Symmetries and Conservation Laws of Higher-order PDEs

R. B. Narain

A thesis submitted to the School of Mathematics,
Faculty of Science,
University of the Witwatersrand,
in fulfilment of the requirements for the degree of
Doctor of Philosophy.

Johannesburg, 2011.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It is been submitted for the degree of Doctor of Philosophy. It has not been submitted before for any degree or examination to any other institution.

R. B. Narain

DEDICATION

To my loving parents,
brother and sister
and to my mentor, Prof A.H. Kara.

ACKNOWLEDGEMENTS

I am highly appreciative to my supervisor Prof. A. H. Kara for his time, encouragement and support. Prof. Kara introduced me to the field of Lie Groups and symmetries. He has not only been my supervisor but also a father figure and a friend of invaluable advice.

My further acknowledgement goes to the School of Mathematics for the support. I would also like to thank the School of Computational and Applied Mathematics, Prof D. Mason, Prof M. Ali and Prof. F.M. Mahomed. My journey thus far would not be possible without the assistance of Prof. D. Sherwell who afforded me the opportunity to study at Wits.

My gratitude goes to my dear parents, brother and sister, thank you for your support through my studies most importantly for always believing in my potential.

Lastly my journey thus far would not be successful without my faith in the Almighty.

ABSTRACT

The construction of conserved vectors using Noether's theorem via a knowledge of a Lagrangian (or via the recently developed concept of partial Lagrangians) is well known. The formulae to determine these for higher-order flows is somewhat cumbersome and becomes more so as the order increases. We carry out these for a class of fourth, fifth and sixth order PDEs. In the latter case, we involve the fifth-order KdV equation using the concept of 'weak' Lagrangians better known for the third-order KdV case.

We then consider the case of a mixed 'high-order' equations working on the Shallow Water Wave and Regularized Long Wave equations. These mixed type equations have not been dealt with thus far using this technique. The construction of conserved vectors using Noether's theorem via a knowledge of a Lagrangian is well known.

In some of the examples, our focus is that the resultant conserved flows display some previously unknown interesting 'divergence properties' owing to the presence of the mixed derivatives.

We then analyse the conserved flows of some multi-variable equations that arise in Relativity. In addition to a larger class of conservation laws than those given by the isometries or Killing vectors, we may conclude what the isometries are and that these form a Lie subalgebra of the Noether symmetry algebra. We perform our analysis on versions of the Vaidya metric yielding some previously unknown information regarding the corresponding manifold. Lastly, with particular reference to this metric, we also show the variations that occur for the unknown functions.

We discuss symmetries of classes of wave equations that arise as a consequence of the Vaidya metric. The objective of this study is to show how the respective geometry is responsible for giving rise to a nonlinear inhomogeneous wave equation as an alternative to assuming the existence of nonlinearities in the wave equation due to physical considerations. We find Lie and Noether point symmetries of the corresponding wave equations and give some reductions. Some interesting physical

conclusions relating to conservation laws such as energy, linear and angular momenta are also determined. We also present some interesting comparisons with the standard wave equations (on a 'flat geometry').

Finally, we pursue the nature of the flow of a third grade fluid with regard to its underlying conservation laws. In particular, the fluid occupying the space over a wall is considered. At the surface of the wall, suction or blowing velocity is applied. By introducing a velocity field, the governing equations are reduced to a class of PDEs. A complete class of conservation laws for the resulting equations are constructed and analysed using the invariance properties of the corresponding multipliers/characteristics.

Contents

Introduction

Marius Sophus Lie was one of the first prominent Norwegian scientists and among the last of the great 19th-century mathematicians. His main contribution was the theory of continuous groups of transformations. Lie produced his finest work in collaboration with Felix Klein and later Friedrich Engel.

Lie worked on transformation groups, which he called finite continuous groups. These groups, later called Lie groups, possessed a fixed number of parameters but could be differentiated in any desired order. Lie applied his theory of transformation groups to show that a majority of the known methods of integration could be introduced all together by means of group theory. He also used transformation groups to help classify ordinary differential equations and to give a unified method of solution using group-theoretic considerations.

In 1884, Lie teamed up with Friedrich Engel, a student recommended by Klein and Mayer, and together they produced a three-volume work on transformation groups, Theorie der Transformationsgruppen published between 1888 and 1893.

Lie's influence continued well after his death, with mathematicians like Wilhelm Killing who continued to work on Lie groups, which Elie Joseph Cartan later revised and based much of his work on and Hermann Weyl contribution into Lie's groups in his papers from 1922 and 1923, and subsequent generalizations of Lie groups gave them a greater role in new fields like quantum physics, particle physics and quantum mechanics.

The notion of symmetries entered the area of conservation law in variational equations through the work of Emmy Noether. Noether was born on the 23 March 1882 in Erlangen, Germany.

Noether received a Ph.D. degree from the University of Erlangen in 1907, with a dissertation on algebraic invariants, under the supervision of Paul Gordan. She worked at the Mathematical Institute of Erlangen. In 1915 she was invited by David Hilbert and Felix Klein to join the mathematics department at the University of Gottingen. While in Gottingen, she was approached by David Hilbert to solve the problem of failure of energy conservation in relativity.

In the paper, Invariante Varlationsprobleme, Noether proved two theorems, known as Noether's theorem [1], which revealed the fundamental connection between symmetries and conservation laws in physics. This led to a deeper understanding of laws such as the principles of conservation of energy, angular momentum, etc. The importance of Noether's theorem in calculus of variations is that the differential equations with variational structures are physical models and their variational symmetries generate conservation laws.

Noether's theorem provides elegant formulae for the construction of conservation laws for Euler-Lagrange equations once the Noether symmetries are known. A Lagrangian of the differential equation is required in order to use the theorem. The central problem in calculus of variations is the determination of a Lagrangian, so that the differential equation is then regarded as the Euler-Lagrange equation. This is regarded as the inverse problem in the calculus of variations, [2, 3].

Applications of Noether's theorem have yielded numerous results in literature, for example, [4, 5, 6]. There are also methods which provide conserved vectors without making use of a Lagrangian. The direct method by [7, 8], which is used to construct conserved quantities. A recent method for constructing conserved vectors without the use of a Lagrangian was provided by [9]. A more systematic way of constructing conserved quantities without the existence of Lagrangians was introduced by [10]. The introduction of partial Lagrangians, partial Euler-Lagrange equations and a modified Noether theorem known as partial Noether theorem, was presented for differential equations which has the same structure as Noether's theorem. Conservation laws are also constructed by utilization of the partial Noether approach.

We discuss the use of these variational techniques on higher-order PDEs. The importance of investigating these sorts of equations, are due to their appearance in different branches of science and engineering, like plasma physics, fluid dynamics, quantum theory, nonlinear optics, solid state physics, relativity and financial mathematics.

Brief Outline of the Chapters

In the first chapter, we introduce the preliminary mathematics that is needed to tackle our investigation. We introduce the concepts of Noether symmetries, Noether operators, Euler-Lagrange operator, Euler-Lagrange equations and conserved quantities.

In the second chapter, we discuss the role of this technique in attaining conservation laws for the fifth-order KdV, and fourth-order Boussinesq equations. For these and any high order PDEs, finding conservation laws by first principles can be extremely tedious. The important point of consideration is the cumbersome formulae that are required due to the order of the Lagrangians and related functions.

In the third chapter, we consider higher order mixed derivatives and their conservation laws using this technique. The Shallow Water Wave and Regularized Long Wave Equations are examined due to their highest derivative term being mixed. These sorts of equations have not been studied before using this technique.

In the fourth chapter, we consider the Vaidya metric that is currently being researched intensively in relativity and astrophysics. We show that a large amount information can be extracted from a knowledge of the vector fields that leave the action integral invariant, viz., Noether symmetries. In addition to a larger class of conservation laws than those given by the isometries or Killing vectors, we may conclude what the isometries are and that these form a Lie subalgebra of the Noether symmetry algebra.

In the fifth chapter, a special case of the Vaidya metric known as the Papapetrou model is discussed. A detailed symmetry analysis and invariance study associated with the Petrov III metric is also carried out.

In the sixth chapter, we consider the classical wave equation in some Lorentzian space-time backgrounds with the point in mind that the wave equation there may naturally inherit nonlinearity from geometry. In this study we look at the wave equation constructed from the Vaidya metric. The wave equation is constructed by using the D'Alembertian operator on the metric tensor. A detailed symmetry analysis is carried out on the wave equation leading to the construction of conserve quantities and higher order symmetries. We construct higher order symmetries using the recursion operators.

In the seventh chapter, the nature of the flow of a third grade fluid with regard to its underlying conservation laws is studied. In particular, the fluid covering a wall is considered. At the surface of the wall, suction or blowing velocity is applied. By introducing a velocity field, the governing equations are reduced to a class of partial differential equations (PDEs). A complete class of conservation laws for the resulting equations are constructed and analyzed using invariance properties of the corresponding multipliers/characteristics.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we introduce the preliminaries and results that are needed to tackle our investigation for higher-order PDEs. We introduce the concepts of Noether symmetries, Noether operators, Euler-Lagrange operator, Euler-Lagrange equations and conserved quantities.

1.2 Main Operators

We first introduce the reader to, the universal space A of differential functions. A locally analytic function $f(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)})$ of a finite number of variables is called a *differential function* of order k. The space A is the vector space of all differential functions of all finite orders and forms an algebra.

A total derivative converts any differential function of order k to a differential function of order $k + 1$. Hence, the space A is closed under total derivations. There are other operators on A and some of the important ones which we will utilize are explained below.

The summation convention is adopted throughout. Let $x = (x^1, \cdots, x^n)$ be the independent variable with co-ordinates x^i, and $u = (u^\alpha, \cdots, u^m)$ the dependent variable with co-ordinates u^α. The derivatives of the u with respect to x are

$$u_i{}^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_{ij}(u^\alpha), \quad \cdots, \tag{1.1}$$

where

$$D = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \cdots, n \tag{1.2}$$

is the *total differential operator.* The collection of all first derivatives $u_i{}^\alpha$ is denoted by $u_{(1)}$. Similarly, the collections of all higher-order derivatives are denoted by $u_{(2)}, u_{(3)}, \cdots$. Following Lie, in group analysis it is expedient to consider all variables $x, u, u_{(1)}, u_{(2)}, u_{(3)}, \cdots$ as functionally independent connected only by the differential relations (1.1). Consequently, the u^α are referred to as differential variables.

We denote by z the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \cdots) \tag{1.3}$$

with elements z^ν,$\nu \geq 1$, for example,

$z^i = x^i, \quad 1 \leq i \leq n, \quad z^{n+\alpha} = u^\alpha, \quad 1 \leq \alpha \leq m,$

with the remaining elements representing the derivatives of u. However, in application one invariably utilizes only infinite subsequences of z which are donated by $[z]$. A locally analytic function $f(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)})$ of a finite number of variables is called a differential function of order k and for brevity is written as $f([z])$. The space A is the vector space of all differential function of all finite orders. A total derivative (1.2) converts any differential function of order k to a differential function of order $k+1$.

Hence, the space A is closed under total derivations D_i. The main operators introduced below are correctly defined in the space A. More concisely, this means that the operators defined as formal sums truncate when they act on differential functions.

Definition 1: The *Euler-Lagrange operator* is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \cdots, m. \tag{1.4}$$

The operator (1.4) is sometimes referred to as the Euler operator, named after Euler (1744) who first introduced it in a geometrical manner for the one-dimensional case. Also, it is called the *Lagrange operator*, bearing the name of Lagrange (1762) who considered the multidimensional case and established its use in a variational sense (see for example, [11] for a history of the calculus of variations). Following Lagrange, equation (1.4) is frequently referred to as a variational derivative. In the modern literature, the terminology Euler-Lagrange and variational derivative are used interchangeably as (1.4) usually arises in considering a variational problem.

Definition 2: The *Lie-Bäcklund operator* is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} \quad \xi^i, u^\alpha \in A. \tag{1.5}$$

This operator is in fact an abbreviated form of the following infinite formal sum,

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i\frac{\partial}{\partial {u_i}^\alpha} + \zeta^\alpha_{ij}\frac{\partial}{\partial u^\alpha_{ij}} + \cdots, \tag{1.6}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}
\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\
\zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{i_1 i_2 j}^\alpha \\
&\cdots
\end{aligned} \tag{1.7}$$

In (1.7), W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.8}$$

One can write the Lie-Bäcklund operator (1.6) in form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_i(W^\alpha) \frac{\partial}{\partial u_i{}^\alpha} + D_{i_1} D_{i_2}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots. \tag{1.9}$$

Definition 3: The *Noether operator* associated with a Lie-Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i{}^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i i_1 \cdots i_s}^\alpha}, \quad i = 1, \cdots, n \tag{1.10}$$

where the Euler-Lagrange operator with respect to derivatives of u^α are obtained from (1.4) by replacing u^α by the corresponding derivatives, for example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s}(W^\alpha) \frac{\partial}{\partial u_{i j_1 \cdots j_s}^\alpha}, \quad i = 1, \cdots, n, \quad \alpha = 1, \cdots, m. \tag{1.11}$$

The operator (1.10) is named the Noether operator. As a consequence of the operator (1.10), the proof of Noether's theorem becomes purely algebraic and independent of variational calculus. The algebraic proof is based on the identity presented in the next section.

1.3 Noether Identity

Theorem 1: The Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.12}$$

Here, $D_i(\xi^i)$ is a differential function which is a sum of functions obtained by total derivations D_i of differential functions ξ_i. That is, $D_i(\xi^i)$ is a divergence of the vector $\xi = (\xi^1, \cdots, \xi^n)$, in other words, $div\xi$ whereas, $D_i N^i$ is an operator obtained as a sum of products of operators D_i on N^i, that is, it is the scalar product of vector operators $D = (D_1, \cdots, D_n)$ and $N = (N^1, \cdots, N^n)$. The identity (1.12) is called the Noether identity because of its close relation to Noether's theorem.

1.4 Noether Generators

Consider a k^{th} order differential equation

$$E^\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m. \tag{1.13}$$

Definition 4: A conserved vector of (1.13) is tuple $T = (T^1, \cdots, T^n)$,
$T^j = T^j(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) \in A, \quad j = 1, ...n$, such that

$$D_i(T^i) = 0 \tag{1.14}$$

is satisfied for all solutions of (1.13).

REMARK. When Definition 4 is satisfied, (1.14) is called a conservation law for (1.13).

We now discuss conservation laws of Euler-Lagrange equations. That is, differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \cdots, m, \tag{1.15}$$

where $L = L(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(l)}) \in A$, $l \leq k$, k being the order of (1.15), are Lagrangians and $\frac{\delta}{\delta u^\alpha}$ is the Euler-Lagrange operator defined by (1.4).

Definition 5: A Lie-Bäcklund operator X of the form (1.6) is called a Noether symmetry corresponding to a Lagrangian $L \in A$ if there exists a vector $B = (B^1, \cdots, B^n)$, $B^i \in A$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{1.16}$$

If equation (1.16), $B^i = 0, \quad i = 1, \cdots, n$, then X is referred to as a strict Noether symmetry corresponding to a Lagrangian $L \in A$.

Theorem 2: For a any Noether symmetry X corresponding to a given Lagrangian $L \in A$, there corresponds a vector $T = (T^1, \cdots, T^n), \quad T^i \in A$, defined by

$$T^i = N^i(L) - B^i, \quad i = 1, \cdots, n, \tag{1.17}$$

which is a conserved vector of equation (1.15), that is, $D_i(T^i) = 0$ on the solutions of (1.15).

1.5 Noether-Type Generators

Consider a k^{th} order differential system

$$E^\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m \tag{1.18}$$

which is of maximal rank and locally solvable. The following definition is well-known.

REMARK. When Definition 4 is satisfied, (1.14) is called the local conservation law of (1.18). Also $D_i T^i = Q^\alpha E_\alpha$ is referred to as the characteristic form of conservation law (1.18) and the function $Q = (Q^1, \cdots, Q^n)$ the associated characteristic form of the conservation law.

Suppose that equations (1.18) are written as

$$E_\alpha \equiv E^0_\alpha + E^1_\alpha = 0, \quad \alpha = 1, \cdots, m. \tag{1.19}$$

We now introduce the definition of a Partial Lagrangian.

Definition 6: If there exists a function $L = L(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(l)}) \in A, l \leq k$ and non-zero functions $f^\beta_\alpha \in A$ such that (1.19) can be written as $\delta L/\delta u^\alpha = f^\beta_\alpha E^1_\beta$ then, provided $E^1_\beta \neq 0$, L is called a partial Lagrangian of equation (1.19) otherwise it is the standard Lagrangian. It is known that differential equations of the form $\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \cdots, m$, are Euler-Lagrange equations. We term differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = f^\beta_\alpha E^1_\beta, \tag{1.20}$$

as Euler-Lagrange-type equations.

Definition 7: A Lie-Bäcklund or generalized operator X of the form (1.6) is called a Noether-type symmetry operator corresponding to a partial Lagrangian $L \in A$ if there exists a vector $B = (B^1, \cdots, B^n), \quad B^i \in A, \quad B^i \neq N^i L + C^i, C^i$ constants, such that

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i), \tag{1.21}$$

where $W = (W^1, \cdots, W^m), W^\alpha \in A$ are characteristics of X.

If the B^i's are identically zero , then the Lie-Bäcklund operator X is called a *strict Noether-type symmetry operator.*

Note that for Euler-Lagrange equations $\delta L/\delta u^\alpha = 0$, if (1.21) is satisfied, X is a *Noether symmetry generator* corresponding to a standard Lagrangian L.

Recall also that for a Noether symmetry generator X corresponding to a standard L, X is said to leave the functional invariant up to gauge B. It is easy to see from (1.21) that if X and Y are Noether-type operators, then so is a linear combination of these operators. Indeed the Noether-type symmetry operators span a vector space.

Theorem 3: A Lie-Bäcklund symmetry operator X of the form (1.9) is a *Noether-type symmetry operator* of a partial Lagrangian L corresponding to an *Euler-Lagrange-type system* of the form (1.20) if and only if the characteristic $W = (W^1, \cdots, W^m), W^\alpha \in A$, of X is also the characteristic of the conservation law $D_i T^i = 0$, where

$$T^i = N^i(L) - B^i, \quad i = 1, \cdots, n, \tag{1.22}$$

of the Euler-Lagrange-type Equations (1.20).

Proof: We use identity (1.12) and act with it on L to obtain

$$XL + D_i(\xi^i)L = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i N^i L \tag{1.23}$$

Since X is a Noether-type symmetry operator of an L corresponding to an Euler-Lagrange system, we can through the use of (1.21) replace the left hand side of the last equation (1.23) with $W^\alpha \delta L/\delta u^\alpha + D_i B^i$ which in turn can be replaced by $W^\alpha f^\beta_\alpha E^1_\beta + D_i B^i$ by utilizing (1.20). We immediately get

$$W^\alpha f^\beta_\alpha E^1_\beta + D_i B^i = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i N^i L. \tag{1.24}$$

From this we have

$$D_i(B^i - N^i L) = W^\alpha (\frac{\delta L}{\delta u^\alpha} - f^\beta_\alpha E^1_\beta) \tag{1.25}$$

and thus

$$D_iT^i = W^\alpha(\frac{\delta L}{\delta u^\alpha} - f_\alpha^\beta E_\beta^1) \qquad (1.26)$$

as a consequence of (1.21) is a conservation law with conserved components $T^i = B^i - N^i(L)$ of the system (1.20) with characteristic W. The steps are reversible. This proves the result. □

A further detailed analysis of the operators is completely given below for the scalar case in two dimensions, viz., (t, x). This discussion is peculiar to our work in the sequel as the Lagrangian and conserved flows are of a high order (third-order). The proofs and finer details of the results are obtainable in [12]. Suppose $X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u$ is a Noether point symmetry generator with gauge (f, g). Then the conserved flow (T^t, T^x) is given by

$$\begin{aligned}
T^t &= L\tau + W\tfrac{\delta L}{\delta u_t} + D_t(W)\tfrac{\delta L}{\delta u_{tt}} + D_x(W)\tfrac{\delta L}{\delta u_{tx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{tx}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{txx}} - f \\
&= L\tau + W(\tfrac{\partial L}{\partial u_t} - D_t\tfrac{\partial L}{\partial u_{tt}} - D_x\tfrac{\partial L}{\partial u_{tx}} + D_t^2\tfrac{\partial L}{\partial u_{ttt}} + D_x^2\tfrac{\partial L}{\partial u_{txx}} + D_tD_x\tfrac{\partial L}{\partial u_{ttx}}) \\
&\quad + D_t(W)\tfrac{\delta L}{\delta u_{tt}} + D_x(W)\tfrac{\delta L}{\delta u_{tx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{tx}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{txx}} - f, \\
T^x &= L\xi + W\tfrac{\delta L}{\delta u_x} + D_t(W)\tfrac{\delta L}{\delta u_{xt}} + D_x(W)\tfrac{\delta L}{\delta u_{xx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{xxt}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{xxx}} - g \\
&= L\xi + W(\tfrac{\partial L}{\partial u_x} - D_t\tfrac{\partial L}{\partial u_{xt}} - D_x\tfrac{\partial L}{\partial u_{xx}} + D_t^2\tfrac{\partial L}{\partial u_{xtt}} + D_x^2\tfrac{\partial L}{\partial u_{xxx}} + D_tD_x\tfrac{\partial L}{\partial u_{txx}}) \\
&\quad + D_t(W)\tfrac{\delta L}{\delta u_{xt}} + D_x(W)\tfrac{\delta L}{\delta u_{xx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{xxt}} - g,
\end{aligned} \qquad (1.27)$$

where

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_t^2\frac{\partial}{\partial v_{tt}} + D_x^2\frac{\partial}{\partial v_{xx}} + D_tD_x\frac{\partial}{\partial v_{tx}} - \dots. \qquad (1.28)$$

A range of literature pertaining to conservation laws is now available mainly presenting the various methods involved, see [13, 14, 15, 16, 17].

1.6 Illustrative example

For simplicity we have looked at point type symmetry operators and we have restricted the gauge terms to be independent of derivatives. One can equally well try to obtain true Lie-Bäcklund type symmetry operators and our method still applies. However, the calculations in this case are quite tedious and best left for a computer algebra package. The illustrative example in [12], is on the classical heat equation. Although simple, it is considered a paradigm for evolution equations and is frequently utilized as a benchmark for one's approach.

Consider the (1+1) linear heat equation

$$u_t = u_{xx}. \tag{1.29}$$

If we invoke the partial Lagrangian $L = u_x^2/2, \delta L/\delta u = -u_{xx}$ so that (1.29) can be written as $u_t = -\delta L/\delta u$ and, therefore, $\delta L/\delta u$ can be replaced by $-u_t$ in (1.29) to determine the Noether-type operators, by Definition 7, $X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$ corresponding to L. That is,

$$\zeta_x u_x + (D_t\tau + D_x\xi)(\frac{1}{2}u_x^2) = (\eta - \tau u_t - \xi u_x)(-u_t) + D_t B^1 + D_x B^2. \tag{1.30}$$

Expansion of the total derivative operators as well as ζ_x and then separation of the derivatives of u yield the over-determined linear system

$$
\begin{aligned}
u_x^3 &\ :\ \xi_u = 0,\\
u_x^2 &\ :\ \eta_u = 0,\\
u_x^2 u_t &\ :\ \tau_u = 0,\\
u_x u_t &\ :\ \xi = -\tau_x,\\
u_t^2 &\ :\ \tau = 0,\\
u_x &\ :\ \eta_x = B_u^2,\\
u_t &\ :\ \eta = B_u^1,\\
1 &\ :\ B_t^1 + B_x^2 = 0.
\end{aligned}
\tag{1.31}
$$

The calculations reveal that $X = \eta(t,x)\frac{\partial}{\partial u}$ where η satisfies the equation

$$\eta_t + \eta_{xx} = 0 \tag{1.32}$$

and $B^1 = \eta u + f(t,x), \quad B^2 = \eta_x u + g(t,x)$,where $f_t + g_x = 0$. We set $f = g = 0$. The corresponding conserved vector components, by Theorem 3, are $T^1 = \eta u$ and $T^2 = -\eta u_x + \eta_x u$. The corresponding conservation law $D_t T^1 + D_x T^2 = 0$ is $\eta(u_t - u_{xx}) = 0$ with characteristic η which is the characteristic of the Noether-type symmetry operator X.

Thus, if for example

(i)$\eta = 1, \quad T^1 = u, \quad T^2 = -u_x,$
(ii)$\eta = t - \frac{1}{2}x^2, \quad T^1 = (t - \frac{1}{2}x^2)u, \quad T^2 = -(t - \frac{1}{2}x)u_x - ux.$

Chapter 2

Conservation laws of Higher order nonlinear PDEs

2.1 Introduction

The fifth-order KdV, and fourth-order Boussinesq equations are well known examples from mathematical physics purported to be of 'high' order. For these and any high order PDEs, finding conservation laws by first principles can be extremely tedious. Thus, one needs to resort to alternate methods appealing to the underlying symmetry generators of the equations. If this means the variational route, then there may be problems such as the existence and determination of a Lagrangian. For the two cases cited here, we construct 'weak' or 'partial' Lagrangians and successfully construct conservation laws. The points to be emphasised is how cumbersome the formulae required in the determination of the conserved flows due to the order of the Lagrangians and related functions [18].

2.2 The fifth-order KdV equation

The particular case that we investigate is the well known generalized fifth-order KdV, (also known as the KdV-5 equation)

$$v_{xxxxx} + \alpha v_x v_{xx} + \frac{\beta}{2} v v_{xxx} + \gamma v^2 v_x + v_t = 0, \tag{2.1}$$

where α, β, γ are arbitrary non-zero constants.

For a variety of combinations of the parameters, (2.1) has been studied using a number of methods, analytical and numerical. Inc [19] and Abbasandy & Zakaria [20] made a detailed numerical study using the Adomian decomposition and homotopy analysis methods, respectively. Several works on the soliton solutions and various analytical methods have been done, for e.g. Lax [21] $(\beta/2 = 10, \alpha = 20, \gamma = 30)$, Sawada-Kotera [22] $(\beta/2 = 5, \alpha = 5, \gamma = 5)$, Ito [23] $(\beta/2 = 3, \alpha = 6, \gamma = 2)$. The well known Kaup-Kuperschmidt equation is based on the case $\beta/2 = -15, \alpha = -15, \gamma = 45$. It can be shown that the equation is Hamiltonian for $\beta = 2\alpha$ on the principle $v_t = D_x(\delta\mathcal{H})$, where $\mathcal{H} = -\int(\alpha u u_{xx} + \alpha/2u_x^2 + \gamma/(12)u^4 + 1/2u_{xx}^2)\mathrm{d}x$.

In this section we determined the Lie algebra of point symmetries of equation (2.1) is given by $\alpha\beta\gamma \neq 0$. This result can be compared to the results obtained in the section that follows,

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_x, \\ X_3 &= -2\partial_u + 5t\partial_t + x\partial_x. \end{aligned} \tag{2.2}$$

The commutator table for the symmetries of equation (2.1),

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3
X_1	0	0	$5X_1$
X_2	0	0	X_2
X_3	$-5X_1$	$-X_2$	0

(2.3)

The standard third-order KdV equation (KdV-3) is an evolution equation but its differential consequence admits a Lagrangian [24] and, thus, the KdV equation itself is construed as a variational equation. We show in section (2.2.1) that one can do this for (2.1) by which some interesting results regarding conservation laws via Noether's theorem are obtained.

2.2.1 The sixth-order expansion of the KdV-5 equation

This analogous study of the KdV-5 equation has not, to the knowledge of the author, been done before. This may be due to the cumbersome forms of the extended Euler-Lagrange operators that need to be used.

If equation (2.1) is differentiated by x or if $v = u_x$ in (2.1), we get the sixth-order equation

$$u_{xxxxxx} + \alpha u_{xx}u_{xxx} + \frac{\beta}{2}u_x u_{xxxx} + \gamma u_x^2 u_{xx} + u_{xt} = 0 \tag{2.4}$$

We firstly determined the Lie algebra of point symmetries of equation (2.4) for $\alpha\beta\gamma \neq 0$, which can be compared to the results obtained via the different methods presented, viz.,

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= F(t)\partial_u, whereF(t)isaarbitraryfunctionoft, \\ X_3 &= \partial_x, \\ X_4 &= 5t\partial_t - u\partial_u + x\partial_x. \end{aligned} \tag{2.5}$$

The commutator table for the symmetries of equation (2.4) is given by,

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	X_2	0	$5X_1$
X_2	$-X_2$	0	0	$-(F[t] + 5tF'[t])\partial_u$
X_3	0	0	0	X_3
X_4	$-5X_1$	$(F[t] + 5tF'[t])\partial_u$	$-X_3$	0

(2.6)

The equation in question has a partial Lagrangian

$$L = -[\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_x u_t + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}] \tag{2.7}$$

so that

$$\frac{\delta L}{\delta u} = u_{xt} + \frac{\beta}{2}u_x u_{xxxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx} = (\beta - \alpha)u_{xx}u_{xxx}. \tag{2.8}$$

Now applying the partial Lagrangian to the Noether-type Identity (1.21), the following expression is obtained

$$X^{[3]}(L) + L(D_t\tau + D_x\xi) = (\phi - u_t\tau - u_x\xi)\frac{\delta L}{\delta u} + (D_t f + D_x g), \tag{2.9}$$

where f and g are gauge functions. The governing equations are obtained from (2.9), by separating the equations by coefficients. These coefficients are derivatives of the dependent variable u.

The separation of monomials are listed as

$$
\begin{aligned}
u_t u_x^2 u_{xxx} &: \tau_u, \\
u_t u_x^2 u_{xx} u_{xxx} &: \tau_{uu}, \\
u_x^2 u_{xx} u_{xxx} &: \xi_{uu}, \\
u_x u_{xxx}^2 &: \xi_u, \\
u_x^3 u_{xxx} &: \phi_{uuu}, \\
u_{xxx} u_{xxt} &: \tau_x, \\
u_{xxx}^2 &: \tfrac{5}{2}\xi_x - \tfrac{1}{2}\tau_t - \phi_u, \\
u_x^2 u_{xxx} &: \tfrac{1}{2}[\tfrac{5\beta}{4}\xi_x - \tfrac{1}{4}\beta(\xi_x + \tau_t) - \tfrac{3\beta}{4}\phi_u - 6\phi_{xuu}], \\
u_x u_{xx} u_{xxx} &: (\beta - \alpha)\xi, \\
u_t u_{xx} u_{xxx} &: (\beta - \alpha)\tau, \\
u_{xx} u_{xxx} &: (\beta - \alpha)\phi, \\
u_x u_{xxx} &: -\tfrac{3}{4}\beta\phi_x\phi + \xi_{xxx} - 3\phi_{xxu}, \\
u_{xxx} &: -\phi_{xxx}, \\
u_x^3 u_{xx} &: -\tfrac{3}{8}\beta\phi_{uu}, \\
u_x^2 u_{xx} &: \tfrac{1}{2}(\tfrac{3}{4}\beta\xi_{xx} - \tfrac{3}{4}\beta\phi_{xu}), \\
u_x^4 &: \tfrac{9}{4}\xi_x - \tfrac{9}{12}\tau_t - \tfrac{9}{3}\phi_u, \\
u_x^3 &: -\tfrac{9}{3}\phi_u + \tfrac{1}{8}\beta\xi_{xxx}, \\
u_x^2 &: \xi_t, \\
u_t u_x &: \tfrac{1}{2}\xi_x + \tfrac{1}{2}(-\xi_x - \tau_t) + \tfrac{1}{2}\tau_t - \phi_u, \\
u_t &: -f_u - \tfrac{1}{2}\phi_x, \\
u_x &: -g_u - \tfrac{1}{2}\phi_t, \\
1 &: -f_t - g_x.
\end{aligned} \tag{2.10}
$$

From the governing equations (2.10), it can be observed that there are two cases appearing (i) $\alpha \neq \beta$ and (ii) $\alpha = \beta$.

In the case (i), we obtain no symmetry generators due to ξ, τ and ϕ being equal to zero, therefore producing trivial solutions.

The case (ii), leads to a nontrivial solution. That is, the partial Lagrangian is, in fact, a Lagrangian of (2.4) due to $\alpha = \beta$, where $\frac{\delta L}{\delta u} = (\beta - \alpha)u_{xx}u_{xxx} = 0$, which therefore changes the Noether-type Identity to Noether Identity,

$$X^{[3]}(L) + L(D_t\tau + D_x\xi) = (D_t f + D_x g). \tag{2.11}$$

The generators are the corresponding Noether symmetries, viz.,

$$X = \partial_t \quad (W = -u_t), \qquad X = \xi\partial_x \quad (W = -\xi u_x). \tag{2.12}$$

We now list the corresponding conserved vectors which are obtained from the given formula in the preliminaries.

(1) $X = \partial_t \quad (W = -u_t)$

$$\begin{aligned}
T^t &= -(\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_x u_t + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}) + (-u_t)(-\frac{1}{2}u_x), \\
&= -\frac{1}{2}u_{xxx}^2 - \frac{1}{2}u_x u_t - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx} + \frac{1}{2}u_x u_t, \\
&= -\frac{1}{2}u_{xxx}^2 - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx},
\end{aligned}$$

$$\begin{aligned}
T^x &= (-u_t)(-\frac{1}{2}u_t - \frac{\gamma}{3}u_x^3 - \frac{\beta}{4}u_x u_{xxx} + \mathrm{D}_x^2(-u_{xxx} - \frac{\beta}{8}u_x^2)) \\
&+ \mathrm{D}_x(-u_t)(-\mathrm{D}_x(-u_{xxx} - \frac{\beta}{8}u_x^2)) + \mathrm{D}_x^2(-u_t)(-u_{xxx} - \frac{\beta}{6}u_x^2), \\
&= (u_t)(\frac{1}{2}u_t + u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}) \\
&+ (-u_{tx})(u_{xxxx} + \frac{\beta}{4}u_x u_{xx}) + (u_{txx})(u_{xxx} + \frac{\beta}{8}u_x^2), \\
&= \frac{1}{2}u_t^2 + u_t u_{xxxxx} + \frac{\beta}{4}u_t u_{xx}^2 + \frac{\beta}{4}u_t u_x u_{xxx} - u_{tx}u_{xxxx} - \frac{\beta}{4}u_{tx}u_x u_{xx} \\
&+ u_{txx}u_{xxx} + \frac{\beta}{8}u_{txx}u_x^2.
\end{aligned}$$

Thus,

$$D_t T^t + D_x T^x = D_t(-\frac{1}{2}u_{xxx}^2 - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx})$$

$$
\begin{aligned}
&+ \quad D_x(\frac{1}{2}u_t^2 + u_t u_{xxxxx} + \frac{\beta}{4}u_t u_{xx}^2 + \frac{\beta}{4}u_t u_x u_{xxx}) \\
&+ \quad D_x(-u_{tx}u_{xxxx} - \frac{\beta}{4}u_{tx}u_x u_{xx} + u_{txx}u_{xxx} + \frac{\beta}{8}u_{txx}u_x^2), \\
&= \quad u_t(u_{xt} + \beta/2 u_x u_{xxxx} + \beta u_{xx}u_{xxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx}), \\
&= \quad 0.
\end{aligned}
$$

(2) $X = \xi \partial_x \quad (W = -\xi u_x)$

$$
\begin{aligned}
T^t &= -\xi u_x(-\frac{1}{2}u_x), \\
&= \frac{1}{2}\xi u_x^2, \\
T^x &= -\xi(\frac{1}{2}u_{xxx}^2 + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}) + \xi u_x(u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}) \\
&- \xi u_{xx}(u_{xxxx} + \frac{\beta}{4}u_x u_{xx}) + \xi u_{xxx}(u_{xxx} + \frac{\beta}{8}u_x^2) + \xi u_x(\frac{\gamma}{3}u_{xxx}^3 + \frac{\gamma}{4}u_x u_{xxx})
\end{aligned}
$$

Thus,

$$
\begin{aligned}
D_t T^t + D_x T^x &= D_t(\frac{1}{2}\xi u_x^2) + D_x(-\xi(\frac{1}{2}u_{xxx}^2 + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx})) \\
&+ D_x(\xi u_x(u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}) - \xi u_{xx}(u_{xxxx} + \frac{\beta}{4}u_x u_{xx})) \\
&+ D_x(\xi u_{xxx}(u_{xxx} + \frac{\beta}{8}u_x^2) + \xi u_x(\frac{\gamma}{3}u_{xxx}^3 + \frac{\gamma}{4}u_x u_{xxx})), \\
&= \xi u_x(u_{xt} + \beta/2 u_x u_{xxxx} + \beta u_{xx}u_{xxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx}), \\
&= 0.
\end{aligned}
$$

REMARK. The conserved vector in (1) is of 'nonlocal' type for the fifth-order KdV equation (2.1) when we substitute back to v since, if $v = u_x$, $u_t = \int v_t \mathrm{d}x$.

2.2.2 Traveling Wave reduction of the KdV-5 Equation

We now reduce the KdV-5 equation to a fourth-order equation by taking the sum of the symmetries from the previous result (2.12). By using the characteristic method of solving PDEs, we obtain $y = x - ct$, that is a traveling wave equation. By taking $v = w$, we find that $v_t = -cw'$ and $v_x = w'$. Then substituting the derivatives into the main equation we end up with

$$w''''' + \frac{\beta}{2}ww''' + \alpha w'w'' + \gamma w^2 w' - cw' = 0. \tag{2.13}$$

We then integrate Equation(2.13), which becomes

$$w'''' + \frac{\beta}{2}ww'' - \frac{\beta}{4}w'^2 + \frac{\alpha}{2}w'^2 + \frac{\gamma}{3}w^3 w' - cw = k, \tag{2.14}$$

which also can be written as

$$w'''' + \frac{\beta}{2}ww'' + (\frac{\alpha}{2} - \frac{\beta}{4})w'^2 + \frac{\gamma}{3}w^3 w' - cw = k, \tag{2.15}$$

for the study of different cases, which has partial Lagrangian

$$L = \frac{1}{2}w''^2 + (\frac{\alpha}{2} - \frac{\beta}{4})ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 - kw \tag{2.16}$$

which has

$$\frac{\delta L}{\delta w} = (\alpha - \beta)ww''. \tag{2.17}$$

(i) For $\alpha \neq \beta$, equation (2.15) has partial Lagrangian (2.16), applying the Noether-type Identity which is

$$X^{[2]}(L) + L(D_y\sigma) = (\eta - w'\sigma)\frac{\delta L}{\delta w} + (D_t f). \tag{2.18}$$

The separation of monomials are:

$$\begin{array}{lll}
w_y^3 w_{yy} & : & \sigma_{ww}, \\
w_y^2 w_{yy} & : & \eta_{ww} - 2\sigma_{yw}, \\
w_y w_{yy} & : & 2\eta_{yw} - \sigma_{yy} + w(\alpha - \beta), \\
w_y w_{yy}^2 & : & \sigma_w, \\
w_{yy}^2 & : & \eta_w - \frac{3}{2}\sigma_y, \\
w_{yy} & : & \eta_{yy} - w\eta(\alpha - \beta), \\
w_y^2 & : & \eta(\frac{\alpha}{2} - \frac{\beta}{4}) + w\eta_w(\alpha - \frac{\beta}{2}) - w\sigma_y(\frac{\alpha}{2} - \frac{\beta}{4}), \\
w_y & : & f_w + (\alpha - \frac{\beta}{2})w\eta_y, \\
1 & : & -cw\eta + \frac{1}{3}w^3\gamma + \frac{\gamma}{12}w^4\sigma_y - \frac{c}{2}w^2\sigma_y - f_y.
\end{array} \tag{2.19}$$

We obtain no symmetry generators since σ and η are equal to zero - this leads to a trivial solution.

(ii) For $\alpha = \beta$, equation (2.15) transforms to

$$w'''' + \frac{\beta}{2}ww'' + \frac{\beta}{4}w'^2 + \frac{\gamma}{3}w^3w' - cw = k, \tag{2.20}$$

which has the partial Lagrangian

$$L = \frac{1}{2}w''^2 - \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 - kw. \tag{2.21}$$

The separation of monomials are:

$$\begin{array}{lll}
w_y^3 w_{yy} & : & \sigma_{ww}, \\
w_y^2 w_{yy} & : & \eta_{ww} - 2\sigma_{yw}, \\
w_y w_{yy} & : & 2\eta_{yw} - \sigma_{yy}, \\
w_y w_{yy}^2 & : & \sigma_w, \\
w_{yy}^2 & : & \eta_w - \frac{3}{2}\sigma_y, \\
w_{yy} & : & \eta_{yy}, \\
w_y^2 & : & -\frac{\beta}{4}\eta - \frac{1}{2}w\beta\eta_w + \frac{1}{4}w\beta\sigma_y, \\
w_y & : & -f_w - \frac{1}{2}w\beta\eta_y, \\
1 & : & -cw\eta + \frac{1}{3}w^3\gamma + \frac{\gamma}{12}w^4\sigma_y - \frac{c}{2}w^2\sigma_y - f_y.
\end{array} \tag{2.22}$$

This leads to a nontrivial solution, that is, the partial Lagrangian is, in fact, a Lagrangian of (2.14) and the generators are the corresponding Noether symmetries, viz.,

$$\partial_y \quad (W = w') \tag{2.23}$$

with the corresponding conserved vector,

$$\begin{aligned} T &= \frac{1}{2}w''^2 - \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 + \frac{\beta}{2}ww'^2 + w'w''' - w''^2, \\ &= -\frac{1}{2}w''^2 + \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 + w'w''', \end{aligned}$$

such that

$$\begin{aligned} D(T) &= -w''w''' + \frac{\beta}{4}(w'^3 + 2ww'w'') + \frac{\gamma}{3}w^3w' - cww' + w''w''' + w'w'''', \\ &= \frac{\beta}{4}(w'^3 + 2ww'w'') + \frac{\gamma}{3}w^3w' - cww' + w'w'''', \\ &= w'(w'''' + \frac{\beta}{4}w'^2 + \frac{\beta}{2}ww'' + \frac{\gamma}{3}w^3 - cw), \\ &= 0. \end{aligned}$$

2.3 The fourth-order Boussinesq equation

The Boussinesq equation which models the behaviour of long waves is sometimes written as the fourth-order equation

$$u_{xxxx} + uu_{xx} + u_x^2 + u_{tt} = 0. \tag{2.24}$$

The highest derivative in the equation (2.24) is a singular independent variable derivative term. Its Noether type symmetries, $X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u$, via the partial Lagrangian

$$L = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}uu_x^2, \tag{2.25}$$

which has

$$\frac{\delta L}{\delta u} = -\frac{1}{2}u_x^2 \tag{2.26}$$

is determined by (1.20). In this case, XL is a second prolongation of X, viz.,

$$\begin{aligned}
&XL = -\tfrac{1}{2}\phi u_x^2 + u_t u_x \xi_t + u_t^2 u_x \xi_u + uu_x^3\xi_u + uu_x^2\xi_x + \\
&u_t^2\tau_t + u_t^3\tau_u + uu_t u_x^2\tau_u + uu_t u_x\tau_x - u_t\phi_t - \\
&u_t^2\phi_u - uu_x^2\phi_u - uu_x\phi_x - 2u_x\tau_u u_{x,t}u_{x,x} - \\
&2\tau_x u_{x,t}u_{x,x} - 3u_x\xi_u u_{x,x}^2 - 2\xi_x u_{x,x}^2 - \\
&u_t\tau_u u_{x,x}^2 + \phi_u u_{x,x}^2 - u_x^3 u_{x,x}\xi_{u,u} - 2u_x^2 u_{x,x}\xi_{x,u} - \\
&u_x u_{x,x}\xi_{x,x} - u_t u_x^2 u_{x,x}\tau_{u,u} - 2u_t u_x u_{x,x}\tau_{x,u} - \\
&u_t u_{x,x}\tau_{x,x} + u_x^2 u_{x,x}\phi_{u,u} + 2u_x u_{x,x}\phi_{x,u} + \\
&u_{x,x}\phi_{x,x}.
\end{aligned} \tag{2.27}$$

The separation of monomials gives rise to

$$
\begin{array}{lcl}
u_t u_{xx} u_x^2 & : & -\tau_{uu}, \\
u_t u_{xx} u_x & : & -2\tau_{xu}, \\
u_{xx} u_x^3 & : & -\xi_{uu}, \\
u_{xx} u_x^2 & : & -2\xi_{xu} + \phi_{uu}, \\
u_{xx} u_x & : & -\xi_{xx} + 2\phi_{xu}, \\
u_{xx} u_{xt} & : & \tau_x, \\
u_{xx}^2 u_x & : & \xi_u, \\
u_{xx}^2 & : & -\frac{3}{2}\xi_x + \frac{1}{2}\tau_t + \phi_u, \\
u_{xx} & : & \phi_{xx}, \\
u_t u_x^2 & : & -\frac{1}{2}\tau + \frac{1}{2}u\tau_u, \\
u_t u_x & : & \xi_t, \\
u_t^3 & : & \frac{1}{2}\tau_u, \\
u_x^3 & : & -\frac{1}{2}\xi + \frac{1}{2}u\xi_u, \\
u_x^2 & : & \frac{1}{2}u\xi_x - \frac{1}{2}u\tau_t - u\phi_u, \\
u_t & : & -f_u - \phi_t, \\
u_x & : & -g_u - u\phi_x, \\
1 & : & -f_t - g_x.
\end{array} \tag{2.28}
$$

The over-determined system has solution

$$
\begin{array}{l}
\tau = 0, \quad \xi = 0, \quad \phi = A + Bt + Cx + Dxt, \\
f = -(B + Dx)u + a(x,t), \quad g = -\frac{1}{2}(C + Dt)u^2 + b(x,t)
\end{array} \tag{2.29}
$$

where $a_t + b_x = 0$ and A, B, C and D are arbitrary constants.

If we choose, for example, $A = D = 0$ (Noether type symmetry $X = (Bt + Cx)\partial_u$, $W = (Bt + Cx)$, $f = -Bu$ and $g = -\frac{1}{2}Cu^2$), we obtain, via a truncated version of (1.27), i.e.,

$$
\begin{array}{l}
T^t = L\tau + W\frac{\partial L}{\partial u_t} + [D_j W - W D_j]\frac{\partial L}{\partial u_{tj}} - f, \\
T^x = L\xi + W\frac{\partial L}{\partial u_x} + [D_j W - W D_j]\frac{\partial L}{\partial u_{xj}} - g,
\end{array} \tag{2.30}
$$

the conserved density and flux

$$
\begin{array}{l}
T^t = -(Bt + Cx)u_t + Bu \\
T^x = -(Bt + Cx)uu_x + Cu_{xx} - (Bt + Cx)u_{xxx} + \frac{1}{2}Cu^2
\end{array} \tag{2.31}
$$

so that $D_tT^t + D_xT^x = -(Bt + Cx)(u_{xxxx} + uu_{xx} + u_x^2 + u_{tt})$.

2.4 A fourth-order non-linear equation

The Lagrangian, $L = \frac{1}{2}u_{xx}^2 - uu_x^2$, of

$$u_{xxxx} + 2uu_{xx} + u_x^2 = 0, \tag{2.32}$$

has been discussed in [17] constructed by the homotopy formula since the Frechet derivative of $u_{xxxx} + 2uu_{xx} + u_x^2$, viz., $D_x^4 + 2uD_x^2 + 2u_xD_x + 2u_{xx}$ is self adjoint.

As before, (1.27) yields the Noether symmetries which are the translations ∂_t and ∂_x. The symmetry $x\partial_x - 2u\partial_u$ is not variational with regard to this Lagrangian. The conservation laws via translations, via (1.27) are with respect to ∂_t and ∂_x,

$$\begin{aligned} T^t &= \tfrac{1}{2}u_{xx}^2 - uu_x^2, \\ T^x &= 2uu_tu_x - u_{xt}u_{xx} + u_tu_{xxx} \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} T^t &= 0, \\ T^x &= \tfrac{1}{2}u_{xx}^2 + uu_x^2 - u_{xx}u_{xxx} + u_xu_{xxx}. \end{aligned} \tag{2.34}$$

Note. If one uses L as a partial Lagrangian for the evolution equation $u_t = u_{xxxx} + 2uu_{xx} + u_x^2$, so that $\frac{\delta L}{\delta u} = u_t$, we obtain no Noether type symmetries. In fact, direct calculations do not yield any either. The point symmetry generators of the equation are, in addition to translations, $4t\partial_t + x\partial_x - 2u\partial_u$.

2.5 Discussion and conclusion

We see that the conserved flows for high-order equations (with Lagrangians and, equivalently, partial Lagrangians of order greater than one in derivatives) support a formula similar to the well known Noether's theorem with the proviso that the higher-order cases have more terms giving rise to the appropriate order of the conserved flow. Also, in the KdV-5 evolution equation, we resorted to variational techniques usually adopted for the KdV-3 equation.

We used the new modified approach of the Noether identity to find symmetries and then conservation laws for the high order equations. We know that when considering the use of the partial Lagrangian, we have to take into account the highest derivative of the equation, where the highest derivative of the equation must be derived from the partial Lagrangian.

Chapter 3

Conservation laws of Higher order PDEs with mixed derivative term

3.1 Introduction

In the previous chapter we had observed the application of variational and 'partial-variational' techniques on higher order equations where no derivatives were of mixed type. When considering the partial Lagrangian formula, a special point of consideration is the term of the highest derivative; the highest derivative term of the equation must be derived from the partial Lagrangian.

In this chapter we consider the equations in which the highest derivative terms are mixed. The equations we investigate are the Camassa-Holms, Hunter-Saxton, Inviscid Burgers, KdV family of equations, the fourth-order Shallow Water Wave and Regularized Long Wave equations which has been discussed in [25]. The importance of the equations lie in many areas of physics, and real world applications, e.g., tsunamis are characterized with long periods and wave lengths and as a result they behave as shallow-water waves, [26].

3.2 Camassa-Holms, Hunter-Saxton, Inviscid Burgers and KdV family of equations

We now consider the family of equations

$$\alpha(v_t + 3vv_x) - \beta(v_{txx} + 2v_x v_{xx} + vv_{xxx}) - \gamma v_{xxx} = 0. \tag{3.1}$$

Even though it represents a class of nonlinear evolution equations, it displays variational/Hamiltonian properties and would then be subject to, amongst other things, Noether's theorem [1]. This is well documented in the case of the KdV equation [24]. Also, it displays interesting soliton or soliton like solutions. Equation (3.1) represents a version of the KdV equation ($\alpha = 1, \beta = 0, \gamma = 1$), the Camassa-Holm equation ($\alpha = 1, \beta = 1, \gamma = 1$), the Hunter-Saxton equation ($\alpha = 0, \beta = 1, \gamma = 1$) and the inviscid Burgers equation ($\alpha = 1, \beta = 1, \gamma = 0$) [28, 29, 30].

We determine the Lie point symmetry generators for equation (3.1) which split into various cases which are also symmetries for the different equations mentioned above. For the case (i) the commutator table is also included.

(i) $\alpha\beta\gamma \neq 0$:

$$\begin{aligned} X_1 &= \partial_t \\ X_2 &= \partial_x \\ X_3 &= (1 - \tfrac{2u\beta}{\gamma})\partial_u + \tfrac{2t\beta}{\gamma}\partial_t + 3t\partial_x \end{aligned} \tag{3.2}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{2\beta X_1}{\gamma} + 3X_2$
X_2	0	0	0
X_3	$-\frac{2\beta X_1}{\gamma} - 3X_2$	0	0

(3.3)

$(ii) \quad \alpha = 0, \quad \beta\gamma \neq 0$:

$$
\begin{aligned}
X_1 &= (u + \tfrac{\gamma}{\beta})\partial_u + x\partial_x \\
X_2 &= F'(t)\partial_u + F(t)\partial_x \\
X_3 &= F(t)\partial_t + xF'(t)\partial_x + xF''(t)\partial_u
\end{aligned}
\tag{3.4}
$$

$(iii) \quad \beta = 0, \quad \alpha\gamma \neq 0$:

$$
\begin{aligned}
X_1 &= F(t)\partial_t, \\
X_2 &= \partial_u \\
X_3 &= u\partial_u \\
X_4 &= (3tu - x)\partial_u \\
X_5 &= F(t)\partial_x \\
X_6 &= xF(t)\partial_x
\end{aligned}
\tag{3.5}
$$

$(iv) \quad \gamma = 0, \quad \alpha\beta \neq 0$:

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= t\partial_t - u\partial_u \\
X_3 &= \partial_x
\end{aligned}
\tag{3.6}
$$

Note. F(t) is an arbitrary function of t.

3.2.1 The higher-order expansion of the Camassa-Holms, Hunter-Saxton, Inviscid Burgers and KdV family of equations

The method of expansion is employed in this section due to the order of the equation. The family of equations is of third order, due to this fact a standard Lagrangian can not be obtained. We seek an alternative approach by modifying equation (3.1). The order of the equation is thus increased by letting $v = u_x$ to obtain

$$\alpha(u_{tx} + 3u_x u_{xx}) - \beta(u_{txxx} + 2u_{xx}u_{xxx} + u_x u_{xxxx}) - \gamma u_{xxxx} = 0. \tag{3.7}$$

We determine the Lie point symmetry generators of the equation and split these into various non-trivial cases.

(i) $\alpha\beta\gamma \neq 0$:

$$\begin{aligned} X_1 &= \partial_t \\ X_2 &= \partial_u \\ X_3 &= F(t)\partial_u \\ X_4 &= \partial_x \\ X_5 &= \tfrac{2t\beta}{\gamma}\partial_t + 3t\partial_x + (x - \tfrac{2u\beta}{\gamma})\partial_u \end{aligned} \tag{3.8}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$X_2 + X_3$	0	$\frac{2\beta X_1}{\gamma} + 3X_4$
X_2	0	0	0	0	$-\frac{2\beta X_2}{\gamma}$
X_3	$-X_2 - X_3$	0	0	0	$X_2 + X_3$
X_4	0	0	0	0	X_2
X_5	$-\frac{2\beta X_1}{\gamma} - 3X_4$	$\frac{2\beta X_2}{\gamma}$	$-X_2 - X_3$	$-X_2$	0

(3.9)

$(ii) \quad \alpha = 0, \quad \beta\gamma \neq 0$:

$$\begin{aligned} X_1 &= \partial_u \\ X_2 &= F(t)\partial_u \\ X_3 &= (2u + \tfrac{x\gamma}{\beta})\partial_u + x\partial_x \\ X_4 &= F(t)\partial_x + xF'(t)\partial_u \\ X_5 &= 2F(t)\partial_t + (2uF'(t) + x^2F''(t))\partial_u + 2xF(t)\partial_x \end{aligned} \tag{3.10}$$

$(iii) \quad \beta = 0, \quad \alpha\gamma \neq 0$:

$$\begin{aligned} X_1 &= F(t)\partial_t \\ X_2 &= u\partial_u \\ X_3 &= F(t)\partial_u \\ X_4 &= \partial_x \\ X_5 &= x\partial_x \\ X_6 &= 3t\partial_x + x\partial_u \\ X_7 &= 6tx\partial_x + x^2\partial_u \end{aligned} \tag{3.11}$$

$(iv) \quad \gamma = 0, \quad \alpha\beta \neq 0$:

$$\begin{aligned} X_1 &= \partial_t \\ X_2 &= \partial_u \\ X_3 &= F(t)\partial_u \\ X_4 &= u\partial_u - t\partial_t \\ X_5 &= \partial_x \end{aligned} \tag{3.12}$$

The modified equation (3.7) displays variational properties with respect to the Lagrangian

$$L = -\frac{\alpha}{2}(u_x u_t + u_x^3) - \frac{\beta}{2}(u_x u_{xx}^2 + u_{tx} u_{xx}) - \frac{\gamma}{2} u_{xx}^2. \tag{3.13}$$

The symmetries and corresponding conserved vectors are

(i) $X = \partial_t, \quad W = -u_t$

The conserved quantities are

$$\begin{aligned} T^1 &= -\tfrac{\alpha}{2}(u_t u_x + u_x^3) - \tfrac{\beta}{2}(u_x u_{xx}^2 + u_{tx} u_{xx}) - \tfrac{\gamma}{2} u_{xx}^2 \\ &\quad + (-u_t)(-\tfrac{\alpha}{2} u_x + \tfrac{\beta}{2} u_{xxx}) + (-u_{tx})(-\tfrac{\beta}{2} u_{xx}), \\ T^2 &= (-u_t)(-\tfrac{\alpha}{2} u_t - \tfrac{3\alpha}{2} u_x^2 + \tfrac{\beta}{2} u_{xx}^2 + \beta u_{txx} + \beta u_x u_{xxx} + \gamma u_{xxx}) \\ &\quad + (-u_{tt})(-\tfrac{\beta}{2} u_{xx}) + (-u_{xx})(-\beta u_x u_{xx} - \tfrac{\beta}{2} u_{tx} - \gamma u_{xx}) \end{aligned} \tag{3.14}$$

The total divergence is

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= 2\gamma u_{xx} u_{xxx} - \gamma u_{tx} u_{xxx} - \gamma u_{xx} u_{txx} + \tfrac{1}{2}\beta u_{xx} u_{txx} \\ &+ \tfrac{1}{2}\beta u_{tx} u_{xxx} - \beta u_{tx} u_{txx} - \tfrac{\beta}{2} u_t u_{txxx} + 2\beta u_x u_{xx} u_{xxx} \\ &+ \beta u_{xx}^3 - \beta u_x u_{xx} u_{txx} - \beta u_x u_{tx} u_{xxx} \\ &+ \tfrac{\beta}{2} u_{xx} u_{ttx} - \tfrac{\beta}{2} u_{tx} u_{txx} \end{aligned} \tag{3.15}$$

Extra terms emerge that require further analysis. By making an adjustment to these terms, they can be absorbed into the conservation law if we note that

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_x(\gamma u_{xx}^2) - D_x(\gamma u_{tx} u_{xx}) + D_x(\tfrac{\beta}{2} u_{tx} u_{xx}) \\ &- D_x(\tfrac{\beta}{2} u_t u_{txx}) + D_x(\beta u_x u_{xx}^2) - D_x(u_x u_{tx} u_{xx}) \\ &- D_x(\tfrac{\beta}{2} u_{tx}^2) + D_t(\tfrac{\beta}{2} u_{tx} u_{xx}). \end{aligned} \tag{3.16}$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law. The modified conserved quantity are now labeled

$\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$ along the equation, viz.,

$$\begin{aligned}\tilde{T^1} &= T^1 - \tfrac{\beta}{2}u_{tx}u_{xx}, \\ \tilde{T^2} &= T^2 - \gamma u_{xx}^2 + \gamma u_{tx}u_{xx} - \tfrac{\beta}{2}u_{tx}u_{xx} \\ &+ \tfrac{\beta}{2}u_t u_{txx} - \beta u_x u_{xx}^2 - u_x u_{tx}u_{xx} + \tfrac{\beta}{2}u_{tx}^2\end{aligned} \tag{3.17}$$

The same applies to the following cases below.

(ii) $X = \partial_x, \quad W = -u_x$

With $T^1 = (-u_x)(-\frac{\alpha}{2}u_x + \frac{\beta}{2}u_{xxx}) + (-u_{xx})(-\frac{\beta}{2}u_{xx})$ and $T^2 = -\frac{\alpha}{2}(u_x u_t + u_x^3) - \frac{\beta}{2}(u_x u_{xx}^2 + u_{tx}u_{xx}) - \frac{\gamma}{2}u_{xx}^2 + (-u_x)(-\frac{\alpha}{2}u_t - \frac{3\alpha}{2}u_x^2 + \frac{\beta}{2}u_{xx}^2 + \beta u_{txx} + \beta u_x u_{xxx} + \gamma u_{xxx}) + (-u_{tx})(-\frac{\beta}{2}u_{xx}) + (-u_{xx})(-\beta u_x u_{xx} - \frac{\beta}{2}u_{tx} - \gamma u_{xx})$ we get

$$D_t(T^1) + D_x(T^2) = -\tfrac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx}), \tag{3.18}$$

so that, since $-\frac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx})$ has derivative consequences,

$$-\tfrac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx}) = -\tfrac{1}{2}\beta(D_x(u_x u_{xx} - D_t(u_{xx}^2)), \tag{3.19}$$

so that a redefinition leads to

$$\begin{aligned}\tilde{T^1} &= T^1 - \tfrac{1}{2}\beta u_{xx}^2, \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}\beta u_x u_{xx},\end{aligned} \tag{3.20}$$

(iii) $X = n(t)\partial_u, \quad W = n(t)$

Here, we get $T^1 = (n(t))(-\frac{\alpha}{2}u_x + \frac{\beta}{2}u_{xxx})$ and $T^2 = (n(t))(-\frac{\alpha}{2}(u_x u_t + u_x^3) - \frac{\beta}{2}(u_x u_{xx}^2 + u_{tx}u_{xx}) - \frac{\gamma}{2}u_{xx}^2) + (n_t(t))(-\frac{\beta}{2}u_{xx}) + \frac{\alpha}{2}n_t(t)u$. so that

$$D_t(T^1) + D_x(T^2) = -\tfrac{1}{2}n(t)\beta u_{txxx}, \tag{3.21}$$

and

$$\begin{aligned} \tilde{T}_2^1 &= T^1, \\ \tilde{T}_2^2 &= T^2 + \tfrac{1}{2}n(t)\beta u_{txx}. \end{aligned} \tag{3.22}$$

3.3 The Shallow Water Wave equation

The shallow water wave equation (SWW), models the simplest water waves which reasonably approximate the behavior of real ocean waves,

$$u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0, \tag{3.23}$$

where α and β are arbitrary constants.

We determine the Lie point symmetry generators of the equation (3.23) and these are split into various cases with each of their commutator tables.

(i) $\alpha\beta \neq 0$:

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_u, \\ X_3 &= F(t)\partial_u + \beta F(t)\partial_t, \\ X_4 &= \partial_x, \\ X_5 &= (2x - u\alpha)\partial_u - t\alpha\partial_t + x\alpha\partial_x. \end{aligned}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$X_1 + X_2 + X_3$	0	$-\alpha X_1$
X_2	0	0	0	0	$-\alpha X_2$
X_3	$-X_1 - X_2 - X_3$	0	0	0	$X_1 + X_2 + X_3$
X_4	0	0	0	0	$2X_2 + \alpha X_4$
X_5	αX_1	αX_2	$-X_1 - X_2 - X_3$	$-2X_2 - \alpha X_4$	0

$(ii) \quad \alpha \neq 0, \quad \beta = 0$:

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= \partial_u, \\ X_3 &= F(t)\partial_u, \\ X_4 &= \partial_x, \\ X_5 &= (2x - u\alpha)\partial_u - t\alpha\partial_t + x\alpha\partial_x. \end{aligned}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$X_2 + X_3$	0	$-\alpha X_1$
X_2	0	0	0	0	$-\alpha X_2$
X_3	$-X_2 - X_3$	0	0	0	$X_2 + X_3$
X_4	0	0	0	0	$2X_2 + \alpha X_4$
X_5	αX_1	αX_2	$-X_2 - X_3$	$-2X_2 - \alpha X_4$	0

$(iii) \quad \alpha = 0, \quad \beta \neq 0$

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= x\partial_u, \\ X_3 &= F(t)\partial_u + \beta F(t)\partial_t, \\ X_4 &= \partial_x. \end{aligned}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$X_1 + X_3$	0
X_2	0	0	0	$-\partial_u$
X_3	$-X_1 - X_3$	0	0	0
X_4	0	∂_u	0	0

From the equation (3.23), there are two cases that emerge, viz., (i) $\alpha \neq \beta$ and (ii) $\alpha = \beta$.

In case (i) $\alpha \neq \beta$, we refer to this equation as shallow water wave-1 (SSW-1), and for case(ii) $\alpha = \beta$, equation (3.23), α is replaced by β, we refer to this case as shallow water wave-2 (SSW-2), which becomes,

$$u_{xxxt} + \beta u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0. \tag{3.24}$$

3.3.1 Shallow Water Wave-1

Here, we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t - \frac{1}{2}\beta u_t u_x^2, \tag{3.25}$$

for which

$$\frac{\delta L}{\delta u} = (2\beta - \alpha)u_{tx}u_x. \tag{3.26}$$

The separation of monomials gives rise to

$$\begin{array}{lcl} u_x u_{tx}^2 & : & \tau_u, \\ u_{tx}^2 & : & \tau_x, \\ u_x u_{xx}^2 & : & \xi_u, \\ u_{xx}^2 & : & \xi_t, \\ u_{tx}u_{xx} & : & \eta_u - \xi_x, \\ u_t u_x u_{tx} & : & (2\beta - \alpha)\tau, \\ u_x^2 u_{tx} & : & (2\beta - \alpha)\xi, \\ u_x u_{tx} & : & (2\beta - \alpha)\eta, \\ u_t u_x^2 & : & \xi_x - 3\eta_u, \\ u_t u_x & : & \eta_u - \beta\eta_x, \\ u_x^2 & : & \eta_u - \frac{1}{2}\beta\eta_t - \frac{1}{2}\xi_x + \frac{1}{2}\tau_t, \\ u_x & : & -g_u + \frac{1}{2}\eta_t + \eta_x, \\ u_t & : & -f_u + \frac{1}{2}\eta_x, \\ 1 & : & f_t + g_x. \end{array} \tag{3.27}$$

From equation (3.27), we observe there are two cases that emerge, (a) $\alpha = 2\beta$ and (b) $\alpha \neq 2\beta$.

Case (a): $\alpha = 2\beta$

(1) $X = \partial_t, \quad W = -u_t$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$

The total divergence is given below.

$$
\begin{aligned}
D_t(T^1) + D_x(T^2) &= D_t(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_{xxx}) \\
&+ D_x(-u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 - \tfrac{1}{2}u_{xx}u_{tt}), \\
&= u_x u_{tx} + \tfrac{1}{2}u_{xxx}u_{tt} + \tfrac{1}{2}u_t u_{xxxt} - u_x u_{tx} - u_t u_{xx} - \tfrac{1}{2}u_{xx}u_{ttx} \\
&- u_t u_{tx} + 2\beta u_t u_x u_{tx} + \beta u_t^2 u_{xx} + u_t u_{xxxt} - \tfrac{1}{2}u_{xxx}u_{tt}, \\
&= u_t(u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) + \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\
&= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}.
\end{aligned} \tag{3.28}
$$

We observe that extra terms emerge. By making an adjustment, these terms can be absorbed into the conservation law. The adjustment of these extra terms can be done by finding differentiable functions that form the extra terms, when they are differentiated,

$$
\begin{aligned}
D_t(T^1) + D_x(T^2) &= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\
&= \tfrac{1}{2}D_t(u_t u_{xxx}) - \tfrac{1}{2}D_x(u_{xx}u_{tt}).
\end{aligned} \tag{3.29}
$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law

$$
D_t(T^1 - \tfrac{1}{2}u_t u_{xxx}) + D_x(T^2 + \tfrac{1}{2}u_{xx}u_{tt}) = 0. \tag{3.30}
$$

The modified conserved quantities are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$.

$$
\begin{aligned}
\tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\
&= \tfrac{1}{2}u_x^2 \\
\tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt} \\
&= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2
\end{aligned} \tag{3.31}
$$

The same consequences apply for the results below.

(2) $X = \partial_x, \quad W = -u_x$

$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$

$T^2 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2\beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2 \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 \\ \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt} \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2\beta + u_x u_{xxt} \end{aligned} \tag{3.32}$$

Case (b): $\alpha \neq 2\beta$

(1) $X = \partial_u, \quad B^1 = \frac{1}{2}u_x^2(2\beta - \alpha), \quad B^2 = 0, \quad W = 1$

$T^1 = \frac{1}{2}u_x - \frac{1}{2}\beta u_x^2 - \frac{1}{2}u_{xxx} + \frac{1}{2}u_x^2(2\beta - \alpha)$

$T^2 = u_x + \frac{1}{2}u_t - u_t u_x\beta - u_{xxt}$

The total divergence is given by,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(\tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha)) \\ &+ D_x(u_x + \tfrac{1}{2}u_t - u_t u_x\beta - u_{xxt}), \\ &= \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - \tfrac{1}{2}u_{xxxt} + u_x u_{tx}(2\beta - \alpha) \\ &+ u_{xx} + \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - u_t u_{xx}\beta - u_{xxxt}, \\ &= (u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) - \tfrac{1}{2}u_{xxxt}, \\ &= -\tfrac{1}{2}u_{xxxt}. \end{aligned} \tag{3.33}$$

From the equation (3.33), u_{xxxt} has two derivative consequences,

$$\begin{aligned} u_{xxxt} &= D_t(u_{xxx}) \\ &= D_x(u_{xxt}) \end{aligned} \tag{3.34}$$

which leads to two pairs of conserved quantities,

(i)

$$\begin{aligned}
\tilde{T_1^1} &= T^1 + \tfrac{1}{2}u_{xxx} \\
&= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 + \tfrac{1}{2}u_x^2(2\beta - \alpha), \\
\tilde{T_1^2} &= T^2 \\
&= u_x + \tfrac{1}{2}u_t - u_t u_x \beta - u_{xxt},
\end{aligned} \tag{3.35}$$

(ii)

$$\begin{aligned}
\tilde{T_2^1} &= T^1 \\
&= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha), \\
\tilde{T_2^2} &= T^2 + \tfrac{1}{2}u_{xxt} \\
&= u_x + \tfrac{1}{2}u_t - u_t u_x \beta - \tfrac{1}{2}u_{xxt}.
\end{aligned} \tag{3.36}$$

(2) $X = \partial_x, \quad B^1 = \frac{1}{3}u_x^3(2\beta - \alpha), \quad B^2 = 0, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 - \frac{1}{3}u_x^3(2\beta - \alpha) - \frac{1}{2}u_x u_{xxx}$

$T^2 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2 \beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned}
\tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2 \\
&= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 - \tfrac{1}{3}u_x^3(2\beta - \alpha), \\
\tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt} \\
&= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2 \beta + u_x u_{xxt}.
\end{aligned} \tag{3.37}$$

3.3.2 Shallow Water Wave-2

For equation (3.24), we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t - \frac{1}{2}\beta u_t u_x^2, \tag{3.38}$$

which has

$$\frac{\delta L}{\delta u} = \beta u_{tx} u_x. \tag{3.39}$$

The separation of monomials gives rise to

$$\begin{array}{lll}
u_x u_{tx}^2 & : & \tau_u, \\
u_{tx}^2 & : & \tau_x, \\
u_x u_{xx}^2 & : & \xi_u, \\
u_{xx}^2 & : & \xi_t, \\
u_{tx} u_{xx} & : & \eta_u - \xi_x, \\
u_t u_x u_{tx} & : & \beta\tau, \\
u_x^2 u_{tx} & : & \beta\xi, \\
u_x u_{tx} & : & \beta\eta, \\
u_t u_x^2 & : & \xi_x - 3\eta_u, \\
u_t u_x & : & \eta_u - \beta\eta_x, \\
u_x^2 & : & \eta_u - \frac{1}{2}\beta\eta_t - \frac{1}{2}\xi_x + \frac{1}{2}\tau_t, \\
u_x & : & -g_u + \frac{1}{2}\eta_t + \eta_x, \\
u_t & : & -f_u + \frac{1}{2}\eta_x, \\
1 & : & f_t + g_x.
\end{array} \tag{3.40}$$

From equation (3.40), it is clear that $\beta \neq 0$ or $\beta = 0$.

If $\beta \neq 0$ then it is a trivial solution, and if $\beta = 0$, then equation (3.24) changes to

$$u_{xxxt} - u_{tx} - u_{xx} = 0, \tag{3.41}$$

and the partial lagrangian (3.38) becomes a standard Lagrangian

$$L = \frac{1}{2} u_{tx} u_{xx} + \frac{1}{2} u_x^2 + \frac{1}{2} u_x u_t, \tag{3.42}$$

and the conserved quantities are as follows:

(i) $X = \partial_t, \quad W = -u_t$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\ &= \tfrac{1}{2}u_x^2 \\ \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt} \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2, \end{aligned} \tag{3.43}$$

(ii) $X = \partial_x, \quad W = -u_x$

$T^1 = -\frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$

$T^2 = -\frac{1}{2}u_x^2 + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2 \\ &= -\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{xx}^2 \\ \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt} \\ &= -\tfrac{1}{2}u_x^2 + u_x u_{xxt}. \end{aligned} \tag{3.44}$$

REMARK. From the conserved quantities attained, there are non-zero divergence terms occur when the conservation law is applied. These extra terms are adjusted by merely taking the extra terms into derivative functions that can be absorbed into the conservation law, therefore producing new conserved quantities.

3.4 The Regularized Long Wave Equation

The Regularized Long Wave Equation (RLW), models soliton waves and is sometimes referred to as the Benjamin-Bona-Mahoney Equation. The regularized long wave (RLW) equation is an important nonlinear wave equation. Solitary waves are wave packets or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons. RLW is an alternative description of nonlinear dispersive waves to the more Korteweg de Vries (KdV) equation,

$$v_{txx} + \alpha v^2 v_x + v_t + v_x = 0. \tag{3.45}$$

The RLW equation is a third order equation and for our purposes of investigation we modify this equation to be compared with the equations we have dealt with thus far. We do this by differentiating the equation by a spatial variable, x and a time variable t.

We refer to the modified RLW equation which has either been differentiated by t or had v replaced with u_t as RLW-1,

$$u_{xxtt} + \alpha u_t^2 u_{tx} + u_{tt} + u_{tx} = 0. \tag{3.46}$$

The symmetry generators obtained for the RLW-1 equation are listed below with two cases for α and its corresponding commutator tables.

(i) $\alpha = 0$

$$\begin{aligned} X_1 &= \partial_t \\ X_2 &= u\partial_u \\ X_3 &= F(t,x)\partial_u, \quad F_{ttxx} + F_{tx} + F_{tt} = 0 \\ X_4 &= \partial_x \end{aligned}$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_3	0
X_2	0	0	$-X_3$	0
X_3	$-X_3$	X_3	0	X_3
X_4	0	0	$-X_3$	0

(3.47)

(ii) $\quad \alpha \neq 0$

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= \partial_u \\
X_3 &= F(x)\partial_u \\
X_4 &= \partial_x
\end{aligned}
$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	0
X_3	0	0	0	$X_2 + X_3$
X_4	0	0	$-X_2 - X_3$	0

(3.48)

We refer to the modified RLW equation which has either been differentiated by x or had v replaced with u_x as RLW-2,

$$u_{xxxt} + \alpha u_x^2 u_{xx} + u_{tx} + u_{xx} = 0. \tag{3.49}$$

The symmetry generators obtained for the RLW-2 equation are listed below with two cases for α and its corresponding commutator tables.

(i) $\quad \alpha = 0$

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= u\partial_u \\
X_3 &= F(t,x)\partial_u, \quad F_{txxx} + F_{xx} + F_{tx} = 0 \\
X_4 &= \partial_x
\end{aligned}
$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_3	0
X_2	0	0	$-X_3$	0
X_3	$-X_3$	X_3	0	X_3
X_4	0	0	$-X_3$	0

(3.50)

$(ii)\quad \alpha \neq 0$

$$
\begin{aligned}
X_1 &= \partial_t \\
X_2 &= F(t)\partial_u \\
X_3 &= \partial_x
\end{aligned}
$$

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3
X_1	0	X_2	0
X_2	$-X_2$	0	0
X_3	0	0	0

(3.51)

3.4.1 Regularized Long Wave-1

Here, we use the partial Lagrangian

$$
L = \frac{1}{2}u_{tx}^2 - \frac{1}{2}u_t u_x - \frac{1}{2}u_t^2, \tag{3.52}
$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_t^2 u_{tx}. \tag{3.53}$$

Using Noether's Identity we substitute (3.52) and (3.53)into the expression below, which we use to find our determining equations and then separate by monomials,

$$X^{[2]}L + L(D_t\xi^1 + D_x\xi^2) = (\eta - u_t\xi^1 - u_x\xi^2)\frac{\delta L}{\delta u} + D_tB^1 + D_xB^2. \tag{3.54}$$

The separation of monomials are:

$$\begin{aligned}
u_{xx}u_{tx} &: \ \xi_t + u_t\xi_u, \\
u_{tt}u_{tx} &: \ \tau_x + u_x\tau_u, \\
u_{tx}^2 &: \ \eta_u - \tfrac{1}{2}\xi_x - \tfrac{1}{2}\tau_t, \\
u_xu_t^2u_{tx} &: \ \alpha\xi, \\
u_tu_t^2u_{tx} &: \ \alpha\tau, \\
u_t^2u_{tx} &: \ \alpha\eta, \\
u_tu_xu_{tx} &: \ \eta_{uu}, \\
u_tu_{tx} &: \ \eta_{tu}, \\
u_xu_{tx} &: \ \eta_{xu}, \\
u_{tx} &: \ \eta_{tx}, \\
u_xu_t &: \ \eta_u, \\
u_t^2 &: \ ,\eta_u + \tfrac{1}{2}\xi_x + \tfrac{1}{2}\tau_t \\
u_x &: \ -g_u - \tfrac{1}{2}\eta_t, \\
u_t &: \ -f_u - \eta_t - \tfrac{1}{2}\eta_x, \\
1 &: \ f_t + g_x.
\end{aligned} \tag{3.55}$$

From equation (3.55), it is clear that $\alpha \neq 0$ or $\alpha = 0$.

If $\alpha \neq 0$ then it is a trivial solution, and if $\alpha = 0$, then equation (3.46) changes to

$$u_{xxtt} + u_{tt} + u_{tx} = 0 \tag{3.56}$$

and the partial Lagrangian becomes a standard Lagrangian. By solving the over-determined system, we get the symmetries. From the symmetries we calculate the conserved vectors,

Case (i) $X = \partial_t, \quad W = -u_t$

$T^1 = -\frac{1}{2}u_{tx}^2 + \frac{1}{2}u_t^2 + u_t u_{txx}$

$T^2 = \frac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}.$

Checking if the conservation law holds, i.e.,

$$
\begin{aligned}
D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 + u_t u_{txx}) + D_x(\tfrac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}), \\
&= -u_{tx}u_{ttx} + u_t u_{tt} + u_{tt}u_{txx} + u_t u_{ttxx} + u_t u_{tx} \\
&+ \; u_{tx}u_{ttx} + u_t u_{ttxx} - u_{tx}u_{ttx} + u_{tt}u_{txx}, \\
&= u_t u_{ttxx} - u_{tx}u_{ttx}.
\end{aligned}
\tag{3.57}
$$

We observe that extra terms emerge. By making an adjustment, these terms can be absorbed into the conservation law. The adjustment of these extra terms can be done by finding differentiable functions that form the extra terms, when they are differentiated.

Thus,

$$
\begin{aligned}
D_t(T^1) + D_x(T^2) &= u_t u_{ttxx} - u_{tx}u_{ttx}, \\
&= D_t(u_t u_{txx}) - D_x(u_{tt}u_{tx}).
\end{aligned}
\tag{3.58}
$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law,

$$
D_t(T^1 - u_t u_{txx}) + D_x(T^2 + u_{tt}u_{tx}) = 0. \tag{3.59}
$$

The modified conserved quantities are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$.

$$\begin{aligned} \tilde{T^1} &= T^1 - u_t u_{txx} \\ &= -\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 \\[1em] \tilde{T^2} &= T^2 + u_{tt}u_{tx} \\ &= \tfrac{1}{2}u_t^2 + u_t u_{ttx}. \end{aligned} \tag{3.60}$$

The same consequences apply for the results below.

Case (ii) $X = \partial_x, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2$

$T^2 = -\frac{1}{2}u_{tx}^2 - \frac{1}{2}u_t^2 + u_x u_{ttx}$

$$\begin{aligned} \tilde{T^1} &= T^1 + u_{tt}u_{tx} \\ &= \tfrac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2 + u_{tt}u_{tx} \\[1em] \tilde{T^2} &= T^2 - u_x u_{ttx} \\ &= -\tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t^2 \end{aligned} \tag{3.61}$$

Case (iii) $X = \partial_u, \quad W = 1$

$T^1 = -\frac{1}{2}u_x - u_t - u_{txx}$

$T^2 = -\frac{1}{2}u_t - u_{ttx}$

The divergence is,

$$
\begin{aligned}
D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}u_x - u_t - u_{txx}) + D_x(-\tfrac{1}{2}u_t - u_{ttx}), \\
&= -\tfrac{1}{2}u_{tx} - u_{tt} - u_{ttxx} - \tfrac{1}{2}u_{tx} - u_{ttxx}, \\
&= -u_{ttxx}.
\end{aligned} \tag{3.62}
$$

From the equation (3.62), u_{ttxx} has two derivative consequences,

$$
\begin{aligned}
u_{ttxx} &= D_t(u_{txx}) \\
&= D_x(u_{ttx})
\end{aligned} \tag{3.63}
$$

which leads to two pairs of conserved quantities

$$
\begin{aligned}
\tilde{T_1^1} &= T^1 + u_{txx} \\
&= -\tfrac{1}{2}u_x - u_t \\
\tilde{T_1^2} &= T^2 \\
&= -\tfrac{1}{2}u_t - u_{ttx}
\end{aligned} \tag{3.64}
$$

$$
\begin{aligned}
\tilde{T_2^1} &= T^1 \\
&= -\tfrac{1}{2}u_x - u_t - u_{txx} \\
\tilde{T_2^2} &= T^2 + u_{ttx} \\
&= -\tfrac{1}{2}u_t
\end{aligned} \tag{3.65}
$$

Case (iv) $X = t\partial_u, \quad W = t, \quad f = -u, \quad g = -\frac{1}{2}u$

$T^1 = -\frac{1}{2}tu_x - tu_t - tu_{txx} - u$

$T^2 = -\frac{1}{2}tu_t - tu_{ttx} + u_{tx} - \frac{1}{2}u$

Checking if the conservation law holds we find that,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}tu_x - tu_t - tu_{txx} - u) + D_x(-\tfrac{1}{2}tu_t - tu_{ttx} + u_{tx} - \tfrac{1}{2}u) \\ &= -tu_{ttxx}. \end{aligned} \tag{3.66}$$

From the equation (3.66), u_{ttxx} has two derivative consequences,

$$\begin{aligned} tu_{ttxx} &= D_x(tu_{ttx}) \\ &= D_t(tu_txx) - D_x(u_{tx}) \end{aligned} \tag{3.67}$$

which leads to two pairs of conserved quantities,

$$\begin{aligned} \tilde{T_1^1} &= T^1 \\ &= -\tfrac{1}{2}tu_x - tu_t - tu_{txx} - u \\ \\ \tilde{T_1^2} &= T^2 + tu_{ttx} \\ &= -\tfrac{1}{2}tu_t + u_{tx} - \tfrac{1}{2}u \end{aligned} \tag{3.68}$$

$$\begin{aligned} \tilde{T_2^1} &= T^1 + tu_{txx} \\ &= -\tfrac{1}{2}tu_x - tu_t - u \\ \\ \tilde{T_2^2} &= T^2 - u_{tx} \\ &= -\tfrac{1}{2}tu_t - tu_{ttx} - \tfrac{1}{2}u. \end{aligned} \tag{3.69}$$

3.4.2 Regularized Long Wave-2

Here, we use the partial Lagrangian

$$L = \frac{1}{2}u_{xx}u_{tx} - \frac{1}{2}u_tu_x - \frac{1}{2}u_x^2 \tag{3.70}$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_x^2 u_{tx}. \tag{3.71}$$

Using Noether's Identity we substitute (3.70) and (3.71)into the expression below, which we use to find our determining equations and then separate by monomials,

$$X^{[2]}L + L(D_t\xi^1 + D_x\xi^2) = (\eta - u_t\xi^1 - u_x\xi^2)\frac{\delta L}{\delta u} + D_tB^1 + D_xB^2. \tag{3.72}$$

The separation of monomials are:

$$\begin{aligned}
u_{xx}u_{tx} &: \xi_t - u_t\xi_u,\\
u_{tt}u_{tx} &: \tau_x - u_x\tau_u,\\
u_{tx}u_{xx} &: \eta_u - \xi_x,\\
u_x^3u_{tx} &: \alpha\xi,\\
u_tu_x^2u_{tx} &: \alpha\tau,\\
u_x^2u_{tx} &: \alpha\eta,\\
u_tu_xu_{xx} &: \eta_{uu},\\
u_xu_{xx} &: \eta_{tu},\\
u_tu_{xx} &: \eta_{xu},\\
u_{xx} &: \eta_{tx},\\
u_xu_{tx} &: \eta_{xu} - \tfrac{1}{2}\xi_{xx},\\
u_{tx} &: \eta_{xx},\\
u_xu_t &: \eta_u,\\
u_x^2 &: -\eta_u + \tfrac{1}{2}\xi_x - \tfrac{1}{2}\tau_t,\\
u_x &: -g_u - \tfrac{1}{2}\eta_t - \eta_x,\\
u_t &: -f_u - \eta_x,\\
1 &: f_t + g_x.
\end{aligned} \tag{3.73}$$

From equation (3.55), it is clear that $\alpha \neq 0$ or $\alpha = 0$.

If $\alpha \neq 0$ then it is a trivial solution, and if $\alpha = 0$, then equation (3.46) changes to

$$u_{xxtt} + u_{tt} + u_{tx} = 0 \tag{3.74}$$

and the partial Lagrangian becomes a standard Lagrangian. By solving the over-determined system, these were the only Noether symmetries which were obtained. From the symmetries we calculate the conserved vectors:

Case (i) $X = \partial_t, \quad W = -u_t$

$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = \frac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \frac{1}{2}u_{tx}^2 - \frac{1}{2}u_{tt}u_{xx}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\ &= -\tfrac{1}{2}u_x^2 \\ \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{tt}u_{xx} \\ &= \tfrac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \tfrac{1}{2}u_{tx}^2. \end{aligned} \tag{3.75}$$

Case (ii) $X = \partial_x, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_{xxx} - \frac{1}{2}u_{xx}^2$

$T^2 = \frac{1}{2}u_x^2 + u_x u_{txx} - \frac{1}{2}u_{tx}u_{xx}$

$$\begin{aligned} \tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2 \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_{xxx} \\ \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{txx} \\ &= \tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x u_{txx} \end{aligned} . \tag{3.76}$$

Case (iii) $X = x\partial_u, \quad W = x, \quad f = -\frac{1}{2}u, \quad g = -u$

$T^1 = -\frac{1}{2}xu_x - \frac{1}{2}xu_{xxx} + \frac{1}{2}u_{xx} + \frac{1}{2}u$

$T^2 = -\frac{1}{2}xu_t - xu_x - xu_{txx} + \frac{1}{2}u_{tx} + u$

$$
\begin{aligned}
\tilde{T^1} &= T^1 \\
&= -\tfrac{1}{2}xu_x - \tfrac{1}{2}xu_{xxx} + \tfrac{1}{2}u_{xx} + \tfrac{1}{2}u \\
\tilde{T^2} &= T^2 - \tfrac{1}{2}u_{tx} + \tfrac{1}{2}xu_{txx} \\
&= -\tfrac{1}{2}xu_t - xu_x - \tfrac{1}{2}xu_{txx} + u
\end{aligned}
\quad . \tag{3.77}
$$

3.5 Discussion and conclusion

We used the Noether identity to find symmetry generators and then conservation laws for high order equations containing mixed derivatives in the highest term. All the conserved vectors in the equations with highest order possessing mixed derivatives produce extra terms that become essential parts of the constructed conserved vector for the equation in question.

Using the variational technique on the Shallow Water Wave equation, we get conserved flows that produce extra terms when the conservation law is applied. These extra terms are adjusted and then merged with the conservation law to form new conserved quantities. These extra terms also occur in the second equation, the Regularized Long Wave equation. An interesting observation obtained from our results in that the mixed derivative equations considered produce extra divergence terms. These extra terms consisted of the product of the characteristic function and the highest derivative term of the equation in question.

Note. We acknowledge the comments made by Sarlet [27] regarding the findings and criticism of this chapter and [18]. Our results are not incorrect and the discussion in [27], in fact, provides a necessary and much needed nontrivial explanation to the findings. Thus, rather than being negative, we believe that [27] endorses our results. What was missing, therefore, in our paper [18] was an explanation. Furthermore, the examples discussed in the chapter are well known and are of interest to wide audience.

Chapter 4

Conservation laws of some Vaidya metrics

4.1 Introduction

In this chapter we will look at the Vaidya metric that is currently being researched intensively in relativity and astrophysics. In the paper by Lindquist et al [31], a detailed description on why the Vaidya metric is the most convenient one for the spherically symmetric solution of the Einstein's field equations. The Vaidya metric is given by [32]

$$\mathrm{d}s^2 = -(1-\frac{2m(u)}{r})\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{4.1}$$

where $m(u)$ is the arbitrary function of the retarded time coordinate u. Following the coordinates introduced by Finkelstein [33], the metric in (4.1) can be construed as

$$\mathrm{d}s^2 = -(1-\frac{2m(u)}{r})\mathrm{d}u^2 - 4\frac{m(u)}{r}\mathrm{d}u\mathrm{d}r + (1+\frac{2m(u)}{r})\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{4.2}$$

Various studies relating to the Vaidya metric have been done, for e.g., the 'nature of naked singularities' [34], the 'Carter constant and Petrov classification' [35] and references therein which include aspects of the nature of the Killing tensors and the well known notion of the 'isometries' of the metric which are the diffeomorphisms of the manifold onto itself which preserve the metric tensor [36].

4.2 Lie point Symmetries of the Vaidya metric

The Euler-Lagrange (geodesic) equations associated with the Lagrangian

$$L = -(1 - \frac{2m(u)}{r})\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2 \tag{4.3}$$

corresponding to (4.1) are

$$\begin{aligned} \ddot{u} &= \frac{-r^3\dot{\phi}^2 + \cos\theta^2 r^3\dot{\phi}^2 - r^3\dot{\theta}^2 + m\dot{u}^2}{r^2}, \\ \ddot{r} &= \frac{-2m\dot{r}\dot{u} + 2mr\ddot{u} - r^2\ddot{u}}{r^2}, \\ \ddot{\theta} &= \frac{r\dot{\phi}^2\cos\theta\sin\theta - 2\dot{r}\dot{\theta}}{r}, \\ \ddot{\phi} &= \frac{-2r\dot{\phi}\dot{\theta}\csc\theta\cos\theta - 2\dot{r}\dot{\phi}}{r\csc\theta\sin\theta}, \end{aligned} \tag{4.4}$$

where $\dot{\alpha}$ is the derivative of α with respect to the arclength parameter s. The geodesic equations are constructed by applying the Euler-Lagrange operator onto the Lagrangian for each variable (t, r, θ, ϕ).

The algebra of Lie point symmetries of (4.4) separate into a number of classes based on $m(u)$. The invariance of differential equations leading to Lie symmetries is now well documented and can be found in, inter alia, [17]. We list the two cases $m(u) = 0$ and $m(u) = k$, with k an arbitrary constant. We note the 'large' 35-dimensional Lie algebra for the first case reduces radically when we consider the Noether symmetries.

Listed below are the Lie symmetries for the two cases.

(i) Case $m = 0$

$$X_1 = \partial_\phi$$

$$X_2 = \partial_s$$

$$X_3 = s\partial_s$$

$$X_4 = (r+u)\partial_s$$

$$X_5 = r\cos\theta\partial_s$$

$$X_6 = r\cos\phi\sin\theta\partial_s$$

$$X_7 = r\sin\phi\sin\theta\partial_s$$

$$X_8 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$$

$$X_9 = \sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi$$

$$X_{10} = \partial_u$$

$$X_{11} = s\partial_u$$

$$X_{12} = (r+u)\partial_u$$

$$X_{13} = r\cos\theta\partial_u$$

$$X_{14} = r\cos\phi\sin\theta\partial_u$$

$$X_{15} = r\sin\phi\sin\theta\partial_u$$

$$X_{16} = u\partial_u + r\partial_r$$

$$X_{17} = s^2\partial_s + su\partial_u + rs\partial_r$$

$$X_{18} = s(r+u)\partial_s + u(r+u)\partial_u + r(r+u)\partial_r$$

$X_{19} = \cos\theta\partial_u - \cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta$

$X_{20} = s\cos\theta\partial_u - s\cos\theta\partial_r + \frac{s}{r}\sin\theta\partial_\theta$

$X_{21} = u\cos\theta\partial_u - (r+u)\cos\theta\partial_r + \frac{(r+u)}{r}\sin\theta\partial_\theta$

$X_{22} = rs\cos\theta\partial_s + ru\cos\theta\partial_u + r^2\cos\theta\partial_r$

$X_{23} = u + r\cos^2 t\partial_u + r\sin^2 t\partial_r + \cos\theta\sin\theta\partial_\theta$

$X_{24} = s\cos\phi\sin\theta\partial_u - s\cos\phi\sin\theta\partial_r - \frac{s}{r}\cos\phi\cos\theta\partial_\theta + \frac{s}{r}\csc\theta\sin\theta\partial_\phi$

$X_{25} = u\cos\phi\sin\theta\partial_u - (r+u)\cos\phi\cos\theta\partial_r - \frac{(r+u)}{r}\cos\phi\cos\theta\partial_\theta + \frac{(r+u)}{r}\csc\theta\sin\theta\partial_\phi$

$X_{26} = rs\cos\phi\sin\theta\partial_s + ru\cos\phi\sin\theta\partial_u + r^2\cos\phi\sin\theta\partial_r$

$X_{27} = r\cos\phi\sin 2\theta\partial_u - 2r\cos\phi\cos\theta\sin\theta\partial_r - \cos\phi\cos 2\theta\partial_\theta + \cot\theta\sin\phi\partial_\phi$

$X_{28} = s\sin\phi\sin\theta\partial_u - s\sin\phi\sin\theta\partial_r - \frac{s}{r}\cos\theta\sin\phi\partial_\theta - \frac{s}{r}\cos\phi\csc\theta\partial_\phi$

$X_{29} = u\sin\phi\sin\theta\partial_u - (r+u)\sin\phi\sin\theta\partial_r - \frac{(r+u)}{r}\cos\theta\sin\phi\partial_\theta - \frac{(r+u)}{r}\cos\phi\csc\theta\partial_\phi$

$X_{30} = rs\sin\phi\sin\theta\partial_s + ru\sin\phi\sin\theta\partial_u + r^2\sin\phi\sin\theta\partial_r$

$X_{31} = r\sin\phi\sin 2\theta\partial_u - r\sin\phi\sin 2\theta\partial_r - \cos 2\theta\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi$

$X_{32} = 2r\cos 2\phi\sin^2\theta\partial_u - 2r\cos 2\phi\sin^2\theta\partial_r - \cos 2\phi\sin 2\theta\partial_\theta + 2\sin 2\phi\partial_\phi$

$X_{33} = 4r\cos\phi\sin\phi\sin^2\theta\partial_u - 4r\cos\phi\sin\phi\sin^2\theta\partial_r - \sin 2\phi\sin 2\theta\partial_\theta - 2\cos 2\phi\partial_\phi$

$X_{34} = -\cos\phi\sin\theta\partial_u + \cos\phi\sin\theta\partial_r + \frac{1}{r}\cos\phi\cos\theta\partial_\theta - \frac{1}{r}\sin\phi\csc\theta\partial_\phi$

$X_{35} = -\sin\phi\sin\theta\partial_u + \sin\phi\sin\theta\partial_r + \frac{1}{r}\sin\phi\cos\theta\partial_\theta - \frac{1}{r}\cos\phi\csc\theta\partial_\phi$

(ii) Case $m = k$, with k an arbitrary constant

$X_1 = \partial_\phi$

$X_2 = \partial_s$

$X_3 = s\partial_s$

$X_4 = \partial_u$

$X_5 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi$

$X_6 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi$

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-X_5$	X_4	0
X_2	0	0	X_2	0	0	0
X_3	0	$-X_2$	0	0	0	0
X_4	X_5	0	0	0	$-X_1$	0
X_5	$-X_4$	0	0	X_1	0	0
X_6	0	0	0	0	0	0

(iii) Case $m = m(u)$, for $m(u)$ an arbitrary function of u

$X_1 = \partial_s$

$X_2 = \partial_\phi$

$X_3 = \partial_u$

$X_4 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi$

$X_5 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi$

4.3 Noether Symmetries of the Vaidya metric

The Noether symmetries $X = \sigma\partial_s + \eta\partial_u + \rho\partial_r + \tau\partial_\theta + \kappa\partial_\phi$ are given by

$$XL + L(\xi_s + u_s\xi_u + r_s\xi_r + \theta_s\xi_\theta + \phi_s\xi_\phi) = (g_s + u_s g_u + r_s g_r + \theta_s g_\theta + \phi_s g_\phi), \quad (4.5)$$

where g is the point dependent gauge term. For the Lagrangian,

$$L = -(1 - \frac{2m(u)}{r})\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2, \quad (4.6)$$

using the formula (4.5) leads to Noether symmetries with variations on $m(u)$.

(i) Case $m = 0$:

$X_1 = -\frac{1}{2}s^2\partial_s - \frac{1}{2}su\partial_u - \frac{1}{2}rs\partial_r, \quad g = \frac{1}{2}u^2$

$X_2 = -\frac{1}{2}s\partial_u, \quad g = \frac{1}{2}u + r$

$X_3 = -\frac{1}{2}\sin\theta\sin\phi\partial_u + \frac{1}{2}(s\csc\theta\sin\phi - \tan\theta\cos\theta\sin\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{2}\frac{s}{r}\csc\theta\cos\phi\partial_\phi$,
$g = r\sin\theta\sin\phi$

$X_4 = -\frac{1}{2}\sin\theta\cos\phi\partial_u + \frac{1}{2}(s\csc\theta\cos\phi - \tan\theta\cos\theta\cos\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\cos\phi\partial_\theta - \frac{1}{2}\frac{s}{r}\csc\theta\sin\phi\partial_\phi$,
$g = r\sin\theta\cos\phi$

$X_5 = -\frac{1}{2}s\cos\theta\partial_u + \frac{1}{2}s\cos\theta\partial_r - \frac{1}{2}\frac{s}{r}\sin\theta\partial_\theta, \quad g = r\cos\theta$

$X_6 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = 0$

$X_7 = \partial_s, \quad g = 0$

$X_{8*} = u\sin\theta\sin\phi\partial_u + \csc\theta(-u\sin\phi - 2r\sin^2\theta\sin\phi + 2u\cos^2\theta\sin\phi)\partial_r + (\frac{u}{r}\cos\theta\sin\phi +$
$\tan\theta\sin\theta\sin\phi - \sec t\sin\phi)\partial_\theta - (\frac{u}{r} + 1)\csc\theta\cos\phi\partial_\phi, \quad g = 0$

$X_{9*} = u\sin\theta\cos\phi\partial_u + (u\tan\theta\cos\theta\cos\phi - r\sin\theta\cos\phi - u\csc\theta\cos\phi)\partial_r + (\tan\theta\sin\theta\cos\phi -$
$\sec\theta\cos\phi - \frac{u}{r}\cos\theta\cos\phi)\partial_\theta + (\frac{u}{r} + 1)\csc\theta\sin\phi\partial_\phi, \quad g = 0$

$X_{10*} = u\cos\theta\partial_u + (-r\cos\theta - u\cos\theta)\partial_r + (\frac{u}{r}+1)\sin\theta\partial_\theta, \quad g=0$

$X_{11*} = \partial_u, \quad g=0$

$X_{12*} = \cos\theta\partial_u - \cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta, \quad g=0$

$X_{13*} = \sin\theta\sin\phi\partial_u + (-\csc\theta\cos\phi + \cos\phi\tan\theta\cos\theta)\partial_r - \frac{1}{r}\cos\theta\sin\phi\partial_\theta - \frac{1}{r}\csc\theta\cos\phi\partial_\phi,$
$g=0$

$X_{14*} = \sin\theta\cos\phi\partial_u - (\csc\theta\cos\phi + \tan\theta\cos\theta\cos\phi)\partial_r - \frac{1}{r}\cos\theta\cos\phi\partial_\theta + \frac{1}{r}\csc\theta\sin\phi\partial_\phi,$
$g=0$

$X_{15*} = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \quad g=0$

$X_{16*} = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi, \quad g=0$

$X_{17*} = \partial_\phi, \quad g=0$

The isometries are selected from Noether symmetries which has no arclength parameter and has a zero gauge function. From this above list, we can easily conclude that $\{X_i : i = 8\dots17\}$ form the 10-dimensional algebra of isometries which with $n=4$ corresponds to the maximal $\frac{1}{2}n(n+1) = 10$-dimensional algebra. That is, the respective manifold is isometric to one of the following:
(a) the 4-dimensional Euclidean space,
(b) the 4-dimensional sphere,
(c) the 4-dimensional projective space,
(d) the 4-dimensional simply connected hyperbolic space (see [36]). This confirms the known result that $m=0$ is equivalent to the Minkowski metric. Each lead to conserved quantities from Noether's theorem, [37]. As a sample case, from X_{15}, we get

$$\begin{aligned} T^{15} &= L\sigma + (\eta - \dot{u}\sigma)\frac{\partial L}{\partial \dot{u}} + (\rho - \dot{r}\sigma)\frac{\partial L}{\partial \dot{r}} + (\tau - \dot{\theta}\sigma)\frac{\partial L}{\partial \dot{\theta}} + (\kappa - \dot{\phi}\sigma)\frac{\partial L}{\partial \dot{\phi}} - g \\ &= 2r^2\dot{\theta}\cos\phi - 2r^2\dot{\phi}\cot\theta\sin^3\theta \end{aligned} \tag{4.7}$$

which, incidentally leads to two of the four Euler-Lagrange equations. The remaining symmetries $\{X_i : i = 1 \dots 7\}$, lead to seven new (previously unknown) conserved quantities. For example, from X_6 we get

$$\begin{aligned} T^6 &= s(-\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2 \sin\theta^2 \dot{\phi}^2) + (\tfrac{1}{2}u - s\dot{u})(-2\dot{u} - 2\dot{r}) \\ &- (r - 2s\dot{r})\dot{u} - 2s\dot{\theta}^2 r^2 - 2s\dot{\phi}^2 r^2 \sin\theta^2 \end{aligned} \tag{4.8}$$

whose total divergence lead to the complete system of Euler-Lagrange equations.

(ii) Case $m = k$, with k an arbitrary constant

$X_1 = \partial_s, \quad g = 0$

$X_2 = \partial_u, \quad g = 0$

$X_3 = \partial_\phi, \quad g = 0$

$X_4 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0$

$X_5 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0$

Here, $\{X_i : i = 2 \dots 5\}$ is the algebra of isometries rendering the metric equivalent to the Schwarzschild metric.

(iii) Case $m = m(u)$, for $m(u)$ an arbitrary function of u

$X_1 = \partial_s, \quad g = 0$

$X_2 = \partial_\phi, \quad g = 0$

$X_3 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0$

$X_4 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0$

Here, a similar conclusion regarding the algebra of isometries as in (iii).

For the Vaidya metric with Finkelstein coordinates given by (4.2), the Noether symmetries are calculated which conclude the subalgebra of isometries. It will be clear that Lie algebra of point symmetries of the geodesic equations will be as large as above. Here the Lagrangian is

$$L = -(1 - \frac{2m(u)}{r})\dot{u}^2 - \frac{4m(u)}{r}\dot{u}\dot{r} + (1 + \frac{2m(u)}{r})\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \qquad (4.9)$$

whose corresponding Noether symmetries separate into the same following cases.

(i) Case $m = 0$:

$X_1 = \frac{1}{2}s^2\partial_s + \frac{1}{2}su\partial_u + \frac{1}{2}rs\partial_r, \quad g = \frac{1}{2}r^2 - \frac{1}{2}u^2$

$X_2 = s\partial_s + \frac{1}{2}u\partial_u + \frac{1}{2}r\partial_r, \quad g = \frac{1}{2}u + r$

$X_3 = \partial_s, \quad g = 0$

$X_4 = -\frac{1}{2}s\partial_u, \quad g = u$

$X_5 = \frac{1}{2}s(-\cot\theta\cos\theta\sin\phi + \csc\theta\sin\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{2}\frac{s}{r}\csc\theta\cos\phi\partial_\phi, \quad g = r\sin\theta\sin\phi$

$X_6 = \frac{1}{2}s(-\cot\theta\cos\theta\cos\phi + \csc\theta\cos\phi)\partial_r + \frac{1}{2}\frac{s}{r}\cos\theta\cos\phi\partial_\theta + \frac{1}{2}\frac{s}{r}\csc\theta\sin\phi\partial_\phi, \quad g = r\sin\theta\cos\phi$

$X_7 = \frac{1}{2}s\cos\theta\partial_r - \frac{1}{2}\frac{s}{r}\sin\theta\partial_\theta, \quad g = r\cos\theta$

$X_8 = -r\sin\theta\cos\phi\partial_u + u\cot\theta\cos\theta\cos\phi - u\csc\theta\cos\phi\partial_r - \frac{u}{r}\cos\theta\cos\phi\partial_\theta + \frac{u}{r}\csc\theta\sin\phi\partial_\phi$, $g = 0$

$X_9 = r\sin\theta\sin\phi\partial_u - u\cot\theta\cos\theta\sin\phi + u\csc\theta\sin\phi\partial_r - \frac{u}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{r}\csc\theta\cos\phi\partial_\phi$, $g = 0$

$X_{10} = -\csc\theta\cos\phi + \cot\theta\cos\theta\cos\phi\partial_r - \frac{u}{r}\cos\theta\cos\phi\partial_\theta + \frac{1}{r}\csc\theta\sin\phi\partial_\phi, \quad g = 0$

$X_{11} = -\csc\theta\sin\phi + \cot\theta\cos\theta\sin\phi\partial_r + \frac{u}{r}\cos\theta\sin\phi\partial_\theta + \frac{1}{r}\csc\theta\cos\phi\partial_\phi, \quad g = 0$

$X_{12} = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0$

$X_{13} = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0$

$X_{14} = \partial_\phi, \quad g = 0$

$X_{15} = -r\cos\theta\partial_u - u\cos\theta\partial_r + \frac{u}{r}\sin\theta\partial_\theta, \quad g = 0$

$X_{16} = -\cos\theta\partial_r + \frac{1}{r}\sin\theta\partial_\theta, \quad g = 0$

$X_{17} = \partial_u, \quad g = 0$

(ii) Case $m = k$, with k an arbitrary constant

$X_1 = \partial_s, \quad g = 0$

$X_2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0$

$X_3 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0$

$X_4 = \partial_\phi, \quad g = 0$

$X_5 = \partial_u, \quad g = 0$

(iii) Case $m = m(u)$, for $m(u)$ an arbitrary function of u

$X_1 = \partial_s, \quad g = 0$

$X_2 = -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \quad g = 0$

$X_3 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad g = 0$

$X_4 = \partial_\phi, \quad g = 0.$

4.4 Discussion and conclusion

We have shown that a large amount information can be extracted from a knowledge of the vector fields that leave the action integral invariant. In addition to obtaining a larger class of conservation laws than those given by the isometries or Killing vectors, we can conclude by identifying the isometries and that these form a Lie subalgebra of the Noether symmetry algebra. We have performed the calculations on versions (4.1) and (4.2) of the Vaidya metric yielding some previously unknown information regarding the corresponding manifold. Lastly, with particular reference to this metric, we concluded that the only variations or classes on $m(u)$ that occur are $m = 0$, $m =$ constant and $m = m(u)$.

Chapter 5

Conservation laws of the Petrov III and Papapetrou metrics

5.1 Introduction

The well known Vaidya metric representing a model for the spherically symmetric solution of the Einstein equations with geometrical optics stress energy tensor of radiation is widely discussed in the literature [31, 32, 33]. A special case of the metric is the well known Papapetrou model [34].

The Petrov classification is a classification of Riemannian spaces according to the algebraic properties of the Weyl tensor (conformal curvature tensor) and the study involving the 'Carter constant and Petrov classification' is conducted in [35]. These classifications are important in the physical interpretations of general relativity. For a non-zero Weyl tensor the various Petrov types are represented by I, II, D, III, N and O [38]. A Weyl tensor is said to be algebraically general if it is of Petrov type I , otherwise it is algebraically special. Also, it is said to be non-degenerate if it is of Petrov type I, II, and III [39].

Physically the most important Petrov classification type is the algebraically special Petrov types. Remembering that symmetries other than the conventional ones (Killing vectors, conformal Killing vectors, symmetries of the Weyl tensor etc) may be of interest to understand the physics of such spacetimes, we find Noether symmetries of the metric [40]. A detailed symmetry analysis and invariance study associated with this particular Petrov III metric, Papapetrou model, will be carried out. The results obtained are compared with other symmetries of the same spacetime metric.

5.2 Lie point Symmetries of the Papapetrou model

In this section we summarize results dealing with the Papapetrou model [34] which is a special case of the Vaidya metric [32], where the mass function is given by $m(u) = u$,

$$\mathrm{d}s^2 = -(1 - \frac{2u}{r})\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{5.1}$$

The Euler-Lagrange (geodesic) equations associated with the natural Lagrangian

$$L = -(1 - \frac{2u}{r})\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{5.2}$$

corresponding to (5.1) are,

$$\begin{aligned} \ddot{u} &= \tfrac{-r^3\dot{\phi}^2 + \cos\theta^2 r^3\dot{\phi}^2 - r^3\dot{\theta}^2 + u\dot{u}^2}{r^2}, \\ \ddot{r} &= \tfrac{-2u\dot{r}\dot{u} + 2ur\ddot{u} - r^2\ddot{u}}{r^2}, \\ \ddot{\theta} &= \tfrac{r\dot{\phi}^2\cos\theta\sin\theta - 2\dot{r}\dot{\theta}}{r}, \\ \ddot{\phi} &= \tfrac{-2r\dot{\phi}\dot{\theta}\csc\theta\cos\theta - 2\dot{r}\dot{\phi}}{r\csc\theta\sin\theta}, \end{aligned} \tag{5.3}$$

where $\dot{\alpha}$ is the derivative of α with respect to the arclength parameter s.

The Papapetrou model leads to six Lie point symmetries given by,

$$
\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_\phi, \\
X_3 &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi \\
X_4 &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \\
X_5 &= s\partial_s, \\
X_6 &= u\partial_u + r\partial_r.
\end{aligned} \tag{5.4}
$$

The algebra of commutators of above symmetries is listed in the table:

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$
$\mathbf{X}_1$	0	0	0	$-\mathbf{X}_5$	$\mathbf{X}_4$	0
$\mathbf{X}_2$	0	0	$\mathbf{X}_2$	0	0	0
$\mathbf{X}_3$	0	$-\mathbf{X}_2$	0	0	0	0
$\mathbf{X}_4$	$\mathbf{X}_5$	0	0	0	$-\mathbf{X}_1$	0
$\mathbf{X}_5$	$-\mathbf{X}_4$	0	0	$\mathbf{X}_1$	0	0
$\mathbf{X}_6$	0	0	0	0	0	0

(5.5)

5.3 Noether Symmetries of the Papapetrou model

In this section we show that we totally recover the information regarding the isometries of the metric from a study of the Noether symmetries associated with the corresponding natural Lagrangian, L, which preserves the action $\mathcal{L} = \int L$ and more. That is, a larger algebra of generators of symmetries is obtained and, hence, more conservation laws classified. We also determine the Lie algebra of symmetry generators of the geodesic equations (Euler-Lagrange equations) which contains all the above as subalgebras.

With the Lagrangian (5.2), the Noether symmetries for $m = 0$ are 17 (of which 10 are isometries) rendering the manifold isomorphic to a Minkowski manifold. The case $m = k$ leads to a metric equivalent to the Schwarzschild metric. The Papapetrou metric coming from $m(u) = u$ admits five Noether symmetries (all gauge functions equal to zero) given by,

$$\begin{aligned} X_1 &= \partial_s, \\ X_2 &= \partial_\phi, \\ X_3 &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \\ X_4 &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \\ X_5 &= s\partial_s + \tfrac{1}{2}u\partial_u + \tfrac{1}{2}r\partial_r. \end{aligned} \tag{5.6}$$

Here, $\{X_2, X_3, X_4\}$ form the basis of a 3-dimensional algebra of isometries and X_5 provides an extra nontrivial conservation law. Notice that X_5 is a linear combination of X_5 and X_6 from the list of Lie symmetries. Separately, these are not Noether symmetries. The extra conserved quantity obtained from this symmetry is given by

$$\begin{aligned} T^5 &= s(-(1-\tfrac{2u}{r})\dot{u}^2 - 2\dot{u}\dot{r} + r^2\dot{\theta}^2 + r^2\sin\theta^2\dot{\phi}^2) + (\tfrac{1}{2}u - s\dot{u})(-2\dot{u}(1-\tfrac{2u}{r}) - 2\dot{r}) \\ &- (r-2s\dot{r})\dot{u} - 2s\dot{\theta}^2r^2 - 2s\dot{\phi}^2r^2\sin\theta^2. \end{aligned} \tag{5.7}$$

5.4 Lie point Symmetries of the Petrov type III metric

In relativity algebraically special Petrov types have interesting interpretations. In order to understand one of the Petrov type III metrics in more depth, we investigate Noether conservation laws admitted by one of the Petrov type III metrics defined by [38],

$$\mathrm{d}s^2 = -\frac{z^2}{1+t^2}\mathrm{d}t^2 + z^2 f\mathrm{d}x^2 + \frac{z^2t^2(1+t^2)}{f}\mathrm{d}y^2 + \mathrm{d}z^2, \tag{5.8}$$

where

$$f = t^{2b}(1+t^2)^{1-b}. \tag{5.9}$$

This is a Ricci, Riemann and Weyl flat metric for both $b = 0$ or $b = 1$. For this metric the Lagrangian is given by,

$$L = -\frac{z^2}{1+t^2}\dot{t}^2 + z^2t^{2b}(1+t^2)^{1-b}\dot{x}^2 + z^2t^{2(1-b)}(1+t^2)^b\dot{y}^2 + \dot{z}^2 \tag{5.10}$$

which gives rise to the geodesic (Euler-Lagrange) equations:

$$-\frac{2zt'^2}{1+t^2}+2zt^{2b}(1+t^2)^{1-b}x'^2+2zt^{2(1-b)}(1+t^2)^by'^2-2z''=0,$$

$$-\left(\frac{2z^2t^{2(1-b)}(2-2b)(1+t^2)^by'}{t}+\frac{4bz^2t^{(2-2b)}(1+t^2)^bty'}{(1+t^2)}\right)t'-2z^2t^{2(1-b)}(1+t^2)^by''$$
$$-4zt^{2(1-b)}(1+t^2)^by'z'=0,$$

$$-\frac{2z^2tt'^2}{(1+t^2)^2}+\frac{2z^2t^{2b}b(1+t^2)^{1-b}x'^2}{t}+\frac{2z^2t^{2b+1}(1-b)(1+t^2)^{1-b}x'^2}{(1+t^2)}+\frac{z^2t^{2(1-b)}(2-2b)(1+t^2)^by'^2}{t}$$
$$+\frac{2z^2t^{2(1-b)+1}b(1+t^2)^by'^2}{(1+t^2)}+\frac{2z^2t''}{(1+t^2)}+\frac{4zt'z'}{(1+t^2)}=0,$$

$$-\left(\frac{4bz^2t^{2b}(1+t^2)^{1-b}x'}{t}+\frac{4(1-b)z^2t^{2b}(1+t^2)^{1-b}tx'}{(1+t^2)}\right)t'-2z^2t^{2b}(1+t^2)^{1-b}x''$$
$$-4zt^{2b}(1+t^2)^{1-b}z'x'=0. \tag{5.11}$$

The Lie point symmetries of the this system are enumerated below for some special cases.

(i) Case $b=0$:

$X_1=\partial_s,$
$X_2=s\partial_s,$
$X_3=e^{-y}tz\partial_s,$
$X_4=e^{y}tz\partial_s,$
$X_5=\sqrt{1+t^2}z\cos x\partial_s,\ X_6=\sqrt{1+t^2}z\sin x\partial_s,$
$X_7=\partial_x,$
$X_8=\partial_y,$
$X_9=e^{-y}\sqrt{1+t^2}\cos x\partial_t+\frac{e^{-y}\sqrt{1+t^2}\cos x}{t}\partial_y-\frac{e^{-y}t\sin x}{\sqrt{1+t^2}}\partial_x,$
$X_{10}=e^{y}\sqrt{1+t^2}\cos x\partial_t-\frac{e^{y}\sqrt{1+t^2}\cos x}{t}\partial_y-\frac{e^{y}t\sin x}{\sqrt{1+t^2}}\partial_x,$
$X_{11}=e^{-y}\sqrt{1+t^2}\sin x\partial_t+\frac{e^{-y}\sqrt{1+t^2}\sin x}{t}\partial_y+\frac{e^{-y}t\cos x}{\sqrt{1+t^2}}\partial_x,$
$X_{12}=e^{y}\sqrt{1+t^2}\sin x\partial_t-\frac{e^{y}\sqrt{1+t^2}\sin x}{t}\partial_y+\frac{e^{y}t\cos x}{\sqrt{1+t^2}}\partial_x,$
$X_{13}=z\partial_z,$
$X_{14}=\frac{e^{-y}(1+t^2)}{z}\partial_t-e^{-y}t\partial_z+\frac{e^{-y}}{tz}\partial_y,\ X_{15}=\frac{e^{y}(1+t^2)}{z}\partial_t-e^{y}t\partial_z-\frac{e^{y}}{tz}\partial_y,$

$X_{16} = \frac{e^{-y}s(1+t^2)}{z}\partial_t - e^{-y}st\partial_z + \frac{e^{-y}s}{tz}\partial_y,$
$X_{17} = \frac{e^{y}s(1+t^2)}{z}\partial_t - e^{y}st\partial_z - \frac{e^{y}s}{tz}\partial_y,$
$X_{18} = s^2\partial_s + sz\partial_z,$
$X_{19} = e^{-2y}\partial_y - e^{-2y}t^2z\partial_z + e^{-2y}(t+t^3)\partial_t,$
$X_{20} = e^{2y}\partial_y + e^{2y}t^2z\partial_z - e^{2y}(t+t^3)\partial_t,$
$X_{21} = e^{-y}stz\partial_s + e^{-y}tz^2t\partial_z,$
$X_{22} = e^{y}stz\partial_s + e^{y}tz^2t\partial_z,$
$X_{23} = -(1+t^2)z\partial_z + (t+t^3)\partial_t,$ $X_{24} = \sqrt{1+t^2}z^2\cos x\partial_z + s\sqrt{1+t^2}z\cos x\partial_s,$
$X_{25} = -\sqrt{1+t^2}\cos x\partial_z + \frac{\sin x}{z\sqrt{1+t^2}}\partial_x + \frac{t\sqrt{1+t^2}\cos x}{z}\partial_t,$
$X_{26} = -s\sqrt{1+t^2}\cos x\partial_z + \frac{s\sin x}{z\sqrt{1+t^2}}\partial_x + \frac{st\sqrt{1+t^2}\cos x}{z}\partial_t,$
$X_{27} = -e^{-y}t\sqrt{1+t^2}z\cos x\partial_z + e^{-y}t^2\sqrt{1+t^2}\cos x\partial_t + \frac{e^{-y}t\sin x}{\sqrt{1+t^2}}\partial_x,$
$X_{28} = -e^{y}t\sqrt{1+t^2}z\cos x\partial_z + e^{y}t^2\sqrt{1+t^2}\cos x\partial_t + \frac{e^{y}t\sin x}{\sqrt{1+t^2}}\partial_x,$
$X_{29} = (1+t^2)z\cos 2x\partial_z - \sin 2x\partial_x - t(1+t^2)\cos 2x\partial_t,$
$X_{30} = \sqrt{1+t^2}z^2\sin x\partial_z + s\sqrt{1+t^2}z\sin x\partial_s$ $X_{31} = -\sqrt{1+t^2}\sin x\partial_z - \frac{\cos x}{z\sqrt{1+t^2}}\partial_x + \frac{t\sqrt{1+t^2}\sin x}{z}\partial_t,$
$X_{32} = -s\sqrt{1+t^2}\sin x\partial_z - \frac{s\cos x}{z\sqrt{1+t^2}}\partial_x + \frac{st\sqrt{1+t^2}\sin x}{z}\partial_t,$
$X_{33} = -e^{-y}t\sqrt{1+t^2}z\sin x\partial_z + e^{-y}t^2\sqrt{1+t^2}\sin x\partial_t - \frac{e^{-y}t\cos x}{\sqrt{1+t^2}}\partial_x,$
$X_{34} = -e^{y}t\sqrt{1+t^2}z\sin x\partial_z + e^{y}t^2\sqrt{1+t^2}\sin x\partial_t - \frac{e^{y}t\cos x}{\sqrt{1+t^2}}\partial_x,$
$X_{35} = -(1+t^2)z\sin 2x\partial_z - \cos 2x\partial_x + t(1+t^2)\sin 2x\partial_t.$

(ii) Case $b = 1$:

$X_1 = \partial_s,$
$X_2 = s\partial_s,$
$X_3 = e^{-x}tz\partial_s,$
$X_4 = e^{x}tz\partial_s,$
$X_5 = \sqrt{1+t^2}z\cos y\partial_s,$
$X_6 = \sqrt{1+t^2}z\sin y\partial_s,$
$X_7 = \partial_x,$
$X_8 = \partial_y,$
$X_9 = e^{-x}\sqrt{1+t^2}\cos y\partial_t + \frac{e^{-x}\sqrt{1+t^2}\cos y}{t}\partial_y - \frac{e^{-x}t\sin y}{\sqrt{1+t^2}}\partial_x,$

$X_{10} = e^x\sqrt{1+t^2}\cos y\partial_t - \frac{e^x\sqrt{1+t^2}\cos y}{t}\partial_y - \frac{e^x t\sin y}{\sqrt{1+t^2}}\partial_x,$
$X_{11} = e^{-x}\sqrt{1+t^2}\sin y\partial_t + \frac{e^{-x}\sqrt{1+t^2}\sin y}{t}\partial_y + \frac{e^{-x} t\cos y}{\sqrt{1+t^2}}\partial_x,$
$X_{12} = e^x\sqrt{1+t^2}\sin y\partial_t - \frac{e^x\sqrt{1+t^2}\sin y}{t}\partial_y + \frac{e^x t\cos y}{\sqrt{1+t^2}}\partial_x,$
$X_{13} = z\partial_z,$
$X_{14} = \frac{e^{-x}(1+t^2)}{z}\partial_t - e^{-x}t\partial_z + \frac{e^{-x}}{tz}\partial_x,$
$X_{15} = \frac{e^{x}(1+t^2)}{z}\partial_t - e^{x}t\partial_z - \frac{e^{x}}{tz}\partial_x,$
$X_{16} = \frac{e^{-x}s(1+t^2)}{z}\partial_t - e^{-x}st\partial_z + \frac{e^{-x}s}{tz}\partial_x,$
$X_{17} = \frac{e^{x}s(1+t^2)}{z}\partial_t - e^{x}st\partial_z - \frac{e^{x}s}{tz}\partial_y,$
$X_{18} = s^2\partial_s + sz\partial_z,$
$X_{19} = e^{-2x}\partial_x - e^{-2x}t^2z\partial_z + e^{-2x}(t+t^3)\partial_t,$
$X_{20} = e^{2x}\partial_x + e^{2x}t^2z\partial_z - e^{2x}(t+t^3)\partial_t,$
$X_{21} = e^{-x}stz\partial_s + e^{-x}tz^2t\partial_z,$
$X_{22} = e^{x}stz\partial_s + e^{x}tz^2t\partial_z,$
$X_{23} = -(1+t^2)z\partial_z + (t+t^3)\partial_t,$
$X_{24} = \sqrt{1+t^2}z^2\cos y\partial_z + s\sqrt{1+t^2}z\cos y\partial_s,$
$X_{25} = -\sqrt{1+t^2}\cos y\partial_z + \frac{\sin y}{z\sqrt{1+t^2}}\partial_y + \frac{t\sqrt{1+t^2}\cos y}{z}\partial_t,$
$X_{26} = -s\sqrt{1+t^2}\cos y\partial_z + \frac{s\sin y}{z\sqrt{1+t^2}}\partial_y + \frac{st\sqrt{1+t^2}\cos y}{z}\partial_t,$
$X_{27} = -e^{-x}t\sqrt{1+t^2}z\cos y\partial_z + e^{-x}t^2\sqrt{1+t^2}\cos y\partial_t + \frac{e^{-x}t\sin y}{\sqrt{1+t^2}}\partial_y,$
$X_{28} = -e^{x}t\sqrt{1+t^2}z\cos y\partial_z + e^{x}t^2\sqrt{1+t^2}\cos y\partial_t + \frac{e^{x}t\sin y}{\sqrt{1+t^2}}\partial_y,$
$X_{29} = (1+t^2)z\cos 2y\partial_z - \sin 2y\partial_y - t(1+t^2)\cos 2y\partial_t,$
$X_{30} = \sqrt{1+t^2}z^2\sin y\partial_z + s\sqrt{1+t^2}z\sin y\partial_s,$
$X_{31} = -\sqrt{1+t^2}\sin y\partial_z - \frac{\cos y}{z\sqrt{1+t^2}}\partial_y + \frac{t\sqrt{1+t^2}\sin y}{z}\partial_t,$
$X_{32} = -s\sqrt{1+t^2}\sin y\partial_z - \frac{s\cos x}{z\sqrt{1+t^2}}\partial_y + \frac{st\sqrt{1+t^2}\sin y}{z}\partial_t,$
$X_{33} = -e^{-x}t\sqrt{1+t^2}z\sin y\partial_z + e^{-x}t^2\sqrt{1+t^2}\sin y\partial_t - \frac{e^{-x}t\cos y}{\sqrt{1+t^2}}\partial_y,$
$X_{34} = -e^{x}t\sqrt{1+t^2}z\sin y\partial_z + e^{x}t^2\sqrt{1+t^2}\sin y\partial_t - \frac{e^{x}t\cos y}{\sqrt{1+t^2}}\partial_y,$
$X_{35} = -(1+t^2)z\sin 2y\partial_z - \cos 2y\partial_y + t(1+t^2)\sin 2y\partial_t.$

(iii) Case $b = \frac{1}{2}$:
$X_1 = \partial_s,$
$X_2 = s\partial_s,$
$X_3 = s^2\partial_s + sz\partial_z,$
$X_4 = z\partial_z,$

$X_5 = -y\partial_x + x\partial_y,$
$X_6 = \partial_x, \quad ,$
$X_7 = \partial_y.$

5.5 Noether Symmetries of the Petrov type III metric

From the Lagrangian defined by (5.10), we generate the Noether symmetries via the defining equation

$$XL + L(\xi_s + t_s\xi_t + x_s\xi_x + y_s\xi_y + z_s\xi_z) = (g_s + t_sg_t + x_sg_x + y_sg_y + z_sg_z), \quad (5.12)$$

where g is a gauge term. Moreover, in the list below, without the need for further calculations involving the Lie derivative of the metric, the vector fields marked with an asterisk (*) are identified as the isometries of the manifold. The isometries are selected from Noether symmetries which has no arclength variable and has a zero gauge function.

(i) Case $b = 0$

$X_1 = \partial_s, \quad g = 0,$
$X_2 = s\partial_s + \frac{1}{2}z\partial_z, \quad g = 0,$
$X_3 = \frac{1}{2}s^2\partial_s + \frac{1}{2}sz\partial_z, \quad g = \frac{1}{2}z^2,$
$X_4 = -\frac{1}{2}\frac{se^y(1+t^2)}{z}\partial_t + \frac{1}{2}\frac{se^y}{zt}\partial_y + \frac{1}{2}e^yst\partial_z, \quad g = e^yzt,$
$X_5 = -\frac{1}{2}\frac{se^{-y}(1+t^2)}{z}\partial_t - \frac{1}{2}\frac{se^{-y}}{zt}\partial_y + \frac{1}{2}e^{-y}st\partial_z, \quad g = e^{-y}zt,$
$X_6 = -\frac{1}{2}\frac{ts\sqrt{1+t^2}\sin x}{z}\partial_t + \frac{1}{2}\frac{s\cos x}{z\sqrt{1+t^2}}\partial_x + \frac{1}{2}s\sin x\sqrt{1+t^2}\partial_z, \quad g = z\sqrt{1+t^2}\sin x,$
$X_7 = -\frac{1}{2}\frac{ts\sqrt{1+t^2}\cos x}{z}\partial_t + \frac{1}{2}\frac{s\sin x}{z\sqrt{1+t^2}}\partial_x + \frac{1}{2}s\cos x\sqrt{1+t^2}\partial_z, \quad g = z\sqrt{1+t^2}\cos x,$
$X_{8*} = -\frac{e^{-y}(1+t^2)}{z}\partial_t - \frac{e^{-y}}{zt}\partial_y + e^{-y}t\partial_z, \quad g = 0,$
$X_{9*} = -\frac{e^{y}(1+t^2)}{z}\partial_t + \frac{e^{y}}{zt}\partial_y + e^{y}t\partial_z, \quad g = 0,$
$X_{10*} = \partial_x, \quad g = 0,$

$$
\begin{aligned}
X_{11*} &= -\tfrac{-t\sqrt{1+t^2}\sin x}{z}\partial_t + \tfrac{\cos x}{z\sqrt{1+t^2}}\partial_x + \sqrt{1+t^2}\sin x\partial_z, \quad g=0,\\
X_{12*} &= -\tfrac{-t\sqrt{1+t^2}\cos x}{z}\partial_t - \tfrac{\sin x}{z\sqrt{1+t^2}}\partial_x + \sqrt{1+t^2}\cos x\partial_z, \quad g=0,\\
X_{13*} &= -e^y\sqrt{1+t^2}\cos x\partial_t + \tfrac{e^y t\sin x}{\sqrt{1+t^2}}\partial_x + \tfrac{e^y\sqrt{1+t^2}\cos x}{t}\partial_y, \quad g=0,\\
X_{14*} &= e^y\sqrt{1+t^2}\sin x\partial_t + \tfrac{e^y t\cos x}{\sqrt{1+t^2}}\partial_x - \tfrac{e^y\sqrt{1+t^2}\sin x}{t}\partial_y, \quad g=0,\\
X_{15*} &= \partial_y, \quad g=0,\\
X_{16*} &= -e^{-y}\sqrt{1+t^2}\cos x\partial_t + \tfrac{e^{-y}t\sin x}{\sqrt{1+t^2}}\partial_x - \tfrac{e^{-y}\sqrt{1+t^2}\cos x}{t}\partial_y, \quad g=0,\\
X_{17*} &= e^{-y}\sqrt{1+t^2}\sin x\partial_t + \tfrac{e^{-y}t\cos x}{\sqrt{1+t^2}}\partial_x + \tfrac{e^{-y}\sqrt{1+t^2}\sin x}{t}\partial_y, \quad g=0
\end{aligned}
$$

From the above list, we can easily conclude that the symmetries form a 10-dimensional algebra of isometries for which with $n = 4$ corresponds to the maximal $\frac{1}{2}n(n+1) =$ 10-dimensional algebra. That is, the respective manifold is isometric to one of the (a) the 4-dimensional Euclidean space, (b) the 4-dimensional sphere, (c) the 4-dimensional projective space or (d) the 4-dimensional simply connected hyperbolic space (see [36]). This implies that for $b = 0$, the Petrov III manifold is isomorphic to the Minkowski manifold.

(ii) Case $b = 1$

$$
\begin{aligned}
X_1 &= \partial_s, \quad g=0,\\
X_2 &= s\partial_s + \tfrac{1}{2}z\partial_z, \quad g=0,\\
X_3 &= \tfrac{1}{2}s^2\partial_s + \tfrac{1}{2}sz\partial_z, \quad g=\tfrac{1}{2}z^2\\
X_4 &= -\tfrac{1}{2}\tfrac{st\sqrt{1+t^2}\sin y}{z}\partial_t + \tfrac{1}{2}\tfrac{s\cos y}{z\sqrt{1+t^2}}\partial_y + \tfrac{1}{2}s\sqrt{1+t^2}\sin y\partial_z, \quad g=z\sqrt{1+t^2}\sin y,\\
X_5 &= -\tfrac{1}{2}\tfrac{st\sqrt{1+t^2}\cos y}{z}\partial_t - \tfrac{1}{2}\tfrac{s\sin y}{z\sqrt{1+t^2}}\partial_y + \tfrac{1}{2}s\sqrt{1+t^2}\cos y\partial_z, \quad g=z\sqrt{1+t^2}\cos y,\\
X_6 &= -\tfrac{1}{2}\tfrac{se^{-x}(1+t^2)}{z}\partial_t - \tfrac{1}{2}\tfrac{se^{-x}}{zt}\partial_x + \tfrac{1}{2}e^{-x}st\partial_z, \quad g=e^{-x}zt,\\
X_7 &= -\tfrac{1}{2}\tfrac{se^{x}(1+t^2)}{z}\partial_t - \tfrac{1}{2}\tfrac{se^{x}}{zt}\partial_x + \tfrac{1}{2}e^{x}st\partial_z, \quad g=e^{x}zt,\\
X_{8*} &= -\tfrac{t\sqrt{1+t^2}\sin y}{z}\partial_t + \tfrac{\cos y}{z\sqrt{1+t^2}}\partial_y + \sin y\sqrt{1+t^2}\partial_z, \quad g=0,\\
X_{9*} &= -\tfrac{t\sqrt{1+t^2}\cos y}{z}\partial_t - \tfrac{\sin y}{z\sqrt{1+t^2}}\partial_y + \cos y\sqrt{1+t^2}\partial_z, \quad g=0,\\
X_{10*} &= -\tfrac{e^{-x}(1+t^2)}{z}\partial_t - \tfrac{e^{-x}}{zt}\partial_x + e^{-x}t\partial_z, \quad g=0,\\
X_{11*} &= -\tfrac{e^{x}(1+t^2)}{z}\partial_t + \tfrac{e^{x}}{zt}\partial_x + e^{x}t\partial_z, \quad g=0,\\
X_{12*} &= \partial_y, \quad g=0,\\
X_{13*} &= -e^{-x}\sqrt{1+t^2}\cos y\partial_t - \tfrac{e^{-x}\sqrt{1+t^2}\cos y}{t}\partial_x + \tfrac{e^{-x}t\sin y}{\sqrt{1+t^2}}\partial_y, \quad g=0,
\end{aligned}
$$

$$X_{14*} = e^{-x}\sqrt{1+t^2}\sin y\partial_t + \frac{e^{-x}\sqrt{1+t^2}\sin y}{t}\partial_x + \frac{e^{-x}t\cos y}{\sqrt{1+t^2}}\partial_y, \quad g = 0,$$
$$X_{15*} = -e^{x}\sqrt{1+t^2}\cos y\partial_t + \frac{e^{x}\sqrt{1+t^2}\cos y}{t}\partial_x + \frac{e^{x}t\sin y}{\sqrt{1+t^2}}\partial_y, \quad g = 0,$$
$$X_{16*} = e^{x}\sqrt{1+t^2}\sin y\partial_t - \frac{e^{x}\sqrt{1+t^2}\sin y}{t}\partial_x + \frac{e^{x}t\cos y}{\sqrt{1+t^2}}\partial_y, \quad g = 0,$$
$$X_{17*} = \partial_x, \quad g = 0,$$

From this list, we again conclude that the symmetries form a 10-dimensional algebra of isometries so that the manifold is again equivalent to the Minkowski manifold (see [36]).

(iii) Case $b = \frac{1}{2}$

$$X_1 = \partial_s, \quad g = 0,$$
$$X_{2*} = -y\partial_x + x\partial_y, \quad g = 0,$$
$$X_{3*} = \partial_x, \quad g = 0,$$
$$X_{4*} = \partial_y, \quad g = 0,$$
$$X_5 = s\partial_s + \frac{1}{2}z\partial_z, \quad g = 0,$$
$$X_6 = \frac{1}{2}s^2\partial_s + \frac{1}{2}sz\partial_z, \quad g = \frac{1}{2}z^2$$

(iv) Case All b

$$X_1 = \partial_s, \quad g = 0,$$
$$X_{2*} = \partial_x, \quad g = 0,$$
$$X_{3*} = \partial_y, \quad g = 0$$

Thus, the most general case of the Petrov III metric admits the two-dimensional algebra of isometries with the basis being translations in x and y. The particular case $b = \frac{1}{2}$ has an additional isometry given by rotation in the $x - y$ plane corresponding to angular momentum there and further conservation laws from X_5 and X_6.

5.6 Discussion and conclusion

We have shown that a large amount of information can be extracted from a knowledge of the vector fields (one parameter Lie group transformations) that leave the action integral invariant. In addition, a larger class of conservation laws are found, other than those given by the isometries or Killing vectors, implying that the isometries form a Lie subalgebra of the Noether point symmetries. Particularly, the metric (5.8) admits 10 Killing vectors for both $b = 0$ and 1 respectively. Note that Ricci curvature tensor, represents the amount by which the volume element of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space. As such, it provides one way of measuring the degree to which the geometry determined by a given Riemannian metric might differ from that of ordinary Euclidean n-space. More generally, the Ricci tensor is defined on any pseudo-Riemannian manifold. Like the metric itself, the Ricci tensor is a symmetric bilinear form on the tangent space of the manifold. We observe that, (5.8) has all the Ricci tensor components are equal to zero except $R_{11} = \frac{2b(b-1)}{t^2(1+t^2)}$ and is Ricci flat for both $b = 0$ and 1. The case $b = 1/2$ admits six Noether point symmetries of which three are Killing vectors.

Chapter 6

Conservation laws of the wave equation on Vaidya manifolds

6.1 Introduction

The Lie and Noether symmetries of the geodesic equations have been discussed in detail, in [40] - the more interesting case being the latter since these lead to conservation laws via Noether's theorem [1]. We recover the isometries of the metric from a study of the Noether symmetries associated with the corresponding natural Lagrangian, L.

The standard wave equation, in (1+3) dimensions, has been extensively studied in the literature from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation is discussed by Ibragimov [41]. It is well known that in three dimensional Euclidean space, the linear wave equation admits a maximal 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry.

In this work, we use a purely geometric consideration to construct the wave equation in a curved background geometry in such a way that the wave equation inherits nonlinearities of the respective geometry in a natural way. Keeping in mind that the wave equation in four dimensional spacetime may be of more physical significance, we use, for our purposes, the Vaidya manifold.

6.2 The Vaidya metric and the wave equation in curved geometry

Firstly, we note that the Vaidya metric [32] is given by,

$$\mathrm{d}s^2 = -(1 - \frac{2m(t)}{r})\mathrm{d}t^2 - 2\mathrm{d}t\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{6.1}$$

for which a special case, $m(t) = t$, is known as the Papapetrou model [34].

A wave equation on a Lorentzian manifold endowed with a metric g_{ij} is given by the expression

$$\Box u(\bar{x}, t) = g^{00}\partial^2_{00} + \frac{1}{2}g^{ij}[g^{00}(\partial_i g_{00})\partial_j + \partial_{ij}\Gamma^k_{ij}\partial_k]u(\bar{x}, t) = 0 \tag{6.2}$$

where $u(\bar{x}, t)$ is some given wave function,

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij}) \tag{6.3}$$

represents the Christoffel symbol, with g^{ij} is the inverse of the metric g_{ij} with polar variables r, θ, ϕ, and

$$g_{ij} = \begin{pmatrix} -(1 - \frac{2m(t)}{r}) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Consequently, the wave equation in a 'curved' Vaidiya background takes the form

$$\begin{aligned} &-r^2\sin\theta u_{rt} - 2r\sin\theta u_t + 2r\sin\theta(1 - \tfrac{2m(t)}{r})u_r + 2\sin\theta m(t)u_r \\ &+r^2\sin\theta(1 - \tfrac{2m(t)}{r})u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + u_{\phi\phi} = 0. \end{aligned} \tag{6.4}$$

We will classify the Lie and Noether point symmetries of this equation and show the effect of the curved background on the respective symmetry algebras.

6.3 Lie Symmetries

We determine the Lie point symmetry generators of the wave equation (6.4) and split these into various cases. The *principle Lie algebra* is stated in (v) below. The most significant result we note is the reduction in the dimension of the algebra of Lie point symmetries when compared with the algebra of the wave equation on a Minkowski manifold. This, as will be seen later, also has consequences on the number of *standard* conservation laws (usually first-order) of (6.4). Furthermore, the number of exact or invariant solutions are reduced drastically. For illustration, we perform a reduction corresponding to some two-dimensional subalgebras.

(i) Case $m = 0$

$X_1 = \partial_t,$

$X_2 = t\partial_t + r\partial_r,$

$X_3 = u\partial_u,$

$X_4 = F(t, r, \theta, \phi)\partial_u$, where the function $F(t, r, \theta, \phi)$ satisfies the equation,
$F_{\phi\phi} + \cos\theta F_\theta + \sin\theta F_{\theta\theta} + 2r \sin\theta F_r + r^2 \sin\theta F_{rr} - 2r \sin\theta F_t - 2r^2 \sin\theta F_{tr} = 0.$

$X_5 = t^2\partial_t + 2r(r + t)\partial_r - 2u(r + t)\partial_u,$

$X_6 = \partial_\phi.$

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	X_1	0	X_4	$2X_2 - 2X_3$	0
X_2	$-X_1$	0	0	$(rF_r + tF_t)\partial_u$	X_5	0
X_3	0	0	0	$-X_4$	0	0
X_4	$-X_4$	$-(rF_r + tF_t)\partial_u$	X_4	0	$(-2(r+t)F - 2r(r+t)F_r - t^2F_t)\partial_u$	X_4
X_5	$-2X_2 + 2X_3$	$-X_5$	0	$-((-2(r+t)F - 2r(r+t)F_r - t^2F_t)\partial_u)$	0	0
X_6	0	0	0	$-X_4$	0	0

It is well known that the case $m = 0$ is 'isomorphic' to a 'flat' manifold and one would expect a maximal 16-dimensional algebra - it is clear that the wave equation is somewhat 'distorted' even in this case and the number of conservation laws will be reduced (see below for a confirmation of this).

(ii) Case $m = k$, with k an arbitrary constant

$X_1 = \partial_t$,

$X_2 = u\partial_u$,

$X_3 = F(t, r, \theta, \phi)\partial_u$, where the function $F(t, r, \theta, \phi)$ satisfies the equation, $F_{\phi\phi} + \cos\theta F_\theta + \sin\theta F_{\theta\theta} - 2k\sin\theta F_r + 2r\sin\theta F_r - 2kr\sin\theta F_{rr} + r^2\sin\theta F_{rr} - 2r\sin\theta F_t - 2r^2\sin\theta F_{tr} = 0$.

$X_4 = \partial_\phi$.

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_3	0
X_2	0	0	$-X_3$	0
X_3	$-X_3$	X_3	0	X_3
X_4	0	0	$-X_3$	0

(6.5)

(iii) Case $m(t) = t$

$X_1 = t\partial_t + r\partial_r,$

$X_2 = u\partial_u,$

$X_3 = F(t,r,\theta,\phi)\partial_u$, where the function $F(t,r,\theta,\phi)$ satisfies the equation, $F_{\phi\phi} + \cos\theta F_\theta + \sin\theta F_{\theta\theta} + 2r\sin\theta F_r - 2t\sin\theta F_r + r^2\sin\theta F_{rr} - 2rt\sin\theta F_{rr} - 2r\sin\theta F_t - 2r^2\sin\theta F_{tr} = 0.$

$X_4 = \partial_\phi.$

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$(rF_r + tF_t)\partial_u$	0
X_2	0	0	$-X_3$	0
X_3	$-(rF_r + tF_t)\partial_u$	X_3	0	X_3
X_4	0	0	$-X_3$	0

(6.6)

(iv) Case $m = e^t$

$X_1 = u\partial_u,$

$X_2 = F(t,r,\theta,\phi)\partial_u$, where the function $F(t,r,\theta,\phi)$ satisfies the equation,
$F_{\phi\phi} + \cos\theta F_\theta + \sin\theta F_{\theta\theta} - 2e^t\sin\theta F_r + 2r\sin\theta F_r - 2e^t r\sin\theta F_{rr} + r^2\sin\theta F_{rr} - 2r\sin\theta F_t - 2r^2\sin\theta F_{tr} = 0.$

$X_3 = \partial_\phi.$

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3
X_1	0	$-X_2$	0
X_2	X_2	0	X_2
X_3	0	$-X_2$	0

(6.7)

(v) Case $m = m(t)$, for $m(t)$ an arbitrary function of t

$X_1 = u\partial_u$,

$X_2 = F(t, r, \theta, \phi)\partial_u$, where the function $F(t, r, \theta, \phi)$ satisfies the equation, $F_{\phi\phi} + \cos\theta F_\theta + \sin\theta F_{\theta\theta} + 2r\sin\theta F_r - 2m(t)\sin\theta F_r + r^2\sin\theta F_{rr} - 2m(t)r\sin\theta F_{rr} - 2r\sin\theta F_t - 2r^2\sin\theta F_{tr} = 0$,

$X_3 = \partial_\phi$.

which yields the following commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	X_1	X_2	X_3
X_1	0	$-X_2$	0
X_2	X_2	0	X_2
X_3	0	$-X_2$	0

(6.8)

6.4 Reduction of order

We demonstrate two reductions of the wave equation using two-dimensional subalgebras for the case $m = t$. In both cases, this leads to a partial differential equation in just two independent variables which can be further analysed using another Lie symmetry reduction or an appropriate, alternative method.

(i). If $\bar{X}_1 = X_2 + X_4 = \partial_\phi + u\partial_u$, $[\bar{X}_1, X_1] = 0$ so that reducing may begin with either $\bar{X}_1$ or X_1. The respective wave equation

$$\begin{aligned}&-2r^2 \sin\theta u_{rt} - 2r\sin\theta u_t + 2r\sin\theta(1-\tfrac{2t}{r})u_r + 2t\sin\theta u_r \\ &+r^2\sin\theta(1-\tfrac{2t}{r})u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + u_{\phi\phi} = 0\end{aligned} \tag{6.9}$$

becomes

$$\begin{aligned}&-2r^2 \sin\theta w_{rt} - 2r\sin\theta w_t + 2r\sin\theta(1-\tfrac{2t}{r})w_r + 2t\sin\theta w_r \\ &+r^2\sin\theta(1-\tfrac{2t}{r})w_{rr} + \cos\theta w_\theta + \sin\theta w_{\theta\theta} + w = 0.\end{aligned} \tag{6.10}$$

via the generator $\bar{X}_1$ since

$$\frac{dt}{0} = \frac{dr}{0} = \frac{d\theta}{0} = \frac{d\phi}{1} = \frac{du}{u}$$

leads to the new dependent variable w defined by $u = w(t, r, \theta)e^\phi$. From $X_1 = t\partial_t + r\partial_r$, we get

$$\frac{dt}{t} = \frac{dr}{r} = \frac{d\theta}{0} = \frac{dw}{0}$$

so that $w = W(\alpha, \theta)$ leads to the partial differential equation

$$\begin{aligned}&(2\alpha^3 + \alpha^2 - \alpha)\sin\theta W_{\alpha\alpha} + (4\alpha^2 + 2\alpha - 2)\sin\theta W_\alpha \\ &+\cos\theta W_\theta + \sin\theta W_{\theta\theta} + W = 0\end{aligned} \tag{6.11}$$

where $\alpha = r/t$.

(ii). If we first reduce using $X_4 = \partial_\phi$ we get

$$\begin{aligned}&-2r^2 \sin\theta u_{rt} - 2r\sin\theta u_t + 2r\sin\theta(1-\tfrac{2t}{r})u_r + 2t\sin\theta u_r \\ &+r^2\sin\theta(1-\tfrac{2t}{r})u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} = 0\end{aligned} \tag{6.12}$$

and then $\bar{X}_2 = X_1 + X_2 = u\partial_u + t\partial_t + r\partial_r$ leads to

$$\frac{dt}{t} = \frac{dr}{r} = \frac{d\theta}{0} = \frac{du}{u}$$

so that with invariants $\alpha = r/t$ and $w(\theta, \alpha) = tu$ we get the reduced partial differential equation

$$\begin{aligned}&(2\alpha^3 + \alpha^2 - 2\alpha)\sin\theta w_{\alpha\alpha} + (2\alpha^2 + 2\alpha - 2)\sin\theta w_\alpha \\ &-2\alpha\sin\theta w + \cos\theta w_\theta + \sin\theta w_{\theta\theta} = 0.\end{aligned} \tag{6.13}$$

6.5 Noether symmetries and conservation laws

Since (6.4) is variational, we determine the Noether symmetries X which are given by a Killing-type equation

$$XL + LD_i\xi_i = D_i g_i. \tag{6.14}$$

and the corresponding conserved flows $(T^t, T^r, T^\theta, T^\phi)$ via Noether's theorem. Firstly, it can be shown that a Lagrangian of (6.4) is given by

$$L = -r^2 \sin\theta u_t u_r + \frac{1}{2} r^2 \sin\theta (1 - \frac{2m(t)}{r}) u_r^2 + \frac{1}{2}\sin\theta u_\theta^2 + \frac{1}{2} u_\phi^2. \tag{6.15}$$

(i). Case $m = 0$.

(a) $X_1 = \partial_t$

$$\begin{aligned} T^t = \; & \tfrac{1}{2}(r^2 \sin\theta u_r u_t - u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} + r(2u_r + ru_{rr} \\ & -2u_t - ru_{tr})))), \\ T^r = \; & -\tfrac{1}{2} r^2 \sin\theta (u_r u_t - {u_t}^2 + u(-u_{tr} + u_{tt})), \\ T^\theta = \; & -\tfrac{1}{2}\sin\theta(u_\theta u_t - u u_{t\theta}), \\ T^\phi = \; & \tfrac{1}{2}(-u_\phi u_t + u u_{t\phi}). \end{aligned}$$

(b) $X_2 = \partial_\phi$

$$T^t = \tfrac{1}{2}r^2\sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\tfrac{1}{2}r^2\sin\theta(u_\phi(u_r - u_t) + u(-u_{r\phi} + u_{t\phi})),$$

$$T^\theta = -\tfrac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \tfrac{1}{2}(-{u_\phi}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + r(2u_r + ru_{rr} -2(u_t + ru_{tr}))))).$$

(c) $X_3 = u\partial_u + t\partial_t + r\partial_r$

$$T^t = \tfrac{1}{2}(r^2\sin\theta u_r(ru_r + tu_t) - u(tu_{\phi\phi} + t\cos\theta u_\theta + \sin\theta(tu_{\theta\theta} +r((r+2t)u_r + r(r+t)u_{rr} - t(2u_t + ru_{tr}))))),$$

$$T^r = -\tfrac{1}{2}r(r\sin\theta(u_r - u_t)(ru_r + tu_t) + u(u_{\phi\phi} + \cos\theta u_\theta +\sin\theta(u_{\theta\theta} - r(-u_r + u_t + ru_{tr} + tu_{tr} - tu_{tt})))),$$

$$T^\theta = -\tfrac{1}{2}\sin\theta(u_\theta(ru_r + tu_t) - u(ru_{r\theta} + tu_{t\theta})),$$

$$T^\phi = \tfrac{1}{2}(-u_\phi(ru_r + tu_t) + u(ru_{r\phi} + tu_{t\phi})).$$

(d) $X_4 = t^2\partial_t + 2r(r+t)\partial_r - 2u(r+t)\partial_u$

$$
\begin{aligned}
T^t = \ & \tfrac{1}{2}(2r^2\sin\theta u^2 - r^2\sin\theta u_r(2r(r+t)u_r + t^2u_t) + u(t^2u_{\phi\phi} \\
& +t^2\cos\theta u_\theta + \sin\theta(t^2u_{\theta\theta} + r(2(2r^2+rt+t^2)u_r + r(2r^2 \\
& +2rt+t^2)u_{rr} - t^2(2u_t + ru_{tr}))))),
\end{aligned}
$$

$$
\begin{aligned}
T^r = \ & \tfrac{1}{2}r(r\sin\theta(u_r - u_t)(2r(r+t)u_r + t^2u_t) + u(2(r+t)u_{\phi\phi} \\
& +2(r+t)\cos\theta u_\theta + \sin\theta(2(r+t)u_{\theta\theta} - r(-2(r+t)u_r \\
& +2(2r+t)u_t + 2r^2u_{tr} + 2rtu_{tr} + t^2u_{tr} - t^2u_{tt})))),
\end{aligned}
$$

$$
T^\theta = \tfrac{1}{2}\sin\theta(u_\theta(2r(r+t)u_r + t^2u_t) - u(2r(r+t)u_{r\theta} + t^2u_{t\theta})),
$$

$$
T^\phi = \tfrac{1}{2}(u_\phi(2r(r+t)u_r + t^2u_t) - u(2r(r+t)u_{r\phi} + t^2u_{t\phi})).
$$

(e) $X_5 = g(t,r,\theta,\phi)\partial_u$, where the function $g(t,r,\theta,\phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + r(2g_r + rg_{rr} - 2(g_t + rg_{tr})))$.

$$
T^t = r^2\sin\theta(ug_r - gu_r),
$$

$$
T^r = r^2\sin\theta(u(-g_r + g_t) + g(u_r - u_t)),
$$

$$
T^\theta = \sin\theta(-ug_\theta + gu_\theta),
$$

$$
T^\phi = -ug_\phi + gu_\phi.
$$

(ii). Case $m = k$, where k is an arbitrary constant.

(a) $X_1 = \partial_t$

$$T^t = \frac{1}{2}(r^2 \sin\theta u_r u_t - u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(k-r)u_r + r((-2k+r)u_{rr} - 2u_t - ru_{tr})))),$$

$$T^r = -\frac{1}{2} r \sin\theta((-2k+r)u_r u_t - r{u_t}^2 + u((2k-r)u_{tr} + ru_{tt})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\theta u_t - uu_{t\theta}),$$

$$T^\phi = \frac{1}{2}(-u_\phi u_t + uu_{t\phi}).$$

(b) $X_2 = \partial_\phi$

$$T^t = \frac{1}{2} r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\frac{1}{2} r \sin\theta(u_\phi((-2k+r)u_r - ru_t) + u((2k-r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-{u_\phi}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(k-r)u_r + r((-2k+r)u_{rr} - 2(u_t + ru_{tr}))))).$$

(c) $X_3 = g(t,r,\theta,\phi)\partial_u$, where the function $g(t,r,\theta,\phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} - 2(k-r)g_r + r((-2k+r)g_{rr} - 2(g_t + rg_{tr})))$.

$$T^t = r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = r\sin\theta(u((2k-r)g_r + rg_t) + g((-2k+r)u_r - ru_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

(iii). Case $m = t$.

(a) $X_1 = \partial_\phi$

$$\begin{aligned}
T^t = & \ \tfrac{1}{2} r^2 \sin\theta (u_\phi u_r - u u_{r\phi}), \\
T^r = & \ -\tfrac{1}{2} r \sin\theta (u_\phi((r-2t)u_r - r u_t) + u(-(r-2t)u_{r\phi} + r u_{t\phi})), \\
T^\theta = & \ -\tfrac{1}{2} \sin\theta (u_\phi u_\theta - u u_{\theta\phi}), \\
T^\phi = & \ \tfrac{1}{2}(-{u_\phi}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r-t)u_r + r((r-2t)u_{rr} \\
& -2(u_t + r u_{tr}))))).
\end{aligned}$$

(b) $X_2 = u\partial_u + t\partial_t + r\partial_r$

$$\begin{aligned}
T^t = & \ \tfrac{1}{2}(r^2 \sin\theta u_r (r u_r + t u_t) + u(-t u_{\phi\phi} - t\cos\theta u_\theta + \sin\theta(-t u_{\theta\theta} \\
& -(r^2 + 2rt - 2t^2)u_r + (2rt^2 - r^3 - r^2 t)u_{rr} + t(2u_t + r u_{tr}))))), \\
T^r = & \ \tfrac{1}{2} r(\sin\theta(r(2t-r){u_r}^2 + (r^2 - rt + 2t^2)u_r u_t + rt{u_t}^2) \\
& -u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} + r(u_r - u_t) \\
& -r^2 u_{tr} - rt u_{tr} + 2t^2 u_{tr} + rt u_{tt}))), \\
T^\theta = & \ -\tfrac{1}{2}\sin\theta(u_\theta(r u_r + t u_t) - u(r u_{r\theta} + t u_{t\theta})), \\
T^\phi = & \ \tfrac{1}{2}(-u_\phi(r u_r + t u_t) + u(r u_{r\phi} + t u_{t\phi})).
\end{aligned}$$

(c) $X_3 = g(t,r,\theta,\phi)\partial_u$, where the function $g(t,r,\theta,\phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + 2(r-t)g_r + r((r-2t)g_{rr} - 2(g_t + r g_{tr})))$.

$$T^t = r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = r\sin\theta(u(-(r-2t)g_r + rg_t) + g((r-2t)u_r - ru_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

(iv). Case $m = e^t$.

(a) $X_1 = \partial_\phi$

$$T^t = \tfrac{1}{2}r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\tfrac{1}{2}r\sin\theta(u_\phi((-2e^t + r)u_r - ru_t) + u((2e^t - r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = -\tfrac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \tfrac{1}{2}(-u_\phi{}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(e^t - r)u_r + r((-2e^t + r)u_{rr} - 2(u_t + ru_{tr}))))).$$

(b) $X_2 = g(t,r,\theta,\phi)\partial_u$, where the function $g(t,r,\theta,\phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} - 2(e^t - r)g_r + r((-2e^t + r)g_{rr} - 2(g_t + rg_{tr})))$.

$$T^t = \quad r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = \quad r\sin\theta(u((2e^t - r)g_r + rg_t) - g((2e^t - r)u_r + ru_t)),$$

$$T^\theta = \quad \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = \quad -ug_\phi + gu_\phi.$$

(v). Case $m = m(t)$, where m(t) is an arbitrary function of t.

(a) $X_1 = \partial_\phi$

$$T^t = \quad \tfrac{1}{2}r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = \quad -\tfrac{1}{2}r\sin\theta(u_\phi((r - 2m(t))u_r - ru_t) + u((2m(t) - r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = \quad -\tfrac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \quad \tfrac{1}{2}(-{u_\phi}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r - m(t))u_r + r((r - 2m(t))u_{rr} - 2(u_t + ru_{tr}))))).$$

(b) $X_2 = g(t, r, \theta, \phi)\partial_u$, where the function $g(t, r, \theta, \phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + 2(r - m(t))g_r + r((r - 2m(t))g_{rr} - 2(g_t + rg_{tr})))$.

$$T^t = \quad r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = \quad r\sin\theta(u(-(r - 2m(t))g_r + rg_t) + g((r - 2m(t))u_r - ru_t)),$$

$$T^\theta = \quad \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = \quad -ug_\phi + gu_\phi.$$

6.6 Higher-order symmetries and conserved densities

In the standard (1+3) wave equation, there exists higher-order variational symmetries which lead to nontrivial conserved flows. Thus, even though there is a radical reduction in the number of variational point symmetries and conservation laws, one could analyse the curved wave equation on a knowledge of the higher order symmetries and conservation laws. In this section, we list some of these variational symmetries $\mathcal{X} = \eta(x, u, u_{(1)}, u_{(2)}, u_{(3)})\partial_u$ and nontrivial conserved flows $(\mathcal{T}^t, \mathcal{T}^r, \mathcal{T}^\theta, \mathcal{T}^\phi)$ wherein the conserved density is given by $\mathcal{T}^t$ (see [17] on the discussion on recursion operators and generalised symmetries).

(i). Case $m = 0$.

$\mathcal{X}_1^1 = (2u_{tt} + tu_{ttt} + ru_{rtt})\partial_u,$

$$\begin{aligned}
\mathcal{T}^t = \;& \tfrac{1}{6}(2\sin\theta u_{\theta\theta}u_t + 2r\sin\theta u_r u_t - ru_{r\phi\phi}u_t - r\cos\theta u_{r\theta}u_t - r\sin\theta u_{r\theta\theta}u_t \\
& -2r^2\sin\theta u_{rr}u_t - r^3\sin\theta u_{rrr}u_t - 2r\sin\theta {u_t}^2 + 2uu_{t\phi\phi} - ru_r u_{t\phi\phi} \\
& -3tu_t u_{t\phi\phi} + 2\cos\theta uu_{t\theta} - r\cos\theta u_r u_{t\theta} - 3t\cos\theta u_t u_{t\theta} + 2\sin\theta uu_{t\theta\theta} \\
& -r\sin\theta u_r u_{t\theta\theta} - 3t\sin\theta u_t u_{t\theta\theta} + 8r\sin\theta uu_{tr} + 2r\sin\theta u_{\theta\theta}u_{tr} \\
& +2r^2\sin\theta u_r u_{tr} + 2r^3\sin\theta u_{rr}u_{tr} - 2r^2\sin\theta u_t u_{tr} - 6rt\sin\theta u_t u_{tr} \\
& -4r^3\sin\theta {u_{tr}}^2 + 2ruu_{tr\phi\phi} + 2r\cos\theta uu_{tr\theta} + 2r\sin\theta uu_{tr\theta\theta} \\
& +10r^2\sin\theta uu_{trr} - r^3\sin\theta u_r u_{trr} + 2r^3\sin\theta u_t u_{trr} - 3r^2t\sin\theta u_t u_{trr} \\
& +2r^3\sin\theta uu_{trrr} - 8r\sin\theta uu_{tt} + 3t\sin\theta u_{\theta\theta}u_{tt} - 4r^2\sin\theta u_r u_{tt} \\
& +6rt\sin\theta u_r u_{tt} + 3r^2t\sin\theta u_{rr}u_{tt} - 6r^2t\sin\theta u_{tr}u_{tt} + u_{\phi\phi}(2u_t + 2ru_{tr} \\
& +3tu_{tt}) + \cos\theta u_\theta(2u_t + 2ru_{tr} + 3tu_{tt}) + 3tuu_{tt\phi\phi} + 3t\cos\theta uu_{tt\theta} \\
& +3t\sin\theta uu_{tt\theta\theta} - 7r^2\sin\theta uu_{ttr} + 6rt\sin\theta uu_{ttr} - r^3\sin\theta u_r u_{ttr} \\
& +6r^2t\sin\theta u_t u_{ttr} - r^3\sin\theta uu_{ttrr} + 3r^2t\sin\theta uu_{ttrr} - 6rt\sin\theta uu_{ttt} \\
& -3r^2t\sin\theta u_r u_{ttt} - 3r^2t\sin\theta uu_{tttr}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^r = \;& \tfrac{1}{6}r(u_{\phi\phi}u_{tt} + \cos\theta u_\theta u_{tt} + \sin\theta u_{\theta\theta}u_{tt} + 8r\sin\theta u_r u_{tt} + r^2\sin\theta u_{rr}u_{tt} \\
& -2r^2\sin\theta u_{tr}u_{tt} + uu_{tt\phi\phi} + \cos\theta uu_{tt\theta} + \sin\theta uu_{tt\theta\theta} - 7r\sin\theta uu_{ttr} \\
& +3r^2\sin\theta u_r u_{ttr} - 2r^2\sin\theta uu_{ttrr} + 7r\sin\theta uu_{ttt} + 3rt\sin\theta u_r u_{ttt} \\
& -u_t(u_{t\phi\phi} + \cos\theta u_{t\theta} + \sin\theta(u_{t\theta\theta} + r(2u_{tr} + ru_{trr} + 6u_{tt} + ru_{ttr} \\
& +3tu_{ttt}))) + r^2\sin\theta uu_{tttr} - 3rt\sin\theta uu_{tttr} + 3rt\sin\theta uu_{tttt}),
\end{aligned}$$

$$\mathcal{T}^\theta = \tfrac{1}{2}\sin\theta(u_\theta(2u_{tt} + ru_{ttr} + tu_{ttt}) - u(2u_{tt\theta} + ru_{ttr\theta} + tu_{ttt\theta})),$$

$$\mathcal{T}^\phi = \tfrac{1}{2}(u_\phi(2u_{tt} + ru_{ttr} + tu_{ttt}) - u(2u_{tt\phi} + ru_{ttr\phi} + tu_{ttt\phi})).$$

(ii). Case $m = k$, where k is an arbitrary constant.

$\mathcal{X}_1^2 = u_{\phi tt}\partial_u,$

$$
\begin{aligned}
\mathcal{T}^t = & \ \tfrac{1}{6}(-u_{\phi\phi\phi}u_t - \cos\theta u_{\theta\phi}u_t - \sin\theta u_{\theta\theta\phi}u_t + 2k\sin\theta u_{r\phi}u_t - 2r\sin\theta u_{r\phi} \\
& +2kr\sin\theta u_{rr\phi}u_t - r^2\sin\theta u_{rr\phi}u_t + 2\cos\theta u_\theta u_{t\phi} + 2r\sin\theta u_\phi u_{tt} \\
& +2\sin\theta u_{\theta\theta}u_{t\phi} - 4k\sin\theta u_r u_{t\phi} + 4r\sin\theta u_r u_{t\phi} - 4kr\sin\theta u_{rr}u_{t\phi} \\
& +2r^2\sin\theta u_{rr}u_{t\phi} - 2r\sin\theta u_t u_{t\phi} - u_\phi u_{t\phi\phi} + 2uu_{t\phi\phi\phi} - 2r\sin\theta u_\phi u_{tr} \\
& +2\cos\theta uu_{t\theta\phi} - \sin\theta u_\phi u_{t\theta\theta} + 2\sin\theta uu_{t\theta\theta\phi} + 2k\sin\theta u_\phi u_{tr} - \cos\theta u_\phi \\
& -4r^2\sin\theta u_{t\phi}u_{tr} - 4k\sin\theta uu_{tr\phi} + 4r\sin\theta uu_{tr\phi} + 2r^2\sin\theta u_t u_{tr\phi} \\
& +2kr\sin\theta u_\phi u_{trr} - r^2\sin\theta u_\phi u_{trr} - 4kr\sin\theta uu_{trr\phi} + 2r^2\sin\theta uu_{trr\phi} \\
& -4r\sin\theta uu_{tt\phi} - 3r^2\sin\theta u_r u_{tt\phi} + 2r^2\sin\theta u_\phi u_{ttr} - r^2\sin\theta uu_{ttr\phi}),
\end{aligned}
$$

$$
\mathcal{T}^r = \tfrac{1}{2}r\sin\theta((-2k+r)u_r u_{tt\phi} - ru_t u_{tt\phi} + u((2k-r)u_{ttr\phi} + ru_{ttt\phi})),
$$

$$
\mathcal{T}^\theta = \tfrac{1}{2}\sin\theta(u_\theta u_{tt\phi} - uu_{tt\theta\phi}),
$$

$$
\begin{aligned}
\mathcal{T}^\phi = & \ \tfrac{1}{6}(u_{\phi\phi}u_{tt} + \cos\theta u_\theta u_{tt} + \sin\theta u_{\theta\theta}u_{tt} - 2k\sin\theta u_r u_{tt} + 2r\sin\theta u_r u_{tt} \\
& -2kr\sin\theta u_{rr}u_{tt} + r^2\sin\theta u_{rr}u_{tt} - 2r^2\sin\theta u_{tr}u_{tt} + 3u_\phi u_{tt\phi} \\
& +\cos\theta uu_{tt\theta} + \sin\theta uu_{tt\theta\theta} - 2k\sin\theta uu_{ttr} + 2r\sin\theta uu_{ttr} - u_t(u_{t\phi\phi} \\
& +\cos\theta u_{t\theta} + \sin\theta(u_{t\theta\theta} - 2(k-r)u_{tr} + r((-2k+r)u_{trr} - 2ru_{ttr}))) \\
& -2kr\sin\theta uu_{ttrr} + r^2\sin\theta uu_{ttrr} - 2r\sin\theta uu_{ttt} - 2r^2\sin\theta uu_{tttr}).
\end{aligned}
$$

(iii). Case $m = t$.

$$
\mathcal{X}_1^3 = (tu_{t\phi\phi} + ru_{r\phi\phi} + u_{\phi\phi})\partial_u,
$$

$$
\begin{aligned}
\mathcal{T}^t = & \ \tfrac{1}{6}(tu_{\phi\phi}{}^2 + tuu_{\phi\phi\phi\phi} + t\cos\theta uu_{\theta\phi\phi} + t\sin\theta uu_{\theta\theta\phi\phi} + 6r^2\sin\theta uu_{r\phi\phi} \\
& +2rt\sin\theta uu_{r\phi\phi} - 2t^2\sin\theta uu_{r\phi\phi} - 3r^3\sin\theta u_r u_{r\phi\phi} + 3r^3\sin\theta uu_{rr\phi\phi} \\
& +r^2t\sin\theta uu_{rr\phi\phi} - 2rt^2\sin\theta uu_{rr\phi\phi} - 2rt\sin\theta uu_{t\phi\phi} - 3r^2t\sin\theta u_{t\phi\phi} \\
& +u_{\phi\phi}(t\cos\theta u_\theta + \sin\theta(tu_{\theta\theta} + (-3r^2 + 2rt - 2t^2)u_r + rt((r-2t)u_{rr} \\
& -2(u_t + ru_{tr})))) + tu_\phi(-u_{\phi\phi\phi} - \cos\theta u_{\theta\phi} - \sin\theta(u_{\theta\theta\phi} + 2(r-t)u_{r\phi} \\
& +r((r-2t)u_{rr\phi} - 2(u_{t\phi} + ru_{tr\phi})))) + r^2t\sin\theta uu_{tr\phi\phi}),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{T}^r = \; & \tfrac{1}{6} r ({u_{\phi\phi}}^2 + u u_{\phi\phi\phi\phi} + \cos\theta u u_{\theta\phi\phi} + \sin\theta u u_{\theta\theta\phi\phi} - 4r \sin\theta u u_{r\phi\phi} \\
& + 10t \sin\theta u u_{r\phi\phi} + 3r^2 \sin\theta u_r u_{r\phi\phi} - 6rt \sin\theta u_r u_{r\phi\phi} - 2r^2 \sin\theta u_{r\phi\phi} \\
& + 4rt \sin\theta u u_{rr\phi\phi} - 3r^2 \sin\theta u_{r\phi\phi} u_t + 4r \sin\theta u u_{t\phi\phi} + 3rt \sin\theta u_r u_{t\phi\phi} \\
& - 6t^2 \sin\theta u_r u_{t\phi\phi} - 3rt \sin\theta u_t u_{t\phi\phi} + u_{\phi\phi}(\cos\theta u_\theta + \sin\theta (u_{\theta\theta} + (5r \\
& - 8t) u_r + r((r-2t) u_{rr} - 5u_t - 2r u_{tr}))) - u_\phi (u_{\phi\phi\phi} + \cos\theta u_{\theta\phi} \\
& + \sin\theta (u_{\theta\theta\phi} + 2(r-t) u_{r\phi} + r((r-2t) u_{rr\phi} - 2(u_{t\phi} + r u_{tr\phi})))) \\
& + r^2 \sin\theta u u_{tr\phi\phi} - 3rt \sin\theta u u_{tr\phi\phi} + 6t^2 \sin\theta u u_{tr\phi\phi} + 3rt \sin\theta u u_{tt\phi\phi}),
\end{aligned}$$

$$\mathcal{T}^\theta = \tfrac{1}{2} \sin\theta (u_{\phi\phi} u_\theta + u_\theta (r u_{r\phi\phi} + t u_{t\phi\phi}) - u(u_{\theta\phi\phi} + r u_{r\theta\phi\phi} + t u_{t\theta\phi\phi})),$$

$$\begin{aligned}
\mathcal{T}^\phi = \; & \tfrac{1}{6}(-r u_{\phi\phi\phi} u_r - r \cos\theta u_{\theta\phi} u_r - r \sin\theta u_{\theta\theta\phi} u_r + 2r u_{\phi\phi} u_{r\phi} - t u_{\phi\phi\phi} u_t \\
& + 2r \sin\theta u_{\theta\theta} u_{r\phi} + 2r^2 \sin\theta u_r u_{r\phi} - 2rt \sin\theta u_r u_{r\phi} + 2r^3 \sin\theta u_{r\phi} u_{rr} \\
& - 4r^2 t \sin\theta u_{r\phi} u_{rr} - r^3 \sin\theta u_r u_{rr\phi} + 2r^2 t \sin\theta u_r u_{rr\phi} + 2t u_{\phi\phi} u_{t\phi} \\
& - t \cos\theta u_{\theta\phi} u_t - t \sin\theta u_{\theta\theta\phi} u_t - 4r^2 \sin\theta u_{r\phi} u_t - 2rt \sin\theta u_{r\phi} u_t \\
& + 2t^2 \sin\theta u_{r\phi} u_t - r^2 t \sin\theta u_{rr\phi} u_t + 2rt^2 \sin\theta u_{rr\phi} u_t + 2r \cos\theta u_\theta u_{r\phi} \\
& + 2t \cos\theta u_\theta u_{t\phi} + 2t \sin\theta u_{\theta\theta} u_{t\phi} + 2r^2 \sin\theta u_r u_{t\phi} + 4rt \sin\theta u_r u_{t\phi} \\
& - 4t^2 \sin\theta u_r u_{t\phi} + 2r^2 t \sin\theta u_{rr} u_{t\phi} - 4rt^2 \sin\theta u_{rr} u_{t\phi} - 2rt \sin\theta u_t u_{t\phi} \\
& - 4r^3 \sin\theta u_{r\phi} u_{tr} - 4r^2 t \sin\theta u_{t\phi} u_{tr} + 2r^3 \sin\theta u_{tr\phi} + 2r^2 t \sin\theta u_t u_{tr\phi} \\
& + u_\phi (4u_{\phi\phi} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + 2r u_{r\phi\phi} - r \cos\theta u_{r\theta} - r \sin\theta u_{r\theta\theta} \\
& - 3r^2 \sin\theta u_{rr} + 4rt \sin\theta u_{rr} - r^3 \sin\theta u_{rrr} + 2r^2 t \sin\theta u_{rrr} + 2t u_{t\phi\phi} \\
& - t \cos\theta u_{t\theta} - t \sin\theta u_{t\theta\theta} + 4r^2 \sin\theta u_{tr} - 2rt \sin\theta u_{tr} + 2t^2 \sin\theta u_{tr} \\
& + 2r^3 \sin\theta u_{trr} + (2rt^2 - r^2 t) \sin\theta u_{trr} + 2rt \sin\theta u_{tt} + 2r^2 t \sin\theta u_{ttr}) \\
& - u(2u_{\phi\phi\phi} - \cos\theta u_{\theta\phi} - \sin\theta u_{\theta\theta\phi} - 6r \sin\theta u_{r\phi} + 6t \sin\theta u_{r\phi} + r u_{r\phi\phi\phi} \\
& - 2r \cos\theta u_{r\theta\phi} - 2r \sin\theta u_{r\theta\theta\phi} - 9r^2 \sin\theta u_{rr\phi} + 14rt \sin\theta u_{rr\phi} \\
& - 2r^3 \sin\theta u_{rrr\phi} + 4r^2 t \sin\theta u_{rrr\phi} + 6r \sin\theta u_{t\phi} + t u_{t\phi\phi\phi} - 2t \cos\theta u_{t\theta\phi} \\
& - 2t \sin\theta u_{t\theta\theta\phi} + 14r^2 \sin\theta u_{tr\phi} + (4t^2 - 4rt) \sin\theta u_{tr\phi} + 4r^3 \sin\theta u_{trr\phi} \\
& - 2r^2 t \sin\theta u_{trr\phi} + 4rt^2 \sin\theta u_{trr\phi} + 4rt \sin\theta u_{tt\phi} + 4r^2 t \sin\theta u_{ttr\phi})).
\end{aligned}$$

(iv). Case $m = e^t$.

$$\mathcal{X}_1^4 = u_{\phi\phi\phi} \partial_u,$$

$$
\begin{aligned}
\mathcal{T}^t = &\ -\tfrac{1}{2}r^2\sin\theta(u_{\phi\phi\phi}u_r - uu_{r\phi\phi\phi}),\\
\mathcal{T}^r = &\ \tfrac{1}{2}r\sin\theta(u_{\phi\phi\phi}((-2e^t + r)u_r - ru_t) + u((2e^t - r)u_{r\phi\phi\phi} + ru_{t\phi\phi\phi})),\\
\mathcal{T}^\theta = &\ \tfrac{1}{2}\sin\theta(u_{\phi\phi\phi}u_\theta - uu_{\theta\phi\phi\phi}),\\
\mathcal{T}^\phi = &\ \tfrac{1}{2}({u_{\phi\phi}}^2 + u_{\phi\phi}(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(e^t - r)u_r + r((-2e^t + r)u_{rr}\\
&-2(u_t + ru_{tr})))) - u_\phi(\cos\theta u_{\theta\phi} + \sin\theta(u_{\theta\theta\phi} - 2(e^t - r)u_{r\phi}\\
&+r((-2e^t + r)u_{rr\phi} - 2(u_{t\phi} + ru_{tr\phi})))) + u(\cos\theta u_{\theta\phi\phi} + \sin\theta(u_{\theta\theta\phi\phi}\\
&-2(e^t - r)u_{r\phi\phi} + r((-2e^t + r)u_{rr\phi\phi} - 2(u_{t\phi\phi} + ru_{tr\phi\phi}))))).
\end{aligned}
$$

(v). Case $m = m(t)$, where m(t) is an arbitrary function of t.

$\mathcal{X}_1^5 = u_{\phi\phi\phi}\partial_u$,

$$
\begin{aligned}
\mathcal{T}^t = &\ -\tfrac{1}{2}r^2\sin\theta(u_{\phi\phi\phi}u_r - uu_{r\phi\phi\phi}),\\
\mathcal{T}^r = &\ \tfrac{1}{2}r\sin\theta(u_{\phi\phi\phi}((r - 2m(t))u_r - ru_t) + (2m(t) - r)uu_{r\phi\phi\phi} + ru_{t\phi\phi\phi})),\\
\mathcal{T}^\theta = &\ \tfrac{1}{2}\sin\theta(u_{\phi\phi\phi}u_\theta - uu_{\theta\phi\phi\phi}),\\
\mathcal{T}^\phi = &\ \tfrac{1}{2}({u_{\phi\phi}}^2 + u_{\phi\phi}(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r - m(t))u_r + r((r - 2m(t))u_{rr}\\
&-2(u_t + ru_{tr})))) - u_\phi(\cos\theta u_{\theta\phi} + \sin\theta(u_{\theta\theta\phi} + 2(r - m(t))u_{r\phi} + r((r\\
&-2m(t))u_{rr\phi} - 2(u_{t\phi} + ru_{tr\phi})))) + u(\cos\theta u_{\theta\phi\phi} + \sin\theta(u_{\theta\theta\phi\phi} + 2(r\\
&-m(t))u_{r\phi\phi} + r((r - 2m(t))u_{rr\phi\phi} - 2(u_{t\phi\phi} + ru_{tr\phi\phi}))))).
\end{aligned}
$$

6.7 Discussion and conclusion

We have considered the classical wave equation in some Lorentzian spacetime background with the point in mind that the wave equation there may naturally inherit nonlinearities from the geometry. In this context, we have considered the Vaidya metric for which a special case is the Papapetrou metric. We have given some symmetry reductions to show how the wave equation there can be either solved or reduced to ordinary differential equations using the method of invariants. Also, some conservation laws were constructed. In the book [41], Ibragimov suggests that in three flat space dimensions the linear wave equation considered admit a 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry. In this study, we show that the wave equations admits fewer symmetries when it is solved on a curved manifold. A special case to note is $m = 0$ when the Vaidya manifold is supposedly 'flat'. Manifolds that are 'flat' need *not* to lead to the wave equation admitting the maximal 16-dimensional Lie algebra of point symmetries. Finally, some higher-order symmetries and associated conservation laws were presented. We conclude that solving or analysing the nonlinear wave equations in a curved spacetime background using the invariance approach may provide some insight into the geometry or relativity for different manifolds.

Chapter 7

Conservation laws of some third-grade fluids

7.1 Introduction

The mechanics of non-linear fluids present a special challenge to engineers, physicists and mathematicians since the non-linearity can manifest itself in a variety of ways. One of the simplest ways in which the viscoelastic fluids have been classified is the methodology given by Rivlin and Ericksen [42] and Truesdell and Noll [43], who present constitutive relations for the stress tensor as a function of the symmetric part of the velocity gradient and its higher (total) derivatives.

In recent years there has been several studies [44]-[51] on flows of non-Newtonian fluids, not only because of their technological significance but also due to the interesting mathematical features presented by the equations governing the flow.

A discussion of the various differential rate type and integral models can be found in the books of Schowalter [52] and Huilgol [53] and in the survay by Rajagopal [54].

The attraction in these models stems largely from the fact that the constitutive relations, whether we consider second- or third-grade fluids, since these have been studied most, are that they are derived on the basis of first principles and unlike many other 'phenomenological' models suggested in [55] by Reiner, there are no curve fittings or parameters to adjust.

Although the second-grade fluid model is able to predict the normal stress differences which are characteristic of non-Newtonian fluids - it does not take into account the shear thinning and thickening phenomena that many show. The third-grade fluid model represents a further, although inconclusive, attempt towards a comprehensive description of of the properties of viscoelastic fluids. With this in mind, the model in the present paper is a third-grade fluid.

In [56], exact analytical solution for the unsteady flow of a third-grade fluid on a porous wall was obtained.

Generally, models in fluids are constructed under the assumption of certain conservation laws. It may turn out that the resultant PDE or PDEs, whether linear or nonlinear, give rise to a number of distinct conserved flows (like density and current) and these may result in a number of different solutions. Thus, a knowledge of the various conservation laws gives one a greater insight into the underlying model. Also, a deeper understanding of the invariance properties are achieved like potential symmetries obtained from respective potential systems, [57]. So, aside from some physical conservation laws, solutions should remain constant along the general conserved flows (see [58]).

7.2 Preliminaries and basic equations

An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration [59]. Its constitutive equation can be written in the form of a functional

$$\mathbf{T}(t) = -p_1\mathbf{I} + \sum_{s=0}^{\infty} \mathbf{F}_t^t(s), \tag{7.1}$$

where $p_1\mathbf{I}$ is the undetermined part of the stress tensor and $\mathbf{F}$ is the deformation gradient.

Coleman and Noll [60] defined the incompressible fluid of differential type of grade n as the simple fluid obeying the constitutive equation

$$\mathbf{T}(t) = -p_1\mathbf{I} + \sum_{s=0}^{\infty} \mathbf{S}_j, \tag{7.2}$$

obtained by asymptotic expansion of the functional in (7.1) through a retardation parameter α. If $n = 3$, the first three tensors S_j are given by

$$\begin{aligned} \mathbf{S}_1 &= \mu\mathbf{A}_1, \\ \mathbf{S}_2 &= \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \\ \mathbf{S}_3 &= \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \end{aligned} \tag{7.3}$$

where μ is the coefficient of viscosity and $\alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3 are the material moduli. $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are kinematical tensors defined by

$$\begin{aligned} \mathbf{A}_1 &= \text{grad}\mathbf{V} + (\text{grad}\mathbf{V})^T, \\ \mathbf{A}_n &= \tfrac{d}{dt}\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\text{grad }\mathbf{V}) + (\text{grad}\mathbf{V})^T\mathbf{A}_{n-1}, \quad n = 2, 3, \cdots \end{aligned} \tag{7.4}$$

where $\mathbf{V}$ denotes the velocity field, grad is the gradient operator and $\frac{d}{dt}$ is the material time derivative which is defined by

$$\frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + [\text{grad}(\cdot)]\mathbf{V}, \tag{7.5}$$

where $\frac{\partial}{\partial t}$ is the partial derivative with respect to time. It was shown that if all the motions of the fluids are to be compatible with thermodynamics in the sense that these motions meet the Clausius-Duhem inequality and if it is assumed that the specific Helmholtz free energy is a minimum when the fluid is locally at rest, then

$$\begin{aligned} \mu &\geq 0, \quad \alpha_1 \geq 0, \\ \mid \alpha_1 + \alpha_2 \mid &\leq \sqrt{24\mu\beta_3}, \\ \beta_1 = \beta_2 = 0, &\quad \beta_3 \geq 0. \end{aligned} \tag{7.6}$$

Therefore, the constitutive relation for a thermodynamically compatible fluid of grade three becomes

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{7.7}$$

If the normal stress parameters α_1 and α_2 are zeros, then

$$\mathbf{T} = -p_1\mathbf{I} + (\mu + \beta_3(tr\mathbf{A}_1^2))\mathbf{A}_1, \tag{7.8}$$

where the quantity in parenthesis can be thought of as an effective shear-dependent viscosity. The basic governing equations are the conservation of mass and linear momentum. These are

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{V}) = 0 \tag{7.9}$$

and

$$\frac{d\rho}{dt} = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \tag{7.10}$$

where ρ is the density and $\mathbf{b}$ is the body force. As we are assuming that the fluid can undergo isochronic motion, (7.9) reduces to

$$\mathrm{div}(\mathbf{V}) = 0. \tag{7.11}$$

On substituting (7.7) into (7.10) and neglecting the body forces, we have

$$\rho\frac{d\mathbf{V}}{dt} + \mathrm{grad}p = \mu\mathrm{div}\mathbf{A}_1 + \alpha_1\mathrm{div}\mathbf{A}_2 + \alpha_2\mathrm{div}\mathbf{A}_1^2 + \beta_3\mathrm{div}[(tr\mathbf{A}_1^2)\mathbf{A}_1]. \tag{7.12}$$

We consider an incompressible fluid flow along an infinite plane porous wall. The x-axis is taken along the wall and the y-axis normal to the wall. Thus, for the flow under consideration, we seek a velocity field of the form

$$\mathbf{V} = [u(x,t), \theta, 0], \tag{7.13}$$

where $\theta \geq 0$ indicates the blowing velocity. From (7.11) and (7.13)

$$\frac{\partial\theta}{\partial x} = 0. \tag{7.14}$$

It is evident from (7.14) that θ is a function of time only. Following Kaloni [61] we have

$$\theta = -W_0, \tag{7.15}$$

where $W_0 \geq 0$ indicates the suction and $W_0 \leq 0$ gives blowing. Substituting (7.4), (7.7), (7.13) and (7.15) into (7.12) and neglecting the modified pressure gradient we have

$$\rho[\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial x}] = \mu\frac{\partial^2 u}{\partial x^2} + \alpha_1[\frac{\partial^3 u}{\partial x^2 \partial t} - W_0\frac{\partial^3 u}{\partial x^3}] + 6\beta_3(\frac{\partial u}{\partial x})^2\frac{\partial^2 u}{\partial x^2}. \tag{7.16}$$

Consider an rth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{7.17}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives, that is, $u^\alpha_i = D_i(u^\alpha), u^\alpha_{ij} = D_jD_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i\frac{\partial}{\partial u^\alpha} + u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_j} + \ldots, \quad i = 1, \ldots, n, \tag{7.18}$$

where the summation convention is used whenever appropriate. A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_iT^i = 0 \tag{7.19}$$

along the solutions of (7.17).

Every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \ldots)$

$$Q_\mu G^\mu = D_iT^i \tag{7.20}$$

such that it holds identically for some current T^i. There is a determining system for finding multipliers (and hence conservation laws) for any given PDE system.

If in (7.17), $G^\mu = G^\mu_0 + G^\mu_1$ such that $G^\mu_0 = \frac{\delta L}{\delta u^\alpha}$ for some function L, we say L is a *partial Lagrangian* for a partially variational system (7.17) - $\frac{\delta L}{\delta u^\alpha}$ is the Lie derivative or Euler operator. That is, for a scalar equation, $G = \delta L + G_1$, where $L = \int L$. Noether type symmetries can then be determined by

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(g^i) \ . \tag{7.21}$$

for some gauge vector g^i and conserved vectors for (7.17) may then be determined by a formula, viz.,

$$T^i = g^i - N^i(L) \ , \qquad i = 1, \ldots, n \ , \tag{7.22}$$

where N^i is the Noether operator (see [16]).

7.3 Conservation laws, multipliers and symmetries

In this section we derive the conservation laws of (7.16) using the notions of the invariance of the multipliers under symmetries of the equation. We separate (7.16) into three cases.

(i) In the first case we suppose that $\rho = 1$, $\beta_3 = W_0 = 0$ so that (7.16) becomes

$$u_t = \alpha u_{xxt} + \beta u_{xx}, \tag{7.23}$$

wherein we have replaced α_1 by α and μ by β. Its algebra of Lie symmetries is spanned by $X = \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial x}, \quad Z = u\frac{\partial}{\partial u}$ excluding the 'infinite' symmetries. The conserved flow (T^t, T^x) of (7.23) satisfies the divergence relation

$$D_t T^t + D_x T^x = Q(t, x, [u])(u_t - \alpha u_{xxt} - \beta u_{xx}), \tag{7.24}$$

so that

$$\frac{\delta}{\delta u} Q(u_t - \alpha u_{xxt} - \beta u_{xx}) = 0, \tag{7.25}$$

since the *Euler operator annihilates a total divergence.* If we suppose that the highest derivative of Q is the second derivative, viz., $Q = Q(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, the cumbersome and detailed calculations which cannot be presented here, reveal that there are no multipliers of order one and two. What we do obtain are the following forms of $Q = Q_i$,

$$\begin{aligned} Q_1 &= 1, \\ Q_2 &= e^{t-\frac{x}{\sqrt{\alpha-\beta}}}, \quad (\alpha \neq \beta) \\ Q_3 &= e^{\frac{\beta t-\alpha x-x}{\alpha-1}}(e^{\frac{2\alpha x}{\alpha-1}} + e^{\frac{2x}{\alpha-1}}), \quad (\alpha \neq 1) \end{aligned} \tag{7.26}$$

The next step is to construct the actual and corresponding conserved density T^t and flux T^x. This may be done by substituting back into (7.24) which again requires detailed calculations of which the results are presented here. The first one, Q_1, is the obvious one giving rise to the conserved vector (T^t, T^x)

$$\begin{aligned} T_1^t &= u, \\ T_1^x &= -\alpha u_{xt} - \beta u_x \end{aligned} \tag{7.27}$$

or

$$\begin{aligned} T_1^t &= u - \alpha u_{xx}, \\ T_1^x &= -\beta u_x. \end{aligned} \tag{7.28}$$

The not-so-obvious conserved vectors are obtained via the remaining multipliers. Corresponding to Q_2, we get

$$\begin{aligned} T_2^t &= (-\tfrac{\alpha u}{\alpha-\beta} - \tfrac{\alpha u_x}{\sqrt{\alpha-\beta}} - \alpha u_{xx} + 3u)e^{t-\frac{x}{\sqrt{\alpha-\beta}}}, \\ T_2^x &= (\tfrac{2\alpha u}{\sqrt{\alpha-\beta}} + \alpha u_x - \tfrac{\alpha u_t}{\sqrt{\alpha-\beta}} - \tfrac{3\beta u}{\sqrt{\alpha-\beta}} - 2\alpha u_{tx} - 3\beta u_x)e^{t-\frac{x}{\sqrt{\alpha-\beta}}}. \end{aligned} \tag{7.29}$$

and corresponding to Q_3 we obtain the density and flux

$$\begin{aligned} T_3^t &= -e^{\frac{\beta t-\alpha x-x}{\alpha-1}}[(\alpha u_{xx} + (\alpha-3)u)(e^{\frac{2\alpha x}{\alpha-1}} + e^{\frac{2x}{\alpha-1}}) + \alpha u_x(e^{\frac{2\alpha x}{\alpha-1}} - e^{\frac{2x}{\alpha-1}})], \\ T_3^x &= \tfrac{1}{\alpha-1}e^{\frac{\beta t-\alpha x-x}{\alpha-1}}[2\alpha u_{xt}(1-\alpha) + \beta u_x(3-2\alpha)](e^{\frac{2\alpha x}{\alpha-1}} + e^{\frac{2x}{\alpha-1}}) \\ &+ [\alpha u_t(\alpha-1) + \beta u(\alpha-3)](e^{\frac{2\alpha x}{\alpha-1}} - e^{\frac{2x}{\alpha-1}})] \end{aligned} \tag{7.30}$$

We have some of the following symmetry or invariance properties of the multipliers based on the symmetries X, Y and Z above. These are

$$\begin{aligned} &XQ_1 = 0, &&YQ_1 = 0, &&ZQ_1 = 0 \\ &XQ_2 = Q_2, &&YQ_2 = -\tfrac{1}{\sqrt{\alpha-\beta}}Q_2, &&ZQ_2 = 0 \\ &XQ_3 = \tfrac{\beta}{\alpha-1}Q_3, &&YQ_3 = e^{\frac{\beta t-\alpha x-x}{\alpha-1}}(e^{\frac{2\alpha x}{\alpha-1}} - e^{\frac{2x}{\alpha-1}}), &&ZQ_3 = 0 \end{aligned} \tag{7.31}$$

Thus, Q_2 is ray invariant with respect to X and Y. The multiplier Q_3 is ray invariant under X but not invariant under Y. Both of these multipliers are strictly invariant under Z. The multiplier Q_1 is obviously, strictly invariant under X, Y, Z.

Invariance of a multiplier under a symmetry implies association of the corresponding conservation law with the symmetry.

(ii) Secondly, we include $W_0 \neq 0$ in which case (7.16), with the usual replacements, becomes

$$u_t = \alpha u_{xxt} + \beta u_{xx} + W_0 u_x - \alpha W_0 u_{xxx} \tag{7.32}$$

which has the same Lie algebra of point symmetries as in case (i). Following the procedure above, the multipliers, when differentiated till second order becomes independent of derivatives. For example, ray invariance under translation in x, viz., Y, requires $Q = e^{\lambda x} f(t, x, u)$ and it turns out that $Q = e^{\lambda x} f(x + W_0 t, u)$. Substituting into (7.25) becomes

$$\frac{\delta}{\delta u}[e^{\lambda x} f(t, x, u)(u_t - \alpha u_{xxt} - \beta u_{xx}) + W_0 u_x - \alpha W_0 u_{xxx})] = 0. \tag{7.33}$$

We obtain the following forms of λ and the corresponding f,
(a) $f = e^{x+W_0 t}$ $(Q_1 = e^{\lambda_1 x} e^{x+W_0 t})$,

$$\lambda_1 = \frac{1}{6W_0\alpha} H_1 + \frac{\frac{2}{3}(W_0^2\alpha^2 - 2W_0\alpha\beta + 3W_0^2\alpha + \beta^2)}{W_0\alpha H_1} - \frac{1}{3}\frac{\beta + 2W_0\alpha}{W_0\alpha}$$

where

$$\begin{aligned} H_1 &= [-24\beta W_0^2\alpha^2 + 8W_0^3\alpha^3 + 24W_0\alpha\beta^2 - 36W_0^2\alpha\beta - 72W_0^3\alpha^2 - 8\beta^3 \\ &+ 12(-\frac{1}{W_0}(3(4W_0^3\alpha - 12W_0^2\alpha^2\beta + 12W_0\alpha\beta^2 - 20W_0^2\alpha\alpha - 4\beta^3 + 4W_0^3\alpha^3 \\ &- 8W_0^3\alpha^2 + W_0\beta^2)))^{\frac{1}{2}} W_0^2\alpha]^{\frac{1}{3}} \end{aligned}$$

(b)$f = 1$ $(Q_2 = e^{\lambda_2 x})$,

$$\lambda_2 = \frac{1}{6W_0\alpha} H_2 + \frac{\frac{2}{3}(3W_0^2\alpha + \beta^2)}{W_0\alpha H_2} - \frac{1}{3}\frac{\beta}{W_0\alpha}$$

where

$$H_2 = [-36W_0^2\alpha\beta - 8\beta^3 + 12(-\frac{1}{W_0}(3(4W_0^3\alpha + W_0\beta^2)))^{\frac{1}{2}}W_0^2\alpha]^{\frac{1}{3}}$$

It is clear that Q_1 is ray invariant under X and Y since $XQ_1 = W_0Q_1$ and $YQ_1 = (\lambda_1 + 1)Q_1$ whilst Q_2 is strictly invariant under X and ray invariant under Y as $XQ_2 = 0$ and $YQ_2 = \lambda_2 Q_2$. Both of these are strictly invariant under Z. The corresponding conserved flows are

(a)

$$\begin{aligned}
T_1^t &= [-(\lambda_1+1)^2\alpha u + (\lambda_1+1)\alpha u_x - \alpha u_{xx} + 3u]Q_1,\\
T_1^x &= [-2W_0(\lambda_1+1)\alpha u + W_0\alpha u_x + 3(\lambda_1+1)^2W_0\alpha u + (\lambda_1+1)\alpha u_t\\
&- 3W_0(\lambda_1+1)\alpha u_x + 3(\lambda_1+1)\beta u - 2\alpha u_{tx} + 3\alpha u_{xx}W_0 - 3u_x\beta - 3uW_0]Q_1
\end{aligned}$$

and

(b)

$$\begin{aligned}
T_2^t &= [-\lambda_2^2\alpha u + \lambda_2\alpha u_x - \alpha u_{xx} + 3u]Q_2,\\
T_2^x &= [3\lambda_2^2\alpha u W_0 + \lambda_2\alpha u_t - 3\lambda_2 u_x\alpha W_0 + 3\lambda_2\beta u - 2\alpha u_{tx} + 3u_{xx}\alpha W_0 - 3u_x\beta - 3uW_0]Q_2.
\end{aligned}$$

This set excludes the obvious conserved vector

$$(T^t, T^x) = (-u, \alpha u_{xt} + \beta u_x + W_0 u - \alpha W_0 u_{xx})$$

with multiplier $Q = 1$.

(iii) It can be shown that the complete version of (7.16) has only the obvious conserved flow with density and flux given by

$$T^t = -u, \qquad T^x = \alpha u_{xt} + \beta u_x + W_0 u - \alpha W_0 u_{xx} + 2\beta_3 u_x^3.$$

7.4 Conservation laws via partial Lagrangians

In this section, we do a 'variational' study of the equations with particular reference to the first case above. Equation (7.23) does not admit a Lagrangian being a scalar evolution equation nor does it lend itself to a partial Lagrangian environment as it is of odd order. One way of getting around this is to make a substitution $u = v_x$ so that the resultant equation is fourth order. Then, however, some or all of the conserved flows that are constructed via a Lagrangian or partial Lagrangian of the higher order equation may not be reducible down to the equation in u in which case we have nonlocal flows of the original equation in u. Making the substitution, (7.23) becomes

$$v_{xt} - \beta v_{xxx} - \alpha v_{xxxt} = 0 \tag{7.34}$$

for which we choose a partial rather than a total Lagrangian

$$L = -\frac{1}{2}v_t v_x + \frac{\alpha}{2}v_t v_{xxx} \tag{7.35}$$

which, when substituted into the appropriate version of (7.21) leads to a number of Noether type generators X_i with gauge vectors (f_i, g_i). When these are put into (7.22), we obtain the conserved flows (T_i^t, T_i^x).

$$X_1 = e^{\frac{2b}{a}t}\partial_v, \quad f_1 = 0, \quad g_1 = -\frac{b}{a}e^{\frac{2b}{a}t}v \tag{7.36}$$

$$\begin{aligned} T_1^t &= e^{\frac{2b}{a}t}(-\tfrac{1}{2}v_x + \tfrac{a}{2}v_{xxx}) \\ T_1^x &= e^{\frac{2b}{a}t}(-\tfrac{1}{2}v_t + \tfrac{a}{2}v_{xxt} + \tfrac{b}{a}v) \end{aligned} \tag{7.37}$$

$$X_2 = e^{\frac{\sqrt{2}}{\sqrt{a}}x}e^{\frac{2b}{a}t}\partial_v, \quad f_2 = \frac{1}{\sqrt{2a}}e^{\frac{\sqrt{2}}{\sqrt{a}}x}e^{\frac{2b}{a}t}v, \quad g_2 = -\frac{b}{a}e^{\frac{\sqrt{2}}{\sqrt{a}}x}e^{\frac{2b}{a}t}v \tag{7.38}$$

$$\begin{aligned} T_2^t &= e^{\frac{\sqrt{2}}{\sqrt{a}}x}e^{\frac{2b}{a}t}(-\tfrac{1}{2}v_x + \tfrac{a}{2}v_{xxx} - \tfrac{1}{\sqrt{2a}}v) \\ T_2^x &= e^{\frac{\sqrt{2}}{\sqrt{a}}x}e^{\frac{2b}{a}t}(\tfrac{1}{2}v_t + \tfrac{a}{2}v_{xxt} - \tfrac{\sqrt{a}}{\sqrt{2}}v_{xt} + \tfrac{b}{a}v) \end{aligned} \tag{7.39}$$

$$X_3 = e^{-\frac{\sqrt{2}}{\sqrt{a}}x} e^{\frac{2b}{a}t}\partial_v, \quad f_3 = -\frac{1}{\sqrt{2a}} e^{-\frac{\sqrt{2}}{\sqrt{a}}x} e^{\frac{2b}{a}t} v, \quad g_3 = -\frac{b}{a} e^{-\frac{\sqrt{2}}{\sqrt{a}}x} e^{\frac{2b}{a}t} v \tag{7.40}$$

$$\begin{aligned} T_3^t &= e^{-\frac{\sqrt{2}}{\sqrt{a}}x} e^{\frac{2b}{a}t}(-\tfrac{1}{2}v_x + \tfrac{a}{2}v_{xxx} + \tfrac{1}{\sqrt{2a}}v) \\ T_3^x &= e^{-\frac{\sqrt{2}}{\sqrt{a}}x} e^{\frac{2b}{a}t}(\tfrac{1}{2}v_t + \tfrac{a}{2}v_{xxt} + \tfrac{\sqrt{a}}{\sqrt{2}}v_{xt} + \tfrac{b}{a}v) \end{aligned} \tag{7.41}$$

For the case $\alpha = 2$ and $\beta = 1$, we obtain an additional result,

$$X_4 = -\frac{1}{2}e^t(e^{x+1}+e^{-(x+1)})\partial_v, \quad f_4 = -\frac{1}{4}ve^t(e^{x+1}-e^{-(x+1)}), \quad g_4 = \frac{1}{4}ve^t(e^{x+1}+e^{-(x+1)}) \tag{7.42}$$

$$\begin{aligned} T_4^t &= -\tfrac{1}{2}e^t(e^{x+1} + e^{-(x+1)})(-\tfrac{1}{2}v_x + v_{xxx}) + \tfrac{1}{4}ve^t(e^{x+1} - e^{-(x+1)}) \\ T_4^x &= -\tfrac{1}{2}e^t(e^{x+1} + e^{-(x+1)})(\tfrac{1}{2}v_t + v_{xxt} + \tfrac{1}{2}v) + \tfrac{1}{2}e^t(e^{x+1} - e^{-(x+1)})(v_{xt}) \end{aligned} \tag{7.43}$$

It can easily be verified that the (T^t, T^x) are conserved flows of (7.34) but are all nonlocal with respect to the original equation (7.23).

We will obtain similar results corresponding to the second case above but the third case does not produce any Noether type generators so that no new conservation laws are obtained via partial Lagrangians.

7.5 Discussion and conclusion

It is clear that the task of determining and classifying all the conservation laws of the reduced equation/s of the third grade fluids considered in this section is a nontrivial one requiring detailed analysis with a number options applied to different situations. We have used the multiplier approach associated with the conserved flow as well as the partial variational approach in all the cases, and thus obtained interesting results which required some tedious calculations.

Conclusion

We see that the conserved flows for high-order equations (with Lagrangians and, equivalently, partial Lagrangians of order greater than one in derivatives) support a formula similar to the well known Noether's theorem with the provisor that the higher-order cases have more terms in the Euler operator giving rise to the appropriate order of the conserved flow. Also, in the fifth-order KdV evolution equation, we resorted to variational techniques usually adopted for the third-order KdV equation. In general, it would be cumbersome to determine the conserved vector for such a high-order equation using first principles. We used the modified approach of the Noether identity to find symmetries and then conservation laws for the high order equations.

In the third chapter, we considered the equations where the highest order derivative was mixed. Using the variational technique on the Shallow Water Wave equation, we get conserved flows that produce extra terms when the conservation law is applied. These extra terms are adjusted and then merged with the conservation law to form new conserved quantities. These extra terms also occur in the Regularized Long Wave equation.

We have shown in the fourth chapter that a large amount of information can be extracted from a knowledge of the vector fields (one parameter Lie group transformations) that leave the action integral invariant. In addition to a larger class of conservation laws than those given by the isometries or Killing vectors, we can conclude what the isometries actually are and that these form a Lie subalgebra of the

Noether symmetry algebra. We have performed the calculations on some versions of the Vaidiya metric yielding some previously unknown symmetries and conserved vectors regarding the corresponding manifold. Lastly, with particular reference to this metric, we concluded that the only variations on $m(u)$ that occur are $m = 0$, $m =$constant, $m = u$ and $m = m(u)$.

Particularly, the Petrov III metric admits 10 Killing vectors for both $b = 0$ and $b = 1$ respectively. Also, the metric has all the Ricci tensor components zero except $R_{11} = \frac{2b(b-1)}{t^2(1+t^2)}$ and is Ricci flat for both $b = 0$ and 1.The case $b = 1/2$ admits six Noether point symmetries of which three are Killing vectors.

We have considered the classical wave equation in some Lorentzian spacetime background with a point in mind that the wave equation there may naturally inherit nonlinearity from the geometry. We have given some symmetry reductions to show how wave equations can be either solved or reduced to ordinary differential equations using the method of invariants. Also, some conservation laws were constructed. In three flat space dimensions the linear wave equation admits a 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry. In this study we show that the wave equation admits fewer symmetries when it is solved on a curved manifold. We note that manifolds that are 'flat' need not lead to the wave equation admitting the maximal 16-dimensional Lie algebra of point symmetries; the case $m = 0$ being noted.

It is hoped that, by fully solving the nonlinear wave equation in curved spacetime background using the invariance approach may provide some insight into the geometry or relativity for different manifolds. We also presented an alternate method in finding some higher-order symmetries and associated conservation laws. Finally, it is clear that the task of determining and classifying all the conservation laws of the reduced equation/s of the third grade fluids considered in this thesis are nontrivial requiring detailed analysis with a number of options applied to different situations. We have done this using the multipliers associated with the conserved flow as well as the partial variational approach. In all the cases, we have results which required some tedious calculations.

Bibliography

[1] E. Noether, Invariante Variationsprobleme, *Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse*, **2** 235-257 (1918) (English translation in *Transport Theory and Statistical Physics*, **1**(3) 186-207 (1971))

[2] J. Douglas, Soultions of the inverse problem of the calculus of variations, *J.Amer. Math Soc.*, **50** 71-128 (1941)

[3] I.M. Anderson and T.E. Duchamp, Variational principles for second-order quasi-linear scalar equations, *J. Diff. Eq.*, **51** 1-47 (1984)

[4] J.M. Levy-Leblond, Conservation laws for gauge-invariant Lagrangians in classical mechanics, *Amer. J. Physics*, **39** 502-506 (1971)

[5] M. Lutzky, Symmetry groups and conserved quantities for the harmonic oscillator, *J. Phys. A: Math. Gen.*, **11** 249-258 (1978)

[6] G.E. Prince and C.J. Eliezer, Symmetries of the time dependent N-dimensional oscillator, *J. Phys. A: Math. Gen.*, **13** 815-823 (1980)

[7] J. Hietarinta, Direct Methods for the search of the second invariant, *Report Series, Dept. of Phys. Sci.*, Univ. of Turku, Finland

[8] H.R. Lewis and P.G.L. Leach, A direct approach to finding exact invariants for one-dimensional time-dependent classical Hamiltonians, *J. Math. Phys.*, **23** 2371 (1982)

[9] N.H. Ibragimov, A new conservation theorem, *J. Math. Anal. Appl.*, **333** 311-328 (2007)

[10] A.H. Kara and F.M. Mahomed, Noether-type symmetries and conservation laws via partial Lagrangians, *J. Nonlinear Dynamics*, **45** 367-383 (2006)

[11] H.H. Goldstine, A History of the Calculus of Variations from the 17th through 19th Century, *Studies in the History of Mathematical and Physical Sciences*, Vol.5, Springer-Verlag, New York, (1980)

[12] N.H. Ibragimov, A.H. Kara and F.M. Mahomed, *Nonlinear Dynamics*, 15 115-136 (1998)

[13] S. Anco and G. Bluman, *European J. of Appl. Maths.*, **9** 245-259 (1998)

[14] S. Anco and G. Bluman, *European J. of Appl. Maths.*, **13** 545-566 (2002)

[15] A.H. Kara and F.M. Mahomed, Relationship between Symmetries and Conservation Laws, *Int. J. Theoretical Physics*, **39**(1) 23-40 (2000)

[16] A.H. Kara and F.M. Mahomed, A basis of Conservation Laws for partial differential equations, *J. Nonlinear Math. Phys.*, **9** 60-72 (2002)

[17] P. Olver, *Application of Lie Groups to Differential Equations*, Springer, New York, (1993)

[18] R. Narain and A.H. Kara, Conservation laws of high-order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives, *J.Phys.A: Math. Theor.*, **43** (2010), 085205 17pp.

[19] M. Inc, On Numerical soliton soultion of the Kaup-Kupershmidt equation and convergence analysis of the decompostion method, *Applied Mathematics and Computation*, **172** 72-85 (2006)

[20] S. Abbasandy and F. Samadian Zakaria, Soliton solutions for the fifth order KdV equation with the homotopy analysis method, *J. Nonlinear Dynamics*, 51 83-87 (2008)

[21] P. Lax, *Communic. Pure. Appl. Maths.*, **21** 467 (1968)

[22] K. Sawada and T. Kotera, *Prog. Theor. Phys.*, **51** 1355 (1974)

[23] M. Ito, *J. Phys. Soc. Jpn*, **49** 771 (1980)

[24] N.H. Ibragimov, *Transformation Groups applied to Mathematical Physics*, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, (1985)

[25] P.A. Clarkson and E.L. Mansfield *On the Shallow Water Wave Equation*, Nonlinearity 7 975-1000 (1994)

[26] R. Narain and A.H. Kara, On the Redefinition of the Variational and 'Partial' Variational Conservation Laws in a Class of Nonlinear PDEs with Mixed Derivatives, *J. Math. and Comp. App.*, Vol. 15, No.4, pp. 732-741, (2010)

[27] W. Sarlet, Comments on 'Conservation laws of higher order nonlinear PDEs and their variational conservation laws in the class with mixed derivatives', *J. Physics A: Math. Theor.*, **43** 458001 (6pp), (2010)

[28] A. Constantin and J. Lennels, On the Inverse Scattering Approach to the Camassa-Holm Equation, *J.Nonlinear Mathematical Physics* **10**(3) 252-255 (2003)

[29] A.N.W. Hone, The associated Camassa-Holm equation and the KdV equation, *J. Phys. A: Math. Gen.*, **32** 307-314 (1999)

[30] J. Lennels, Weak geodesic flow and global solutions of the Hunter-Saxton equation *Discrete and continuous dynamical systems*, 18 (4) 643-656 (2007)

[31] R.W. Lindquist, R.A. Schwartz and C.W. Misner, Vaidiya's radiating Schwartzchild metric, *Physical Review*, **137**(5B) B1364-B1368 (1965)

[32] P.C. Vaidiya, *Proc. Indian Acad. Sc.*, **A33** 264 (1951)

[33] D. Finkelstein, *Physical Review*, **110** (1958)

[34] I.H. Dwivedit and P.S. Joshi, On the nature of naked singularities in Vaidya spacetimes, *Class. Quantum Grav.*, **6** 1599-1606 (1989)

[35] B.S. Ramachandra, The Carter constant and Petrov classification of the VaidyaEinsteinKerr spacetime, *Gen Relativ. Gravit.*, DOI 10.1007/s10714-009-0805-y

[36] S. Kobayashi, *Transformation Groups in Differential Geometry*, Classics in Mathematics, Springer (1972)

[37] H. Stephani, *Differential Equations: their solution using symmetries*, Cambridge University Press, Cambridge, (1989)

[38] D. Kramer, H. Stephani, E. Herlt, and M. A. H. MacCallum, *Exact Solutions of Einstein Field Equations*, Cambridge University Press, Cambridge, (1980)

[39] C.B.G. McIntosh, *Phys. Lett.* 50A, 429 (1975); A.H. Taub, *In General Relativity: Papers in Honor of J.L. Synge*, Oxford University Press, London, Chap.VIII, p.133 (1972); M.E. Cahill and A.H. Tuab *Commun. Math. Phys.*, 21 1 (1971); J. Carot and A. Sintes, M. *Class. Quantum Grav.*, 14 1183 (1997)

[40] R. Narain and A. H. Kara, The Noether conservation laws of some Vaidya metrics, *Int J. Theor. Phys.*, 49 (2010), 260-269

[41] N.H. Ibragimov (Ed.), *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 1, CRC Press, Boca Raton, Florida, (1996)

[42] R.S. Rivlin and J.L. Ericksen, Stress deformation relations for isotropic materials, *J. Rational Mech. Anal.*, 4 323 (1955)

[43] C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*, 2nd Edition, Springer, New York, (1992)

[44] C.H. Hsu and K.L. Hsiao, Conjugate heat transfer of a plate in a second-grade fluid flow, *Int. J. Heat Mass Transfer*, 41 1087 (1998)

[45] K.R. Rajagopal and A.S. Gupta, A class of exact solutions to the equations of motions of a second-grade fluid, *Int. J. Eng. Sci.*19 1014 (1981)

[46] A.M. Benharbit and A.M. Siddiqui, Certain solutions of the planar motion of a second-grade fluid for steady and unsteady cases, *Acta Mech.*, 94 96 (1992)

[47] V.K. Garg and K.R. Rajagopal, Flow of a non-Newtonian fluid past a wedge, *Acta Mech.*, 88 113 (1991)

[48] M. Akcay, M.A. Yukselen, Drag reduction of a non-Newtonian fluid by fluid injection on a moving wall, *Arch. Appl. Mech.*, 69 215 (1999)

[49] T. Hayat, S. Asghar and A.M. Siddiqui, Periodic flows of a non-Newtonian fluid, *Acta Mech.*, 131 169 (1998)

[50] T. Hayat, S. Asghar and A.M. Siddiqui, On the moment of a plane disk in a non-Newtonian fluid, *Acta Mech.*, 136 125 (1999)

[51] T. Hayat, S. Asghar and A.M. Siddiqui, Periodic flows of a non-Newtonian fluid between two parallel plates, *Int. J. Non-LinearMech.*, 34 895 (1999)

[52] W.R. Schowalter, *Mechanics of Non-Newtonian Fluids*, Oxford University Press, Pergamon Press, Oxford, (1978)

[53] R.R. Huilgol, *Continuum Mechanics of Visco Elastic Liquids*, Hindustan Publishing Corporation, Delhi, (1975)

[54] K.R. Rajagopal, Mechanics of non-Newtonian fluids, *Recent Developments in Theorectical Fluid Mechanics*, Pitman Research Notes in Mathematical Series, Vol.291, Longman Science Technology, Marlow, 129, (1993)

[55] M. Reiner, and S. Flügge (Ed.), *Handbuch der Physik*, Vol. VI, Springer, Berlin, (1958)

[56] T. Hayat, A.H. Kara and E. Momoniat, Exact flow of a third-grade fluid on a porous wall, *Int. J. Non-Linear Mech.*, 38 1533 (2003)

[57] U. Göktas and W. Hereman, Computation of conservation laws for nonlinear lattices, *Physica D*, 123 425 (1998)

[58] R. Narain and A.H. Kara, An analysis of the conservation laws for certain third-grade fluids, *J. Nonlinear Analysis: Real World Applications*, 11 (2010), 3236-3241

[59] C. Truesdell and W. Noll, The non-linear theories of mechanics, *Handbuch der Physik*, Vol. III/3, Springer, Berlin, (1965)

[60] B.D. Coleman and W. Noll, An approximation theorem for functionals, with applications in continuum mechanics, *Arch.Rational Mech. Anal.*, 56 191 (1965)

[61] P.N. Kaloni, Fluctuating flow of an elastico-viscous fluid past a porous at plate, *J. Phys. Fluids*, 10 1344 (1967)