Generating Functions and the Enumeration of Lattice Paths

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Abstract

Our main focus in this research is to compute formulae for the generating function of lattice paths. We will only concentrate on two types of lattice paths, Dyck paths and Motzkin paths. We investigate different ways to enumerate these paths according to various parameters. We start off by studying the relationship between the Catalan numbers C_n, Fine numbers F_n and the Narayana numbers $v_{n,k}$ together with their corresponding generating functions. It is here where we see how the the Lagrange Inversion Formula is applied to complex generating functions to simplify computations. We then study the enumeration of Dyck paths according to the semilength and parameters such as, number of peaks, height of first peak, number of return steps, e.t.c. We also show how some of these Dyck paths are related. We then make use of Krattenhaler's bijection between 123-avoiding permutations of length n, denoted by $S_n(123)$, and Dyck paths of semilength n. Using this bijective relationship over $S_n(123)$ with k descents and Dyck paths of semilength n with sum of valleys and triple falls equal to k, we get recurrence relationships between ordinary Dyck paths of semilength n and primitive Dyck paths of the same length. From these relationships, we get the generating function for Dyck paths according to semilength, number of valleys and number of triple falls. A plateau in a Motzkin path is the occurrence of a sequence of an up step $(1, 1)$ a horizontal step $(1, 0)$ and a down step $(1, -1)$ in that order. We find different forms of the generating function for Motzkin paths according to length and number of plateaus with one horizontal step, then extend the discussion to the case where we have more than one horizontal step. We also study Motzkin paths where the horizontal steps have different colours, called the k-coloured Motzkin paths and then the k-coloured Motzkin paths which don't have any of their horizontal steps lying on the x-axis, called the k-coloured c-Motzkin paths. We find that these two types of paths have a special relationship which can be seen from their generating functions. We use this relationship to simplify our enumeration problems.

Declaration

I declare that *Generating Functions and the Enumeration of Lattice Paths* is my own work, composed while registered as a fulltime student at the University of the Witwatersrand, Johannesburg.

Phumudzo Hector Mutengwe 14 May 2013

Signed: ....

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Contents

Chapter 1

INTRODUCTION

In our research, we try to understand different relationships between the Catalan numbers, Fine numbers and Motzkin numbers. To do this, we rely heavily on generating functions and the different techniques used in the enumeration of lattice paths. We will also make use of the Binomial Coefficients because they are easy to use and have important applications.

Below is a list of some of the important binomial identities we will apply when solving the enumeration problems. These expressions hold for $n, k, m, r \in \{0\} \cup \mathbb{N}$

$$\sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^{i} = (x+y)^{n} = \sum_{i=0}^{n} \binom{n}{i} x^{i} y^{n-i},$$

$$\binom{n}{k} = \binom{n}{n-k} \qquad \text{where } n \geq k,$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{where } n \geq k \geq 1,$$

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{n+k+1}{n},$$

$$\sum_{i=0}^{n} \binom{i}{m} = \binom{n+1}{m+1} \qquad \text{where } n \geq m,$$

$$\binom{r}{k} = (-1)^{k} \binom{k-r-1}{k} \qquad \text{where } r \geq k \text{ with } r \geq 1,$$

and lastly Vandermonde identity with its many variations,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} \text{ where } r \geq m \geq k.$$

It should further be noted that $\binom{0}{0} = 1$ and $\binom{n}{k} = 0$ whenever $n < k$.

We will mainly focus on the papers [3], [7], [10] and [16]. We will start with the study of the different relationships between generating functions for Catalan numbers, the Narayana function and generating functions for Fine numbers. Then we will study the different ways to enumerate Dyck paths according to various parameters. We also investigate the bijective relationship between permutation of length n and Dyck paths of semilength n.

Chapter 2

GENERATING FUNCTIONS

2.1 Introduction

Definition 1 *A Dyck path is a lattice path in the first quadrant which begins at the origin and has up steps,* $(1, 1)$ *denoted by* u*, and down steps,* $(1, -1)$ *denoted by* d *and ends at the point* $(2n, 0)$*. It never goes below the* $x-axis$ *and* n *is the semilength of the path (i.e the number of up steps).*

e.g. The Dyck path below can be represented by *uuduuudddduded*

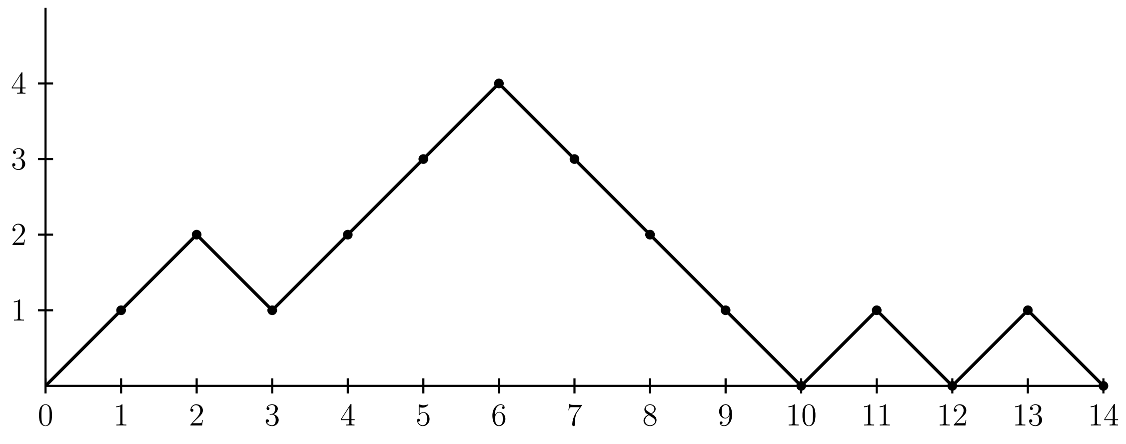

Figure 1.

Definition 2 *A generating function is a formal power series in one indeterminate* *whose coefficients encode information about a sequence of numbers* (a_n).

Following the dictum by Herbert Wilf [18], "A generating function is a clothesline on which we hang up a sequence of numbers for display". We will use sequences and their generating functions interchangeably. For example, the generating function for the sequence $(1, 1, 1, \cdots)$ is $A(x) = 1 + x + x^2 + x^3 + \cdots$. Here we see that the coefficient of x^n, where $n \geq 0$, represents a corresponding element in our sequence. The *multivariate* generating functions of a sequence are the formal power series in multiple variables. If $A(z)$ is a generating function then we use $[z^n]A(z)$ to represent the coefficient of z^n in the power series expansion of $A(z)$. From our generating function above we get $[x^n]A(x) = 1$.
We begin by reading *E. Deutsch*'s paper, Dyck path enumeration [7].

2.2 The Lagrange Inversion Formula

The Lagrange Inversion Formula (LIF) is a remarkable tool for solving functional equations and can sometimes give explicit formulas. To apply LIF, our functional equation must be of the form,

$$F = p\Phi(F).$$

Here Φ is a function of F and we are solving for F in terms of p.

Theorem 1 *Let $A(z)$ be a generating function satisfying the equation*

$$A(z) = 1 + zH(A(z)) \tag{2.2.1}$$

where $H(\lambda)$ is a polynomial in λ, then (2.2.1) has a unique solution $A(z)$. Furthermore, if $G(\lambda)$ is a polynomial in λ, then

$$[z^n]G(A(z)) = \frac{1}{n}[\lambda^{n-1}]G'(1+\lambda)(H(1+\lambda))^n \quad \textit{for} \quad n \geq 1. \tag{2.2.2}$$

As an example, consider the functional equation

$$D^s(z) = \left(ze^{D(z)}\right)^s.$$

If we let $A(z) = D(z)$, $H(z) = e^z$ and $G(z) = z^s$, then applying the LIF to get the coefficient of z^n in $D^s(z)$ gives,

$$\begin{aligned}
[z^n]D^s(z) &= \frac{1}{n}[\lambda^{n-1}]s\lambda^{s-1}e^{\lambda n} \\
&= \frac{s}{n}[\lambda^{n-s}]\left(1 + n\lambda + \frac{(n\lambda)^2}{2!} + \frac{(n\lambda)^3}{3!} + \cdots\right) \\
&= \frac{s}{n}\frac{n^{n-s}}{(n-s)!}.
\end{aligned}$$

2.3 The Catalan Numbers

These numbers form a sequence of natural numbers that appear frequently in combinatorics. The first few Catalan numbers are $1, 1, 2, 5, 14, 42, \cdots$. We will often use them in our study of lattice paths. The Catalan numbers have a functional equation given by

$$zC^2(z) - C(z) + 1 = 0, \qquad C(0) = 1. \tag{2.3.1}$$

Making $C(z)$ the subject of the formula we get

$$\begin{aligned}
\sum_{n\geq 0} C_n z^n &= 1 + z\sum_{n\geq 0} C_n z^n \sum_{n\geq 0} C_n z^n \\
&= 1 + z\sum_{n\geq 0}\sum_{k=0}^{n} C_k C_{n-k} z^n.
\end{aligned}$$

Equating the coefficient of the equation above we get, $C_0 = 1$,

$$\begin{aligned}
C_n &= \sum_{k=0}^{n-1} C_k C_{n-1-k} \\
&= C_0C_{n-1} + C_1C_{n-2}\cdots + C_{n-1}C_0 \\
&= C_{n-1} + C_1C_{n-2}\cdots + C_{n-1}, \qquad \text{for} \quad n \geq 1.
\end{aligned}$$

Solving the quadratic equation (2.3.1), we get

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{2.3.2}$$

Now

$$\begin{aligned}
[z^n]C(z) &= [z^n]\frac{1-\sqrt{1-4z}}{2z} \\
&= [z^{n+1}]\frac{1-\sqrt{1-4z}}{2} \\
&= [z^{n+1}]\frac{1-\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4z)^n}{2} \\
&= [z^{n+1}](-1)\frac{\sum_{n\geq 1}\binom{\frac{1}{2}}{n}(-4z)^n}{2} \\
&= -\frac{1}{2}[z^{n+1}]\sum_{n\geq 1}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-4z)^n \\
&= -\frac{1}{2}[z^{n+1}]\sum_{n\geq 1}\frac{(-1)(2n-2)!}{2^{2n-1}n!(n-1)!}4^n z^n \\
&= \frac{1}{2}\frac{(2n)!}{2^{2n+1}(n+1)!n!}4^{n+1} \\
&= \frac{1}{n+1}\binom{2n}{n}. \qquad (2.3.3)
\end{aligned}$$

Thus,

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.3.4}$$

Making $C(z)$ the subject of the formula in (2.3.1) and raising it to the exponent s we get

$$C^s(z) = \left(1 + zC^2(z)\right)^s.$$

Applying the LIF to the equation above, with $A(z)$ being $C(z)$, $H(A(z))$ being $(C(z))^2$ and $G(A(z))$ being $(C(z))^s$ gives,

$$[z^n]C^s(z) = \frac{1}{n}[\lambda^{n-1}]s(1+\lambda)^{s-1}((1+\lambda)^2)^n$$

$$
\begin{aligned}
&= \frac{1}{n}[\lambda^{n-1}]s(1+\lambda)^{2n+s-1} \\
&= \frac{s}{n}[\lambda^{n-1}]\sum_{k=0}^{2n+s-1}\binom{2n+s-1}{k}\lambda^k \\
&= \frac{s}{n}\binom{2n+s-1}{n-1} \\
&= \frac{s}{n}\frac{(2n+s-1)!}{(n-1)!(n+s)!} \\
&= \frac{s}{2n+s}\frac{(2n+s)!}{n!(n+s)!} \\
&= \frac{s}{2n+s}\binom{2n+s}{n}, \qquad \text{where } n, s \text{ are not both equal to 0.} \qquad (2.3.5)
\end{aligned}
$$

If $s = 1$, we see that $[z^n]C(z) = \dfrac{1}{n+1}\binom{2n}{n}$ as expected and if $s = 2$, $[z^n]C^2(z) = \frac{1}{n+2}\binom{2(n+1)}{n+1}$ which is equal to C_{n+1}.

2.4 The Fine Numbers

There are many combinatorial interpretations of these numbers. Some of them are:
i.) The number of Dyck paths where no u step starting at the x-axis is immediately followed by a d step.
ii.) The number of Dyck paths where the first ud occurs such that the vertex between the two steps corresponds to a y coordinate that is even in the Cartesian plane.

The first few Fine numbers are $1, 0, 1, 2, 6, 18, 57, \cdots$. The Fine function is defined by

$$F(z) = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}. \qquad (2.4.1)$$

From the equation above we get

$$
\begin{aligned}
F(z) &= \frac{\dfrac{1-\sqrt{1-4z}}{2z}}{z\left(\dfrac{1}{z}+\dfrac{1-\sqrt{1-4z}}{2z}\right)} \\
&= \frac{C(z)}{1+zC(z)} \qquad (2.4.2)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{C(z)(1-zC(z))}{(1+zC(z))(1-zC(z))} \\
&= \frac{C(z)-zC^2(z)}{1-z^2C^2(z)} \\
&= \frac{1}{1-z^2C^2(z)}, \qquad \text{from (2.3.1).} \qquad (2.4.3)
\end{aligned}$$

From (2.4.2) we get

$$F(z) = C(z)\textstyle\sum_{s\geq 0}(-1)^s z^s C^s(z) = \sum_{s\geq 0}(-1)^s z^s C^{s+1}(z).$$

Applying (2.3.5) on C^{s+1} in the equation above gives,

$$\begin{aligned}
F(z) &= \sum_{s\geq 0}(-1)^s z^s \sum_{n\geq 0}\frac{s+1}{2n+s+1}\binom{2n+s+1}{n}z^n \\
&= \sum_{n\geq 0}\sum_{s\geq 0}(-1)^s \frac{s+1}{2n+s+1}\binom{2n+s+1}{n}z^{n+s}.
\end{aligned}$$

Extracting the coefficient of $F(z)$ from the above equation we get,

$$\begin{aligned}
[z^n]F(z) = F_n &= \sum_{s\geq 0}(-1)^s \frac{s+1}{2(n-s)+s+1}\binom{2(n-s)+s+1}{n-s} \\
&= \sum_{s\geq 0}(-1)^s \frac{s+1}{2n-s+1}\frac{(2n-s+1)!}{(n-s)!(n+1)!} \\
&= \frac{1}{n+1}\sum_{s\geq 0}(-1)^s(s+1)\binom{2n-s}{n} \\
&= \frac{1}{n+1}\sum_{s=0}^{n}(-1)^s(s+1)\binom{2n-s}{n}.
\end{aligned}$$

The upper limit in the last step follows since $\binom{2n-s}{n} = 0$ for $s \geq n+1$. From (2.4.3) we have,

$$\begin{aligned}
F(z) &= \sum_{s\geq 0} z^{2s}C^{2s}(z) \\
&= \sum_{s\geq 0} z^{2s}\sum_{n\geq 0}\frac{2s}{2n+2s}\binom{2n+2s}{n}z^n, \text{ using (2.3.5)}
\end{aligned}$$

$$= \sum_{n\geq 0}\sum_{s\geq 0} \frac{s}{n+s}\frac{(2n+2s)!}{n!(n+2s)!} z^{n+2s}.$$

Thus,

$$F_n = \sum_{s\geq 0} \left(\frac{s}{(n-2s)+s}\right)\left(\frac{(2(n-2s)+2s)!}{(n-2s)!((n-2s)+2s)!}\right)$$
$$= \sum_{s\geq 0} \frac{s}{n-s}\binom{2n-2s}{n}, \text{ for } n \geq 2. \tag{2.4.4}$$

2.5 The Narayana numbers

The generating functions associated with these numbers will be used extensively in our enumeration problems. The bivariate generating function for the Narayana numbers, $\rho(t,z)$, has implicit formula

$$(1+\rho)(1+t\rho)z = \rho, \quad \text{where} \quad \rho(t,0) = 0. \tag{2.5.1}$$

To get the explicit formula we multiply out the brackets to get,

$$zt\rho^2 + \rho(zt + z - 1) + z = 0, \qquad \text{where } \rho = \rho(t,z).$$

Using the quadratic formula and then simplifying we get,

$$\rho(t,z) = \frac{1 - z - zt - \sqrt{(zt+z-1)^2 - 4z^2t}}{2zt}$$
$$= \frac{1 - z - tz - \sqrt{1 - 2z + z^2 - 2tz - 2tz^2 + t^2z^2}}{2tz}. \tag{2.5.2}$$

We then extract the coefficients of $t^k z^n$ in ρ^m, $(1+\rho)^m$, and $(1+t\rho)^m$ using LIF and then simplify the expressions.

If $m \geq 1$ and $n \geq 1$, then

$$[t^k z^n]\rho^m = [t^k]\frac{1}{n}[\lambda^{n-1}]m\lambda^{m-1}\left((1+\lambda)(1+t\lambda)\right)^n$$

$$
\begin{aligned}
&= [t^k]\frac{m}{n}[\lambda^{n-m}]\sum_{i=0}^{n}\binom{n}{i}\lambda^i\sum_{j=0}^{n}t^j\binom{n}{j}\lambda^j \\
&= [t^k]\frac{m}{n}[\lambda^{n-m}]\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}t^j\lambda^{i+j} \\
&= [t^k]\frac{m}{n}\sum_{j=0}^{n}\binom{n}{n-m-j}\binom{n}{j}t^j \\
&= \frac{m}{n}\binom{n}{n-m-k}\binom{n}{k} \\
&= \frac{m}{n}\binom{n}{m+k}\binom{n}{k}. \qquad (2.5.3)
\end{aligned}
$$

Thus,

$$
[t^kz^n]\rho^m = \begin{cases} \frac{m}{n}\binom{n}{m+k}\binom{n}{k} & \text{if } n \geq 1 \text{ and } m \geq 1, \\ 1 & \text{if } n = m = k = 0, \\ 0 & \text{otherwise.} \end{cases}
$$

If $n \geq 1$, then

$$
\begin{aligned}
[t^kz^n](1+\rho)^m &= [t^kz^n]\sum_{s=0}^{m}\binom{m}{s}\rho^s \\
&= [t^kz^n]\sum_{s=0}^{m}\binom{m}{s}\sum_{i\geq 1}\sum_{j\geq 0}\frac{s}{i}\binom{i}{s+j}\binom{i}{j}t^jz^i \quad \text{using (2.5.3)} \\
&= \sum_{s=0}^{m}\binom{m}{s}\frac{s}{n}\binom{n}{s+k}\binom{n}{k} \\
&= \frac{m}{n}\binom{n}{k}\sum_{s=1}^{m}\binom{m-1}{m-s}\binom{n}{k+s} \\
&= \frac{m}{n}\binom{n}{k}\binom{m+n-1}{m+k} \quad \text{after using Vandermonde's identity.}
\end{aligned}
\qquad (2.5.4)
$$

Thus,

$$
[t^kz^n](1+\rho)^m = \begin{cases} \frac{m}{n}\binom{n}{m+k}\binom{n+m-1}{m+k} & \text{if } n \geq 1, \\ 1 & \text{if } n = k = 0, \\ 0 & \text{otherwise.} \end{cases}
$$

If $n \geq 1$, then

$$\begin{aligned}
[t^k z^n](1+t\rho)^m &= [t^k z^n] \sum_{s=0}^{m} \binom{m}{s} t^s \rho^s \\
&= [t^k z^n] \sum_{s=0}^{m} \binom{m}{s} \sum_{i\geq 1} \sum_{j \geq 0} \frac{s}{i} \binom{i}{s+j} \binom{i}{j} t^{j+s} z^i \quad \text{using (2.5.3)} \\
&= \sum_{s=0}^{m} \binom{m}{s} \frac{s}{n} \binom{n}{k} \binom{n}{k-s} \\
&= \frac{m}{n} \binom{n}{k} \sum_{s=1}^{m} \binom{m-1}{m-s} \binom{n}{n-k+s} \\
&= \frac{m}{n} \binom{n}{k} \binom{n+m-1}{n+m-k} \quad \text{after using Vandermonde's identity} \\
&= \frac{m}{n} \binom{n}{k} \binom{n+m-1}{k-1}. \qquad (2.5.5)
\end{aligned}$$

Thus,

$$[t^k z^n](1+t\rho)^m = \begin{cases} \frac{m}{n}\binom{n}{m+k}\binom{n+m-1}{m+k} & \text{if } \ n \geq 1, \\ 1 & \text{if } \ n = k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The Narayana numbers $v_{n,k}$ are defined by $v_{n,k} = [t^k z^n]\rho(t,z)$. If in (2.5.3) we let $m = 1$ then we find that the Narayana numbers are given by

$$\frac{1}{n}\binom{n}{k+1}\binom{n}{k} \quad \text{for } \ n \geq 1.$$

From (2.5.1) and (2.5.2) we then derive some relationships satisfied by the Narayana function.
We have

$$\begin{aligned}
\rho(\frac{1}{t}, tz) &= \frac{1-(tz)-(\frac{1}{t})(tz) - \sqrt{1-2(tz)+(tz)^2-2(\frac{1}{t})(tz)-2(\frac{1}{t})(tz)^2+(\frac{1}{t})^2(tz)^2}}{2(\frac{1}{t})(tz)} \\
&= \frac{1-z-tz-\sqrt{1-2tz+(tz)^2-2z-2tz^2+z^2}}{2z} \\
&= t\left(\frac{1-z-tz-\sqrt{1-2tz+(tz)^2-2z-2tz^2+z^2}}{2tz}\right)
\end{aligned}$$

$$= t\rho(t,z). \tag{2.5.6}$$

Also

$$\begin{aligned}
\frac{1-z}{z}\rho(t,z) &= \frac{1-z}{z}\left(\frac{1-z-tz-\sqrt{1-2tz+(tz)^2-2z-2tz^2+z^2}}{2tz}\right)\\
&= \frac{(1-z)^2-tz+tz^2-(1-z)\sqrt{1-2tz+(tz)^2-2z-2tz^2+z^2}}{2tz^2}\\
&= 1+\frac{1}{2tz^2}\big((1-z)^2-tz-tz^2\\
&\qquad -(1-z)\sqrt{1-2tz+(tz)^2-2z-2tz^2+z^2}\big)\\
&= 1+\frac{(1-z)^2}{2tz^2}\left(1-\frac{tz}{(1-z)^2}-\frac{ztz}{(1-z)^2}\right.\\
&\qquad \left.-\sqrt{1-\frac{2tz}{(1-z)^2}-\frac{2ztz}{(1-z)^2}+\frac{(tz)^2(1-z)^2}{(1-z)^4}}\right)\\
&= 1+\frac{1}{2z\frac{tz}{(1-z)^2}}\left(1-\frac{tz}{(1-z)^2}-z\frac{tz}{(1-z)^2}\right)\\
&\quad -\frac{\sqrt{1-2\frac{tz}{(1-z)^2}-2z\frac{tz}{(1-z)^2}+(\frac{tz}{(1-z)^2})^2-2z\left(\frac{tz}{(1-z)^2}\right)^2+z^2\left(\frac{tz}{(1-z)^2}\right)^2}}{2z\frac{tz}{(1-z)^2}}\\
&= 1+\rho\left(z,\frac{tz}{(1-z)^2}\right).
\end{aligned}$$

Thus,

$$\rho(t,z) = \frac{z}{1-z}\left(1+\rho(z,\frac{tz}{(1-z)^2})\right). \tag{2.5.7}$$

Now

$$\begin{aligned}
1+\rho\left(\frac{1}{z},\frac{tz^2}{(1-z)^2}\right) &= 1+z\rho\left(z,\frac{tz}{(1-z)^2}\right) \quad \text{by } (2.5.6)\\
&= 1+z\left(\rho(t,z)\frac{1-z}{z}-1\right) \quad \text{by } (2.5.7)\\
&= 1-z+z\rho(t,z)(1-z)\\
&= (1-z)(\rho(t,z)+1). \qquad (2.5.8)
\end{aligned}$$

2.6 Terminology and notations

A *peak*, *valley* and *doublerise* in a Dyck path is the occurrence of ud, du and uu respectively. A *triple fall* is the occurrence of three consecutive down steps ddd.
The *level* of a vertex in a Dyck path is the y coordinate corresponding to that vertex. Thus, if a vertex of a Dyck path has coordinates (n, k) then, we say it is at level k. A step of a Dyck path having extremeties of co-ordinates $(n, k-1)$ and (s, k), where n, s are natural numbers and $|n - s| = 1$ with $k \geq 1$, is said to be at level k. The level of a peak, valley or doublerise is the level of the vertex between its two steps. A *low peak* is a peak at level 1 and a *high peak* is a peak at a level greater than 1. A *low valley* is a valley at level 0 and a *high valley* is one at level 1 or greater.
A *return* step is a d step that touches the x-axis. If a Dyck path only has one return step, we say the Dyck path is *primitive.* In a Dyck path, we call a maximal string of u's an *ascent* and a maximal string of d's a *descent.* If a descent ends on the x-axis, it is called a return descent.

Examples: We consider the diagram below to explain some of the definitions above.

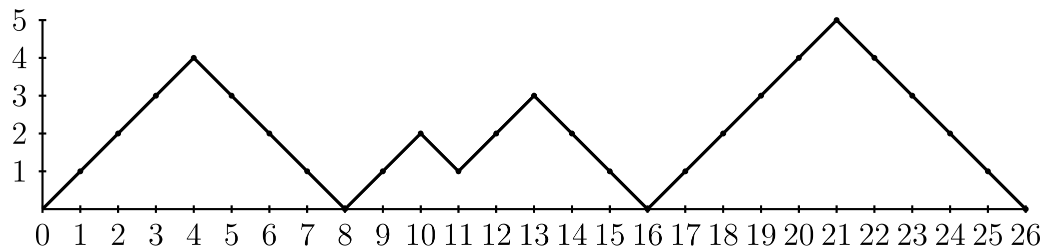

Figure 2.

In this diagram, the vertices $(4, 4), (10, 2), (13, 3)$ and $(21, 5)$ are the peaks and the vertices $(8, 0), (11, 1)$ and $(16, 0)$ are the valleys of the path. The doublerises in this diagram are the subpaths that are formed by any combination of two consecutive u steps in $(0, 0) - (4, 4), (8, 0) - (10, 2), (11, 1) - (13, 3)$ and $(16, 0) - (21, 5)$. Likewise, the triple falls are any three consecutive d steps in $(4, 4) - (8, 0), (13, 3) - (16, 0)$ and $(21, 5) - (26, 0)$. This Dyck path has return steps at $(8, 0), (16, 0)$ and $(26, 0)$. If we cut off our path at the point $(8, 0)$, then the left part becomes a primitive Dyck path. The ascents of this path are $(0, 0) - (4, 4), (8, 0) - (10, 2), (11, 1) - (13, 3)$ and

$(16,0)-(21,5)$. Similarly, the descents are $(4,4)-(8,0), (10,2)-(11,1), (13,3)-(16,0)$ and $(21,5)-(26,0)$.

We define $\alpha\beta$ to be the *concatenation* of two Dyck paths α and β and $\hat{\alpha} = u\alpha d$ to be the *elevation* of the Dyck path α. From these definitions, it is clear that all elevated paths are primitive and non-empty since they always start with an u step and end with a d step. We also note that the concatenation of two non-empty paths cannot give a primitive path since the two paths will be joined on the x-axis which means the path will have more than one return step. The empty path is denoted by ϵ.
Let's consider the two diagrams below.

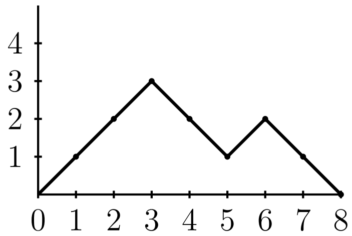

Figure 3(a).

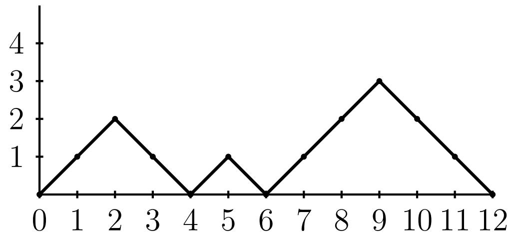

Figure 3(b).

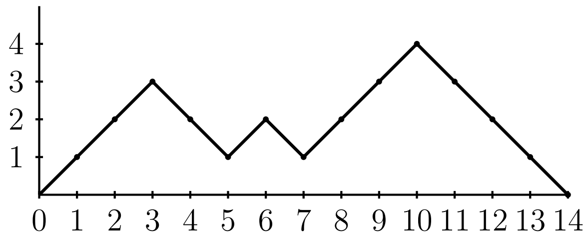

Figure 3(c).

To elevate the Dyck path in figure $3(b)$, we prepend a u step and append a d step. This leads to the primitive Dyck path in figure $3(c)$ with the semilength increased by one.
For concatenation, we must first relax the condition that all Dyck paths must start at the origin and end at the vertex $(2n,0)$. Essentially, we shift the Dyck path which will appear on the right of the concatenation to the right by the number of steps in the Dyck path on the left of the concatenation. After this shift, we insert the left Dyck path on the left of the shifted Dyck path such that the terminal vertex of the

left path joins the initial vertex of the shifted path. The concatenation of path $3(a)$ and $3(b)$ gives either one of the paths below, depending on which of the two was shifted,

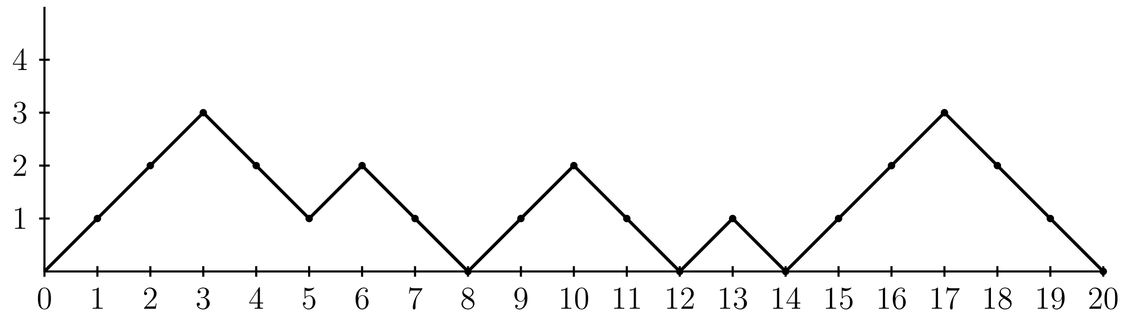

Figure 4(a).

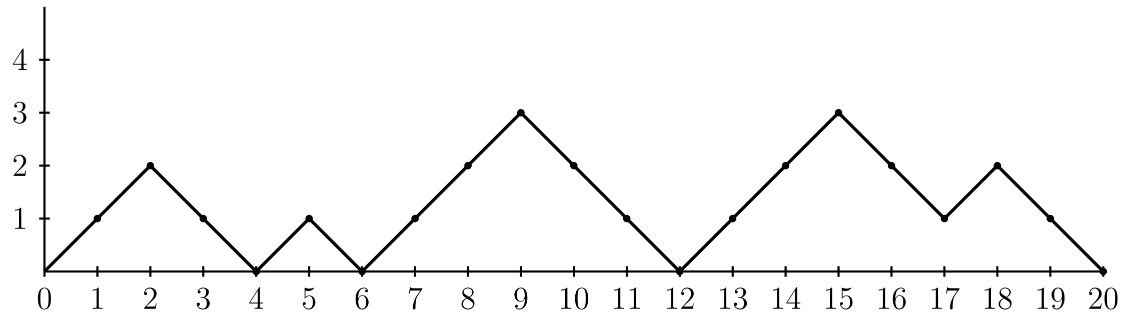

Figure 4(b).

If A and B are finite sets of Dyck paths, then we define the concatenation of A and B by

$$AB := \{\alpha\beta : \alpha \in A, \beta \in B\}$$

and the elevation of A by

$$\hat{A} := \{\hat{\alpha}, \alpha \in A\}.$$

We will denote the set of all Dyck paths of semilength n by D_n. The set containing the empty path is $D_0 = \{\epsilon\}$. The set of elevated Dyck paths of semilength n will be represented by $\hat{D_n}$.

Every nonempty Dyck path α can be uniquely decomposed into the form

$$\alpha = u\beta_1 d\gamma_1 \quad i.e. \quad \alpha = \hat{\beta}_1\gamma_1, \tag{2.6.1}$$

where β_1 and γ_1 are Dyck paths that can be empty. This is called the *first return decomposition*, since the d step in $u\beta_1 d$ is the first return step of α. We also note that both $\hat{\beta}_1$ and γ_1 must be shorter that α. Alternatively, we can uniquely decompose α into

$$\alpha = \beta_2 u\gamma_2 d \quad i.e. \quad \alpha = \beta_2\hat{\gamma}_2 \tag{2.6.2}$$

where β_2 and γ_2 are Dyck paths that can be empty.
From (2.6.1) we get

$$D_n = \hat{D}_0 D_{n-1} \cup \hat{D}_1 D_{n-2} \cup \cdots \cup \hat{D}_{n-1} D_0, \quad n \geq 1. \tag{2.6.3}$$

This is because the concatenation of $\hat{D}_a D_{n-1-a}$, where $n-1 \geqslant a \geqslant 0$, gives a Dyck path with semilength equal to $a+1$ (due to elevation) before the first return and $n-1-a$ after the first return which makes the total semilength of the path equal to n . If we take the union of all such sets we get all the paths in D_n except for the empty path when $n=0$. From (2.6.2) and using a similar argument to the one above we get

$$D_n = D_0\hat{D}_{n-1} \cup D_1\hat{D}_{n-2} \cup \cdots D_{n-1} \cup \hat{D}_0, \quad n \geq 1. \tag{2.6.4}$$

These are both unions of disjoint sets because if we consider the paths in $\hat{D}_a D_{n-1-a}$ and $\hat{D}_{1+a} D_{n-2-a}$, where $n-1 \geq a \geq 0$, we have:
$\alpha \in \hat{D}_a D_{n-1-a} \;\Rightarrow\; \alpha = u\alpha_1 d\alpha_2$ where $\alpha_1 \in D_{a-1}$ (for the case when $a=0$, α_1 is the empty path) with $\alpha_2 \in D_{n-1-a}$ and $\beta \in \hat{D}_{a+1} D_{n-2-a} \;\Rightarrow\; \beta = u\beta_1 d\beta_2$ where $\beta_1 \in D_a$ with $\beta_2 \in D_{n-2-a}$.
This shows that $\alpha \neq \beta$ because $\alpha_1 \neq \beta_1$ since they have unequal semilengths and hence α is not an element of $\hat{D}_{1+a} D_{n-2-a}$ which means $\hat{D}_a D_{n-1-a} \neq \hat{D}_{1+a} D_{n-2-a}$, so the unions must be disjoint which makes the sets in (2.6.3) and (2.6.4) mutually exclusive.

Example: Let's consider the paths in D_3, these are,

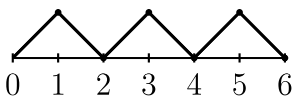

Figure 5(a)

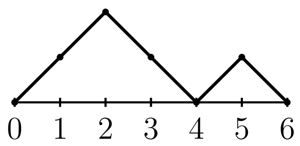

Figure 5(b)

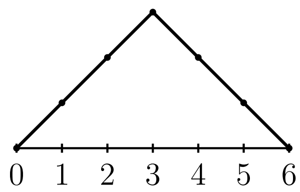

Figure 5(c)

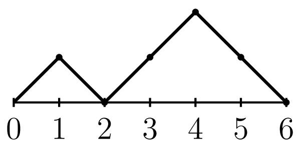

Figure 5(d)

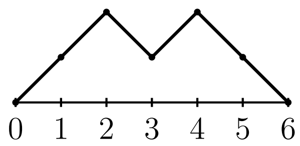

Figure 5(e)

From (2.6.3), we have

$$D_3 = \hat{D}_0 D_2 \cup \hat{D}_1 D_1 \cup \hat{D}_2 D_0.$$

We then look at the Dyck paths that are in each of the sets in the union. The elements of $\hat{D}_0 D_2$ are Figure $5(a)$ and Figure $5(d)$, those for $\hat{D}_2 D_0$ are Figure $5(c)$ and Figure $5(e)$, and finally, the element of $\hat{D}_1 D_1$ is Figure $5(b)$. From this, we see that the sets $\hat{D}_0 D_2$, $\hat{D}_1 D_1$ and $\hat{D}_2 D_0$ are mutually exclusive and if we take the union of these sets we get exactly the elements in D_3.

We have that $|\hat{D}_n| = |D_n|$ since $\hat{D}_n$ is the set of the Dyck paths in D_n raised by one level.
Now

$$\begin{aligned}
|D_n| &= |\hat{D}_0 D_{n-1} \cup \hat{D}_1 D_{n-2} \cup \cdots \cup \hat{D}_{n-1} D_0|, \ n \geq 1 \\
&= |\hat{D}_0 D_{n-1}| + |\hat{D}_1 D_{n-2}| + \cdots + |\hat{D}_{n-1} D_0|, \ n \geq 1 \text{ by mutual exclusivity} \\
&= |D_0||D_{n-1}| + |D_1||D_{n-2}| + \cdots + |D_{n-1}||D_0|.
\end{aligned}$$

Since $|D_0| = 1$, it follows that $|D_n|$ satisfies the same recurrence and initial condition as the Catalan numbers. This shows that $|D_n| = C_n$ for $n \geq 0$.

2.7 Enumeration and special types of parameters

Let p be a fixed non-negative integer-valued parameter of Dyck paths , *i.e.* a mapping from $\cup_{n\geq 0} D_n$ into $\{0, 1, 2, \cdots\}$. We can use p to represent the number of peaks, valleys, doublerises etc.

If A is a finite set of Dyck paths, then $P_A(t)$ denotes the *enumerating polynomial* of A relative to parameter p and

$$P_A(t) := \sum_{x \in A} t^{p(x)}.$$

If A and B are disjoint finite sets of Dyck paths, then

$$\begin{aligned} P_{A\cup B}(t) &= \sum_{x \in A \cup B} t^{p(x)} \\ &= \sum_{x \in A} t^{p(x)} + \sum_{x \in B} t^{p(x)} \\ &= P_A(t) + P_B(t). \end{aligned} \tag{2.7.1}$$

We denote $P_{D_n}(t)$ as $P_n(t)$ and $P_{\hat{D}_n}(t)$ as $\hat{P}_n(t)$ for simplicity. The generating function for the enumeration of Dyck paths according to semilength (coded by z) and the parameter p (coded by t) is

$$D(t, z) := \sum_{n \geq 0} P_n z^n \tag{2.7.2}$$

and the generating function for the enumeration of elevated Dyck paths according to semilength (coded by z) and the parameter p (coded by t) is

$$\hat{D}(t, z) := \sum_{n \geq 0} \hat{P}_n z^n. \tag{2.7.3}$$

For any given parameter p, we will denote the sum of all the values of the parameter p on all the Dyck paths of semilength n by σ_n.

Now we have, $P_n(t) = \sum_{x \in D_n} t^{p(x)}$, so $P_n'(t) = \sum_{x \in D_n} p(x) t^{p(x)-1}$ hence

$$\begin{aligned} P_n'(t)|_{t=1} &= \sum_{x \in D_n} p(x)(1)^{p(x)-1} \\ &= \sum_{x \in D_n} p(x) \end{aligned}$$

$$= \sigma_n.$$

We also have

$$\begin{aligned}\frac{\partial}{\partial t} D(t,z)|_{t=1} &= \sum_{n\geq 0} \frac{\partial}{\partial t} P_n(t)|_{t=1} z^n \\ &= \sum_{n\geq 0} \sigma_n z^n.\end{aligned}$$

The above implies that $\frac{\partial}{\partial t} D(t,z)|_{t=1}$ is the generating function of the sequence (σ_n) for $n \geq 0$. If we assume that the Dyck paths are equally likely to occur then the expected value of the parameter p is

$$\frac{\sigma_n}{|D_n|} = \frac{\sigma_n}{C_n}.$$

A parameter p is said to be *additive* if $p(\alpha\beta) = p(\alpha) + p(\beta)$ for all Dyck paths α and β. The number of peaks, number of doublerises are examples of additive parameters since the concatenation of two Dyck paths does not add or remove a peak or doublerise in the new path. The parameter number of valleys (du) is not additive because the concatenation of two non-empty Dyck paths will give one more du.

If the parameter p is additive, then

$$\begin{aligned}P_{AB}(t) &= \sum_{\alpha\beta\in AB} t^{p(\alpha\beta)} \\ &= \sum_{\alpha\in A\cup\beta\in B} t^{p(\alpha)+p(\beta)} \\ &= P_A(t)P_B(t).\end{aligned}$$

Using this result and (2.6.3) with an additive parameter p we get

$$\begin{aligned}P_n(t) &= \sum_{x\in D_n} t^{p(x)} \\ &= \sum_{x\in \hat{D}_0 D_{n-1}\cup\hat{D}_1 D_{n-2}\cup\cdots\cup\hat{D}_{n-1}D_0} t^{p(x)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha \in \hat{D}_0 D_{n-1}} t^{p(\alpha)} + \sum_{\beta \in \hat{D}_1 D_{n-2}} t^{p(\beta)} + \cdots + \sum_{\gamma \in \hat{D}_{n-1} D_0} t^{p(\gamma)} \quad \text{where} \quad x = \alpha\beta\cdots\gamma \\
&= P_{\hat{D}_0 D_{n-1}}(t) + P_{\hat{D}_1 D_{n-2}}(t) + \cdots + P_{\hat{D}_{n-1} D_0}(t) \\
&= P_{\hat{D}_0}(t) P_{D_{n-1}}(t) + P_{\hat{D}_1}(t) P_{D_{n-2}}(t) + \cdots + P_{\hat{D}_{n-1}}(t) P_{D_0}(t) \\
&= \hat{P}_0(t) P_{n-1}(t) + \hat{P}_1(t) P_{n-2}(t) + \cdots + \hat{P}_{n-1}(t) P_0(t). \qquad (2.7.4)
\end{aligned}
$$

Multiplying the above expression by z^n and summing over n where $n \geq 1$ we get

$$
\begin{aligned}
\sum_{n\geq 1} P_n(t) z^n &= \sum_{n \geq 1} [\hat{P}_0(t) P_{n-1}(t) + \hat{P}_1(t) P_{n-2}(t) + \cdots + \hat{P}_{n-1}(t) P_0(t)] z^n \\
&= \sum_{n\geq 1} \sum_{k=0}^{n-1} \hat{P}_k P_{n-1-k} z^n \\
&= z \sum_{k \geq o} \sum_{n \geq 1+k} \hat{P}_k z^k P_{n-1-k} z^{n-1-k} \\
&= z \sum_{k \geq o} \hat{P}_k z^k \sum_{r \geq 0} P_r z^r, \text{ by making the substitution } r = n-1-k \\
&= z \hat{D}(t,z) D(t,z).
\end{aligned}
$$

Now

$$
\begin{aligned}
\sum_{n\geq 1} P_n(t) z^n &= \sum_{n \geq 0} P_n(t) z^n - P_0 z^0 \\
&= \sum_{n \geq 0} P_n(t) z^n - 1 \\
&= D(t,z) - 1.
\end{aligned}
$$

Combining the two we get

$$
D(t,z) - 1 = z\hat{D}(t,z) D(t,z). \qquad (2.7.5)
$$

We say a parameter p is *quasiadditive* when for any two Dyck paths α and β, α contributes $p(\alpha)$ and β contributes $p(\beta)$ to $p(\alpha\beta)$ and due to concatenation, for some α and β we get $p(\alpha\beta) > p(\alpha) + p(\beta)$. The number of valleys and the number of duu's are examples of quasiadditive parameters. A parameter p is said to be a *left parameter* if for any two Dyck paths α and β we have

$$p(\alpha\beta) = \begin{cases} p(\alpha), & \text{if } \alpha \neq \epsilon \\ p(\beta), & \text{if } \alpha = \epsilon. \end{cases}$$

The number of peaks before the first return and the height of the first valley are examples of such parameters. In the case of a left parameter for two finite sets A and B of Dyck paths such that ϵ is not in A, we have

$$\begin{aligned} P_{AB}(t) &= \sum_{\alpha\beta \in AB} t^{p(\alpha\beta)} \\ &= \sum_{\alpha\beta \in AB} t^{p(\alpha)} \\ &= |B| \sum_{\alpha \in A} t^{p(\alpha)} \\ &= |B| P_A(t). \end{aligned} \tag{2.7.6}$$

From (2.6.3) and considering a left parameter p we get

$$\sum_{\alpha \in D_n} t^{p(\alpha)} = \sum_{\alpha \in \hat{D}_0 D_{n-1} \cup \hat{D}_1 D_{n-2} \cup \cdots \cup \hat{D}_{n-1} D_0} t^{p(\alpha)}.$$

Now, we have that the elevated sets do not contain the empty path and the unions are disjoint so the equation above becomes

$$\begin{aligned} P_n(t) &= \sum_{\alpha_1\beta_1 \in \hat{D}_0 D_{n-1}} t^{p(\alpha_1\beta_1)} + \sum_{\alpha_2\beta_2 \in \hat{D}_1 D_{n-2}} t^{p(\alpha_2\beta_2)} + \cdots + \sum_{\alpha_n\beta_n \in \hat{D}_{n-1} D_0} t^{p(\alpha_n\beta_n)} \\ &= |D_{n-1}| \sum_{\alpha_1 \in \hat{D}_0} t^{p(\alpha_1)} + |D_{n-2}| \sum_{\alpha_2 \in \hat{D}_1} t^{p(\alpha_2)} + \cdots + |D_0| \sum_{\alpha_n \in \hat{D}_{n-1}} t^{p(\alpha_n)} \\ &= C_{n-1}\hat{P}_0(t) + C_{n-2}\hat{P}_2(t) + \cdots + C_0\hat{P}_{n-1}(t), \quad n \geq 1, \end{aligned}$$

multiplying this by z^n and summing over n for $n \geq 1$ we get

$$\begin{aligned} \sum_{n\geq 1} P_n(t) z^n &= \sum_{n \geq 1} [C_{n-1}\hat{P}_0(t) + C_{n-2}\hat{P}_2(t) + \cdots + C_0\hat{P}_{n-1}(t)] z^n \\ &= z \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k z^k \hat{P}_{n-1-k} z^{n-1-k} \end{aligned}$$

$$
\begin{aligned}
&= z \sum_{k\geq 0} \sum_{n \geq k+1} C_k z^k \hat{P}_{n-1-k} z^{n-1-k} \\
&= z \sum_{k\geq 0} \sum_{r\geq 0} C_k z^k \hat{P}_r z^r \text{ by making the substitution } r = n-1-k \\
&= z \sum_{k\geq 0} C_k z^k \sum_{r\geq 0} \hat{P}_r z^r \\
&= zC(z)\hat{D}(t,z). \qquad (2.7.7)
\end{aligned}
$$

Chapter 3

ENUMERATION OF DYCK PATHS ACCORDING TO DIFFERENT PARAMETERS

In this chapter, we continue with E. Deutsch's paper were we will derive generating functions for Dyck paths according to semilength and various other parameters. We will then extract the coefficient of these generating functions to get the total number of ways to enumerate the associated Dyck paths according to the relevant parameters.

3.1 Enumeration of Dyck paths according to semilength and number of peaks

The parameter number of peaks is additive, since the concatenation of two Dyck paths does not affect the total number of peaks.
If we elevate an empty Dyck path, this contributes one peak to the elevated path and if we elevate a nonempty path we see that the number of peaks remains the same. Thus, if we let $p(x)$ be the number of peaks in a Dyck path x with length n and $p_1(x)$ be the number of peaks in the same path after elevation, we get

$$\hat{P}_0(t) = \sum_{x \in \hat{D}_0} t^{p_1(x)} = \sum_{x \in \hat{D}_0} t = t,$$

and

$$\hat{P}_n(t) = \sum_{x \in \hat{D}_n} t^{p_1(x)} = \sum_{x \in \hat{D}_n} t^{p(x)} = P_n(t), \quad n \geq 1.$$

Multiplying the $\hat{P}_n$ by z^n and summing over $n \geq 0$ we get

$$\sum_{n \geq 0} \hat{P}_n(t) z^n = t + \sum_{n \geq 1} P_n(t) z^n,$$

which becomes

$$\hat{D}(t, z) = t + D(t, z) - 1. \tag{3.1.1}$$

By dividing both sides of (2.7.5) by $zD(t, z)$ and substituting what we get into (3.1.1) we find that

$$\frac{D(t, z) - 1}{zD(t, z)} = D(t, z) + (t - 1),$$

which simplifies to

$$zD^2(t, z) - (1 - tz + z)D(t, z) + 1 = 0.$$

Using the quadratic formula to solve for $D(t, z)$ gives

$$\begin{aligned} D(t, z) &= \frac{1 - tz + z \pm \sqrt{(1 - tz + z)^2 - 4z}}{2z} \\ &= \frac{1 - tz + z \pm \sqrt{1 - 2z + z^2 - 2tz - 2tz^2 + t^2 z^2}}{2z}, \end{aligned}$$

we know $D(0, 0) = 1$, so from the expression above we only consider the negative sign since it gives an indeterminate form $D(0, 0) = \frac{0}{0}$. Thus,

$$D(t, z) = \frac{1 - tz + z - \sqrt{1 - 2z + z^2 - 2tz - 2tz^2 + t^2 z^2}}{2z}$$

$$\begin{aligned} &= \frac{2tz}{2tz} + t \frac{1 - tz - z - \sqrt{1 - 2z + z^2 - 2tz - 2tz^2 + t^2 z^2}}{2tz} \\ &= 1 + t\rho(t, z) \ \text{ from } \ (2.5.2). \end{aligned} \tag{3.1.2}$$

We then extract the coefficient of $D(t,z)$ by making use of (2.5.5) with $m = 1$ to get the number of Dyck paths with k peaks and semilength n.

$$\begin{aligned}[t^k z^n]D(t,z) &= [t^k z^n](1 + t\rho(t,z)) \\ &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.\end{aligned}$$

We know that $P_n(t)$ is the generating function for the number of Dyck paths according to peaks which have fixed semilength n and $D(t,z) = \sum_{n\geq 0} P_n(t)z^n$. If we let $D(t,z) = \sum_{k\geq 0} G_k(z)t^k$, then we see $G_k(z)$ is the generating function for Dyck paths according to semilength, which have a fixed number of peaks equal to k. Now, we also know that the empty path is the only Dyck path that has number of peaks equal to 0, so $G_0(z) = 1$. We see that

$$\begin{aligned}D(t,z) &= 1 + t\rho(t,z) \quad \text{from} \quad (3.1.2) \\ &= 1 + t\left(\frac{z}{1-z}\left(1 + \rho\left(z, \frac{tz}{(1-z)^2}\right)\right)\right) \quad \text{from} \quad (2.5.7) \\ &= 1 + \frac{tz}{1-z}\left(1 + \frac{D\left(z, \frac{tz}{(1-z)^2}\right) - 1}{z}\right) \quad \text{from} \quad (3.1.2) \\ &= 1 - t + \frac{tD\left(z, \frac{tz}{(1-z)^2}\right)}{1-z} \\ &= 1 - t + \frac{t}{1-z}\sum_{n\geq 0} P_n(z)\frac{(tz)^n}{(1-z)^{2n}} \quad \text{from} \quad (2.7.2).\end{aligned}$$

Thus,

$$\sum_{k\geq 0} G_k(z)t^k = 1 - t + \sum_{n\geq 0}\frac{P_n(z)z^n t^{n+1}}{(1-z)^{2n+1}}.$$

When $k = 0$, the right hand side of this equation is 1 and when $k = 1$ it is $\frac{z}{1-z}$. For $k \geq 2$, the summands have the same coefficient of t. So we can write this as,

$$G_k(z) = \begin{cases} 1 & if \quad k = 0 \\ \dfrac{z}{1-z} & if \;\; k = 1 \\ \dfrac{P_{k-1}(z)z^{k-1}}{(1-z)^{2k-1}} & if \;\; k \geq 2. \end{cases}$$

If we look at a Dyck path we notice that each u step is followed by either another u step or a d step. This tells us that for each u step, we either have a double rise or a peak. Since a Dyck path with semilength n has n number of u steps, it follows that the sum of peaks and double rises in a Dyck path should equal the semilength of the path. We also have that in a Dyck path every peak is followed by a descent and every descent leads to either a valley or ends with a return step that is not followed by a u step $i.e$, the last step of the path. Thus, every peak except the last one produces a valley. This implies that the number of valleys is one less than the number of peaks.

3.2 Enumeration of Dyck paths according to number of low peaks and number of high peaks

Both these parameters are additive. Let's consider the Dyck path below,

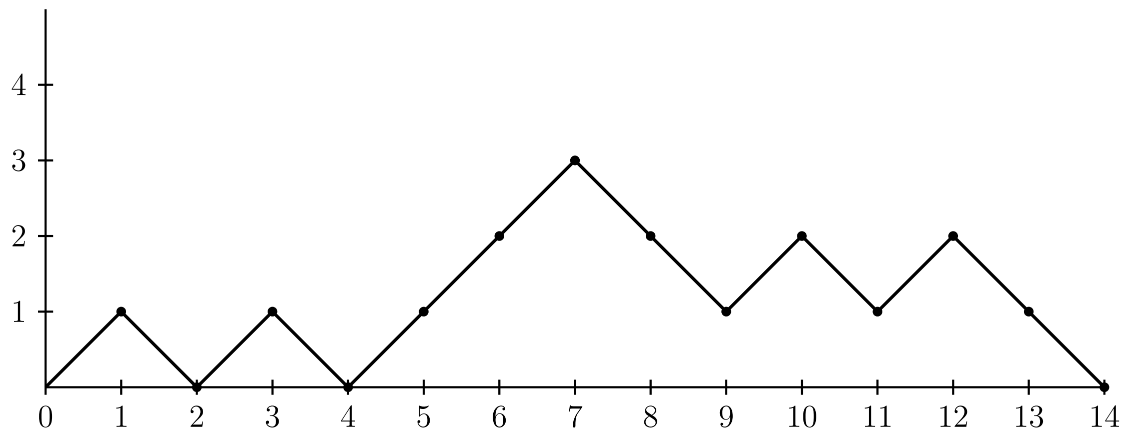

Figure 6.

We see that in Figure 6, our path has two low peaks $(1,1)$ and $(3,1)$ and three high peaks $(7,3)$, $(10,2)$ and $(12,2)$. If we prepend a u step and then append a d step, we get the Dyck path below,

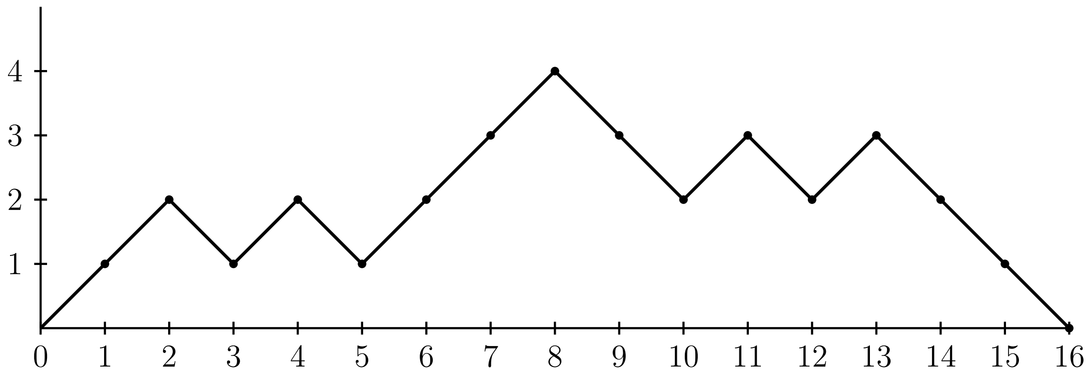

Figure 7.

the low peaks will be raised to level two, thus, all the peaks become high peaks. More generally, if we elevate a nonempty Dyck path we get a primitive Dyck path with no low peaks.

Let s be the variable coding the parameter number of low peaks and t be the variable coding the parameter number of high peaks. If we elevate an empty path we get a primitive path with one low peak and if we elevate a nonempty path we get a primitive Dyck path with no low peaks because all the low peaks become high peaks, *i.e.*,

$$\hat{P}_n(s,t) = \begin{cases} s & \text{if } \ n = 0 \\ P_n(t,t) & \text{if } \ n \geq 1. \end{cases}$$

Multiplying $\hat{P}_n$ by z^n and summing over $n \geq 0$ we get

$$\sum_{n\geq 0} \hat{P}_n(s,t)z^n = s + \sum_{n\geq 1} P_n(t,t)z^n$$

which becomes

$$\hat{D}(s,t,z) = s - 1 + D(t,t,z) = s - 1 + D(t,z). \tag{3.2.1}$$

Substituting this into (2.7.5), we obtain

$$D(s,t,z) - 1 = zD(s,t,z)(s - 1 + D(t,z)),$$

solving for $D(s,t,z)$, we get

$$\begin{aligned}
D(s,t,z) &= \frac{1}{1 - z(s - 1 + D(t,z))} \\
&= \frac{1}{1 - z(s + t\rho(t,z))} \quad \text{from} \quad (3.1.2) \\
&= \sum_{a\geq 0} (z(s + t\rho(t,z)))^a \\
&= \sum_{a\geq 0} \sum_{i=0}^{a} \binom{a}{i} t^i \rho^i(t,z) s^{a-i} z^a \\
&= \sum_{a\geq 0} \sum_{i=0}^{a} \binom{a}{i} t^i \sum_{m\geq 0} \sum_{u\geq 0} \frac{i}{m} \binom{m}{i+u} \binom{m}{u} t^u z^{a+m} s^{a-i} z^a \quad \text{from} \quad (2.5.6) \\
&= \sum_{i\geq 0} \sum_{a-i\geq 0} \sum_{m\geq 0} \sum_{u\geq 0} \binom{a}{i} \frac{i}{m} \binom{m}{i+u} \binom{m}{u} t^{u+i} s^{a-i} z^{a+m}.
\end{aligned}$$

To get the number of Dyck paths of length n with j low peaks and k high peaks, we extract the coefficient of $s^j t^k z^n$ in $D(s,t,z)$ above. Thus, by making the substitutions $u + i = k$, $a - i = j$ and $a + m = n$ in the coefficient of $s^{a-i} t^{u+i} z^{a+m}$ gives

$$\begin{aligned}
[t^k s^j z^n] D(s,t,z) &= \sum_{i\geq 1} \frac{i}{n - (i+j)} \binom{i+j}{j} \binom{n - (i+j)}{k} \binom{n - (i+j)}{k - i} \\
&= \sum_{h\geq j+1} \frac{h - j}{n - h} \binom{h}{j} \binom{n-h}{k} \binom{n-h}{j+k-h} \quad \text{where} \quad i + j = h,
\end{aligned}$$

with $n - h \geq k \Rightarrow n - k \geq h$ and $j + k \geq h$. It must be the case that in the above equality, $n > h$ and $n - k \geq h$ implies that $k > 0$ and $n \geq h + k = i + j + k > j + k$. When $j = n$ or $k = 0$, we either have an empty path or the trivial path which only has low peaks, so $[t^k s^j z^n] D(s,t,z) = 1$.

3.3 Enumeration of Dyck paths according to semilength and number of high peaks

From above we know that the parameter number of high peaks is additive and that the generating function for Dyck paths according to semilength, number of low peaks and number of high peaks is

$$D(t,s,z) = \frac{1}{1 - z(s + t\rho(t,z))}.$$

Since we just want the generating function for Dyck paths according to semilength and number of high peaks, we let $s = 1$ in the equation above. This gives

$$\begin{aligned} D(t,z) &= \frac{1}{1 - z(1 + t\rho(t,z))} \\ &= \frac{1}{1 - z(\frac{\rho(t,z)}{z\rho(t,z)})} \quad \text{from} \quad (2.5.1) \\ &= 1 + \rho(t,z). \end{aligned}$$

To get the number of Dyck paths with semilength n and k high peaks, we make use of (2.5.4) with $m = 1$ to get

$$[t^k z^n]D(t,z) = \begin{cases} \frac{1}{n}\binom{n}{k}\binom{n}{k+1} & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Enumeration of Dyck paths according to semilength and number of low peaks

This case is similar to the one in section 3.3 above, however, instead of letting $s = 1$ we let $t = 1$ in our generating function. Thus,

$$D(s,z) = \frac{1}{1 - z(s - 1 + D(z))}$$

$$
\begin{aligned}
&= \frac{1}{1 - z(s + C(z) - 1)} \\
&= \sum_{i \geq 0} z^i (s + C(z) - 1)^i \\
&= \sum_{i \geq 0} z^i \sum_{r=0}^{i} \binom{i}{r} z^r C^{2r}(z) s^{i-r} \\
&= \sum_{r \geq 0} \sum_{i \geq r} \binom{i}{r} z^{r+i} C^{2r}(z) s^{i-r} \\
&= \sum_{r \geq 0} \sum_{i \geq r} \sum_{j \geq 0} \binom{i}{r} \frac{2r}{2j + 2r} \binom{2j + 2r}{j} z^{r+i+j} s^{i-r}.
\end{aligned}
$$

To get the number of Dyck paths with semilength n and k low peaks, we extract the coefficient of $D(s, z)$ to get

$$
[s^k z^n] D(s, z) = \sum_{r \geq 0} \frac{r}{n - r - k} \binom{r + k}{r} \binom{2(n - r - k)}{n - 2r - k},
$$

with $2(n - r - k) \geq n - k$, that is, $\frac{n-k}{2} \geq r$. So for $n > k$

$$
[s^k z^n] D(s, z) = \sum_{r=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{r}{n - r - k} \binom{r + k}{r} \binom{2(n - r - k)}{n - k}.
$$

If we let $k = 0$ in our expression above, we see that it is the same as that of the Fine numbers (2.4.4). This shows that the number of Dyck paths with no low peaks has the same distribution as the Fine numbers.

3.5 Enumeration of Dyck paths according to semilength and height of first peak

The parameter height of first peak is a left parameter. Let t be the variable coding the number of peaks and s be the variable coding the height of the first peak.
If we elevate an empty path, this contributes a peak and the height of the first peak will be one. If we elevate a nonempty path, we will have the same number of peaks but the height of the first peak increases by one.
Thus, if we let $q(x)$ $(p(x))$ be the height of the first peak (number of peaks) in a

Dyck path x with semilength n, then

$$\hat{P}_0(t,s) = \sum_{x \in D_0} t^{p(uxd)} s^{q(uxd)} = \sum_{x \in D_0} ts = ts,$$

and for $n \geq 1$

$$\hat{P}_n(t,s) = \sum_{x \in D_n} t^{p(uxd)} s^{q(uxd)} = \sum_{x \in D_n} t^{p(x)} s^{q(x)+1} = sP_n(t,s).$$

Multiplying $\hat{P}_n(t,s)$ by z^n and summing over $n \geq 0$ gives

$$\hat{D}(t,s,z) = ts - s + sD(t,s,z). \tag{3.5.1}$$

Since we have an additive parameter and a left parameter, we combine (2.7.5) and (2.7.7). Let p be an additive parameter and q a left parameter then

$$\begin{aligned}
P_n(t,s) &= \sum_{x \in D_n} t^{p(x)} s^{q(x)} \\
&= \sum_{x \in \hat{D}_0 D_{n-1} \cup \hat{D}_1 D_{n-2} \cup \cdots \cup \hat{D}_{n-1} D_0} t^{p(x)} s^{q(x)} \\
&= \sum_{\alpha_1\beta_1 \in \hat{D}_0 D_{n-1}} t^{p(\alpha_1\beta_1)} s^{q(\alpha_1\beta_1)} + \cdots + \sum_{\alpha_n\beta_n \in \hat{D}_{n-1} D_0} t^{p(\alpha_n\beta_n)} s^{q(\alpha_n\beta_n)} \\
&= \sum_{\alpha_1 \in \hat{D}_0} \sum_{\beta_1 \in D_{n-1}} t^{p(\alpha_1)} t^{p(\beta_1)} s^{q(\alpha_1)} + \cdots + \sum_{\alpha_n \in \hat{D}_{n-1}} \sum_{\beta_n \in D_0} t^{p(\alpha_n)} t^{p(\beta_n)} s^{q(\alpha_n)} \\
&= \hat{P}_0(t,s) P_{n-1}(t) + \cdots + \hat{P}_{n-1}(t,s) P_0(t).
\end{aligned}$$

Multiplying by z^n and summing over n for $n \geq 1$ gives

$$\begin{aligned}
D(t,s,z) &= 1 + \sum_{n \geq 1} \sum_{k=0}^{n-1} \hat{P}_k(t,s) P_{n-1-k}(t) z^n \\
&= 1 + z \sum_{k \geq 0} \hat{P}_k(t,s) z^k \sum_{n \geq 1+k} P_{n-1-k}(t) z^{n-1-k} \\
&= 1 + z\hat{D}(t,s,z) D(t,z).
\end{aligned}$$

Substituting (3.5.1) into the equation above gives

$$\begin{aligned}
D(t,s,z) &= 1 + szD(t,z)(t-1+D(t,s,z)) \\
&= \frac{1+szD(t,s)t - szD(t,s)}{1-szD(t,z)} \\
&= \frac{1+tsz(1+t\rho(t,z)) - sz(1+t\rho(t,z))}{1-sz(1+t\rho(t,z))} \quad \text{from} \quad (3.1.2) \\
&= \frac{1+\dfrac{ts\rho(t,z)}{1+\rho(t,z)} - \dfrac{s\rho(t,z)}{1+\rho(t,z)}}{1-\dfrac{s\rho(t,z)}{1+\rho(t,z)}} \\
&= 1 + \frac{ts\rho(t,z)}{1+(1-s)\rho(t,z)} \\
&= 1 + ts\rho(t,z)\sum_{i\geq 0}(1-s)^i(-\rho(t,z))^i \\
&= 1 + ts\rho(t,z)\sum_{i\geq 0}\sum_{r=0}^{i}\binom{i}{r}(-s)^r(-\rho(t,z))^i \\
&= 1 + ts\rho(t,z)\sum_{r\geq 0}s^r\rho^r(t,z)\sum_{i\geq r}\binom{i}{i-r}(-\rho(t,z))^{i-r} \\
&= 1 + \sum_{r\geq 0}ts^{r+1}\rho^{r+1}(t,z)\sum_{a\geq 0}\binom{a+r}{a}(-\rho(t,z))^a \\
&= 1 + \sum_{r\geq 0}\rho^{r+1}(t,z)\frac{1}{(1+\rho(t,z))^{r+1}}ts^{r+1} \\
&= 1 + \sum_{r\geq 0}z^{r+1}(1+t\rho(t,z))^{r+1}ts^{r+1} \quad \text{from} \quad (2.5.1).
\end{aligned}$$

To get the number of Dyck paths of semilength n with i peaks and height of first peak k, we must extract the coefficient of $D(t,s,z)$ above. For i,j,n not all equal to zero, this gives

$$\begin{aligned}
[t^is^kz^n]D(t,s,z) &= [t^{i-1}z^{n-k}](1+t\rho(t,z))^k \\
&= \frac{k}{n-k}\binom{n-k}{i-1}\binom{n-1}{i-2},
\end{aligned}$$

with $n-1 \geq i-2$ and $n-k > 0$, and thus, $0 < i < n$ and $0 < k < n$. We note that $[t^is^kz^n]D(t,s,z) = 1$ if $j = k = n$ since this implies that we either have an empty path or the trivial path with heights of all peaks equal to one. Now, if we let $t = 1$ in our trivariate generating function above, we get

$$\begin{aligned} D(s,z) &= 1 + szD(z)D(t,s,z) \\ &= \frac{1}{1 - szC(z)} \\ &= \sum_{r\geq 0}\sum_{i\geq 0} \frac{r}{2i+r}\binom{2i+r}{i} s^r z^{r+i} \quad \text{from} \quad (2.3.5). \end{aligned}$$

Thus, to get the number of Dyck paths with semilength n and height of first peak k, we extract the coefficient of $s^k z^n$ to get

$$\frac{k}{2n-k}\binom{2n-k}{n-k}.$$

3.6 Enumeration of Dyck paths according to semilength and number of return steps

The parameters number of peaks and number of return steps are additive. Let t be the variable coding the number of peaks and s be the variable coding the number of return steps. If we elevate an empty path, this contributes one peak and one return step to the elevated path and if we elevate a nonempty Dyck path, we get a primitive Dyck path with the same number of peaks and one return step. Thus, if we let $p(x)$ $(q(x))$ be the number of peaks (return steps) in Dyck path x, with semilength n, then

$$\hat{P}_0(t,s) = \sum_{x\in D_0} t^{p(uxd)} s^{q(uxd)} = \sum_{x\in D_0} ts = ts,$$

and for $n \geq 1$

$$\hat{P}_n(t,s) = \sum_{x\in D_n} t^{p(uxd)} s^{q(uxd)} = \sum_{x\in D_n} t^{p(x)} s = sP_n(t).$$

Multiplying $\hat{P}_n(t,s)$ by z^n and summing over $n \geq 0$ gives

$$\sum_{n\geq 0} \hat{P}_n(t,s) z^n = ts + \sum_{n\geq 1} sP_n(t) z^n$$

$$= ts - s + sD(t,z).$$

Substituting this into (2.7.5) gives

$$\begin{aligned}
D(t,s,z) &= 1 + szD(t,s,z)(t - 1 + D(t,z)) \\
&= 1 + tszD(t,s,z)(1 + \rho(t,z)) \quad \text{from} \quad (3.1.2) \\
&= \frac{1}{1 - tsz(1 + \rho(t,z))} \\
&= \sum_{i \geq 0} (tsz)^i (1 + \rho(t,z))^i \\
&= \sum_{i \geq 0} \sum_{l \geq 0} \sum_{a \geq 0} \frac{i}{l} \binom{l}{a} \binom{l + i - 1}{a + i} s^i t^{i+a} z^{i+l}.
\end{aligned}$$

To get the number of Dyck paths with semilength n, j peaks and k returns, we extract the coefficient of $D(t,s,z)$ to get

$$\begin{aligned}
[t^j s^k z^n] D(t,s,z) &= \frac{k}{n-k} \binom{n-k}{j-k} \binom{n-1}{j} \\
&= \frac{k}{j} \binom{n-k-1}{j-k} \binom{n-1}{n-j},
\end{aligned}$$

with $j > 0$ and $n > k$. We note that $[t^j s^k z^n] D(t,s,z) = 1$ if $j = k = n$ since this implies that we either have an empty path or the trivial path with heights of all peaks equal to one. Now, if we let $t = 1$ in our trivariate generating function above, we get

$$D(s,z) = \frac{1}{1 - sz(1 + \rho(z))} = \frac{1}{1 - szC(z)} \qquad \text{from} \quad (3.1.2).$$

This shows that the number of Dyck paths with semilength n and height of first peak k has the same distribution as the number of Dyck paths of the same length with k return steps.

Chapter 4

A BIJECTIVE MAPPING FROM PERMUTATIONS TO DYCK PATHS

We now look at a paper by M. Barnabei, F. Bonetti and M Silimbani, The descent statistic over (123)-avoiding permutations [3], on the relationship between (123)-*avoiding* permutations and Dyck paths.

4.1 Introduction

We say a permutation $\sigma \in S_n$ *avoids* a pattern $\tau \in S_k$ if it does not contain a subsequence that is order-isomorphic to τ. For example, a permutation $\sigma = u_1 u_2 \cdots u_n$ avoids a pattern (123) if there does not exist a subsequence in σ such that $u_i < u_j < u_k$ and $i < j < k$. From here onwards, a set of all permutations with n elements avoiding a pattern τ will be denoted by $S_n(\tau)$.
As an example, consider the permutation

$$\sigma = 7\ 10\ 5\ 9\ 8\ 3\ 6\ 4\ 2\ 1. \tag{4.1.1}$$

This permutation is in $S_{10}(123)$ since you cannot find a subsequence of three or more strictly increasing numbers. The only increasing subsequences in this permutation are, $7\ 8, 7\ 10, 7\ 9, 5\ 9, 5\ 8, 5\ 6, 3\ 4$ and $3\ 6$.
A *reverse* permutation is one where the last term becomes the first, the second last

term becomes the second and continuing the process until we get to the first term. For example, the reverse of the permutation 1234 is 4321 and we denote it by *rev*. To get the *complement* of a permutation with length n, we add 1 to its length and then from $n+1$ we subtract each element in the permutation. This operation is denoted by c. For example, from the permutation 4213, we see that it has length 4, adding 1 to 4 and subtracting each element of 4213 from 5 gives 1342.
We say a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ has a *descent* at i if $\sigma(i) > \sigma(i+1)$. We know that S_3 has six elements and their relationships are as follows:

- $123 = 321^{rev}$
- $132 = (213^{rev})^{c}$
- $132 = 231^{rev}$
- $132 = 312^{c}$

where *rev* and c respectively denote the reverse and complement operations. This shows that in order to determine the distribution of the descent statistic over $S_n(\tau)$ for every $\tau \in S_3$, it is sufficient to look at the distribution of descents over the sets $S_n(132)$ and $S_n(123)$. We will investigate the case $\tau = 123$.

4.2 Relationship between Dyck paths and permutations

The generating function for Dyck paths according to semilength, number of valleys $v(D)$ and triple falls $tf(D)$ is

$$\begin{aligned} D(x,y,z) &= \sum_{n\geq 0}\sum_{D\in D_n} x^n y^{v(D)} z^{tf(D)} \\ &= \sum_{n,v,t\geq 0} d_{n,v,t} x^n y^v z^t \end{aligned}$$

where $d_{n,v,t}$ represents the number of Dyck paths with semilength n, v valleys and t triple falls.

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be a (123)-avoiding permutation. A *left-to-right minimum* of σ is an element $\sigma(i)$ which is smaller than $\sigma(j)$, with $i > j$. It is further assumed that the first element in a permutation is a left-to-right minimum. In the permutation $\sigma = 7\ 10\ 5\ 9\ 8\ 3\ 6\ 4\ 2\ 1$, we find that 7, 5, 3, 2, and 1 are the left-to-right minima. Thus, with the exception of the first element, for an element to be a left-to-right minimum in a permutation, it must be smaller than all the element to its left. If we let $x_1, x_2 \cdots x_k$ be the left-to-right minima in σ, then, we can write

$$\sigma = x_1 w_1 x_2 w_2 \cdots x_s w_s$$

where w_i represents the elements between the left-to-right minima x_i and x_{i+1}.

Now, we note that the elements in w_i must be decreasing. To see this, let's consider w_i, obviously all elements in w_i must be greater than x_i otherwise w_i has a left-to-right minimum, which is impossible since w_i is all the elements between two consecutive left-to-right minima. If we assume that w_i has a subsequence of two or more increasing elements, then, $x_i w_i$ will have a subsequence of at least three strictly increasing elements. This contradicts σ being a (123)-avoiding permutation. So clearly all the elements in w_i must be decreasing.

We will construct a Dyck path from a permutation σ. To do this we let x_i represent a left-to-right minimum where $x_0 = n+1$ and w_i be a word representing the elements between x_i and x_{i+1}. Starting from the left and going to the right of a permutation whenever there is left-to-right minimum, x_i, we will translate this to $x_{i-1} - x_i$ up steps denoted by $u^{(x_{i-1}-x_i)}$ in our Dyck path and any word w_i is translated to $l_i + 1$ down steps denoted by $d^{(l_i+1)}$ where l_i is the number of elements in w_i.

Example: Let's consider our permutation $\sigma = 7\ 10\ 5\ 9\ 8\ 3\ 6\ 4\ 2\ 1$. This is a (123)-avoiding permutation of length 10, so we can construct a Dyck path from it. The left-to-right minima of this permutation are 7, 5, 3, 2, 1. To construct the corresponding Dyck path, we need to translate $x_0 - x_1 = 11 - 7 = 4$ into 4 up steps, then translate $l_1 + 1 = 1 + 1 = 2$ into 2 down steps, repeating the same process every time we have a left-to-right minimum. This procedure yields the Dyck path below,

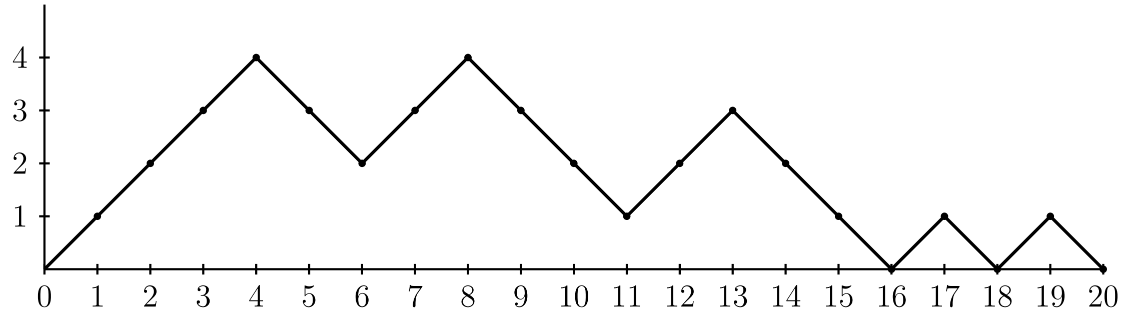

Figure 8.

Proposition 1 *Let σ be a permutation in $S_n(123)$, and D be a Dyck path created from σ.*
The number of descents of σ is

$$des(\sigma) = v(D) + tf(D),$$

where $des(\sigma)$ is the number of descents in σ, $v(D)$ is the number of valleys in D and $tf(D)$ is the number of triple falls in D.

Proof 2 *Let $\sigma = x_1 w_1 \cdots x_s w_s$ be a (123)-avoiding permutation.*
To find the descents of σ we note the following:

i) since w_i has l_i elements and it is decreasing, it must have $l_i - 1$ descents,
ii) whenever we make a transition from w_i to x_{i+1} we get another descent.

Now, to see the bijective mapping, we also note the following:

i) for every nonempty w_i, we get $l_i + 1$ consecutive down steps in our created Dyck path; these $l_i + 1$ down steps translate to $l_i - 1$ triple falls which are in bijection with the descents in w_i,
ii) with the exception of the first left-to-right minimum, every x_i results in up steps immediately after a descent, whenever this happens, we get a valley.
This shows that descents in σ can be translated into valleys and triple falls in $D(\sigma)$, where $D(\sigma)$ is the Dyck path built from the (123)-avoiding permutation σ.

The above proposition allows us to switch our attention from permutations in $S_n(123)$ with k descents to Dyck paths of semilength n where k is the sum of valleys and triple falls. We will focus on generating functions of Dyck paths according to semilength, valleys and triple falls.
Let the generating function for primitive paths according to semilength, valleys and triple falls be,

$$\begin{aligned} PD(x,y,z) &= \sum_{n\geq 0}\sum_{D\in PD_n} x^n y^{v(D)} z^{tf(D)} \\ &= \sum_{n,v,t\geq 0} pd_{n,v,t} x^n y^v z^t, \end{aligned}$$

where PD_n is the set of primitive Dyck paths.

To see the relationship between Dyck paths and primitive Dyck paths, we look at the propositions below.

Proposition 2 *Let $n \geq 3$, then,*

$$pd_{n,v,t} = d_{n-1,v,t-1} - d_{n-2,v-1,t-1} + d_{n-2,v-1,t}. \tag{4.2.1}$$

Proof 3 *A primitive Dyck path of semilength n, with v valleys and t triple falls can be obtained in two ways.*
i) By elevating a Dyck path of semilength $n-1$ with v valleys and t triple falls that ends with a ud. Elevating this path gives a primitive Dyck path with v valleys and t triple falls. We also note that this path is in bijection with Dyck paths of semilength $n-2$, with $v-1$ valleys and t triple falls, $d_{n-2,v-1,t}$.
ii) By elevating a Dyck path of semilength $n-1$ with v valleys and $t-1$ triple falls that does not end with ud. In order to get these paths, we must remove, from the number counting Dyck paths of semilength $n-1$ with v valleys and $t-1$ triple falls, $d_{n-1,v,t-1}$, all those paths that end with ud. These paths that end with ud are in bijection with Dyck paths of semilength $n-2$ with $v-1$ valleys and $t-1$ triple falls, $d_{n-2,v-1,t-1}$.
Combining these two cases, we get (4.2.1).

We also note that Dyck paths and primitive Dyck paths satisfy the proposition below.

Proposition 3 *For every $n \geq 1$, we have*

$$d_{n,v,t} = pd_{n,v,t} + \sum_{i=0}^{n-1} \sum_{j,k \geq 0} pd_{i,j,k} d_{n-i,v-j-1,t-k}. \qquad (4.2.2)$$

Proof 4 *Let D be a Dyck path of semilength n and let's consider its last return decomposition, $D = \alpha\beta$ where $\alpha \in D_n$ and $\beta \in PD_n$.*
If α is the empty path, then D is primitive, otherwise,

$$v(D) = v(\alpha) + v(\beta) + 1$$

and

$$tf(D) = tf(\alpha) + tf(\beta).$$

$$\begin{aligned}
D(x,y,z) &= \sum_{n\geq 0} \sum_{D \in D_n} x^n y^{v(D)} z^{tf(D)} \\
&= \sum_{n\geq 0} \sum_{\beta \in PD_n} x^n y^{v(\beta)} z^{tf(\beta)} \\
&\quad + y \sum_{n\geq 0} \sum_{k=1}^{n-1} \sum_{\alpha \in D_{n-k}} x^{n-k} y^{v(\alpha)} z^{tf(\alpha)} \sum_{\beta \in PD_k} x^k y^{v(\beta)} z^{tf(\beta)} \\
&= PD(x,y,z) + y \sum_{k\geq 1} \sum_{n-k\geq 1} \sum_{\alpha \in D_{n-k}} x^{n-k} y^{v(\alpha)} z^{tf(\alpha)} \sum_{\beta \in PD_k} x^k y^{v(\beta)} z^{tf(\beta)} \\
&= PD(x,y,z) + y \sum_{k\geq 1} \sum_{\beta \in PD_k} x^k y^{v(\beta)} z^{tf(\beta)} \sum_{n-k\geq 1} \sum_{\alpha \in D_{n-k}} x^{n-k} y^{v(\alpha)} z^{tf(\alpha)} \\
&= PD(x,y,z) \\
&\quad + y \left(\sum_{k\geq 0} \sum_{\beta \in PD_k} x^k y^{v(\beta)} z^{tf(\beta)} - 1 \right) \left(\sum_{n-k\geq 0} \sum_{\alpha \in D_{n-k}} x^{n-k} y^{v(\alpha)} z^{tf(\alpha)} - 1 \right) \\
&= PD(x,y,z) + y \left(PD(x,y,z) - 1 \right) \left(D(x,y,z) - 1 \right). \qquad (4.2.3)
\end{aligned}$$

Extracting the coefficient of $D(x,y,z)$ in this functional equation gives us the desired result.

Multiplying proposition (2) by $x^n y^v z^t$ and summing over $v, t \geq 0$ and $n \geq 3$, we get

$$\begin{aligned}
&\sum_{n\geq 3}\sum_{v,t\geq 0} pd_{n,v,t}x^n y^v z^t \\
&= \sum_{n\geq 3}\sum_{v,t\geq 0} d_{n-1,v,t-1}x^n y^v z^t - \sum_{n\geq 3}\sum_{v,t\geq 0} d_{n-2,v-1,t-1}x^n y^v z^t + \sum_{n\geq 3}\sum_{v,t\geq 0} d_{n-2,v-1,t}x^n y^v z^t \\
&= xz\left(\sum_{n\geq 1}\sum_{v\geq 0}\sum_{t\geq 1} d_{n-1,v,t-1}x^{n-1}y^v z^{t-1} - 1 - x\right) \\
&\quad - x^2yz\left(\sum_{n\geq 2}\sum_{v,t\geq 1} d_{n-2,v-1,t-1}x^{n-2}y^{v-1}z^{t-1} - 1\right) \\
&\quad + x^2y\left(\sum_{n\geq 2}\sum_{v\geq 1}\sum_{t\geq 0} d_{n-2,v-1,t}x^{n-2}y^{v-1}z^{t} - 1\right) \\
&= xz(D(x,y,z) - 1 - x) - x^2yz(D(x,y,z) - 1) + x^2y(D(x,y,z) - 1) \\
&= (D(x,y,z) - 1)(xz + x^2y - x^2yz) - x^2z.
\end{aligned}$$

Now, on the left hand side we have,

$$\sum_{n\geq 3}\sum_{v,t\geq 0} pd_{n,v,t}x^n y^v z^t = \sum_{n\geq 0}\sum_{v,t\geq 0} pd_{n,v,t}x^n y^v z^t - 1 - x - x^2 \quad = PD(x,y,z) - 1 - x - x^2.$$

Note, we do not subtract x^2y because the path is primitive. Equating the two sides gives

$$PD(x,y,z) = (D(x,y,z) - 1)(xz + x^2y - x^2yz) + 1 + x + x^2 - x^2z. \qquad (4.2.4)$$

We then solve (4.2.3) and (4.2.4) simultaneously to make $D(x,y,z)$ the subject of the formula as follows:
Let $D = D(x,y,z)$, $PD = PD(x,y,z)$, $a = 1 + x + x^2 - x^2z$ and $b = xz + x^2y - x^2yz$. Substituting (4.2.4) into (4.2.3) gives

$$\begin{aligned}
D &= (D-1)b + a + y((D-1)b + a - 1)(D-1) \\
&= bD - b + a + ybD^2 - 2ybD + 2yb + y(a-1)D - y(a-1),
\end{aligned}$$

which leads to the quadratic equation

$$ybD^2 + (b - 1 - 2yb + y(a-1))D + a - b + 2yb - y(a-1) = 0.$$

Using the quadratic formula, we get

$$\begin{aligned}
D &= \frac{-b + 1 + 2yb - y(a-1)}{2yb} \\
&\quad \pm \frac{\sqrt{(b - 1 - 2yb + y(a-1))^2 - 4yb(a - b + 2yb - y(a-1))}}{2yb} \\
&= \frac{-1 + xz + xy + 2x^2y - 2xzy - 2x^2y^2 - 2x^2yz + 2x^2y^2z}{2xy(xyz - xy - z)} \\
&\quad \pm (1 - 2xz + xz^2 - 2xy - 4x^2y + x^2y^2 - 4xy^2z^2 - 8x^3y^3z - 4x^4y^4 \\
&\qquad + 2x^3y^2z + 8x^3y^3z^2 + 8x^4y^4z + x^4yz^2 - 4x^4y^3z^2 + 2x^2yz - 2x^3y^2z \\
&\qquad - x^4y^2z^2)^{\frac{1}{2}} \frac{1}{2xy(xyz - xy - z)}.
\end{aligned}$$

Chapter 5

MOTZKIN PATHS

We now start with the study of Motzkin paths. These are lattice paths constructed from the same step set as the Dyck paths, $\{u, d\}$, but might also have the horizontal step h. We give a sketch of the Motzkin paths below followed by the formal definition.

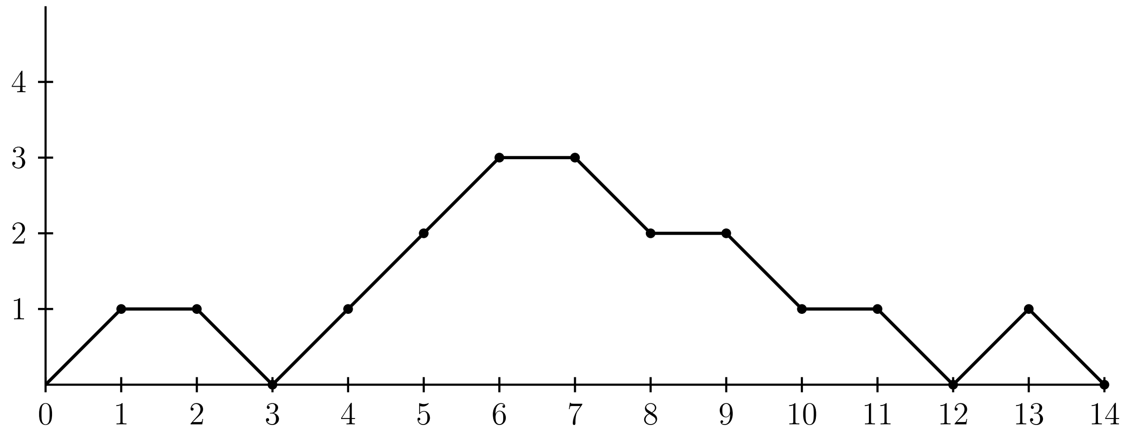

Figure 9.

Definition 3 *A Motzkin path is a lattice path in the first quadrant which begins at the origin and has up steps,* $(1, 1)$*, horizontal steps,* $(1, 0)$*, and down steps,* $(1, -1)$*, respectively denoted by* u, h *and* d*, that ends at the point* $(n, 0)$*.*

At any point in the path, the number of down steps is at most equal to the number of up steps. Hence, the steps of a Motzkin path never go below the x-axis. Using the symbolic method, we note that Motzkin paths are of the form

$$M = \epsilon + hM + uMdM,$$

where ϵ denotes the empty path and h, u, d represent the horizontal, up and down steps respectively, in a Motzkin path. Thus, a Motzkin path is either empty or a horizontal step followed by a Motzkin path or an up step followed by a Motzkin path then a down step and then another Motzkin path.
Translating the equation above to generating functions, we get

$$M(x) = 1 + xM(x) + x^2M^2(x).$$

This is a quadratic equation which we solve to get,

$$M(x) = \frac{1 - x \pm \sqrt{(x-1)^2 - 4x^2}}{2x^2}.$$

Since the limit of $M(x)$ as x goes to 0 is 1, and we get $M(0) = \frac{0}{0}$ only when we take the negative solution, then we discard the positive solution since it gives $M(0) = \frac{2}{0}$.
We now study the paper by D. Drake and R. Gantner, Generating functions for plateaus in Motzkin paths [10].

Definition 4 *A plateau in a Motzkin path is any part of the Motzkin path that has the subsequence uhd, thus, a u step immediately followed by an h step immediately followed by a d step.*

5.1 Generating functions for Motzkin paths

Let M_n^p be the number of Motzkin paths of length n with p plateaus and

$$M(x, y) = \sum_{n\geq 0} \sum_{p=0}^{\lfloor \frac{n}{3} \rfloor} M_n^p x^n y^p,$$

be the generating function for Motzkin paths according to length and number of plateaus. In this generating function, p has an upper limit of $\lfloor \frac{n}{3} \rfloor$ because each plateau has three steps and if all steps of the Motzkin path form part of a plateau,

then we see that the total number plateaus is $\lfloor \frac{n}{3} \rfloor$.
We state and prove some theorems related to generating functions for Motzkin paths according to length and number of plateaus.
Our plateau-counting formulae depend on a recursion among the M_n^p. To get these generating functions we will first prove the following lemmas:

Lemma 1 *If we set $M_n^p = 0$ when p is negative or n is negative, then the M_n^p satisfy*

$$M_n^p = \frac{n-2p}{p} M_{n-3}^{p-1} + 2M_{n-3}^{p} \tag{5.1.1}$$

for all n, and for all $p > 0$.

Proof 5 *To get a path with p plateaus and length n, we can do one of two things:*

i) We start with a Motzkin path of length $n-3$ with $p-1$ plateaus. We then insert a plateau in the path at a vertex that is not adjacent to a horizontal step in a plateau. There are $2(p-1)$ vertices which are adjacent to the horizontal step in the plateaus of the path and the total number of vertices in the path is $n-3+1 = n-2$. So, we choose one vertex from $n-2-2(p-1) = n-2p$ and insert a plateau.
ii) Secondly, we take a path of length $n-3$ with p plateaus and insert a plateau at one of the vertices adjacent to the horizontal step in a plateau. This destroys one plateau and creates a new one. Thus, we choose one vertex from one of the $2p$ and insert a plateau.

If we insert a plateau in a path of length $n-3$ and $p-1$ plateaus we get a specific path with length n and p plateaus. However, we get the same path by removing and then reinserting any of the other $p-1$ plateaus. Thus, the two procedures create each path p times.

Therefore,

$$pM_n^p = (n-2p)M_{n-3}^{p-1} + 2pM_{n-3}^{p-1},$$

dividing both sides by p gives the result.

Let $f_p(x)$ be the generating function for Motzkin paths with p plateaus.

Thus,

$$f_p(x) = \sum_{n\geq 0} M_n^p x^n.$$

Lemma 2 *The generating function for Motzkin paths with no plateaus is*

$$f_0(x) = \frac{1 - x + x^3 - \sqrt{(1 - x + x^3)^2 - 4x^2}}{2x^2},$$

and the number of Motzkin paths with no plateaus satisfy the recurrence

$$M_n^0 = M_{n-1}^0 + M_{n-2}^0 + \sum_{k=2}^{n-2} M_{n-k-2}^0 M_k^0.$$

Proof 6 *To get a Motzkin path with no plateaus, the path must be one of the following:*

i) An empty path, this contributes a 1 *to the generating function.*
ii) A horizontal step followed by a zero plateau Motzkin path, this contributes $xf_0(x)$ *to the generating function.*
iii) An up step, followed by a zero plateau Motzkin path excluding the case of a single horizontal step, then a down step followed by a zero plateau Motzkin path.
This contributes $x(f_0(x) - x)xf_0(x)$ *to the generating function.*

Thus,

$$\begin{aligned} f_0(x) &= 1 + xf_0(x) + x^2(f_0(x) - x)f_0(x) \\ &= 1 + (x - x^3)f_0(x) + x^2 f_0^2(x). \end{aligned}$$

Solving the quadratic equation we get

$$f_0(x) = \frac{1 - x + x^3 \pm \sqrt{(1 - x + x^3)^2 - 4x^2}}{2x^2}.$$

Now, from our functional equation for $f_0(x)$*, the limit of* $f_0(x)$ *as* x *tends to* 0 *is* 1*. So, from the solutions above we will only consider the negative sign case since it is* *the one that gives us $\frac{0}{0}$, which is what we want.*

To get the recurrence relationship we first note that,

$$\begin{aligned} f_0^2(x) &= \sum_{n\geq 0} M_n^0 x^n \sum_{n\geq 0} M_n^0 x^n \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} M_k^0 M_{n-k}^0 x^n. \end{aligned}$$

Now, extracting coefficients

$$\begin{aligned} [x^n]f_0(x) &= [x^n]\left(1 + xf_0(x) - x^3 f_0(x) + x^2 f_0^2(x)\right) \\ &= \begin{cases} 1 & \text{if } n = 0 \\ [x^{n-1}]f_0(x) - [x^{n-3}]f_0(x) + [x^{n-2}]f_0^2(x) & \text{if } n \geq 1. \end{cases} \end{aligned}$$

Thus, if $n \geq 1$, then

$$\begin{aligned} M_n^0 &= M_{n-1}^0 - M_{n-3}^0 + \sum_{k=0}^{n-2} M_k^0 M_{n-k-2}^0 \\ &= M_{n-1}^0 - M_{n-3}^0 + M_0^0 M_{n-2}^0 + M_1^0 M_{n-3}^0 + \sum_{k=2}^{n-2} M_k^0 M_{n-k-2}^0 \\ &= M_{n-1}^0 + M_{n-2}^0 + \sum_{k=2}^{n-2} M_k^0 M_{n-k-2}^0. \end{aligned}$$

We then use these properties of $f_0(x)$ to get a general expression of $M(x, y)$.

Theorem 7 *The generating function of Motzkin paths according to length and number of plateaus is given by,*

$$M(x,y) = \frac{1-2x^3}{1-2x^3(1-y)}\left(f_0(x) + \frac{x}{1-2x^3}\frac{\partial}{\partial x}x^3\int_0^y M(x,s)ds\right),$$

where, $f_0(x)$ is as defined above.

Proof 8 *We know that*

$$M_n^p = \frac{n-2p}{p} M_{n-3}^{p-1} + 2M_{n-3}^p.$$

Multiplying this equation by x^n and summing over $n \geq 0$ we get,

$$\begin{aligned}
f_p(x) &= \sum_{n\geq 0} M_n^p x^n \\
&= \sum_{n\geq 0} \frac{n-2p}{p} M_{n-3}^{p-1} x^n + \sum_{n\geq 0} 2M_{n-3}^p x^n \\
&= \frac{x}{p} \sum_{n\geq 0} n M_{n-3}^{p-1} x^{n-1} - 2x^3 \sum_{n\geq 0} M_{n-3}^{p-1} x^{n-3} + 2x^3 \sum_{n\geq 0} M_{n-3}^p x^{n-3} \\
&= \frac{x}{p} \sum_{n\geq 0} M_{n-3}^{p-1} \frac{\partial}{\partial x} x^n - 2x^3 f_{p-1}(x) + 2x^3 f_p(x) \\
&= \frac{x}{p} \frac{\partial}{\partial x} \left(x^3 \sum_{n\geq 0} M_{n-3}^{p-1} x^{n-3} \right) - 2x^3 f_{p-1}(x) + 2x^3 f_p(x) \\
&= \frac{x}{p} \frac{\partial}{\partial x} \left(x^3 f_{p-1}(x) \right) - 2x^3 f_{p-1}(x) + 2x^3 f_p(x).
\end{aligned}$$

By making $f_p(x)$ the subject of the formula in the above equation, we get

$$f_p(x) = \frac{1}{1-2x^3} \left(\frac{x}{p} \frac{\partial}{\partial x} \left(x^3 f_{p-1}(x) \right) - 2x^3 f_{p-1}(x) \right).$$

Multiplying this equation by y^p and summing over $p \geq 1$ we get

$$\sum_{p\geq 1} \sum_{n\geq 0} M_n^p x^n y^p = \sum_{p\geq 1} \frac{1}{1-2x^3} \left(\frac{x}{p} \frac{\partial}{\partial x} \left(x^3 f_{p-1}(x) \right) - 2x^3 f_{p-1}(x) \right) y^p,$$

thus,

$$\begin{aligned}
M(x,y) - f_0(x) &= \frac{1}{1-2x^3} \sum_{p\geq 1} \frac{x}{p} \frac{\partial}{\partial x} \left(x^3 f_{p-1}(x) \right) y^p - \frac{1}{1-2x^3} \sum_{p\geq 1} 2x^3 f_{p-1}(x) y^p \\
&= \frac{x}{1-2x^3} \frac{\partial}{\partial x} x^3 \sum_{p\geq 1} \frac{1}{p} f_{p-1}(x) y^p - \frac{2x^3 y}{1-2x^3} \sum_{p\geq 1} f_{p-1}(x) y^{p-1} \\
&= \frac{x}{1-2x^3} \frac{\partial}{\partial x} x^3 \sum_{p\geq 1} f_{p-1}(x) \int_0^y s^{p-1} ds - \frac{2x^3 y}{1-2x^3} M(x,y)
\end{aligned}$$

$$= \frac{x}{1-2x^3}\frac{\partial}{\partial x}x^3\int_0^y M(x,s)ds - \frac{2x^3y}{1-2x^3}M(x,y).$$

By making $M(x,y)$ the subject of the formula, we get,

$$M(x,y)\left(1+\frac{2x^3y}{1-2x^3}\right) = f_0(x) + \frac{x}{1-2x^3}\frac{\partial}{\partial x}x^3\int_0^y M(x,s)ds$$

thus,

$$M(x,y)\left(\frac{1-2x^3(1-y)}{1-2x^3}\right) = f_0(x) + \frac{x}{1-2x^3}\frac{\partial}{\partial x}x^3\int_0^y M(x,s)ds.$$

Dividing both sides by $\frac{1-2x^3(1-y)}{1-2x^3}$ *gives us the desired form.*

This generating function can be written in different forms:

Theorem 9 *The bivariate generating function $M(x,y)$ has a differential form given by,*

$$\frac{\partial}{\partial x}\left(xM\left(x,\frac{z}{x^3}\right)\right) = (1-z-2x^3)\frac{\partial}{\partial z}M(x,\frac{z}{x^3}).$$

Proof 10 *Let $d_m = M^m_{3m+k}$ and*

$$h_k(z) = \sum_{m\geq 0} d_m z^m.$$

So, we have

$$h_0(z) = \sum_{m\geq 0} M^m_{3m} z^m = \sum_{m\geq 0} z^m = \frac{1}{1-z}.$$

Since all the steps in M^m_{3m} are inside some plateau, then there is only one path of this kind.
From (5.1.1)*, for $m \geq 1$ we have,*

$$M^m_{3m+k} = \frac{3m+k-2m}{m}M^{m-1}_{3m+k-3} + 2M^m_{3m+k-3}$$

$$= \frac{m+k}{m} M^{m-1}_{3m+k-3} + 2M^{m}_{3m+k-3}.$$

Substituting this into the generating function $h_k(z)$, and then expanding, we get,

$$\begin{aligned}
\sum_{m\geq 0} d_m z^m &= M^0_k + \sum_{m\geq 1} M^m_{3m+k} z^m \\
&= M^0_k + \sum_{m\geq 1} M^{m-1}_{3(m-1)+k} z^m + \sum_{m\geq 1} \frac{k}{m} M^{m-1}_{3(m-1)+k} z^m + 2\sum_{m\geq 1} M^m_{3(m)+(k-3)} z^m \\
&= M^0_k + zh_k(z) + 2h_{k-3}(z) - 2M^0_{k-3} + k\sum_{m\geq 1} M^{m-1}_{3(m-1)+k} \int_0^z s^{m-1} ds \\
&= M^0_k + zh_k(z) + 2h_{k-3}(z) - 2M^0_{k-3} + k\int_0^z h_k(s) ds.
\end{aligned}$$

Thus,

$$h_k(z) = M^0_k + zh_k(z) + 2h_{k-3}(z) - 2M^0_{k-3} + k\int_0^z h_k(s) ds.$$

Differentiating this equation with respect to z we get,

$$h'_k(z) = h_k(z) + zh'_k(z) + 2h'_{k-3}(z) + kh_k(z),$$

which simplifies to,

$$(k+1)h_k(z) = (1-z)h'_k(z) - 2h'_{k-3}(z). \tag{5.1.2}$$

Now, replacing z with x^3y in $h_k(z)$ then multiplying by x^k and summing over $k \geq 0$, we get,

$$\begin{aligned}
\sum_{k\geq 0} h_k x^3 y x^k &= \sum_{k\geq 0}\sum_{m\geq 0} M^m_{3m+k} x^{3m} y^m x^k \\
&= \sum_{k\geq 0}\sum_{m\geq 0} M^m_{3m+k} x^{3m+k} y^m \\
&= \sum_{n\geq 0}\sum_{p=0}^{\lfloor \frac{n}{3} \rfloor} M^p_n x^n y^p \\
&= M(x,y).
\end{aligned}$$

So,

$$\frac{\partial}{\partial x}xM\left(x,\frac{z}{x^3}\right)=\frac{\partial}{\partial x}\sum_{k\geq 0}h_k\left(x^3\frac{z}{x^3}\right)x^{k+1}$$
$$=\sum_{k\geq 0}h_k(z)(k+1)x^k,$$

and

$$\frac{\partial}{\partial z}M\left(x,\frac{z}{x^3}\right)=\frac{\partial}{\partial z}\sum_{k\geq 0}h_k\left(x^3\frac{z}{x^3}\right)x^k$$
$$=\sum_{k\geq 0}h'_k(z)x^k.$$

Multiplying (5.1.2) *by* x^n *and summing over* $k\geq 0$ *we find,*

$$\sum_{k\geq 0}(k+1)h_k(z)x^k=(1-z)\sum_{k\geq 0}h'_k(z)x^k-2\sum_{k\geq 0}h'_{k-3}(z)x^k,$$

which leads to,

$$\frac{\partial}{\partial x}\left(xM\left(x,\frac{z}{x^3}\right)\right)=(1-z)\frac{\partial}{\partial z}M\left(x,\frac{z}{x^3}\right)-2x^3\frac{\partial}{\partial z}M\left(x,\frac{z}{x^3}\right)$$
$$=(1-z-2x^3)\frac{\partial}{\partial z}M\left(x,\frac{z}{x^3}\right).$$

We also find that the generating function for Motzkin paths has an explicit form given by the theorem below.

Theorem 11 *The generating function for Motzkin paths according to length and number of plateaus has explicit form*

$$M(x,y)=\frac{1-x+x^2-x^3y-\sqrt{(1-3x+x^3-x^3y)(1+x+x^3-x^3y)}}{2x^2}.$$

Proof 12 *To get a Motzkin path, it must be one of the following forms:*
i) An empty path, this contributes a 1 *to the generating function.*
ii) A horizontal step followed by a Motzkin path, this contributes $xM(x,y)$ *to the generating function.*

iii) An up step, followed by a Motzkin path, followed by a down step, followed by a Motzkin path. In this case, if an up step is followed by a single step Motzkin path, i.e., a Motzkin path that is just a horizontal step, then we have a plateau that will not be counted in our generating function. To solve this problem, we subtract the case where $M(x,y)$ *is a horizontal step,* x*, and add* xy *to put back the horizontal step and count plateau that will be formed. This contributes* $x\left(M(x,y)-x+xy\right)xM(x,y)$*. Combining these three forms we get*

$$M(x,y) = 1 + (x - x^3 + x^3y)M(x,y) + x^2M^2(x,y).$$

Thus,

$$x^2M^2(x,y) - (1 - x + x^3 - x^3y)M(x,y) + 1 = 0.$$

We note that this is a quadratic equation in $M(x,y)$*; using the quadratic formula we get*

$$\begin{aligned}
M(x,y) &= \frac{(1-x+x^3-x^3y) \pm \sqrt{(1-x+x^3-x^3y)^2 - 4x^2}}{2x^2} \\
&= \frac{(1-x+x^3-x^3y)}{2x^2} \\
&\quad \pm \frac{\sqrt{1-2x-3x^2+2x^3-2x^4+x^6-2x^3y(1-x+x^3)+(x^3y)^2}}{2x^2} \\
&= \frac{(1-x+x^3-x^3y)}{2x^2} \\
&\quad \pm \frac{\sqrt{1-2x-3x^2+2x^3-2x^4+x^6-2x^3y(1-x+x^3)+(x^3y)^2}}{2x^2} \\
&= \frac{(1-x+x^3-x^3y)}{2x^2} \\
&\quad \pm \frac{\sqrt{(1+x+x^3)(1-3x+x^3)-2x^3y(1-x+x^3)+(x^3y)^2}}{2x^2} \\
&= \frac{(1-x+x^3-x^3y) \pm \sqrt{(1+x+x^3-x^3y)(1-3x+x^3-x^3y)}}{2x^2}.
\end{aligned}$$

Now, the limit as x *tends to* 0 *of* $M(x,y)$ *is* 1 *from our functional equation. In our equation above, only the negative sign leads to the same answer when we take limits. Hence, we get the result we require.*

5.2 Generating functions for Motzkin paths with plateaus of length $r > 1$

We now generalize our generating functions to cases where the plateaus are longer, that is, of length greater than 1. In the previous section we considered the case $r = 1$.

Definition 5 *A plateau of length r in a Motzkin path is any part of the Motzkin path that has the subword $uh^r d$ where $r \geq 1$, thus, a u step immediately followed by r h steps immediately followed by a d step.*

From this definition, we see that the number of horizontal steps in each plateau of length r is r. Let ${}_rM_n^p$ be the number of Motzkin paths of length n with p plateaus of length r and

$$ {}_rM(x, y) = \sum_{n \geq 0} \sum_{p=0}^{\lfloor \frac{n}{2+r} \rfloor} {}_rM_n^p x^n y^p, $$

be the generating function for Motzkin paths according to length and number of plateaus of length r. In this generating function, p has an upper limit of $\lfloor \frac{n}{2+r} \rfloor$ because each plateau is made up of $2 + r$ steps and if all steps of the Motzkin path form part of a plateau, then we see that the total number of plateaus is $\lfloor \frac{n}{2+r} \rfloor$. To get the generating functions, we must first prove recursions for ${}_rM_n^p$.

Lemma 3 *If we set ${}_rM_n^p = 0$ when p is negative or n is negative, then the ${}_rM_n^p$ satisfy*

$$ {}_rM_n^p = \frac{n - (r+1)p}{p} \, {}_rM_{n-(2+r)}^{p-1} + (r+1) \, {}_rM_{n-(2+r)}^{p} \qquad (5.2.1) $$

for all n, and for all $p > 0$.

Proof 13 *To get a path of length n with p plateaus of length r, we can do one of two things:*

i) We start with a Motzkin path of length $n - (2 + r)$ with $p - 1$ plateaus of length r. We then insert a plateau of length r in the path at a vertex that is not adjacent to

any horizontal step in a plateau of length r*. There are* $(1+r)(p-1)$ *vertices which are adjacent to the horizontal steps in the plateaus of length* r *in the path and the total number of vertices in the path is* $n-(2+r)+1 = n-r-1$*. So, we choose one vertex from* $n-(1+r)-(1+r)(p-1)$ *and insert a plateau.*
ii) Secondly, we take a path of length $n-(2+r)$ *with* p *plateaus of length* r *and insert a plateau of length* r *at one of the vertices adjacent to the horizontal step in a plateau of length* r*. This destroys one plateau of length* r *and creates a new one. Thus, we choose one vertex from one of the* $(1+r)p$ *and insert a plateau.*

If we insert a plateau of length r *in a path of length* $n-(1+r)$ *and* $p-1$ *plateaus of length* r *we get a specific path with length* n *and* p *plateaus of length* r*. However, we get the same path by removing and then reinserting any of the other* $p-1$ *plateaus. Thus, the two procedures create each path* p *times.*

Therefore,

$$p\, {}_rM_n^p = (n-(1+r)-(1+r)(p-1))\, {}_rM_{n-(2+r)}^{p-1} + (1+r)p\, {}_rM_{n-(2+r)}^{p-1}.$$

Dividing both sides by p *gives the result.*

Let ${}_rf_p(x)$ be the generating function for Motzkin paths with p plateaus of length r.
Thus,

$${}_rf_p(x) = \sum_{n\geq 0} {}_rM_n^p x^n.$$

Lemma 4 *The generating function for Motzkin paths with no plateaus of length* r *is*

$${}_rf_0(x) = \frac{1-x+x^{2+r}-\sqrt{(1-x+x^{2+r})^2-4x^2}}{2x^2},$$

and the number of Motzkin paths with no plateaus of length r *satisfy the recurrence*

$${}_rM_n^0 = {}_rM_{n-1}^0 - {}_rM_{n-(2+r)}^0 + \sum_{k=0}^{n-2} {}_rM_{n-k-2}^0\, {}_rM_k^0.$$

Proof 14 *To get a Motzkin path with no plateaus of length* r*, it must be one of the*

following:
i) An empty path, this contributes a 1 *to the generating function.*
ii) A horizontal step followed by a Motzkin path with no plateaus of length r*, this contributes* $x\ {}_rf_0(x)$ *to the generating function.*
iii) An up step, followed by a Motzkin path with no plateaus of length r *excluding the case counted by* x^r*, then a down step followed by a Motzkin path with no plateaus of length* r*. This contributes* $x(\ {}_rf_0(x) - x^r)x\ {}_rf_0(x)$ *to the generating function.*

Thus,

$$\begin{aligned} {}_rf_0(x) &= 1 + x\ {}_rf_0(x) + x^2(\ {}_rf_0(x) - x^r)\ {}_rf_0(x) \\ &= 1 + (x - x^{2+r})\ {}_rf_0(x) + x^2 f_0^2(x). \end{aligned}$$

Solving the quadratic equation we get

$${}_rf_0(x) = \frac{1 - x + x^{2+r} \pm \sqrt{(1 - x + x^{2+r})^2 - 4x^2}}{2x^2}.$$

Discarding the spurious solution, we get the desired result.

To get the recurrence relationship we first note that,

$$\begin{aligned} {}_rf_0^2(x) &= \sum_{n\geq 0} {}_rM_n^0 x^n \sum_{n\geq 0} {}_rM_n^0 x^n \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} {}_rM_k^0\ {}_rM_{n-k}^0 x^n. \end{aligned}$$

Now,

$$\begin{aligned} [x^n]\ {}_rf_0(x) &= [x^n]\left(1 + x\ {}_rf_0(x) - x^{2+r}\ {}_rf_0(x) + x^2\ {}_rf_0^2(x)\right) \\ &= \begin{cases} 1 & if\ \ n = 0 \\ [x^{n-1}]\ {}_rf_0(x) - [x^{n-(2+r)}]\ {}_rf_0(x) + [x^{n-2}]\ {}_rf_0^2(x) & if\ \ n \geq 1. \end{cases} \end{aligned}$$

Thus, if $n \geq 1$*, then*

$${}_rM_n^0 = {}_rM_{n-1}^0 - {}_rM_{n-(2+r)}^0 + \sum_{k=0}^{n-2} {}_rM_k^0\ {}_rM_{n-k-2}^0.$$

We then use these properties of ${}_rf_o(x)$ to get a general expression of ${}_rM(x,y)$.

Theorem 15 *The generating function of Motzkin paths according to length and number of plateaus of length r is given by,*

$$ {}_rM(x,y) = \frac{1-2x^{2+r}}{1-2x^{2+r}(1-y)} \left({}_rf_0(x) + \frac{x}{1-2x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \int_0^y {}_rM(x,s)ds \right), $$

where ${}_rf_0(x)$ is as defined above.

Proof 16 *We know that*

$$ {}_rM_n^p = \frac{n-(r+1)p}{p} \, {}_rM_{n-(2+r)}^{p-1} + (r+1) \, {}_rM_{n-(2+r)}^p. $$

Multiplying this equation by x^n and summing over $n \geq 0$ we get,

$$ \begin{aligned} {}_rf_p(x) &= \sum_{n\geq 0} {}_rM_n^p x^n \\ &= \sum_{n\geq 0} \frac{n-(r+1)p}{p} \, {}_rM_{n-(2+r)}^{p-1} x^n + \sum_{n\geq 0} (r+1) \, {}_rM_{n-(2+r)}^p x^n \\ &= \frac{x}{p} \sum_{n\geq 0} n \, {}_rM_{n-(2+r)}^{p-1} x^{n-1} - (1+r)x^{2+r} \sum_{n\geq 0} {}_rM_{n-(2+r)}^{p-1} x^{n-(2+r)} \\ &\quad + (1+r)x^{2+r} \sum_{n\geq 0} {}_rM_{n-(2+r)}^p x^{n-(2+r)} \\ &= \frac{x}{p} \sum_{n\geq 0} {}_rM_{n-(2+r)}^{p-1} \frac{\partial}{\partial x} x^n - (1+r)x^{2+r} \, {}_rf_{p-1}(x) + (1+r)x^{2+r} \, {}_rf_p(x) \\ &= \frac{x}{p} \frac{\partial}{\partial x} \left(x^{2+r} \sum_{n\geq 0} {}_rM_{n-(2+r)}^{p-1} x^{n-(2+r)} \right) - (1+r)x^{2+r} \, {}_rf_{p-1}(x) + (1+r)x^{2+r} \, {}_rf_p(x) \\ &= \frac{x}{p} \frac{\partial}{\partial x} \left(x^{2+r} \, {}_rf_{p-1}(x) \right) - (1+r)x^{2+r} \, {}_rf_{p-1}(x) + (1+r)x^{2+r} \, {}_rf_p(x). \end{aligned} $$

By making ${}_rf_p(x)$ the subject of the formula in the above equation, we get

$$ {}_rf_p(x) = \frac{1}{1-(1+r)x^{2+r}} \left(\frac{x}{p} \frac{\partial}{\partial x} \left(x^{2+r} \, {}_rf_{p-1}(x) \right) - (1+r)x^{2+r} \, {}_rf_{p-1}(x) \right). $$

Multiplying this equation by y^p and summing over $p \geq 1$ we get

$$\sum_{p\geq 1}\sum_{n\geq 0} {}_rM_n^p x^n y^p = \sum_{p\geq 1} \frac{1}{1-(1+r)x^{2+r}} \left(\frac{x}{p}\frac{\partial}{\partial x}\left(x^{2+r}\ {}_rf_{p-1}(x)\right) - (1+r)x^{2+r}\ {}_rf_{p-1}(x)\right) y^p.$$

Thus,

$$\begin{aligned} {}_rM(x,y) - {}_rf_0(x) &= \frac{1}{1-(1+r)x^{2+r}} \sum_{p\geq 1} \frac{x}{p}\frac{\partial}{\partial x}\left(x^{2+r}\ {}_rf_{p-1}(x)\right) y^p \\ &\quad - \frac{1}{1-(1+r)x^{2+r}} \sum_{p\geq 1} (1+r)x^{2+r}\ {}_rf_{p-1}(x) y^p \\ &= \frac{x}{1-(1+r)x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \sum_{p\geq 1} \frac{1}{p}\ {}_rf_{p-1}(x) y^p \\ &\quad - \frac{(1+r)x^{2+r}y}{1-(1+r)x^{2+r}} \sum_{p\geq 1} {}_rf_{p-1}(x) y^{p-1} \\ &= \frac{x}{1-(1+r)x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \sum_{p\geq 1} {}_rf_{p-1}(x) \int_0^y s^{p-1} ds \\ &\quad - \frac{(1+r)x^{2+r}y}{1-(1+r)x^{2+r}}\ {}_rM(x,y) \\ &= \frac{x}{1-(1+r)x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \int_0^y {}_rM(x,s) ds - \frac{(1+r)x^{2+r}y}{1-(1+r)x^{2+r}}\ {}_rM(x,y). \end{aligned}$$

By making ${}_rM(x,y)$ the subject of the formula, we get,

$${}_rM(x,y)\left(1 + \frac{(1+r)x^{2+r}y}{1-(1+r)x^{2+r}}\right) = {}_rf_0(x) + \frac{x}{1-(1+r)x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \int_0^y {}_rM(x,s) ds.$$

Thus,

$${}_rM(x,y)\left(\frac{1-(1+r)x^{2+r}(1-y)}{1-(1+r)x^{2+r}}\right) = {}_rf_0(x) + \frac{x}{1-(1+r)x^{2+r}} \frac{\partial}{\partial x} x^{2+r} \int_0^y {}_rM(x,s) ds.$$

Dividing both sides by $\dfrac{1-(1+r)x^{2+r}(1-y)}{1-(1+r)x^{2+r}}$ gives us the desired form.

This generating function can be written in different forms.

Theorem 17 *The bivariate generating function ${}_rM(x,y)$ has a differential form* *given by,*

$$\frac{\partial}{\partial x}\left(x\,{}_rM\left(x,\frac{z}{x^{2+r}}\right)\right)=(1-z-(1+r)x^{2+r})\frac{\partial}{\partial z}\,{}_rM\left(x,\frac{z}{x^{2+r}}\right).$$

Proof 18 *Let* ${}_rd_m = {}_rM^m_{3m+k}$ *and*

$${}_rh_k(z)=\sum_{m\geq 0}{}_rd_m z^m.$$

So, we have

$${}_rh_0(z)=\sum_{m\geq 0}{}_rM^m_{(2+r)m}z^m=\sum_{m\geq 0}z^m=\frac{1}{1-z}.$$

Since all the steps in $M^m_{(2+r)m}$ *are inside some plateau, then there is only one path of this kind.*
From (5.2.1)*, for* $m\geq 1$ *we have,*

$$\begin{aligned}{}_rM^m_{(2+r)m+k} &= \frac{(2+r)m+k-(1+r)m}{m}\,{}_rM^{m-1}_{(2+r)m+k-(2+r)}+(1+r)\,{}_rM^m_{(2+r)m+k-(2+r)}\\ &= \frac{m+k}{m}\,{}_rM^{m-1}_{(2+r)m+k-(2+r)}+(1+r)\,{}_rM^m_{(2+r)m+k-(2+r)}.\end{aligned}$$

Substituting this into the generating function ${}_rh_k(z)$*, and then expanding, we get,*

$$\begin{aligned}\sum_{m\geq 0}{}_rd_m z^m &= {}_rM^0_k+\sum_{m\geq 1}{}_rM^m_{(2+r)m+k}z^m\\ &= {}_rM^0_k+\sum_{m\geq 1}{}_rM^{m-1}_{(2+r)(m-1)+k}z^m+\sum_{m\geq 1}\frac{k}{m}\,{}_rM^{m-1}_{(2+r)(m-1)+k}z^m\\ &\quad +(1+r)\sum_{m\geq 1}{}_rM^m_{(2+r)(m)+(k-(2+r))}z^m\\ &= {}_rM^0_k+z\,{}_rh_k(z)+(1+r)\,{}_rh_{k-(2+r)}(z)-(1+r)\,{}_rM^0_{k-(2+r)}\\ &\quad +k\sum_{m\geq 1}{}_rM^{m-1}_{(2+r)(m-1)+k}\int_0^z s^{m-1}ds\\ &= {}_rM^0_k+z\,{}_rh_k(z)+(1+r)\,{}_rh_{k-(2+r)}(z)-(1+r)\,{}_rM^0_{k-(2+r)}+k\int_0^z {}_rh_k(s)ds.\end{aligned}$$

Thus,

$$ {}_r h_k(z) = {}_r M_k^0 + z\, {}_r h_k(z) + (1+r)\, {}_r h_{k-(2+r)}(z) - (1+r)\, {}_r M_{k-3}^0 + k \int_0^z {}_r h_k(s) ds. $$

Differentiating this equation with respect to z we get,

$$ {}_r h_k'(z) = {}_r h_k(z) + z\, {}_r h_k'(z) + (1+r)\, {}_r h_{k-(2+r)}'(z) + k\, {}_r h_k(z), $$

which simplifies to

$$ (k+1)\, {}_r h_k(z) = (1-z)\, {}_r h_k'(z) - (1+r)\, {}_r h_{k-(2+r)}'(z). \tag{5.2.2} $$

Now, replacing z with $x^3 y$ in ${}_r h_k(z)$ then multiplying by x^k and summing over $k \geq 0$, we get,

$$ \begin{aligned} \sum_{k\geq 0} {}_r h_k(x^{2+r} y) x^k &= \sum_{k\geq 0} \sum_{m\geq 0} {}_r M_{(2+r)m+k}^m x^{(2+r)m} y^m x^k \\ &= \sum_{k\geq 0} \sum_{m\geq 0} {}_r M_{(2+r)m+k}^m x^{(2+r)m+k} y^m \\ &= \sum_{n\geq 0} \sum_{p=0}^{\lfloor \frac{n}{2+r} \rfloor} {}_r M_n^p x^n y^p \\ &= {}_r M(x,y). \end{aligned} $$

So,

$$ \begin{aligned} \frac{\partial}{\partial x} x\, {}_r M\left(x, \frac{z}{x^{2+r}}\right) &= \frac{\partial}{\partial x} \sum_{k\geq 0} {}_r h_k\left(x^{2+r} \frac{z}{x^{2+r}}\right) x^{k+1} \\ &= \sum_{k\geq 0} {}_r h_k(z)(k+1) x^k, \end{aligned} $$

and

$$ \begin{aligned} \frac{\partial}{\partial z} {}_r M\left(x, \frac{z}{x^{2+r}}\right) &= \frac{\partial}{\partial z} \sum_{k\geq 0} {}_r h_k\left(x^{2+r} \frac{z}{x^{2+r}}\right) x^k \\ &= \sum_{k\geq 0} {}_r h_k'(z) x^k. \end{aligned} $$

Multiplying (5.2.2) *by* x^k *and summing over* $k \geq 0$ *we find,*

$$\sum_{k\geq 0}(k+1)\ {}_rh_k(z)x^k = (1-z)\sum_{k\geq 0}\ {}_rh'_k(z)x^k - (1+2)\sum_{k\geq 0}\ {}_rh'_{k-(2+r)}(z)x^k,$$

which leads to,

$$\begin{aligned}\frac{\partial}{\partial x}\left(x\ {}_rM\left(x, \frac{z}{x^{2+r}}\right)\right) &= (1-z)\frac{\partial}{\partial z}\ {}_rM\left(x, \frac{z}{x^{2+r}}\right) - (1+r)x^{2+r}\frac{\partial}{\partial z}\ {}_rM\left(x, \frac{z}{x^{2+r}}\right)\\ &= (1-z-(1+r)x^{2+r})\frac{\partial}{\partial z}\ {}_rM\left(x, \frac{z}{x^{2+r}}\right).\end{aligned}$$

We also find that the generating function for Motzkin paths has an explicit form given by the theorem below.

Theorem 19 *The generating function for Motzkin paths according to length and number of plateaus of length* r *has explicit form*

$${}_rM(x,y) = \frac{1-x+x^{2+r}-x^{2+r}y-\sqrt{(1-x+x^{2+r}-x^{2+r}y)^2-4x^2}}{2x^2}.$$

Proof 20 *To get a Motzkin path, it must be one of the following forms:*
i) An empty path, this contributes a 1 *to the generating function.*
ii) A horizontal step followed by a Motzkin path, this contributes $x\ {}_rM(x,y)$ *to the generating function.*
iii) An up step, followed by a Motzkin path, followed by a down step, followed by a Motzkin path. In this case, if an up step is followed by a Motzkin path which is a sequence of r *horizontal steps, then we have a plateau that will not be counted in our generating function. To solve this problem, we subtract the case where* ${}_rM(x,y)$ *is* x^r*, and add* x^ry *to put back the horizontal steps and count the plateau of length* r *that will be formed. This contributes* $x\,({}_rM(x,y)-x^r+x^ry)\,x\ {}_rM(x,y)$*.*
Combining these three forms we get

$$\begin{aligned}{}_rM(x,y) &= 1+x\ {}_rM(x,y)+x\,({}_rM(x,y)-x^r+x^ry)\,x\ {}_rM(x,y) \qquad (5.2.3)\\ &= 1+(x-x^{2+r}+x^{2+r}y)\ {}_rM(x,y)+x^2\ {}_rM^2(x,y).\end{aligned}$$

Thus,

$$x^2 \,{}_rM^2(x,y) - (1 - x + x^{2+r} - x^{2+r}y)\,{}_rM(x,y) + 1 = 0.$$

We note that this is a quadratic equation in ${}_rM(x,y)$*; using the quadratic formula we get*

$${}_rM(x,y) = \frac{(1 - x + x^{2+r} - x^{2+r}y) \pm \sqrt{(1 - x + x^{2+r} - x^{2+r}y)^2 - 4x^2}}{2x^2}.$$

Discarding the spurious solution, we get the result we require.

5.3 Continued fractions

We then generalize (5.2.3) into a continued fraction. To do this, we first replace ${}_rM(x,y)$ with $m(x,y)$ and the correction we make for the plateaus, $(x^r y - x^r)$, with P, where $P := P(x,y)$. Thus,

$$\begin{aligned} m(x,y) &= 1 + xm(x,y) + x\,(m(x,y) + P)\,xm(x,y) \\ &= 1 + m(x,y)\left(x + x^2 m(x,y) + x^2 P\right) \\ &= \frac{1}{1 - x - x^2P - x^2 m(x,y)}. \end{aligned}$$

If we inductively replace $m(x,y)$ with the right hand side of the equation, we have

$$m(x,y) = \cfrac{1}{1 - x - x^2P - \cfrac{x^2}{1 - x - x^2P - \cfrac{x^2}{1 - x - x^2P - \cfrac{x^2}{1 - \cdots}}}}$$

Now, replacing P with $xy - x$, we find that the generating function for Motzkin paths according to length and number of plateaus has a continued fraction expansion form given by,

$$M(x,y) = \cfrac{1}{1 - x - x^2(xy - x) - \cfrac{x^2}{1 - x - x^2(xy - x) - \cfrac{x^2}{1 - x - x^2(xy - x) - \cfrac{x^2}{1 - \cdots}}}}.$$

Similarly, we find

$$_rM(x,y) = \cfrac{1}{1 - x - x^2(x^r y - x^r) - \cfrac{x^2}{1 - x - x^2(x^r y - x^r) - \cfrac{x^2}{1 - \cdots}}}.$$

From, P. Flajolet, Combinatorial aspects of continued fractions [11], Theorem 1 , we see that the P appearing at the ith level of our continued fraction corresponds to the correction we make at height i. Looking at our continued fraction and taking note of P. Flajolet, Combinatorial aspects of continued fractions [11], Theorem 1, we see that the corrections at different heights do not have to be the same. This allows us to write our continued fraction in the following form,

$$M(x,y) = \cfrac{1}{1 - x - x^2 P_1 - \cfrac{x^2}{1 - x - x^2 P_2 - \cfrac{x^2}{1 - x - x^2 P_3 - \cfrac{x^2}{1 - \cdots}}}}. \tag{5.3.1}$$

Using this information, we then look at various examples where we make adjustments at different heights.

Example 1: The generating function for Motzkin paths with no peaks at even levels is,

$$S(x) = \frac{1 - 2x + 2x^2 - x^3 - \sqrt{1 - 4x + 4x^2 - 2x^3 + x^6}}{2(1 - x)x^2}.$$

In this type of path, at each even height, we can have any nonempty Motzkin path with no peaks at odd heights. So, the correction we make will be to exclude the empty path, that is, -1. This leads to a path with at least one horizontal step between any u and the corresponding d step.

We find that our correction terms must be zero whenever we have an odd height and -1 whenever we have an even height. That is, $P_k = 0$ when k is odd and $P_k = -1$ when k is even. Thus,

$$\begin{aligned} S(x) &= \cfrac{1}{1 - x - \cfrac{x^2}{1 - x + x^2 - x^2 S(x)}} \\ &= \frac{1 - x + x^2 - x^2 S(x)}{1 - 2x + x^2 - x^3 - (1-x)x^2 S(x)}, \end{aligned}$$

which gives the quadratic equation

$$(1-x)x^2\,(S(x))^2 - (1 - 2x + 2x^2 - x^3)S(x) + 1 - x + x^2 = 0.$$

Using the quadratic formula leads to

$$\begin{aligned} S(x) &= \frac{1 - 2x + 2x^2 - x^3 \pm \sqrt{(1 - 2x + 2x^2 - x^3)^2 - 4(1-x)x^2(1 - x + x^2)}}{2(1-x)x^2} \\ &= \frac{1 - 2x + 2x^2 - x^3 \pm \sqrt{1 - 4x + 4x^2 - 2x^3 + x^6}}{2(1-x)x^2}, \end{aligned}$$

discarding the spurious solution gives the desired result.

Example 2: The generating function for Motzkin paths in which the *uhd*'s have weight y and no plateaus are of length greater than one is

$$\begin{aligned} S(x,y) = &\frac{(1-x)^2 - x^2(xy(1-x) - x)}{2x^2(1-x)} \\ &- \frac{\sqrt{(1 + (x-1)x^2(xy+1))(1 + (x-1)x(4 + x + x^2 y))}}{2x^2(1-x)}. \end{aligned}$$

In this type of path, at each height ≥ 1, we must exclude the possibility of having a Motzkin path that is just a sequence of two or more horizontal steps. So, for each height ≥ 1, we must have the correction term being

$$P_k = xy - (x + x^2 + x^3 + \cdots) = xy - \frac{x}{1-x}.$$

Plugging this in our continued fraction we find

$$S(x,y) = \frac{1}{1 - x - x^2\left(xy - \dfrac{x}{1-x}\right) - x^2 S(x,y)},$$

which leads to the quadratic equation

$$x^2S^2(x,y) - \left(1 - x - x^2\left(xy - \frac{x}{1-x}\right)\right)S(x,y) + 1 = 0.$$

Using the quadratic formula we get

$$\begin{aligned} S(x,y) &= \frac{1 - x - x^2\left(xy - \dfrac{x}{1-x}\right) \pm \sqrt{\left(1 - x - x^2\left(xy - \dfrac{x}{1-x}\right)\right)^2 - 4x^2}}{2x^2} \\ &= \frac{(1-x)^2 - x^2(xy(1-x) - x)}{2x^2(1-x)} \\ &\quad \pm \frac{\sqrt{(1 + (x-1)x^2(xy+1))(1 + (x-1)x(4 + x + x^2y))}}{2x^2(1-x)}, \end{aligned}$$

which gives us the solution.
In this example, setting $y = 1$ we get a generating function for the number of Motzkin paths which only have plateaus of length one.

Example 3: We now find he generating function which counts Motzkin paths in which *uhd*'s at height three or more have weight y and *uhhd*'s at heights that are a multiple of two have weight z.
To find it, we first note that at all heights ≥ 3 we make the correction $Y = xy - x$ and at heights ≥ 2 which are a multiple of 2 we make the correction $Z = x^2z - x^2$. These adjustments are shown in the table below

k	1	2	3	4	5	6	7	8	9	10
P_k	0	Z	Y	$Y+Z$	Y	$Y+Z$	Y	$Y+Z$	Y	$Y+Z$

Plugging these corrections into our continued fraction (5.3.1), we get,

$$S(x,y,z) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - x^2 Z - \cfrac{x^2}{1 - x - x^2 Y - \cfrac{x^2}{1 - \cdots}}}}. \tag{5.3.2}$$

Now, we note that from the third height upwards the corrections we make are periodic. So, we let

$$\begin{aligned} s &= \cfrac{1}{1 - x - x^2 Y - \cfrac{x^2}{1 - x - x^2(Y+Z) - x^2 s}} \\ &= \frac{1 - x - x^2(Y+Z) - x^2 s}{(1 - x - x^2 Y)(1 - x - x^2(Y+Z) - x^2 s) - x^2} \\ &= \frac{-x^2 - A(A - x^2 Z) \pm \sqrt{(x^2 + A(A - x^2 Z))^2 + 4(A - x^2 Z)x^2 A}}{2x^2 A}, \end{aligned}$$

where $A = 1 - x - x^2 Y$. Substituting $B = A(A - x^2 Z)$ and discarding the solution that does not lead to the form $\frac{0}{0}$ we get,

$$s = -\frac{B + x^2 - \sqrt{(B + x^2)^2 + 4x^2 B}}{2x^2 A}.$$

Substituting the expression above into (5.3.2) gives

$$\begin{aligned} S(x,y,z) &= \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - x^2 Z - x^2 s}} \\ &= \frac{1 - x - x^2 Z - x^2 s}{(1 - x)(1 - x - x^2 Z - x^2 s) - x^2}. \end{aligned}$$

Chapter 6

MOTZKIN PATHS WITH HORIZONTAL STEPS OF DIFFERENT COLOURS

Now we move on to the paper by, A. Sapounakis and P. Tsikouras, On k-coloured Motzkin words [16], where we investigate the k-coloured Motzkin paths. These are Motzkin paths whose horizontal steps are coloured by means of k different colours.

Let S be a set of $k+2$ different steps, where, $k \in \mathbb{N}$ and u, the up step, and d, the down step, are elements of S. If $k = 0$ then S will be a step set for the Dyck paths, for $k \neq 0$, then $S \backslash \{u, d\} = \{\mu_1, \mu_2 \cdots \mu_k\}$ are the k differently coloured horizontal steps. Let S^* be the set of lattice paths created from the step set S and let ϵ, be the empty path, also be in S^*. If c is a path in S^*, then $|c|_x$ denotes the number of occurrences of the step x in c, where $x \in S$.
This shows that for any path c in S^* to be a k-coloured Motzkin path, we must have $|s|_u = |s|_d$ and for any factorisation, $s = wv$, we must have $|w|_u \geq |w|_d$. Below is an illustration of a 3-coloured Motzkin path of length 21, where b, r and y respectively represent the horizontal steps of blue, red and yellow colours.

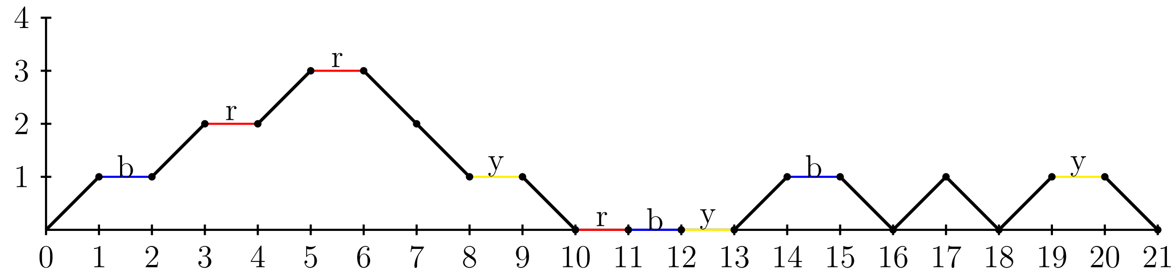

Figure 10.

Looking at the diagram above, we see that it has length 21 with 9 horizontal steps created from a set of 3-coloured horizontal steps.
We will denote the number of k-coloured Motzkin paths of length n by $M_{k,r}$ and the number of k-coloured Motzkin paths of length n and with r number of u by $M_{k,n,r}$. If $k = 0$, then we get that $M_{0,n}$ is the number of Dyck paths of semilength $\frac{n}{2}$, since there won't be any horizontal steps. We also note that if $k = 0$ and $n = 2s + 1$, where $s \in \mathbb{N}$, then $M_{0,n}$ will neither be a Dyck path nor a Motzkin path since this path will not terminate on the x-axis. If $k = 1$, then we get that $M_{1,n}$ is equal to the number of Motzkin paths of length n.

6.1 Enumeration of k-coloured Motzkin paths

Let $s = s_1 s_2 \cdots s_n \in S^*$. We say two indices, $i, j \in [n] = 1, 2, \cdots n$ with $i < j$, are called *conjugates* with respect to s if and only if j is the smallest element in $\{i+1, i+2, \cdots n\}$ for which the subpath $s_i s_{i+1} \cdots s_j$ is a k-coloured Motzkin path. From this definition we see that the subpath must have horizontal step coloured by k-colours and the conjugate steps must also be in this path.
A k-coloured Motzkin path, $s \in S^*$, is called a *k-coloured c-Motzkin* path if and only if every $i \in [n]$, with s_i neither a u nor a d, lies between two conjugate indices. We will denote the number of these paths by ${}^{c}M_{k,n}$.

Note: what the definition is saying is that all the level steps in a k-coloured Motzkin path must lie between two conjugates. Clearly the first step cannot be a level step because there won't be a step before it and hence it won't lie between two conjugates. From this restriction, we also have that for the first level step that lies on the x-axis,

there must be a d before it. This d step will not have a conjugate because you cannot have a Motzkin path which starts with a d step.
From this note, we see that the k-coloured c-Motzkin paths don't have any level steps on the x-axis.

Let $\mathbb{M}^k$ be the set of k-coloured Motzkin paths and ${}^c\mathbb{M}^k$ be the set of k-coloured c-Motzkin paths. The *initial rise* of a nonempty path $s = s_1 s_2 \cdots s_n \in S^*$, with $s_1 = u$, is the segment $s_1 s_2 \cdots s_j$ where $s_v = u$ for all $v \in [j]$ and $s_{j+1} \neq u$. If $s_1 \neq u$ then the initial rise of s is the empty step, *i.e.*, the path is either empty or it starts with a horizontal step. We will denote the length of the initial rise of path s by $i(s)$. It is easy to see that $i(s) \geq |s|_d$.

Let $M^k(x,y,z)$ be the generating function for k-coloured Motzkin paths and ${}^cM^k(x,y,z)$ be the generating function for k-coloured c-Motzkin paths according to length, l, number of u steps, r, and length of the initial rise, i.
Thus,

$$M^k(x,y,z) = \sum_{s \in \mathbb{M}^k} x^{l(s)} y^{r(s)} z^{i(s)} \quad \text{and} \quad {}^cM^k(x,y,z) = \sum_{s \in {}^c\mathbb{M}^k} x^{l(s)} y^{r(s)} z^{i(s)}.$$

Proposition 4 *The generating functions for $M^k(x,y,z)$ and ${}^cM^k(x,y,z)$ are given by*

$$M^k(x,y,z) = \frac{1 + kxM^k(x,y)}{1 - x^2yzM^k(x,y)}, \tag{6.1.1}$$

and

$${}^cM^k(x,y,z) = \frac{1}{1 - x^2yzM^k(x,y)}. \tag{6.1.2}$$

Proof 21 *For* (6.1.1)*, there are two cases we need to consider, when $k = 0$ and secondly, when $k \neq 0$.*
i) In this case, any nonempty path $s \in \mathbb{M}^0$ can be uniquely decomposed into the form $s = uwdv$, where $w, v \in \mathbb{M}^0$. From this decomposition, we find the following relationships:

$$l(s) = l(w) + 2 + l(v)$$

$$\begin{aligned} i(s) &= 1 + i(w) \\ r(s) &= 1 + r(w) + r(v). \end{aligned}$$

Thus,

$$\begin{aligned} M^k(x,y,z) &= 1 + \sum_{s \in \mathbb{M}^k} x^{l(s)} y^{r(s)} z^{i(s)} \\ &= 1 + \sum_{w,v \in \mathbb{M}^k} x^{l(w)+2+l(v)} y^{1+r(w)+r(v)} z^{1+i(w)} \\ &= 1 + x^2yz \sum_{w \in \mathbb{M}^k} x^{l(w)} y^{r(w)} z^{i(w)} \sum_{v \in \mathbb{M}^k} x^{l(v)} y^{r(v)} \\ &= 1 + x^2yzM^k(x,y,z)M^k(x,y) \\ &= \frac{1}{1 - x^2yzM^k(x,y)}. \end{aligned}$$

ii) In this case, every nonempty $s \in \mathbb{M}^k$ can be uniquely decomposed into either $s = \mu_t w$, where $w \in \mathbb{M}^k$ and μ_t can be any of the k-coloured horizontal steps, i.e., $\mu_t \in S \setminus \{u, d\}$, or $s = uwdv$, where $w, v \in \mathbb{M}^k$.
From the first decomposition we get,

$$\begin{aligned} l(s) &= l(w) + 1 \\ i(s) &= 0 \\ r(s) &= r(w), \end{aligned}$$

and from the second, we get,

$$\begin{aligned} l(s) &= l(w) + 2 + l(v) \\ i(s) &= 1 + i(w) \\ r(s) &= 1 + r(w) + r(v). \end{aligned}$$

Combining these two decompositions into one generating function gives,

$$M^k(x,y,z) = 1 + \sum_{t=1}^{k} \sum_{w \in \mathbb{M}^k} x^{l(\mu_t w)} y^{r(w)} + \sum_{w,v \in \mathbb{M}^k} x^{l(w)+2+l(v)} y^{1+r(w)+r(v)} z^{1+i(w)}$$

$$
\begin{aligned}
&= 1 + x\sum_{t=1}^{k}\sum_{w\in\mathbb{M}^k} x^{l(w)}y^{r(w)} + x^2yz\sum_{w\in\mathbb{M}^k} x^{l(w)}y^{r(w)}z^{i(w)}\sum_{v\in\mathbb{M}^k} x^{l(v)}y^{r(v)} \\
&= 1 + xkM^k(x,y) + x^2yzM^k(x,y,z)M^k(x,y) \\
&= \frac{1+xkM^k(x,y)}{1-x^2yzM^k(x,y)}.
\end{aligned}
$$

This proves the first part of the proposition. Now for the second part, we have, every nonempty $s \in {}^c\mathbb{M}^k$ can be uniquely decomposed into $s = uwdv$, where $w \in \mathbb{M}^k$ and $v \in {}^c\mathbb{M}^k$.

From this decomposition, we find the following relationships:

$$
\begin{aligned}
l(s) &= l(w) + 2 + l(v) \\
i(s) &= 1 + i(w) \\
r(s) &= 1 + r(w) + r(v).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^cM^k(x,y,z) &= 1 + \sum_{s\in {}^c\mathbb{M}^k} x^{l(s)}y^{r(s)}z^{i(s)} \\
&= 1 + \sum_{v\in {}^c\mathbb{M}^k}\sum_{w\in\mathbb{M}^k} x^{l(w)+2+l(v)}y^{1+r(w)+r(v)}z^{1+i(w)} \\
&= 1 + x^2yz\sum_{w\in\mathbb{M}^k} x^{l(w)}y^{r(w)}z^{i(w)}\sum_{v\in {}^c\mathbb{M}^k} x^{l(v)}y^{r(v)} \\
&= 1 + x^2yzM^k(x,y,z)\,{}^cM^k(x,y), \qquad (6.1.3)
\end{aligned}
$$

substituting $z = 1$, we get,

$$
{}^cM^k(x,y) = \frac{1}{1-x^2yM^k(x,y,z)}.
$$

We then substitute our equation above and (6.1.1) *into* (6.1.3) *to get*

$$
\begin{aligned}
{}^cM^k(x,y,z) &= 1 + x^2yzM^k(x,y,z)\frac{1}{1-x^2yM^k(x,y)} \\
&= \frac{1-x^2y\left(M^k(x,y)-zM^k(x,y,z)\right)}{1-x^2yM^k(x,y)}
\end{aligned}
$$

$$= \frac{1 - x^2y\left(M^k(x,y) - z\dfrac{1 + xkM^k(x,y)}{1 - x^2yzM^k(x,y)}\right)}{1 - x^2yM^k(x,y)}$$

$$= \frac{1 - x^2y\dfrac{\left(M^k(x,y) - zx^2y\left(M^k(x,y)\right)^2 - z - zkxM^k(x,y)\right)}{1 - x^2yzM^k(x,y)}}{1 - x^2yM^k(x,y)}, \tag{6.1.4}$$

after making the substitution $z = 1$ *and cross multiplying* (6.1.1)*, we find that it has the quadratic form given by,*

$$x^2y\left(M^k(x,y)\right)^2 - (1 - xk)M^k(x,y) + 1 = 0, \tag{6.1.5}$$

substituting this equation into (6.1.4) *gives*

$$^cM^k(x,y,z) = \frac{1 - x^2y\dfrac{\left(M^k(x,y) - z((1 - xk)M^k(x,y,z) - 1) - z - zkxM^k(x,y)\right)}{1 - x^2yzM^k(x,y)}}{1 - x^2yM^k(x,y)}$$

$$= \frac{1 - x^2y\dfrac{M^k(x,y) - zM^k(x,y)}{1 - x^2yzM^k(x,y)}}{1 - x^2yM^k(x,y)}$$

$$= \frac{1}{1 - x^2yzM^k(x,y)},$$

which is what we wanted to show.

Solving the quadratic formula, (6.1.5), and discarding the spurious solution, we get,

$$M^k(x,y) = \frac{1 - xk - \sqrt{(1 - xk)^2 - 4x^2y}}{2x^2y}. \tag{6.1.6}$$

In the above formula, making the substitutions $k = 0$ and $x = 1$, we correctly get the generating function for Dyck paths according to semilength, coded by y. Thus,

$$M^k(y) = \frac{1 - \sqrt{1 - 4y}}{2y}.$$

If we make the substitutions $k = 1$ and $y = 1$, we correctly get the generating function for Motzkin paths according to length, *i.e.*,

$$M^1(x) = \frac{1 - x - \sqrt{(1-x)^2 - 4x^2}}{2x^2}.$$

If we make the substitutions $k = 2$ and $y = 1$, we correctly get the generating function for Motzkin paths according to length, *i.e.*,

$$\begin{aligned} M^2(x) &= \frac{1 - 2x - \sqrt{(1-2x)^2 - 4x^2}}{2x^2} \\ &= \frac{1 - 2x - \sqrt{1-4x}}{2x^2} \\ &= \frac{-1}{x} + \frac{D(x)}{x}. \end{aligned}$$

Thus,

$$[x^n]M^2(x) = [x^{n+1}](D(x) - 1) = C_{n+1}.$$

This shows us that the number of Motzkin paths of length n, whose horizontal steps are coloured by means of two colours, is equal to the number of Dyck paths with $n+1$ up steps.
Now, from above, we see that

$$M^2(x) = \frac{D(x) - 1}{x} = D^2(x) = C^2(x),$$

which leads to,

$${}^cM^2(x) = \frac{1}{1 - x^2 C^2(x)}.$$

Thus, the generating function for 2-coloured c-Motzkin paths is equal to the generating function for Fine numbers, (2.4.3). So, we conclude that the number of 2-coloured c-Motzkin paths of length n is equal to the nth fine number, f_n.
Let M_n^k be the number of k-coloured Motzkin paths of length n and ${}^cM_n^k$ be the number of k-coloured c-Motzkin paths of length n.

Proposition 5 *For every $k, h, n, r \in \mathbb{N}$ with $r \leq \lfloor \frac{n}{2} \rfloor$, we have*

$$m_{n,r}^{k+h} = \sum_{j=2r}^{n} \binom{n}{j} m_{j,r}^{k} h^{n-j} = \sum_{j=2r}^{n} \binom{n}{j} m_{j,r}^{h} k^{n-j},$$

and

$$m_n^{k+h} = \sum_{j=0}^{n} \binom{n}{j} m_j^k h^{n-j} = \sum_{j=0}^{n} \binom{n}{j} m_j^h k^{n-j}.$$

Proof 22 *Considering* (6.1.6), *we find that for every* $k, h \in \mathbb{N}$,

$$\begin{aligned}
M^{k+h}(x,y) &= \frac{1-(k+h)x-\sqrt{(1-(k+h)x)^2-4x^2y}}{2x^2y} \\
&= \frac{1-xh-xk-(1-xh)\sqrt{1-\dfrac{2xk}{1-xh}+\dfrac{x^2k^2-4x^2y}{(1-xh)^2}}}{2x^2y} \\
&= (1-xh)\frac{1-\dfrac{x}{1-xh}k-\sqrt{\left(1-k\left(\dfrac{x}{1-xh}\right)\right)^2-4\left(\dfrac{x}{1-xh}\right)^2 y}}{2\left(\dfrac{x}{1-xh}\right)^2 y(1-xh)^2} \\
&= \frac{M^k\left(\dfrac{x}{1-xh}, y\right)}{1-xh}.
\end{aligned}$$

Similarly

$$M^{k+h}(x,y) = \frac{M^h\left(\dfrac{x}{1-xk}, y\right)}{1-xk}.$$

Now, we have

$$\begin{aligned}
\frac{M^k\left(\dfrac{x}{1-xh}, y\right)}{1-xh} &= \frac{1}{1-xh}\sum_{s\in\mathbb{M}^k}\left(\frac{x}{1-xh}\right)^{l(s)} y^{r(s)} \\
&= \sum_{j\geq 0}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor} m_{j,r}^k x^j y^r \frac{1}{(1-xh)^{j+1}} \\
&= \sum_{j\geq 0}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor} m_{j,r}^k x^j y^r \sum_{t\geq 0}\binom{-j-1}{t}(-xh)^t \\
&= \sum_{j\geq 0}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor}\sum_{t\geq 0} m_{j,r}^k \binom{j+t}{j} x^{j+t} y^r h^t \\
&= \sum_{j\geq 0}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor}\sum_{n\geq j} m_{j,r}^k \binom{n}{j} x^n y^r h^{n-j} \quad \textit{substituting} \quad j+t=n
\end{aligned}$$

$$= \sum_{n\geq 0}\sum_{j=0}^{n}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor} m_{j,r}^{k}\binom{n}{j}x^n y^r h^{n-j}$$
$$= \sum_{n\geq 0}\sum_{r=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=2r}^{n} m_{j,r}^{k}\binom{n}{j}x^n y^r h^{n-j}.$$

So, extracting the coefficients of x and y we get,

$$m_{n,r}^{k+h} = \sum_{j=2r}^{n} m_{j,r}^{k}\binom{n}{j}h^{n-j} = \sum_{j=2r}^{n} m_{j,r}^{h}\binom{n}{j}k^{n-j},$$

since $m_{n,r}^{k+h} = m_{n,r}^{h+k}$. This proves the first part of the proposition. For the second part, we have,

$$\begin{aligned}
m_n^{k+h} &= \sum_{r=0}^{\lfloor \frac{n}{2}\rfloor} m_{n,r}^{k+h}\\
&= \sum_{r=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=2r}^{n} m_{j,r}^{k}\binom{n}{j}h^{n-j}\\
&= \sum_{j=2r}^{n}\binom{n}{j}h^{n-j}m_j^k\\
&= \sum_{j=2r}^{n}\binom{n}{j}k^{n-j}m_j^h \quad \textit{since} \quad m_n^{k+h} = m_n^{h+k}.
\end{aligned}$$

Now, we know

$$m_{j,r}^{0} = \begin{cases} C_r & \text{if } j = 2r\\ 0 & \text{if } j \neq 2r\end{cases}$$

and

$$m_j^0 = \begin{cases} C_{\lfloor \frac{j}{2}\rfloor} & \text{if } j \text{ is even}\\ 0 & \text{if } j \text{ is odd.}\end{cases}$$

So letting $h = 0$ in our equation above, we get

$$m_{n,r}^{k} = \sum_{j=2r}^{n}\binom{n}{j}m_{j,r}^{0}k^{n-j}$$

$$= \binom{n}{2r} C_r k^{n-2r} + \sum_{j>2r}^{n} \binom{n}{j} m_{j,r}^0 k^{n-j}$$
$$= \frac{n!}{(2r)!(n-2r)!} \frac{(2r)!}{(r!)^2} \frac{k^{n-2r}}{r+1}$$
$$= \frac{k^{n-2r}}{n+1} \binom{n+1}{r+1, r, n-2r}.$$

Now, noting that for $k = 1$, then $m_{n,r}^1$ counts the number of Motzkin paths of length n with r up steps, we have,

$$m_{n,r}^1 = \binom{n}{2r} C_r \quad \text{and} \quad m_n^1 = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} C_r,$$

thus giving us a relationship between the enumeration of Motzkin paths and the enumeration of Dyck paths.

6.2 Powers of generating functions for Motzkin paths

We now turn our attention to powers of the generating functions for k-coloured Motzkin paths and k-coloured c-Motzkin paths

Proposition 6 *The coefficients of $(M^k(x,y))^s$, with $s \in \mathbb{N}$, are given by the formula*

$$[x^n y^r](M^k(x,y))^s = \frac{k^{n-2r} s}{n+s} \binom{n+s}{s+r, r, n-2r}$$

where, $n, r \in \mathbb{N}$ and $r \geq \lfloor \frac{n}{2} \rfloor$.

Proof 23 *Define the function $S(x) = xM^k(x,y)$. From* (6.1.5)*, we know $M^k(x,y) = x^2 y (M^k(x,y))^2 + xkM^k(x,y) + 1$. By making use of our definition, this shows that*

$$S(x) = x[yS^2(x) + kS(x) + 1].$$

Thus, $S(x) = x\Phi(S(x))$ and we can apply LIF (1)*, with $A(z) = S(z)$, $H(\lambda) = (y\lambda^2 + k\lambda + 1)$ and $G(z) = z^s$. We start by finding the coefficient of $x^n y^r$ in $S^s(x)$.*

By applying LIF we have

$$\begin{aligned}
[x^n]S^s(x) &= \frac{1}{n}[\lambda^{n-1}]s\lambda^{s-1}(y\lambda^2+k\lambda+1)^n \\
&= \frac{s}{n}[\lambda^{n-s}]\sum_{i=0}^{n}\binom{n}{i}\lambda^i(y\lambda+k)^i \\
&= \frac{s}{n}[\lambda^{n-s}]\sum_{i=0}^{n}\sum_{u=0}^{i}\binom{n}{i}\binom{i}{u}\lambda^{i+u}y^uk^{i-u} \\
&= \frac{s}{n}[\lambda^{n-s}]\sum_{u=0}^{n}\sum_{i=u}^{n}\binom{n}{i}\binom{i}{u}\lambda^{i+u}y^uk^{i-u} \\
&= \frac{s}{n}[\lambda^{n-s}]\sum_{u=0}^{n}\sum_{m=2u}^{2n}\binom{n}{m-u}\binom{m-u}{u}\lambda^m y^u k^{m-2u} \quad \textit{where} \quad i+u=m \\
&= \frac{s}{n}[\lambda^{n-s}]\sum_{m=0}^{2n}\sum_{u=0}^{\lfloor\frac{m}{2}\rfloor}\binom{n}{m-u}\binom{m-u}{u}\lambda^m y^u k^{m-2u} \\
&= \frac{s}{n}\sum_{u=0}^{\lfloor\frac{n-s}{2}\rfloor}\binom{n}{s+u,u,n-s-2u}y^uk^{n-s-2u},
\end{aligned}$$

with $n \geq m-u$, *which implies that* $r \geq max(0, n-m)$. *Now, for the desired coefficient,*

$$\begin{aligned}
[x^ny^r](M^k(x,y))^s &= [x^ny^r]x^{-s}S^s(x) \\
&= [y^r]\frac{s}{n+s}\sum_{u=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+s}{s+u,u,n-2u}y^uk^{n-2u} \\
&= \frac{k^{n-2r}s}{n+s}\binom{n+s}{s+r,r,n-2r}.
\end{aligned}$$

When $s=1$ we get $m_{n,r}^k$ as expected. To find the coefficients of $({}^cM^k(x,y))^s$ we need to make use of (6.1.2) and the proposition above as can be seen below,

$$\begin{aligned}
[x^ny^r]({}^cM^k(x,y))^s &= [x^ny^r]\left(\frac{1}{1-x^2yM^k(x,y)}\right)^s \quad \text{from} \quad (6.1.2) \\
&= [x^ny^r]\sum_{i\geq 0}\binom{s+i-1}{i}x^{2i}y^i(M^k(x,y))^i \\
&= [x^ny^r]\sum_{i\geq 0}\binom{s+i-1}{i}x^{2i}y^i\sum_{a\geq 0}\sum_{u\geq 0}\frac{i}{a+i}\binom{a-i}{i+u,u,a-2u}k^{a-2u}x^ay^u
\end{aligned}$$

$$= \sum_{i\geq 0} \binom{s+i-1}{i} \frac{i}{n-i} \binom{n-i}{r, r-i, n-2r} k^{n-2r},$$

with $i \leq r$ and $2r \leq n$.

Proposition 7 *The number of all k-coloured C-Motzkin with initial rise of length i is*

$$[x^n y^r z^i]\ {}^c M^k(x,y) = \frac{k^{n-2r} i}{n-i} \binom{n-i}{r, r-i, n-2r},$$

where $1 \leq s \leq r \leq \lfloor \frac{n}{2} \rfloor$

Proof 24

$$\begin{aligned}
[x^n y^r z^i]\ {}^c M^k(x,y,z) &= [x^n y^r z^i] \sum_{u\geq 0} x^{2u} y^u z^u (M^k(x,y))^u \quad \textit{from} \ \ 6.1.2 \\
&= [x^n y^r z^i] \sum_{u\geq 0} x^{2u} y^u z^u \sum_{a\geq 0} \sum_{v\geq 0} \frac{u}{a+u} \binom{a-u}{v+u, v, a-2v} k^{a-2v} x^a y^v,
\end{aligned}$$

making the substitutions $u = i, r = v + i, n = a + 2i$ *with* $i > 0, n - i \geq 2r - i$ *and* $n - i \geq n - r$ *gives the result.*

Now, extending our study to all Motzkin paths, we get that the number of all Motzkin paths of length n, r up steps and with initial rise of length i is

$$\begin{aligned}
[x^n y^r z^i] M^k(x,y,z) &= [x^n y^r z^i] \left({}^c M^k(x,y,z) + kx\ {}^c M^k(x,y,z) M^k(x,y) \right) \\
&= [x^n y^r z^i] \left(\sum_{u\geq 0} x^{2u} y^u z^u (M^k(x,y))^u + kx \sum_{u\geq 0} x^{2u} y^u z^u (M^k(x,y))^{u+1} \right) \\
&= \frac{k^{n-2r} i}{n-i} \binom{n-i}{r, r-i, n-2r} + k[x^{n-2i-1} y^{r-i}](M^k(x,y))^{i+1} \\
&= \left(\frac{k^{n-2r} i(r+1)}{(n-i)(n-i+1)} \right) \left(\frac{(n-i+1)!}{(r+1)!(r-i)!(n-2r)!} \right) \\
&\quad + \left(\frac{k^{n-2r} (i+1)(n-2r)}{(n-i)(n-i+1)} \right) \left(\frac{(n-i+1)!}{(r+1)!(r-i)!(n-2r)!} \right) \\
&= \frac{k^{n-2r} (ni - ri + i + n - 2r)}{(n-i)(n-i+1)} \binom{n-i+1}{r+1, r-i, n-2r}.
\end{aligned}$$

When $n = 2r$ then the result becomes

$$\frac{i}{2r-i}\binom{2r-i}{r-i},$$

which is equal to the number of Dyck paths of semilength r with height of first peak i as expected. This is because if the length of our Motzkin path is $n = 2r$, then it would be a Dyck path.

Chapter 7

CONCLUSION

This thesis studies various properties and parameters in Dyck and Motzkin paths and shows their relationship to other combinatorial objects such as permutations. This was done using the methodology of generating functions and extraction of coefficients using various techniques. For further reading on one of these topics, one can consult the papers [1], [2], [4], [5], [6], [8], [9], [11], [12], [13], [14], [15], [17], [18].

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