., $q=4$. Hence there are at most two 1's in each row of E_4.

Since E_4 has 5 columns and E_4 has no zero column, we see that

$b(E_4) \geq \left\lceil \frac{5}{2} \right\rceil = 3$ by Corollary 2.2.6.

Thus $b(B_3) \geq 4$, a contradiction. $\square$

Similarly, we have

Claim 3.1.11. E_4 has no all 1's row.

We see that the number of 1's in any one column in the third column block of B_3 is at least $6-p+5-p$. Then $6-p+5-p \leq 6$, i.e., $2p \geq 5$.

Therefore $p \geq 3$. Hence p is 3 or 4.

Claim 3.1.12. $\begin{bmatrix} E_2 \\ E_4 \end{bmatrix}$ has no all 1's column.

Proof of the claim. Suppose $\begin{bmatrix} E_2 \\ E_4 \end{bmatrix}$ has an all 1's column, then $6-p$=5, i.e., p=1, a contradiction. $\square$

By Theorem 2.1.2, $\begin{bmatrix} E_2 \\ E_4 \end{bmatrix} \cong O_{p-1,t_1} \overline{\oplus} ... \overline{\oplus} O_{p-1,t_k}$, where $t_1 + ... + t_k = 5$.

Then $p-1 \mid 2p$.

Therefore $p \neq 4$. Hence p =3, and $\begin{bmatrix} E_2 \\ E_4 \end{bmatrix} \cong O_{2,t_1} \overline{\oplus} O_{2,t_2} \overline{\oplus} O_{2,t_3}$.

Due to claim 3.1.8, we have $b(E_2)$=2 and $b(E_4)$=2.

We see that there is only one 1's in $\begin{bmatrix} E_1 \\ E_3 \end{bmatrix}$. So we may assume $E_1 = O$ or $E_3 = O$.

If $E_1 = O$, then $b([J_{p,q}\ E_3])$=2.

Since $b(E_2)$=2, we see $b(\begin{bmatrix} O & E_1 & E_2 \\ J_{p,q} & E_3 & E_4 \end{bmatrix}) \geq 4$, thus $b(B_3) \geq 4$, a contradiction.

If $E_3 = O$, then $b(\begin{bmatrix} E_3 & E_4 \\ J_{5-p,1} & J_{5-p,5} \end{bmatrix}) \geq 3$.

Therefore $b(B_3) \geq 4$, a contradiction.

We complete the proof of Lemma 3.1.3. □

Lemma 3.1.13. In $\Lambda(11,6)$, there does not exist a matrix of the form (3.1) whose stair number is 3, and $b(A_{12})=b(A_{23})=2$.

Proof. Suppose there exists a matrix A of the form (3.1) and $A \in \Lambda(11,6)$, $b(A)=3$ and $b(A_{12})=b(A_{23})=2$.

Let A_{22} be a $p \times q$ matrix. Since $b(A)=b(A^T)$, without loss of generality, we may assume $p \leq q$. (3.2)

A_{12} has no zero row, otherwise we contradict $b(A)=3$.

Suppose A_{12} has exactly s rows which consists of all 1's.

Then A$\cong$ $B_1 = \begin{bmatrix} J_{s,5-q} & J_{s,q} & C_2 \\ J_{6-s,5-q} & C_1 & C_3 \\ O & J_{p,q} & A_{23} \\ O & O & J_{5-p,6} \end{bmatrix}$, where $C_1 = O_{p,q_1}\overline{\oplus}...\overline{\oplus}O_{p,q_k}$, $kp + s = 6$ and $k \geq 2$.

Hence $2p \leq 6$, thus $p \leq 3$.

Since b(A_{23})=2, we see that $p \geq 2$.

Hence p=2 or 3.

Similarly, we can have q=2 or 3.

Distinguish the following two cases.

Case 1. p=2. Then k=2 or 3, hence s=0 or 2.

The following two subcases will be discussed.

Subcase 1.1. s=0. Then k=3.

Since 2$\leq q \leq$3, we see that $C_1 = O_{2,1}\overline{\oplus}O_{2,1}\overline{\oplus}O_{2,1}$ and q=3.

Thus $A_{23} \cong O_{1,3}\overline{\oplus}O_{1,3}$.

Then A$\cong$ $B_2 = \begin{bmatrix} J_{6,2} & O_{2,1}\overline{\oplus}O_{2,1}\overline{\oplus}O_{2,1} & D \\ O & J_{2,3} & O_{1,3}\overline{\oplus}O_{1,3} \\ O & O & J_{3,6} \end{bmatrix}$.

Then D$\in \Lambda(6,2)$.

By Corollary 2.2.5, we have b(D)$\geq$ 3.

Since b(D)≤ 3, we see b(D)$= 3$.

Thus D$\cong \begin{bmatrix} J_{m_1,n_1} & D_{12} & D_{13} \\ O & J_{m_2,n_2} & D_{23} \\ O & O & J_{m_3,n_3} \end{bmatrix}$ by Theorem 2.2.1.

Since D$\in \Lambda(6,2)$, we have $m_1 = m_2 = m_3 = n_1 = n_2 = n_3 = 2$

and $\mathrm{D}_{12} = D_{13} = D_{23} = O$.

Thus D$\cong J_2 \oplus J_2 \oplus J_2$, therefore b($\begin{bmatrix} D \\ O_{1,3}\overline{\oplus}O_{1,3} \end{bmatrix}$)$\geq 4$, a contradiction.

Subcase 1.2. s=2. Then k=2.

Thus $\mathrm{C}_1 = O_{2,q_1}\overline{\oplus}O_{2,q_2}$.

Claim 3.1.14. C_2 has zero column.

Proof of the claim. Similar to the proof of claim 3.1.4. □

Thus we have $\mathrm{b}(C_2) = 1$ or 2.

Claim 3.1.15. b(C_2)=1.

Proof of the claim. Suppose b(C_2)$\neq$1, then b(C_2)=2.

Without loss of generality, we may assume $\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} J_{1,1} \oplus J_{1,1} & O_{2,4} \\ * & E \end{bmatrix}$.

Due to the fact that the column sum of B_1 is 6, we have E has no zero column.

Since b(E)=1, we see that E has at least one all 1's row, and we may assume this row is the i-th row of B_1.

Since C_1 has no zero row, the number of 1's in the i-th row of $B_1 > 5 - q + 4 = 9 - q \geq 6$, a contradiction. □

Then $A \cong B_3 = \begin{bmatrix} J_{2,5-q} & J_{2,q} & J_{2,1} & O_{2,5} \\ J_{4,5-q} & O_{2,q_1} \overline{\oplus} O_{2,q_2} & F_1 & F_2 \\ O & J_{2,q} & F_3 & F_4 \\ O & O & J_{3,1} & J_{3,5} \end{bmatrix}$.

Claim 3.1.16. $q = 3$.

Proof of claim. Suppose $q = 2$, then $q_1 = q_2 = 1$.

Therefore there are at most two 1's in each row of F_2.

Since the column sum of B_3 is 6, it implies that F_2 has no zero column. By Corollary 2.2.6, $b(F_2) \geq \lceil \frac{5}{2} \rceil = 3$, thus $b(B_3) \geq 4$, a contradiction. □

Now we may assume $q_1 = 1$, $q_2 = 2$ in B_3.

Consider the following sub-matrix B of B_3,

$$B = \begin{bmatrix} O_{2,1} \overline{\oplus} O_{2,2} & F_1 & F_2 \\ J_{2,q} & F_3 & F_4 \end{bmatrix}.$$

It is clear that $\begin{bmatrix} F_2 \\ F_4 \end{bmatrix}$ has no rows or columns consisting of all 0's or all 1's.

Then $b(\begin{bmatrix} F_2 \\ F_4 \end{bmatrix}) \neq 1$. Thus $b(\begin{bmatrix} F_2 \\ F_4 \end{bmatrix}) = 2$. Hence, by Theorem 2.1.2,

$$\begin{bmatrix} F_2 \\ \\ F_4 \end{bmatrix} \cong O_{3,t_1} \overline{\oplus} O_{3,t_2}.$$

Then B$\cong (G_1\ G_2\ O_{3,t_1}\overline{\oplus}O_{3,t_2})$, where G_1 is obtained from $\begin{bmatrix} O_{2,1}\overline{\oplus}O_{2,2} \\ \\ J_{2,q} \end{bmatrix}$

by independent row permutations and $G_2 \cong \begin{bmatrix} F_1 \\ \\ F_3 \end{bmatrix}$.

There is only one 1's in G_2, hence we have

$$\text{B}\cong B' = \begin{bmatrix} * & 1 & J_{1,t_1'} & O_{1,t_2'} \\ \\ L_1 & O_{2,1} & J_{2,t_1'} & O_{2,t_2'} \\ \\ L_2 & O_{3,1} & O_{3,t_1'} & J_{3,t_2'} \end{bmatrix} \text{ where } \begin{bmatrix} L_1 \\ \\ L_2 \end{bmatrix} \text{ consists of 5 rows of}$$

the matrix $\begin{bmatrix} O_{2,1}\overline{\oplus}O_{2,2} \\ \\ J_{2,q} \end{bmatrix}$, and $\begin{cases} t_1' = t_1 \\ \\ t_2' = t_2 \end{cases}$ or $\begin{cases} t_1' = t_2 \\ \\ t_2' = t_1 \end{cases}$.

Since L_2 consists of three rows of the matrix $\begin{bmatrix} O_{2,1}\overline{\oplus}O_{2,2} \\ \\ J_{2,q} \end{bmatrix}$, we see that b($L_2$)$\geq 2$, thus b(B$'$)$\geq 4$, therefore b($B_3$)$\geq 4$, a contradiction.

Case 2. p =3.

By (3.2), we have q =3.

Then we may assume A$\cong B_4 = \begin{bmatrix} J_{6,2} & O_{3,2}\overline{\oplus}O_{3,1} & A_{13} \\ \\ O & J_{3,3} & O_{1,3}\overline{\oplus}O_{2,3} \\ \\ O & O & J_{2,6} \end{bmatrix}$. Here we may suppose A_{13} is

$$\text{(i)} \quad \begin{bmatrix} O_{1,3} & 1 \ 1 \ 0 \\ H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \quad \text{or}$$

$$\text{(ii)} \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ H_5 & H_6 & * & * & * & * \end{bmatrix} \quad \text{or}$$

$$\text{(iii)} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ H_7 & * & H_8 & * & * & * \end{bmatrix},$$ where H_1 and H_2 consist of two rows.

In case (i), we see that any row of $\begin{bmatrix} H_1 \\ H_3 \end{bmatrix}$ must be (0 0 0) or (1 1 1), otherwise

$$\begin{bmatrix} A_{13} \\ O_{1,3}\overline{\oplus}O_{2,3} \end{bmatrix} \cong \begin{bmatrix} H_1 & & & H_2 & & \\ H_3 & & & H_4 & & \\ O_{1,3} & & & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cong \begin{bmatrix} H_1 & & & H_2 & & \\ H_3 & & & H_4 & & \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

thus $b\left(\begin{bmatrix} A_{13} \\ O_{1,3}\overline{\oplus}O_{2,3} \end{bmatrix}\right)>3$, then $b(B_4)>3$, a contradiction.

Due to the fact that the row sum of B_4 is 6, then we have $H_1 = O_{2,3}$.

Without loss of generality, we may assume $H_3 = \begin{bmatrix} J_{2,3} \\ O_{1,3} \end{bmatrix}$, then $H_4 =$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Thus $b(H_2) \neq 1$, hence $b(H_2) \geq 2$.

Therefore $b(\begin{bmatrix} O_{2,1} & H_1 & H_2 \\ J_{3,1} & H_3 & H_4 \end{bmatrix}) \geq 4$.

So $b(B_4) \geq 4$, a contradiction.

In case (ii), there is only one 1's in H_5, thus there exists one row in $\begin{bmatrix} H_5 & H_6 \end{bmatrix}$ of the form $(0 \quad 0)$, otherwise the number of 1's in the seventh column of B_4 is greater than 6, a contradiction. Assume this row is the i-th row of B_4.

Therefore B_4 has the sub-matrix

$$\begin{array}{rc}
L= & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ l_1 & l_2 & l_3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \textit{the first row} \\ \textit{the } i\text{-th } \textit{row} \\ \textit{the } 8\text{-th } \textit{row} \\ \textit{the } 9\text{-th } \textit{row} \end{array} \\
\textit{column} & \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array}
\end{array}$$

where $(l_1 \; l_2 \; l_3) = (1 \; 1 \; 0)$ or $(0 \; 0 \; 1)$.

Therefore b(L)$\geq$ 4, thus b(B_4)$\geq$ 4, a contradiction.

In case (iii), similar to the case (ii), discuss column H_7 and H_8 instead of H_5 and H_6, then we also have b(B_4)$\geq$ 4, a contradiction.

We complete the proof of Lemma 3.1.13. □

By Lemma 3.1.3 and Lemma 3.1.13, we see there does not exist a matrix in Λ(11,6) whose stair number is 3. Hence M(11,6)$\neq$ 18. Therefore M(11,6)=17.

□

Bruualdi and Jung stated the following conjecture in [BrJu1992].

Conjecture 3.1.17. $M(2k+1, k+1)$=$4k$-$\lceil\sqrt{k}\rceil$.

The fact that M(11,6)=17 demonstrates that conjecture 3.1.17 holds for k=5.

3.2 Recursive constructions and some inequalities

Our recursive constructions are based on the next two facts.

Fact 1. If $A_1 \in \Lambda(n_1, k)$ and $A_2 \in \Lambda(n_2, k)$, then $A_1 \oplus A_2 \in \Lambda(n_1+n_2, k)$.

Fact 2. If A$\in \Lambda(k, 2k-n)$ where $k < n < 2k$, then A$\overline{\oplus}$O$_{n-k} \in \Lambda(n,k)$.

From Fact 1, we have the following theorem.

Theorem 3.2.1. If $n_1 \geq k \geq 1$ and $n_2 \geq k \geq 1$, then $M(n_1+n_2,k) \geq M(n_1,k)+M(n_2,k)+1$.

Proof. By definition, there exists a matrix A$_1 \in \Lambda(n_1,k)$ such that s(A$_1$)=$M(n_1,k)$.

Similarly, there exists a matrix A$_2 \in \Lambda$(n$_2$,k) such that s(A$_2$)=$M(n_2,k)$.

Since s($A_1 \oplus A_2$)=$M(n_1,k)+M(n_2,k)+1$ and $A_1 \oplus A_2 \in \Lambda(n_1+n_2,k)$, we have $M(n_1+n_2,k) \geq M(n_1,k)+M(n_2,k)+1$. □

Corollary 3.2.2. M(17,6)= 29.

Proof. M(17,6)$\geq M(9,6)+M(8,6)+1$

$$= 15+13+1$$

$$= 29.$$

By Corollary 2.4.13, $M(17,6) < 2 \times 17 - 1 - 3$

$$= 30.$$

Then M(17,6)=29. □

Corollary 3.2.3 ([BrJu1992]). M(n,k)$\geq 2k-1+M(n-k,k)$ where $n \geq 2k$.

Brualdi and Jung conjectured that equality in Corollary 3.2.3 holds in general. The following conjecture appeared in [BrJu1992].

Conjecture 3.2.4 ([BrJu1992]). Suppose $n \geq 2k$. Let A$\in \Lambda(n,k)$ satisfy s(A)=M(n,k). Then there exists B$\in \Lambda(n-k,k)$ such that A is permutation equivalent to $J_k \oplus B$ where s(B)=M$(n-k,k)$.

Proposition 3.2.5. Conjecture 3.2.4 does not hold.

Proof. If Conjecture 3.2.4 holds, then M(n,k)=s(J_k)+$M(n-k,k)+1$

$$= 2k-1+M(n-k,k).$$

Let $n=17$ and $k=6$. Then M(17,6)= $11+M(11,6)=28$, a contradiction to Corollary 3.2.2. □

Therefore the equality in Corollary 3.2.3 does not hold in general.

From Fact 2, we have the following theorem.

Theorem 3.2.6. M$(n,k)$$\geq 2n-2k+M(k,2k-n)$, where $k<n<2k$.

Proof. By definition, there exists a matrix A$\in \Lambda(k,2k-n)$ such that s(A)=$M(k,2k-n)$, *i.e.*, b(A)=$2k-1-M(k,2k-n)$.

Due to Lemma 2.3.3, b(A$\overline{\oplus}$O$_{n-k}$) =b$(A)=2k-1-M(k,2k-n)$.

Then M$(n,k)$$\geq 2n-1-b(A\overline{\oplus}O_{n-k})$

$\geq 2n-2k+M(k,2k-n)$. □

We note that equality in Theorem 3.2.6 does not hold in general. For example, M(11,6)=17>2×5 + $M(6,1)$.

3.3 Some exact values

From Corollary 2.4.13, Theorems 3.2.1 and 3.2.6, we can get exact values for $M(n,6)$, $M(n,n-3)$, $M(n,n-4)$, $M(n,n-2)$ and $M(qk+2,k)$.

Theorem 3.3.1. Suppose $0<k<\frac{n}{2}$, $k \nmid n$ and $(n \bmod k) \nmid k$. If there exist natural numbers n_1 and n_2 such that $n = n_1 + n_2$, $(n_1 \bmod k) | k$ and $(n_2 \bmod k) | k$, then $M(n,k) = 2n - 2 - \lceil n/k \rceil$.

Proof. Since $n > 2k$, without loss of generality, we may assume $n_1 > k$ and $n_2 > k$.

Then $M(n,k) \geq M(n_1,k) + M(n_2,k) + 1$

$$= 2n_1 - 1 - \left\lceil \frac{n_1}{k} \right\rceil + 2n_2 - 1 - \left\lceil \frac{n_2}{k} \right\rceil + 1$$

$$= 2n - 2 - \lceil n/k \rceil .$$

On the other hand, $M(n,k) \leq 2n - 2 - \lceil n/k \rceil$ by Corollary 2.4.13.

Hence $M(n,k) = 2n - 2 - \lceil n/k \rceil$. □

Corolary 3.3.2. $M(n,6) = 2n - 1 - \lceil n/6 \rceil - a$,

$$\text{where } a = \begin{cases} 0 & \text{if } 6 \mid n \text{ or } (n \bmod 6) \mid 6 \\ 1 & \text{if } (n \bmod 6) = 4 \text{ or } 5, \text{ and } n \neq 11 \\ 2 & \text{if } n = 11 \end{cases} .$$

Proof. If $6 \mid n$ or $(n \bmod 6) \mid 6$, it is clear.

For n=10 and 11, we can verify it directly.

If $n \geq 16$ and $(n \bmod 6) = 4$, let $n_1 = 8$ and $n_2 = n - 8$. Therefore we get $M(n, 6) = 2n - 2 - \lceil n/6 \rceil$ by Theorem 3.3.1.

If $n \geq 17$ and $(n \bmod 6) = 5$, it is similar to the above case. □

Corollary 3.3.3 ([BrJu1992]). $M(n, 3) = 2n - 1 - \lfloor n/3 \rfloor - a$ where $a \equiv n \pmod 3$ and $0 \leq a \leq 2$.

Corollary 3.3.4 ([BrJu1992]). $M(n, 4) = 2n - 1 - \lceil n/4 \rceil - a$ where $a = 1$ if $4 \mid n - 3$ and 0 otherwise.

Theorem 3.3.5. If $q \geq 1$ and $k > 1$, then

$$\mathrm{M}(qk+2, k) = \begin{cases} 2qk - q + 2 & \text{if } k \text{ is even} \\ 2qk - q + 1 & \text{if } k \text{ is odd} \end{cases}.$$

Proof. If k is even, by Corollary 2.4.15, then

$$\mathrm{M}(qk+2, k) = 2(qk+2) - 1 - (q+1)$$

$$= 2qk - q + 2.$$

If k is odd and $q \geq 2$, let $n_1 = k + 1$ and $n_2 = (q-1)k + 1$.

Then $\mathrm{M}(qk+2, k) = 2(qk+2) - 2 - (q+1)$

$$= 2qk - q + 1 \qquad \text{by Theorem 3.3.1.}$$

If k is odd and q=1, then the conclusion follows by formula (1.7) in Chapter 1.

Hence if k is odd, $M(qk+2,k) = 2qk - q + 1$. $\square$

Let q=2, we have a corollary.

Corollary 3.3.6 ([BrJu1992]). $M(2k+2,k)=\begin{cases} 4k & \text{if } k \text{ is even} \\ 4k-1 & \text{if } k \text{ is odd} \end{cases}$.

Theorem 3.3.7. If $M(m,m-k)=\begin{cases} 2m-3 & \text{if } k \mid m \\ 2m-4 & \text{if } k \nmid m \end{cases}$, where $1 \leq k \leq m-1$,

then $M(m+qk, m+(q-1)k)=\begin{cases} 2(m+qk)-3 & \text{if } k \mid m \\ 2(m+qk)-4 & \text{if } k \nmid m \end{cases}$ for $q \geq 0$.

Proof. If $k \mid m$, then

$$M(m+qk, m+(q-1)k)=2(m+qk)-1-\left\lceil \frac{m+qk}{m+(q-1)k} \right\rceil$$

$$= 2(m+qk)-3 .$$

If $k \nmid m$, we prove it by induction.

When $q=1$, $M(m+k,m) \geq 2k + M(m,m-k)$ (by Theorem 3.2.6)

$$\geq 2k + 2m - 4 .$$

On the other hand, $M(m+k,m) < 2(m+k)-1-\left\lceil \frac{m+k}{m} \right\rceil$

$$= 2(m+k)-3.$$

Hence $M(m+k,m) = 2(m+k)-4$.

Assume when $q=p$, the statement holds.

Then when $q = p+1$, $\mathrm{M}(m+(p+1)k, m+pk) \geq 2k+\mathrm{M}(m+pk, m+(p-1)k)$

$$=2(m+(p+1)k)-4.$$

On the other hand, $\mathrm{M}(m+(p+1)k, m+pk) < 2(m+(p+1)k)-1-\left\lceil \frac{m+(p+1)k}{m+pk} \right\rceil$

$$=2(m+(p+1)k)-3.$$

Hence $\mathrm{M}(m+(p+1)k, m+pk)=2(m+(p+1)k)-4$.

Therefore if $k \nmid m$, we get $\mathrm{M}(m+qk, m+(q-1)k)=2(m+qk)-4$. □

Corollary 3.3.8 ([BrJu1992]). $M(n, n-2) = \begin{cases} 2n-3 & \text{if } n \text{ is even} \\ 2n-4 & \text{if } n \text{ is odd} \end{cases}$.

Corollary 3.3.9. $M(n, n-3) = \begin{cases} 2n-3 & \text{if } 3 \mid n \\ 2n-4 & \text{if } 3 \nmid n \end{cases}$ for $n>4$.

Proof. Let k=3 in Theorem 3.3.7. Due to the fact that

$\mathrm{M}(5,2)=6=2 \times 5-4$,

$M(6,3)=9=2 \times 6-3$, and

$M(7,4)=10=2 \times 7-4$,

we get $M(n, n-3) = \begin{cases} 2n-3 & \text{if } 3 \mid n \\ 2n-4 & \text{if } 3 \nmid n \end{cases}$ for $n>4$. □

Corollary 3.3.10. $M(n, n-4) = \begin{cases} 2n-3 & \text{if } 4 \mid n \\ 2n-4 & \text{if } 4 \nmid n \end{cases}$ for $n>5$.

Proof. Let k=4 in Theorem 3.3.7. Due to the fact that

$\mathrm{M}(6,2)=8=2 \times 6-4$,

$M(7,3) = 10 = 2 \times 7 - 4,$

$M(8,4) = 13 = 2 \times 8 - 3$ and

$M(9,5) = 14 = 2 \times 9 - 4,$

we get $M(n, n-4) = \begin{cases} 2n-3 & \text{if } 4 \mid n \\ 2n-4 & \text{if } 4 \nmid n \end{cases}$ for $n>5$. □

Chapter 4

Maximum Number of Mutually Orthogonal Frequency Hyperrectangles

In this chapter, we derive a bound for the maximum number of mutually orthogonal frequency hyperrectangles that simultaneously generalizes Suchower's bound [Su1989] and LMW 's bound [LaMuWh1995]. Before it we state a simple necessary condition for an F-hyperrectangle of size $n_1 \times ... \times n_d$ and type t to exist.

4.1 A necessary condition

Firstly we give the following necessary condition for an F-hyperrectangle of size $n_1 \times \ldots \times n_d$ and type t to exist.

Theorem 4.1.1. If there exists an F-hyperrectangle HR$(n_1, \ldots, n_d; t; \lambda_1, \ldots, \lambda_m)$, where $1 \leq t \leq d-1$, then $l.c.m.\{\prod_{k=1}^{t} n_{i_k} \mid \{i_1, \ldots, i_t\} \in P_t(\underline{d})\} < \prod_{i=1}^{d} n_i$.

Proof. Suppose not, then $l.c.m.\{\prod_{k=1}^{t} n_{i_k} \mid \{i_1, \ldots, i_t\} \in P_t(\underline{d})\} = \prod_{i=1}^{d} n_i$, by the fact that

$$l.c.m.\{\prod_{k=1}^{t} n_{i_k} \mid \{i_1, \ldots, i_t\} \in P_t(\underline{d})\} \mid \prod_{i=1}^{d} n_i.$$

Set $Q = (\prod_{i=1}^{t} n_i)\lambda_1 = (\prod_{i=1}^{t} n_i)\Phi((1, \ldots, t), 1)$.

By formula (1.9) in Chapter 1, we have

$$l.c.m.\{\prod_{k=1}^{t} n_{i_k} \mid \{i_1, \ldots, i_t\} \in P_t(\underline{d})\} \mid Q,$$

then $\prod_{i=1}^{d} n_i \mid Q$.

Therefore $Q \geq \prod_{i=1}^{d} n_i$. (4.1)

Since $\lambda_1 + \ldots + \lambda_m = \prod_{j \neq 1, \ldots, t} n_j$, we have

$Q = \prod_{i=1}^{d} n_i(\lambda_1 / \prod_{j \neq 1, \ldots, t} n_j) < \prod_{i=1}^{d} n_i$, which contradicts (4.1).

□

Let $t = 1$, then we have a corollary.

Corollary 4.1.2. If there exists an F-hyperrectangle F$(n_1, \ldots, n_d; \lambda_1, \ldots, \lambda_m)$, then $l.c.m.\{n_1, \ldots, n_d\} < \prod_{i=1}^{d} n_i$.

Secondly we correct some errors in [Su1989].

In order to generalize Definition 1.2.3, the following definition was stated in [Su1989].

Definition 4.1.3. An F-hyperrectangle of size $n_1 \times ... \times n_d$, denoted by $F(n_1, ..., n_d; \lambda_{1,1}, ..., \lambda_{1,m}; ...; \lambda_{d,1}, ..., \lambda_{d,m})$ where for each i, $1 \leq i \leq d$, $n_i = \lambda_{i,1} + ... + \lambda_{i,m}$, is an $n_1 \times ... \times n_d$ array consisting of $m \geq 2$ symbols, say $\{1, ..., m\}$, with the property that for each i and j, $1 \leq i \leq d$, $1 \leq j \leq m$, the symbol j occurs exactly $\lambda_{i,j}$ times in every subarray consisting of the n_i cells $(k_1,...,k_d)$ where all coordinates but the i-th coordinate are fixed.

Let's look at an example.

$$
F_0: \quad
\begin{array}{cccc}
0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 \\
0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 \\
1\ 1\ 0\ 0 & 1\ 1\ 0\ 0 & 1\ 1\ 0\ 0 & 1\ 1\ 0\ 0 \\
1\ 1\ 0\ 0 & 1\ 1\ 0\ 0 & 1\ 1\ 0\ 0 & 1\ 1\ 0\ 0
\end{array}
$$

It is easy to verify that F_0 is an F-hyperrectangle of size $4 \times 4 \times 4$ by Definition 1.2.3. But according to Definition 4.1.3, it is not an F-hyperrectangle. So Definition 4.1.3 is not a generalization of Definition 1.2.3. Definition 1.2.4, which is a generalization of Definition 1.2.3, corrects Definition 4.1.3.

Concerning the definition of mutually orthogonal F-hyperrectangles, the following one corrects Definition 2 in [Su1989].

Definition 4.1.4. Two F-hyperrectangles $F(n_1, ..., n_d; \lambda_1, ..., \lambda_{m_1})$ and $F(n_1, ..., n_d; \mu_1, ..., \mu_{m_2})$ are orthogonal if upon superposition, each ordered pair (i, j), $1 \leq i \leq m_1$, $1 \leq j \leq m_2$, appears exactly $n_1^2\lambda_i\mu_j/\prod_{i=1}^{d} n_i$ times. A set of F-hyperrectangles is called mutually orthogonal if every pair of F-hyperrectangles are orthogonal.

4.2 An upper bound

Now we give the main result of this chapter.

Theorem 4.2.1. If there is a set of mutually orthogonal F-hyperrectangles $F_1 = HR(n_1, ..., n_d; t; \lambda_1^{(1)}, ..., \lambda_{m_1}^{(1)}), ..., F_r = HR(n_1, ..., n_d; t; \lambda_1^{(r)}, ..., \lambda_{m_r}^{(r)})$, then

$$\sum_{j=1}^{r}(m_j - 1) \leq \prod_{i=1}^{d} n_i - \sum_{k=1}^{t} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1.$$

Proof. Set $N = n_1 \times ... \times n_d$ and $M = m_1 + ... + m_r$. For the F-hyperrectangle F_k, $1 \leq k \leq r$, we define an $N \times m_k$ matrix $A_k = (a^{(k)}_{(i_1,...,i_d),j})$ by

$$a^{(k)}_{(i_1,...,i_d),j} = \begin{cases} 1 & \text{if } j \text{ occurs in position } (i_1, ..., i_d) \text{ of } F_k \\ 0 & \text{otherwise} \end{cases}.$$

Construct an $N \times M$ matrix $B = (A_1|A_2|...|A_r)$.

In any subarray of an F-hyperrectangle of type t defined by fixed k coordinates, say $i_1, i_2, \ldots i_k$, $1 \leq k \leq t$, one element will be determined if the others are known. For example, after we determine a subarray by the values of coordinates $i_1, i_2, \ldots i_k$, the element with the rest $d-k$ coordinates equal to their respective dimensions n_h,where $h \neq i_1, i_2, \ldots i_k$, can be interpreted as the last element of the above subarray. There are $\sum_{\{i_1,\ldots,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1)$ element of B are dependent for a fixed k.

Summing over k, there are

$\sum_{k=1}^{t} \sum_{\{i_1,\ldots,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1)$ such elements.

Then $\text{rank}(B) \leq \prod_{i=1}^{d} n_i - \sum_{k=1}^{t} \sum_{\{i_1,\ldots,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1)$.

Now we see that for $1 \leq k, l \leq r$. $A_k^T A_l = (f_{ij})$ is an $m_k \times m_l$ matrix where f_{ij}=number of times the ordered pair (i, j) occurs when F_k is superimposed onto F_l. Then

$$A_k^T A_l = \begin{cases} (N/\prod_{i=t+1}^{d} n_i) L_k & \text{if } k = l \\ (N/(\prod_{i=t+1}^{d} n_i)^2) L_k J_{m_k \times m_l} L_l & \text{if } k \neq l \end{cases},$$

where $\mathrm{L}_k = diag(\lambda_1^{(k)}, \ldots, \lambda_{m_k}^{(k)})$ and $J_{m_k \times m_l}$ is an $m_k \times m_l$ matrix consisting of all 1's.

Thus the $M \times M$ matrix $B^T B$ is given by

$$B^TB=\begin{pmatrix} A_1^TA_1 & A_1^TA_2 & \dots & A_1^TA_r \\ A_2^TA_1 & A_2^TA_2 & \dots & A_2^TA_r \\ \dots & \dots & \dots & \dots \\ A_r^TA_1 & A_r^TA_2 & \dots & A_r^TA_r \end{pmatrix}$$

$$= (N/(\prod_{i=t+1}^d n_i)^2)\begin{pmatrix} \prod_{i=t+1}^d n_iL_1 & L_1J_{m_1\times m_2}L_2 & \dots & L_1J_{m_1\times m_r}L_r \\ L_2J_{m_2\times m_1}L_1 & \prod_{i=t+1}^d n_iL_2 & \dots & L_2J_{m_2\times m_r}L_r \\ \dots & \dots & \dots & \dots \\ L_rJ_{m_r\times m_1}L_1 & L_rJ_{m_r\times m_2}L_2 & \dots & \prod_{i=t+1}^d n_iL_r \end{pmatrix}.$$

Set the $M \times M$ nonsingular diagonal matrix $C=\begin{pmatrix} L_1 & & & 0 \\ & L_2 & & \\ & & \dots & \\ 0 & & & L_r \end{pmatrix}.$

Then

$$B^TBC^{-1}=(N/(\prod_{i=t+1}^d n_i)^2)\begin{pmatrix} \prod_{i=t+1}^d n_iI_{m_1} & L_1J_{m_1\times m_2} & \dots & L_1J_{m_1\times m_r} \\ L_2J_{m_2\times m_1} & \prod_{i=t+1}^d n_iI_{m_2} & \dots & L_2J_{m_2\times m_r} \\ \dots & \dots & \dots & \dots \\ L_rJ_{m_r\times m_1} & L_rJ_{m_r\times m_2} & \dots & \prod_{i=t+1}^d n_iI_{m_r} \end{pmatrix}.$$

Next we find the eigenvalues of B^TBC^{-1}.

Observe that the sum of each column of B^TBC^{-1} is $rN/(\prod_{i=t+1}^d n_i)$, then $rN/(\prod_{i=t+1}^d n_i)$ is one eigenvalue of B^TBC^{-1}. For the eigenvalue $N/(\prod_{i=t+1}^d n_i)$, there are $m_1 - 1$ eigenvectors $(\alpha_{m_1}, \theta_{m_2}, ..., \theta_{m_r})^T$,..., $m_r - 1$ eigenvectors $(\theta_{m_1}, \theta_{m_2}, ..., \alpha_{m_r})^T$ where $\theta_{m_k} = (0, ..., 0)$ and α_{m_k} have m_k coordinates, and the coordinates of α_{m_k} sum to 0.

Hence rank $(B^T B C^{-1}) \geq 1 + M - r$.

Then rank$(B^T B)$=rank $(B^T B C^{-1}) \geq 1 + M - r$.

Due to the fact that rank$(B^T B) \leq$rank(B),

then we have

$$1 + M - r \leq \prod_{i=1}^d n_i - \sum_{k=1}^t \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^k (n_{i_j} - 1),$$

i.e., $\sum_{j=1}^r (m_j - 1) \leq \prod_{i=1}^d n_i - \sum_{k=1}^t \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^k (n_{i_j} - 1) - 1$. □

The above theorem generalizes many previous results.

When we reduce to the following cases:

(1) $t = 1$;

(2) $m_1 = ... = m_r$;

(3) (hypercubes case) $m_1 = ... = m_r$, $n_1 = ... = n_d$, $\lambda_1^{(i)} = ... = \lambda_{m_i}^{(i)}$, where $i = 1, ..., r$,

(4) $t = 1$, $m_1 = ... = m_r$, $\lambda_1^{(i)} = ... = \lambda_{m_i}^{(i)}$ where $i = 1, ..., r$;

(5) (F-squares of type 1) $d = 2$, $n_1 = n_2$, $t = 1$;

(6) (F-squares of type 1 with constant frequency vectors) $d = 2$, $n_1 = n_2$, $t = 1$, $\lambda_1^{(i)} = ... = \lambda_{m_i}^{(i)}$ for $i = 1, ..., r$;

(7) (F-squares of type 1 with constant frequency vectors and based on the same number of symbols) $d = 2$, $n_1 = n_2$, $t = 1$, $m_1 = ... = m_r$, $\lambda_1^{(i)} = ... = \lambda_{m_i}^{(i)}$ where $i = 1, ..., r$;

we have some corollaries, which appeared in the literature.

Corollary 4.2.2 ([Su1989]). If there is a set of r mutually orthogonal F-hyperrectangles of type 1

$F_1 = F(n_1, ..., n_d; \lambda_1^{(1)}, ..., \lambda_{m_1}^{(1)}), ..., F_r \quad F(n_1, ..., n_d; \lambda_1^{(r)}, ..., \lambda_{m_r}^{(r)})$,

then $\sum_{j=1}^{r}(m_j - 1) \leq \prod_{i=1}^{d} n_i - \sum_{i=1}^{d}(n_i - 1) - 1$.

Corollary 4.2.3. If there is a set of r mutually orthogonal F-hyperrectangles

$F_1 = HR(n_1, ..., n_d; t; \lambda_1^{(1)}, ..., \lambda_m^{(1)}), ..., F_r = HR(n_1, ..., n_d; t; \lambda_1^{(r)}, ..., \lambda_m^{(r)})$,

then $r \leq \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{k=1}^{t} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_k} - 1) - 1)$.

Corollary 4.2.4 ([LaMuWh1995]). If there is a set of r d-dimensional mutually orthogonal hypercubes

$F_1 = HR(n, ..., n; t; n^{d-t-1}, ..., n^{d-t-1}), ..., F_r = HR(n, ..., n; t; n^{d-t-1}, ..., n^{d-t-1})$,

then $\quad r \leq \frac{1}{n-1}(n^d - \sum_{i=1}^{t} \binom{d}{i}(n-1)^i - 1)$.

Corollary 4.2.5 ([Ch1980]). If there is a set of r mutually orthogonal F-hyperrectangles of type 1 based on the same number of symbols, m,

$F_1 = HR(n_1, ..., n_d; 1; \lambda^{(1)}, ..., \lambda^{(1)}), ..., F_r = HR(n_1, ..., n_d; 1; \lambda^{(r)}, ..., \lambda^{(r)})$,

then $\quad r \leq \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{i=1}^{d}(n_i - 1) - 1)$.

Corollary 4.2.6 ([PeMa1986]). If there is a set of r mutually orthogonal F-squares of type 1

F_1 =HR$(n, n; 1; \lambda_1^{(1)}, ..., \lambda_{m_1}^{(1)})$,..., F_r =HR$(n, n; 1; \lambda_1^{(r)}, ..., \lambda_{m_r}^{(r)})$,

then $\sum_{j=1}^{r}(m_j - 1) \leq (n-1)^2$.

Corollary 4.2.7 ([MaLeFe1981]). If there is a set of r mutually orthogonal F-squares of type 1

F_1 =HR$(n, n; 1; \lambda_1^{(1)}, ..., \lambda_{m_1}^{(1)})$,..., F_r =HR$(n, n; 1; \lambda_1^{(r)}, ..., \lambda_{m_r}^{(r)})$, where $\lambda_1^{(i)} = ... = \lambda_{m_i}^{(i)}$ for $i = 1, ..., r$,

then $\sum_{j=1}^{r}(m_j - 1) \leq (n-1)^2$.

Corollary 4.2.8 ([HeRaSe1975]). If there is a set of r mutually orthogonal F-squares of type 1 based on the same number of symbols, m,

F_1 =HR$(n, n; 1; \lambda, ..., \lambda)$,..., F_r =HR$(n, n; 1; \lambda, ..., \lambda)$, where $\lambda = \frac{n}{m}$,

then $r \leq \frac{1}{m-1}(n-1)^2$.

Theorem 4.2.1 also motivates the following definition.

Definition 4.2.9. A set of mutually orthogonal F-hyperrectangles F_1 =HR$(n_1, ..., n_d; t; \lambda_1^{(1)}, ..., \lambda_{m_1}^{(1)})$,..., F_r =HR$(n_1, ..., n_d; t; \lambda_1^{(r)}, ..., \lambda_{m_r}^{(r)})$ is called complete if the equality holds in Theorem 4.2.1.

At this time, complete sets of F-hyperrectangles are only known to exist when the frequency vectors are constant and, except for a few sporadic cases, when $N = n_1 \ldots n_d$ is a prime power.

Chapter 5

Constructions for Mutually Orthogonal Frequency Hyperrectangles

In this chapter, we provide two different ways of constructing mutually orthogonal frequency hyperrectangles of a prescribed type. We concentrate on frequency hyperrectangles based on m symbols with constant frequency vector, $\mathrm{HR}(n_1, \ldots, n_d; t; \frac{1}{m}\prod_{i=t+1}^{d} n_i, \ldots, \frac{1}{m}\prod_{i=t+1}^{d} n_i)$. So we simplify $\mathrm{HR}(n_1, \ldots, n_d; t; \frac{1}{m}\prod_{i=t+1}^{d} n_i, \ldots, \frac{1}{m}\prod_{i=t+1}^{d} n_i)$ to $\mathrm{FHR}(n_1, \ldots, n_d; t; m)$.

Firstly, we exhibit sets of linear polynomials over finite fields that represent complete sets of mutually orthogonal frequency hyperrectangles(MOFHR) of a prescribed type and of prime power order, which generalize Bose [Bo1938] and Mullen [Mu1988] construction.

Secondly, we give a recursive algorithm to construct $(d+1)$-dimensional MOFHR of type $t+1$ from d-dimensional MOFHR of type t, which generalizes a recursive procedure described in [LaMuWh1995].

5.1 Polynomial representation of orthogonal F-hyperrectangles

Let F_q denote the finite field of order q, where q is a prime power. Following Niederreiter in [Ni1971], we say that a polynomial $f(x_1, ..., x_n)$ with coefficients in F_q is a *permutation polynomial* in n variables over F_q if the equation $f(x_1, ..., x_n) = \alpha$ has exactly q^{n-1} solutions in F_q^n for each $\alpha \in F_q$. More generally, we say that a system $f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)$ of polynomials with $1 \leq m \leq n$ is *orthogonal* in F_q if the system of equations $f_i(x_1, ..., x_n) = \alpha_i$ $(i = 1, ..., m)$ has exactly q^{n-m} solutions in F_q^n for each $(\alpha_1, ..., \alpha_m) \in F_q^m$.

As indicated by Niederreiter in [Ni1971], the system $f_1(x_1, ..., x_n), ...,$

$f_m(x_1, ..., x_n)$ is orthogonal if and only if for all $(b_1, ..., b_m) \in F_q^m$ with $(b_1, ..., b_m) \neq (0, ..., 0)$, the polynomial $b_1 f_1(x_1, ..., x_n) + ... + b_m f_m(x_1, ..., x_n)$ is a permutation polynomial in n variables over F_q.

Let $m = q$, a prime power, and let $n_i = q^{s_i}$, where $s_i \geq 1$ is an integer. Now we have the following theorem.

Theorem 5.1.1. The $\frac{1}{q-1}(q^{\sum_{i=1}^{d} s_i} - \sum_{k=1}^{t} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(q^{s_{i_j}} - 1) - 1)$ polynomials

$$f_{(a_{11},...,a_{1s_1},...,a_{d1},...a_{ds_d})}(x_{11}, ..., x_{1s_1}, ..., x_{d1}, ..., x_{ds_d}) = \sum_{i=1}^{d} \sum_{j=1}^{s_i} a_{ij} x_{ij} \quad (5.1)$$

over F_q, where

(a) at least t+1 of the subvectors $(a_{11}, ..., a_{1s_1}), ..., (a_{d1}, ...a_{ds_d})$ are nonzero;

(b) No two sets of a's are nonzero F_q multiples of each other, i.e.,

$(a'_{11}, ..., a'_{1s_1}, ..., a'_{d1}, ...a'_{ds_d}) \neq e(a_{11}, ..., a_{1s_1}, ..., a_{d1}, ...a_{ds_d})$ for any $e \neq 0 \in F_q$

represent a complete set of MOFHR$(q^{s_1}, ..., q^{s_d}; t; q)$ of dimension d and type t.

Proof. There are $\frac{1}{q-1}(q^{\sum_{i=1}^{d} s_i} - \sum_{k=1}^{t} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(q^{s_{i_j}} - 1) - 1)$ polynomials over F_q defined by (5.1) and conditions (a), (b).

Label the i-th coordinate with all s_i-tuples $(j_{i1}, ..., j_{is_i})$ over F_q, for $1 \leq i \leq d$. Now we may view an FHR$(n_1, ..., n_d; t; m)$ as a function $f : F_q^{\sum_{i=1}^{d} s_i} \to F_q$, where the element $(j_{11}, ..., j_{1s_1}, ..., j_{d1}, ..., j_{ds_d})$ becomes the element

$$f(j_{11}, ..., j_{1s_1}, ..., j_{d1}, ..., j_{ds_d}) \in F_q.$$

If $(j_{i_k,1}, ..., j_{i_k,s_{i_k}})$, for $k = 1, ..., t$, is fixed, then

$$f_{(a)}(x_{11}, \ldots, x_{1s_1}, \ldots, x_{d1}, \ldots, x_{ds_d}) \ |_{(x_{i_k,1},\ldots,x_{i_k,s_{i_k}})=(j_{i_k,1},\ldots,j_{i_k,s_{i_k}}),\ k=1,\ldots,t} = \alpha$$

has the same number of solutions in $F_q^{\sum_{w\neq i_1,\ldots,i_k} s_w}$ for each $\alpha \in F_q$. so that in the subarray obtained by fixing the i_1-th, ..., i_t-th coordinates, each element of F_q is picked up equally often. Hence $f_{(a)}(x_{11}, ..., x_{1s_1}, ..., x_{d1}, ..., x_{ds_d})$ represents an FHR$(n_1, ..., n_d; t; m)$.

Clearly the F-hyperrectangles represented by $f_1 = f_{(a)}(x_{11}, ..., x_{1s_1}, ..., x_{d1}, ..., x_{ds_d})$ and $f_2 = f_{(a')}(x_{11}, ..., x_{1s_1}, ..., x_{d1}, ..., x_{ds_d})$ are orthogonal if and only of f_1 and f_2 form an orthogonal system of polynomials in $\sum_{i=1}^{d} s_i$ variables over F_q. By the Corollary of [Ni1971, p.417], f_1 and f_2 form an orthogonal system over F_q if and only if for all $(b_1, b_2) \neq (0,0) \in F_q^2$, the polynomial $b_1 f_1 + b_2 f_2$ is a permutation polynomial in $\sum_{i=1}^{d} s_i$ variables over F_q. Any linear polynomial of the form $\sum_{j=1}^{r} c_j x_j$ is a permutation polynomial in r variables provided at least one $c_j \neq 0$.

Let $(b_1, b_2) \neq (0,0) \in F_q^2$. If $b_1 = 0$, then $b_2 f_2$ is a permutation polynomial since $b_2 \neq 0$ while if $b_2 = 0$, then $b_1 f_1$ is a permutation polynomial. Suppose $b_1 b_2 \neq 0$, so $b_1 f_1 + b_2 f_2$ is a permutation polynonial unless all

$\sum_{i=1}^{d} s_i$ coefficients vanish, in which case $b_1 a_j' = -b_2 a_j$ for $j = 1, ..., \sum_{i=1}^{d} s_i$, i.e., unless $a_j' = -b_2 a_j / b_1$ for $j = 1, ..., \sum_{i=1}^{d} s_i$, a contradiction of condition (b). Hence f_1 and f_2 form an orthogonal system and the proof is complete.

$\square$

Theorem 5.1.1 is a generalization of the Mullen and Bose construction.

Corollary 5.1.2 ([Mu1988]) The $\frac{1}{q-1}(q^s - 1)^2$ polynomials

$f_{(a_1,...,a_{2s})}(x_1, ..., x_{2s}) = a_1 x_1 + ... + a_{2s} x_{2s}$ over F_q, where

(a) $(a_1, ..., a_s) \neq (0, ..., 0)$;

(b) $(a_{s+1}, ..., a_{2s}) \neq (0, ..., 0)$;

(c) No two sets of a's are nonzero F_q multiples of each other, i.e.,

$(a_1', ..., a_{2s}') \neq e(a_1, ..., a_{2s})$ for any $e \neq 0 \in F_q$

represent a complete set of mutually orthogonal F-squares $F(q^s, q^s; \lambda, ..., \lambda)$, where $\lambda = q^{s-1}$.

Corollary 5.1.3 ([Bo1938]) The $q-1$ polynomials $f_{(a)}(x_1, x_2) = ax_1 + x_2$ with $a \neq 0 \in F_q$ represent a complete set of $q-1$ mutually orthonogal Latin squares of order q.

For example, Theorem 5.1.1 gives the complete sets Ξ_1 of 9 MOFHR(4,4;1;2);

Ξ_2 of 3 MOFHR(4,2;1;2); and Ξ_3 of 3 MOFHR(4;0;2).

Consider the 9 polynomials over GF(2) given by

$$f_1(x_1,x_2,x_3,x_4) = x_1 \quad + x_3$$

$$f_2(x_1,x_2,x_3,x_4) = \quad x_2 + x_3$$

$$f_3(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_3$$

$$f_4(x_1,x_2,x_3,x_4) = x_1 \qquad + x_4$$

$$f_5(x_1,x_2,x_3,x_4) = x_1 \quad + x_3 + x_4$$

$$f_6(x_1,x_2,x_3,x_4) = \quad x_2 \quad + x_4$$

$$f_7(x_1,x_2,x_3,x_4) = \quad x_2 + x_3 + x_4$$

$$f_8(x_1,x_2,x_3,x_4) = x_1 + x_2 \quad + x_4$$

$$f_9(x_1,x_2,x_3,x_4) = x_1 + x_2 + x_3 + x_4.$$

These 9 polynomials represent the complete set of 9 MOFHR(4,4;1;2):

$$\Xi_1: \quad H_1 = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad H_2 = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad H_3 = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$$H_4 = \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \qquad H_5 = \begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array} \qquad H_6 = \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}$$

$$H_7 = \begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \qquad H_8 = \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \qquad H_9 = \begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}.$$

Consider the 3 polynomials over GF(2) given by

$$f_1(x_1, x_2, x_3) = x_1 \quad + x_3$$

$$f_2(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$f_3(x_1, x_2, x_3) = \quad x_2 + x_3.$$

These 3 polynomials represent the complete set of 3 MOFHR(4,2;1;2):

$$\Xi_2 : \; Q_1 = \begin{array}{cc} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{array} \qquad Q_2 = \begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \qquad Q_3 = \begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}.$$

Consider the 3 polynomials over GF(2) given by

$$f_1(x_1, x_2) = x_1$$

$$f_2(x_1, x_2) = x_2$$

$$f_3(x_1, x_2) = x_1 + x_2.$$

These 3 polynomials represent the complete set of 3 MOFHR(4;0;2):

Ξ_3 : 0 0 1 1 0 1 0 1 0 1 1 0 .

5.2 Type 0 canonical F-hyperrectangles

The following construction gives type 0 canonical FHR$(n_1, ..., n_d; 0; m)$ from MOFHR$(n_1; 0; m)$,...,MOFHR$(n_d; 0; m)$. Furthermore, adding a set of MOFHR of type 1, we will have an enlarged set of MOFHR of type 0. If the set of MOFHR of type 1 and the sets of MOFHR$(n_1; 0; m)$,...,MOFHR$(n_d; 0; m)$ are both complete, then the enlarged set of MOFHR of type 0 is also complete.

Suppose X is an FHR$(n_1, ..., n_d; t; m)$. Let $X(x_1, x_2, ..., x_d)$ denote the entry in position $(x_1, x_2, ..., x_d)$. The subarray obtained by assigning some fixed values $a_1, ..., a_t$ to the i_1-th, ..., i_t-th coordinates, where $0 \leq a_j \leq n_{i_j} - 1$ for $1 \leq j \leq t$, will be called a *hyperplane* and denoted by $X(x_{i_j} = a_j,\ j = 1, ..., t)$. The class of hyperplanes into which X is partitioned by coordinates

$x_{i_1}, ..., x_{i_t}$ is denoted by $\{X(x_{i_1} = a_1, ..., x_{i_t} = a_t) | \ 0 \leq a_j \leq n_{i_j} - 1 \ \}$.

A set Ψ_i of type 0 canonical FHR can be constructed from a set Δ of MOFHR$(n_i; 0; m)$.

Suppose L$\in$ Δ, then define a size $n_1 \times ... \times n_d$ FHR, L* as follows:

L*$(x_1, x_2, ..., x_d)$=L(x_i).

It is clear that L* is a type 0, d-dimensional FHR. The set $\cup_{i=1}^{d} \Psi_i$ is called the set of type 0 *canonical* FHR.

For example, from the set Ξ_3 of MOFHR(4;0;2) in section 5.1, we can construct two sets Ψ_1 and Ψ_2. The set $\Psi_1 \cup \Psi_2$ is the set of type 0 cononical FHR(4,4;0;2). Furthmore, $(\Psi_1 \cup \Psi_2) \cup \Xi_1$ is a complete set of MOFHR(4,4;0;2).

$$\Psi_1 : C_1 = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \qquad C_2 = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix} \qquad C_3 = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\Psi_2 : \ C_4 = \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \qquad C_5 = \begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{matrix} \qquad C_6 = \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}.$$

Theorem 5.2.1. Given a set of l_1 MOFHR$(n_1; 0; m)$, ..., a set of l_d MOFHR$(n_d; 0; m)$, there exists a set Ψ of $\sum_{i=1}^{d} l_i$ type 0 canonical FHR$(n_1, ..., n_d; 0; m)$. Furthermore, adding a set Λ of h MOFHR$(n_1, ..., n_d; 1; m)$, the enlarged set $\Psi \cup \Lambda$ is a set of $\sum_{i=1}^{d} l_i + h$ MOFHR$(n_1, ..., n_d; 0; m)$.

Proof. We only need to show that the members of $\Psi \cup \Lambda$ are orthogonal. Let X and Y be members of $\Psi \cup \Lambda$. If X, Y$\in \Lambda$, then X and Y are orthogonal.

Otherwise, we may assume X$\in \Psi$, which implies that there exists k, where $1 \leq k \leq d$, such that X$\in \Psi_k$.

If Y$\in \Psi_k$, we assume that X is constructed from X', Y is constructed from Y', where X', Y' are MOFHR$(n_k; 0; m)$. By the fact that each ordered pair occurs $\frac{n_k}{m^2}$ times in (X', Y'), where (X', Y') denotes the F-hyperrectangle obtained by superimposing X' and Y', we have that each ordered pair occurs $\frac{n_k}{m^2} \frac{\prod_{i=1}^{d} n_i}{n_k} = \frac{\prod_{i=1}^{d} n_i}{m^2}$ times in (X,Y). Hence X and Y are orthogonal.

If Y$\notin \Psi_k$, then each element occurs $\frac{\prod_{i=1}^{d} n_i}{m \times n_k}$ times in each hyperplane $Y(x_k = a)$, $0 \leq a \leq n_k - 1$. Hence each ordered pair occurs exactly $\frac{n_k}{m} \frac{\prod_{i=1}^{d} n_i}{m \times n_k} = \frac{\prod_{i=1}^{d} n_i}{m^2}$ times in (X,Y). Therefore X and Y are orthogonal. □

Corollary 5.2.2. If the initial sets of MOFHR$(n_1; 0; m)$, ...,MOFHR$(n_d; 0; m)$ and the set of MOFHR$(n_1, ..., n_d; 1; m)$ are both complete, so is the enlarged set of MOFHR$(n_1, ..., n_d; 0; m)$.

Proof. If $l_i = \frac{n_i - 1}{m-1}$ and $h = \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{i=1}^{d}(n_i - 1) - 1)$, then $\sum_{i=1}^{d} l_i + h = \frac{1}{m-1}(\prod_{i=1}^{d} n_i - 1)$. □

5.3 A recursive procedure

The following procedure constructs MOFHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$ from MOFHR$(n_1, ..., n_d; t; m)$ and MOFHR$(n_{d+1}, m; 1; m)$.

Given a set Ω of h MOFHR$(n_1, ..., n_d; t; m)$, we can divide the set into two classes, Ω_1 and Ω_2. The class Ω_1 consists of all FHR$(n_1, ..., n_d; t+1; m)$, and Ω_2 consists of the rest. Let h_1 be the cardinality of Ω_1.

Given a set Γ of l MOFHR$(n_{d+1}, m; 1; m)$, we append the following $n_{d+1} \times$

m rectangle

$$R= \begin{array}{cccc} 0 & 1 & \dots & m-1 \\ 0 & 1 & \dots & m-1 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & m-1 \end{array}$$ to Γ. We denote this new set as Γ^+.

Using this we now construct $hl + h_1$ MOFHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$. Suppose $X \in \Omega$, and $L \in \Gamma^+$, then define the $(d+1)$-dimensional hyperrectangle X^L as follows:

$X^L(x_1, x_2, ..., x_d, x_{d+1}) = L(X(x_1, x_2, ..., x_d), x_{d+1})$. The expression is interpreted to mean that X^L is partitioned into the hyperplanes $\{X^L(x_{d+1} = 0), X^L(x_{d+1} = 1), ..., X^L(x_{d+1} = n_{d+1} - 1)\}$, where $X^L(x_{d+1} = i)$ is X with a permutation applied to its symbols as defined by the i-th row of L. We can view each row of L as the image of a permutation from S_m, the symmetric group on m letters, 0,1,...,$m-1$.

The construction gives a new set $\Phi = \{X^L : X \in \Omega_1,\ L \in \Gamma^+\} \cup \{X^L : X \in \Omega_2,\ L \in \Gamma\}$.

Note that $|\Phi| = |\Omega_1| \times (|\Gamma| + 1) + |\Omega_2| \times |\Gamma| = h_1(l+1) + (h - h_1)l = hl + h_1$, as earlier claimed.

Let us look at an example. Set $\Omega = \{H_1, ..., H_9, C_1, ..., C_6\}$, and $\Gamma =$

$\{Q_1, Q_2, Q_3\}$. From the above procedure we can construct a complete set of 54 MOFHR(4,4,4;1;2). The following is $H_5^{Q_2}$.

	$x_3 = 0$				$x_3 = 1$				$x_3 = 2$				$x_3 = 3$			
$x_2 = 0, 1, 2, 3$																
$x_1 = 0$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
$x_1 = 1$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
$x_1 = 2$	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
$x_1 = 3$	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1

Before showing that Φ is a set of MOFHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$, we make some observations about the hyperplanes of members of Φ.

Let $(i_1, ..., i_t, i_{t+1})$ be an arbitrary element in $P_{t+1}(\underline{d+1})$. Now we consider two classes of hyperplanes in X^L, a typical member of Φ.

Class 1 hyperplanes are of the form $X^L(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$, where $i_{t+1} = d+1$. By definition this is $X(x_{i_1} = a_1, ..., x_{i_t} = a_t)$ with the permutation determined by row a_{t+1} of L applied to the symbols.

Class 2 hyperplanes are of the form $X^L(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$, where $i_{t+1} < d+1$. Say $P = X^L(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$. Partition P into $\{P(x_{d+1} = 0), ..., P(x_{d+1} = n_{d+1} - 1)\}$. Further $P(x_{d+1} = k)$ is obtained from $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ by permuting the symbols in $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ according to the permutation defined by the k-th row of L.

Lemma 5.3.1. The members of Φ are of type $t+1$.

Proof. Suppose $X^L \in \Phi$, and let $(i_1, ..., i_t, i_{t+1})$ be an arbitrary element in $P_{t+1}(\underline{d+1})$. We have to show that each symbol occurs an equal number of times in the hyperplane $X^L(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$, where $0 \leq a_k \leq n_{i_k} - 1$, $1 \leq k \leq t+1$.

This is obvious if the hyperplane is in class 1.

Consider a hyperplane P in class 2. Then P consists of n_{d+1} copies of hyperplane $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ with the k-th copy having the symbols permuted by the k-th row of L.

If X$\in \Omega_1$, then each symbol occurs equally often in $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ and therefore each symbol occurs equally often in P since permutations of the symbols of $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ leave the number of occurrences of each symbol unchanged. So if X$\in \Omega_1$, then X^L is an FHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$.

If X$\notin \Omega_1$, then X$\in \Omega_2$. Hence L$\in \Gamma$. Hyperplane P has partition

$$\{P(x_{d+1}=0), P(x_{d+1}=1), ..., P(x_{d+1}=n_{d+1}-1)\},$$

where $P(x_{d+1} = k)$ is $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})$ with the symbols replaced according to row k of L. For any element e, $0 \leq e \leq m-1$, the first row of L permutes e_1 to e,..., the n_{d+1} row of L permutes $e_{n_{d+1}}$ to e, where $0 \leq e_1, ..., e_{n_{d+1}} \leq m-1$.

By the fact that L$\in \Gamma$, we see that the multi-set

$\{e_1, ..., e_{n_{d+1}}\}$=$\{0,...,0,1,...,1,m-1,...,m-1\}$ (each element with multiplicity $\frac{n_{d+1}}{m}$). Hence the number of times that symbol e appears in P is $\sum_{k=1}^{n_{d+1}}$(the number of times that symbol e_k appears in $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})) = \frac{n_{d+1}}{m} \times \sum_{j=0}^{m-1}$(the number of times that symbol j appears in $X(x_{i_1} = a_1, ..., x_{i_t} = a_t, x_{i_{t+1}} = a_{t+1})) = \frac{n_{d+1}}{m} \times \frac{\prod_{k=1}^{d} n_k}{\prod_{k=1}^{t+1} n_{i_k}}$. Thus each symbol occurs equally often in P. □

Lemma 5.3.2. The members of Φ are mutually orthogonal.

Proof. Let X^L and Y^M be members of Φ, and assume X$\neq$Y. Then X and Y are orthogonal, and X^L and Y^M, respectively, have partitions $\{X^L(x_{d+1} = 0), ..., X^L(x_{d+1} = n_{d+1}-1)\}$ and $\{Y^M(x_{d+1} = 0), ..., Y^M(x_{d+1} = n_{d+1}-1)\}$. Each member of these partitions is obtained from X or Y by a permutation of the symbols, an operation that does not affect orthogonality. Hence

$X^L(x_{d+1} = k)$ is orthogonal to $Y^M(x_{d+1} = k)$ since X is orthogonal to Y. Therefore X^L is orthogonal to Y^M.

Assume X=Y. Then L is orthogonal to M. Let (L,M) denote the F-hyperrectangle obtained by superimposing L and M.

If the ordered pair (α, β) appears in the position (i, j) of (L,M), then (α, β) appears in $(X^L, X^M)(x_{d+1} = i)$ $\frac{\prod_{k=1}^{d} n_k}{m}$ times by the fact that element j appears $\frac{\prod_{k=1}^{d} n_k}{m}$ times in X.

Since L and M are orthogonal F-hyperrectangles, we see that each ordered pair occurs exactly $\frac{n_{d+1} \times m}{m^2} = \frac{n_{d+1}}{m}$ times in (L,M). Hence each pair occurs exactly $\frac{\prod_{k=1}^{d+1} n_k}{m^2}$ times in (X^L, X^M). $\square$

The following theorem follows from Lemmas 5.3.1 and 5.3.2.

Theorem 5.3.3. Given a set of h MOFHR$(n_1, ..., n_d; t; m)$, which consists of h_1 MOFHR$(n_1, ..., n_d; t+1; m)$, and a set of l MOFHR$(n_{d+1}, m; 1; m)$, there exists a set of $hl + h_1$ MOFHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$.

We note that Theorem 5.3.3 provides a generalization of Theorem 3.6 of [LaMuWh1995].

Given a complete set Φ of MOFHR$(n_1, ..., n_d; t; m)$, where $0 \leq t \leq d-2$, if the subset of Φ, consisting of all type $(t+1)$ F-hyperrectangles, is also complete, then we call the set Φ is *strongly complete*.

Corollary 5.3.4. (1) Given a complete set of MOFHR$(n_1, ..., n_d; d-1; m)$ and a complete set of MOFHR$(n_{d+1}, m; 1; m)$, then the above recursive algorithm gives a complete set of MOFHR$(n_1, ..., n_d, n_{d+1}; d; m)$.

(2) Given a strongly complete set of MOFHR$(n_1, ..., n_d; t; m)$, where $0 \leq t \leq d-2$, and a complete set of MOFHR$(n_{d+1}, m; 1; m)$, then the above recursive algorithm gives a complete set of MOFHR$(n_1, ..., n_d, n_{d+1}; t+1; m)$.

Proof. (1) If $h = \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{k=1}^{d-1} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1) = \frac{1}{m-1}(\prod_{i=1}^{d}(n_i - 1))$, $h_1 = 0$ and $l = n_{d+1} - 1$, then

$$\begin{aligned} hl + h_1 &= \frac{1}{m-1}(\prod_{i=1}^{d+1}(n_i - 1)) \\ &= \frac{1}{m-1}(\prod_{i=1}^{d+1} n_i - \sum_{k=1}^{d} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d+1})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1). \end{aligned}$$

(2) If $h = \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{k=1}^{t} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1)$, $h_1 = \frac{1}{m-1}(\prod_{i=1}^{d} n_i - \sum_{k=1}^{t+1} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1)$ and $l = n_{d+1} - 1$, then

$$hl + h_1 = \frac{1}{m-1}(\prod_{i=1}^{d+1} n_i - \sum_{k=1}^{t+1} \sum_{\{i_1,...,i_k\} \in P_k(\underline{d+1})} \prod_{j=1}^{k}(n_{i_j} - 1) - 1). \quad \square$$

It is easily seen that the complete set of MOFHR$(q^{s_1}, ..., q^{s_d}; t; m)$ constructed by Theorem 2.1.1 is strongly complete. So the polynomial construction (Theorem 2.1.1) gives a strongly complete set of MOFHR of prime power order.

Chapter 6

Carlitz Compositions

In the first two sections, we discuss Carlitz compositions with zeros allowed (CCWZA). The following two parameters: number of summands, largest summand are analyzed. In the third section, we introduce the Carlitz word which is a generalization of Carlitz composition.

6.1 The number of summands

Firstly we recall some results in [KnPr1998].

Let $d(n)$ be the number of CCWZA of size n. Then the generating function $D(z) := \sum_{n\geq 0} d(n)z^n = (1+2\sigma(z))/(1-2\sigma(z))$,

where $\sigma(z) = \sum_{j\geq 1}(-1)^{j-1}z^j/(1-z^j)$.

There is a dominant singularity τ, which is the solution in the interval [0,1] of the equation $\sigma(z) = 1/2$. Numerically $\tau = 0.386960$.

Therefore $d(n) \sim A\tau^{-n} = 1.337604 \cdot (2.584243)^n$ where $A = (1+2\sigma(\tau))/(2\tau\sigma'(\tau))$.

Now we consider the average number of summands in a random CCWZA of size n by a method that has appeared in [FlPr1987] under the nickname "adding a new slice".

Let $s(n)$ denote the summands number of CCWZA of size n.

We proceed from a CCWZA with k summands to one with $k+1$ summands by allowing the $(k+1)$-th summand x_{k+1} to be any positive integer and subtracting the forbidden case $x_{k+1} = x_k$. In terms of generating functions this reads as follows.

Let $f_k(z,u)$ be the generating function of those CCWZA with k summands where the coefficient of $z^n u^j$ refers to size n and the last summand $x_k = j$. We dissect the set of compositions into those with the last summand ≥ 1 (counted by $g_k(z,u)$) and those with the last summand $=0$ (counted by $h_k(z)$). Clearly, $h_k(z) = g_{k-1}(z,1) \quad (k \geq 2)$.

Then $g_{k+1}(z,u) = g_k(z,1)\frac{zu}{1-zu} + h_k(z)\frac{zu}{1-zu} - g_k(z,zu)$

$$\text{for } k \geq 1,\ g_1(z,u) = \frac{zu}{1-zu} \text{ and } h_1(z) = 1. \qquad (6.1)$$

We use another variable, w, to label the number of summands. Let

$G(z,u,w){:=}\sum_{k\geq 1} w^k g_k(z,u)$ and $F(z,u,w){:=}\sum_{k\geq 1} w^k f_k(z,u)$.

Multiplying (6.1) by w^{k+1} and summing over $k \geq 1$, we get

$$G(z,u,w){=}(w+w^2)G(z,1,w)\cdot\tfrac{zu}{1-zu}+(w+w^2)\tfrac{zu}{1-zu}-wG(z,zu,w).$$

Iterate it, and consequently we have

$$G(z,1,w){=}(w+1)r(z,w)/(1-(w+1)r(z,w)),$$

$$\text{where } r(z,w){=}\sum_{i\geq 1} z^i w^i(-1)^{i-1}/(1-z^i).$$

Therefore $F(z,1,w)=w+(w+1)G(z,1,w)$

$$={(w+(w+1)r(z,w))/(1-(w+1)r(z,w))}.$$

Hence the generating function $F(z){:=}\sum_{n\geq 0} s(n)\cdot z^n = \frac{\partial}{\partial w}F(z,1,w)\,|_{w=1}$

$$=(1+4u(z))/[1-2\sigma(z)]^2 \qquad \text{with}$$

$u(z)=\sum_{j\geq 1}(-1)^{j-1}jz^j/(1-z^j)$.

Therefore $F(z)\sim B/(1-z/\tau)^2 = 1.16589/(1-z/\tau)^2$,

where $B=(1+4u(\tau))/(2\tau\sigma'(\tau))^2$.

Hence $s(n)\sim Bn\cdot\tau^{-n}$.

Then the average number of summands in a random CCWZA of size n is $\sim \frac{Bn\cdot\tau^{-n}}{A\tau^{-n}} = 0.871626n$.

6.2 The largest summand

First we consider $F^{(h)}(z)$, the generating function of those CCWZA with the largest summand $\leq h$, where the coefficient of z^n refers to size n.

Let $f_k^{(h)}(z,u)$ be the generating function of those CCWZA with k summands and all summands $\leq h$, where the coefficient of $z^n u^j$ refers to size n and the last summand $x_k = j$. We dissect the set of compositions into those with the last summand ≥ 1 (counted by $g_k^{(h)}(z,u)$) and those with the last summand =0 (counted by $l_k^{(h)}(z)$). Clearly, $l_k^{(h)}(z) = g_{k-1}^{(h)}(z,1) \quad (k \geq 2)$.

Then essentially the same idea as in (6.1) works, except that we only use a factor

$(zu) + ... + (zu)^h = zu(1-(zu)^h)/(1-zu)$ instead of the full geometric series.

Hence $g_{k+1}^{(h)}(z,u) = g_k^{(h)}(z,1)zu(1-(zu)^h)/(1-zu)+l_k^{(h)}(z)zu(1-(zu)^h)/(1-zu) - g_k^{(h)}(z,zu)$

for $k \geq 1$, $g_1^{(h)}(z,u) = zu(1-(zu)^h)/(1-zu)$ and $l_1^{(h)}(z) = 1$.

Setting $G^{(h)}(z,u){:=}\sum_{k\geq 1} g_k^{(h)}(z,u)$, we get

$G^{(h)}(z,1){=}2\sigma^{(h)}(z)/(1-2\sigma^{(h)}(z))$,

where $\sigma^{(h)}(z) = \sum_{i\geq 1}(-1)^{i-1}(z^i - z^{i(h+1)})/(1-z^i)$.

Therefore $F^{(h)}(z) = 1 + 2G^{(h)}(z,1)$

$$=(1+2\sigma^{(h)}(z))/(1-2\sigma^{(h)}(z)).$$

The dominant pole τ_h is now the real solution of the equation

$$\sigma^{(h)}(z)=1/2.$$

It is clear that τ_h tends to τ, but we have to determine how fast. We will use the "bootstrapping method" from [Kn1978].

Now, around $z=\tau$ we have approximate equation

$$1/2\approx \sigma(\tau_h)-\tau^{h+1}/(1-\tau).$$

Let $\tau_h=\tau(1+\epsilon_h)$, and use Taylor's Theorem, then we get

$$0\approx \tau\epsilon_h\sigma'(\tau)-\tau^{h+1}/(1-\tau) \qquad \text{or}$$

$$\epsilon_h\approx \tau^h/((1-\tau)\sigma'(\tau)).$$

Therefore the number of CCWZA with the largest summand $\leq h$ is approximated by

$$[(1+2\sigma(\tau_h))/(2\tau_h\sigma'(\tau_h))]\cdot\tau_h^{-n}\approx [(1+2\sigma(\tau))/(2\tau\sigma'(\tau))]\tau^{-n}(1+\epsilon_h)^{-n}.$$

Thus the probability that the largest summand is $\leq h$ is approximated by

$$[(1+2\sigma(\tau))/(2\tau\sigma'(\tau))]\tau^{-n}(1+\epsilon_h)^{-n}/(A\tau^{-n})$$

$$=(1+\epsilon_h)^{-n}\approx(1-\tau^h/((1-\tau)\sigma'(\tau)))^n.$$

For the probability that the largest summand is $>h$, we have approximately $1-(1-\tau^h/((1-\tau)\sigma'(\tau)))^n$.

Summing this up over $h \geq 0$, we get the desired average value E_n, the average number of the largest summand in a random CCWZA of size n.

The next step is to use the exponential approximation $(1-\alpha)^n \approx e^{-\alpha n}$. Then $E_n \approx \sum_{h \geq 0}(1 - e^{-n\tau^h/((1-\tau)\sigma'(\tau))})$.

This quantity is quite well studied in [FlGoDu1995]. Set N:=$n/((1-\tau)\sigma'(\tau))$. Therefore

$E_n \sim log_{1/\tau}N - \gamma/ln\tau + 1/2 + \delta(log_{1/\tau}N)$, with a certain periodic function $\delta(x)$ that has period 1, mean 0, and small amplitude.

Rewriting this we get

$E_n \sim log_{1/\tau}n - log_{1/\tau}\sigma'(\tau) - log_{1/\tau}(1-\tau) - \gamma/ln\tau + 1/2 + \overline{\delta}(log_{1/\tau}n)$, where $\overline{\delta}(x) = \delta(x - log_{1/\tau}\sigma'(\tau) - log_{1/\tau}(1-\tau))$, which has the same property as $\delta(x)$.

The numerical constant is

$-\log_{1/\tau}\sigma'(\tau) - \log_{1/\tau}(1-\tau) - \gamma/\ln\tau + 1/2 = 0.929718.$

6.3 Carlitz words

In this section we discuss words (strings) over an alphabet with symbols $\beta_1, \beta_2, ..., \beta_m$ $(m \geq 2)$.

The number of words of length n over an alphabet with m symbols is found to be m^n. Each word may be decomposed into a succession of blocks each formed with a single symbol maximally (as long as possible). Each block is called a run. For each block, the block length is called run length.

If a word has run lengths $l_1, l_{2,}..., l_k$ sequencely, then we call it a $(l_1, l_{2,}..., l_k)-$ *word*. For example, $m = 3$; $\beta_2\beta_2\beta_2\beta_2\beta_1\beta_3\beta_1\beta_1\beta_2\beta_3\beta_3\beta_1$ is a word of length 12. $\beta_2\beta_2\beta_2\beta_2$, β_1, β_3, $\beta_1\beta_1$, β_2, $\beta_3\beta_3$, β_1 are its runs. Their lengths are 4, 1, 1, 2, 1, 2, 1 respectively. It is a (4, 1, 1, 2, 1, 2, 1)-words.

For a $(l_1, l_{2,}..., l_k)$-word, if $l_i \neq l_{i+1}$ $(i = 1, ..., k-1)$, then we call it a *Carlitz word.*

Let $c_m(n)$ be the number of such Carlitz words of length n over an alphabet with symbols $\beta_1, \beta_2, ..., \beta_m$. For convenience, we define $c_m(0) = 1$.

From the definition, we see that $c_2(n) = 2c(n)$, i.e., $c(n) = \frac{1}{2}c_2(n)$ for $n \geq 1$ where $c(n)$ is the number of Carlitz compositions defined in section 1.3.

$c_m(n)$

n\m	2	3	4
0	1	1	1
1	2	3	4
2	2	3	4
3	6	15	28
4	8	27	64
5	14	51	124
6	28	159	520
7	46	339	1372
8	78	699	3100

Fig.4

In this section, we deduce the generating function

$$C_m(z) := \sum_{n\geq 0} c_m(n) z^n$$

$$= \tfrac{m}{m-1} \cdot 1/(1 - \sum_{i=1}^{\infty}(-1)^{i-1}(m-1)^i z^i/(1-z^i)) - \tfrac{1}{m-1} \qquad (6.2)$$

$$= \tfrac{m}{m-1} \cdot 1/(1 - \sum_{i=1}^{\infty}(m-1)z^i/(1+(m-1)z^i)) - \tfrac{1}{m-1} \qquad (6.2')$$

and the recurrence $c_m(n) = \sum_{j=1}^{n-1} P_m(j)\cdot c_m(n-j) + P_m(n)\cdot \frac{m}{m-1}$ $(n \geq 1)$,

where $c_m(0) = 1$ and $P_m(n) = \sum_{i|n}(-1)^{i-1}(m-1)^i$. (6.3)

Now we deduce the generating function (6.2).

Let $f_{k,m}(z,u)$ be the generating function of those Carlitz words with k runs where the coefficient of $z^n u^j$ refers to size n and the last run length $x_k = j$.

Then $f_{k+1,m}(z,u) = f_{k,m}(z,1)\frac{(m-1)zu}{1-zu} - (m-1)f_{k,m}(z,zu)$

for $k \geq 1$ and $f_{1,m}(z,u) = m \cdot \frac{zu}{1-zu}$. (6.4)

Define $F_m(z,u) := \sum_{k\geq 1} f_{k,m}(z,u)$.

From (6.4), we get

$$F_m(z,u) = F_m(z,1) \cdot \frac{(m-1)zu}{1-zu} + m \cdot \frac{zu}{1-zu} - (m-1)F_m(z,zu).$$

Iterate it, and consequently we have

$$F_m(z,1) = \frac{m}{m-1}\sigma_m(z) + F_m(z,1)\sigma_m(z),$$

where $\sigma_m(z) = \sum_{i\geq 1}(-1)^{i-1}(m-1)^i z^i/(1-z^i)$.

Hence $C_m(z) = 1 + F_m(z,1)$

$$= \frac{m}{m-1} \cdot 1/(1-\sigma_m(z)) - \frac{1}{m-1} \quad .$$

This finishes the proof of (6.2).

Since $\sum_{i\geq 1}(-1)^{i-1}(m-1)^i z^i/(1-z^i)$

$= \sum_{i=1}^{\infty}(-1)^{i-1}(m-1)^i \sum_{j=1}^{\infty} z^{ij}$

$= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{i-1}(m-1)^i z^{ij}$

$= \sum_{j=1}^{\infty}(m-1)z^j/(1+(m-1)z^j)$, we get (6.2′) from (6.2).

In the following, we derive (6.3).

$$\sum_{i=1}^{\infty}(-1)^{i-1}(m-1)^i z^i/(1-z^i) = \sum_{i=1}^{\infty}(-1)^{i-1}(m-1)^i \sum_{j=1}^{\infty} z^{ij}$$

$$= \sum_{n=1}^{\infty} z^n \sum_{i|n}(-1)^{i-1}(m-1)^i.$$

Define $P_m(n) = \sum_{i|n}(-1)^{i-1}(m-1)^i$.

Then $1+\sum_{n\geq 1} c_m(n)z^n = \frac{m}{m-1} \cdot 1/(1-\sum_{n\geq 1} P_m(n)z^n) - \frac{1}{m-1}$.

Hence $1+\frac{m-1}{m}\sum_{n\geq 1} c_m(n)z^n = 1/(1-\sum_{n\geq 1} P_m(n)z^n)$.

It follows that $c_m(n)$ satisfies the recurrence

$$c_m(n) = \sum_{j=1}^{n-1} P_m(j)\cdot c_m(n-j) + P_m(n)\cdot \frac{m}{m-1} \; (n \geq 1), \text{ where } c_m(0) = 1$$

and $P_m(n) = \sum_{i|n}(-1)^{i-1}(m-1)^i$.

Finally, we state some asymptotic results for $c_m(n)$, $2 \leq m \leq 10$.

m	ρ_m	A_m
2	0.571349	0.912774
3	0.434461	0.920266
4	0.360573	1.012713
5	0.312720	1.061314
6	0.278582	1.105405
7	0.252707	1.145457
8	0.232258	1.182073
9	0.215596	1.215771
10	0.201697	1.246993

Fig.5

Here ρ_m is the dominant pole of the generating function $C_m(z)$. Therefore $c_m(n) \sim A_m \rho_m^{-n}$, where $A_m = \frac{m}{m-1}/(\rho_m \sigma'_m(\rho_m))$.

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