Identities for the gamma and hypergeometric functions: an overview from Euler to the present

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other university.

Signed:

Julie Hannah

_____________________day of _______________________ , 2013 in Johannesburg

Abstract

Equations involving the gamma and hypergeometric functions are of great interest to mathematicians and scientists, and newly proven identities for these functions assist in finding solutions to differential and integral equations.

In this work we trace a brief history of the development of the gamma and hypergeometric functions, illustrate the close relationship between them and present a range of their most useful properties and identities, from the earliest ones to those developed in more recent years. Our literature review will show that while continued research into hypergeometric identities has generated many new results, some of these can be shown to be variations of known identities. Hence, we will also discuss computer based methods that have been developed for creating and analysing such identities, in order to check for originality and for numerical validity.

In reverence and gratitude to the

One Source from Whom

all proceeds.

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Contents

List of Figures

Introduction

Functions which are important enough to be given their own name are known as 'special functions'. These include the well known logarithmic, exponential and trigonometric functions, and extend to cover the gamma, beta and zeta functions, spherical and parabolic cylinder functions, and the class of orthogonal polynomials, among many others. The vast field of these functions contains many formulae and identities used by mathematicians, engineers and physicists. Special functions have extensive applications in pure mathematics, as well as in applied areas such as acoustics, electrical current, fluid dynamics, heat conduction, solutions of wave equations, moments of inertia and quantum mechanics (cf. [11], [14], [60], [78], [84]).

At the heart of the theory of special functions lies the hypergeometric function, in that all of the classical special functions can be expressed in terms of this powerful function. Hypergeometric functions have explicit series and integral representations, and thus provide ideal tools for establishing useful summation and transformation formulae. In addition, applied problems frequently require solutions of a function in terms of parameters, rather than merely in terms of a variable, and such a solution is perfectly provided for by the parametric nature of the hypergeometric function. As a result, the hypergeometric function can be used to solve physical problems in diverse areas of applied mathematics.

In the arena of pure mathematics, Graham et al. [45] and Koepf [52] point out that hypergeometric identities provide a unifying principle for handling a variety of binomial coefficient summations, thus playing a particularly useful role in combinatorics. Hypergeometric functions have also been shown to have applications in group theory, algebraic geometry, algebraic K-theory, and conformal field theory. The extended q-hypergeometric series are related to elliptic and theta functions, and are thus useful in partition theory, difference equations and Lie algebras (cf. [52], [4]).

Equations involving hypergeometric functions are of great interest to mathematicians and scientists, and newly proven identities for these functions assist in finding solutions for many differential and integral equations. There exist a vast number of such identities, representations and transformations for the hypergeometric function, the comprehensive text by Prudnikov et al. [74] providing over 400 integral and series representations for these functions. Hypergeometric functions thus provide a rich field for ongoing research, which continues to produce new results.

If the hypergeometric function is at the heart of special function theory, the gamma function is central to the theory of hypergeometric functions. Davis goes so far as to state that "Of the so-called 'higher mathematical functions', the gamma function is undoubtedly the most fundamental" (cf. [19], p.850). The rising factorial provides a direct link between the gamma and hypergeometric functions, and most hypergeometric identities can be more elegantly expressed in terms of the gamma function. In the words of Andrews et al., "the gamma function and beta integrals … are essential to understanding hypergeometric functions" (cf. [4], p.xiv). It is thus enlightening and rewarding to explore the various representations and relations of the gamma function.

In this work we aim to trace a brief history of the development of the gamma and hypergeometric functions, to illustrate the close relationship between them, and to present a range of their most useful properties and identities from the earliest ones to those developed in more recent years.

In Chapter 1 we briefly present some preliminary concepts and theorems drawn from other areas of mathematics, which will provide the necessary foundation for establishing results involving the gamma and hypergeometric functions. The gamma function has a long and interesting history and is used in the sciences almost as often as the factorial function. Chapter 2 describes how this function had its humble birth in Euler's extrapolation of the factorial of natural numbers, and traces its further development. We provide some of its most widely used representations and identities, particularly those which relate to the hypergeometric function, and also discuss a few interesting applications.

In Chapter 3 we define the Gauss hypergeometric function and present certain classical results dating from the 19th and early 20th centuries, which draw heavily on gamma function relations. In Chapter 4 we extend this definition to the generalised hypergeometric function: here we present many core identities and transformations for this generalisation, and provide some insights into its applications. As there exist vast numbers of such identities, we have chosen to focus mainly (although not exclusively) on the generalised series ${}_{p+1}F_p$, which has one more parameter in the numerator than in the denominator.

Having provided these classical results for the hypergeometric function, in Chapter 5 we present a literature review of more recent findings. This review will illustrate the extent to which new hypergeometric identities continue to be proposed. In the face of such proliferation, research since the 1970s has tended towards a search for systematic computer algorithms to generate, compare and test such identities. We present some of these algorithmic methods in Chapter 6, together with a recent computer based analysis by Michael Milgram, which provides a valuable insight into the powerful role which such approaches can play in the field of hypergeometric identities.

Chapter 1

Preliminary results

1.1 Introduction

In order to enhance the flow of this work, we provide here results required for later chapters, including useful definitions and theorems related to infinite products and series, the rising factorial, log convexity and some relevant results from complex analysis. In the interests of conciseness, proofs are omitted but can be found in the textbooks referred to.

1.2 Working with products and series

We first provide some central results for infinite products and series. When working with hypergeometric series, we are often required to interchange the order of summation and integration. To justify such procedures, we draw on the following theorem of integration (cf. [89], Th 6.10).

Theorem 1.2.1 *Tonelli's Theorem*: *If* $f(x,y) \geq 0$ over the domain $E \times F = \{(x,y) \in \mathbb{R}^{m+n} : x \in E, y \in F\}$, *then*

$$\int_E \int_F f(x,y)dydx = \int_F \int_E f(x,y)dxdy.$$

Absolutely convergent series can be considered to be special cases of Lebesgue integrals over the measure space $(0, 1, 2, \dots)$. Hence, the interchange of summation and integration can be applied to functions which are Lebesgue integrable, including those not necessarily non-negative. Analogues for Tonelli's theorem are given in the corollaries below.

Corollary 1.2.2 *Let $E \subseteq R^d$ be given, and suppose that $f_n : E \to [0, \infty)$ for $n \in \mathbb{N}$. Then, for integrals and series which converge absolutely,*

$$\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_E f_n \,.$$

Corollary 1.2.3 *If $\sum_{n=1}^{\infty} A(k, n)$ and $\sum_{k=1}^{\infty} A(k, n)$ are both absolutely convergent for every $n, k \in \mathbb{N}$, then the iterated series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A(k, n)$ and $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} A(k, n)$ are equal and absolutely convergent.*

The following theorem provides useful techniques for manipulating iterated series (cf. [75], p.56).

Theorem 1.2.4

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \tag{1.1}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k) \tag{1.2}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n-k) \tag{1.3}$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(k, m, n) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{k-m} A(k-m-n, m, n) \tag{1.4}$$

We now consider the convergence of an infinite product (cf. [75], p.2).

Theorem 1.2.5 *If* $\prod_{k=1}^{\infty}(1+a_k)$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Definition 1.2.6 *Consider the product* $P_n = \prod_{k=1}^{n}(1+a_k)$.

i) *If* $\lim_{n\to\infty} P_n$ *exists and is equal to* $P \neq 0$*, we say that the infinite product converges to P.*

ii) *If at least one factor of* P_n *is zero, only a finite number of factors are zero, and the product with zero factors deleted converges to* $P \neq 0$*, then the product converges to zero.*

iii) *If the product does not converge because* $lim_{n\to\infty} P_n$ *does not exist, the product is said to be divergent.*

iv) *If* $\lim_{n\to\infty} P_n$ *is zero, the product is said to diverge to zero.*

The important role that zero plays in the above conditions is the reason for the difference in the definition of convergence for an infinite product compared to an infinite series. The following three theorems provide ways to establish uniform and absolute convergence of an infinite product (cf. [1], pp.159-160, [70], p.223).

Theorem 1.2.7 *The Weierstrass M-test for convergence*: *If there exist positive constants* M_n *so that* $\sum_{n=1}^{\infty} M_n$ *is convergent and* $|a_n(x)| < M_n$ *for all x in the closed region R, then the product* $\prod_{n=1}^{\infty}(1+a_n)$ *is uniformly convergent.*

Theorem 1.2.8 *In order that the infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *be absolutely convergent, it is a necessary and sufficient condition that the series* $\sum_{n=1}^{\infty} a_n$ *be absolutely convergent.*

Infinite products without zero factors are often simplified by using the principal values of the logarithms of the factors. This allows for conversion to a more convenient infinite series (cf. [75], p.3).

Theorem 1.2.9 *If no* $a_n = -1$, *then* $\prod_{n=1}^{\infty}(1+a_n)$ *and* $\sum_{n=1}^{\infty}\log(1+a_n)$ *diverge or converge together, when the logarithms have their principal values.*

1.3 The rising factorial

Many results involving special functions can be expressed more concisely through the use of the rising (shifted) factorial. This is denoted by Pochhammer's symbol, as defined by the German mathematician Leo Pochhammer (1841-1920), [4], p.2.

Definition 1.3.1 *Pochhammer's symbol for the rising factorial is* $(a)_n$, *where* a *is any complex number and*

$$(a)_n = \begin{cases} a(a+1)(a+2)\dots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0,\ a \neq 0 \end{cases}. \tag{1.5}$$

It follows that $(1)_n = n!$, and it is a simple matter to derive the following expression for the rising factorial of a negative integer.

Theorem 1.3.2 For $k, n \in \mathbb{N}$,

$$(-n)_k = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!}, & 1 \leq k \leq n \\ 0, & k \geq n+1 \end{cases}. \tag{1.6}$$

This result can also be written in the equivalent form $\binom{n}{k} = \frac{(-1)^k(-n)_k}{k!}$.

Manipulation of factorials easily establishes the following useful properties. Proofs can be found in [14], [75], [78] and [84].

$$(a)_n = \frac{(a+n-1)!}{(a-1)!} = \frac{a!}{(a-n)!} = \binom{a}{n} n!, n \in \mathbb{N} \tag{1.7}$$

$$a(a+1)_n = (a)_{n+1} = (a+n)(a)_n \tag{1.8}$$

$$(a)_k(a+k)_n = (a)_{n+k} \tag{1.9}$$

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n, a \neq 0, n \in \mathbb{N} \tag{1.10}$$

$$(a)_{kn} = k^{kn}\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n \cdots \left(\frac{a+k-1}{k}\right)_n, a \neq 0, n \in \mathbb{N} \tag{1.11}$$

$$(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}, \qquad 0 \leq k \leq n \tag{1.12}$$

$$(a)_{-n} = (-1)^n/(1-a)_n, n \in \mathbb{Z} \tag{1.13}$$

$$(-a)_n = (-1)^n (a-n+1)_n, n \in \mathbb{N}, 0 \leq n \leq a \tag{1.14}$$

$$(a+n)_{k-n} . (a+k)_{n-k} = 1 \tag{1.15}$$

$$(2n)! = 2^{2n} n! \left(\frac{1}{2}\right)_n \tag{1.16}$$

$$\frac{\left(\frac{a}{2}+1\right)_n}{\left(\frac{a}{2}\right)_n} = \frac{a+2n}{a} \tag{1.17}$$

$$(1+y)^{-a} = \sum_{n=0}^{\infty} \binom{-a}{n} y^n = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n y^n}{n!} \tag{1.18}$$

Alexandre-Théophile Vandermonde (1735-1796) derived the following relation also known as Vandermonde's convolution theorem (cf. [14], p.12]).

Theorem 1.3.3 *Vandermonde's Identity*: *For* $n \in \mathbb{N}$, *a, b* $\in \mathbb{C}$,

$$(a+b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k} \,. \tag{1.19}$$

Proof: We use (1.18) to obtain

$$(1-x)^{-a-b} = \sum_{n=0}^{\infty} \frac{(a+b)_n x^n}{n!}, \tag{1.20}$$

and together with the summation formula (1.1), to obtain

$$(1-x)^{-a}(1-x)^{-b} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(b)_{n-k}x^n}{(n-k)!}\frac{(a)_k}{k!} \,. \tag{1.21}$$

The theorem[1] follows by equating the coefficients of x^n in (1.20) and (1.21). ■

1.4 Log-convexity of a function

The concept of log-convexity plays a central role in establishing the uniqueness of the gamma function. In this section we define this notion and present some important related results found in [4], [6] and [14]. The definitions below are given by Andrews et al. [4], p.34. (Throughout this work we use the notation $\log x$ to represent the natural logarithmic function $\ln x$.)

Definition 1.4.1 *A function $f\colon(a,b)\to\mathbb{R}$ is convex if for any $x,y\in(a,b)$, we have $f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$ for all $\lambda\in(0,1)$.*

By the parametrisation $\{\lambda x+(1-\lambda)y : 0<\lambda<1\}$, we see that informally a function is convex over an interval (x, y) if the line joining $(x,f(x))$ to $(y,f(y))$ lies above the graph of f.[2] It is intuitively clear that for a convex function, the difference quotient of two points on the graph increases with increasing values of the domain. This is summarised in the alternative definition given below.

Definition 1.4.2 *A function $f\colon(a,b)\to\mathbb{R}$ is convex if the function $\varphi(x_1,x_2)=\frac{f(x_1)-f(x_2)}{x_1-x_2}=\varphi(x_2,x_1)$ is monotonically increasing. That is, for $a<s<t<u<b$,*

$$\frac{f(t)-f(s)}{t-s}\le\frac{f(u)-f(s)}{u-s}\le\frac{f(u)-f(t)}{u-t}\,. \tag{1.22}$$

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[1] In 1303, Chu Shih-Chieh (Ssu Yü Chien) wrote *Precious Mirror of the Four Elements* in which he stated the combinatorial result $\sum_{i=1}^{k}\binom{a}{i}\binom{b}{k-i}=\binom{a+b}{k}, a,b,k,\in\mathbb{N}$. This is equivalent to (1.19) and was also re-stated in hypergeometric terms in 1772 by Vandermonde, given as the Chu-Vandermonde equation (3.15) in Chapter 3.

[2] As a particular example, the inequality $f\left(\frac{x+y}{2}\right)\le\frac{1}{2}(f(x)+f(y))$ holds for a convex function f. Some texts use the term 'concave up' for convexity.

We also have the following useful corollary (cf. [6], p.4).

Corollary 1.4.3 *If a function* $f: (a, b) \to \mathbb{R}$ *is twice differentiable, then it is convex if and only if* $f''(x) > 0$ *for* $a < x < b$.

The logarithmic function of a convex function might not be convex, as with the function $f(x) = x^2$. If $\log f(x)$ is convex, we say that f is a log-convex function. Log-convexity is a stronger condition than function convexity, as the convexity of the logarithm of a function guarantees the convexity of the function itself over the specified interval.

To complete this section, we state Holder's inequality which will be required for establishing the log-convexity of the gamma function in Chapter 2 (cf. [3], p.11).

Theorem 1.4.4 *Holder's inequality*: *If p and q are positive numbers so that* $\frac{1}{p} + \frac{1}{q} = 1$, *then for any integrable functions* $f, g: (a, b) \to \mathbb{R}$, *the following inequality holds:*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f|^p dx\right)^{1/p} \left(\int_a^b |g|^q dx\right)^{1/q}. \tag{1.23}$$

1.5 Some complex results

In Chapter 2 the domain of definition of the gamma function will be extended to include complex arguments, through the process of analytic continuation. For this reason, we state here some required techniques of complex analysis.

1.5.1 Contour integration

We recall that a function $f(z)$ is said to be analytic at z_0 if it has a Taylor expansion $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ which converges to $f(z)$ in some interval about z_0. For a given contour in a specified domain, the following remarkable result enables us to express the value of an analytic function at any point *a* within the contour, in terms of an integral which depends only on the value of the function at points on the contour itself (cf. [93], p.89).

Theorem 1.5.1 *If a function $f(z)$ is analytic at all points on or inside a contour C, and a is any point within the contour, then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}.$$

Such a contour integral can be evaluated by using the residue of a function, as defined in [77], p.308.

Definition 1.5.2 *If a function $f(z)$ has a pole of order m at z = a, then for values of z near a, we can write $f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + \phi(z)$, where $\phi(z)$ is analytic near and at x = a. The coefficient a_{-1} is the residue of the function $f(z)$ relative to the pole a.*

The residue of a function can also be evaluated by the following theorem (cf. [77], p.309, Eq. 3).

Theorem 1.5.3 *If a is a simple pole of $f(z)$, the residue of the function at that pole is given by $Res(f, a) = \lim_{z\to a}\{(z-a).f(z)\}$.*

Cauchy's Residue theorem below states how we can use residues to evaluate complex contour integrals (cf. [50], p.193, Th. A.6) and [37], p.154.

Theorem 1.5.4 *If $f(z)$ is analytic throughout a contour C and its interior, except at a finite number of poles inside the contour, then $\int_C f(z)dz = 2\pi i \sum R$, where $\sum R$ denotes the sum of the residues of the function at the poles within the contour C.*

1.5.2 Multi-valued functions and branches

When working with the complex gamma function, we will be required to consider certain restrictions in the domain. Functions which are single-valued in the set of real numbers might be multi-valued for complex arguments. For example, $f(x) = (1+x)^{1/3}$ has a single value for $x = 0, f(x) \in \mathbb{R}$, but in the complex plane we have the infinite solutions $f(x) = e^{2\pi[1+n]i/3}, n \in \mathbb{Z}$. Similarly, while the logarithmic function is single-valued in the set of reals, the complex logarithmic function is defined by $\log x = \ln|x| + i\arg x, x \neq 0$, thus representing a set of complex numbers in which each pair differs by an integral multiple of 2π. As various multi-valued functions occur frequently in definitions of special mathematical functions, such as the gamma and hypergeometric functions, we allow only single-valued outcomes by restricting them to certain branches. Norton [70], p.23 provides a succinct explanation of this phenomena, as summarised below.

In general, if $f(z)$ is a multi-valued function with one form being $f_1(z)$, then the function $f_1(z)$ will be single-valued when z is restricted to a sufficiently small region of the complex plane. Let z_1 be a point in the restricted region, and let it trace a simple closed curve starting and returning to z_1. If the closed curve lies completely inside the restricted region, then the difference between the complex values of $f_1(z_1)$ at the beginning and the end of the circuit will be zero. If it is not zero, we imagine continuously contracting the curve so that at some point the difference will change discontinuously. If this happens when the contracting curve crosses a point z_0, then we define that to be a branch point of $f(z)$. (For this reason, functions which contain fractional powers and logarithms have a branch point at $z = 0$.) In order to avoid branch points, we draw a branch cut in the complex plane from each branch point to infinity in any direction, so that in this cut plane $f(z)$ is a single-valued branch of the original multi-valued function. When specifying domains, we thus use branch cuts to create single-valued complex functions.

1.5.3 Analytic continuation

Davis [19] points out that analytic functions "exhibit the remarkable phenomenon of 'action at a distance'. This means that the behaviour of an analytic function over an interval, no matter how small, is sufficient to determine completely its behaviour everywhere else." Furthermore, if the function satisfies a certain functional relationship in one part of its domain, then this relationship can be used to extend its definition to a larger domain. This process is known as analytic continuation, and it is used to extend the domain of definition of the gamma and hypergeometric functions.

If a function g is analytic in a larger domain than the domain D of an analytic function f, and yet its values agree with those of f within D, we say that g is an analytic continuation of f, as defined below (cf. [77], p.294).

Definition 1.5.5 *Suppose that f is analytic in a domain D_1, and that g is analytic in a domain D_2. Then g is said to be the direct analytic continuation of f to D_2 if $D_1 \cap D_2$ is nonempty and $f(z) = g(z)$ for all z in $D_1 \cap D_2$.*

Another way to view the process of analytic continuation is to consider whether "given a power series which converges and represents a function only at points within a circle, … [we can] define by means of it the values of the function at points outside the circle" (cf. [93], p.97). To do this, we first construct a Taylor series for the given function $f(z)$ for points inside the circle of convergence C_1 centred at z_1. We then choose a point z_2 inside C_1 and construct a new Taylor series to represent $f(z)$ inside a circle C_2 centred at z_2. This new circle of convergence will extend as far as the nearest singularity to z_2, and will usually lie partly outside the original circle of convergence (cf. [78], p.124, Fig. 7.12). For points which lie within the new circle but outside the original circle, the new series may be used to define the values of the function not defined by the original series. By repeating this process, we can find many power series which between them define the value of the function at all points of a domain which can be reached without passing through a singularity of the function. The collection of these power series then forms the analytic expression of the function.

For complex functions, two different analytic expressions are said to define the same function if they represent power functions which are derivable from each other by analytic continuation (cf. [93], p.99).

Havil [46] provides us with the following powerful result.

Theorem 1.5.6 *If, in some complex domain D, two analytic functions are defined and are equal at all points within a curve C lying inside D, they are equal throughout D.*

Havil comments further on "the enormity of what is being said" in the above theorem: if two analytic functions are defined on the whole of $\mathbb{C}$ and coincide over an interval, say (-1, 1) on the real axis, then they will be *equal everywhere else.*[3] For example, given the complex function $f(z) = \sum_{n=0}^{\infty} z^n$ defined within the unit circle centred at the origin, the function $v(z) = \frac{1}{1-z}$ which is analytic everywhere in the complex plane except $Re(z) = 1$ is regarded to be *the extension* of $f(z)$.

To complete this chapter, we use complex number theory to establish the result below (cf. [75], p.26, Lemma 8). This will be used in Chapter 3 to prove the multiplication theorem for the gamma function.

Theorem 1.5.7 *For* $k \geq 2$,

$$\prod_{s=1}^{k-1} \sin\frac{\pi s}{k} = \frac{k}{2^{k-1}}.$$

Proof: If $\alpha = \exp(2\pi i/k)$ is the primitive k-th root of unity, then

[3] Havel [46], p.191 makes the striking observation that "analytic continuation is initially unbelievable"!

$$x^k - 1 = (x-1)\prod_{s=1}^{k-1}(x - \alpha^s).$$

By differentiation, this result becomes $k = \prod_{s=1}^{k-1}(1-\alpha^s)$ when $x = 1$. It also follows that

$$\begin{aligned}1 - \alpha^s &= 1 - \exp\left(\frac{2\pi i s}{k}\right), \\ &= -\exp\left(\frac{\pi i s}{k}\right)\left[\exp\left(\frac{\pi i s}{k}\right) - \exp\left(\frac{-\pi i s}{k}\right)\right] \\ &= -2i\exp\left(\frac{\pi i s}{k}\right)\sin\frac{\pi s}{k}.\end{aligned}$$

Hence, we have

$$\begin{aligned}k &= \prod_{s=1}^{k-1}\left(-2i\exp\left(\frac{\pi i s}{k}\right)\sin\frac{\pi s}{k}\right) \\ &= (-2i)^{k-1}\left[\exp\left(\frac{\pi i}{k} + \frac{2\pi i}{k} + \cdots + \frac{(k-1)\pi i}{k}\right)\right]\prod_{s=1}^{k-1}\left(\sin\frac{\pi s}{k}\right) \\ &= 2^{k-1}.\exp\left(\frac{-\pi i(k-1)}{2}\right)\left[\exp\left(\pi i.\frac{k-1}{2}\right)\right]\prod_{s=1}^{k-1}\left(\sin\frac{\pi s}{k}\right) \\ &= 2^{k-1}\prod_{s=1}^{k-1}\left(\sin\frac{\pi s}{k}\right),\end{aligned}$$

and the desired result follows directly. ■

Having established some required preliminary results from various areas of mathematics, we now turn our attention to the gamma function. This function arises in many applications of special functions, and plays a central role in proving various identities and transformations for the hypergeometric function.

Chapter 2

The gamma function

2.1 Introduction

The gamma function has its roots in attempts to extend the factorial function to non-natural arguments. From that initial impetus we can trace a long and fascinating history of the development of the gamma and related functions, with contributions from a wide range of mathematicians. The function is thus a spectacular example of how a rich tapestry of mathematical concepts is developed through collaboration. As a result of this broad input, the gamma function has an impressive number of different representations, including series, limit and integral forms, each offering their own particular advantage in different applications.

Apart from its central role in pure mathematics, the gamma function also arises in applied fields as varied as fluid dynamics, astrophysics, quantum physics and statistics. Its vast array of applications include modeling the time between earthquakes, evaluating infinite products and integrals of expressions of the form $f(t)e^{-gt}$ (which describe processes of exponential decay), and evaluating arc lengths, areas and volumes (cf. [11], [33], [78], [84]). Davis [19], p.863 lists further applications of the complex gamma function in problems involving radial wave functions for energy states in a Coulomb field, formulae for the scattering of charged particles, for nuclear forces between protons, and the probability of β-radiation.

Apart from having a wider range of powerful applications in its own right, this function also provides a convenient representation for many special functions, and plays a particularly central role in the statement and proof of numerous hypergeometric identities.

In the next section of this chapter we provide a brief historical overview of the development of the gamma function. In Section 2.3 we then discuss its most familiar representations, together with some proofs of their equivalence, while Section 2.4 presents some gamma function identities which are of central importance to later work with the hypergeometric function. In Section 2.5 we present some useful functions closely related to the gamma function, concentrating particularly on the beta function which arises frequently in hypergeometric identities and their proofs. Section 2.6 extends the domain of the gamma function to include complex values, and finally in Section 2.7 we briefly mention a few interesting applications of this intriguing and ubiquitous function.

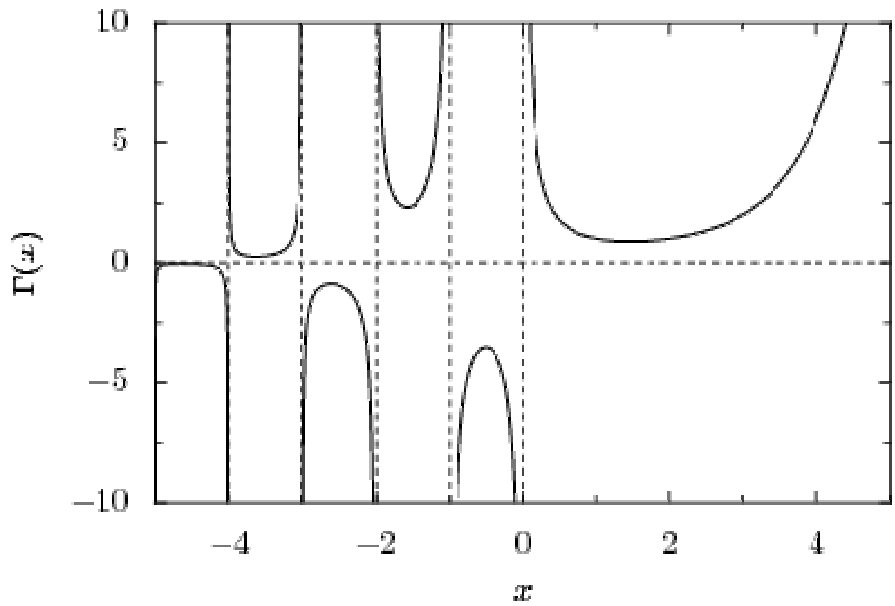

Figure 2.1 The graph of the gamma function in the real plane

2.2 A brief history of the gamma function

As Davis [19] asserts: "It is difficult to chronicle the exact course of scientific discovery". However, in this section we hope to provide a reasonably accurate overview of how the various representations of the gamma function developed over time.

In the 18th century, the problem of interpolation had become a popular issue. The problem is stated generally as follows: *Given a sequence* P_k *defined for natural values of k, find the meaning of* P_α *where* α *is a non-natural number.*

Christian Goldbach (1690-1764) considered particularly the interpolation of the sequence of factorial numbers 1, 2, 6, 24, 120, 720, … , seeking a function that would give meaning to an expression such as the factorial of $2\frac{1}{2}$. He enlisted the assistance of several mathematicians, including Daniel Bernoulli (1700-1782) and Leonhard Euler (1707-1783). In a letter to Goldbach dated October 6, 1729, Daniel Bernouli proposed the expression

$$\left(A+\frac{x}{2}\right)^{x-1}\left(\frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\cdots\frac{A}{A-1+x}\right)$$

as an interpolating product for the factorial of any real number x, where the accuracy of the result increases with larger positive integral values of *A*. For example, the values $x=3,\ A=5$ yield the approximation 6.04 for the factorial of 3.

At the same time, Euler was independently developing his own product representation and, on the encouragement of Bernoulli, he wrote a letter to Goldbach dated October 13, 1729 (cf. [39], pp.1-18) in which he represented the *m*-th term of the sequence 1, 2, 6, 24, 120, … by the product

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}\cdot 3^m}{2+m}\cdot\frac{3^{1-m}\cdot 4^m}{3+m}\cdot\frac{4^{1-m}\cdot 5^m}{4+m}\cdot \dots.$$

Euler stated that his product should be determined close to infinity, so that increasing values of n would provide results closer to $m!$.[4] While Euler's and Bernoulli's products are formally different, they both yield the same value in the limit, although Bernoulli's formula converges faster.

Euler's discussion of his factorial interpolation was brief in his 1729 letter, but he provided more details in his article [27] written later that year and published in 1730, entitled '*On transcendental progressions, or those for which the general term is not given algebraically*'.[5] In this article Euler claimed that his product was "wonderfully suitable for interpolating terms whose indices are fractional numbers", but that he intended to present more convenient methods which would provide exact values.

These alternative methods would lead to his second interpolation, given in terms of a definite integral. The impetus for this new direction seems to have been provided by the factorial value of ½. In his article, Euler substituted m = ½ into his product $\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}.3^m}{2+m}\cdot\frac{3^{1-m}.4^m}{3+m}\cdot\frac{4^{1-m}.5^m}{4+m}\cdot \ldots$, to obtain $\sqrt{\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot \ldots}$, and expression which was directly related to an earlier result of John Wallis (1616-1703). In his '*Arithmetica infinitorum'* of 1656, Wallis had established that the area of the circle with unit diameter can be expressed as the infinite product $\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot \ldots$. Euler thus concluded (in Point 2 of his article) that his term of index ½ was equal to the square root of this area, hence establishing a central result: $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$.

This was a turning point in Euler's thinking. He stated in Point 3: "I had previously supposed that the general term of the series 1, 2, 6, 24, etc. could be given, if not algebraically, at least exponentially. But after I had seen that some intermediate terms depended on the quadrature of the circle, I recognised that neither algebraic nor

[4] For the factorial product *n(n – 1)(n – 2) ... 3.2.1*, Euler used the notations [n] and $\Delta(n)$. The notation $n!$ was introduced in 1808 by Christian Kramp (1760-1826).

[5] This inspiring work can be read in Stacy Langton's English translation of Euler's 1730 article E19, available at http://www.math.dartmouth.edu/~euler/.

exponential quantities were suitable for expressing it". This insight suggested to Euler that his infinite product representation could perhaps be expressed in integral form. In attempting to compute the area of a circle, Wallis had worked with integrals of the form $\int_0^1 x^p(1-x)^q\,dx$. Euler thus let the general term of his progression be represented by the integral $\int_0^1 x^e(1-x)^n\,dx$ (in which the letter *e* represented any exponent, not the special constant associated with this letter today).[6] He then manipulated his integral using a binomial expansion, some ingenious substitutions and an informal application of L'Hôpital's limit rule to finally establish his landmark result, that $n! = \int_0^1 (-\log x)^n\,dx$, where n is any positive real number and $\log x$ is the natural logarithm with base e.[7]

At this stage Euler's integral was not viewed as a function in its own right, but merely as a tool for evaluating and representing factorials of non-natural arguments. The change to a function role was initiated by Adrien-Marie Legendre (1752-1833). In his '*Exercices de Calcul Intégral'*, Vol. 1 (1811) [61], Legendre introduced a unit shift to denote Euler's integral by

$$\Gamma(a) = (a-1)! = \int_0^1 \left(\log\frac{1}{x}\right)^{a-1} dx,\ a > 0\,, \tag{2.1}$$

and the gamma function was born.

By using the substitution $t = \log\frac{1}{x}$, this early integral representation for the gamma function is more conveniently used in the form

$$\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}dt\,, a > 0\,, \tag{2.2}$$

[6] Legendre would later call this integral the Euler integral of the first kind, and it would provide the basis for the beta function. (See Eq. (2.25) in Section 2.5.)

[7] As an illustration, Euler showed that for $n = 3$, integration by parts yields $\int(-\log x)^3 dx = -x\log x^3 + 3x\log x^2 + 6x$ (with no constant as the integral is zero when $x = 0$), giving the result 6 when $x = 1$.

commonly referred to as Euler's integral. Legendre further used his integral definition to show that $\Gamma(1) = 1$ (so that 0!=1), and to establish the central relation[8]

$$\Gamma(a+1) = a\Gamma(a) = a!, \quad a > 0. \tag{2.3}$$

This defining property is known variously as the functional equation, the difference equation or the recurrence relation. It represents a generalisation of the identity $n! = n(n-1)!$ for natural numbers, and provides the basis for the development of the theory of the gamma function. Repeated application of the recurrence relation (2.3) provides values for the gamma function for all natural arguments. As this relation is also valid for non-natural numbers, the values of the gamma function in any range $[a, a+1]$ determine the gamma function on the whole real line (excluding non-negative integers as shown below). For example, using (2.3) together with Euler's result for the factorial of ½, we have $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$. From this follows the important result which states that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.[9] In a similar way, we have $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$.

It can also be shown that for $n \in \mathbb{N}$,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi},$$
$$\Gamma\left(n+\frac{1}{3}\right) = \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right),$$
$$\Gamma\left(n+\frac{1}{4}\right) = \frac{1\cdot 5\cdot 9\cdots(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right).$$

[8] In an alternative notation known as the Pi function, $\Pi(z) = \Gamma(z+1) = z\Gamma(z)$, so that $\Pi(n) = n!$ (cf. [36], p.159).

[9] This result can also be obtained by substituting $x = \sqrt{t}$ into (2.2), and forming a double integral in polar coordinates (cf. [50], p.33). It thus provides a value for the Gaussian probability integral for standard distribution: $\int_{-\infty}^{\infty} e^{-u^2} du = 2\int_0^{\infty} e^{-u^2} du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

In order to evaluate gamma for any positive real number, many tables of gamma values for fractional arguments have been drawn up, samples of which can be found in [24], p.58 and [33], p.3.

For negative non-integral arguments we use (2.3) in the form $\Gamma(a) = \frac{\Gamma(a+1)}{a}$, so that for example, $\Gamma(-1.8) = \frac{\Gamma(0.2)}{(-1.8)(-0.8)} \approx \frac{4.5909}{(1.8)(0.8)} \approx 3.188125$ (from tables of values). Iteration of this process for $a < 0$, $a \neq -1, -2, -3, \ldots$, leads to the generalisation

$$\Gamma(a) = \frac{\Gamma(a+n)}{a(a+1)(a+2)\ldots(a+n-1)}, \text{for } -n < a < -n+1, n \in \mathbb{N}\,. \tag{2.4}$$

Example 2.2.1 By applying (2.3), we have $\Gamma(-2.5) = \frac{\Gamma(-1.5)}{-2.5} = \frac{\Gamma(-0.5)}{-2.5(-1.5)} = \frac{\Gamma(0.5)}{-2.5(-1.5)(-0.5)} = \frac{-8\sqrt{\pi}}{15}$. We can derive the same result more directly from (2.4), with $a = -3 + 0.5, n = 3$ so that $\Gamma(-2.5) = \frac{\Gamma(-2.5+3)}{-2.5(-1.5)(-0.5)} = \frac{-8\sqrt{\pi}}{15}$.

Applying (2.3) to non-positive integral arguments, we have $\Gamma(0) = \frac{\Gamma(1)}{0} = \infty$, $\Gamma(-1) = \frac{\Gamma(0)}{-1} = -\infty$, and so on. It is thus clear that for n a non-positive integer, $\Gamma(n) \rightarrow \pm\infty$, from which it follows that $\frac{1}{\Gamma(n)} = \frac{1}{(n-1)!} = 0$. An important consequence of this result is that a quotient of gamma functions is interpreted to be zero if any denominator argument is a non-positive integer. It also follows that the expression $\binom{a}{n} = \frac{1}{n!}.\frac{a!}{(a-n)!}$ is considered to be zero for n a negative integer.

Relation (2.4) implies that the sign of $\Gamma(a)$ on the interval $-n < a < -n+1,\ n \in \mathbb{N}$, will be $(-1)^n$, and hence the graph of the real gamma function consists of an infinite number of disconnected portions which alternately open upwards and downwards for negative arguments, and opens upwards for positive arguments. The portions with negative arguments lie within strips of unit width, tending to positive or negative infinity at the non-positive integers, while the portion with positive arguments contains the factorials and is of infinite width. Figure 2.1 shows the graph of the gamma function in the real plane.

The gamma function thus extends the factorial function from the natural numbers to all real numbers excluding non-positive integers. However, any smooth curve drawn through the discrete points (n, n!) for n a natural number will provide values for the factorial for non-natural arguments. Indeed, other functions besides the gamma function satisfy the recurrence relation (2.3).[10] In what way then, can the gamma function be considered to be the unique interpolating function for the factorial?

After almost 200 years of investigation, the answer was found to lie squarely with the concept of convexity. In 1922, Harald Bohr (1887-1951) and Johannes Mollerup (1872-1937) proved that among all possible functions that extend the factorial function to the positive real numbers, the gamma function is the only function that satisfies the recurrence relation and is logarithmically convex. We first prove that the gamma function is log-convex for positive arguments, and then provide the Bohr-Mollerup theorem.

Theorem 2.2.2 $\Gamma : (0, \infty) \to \mathbb{R}$ *is log-convex.*

Proof: Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. From Euler's integral (2.2), we have

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) = \int_0^\infty (t^{x-1}e^{-t})^{1/p}(t^{y-1}e^{-t})^{1/q} dt \,.$$

Hence, by Holder's inequality (1.23) we obtain

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leq \left(\int_0^\infty t^{x-1}e^{-t} dt\right)^{1/p} \left(\int_0^\infty t^{y-1}e^{-t} dt\right)^{1/q}$$
$$= [\Gamma(x)]^{1/p}[\Gamma(y)]^{1/q} \,.$$

[10] For example, Hadamard's factorial function $F(x) = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \log\left[\Gamma\left(\frac{1-x}{2}\right) / \Gamma\left(1 - \frac{x}{2}\right)\right]$ interpolates the factorials at the positive integers. While its definition relies on the gamma function (and its log-derivative), it does have the advantage of possessing no singularities anywhere in the finite complex plane, and hence from a function theoretic point of view can be considered to be a simpler solution to the interpolation problem (cf. [19], p.865).

We now let $\lambda = \frac{1}{p}$ so that $1 - \lambda = \frac{1}{q}$, and take the logarithms of both sides to obtain

$$\log\Gamma(\lambda x + (1-\lambda)y) \leq \lambda\log\Gamma(x) + (1-\lambda)\log\Gamma(y) .$$

This result holds for all $x, y \in (0, \infty)$ and hence $\log\Gamma$ is convex for positive arguments, according to Definition 1.4.1. ■

The Bohr-Mullerup Theorem given below states that the uniqueness of the gamma function lies in its log-convexity property (cf. [4], p.35).

Theorem 2.2.3 *The Bohr-Mollerup Theorem*: *If a function* $f: (0, \infty) \to (0, \infty)$ *satisfies the following three conditions*:

1. $f(1) = 1$,
2. $f(x+1) = xf(x)$,
3. $\log f$ *is convex,*

then $f(x) = \Gamma(x)$ *for all* $x \in (0, \infty)$.

Proof: We first consider $0 < x < 1$ and *n* a positive integer. As f is log-convex, we can apply Definition 1.4.2 over intervals $[n, n+1]$, $[n+1, n+1+x]$ and $[n+1, n+2]$ to obtain

$$\log\frac{f(n+1)}{f(n)} \leq \frac{\log f(n+1+x) - \log f(n+1)}{x} \leq \log\frac{f(n+2)}{f(n+1)} .$$

From Condition 2 we have $\log f(n+1) = \log n!$, and hence, we can write the above result in the form

$$x\log n \leq \log\frac{(x+n)(x+n-1)\dots xf(x)}{n!} \leq x\log(n+1) ,$$

$$\Rightarrow 0 \leq \log\frac{(x+n)(x+n-1)\dots x}{n!\, n^x} + \log f(x) \leq x\log\left(1 + \frac{1}{n}\right).$$

Taking limits as $n \to \infty$ shows that $f(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\dots(x+n)}$. This is an equivalent form of Euler's infinite product for the gamma function (see equation (2.6) in Section 2.3), and thus the equivalence between f and the gamma function is established. It can further be shown that for $x \in (0,1)$, f is uniquely determined by the three given conditions, and Condition 2 then extends the result to values greater than 1. ■

2.3 Representations of the gamma function

Since Euler's early work on the interpolated factorial, mathematicians have developed various representations of the gamma function, each offering its own advantage depending on the context of the application. In this section we present some of the more common forms of the gamma function, and demonstrate their equivalence. Representations and properties of the gamma function were initially developed in the realm of real numbers. For this reason, we restrict our present work to real arguments, and extend the domain to include complex numbers in Section 2.6.

2.3.1 Product and limit representations

The original product representations of Bernoulli and Euler, as discussed in Section 2.2, are equivalent to the limits given below in modern notation, found in [4], p.3.

Definition 2.3.1 *Euler's limit formula*: *For* $x > 0$,

$$\Gamma(x+1) = \lim_{n\to\infty}\left[(n+1)^x \prod_{i=1}^{n} \frac{i}{i+x}\right] = \lim_{n\to\infty} \frac{n!\,(n+1)^x}{(x+1)(x+2)\dots(x+n)}. \tag{2.5}$$

Carl Friedrich Gauss (1777-1855) used the equivalent limit

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\dots(x+n)} = \lim_{n\to\infty} \frac{n!\,n^x}{(x)_{n+1}} \tag{2.6}$$

as the fundamental definition of the gamma function (cf. [50], p.41, Eq. (2.37)). The gamma function can also be expressed in the useful product form given below.

Theorem 2.3.2 *Euler's infinite product*: *For* $x \neq 0, -1, -2, \dots$

$$\Gamma(x) = \frac{1}{x} \prod_{1 \le n < \infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}} . \tag{2.7}$$

Proof: As $\lim_{n\to\infty} \frac{n^x}{(n+1)^x} = 1$, we can write equation (2.6) in the form $\Gamma(x) = \lim_{n\to\infty} \frac{n!(n+1)^x}{x(x+1)\dots(x+n)}$. From the telescoping product $\frac{n+1}{n} \cdot \frac{n}{n-1} \cdots \frac{3}{2} \cdot \frac{2}{1} = n+1$, this can then be written as

$$\Gamma(x) = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{1}{1+x} \cdot \frac{2}{2+x} \cdots \frac{n}{n+x} \left[\left(\frac{2}{1}\right)^x \cdot \left(\frac{3}{2}\right)^x \cdots \left(\frac{n+1}{n}\right)^x \right]$$

$$= \frac{1}{x} \lim_{N\to\infty} \prod_{n=1}^{N} \left(\frac{n}{n+x}\right) \left(\frac{n+1}{n}\right)^x ,$$

and the desired result follows directly. ■

Equivalent limit forms include $\Gamma(x) = \lim_{n\to\infty} \frac{n! n^{x-1}}{(x)_n}$ and $\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)! n^x}{(x)_n}$ (cf. [75], p.11, [93], p.237).

In the *Journal für Math*. L1. (1856), Karl Weierstrass (1815-1897) provided a further useful product formula for the reciprocal of the gamma function, in terms of the Euler-Mascheroni gamma constant defined by $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$.[11] Weierstrass' product is frequently used as a fundamental definition of the gamma function. Various proofs of this result exist (cf. [33], p.27, [46], p.57, [75], p.11). We present here a method involving logarithms, as certain of the results will be used to prove Proposition 2.3.6.

[11] For convergence of this limit see [93], p.235.

Definition 2.3.3 *The Weierstrass product*: *For* $x \neq 0, -1, -2, \dots$ *and* $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$,

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left[\left(1+\frac{x}{n}\right)e^{-x/n}\right]. \tag{2.8}$$

Proof: We take the logarithms of both sides of Euler's product (2.7) to obtain

$$\log\Gamma(x) = -\log x + x\lim_{n\to\infty}\log\left[\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+1}{n}\right] - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{x}{n}\right)\right]$$

$$= -\log x + x\lim_{n\to\infty}\log(n+1) - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{x}{n}\right)\right]. \tag{2.9}$$

Now by taking the logarithms of both sides of (2.8), we have

$$\log\Gamma(x) = -\gamma x - \log x + \sum_{n=1}^{\infty}\left[\frac{x}{n} - \log\left(1+\frac{x}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{-x}{1}+\frac{-x}{2}+\cdots+\frac{-x}{n}+x\log n+\frac{x}{1}+\frac{x}{2}+\cdots+\frac{x}{n}\right] - \log x$$
$$- \sum_{n=1}^{\infty}\left[\log\left(1+\frac{x}{n}\right)\right]$$

$$= x\lim_{n\to\infty}\log n - \log x - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{x}{n}\right)\right]. \tag{2.10}$$

The theorem is then established by the equivalence of (2.9 and (2.10). ■

An alternative direct proof of the Weierstrass product involves expanding the gamma constant in the reciprocal of the right side of (2.8), to show that

$$\lim_{n\to\infty}\frac{1}{x}\exp\left(-x\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right]\right)\prod_{k=1}^{n}\frac{e^{\frac{x}{k}}}{\frac{k+z}{k}}$$

$$= \lim_{n\to\infty}\frac{1}{x}\prod_{k=1}^{n}\exp\left(\frac{-x}{k}\right)\exp(\log n^{x})\prod_{k=1}^{n}\exp\left(\frac{x}{k}\right)\prod_{k=1}^{n}\frac{k}{x+k}$$

$$= \lim_{n\to\infty} \left[\frac{1}{x} \cdot n^x \cdot \frac{1}{x+1} \cdot \frac{2}{x+2} \cdots \frac{n}{x+n}\right]$$
$$= \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\dots(x+n)}$$
$$= \Gamma(x).$$

The following theorem, found on p.235 of [93], establishes the analyticity of the Weierstrass product, and hence of the gamma function.

Theorem 2.3.4 *The Weierstrass' product is analytic for all finite values of* x.

Proof: By the Taylor expansion for the logarithmic function,[12] we have

$$\left|\log\left(1+\frac{x}{n}\right)-\frac{x}{n}\right| = \left|\frac{-x^2}{2n^2}+\frac{x^3}{3n^3}-\frac{x^4}{4n^4}+\cdots\right|.$$

Hence, for any integer N such that $|x| \le N/2$ and $n > N$, we have

$$\left|\log\left(1+\frac{x}{n}\right)-\frac{x}{n}\right| \le \left|\frac{x}{n}\right|^2 \left\{\frac{1}{2}+\left|\frac{x}{3n}\right|+\left|\frac{x^2}{4n^2}\right|+\cdots\right\}$$
$$\le \left|\frac{x}{n}\right|^2 \left\{1+\left|\frac{x}{n}\right|+\left|\frac{x^2}{n^2}\right|+\cdots\right\}$$
$$\le \frac{N^2}{4n^2}\left\{1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right\} \le \frac{1}{2}\frac{N^2}{n^2}.$$

Since $\sum_{n=N+1}^{\infty} \frac{N^2}{2n^2}$ converges, it follows that $\sum_{n=N+1}^{\infty}\left\{\log\left(1+\frac{x}{n}\right)-\frac{x}{n}\right\}$ is an absolutely and uniformly convergent series of analytic functions, and is thus an analytic function. Hence its exponential is an analytic function, which ensures the analyticity of $xe^{\gamma x}\prod_{n=1}^{\infty}\left[\left(1+\frac{x}{n}\right)e^{-x/n}\right]$ for all finite values of x. ■

As the right side of the Weierstrass product (2.8) is zero when $x = 0$ or a negative integer, this also confirms that the gamma function, as the reciprocal of an analytic

[12] For complex arguments, we take the principle value of $\log(1 + x/n)$.)

function, is analytic except at $x = 0, -1, -2, \ldots$ where it has singularities (simple poles in the complex plane). The Weierstrass product also shows that $\frac{1}{\Gamma(x)}$ is defined for all finite values of x, which confirms that the gamma function is never zero.

2.3.2 Integral representations

In Section 2.2 we introduced the gamma representation (2.2) known as Euler's integral. This representation has advantages over other forms through its frequency of appearance and its simplicity, and many other useful integral representations follow from this early definition. To prove that this integral is well-defined, we follow Artin [6] by expressing the gamma function as the sum of its principal parts, known as the incomplete gamma functions.

Theorem 2.3.5 *The integral* $\int_0^\infty e^{-t}t^{x-1}dt$ *converges for* $x > 0$.

Proof: Let $\Gamma(x) = \int_0^1 e^{-t}t^{x-1}dt + \int_1^\infty e^{-t}t^{x-1}dt = I_1 + I_2$. As $t > 0$ in the first integral, we have

$$I_1 = \int_0^1 e^{-t}t^{x-1}dt < \int_0^1 t^{x-1}dt = \lim_{a\to 0^+}\int_a^1 t^{x-1}dt = \lim_{a\to 0^+}\left(\frac{1}{x} - \frac{a^x}{x}\right).$$

For $x > 0$, I_1 is thus bounded above by $1/x$, and if we hold x fixed and let a decrease, the value of the integral will increase monotonically. Hence this integral exists for $x > 0$. For the integral I_2, we note that by the Taylor expansion of e^t we have $e^t > \frac{t^n}{n!}$ for t positive and any $n \in \mathbb{N}$, and hence $e^{-t} < \frac{n!}{t^n}$. It follows that

$$I_2 = \int_1^\infty e^{-t}t^{x-1}dt < \lim_{b\to\infty}\int_1^b n!\, t^{x-n-1}dt = \frac{n!}{x-n}\lim_{b\to\infty}\left\{\left(\frac{1}{b}\right)^{n-x} - 1\right\}.$$

Thus, if we hold x fixed and choose $n > x + 1$, we can make $n!/(n-x)$ an upper bound for $\int_1^b e^{-t}t^{x-1}dt$. This integral increases as b increases, and thus the integral I_2 exists. We can thus conclude that the gamma integral exists for $x > 0$. ■

There exist interesting proofs for the equivalence of Euler's integral and limit representations for the gamma function. For example, Norton [70], p.223 uses $e^{-t} = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$ to write (2.2) in the form $\Gamma(x) = \lim_{n\to\infty}\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt,$ for $n \geq 1$. The desired limit (2.5) then follows by substituting *nt* for *t*, integrating by parts and iterating the result.

The fundamental integral (2.2) can be adapted to many alternative forms through substitutions such as those given below.

- $\Gamma(x) = 2\int_0^\infty e^{-t^2} t^{2x-1} dt$, from the substitution $t \to t^2$.
- $\Gamma(x) = \alpha^x \int_0^\infty e^{-\alpha t} t^{x-1} dt$, from the substitution $t \to \alpha t$.
- $\Gamma(x+1) = \int_0^\infty e^{-y^{1/x}} dy$ from the substitution $y = t^x$.
- $\Gamma(x) = \int_{-\infty}^\infty e^{xz} e^{-e^z} dz$ from the substitution $t = e^z$.

Temme [84], p.44 provides yet another interesting integral representation. Using integration by parts, he obtains for $-1 < x < 0$,

$$\int_0^\infty t^{x-1}(e^{-t}-1)dt = \frac{1}{x}\int_0^\infty t^x e^{-t} dt = \frac{1}{x}\Gamma(x+1) = \Gamma(x).$$

The left integral thus defines the gamma function in the strip $-1 < x < 0$. In general, the Cauchy-Saalschütz representation for the gamma function is

$$\Gamma(x) = \int_0^\infty t^{x-1}\left[e^{-t} - 1 + t - \frac{t^2}{2!} + \cdots + (-1)^{n+1}\frac{t^n}{n!}\right] dt, n \geq 0.$$

2.3.3 Gamma representations and the Bohr-Mollerup theorem

One way to establish the validity of a proposed representation for the gamma function, is to show that it satisfies the conditions of the Bohr-Mollerup theorem. To complete this section, we illustrate this approach for the Weierstrass product.

Proposition 2.3.6 *Weierstrass' product* (2.8) *satisfies the conditions of the Bohr-Mollerup theorem.*

Proof: *Condition* 1 $\Gamma(1) = \frac{e^{-\gamma}}{1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1} e^{1/n}$

$$= \lim_{n\to\infty} \exp\left(-\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right]\right)\prod_{k=1}^{n} e^{1/k}\frac{k}{k+1}$$

$$= \lim_{n\to\infty}\left[n\cdot\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}\cdot\frac{n}{n+1}\right] = 1$$

Condition 2 From (2.8) and $e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$, we have

$$\frac{\Gamma(x+1)}{\Gamma(x)} = \frac{xe^{-\gamma}}{x+1}\frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{x+1}\left(1+\frac{x}{n}\right)\right]}{\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{x+1}{n}\right)\right]}.$$

As $e^{-\gamma} = \lim_{n\to\infty} n\prod_{k=1}^{n} e^{-1/k}$, this becomes

$$\frac{\Gamma(x+1)}{\Gamma(x)} = \frac{x}{x+1}\lim_{n\to\infty} n\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{-1}\left(\frac{k+1}{k}\right)\left(\frac{k+x}{k+x+1}\right)$$

$$= \frac{x}{x+1}\lim_{n\to\infty}\left(n.\frac{x+1}{x+2}.\frac{x+2}{x+3}\cdots\frac{x+n}{x+n+1}\right) = x\,.$$

Condition 3 By differentiating our earlier result (2.10) twice with respect x, we can show that $\frac{d^2}{dx^2}[\log\Gamma(x)] = \frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{1}{(n+x)^2} > 0$. Hence the log-gamma function is convex, and the proof is complete. ■

2.4 Central properties of the gamma function

The gamma function has a wide variety of useful properties which make it a powerful tool for simplifying various combinatoric and hypergeometric identities. In this section we provide some central properties that will be used extensively in later chapters.

2.4.1 The gamma function and factorials

The gamma function can be expressed as various combinations of binomial coefficients and the rising factorial $(a)_n$ defined in Chapter 1. The identities below can be found in standard texts such as [52] and [84], and are easily established by basic manipulation of factorials and use of the gamma recurrence relation (2.3). These are invaluable in simplifications involving hypergeometric functions.[13]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a \neq 0, -1, -2, \dots, n \in \mathbb{N} \tag{2.11}$$

$$\binom{a}{n} = (-1)^n \frac{(-a)_n}{n!} = (-1)^n \frac{\Gamma(n-a)}{n!\,\Gamma(-a)}, a \neq 1, 2, 3, \dots, n \in \mathbb{N}, n \leq a \tag{2.12}$$

$$\binom{a}{n} = \frac{\Gamma(a+1)}{n!\,\Gamma(a+1-n)}, a \neq -1, -2, -3, \dots, n \in \mathbb{N}, n \leq a \tag{2.13}$$

$$(a)_n = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}, a \neq 1, 2, 3, \dots, n \in \mathbb{N} \tag{2.14}$$

$$\Gamma(a) = \frac{(n-1)!\, n^a}{(a)_n}, a \neq 0, -1, -2, \dots, n \in \mathbb{N} \tag{2.15}$$

[13] Identity (2.11) implies that $\frac{(a+1)_n}{(a)_n}$ can be simplified to $\frac{a+n}{a}$ even if a is a negative integer, and to 1 if $n = a = 0$.

As $\frac{1}{\Gamma(-n)}$ is zero for $n = 0, 1, 2, \ldots$, an important consequence of (2.13) is that for $n, a \in \mathbb{N}_0$, $\binom{a}{n} = 0$ when $n > a$. This 'vanishing' of terms will be important when working with sums of binomial coefficients in the following chapters.

2.4.2 Euler's reflection formula

In 1771, Euler proved the powerful result $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$. This useful reflection formula[14] establishes a fundamental but perhaps unexpected link between the gamma function and the sine function. Davis [19] points out that "from the complex point of view, a partial reason for the identity lies in the similarity between the zeros [of $\sin(\pi z)$] and the poles of the gamma function". However, while both functions are periodic, it is not immediately obvious that they can be expressed in terms of one another. Davis poetically describes this relation as "a fine example of the delicate patterns which make the mathematics of the period so magical".

There are various approaches to proving Euler's reflection formula. Some texts (cf. [11], p.35, [24], p.65, [46], p.59) first prove that $\sin(\pi x) = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$ and then use this result, while other texts first prove the reflection formula directly, and then deduce the infinite sine product as a consequence (cf. [4], p.10, [6], p.27, [75], p.20). We will follow the former approach, basing the proof on the Euler-Wallis formula for the sine function, as proved below (cf. [11], pp.32-37).

Theorem 2.4.1 *The Euler-Wallis formula*: *For* $n \in \mathbb{N}$,

$$\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

Proof: By repeated application of the identity $\sin 2x = 2\sin x \cos x$, we obtain

[14] Havil [46] points out that a reflection formula relates $f(x)$ to $f(a-x)$ for some constant a. Euler's formula is thus properly a complement formula, i.e. a special case of the general reflection formula, with $a = 1$.

$$\sin x = 2^3 \sin\frac{x}{2^2}\sin\frac{\pi + x}{2^2}\sin\frac{2\pi + x}{2^2}\sin\frac{3\pi + x}{2^2}.$$

Applying the same identity again to each of the four factors on the right yields

$$\sin x = 2^7 \sin\frac{x}{2^3}\sin\left(\frac{\pi + x}{2^3}\right)\sin\left(\frac{2\pi + x}{2^3}\right)\sin\left(\frac{3\pi + x}{2^3}\right)\sin\left(\frac{4\pi + x}{2^3}\right)$$
$$\times \sin\left(\frac{5\pi + x}{2^3}\right)\sin\left(\frac{6\pi + x}{2^3}\right)\sin\left(\frac{7\pi + x}{2^3}\right).$$

By repeating this process *n* times, we obtain the general expansion

$$\sin x = 2^{2^n - 1} \sin\frac{x}{2^n}\sin\left(\frac{\pi + x}{2^n}\right)\sin\left(\frac{2\pi + x}{2^n}\right)\ldots\sin\left(\frac{[2^n - 1]\pi + x}{2^n}\right).$$

It is possible to rewrite the factors in alternative forms, starting from the last term so that $\sin\left(\frac{[2^n-1]\pi+x}{2^n}\right) = \sin\left(\pi - \frac{\pi - x}{2^n}\right) = \sin\left(\frac{\pi - x}{2^n}\right)$, $\sin\left(\frac{[2^n-2]\pi+x}{2^n}\right) = \sin\left(\frac{2\pi - x}{2^n}\right)$, and so on. In general, the *r*-th factor from the end can be written as $\sin\left(\frac{r\pi - x}{2^n}\right)$. Then by pairing the second factor with the last, the third with the second last, and so on, we obtain

$$\sin x = 2^{2^n - 1} \sin\frac{x}{2^n}\left\{\sin\left(\frac{\pi + x}{2^n}\right)\sin\left(\frac{\pi - x}{2^n}\right)\right\}\left\{\sin\left(\frac{2\pi + x}{2^n}\right)\sin\left(\frac{2\pi - x}{2^n}\right)\right\}\ldots$$
$$\ldots\left\{\sin\left(\frac{\left[\frac{1}{2}2^n - 1\right]\pi + x}{2^n}\right)\sin\left(\frac{\left[\frac{1}{2}2^n - 1\right]\pi - x}{2^n}\right)\right\}\sin\left(\frac{\frac{1}{2}2^n\pi + x}{2^n}\right)$$
$$= 2^{2^n - 1} \sin\frac{x}{2^n}\left\{\sin^2\frac{\pi}{2^n} - \sin^2\frac{x}{2^n}\right\}\left\{\sin^2\frac{2\pi}{2^n} - \sin^2\frac{x}{2^n}\right\}\ldots$$
$$\ldots\left\{\sin^2\frac{\left[\frac{1}{2}2^n - 1\right]\pi}{2^n} - \sin^2\frac{x}{2^n}\right\}\cos\left(\frac{x}{2^n}\right),$$

by the identity $\sin(A + B)\sin(A - B) = \frac{1}{2}(2\sin^2 A - 2\sin^2 B)$. As $\lim_{x\to 0}\frac{\sin x}{\sin\frac{x}{2^n}} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{x/2^n}{\sin(x/2^n)}\cdot 2^n = 2^n$, we can divide both sides of the above result by $\sin\frac{x}{2^n}$ and take the limit as $x \to 0$, to obtain

$$2^n = 2^{2^n - 1}\sin^2\frac{\pi}{2^n}\sin^2\frac{2\pi}{2^n}\ldots\sin^2\frac{\left[\frac{1}{2}2^n - 1\right]\pi}{2^n}.$$

Division of this result into the previous expansion for $\sin x$ yields

$$\sin x = 2^n \sin\frac{x}{2^n}\left\{1-\frac{\sin^2(x/2^n)}{\sin^2(\pi/2^n)}\right\}\left\{1-\frac{\sin^2(x/2^n)}{\sin^2(2\pi/2^n)}\right\}\cdots$$

$$\cdots\left\{1-\frac{\sin^2(x/2^n)}{\sin^2\left(\left[\frac{1}{2}2^n-1\right]\pi/2^n\right)}\right\}\cos\frac{x}{2^n}\ .$$

We again take limits of both sides, this time as $n \to \infty$, to obtain the final result,

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right),$$

and the theorem is proved. ■

We can now use this result to prove Euler's reflection formula.

Theorem 2.4.2 *Euler's reflection formula*: *For* $0 < x < 1$,

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}. \qquad (2.16)$$

Proof: By the recurrence relation (2.3) and Euler's product (2.7), we have

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = \frac{1}{\Gamma(x)(-x)\Gamma(-x)}$$

$$= \frac{1}{-x}\prod_{n=1}^{\infty}\frac{x\left(1+\frac{x}{n}\right)(-x)\left(1-\frac{x}{n}\right)}{\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{1}{n}\right)^{-x}}$$

$$= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right).$$

By the Euler-Wallis formula, $\sin(\pi x) = \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$, and the desired result follows directly. ■

Euler's reflection formula provides an elegant way to confirm the well-known result $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, by simply setting $x = \frac{1}{2}$ in (2.16) to obtain $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{\pi}{\sin^{\pi}/_{2}} = \pi$. It also leads to further useful results such as those given below (cf. [4], p.11, [11], p.41, [84], p.74).

$$\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right)=\frac{\pi}{\cos(\pi x)}\;, x-\frac{1}{2}\notin\mathbb{Z}$$

$$\pi\cot(\pi x)=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{x+n}+\frac{1}{x-n}\right)=\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{x-k}$$

$$\pi\tan(\pi x)=\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{k+\frac{1}{2}-x}$$

Using the recurrence relation (2.3), the reflection formula can also be generalised to $\Gamma(n+z)\Gamma(n-z)=[(n-1)!]^2\frac{\pi z}{\sin(\pi z)}\prod_{m=1}^{n-1}\left(1-\frac{z^2}{m^2}\right), n\in\mathbb{N}$ (cf. [84], p.48).

2.4.3 Multiplication and duplication formulae

These formulae provide powerful tools for simplifying gamma expressions, and are used extensively in work with hypergeometric identities. A proof of the Legendre duplication formula requires the lemma below.

Lemma 2.4.3 *For* $n\in\mathbb{Z}$ *,* $x\neq -1,-2,-3,\ldots$ *,* $\lim_{n\to\infty}\frac{(n-1)!n^x}{\Gamma(x+n)}=1.$[15]

Proof: The result follows from writing (2.6) in the equivalent form $\Gamma(x)=\lim_{n\to\infty}\frac{(n-1)!n^x}{(x)_n}=\lim_{n\to\infty}\frac{(n-1)!n^x}{\Gamma(x+n)}.\Gamma(x).$ ■

We now prove Legendre's duplication formula as found in [75], p.23, which is useful for reducing the argument of a gamma function.

Theorem 2.4.4 *Legendre's duplication formula*: *For* $x\neq 0,-\frac{1}{2};\ -1,-1\frac{1}{2},\ldots,$

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\sqrt{\pi}\,\Gamma(2x). \tag{2.17}$$

[15] Euler derived a similar result in his 1776 article '*De termino generali serium hypergeometricarum'*, by showing that $\Delta\colon (i+n)=1.2.3.4\ldots\ldots i(i+\alpha)^n$, where Δ: represents the factorial operation and *i* is an infinite number. In modern notation, his result would be written as $(n+m)!\sim(n+1)^m n!$ for large values of *n*.

Proof: By using (2.11), the Pochhammer identity (1.10) can be written in the form

$$\frac{\Gamma(2x)}{\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)} = \frac{\Gamma(2x+2n)}{2^{2n}\Gamma(x+n)\Gamma\left(x+\frac{1}{2}+n\right)}.$$

We now take limits of both sides as $n \to \infty$, and introduce suitable factors so that we can apply Lemma 2.4.3. The result is

$$\frac{\Gamma(2x)}{2^{2x}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)}$$

$$= \lim_{n\to\infty}\left(\frac{\Gamma(2x+2n)}{(2n-1)!\,(2n)^{2x}}\right)\left(\frac{(n-1)!\,n^{x}}{\Gamma(x+n)}\right)\left(\frac{(n-1)!\,n^{x+1/2}}{\Gamma\left(x+\frac{1}{2}+n\right)}\right)\left(\frac{(2n-1)!}{2^{2n}n^{1/2}[(n-1)!]^2}\right).$$

$$= \lim_{n\to\infty}(1)(1)(1)\left(\frac{(2n-1)!}{2^{2n}n^{1/2}[(n-1)!]^2}\right) = c,$$

where c is independent of x. Finally, to evaluate c we substitute $x = \frac{1}{2}$ to find that $\frac{\Gamma(1)}{2\Gamma\left(\frac{1}{2}\right)\Gamma(1)} = \frac{1}{2\sqrt{\pi}} = c$, and the desired result follows directly. ■

Koepf [52], p.16 provides an impressively elegant proof of (2.17) for $k \in \mathbb{N}$. He shows that if $a_k = \frac{\Gamma(2k)}{4^k\Gamma(k)\Gamma(k+1/2)}, \frac{a_{k+1}}{a_k} = 1$. Hence $a_k = a_1 = \frac{1}{2\sqrt{\pi}}$, and the duplication formula follows directly. The substitution of $x = \frac{1}{2}$ into (2.17) once again confirms that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The duplication formula can also be extended to higher multiples of the argument. For example, the triplication formula found in [24], p.65, is given by

$$\Gamma(3x) = \frac{3^{3x-1/2}}{2\pi}\,\Gamma(x)\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right).$$

Formulae such as the duplication and triplication formulas are particular cases of the general multiplication formula for the argument $kx,\ k \in \mathbb{N},$ as proven by Gauss in 1812. A proof can be found in [75], pp.24-26, Theorem 10.

Theorem 2.4.5 *Gauss' multiplication formula*: *For* $x \neq 0, -1, -2, \ldots,$

$$\Gamma(x)\Gamma\left(x+\frac{1}{k}\right)\Gamma\left(x+\frac{2}{k}\right)\ldots\Gamma\left(x+\frac{k-1}{k}\right) = (2\pi)^{\frac{k-1}{2}}k^{\frac{1}{2}-kx}\Gamma(kx). \tag{2.18}$$

2.4.4 Asymptotic behaviour of the gamma function

It is useful to know how the gamma function behaves for large values of the argument. As $\Gamma(n+1) = n!$ for natural numbers, we can use an approximation for the factorial as a measure of the asymptotic behaviour of the gamma function. The remarkable asymptotic formula below, developed by James Stirling (1692-1730), combines the important analytical constants $\sqrt{2}$, e and π in order to approximate the factorial of a large natural number (cf. [24], p.66).

Theorem 2.4.6 *Stirling's approximation formula*: *For large natural values of n,*

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n}n^n e^{-n}\,. \tag{2.19}$$

Proof: For $x > 0,\ n! = \Gamma(n+1) = \int_0^\infty e^{-x}x^n dx = \int_0^\infty e^{n\ln x - x}dx$. By the Taylor expansion of $n\ln x - x$ about $x = n$, this integral can be written in the form $\int_0^\infty \exp\left[-n + n\ln n - \frac{(x-n)^2}{2n} + \cdots\right]dx$. As a very large n makes the resulting error negligible, we can let this integral run from $-\infty$ to ∞, to obtain

$$n! \sim e^{-n}n^n \int_{-\infty}^{\infty} \exp\left(\frac{-(x-n)^2}{2n}\right)dx = e^{-n}n^n \int_{-\infty}^{\infty} \exp\left(\frac{-t^2}{2n}\right)dt.$$

The further substitution , together with the fact that ,[16] enables us to conclude that . ■

2.5 Functions related to the gamma function

There are many useful functions closely associated with the gamma function, such as its reciprocal function, the digamma and polygamma functions which involve log-derivatives, the incomplete gamma function, and the beta function, which plays a central role in establishing hypergeometric identities. In this section we present some related functions which will be useful in later chapters.

2.5.1 The reciprocal gamma function

As the gamma function is not defined for non-positive integers, it is often more practical to work with the reciprocal gamma function, which is defined for all finite arguments and has zeros when *x* is a non-positive integer (see Figure 2.2). In the complex domain, the reciprocal gamma function is an entire function with no singularities. Every entire function has a product representation, which is provided in this case by Weierstrass' definition (2.8).

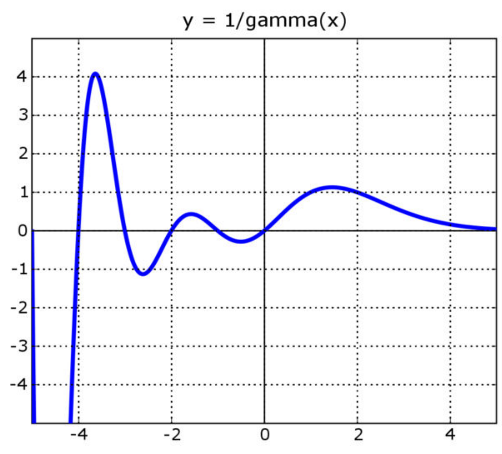

Figure 2.2 The reciprocal gamma function in the real plane

[16] See the footnote on page 21.

2.5.2 Incomplete and multiple gamma functions

It is frequently useful to take limits of integration other than 0 and ∞ to describe the accumulation of a finite process. In some cases this leads to the upper and lower incomplete gamma functions, denoted by $\Gamma(a, x)$ and $\gamma(a, x)$ respectively, as defined in [75], p.127. These functions were introduced in 1811 by Legendre, and arise in many areas of engineering and physics (cf. [15], [78]). For example Seaborne [78], p.9 shows the use of the lower incomplete gamma function in finding velocity distribution in an ideal gas.

Definition 2.5.1 *For* $x > 0$, $\Gamma(x) = \Gamma(a, x) + \gamma(a, x)$, *where*

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt\,, \quad \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt\,. \tag{2.20}$$

The multiple gamma function ($\Gamma_n(z)$) was developed by Ernest William Barnes (1904-1010) around 1900.[17] This function generalises the Euler gamma function through a recurrence-functional equation, and has applications to infinite series and products, and in connection with the Riemann hypothesis.

Definition 2.5.2 *For* $z \in \mathbb{C}$, $n \in \mathbb{N}$, *the multiple gamma function is defined by* $\Gamma_n(z)$, *where*

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \Gamma_1(z) = \Gamma(z) \text{ and } \Gamma_n(1) = 1.$$

[17] See particularly his article '*On the theory of the multiple gamma function*' of 1904.

2.5.3 The digamma function

In the same way as Stirling originally used the series $\sum_{n=1}^{\infty} \log(n!)$ to derive his asymptotic formula for $n!$, mathematicians have used the logarithm of the gamma function to derive useful properties of the gamma function. A powerful tool for such investigation is the logarithmic-derivative of the gamma function, also called the psi or digamma function, (cf. [84], p.53).

Definition 2.5.3 *The digamma function is defined for $x \neq 0, -1, -2, \ldots$ by $\psi(x)$, where*

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{2.21}$$

From (2.21) we have the useful result $\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \Gamma'(1)$.

The recurrence relation for the gamma function has its parallel for the digamma function. By logarithmic differentiation of $\Gamma(x+1) = x\Gamma(x)$, we have

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{2.22}$$

Havil [46], p.58, points out that the recurrence (2.22), leads to the pleasing relation $\psi(n) = -\gamma + H_{n-1}$, where the H_n are terms of the harmonic series.

In general we have

$$\psi(x+n) = \psi(x) + \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n-1}, n \geq 1\,. \tag{2.23}$$

Through its relation with the gamma function, the digamma function provides a useful tool for evaluating some awkward definite integrals such as the Dirichlet and Gauss integrals below (cf. [4], p.26, Th. 1.6.1).

$$\psi(x) = \int_0^\infty \frac{1}{z}\left(e^{-z} - \frac{1}{(1+z)^x}\right) dz, x > 0$$

$$\psi(x) = \int_0^\infty \left(\frac{e^{-z}}{z} - \frac{e^{-xz}}{1-e^{-z}}\right) dz, x > 0$$

The digamma function also has useful series representations, such as that obtained below by differentiating the logarithm of the Weierstrass product (2.8). Convergence of the digamma series is established by the Weierstrass M-test and the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, as discussed in [75], p.10.

Definition 2.5.4 *For* $x \neq 0, -1, -2, \dots$,

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n-1}\right) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{x}{n(n+x)}\right). \qquad (2.24)$$

This series expansion can be useful in evaluating certain series with rational terms, as illustrated in the following example.

Example 2.5.5 The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)}$ can be written as

$$\sum_{n=1}^{\infty}\left[\frac{1}{3(n+1)} - \frac{1}{n+\frac{1}{2}} + \frac{2}{3\left(n+\frac{1}{4}\right)}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{3}\left(\frac{1}{n+1} - \frac{1}{n}\right) - \left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n}\right) + \frac{2}{3}\left(\frac{1}{n+\frac{1}{4}} - \frac{1}{n}\right)\right]$$

$$= -\frac{1}{3}\psi(2) + \psi\left(\frac{3}{2}\right) - \frac{2}{3}\psi\left(\frac{5}{4}\right).$$

As $\psi\left(\frac{1}{2}\right)=\frac{\Gamma\prime(1/2)}{\Gamma(1/2)}$, we can also use the digamma series expansion to find a value for $\Gamma'\left(\frac{1}{2}\right)$. By (2.24), $\Gamma'(1/2)=-\sqrt{\pi}\left\{\gamma-\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\right)-2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right)\right]\right\}$, and hence $\Gamma'\left(\frac{1}{2}\right)=-\sqrt{\pi}\left\{\gamma+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)\right\}=-\sqrt{\pi}\{\gamma+\log 2\}$.

Another interesting result which follows from the series definition (2.24) is that $\psi(1)=\Gamma'(1)=-\gamma.$ Hence, the Euler-Mascheroni constant is the negative of the gradient of the gamma function at the point where $x=1$. This result also provides the alternative definition $\gamma=-\int_0^\infty e^{-t}\ln t\; dt$, through differentiating Euler's gamma integral (2.2). Havil [46], p. 108 further shows that the Euler-Mascheroni constant is the value of the "fearsome integral" $\int_0^1 \frac{1-e^{-u}-e^{-1/u}}{u}\,du$, and of $\int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)dt$.

The digamma function confirms certain behaviours of the gamma function. For example, by further differentiation, we obtain

$$\psi'(x)=\sum_{n=1}^{\infty}\frac{1}{(n+x-1)^2}=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\cdots>0\,,$$

which by Corollary 1.4.3 proves that $\log\Gamma(x)$, and hence $\Gamma(x)$, is convex on the positive real axis. This result also shows that $\frac{d}{dx}\left(\frac{\Gamma'(x)}{\Gamma(x)}\right)=\frac{\Gamma''(x)\Gamma(x)-\Gamma'(x)\Gamma'(x)}{(\Gamma(x))^2}>0,$ from which we conclude that $\Gamma''(x)\Gamma(x)>(\Gamma'(x))^2\geq 0$. Hence, the functions $\Gamma(x)$ and $\Gamma''(x)$ are either both positive or both negative, which is consistent with the concavity of the graph given in Figure 2.1. Artin [6], p.16 uses series convergence to show that the digamma function is differentiable to any order, and the resulting polygamma functions are defined by

$$\psi_n(x)=\psi^{(n)}(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{(p+x-1)^{n+1}}.$$

2.5.4 Fractional derivatives

The gamma function provides a natural way to interpolate the derivative function to include fractional derivatives. For $\in \mathbb{N}$, $k \leq n$, the *k*-th derivative of $f(x) = x^n$ is $n(n-1)(n-2)\ldots(n-k+1)x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$. We can now interpolate this *k*-th derivative to include fractional values of *k*, by writing the result in the form

$$\frac{d^k}{x^k}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-k+1)}x^{n-k}, k \in (0, \infty).$$

This result was proposed by Euler in 1730 in his article '*On transcendental progressions'* [27], accompanied by his modest comment that this result was "certainly more curious than useful". In this article, Euler established that the ½-th derivative of $f(x) = x$ is $\frac{\Gamma(2)}{\Gamma(3/2)}\sqrt{x} = \frac{2\sqrt{x}}{\sqrt{\pi}}$.

2.5.5 The beta function

The relationship between the beta and gamma functions forms a basis for many useful identities related to hypergeometric functions. The beta function originated as Euler's integral of the form $\int x^e dx(1-x)^n$, which Legendre called "Euler's first integral" in his '*Exercices de Calcul Intégral*', Vol. 1 (1811). The modern form of the beta function is given below (cf. [24], p.77).

Definition 2.5.6 *For* $x > 0,\ y > 0$ *, the beta function is defined by* $\beta(x, y)$ *where*

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt\,. \tag{2.25}$$

We verify below that the beta integral converges for $x > 0$, $y > 0$.

Theorem 2.5.7 *The integral* $\int_0^1 t^{x-1}(1-t)^{y-1}dt$ *converges for* $x > 0$, $y > 0$.

Proof: As with the gamma integral, we express the integral as the sum of two integrals and explore the convergence of each one separately. Let

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{1/2} t^{x-1}(1-t)^{y-1}dt + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}dt = I_1 + I_2.$$

For the first integral, when $0 < t \leq \frac{1}{2}, (1-t)^{y-1} \leq t^{x-1}$, so that

$$I_1 \leq \int_0^{1/2} t^{x-1}dt = \lim_{a\to 0^+}\left[\frac{t^x}{x}\right]_a^{1/2} = \lim_{a\to 0^+}\frac{2^{-x}-a^x}{x} = \frac{2^{-x}}{x}.$$

For the second integral, when $\frac{1}{2} \leq t < 1$, we have

$$I_2 \leq \int_{1/2}^1 (1-t)^{y-1}dt = \lim_{a\to 1^-}\left[\frac{(1-t)^y}{-y}\right]_{1/2}^a = \lim_{a\to 1^-}\frac{2^{-y}-(1-a)^y}{y} = \frac{2^{-y}}{y}.$$

This establishes the convergence of the beta function integral for $x > 0, y > 0$. ■

As with the gamma function, the beta function has a vast array of alternative representations and useful properties, found in texts such as [4], [6] and [75]. In this section we restrict ourselves to those properties most relevant to our work with hypergeometric functions.

By suitable substitutions into definition (2.25), the following integral representations can be obtained for $x, y > 0$.

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta \tag{2.26}$$

$$\beta(x,y) = \int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}}dt \tag{2.27}$$

Chaudry [15] provides eight such integral representations in equations (5.17) to (5.24).

The symmetry of the beta function is clear from definition (2.25), and is stated below.

Theorem 2.5.8 *For* $x, y > 0,\ \beta(x,y) = \beta(y,x).$ (2.28)

Although the beta function is a function of two variables while the gamma function is a function of only one variable, there exists a powerful direct relation between the two functions which is frequently used in establishing hypergeometric identities, and which we prove below (cf. [75], p.19, Th. 7).

Theorem 2.5.9 *For* $x > 0, y > 0$,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.29}$$

Proof: We use the integral definition (2.2) for the gamma function, together with various substitutions, to obtain

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-t}t^{x-1}dt \int_0^\infty e^{-u}u^{y-1}du$$

$$= 4\int_0^\infty \int_0^\infty e^{-a^2-b^2}x^{2x-1}y^{2y-1}dadb$$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2}(r\cos\theta)^{2x-1}(r\sin\theta)^{2y-1}rd\theta dr$$

$$= 2\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta \cdot 2\int_0^\infty e^{-r^2}r^{2x+2y-1}dr.$$

By (2.26), the first integral of this result is $\beta(y,x) = \beta(x,y)$. Also, from the integral $\Gamma(x) = 2\int_0^\infty e^{-t^2}t^{2x-1}dt$ established in Section 2.3, $2\int_0^\infty e^{-r^2}r^{2x+2y-1}dr = \Gamma(x+y)$, and the theorem is proved. ■

For $x, y, z, w > 0$, this result can be generalised to the identity

$$\beta(x,y)\beta(x+y,z)\beta(x+y+z,w) = \frac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x+y+z+w)}.$$

Remark 2.5.10 The following are useful consequences of the gamma-beta relation (2.29).

- It provides elegant proofs for certain properties of the gamma function. For example, proofs for Legendre's duplication formula based on this relation can be found in [11], p.29, [24], p.80 and [33], p.16.
- Through Euler's reflection formula (2.15), it provides for integral evaluations such as $\int_0^\infty \frac{t^{p-1}}{1+t}dt = \beta(p,1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$, $0 < p < 1$.
- It establishes a useful functional equation for beta: $\frac{\beta(x,y+1)}{y} = \frac{\Gamma(x)\Gamma(y+1)}{y\Gamma(x+y+1)} = \frac{\Gamma(x)y\Gamma(y)}{y(x+y)\Gamma(x+y)} = \frac{\beta(x,y)}{x+y}$, which can also be expressed in the factorial form $\beta(x,y) = \frac{x+y}{xy} \cdot \frac{x!y!}{(x+y)!}$.
- It provides an elegant proof for Wallis' integral $\int_0^{\pi/2} \sin^n x \, dx$ (cf. [33], p.73).

2.6 The complex gamma function

We have until now restricted our discussion of the gamma function to the domain of real numbers, in agreement with Andrews et al. [4] that "the gamma function is a real variable function in the sense that many of its important characterisations occur within that theory". However, the early work on the real gamma function has since been extended to include complex arguments. In this section we look briefly at the extension of the domain of definition of the gamma function to complex arguments, and offer some insights into the application of complex analysis to this function.

While Euler was a pioneer in the theory of complex variables, he does not appear to have considered the factorial of a complex number. According to Davis [19], the move to the complex plane was initiated by Gauss, although until the early 1930s the complex values of the gamma function remained largely untouched. Tables of complex values and a hand-drawn three-dimensional graph of the complex gamma function were provided as early as 1909 by Jahnke and Emde [48], (see Figure 2.3), but the tables for real values produced by Gauss in 1813 and Legendre in 1825 were all that were needed for almost a century. Then in the 1930s, applications for the gamma function were discovered in theoretical physics, and through the use of computers in the early 1950s, extensive tables were published of values of the gamma function in the complex plane.

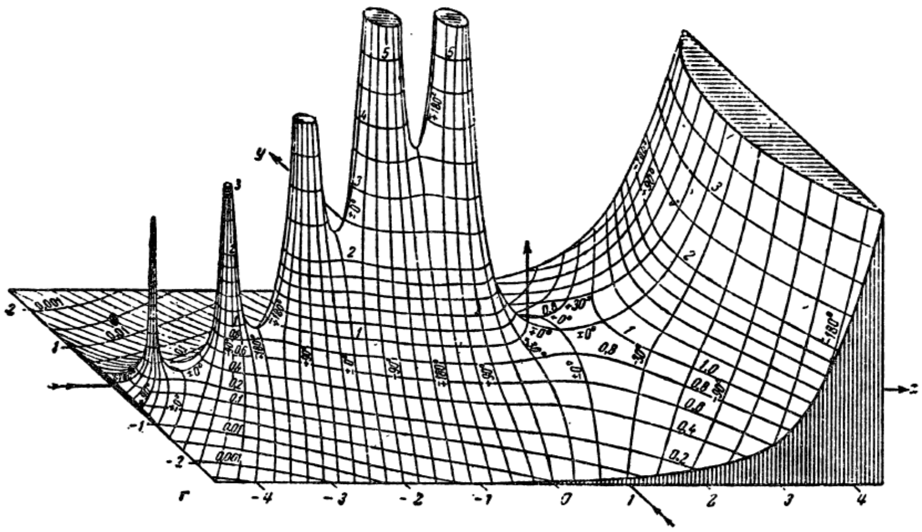

Figure 2.3 The complex gamma function of Jahnke and Emde (1909)

2.6.1 Analytic continuation of the gamma function

Davis [19] notes that "we have at our disposal a number of methods, conceptually and operationally different, for extending the domain of definition of the gamma function. Do these different methods yield the same result?" He assures us that they do, because of the notion of analytic continuation, as discussed in Chapter 1. By using the recurrence relation (2.3) and analytic continuation, we can extend the gamma function to include all portions of the complex plane excluding the non-positive integers. The gamma function also has an essential singularity at complex infinity, because $\Gamma\left(\frac{1}{z}\right)$ has a non-defined limit when $z \to \infty$, and hence the gamma function is not well defined in the compactified complex plane.

As one extension, Euler's integral formula (2.2) can be defined for complex numbers in the right-half of the complex plane. The integral then yields the complex-valued function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt\,, \quad \text{for } Re(z) > 0,$$

where the many-valued function t^{z-1} is made precise by using $t^{z-1} = e^{(z-1)\log t}$ ($\log t$ being purely real). This complex function coincides with the ordinary gamma function for real values. In a similar way, the beta function can be extended to include complex arguments, using the definition $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ Re(x) > 0, Re(y) > 0,$ where again $t^{x-1} = e^{(x-1)\log t}$ and $(1-t)^{y-1} = e^{(y-1)\log(1-t)}$ (cf. [93], p.253).

Euler's limit form (2.5) and Weierstrass' product (2.8) are well defined in the whole complex plane except at the non-positive integers, and hence provide analytic continuations of the gamma function. To show that the complex gamma function has simple poles at the non-positive integers, we again make use of the incomplete gamma functions.

Theorem 2.6.1 *For* $z \in \mathbb{C}$, $\Gamma(z)$ *has simple poles at* $z = -n\,, n = 0, 1, 2, \ldots$ *with residues* $Res(\Gamma, -n) = \frac{(-1)^n}{n!}$.

Proof: Through series expansion of the exponential function, and the interchange of summation and integration, we obtain

$$\Gamma(z) = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{n!} \left[\frac{t^{n+z}}{n+z} \right]_0^1 \right\} + \int_1^{\infty} e^{-t} t^{z-1} dt$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{n!} \frac{1}{n+z} \right\} + \int_1^{\infty} e^{-t} t^{z-1} dt.$$

The remaining integral is an analytic (entire) function of *z* because the convergence problem for *t* near 0^+ has disappeared, so there are no singularities over the range of integration. Thus by Definition 1.5.2, we have the positions of the poles and their residues. ■

This result is confirmed by applying the residue calculation in Theorem 1.5.3 to $\Gamma(z) = \frac{\Gamma(z+n+1)}{(z)_{n+1}}$, where *n* is a non-negative integer. Then $\mathrm{Res}(\Gamma, -n) = \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z(z+1)\ldots(z+n-1)} = \frac{(-1)^n}{n!}$, $n \in \mathbb{N}$. Hence, Γ(*z*) is a meromorphic function, defined for all complex numbers except the non-positive integers, and it is this resulting analytic extension of Euler's integral to complex values that is now referred to as the gamma function.

2.6.2 Complex theory and the gamma function

Having extended the gamma function to include complex arguments, we can harness the powerful methods of complex analysis to establish many fundamental results already proved for real numbers. To illustrate this, we will use contour integration and residue theory to prove Euler's reflection formula for complex numbers, following the work in [75], pp.20-21, Theorem 8, and in [78], pp.155-157.

Theorem 2.6.2 *For* $z \in \mathbb{C}$, *z not an integer,* $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.

Proof: For $0 < Re(z) < 1$, from (2.27) and (2.29) we have $\Gamma(z)\Gamma(1-z) = \beta(z, 1-z) = \int_0^{\infty} \frac{y^{z-1}}{1+y} dy$. We can evaluate this integral using contour integration in an α-plane, where $\alpha = re^{i\theta}$, $Re(\alpha) = y$, and the contour *C* encircles the simple pole $\alpha = -1$ of the integrand in $\int_C \frac{\alpha^{z-1}}{1+\alpha} d\alpha$. The origin is a branch point for this multi-valued integrand. We choose a cut along the positive real axis, across which the integrand is discontinuous, and define the contour C in the α-plane to consist of two circles centred at the origin, radii r (clockwise) and R (anticlockwise) with $r < R$, joined along the positive real axis from R to r, as shown in Figure 2.4 (found in [75], p.20).

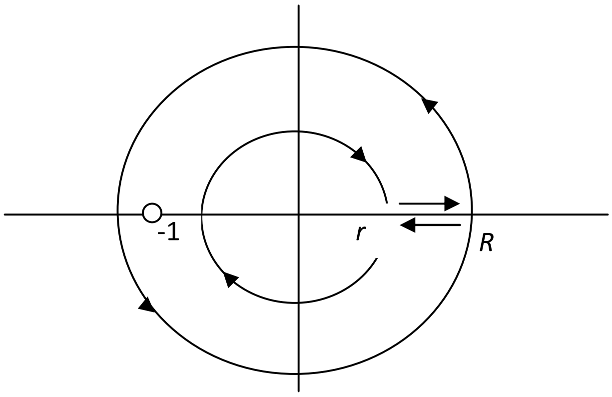

Figure 2.4 The contour C encircling a simple pole at -1

The contour C thus consists of the four parts $\alpha = Re^{i\theta}$(θ from 0 to 2π), $\alpha = ye^{2\pi i}$(y from R to r), $\alpha = re^{i\theta}$ (θ from 2π to 0) and $\alpha = ye^{0i} = y$ (y from r to R). Hence, we can write the contour integral as the sum

$$iR^z \int_0^{2\pi} \frac{e^{iz\theta} d\theta}{1 + Re^{i\theta}} + e^{2\pi iz} \int_R^r \frac{y^{z-1} dy}{1+y} + \int_{2\pi}^0 \frac{ir^z e^{iz\theta} d\theta}{1 + re^{i\theta}} + e^{0iz} \int_r^R \frac{y^{z-1} dy}{1+y}.$$

For $0 < Re(z) < 1$, we have $\lim_{R\to\infty} \int_0^{2\pi} \frac{e^{iz\theta} d\theta}{1+Re^{i\theta}} = \lim_{r\to 0^+} \int_{2\pi}^0 \frac{ir^z e^{iz\theta} d\theta}{1+re^{i\theta}} = 0$. The limiting form of the contour integral will thus be

$$e^{2\pi iz} \int_\infty^0 \frac{y^{z-1} dy}{1+y} + \int_0^\infty \frac{y^{z-1} dy}{1+y}.$$

Now from the residue theorems 1.5.3 and 1.5.4, we have that for $f(z) = \frac{\alpha^{z-1}}{1+\alpha}$,

$$\int_C \frac{\alpha^{z-1}}{1+\alpha} dz = 2\pi i.Res(f, -1) = 2\pi i(-1)^{z-1} = 2\pi i e^{\pi i(z-1)} = -2\pi i e^{\pi iz},$$

from which it follows that

$$e^{2\pi iz}\int_{\infty}^{0}\frac{y^{z-1}dy}{1+y}+\int_{0}^{\infty}\frac{y^{z-1}dy}{1+y}=-2\pi ie^{\pi iz}\,,$$

and hence

$$\int_{0}^{\infty}\frac{y^{z-1}dy}{1+y}=\frac{2\pi ie^{\pi iz}}{e^{2\pi iz}-1}=\frac{2\pi i}{e^{\pi iz}-e^{-\pi iz}}\,.$$

From the definition $\sin A=\frac{e^{Ai}-e^{-Ai}}{2i}$, we thus conclude that for $0<Re(z)<1$,

$$\int_{0}^{\infty}\frac{y^{z-1}dy}{1+y}=\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\,,$$

and by analytic continuation we can apply the result to all non-integral complex numbers. ■

Complex analysis also facilitates the development of various contour integral representations for the gamma function, such as Hankel's contour integral provided below, as discussed in [78], p.172.

Theorem 2.6.3 *Hankel's contour integral*: *For z not an integer,*

$$\Gamma(z)=\frac{1}{2i\sin\pi z}\int_{C}t^{z-1}e^{t}\,dt\,. \tag{2.30}$$

Proof: Consider the contour integral $\int_C t^{z-1}e^{t}\,dt$. For $0<Re(z)<1$, the integrand is multivalued with a branch point at the origin. We thus make a branch cut along the negative real axis, and choose a contour *C* which starts from below the branch cut, circles the origin once anticlockwise and returns parallel to the negative real axis. We deform this contour into the real axis with $\arg t=-\pi$, the circle centred at the origin with radius r, and again the real axis with $\arg t=\pi$. With $t=ue^{i\theta}$, we consider the integrals on the three parts of the contour:

$$I_1=\int_{\infty}^{r}e^{-u}(ue^{-i\pi})^{z-1}e^{-i\pi}du\,,$$

$$I_2 = ir^z \int_{arc} e^{re^{i\theta}} (e^{i\theta})^{z-1} e^{i\theta} d\theta \,,$$

$$I_3 = \int_r^\infty e^{-u} (ue^{i\pi})^{z-1} e^{i\pi} du \,.$$

For $Re(z) > 0$, we now take the limit as r tends to zero, so that I_2 tends to zero. We then have $\int_C t^{z-1} e^t \, dt = e^{-i\pi z} \int_\infty^0 e^{-u} u^{z-1} du + e^{i\pi z} \int_0^\infty e^{-u} u^{z-1} du$

$$= (e^{i\pi z} - e^{-i\pi z}) \int_0^\infty e^{-u} u^{z-1} du = 2i \sin(\pi z) \Gamma(z),$$

and the desired result follows directly. ■

The function $\frac{1}{2i\sin\pi z} \int_C t^{z-1} e^t \, dt$ is analytic for all non-integral values of z, and hence Henkel's contour integral provides an analytic continuation for $\Gamma(z)$ into the left half plane. It is possible to relax domain restrictions by choosing a more complicated double contour of integration, such as a Pochhammer contour (cf. [88], p.105).

2.7 Some interesting applications of the gamma function

Through its various integral and series representations, the gamma function provides a powerful calculation tool in a variety of contexts. The gamma function is a truly ubiquitous function in the world of pure and applied mathematics. While our prime focus is on its role in establishing hypergeometric identities, we complete this chapter by presenting a few other interesting applications of this function.

2.7.1 Integral evaluation

Some intimidating integrals become more tractable through judicious application of the gamma and beta integrals. Many illuminating and detailed examples can be found in texts such as [24] and [33], a few of which are provided below.

$$\int_0^\infty x^3 e^{-2x} dx = \frac{\Gamma(4)}{2^4} = \frac{3!}{16} = \frac{3}{8}$$

$$\int_0^1 x^m (\log x)^n \; dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_0^\infty t^{-1/2} e^{-t} \, dt = \Gamma\left(\frac{1}{2}\right) = \pi$$

$$\int_0^{\pi/2} \left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{1/4} \cos x \, dx \; = \; \beta\left(\frac{1}{4}, \frac{5}{4}\right) = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{2\sqrt{\pi}}$$

$$\int_0^1 x^m \log^n x \, dx = (-1)^n \int_0^\infty t^{-n} e^{-(m+1)t} \, dt = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}} = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

2.7.2 Infinite products

The gamma function provides a finite method for evaluating a certain class of converging infinite products (cf. [93], p.239).

Theorem 2.7.1 $\quad P = \prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\ldots(n-a_k)}{(n-b_1)(n-b_2)\ldots(n-b_k)} = \prod_{m=1}^{k} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$

Proof: The general term of the infinite product can be written in the form

$$\left(1-\frac{a_1}{n}\right)\ldots\left(1-\frac{a_k}{n}\right)\left(1-\frac{b_1}{n}\right)^{-1}\ldots\left(1-\frac{b_k}{n}\right)^{-1} = \; 1 - \frac{a_1+a_2+\cdots+a_k-b_1-\cdots-b_k}{n} + O(n^{-2})$$

when *n* is large. Absolute convergence requires that $a_1 + a_2 + \cdots + a_k - b_1 - \cdots - b_k = 0$, and thus we can multiply the product *P* by $\exp\left(\frac{a_1+a_2+\cdots+a_k-b_1-\cdots-b_k}{n}\right)$ without changing its value. Hence, we have

$$P = \prod_{n=1}^{\infty} \frac{\left(1-\frac{a_1}{n}\right) e^{\frac{a_1}{n}} \left(1-\frac{a_2}{n}\right) e^{\frac{a_2}{n}} \ldots \left(1-\frac{a_k}{n}\right) e^{\frac{a_k}{n}}}{\left(1-\frac{b_1}{n}\right) e^{\frac{b_1}{n}} \left(1-\frac{b_2}{n}\right) e^{\frac{b_2}{n}} \ldots \left(1-\frac{b_k}{n}\right) e^{\frac{b_k}{n}}}.$$

From the Weierstrass product (2.8), $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right) e^{\frac{z}{n}} = \frac{1}{-z\Gamma(-z)e^{-\gamma z}}$, and hence $P = \frac{b_1\Gamma(-b_1)b_2\Gamma(-b_2)\ldots b_k\Gamma(-b_k)}{a_1\Gamma(-a_1)a_2\Gamma(-a_2)\ldots a_k\Gamma(-a_k)} = \prod_{m=1}^{k} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$, which is the desired result. ■

Example 2.7.2 The above result allows us to establish that

$$\prod_{n=1}^{\infty}\frac{n(a+b+n)}{(a+n)(b+n)}=\frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

2.7.3 In relation to other special functions

The gamma function is used extensively in work with classical special functions. For example, the solutions of the differential equation $y''+\left(a+\frac{1}{4}z^2\right)y=0$ are called the parabolic cylinder functions, with one pair of solutions given $U(a,z)=\sqrt{\pi}2^{-1/4-a/2}\left[\frac{y_1}{\Gamma\left(\frac{3}{4}+\frac{a}{2}\right)}-\frac{\sqrt{2}y_2}{\Gamma\left(\frac{1}{4}+\frac{a}{2}\right)}\right]$ and $V(a,z)=\frac{1}{\pi}\Gamma\left(\frac{1}{2}+a\right)[\sin\pi a U(a,z)+U(a,-z)]$ (cf. [84], p.179). There also exist gamma-based definitions for the classical orthogonal polynomials such as the Laguerre, Hermite, Jacobi and Chebyshev polynomials (cf. [78], [84]).

2.7.4 Physical applications

The gamma function plays a central role in many calculations in applied mathematics. For example, the probability density function can be defined by $f(x,a,b)=\frac{1}{\Gamma(a)}x^{a-1}e^{-x/b}$, and is used to determine time-based occurrences such as the remaining life of a component. In his intriguing article of 1964, David Singmaster [80] used the gamma function (and its asymptotic approximation) to show that while a round peg fits better into a square hole than a square peg into a round hole, for values of *n* greater than 8, an *n*-cube fits better into an *n*-ball. Through its integral representation, the gamma function also provides a convenient method for evaluating physical quantities such as arc length, area and volume. Farrell and Ross [33], pp.84-86 provide the following illuminating example.

Example 2.7.3 Let R denote the region in the first quadrant bounded by the coordinate planes $x=0, y=0, z=0$, and the surface defined by $(x/5)^{1/2}+(y/6)^{2/3}+(z/7)^{3/4}=1$. The volume V of R, the mass M of R with density

$D = \sqrt{x^3 y^5 z}$, and the moment of inertia I_z of a homogeneous solid occupying R are given respectively by:

$$V = 840 \frac{\frac{\sqrt{\pi}}{2}\Gamma(4/3)}{\Gamma(3 + 3/2 + 4/3)},$$

$$M = 4\sqrt{5^5 6^7 7^3}\frac{\Gamma(5)\Gamma(21/4)}{\Gamma(8 + 21/4)},$$

$$I_z = 4\sqrt{5^9 6^7 7^3}\frac{\Gamma(9)\Gamma(21/4)}{\Gamma(69/4)} + 4\sqrt{5^5 6^{11} 7^3}\frac{\Gamma(59)\Gamma(33/4)}{\Gamma(65/4)}.$$

Having in this chapter established certain central definitions and identities of the gamma function, we will now proceed to a discussion of the hypergeometric function, in which we will draw heavily on its relationship with the gamma function.

Chapter 3

The Gauss hypergeometric function

3.1 Introduction

The hypergeometric function plays a central role in the realm of special mathematical functions, as all special functions can be expressed in terms of these functions. It is thus frequently encountered in pure mathematics, and its parametric nature provides a powerful tool for the solution of a wide range of applied problems.

The term "hypergeometric series" was first used by John Wallis in his '*Arithmetica Infinitorum*' (1655) to describe infinite series of the form $1 + a + a(a+1) + a(a+1)(a+2) + \cdots$. In 1836 this term was used by Ernst Eduard Kummer (1810-1898) for the series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

which is the present form of the hypergeometric series.

The term 'hypergeometric' can be applied to three objects: the hypergeometric equation (a linear second order differential equation), the hypergeometric series (a particular solution of the hypergeometric equation), and the hypergeometric function (the sum of the hypergeometric series). This type of function has been extensively studied by many mathematicians, including Johann Friedrich Pfaff (1765-1825), Kummer, Euler and Gauss. Srinisvasa Ramanujan (1887-1920) also independently discovered many of the classical theorems of this function. Much of the theory of hypergeometric series follows from the work done by Gauss, who presented some of his early results in his 1812 publication '*Disquisitiones generales circa seriem infinitam*' [41]. In this chapter we consider in particular the Gauss hypergeometric series which contains three parameters, as well as its associated hypergeometric equation and hypergeometric function. In the following chapter we will then extend this definition to generalised hypergeometric functions which can contain any number of parameters.

After defining the Gauss function in Section 3.2, we will state and prove some of its fundamental properties in Section 3.3, including its role as the solution to the hypergeometric differential equation. In Section 3.4 we present some of its real and complex integral representations, using methods which reveal the close relation between the Gauss function and the gamma and beta functions. We will then build on these established relations to prove some fundamental summation theorems for the Gauss function in Section 3.5, and some well-known linear and quadratic transformations in Section 3.6. These summation and transformation identities are central to classical developments as well as to more recent work in this field.

3.2 Defining the Gauss hypergeometic function

The Gauss hypergeometric series contains three parameters, and can be written in terms of the Pochhammer symbol $(a)_k$ as defined in Chapter 1.

Definition 3.2.1 *For* $z \in \mathbb{C}$, $|z| < 1$, *and where a, b and c are real or complex parameters with* $c \neq 0, -1, -2, \ldots$ *,the Gauss hypergeometric series is the infinite series denoted by* $F(a, b; c; z)$, *and given by*

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\,(c)_k} z^k = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{k!\,(c)_k} z^k. \tag{3.1}$$

When $a = c$ and $b = 1$ this series becomes the elementary geometric series, which explains the terminology, but it is not defined when the denominator parameter is a non-positive integer, as this results in division by zero (by property (1.6) of the rising factorial). The Gauss series is clearly symmetrical with respect to its numerator parameters, so that $F(a,b;c;z) = F(b,a;c;z)$, and reduces to unity if one or more of the numerator parameters is zero. If one or more of the numerator parameters is a negative integer $-n, n \in \mathbb{N}$, the series reduces to a hypergeometric polynomial which terminates at the $(n+1)$-th term. In such a case, by property (1.6), we have

$$F(-n,b;c;z) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{(b)_k}{k!\,(c)_k} z^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)_k}{(c)_k} z^k,$$

which is a polynomial of degree *n* in *z,* so that convergence is not an issue. In [16], p.56, Chaundy refers to $(-n)$ as the 'parameter of closure' of the series. Binomial sums are thus usually a form of terminating hypergeometric series.

While a hypergeometric series is not defined if the denominator parameter alone is zero or a negative integer, the series may be defined if a numerator parameter is also a non-positive integer. For example, in $F(a,b;c;z)$ let $a = -m$ and $c = -m-n$, $n, m \in \mathbb{N}_0$. If $n = 0$, then $a = c$ and the series reduces to $\sum_{k=0}^{\infty} \frac{(b)_k}{k!} z^k$, which is defined. If $n > 0$, then $F(-m,b;-m-n;z)$ reduces to the terminating hypergeometric polynomial $\sum_{k=0}^{m} \binom{m}{k} \frac{(b)_k (m+n-k)!}{(m+n)!} z^k$, which is also defined.

It is also interesting to note that a Gauss series can have terms beyond a zero term. For example, if $m, n \in \mathbb{N}$, $k \le m$, we can write $\frac{(-m)_k}{(-m-n)_k}$ in the form $\frac{m!}{(m+n)!}(m+n-k)(m+n-k-1)\dots(m-k+1)$, and it thus follows that $F(-m,b;-m-n;z) = \sum_{k=0}^{\infty} \left(1-\frac{k}{m+n}\right)\left(1-\frac{k}{m+n-1}\right)\dots\left(1-\frac{k}{m+1}\right)\frac{(b)_k z^k}{k!}$ for $b \neq 0,-1,-2,\dots$. As a result, although the series terminates at the m-th power, it starts up again at the $(m+n+$

1)-th power.[18] As an illustration, $F(-2,1;-5;z) = 1 + \frac{2}{5}z + \frac{1}{10}z^2 - \frac{1}{10}z^6 - \frac{2}{5}z^7 - z^8 + \cdots$.

For values of the parameters for which the Gauss series is well defined and non-terminating, we must consider conditions of convergence, which involve a unit radius of convergence (cf. [75], p.46). When using *z* as a formal symbol we need not consider convergence of the hypergeometric series, as any identities derived from processes such as multiplication, differentiation, composition and so on will still be formally true. However, when z is replaced by a particular value it is essential that the infinite sum is well defined.

Theorem 3.2.2 *The Gauss hypergeometric series $F(a,b;c;z)$ is absolutely convergent when $|z| < 1$, divergent when $|z| > 1$, and converges on the boundary $|z| = 1$ for $Re(c-a-b) > 0$.*

Proof: Consider $F(a,b;c;z) = \sum_{k=0}^{\infty} t_k$. As $\frac{(\alpha)_{k+1}}{(\alpha)_k} = \alpha + k$, by definition of the Gauss function we have $\left|\frac{t_{k+1}}{t_k}\right| = \frac{|z|(1+|a|/k)(1+|b|/k)}{(1+|c|/k)(1+1/k)}$, so that the ratio test establishes absolute convergence for $|z| < 1$ and divergence for $|z| > 1$. To investigate the case when $|z| = 1$, we let $\delta = \frac{1}{2}Re(c-a-b) > 0$, and compare the terms of the series $1 + \sum_{k=1}^{\infty}\left|\frac{(a)_k(b)_k z^k}{(c)_k k!}\right|$ with those of the convergent series $\sum_{k=1}^{\infty}\frac{1}{k^{1+\delta}}$. If $|z| = 1$, then by introducing new factors we have

$$\lim_{k\to\infty}\left|\frac{k^{1+\delta}(a)_k(b)_k}{(c)_k k!}\right| = \lim_{k\to\infty}\left|\frac{(a)_k}{(k-1)!\,k^a}.\frac{(b)_k}{(k-1)!\,k^b}.\frac{(k-1)!\,k^c}{(c)_k}.\frac{(k-1)!\,k^{1+\delta}}{k!\,k^{c-a-b}}\right|$$

$$= \left|\frac{1}{\Gamma(a)}.\frac{1}{\Gamma(b)}.\Gamma(c)\right| \lim_{k\to\infty}\left|\frac{1}{k^{c-a-b-\delta}}\right|,$$

[18] Sneddon [83], p.20 incorrectly gives these as the m-th *term* and the $(m+n+1)$-th *term*.

from (2.15). As $Re(c-a-b-\delta) = 2\delta - \delta > 0$, this last limit is zero, and hence by the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{k^{1+\delta}}$, the hypergeometric series $F(a,b;c;z)$ is convergent on the boundary $|z| = 1$ when $Re(c-a-b) > 0$. ■

Slater [82] provides the following additional convergence criteria. For $|z| = 1,\ z \neq 1$, the Gauss series is divergent when $Re(c-a-b) < -1$ and convergent (but not absolutely) when $-1 < Re(c-a-b) \leq 0$. Also, if $Re(c-a-b = -1)$ then the series converges if $Re(a+b) > Re(ab)$ otherwise it diverges. For example, $1 - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \cdots = \frac{1}{2}\{1 + F(2,2;3;-1)\}$ is divergent.

The sum of the convergent Gauss series (3.1) is called the hypergeometric function.

Definition 3.2.3 *For* $c \neq 0, -1, -2, \ldots$, *the sum of the Gauss hypergeometric series is termed the Gauss hypergeometric function, defined by* $F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$ *for* $|z| < 1$, *and by continuation elsewhere.*[19]

The Gauss hypergeometric function is also denoted by alternative notations such as ${}_2F_1(z)$ or $F\left(\begin{matrix} a, b; \\ c; \end{matrix}\ z\right)$, and is regarded as a function of four complex variables, rather than as a function in only *z*. The power of the hypergeometric function lies in its ability to represent many standard functions. Some examples are provided below and many more can be found in standard texts such as [4], [60] and [82].

$$\log(1-z) = -\sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = -z \sum_{k=0}^{\infty} \frac{(1)_k (1)_k z^k}{(2)_k k!} = -zF(1,1;2;z),\ |\arg(1-z)| < \pi$$

$$(1-z)^{-a} = F(a,b;b;z),\ |z| < 1$$

$$\cos z = F\left(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \sin^2 zx\right)$$

$$\arcsin z = zF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right),\ |\arg(1 \pm zi)| < \pi$$

[19] We provide details of this analytic continuation in Section 3.4.

$$\arctan z = zF\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right), |\arg(1 \pm zi)| < \pi$$

$$\cosh z = \lim_{a,b\to\infty} F\left(a, b; \frac{1}{2}; \frac{x^2}{4ab}\right)$$

Complete elliptic integrals are defined by $K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, and through binomial expansion and integrating term-by-term it can be shown that $K(k) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ (cf. [4], p.132). Orthogonal polynomials also have representations in terms of the hypergeometric function. One example is the Legendre polynomial, defined as the coefficient of z^n in the ascending power expansion of $(1 - 2xz + z^2)^{-1/2}$; by direct expansion it can be shown that the coefficient is given by $P_n(x) = F\left(-n, 1+n; 1; \frac{1}{2} - \frac{x}{2}\right)$ (cf. [82], p.17).

While in a geometric series $\sum ar^k$ the ratio of any two consecutive terms is constant, in a hypergeometric series $F(a, b; c; z) = \sum_{k=0}^{\infty} t_k$, the first term is unity and the term ratio t_{k+1}/t_k is a rational function of the summation index k. Hence, any term t_k for which t_{k+1}/t_k is a rational function of k is called a hypergeometric term. The term ratio for the hypergeometric function $F(a, b; c; z)$ can be simplified to

$$\frac{t_{k+1}}{t_k} = \frac{(a)_{k+1}(b)_{k+1}z}{(c)_{k+1}(k+1)!}\frac{(c)_k k!}{(a)_k(b)_k} = \frac{(k+a)(k+b)z}{(k+c)(k+1)}, \tag{3.2}$$

which also provides the recurrence relation

$$t_k = \frac{(k-1+a)(k-1+b)z}{k(k-1+c)} t_{k-1}\,. \tag{3.3}$$

If a term ratio t_{k+1}/t_k cannot be written in the form $P(k)/Q(k)$ where P and Q are polynomials in k (and z is a constant), then the given function is not hypergeometric. This enables us to establish when a given series consists of hypergeometric terms, to identify the particular hypergeometric function, and hence to use known results about that function.

Example 3.2.4 Consider that the term ratio of a given series is known to be $\frac{t_{k+1}}{t_k} = \frac{(k+2)(k-1)z}{(k+3)(k+1)}$, where z is a constant. If we normalise to ensure $t_0 = 1$, then from (3.2) it follows that t_k is a hypergeometric term in the Gauss series $F(2, -1; 3; z)$.

Example 3.2.5 Consider the series $\sum_{k=-\infty}^{\infty} t_k$, where $t_k = \left\{\frac{1}{2^{n+1}}\binom{n+1}{k} - \frac{1}{2^n}\binom{n}{k}\right\}, n \in \mathbb{N}_0$. In this case t_k is a sum of ratios, but the preceding method still applies. The term ratio is $\frac{t_{k+1}}{t_k} = -\frac{(k-n-1)(k-n/2+1/2)}{(k-n/2-1/2)(k+1)}$. As $t_0 = -\frac{1}{2^{n+1}}$ and $t_k = 0$ for $k = -1, -2, \ldots$, we have $\sum_{k=-\infty}^{\infty} t_k = \frac{-1}{2^{n+1}} F\left(-n-1, -\frac{n}{2}+\frac{1}{2}; -\frac{n}{2}-\frac{1}{2}; -1\right)$, when n is an even natural number (to avoid negative integers in the lower parameter).

Petkovšek et al. [73], Section 3.4, provide details of how to use Mathematica and Maple programs to convert from the k-th term of a series to a description of the corresponding hypergeometric function. In general, it follows that a summand which comprises ratios of rational functions, binomial coefficients, gamma functions, factorials, powers and so on, is extremely likely to be a hypergeometric term.

3.3 Some fundamental properties of the Gauss function

Many classical texts have been written on the vast number of properties and relations of the Gauss hypergeometric function (cf. [8], [82], [74]). While it is not in the scope of this work to attempt a summary of all existing properties, in this section we provide those which are central to later sections of this work.

3.3.1 Analyticity

The theorem below shows that the Gauss function is analytic except at a countable number of poles, which allows us to apply properties of well-behaved functions in later proofs. The proof below follows Rainville (cf. [75], p.56, Theorem 19).

Theorem 3.3.1 *For $|z| < 1$, the function F(a,b;c;z) is analytic in a, b and c for all finite a, b and c except for simple poles at* $c = 0, -1, -2, \dots$.

Proof: Consider the entire function $\frac{1}{\Gamma(c)} F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{\Gamma(c+k) k!}$, in which there is no longer a possibility of division by zero. As in the proof for Theorem 3.2.2, we use the identity $\Gamma(a) = \frac{(k-1)! k^a}{(a)_k}$ to obtain

$$\lim_{k\to\infty} \left| \frac{(a)_k (b)_k z^{k/2}}{\Gamma(c+k) k!} \right| = \lim_{k\to\infty} \left| \frac{(a)_k}{(k-1)!\, k^a} \frac{(b)_k}{(k-1)!\, k^b} \frac{(k-1)!\, k^c}{\Gamma(c+k)} \frac{z^{k/2}}{k^{1-c-a-b}} \right|$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \lim_{k\to\infty} \left| \frac{z^{k/2}}{k^{1-c-a-b}} \right| = 0, \text{for } |z| < 1.$$

Hence, for any fixed *z* in the unit circle there exists a constant *K* independent of a, b, c such that $\left| \frac{(a)_k (b)_k z^k}{\Gamma(c+k) k!} \right| < K |z|^{k/2}$, and thus by the comparison test the series $\frac{1}{\Gamma(c)} F(a, b; c; z)$ is absolutely and uniformly convergent. The complete result follows from the location of the poles of $\Gamma(c)$. ■

3.3.2 Contiguous function relations

Two Gauss hypergeometric functions are said to be contiguous if one pair of corresponding parameters differs by unity (cf. [75], p.50). These contiguous relations have many applications, including establishing three-term recurrence relations for hypergeometric orthogonal polynomials

Definition 3.3.2 *The hypergeometric function contiguous to* $F(a, b; c; z)$, *are* $F(a \pm 1) = F(a \pm 1, b; c; z)$, $F(b \pm 1) = F(a, b \pm 1; c; z)$, $F(c \pm 1) = F(a, b; c \pm 1; z)$.

Gauss showed that the hypergeometric function $F(a, b; c; z)$ can be written as a linear combination of any two of its contiguous functions, with rational coefficients given in terms of *a*, *b*, *c* and *z*. A total of $\binom{6}{2} = 15$ such recurrence relations can be established and are listed in [82], pp.13-14, although only four are independent, with the others being obtained through combinations and symmetry. These recurrence relations are useful for extending numerical tables of the Gauss function, since for one fixed value of z it is necessary only to calculate the values of the function over two units in a, b and c, and then apply some recurrence relations in order to find the function values over a larger range of the parameters in the particular z-plane. Contiguous recurrence relations can be established from first principles, by using power series expansions and factorial identities as shown in the proof below (cf. [9]).

Theorem 3.3.3 *For* $c \neq 0, -1, -2, \dots,$

$$cF(b-1) + (a-b)zF(c+1) = cF(a-1)\,.$$

Proof: By definition, the coefficient of z^n on the left side of the theorem is

$$c\frac{(a)_n(b-1)_n}{(c)_n n!} + (a-b)\frac{(a)_{n-1}(b)_{n-1}}{(c+1)_{n-1}(n-1)!}$$

$$= \frac{(a)_{n-1}(b)_{n-1}}{(c+1)_{n-1}(n-1)!}\left[\frac{(b-1)(a+n-1)}{n} + (a-b)\right]$$

$$= \frac{c(a)_{n-1}(b)_{n-1}}{(c)_n n!}(a-1)(b+n-1) = c\frac{(a-1)_n(b)_n}{(c)_n n!}.$$

This is also the coefficient of z^n on the right side, which proves the theorem. ■

Rainville [75] provides an alternative method for proving recurrence relations. By applying standard simplifications to $F(a,b;c;z) = \sum_{n=0}^{\infty} t_n$, he first shows that $F(a+1) = \sum_{n=0}^{\infty} \frac{(a+1)_n\, t_n}{(a)_n} = \sum_{n=0}^{\infty} \frac{(a+n)\, t_n}{a}$, $F(b+1) = \sum_{n=0}^{\infty} \frac{(b+n)\, t_n}{b}$ and $F(c+1) = \sum_{n=0}^{\infty} \frac{c t_n}{c+n}$. It also follows that $F(a-1) = \sum_{n=0}^{\infty} \frac{(a-1)\, t_n}{a-1+n}$, $F(b-1) = \sum_{n=0}^{\infty} \frac{(b-1)\, t_n}{b-1+n}$ and $F(c-1) = \sum_{n=0}^{\infty} \frac{(c-1+n)\, t_n}{c-1}$. By defining the differential operator ϑ as

$$\vartheta = z\left(\frac{d}{dz}\right), \tag{3.4}$$

so that $\vartheta z^n = nz^n$, it is then a simple matter to establish that

$$(\vartheta + a)F(a, b; c; z) = \sum_{n-0}^{\infty} \frac{(a+n)(a)_n(b)_n z^n}{(c)_n n!}. \tag{3.5}$$

From these results we thus have that for $F \equiv F(a, b; c; z)$, $(\vartheta + a)F = aF(a + 1)$, $(\vartheta + b)F = bF(b + 1)$ and $(\vartheta + c - 1)F = (c - 1)F(c - 1)$. These relations form the basis of the four contiguous relations given below, the first two following directly from addition of the above results and the others by simple manipulation.

Theorem 3.3.4 *For* $F \equiv F(a, b; c; z)$,

$$(a - b)F = aF(a + 1) - bF(b + 1),$$

$$(a - c + 1)F = aF(a + 1) - (c - 1)F(c - 1),$$

$$c[a + (b - c)z]F = ac(1 - z)F(a + 1) - (c - a)(c - b)zF(c + 1),$$

$$c(1 - z)F = cF(a - 1) - (c - b)zF(c + 1).$$

In general, the functions $F(a, b; c; z)$ and $F(a + m, b + n; c + l; z)$, $m, n, l \in \mathbb{Z}$ are called associated series. By repeated application of the recurrence relations for contiguous functions, we can express any associated series as a linear combination of $F(a, b; c; z)$ and one of its contiguous functions, with coefficients which are rational functions of a, b, c, z (assuming as always that $c + l$ is not a non-positive integer). Many such linear relations can be established through simplification techniques such as those used to prove the following recurrence relation, found in [82], p.14.

Theorem 3.3.5 For $c \neq 0, -1, -2, \ldots$,

$$(c - a)(c - b)F(a, b; c + 1; z)$$
$$= c(c - a - b)F(a, b; c; z) + ab(1 - z)F(a + 1, b + 1; c + 1; z).$$

Proof: By definition (3.1) and the identity $\alpha(\alpha+1)_n = (\alpha+n)(\alpha)_n$, the coefficient of z^n on the right side of the theorem is

$$c(c-a-b)\frac{(a)_n(b)_n}{(c)_n n!} + ab\frac{(a+1)_n(b+1)_n}{(c+1)_n n!} - ab\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(n-1)!}$$

$$= \frac{(a)_n(b)_n}{(c+1)_n n!}\{(c-a-b)(c+n) + (a+n)(b+n) - n(c+n)\}$$

$$= \frac{(a)_n(b)_n}{(c+1)_n n!}(c-a)(c-b),$$

which is also the coefficient of z^n on the left side of the theorem. ■

It can similarly be shown that $F(a,b+1;c;z) = F(a,b;c;z) + \frac{az}{c}F(a+1,b+1;c+1;z)$ by using the fact that $\frac{(a)_n(b+1)_n}{(c)_n n!} - \frac{(a)_n(b)_n}{(c)_n n!} = \frac{a(a+1)_{n-1}(b+1)_{n-1}}{c(c+1)_{n-1}(n-1)!}$. There exist many such useful relations involving associated series.

3.3.3 Differential properties of the Gauss function

Using the identity $(a)_{n+1} = a(a+1)_n$, we can establish the following useful derivative formula for the Gauss hypergeometric function, involving associated series.

Theorem 3.3.6 *For* $c \neq 0, -1, -2, \dots,$

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z).$$

Repeated application of this result yields the generalised formula below (cf. [60], p.241, Eq. (9.2.3)).

$$\frac{d^m}{dz^m}F(a,b;c;z) = \frac{(a)_m(b)_m}{(c)_m}F(a+m,b+m;c+m;z), m \in \mathbb{N}.$$

Nine further differential relations can be found in [82], p.16.

3.3.4 The hypergeometric differential equation

A central property of the hypergeometric series is its role as a solution to the hypergeometric differential equation. Consider a homogeneous, linear second-order differential equation of the form

$$u''(z) + P(z)u'(z) + Q(z)u(z) = 0\,.$$

A point z_0 for which P and Q are analytic is called a regular point of the differential equation. When z_0 is not regular, it is called a singular point. If z_0 is a singular point but both $(z - z_0)P(z)$ and $(z - z_0)^2Q(z)$ are analytic, then z_0 is called a regular singular point (cf. [4], p.639). For rational functions $P(z)$ and $Q(z)$, Slater [82] has shown that through a change of variable, the three regular singular points of a homogenous linear second-order differential equation can be transformed to the points 0, 1 and ∞, and the differential equation takes on the form

$$z(1 - z)u''(z) + [c - (a + b + 1)z]u'(z) - abu = 0\,, \tag{3.6}$$

which is known as the Gauss hypergeometric equation. This equation was first established by Euler in 1769, was extensively studied by Gauss and Kummer, and given a more abstract treatment by Bernhard Riemannn (1826-1866). Dividing equation (3.6) by $z(1 - z)$ reveals regular singular points at 0 and 1, and also at infinity (by replacing z with $1/z$).

From the theory of linear differential equations, the hypergeometric equation has a particular series solution of the form $u = z^s \sum_{k=0}^{\infty} c_k z^k$, where $c_0 \neq 0$, s is suitably chosen, and the power series converges for $|z| < 1$. We can use direct substitution and brute force to show that the hypergeometric series (3.1) satisfies equation (3.6), (cf. [60], p.162), or we can use series methods to derive the Gauss series as a solution to the hypergeometric equation (cf. [78], p.26, [83], p.26]). However, it is more elegant to prove this by using properties of the rising factorial together with the differential operator $\vartheta = z\left(\frac{d}{dz}\right)$, as shown below (cf. [75], p.53, [82], p.6, [84], p.112).

Theorem 3.3.7 *For* $|z| < 1$, *the series* $F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$ *is a particular solution of the hypergeometric equation* $z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0$.

Proof: Let $u = F(a,b;c;z)$ and $\vartheta = z\left(\frac{d}{dz}\right)$. By property (3.5), we can write

$$(\vartheta + c - 1)u = \sum_{n=0}^{\infty} \frac{(c-1+n)(a)_n (b)_n z^n}{(c)_n n!},$$

and hence

$$\begin{aligned}
\vartheta(\vartheta + c - 1)u &= z\sum_{n=0}^{\infty} \frac{n(c-1+n)(a)_n (b)_n z^{n-1}}{(c)_n n!} \\
&= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_{n-1}(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1} z^{n+1}}{(c)_n n!} \\
&= z\sum_{n=0}^{\infty} \frac{(a+n)(b+n)(a)_n (b)_n z^n}{(c)_n n!} \\
&= z(\vartheta + a)(\vartheta + b)u\,.
\end{aligned}$$

Hence, $u = F(a,b;c;z)$ is a solution of the differential equation

$$\vartheta(\vartheta + c - 1)u = z(\vartheta + a)(\vartheta + b)u, \tag{3.7}$$

or equivalently, $z\frac{d}{dz}\left(z\frac{d}{dz} + c - 1\right)u = z\left(z\frac{d}{dz} + a\right)\left(z\frac{d}{dz} + b\right)u$. This equation can again be rewritten in the form $z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0$, and the theorem is proved. ■

As the hypergeometic equation (3.6) is of the second order, we require a second linearly independent solution. Following Temme [84], pp.112-113 we consider a function of the form $z^{1-c}G$, where G is again a hypergeometric function and $\vartheta =$

$z\left(\frac{d}{dz}\right)$. Substituting $u = z^{1-c}G$ into equation (3.7) and applying (3.5) gives on the left side the result

$$\vartheta(\vartheta + c - 1)z^{1-c}G = \vartheta \sum_{n=0}^{\infty} \frac{(n + c - 1)(a)_n(b)_n z^{n+1-c}}{(c)_n n!} = z^{1-c}\vartheta(\vartheta + 1 - c)G\,,$$

while the right side becomes

$$z(\vartheta + a)(\vartheta + b)z^{1-c}G = z.z^{1-c}(\vartheta + a - c + 1)(\vartheta + b - c + 1)G\,,$$

from which we can conclude that

$$\vartheta(\vartheta + 1 - c)G = z(\vartheta + a - c + 1)(\vartheta + b - c + 1)G\,.$$

This result is a reparametrisation of the hypergeometric differential equation (3.7), with solution $G = F(a - c + 1, b - c + 1; 2 - c; z)$. Hence, a second solution to (3.6) is provided by $z^{1-c}F(a - c + 1, b - c + 1; 2 - c; z)$. As the two solutions are linearly independent, one complete solution to the hypergeometric equation is

$$u(z) = AF(a, b; c; z) + Bz^{1-c}F(a - c + 1, b1 - c + 1; 2 - c; z),$$

where $|z| < 1, |\arg z| < \pi$, A and B are arbitrary constants determined by boundary conditions, and c is not an integer. When $c = 1$ we do not obtain a new solution to (3.6), and in general if c is an integer the two solutions $F(a, b; c; z)$ and $z^{1-c}F(a - c + 1, b - c + 1; 2 - c; z)$ might not be linearly independent, and one solution may become logarithmic (cf. [78], p.29).

Slater [82] lists the 24 solutions of the hypergeometric equation (3.6) due to Kummer. These are of the form $z^{\rho}(1 - z)^{\sigma}F(a', b'; c'; z')$, where ρ, σ, a', b', c' are linear functions of a, b, c, and z and z' are connected by a linear fractional transformation of the form $\frac{a+bz}{c+dz}$. As the Gauss equation can only have two linearly independent solutions in any one domain, any three of these solutions can be connected by a linear relation with constant coefficients. Below are six of Kummer's solutions around the singular points 0, 1 and infinity, as listed in [3], p.563.

$$u_1(x) = F(a, b; c; z)$$

$$u_2(x) = F(a, b; a + b + 1 - c; 1 - z)$$

$$u_3(x) = z^{-a} F(a, a + 1 - c; a - b + 1; z^{-1})$$

$$u_4(x) = z^{-b} F(b + 1 - c, b; b + 1 - a; z^{-1})$$

$$u_5(x) = z^{1-c} F(b + 1 - c, a + 1 - c; 2 - c; z)$$

$$u_6(x) = (1 - z)^{c-a-b} F(c - a, c - b; c + 1 - a - b; 1 - z)$$

The transformation identities between these solutions are discussed in Section 3.6.

3.4 Integral representations of the Gauss function

Integral representations of the Gauss hypergeometric function provide a powerful tool for developing transformation relations and other hypergeometric identities. These can often provide a more useful approach than series expressions which only hold for $|z| < 1$. They also illustrate the close relationship that exists between the hypergeometric and gamma functions. In 1748, Euler developed his famous integral representation for the Gauss hypergeometric function (cf. [84], p.110), sometimes referred to as Pochhammer's integral. An interesting observation is that while the hypergeometric function is clearly symmetrical with respect to its numerator parameters, this is not immediately obvious in Euler's integral representation.

Theorem 3.4.1 *Euler's integral*: *For* $|z| < 1$ *and* $Re(c) > Re(b) > 0$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt \,, \tag{3.8}$$

in the z plane cut along the real axis from 1 *to* ∞, *where it is understood that* $\arg t = \arg(1-t) = 0$, *and* $(1-zt)^{-a}$ *has its principal value.*[20]

Proof: We first write

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)\Gamma(c)}{\Gamma(c+n)\Gamma(b)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} . \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} .$$

From the beta-gamma relation (2.29), if $Re(c-b) > 0$ we also have

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \beta(b+n, c-b) = \int_0^1 t^{b+n-1}(1-t)^{c-b-1} dt .$$

By substituting these results into the definition of the Gauss hypergeometric function, and then interchanging the order of summation and integration, we obtain

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1} z^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n (zt)^n \, dt}{n!} .$$

Finally we have $(1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n (zt)^n}{n!}$, which proves the theorem for $Re(c) > Re(b) > 0$ and $|z| < 1$. ■

Euler's integral shows that the hypergeometric function has a branch point at $z = 1$ and at $z = \infty$. Consider $t = \frac{1}{z}$ as a singularity of $(1-zt)^{-a}$ (a not a negative integer). If z varies continually along a closed path around the point $z = 1$, the path of integration must be deformed to ensure that it does not cross the point $\frac{1}{z}$. However, when z returns to its initial position, the integral does not necessarily take its initial value, and thus $z = 1$ is a branch point of $F(a,b;c;z)$. Similarly, $z = \infty$ is a branch point, as the varying of z around the point at infinity is equivalent to that of $\frac{1}{z}$ around

[20] These restrictions ensure that t and $(1-t)$ are pure real, and $(1-zt)^{-a} \to 1$ as $t \to 0$ so that the integrand is a single-valued function of z.

the point $z = 0$. Thus, Euler's integral is a single-valued branch of $F(a, b; c; z)$ which takes on the value 1 when $z = 0$. Since Euler's integral is analytic in the z-plane cut along $[1, \infty)$, it provides the analytic continuation of $F(a, b; c; z)$ to the domain $|\arg(1 - z)| < \pi$, when $Re(c) > Re(b) > 0$. Thus, under these given conditions, $F(a, b; c; z)$ is a single-valued analytic function in the z-plane with a branch cut $[1, \infty)$ along the real axis, and hence by the principle of analytic continuation, formulae proven for the hypergeometric series under the restriction $|z| < 1$ also apply to the whole domain of definition.

Slater provides eight further integral representations for the Gauss function, through suitable substitutions into Euler's integral (3.8), (cf. [82], p.20). For example, for $Re(c) > Re(b) > 0, |\arg z| < \pi$, substituting $s = \frac{t}{1-t}$ yields

$$F(a, b; c; 1 - z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^\infty s^{b-1}(1 + s)^{a-c}\,(1 + sz)^{-a}\, ds\,.$$

For $Re(c) > Re(b) > 0$, substituting e^{-t} for t yields

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^\infty e^{-bt}(1 - e^{-t})^{c-b-1}\,(1 - ze^{-t})^{-a}\, dt\,.$$

For $1 + Re(a) > Re(c) > Re(b), |\arg(z - 1)| < \pi$, substituting $\frac{1}{s}$ for t yields

$$F(a, b; c; 1/z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_1^\infty (s - 1)^{c-b-1}\, s^{a-c}(s - 1/z)^{-a}\, ds\,,$$

and for $Re(c) > Re(b) > 0$, substituting $\sin^2 t$ for t yields

$$F(a, b, c; z) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^{\pi/2} \frac{(\sin t)^{2b-1}\,(\cos t)^{2c-2b-1}}{(1 - z\sin^2 t)^a}\, dt\,.$$

H. Bateman (1882-1944) provided the following extension of Euler's integral to a larger domain, given as Theorem 2.2.4 in [4], p.68.

Theorem 3.4.2 *For $Re(c) > Re(d) > 0$, $x \neq 1$, and* $|\arg(1-x) < \pi|$,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 t^{d-1}(1-t)^{c-d-1} F(a,b;d;zt)dt \,. \tag{3.9}$$

Erdéyli [25] provides three further extensions of the integral representation.

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 t^{d-1}(1-t)^{c-d-1}(1-tz)^{c-a-b}$$

$$\times F\left({\lambda-a, \lambda-b; \atop d;} tz\right) F\left({a+b-\lambda, \lambda-d; \atop \gamma-d;} \frac{(1-t)z}{1-tz}\right) dt, Re(c) > Re(d) > 0 \,, \tag{3.10}$$

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 t^{d-1}(1-t)^{c-d-1}(1-tz)^{-a'}$$

$$\times F\left({a-a', b; \atop d;} tz\right) F\left({a', b-d; \atop \gamma-d;} \frac{(1-t)z}{1-tz}\right) dt, Re(c) > Re(d) > 0 \,, \tag{3.11}$$

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\gamma+d-\lambda-\nu)} \int_0^1 t^{\nu-1}(1-t)^{\gamma+d-\lambda-\nu-1}$$

$$\times F\left({d-\lambda, \gamma-\lambda; \atop \gamma+d-\lambda-\nu;} 1-t\right) {}_3F_2\left({a,b,d; \atop \lambda,\nu;} tz\right) dt, Re(\lambda,\nu,\gamma+d-\lambda-\nu) > 0 \,. \tag{3.12}$$

Remark 3.4.3

- When $d = b$, Bateman's extension (3.9) becomes Euler's integral (3.8).
- The substitutions $\lambda = a + b$, $a' = 0$, and $\lambda = d$ into the first and second Erdéyli integrals (3.10) and (3.11) respectively, yield Bateman's extension (3.9).
- Erdéyli used fractional integration by parts to derive the integral extensions (3.10)-(3.12), but more recently Joshi and Vyas [49] used series manipulations and classical summation theorems to prove and generalise these results, as we discuss in Section 5.2.

Slater [82] showed that convergence conditions for the Euler integral can be relaxed with more complicated contours. She provided three examples, one of which is given below.

Theorem 3.4.4 *For* $c-b \neq 1,2,3,\ldots, |\arg(1-z)| < \pi, Re(b) > 0,$

$$F(a,b;c;z) = \frac{i\Gamma(c)e^{i\pi(b-c)}}{2\Gamma(b)\Gamma(c-b)\sin\{\pi(c-b)\}}\int_0^{(1+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,.$$

Proof: Consider the integral $I = \int_0^{(1+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$, where the contour starts at the origin, encircles the point +1 once in an anticlockwise direction and returns to the origin. We deform this contour into the real axis $(0, 1-\varepsilon)$ with $\arg t = 0$, the circle C centred at +1 with radius ε, and again the real axis $(1-\varepsilon, 0)$ with $\arg t = 2\pi$. Then

$$I = [1 - e^{2\pi i(c-b)}]\int_0^{1-\varepsilon} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

$$+\int_C t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,.$$

As $\varepsilon \to 0$, we have $\int_{1-\varepsilon}^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \to 0$ and $\int_C t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \to 0$, so that

$$I = \left[1 - e^{2\pi i(c-b)}\right]\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,.$$

The desired result then follows by substituting this result into Euler's integral (3.8) and using the relation $\sin A = \frac{e^{Ai}-e^{-Ai}}{2i}$. ■

In a sequence of papers published during the period 1904-1910, Barnes developed alternative methods of treating the Gauss hypergeometric function based on contour integration and the theory of residues. While Euler-type integral representations of the hypergeometric series are useful in numerical work, contour integrals of the

Barnes (or Mellin-Barnes) type are extremely useful in deriving transformation relations between hypergeometric functions and for studying their asymptotics.

The Mellin transform of a function $F(x)$ is defined by the integral $\int_0^\infty x^{s-1}F(x)\,dx$. For a wide class of functions, if $f(x) = \int_0^\infty x^{s-1}F(x)dx$, then its inversion is given by $F(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}f(s)ds$ (cf. [4], p.85). In order to develop a complex integral representation for the hypergeometric function, Barnes considered its Mellin transform and inverse, integrating at $-x$ to avoid the branch point at $x = 1$. To illustrate this approach, Andrews et al. [4] use Euler's integral (3.8) to obtain

$$\begin{aligned}
I &= \int_0^\infty x^{s-1}F(a,b;c;-x)dx \\
&= \int_0^\infty x^{s-1}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+tx)^{-a}dt\,dx \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\int_0^\infty \frac{x^{s-1}}{(1+tx)^a}dxdt\,.
\end{aligned}$$

By (2.28), $\beta(s,a-s) = \int_0^\infty x^{s-1}(1+x)^{-a}dx$, and hence from the substitution $xt \to x$ the above result becomes

$$\begin{aligned}
I &= \frac{\Gamma(c)\Gamma(s)\Gamma(a-s)}{\Gamma(b)\Gamma(c-b)\Gamma(a)}\int_0^1 t^{b-s-1}(1-t)^{c-b-1}dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}.
\end{aligned}$$

By inversion, we thus expect that

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}(-x)^{-s}ds\,,$$

where $\min\big(Re(a), Re(b)\big) > k > 0,$ and $c \neq 0, -1, -2, \dots$. This is Barnes' contour integral representation (cf. [4], p.86, Th. 2.4.1), and it provides the basis for an alternative development of the theory of hypergeometric functions. In general, a

Barnes-type integral contains gamma functions in the integrand. This type of integral is evaluated by using the sum of the residues at the sequences of poles within the Barnes integration contour, which starts at $-i\infty$ and runs to $+i\infty$ in the s-plane, curving where necessary so that for $m = 1, 2, \ldots., p$, the poles of $\Gamma(a_m + s)$ lie to the left of the path and the poles of $\Gamma(-s)$ lie to the right of the path. This is shown in Figure 3.1 (cf. [75], p.95, Figure 5). The existence of such a path requires that no a_m is zero or a negative integer, and we assume that $|\arg z| < \pi$.

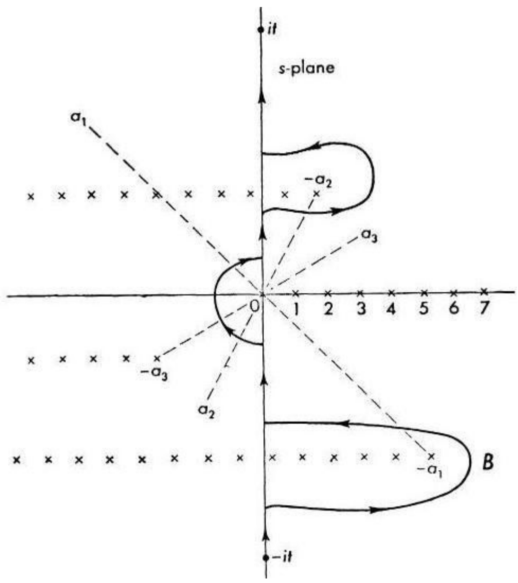

Figure 3.1 A Barnes contour of integration

In 1910, Barnes proved the integral analogue of Gauss' theorem (also known as Barnes' first lemma), stated below. A full proof is found in [4], p.89, Th. 2.4.2, using standard techniques of contour integration.

Theorem 3.4.5 *Barnes' first lemma: If the path of integration is curved to separate the poles of* $\Gamma(a+s)\Gamma(b+s)$ *from the poles of* $\Gamma(c-s)\Gamma(d-s)$*, then*

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)ds \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \end{aligned}$$

for $Re(a+b+c+d) < 1$, *and none of* $a+c, a+d, b+c, b+d$ *equal to zero or a negative integer.*

Andrews et al. [4] use this result to prove that

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z).$$

3.5 Evaluating the Gauss function for $z = \pm 1$

Over the years, many useful formulae have been developed for evaluating the hypergeometric function for particular values of *z*. Berndt [12] provides many such evaluations in his summary of Ramanujan's work. Wilfred N. Bailey (1893-1961) provided an extensive list of such summation formulae in [8], which has become the standard reference work for hypergeometric identities, and he later collaborated with Slater [82] to extend this list. Many of these formulae involve unit arguments, and express hypergeometric series as ratios of gamma functions. In this section we provide some of the most common and useful of these identities.

In 1812, Gauss established a central identity evaluating the hypergeometric series when $z = 1$. Below we follow the proof provided by Rainville [75], p. 49, Th. 18. Alternative proofs exist for Gauss' summation theorem. For example, Slater [82], p.27 lets $z \to 1^-$ in the recurrence relation $(c-a)(c-b)F(a,b;c+1;z) = c(c-a-b)F(a,b;c;z) + ab(1-z)F(a+1,b+1;c+1;z)$. Andrews et al. [4], p.66 base their proof on the relation $F(a,b;c;1) = \frac{(c-a)(c-b)}{c(c-a-b)} \times F(a,b;c+1;1)$.

Theorem 3.5.1 *Gauss' Summation Theorem*: *For* $c \neq 0, -1, -2, \ldots$, $Re(c-a-b) > 0$, *and* $Re(c) > Re(b) > 0$,

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.13}$$

Proof: Let $z \to 1^-$ in Euler's integral (3.8), and use the relation (2.29) between the gamma and beta functions to obtain

$$F(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} . \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

when $Re(c) > Re(b) > 0$ (a condition removable by analytic continuation). ■

Rainville [75], p.49 uses Gauss' summation theorem to establish the following useful result.

Theorem 3.5.2 *For* $Re(b) > 0$ *and* $n \in \mathbb{N}$,

$$F\left(\begin{matrix} -\frac{n}{2}, \frac{1}{2} - \frac{n}{2}; \\ b + \frac{1}{2}; \end{matrix} 1\right) = \frac{(b)_n 2^n}{(2b)_n}. \tag{3.14}$$

Proof: Applying Gauss' summation (3.13) to the left side of (3.14) yields

$$F\left(\begin{matrix} -\frac{n}{2}, \frac{1}{2} - \frac{n}{2}; \\ b + \frac{1}{2}; \end{matrix} 1\right) = \frac{\Gamma\left(b+\frac{1}{2}\right)\Gamma(b+n)}{\Gamma\left(b+\frac{n}{2}\right)\Gamma\left(b+\frac{1}{2}+\frac{n}{2}\right)}$$

$$= \frac{(b)_n \Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}{\Gamma\left(b+\frac{n}{2}\right)\Gamma\left(b+\frac{1}{2}+\frac{n}{2}\right)}.$$

Now from Legendre's duplication formula (2.17), we have $\Gamma(b)\Gamma\left(b+\frac{1}{2}\right) = 2^{1-2b}\sqrt{\pi}\Gamma(2b)$, and $\Gamma\left(b+\frac{n}{2}\right)\Gamma\left(b+\frac{1}{2}+\frac{n}{2}\right) = 2^{1-2b-n}\sqrt{\pi}\Gamma(2b+n)$. Hence,

$$F\left(\begin{matrix} -\frac{n}{2}, \frac{1}{2} - \frac{n}{2}; \\ b + \frac{1}{2}; \end{matrix} 1\right) = \frac{(b)_n 2^n \Gamma(2b)}{\Gamma(2b+n)} = \frac{(b)_n 2^n}{(2b)_n},$$

and the result is proved. ■

For argument $z = 1$, if a numerator parameter in the Gauss hypergeometric function is a negative integer, the resulting terminating series can be evaluated by the useful Chu-Vandermonde identity, given below. This result follows directly from Gauss' summation formula (3.13) and the identity $\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n$.

Theorem 3.5.3 *The Chu-Vandermonde identity*: *For* $n \in \mathbb{N}$,

$$F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}. \tag{3.15}$$

This result will clearly not apply if c is a negative integer $(-m)$ such that $m < n$, as then $(c)_n = 0$. For example, $F(-3, -4; -2; 1) = \frac{(-2+4)_3}{(-2)_3} = 24/\{(-2)(-1)(0)\}$, which is not defined. The Chu-Vandermonde identity is a powerful tool for establishing identities involving binomial coefficients as illustrated in the example below, found in [52], p.33.

Example 3.5.4 To prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$, we define $t_k = \binom{n}{k}^2$ so that $\frac{t_{k+1}}{t_k} = \frac{(k-n)^2}{(k+1)^2}$, and hence from (3.2) we have $\sum_{k=0}^{n} \binom{n}{k}^2 = F(-n, -n; 1; 1)$. This is a special case of (3.15) with $b = -n, c = 1$, from which we obtain $\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{(1+n)_n}{(1)_n}$ $= \frac{(2n)!}{n!n!} = \binom{2n}{n}$.

Kummer's central theorem below provides an evaluation for the Gauss function at $z = -1$, and is Bailey's Theorem 2.3 (1) in [8], p.9.[21]

Theorem 3.5.5 *Kummer's theorem*: *For* $Re(b) < 1,\ b - a \neq -1, -2, \dots,$

$$F(a, b; 1 + b - a; -1) = \frac{\Gamma(1 + b - a)\Gamma(1 + b/2)}{\Gamma(1 + b)\Gamma(1 + b/2 - a)}. \tag{3.16}$$

Proof: In Euler's integral (3.8), put $z = -1$ and $c = b - a + 1$ to obtain

[21] A typographical error in Slater [82], p.32, Equation 1.7.1.6, incorrectly includes 2^{-a} in the right side of Kummer's theorem.

$$F(a,b;1+b-a;-1)=\frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)}\int_0^1 t^{b-1}(1-t^2)^{-a}dt\,.$$

By the substitution $u=t^2$, the integral becomes $\frac{1}{2}\beta\left(\frac{b}{2},1-a\right)$, so that we obtain

$$F(a,b;1+b-a;-1)=\frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)}\frac{1/2\Gamma(b/2)\Gamma(1-a)}{\Gamma(1-a+b/2)}\,.$$

As $\frac{\Gamma(1+b)}{\Gamma(b))}=\frac{2\Gamma(1+b/2)}{\Gamma(b/2)}$, the desired result follows directly. ■

Below is an equivalent form of Kummer's theorem, found in [8], p.10.

$$F(a,b;1+b-a;-1)=\frac{\Gamma(1+b-a)\Gamma(1/2)}{2^a\Gamma(1/2+b/2)\Gamma(1+b/2-a)} \tag{3.17}$$

As an illustration of the power of Kummer's theorem, the following Ramanujan summation identities can be derived directly from (3.17). By substituting $a=b=\frac{1}{2}$, and $a=1, b=1-x$ respectively and then simplifying, we obtain

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\cdots+(-1)^n\left(\frac{\left(\frac{1}{2}\right)_n}{n!}\right)^2=\frac{\sqrt{\pi}}{\sqrt{2}\Gamma^2(3/4)}\,,$$

and for $Re(x)>0$,

$$1+\frac{x-1}{x+1}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\cdots+(-1)^n\frac{(1-x)_n}{(x+1)_n}=\frac{2^{2x-1}\Gamma^2(x+1)}{\Gamma(2x+1)}\,.$$

Petkovšek et al. [73], p.43 provide the following form of Kummer's theorem for *b* a negative integer. The result follows from (3.16) by using the reflection formula $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$ and taking the limit as *b* approaches a negative integer.

Theorem 3.5.6 *For b a negative integer,*

$$F(a,b;1+b-a;-1)=2\cos\frac{\pi b}{2}\frac{\Gamma(|b|)\Gamma(b-a+1)}{\Gamma\left(\frac{|b|}{2}\right)\Gamma(1+b/2-a)}\,. \tag{3.18}$$

Petkovšek et al. also provide the following instructive example of how to use the above form of Kummer's theorem to evaluate a given series.

Example 3.5.7 Consider the series $f(n) = \sum_{k=0}^{\infty}(-1)^k \binom{2n}{k}^2$ for a non-negative integer n. The term ratio is

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1}\binom{2n}{k+1}^2}{(-1)^k\binom{2n}{k}^2} = \frac{(k-2n)^2(-1)}{(k+1)^2}.$$

The series is thus $f(n) = F(-2n, -2n; 1; -1)$, which by application of Kummer's identity (3.18) for the negative integer $(-2n)$ becomes the pleasing closed form $f(n) = \frac{2(-1)^n\Gamma(2n)\Gamma(1)}{\Gamma(n)\Gamma(1+n)} = \frac{(2n-1)!2(-1)^n}{(n-1)!n!} = (-1)^n\binom{2n}{n}$.

Related to Kummer's theorem is Kummer's identity, proven below (cf. [4], p.144, Remark 3.4.1).

Theorem 3.5.8 *For* $n \in \mathbb{Z}$,

$$F(-2n, b; 1-2n-b; -1) = \frac{(b)_n(2n)!}{n!\,(b)_{2n}}. \tag{3.19}$$

Proof: From the identity $(1-x)^{-b}(1+x)^{-b} = (1-x^2)^{-b}$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(b)_n x^n}{n!}\frac{(b)_k(-x)^k}{k!} = \sum_{n=0}^{\infty}\frac{(b)_n x^{2n}}{n!},$$

which by property (1.1) can be written in the form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(b)_{n-k}x^n}{(n-k)!}\frac{(b)_k(-1)^k}{k!} = \sum_{n=0}^{\infty}\frac{(b)_n x^{2n}}{n!}.$$

We now equate the coefficients of x^{2n} on both sides to obtain

$$\sum_{k=0}^{2n}\frac{(b)_{2n-k}}{(2n-k)!}\frac{(b)_k(-1)^k}{k!} = \frac{(b)_n}{n!}.$$

By introducing appropriate factors and using the identities $(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}$ and $(-n)_k = \frac{(-1)^k n!}{(n-k)!}$, this result becomes

$$\sum_{k=0}^{2n} \frac{(2n)!\,(b)_{2n-k}}{(b)_{2n}(2n-k)!}\frac{(b)_k(-1)^k}{k!} = \frac{(b)_n(2n)!}{n!\,(b)_{2n}}$$

$$\Rightarrow \sum_{k=0}^{2n} \frac{(-2n)_k(b)_{2n}}{(b)_{2n}(1-b-2n)_k}\frac{(b)_k(-1)^k}{k!} = \frac{(b)_n(2n)!}{n!\,(b)_{2n}},$$

which proves the theorem. ■

Whipple [92] provided the following evaluation for the hypergeometric series with argument $z = -1$, given as his formula (8.41).

Theorem 3.5.9 *For* $c \neq 0, -1, -2, \dots,$

$$F(a, b; c; -1) = \frac{\Gamma(c)}{2\Gamma(a)} \sum_{k=0}^{\infty} (-1)^k \frac{(c-a+b-1)_k}{k!} \frac{\Gamma\left(\frac{a+k}{2}\right)}{\Gamma\left(c - \frac{a-k}{2}\right)}. \tag{3.20}$$

Proof: In Euler's integral (3.8), let $z = -1$ to obtain

$$F(b, a; c; -1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t^2)^{c-a-1}(1+t)^{-c+a-b+1} dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t^2)^{c-a-1} \sum_{k=0}^{\infty} \frac{(-1)^k(c-a+b-1)_k t^k}{k!} dt\,.$$

Now by interchanging the order of integration and summation and using the substitution $t = \sqrt{s}$, we have

$$F(b, a; c; -1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(-1)^k(c-a+b-1)_k}{k!}$$

$$\times \int_0^1 \frac{1}{2} s^{(a+k)/2-1}(1-s)^{c-a-1} ds$$

$$= \frac{\Gamma(c)}{2\Gamma(a)\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(-1)^k (c-a+b-1)_k}{k!} \frac{\Gamma\left(\frac{a}{2}+\frac{k}{2}\right)\Gamma(c-a)}{\Gamma\left(\frac{a}{2}+\frac{k}{2}+(c-a)\right)},$$

by the presence of a beta integral, and the desired result follows directly. ■

3.6 Transformation identities

We are now able to state and prove some of the classical transformation identities for the Gauss hypergeometric function, which provide the foundation for more recent developments in this field. There exist vast numbers of these identities, which express one Gauss hypergeometric function in terms of one or more others, and extensive lists can be found in texts such as [3], [74] and [82]. In this section we discuss the most well-known linear and quadratic transformations.

3.6.1 Linear transformations

The 24 Kummer solutions of the hypergeometric differential equation (3.6) are derived from transformations of this equation into itself, under the linear fractional transformations $z' = \frac{az+b}{cz+d}$ of the independent variable (cf. [82], pp.11-12). This class consists of the transformations $z' = z$, $z' = \frac{z}{z-1}$, $z' = 1 - z$, $z' = \frac{1}{1-z}$, $z' = \frac{1}{z}$ and $z' = \frac{z-1}{z}$, which map the set $\{0, 1, \infty\}$ to itself. By changing the independent variable in the hypergeometric differential equation to any of these forms, equation (3.6) transforms to one of the same type with different parameters, and hence with a different hypergeometric series solution. In this way, it is possible to produce twelve solutions to the hypergeometric equation (two for each independent variable, convergent within the unit circle), and twelve more solutions through symmetry. Hence of Kummer's 24 series solutions, four have the same argument, say z, another four the argument $1/z$ and so on. Linear transformations connect hypergeometric functions which contain the variables z and z', thus creating relations which are central to the theory of the hypergeometric functions. In particular, they facilitate the analytic continuation of $F(a,b;c;z)$ into any part of the complex plane cut along $[1,\infty]$. Pfaff provided the following transformation from z to $z/(z-1)$, found in [60], p. 247, Eq. 9.5.1.

Theorem 3.6.1 *Pfaff's transformation*: *For* $|\arg(1-z) < \pi|, |z| < 1$ *and* $\left|\frac{z}{1-z}\right| < 1,$

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right). \tag{3.21}$$

Proof: By identities (1.9) and (1.18), the right side of the theorem becomes

$$(1-z)^{-a} F\left(a, c-b; c; \frac{-z}{1-z}\right) = \sum_{r=0}^{\infty} \frac{(a)_r (c-b)_r (-1)^r z^r (1-z)^{-(a+r)}}{(c)_r r!}$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{r+s} (c-b)_r (-1)^r z^{r+s}}{(c)_r r!\, s!}.$$

As $(n-r)! = \frac{(-1)^r n!}{(-n)_r}$ for $0 \le r \le n$, the coefficient of z^n in this double series is

$$\sum_{r=0}^{\infty} \frac{(a)_n (c-b)_r (-1)^r}{(c)_r r!\,(n-r)!} = \frac{(a)_n}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r (c-b)_r}{(c)_r r!} = \frac{(a)_n}{n!} F(-n, c-b; c; 1)\,.$$

By Gauss' summation theorem (3.13), this result can be written in the form

$$\sum_{n=0}^{\infty} \frac{\Gamma(c)\Gamma(b+n)}{\Gamma(c+n)\Gamma(b)} \frac{(a)_n}{n!} = \sum_{r=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!},$$

and the theorem is proved. ■

Pfaff's transformation (3.21) can also be elegantly established by setting $t = 1-s$ in Euler's integral (3.8). Pfaff's result follows from the simplification $\int_0^1 s^{c-b-1}(1-s)^{b-1}\,[(1-z)+sz]^{-a} ds = (1-z)^{-a} \int_0^1 s^{c-b-1}(1-s)^{b-1}\left(1-\frac{sz}{z-1}\right)^{-a} ds.$

The restriction $\left|\frac{z}{1-z}\right| < 1$ is satisfied when $Re(z) < \frac{1}{2}$, so Pfaff's transformation provides a continuation of the Gauss hypergeometric series from the unit circle to the left of the vertical line $Re(z) = \frac{1}{2}$. This continuation into the half plane is valid only if $|z| < 1$, unless $a, b, c - a$ or $c - b$ is a non-positive integer (cf. [9], p.64).

There exist various forms of Pfaff's transformation. The two provided below are found in [8], p.10, Eq. 2.4 (1) and [60], p.247, Eq. 9.5.2 respectively.

$$F(a, b; c; z) = (1 - z)^{-b} F\left(c - a, b; c; \frac{z}{z - 1}\right) \tag{3.22}$$

$$F(a, c - b; c; z) = (1 - z)^{-a} F\left(a, b; c; -\frac{z}{1 - z}\right) \tag{3.23}$$

Pfaff's transformation (3.21) provides an elegant approach for proving various relations. For example, it allows us to show that

$$\arctan z = zF\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{z}{\sqrt{1 + z^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{z^2}{1 + z^2}\right) = \arcsin \frac{z}{\sqrt{1 + z^2}}.$$

We can also use Pfaff's transformation together with Kummer's Theorem (3.16) to evaluate Gauss hypergeometric functions with argument $z = \frac{1}{2}$. This leads to Gauss' second summation theorem and Bailey's summation theorem, as detailed below.

By substituting $z = \frac{1}{2}$ into Pfaff's transformation (3.21), we find that $F\left(a, b; c; \frac{1}{2}\right) = 2^a F(a, c - b; c; -1)$. We can now apply Kummer's theorem (3.16) to the right side of this result in two ways: either applying the identity to $2^a F(c - b, a; c; -1)$ so that $c = a - (c - b) + 1$ (i.e. $c = (a + b + 1)/2$), or to $F(a, c - b; c; -1)$ so that $c = c - b - a + 1$ (i.e. $b = 1 - a$). We consider both cases.

Case 1: When $c = \frac{a+b+1}{2}$, Kummer's theorem (3.16) yields

$$2^a F(c - b, a; c; -1) = \frac{\Gamma(c)\Gamma(1 + a/2)2^a}{\Gamma(1 + a)\Gamma(1 + a/2 - c + b)}.$$

From Legendre's duplication (2.17) in the form $\frac{\Gamma(1+a/2)2^a}{\Gamma(1+a)} = \frac{\Gamma(1/2)}{\Gamma(1/2+a/2)}$, we have

$$F(a, b; c; 1/2) = 2^a F(c - b, a; c; -1) = \frac{\Gamma(1/2)\Gamma(c)}{\Gamma(1/2 + a/2)\Gamma(1 + a/2 - c + b)},$$

and hence

$$F(a,b,(1+a+b)/2;1/2)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}. \tag{3.24}$$

This result is often called Gauss's second summation theorem, and is Bailey's equation 2.4 (2) in [8], p.11.

Case 2: When $b=1-a$, Kummer's theorem (3.16) gives us

$$2^{a}F(a,c-b;c;-1)=\frac{\Gamma(c)\Gamma\left(1+\frac{c-b}{2}\right)2^{c-b}2^{1-c}}{\Gamma(1+[c-b])\Gamma\left(1+\frac{c-b}{2}-a\right)}.$$

Once again applying Legendre's duplication formula, we obtain

$$2^{a}F(a,c-b;c;-1)=\frac{\Gamma(c)\Gamma\left(\frac{1}{2}\right)2^{1-c}}{\Gamma\left(\frac{1}{2}+\frac{c-b}{2}\right)\Gamma\left(1+\frac{c-b}{2}-a\right)},$$

which by a second application of the duplication formula and the substitution $b=1-a$, yields

$$F(a,1-a;c;1/2)=\frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)}. \tag{3.25}$$

This result is sometimes called Bailey's summation theorem, and can also be derived directly from Euler's integral (cf. [82], p.32). It is given as equation 2.4 (3) in [8], p.11, but can also be found earlier in Kummer's 1836 paper '*Über die hypergeometrische Reihe'*.

Using Bailey's summation theorem (3.25) and Legendre's duplication formula (2.17), Bateman [9], p.68, also showed that

$$F\left(2a,1-2a;2c;\frac{1}{2}\right)=2^{1-2c}\frac{\Gamma(2c)\Gamma\left(\frac{1}{2}\right)}{\Gamma(a+c)\Gamma\left(c-a+\frac{1}{2}\right)}. \tag{3.26}$$

By applying Pfaff's transformation to itself, we obtain Euler's linear transformation, found in [60], p.248, Eq. 9.5.3.

Theorem 3.6.2 *Euler's transformation*: *For* $|z| < 1$, $|\arg (1 - z)| < \pi$,

$$F(a, b; c; z) = (1 - z)^{c-a-b} F(c - a, c - b; c; z)\,. \tag{3.27}$$

Proof: From Pfaff's transformation (3.21) and the symmetry of the Gauss function, we have

$$F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; -\frac{z}{1 - z}\right)$$

$$= (1 - z)^{-a} F\left(c - b, a; c; -\frac{z}{1 - z}\right).$$

We now apply Pfaff's transformation again to the right side of this result, using $w = \frac{-z}{1-z}$ so that $1 - w = \frac{1}{1-z}$ and $-\frac{w}{1-w} = z$. We then obtain

$$F(a, b; c; z) = (1 - z)^{-a}(1 - w)^{b-c} F\left(c - b, c - a; c; \tfrac{-w}{1-w}\right)$$

$$= (1 - z)^{c-a-b} F(c - a, c - b; c; z),$$

as required. ■

Two further linear transformations are provided below. Theorem 3.6.3 can be proved by induction on *n*.

Theorem 3.6.3 For $n \in \mathbb{N}$ and $c, b + 1 - c - n \neq 0, -1, \dots, -n + 1$,

$$F(-n, b; c; 1 - z) = \frac{(c - b)_n}{(c)_n} F(-n, b; -n + b + 1 - c; z). \tag{3.28}$$

Theorem 3.6.4 *For* $n \in \mathbb{N}$,

$$F(-n, b; c; z) = \frac{(b)_n}{(c)_n} (-z)^n F(-n, 1 - c - n; 1 - b - n; 1/z)\,. \tag{3.29}$$

Proof: Using the identity (2.14) in the form $(1-\alpha-n)_k = \frac{(-1)^k\Gamma(\alpha+n)}{\Gamma(\alpha+n-k)}$, we expand the finite hypergeometric series on the right side of the theorem to obtain

$$\frac{(b)_n}{(c)_n}(-z)^n \sum_{k=0}^{n} \frac{(-n)_k(1-c-n)_k z^{-k}}{(1-b-n)_k k!}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (b)_n (-1)^n \Gamma(c+n)\Gamma(b+n-k) z^{n-k}}{(c)_n k!\, \Gamma(c+n-k)\Gamma(b+n)}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (-1)^n}{k!} \frac{\Gamma(c)\Gamma(b+n-k) z^{n-k}}{\Gamma(c+n-k)\Gamma(b)}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (-1)^n}{k!} \frac{(b)_{n-k} z^{n-k}}{(c)_{n-k}},$$

and the desired result follows by replacing $n-k$ with j and applying identity (1.6). ■

3.6.2 Quadratic transformations

If any two of the values $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ are equal or if one of them is ½, then there exists a quadratic transformation of the hypergeometric function $F(a,b;c;z)$, in which the arguments are related by a quadratic equation. The first such transformations were given by Kummer in *Crelle's Journal* (*Journal für die reine und angewandte Mathematik*), Bd 15 (1836), and comprehensive lists can be found in [9] and [84]. There exist various techniques for deriving these transformation identities. One approach is to use the definition of the Gauss function and properties of the rising factorial, as in the proof below (cf. [75], p.65, Th. 24).

Theorem 3.6.5 *If $2b$ is neither zero nor a negative integer, and if $|y| < \frac{1}{2}$ and $\left|\frac{y}{1-y}\right| < 1$, then*

$$F(a,b;2b;2z) = (1-z)^{-a} F\left(\frac{a}{2},\frac{1+a}{2};b+\frac{1}{2};\frac{z^2}{(1-z)^2}\right). \tag{3.30}$$

Proof: By expanding the right side according to the definition of the Gaussian series and then applying the identity $(a)_{2k} = 2^{2k}\left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k$, we have

$$RHS = \sum_{k=0}^{\infty} \frac{(a)_{2k} z^{2k}}{2^{2k}\left(b+\frac{1}{2}\right)_k k!\,(1-z)^{a+2k}}.$$

We now use the series expansion for $(1-z)^{-(a+2k)}$ and then identity (1.9) in the form $(a)_{2k}(a+2k)_n = (a)_{n+2k}$ to write this result as

$$RHS = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{2k} z^{2k+n}(a+2k)_n}{2^{2k}\left(b+\frac{1}{2}\right)_k k!\,n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{n+2k} z^{2k+n}}{2^{2k}\left(b+\frac{1}{2}\right)_k k!\,n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(a)_n z^n}{2^{2k}\left(b+\frac{1}{2}\right)_k k!\,(n-2k)!},$$

by application of the identity (1.2) for iterated series. Now by applying identity (1.10) in the form $(-n)_{2k} = 2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{1-n}{2}\right)_k$, and the identity $(n-2k)! = n!/(-n)_{2k}$, this result simplifies to

$$RHS = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{1-n}{2}\right)_k (a)_n z^n}{\left(b+\frac{1}{2}\right)_k n!} = \sum_{n=0}^{\infty} \frac{(b)_n}{(2b)_n}\frac{(a)_n (2z)^n}{n!},$$

by (3.14), and the theorem is proved. ■

The useful quadratic transformation below was established by Kummer [8] as Equation 2.3.2, and provides a powerful tool for deriving further transformations.

Theorem 3.6.6 *Kummer's quadratic transformation: For* $\left|\frac{4z}{(1-z)^2}\right| < 1$ *and* $|z| < 1$*, with* $1 + a - b$ *not zero or a negative integer,*

$$F(a, b; 1 + a - b; z) = (1 - z)^{-a} F\left(\frac{a}{2}, \frac{1 + a}{2} - b; 1 + a - b; -\frac{4z}{(1 - z)^2}\right). \tag{3.31}$$

We reserve the proof of this theorem for the next chapter, in order to make use of generalised hypergeometric functions. An alternative form of Kummer's quadratic transformation, given as equation 2.11(17) in [26], is

$$F\left(a, a + \frac{1}{2}; b + \frac{1}{2}; z^2\right) = (1 + z)^{-2a} F\left(2a, b; 2b; \frac{2z}{1 + z}\right), |z| < 1. \tag{3.32}$$

Letting $b \to 0$, $b = 1$ and $b = 2$ respectively in (3.32) yields three further results:

$$F\left(a, a + \frac{1}{2}; \frac{1}{2}; z^2\right) = \frac{1}{2}[(1 + z)^{-2a} + (1 - z)^{-2a}],$$

$$F\left(a, a + \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{1}{2(1 - 2a)z}[(1 + z)^{1-2a} - (1 - z)^{1-2a}],$$

$$F\left(a, a + \frac{1}{2}; \frac{5}{2}; z^2\right) = \frac{3}{2(1 - 2a)(2 - 2a)(3 - 2a)z^3}$$

$$\times [\{2 - 2a)z - 1\}(1 + z)^{2-2a} + \{2 - 2a)z + 1\}(1 - z)^{2-2a}].$$

Kummer's theorem (3.16) can also be established from his quadratic transformation. If we let $z \to -1$ in his transformation (3.31), and then apply Gauss' summation formula (3.13) we obtain

$$F(a, b; 1 + a - b; -1) = 2^{-a} F\left(\frac{a}{2}, \frac{1 + a}{2} - b; 1 + a - b; 1\right)$$

$$= 2^{-a} \frac{\Gamma(1 + a - b)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{a}{2} - b\right)\Gamma\left(\frac{1}{2} + \frac{a}{2}\right)}.$$

Applying Legendre's duplication formula $2^{-a}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}=\frac{\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)}$ to this result, yields Kummer's theorem in the form $F(a,b;1+a-b;-1)=\frac{\Gamma(1+a-b)\Gamma(1+a/2\,)}{\Gamma(1+a/2-b)\Gamma(1+a)}$.

This is one of many cases where a hypergeometric function can be evaluated at $z=-1$ by first using a quadratic transformation to change the argument to $z=1$, and then using Gauss' summation theorem to evaluate the result.

Euler's integral representation for the Gauss hypergeometric function provides a useful base for establishing further quadratic transformations such as the one below, given as Theorem 3.1.3 in [4], p.127.

Theorem 3.6.7 *For all z such that the series converge,*

$$F(a,b;2a;z)=\left(1-\frac{z}{2}\right)^{-b}F\left(\frac{b}{2},\frac{1+b}{2};a+\frac{1}{2};\left(\frac{z}{2-z}\right)^{2}\right). \tag{3.33}$$

Proof: By Euler's integral (3.8), the Gauss function on the left side of the theorem becomes[22]

$$\frac{\Gamma(2a)}{4^{a-1}\Gamma(a)\Gamma(a)}\int_0^1(1-xt)^{-b}[1-(1-2t)^2]^{a-1}dt\,.$$

Substituting $t=\frac{1-s}{2}$, so that $dt=-\frac{1}{2}ds$, this integral becomes

$$\frac{\Gamma(2a)}{2^{2a-1}\Gamma(a)\Gamma(a)}\int_{-1}^1\left(1-\frac{x(1-s)}{2}\right)^{-b}(1-s^2)^{a-1}ds$$

$$=\frac{\Gamma(2a)\left(1-\frac{x}{2}\right)^{-b}}{2^{2a-1}[\Gamma(a)]^2}\int_{-1}^1\left(1-\frac{sx}{x-2}\right)^{-b}(1-s^2)^{a-1}ds$$

$$=\frac{\Gamma(2a)\left(1-\frac{x}{2}\right)^{-b}}{2^{2a-1}[\Gamma(a)]^2}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left(\frac{x}{x-2}\right)^n\int_{-1}^1 s^n(1-s^2)^{a-1}ds.$$

[22] A typographical error in Andrews et al. [4], p.127 omits the power of 2 in the term $\left(\frac{1}{2}-t\right)$ in the second line of the proof .

If n is odd the last integral is zero, but for $n = 2m, m \in \mathbb{N}$, we use a substitution and the beta-gamma relation (2.29) to show that

$$\int_{-1}^{1} s^n (1 - s^2)^{a-1} ds \ = \int_0^1 u^{m-1/2}(1-u)^{a-1} du = \frac{\Gamma(m + 1/2)\Gamma(a)}{\Gamma(m + a + 1/2)}.$$

We thus obtain

$$F(a, b; 2a; z) = \frac{\Gamma(2a)\left(1 - \frac{x}{2}\right)^{-b}}{2^{2a-1}[\Gamma(a)]^2} \sum_{m=0}^{\infty} \frac{(b)_{2m}}{(2m)!} \frac{\Gamma(m + 1/2)\Gamma(a)}{\Gamma(m + a + 1/2)} \left(\frac{x}{x-2}\right)^{2m},$$

which by the identities $(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$, $\Gamma(\alpha + m) = \Gamma(\alpha)(\alpha)_m$ and (1.10) in the form $\frac{(1)_{2n}}{\left(\frac{1}{2}\right)_n} = 2^{2n} n!$, becomes

$$F(a, b; 2a; z) = \frac{\Gamma(2a)\Gamma\left(\frac{1}{2}\right)\left(1 - \frac{x}{2}\right)^{-b}}{\Gamma(a) 2^{2a-1} \Gamma\left(a + \frac{1}{2}\right)} \ F\left(\frac{b}{2}, \frac{1+b}{2}; a + \frac{1}{2}; \left(\frac{z}{2-z}\right)^2\right),$$

and the desired result follows by applying Legendre's duplication formula (2.17) to the gamma ratio on the right side. ■

Equations (3.31) and (3.33) contain the fundamental quadratic transformations, and many others can be derived from these by using fractional linear transformations or three-term relations connecting the solutions of the hypergeometric differential equation. The literature abounds in such transformations, and we provide a sample of these below, found in [4] and [75].

Recall Pfaff's transformation (3.21): $F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z-1}\right)$. For argument $z' = \frac{-4z}{(1-z)^2}$, we have $1 - z' = \frac{(1+z)^2}{(1-z)^2}$ and $\frac{z'}{z'-1} = \frac{4z}{(1+z)^2}$, so by applying Pfaff's transformation to the Gauss series on the right side of Kummer's quadratic transformation (3.31), we obtain

$$F(a,b;a-b+1;z)=(1-z)^{-a}\left[\frac{(1+z)^2}{(1-z)^2}\right]^{-\frac{a}{2}}F\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(1+z)^2}\right)$$

$$=(1+z)^{-a}F\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(1+z)^2}\right). \tag{3.34}$$

Now replacing $\frac{4z}{(1+z)^2}$ with z', we have $z=\frac{2-z'-2\sqrt{1-z\prime}}{z\prime}=\frac{1-\sqrt{1-z\prime}}{1+\sqrt{1-z\prime}}$, so that $1+z=\frac{2}{1+\sqrt{1-z\prime}}$, and the above result yields the equivalent formula:

$$F\left(\frac{a}{2},\frac{a+1}{2};a-b+1;z\right)=2^a\left(1+\sqrt{1-z}\right)^{-a}F\left(a,b;a-b+1;\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right).$$

For further transformations, we consider Theorem 3.6.7 in the form

$$F(a,b;2b;z)=\left(1-\frac{z}{2}\right)^{-a}F\left(\frac{a}{2},\frac{1+a}{2};b+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right),$$

and the transformation (3.34). The Gauss functions on the left of these identities become equal when $2b=a-b+1$, or $b=(a+1)/3$. Thus we have the following transformation formulae:

$$\left(1-\frac{z}{2}\right)^{-a}F\left(\frac{a}{2},\frac{1+a}{2};b+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right)=F\left(a,\frac{a+1}{3};\frac{2(a+1)}{3};z\right)$$

$$=(1+z)^{-a}F\left(\frac{a}{2},\frac{1+a}{2};\frac{2(a+1)}{3};\frac{4z}{(1+z)^2}\right)$$

$$=(1-z)^{-a}F\left(\frac{a}{2},\frac{1+a}{6};\frac{2(a+1)}{3};\frac{-4z}{(1+z)^2}\right),$$

where the last result is given by Pfaff's linear transformation (3.21). By manipulation of the above formulae, Andrews et al. [4] derive the further results below, given as equations 3.1.17 and 3.1.20 respectively.

$$F\left(\frac{a}{2},\frac{1+a}{2};\frac{2a+5}{6};\frac{1}{9}\right)=\left(\frac{3}{4}\right)^a\frac{\sqrt{\pi}\Gamma\left(\frac{2a+2}{3}\right)}{\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+1}{2}\right)}\text{ ,and}$$

$$F\left(\frac{a}{2},\frac{1+a}{2};\frac{2(a+1)}{3};\frac{8}{9}\right)=\left(\frac{3}{2}\right)^{a}\frac{\sqrt{\pi}\Gamma\left(\frac{2a+2}{3}\right)}{\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+1}{2}\right)}.$$

While it is convenient to use existing transformations to derive further ones, we can also establish new transformations independently by substitution into the hypergeometic differential equation. Rainville [75] provides the following illustration of this method in proving his Theorem 25.

Theorem 3.6.8 *For* $|x|<1, |4x(1-x)|<1$ *and* $a+b+\frac{1}{2}$ *not zero or a negative integer,*

$$F\left(a,b;a+b+\frac{1}{2};4x(1-x)\right)=F\left(2a,2b;a+b+\frac{1}{2};x\right). \tag{3.35}$$

Proof: The function $F\left(a,b;a+b+\frac{1}{2};z\right)$ is a solution to the differential equation $z(1-z)\frac{d^2y}{dz^2}+[c-(a+b+1)z]\frac{dy}{dz}-aby=0$, with $c=a+b+\frac{1}{2}$. By letting $z=4x(1-x)$, we obtain $\frac{dy}{dz}=\frac{dy}{dx}(4-8x)^{-1}$ and $\frac{d^2y}{dz^2}=\frac{d^2y}{dx^2}(4-8x)^{-2}+8\frac{dy}{dx}(4-8x)^{-3}$. Substitution into the differential equation then yields

$$x(1-x)\frac{d^2y}{dx^2}+\left[a+b+\frac{1}{2}-(2a+2b+1)x\right]\frac{dy}{dx}-4aby=0,$$

which is a hypergeometric equation with solution $y=F\left(2a,2b;a+b+\frac{1}{2};x\right)$, and the theorem is proved. ■

Similar substitutions yield the following transformations, respectively found in [84], p.122, Eq. (5.28) and its inverse, and in [60], p.250, Eq. 9.6.1:

$$F(2a,2a+1-c;c;z)=(1+z)^{-2a}F\left(a,a+\frac{1}{2};c;\frac{4z}{(1+z)^2}\right), \tag{3.36}$$

and its inverse

$$F\left(a,a+\frac{1}{2};c;z\right)=\left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right)^{-2a}F\left(2a,2a-c+1;c;\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right), \tag{3.37}$$

and for $|\arg(1-z)| < \pi,\ a+b+\frac{1}{2} \neq 0,-1,-2,\dots,$

$$F\left(a,b;a+b+\frac{1}{2};z\right) = F\left(2a,2b;a+b+\frac{1}{2};\frac{1-\sqrt{1-z}}{2}\right). \tag{3.38}$$

In summary, a quadratic transformation exists for each of the hypergeometric functions given below (cf. [84], p.124]. The results hold in an unspecified neighbourhood of the origin, and the domain of validity can be extended by analytic continuation.

$$\begin{array}{lll} F(a,b;1\backslash 2;z), & F(a,a+1\backslash 2;c;z), & F(a,b;a+b+1\backslash 2;z) \\ F(a,b;3\backslash 2;z), & F(a,a-1\backslash 2;c;z), & F(a,b;a+b-1\backslash 2;z) \\ F(a,b;2a;z), & F(a,b;b-a+1;z), & F(a,1-a;c;z) \\ F(a,b;2b;z), & F(a,b;a-b+1;z), & F(a,b;(a+b+1)/2;z) \end{array}$$

In the case that $1-c,\ a-b,$ and $a+b-c$ differ only by signs, or if two of these numbers equal $\pm\frac{1}{3}$, then there exists a cubic transformation of the hypergeometric function $F(a,b;c,z)$. A complete list of these is provided in [21]. There also exist hypergeometric transformations of degree 4 and 6.

Having defined the Gauss hypergeometric function and explored some of its most useful identities, we will now extend this concept to the generalised hypergeometric function.

Chapter 4

Generalised hypergeometric functions

4.1 Introduction

The Gauss hypergeometric function can be extended to form the generalised hypergeometric function, which can contain any number of numerator and denominator parameters. According to Slater [2] this extension was first done by Thomas Clausen (1801-1885) in 1828, using three numerator and two denominator parameters. Over the following hundred years a well-known set of summation theorems were developed for these functions, including famous identities established by Louis Saalschütz (1835-1913), Alfred. Dixon (1865-1936), Henry Watson (1827-1903), John Dougall (1867-1960) and Whipple. The theory was exhaustively covered by Bailey in a series of papers between 1920 and 1950.

The generalised hypergeometric function is exceptionally useful, as all the special functions of mathematical physics can be expressed in terms of these functions, increasing in complexity as the number of parameters increases. In Section 4.2 we define this function, then we present in Section 4.3 certain classical identities and transformations, particularly those which play a central role in more recent related work. Section 4.4 presents a brief discussion of the very useful confluent hypergeometric function, and finally we mention a few interesting applications and extensions of the generalised hypergeometric function, in Section 4.5.

4.2 Defining the generalised hypergeometric function

The generalised hypergeometric function ${}_pF_q$ is the sum of a series which contains p numerator parameters and q denominator parameters (cf. [75], p.73).

Definition 4.2.1 *For $b_j \neq 0, -1, -2, \ldots,$ the generalised hypergeometric function ${}_pF_q$ is defined by*

$$ {}_pF_q(a_1, a_2, \ldots, a_p;\ b_1, b_2, \ldots, b_q; z) = {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ z\right) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{z^k}{k!}. \tag{4.1} $$

According to this notation, the Gauss hypergeometric function is represented by ${}_2F_1(a, b; c; z)$ or ${}_2F_1\left(\begin{matrix} a, b; \\ c; \end{matrix}\ z\right)$, which notations we will use from this point on. As in the case of the Gauss hypergeometric function, it is assumed that no denominator parameter is zero or a negative integer, and if any numerator parameter is zero or a negative integer the function is a terminating hypergeometric polynomial. A dash is usually used to indicate when there is no parameter in either the numerator or the denominator, as in ${}_0F_1(-; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!}$.

We have seen that for the Gauss hypergeometric series, the term ratio is a rational function in k . Through properties of the rising factorial, it can similarly be shown that for ${}_pF_q(a_1, a_2, \ldots, a_p;\ b_1, b_2, \ldots, b_q; z) = \sum_{k=0}^{\infty} t_k$, the term ratio is the rational function $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)(k+a_2)\ldots(k+a_p)z}{(k+b_1)(k+b_2)\ldots(k+b_q)(k+1)}$. According to the Fundamental Theorem of Algebra, any rational function of k can be factorised over the complex numbers and written in this form, where z can be some constant factor. Hence, we can reconstruct the generalized hypergeometric series from its term ratio as shown in the example below, provided by Graham et al. [45], p. 208.

Example 4.2.2 Consider the term ratio $\frac{t_{k+1}}{t_k} = \frac{k^2+7k+10}{4k^2+1}$. By writing this rational function in the form $\frac{t_{k+1}}{t_k} = \frac{(k+2)(k+5)(k+1)(\frac{1}{4})}{(k+i/2)(k-i/2)(k+1)}$, we can conclude that the term ratio proceeds from the generalised hypergeometric series ${}_3F_2\left(2, 5, 1; \frac{i}{2}, -\frac{i}{2}; \frac{1}{4}\right)$.

As with the Gauss function, when the hypergeometric series does not terminate, we need to consider conditions of convergence (cf. [4], p.62, Th. 2.1.1).

Theorem 4.2.3 *The hypergeometric series* ${}_pF_q\left(a_1, a_2, \dots, a_p;\ b_1, b_2, \dots, b_q; z\right)$ *converges for all finite* z *if* $p \leq q$, *converges for* $|z| < 1$ *if* $p = q + 1$, *diverges for* $|z| > 1$, *and diverges for* $z \neq 0$ *if* $p > q + 1$ (*and the series does not terminate*).

Proof: By factorial properties, we have $\left|\frac{t_{k+1}}{t_k}\right| = \frac{|z|k^{p-q-1}(1+|a_1|/k)\dots(1+|a_p|/k)}{|(1+|b_1|/k)\dots(1+|b_q|/k)(1+1/k)|}$. Using the ratio test, the stated convergence conditions then follow by considering $|z|$ and the relationships $p - q - 1 = 0, p - q - 1 < 0$ and $p - q - 1 > 0$ in k^{p-q-1}. ■

The hypergeometric function ${}_{q+1}F_q$ converges on the unit circle under the following condition, where the parametric excess is defined by $\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i$ (cf. [75], p.74).

Theorem 4.2.4 *For* $p = q + 1$, *the hypergeometric series* ${}_pF_q$ *is absolutely convergent on the circle* $|z| = 1$ *if* $Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$.

Most standard mathematical functions can be represented by hypergeometric functions, and lists of these representations can be found in [52], p.13, [60], p.275, [75], p.74 and [82] p.46. These include orthogonal polynomials, a list of which can be found in Table 6.1 of [78], p.97. We provide an illustration in the example below, found as Equation 3.3.5 in [73].

Example 4.2.5 For the Bessel function defined by $J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!(k+p)!}$, the term ratio is given by $\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1}\left(\frac{x}{2}\right)^{2k+2+p} k!(k+p)!}{(k+1)!(k+p+1)!(-1)^k\left(\frac{x}{2}\right)^{2k+p}} = \frac{\frac{-x^2}{4}}{(k+1)(k+p+1)}$. If we normalise the function so that $t_0 = 1$, we obtain $J_p(x) = \frac{\left(\frac{x}{2}\right)^p}{p!}\ {}_0F_1\left(-; p+1; \frac{-x^2}{4}\right)$.

Below are further examples of mathematical functions expressed in terms of ${}_pF_q$ functions.[23]

$$ {}_0F_0(-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z $$

$$ {}_0F_1\left(-; \frac{3}{2}; \frac{-z^2}{4}\right) = \sin z $$

$$ {}_0F_1\left(-; \frac{1}{2}; \frac{-z^2}{4}\right) = \cos z $$

$$ {}_1F_0(a;-;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = (1-z)^{-a} $$

The ${}_pF_q$ function also has an integral representation, analogous to Euler's integral for the Gauss hypergeometric function. We first notice that Euler's integral (3.8) can be written in the form ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1F_0(a;-;zt)dt$, from which we see that integrating a ${}_1F_0$ with respect to the beta distribution adds a parameter to the numerator and also to the denominator of the original series. This result can be generalised to the formula below, with conditions which ensure the convergence of the integral, and where z is restricted to a domain for which the ${}_pF_q$ in the integrand is single valued (cf. [4], p.67, Eg. 2.2.2).

[23] The generalised hypergeometric function also has links to the Riemann zeta function, defined by $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$. For example, ${}_{k+1}F_k(1,\ldots,1; 2,\ldots,2; 1) = \sum_{n=0}^{\infty} \frac{((1)_n)^{k+1}}{((2)_n)^k\, n!} = \sum_{n=0}^{\infty} \left(\frac{(1)_n}{(2)_n}\right)^k = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^k = \zeta(k)$.

Theorem 4.2.6 *For* $Re(b_{q+1}) > Re(a_{p+1}) > 0$,

$$
{}_{p+1}F_{q+1}\left(\begin{matrix} a_1, \dots, a_p, a_{p+1}; \\ b_1, \dots, b_q, b_{q+1}; \end{matrix} z\right)
$$

$$
= \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})\Gamma(b_{q+1} - a_{p+1})} \int_0^1 t^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1} \;{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; zt\right) dt
$$

$$
= \frac{\Gamma(b_{q+1}) z^{1-b_{q+1}}}{\Gamma(a_{p+1})\Gamma(b_{q+1} - a_{p+1})} \int_0^x t^{a_{p+1}-1}(z-t)^{b_{q+1}-a_{p+1}-1} \;{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; t\right) dt\,.
$$

Apart from reducing the number of parameters in a hypergeometric function, the above result is also useful for changing the value of a numerator or denominator parameter. For example, if we take $a_{p+1} = b_q$ in the above theorem, we obtain

$$
{}_pF_q\left(\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_{q-1}, b_{q+1}; \end{matrix} z\right)
$$

$$
= \frac{\Gamma(b_{q+1})}{\Gamma(b_q)\Gamma(b_{q+1} - b_q)} \int_0^1 t^{b_q-1}(1-t)^{b_{q+1}-b_q-1} \;{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; zt\right) dt.
$$

A special case of this result is Bateman's integral (3.9), given in Chapter 3.

4.3 Classical identities for certain ${}_{p+1}F_p$ series

We are now ready to present some classical identities and transformations related to the generalised hypergeometric function, most of which were established in the first half of the 20th century. In the next chapter we review more recent related results. As there exists a vast number of such identities, we have chosen to focus on series of the form ${}_{p+1}F_p$, which have one more parameter in the numerator than in the denominator, and in particular on ${}_3F_2$ series.

There are two special classes of ${}_{p+1}F_p$ hypergeometric series for which many interesting results can be obtained. These are referred to as balanced and poised series (cf. [9], p.188).

Definition 4.3.1 *A series ${}_{p+1}F_p(z)$ is called balanced (or Saalschützian) if $z = 1$, one of the numerator parameters is a negative integer, and the parametric excess is unity.*

A series is called k-balanced if the parametric excess is k (cf. [4], p.140).

Definition 4.3.2 *A series ${}_{p+1}F_p\left(\begin{matrix} a_1, \dots, a_{p+1}; \\ b_1, \dots, b_p; \end{matrix} z\right)$ is said to be well-poised if $1 + a_1 = a_2 + b_1 = \cdots = a_{p+1} + b_p$.*

A series is called nearly-poised if all but one of the pairs of parameters have the same sum (cf. [8], p.11).

4.3.1 Evaluation and transformation identities for ${}_3F_2(1)$

The function ${}_3F_2(1)$ is a fundamental special function, as it occurs in a wide range of pure and applied mathematics contexts, as well as in other disciplines. There exist many useful identities for summing these hypergeometric series, including the well-known theorems of Dixon, Whipple and Watson.

For certain values of the parameters, the function ${}_3F_2(1)$ can be expressed in closed form, that is, as a ratio of finite products of gamma functions. Transformation identities also exist which express one such series in terms of one or more other hypergeometric functions, sometimes with a different number of parameters. Carl Thomae (1840-1921) [85] first derived transformations of general ${}_3F_2(1)$ series in his classical work of 1879. These transformations have been shown to have a group structure with, for example, an action of the symmetric group S_5 on ${}_3F_2(1)$. Details can be found in [90], [64] and in Sections 3.5-3.7 of [8].

We first consider the central Pfaff-Saalschütz theorem, which expresses every terminating Saalschützian ${}_3F_2$ as a ratio of rising factorials (or gamma functions). This theorem was first discovered by J. F. Pfaff in 1797, and was rediscovered independently by Saalschütz in 1890 (cf. [82], p.48, [12], p.9).

Theorem 4.3.3 *The Pfaff-Saalschütz Theorem*: *If n is a non-negative integer and a, b, c are independent of n, then*

$$ {}_3F_2\left(\begin{matrix}-n, & a, & b; \\ c, 1-c+a+b-n; & & \end{matrix} 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{4.2}$$

Proof: From Euler's linear transformation (3.27), we have

$$ {}_2F_1(c-a,c-b;c;z) = (1-z)^{a+b-c}\,{}_2F_1(a,b;c;z)$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r(b)_r z^r}{(c)_r r!}\sum_{s=0}^{\infty}\frac{(c-a-b)_s z^s}{s!}.$$

If we now put $s = n - r$ and then use identities (1.6) and (1.12), we find the coefficient of z^n in the above double series to be

$$\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(c-a-b)_{n-r}}{(c)_r r!\,(n-r)!} = \frac{(c-a-b)_n}{n!}\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(-n)_r}{(c)_r r!\,(1-c+a+b-n)_r}$$

$$= \frac{(c-a-b)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n, & a, & b; \\ c, 1-c+a+b-n; & & \end{matrix} 1\right),$$

while the coefficient of z^n in ${}_2F_1(c-a,c-b;c;z)$ is $\frac{(c-a)_n(c-b)_n}{(c)_n n!}$. Equating these two coefficient expressions establishes the Pfaff-Saalchütz identity. ∎

The above theorem provides the most useful form of the Pfaff-Saalschütz identity, but through appropriate substitution of parameters, this result can also be written in alternative forms. For example, Graham et al. [45] provide an equivalent form involving only factorials, in their Equation 5.134. A further alternative form is Bailey's equation 2.2 (2) in [8], which states that

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ & d, & e; \end{matrix}\ 1\right) = \frac{\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)}{\Gamma(1-e)\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)}, \tag{4.3} $$

provided that $d+e=a+b+c+1$, and one of the numerator parameters is a negative integer. Andrews et al. [4], Corollary 2.4.5, provide the following non-terminating form of the Pfaff- Saalschütz identity.

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a, & b, & c; \\ & d, & e; \end{matrix}\ 1\right) &= \frac{\Gamma(d)\Gamma(e)\Gamma(d-a-b)\Gamma(e-a-b)}{\Gamma(d-a)\Gamma(d-b)\Gamma(e-a)\Gamma(e-b)} \\ &+ \frac{1}{a+b-d}\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(d+e-a-b)} \\ &\times {}_3F_2\left(\begin{matrix} d-a, & d-b, & 1 \\ d+e-a-b, & & d+1-a-b; \end{matrix}\ 1\right) \end{aligned} \tag{4.4} $$

Using the Pfaff-Saalschütz identity (4.2), we can now prove Kummer's quadratic transformation (3.31) for the Gauss hypergeometric function, which was stated without proof in Section 3.6. We recall Kummer’s transformation, which states that given suitable restrictions, $F(a,b;1+a-b;z) = (1-z)^{-a}F\left(\frac{a}{2},\frac{1+a}{2}-b;1+a-b;-\frac{4z}{(1-z)^2}\right)$. By definition, the right side of the identity can be written in the form [24]

$$ RHS = \sum_{r=0}^{\infty}\frac{(a/2)_r(a/2+1/2-b)_r}{r!\,(1+a-b)_r}(-4z)^r(1-z)^{-a-2r}\,. \tag{4.5} $$

Within the loop of the curve $|4z| = |(1-z)^2|$ which encloses the origin, this sum is analytic and can hence be expanded in powers of z for $|z| < 3-2\sqrt{2}$. Using the identity $(1-y)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n y^n}{n!}$, we can thus rewrite (4.5) in the form

$$ RHS = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-4)^r(a/2)_r(a/2+1/2-b)_r}{r!\,(1+a-b)_r}\frac{(a+2r)_n}{n!}z^{n+r}\,. $$

[24] Andrews et al. [4], p.125 mistakenly have $(1-z)^{-a+2r}$ in this line of their proof.

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-4)^r (a/2)_r (a/2 + 1/2 - b)_r}{r!\,(1 + a - b)_r} \frac{(a + 2r)_{n-r}}{(n - r)!} z^n ,$$

by (1.1). Applying identity (1.10) in the form $(a/2)_r = \frac{(a)_{2r}}{2^{2r}(a/2+1/2)_r}$, we can write the coefficient C of z^n in the above expansion as

$$C = \sum_{r=0}^{n} \frac{(-1)^r (a)_{2r} (a/2 + 1/2 - b)_r}{r!\,(1 + a - b)_r} \frac{(a + 2r)_{n-r}}{(a/2 + 1/2)_r (n - r)!} ,$$

which by the identities $(a)_{n+r} = (a)_n (a + n)_r$, $\frac{(-1)^r}{(n-r)!} = \frac{(-n)_r}{n!}$ and $(a)_{2r}(a + 2r)_k = (a)_{k+2r}$ becomes

$$C = \sum_{r=0}^{n} \frac{(-n)_r (a)_{2r} (a/2 + 1/2 - b)_r}{n!\,r!\,(1 + a - b)_r} \frac{(a + 2r)_{n-r}}{(a/2 + 1/2)_r}$$

$$= \frac{(a)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r (a/2 + 1/2 - b)_r)(a + n)_r}{r!\,(1 + a - b)_r (a/2 + 1/2)_r}$$

$$= \frac{(a)_n}{n!}\ {}_3F_2\left(\begin{matrix} -n, a/2 + 1/2 - b, a + n; \\ 1 + a - b, a/2 + 1/2; \end{matrix}\ 1\right).$$

The Pfaff-Saalschütz Theorem (4.2) evaluates the resulting balanced ${}_3F_2(1)$, and hence this result can be written in the form

$$C = \frac{(a)_n}{n!} \frac{(1/2 + a/2\,)_n (1 - b - n)_n}{(1 + a - b)_n (1/2 - a/2 - n)_n} ,$$

which by identity (1.12) in the form $(1 - A - n)_n = (-1)^n (A)_n$, with $A = b$ and $A = \frac{1+a}{2}$, can finally be written as

$$C = \frac{(a)_n}{n!} \frac{(b)_n}{(1 + a - b)_n} .$$

This is also the coefficient of z^n in the expansion of ${}_2F_1(a, b; 1 + a - b; z)$, so

$${}_2F_1(a, b; 1 + a - b; z) = (1 - z)^{-a}\ {}_2F_1\left(\frac{a}{2}, \frac{1 + a}{2} - b; 1 + a - b; -\frac{4z}{(1 - z)^2}\right)$$

for $|z| < 3 - 2\sqrt{2}$, and the complete result follows by analytic continuation. ∎

Apart from the Pfaff-Saalschütz identity, there exist many other useful formulae for evaluating ${}_3F_2(1)$ series. H. Bateman established the following formula for a Saalschützian ${}_3F_2$. Proofs based on standard simplification techniques can be found in [75], p.87, Th. 30, and [82], p.50.

Theorem 4.3.4 *For n a non-negative integer, and a and b independent of n,*

$$ {}_3F_2\left(\begin{matrix} -n, & a+n, & \frac{1}{2}+\frac{a}{2}-b; \\ & 1+a-b, & \frac{1}{2}+\frac{a}{2}\;; \end{matrix}\; 1\right) = \frac{(b)_n}{(1+a-b)_n}. \tag{4.6} $$

In 1903, Dixon produced the theorem below which sums the well-poised ${}_3F_2$ series with unit argument (cf. [75], p. 92, Th. 33).

Theorem 4.3.5 *Dixon's Theorem*: *For Re*$(a-2b-2c) > -2$,

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c; & \end{matrix}\; 1\right) \\ &\quad = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma(1+a-b-c)}. \end{aligned} \tag{4.7} $$

Proof: Using the identity $\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n$, we have

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c; & \end{matrix}\; 1\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k\Gamma(1+a-b)\Gamma(1+a-c)}{k!\,\Gamma(1+a-b+k)\Gamma(1+a-c+k)} $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!\,(1+a)_{2k}} \frac{\Gamma(1+a+2k)\Gamma(1+a-b-c)}{\Gamma(1+a-b+k)\Gamma(1+a-c+k)}. $$

By Gauss' summation (3.13), $\frac{\Gamma(1+a+2k)\Gamma(1+a-b-c)}{\Gamma(1+a-b+k)\Gamma(1+a-c+k)} = {}_2F_1\left(\begin{matrix} b+k, & c+k; \\ 1+a+2k; & \end{matrix}\; 1\right)$, and hence by interchanging the order of summation we can write the above result in the form

$$
{}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c; \end{matrix}\ 1\right)
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_k (b)_{n+k} (c)_{n+k}}{k!\, n!\, (1+a)_{n+2k}}
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a)_k (b)_n (c)_n}{k!\, (n-k)!\, (1+a)_{n+k}}
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (-n)_k (a)_k (b)_n (c)_n}{k!\, n!\, (1+a+n)_k (1+a)_n}
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} {}_2F_1\left(\begin{matrix} -n, & a; \\ 1+a+n; \end{matrix}\ -1\right) \frac{(b)_n (c)_n}{n!\, (1+a)_n}.
$$

By applying Kummer's theorem (3.16) to this result, we obtain

$$
{}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c; \end{matrix}\ 1\right)
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(1+a+n)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1+\frac{a}{2}+n\right)\Gamma(1+a)} \frac{(b)_n (c)_n}{(1+a)_n}
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(1+a)_n}{\left(1+\frac{a}{2}\right)_n} \frac{(b)_n (c)_n}{(1+a)_n}
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b-c)\Gamma(1+a)} \, {}_2F_1\left(\begin{matrix} b, & c; \\ 1+\frac{a}{2}; \end{matrix}\ 1\right),
$$

and the desired result follows directly by applying Gauss' summation formula (3.13) to the Gauss hypergeometric function on the right side. ∎

When $c \to -\infty$, (4.7) reduces to Kummer's Theorem (3.16), and if $c = -n$, it yields the particular result

$$ {}_3F_2\left(\begin{matrix} a, & b, & -n; \\ 1+a-b, & 1+a+n; & \end{matrix} 1\right) = \frac{(1+a)_n \left(1+\frac{a}{2}-b\right)_n}{\left(1+\frac{a}{2}\right)_n (1+a-b)_n}. \tag{4.8} $$

By applying standard identities, Dixon's theorem (4.7) can also be written in the form

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c; & \end{matrix} 1\right) = \frac{\left(\frac{a}{2}\right)!\,(a-b)!\,(a-c)!\left(\frac{a}{2}-b-c\right)!}{(a)!\left(\frac{a}{2}-b\right)!\left(\frac{a}{2}-c\right)!\,(a-b-c)!}, \tag{4.9} $$

given as equation (IV) in [73], p.43.[25]

The generalisation of Dixon's theorem (4.7) given below, is Bailey's Equation (3.2.1) in [8], and provides a transformation between two ${}_3F_2(1)$ series. If one of the ${}_3F_2(1)$ series is well-poised, then so is the other. This result is originally due to Thomae.

Theorem 4.3.6 *If* $s = e + f - a - b - c$, *and* $Re(s) > 0,\ Re(a) > 0$, *then*

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ e, & & f; \end{matrix} 1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\, {}_3F_2\left(\begin{matrix} e-a, & f-a, & s; \\ s+b, & & s+c; \end{matrix} 1\right). \tag{4.10} $$

Proof: We first define $F \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}\, {}_3F_2\left(\begin{matrix} a, & b, & c; \\ & e, & f; \end{matrix} 1\right)$. Then

$$ F = \sum_{n=0}^{\infty} \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} \frac{(a)_n (b)_n (c)_n}{(e)_n (f)_n n!} = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n) n!}. $$

By Gauss' first summation theorem (3.13), ${}_2F_1(e-a, f-a; e+f-a+n; 1) = \frac{\Gamma(e+f-a+n)\Gamma(a+n)}{\Gamma(f+n)\Gamma(e+n)}$, and hence we can write

[25] In their Example 3.6.2, Petkovšek et al. use (4.9) to evaluate the combinatorial series $f(n) = \sum_k (-1)^k \binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k}$, by first writing it as $f(n) = \binom{4n}{2n}\, {}_3F_2\left(\begin{matrix} -2n, & -2n, & \frac{1}{2}; \\ 1, & -2n+\frac{1}{2}; & \end{matrix} 1\right)$. They also provide a "prettier and easier to remember form" of (4.9): $\sum_k (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$.

$$F = \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+f-a+n)n!} \, {}_2F_1\left(\begin{matrix} e-a, & f-a; \\ e+f-a+n; & \end{matrix} 1\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c+n)\Gamma(e-a+m)\Gamma(f-a+m)}{\Gamma(e+f-a+n+m)\Gamma(e-a)\Gamma(f-a)m!\,n!}.$$

When m and n are large enough, the above factors will ultimately all have the same sign, so that the double series is absolutely convergent and the order of summation can be interchanged to yield

$$F = \sum_{m=0}^{\infty} \frac{\Gamma(b)\Gamma(c)\Gamma(e-a+m)\Gamma(f-a+m)}{\Gamma(e+f-a+m)\Gamma(e-a)\Gamma(f-a)m!} \, {}_2F_1\left(\begin{matrix} b, & c; \\ e+f-a+m; & \end{matrix} 1\right).$$

By applying Gauss' theorem (3.13) to the Gauss function, we can write *F* in the form

$$\frac{\Gamma(b)\Gamma(c)}{\Gamma(e-a)\Gamma(f-a)} \sum_{m=0}^{\infty} \frac{\Gamma(e-a+m)\Gamma(f-a+m)}{\Gamma(e+f-a+m)m!} \frac{\Gamma(e+f-a+m)\Gamma(e+f-a+m-b-c)}{\Gamma(e+f-a+m-b)\Gamma(e+f-a+m-c)}$$

$$= \frac{\Gamma(b)\Gamma(c)\Gamma(e+f-a-b-c)}{\Gamma(e+f-a-b)\Gamma(e+f-a-c)} \, {}_3F_2\left(\begin{matrix} e-a, f-a, e+f-a-b-c; \\ e+f-a-b, e+f-a-c; \end{matrix} 1\right),$$

from which, by the original definition of *F*, we obtain

$${}_3F_2\left(\begin{matrix} a, & b, & c; \\ & e, & f; \end{matrix} 1\right)$$

$$= \frac{\Gamma(e)\Gamma(f)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(e+f-a-b)\Gamma(e+f-a-c)} \, {}_3F_2\left(\begin{matrix} e-a, f-a, e+f-a-b-c; \\ e+f-a-b, e+f-a-c; \end{matrix} 1\right),$$

and the theorem follows from the given relation $s = e+f-a-b-c$. ∎

We can use the above result to prove Watson's summation theorem, which is given as Bailey's Equation 3.3 (1) in [8].

Theorem 4.3.7 *Watson's theorem: For values for which the series is defined,*

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ \frac{a+b+1}{2}, & 2c; & \end{matrix} 1\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1+a+b}{2}\right)\Gamma\left(\frac{1-a-b}{2}+c\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1-a}{2}+c\right)\Gamma\left(\frac{1-b}{2}+c\right)}. \tag{4.11}$$

Proof: We first apply transformation (4.10) to the left side of the theorem, with $s = \frac{a+b+1}{2} + 2c - a - b - c = c - \frac{a+b-1}{2}$, to obtain

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ \frac{a+b+1}{2}, & 2c; & \end{matrix} 1\right)$$

$$= \frac{\Gamma\left(\frac{1+a+b}{2}\right)\Gamma(2c)\Gamma\left(\frac{1-a-b}{2}+c\right)}{\Gamma(a)\Gamma\left(\frac{1-a+b}{2}+c\right)\Gamma\left(\frac{1-a-b}{2}+2c\right)} \cdot {}_3F_2\left(\begin{matrix} 2c-a, & \frac{1+b-a}{2}, & \frac{1-a-b}{2}+c; \\ \frac{1-a-b}{2}+2c, & \frac{1-a+b}{2}+c; & \end{matrix} 1\right)$$

$$= \frac{\Gamma\left(\frac{1+a+b}{2}\right)\Gamma\left(\frac{1-a-b}{2}+c\right)}{\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1-b}{2}+c\right)}\left\{\frac{\Gamma(2c)\Gamma\left(1+c-\frac{a}{2}\right)\Gamma\left(\frac{a}{2}\right)}{\Gamma(c)\Gamma(1+2c-a)\Gamma(a)}\right\},$$

by applying Dixon's Theorem (4.7) to the well-poised ${}_3F_2$. To simplify the gamma ratio in parenthesis, we now apply Legendre's duplication theorem (2.17) in the form $\frac{\Gamma(x)}{\Gamma(2x)} = \frac{\sqrt{\pi}}{2^{2x-1}\Gamma(x+\frac{1}{2})}$, with $2x = 2c, 2x = 1 + 2c - a$ and $2x = a$, and the desired result follows. ■

Whipple also provided the following identity for evaluating a ${}_3F_2(1)$ in terms of a ratio of gamma functions, given as Bailey's Equation 3.4 (1) in [8].

Theorem 4.3.8 *Whipple's theorem: For $a + b = 1$ and $e + f = 2c + 1$,*

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ & e, & f; \end{matrix} 1\right) = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma\left(\frac{a+e}{2}\right)\Gamma\left(\frac{a+f}{2}\right)\Gamma\left(\frac{b+e}{2}\right)\Gamma\left(\frac{b+f}{2}\right)}. \tag{4.12}$$

Proof: Under the given conditions, we apply transformation (4.10) with $s = c$, to obtain

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ e, & f; & \end{matrix} 1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(c)}{\Gamma(a)\Gamma(c+b)\Gamma(2c)} \, {}_3F_2\left(\begin{matrix} e-a, & f-a, & c; \\ c+b, & 2c; & \end{matrix} 1\right). \tag{4.13} $$

The resulting ${}_3F_2$ on the right can be summed by Watson's theorem (4.11), as the given conditions ensure that $\frac{(e-a)+(f-a)+1}{2} = c+b$. Hence, we can write (4.13) in the form

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ e, & f; & \end{matrix} 1\right) $$

$$ = \frac{\Gamma(e)\Gamma(f)\Gamma(c)}{\Gamma(a)\Gamma(c+b)\Gamma(2c)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{e+f-2a+1}{2}\right)\Gamma\left(\frac{1-e-f+2a}{2}+c\right)}{\Gamma\left(\frac{1+e-a}{2}\right)\Gamma\left(\frac{1+f-a}{2}\right)\Gamma\left(\frac{1-e+a}{2}+c\right)\Gamma\left(\frac{1-f+a}{2}+c\right)} $$

$$ = \frac{\Gamma(e)\Gamma(f)\pi}{\Gamma(a)\Gamma(c+b)2^{2c-1}} \frac{\Gamma\left(\frac{e+f-2a+1}{2}\right)\Gamma\left(\frac{1-e-f+2a}{2}+c\right)}{\Gamma\left(\frac{1+e-a}{2}\right)\Gamma\left(\frac{1+f-a}{2}\right)\Gamma\left(\frac{1-e+a}{2}+c\right)\Gamma\left(\frac{1-f+a}{2}+c\right)}, $$

by Legendre's duplication theorem (2.17). The final result follows from substitutions based on the given relations $a+b=1$ and $e+f=2c+1$. ∎

An alternative form for Whipple's formula (4.12) is given in [64], p.45:

$$ {}_3F_2\left(\begin{matrix} a, & 1-a, & c; \\ e, & 2c+1-e; & \end{matrix} 1\right) = \frac{\Gamma(e)\Gamma(2c+1-e)\Gamma\left(\frac{1+a+e}{2}\right)\Gamma\left(\frac{1-a-e}{2}+c\right)}{\Gamma(a+e)\Gamma\left(\frac{1+a-e}{2}+c\right)\Gamma\left(\frac{1-a+e}{2}\right)\Gamma(1-a+2c-e)}. \tag{4.14} $$

Many recent results regarding hypergeometric identities are related to the classical identities of Dixon, Whipple and Watson for ${}_3F_2(1)$. These newer results, discussed in Chapter 5, in some cases generalise these classical results, and in other cases present summation formulae contiguous to these established identities.[26]

[26] Using the usual definition of contiguous hypergeometric functions, there exist $(2p+2q)$ functions contiguous to ${}_pF_q$. An extensive list of such relations can be found in [75], pp.80-85.

4.3.2 Evaluation and transformation identities for ${}_{p+1}F_p(\pm 1)$

While the list of identities for ${}_{p+1}F_p(\pm 1)$ is too extensive to address exhaustively in this work, we provide a sample of such relations. We first provide some evaluation formulae for functions with arguments ± 1. Berndt [12], Entry 5, proves Dixon's Theorem (4.7) by substituting $d = -\frac{a}{2}$ into Dougall's formula below for a ${}_5F_4$ series.

Theorem 4.3.9 *Dougall's formula*: *For* $Re(a+b+c+d+1) > 0$,

$$
\begin{aligned}
&{}_5F_4\left(\begin{matrix} a, & \frac{a}{2}+1, & -b,-c,-d; \\ \frac{a}{2}, a+b+1, a+c+1, a+d+1; \end{matrix} \; 1\right) \\
&= \frac{\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(a+d+1)\Gamma(a+b+c+d+1)}{\Gamma(a+1)\Gamma(a+b+c+1)\Gamma(a+b+d+1)\Gamma(a+c+d+1n)}.
\end{aligned} \tag{4.15}
$$

The identity below transforms a terminating nearly-poised ${}_4F_3$ into a ${}_5F_4$ series, and is found in [9], p.189.

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} -n, & b, & c, & d; \\ 1-n-b, 1-n-c, w; \end{matrix} \; 1\right) \\
&= \frac{(w-d)_n}{w_n} \; {}_5F_4\left(\begin{matrix} d, & 1-n-b-c, & -\frac{n}{2}, \frac{1-n}{2}, 1-n-w; \\ 1-n-b, 1-n-c, \frac{1+d-w-n}{2}, 1+(d-w-n)/2; \end{matrix} \; 1\right)
\end{aligned} \tag{4.16}
$$

Andrews et al. [4], Theorem 3.3.3, provide the result below, which was established by Whipple in 1926. This identity transforms one balanced ${}_4F_3$ series to another, and provides a useful basis for further transformation results.

Theorem 4.3.10 *For* $a+b+c-n+1 = d+e+f$,

$$
{}_4F_3\left(\begin{matrix} -n, a, b, c; \\ d, e, f; \end{matrix} \; 1\right) = \frac{(e-a)_n (f-a)_n}{(e)_n (f)_n} \; {}_4F_3\left(\begin{matrix} -n, a, d-b, d-c; \\ d, a+1-n-e, a+1-n-f; \end{matrix} \; 1\right). \tag{4.17}
$$

Proof: We first write Euler's transformation (3.27) in the forms ${}_2F_1\left(\begin{matrix} a, b; \\ c; \end{matrix} z\right) = (1-z)^{c-a-b}\ {}_2F_1\left(\begin{matrix} c-a, c-b; \\ c; \end{matrix} z\right)$, and ${}_2F_1\left(\begin{matrix} d, e; \\ f; \end{matrix} z\right) = (1-z)^{f-d-e} \times {}_2F_1\left(\begin{matrix} f-d, f-e; \\ f; \end{matrix} z\right)$. We then assume that $c-a-b = f-d-e$, and equate the two results to obtain

$$ {}_2F_1\left(\begin{matrix} a, b; \\ c; \end{matrix} z\right) {}_2F_1\left(\begin{matrix} f-d, f-e; \\ f; \end{matrix} z\right) = {}_2F_1\left(\begin{matrix} c-a, c-b; \\ c; \end{matrix} z\right) {}_2F_1\left(\begin{matrix} d, e; \\ f; \end{matrix} z\right). \tag{4.18} $$

We can write the left side of this equation as

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_k(b)_k z^k (f-d)_n (f-e)_n z^n}{(c)_k k!\,(f)_n (n)!} $$
$$ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_k(b)_k z^k (f-d)_{n-k} (f-e)_{n-k} z^{n-k}}{(c)_k k!\,(f)_{n-k} (n-k)!}. $$

Hence, by applying identities (1.6) and (1.12) we can write the coefficient of z^n on the left side of (4.18) as

$$ \frac{(f-d)_n (f-e)_n}{n!\,(f)_n}\ {}_4F_3\left(\begin{matrix} a, b, 1-f-n, -n; \\ c, d-f-n+1, e-f-n+1; \end{matrix} 1\right). \tag{4.19} $$

Similarly, the coefficient of z^n on the right side of (4.18) can be written as

$$ \frac{(d)_n (e)_n}{n!\,(f)_n}\ {}_4F_3\left(\begin{matrix} c-a, c-b, 1-f-n, -n; \\ c, 1-d-n, 1-e-n; \end{matrix} 1\right). \tag{4.20} $$

By equating expressions (4.19) and (4.20), we obtain

$$ {}_4F_3\left(\begin{matrix} a, b, 1-f-n, -n; \\ c, d-f-n+1, e-f-n+1; \end{matrix} 1\right) $$
$$ = \frac{(d)_n (e)_n}{(f-d)_n (f-e)_n}\ {}_4F_3\left(\begin{matrix} c-a, c-b, 1-f-n, -n; \\ c, 1-d-n, 1-e-n; \end{matrix} 1\right). $$

The desired result then follows by a change of parameters and by applying identity (1.12) in the form $(1-\alpha-n)_n = (-1)^n(\alpha)_n$. ■

In 1912, William F. Sheppard (1863-1936) used the above theorem to establish the following result, which provides a transformation between two terminating ${}_3F_2(1)$ series (cf. [4], p.141, Corollary 3.3.4).

Corollary 4.3.11 *For n a non-negative integer,*

$$ {}_3F_2\left(\begin{matrix}-n, a, b; \\ d, e;\end{matrix} 1\right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} \, {}_3F_2\left(\begin{matrix}-n, a, a+b-n-d-e+1; \\ a-n-d+1; a-n-e+1;\end{matrix} 1\right). \tag{4.21}$$

Proof: In transformation (4.17), we keep $f-c$ fixed and let $f \to \infty$ to obtain

$$ {}_3F_2\left(\begin{matrix}-n, a, b; \\ d, e;\end{matrix} 1\right) = \frac{(e-a)_n}{(e)_n} \, {}_3F_2\left(\begin{matrix}-n, a, d-b; \\ d; a-n-e+1;\end{matrix} 1\right),$$

and the corollary is proved by applying this transformation again to the result ${}_3F_2\left(\begin{matrix}-n, a, d-b; \\ d; a-n-e+1;\end{matrix} 1\right)$. ■

The above identity provides elegant confirmation of the Pfaff-Saalschütz identity. If the series ${}_3F_2\left(\begin{matrix}-n, a, b; \\ d, e;\end{matrix} 1\right)$ is balanced, so that $e = 1 — d - n + a + b$, then (4.21) becomes

$$ {}_3F_2\left(\begin{matrix}-n, a, b; \\ d, 1—d-n+a+b;\end{matrix} 1\right) $$
$$ = \frac{(d-a)_n(1—d-n+b)_n}{(d)_n(1—d-n+a+b)_n} \, {}_3F_2\left(\begin{matrix}-n, a, 0; \\ a-n-d+1; d-b;\end{matrix} 1\right).$$

By applying the identity $(1-\alpha-n)_n = (-1)^n(\alpha)_n$, we can then rewrite this result in the form

$$ {}_3F_2\left(\begin{matrix}-n, \quad a, \quad b; \\ d, 1-d+a+b-n;\end{matrix} 1\right) = \frac{(d-a)_n(d-b)_n}{(d)_n(d-a-b)_n},$$

which is the Pfaff-Saalschütz identity (4.2).

In 1836, Kummer provided the further result below, which follows from (4.17), and is given as Corollary 3.3.5 in [4].

Theorem 4.3.12 *For values of the parameters for which the two series converge,*

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ d, & & e; \end{matrix}\ 1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}\ {}_3F_2\left(\begin{matrix} a, d-b, d-c; \\ d, d+e-b-c; \end{matrix}\ 1\right). \tag{4.22} $$

Proof: Let $n \to \infty$ and keep $f + n$ fixed, so that the series on the left side of (4.17) becomes ${}_3F_2\left(\begin{matrix} a, & b, & c; \\ & d, & e; \end{matrix}\ 1\right)$. To evaluate the right side of (4.17), we write

$$ \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n} = \frac{\Gamma(e-a+n)\Gamma(f-a+n)}{\Gamma(e-a)\Gamma(f-a)} \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+n)\Gamma(f+n)}. \tag{4.23} $$

By Euler's reflection formula (2.16), we can establish that $\Gamma(f) = \frac{\pi}{\Gamma(1-f)\sin{(\pi f)}}$, $\frac{1}{\Gamma(f+n)} = \frac{\Gamma(-n-f+1)\sin[\pi(f+n)]}{\pi}$, $\frac{1}{\Gamma(f-a)} = \frac{\Gamma(a-f+1)\sin[\pi(f-a)]}{\pi}$ and $\Gamma(f-a+n) = \frac{\pi}{\Gamma(a-f-n+1)\sin{[\pi(f-a+n)]}}$. We further note that for $n \in \mathbb{N}$, $\sin[\pi(A+n)] = \begin{cases} \sin \pi A, & \text{if } n \text{ is even} \\ -\sin \pi A, & \text{if } n \text{ is odd} \end{cases}$, and hence $\frac{\sin[\pi(f+n)]}{\sin{(\pi f)}} \frac{\sin[\pi(f-a)]}{\sin{[\pi(f-a+n)]}} = 1$. By using these relations in (4.23), we obtain

$$ \begin{aligned} \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n} &= \frac{\Gamma(e)\Gamma(a-f+1)}{\Gamma(e-a)\Gamma(a-f-n+1)} \frac{\Gamma(-n-f+1)\Gamma(e-a+n)}{\Gamma(1-f)\Gamma(e+n)} \\ &= \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} \left(\frac{\Gamma(a-f+1)\Gamma(e-a+n)}{\Gamma(1-f)\Gamma(e+n)}\right), \end{aligned} $$

from the restriction $a+b+c-n+1 = d+e+f$ in (4.17). Now as $n \to \infty$ and $-f \to \infty$, the above ratio in brackets tends to unity, and the ${}_4F_3$ series on the right side of (4.17) tends to ${}_3F_2\left(\begin{matrix} a, d-b, d-c; \\ d, a+1-n-f; \end{matrix}\ 1\right) = {}_3F_2\left(\begin{matrix} a, d-b, d-c; \\ d, d+e-b-c; \end{matrix}\ 1\right)$, and the desired result follows. ■

By applying Kummer's result (4.22) to itself, we obtain the following transformation of Thomae, given as Corollary 3.3.6 in [4].

Theorem 4.3.13 *For $s = d + e - a - b - c$,*

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ d, & e; & \end{matrix}\; 1\right) = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\; {}_3F_2\left(\begin{matrix} d-a, e-a, s; \\ s+b, s+c; \end{matrix}\; 1\right). \tag{4.24} $$

Proof: By (4.22), we have

$$ {}_3F_2\left(\begin{matrix} d-b, d-c, a; \\ d+e-b-c, d; \end{matrix}\; 1\right) $$
$$ = \frac{\Gamma(d)\Gamma(e-a)}{\Gamma(b)\Gamma(d+e-b-a)}\; {}_3F_2\left(\begin{matrix} d-b, e-b, d+e-a-b-c; \\ d+e-b-c, d+e-b-a; \end{matrix}\; 1\right). $$

Substituting this result into the right side of (4.22), we obtain

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ d, & e; & \end{matrix}\; 1\right) $$
$$ = \frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(b)\Gamma(d+e-b-c)\Gamma(d+e-b-a)}\; {}_3F_2\left(\begin{matrix} d-b, e-b, d+e-a-b-c; \\ d+e-b-c, d+e-b-a; \end{matrix}\; 1\right). $$

By an exchange of numerator parameters $a \leftrightarrow b$, this result can be written in the form

$$ {}_3F_2\left(\begin{matrix} b, & a, & c; \\ d, & e; & \end{matrix}\; 1\right) $$
$$ = \frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(d+e-a-c)\Gamma(d+e-a-b)}\; {}_3F_2\left(\begin{matrix} d-a, e-a, d+e-a-b-c; \\ d+e-a-c, d+e-a-b; \end{matrix}\; 1\right), $$

and the theorem is proved for $s = d + e - a - b - c$. ■

Whipple also provides an important formula which transforms a terminating well-poised ${}_7F_6$ to a balanced ${}_4F_3$. We first require the following lemma and its corollary (cf. [4], p.144, Lemma 3.4.2).

Lemma 4.3.14 *For m a non-negative integer,*

$$ {}_5F_4\left(\begin{matrix} a, b, c, d, -m; \\ a-b+1, a-c+1, a-d+1, a+m+1; \end{matrix}\; 1\right) $$
$$ = \frac{(a+1)_m(a/2-d+1)_m}{(a/2+1)_m(a-d+1)_m}\; {}_4F_3\left(\begin{matrix} \frac{a}{2}, a-b-c+1, d, -m; \\ a-b+1, a-c+1, d-m-\frac{a}{2}; \end{matrix}\; 1\right). \tag{4.25} $$

Proof: By the Pfaff-Saalchütz identity (4.2) for a terminating, balanced ${}_3F_2(1)$, we have

$$\sum_{r=0}^{n}\frac{(-n)_r(a-b-c+1)_r(a+n)_r}{r!\,(a-b+1)_r(a-c+1)_r}=\frac{(c)_n(1-b-n)_n}{(a-b+1)_n(c-a-n)_n}$$

$$=\frac{(c)_n(b)_n}{(a-b+1)_n(a-c+1)_n}.$$

Using this result and the relation $\sum_{j=i}^{n}\sum_{k=j}^{n}a_{j,k}=\sum_{1\le j\le k\le n}a_{j,k}=\sum_{k=i}^{n}\sum_{j=i}^{k}a_{j,k}$ (cf. [45], p.36), we can write the left side of (4.25) in the form

$$LHS=\sum_{n=0}^{m}\frac{(-m)_n(a)_n(d)_n}{n!\,(a-d+1)_n(a+m+1)_n}\sum_{r=0}^{n}\frac{(-n)_r(a-b-c+1)_r(a+n)_r}{r!\,(a-b+1)_r(a-c+1)_r}$$

$$=\sum_{r=0}^{m}\sum_{n=r}^{m}\frac{(-1)^r(a)_{n+r}(d)_n(a-b-c+1)_r(-m)_n}{(n-r)!\,r!\,(a-d+1)_n(a+m+1)_n(a-b+1)_r(a-c+1)_r},$$

$$=\sum_{r=0}^{m}\sum_{t=0}^{m-r}\frac{(-1)^r(a)_{t+2r}(d)_{t+r}(-m)_{t+r}(a-b-c+1)_r}{t!\,r!\,(a-b+1)_r(a-c+1)_r r!\,(a-d+1)_{t+r}(a+m+1)_{t+r}}$$

$$=\sum_{r=0}^{m}\frac{(-1)^r(a)_{2r}(d)_r(-m)_r(a-b-c+1)_r}{r!\,(a-b+1)_r(a-c+1)_r(a-d+1)_r(a+m+1)_r}$$

$$\times\sum_{t=0}^{m-r}\frac{(a+2r)_t(d+r)_t(-m+r)_t}{t!\,(a-d+1+r)_t(a+m+1+r)_t}=S_1\times S_2.$$

Considering each sum separately, we have

$$S_1=\sum_{r=0}^{m}\frac{2^{2r}(-1)^r(a/2)_r(a/2+1/2)_r(d)_r(-m)_r(a-b-c+1)_r}{r!\,(a-b+1)_r(a-c+1)_r(a-d+1)_r(a+m+1)_r}.$$

We now apply Dixon's theorem (4.7) to S_2 with $a \to a+2r, b \to -(d+r), c \to m-r$, to obtain

$$S_2 = {}_3F_2\left(\begin{matrix} a+2r, & d+r, & -m+r; \\ a-d+1+r, a+m+1+r; & & \end{matrix} 1\right)$$

$$= \frac{\Gamma\left(1+\frac{a}{2}+r\right)\Gamma(1+a+r-d)\Gamma(1+a+m+r)\Gamma\left(1+\frac{a}{2}-d+m-r\right)}{\Gamma(1+a+2r)\Gamma\left(1+\frac{a}{2}-d\right)\Gamma\left(1+\frac{a}{2}+m\right)\Gamma(1+a-d+m)},$$

and the result follows from simplification through standard factorial identities. ■

A useful corollary follows from the above lemma (cf. [4], p.145, Corollary 3.4.3).

Corollary 4.3.15

$${}_5F_4\left(\begin{matrix} a, \frac{a}{2}+1, c, d, -m; \\ \frac{a}{2}, a-c+1, a-d+1, a+m+1; \end{matrix} 1\right) = \frac{(a+1)_m(a-c-d+1)_m}{(a-c+1)_m(a-d+1)_m} \tag{4.26}$$

Proof: In Lemma 4.3.14, we let $b = \frac{a}{2}+1$ so that the left side of theorem is

$$LHS = \frac{(a+1)_m(a/2-d+1)_m}{(a/2+1)_m(a-d+1)_m}\ {}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a}{2}-c, d, -m; \\ \frac{a}{2}, a-c+1, d-m-\frac{a}{2}; \end{matrix} 1\right).$$

The resulting ${}_4F_3$ thus reduces to a balanced ${}_3F_2$ so that by (4.2), we have

$$LHS = {}_4F_3\left(\begin{matrix} a, \frac{a}{2}+1, c, d, -m; \\ \frac{a}{2}, a-c+1, a-d+1, a+m+1; \end{matrix} 1\right)$$

$$= \frac{(a+1)_m\left(\frac{a}{2}-d+1\right)_m}{\left(\frac{a}{2}+1\right)_m(a-d+1)_m} \times \frac{\left(\frac{a}{2}+1\right)_m(a-c+1-d)_m}{(a-c+1)_m\left(\frac{a}{2}-d+1\right)_m}$$

$$= \frac{(a+1)_m(a-c-d+1)_m}{(a-c+1)_m(a-d+1)_m},$$

which proves the corollary. ■

We are now ready to prove Whipple's theorem, which transforms a terminating well-poised ${}_7F_6$ to a balanced ${}_4F_3$ (cf. [4], p.145, Th. 3.4.4).

Theorem 4.3.16 *Whipple's theorem: For m a non-negative integer,*

$$ {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, -m; \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1, a+m+1; \end{matrix} 1\right) $$

$$ = \frac{(a+1)_m(a-d-e+1)_m}{(a-d+1)_m(a-e+1)_m}\ {}_4F_3\left(\begin{matrix} a-b-c+1, d, e, -m; \\ a-b+1, a-c+1, d+e-a-m; \end{matrix} 1\right). \tag{4.27} $$

Proof: Using the same techniques as in Lemma 4.3.14, we obtain

$$ {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, -m; \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1, a+m+1; \end{matrix} 1\right) $$

$$ = \sum_{n=0}^{m} \frac{(a)_n(a/2+1)_n(d)_n(e)_n(-m)_n}{n!\,(a/2)_n(a-d+1)_n(a-e+1)_n(a+m+1)_n} \sum_{r=0}^{n} \frac{(-n)_r(a-b-c+1)_r(a+n)_r}{r!\,(a-b+1)_r(a-c+1)_r} $$

$$ = \sum_{r=0}^{m} \frac{(a)_{2r}\left(\frac{a}{2}+1\right)_r (d)_r(e)_r(-m)_r}{r!\,(a-b+1)_n(a-c+1)_n\left(\frac{a}{2}\right)_n (a-d+1)_r(a-e+1)_r(a+m+1)_r} $$

$$ \times \sum_{t=0}^{m-r} \frac{(a+2r)_t\left(\frac{a}{2}+r+1\right)_t (d+r)_t\,(e+r)_t(-m+r)_t}{t!\left(\frac{a}{2}+r\right)_t (a-d+r+1)_t(a+m+r+1)_t}. $$

The inner sum can be evaluated by Corollary 4.3.15 and the final result follows after further series manipulation. ■

Setting $1+2a = b+c+d+e-m$ in Whipple's Theorem (4.27) reduces the ${}_4F_3$ to a balanced ${}_3F_2$ which can be summed, and the result is another formula of Dougall, given as Theorem 3.5.1 in [4].

Theorem 4.3.17 *Dougalls' theorem*: *For* $1 + 2a = b + c + d + e - m, m \in \mathbb{N}_0$,

$$ {}_7F_6\left(\begin{matrix} a, & \frac{a}{2}+1, & b, & c, & d, & e, & -m; \\ \frac{a}{2}, & 1+a-b, & 1+a-c; & 1+a-d, & 1+a-e, & 1+a+m; & \end{matrix} 1\right) = \frac{(a+1)_m(a-b-c+1)_m(a-b-d+1)_m(a-c-d+1)_m}{(a-b+1)_m(a-c+1)_m(a-d+1)_m(a-b-c-d+1)_m}. \tag{4.28} $$

Substituting $b = 2a - c - d - e + m + 1$ in Dougall's theorem (4.28) and letting $m \to \infty$ yields the following corollary.

Corollary 4.3.18

$$ {}_5F_4\left(\begin{matrix} a, & \frac{a}{2}+1, & c, & d, & e; \\ \frac{a}{2}, & a-c+1, & a-d+1, & a-e+1; & \end{matrix} 1\right) = \frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+1)}{\Gamma(1+a)\Gamma(a-d-e+1)\Gamma(a-c-e+1)\Gamma(a-c-d+1)} \tag{4.29} $$

Dixon's formula (4.7) follows from this corollary by taking $e = a/2$. Taking $b + c = a + 1$ in (4.29) yields Bailey's 1935 identity below, which evaluates a well-poised ${}_4F_3$ series for argument $z = -1$ (cf. [4], p.148, Corollary 3.5.3).

Corollary 4.3.19

$$ {}_4F_3\left(\begin{matrix} a, & \frac{a}{2}+1, & c, & d; \\ \frac{a}{2}, & 1+a-c, & 1+a-d; & \end{matrix} -1\right) = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}. \tag{4.30} $$

Apart from evaluation identities, there also exist linear, quadratic and cubic transformations for ${}_{p+1}F_p$ series for certain values of the parameters. The following linear transformation of a nearly-poised ${}_3F_2$ is provided by Bailey [9], p.190.

$$(1-z)^{2a-1}{}_3F_2\left(\begin{matrix}2a-1, a+\frac{1}{2}, a-b-\frac{1}{2}; \\ a-\frac{1}{2}, a+b+\frac{1}{2};\end{matrix} z\right)$$
$$= \quad (1-z)^{2b-1}{}_3F_2\left(\begin{matrix}2b-1, b+\frac{1}{2}, b-a-\frac{1}{2}; \\ b-\frac{1}{2}, a+b+\frac{1}{2};\end{matrix} z\right). \tag{4.31}$$

Whipple produced a great many transformation identities, one of which is given as Theorem 31 in [75].

Theorem 4.3.20 *Whipple's theorem*: *For $n \epsilon \mathbb{N}$ and b and c independent of n,*

$${}_3F_2\left(\begin{matrix}-n, & b, & c; \\ 1-b-n, 1-c-n;\end{matrix} z\right)$$
$$= (1-z)^n\, {}_3F_2\left(\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-b-c-n; \\ 1-b-n, 1-c-n;\end{matrix} \frac{-4z}{(1-z)^2}\right). \tag{4.32}$$

Proof: Starting with a Gauss hypergeometric function, we have

$${}_2F_1\left(\begin{matrix}b, c; \\ b+c;\end{matrix} t[(1-z)+zt]\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(b)_n(c)_n(1-z)^{n-k}t^{n+k}z^k}{k!\,(n-k)!\,(b+c)_n}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(b)_{n-k}(c)_{n-k}(1-z)^{n-2k}t^n z^k}{k!\,(n-2k)!\,(b+c)_{n-k}},$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-n)_{2k} (b)_n (c)_n (1-b-c-n)_k (-1)^k (1-z)^{n-2k} t^n z^k}{k!\, n!\, (1-b-n)_k (1-c-n)_k (b+c)_n}$$

$$= \sum_{n=0}^{\infty} (1-z)^n \, {}_3F_2\left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-b-c-n; \\ 1-b-n, 1-c-n; \end{matrix} \frac{-4z}{(1-z)^2} \right) \frac{(b)_n (c)_n t^n}{n!\, (b+c)_n}. \tag{4.33}$$

Using similar techniques, we can expand the original Gauss function in different powers of t, to obtain

$${}_2F_1\left(\begin{matrix} b, c; \\ b+c; \end{matrix} t[1-z(1-t)] \right) = \sum_{n=0}^{\infty} {}_3F_2\left(\begin{matrix} -n, \ b, \ c; \\ 1-b-n, 1-c-n; \end{matrix} z \right) \frac{(b)_n (c)_n t^n}{n!\, (b+c)_n}, \tag{4.34}$$

and the theorem is proved by comparing results (4.33) and (4.34). ■

Having established some of the best-known classical results for hypergeometric functions, we will finally consider one more particular such function before presenting more recent findings in the field of hypergeometric identities.

4.4 The confluent hypergeometric function

We have so far discussed the function ${}_pF_q$ when $p \leq q+1$ so that the series has a radius of convergence. For $q \in \{0, 1\}$ we have encountered the exponential function ${}_0F_0$, the binomial function ${}_1F_0$, the Bessel function ${}_0F_1$ and the Gauss function ${}_2F_1$. We now complete this set by considering the important function ${}_1F_1$.

The ${}_1F_1$ function has been named the confluent hypergeometric function, also known as the Pochhammer-Barnes function or Kummer series, and it is very useful in analysis as many special functions can be obtained from this function through a suitable choice of parameters. Some cases are exponential integrals, error functions, Hermite and Laguerre polynomials, Coulomb wave functions, parabolic cylinder functions and Bessel functions (cf. [81], [84], p.171). The confluent hypergeometric function thus has a wide range of applications in fields such as wave mechanics, quantum theory, hydrodynamics, acoustics, optics, random walk theory and statistics.

The confluent hypergeometric function is the result of a limiting process known as confluence, in which one differential equation is derived from another by making two or more singularities tend to coincide. Consider again the Gauss hypergeometric equation $z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0$, with regular single points $z=0, z=1$ and $z=\infty$, having $u = {}_2F_1(a,b;c;z)$ as one solution. The confluent hypergeometric function arises when two of these regular single points merge into one, by replacing z by z/b and letting $b \to \infty$. As $\frac{(b)_n}{b^n} \to 1$, we then have $\lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}$, which is convergent for finite z and for $c \neq 0,-1,-2,\dots$. This result is a ${}_1F_1$ function, commonly denoted by Humbert's symbol $\Phi(a;c;z)$, or by $\mathrm{M}(a;c;z)$ (cf. [9], p.248). The elementary properties of this function were given by Kummer in 1836.

Definition 4.4.1 *For* $|z| < \infty$, $c \neq 0,-1,-2,\dots$, *the confluent hypergeometric function is defined to be*

$$\Phi(a;c;z) = {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!\,(c)_k} z^k. \tag{4.35}$$

In the hypergeometric equation (3.6), we similarly replace z by z/b to show that the hypergeometric function ${}_2F_1\left(a,b;c;\frac{z}{b}\right)$ is a solution of the differential equation $z/b\left(1-\frac{z}{b}\right)u'' + \left\{c - \left(1+\frac{a+1}{b}\right)z\right\}u' - au = 0$. Again letting $b \to \infty$, we have that $\mathrm{u} = \Phi(a;c;z)$ is a solution to Kummer's confluent hypergeometric equation

$$zu'' + (c-z)u' - au = 0, \tag{4.36}$$

with a regular singular point at the origin (cf. [78], p.42, Eq. (3.1)). It can also be shown from first principle substitution that the confluent hypergeometric function is a solution to Kummer's equation (4.36) (cf. [60], p.262, [83] p.33).

Analogous to the earlier work with the Gauss hypergeometric equation, we can obtain a second linear independent solution of Kummer's confluent hypergeometric equation

(4.36) by assuming $|\arg z| < \pi$ and using the substitution $u = z^{1-c}v$. It is then easy to show that Kummer's equation becomes an equation of the same form, i.e. $zv'' + (c' - z)v' - a'v = 0$, with parameters $a' = 1 + a - c$ and $c' = 2 - c$, so that the function $z^{1-c}\Phi(1 + a - c; 2 - c; z)$ is also a solution of Kummer's equation if $c \neq 2, 3, \ldots$ (cf. [60], p.263, [78], p.42). Thus, if $c \neq 0, \pm 1, \pm 2, \ldots$ the general solution of the confluent hypergeometric equation can be written in the form
$u = A\Phi(a; c; z) + Bz^{1-c}\Phi(1 + a - c; 2 - c; z)$, for $|\arg z| < \pi$.[27]

The confluent hypergeometric function is not an analytic function of *c*, as the function has simple poles at $c = 0, -1, -2, \ldots$. However, from the result

$$\lim_{c \to 1-m} \frac{1}{\Gamma(c+n)} = \begin{cases} 0, & n = 0, 1, 2, \ldots, m-1 \\ \dfrac{1}{(n-m)!}, & n = m, m+1, m+2, \ldots \end{cases},$$

we can conclude that

$$\lim_{c \to 1-n} \frac{\Phi(a; c; z)}{\Gamma(c)} = \frac{(a)_n x^n}{n!} \Phi(a + n; 1 + n; z),$$

so that $\Phi^* = \frac{\Phi(a;c;z)}{\Gamma(c)}$ is an analytic function of *c* as well as of *a* and *x* (cf. [81], p.4).

Some common functions that can be expressed as confluent hypergeometric functions are given below.

$$\Phi(a; a; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\Phi(1; 2; z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{e^z - 1}{z}$$

$$\Phi(-2; 1; z) = 1 - 2z + \frac{z^2}{2}$$

[27] Slater [81] classifies four confluent hypergeometric functions: Kummer's function $\Phi(a; b; z)$, its associated solution $z^{1-c}\Phi(1 + a - c; 2 - c; z)$, and the two Whittaker functions $M_{k,m}(x) = e^{-\frac{x}{2}} x^{\frac{1}{2}+m} {}_1F_1\left(\frac{1}{2} + m - k; 1 + 2m; z\right)$, $M_{k,-m}(x) = e^{-\frac{x}{2}} x^{\frac{1}{2}-m} {}_1F_1\left(\frac{1}{2} - m - k; 12m; z\right)$, often encountered in applications.

As with the Gauss hypergeometric function, we can use term-by-term differentiation, together with properties of the rising factorial, to establish the following derivative properties for the confluent function.

$$\frac{d}{dz}\,\Phi(a;c;z) = \frac{a}{c}\Phi(a+1;c+1;z)$$

$$\frac{d^n}{dz^n}\,\Phi(a;c;z) = \frac{(a)_n}{(c)_n}\Phi(a+n;c+n;z)$$

Slater [81], pp.15-16, provides a list of 23 such derivative identities, and from them deduces seven addition theorems for the confluent function, three of which are given below.

$$\Phi(a;b;x+y) = \sum_{n=0}^{\infty}\frac{(a)_n y^n}{(b)_n n!}\,\Phi(a+n;b+n;x) \tag{4.37}$$

$$= \left(\frac{x}{x+y}\right)^a \sum_{n=0}^{\infty}\frac{(a)_n y^n}{(x+y)_n n!}\,\Phi(a+n;b;x) \tag{4.38}$$

$$= e^y \sum_{n=0}^{\infty}\frac{(b-a)_n(-y)^n}{(b)_n n!}\,\Phi(a;b+n;x) \tag{4.39}$$

Several properties for the Gauss hypergeometric function have analogues for the confluent function, such as the contiguous relations which are discussed in detail in [9], p.254, [60], p.262, [75], p.124 and [83], p.35. The three contiguous relations given below form a canonical set.

$$(a-c+1)\Phi(a;c;z) = a\Phi(a+1;c;z) - (b-1)\Phi(a;c-1;z) \tag{4.40}$$

$$c(a+z)\Phi(a;c;z) = ac\Phi(a+1;c;z) - (a-c)z\Phi(a;c+1;z) \tag{4.41}$$

$$c\Phi(a;c;z) = c\;\Phi(a-1;c;z) + z\Phi(a;c+1;z) \tag{4.42}$$

There also exists the following useful integral representation for the confluent function (cf. [81], p.34).

Theorem 4.4.2 *For* $Re(c) > Re(a) > 0$,

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt. \tag{4.43}$$

Proof: By expanding the exponential in the integral on the right side of the theorem, we obtain a beta integral.

$$\int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^1 t^{a+n-1} (1-t)^{c-a-1} dt$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(a)(a)_n \Gamma(c-a)}{\Gamma(c)(c)_n}.$$

As $\Phi(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}$, the desired result follows directly. ■

The confluent function also has a complex contour integral representation (cf. [75], p.124, Eq. 10).

Theorem 4.4.3 *If* $Re(z) < 0$ *and neither* a *nor* b *is a non-positive integer, then using a Barnes path of integration we have*

$$\Phi(a;c;z) = \frac{\Gamma(c)}{2\pi i \Gamma(a)} \int_B \frac{\Gamma(a+s)\Gamma(-s)(-z)^s ds}{\Gamma(c+s)}. \tag{4.44}$$

As is to be expected, there exist many transformation identities for the confluent function. We provide below two important transformations of Kummer, given as Equations (4.1.11) and (4.1.12) respectively in [4].

Theorem 4.4.4 *Kummer's first formula*: *For* $c \neq 0, -1, -2, \dots$,

$$\Phi(a; c; z) = e^z \Phi(c-a; c; -z)\,. \tag{4.45}$$

Proof: In the integral form in (4.37), we substitute $t \to 1-t$ to obtain

$$\Phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} e^z \int_0^1 e^{-zt} t^{c-a-1} (1-t)^{a-1} dt\,,$$

and the result follows by the integral definition of $\Phi(c-a; c; -z)$. ■

The above result can also be established by applying the limiting process $b \to \infty$ to Pfaff's quadratic transformation ${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1(a, c-b; c; z/(z-1))$ (cf. [84], p.173). Kummer's second formula below can similarly be established by applying the same limiting process to the transformation ${}_2F_1(a, b; 2a; 4z/(1+z)^2) = (1+z)^{2a}\, {}_2F_1\left(a, a+\frac{1}{2}-b; b+\frac{1}{2}; z^2\right)$.

Theorem 4.4.5 *Kummer's second formula*: *For 2a not a negative odd integer,*

$$\Phi(a; 2a; 4z) = e^{2z}\, {}_0F_1\left(-; a+\frac{1}{2}; z^2\right). \tag{4.46}$$

In Chapter 5 we discuss a recent paper by Miller [68] (see Article 1 of Section 5.2), in which the author makes use of a recurrence relation for the confluent function. In preparation, we prove the required relation below.

Theorem 4.4.6 *For* $c+1 \neq 0, -1, -2, \dots$,

$$\Phi(a; c; z) + \frac{z}{c}\Phi(a+1; c+1; z) = \Phi(a+1; c; z)\,. \tag{4.47}$$

Proof: By expanding the confluent functions on the left side, we obtain

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!\,(c)_k}z^k + \sum_{k=0}^{\infty}\frac{(a+1)_k}{k!\,(c)_{k+1}}z^{k+1}$$

$$= \left\{1+\frac{a}{c}z+\frac{a(a+1)}{2!\,c(c+1)}z^2+\frac{a(a+1)(a+2)}{3!\,c(c+1)(c+2)}z^3+\cdots\right\}$$

$$+\left\{\frac{1}{c}z+\frac{a+1}{c(c+1)}z^2+\frac{(a+1)(a+2)}{2!\,c(c+1)(c+2)}z^3+\cdots\right\}$$

$$= 1+\frac{a+1}{c}z+\frac{a(a+1)(a+2)}{2!\,c(c+1)}z^2+\frac{a(a+1)(a+2)(a+3)}{3!\,c(c+1)(c+2)}z^3+\cdots,$$

which is the expansion of the right side of the theorem. ■

Many other identities of the confluent and other generalised hypergeometric function can be found in classical texts such as [8], [9], [26], [74] and [81]. To complete this chapter, we briefly present a sample of interesting applications of the generalised hypergeometric function.

4.5 Some applications and extensions of hypergeometric functions

It has been mentioned that hypergeometric functions play a powerful role in a wide variety of pure and applied mathematics contexts. In this section we mention a few such applications, as well as some ways in which the fundamental definition of the hypergeometric function has been extended.

The Weierstrass $\wp$-function (elliptic function) is defined by

$$\wp(z,\tau) = z^{-2} + \sum_{\omega\neq 0}\{(z+\omega)^{-2}-\omega^{-2}\},$$

for $z \in \mathbb{C}$ and τ in the upper half-plane, where ω runs over the lattice $\mathbb{Z}+\tau\mathbb{Z}$. Duke [23] shows that the zeros of this function are given by $\pm z_0$, where $x = 1 - 1728/j$ and

$$z_0 = \frac{1+\tau}{2} - \frac{i\sqrt{6}x^{1/4}\ {}_3F_2\left(\frac{1}{3},\frac{2}{3},1;\frac{3}{4},\frac{5}{4};x\right)}{3\pi\ {}_2F_1\left(\frac{1}{12},\frac{5}{12};1;1-x\right)}.$$

The Fibonacci numbers can be defined through Binet's formula as

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

By using the identity ${}_2F_1\left(a,\frac{1}{2}+a;\frac{3}{2};z^2\right) = \frac{1}{2z(1-2a)}\left[(1+z)^{1-2a} - (1-z)^{1-2a}\right]$, Dilcher [20] obtains $F_n = \frac{n}{2^{n-1}}\ {}_2F_1\left(\frac{1-n}{2},\frac{2-n}{2};\frac{3}{2};5\right)$.

Confluent hypergeometric functions are connected with representations of the group of third-order upper triangular matrices, in that group elements which correspond to integral operators have kernels that can be expressed in terms of Whittaker functions (cf. [87]). The reduced wave equation $\nabla^2\omega = k^2\omega$ also has solutions involving Whittaker functions (cf. [47], Chapter 7).

The steady-state probability of there being n persons in a system, when the probabilities of birth and death in the time interval $(t, t+\Delta t)$ are $\lambda_n = na + b$ and $\mu_n = nc + d$ respectively, is

$$p_n = \frac{(b/c)_n}{(d/c+1)_n}\left\{{}_2F_1\left(1,\frac{b}{c};\frac{d}{c}+1;\frac{a}{c}\right)\right\}^{-1}.$$

In the generalised case, birth and death processes are said to be contiguous if either b_i is replaced by $b_i \pm a_i$ or if d_j is replaced by $d_j \pm c_j$ for one value of i or j. Since contiguous hypergeometric function are related, contiguous birth and death processes are also related, in particular as regards the probabilities of extinction (cf. [51]).

In helicopter rotor blade theory, calculating the induced velocity involves the integral $S_n = \frac{2}{\pi}\int_0^{\pi} \frac{\cos n\varphi}{L(\rho,\theta)}\, d\theta$. This integral satisfies the Gauss differential equation $z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0$, with certain boundary conditions. If

$z = \rho^2, a = -m, b = m + 1, c = 1,$ it can be shown that $S_{2m+1} = -2\ {}_2F_1(-m, m + 1; 1; \rho^2)$ (cf. [10]).

There also exist extensions of the hypergeometric functions. One such extension is the basic or q-hypergeometric function, in which the rising factorial is replaced by $(q^a)_{q,n} = (1 - q^a)(1 - q^{a+1}) \ldots (1 - q^{a+n-1})$ and the term ratio $\frac{a_{n+1}}{a_n}$ is a rational function of q^n (cf. [4], [40]). These q-hypergeometric series are related to elliptic and theta functions, and are thus useful in partition theory, difference equations and Lie algebras. Hypergeometric functions can also have matrix arguments, and are thus used to express certain distributions occurring in multivariate analysis (cf. [69]).

A further extension is the Kampé de Fériet double hypergeometric function, defined by

$$F^{A:\ B;\ B\prime}_{C:\ D;\ D\prime}\begin{bmatrix} a_1, \ldots, a_A: b_1, \ldots, b_B;\ b_1', \ldots, b_B'; & x\ y \\ c_1, \ldots, c_C: d_1, \ldots, d_D;\ d_1', \ldots, d_B'; & \end{bmatrix}$$

$$= \sum [(a_1, m + n)..(a_A, m + n)(b_1, m)..(b_B, m)(b'_1, n)..(b'_{B\prime}, n)x^m y^n]/$$
$$[(c_1, m + n)..(c_C, m + n)(d_1, m)..(d_D, m)(d'_1, n)..(d'_{D\prime}, n)m!\,n!],$$

where $(a, n) = (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ (cf. [28]).

Having discussed certain classical identities and transformations for the generalised hypergeometric function, we will now present findings from a literature review of more recent research in this area.

Chapter 5

Recent results involving hypergeometric functions

5.1 Introduction

This work has so far concerned itself with classical results related to the hypergeometric function, generally established by the first half of the 20^{th} century. In this chapter we now present the results of a literature review of more recent identities for these functions, developed during this century and the late 20^{th} century. In Section 5.2 we discuss six interesting results and their proofs in some detail, then in Section 5.3 we present a brief summary of further relevant results and their methods of derivation.

The findings of this review support the view that there is almost no end to the possible permutations of the open-ended class of hypergeometric identities, so that some apparently new results can often be shown to be variations of existing identities. Furthermore, some proposed new results are shown to be invalid when numerically checked. Consequently, computer algorithmic techniques have been developed for testing and comparing various hypergeometric identities, as well as for generating new ones. In Chapter 6 we thus discuss some of these algorithmic procedures, together with their implications for future work with hypergeometric functions.

As the field of hypergeometric identities is so vast, this review chapter will concentrate mainly (although not exclusively) on ${}_{p+1}F_p$ functions, and in particular on the ${}_3F_2(1)$ function in the light of a useful and exhaustive computer-based analysis conducted by Milgram [66], which we present at the last section of Chapter 6.

5.2 Some recent results in detail

In this section we discuss, in chronological order, six recent articles which are interesting with respect to either their methods or results. For ease of reference, in this chapter we label equations and identities according to their original numbering in each article, and we also provide additional details in the proofs where useful.

***Article 1*: *On a Kummer-type transformation for the generalized hypergeometric function ${}_2F_2$*,** Miller [68]

In 1997, H. Exton [28] used Kampé de Fériet functions to prove the relation

$$ {}_2F_2\left(a, 1+\frac{a}{2}; b, \frac{a}{2}; y\right) = e^{y}\, {}_2F_2(b-a-1, 2+a-b; b, 1+a-b; -y)\,, \tag{2}$$

in which the ${}_2F_2(y)$ hypergeometric function is expressed in terms of a ${}_2F_2(-y)$ function. In his 2003 article Miller uses summation methods as a more convenient approach to establish the same result. Using series techniques, he shows that

$$\begin{aligned} e^{-y}\, {}_2F_2(a, c; b, d; y) &= \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} \sum_{k=0}^{\infty} \frac{(a)_k (c)_k y^k}{(b)_k (d)_k k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a)_k (c)_k (-1)^k}{(b)_k (d)_k k!} \frac{(-y)^n n!}{(n-k)!\, n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a)_k (c)_k (-n)_k}{(b)_k (d)_k k!} \frac{(-y)^n}{n!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} {}_3F_2(-n, a, c; b, d; 1) \frac{(-y)^n}{n!}.$$

Miller then sets $c = 1 + \frac{a}{2}$ and $d = \frac{a}{2}$ in the above result and uses Slater's formula 2.4.2.3 in [82], p.65,[28] to obtain

$$\sum_{n=0}^{\infty} {}_3F_2(-n, a, c; b, d; 1) \frac{(-y)^n}{n!} = \sum_{n=0}^{\infty} \frac{(b-a-1-n)(b-a)_{n-1}}{(b)_n} \frac{(-y)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(b-a-1)_n (2+a-b)_n}{(b)_n (1+a-b)_n} \frac{(-y)^n}{n!},$$

by expansion of the rising factorials, and identity (2) follows directly. ■

In the same paper, the author also shows that (2) is a direct consequence of Kummer's first transformation: ${}_1F_1(a; b; y) = e^y \, {}_1F_1(b-a; b; -y)$. (We provide some additional steps in Miller's proof below.)

Proof: From the identity (1.8) in the form $\frac{(1+\alpha)_n}{(\alpha)_n} = 1 + \frac{n}{\alpha}$, it follows that the left side of (2) can be written as

$${}_2F_2\left(a, 1 + \frac{a}{2}; b, \frac{a}{2}; y\right) = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{(b)_n n!} [1 + \frac{2n}{a}]$$

$$= {}_1F_1(a; b; y) + \sum_{n=0}^{\infty} \frac{2(a)_{n+1} y^{n+1}}{a(b)_{n+1} n!}.$$

$$= {}_1F_1(a; b; y) + \frac{2y}{b} {}_1F_1(a+1; b+1; y). \qquad (6)$$

Using similar simplifications with $\alpha = 1 + a - b$ in identity (1.8), the right side of (2) becomes

$$e^y \, {}_2F_2(b-a-1, 2+a-b; b, 1+a-b; -y)$$

---

[28] Slater's result is ${}_3F_2\left(-n, a, 1 + \frac{a}{2}; b, \frac{a}{2}; 1\right) = \frac{(b-a-1-n)(b-a)_{n-1}}{(b)_n}$.

$$= e^{y}\ {}_1F_1(b-a-1;b;-y) + \frac{y}{b} e^{y}\ {}_1F_1(b-a;b+1;-y).$$

$$= {}_1F_1(a+1;b;y) + \frac{y}{b}\ {}_1F_1(a+1;b+1;y)\,, \tag{8}$$

by Kummer's formula (4.38) in our Chapter 4. It remains to show that the right sides of identities (6) and (8) are equal, in order to establish the desired result. This is done by drawing on the recurrence relation (4.47) in Chapter 4, in the form ${}_1F_1(a;b;y) + \frac{y}{b}\ {}_1F_1(a+1;b+1;y) = {}_1F_1(a+1;b;y)$, and identity (2) follows directly. ■

Article 2*: *Extensions of certain classical integrals of Erdélyi for Gauss hypergeometric functions, Joshi and Vyas [49]

In Chapter 3 we provided Erdélyi's integral representations (3.10)-(3.12) for the Gauss hypergeometric function, which Erdélyi [25] proved in 1939 using fractional calculus. In this article the authors establish these same results (their equations 1.3-1.5) in a more convenient way, using series manipulation and certain classical summation theorems. Through this approach they are also able to generalise the result, and hence to generate new integrals of Erdélyi type. (We have provided additional details where useful.)

Equation 1.3 *For* $Re(c) > Re(d) > 0$,

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 t^{d-1}(1-t)^{c-d-1}(1-tz)^{\lambda-a-b}$$

$$\times\ {}_2F_1(\lambda-a,\lambda-b;d;tz)\ {}_2F_1\left(a+b-\lambda,\lambda-d;c-d;\frac{(1-t)z}{1-tz}\right) dt\,.$$

Proof: By expressing the Gauss hypergeometric functions in their series form and interchanging the order of summation and integration (valid for $|z| < 1$), the right side of the theorem becomes

$$\frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)}\sum_{m,n=0}^{\infty}\frac{(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n(a+b-\lambda)_n}{(d)_m(c-d)_n}\frac{z^{m+n}}{m!\,n!}$$

$$\times\int_0^1 t^{d+m-1}(1-t)^{c-d+n-1}(1-tz)^{\lambda-a-b-n}dt\,.$$

Now by applying Euler's integral (3.8) of our Chapter 3, this result becomes

$$\frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)}\sum_{m,n=0}^{\infty}\frac{\Gamma(d+m)\Gamma(c-d+n)}{\Gamma(c+m+n)}\frac{(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n(a+b-\lambda)_n}{(d)_m(c-d)_n}\frac{z^{m+n}}{m!\,n!}$$

$$\times\ {}_2F_1(a+b-\lambda+n,d+m;c+m+n;z)$$

$$=\sum_{m,n=0}^{\infty}\frac{(d)_m(c-d)_n}{(c)_{m+n}}\frac{(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n(a+b-\lambda)_n}{(d)_m(c-d)_n}\frac{z^{m+n}}{m!\,n!}$$

$$\times\ {}_2F_1(a+b-\lambda+n,d+m;c+m+n;z)$$

$$=\sum_{k,m,n=0}^{\infty}\frac{(d)_m(c-d)_n}{(c)_{m+n}}\frac{(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n(a+b-\lambda)_n}{(d)_m(c-d)_n}$$

$$\times\frac{(a+b-\lambda+n)_k(d+m)_k}{(c+m+n)_k}\frac{z^{m+n+k}}{m!\,n!\,k!}$$

$$=\sum_{k,m,n=0}^{\infty}\frac{(a+b-\lambda)_{k+n}(d)_{k+m}(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n}{(c)_{k+m+n}(d)_m}\frac{z^{m+n+k}}{m!\,n!\,k!},$$

by applying identity (1.9) of Chapter 1. At this stage, the authors use iterated series manipulation techniques to write $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(k,m,n)$ in the form $\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k-m}A(k-m-n,m,n)$. The above triple series can thus be written as

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k-m}\frac{(a+b-\lambda)_{k-m}(d)_{k-n}(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n}{(c)_k(d)_m}\frac{z^k}{m!\,n!\,(k-m-n)!}.$$

Applying the identity for $(n-k)!$ twice yields $([k-m]-n)!=\frac{(-1)^n(k-m)!}{(m-k)_n}=\frac{(-1)^{m+n}k!}{(m-k)_n(-k)_m}$. Together with the identity (1.12) for $(d)_{k-n}$, this becomes

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k-m}\frac{(a+b-\lambda)_{k-m}(\lambda-a)_m(\lambda-b)_m(\lambda-d)_n}{(c)_k(d)_m}\frac{(d)_k(m-k)_n(-k)_m}{(1-d-k)_n}\frac{z^k}{m!\,n!\,k!\,(-1)^m}$$

$$=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(a+b-\lambda)_{k-m}(\lambda-a)_m(\lambda-b)_m}{(c)_k(d)_m}\frac{(d)_k(-k)_m(-1)^m z^k}{m!\,k!}$$

$$\times\ {}_2F_1(\lambda-d,m-k;1-d-k;1)\,.$$

By applying the Chu-Vandermonde identity (3.15) to the inner ${}_2F_1$ (with $-n=m-k$), the authors rewrite the above result in the form

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(a+b-\lambda)_{k-m}(\lambda-a)_m(\lambda-b)_m}{(c)_k(d)_m}\frac{(d)_k(-k)_m(-1)^m z^k}{m!\,k!}\frac{(1-\lambda-k)_{k-m}}{(1-d-k)_{k-m}},$$

which by identities $(\alpha)_{k-m}=\frac{(-1)^m(\alpha)_k}{(1-\alpha-k)_m}$ and $(1-\alpha-k)_k=(-1)^k(\alpha)_k$, becomes

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(\lambda)_k(a+b-\lambda)_k z^k}{(c)_k k!}.\frac{(\lambda-a)_m(\lambda-b)_m(-k)_m}{(1+\lambda-a-b-k)_m(\lambda)_m m!}$$

$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k(a+b-\lambda)_k}{(c)_k}\frac{z^k}{k!}\times\ {}_3F_2\left(\begin{matrix}\lambda-a,\lambda-b,-k;\\1+\lambda-a-b-k,\lambda;\end{matrix}\,1\right)$$

$$=\ {}_2F_1\left(\begin{matrix}a,b;\\c;\end{matrix}\,z\right),$$

where the final result follows from the Pfaff-Saalschütz theorem (4.2), and the proof is complete. ■

The authors then summarise how they use similar techniques to prove the other Erdélyi integrals (1.4) and (1.5), and go on to establish seven similar integrals which they term 'integrals of Erdélyi type', connecting certain ${}_{q+1}F_q$ and ${}_2F_1$series. The authors also state that such series methods can be used to prove generalised integral formulas which yield the earlier integrals as special cases. Two of their nine generalisations are given below.

For $|\arg(1-z)|<\pi, Re(\gamma)>Re(d)$ and $z\neq 1$,

$$
{}_3F_2(\nu,\xi,\lambda;\gamma,\delta;z)=\frac{\Gamma(\gamma)}{\Gamma(d)\Gamma(\gamma-d)}\int_0^1 t^{d-1}(1-t)^{\gamma-d-1}(1-tz)^{\delta-\nu-\xi}
$$

$$
\times\ {}_3F_2(\delta-\xi,\delta-\nu,\lambda;d,\delta;tz)\ {}_2F_1\left(\lambda-d,\nu+\xi-\delta;\gamma-d;\frac{(1-t)z}{1-tz}\right)dt. \tag{4.1}
$$

For $Re(\gamma,\nu,\gamma+d-\lambda-\nu)>0$, and $\phi(z)=\sum_{n=0}^{\infty}c_n z^n$ a convergent series with c_n a bounded sequence of complex numbers,

$$
\sum_{n=0}^{\infty}c_n\frac{(\lambda)_n(\nu)_n}{(\gamma)_n(d)_n}z^n=\frac{\Gamma(\gamma)\Gamma(d)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+d-\lambda-\nu)}\int_0^1 t^{\nu-1}(1-t)^{\gamma+d-\lambda-\nu-1}
$$

$$
\times_2 F_1(d-\lambda,\gamma-\lambda;\gamma+d-\lambda-\nu;1-t)\phi(zt)dt. \tag{4.5}
$$

Article 3*: *A Kummer-type transformation for a ${}_2F_2$ hypergeometric function, Paris [71]

As discussed in Article 1, Miller [68] established the Kummer-type transformation:

$$
{}_2F_2\left(a,1+\frac{a}{2};b,\frac{a}{2};x\right)=e^y\ {}_2F_2(b-a-1,2+a-b;b,1+a-b;-x)\,. \tag{1}
$$

In this article Paris first points out that Miller's transformation contains only two free parameters a and b, and then proceeds to obtain a general result expressing ${}_2F_2(a,d;b,c;x)$ with four unrelated parameters $a,b,c,d,$ in terms of ${}_2F_2(-x)$ functions. This yields a Kummer-type transformation for ${}_2F_2(a,c+1;b,c;x)$, of which Miller's result (1) is a special case.

Paris bases his proof on Euler's integral representation

$$
{}_2F_2(a,d;b,c;x)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\ {}_1F_1(d;c;xt)dt
$$

$$
=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{b-a-1}(1-t)^{a-1}\ {}_1F_1(d;c;x(1-t))dt\,, \tag{2}
$$

for $Re(b)>Re(a)>0$.

Slater [81] provides addition theorem (2.3.5) for the confluent hypergeometric function: $\Phi(a;b;x+y) = e^{y}\sum_{n=0}^{\infty}\frac{(b-a)_n(-y)^n}{(b)_n n!}\ \Phi(a;b+n;x)$ (equation 4.39 in our Chapter 4). The authors use this theorem in the form ${}_1F_1(d;c;x(1-t)) = e^{x}\sum_{n=0}^{\infty}\frac{(c-d)_n}{(c)_n n!}(-x)^n\ {}_1F_1(d;c+n;-xt)$, so that by substitution and interchanging of summation and integration, equation (2) can be written as

$${}_2F_2(a,d;b,c;x)$$

$$= e^{x}\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{n=0}^{\infty}\frac{(c-d)_n}{(c)_n n!}(-x)^n\int_0^1 t^{b-a-1}(1-t)^{a-1}\ {}_1F_1(d;c+n;-xt)dt$$

$$= e^{x}\sum_{n=0}^{\infty}\frac{(c-d)_n}{(c)_n n!}(-x)^n\ {}_2F_2(b-a,d;b,c+n;-x), \tag{3}$$

by again applying Euler's integral. This is the desired general result, as it provides a formula for ${}_2F_2(a,d;b,c;x)$ in terms of a ${}_2F_2(-x)$ function with four unrelated parameters. Since both sides of the final result are analytic functions of a and b, the original parameter restrictions can be removed by analytic continuation.

The author then illustrates how this general result yields further particular results. When $c = d$, the right side of the general identity (3) contains only the term with $n = 0$, and hence the identity reduces to Kummer's first transformation for the confluent hypergeometric function: ${}_1F_1(a;b;x) = e^{x}\ {}_1F_1(b-1;b;-x)$. If $d = c + 1$, the right side of (3) contains only terms resulting from $n = 0, 1$, and the result is

$${}_2F_2(a,c+1;b,c;x) = e^{x}\left\{{}_2F_2(b-a,c+1;b,c;-x) + \frac{x}{c}\ {}_1F_1(b-a;b;-x)\right\}. \tag{6}$$

In order to simplify this further, Paris makes use of Pochhammer identities and standard simplifications to establish the summation relation

$${}_2F_2(\alpha-1,\delta+1;\beta,\delta;x) + \frac{x}{c}\ {}_1F_1(\alpha;\beta;x) = {}_2F_2(\alpha,\gamma+1;\beta,\gamma;x),$$

for $\beta, \gamma \neq 0, -1, -2, \ldots$, and $\delta = \frac{\gamma(\alpha-1)}{\alpha-\beta+\gamma}$. He thus obtains the desired result

$$ {}_2F_2(a, c+1; b, c; x) = e^x \, {}_2F_2(b-a-1, f+1; b, f; -x)\,, \tag{4} $$

where $f = \frac{c(1+a-b)}{a-c}$.

Miller's identity (1) then follows by the substitution $c = \frac{a}{2}$ into (4). On p.381, Paris also notes that when $d = c + m, m \in \mathbb{N}$, the right side of (3) reduces to a finite sum of ${}_2F_2$ functions in the form

$$ {}_2F_2(a, c+m; b, c; x) = e^x \sum_{n=0}^{m} \binom{m}{n} \frac{x^n}{(c)_n} \, {}_2F_2(b-a, c+m; b, c+n; -x). $$

Article 4*: *A generalization of Euler's hypergeometric transformation, Maier [64]

In this paper the author extends Euler's linear transformation for the Gauss hypergeometric function to more general series of the form ${}_{r+1}F_r(x)$. From this formula, new one-term evaluations of ${}_3F_2(-1)$ and ${}_3F_2(1)$ are derived.

Recall Euler's transformation: *For* $|z| < 1$, $|\arg(1-z)| < \pi$,

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, {}_2F_1(c-a, c-b; c; z)\,. $$

Maier asserts that while cubic and quadratic transformations of ${}_2F_1$ have been shown to have analogues for ${}_3F_2$, the more generalised form of Euler's transformation had not been previously noticed. As a reason for this oversight, he suggests the fact that in this case the hypergeometric function parameters are not linearly constrained, in contrast to the listed hypergeometric identities. Maier provides the generalised form of Euler's transformation in his Theorem 2.1, which reduces to Euler's transformation when $r = 1$. The theorem requires the definition below.

Definition *For each $r \geq 1$, the algebraic variety $U_r \subset \mathbb{C}^{r+1} \times \mathbb{C}^r$, which is $(r+1)$-dimensional, comprises all (a, b) for which the r equations*

$$\begin{cases} \displaystyle\sum_{1 \leq i \leq r+1} a_i = \sum_{1 \leq i \leq r} b_i \\ \displaystyle\sum_{1 \leq i < j \leq r+1} a_i a_j = \sum_{1 \leq i < j \leq r} b_i b_j \\ \displaystyle\sum_{1 \leq i < j < k \leq r+1} a_i a_j a_k = \sum_{1 \leq i < j < k \leq r} b_i b_j b_k \\ \vdots \end{cases}$$

hold. In the k-th equation, the left and right sides are the k-th elementary symmetric polynomials in $a_1 \dots a_{r+1}$ and $b_1 \dots b_r$ respectively.

Theorem 2.1 *For $r \geq 1$ and $(a, b) \in U_r$ for which no $b_i + 1$ is a non-positive integer,*

$${}_{r+1}F_r\left(\begin{matrix} a_1, \dots, a_{r+1}; \\ b_1 + 1, \dots, b_r + 1; \end{matrix} x\right) = (1 - x)\, {}_{r+1}F_r\left(\begin{matrix} a_1 + 1, \dots, a_{r+1} + 1; \\ b_1 + 1, \dots, b_r + 1; \end{matrix} x\right).$$

Proof: The above result will hold if it can be shown that for all $k \geq 1$,

$$\frac{(a_1)_k \dots (a_{r+1})_k}{(b_1 + 1)_k \dots (b_r + 1)_k k!} = \frac{(a_1 + 1)_k \dots (a_{r+1} + 1)_k}{(b_1 + 1)_k \dots (b_r + 1)_k k!} - \frac{(a_1 + 1)_{k-1} \dots (a_{r+1} + 1)_{k-1}}{(b_1 + 1)_{k-1} \dots (b_r + 1)_{k-1} (k-1)!},$$

or if $a_1 \dots a_{r+1} = (a_1 + k) \dots (a_{r+1} + k) - (b_1 + k) \dots (b_r + k)k$. (2.2)

The left side of (2.2) is independent of k, and the right side is a polynomial in k of degree r. For $n = 1, \dots, r$, the coefficient of k^n is proportional to the sum of all monomials $a_{i_1} \dots a_{i_{r+1-n}}$ minus the sum of all monomials $b_{j_1} \dots b_{j_{r+1-n}}$. Since $(a, b) \in U_r$, each coefficient is zero, from which it follows that the right side of (2.2) reduces to the left side and the theorem is proved. ■

According to Maier, Theorem 2.1 is surprising as it provides the first two-term relation for contiguous hypergeometric functions with general r. This theorem leads to two further results.[29] By multiplying both sides of the theorem by $(1-x)^{-1}$ and equating coefficients of x^n, it can be shown that for $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} \frac{(a_1)_j \dots (a_{r+1})_j}{(b_1+1)_j \dots (b_r+1)_j j!} = \frac{(a_1+1)_n \dots (a_{r+1}+1)_n}{(b_1+1)_n \dots (b_r+1)_n n!}, \tag{2.3}$$

from which it follows that for $a_1 = -n, n \in \mathbb{N}_0$,

$${}_{r+1}F_r\left({-n, a_2, \dots, a_{r+1}; \atop b_1+1, \dots, b_r+1;}\ 1\right) = \frac{(a_1+1)_n \dots (a_{r+1}+1)_n}{(b_1+1)_n \dots (b_r+1)_n n!}. \tag{2.4}$$

The result in (2.4) is extended in Theorem 2.3 to non-terminating series, and in Theorem 2.5 to two series which are $(B-A+r)$-balanced and $(B-A)$-balanced.

Theorem 2.3 *For $r \geq 1$ and $(a,b) \in U_r$ for which no $b_i + 1$ is a non-positive integer,*

$${}_{r+1}F_r\left({a_1, \dots, a_{r+1}; \atop b_1+1, \dots, b_r+1;}\ 1\right) = \frac{\Gamma(b_1+1) \dots \Gamma(b_r+1)}{\Gamma(a_1+1) \dots \Gamma(a_{r+1}+1)}.$$

Theorem 2.5 *For $r \geq 1$ and $(a,b) \in U_r$ for which no $b_i + 1$ is a non-positive integer, and provided that $Re(B-A) > 0$ and B is not a non-positive integer,*

$${}_{r+2}F_{r+1}\left({a_1, \dots, a_{r+1}, A; \atop b_1+1, \dots, b_r+1, B;}\ 1\right) = \frac{B-A}{B}\ {}_{r+2}F_{r+1}\left({a_1+1, \dots, a_{r+1}+1, A; \atop b_1+1, \dots, b_r+1, B+1;}\ 1\right).$$

In Theorem 3.4 of his paper, Maier uses a group-theoretic approach to derive three ${}_3F_2(1)$ identities in which the parameters are not linearly but quadratically constrained.

Theorem 3.4 (i) *For $ab + bc + ca = (d-1)(c-1)$, and $d + e - a - b - c = 2$,*

$${}_3F_2\left({a, b, c; \atop d, e;}\ 1\right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)}.$$

[29] Maier points out that results (2.3) and (2.4) can be found in Slater [82], p.84.

(ii) *For* $(a-1)(b-1) = [(a-1)+(b-1)-(e-1)]c$,

$$ {}_3F_2\left(\begin{matrix} a,b,c; \\ c+2,e; \end{matrix} 1\right) = \frac{\Gamma(e)\Gamma(e-a-b+2)\Gamma(c+2)}{\Gamma(e-a+1)\Gamma(e-b+1)\Gamma(c+1)}, $$

provided that $Re(e-a-b+2) > 0$.

(iii) *For* $(a-1)(b-1) = (d-2)(e-2)$,

$$ {}_3F_2\left(\begin{matrix} a,b,2; \\ d,e; \end{matrix} 1\right) = \frac{(d-1)(e-1)}{d+e-a-b-2}. $$

provided that $Re(d+e-a-b-2) > 0$.

Numerical investigation shows that the identities (ii) and (iii) hold whenever the series terminates, even if the parametric excess has a non-positive real part, thus differing from the related formulas of Dixon, Watson and Whipple.[30] The author points out that the associated result

$$ {}_3F_2\left(\begin{matrix} 2,b,c; \\ \frac{3+b}{2}, 2c; \end{matrix} 1\right) = \frac{(1+b)(1-2c)}{1+b-2c} $$

is a specialisation of Theorem 3.4 (iii) and of Watson's formula (equation (4.11) of our Chapter 4), while the associated result

$$ {}_3F_2\left(\begin{matrix} a,1-a,c; \\ 2+a,2c-a-1; \end{matrix} 1\right) = \frac{\Gamma(2c-a-1)\Gamma(c)\Gamma(2+a)}{\Gamma(c-a)\Gamma(2c-1)\Gamma(1+a)} $$

is a specialisation of Theorem 3.4 (ii) and of Whipple's formula (4.12) of Chapter 4. In Theorems 4.1 and 4.2 of the final section of his paper, Maier deduces six similar gamma function identities for ${}_3F_2(-1)$.

Article 5*: *An integral representation of some hypergeometric functions, Driver and Johnston [22]

In this article the authors provide an elegant proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions with $q \geq 2$. From this result, they then deduce identities for certain ${}_3F_2$ and ${}_4F_3$ functions with unit argument.

[30] See theorems (4.7), (4.11) and (4.12) respectively in Chapter 4.

Theorem 2.1 *For* $Re(c) > Re(b) > 0$,

$$ {}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^{-a}dt\,. $$

Proof: By the use of the identities $(\alpha)_{2k} = 2^{2k}\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha+1}{2}\right)_k$ and $\frac{(b)_{2k}}{(c)_{2k}} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\beta(b+2k,c-b), Re(c) > Re(b) > 0$, the left side of the theorem can be written in the form

$$ LHS = \sum_{k=0}^{\infty}\frac{(a)_k(b)_{2k}}{(c)_{2k}k!}x^k = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}x^k\int_0^1 t^{b+2k-1}(1-t)^{c-b-1}dt\,. $$

The desired result then follows for $|x| < 1$ by interchanging the order of summation and integration, and applying the identity $(1-xt^2)^{-a} = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}(xt^2)^k$. As the integral is analytic in the cut plane (cut along the real axis from 1 to infinity), the result also holds for all *x* in this region. ■

The authors then use the result in Theorem 2.1 to express a ${}_3F_2(1)$ series of particular form in terms of a Gauss hypergeometric series with argument $x = -1$, proven in Theorem 2.3. This is a more concise proof than Whipple [92] provided for his equivalent formula (3.7).

Theorem 2.3 *For* $Re(c) > Re(b) > 0$, *and* $Re(c-a-b) > 0$,

$$ {}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}\,{}_2F_1(a,b;c-a;-1). $$

Proof: The authors let $x = 1$ in Theorem 2.1, factorise $(1-t^2)$ and then apply identity (1.18) in the form $(1+t)^{-a} = \sum_{k=0}^{\infty}\frac{(-1)^k(a)_n t^k}{k!}$ to obtain

$$ {}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right) $$
$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b+k-1}(1-t)^{c-b-a-1}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k}{k!}dt\,. $$

Interchanging the order of summation and integration yields a beta integral, so that the right side of the above result can be written in the form

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k}{k!}\frac{\Gamma(b+k)\Gamma(c-b-a)}{\Gamma(c-a+k)},$$

$$=\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k}{k!}\frac{(b)_k}{(c-a)_k},$$

and the result is proved. ■

The authors then use the identity $\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}=(\alpha)_k$ to obtain the corollary below for a terminating ${}_3F_2(1)$ series.

Corollary 2.4 $\quad {}_3F_2\left(-n,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right)=\frac{(c-b)_n}{(c)_n}\,{}_2F_1(-n,b;c-a;-1)$

Using the identity $(\alpha)_{qk}=2^{qk}\left(\frac{\alpha}{q}\right)_k\left(\frac{\alpha+1}{q}\right)_k\dots\left(\frac{\alpha+q-1}{q}\right)_k$, the integral identity in Theorem 2.1 can then be generalised to apply to ${}_{q+1}F_q(x)$, as given below.

Theorem 2.5 *For* $Re(c)>Re(b)>0$,

$$\begin{aligned}&{}_{q+1}F_q\left(a,\frac{b}{q},\frac{b+1}{q},\dots\frac{b+q-1}{q};\frac{c}{q},\frac{c+1}{q}\dots\frac{c+q-1}{q};x\right)\\&\qquad=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}dt\,.\end{aligned}$$

By substitution into this result, the authors obtain the following identity for a certain class of ${}_4F_3(1)$ functions.

Theorem 2.6 *For* $Re(c-a-b)>0$,

$$\begin{aligned}&{}_4F_3\left(a,\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3};\frac{c}{3},\frac{c+1}{3},\frac{c+2}{3};1\right)\\&\qquad=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{k!\,(c-a)_k}\,{}_2F_1(-k,b+k;c-a+k;-1).\end{aligned}$$

Proof: Letting $q=3$ and $x=1$ in Theorem 2.5 yields

$${}_4F_3\left(a,\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3};\frac{c}{3},\frac{c+1}{3},\frac{c+2}{3};1\right)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t^3)^{-a} dt .$$

By factorising $(1-t^3)$ and expanding $(1+[t+t^2])^{-a}$, this result becomes

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k}{k!} (1+t)^k \, t^k dt .$$

The authors then expand the power $(1+t)^k$ and apply standard identities to obtain

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^k (a)_k}{k!} \binom{k}{r} \frac{\Gamma(b+r+k)\Gamma(c-b-a)}{\Gamma(c-a+k+r)}$$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^k (a)_k}{k!} \binom{k}{r} \frac{(b)_{r+k}}{(c-a)_{r+k}}$$

$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (b)_k}{k! \, (c-a)_k} \sum_{r=0}^{k} \binom{k}{r} \frac{(b+k)_r}{(c-a+k)_r} .$$

Since $\sum_{r=0}^{k} \binom{k}{r} = \sum_{r=0}^{k} \frac{(-1)^r (-k)_r}{r!}$, the desired result follows directly. ■

Using similar techniques, the authors also conclude in Theorem 2.7 that

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k} \, {}_2F_1(-k, b; c+k; -1) ,$$

which together with Corollary 2.4 yields the transformation equation

$${}_3F_2\left(a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; \frac{1}{2}\right) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \, {}_3F_2\left(-k, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right) .$$

The authors note that this final result is a special case of a transformation given by Chaundy.

Article* 6: *Some results involving series representations of hypergeometric functions Coffey and Johnston [18]

This paper contains further generalisations following from the earlier work in [22], discussed as Article 5 above.

Theorem 2.1 *For* $Re(c) > Re(b) > 0$ *and* $r \in \mathbb{R}\backslash\{0,-1\}$,

$$ {}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};\frac{1}{r+1}\right) = \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\ {}_3F_2\left(-k,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right). $$

Proof: The authors first set $x = \frac{1}{r+1}$ in Theorem 2.1 of [22], for $r \in \mathbb{R}\backslash\{0,-1\}$. Then by writing $(1-\frac{1}{r+1}t^2)$ in the form $\left(\frac{r+1}{r}\right)^a \left(1+\frac{1-t^2}{r}\right)^{-a}$ and using the binomial theorem, they obtain

$$ {}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};\frac{1}{r+1}\right) $$

$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1-\frac{1}{r+1}t^2\right)^{-a} dt $$

$$ = \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\int_0^1 t^{b-1}(1-t)^{c-b-1+k}(1+t)^k dt $$

$$ = \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\sum_{m=0}^{k}\binom{k}{m}\int_0^1 t^{b+m-1}(1-t)^{c-b-1+k} dt $$

$$ = \left(\frac{r+1}{r}\right)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{-a}{k}\binom{k}{m} t^m r^{-k}\beta(b+m,c-b+k) $$

$$ = \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\frac{(c-b)_k}{(c)_k}\sum_{m=0}^{k}\frac{(-1)^m (b)_m(-k)_m}{m!\,(c+k)_m} $$

$$ = \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\frac{(c-b)_k}{(c)_k}\ {}_2F_1(-k,b;c+k;-1) $$

$$ = \left(\frac{r+1}{r}\right)^a \sum_{k=0}^{\infty}\binom{-a}{k} r^{-k}\ {}_3F_2\left(-k,\frac{b}{2},\frac{b+1}{2};\frac{c}{2},\frac{c+1}{2};1\right), $$

by using Theoreom 2.3 of [22], and the result is proved. ■

This result is a special case of a transformation of Chaundy's (cf. [16], Eq. (25)]:

$$(1-z)^{-a}\ {}_{q+1}F_q\left(a, b_1, \dots, b_q; c_1, \dots, c_q; \frac{xz}{z-1}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!}\ {}_{q+1}F_q\left(-k, b_1, \dots, b_q; c_1, \dots, c_q; x\right),$$

with $x = 1, z = -1/r$ and $q = 2$. The authors also apply Chaundy's transformation to create an interesting identity involving ${}_{q+1}F_q$ and the zeta function (Corollary 2.3).

Theorem 2.4 *For* $Re(c) > Re(b) > 0,$

$${}_2F_1\left(\frac{b}{2}, \frac{b+1}{2}; \frac{c+1}{2}; 1\right) = \frac{\Gamma(c)\Gamma(c/2-b)}{\Gamma(c-b)\Gamma(c/2)} 2^{-b}.$$

Proof: By introducing the same additional parameter in the numerator and denominator of the right side and applying Theorem 2.3 of [22], the authors obtain

$${}_2F_1\left(\frac{b}{2}, \frac{b+1}{2}; \frac{c+1}{2}; 1\right) = {}_3F_2\left(\frac{c}{2}, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; 1\right)$$

$$= \frac{\Gamma(c)\Gamma(c/2-b)}{\Gamma(c-b)\Gamma(c/2)}\ {}_2F_1(c/2, b; c/2; -1)$$

$$= \frac{\Gamma(c)\Gamma(c/2-b)}{\Gamma(c-b)\Gamma(c/2)}\ {}_1F_0(b; -; -1),$$

which is the right side of the theorem. ■

The authors provide a well-known special case of this result, by letting $b = 1$ and $c = 3$ so that ${}_2F_1(1/2,1;2;1) = \frac{\Gamma(3)\Gamma(1/2)}{\Gamma(2)\Gamma(3/2)} 2^{-1} = \frac{2\Gamma(2)\Gamma(1/2)}{2\Gamma(2)\Gamma(1/2)1/2} = 2.$

The authors also obtain the following useful result for writing certain ${}_{q+1}F_q$ series in terms of the Gauss hypergeometric series, again using a theorem from [22].

Theorem 2.5 *If* $Re(c) > Re(b) > 0$, *then*

$${}_{q+1}F_q\left(a, \frac{b}{q}, \frac{b+1}{q}, \dots, \frac{b+q-1}{q}; \frac{c}{q}, \frac{c+1}{q}, \dots, \frac{c+q-1}{q}; x\right)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{c-b-1}{k} \frac{(-1)^k}{b+k}\ {}_2F_1\left(a, \frac{b+k}{q}; \frac{b+k}{q}+1; x\right).$$

Proof: By the power series expansion for $(1-t)^{c-b-1}$ and the substitution $v = t^q$, Theorem 2.5 of [22] can be written in the form

$$
{}_{q+1}F_q\left(a,\frac{b}{q},\frac{b+1}{q},\dots\frac{b+q-1}{q};\frac{c}{q},\frac{c+1}{q}\dots\frac{c+q-1}{q};x\right)
$$

$$
=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}dt
$$

$$
=\frac{1}{q}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k\int_0^1 v^{(b+k)/q-1}(1-xv)^{-a}dv\,.
$$

To this result, the authors apply Euler's integral formula (3.8) of Chapter 3 and the recurrence relation $\Gamma(a+1)=a\Gamma(a)$, to obtain

$$
{}_{q+1}F_q\left(a,\frac{b}{q},\frac{b+1}{q},\dots,\frac{b+q-1}{q};\frac{c}{q},\frac{c+1}{q},\dots,\frac{c+q-1}{q};x\right)
$$

$$
=\frac{1}{q}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k\frac{\Gamma\left(\frac{b+k}{q}\right)\Gamma(1)}{\Gamma\left(\frac{b+k}{q}+1\right)}
$$

$$
\times\ {}_2F_1\left(a,\frac{b+k}{q};\frac{b+k}{q}+1;x\right)
$$

$$
=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k=0}^{\infty}\binom{c-b-1}{k}(-1)^k\frac{1}{b+k}\,{}_2F_1\left(a,\frac{b+k}{q};\frac{b+k}{q}+1;x\right),
$$

and the desired result follows. ■

The following special result for a ${}_5F_4$ series of the above form with $q=4$ and $x=1$ is also established, by using Theorem 2.5 of [22] in a slightly different way.

Theorem 2.6 *For* $Re(c) > Re(b) > 0$,

$$
{}_5F_4\left(a,\frac{b}{4},\frac{b+1}{4},\frac{b+2}{4},\frac{b+3}{4};\frac{c}{4},\frac{c+1}{4},\frac{c+2}{4},\frac{c+3}{4};1\right)
$$

$$
=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{r=0}^{\infty}\binom{-a}{r}\frac{\Gamma(b+2r)\Gamma(c-a-b)}{\Gamma(c-a+2r)}\,{}_2F_1(a,b+2r;c-a+2r;-1)\,.
$$

Proof: Letting $q = 4$ and $x = 1$ in Theorem 2.5 of [22] yields

$$
{}_5F_4\left(a, \frac{b}{4}, \frac{b+1}{4}, \frac{b+2}{4}, \frac{b+3}{4}; \frac{c}{4}, \frac{c+1}{4}, \frac{c+2}{4}, \frac{c+3}{4}; 1\right)
$$

$$
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t^4)^{-a} dt
$$

$$
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \binom{-a}{r} \int_0^1 t^{2r+b-1}(1-t)^{c-b-1-a}(1+t)^{-a} dt
$$

$$
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \binom{-a}{r} \frac{\Gamma(b+2r)\Gamma(c-a-b)}{\Gamma(c-a+2r)} {}_2F_1(a, b+2r; c-a+2r; -1),
$$

by the application of Euler's integral (3.8). ■

5.3 An overview of further results

The extensive literature of the 21st and late 20th centuries abounds in results related to hypergeometric identities, such as those discussed in Section 5.2. While it is not feasible to present all such findings, in this section we summarise some interesting results, providing the main findings together with brief comments on the methods used and the significance of the results. This overview also provides some insight into the overlap and interchange of ideas between researchers in this field.

1987 ***Some Summation Formulas for the series $_3F_2(1)$***, Lavoie [56]

In this paper the authors obtain summation formulas (1) and (2) contiguous to Watson's theorem (4.11) in Chapter 4, and summation formulas (3) and (4) contiguous to Whipple's theorem (4.12).

(1) *For* $Re(2c-a-b)>1$,

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ \dfrac{a+b+1}{2}, & & 2c-1; \end{matrix}\, 1\right) = \frac{2^{a+b}\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)\Gamma\left(\frac{-1-a-b}{2}+c\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a+1)\Gamma(b+1)} $$

$$ \times\left(\frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right)}{\Gamma\left(\frac{-1-a}{2}+c\right)\Gamma\left(\frac{-1-b}{2}+c\right)}+\frac{ab\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{4\Gamma\left(c-\frac{a}{2}\right)\Gamma\left(c-\frac{b}{2}\right)}\right). $$

(2) *For* $Re(2c-a-b)>-3$,

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ \dfrac{a+b+1}{2}, & & 2c+1; \end{matrix}\, 1\right) = \frac{2^{a+b}\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)\Gamma\left(\frac{1-a-b}{2}+c\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a+1)\Gamma(b+1)} $$

$$ \times\left(\frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{b}{2}+1\right)}{\Gamma\left(\frac{1-a}{2}+c\right)\Gamma\left(\frac{1-b}{2}+c\right)}-\frac{ab\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{4\Gamma\left(c-\frac{a}{2}+1\right)\Gamma\left(c-\frac{b}{2}+1\right)}\right). $$

(3) *For* $Re(c)>-1,\ a+b=0$ *and* $e+f=1+2c$,

$$ {}_3F_2\left(\begin{matrix} a,b,c; \\ e,f; \end{matrix}\, 1\right) = \frac{\Gamma(e)\Gamma(f)}{2^{2a+1}\Gamma(e-a)\Gamma(f-a)} $$

$$ \times\left(\frac{\Gamma\left(\frac{e-a}{2}\right)\Gamma\left(\frac{f-a}{2}\right)}{\Gamma\left(\frac{e-b}{2}\right)\Gamma\left(\frac{f-b}{2}\right)}+\frac{\Gamma\left(\frac{e-a+1}{2}\right)\Gamma\left(\frac{f-a+1}{2}\right)}{\Gamma\left(\frac{e-b+1}{2}\right)\Gamma\left(\frac{f-b+1}{2}\right)}\right). $$

(4) *For* $Re(c)>1, a+b=2$ *and* $e+f=1+2c$,

$$ {}_3F_2\left(\begin{matrix} a,b,c; \\ e,f; \end{matrix}\, 1\right) = \frac{\Gamma(e)\Gamma(f)}{2^{2a-1}(a-1)(c-1)\Gamma(e-a)\Gamma(f-a)} $$

$$ \times\left(\frac{\Gamma\left(\frac{e-a}{2}\right)\Gamma\left(\frac{f-a}{2}\right)}{\Gamma\left(\frac{e-b}{2}\right)\Gamma\left(\frac{f-b}{2}\right)}-\frac{\Gamma\left(\frac{e-a+1}{2}\right)\Gamma\left(\frac{f-a+1}{2}\right)}{\Gamma\left(\frac{e-b+1}{2}\right)\Gamma\left(\frac{f-b+1}{2}\right)}\right). $$

Comment: The proofs use classical results, including gamma identities and three-term contiguous relations for ${}_3F_2(1)$ found in [75], p.80. The author presents his findings as 'probably new', but Milgram [67] has found that only equation (2) is new. This is discussed in more detail in the last section of Chapter 6.

1992 ***Generalizations of Watson's Theorem on the Sum of a*** $\boldsymbol{{}_3F_2}$, Lavoie et al. [57]

This work follows on Lavoie's 1987 published work on summation formulae for the series ${}_3\mathrm{F}_2(1)$ [56]. The authors repeatedly apply contiguous hypergeometric function relations to Watson's Theorem (4.11) in order to obtain 25 sums for associated ${}_3F_2(1)$ series. From these they develop a general formula for series of the form ${}_3F_2\left(\begin{matrix} a, & b, & c; \\ \frac{a+b+i+1}{2}, & 2c+j; & 1\end{matrix}\right)$, as an extension of Watson's theorem (4.11), with coefficients provided in a table for $i, j = -2, -1, 0, 1, 2$.

1994 ***Generalizations of Dixon's Theorem on the Sum of a*** $\boldsymbol{{}_3F_2}$, Lavoie et al. [58]

Continuing from their 1992 work [57], the authors systematically use the contiguous relations for hypergeometric functions in this article to create 38 distinct formulas contiguous to Dixon's summation theorem (4.7) of Chapter 4. These separate results are then incorporated into a single artificially constructed formula, which provides the generalised form for Dixon's theorem, given as

$$ {}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+i+a-b, & 1+i+j+a-c; & 1\end{matrix}\right)$$

$$= \frac{2^{-2c+i+j}\Gamma(1+i+a-b)\Gamma(1+i+j+a-c)\Gamma\left(b-\frac{i}{2}-\frac{|i|}{2}\right)\Gamma\left(c-\frac{1}{2}[i+j+|i+j|]\right)}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)}$$

$$\times\left\{A_{i,j}\frac{\Gamma\left(\frac{a}{2}-c+\frac{1}{2}+\left\lfloor\frac{i+j+1}{2}\right\rfloor\right)\Gamma\left(\frac{a}{2}-b-c+1+i+\left\lfloor\frac{j+1}{2}\right\rfloor\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1+\left\lfloor\frac{i}{2}\right\rfloor\right)}\right.$$

$$\left.+B_{i,j}\frac{\Gamma\left(\frac{a}{2}-c+1+\left\lfloor\frac{i+j}{2}\right\rfloor\right)\Gamma\left(\frac{a}{2}-b-c+\frac{3}{2}+i+\left\lfloor\frac{j}{2}\right\rfloor\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}+\left\lfloor\frac{i+1}{2}\right\rfloor\right)}\right\}, \qquad (2)$$

for $Re(a-2b-2c) > -2-2i-j$, $i = -3,-2,-1,0,1,2$, $j = 0,1,2,3$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and the polynomial expressions in parameters a, b, c for $A_{i,j}$ and $B_{i,j}$ are provided in a table.

Comment: Many classical summation formulae are found to be special cases of this general form; for example, Dixon's result (4.7) is obtained when $i = j = 0$. The authors evaluate many new sums of a certain class of generalised hypergeometric series from their main result, and provide some limiting cases involving the digamma function. They acknowledge the use of Mathematica in obtaining and checking results by computer.

1996 Generalizations of Whipple's theorem on the sum of a ${}_3F_2$, Lavoie et al. [59]

Using similar procedures to their 1994 work [58], the authors in this paper obtain the following generalised formula for Whipple's theorem (4.12) of Chapter 4. For $a+b = 1+i+j, e+f = 2c+1+i$,

$${}_3F_2\left(\begin{matrix}a, & b, & c; \\ & e, & f;\end{matrix}\ 1\right)$$

$$=\frac{\Gamma(e)\Gamma(f)\Gamma\left(c-\frac{j}{2}-\frac{|j|}{2}\right)\Gamma\left(e-c-\frac{i}{2}-\frac{|i|}{2}\right)\Gamma\left(a-\frac{1}{2}[i+j+|i+j|]\right)}{2^{2a-i-j}\Gamma(e-a)\Gamma(f-a)\Gamma(e-c)\Gamma(a)\Gamma(c)}$$

$$\times\left\{A_{i,j}\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{4}(1-(-1)^i)\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{i}{2}+\left\lfloor\frac{-j}{2}\right\rfloor\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}-\frac{i}{2}+((-1^j)/4)((-1)^i-1)+\left\lfloor\frac{-j}{2}\right\rfloor\right)}\right. \qquad (4)$$

$$+B_{i,j}\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{4}\left(1+(-1)^i\right)\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{1}{2}-\frac{i}{2}+\left\lfloor\frac{-j+1}{2}\right\rfloor\right)}\frac{\Gamma\left(\frac{f}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{f}{2}+\frac{a}{2}-\frac{1}{2}-\frac{i}{2}+((-1)^j/4)(1-(-1)^i)+\left\lfloor\frac{-j+1}{2}\right\rfloor\right)}\Bigg\},$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, $i,j=\pm3,\pm2,\pm1,0$, and the coefficients $A_{i,j}$ and $B_{i,j}$ are various polynomial expressions in parameters a,b,c provided in a table.

Comment: The authors first apply Bailey's equation (3.2.1) of [8] to itself, and then use a change of variables to obtain a two-term identity for a ${}_3F_2(1)$ series. Then by applying Dixon's theorem (4.7) of Chapter 4 and Legendre's duplication formula (2.17) for the gamma function, they obtain a form of Whipple's theorem. They then use the other 38 forms of Dixon's theorem derived in [58] in a similar way, and incorporate these into a single artificial formula to generalise Whipple's theorem. A limiting case provides similar identities for ${}_2F_1(a,1+i+j-a;e;1/2)$.

1997 ***Generalized Watson's summation formula for*** $\boldsymbol{{}_3F_2(1)}$, Lewanowicz [62]

Following the previous works of Lavoie et al. [56]-[59], the author presents an analytic formula in Theorem 2.4, for ${}_3F_2(a,b,c;d+\frac{i}{2},2c+j;1)$ with fixed j and arbitrary $i\in\mathbb{Z}$. The author also shows in Theorems 3.1 and 4.1 that the evaluationof ${}_3F_2(a,1+i+j-a,c;e,1+i+2c-e;1)$ and ${}_3F_2(a,b,c;1+i+a-b,1+i+j+a-c;1)$ as undertaken by Lavoie et al., can be reduced to the evaluation of ${}_3F_2(a,b,c;d+\frac{i}{2},2c+j;1)$.

Comment: In contrast to the approach of Lavoie et al., the author considers his result to be a natural formula which does not require the storage of many coefficients. The method used can also be implemented in a computer algebra language such as Maple or Mathematica.

1997 ***On the reducibility of the Kampé de Fériet function***, Exton [28]

In this paper the author applies techniques of elementary series manipulation and summation formulae for nearly poised hypergeometric series to derive new reducible cases of Kampé de Fériet double hypergeometric functions. He further obtains as special cases the following new relations for single hypergeometric functions.

$$e^y \, {}_2F_2\left(a, 1+\frac{a}{2}; \frac{a}{2}, b; -y\right) = {}_2F_2(2+a-b, b-a-1; b, 1+a-b; y) \quad (12)$$

$$(1-y)^{-h} {}_3F_2\left(h, a, 1+\frac{a}{2}; \frac{a}{2}, b; \frac{y}{1-y}\right) = {}_3F_2(h, 2+a-b, b-a-1; b, 1+a-b; y) \quad (13)$$

$${}_2F_1\left(a, -\frac{e}{2}; 1+a+\frac{e}{2}; x\right) = (1-x)^e \, {}_3F_2\left(a+e, 1+\frac{a}{2}+\frac{e}{2}, \frac{e}{2}; 1+a+\frac{e}{2}, \frac{a}{2}+\frac{e}{2}; x\right) \quad (14)$$

$${}_2F_1(a+d-1, -2a; d; y) = (1-y)^{-a} \, {}_4F_3\left(\begin{matrix} a+d-1, -a, 2-d-2a, d+2a-1; \\ d, d+a, 1-d-2a; \end{matrix} \; y\right) \quad (15)$$

Comment: Identity (12) is the result used by Miller in his 2003 paper [68], as discussed in Section 5.2.

1997 ***New hypergeometric identities arising from Gauss' second summation theorem***, Exton [30]

In this paper Exton uses elementary manipulation of a double series and relations such as Gauss' summation theorem to obtain the general hypergeometric transformation

$$\sum_{n=0}^{\infty} \frac{(c_C)_n \left(\frac{a}{2}\right)_n (-2x)^n}{(d_D)_n n!} {}_{C+1}F_D\left(\begin{matrix} c_1+n, \dots, c_C+n, a+2n; \\ d_1+n, \dots, d_D+n; \end{matrix} x\right)$$

$$= {}_{2C+1}F_{2D}\left(\begin{matrix} \frac{c_C}{2}, \frac{1}{2}+\frac{c_C}{2}, \frac{a}{2}; \\ \frac{d_D}{2}, \frac{1}{2}+\frac{d_D}{2}; \end{matrix} -4^{C-D}x^2\right). \tag{1.8}$$

By applying to (1.8) various known summation theorems found in Slater [82], numbered here as (1.9)-(1.19), he then deduces a number of proposed new identities, numbered as (2.1)-(2.9) and (3.1)-(3.9), involving various ${}_{C+1}F_D$ functions with arguments -1 , ½ and 27/8. Two of these are given below.

For $n = 0, 1, 2, \dots,$

$$\frac{(1+a)_n}{1+\frac{a}{2}} {}_2F_1\left(\begin{matrix} \frac{a}{2}, -n; \\ \frac{1+a}{2}; \end{matrix} -\frac{1}{2}\right) = {}_3F_2\left(\begin{matrix} \frac{-n}{2}, \frac{1-n}{2}, \frac{a}{2}; \\ \frac{1+a+n}{2}, 1+\frac{a+n}{2}; \end{matrix} -1\right), \tag{3.1}$$

and

$$\frac{(1+a)_n}{\left(\frac{1+a}{2}\right)_n} {}_1F_0\left(\begin{matrix} -n; \\ -; \end{matrix} 2^{-n}\right) = {}_4F_3\left(\begin{matrix} 1+\frac{a}{4}, \frac{1-n}{2}, \frac{-n}{2}, \frac{a}{2}; \\ \frac{a}{4}, \frac{1+a+n}{2}, 1+\frac{a+n}{2}; \end{matrix} -1\right). \tag{3.3}$$

1999 ***A new two-term relation for the ${}_3F_2$ hypergeometric function of unit argument***, Exton [32]

The author obtains a new two-term relation and a new resulting summation formula for a nearly-poised ${}_3F_2(1)$ series in terms of gamma functions. His two term relation is

$${}_3F_2\left(\begin{matrix} a, b, c; \\ d, e; \end{matrix} 1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(e-c)} {}_3F_2\left(\begin{matrix} d-a, \quad d-b, \quad c; \\ d, d+e-a-b; \end{matrix} 1\right). \tag{11}$$

This is similar to Thomae's theorem (4.2.8) in [85], but differs in that if one of the ${}_3F_2(1)$ functions is well-poised, this does not necessarily hold for the other. With $d = 1+a-b$ and $e = 1+a-c$, this result yields the summation formula

$$
{}_3F_2\left(\begin{matrix} a, & b, & c; \\ 1+a-b, & a+2b-c-1; & \end{matrix} 1\right)
$$

$$
= \frac{\Gamma(1+2a-b-c)\Gamma(2-b-2c)\Gamma\left(\frac{3-b}{2}\right)\Gamma(1+a-b)\Gamma\left(a-c+\frac{1-b}{2}\right)}{\Gamma(2-b)\Gamma\left(a+\frac{1-b}{2}\right)\Gamma\left(\frac{3+b}{2}-c\right)\Gamma(1+a-b-c)\Gamma(1+2a-b-2c)}. \quad (13)
$$

Comment: Identity (11) is established by manipulation of series, together with the Gauss summation formula (3.13), the Pfaff-Saalschütz' theorem (4.2) and Dixon's theorem (4.7). Milgram's comments on these findings are presented in Section 6.6.

1999 ***A new hypergeometric generating relation,*** Exton [31]

In [74], p.461, Entry 7.3.1.105 is ${}_2F_1\left(\lambda,\lambda+\frac{1}{2};\frac{1}{2};x^2\right)=\frac{1}{2}\left[(1+x)^{-2\lambda}+(1-x)^{-2\lambda}\right]$. In this article Exton uses this reduction formula to derive the following hypergeometric generating relation for $|t|<1$, given as equation (8), p.55.

$$
\sum_{n=0}^{\infty}\frac{(\lambda)_n(\lambda+1/2)_n}{(1/2)_n}\,{}_{p+2}F_q\left(\begin{matrix}-n,\frac{1}{2}-n,\alpha_1,\dots,\alpha_p; \\ \beta_1\dots,\beta_q;\end{matrix}\;x\right)\frac{t^{2n}}{n!}
$$

$$
=\frac{1}{2}(1+t)^{-2\lambda}\,{}_{p+2}F_q\left(\begin{matrix}\lambda,\lambda+\frac{1}{2},\alpha_1,\dots,\alpha_p; \\ \beta_1\dots,\beta_q;\end{matrix}\;\frac{xt^2}{(1+t)^2}\right)
$$

$$
+\frac{1}{2}(1-t)^{-2\lambda}\,{}_{p+2}F_q\left(\begin{matrix}\lambda,\lambda+\frac{1}{2},\alpha_1,\dots,\alpha_p; \\ \beta_1\dots,\beta_q;\end{matrix}\;\frac{xt^2}{(1-t)^2}\right)
$$

2002 ***Some families of Hypergeometric Transformations and Generating Relations***, Lin et al. [63]

The authors refer to Exton's hypergeometric generating relation, equation (8) of [31]. The authors extend this result to derive a family of generating relations for a general polynomial system, including generating relations associated with the Laguerre polynomials. They first define polynomials $S_n^{l,m}(z;\mu)$ by

$$S_n^{l,m}(z;\mu) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!}\left(\frac{1}{2}-\mu-n\right)_{lk} A_k z^k, \tag{3.2}$$

where $l, m \in \mathbb{N}, n \in \mathbb{N}_0$, $[\nu]$ denotes the largest integer less than or equal to ν, and $\{A_n\}_{n=0}^{\infty}$ is a bounded sequence of real or complex parameters. Through standard series techniques, the authors then derive a family of generating relations for $|t| < 1; l, m \in \mathbb{N}$, given by

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\lambda+1/2)_n}{(\mu+1/2)_n} S_n^{l,m}(z;\mu)\frac{t^{2n}}{n!} = (1+t)^{-2\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_{mk}(\lambda+1/2)_{mk}}{k!\,(\mu+1/2)_{(m-l)/k}}$$

$$\times\left[(-1)^{l+m}z\left(\frac{t}{1+t}\right)^{2m}\right]^k {}_2F_1\left(\begin{matrix}2\lambda+2mk, \mu+(m-l)k;\\ 2\mu+2(m-l)k;\end{matrix}\ \frac{2t}{1+t}\right). \tag{3.3}$$

This generating relation is then used to derive new families of generating relations for polynomials in equations (4.5), (4.6) and (4.9), together with particular results which follow from the generalisations. This includes special cases involving the Laguerre polynomials $L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$. It also leads to the elegant results $\sum_{n=0}^{\infty} L_n^{(n)}(x)\frac{x^n}{n!} = e^{2x}$ in equation (5.20), which is a special case of equation 152 (4) found in Rainville [75], and $\sum_{n=0}^{\infty} L_n^{(n)}(x)\frac{x^n}{(n+1)!} = e^x$ in equation (5.21).

2002 ***A generalization of Kummer's identity***, Vidunas [86]

Kummer's identity ${}_2F_1\left(\begin{matrix}a, & b;\\ & c;\end{matrix} -1\right) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}$ holds for $c-a+b=1$. In this article, the author presents a more general evaluation for ${}_2F_1(-1)$ when $c-a+b$ is any integer. The generalisation applies to ${}_2F_1(-1)$ series which are contiguous to a series for Kummer's formula, and is given as

$$ {}_2F_1\left(\begin{matrix}a+n,b;\\a-b;\end{matrix}-1\right)=P(n)\frac{\Gamma(a-b)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{a+1}{2}-b\right)}+Q(n)\frac{\Gamma(a-b)\Gamma\left(\frac{a}{2}\right)}{\Gamma(a)\Gamma\left(\frac{a}{2}-b\right)}, \tag{2} $$

where n is an integer and $P(n), Q(n)$ are rational functions in a, b for every integer n. For n a non-negative integer or -1, Theorem 1 provides the functions

$$ P(n)=\frac{1}{2}\,{}_3F_2\left(\begin{matrix}-\frac{n}{2},-\frac{n+1}{2},b;\\-n,\frac{a}{2};\end{matrix}1\right), Q(n)=\frac{1}{2}\,{}_3F_2\left(\begin{matrix}-\frac{n}{2},-\frac{n+1}{2},b;\\-n,\frac{a+1}{2};\end{matrix}1\right), $$

where $(a)_n \neq 0$, and $a-b$ is not zero or a negative integer. Other possible expressions for $P(n), Q(n)$ are provided in equations (5) – (7) and (16) – (19).

Comment: The generalisating formula (2) involves series contiguous to a series for Kummer's formula, as the two gamma-terms are respectively equal to ${}_2F_1\left(\begin{matrix}a-1,b;\\a-b;\end{matrix}-1\right)$ and $\frac{a-b}{a-2b}\,{}_2F_1\left(\begin{matrix}a,b;\\1+a-b;\end{matrix}-1\right)$.[31] To derive various expressions for the rational functions $P(n), Q(n)$, the authors use standard transformations such as Whipple's formula (8.41) in [92] (Theorem 3.5.9 in our Chapter 3). In special cases, $P(n)$ or $Q(n)$ vanishes, giving an evaluation of ${}_2F_1(-1)$ with a single gamma-term. The authors use classical techniques to derive their results, as well as Zeilberger's computer algorithmic approach, which we discuss in Chapter 6.[32]

[31] Strictly speaking, the functions ${}_2F_1\left(\begin{matrix}a+n,b;\\a-b;\end{matrix}-1\right)$ and ${}_2F_1\left(\begin{matrix}a-1,b;\\a-b;\end{matrix}-1\right)$ are associated series as defined in Section 3.3 of this work.

[32] In their references, the authors give the web address for Zeilberger's EKHAD package as http://www.math.temple.edu/~zeilberger/programs.html, but it is now available at http://www.math.rutgers.edu/~zeilberg/programsAB.html.

2004 ***On Some Sums of Digamma and Polygamma Functions***, Milgram [66]

In this useful piece of work the author collects a number of summation formulae involving digamma and polygamma functions, many of which he asserts to be newly published results. These results include relations with harmonic numbers and Riemann's zeta function. His identities (9), (11) and (12) for ${}_3F_2(1)$ are special cases of more general results but are not listed in the standard tables. He then goes on to obtain numerous identities involving the digamma and polygamma functions.

2010 ***Contiguous extensions of Dixon's theorem on the sum of a*** $\boldsymbol{{}_3F_2}$, Choi [17]

In their 1994 paper [58], Lavoie et al. constructed a formula closely related to the classical Dixon's theorem, with results which excluded 5 cases. In this paper the author presents summation formulas for the remaining 5 cases: $(i,j) = (3,1),(3,2),(3,3),(2,3),(1,3)$.

In this brief overview we have summarised a selection of recently published results related to hypergeometric identities. While this is not designed to be a comprehensive review, the literature provides the overall impression that a large number of such identities continue to be produced, some of which are likely to be adaptations of existing identities

In response to this proliferation of identities, many researches have looked to computer-based techniques to analyse existing and proposed results, as well as to systematically produce new ones. Hence, at this point we turn from classical methods to consider the role that computer technology has come to play in the field of hypergeometric identities.

Chapter 6

Using computer algorithms to investigate hypergeometric identities

6.1 Introduction

As Petkovšek et al. [73], p.36 so aptly state: If we have a hypergeometric series that interests us, we might wonder what is known about it. Can the series be summed in a simple closed form?[33]Can it be transformed to another result that is easier to work with? Is the result that we have discovered about this series really new or a rephrasing of an established result?

These questions have traditionally been addressed by referring to extensive lists in classical works on hypergeometric identities by authors such as Bailey [8]. However, this is far from a simple process. Petkovšek et al. even provocatively assert that a hypergeometric database cannot really be said to exist. They argue that from a computer science point of view a database should consist of a collection of information, a collection of questions that can be addressed to the database by the user, and a collection of algorithms by which the system responds to the queries and searches the data in order to find the answers. By contrast, we cannot state that there

[33] That is, as a linear combination of a fixed number of hypergeometric terms or a ratio of gamma products. Chaundy [16] labels such hypergeometric series as 'reducible'. Maier [64] points out that characterising the hypergeometric series that are summable in finite terms is still an unsolved problem.

exists an exhaustive list of all known hypergeometric identities, as it is always possible to produce one that is not on that list. Even assuming the existence of such a hypothetical list, there does not exist an algorithm which can ascertain whether or not any given sum is transformable into an identity within the database, as the number of required transformations and parametric substitutions might be enormous.

As the results in Chapter 5 illustrate, ongoing research continues to produce hypergeometric identities, some of which can be shown to be alternative forms of existing identities. In order to check for such permutations, as well as to test the validity of proposed new identities, a more sophisticated approach than database look-up is required. As suggested in [95]: "more and more hypergeometric identities are still being conjectured and proved, so the need remains for mechanizing the proofs".

Hence, many researchers are now using computer techniques for comparing, testing and systematically developing hypergeometric transformations and identities. These computer algorithms can evaluate hypergeometric sums under broader conditions than a database look-up, with more clearly stated conditions under which they hold. Furthermore these algorithms are exhaustive in that if they produce nothing, this establishes that nothing exists to find, rather than that the approach has failed. Thus, while our earlier chapters have dealt with classical methods of establishing hypergeometric identities, we now shift our focus to more recent computer-based approaches.

Petkovšek et al. [73] identify three major developmental phases in the proof theory of hypergeometric identities. There was initially the construction of individual proofs, based on methods of combinatorics, generating functions and other useful techniques, as illustrated in Chapters 3 and 4. Then in 1974 it was realised that many combinatorial identities, involving elements such as binomial coefficients and factorials, were special cases of a few general hypergeometric identities. Finally in 1982, Doron Zeilberger realised that algorithmic techniques which Sister Mary Celine

Fasenmyer (1906-1996) developed in the 1940s for discovering recurrence relations for hypergeometric sums, also provide ideal tools for automated proofs of hypergeometric identities. It thus became clear that a priority in analysing hypergeometric identities was to find recurrence relations satisfied by these sums, and researchers turned to computer procedures in the quest for efficiency.

While it is not within the scope of this work to provide comprehensive details of all available algorithmic and computer based techniques, we present in this chapter an overview of some of the most widely used procedures. In Section 6.2 we summarise Sister Celine's fundamental algorithmic methods for obtaining recurrence relations for certain polynomial sequences. In Section 6.3 we discuss how in 1978, Ralph (Bill) Gosper, Jr. provided an algorithmic solution to the problem of indefinite hypergeometric summation. In 1982, Doron Zeilberger developed his 'creative telescoping' algorithm, which we illustrate in Section 6.4, and in Section 6.5 we present the powerful WZ method developed by Herbert Wilf (1931-2012) and Zeilberger in the early 1990s. This elegant algorithm provides extremely short proofs for known hypergeometric identities, as well as creating new identities from old ones.

Finally, in Section 6.6 we illustrate the power of such computer-based strategies by describing how Milgram recently used the Wilf-Zeilberger computer algorithmic technique to conduct a comprehensive analysis of the many existing results for ${}_3F_2(1)$ summations.

6.2 Celine Fasenmyer's algorithmic method

The fundamental algorithmic process for finding hypergeometric term representations for hypergeometric series can be traced to Sister Mary Celine Fasenmyer. While studying a certain class of hypergeometric polynomials in her landmark doctoral work under Earl Rainville in the forties, Sister Celine developed a systematic method for finding a recurrence equation for a given hypergeometric sum. (For details see [34, [35], [75], [73] and [52].) These recurrence equations were found to be holonomic (homogeneous and linear) with polynomial coefficients. Her technique facilitates the evaluation of sums involving binomial coefficients, and is thus ideal for establishing hypergeometric identities. Her algorithm has also provided general existence theorems for recurrence relations satisfied by hypergeometric sums. Sister Celine's method lends itself perfectly to computer automation, although it is substantially slower than more modern algorithms.

Her approach considers the sum $s_n = \sum_{k=-\infty}^{\infty} F(n,k)$ taken over all integers k, where the summand $F(n,k)$ is a doubly hypergeometric term, so that $\frac{F(n+1,k)}{F(n,k)}$ and $\frac{F(n,k+1)}{F(n,k)}$ are both rational functions of n and k (disregarding singularities)[34]. $F(n,k)$ is said to have finite support with respect to k if for each fixed n all summands vanish outside a finite integer interval, often the interval $k \in [0,n]$, $n \in \mathbb{N}$. Wilf and Zeilberger [97] proved that Sister Celine's method can be applied to every proper hypergeometric term $F(n,k)$ as defined below (cf. [52], p.110).

Definition 6.2.1 A hypergeometric term $F(n,k)$ is proper if it has finite support and can be written in the form $F(n,k) = P(n,k)\frac{Q(n,k)}{R(n,k)}\varepsilon^k$, where ε is a parameter, P is a polynomial (the polynomial part), and Q and R are gamma-term products with integer-linear arguments (the factorial part).[35]

In the first step of Sister Celine's method, we seek a mixed recurrence equation satisfied by $F(n,k)$, in the form

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}F(n+j,k+i) = 0\,,^{36} \tag{6.1}$$

such that the coefficients a_{ij} are polynomials in n, independent of k. Dividing the recurrence by $F(n,k)$ will express the left side as a sum of rational functions, and

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[34] Wilf and Zeilberger [97] point out that orthogonal polynomials such as the Hermite, Laguerre, Jacobi and Legendre polynomials, can be expressed as sums of doubly hypergeometric summand functions $F(n,k)$.

[35] $F(n,k)$ thus has no denominator polynomial.

[36] Or alternatively, $\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}F(n-j,k-i) = 0$ (cf. [73], p.58).

with appropriate simplification this can be written as a polynomial in k. By equating the coefficients of this polynomial equation to zero, we then obtain a homogeneous system of linear equations, the solutions of which yield a k-free recurrence equation for F. Wilf and Zeilberger [97] provide upper bounds for the order of the recurrence required to guarantee non-trivial solutions to the resulting system of linear equations.

The second step involves deducing a recurrence equation for the series itself, by summing the recurrence equation for $F(n,k)$ over all integers k. As the coefficients in the recurrence are now independent of k, the summation can operate directly on F in each term, and we can deduce the required sum formula. We illustrate Sister Celine's approach in the examples below.

Example 6.2.2 Consider $s_n = \sum_{k=-\infty}^{\infty} F(n,k)$ where $F(n,k) = k\binom{n}{k}, n \in \mathbb{N}_0$ (so that the summand vanishes for $k < 0$ and $k > n$). We first seek a low order recurrence equation $\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}F(n+j,k+i) = 0$. By setting $I = J = 1$ and $a_{00} = 1$, we obtain

$$F(n,k) + a_{01}F(n+1,k) + a_{10}F(n,k+1) + a_{11}F(n+1,k+1) = 0. \tag{6.2}$$

If we now divide throughout by the definition of $F(n,k)$, (6.2) simplifies to

$$(n+1-k)k + a_{01}(n+1)k + a_{10}(n-k)(n+1-k)$$
$$+a_{11}(n+1)(n+1-k) = 0.$$

By equating the coefficients of this polynomial in k to zero, we obtain

$$\begin{pmatrix} 0 & 1 & 0 \\ n+1 & -2n-1 & -n-1 \\ 0 & n^2+n & n^2+2n+1 \end{pmatrix}\begin{pmatrix} a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ -n-1 \\ 0 \end{pmatrix}.$$

The solution to this system of linear equations is $a_{01} = 0, a_{10} = 1, a_{11} = -\frac{n}{n+1}$, and (6.2) thus becomes the k-free recurrence equation

$$(n+1)F(n,k)+(n+1)F(n,k+1)-nF(n+1,k+1)=0. \qquad (6.3)$$

Now to obtain a recurrence equation for the series itself, we sum (6.3) over all integers k. As $s_n=\sum_{k=-\infty}^{\infty}F(n,k)=\sum_{k=-\infty}^{\infty}F(n,k+1)$, the resulting recurrence is $2(n+1)s_n-ns_{n+1}=0$. From this holonomic recurrence equation, we deduce that $\frac{s_{n+1}}{s_n}=2\frac{n+1}{n}$. To write this result in terms of a generalised hypergeometric series, we introduce a shift by one to obtain $t_n=s_{n+1}$, so that $\frac{t_{n+1}}{t_n}=2\frac{n+2}{n+1}$. The initial value is $t_0=s_1=1$, and hence t_n is a hypergeometric term of the series ${}_1F_0(2;-;2)=\sum_{n=0}^{\infty}\frac{(2)_n}{n!}2^n$. We thus have that $t_n=\frac{(2)_n}{n!}2^n=(n+1)2^n$, and the desired sum is $s_n=n2^{n-1}$ for $n\geq 0$. (This result can be confirmed by the recursion $s_{n+1}=2\frac{n+1}{n}s_n=2^2\frac{n+1}{n}\frac{n}{n-1}s_{n-1}=\cdots=2^n(n+1)s_1$).

Example 6.2.3 Consider the sum $s_n=\sum_{k=0}^{n}F(n,k)$ where $F(n,k)=2^k\binom{n}{k}$. We assume that the summand satisfies the recurrence equation

$$a_{00}F(n,k)+a_{01}F(n+1,k)+a_{10}F(n,k+1)+a_{11}F(n+1,k+1)=0, \qquad (6.4)$$

with k-free polynomial coefficients. After substitution and simplification this equation becomes $a_{00}+a_{01}\frac{n+1}{n+1-k}+2a_{10}\frac{n-k}{k+1}+2a_{11}\frac{n+1}{k+1}=0$, which by equating to zero the coefficients of the polynomial in k, yields a system of linear equations with solutions $a_{00}=2a_{10}, a_{01}=0, a_{11}=-a_{10}$. Recurrence equation (6.4) thus becomes

$$2F(n,k)+F(n,k+1)-F(n+1,k+1)=0, \qquad (6.5)$$

which we then sum on both sides with respect to k in order to get a recurrence equation for the original sum s_n. By the relations

$$\sum_{k=0}^{n} F(n+1,k+1) = \sum_{k=0}^{n+1} F(n+1,k) - F(n+1,0) = s_{n+1} - F(n+1,0), \text{and}$$

$$\sum_{k=0}^{n} F(n,k+1) = \sum_{k=1}^{n+1} F(n,k) = \sum_{k=0}^{n} F(n,k) - F(n,0) + F(n,n+1) = s_n - F(n,0),$$

summing (6.5) yields the relation $s_{n+1} = 3s_n$. As $s_0 = 1$, it follows from iteration that our desired sum has the form $s_n = 3^n$.

Sister Celine's algorithmic process can be run efficiently on computer packages such as EKHAD in Maple.[37] However, there are limitations of this method. Firstly, it does not state for which type of inputs it will be successful, and there is no guarantee that the process will find the holonomic recurrence equation of the lowest order for s_n. Even more severe is the complexity issue, as the resulting system of linear equations contains $(I+1)(J+1)$ variables. A better algorithm of lower complexity has since been developed by Doron Zeilberger, which will be discussed in Section 6.4.

6.3 Gosper's algorithm for indefinite summation

A major landmark in computerising the search for closed form summations is Gosper's algorithm, developed by Gosper in the 1970s at MIT.[38] This algorithm decides whether a partial sum of a hypergeometric series can itself be expressed as a hypergeometric term, and gives its value if it does. It therefore answers the question of whether a given sum involving factorials and binomial coefficients can be expressed in a simple closed form or not. This method addresses the problem of indefinite summation, and has been used extensively in various computer packages for investigating hypergeometric identities.

[37] The process can also in principle be extended to find holonomic recurrence equations for multiple sums $s_n = \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_m=-\infty}^{\infty} F(n, k_1, \dots, k_m)$.

[38] Gosper's algorithm is invoked by the *nusum* command in Macsyma (cf. [94], p.135).

Gosper's systematic procedure is presented in his classical work of 1978 [44], and can be considered to be a discrete analogue of symbolic integration. In his article, Gosper considered a sum of the form $\sum_{n=1}^{m} a_n$, where a_n is a hypergeometric term so that $\frac{a_n}{a_{n-1}}$ is a rational function of n. In indefinite integration we search for a function which has the integrand as its derivative; in Gosper's case we wish to know if for the given hypergeometric term a_n there exists another hypergeometric term, say $S(n)$, such that its difference is the summand, expressed in the form

$$S(n) - S(n-1) = a_n, \tag{6.6}$$

(equation [1] in Gosper's article). Informally, the role of the differential operator in integration is taken over by the difference operator $\Delta S \coloneqq S(n) - S(n-1)$. Finding an 'antidifference' $S(n)$ is called indefinite summation, and Gosper's algorithm finds those $S(n)$ such that $\frac{S(n)}{S(n-1)}$ is a rational function of n. Definite summation then follows easily, since by telescoping we obtain $\sum_{n=1}^{m} a_n = \sum_{n=1}^{m}(S(n) - S(n-1)) = S(m) - S(0)$, where $S(0)$ is a constant.

If there exists a hypergeometric term $S(n)$ which satisfies (6.6), the algorithm outputs this term, a_n is said to be Gosper-summable, and we can express the sum $\sum_{n=1}^{m} a_n$ in the simple closed form of a single hypergeometric term plus a constant. If no such term exists, this proves that (6.6) has no hypergeometric solution. Below we provide further details of Gosper's approach, then present the work in more recent notation, and finally illustrate the method through two examples.

Gosper [44] first assumes that $\frac{S(n)}{S(n-1)}$ is a rational function of n (so that $S(n)$ is hypergeometric). Then by equation (6.6), the term ratio becomes

$$\frac{a_n}{a_{n-1}} = \frac{S(n) - S(n-1)}{S(n-1) - S(n-2)} = \frac{\dfrac{S(n)}{S(n-1)} - 1}{1 - \dfrac{S(n-2)}{S(n-1)}},$$

which is also a rational function of n, so that a_n itself is a hypergeometric term. This result can be written in the form

$$\frac{a_n}{a_{n-1}} = \frac{p_n}{p_{n-1}}\frac{q_n}{r_n}, \tag{6.7}$$

where p_n, q_n and r_n are polynomials in n with $\gcd(q_n, r_{n+j}) = 1$ for all nonnegative integers j. (Gosper showed that it is always possible to put a rational function in this form, using a change of variables to eliminate common factors where necessary.)
After studying many particular cases, Gosper was led to write $S(n)$ in the form

$$S(n) = \frac{q_{n+1}}{p_n} f(n) a_n\,, \tag{6.8}$$

where the function $f(n)$ must yet be found. By using equation (6.6) this yields the result

$$f(n) = \frac{p_n}{q_{n+1}} \frac{1}{1 - \frac{S(n-1)}{S(n)}},$$

and hence $f(n)$ will be a rational function of n whenever $\frac{S(n-1)}{S(n)}$ is. By substituting (6.8) into (6.6), Gosper obtained $a_n = \frac{q_{n+1}}{p_n} f(n) a_n - \frac{q_n}{p_{n-1}} f(n-1) a_{n-1}$, which after multiplying by p_n/a_n and using equation (6.7) becomes the recurrence equation

$$p_n = q_{n+1} f(n) - r_n f(n-1), \tag{6.9}$$

with polynomial coefficients. Gosper further established that if $\frac{S(n)}{S(n-1)}$ is a rational function of n, then $f(n)$ is a polynomial with a maximum degree determined by the degrees of p_n, q_n and r_n .

The problem of finding a hypergeometric solution $S(n)$ of (6.6) is thus ingeniously reduced to finding polynomial solutions $f(n)$ of (6.9). In other words, the original problem becomes one of solving the system of linear equations which results from introducing an appropriate generic polynomial and equating coefficients. The consistency of such a system is thus equivalent to the existence of the required hypergeometric term $S(n)$, and the desired indefinite sum can then be found through equation (6.8). If any of the steps to find the polynomial $f(n)$ fails, then no hypergeometric antidifference exists for the given a_n. Gosper's algorithm is thus a decision procedure which returns the hypergeometric term $S(n)$, or returns the response "No closed form (hypergeometric) antidifference exists".

While Gosper's original work was based on the backward antidifference $S(n) - S(n-1)$, recent related work usually uses a forward antidifference, as in [4], [73], [52] and [75]. In this approach, Gosper's algorithm searches for a hypergeometric term z_n such that $z_{n+1} - z_n = t_n$ for a given hypergeometric term t_n, and evaluates the sum $S_n = \sum_{k=0}^{n-1} t_k$ using the steps below (cf. [73], p.79).

1. Form the term ratio $r_n = \frac{t_{n+1}}{t_n}$.
2. Write this ratio in the form

$$r_n = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}, \tag{6.10}$$

   where $a(n), b(n), c(n)$ are polynomials such that $\gcd\big(a(n), b(n+j)\big) = 1$ for all nonnegative integers j.
3. Find a nonzero polynomial solution $f(n)$ for the recurrence equation

$$a(n)f(n+1) - b(n-1)f(n) = c(n). \tag{6.11}$$

4. Then

$$z_n = \frac{b(n-1)f(n)}{c(n)} t_n, \tag{6.12}$$

   and the required sum is

$$s_n = \sum_{k=0}^{n-1} t_k = \sum_{k=0}^{n-1} (z_{k+1} - z_k) = z_n - z_0 \,, \tag{6.13}$$

   where z_0 is a constant, or $s_n = z_n - z_{k_0}$ if the lower limit of summation is not zero.

Example 6.3.1 Given the sum $S_n = \sum_{k=1}^{n} k.k!$, $r_n = \frac{(n+1)(n+1)!}{n.n!} = \frac{(n+1)^2}{n}$, so that in (6.10) we have $a(n) = n+1$, $b(n) = 1$, $c(n) = n$. The related polynomial equation (6.11) thus becomes $(n+1)f(n+1) - f(n) = n$, which is satisfied by the constant function $f(n) = 1$. Hence, by (6.12) we have $z_n = \frac{1}{n}.n.n! = n!$, so that according to (6.13), $s_n = z_n - z_1 = n! - 1$, and our desired sum is $S_n = s_{n+1} = (n+1)! - 1$.

Example 6.3.2 Consider the sum $S_n = \sum_{k=0}^{n}(4k+1)\frac{k!}{(2k+1)!}$, in which the summand is a hypergeometric term with term ratio $r_n = \frac{4n+5}{2(4n+1)(2n+3)}$. We choose $a(n) = 1, b(n) = 2(2n+3), c(n) = 4n+1$, so that the associated equation (6.11) becomes $f(n+1) - 2(2n+1)f(n) = 4n+1$, one solution of which is the constant function $f(n) = -1$. Hence, we have $z_n = \frac{-2(2n+1)}{4n+1}.\frac{(4n+1)n!}{(2n+1)!} = \frac{-2n!}{(2n)!}$, so that $s_n = z_n - z_0 = 2 - \frac{2n!}{(2n)!}$, and the closed form for our sum is $S_n = s_{n+1} = 2 - \frac{n!}{(2n+1)!}$. (While S_n is not itself a hypergeometric term, it is the sum of two such terms.)

Example 6.3.3 For $S_n = \sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(k+1)4^{2k}}$, the term ratio can be shown to be $r_n = \frac{t_{n+1}}{t_n} = \frac{(n+1/2)^2}{(n+1)(n+2)}$. It follows that our relatively prime polynomials are $a(n) = (n+1/2)^2$, $b(n) = (n+1)(n+2)$, $c(n) = 1$, and we seek a function $f(n)$ such that $(n+1/2)^2 f(n+1) - n(n+1)f(n) = 1$. This is easily found to be the constant function $f(n) = 4$, from which it follows that $z_n = \frac{b(n-1)f(n)}{c(n)}.t_n = \frac{4n\binom{2n}{n}^2}{4^{2n}}$. Then $s_n = \sum_{k=0}^{n-1} t_k = z_n - z_0 = z_n - 0$, and our sum is $S_n = s_{n+1} = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2+2n}}$.

The above examples have clearly been carefully chosen so as to be suitable for manual calculation. For summations which are not as amenable to human efforts, Gosper's algorithm is effectively run by computer packages such as the *Gosper* command in Maple's *SumTools* [*Hypergeometric*], and *gosper.m* in Mathematica (cf. [73], p.87). For any given input hypergeometric term t_n, Gosper's algorithm will then construct where possible another hypergeometric term z_n such that $z_{n+1} - z_n = t_n$. As Graham et al. [45] point out, we now do not have to compile a list of indefinitely summable hypergeometric terms, because Gosper's algorithm provides a quick method that works in all summable cases. Gosper's algorithm also plays a central role in other algorithmic approaches to hypergeometric summation, such as that developed by Zeilberger, which we discuss below.

6.4 Zeilberger's telescoping algorithm

When Zeilberger was awarded the Euler Medal in 2004, the citation referred to him as "a champion of using computers and algorithms to do mathematics quickly and efficiently". In 1990 he developed an algorithm for finding the recurrence relation for a hypergeometric term, which is a faster alternative to Fasenmyer's approach. He published his result in his well-known paper [99], entitled '*The Method of Creative Telescoping*'.[39]

In developing his method, Zeilberger extended Gosper's algorithm to apply to a far wider range of cases. Many summands are not indefinitely summable, in which case Gosper's algorithm returns a 'No' result. However, the same sum might be expressible in simple terms when the index runs over all integers.[40] Thus, while Gosper's method can establish whether or not a given hypergeometric term can be indefinitely summed, Zeilberger's algorithm plays a similarly central role in the study of definite summation.

While Gosper's algorithm deals with a summand t_k of one variable, Zeilberger follows Sister Celine in considering a sum of the form $S_n = \sum_k F(n,k)$, where the summation index runs through all the integers if not explicitly specified, and $F(n,k)$ is doubly hypergeometric with finite support with respect to k, so that the sum $\sum_k F(n,k)$ is finite. Zeilberger's method first produces a discrete function $G(n,k)$ which satisfies a recurrence relation of the form

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[39] The term 'creative telescoping' was first coined by A. van Poorten in his 1979 paper: '*A proof that Euler missed … Apery's proof of the irrationality of* $\zeta(3)$', in which he applied a similar approach to Apery numbers.

[40] In a similar way, while the indefinite integral $\int e^{-t^2} dt$ has no simple form, the definite integral $\int_{-\infty}^{\infty} e^{-t^2} dt$ has the value $\sqrt{\pi}$.

$$\sum_{j=0}^{J} a_j(n)\, F(n+j,k) = G(n,k+1) - G(n,k), \tag{6.14}$$

where $G(n,k)/F(n,k)$ is a rational function of n, k.

The Fundamental Theorem of Algorithmic Hypergeometric Proof Theory states that the existence of such a recurrence is assured when $F(n,k)$ is a proper hypergeometric term. This relies on the theory of holonomic systems, initiated by Bernstein [13]. See also [100] and the concise proof of Theorem 3.2A in [97]. However Abramov and Stegun [2] point out that Zeilberger's algorithm does not necessarily fail if a hypergeometric term is not proper.

The telescoping algorithm begins with an assumed order J for the desired recurrence, and then searches for a recurrence of higher order if necessary. For that fixed J, we denote the left side of (6.14) by

$$t_k = a_0 F(n,k) + a_1 F(n+1,k) + \cdots + a_J F(n+J,k), \tag{6.15}$$

and then use appropriate substitutions to write the term ratio $\frac{t_{k+1}}{t_k}$ in the form

$$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{p_2(k)}{p_3(k)}, \tag{6.16}$$

as a combination of polynomials in k, where p_2 and p_3 are coprime. This is a standard form for taking to Gosper's algorithm, and hence from that theory we have that t_k will be an indefinitely summable hypergeometric term if and only if there exists a polynomial solution $f(k)$ to the recurrence relation

$$p_2(k) f(k+1) - p_3(k-1) f(k) = p(k). \tag{6.17}$$

As before, this leads to a system of simultaneous linear equations, and its solution (should it exist) will produce Gosper's expression in the form

$$G(n,k) = \frac{p_3(k-1)}{p(k)} f(k) t_k. \tag{6.18}$$

After substituting this result into (6.14) and summing both sides over k, the right side telescopes to zero and we find an expression for the given sum $S_n = \sum_k F(n,k)$.

Example 6.4.1 To evaluate the familiar binomial sum $S_n = \sum_{k=0}^{n} \binom{n}{k}$, we set $F(n,k) = \binom{n}{k}$. Then for $J = 1,\ a_0 = 1,$ equation (6.15) becomes $t_k = F(n,k) + a_1 F(n+1,k) = \binom{n}{k} + a_1 \binom{n+1}{k}$, and we have $\frac{t_{k+1}}{t_k} = \frac{(n-k+a_1 n+a_1)}{(n+1-k+a_1 n+a_1)} \frac{(n+1-k)}{(k+1)}$. This is of the form (6.16) with $p(k) = n+1-k+a_1 n+a_1$, $p_2(k) = n+1-k$ and $p_3(k) = k+1$. Hence, (6.17) is $(n+1-k)f(k+1) - kf(k) = n+1-k+a_1 n+a_1$. According to Gosper's algorithm, the polynomial solution of this recurrence will be of degree zero, so we substitute $f(k) = C$ into the recurrence equation and equate the coefficients of like powers of k to find that the solution is given by $C = \frac{1}{2}, a_1 = -\frac{1}{2}$. It follows that $F(n,k)$ satisfies (6.14) in the form

$$F(n,k) - \frac{F(n+1,k)}{2} = G(n,k+1) - G(n,k).$$

By summing and telescoping we obtain $S_{n+1} = 2S_n$, which with $S_0 = 1$ and recursive evaluation yields the result $S_n = \sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Example 6.4.2 To evaluate $S_n = \sum_{k=0}^{n} \binom{n}{k}^2$, we first find a recurrence of order $J = 1$ for the summand $F(n,k)$. Equation (6.15) becomes $t_k = a_0 F(n,k) + a_1 F(n+1,k)$, and hence

$$\frac{t_{k+1}}{t_k} = \frac{a_0 \binom{n}{k+1}^2 + a_1 \binom{n+1}{k+1}^2}{a_0 \binom{n}{k}^2 + a_1 \binom{n+1}{k}^2} = \frac{a_0 (n-k)^2 + a_1 (n+1)^2}{a_0 (n+1-k)^2 + a_1 (n+1)^2} \frac{(n+1-k)^2}{(k+1)^2}.$$

This result is of the form (6.16) with $p(k) = a_0(n+1-k)^2 + a_1(n+1)^2$, $p_2(k) = (n+1-k)^2$ and $p_3(k) = (k+1)^2$. Hence, our recurrence equation (6.17) becomes

$$(n+1-k)^2 f(k+1) - k^2 f(k) = a_0(n+1-k)^2 + a_1(n+1)^2. \tag{6.19}$$

According to Gosper's algorithm, the polynomial solution of this recurrence will be of degree one, so we substitute $f(k) = \alpha + \beta k$ into (6.19) and equate the coefficients of like powers of k to find that

$$\alpha = -3(n+1), \beta = 2, a_0 = -2(2n+1), a_1 = n+1 .$$

By (6.14), $F(n,k)$ then satisfies the recurrence equation

$$-2(2n+1)F(n,k) + (n+1)F(n+1,k) = G(n,k+1) - G(n,k).$$

When summing this equation over all integers k, the right side telescopes to zero and we have $S_{n+1} = \frac{2(2n+1)}{n+1} S_n$, which with $S_0 = 1$ and recursive evaluation yields the final result: $S_n = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

For more complicated identities, Zeilberger's creative telescoping algorithm is effectively applied by programs such as his Maple program *ct* in the package EKHAD, and a Mathematica program written by Peter Paule and Markus Schorn.[41] For example, the algorithm has been used to generate identities of Dixon (cf. [73], p.11), Saalschütz (cf. [45], p.235) and Clausen (cf. [52], p.106). Paule and Shorn [72] point out that when running the Gosper algorithm as part of Zeilberger's method, the built-in Mathematica functions are far too slow for solving the resulting system of homogenous linear equations with polynomial coefficients. They instead recommend an alternative algorithm written by K. Eichhorn in Mathematica code.[42] Various techniques at programming and user level are also being developed to reduce computational time and memory requirements, including automatic filtering of factors, substitution of parameters and shifting of the summation interval (cf. [76]). Maple routines for deriving linear relations between contiguous Gauss

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[41] These programs are available at http://www.math.rutgers.edu/~zeilberg/programsAB.html, and http://www.cis.upenn.edu/~wilf/progs.html.

[42] This is available along with other relevant algorithms on request to Peter.Paule@risc.uni-linz.ac.at.

hypergeometric functions can also be found in the postdoctoral work by Vidunas, coordinated by T. H. Koornwinder and N. Temme, for the NWO project: '*Algorithmic methods for special functions by computer algebra*'.[43]

The Gosper-Zeilberger algorithm does not work for all sums. For example, when the summand is $F(n,k) = \binom{n}{k} n^k$, the term ratio $\frac{F(n+1,k)}{F(n,k)}$ is not a rational function of k. However, it does succeed in the enormous number of cases when the summand is a proper hypergeometric term, so that the algorithm does terminate, and unlike Sister Celine's method it almost always generates the lowest order recurrence equation in reasonable time. Koepf [52], p.102 does, however, provide a sum for which Zeilberger's algorithm returns a second order recurrence equation even though a first order recurrence is satisfied.

6.5 WZ pairs

In 1990, Wilf and Zeilberger [96] published an amazingly short method for certifying combinatorial, and hence hypergeometric identities. To certify an identity entails providing some information additional to the identity itself, which will make it easier to independently verify that identity. Once the independent information is verified, it must then be shown that this implies the validity of the given identity. This approach, termed the WZ method, thus reduces the proof of a given identity to that of a finite identity between polynomials. This theory is directly concerned only with terminating identities, but the developers point out in [97] that many non-terminating identities are immediate consequences of terminating ones, usually with extra parameters.

While the Zeilberger algorithm ascertains whether or not an unknown hypergeometric sum can be evaluated in closed form, and can also discover an unknown closed form solution, the remarkable WZ approach provides a proof for every known

[43] Available at http://staff.science.uva.nl/~thk/specfun/compalg.html.

hypergeometric identity, as well as creating new identities from a known one. Each of these two methods thus has a slightly different application. In this section we provide a brief overview of the WZ method, which is a clever application of Gosper's algorithm to the difference $F(n+1,k)-F(n,k)$ rather than to $F(n,k)$ alone.

When developing standard proof procedures for certain categories of theorems, allowance must be made for the slight variations in each example. This small varying detail in the standardised proof is known as the proof certificate for that theorem. In the WZ method, this proof certificate contains just one rational function $R(n,k)$ of two variables, which certifies the proof of that specific identity. The WZ proof algorithm comprises the five steps summarised below (cf. [73], p.124), together with the following conditions: for each integer k, $\lim_{n\to\infty} F(n,k)$ exists and is finite; for each integer $n \geq 0$, $\lim_{k\to\pm\infty} G(n,k) = 0$.

Assume that we wish to prove an identity of the form $\sum_k f(n,k) = r(n)$, where n is some positive integer, k ranges over the set of integers, and the summand has finite support with respect to k.

1. Divide to write the identity in the form $\sum_k F(n,k) = 1$, where $F(n,k) = f(n,k)/r(n)$, and $r(n)$ is non-zero.
2. Apply Gosper's algorithm to the difference $F(n+1,k)-F(n,k)$, with respect to k. This process is not always trivial and might involve a change of variables, but if successful it generates the function $G(n,k)$ such that

$$F(n+1,k)-F(n,k) = G(n,k+1)-G(n,k), \tag{6.20}$$

in which case the pair of functions (F,G) is called a WZ pair, and $F(n+1,k)-F(n,k)$ is Gosper-summable.[44] The function $G(n,k)$ is given in the form $G(n,k) = R(n,k)F(n,k)$, where the rational function $R(n,k)$ is the WZ proof certificate for the

[44] Ira Gessel provides an intriguing relation between equation (6.20) and path invariance of a weighted grid graph, in his 2011 workshop presentation '*On the WZ Method*'. A pdf of his presentation slides is available at http://people.brandeis.edu/~gessel/homepage/slides/wilf80-slides.pdf.

given identity. Should there exist a denominator zero of $R(n,k)$ for some $k \in \mathbb{Z}$, this must be compensated for by a zero of $F(n,k)$.

3. Check the rational certificate independently by confirming the validity of (6.20), or equivalently by proving the rational identity
$$\frac{F(n+1,k)}{F(n,k)} - 1 + R(n,k) = R(n,k+1)\frac{F(n,k+1)}{F(n,k)}.$$

4. Sum equation (6.20) over all integers k. Then
$$\begin{aligned}\sum_{k=-\infty}^{\infty}[F(n+1,k)-F(n,k)] &= \lim_{M\to\infty}\sum_{k=-M}^{M}[G(n,k+1)-G(n,k)]\\ &= \lim_{M\to\infty}[G(n,M+1)-G(n,-M)]\\ &= 0-0\\ &= 0.\end{aligned}$$
Through this telescoping result, (6.20) becomes $\sum_k F(n+1,k) = \sum_k F(n,k)$, and hence $\sum_k F(n,k)$ is a constant independent of n (cf. [94], Th. 4.4.1).
5. Verify that the constant is 1 by checking that $\sum_k F(0,k) = 1$, from which it follows that $\sum_k F(n,k) = 1$, and the given identity is verified.

We illustrate the method in the example below.

Example 6.5.1 Consider again the identity $\sum_{k=0}^{n}\binom{n}{k} = 2^n, n \geq 0$. We are required to show that $\sum_{k=0}^{n} F(n,k) = \sum_{k=0}^{n}\binom{n}{k}2^{-n} = 1$. Let $t_k = F(n+1,k) - F(n,k)$, so that after simplifying we have $\frac{t_{k+1}}{t_k} = \frac{n-k+1}{k+1}.\frac{n-2k-1}{n-2k+1} = \frac{a(k)}{b(k)}.\frac{c(k+1)}{c(k)}$. This form can be taken to Gosper's algorithm, which returns the result $G(n,k) = -\frac{\binom{n}{k-1}}{2^{n+1}}$. The WZ certificate is thus $R(n,k) = \frac{G(n,k)}{F(n,k)} = k/(2(k-n-1))$. To check this certificate we are required to confirm that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$, i.e. $\binom{n+1}{k}2^{-n-1} - \binom{n}{k}2^{-n} = -\binom{n}{k}2^{-n-1} + \binom{n}{k-1}2^{-n-1}$. By standard factorial simplification, this becomes $\frac{n+1}{2} - (n+1-k) = -\frac{n+1-k}{2} + \frac{k}{2}$, which is easy to prove as an identity. It is clear that $\lim_{k\to\pm\infty} G(n,k) = 0$ so that $\sum_k [G(n,k+1) - G(n,k)]$ telescopes to zero. In addition, we have $\sum_{k=0}^{0} F(0,k) = 1$ and the given identity is thus verified.

Wilf [94], p.133 shows how the algorithm uses the proof certificate $R(n,k) = \frac{2k-1}{2n+1}$ to confirm the sum $\sum_k (-1)^k \binom{n}{k}\binom{2k}{k} 4^{n-k} = \binom{2n}{n}$. Andrews et al. [4], p.169, similarly show how the WZ method can be used to confirm the Pfaff-Saalschütz identity (4.2) of our Chapter 4 and also a ${}_4F_3$ summation formula of Bailey's, while Koepf [52], Table 6.2, provides a complete list of Bailey's identities which can be treated by this method, together with the associated rational certificates. Identities such as Watson's and Whipple's, for which the term ratio is not rational, cannot be dealt with by this approach.

The power of this method is that computer packages can perform the algorithm in cases requiring more complicated simplification, using Gosper's algorithm to find the WZ mate for the given summand $F(n,k)$. Petkovšek et al. [73], Sections 2.4 and 2.5, provide details of the Mathematica and Maple commands[45] which create the function $G(n,k) = R(n,k)F(n,k)$ using a given certificate function $R(n,k)$, and which run the algorithm to verify a given identity. They also provide one-line proofs for six standard hypergeometric identities, using the appropriate rational function certificate $R(n,k)$ in each case. In their article '*Symbolic summation - some recent developments*' published in [38], Paule and Strehle provide full algorithmic details for implementing this method in Mathematica in order to prove the identity $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}\binom{2k}{k+a}\binom{2k}{k}^{-1} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n}{k+a}$.

[45] Wilf and Zeilberger make their Maple programme available upon request from zeilberger@euclid.math.temple.edu.

The main purpose of the WZ method is the certification of identities. However, for each identity that it certifies, this approach also provides a systematic way to find related identities. Wilf and Zeilberger [96] have shown that if $(F(n,k),G(n,k))$ is a WZ pair, then for complex numbers a,b,c, the following are also WZ pairs: $\big(F(n+a,k+b),G(n+a,k+b)\big)$, $\big(cF(n,k),cG(n,k)\big)$, $(F(-n,k),-G(-n-1,k))$, $(F(n,-k,),-G(n,-k+1))$ and $(G(k,n),F(k,n))$. The first four WZ pairs are known as the associates (or companions) of $(F(n,k),G(n,k))$, and the last WZ pair and its associates are known as the duals of $(F(n,k),G(n,k))$. Using this approach together with Maple computer programs which involve substitutions such as $a \to a+n$, Gessel [42] derives an extensive list of terminating hypergeometric series identities, three of which are provided below.[46]

$$ {}_3F_2\left(\begin{matrix} -3n, \frac{2}{3}-c, 3n+2; \\ \frac{3}{2}, 1-3c; \end{matrix} \; \frac{3}{4}\right) = \frac{\left(c+\frac{2}{3}\right)_n \left(\frac{1}{3}\right)_n}{(1-c)_n \left(\frac{4}{3}\right)_n} $$

$$ {}_3F_2\left(\begin{matrix} -3b, \frac{-3n}{2}, \frac{1-3n}{2}; \\ \frac{2}{3}-b-n, -3n; \end{matrix} \; \frac{4}{3}\right) = \frac{\left(\frac{1}{3}-b\right)_n}{\left(\frac{1}{3}+b\right)_n} $$

$$ {}_4F_3\left(\begin{matrix} -n, \frac{3}{2}+\frac{n}{5}, \frac{2}{3}, 2n+2; \\ \frac{4}{3}, n+\frac{11}{6}, \frac{n}{5}+\frac{1}{2}; \end{matrix} \; \frac{2}{27}\right) = \frac{\left(\frac{5}{2}\right)_n \left(\frac{11}{6}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{7}{2}\right)_n} $$

These identities are new in the sense of not being found in the existing databases of identities, although they might be obtainable by a transformation of some known identity. Many other interesting WZ forms of linear hypergeometric transformations have also been found by Zhou [102].

[46] Gessel [42] makes the interesting observation that in the identities such as those given here, the fractional arguments contain powers of 2, 3 or 5 (the only exception in his article being an identity with argument $z = 16/27$).

A further useful aspect of this method for generating related identities is that when we specialise the free parameters in an existing hypergeometric identity, the dual of a specialisation is not in general the specialisation of the dual, and thus countless new identities can be created. Hence, many of Gosper's 'strange identities' tackled individually by Gessel and Stanton [43] turn out to be duals of specialisations of classical identities such as those of Saalschütz and Dougall, as shown in [96]. This insight has also established that Dixon's classical well-poised ${}_3F_2(1)$ identity (4.7) of Chapter 4 is the dual of a specialisation of the balanced Saalschütz identity (4.2).

A particularly dramatic illustration of the power of the WZ method is given by Zeilberger [101] regarding recent investigation by George Andrews into a conjecture about self complementary plane partitions. The work required the evaluation of the rather tricky hypergeometric sum

$$ {}_4F_3\left(\begin{matrix} -i, \frac{8}{3}, j+3, 5/2; \\ \frac{5}{3}, -2j, 2j+6; \end{matrix} -8\right), $$

which the WZ method solved in a few seconds.

There continues to be extensive ongoing work involving the use of computer algorithms to create and investigate hypergeometric identities. For example, in their paper of 2003, Krattenthaler and Rao [54] describe a completely automatised algorithim implemented in *Mathematica*, which uses a beta integral method to systematically derive additional identities from existing ones, with some interesting and new results. These various methods have also been extended to multisum terminating hypergeometric identities, and to single- and multi- (terminating) q-hypergeometric identities with continuous or discrete variables (cf. [5], [97], [101]).

While the main focus of this algorithmically-based research has been on the development of theory, in their 2006 article '*Five applications of Wilf-Zeilberger theory to enumeration and probability*' published in [53], Apagodu and Zeilberger also illustrate some practical applications of the WZ theory in computational contexts involving dice rolling, generating functions, Markov models and lattice path counting.

To illustrate the power of applying algorithmic methods to hypergeometric identities, we now present the findings of a computer-based investigation into the many identities that exist for ${}_3F_2(1)$ summations.

6.6 Refining the database of ${}_3F_2(1)$ sums

We have seen that computerised algorithmic procedures offer an elegant way forward in establishing and validating hypergeometric identities. They also provide a powerful tool for establishing equivalence between alternative versions of an identity. In 2006, Milgram [67] made use of the WZ method to investigate and analyse all published hypergeometric sums ${}_3F_2(1)$.

Of the 120 Thomae transformations for ${}_3F_2(1)$, there exist ten inequivalent forms (see Milgram's Appendix A in [67]). Hence, according to Milgram, the 'potential universe' of knowable closed-form results for ${}_3F_2(1)$ identities is nine times larger than that provided in existing tables. However, computer algebra systems have made it possible to explore this universe for potentially new results, and Milgram's paper provides a useful insight into the results of such analysis.

To initiate his review, Milgram first used computer algorithms to numerically test each closed form ${}_3F_2(1)$ identity provided by Prudnikov et al. [74]. As a result of this process, he omitted from consideration equations [7.4.4.38] and [7.7.7.73] as not satisfying non-trivial numerical tests, and amended [7.4.4.19] to reflect the particular case which it satisfies. He further corrected misprints in equations [7.4.4.43], [7.4.4.55], [7.4.4.67] and [7.4.4.71], and developed three additional summations based on equation [7.4.4.25], labelled as (6), (7) and (8) in his paper.

Milgram then applied the 10 Thomae relations to the resulting closed form ${}_3F_2(1)$ identities, to obtain 630 possibly new or different closed form sums. Each result was then systematically compared against all the others by using the WZ method to search for a valid transformation among all parameters (taking symmetry into account), and equivalent terminating sums were removed. This procedure yielded a base set of fundamental results. The list was then expanded to include all closed form ${}_3F_2(1)$ results found in the literature, for which a transformation from the existing base set parameters could not be found. The comparing and eliminating procedure was then applied for a second time.

After this rigorous process 89 identities finally remained, 23 of which were in the original set and 66 of which were possibly new, which Milgram lists in his Appendix B. These entries were finally tested for novelty using computer algebra simplification commands, and also tested numerically for validity.

In Section 3.3 of his work, Milgram reports on the status of existing ${}_3F_2(1)$ summations, particularly those found in more recent literature. Two of Milgram's own results from his 2004 paper [66] survived the algorithmic-based tests, and he presents these as equations (9) and (10) on page 5.

In 1975, Sharma [79] presented two new closed forms for a particular ${}_3F_2(1)$ based on parameter choices in a new ${}_4F_3(1)$ series which he obtained by evaluating a double series. However, this new series does not satisfy numerical tests and Sharma's results are omitted.

All results given by Krupnikov and Kölbig [55], Table 2 are special cases of existing results in Milgram's Appendix B.

In 1999, Exton [32] proposed a new two-term transformation between ${}_3F_2(1)$ series, with a new result following as a special case (discussed briefly in our Section 5.3). However, Milgram states that Exton's proposed transformation (11) is in fact a symmetric permutation of Thomae's third relation (T3 in Milgram's Appendix A), while his special case in equation (13) is obtained by incorrect applications of transformations, and fails numerical tests. Another of Exton's new results, Case 3.3 with $q = 2$ in [29], is a special case of Prudnikov's equation 7.4.4.14 (B1 in Milgram's Appendix B).

Two results that Milgram confirms to be new, are equations (1.6) and (1.9) of Gessel and Stanton [43]. Milgram provides these as equations (11) and (12).

Regarding the proliferation of results contiguous to the ${}_3F_2(1)$ summations of Watson, Dixon and Whipple, Milgram first comments on the work of Lavoie, in Section 3.4. He states that in Lavoie's article [56], equation (2) is new and is thus added to the database as equation (15), while equation (1) can be obtained from equation (2) by a Thomae transformation (T1 in Appendix A). All other relations in Lavoie's paper [56] have been shown to be special cases of results already in the database. (Milgram does, however, pose the question whether Lavoie's results might not in fact be the source of the relevant entries in the famous work of Prudnikov et al. [74]).

In their later papers, Lavoie et al. presented generalisations for cases contiguous to Dixon's [58], Watson's [57] and Whipple's [59] theorems. The results for Whipple's and Dixon's theorems are limited to a small subset (38 and 39 cases respectively), whereas Lewanowicz [62] independently gave a far more general result valid for Watson's theorem. In this sense, Milgram considers Watson's theorem as being more fundamental than Dixon's and Whipple's, and hence only Lewanowicz's general result for contiguous elements of the generalised Watson's theorem has been retained in Milgram's database, together with six other relations obtained through applying Thomae transformations, which are listed in Milgram's Appendix C. The general problem of finding elements contiguous to Watson's, Dixon's and Whipple's theorems is thus reducable to a simple computer algorithm.

In his Section 3.5, Milgram reports on the results of other computer algebra methods for obtaining transformations and identities between hypergeometric series ${}_3\mathrm{F}_2(1)$, such as Krattenthaler's Mathematica package HYP and Gauthier's Maple package HYPERG. He concludes that these have produced no new sums for his database. For example, equation (30.1) in Gessel [42] is a special case of Bailey's equation 7.4.4.10 in [8].

Milgram's original work was conducted in 2006. In page 11 of his 2010 update, he extends his analysis to include the more recent work by Maier [65]. This has yielded several new cases of ${}_3\mathrm{F}_2(1)$ such as Maier's equation (1.2):

$$ {}_3\mathrm{F}_2\left(\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3}; \\ 3a+1; \quad \frac{1}{2} \end{matrix} 1\right) = \frac{3^{3a}(1+4^{-3a})}{2}, $$

and Theorem 7.1, which with $l = 0$ and $l = -1$ respectively, yield

$$ {}_3\mathrm{F}_2\left(\begin{matrix} 2a, 2a-\frac{1}{3}, 2a+\frac{2}{3}; \\ 3a; \quad 3a+\frac{1}{2} \end{matrix} 1\right) = \left(\frac{3}{2}\right)^{6a-1} $$

and

$$ {}_3\mathrm{F}_2\left(\begin{matrix} 2a, 2a-\frac{1}{3}, a+\frac{2}{3}; \\ 1+3a; \quad 3a+\frac{1}{2} \end{matrix} 1\right) = \left(\frac{3}{2}\right)^{6a-1} \Big/ \left(2a+\frac{2}{3}\right). $$

These new results are listed in Milgram's Appendix D of 2010.

Milgram finally concludes that as a result of his computer based analysis, 66 new ${}_3\mathrm{F}_2(1)$ sums have been found, and his review provides a comprehensive list of all 89 such summations. His paper thus provides a useful database for possible computerised identification of any desired sum of this form, by seeking a transformation between the parameters of the candidate and elements of the database.

Milgram's comprehensive work suggests a possible role for similar systematic computer-based analysis of identities and transformations of other hypergeometric series.

Conclusion

Hypergeometric functions, together with the gamma function, provide a powerful tool in a wide variety of mathematical fields and related sciences. In this work we have aimed to present a useful synopsis of the fundamental representations, identities and transformations for the gamma and hypergeometric functions, including both classical and more recent results.

The world of hypergeometric identities is as exciting as it is vast, and its very immensity can be somewhat daunting. However, the increasing use of computer algorithms to systematically explore, create and test these identities provides an effective way forward. These recent techniques have proven to be more efficient than laborious reference to extensive lists of identities and transformations. Regarding the use of automated methods, Andrews et al. [4], p.140 make the following observation: “The ease with which such impressive identities can be manufactured shows again the inadequacy of relying solely on some fixed database of identities and underscores the flexibility and comprehensiveness of a computer-based algorithmic approach”. Zeilberger [100] and [98] quips: “now that the silicon saviour has arrived, a new testament is going to be written”, and on a more serious note he observes that “the team human-computer is a mighty one, and an open-minded human can draw inspiration from all sources, even from a machine”.

It thus seems likely that computer algorithmic packages will become a standard tool of the trade for those working in number theory and combinatorics, as well as other areas relying heavily on special functions and their properties. In this dissertation, we hope to have illustrated the value of such automated procedures as applied to special functions in general and to the hypergeometric function in particular, and highlighted the increasingly central role that they are likely to play in future developments in this field.

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