SPECTRAL ANALYSIS AND RIESZ BASIS PROPERTY FOR VIBRATING SYSTEMS WITH DAMPING

Thesis by

Guo-Dong Zhang

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

School of Computational and Applied Mathematics,

University of the Witwatersrand,

Private Bag-3, Wits-2050, Johannesburg,

South Africa

(Submitted February 16, 2011)

Declaration

This thesis contains the work done by the author under the supervision of Professor Bao-Zhu Guo during the period 2008-2010 for the degree of Doctor of Philosophy at the University of Witwatersrand. The work submitting has not been previously included in any thesis, dissertation or report submitted to this and any institution for a degree, diploma or other qualification.

(Signature)

(Date)

Acknowledgements

I wish to express my sincere gratitude to my supervisor Professor Bao-Zhu Guo, for his continual guidance and constant encouragement throughout the period of my postgraduate studies, and in the preparation of this thesis. This work was supported by the project "Riesz Basis Property for Vibrating Systems" from the National Research Foundation of South Africa with grant No. FA2006031000006.

I am grateful to Professor Jun-Min Wang for many useful discussions.

I would also like to thank my parents and my friends for their support and encouragement.

Abstract

In this thesis, we study one-dimensional wave and Euler-Bernoulli beam equations with Kelvin-Voigt damping, and one-dimensional wave equation with Boltzmann damping.

The spectral property of equations with clamped boundary conditions and internal Kelvin-Voigt damping are considered. Under some assumptions on the coefficients, it is shown that the spectrum of the system operator is composed of two parts: point spectrum and continuous spectrum. The point spectrum consists of isolated eigenvalues of finite algebraic multiplicity, and the continuous spectrum that is identical to the essential spectrum is an interval on the left real axis. The asymptotic behavior of eigenvalues is also presented.

Two different Boltzmann integrals that represent the memory of materials are considered. The spectral properties for both cases are thoroughly analyzed. It is found that when the memory of system is counted from the infinity, the spectrum of system contains a left half complex plane, which is sharp contrast to most results in elastic vibration systems that the vibrating dynamics can be considered from the vibration frequency point of view. This suggests us to investigate the system with memory counted from the vibrating starting moment. In the later case, it is shown that the spectrum of system determines completely the dynamic behavior of the vibration: There is a sequence of generalized eigenfunctions of the system, which forms a Riesz basis for the state space. As the consequences, the spectrum-determined growth condition and exponential stability are concluded.

Contents

Publications During the Candidate

1. B.Z. Guo, J.M. Wang and **G.D. Zhang**, Frequency analysis of a wave equation with Kelvin-Voigt damping, *48th IEEE Conference on Decision and Control*, Shanghai, P.R. China, (2009), 4471-4476.

2. B.Z. Guo, J.M. Wang and **G.D. Zhang**, Spectral analysis of a wave equation with Kelvin-Voigt damping, *Z. Angew. Math. Mech.*, 90(2010), 323-342.

3. **G.D. Zhang** and B.Z. Guo, On the spectrum of Euler-Bernoulli beam equation with Kelvin-Voigt damping, *J. Math. Anal. Appl.*, 374(2011), 210-229.

4. B.Z. Guo and **G.D. Zhang**, Spectral analysis and Riesz basis property for wave equation with Boltzmann damping, *18th IFAC World Congress, August 28 - September 2, 2011, Milan, Italy, Accepted.*

5. B.Z. Guo and **G.D. Zhang**, On spectrum and Riesz basis property for one-dimensional wave equation with Boltzmann damping, submitted.

Chapter 1

Introduction

1.1 Background of the thesis and literature review

Mechanical vibration is one of the most frequent movements in physical world. The earlier study on vibration is closely related to sound and music instruments. However, little progress was made before 16^{th} century ([77]).

Spurred by the large applications of smart materials, there has been, starting from 1990, an increasing research on elastic system with viscoelastic dampings ([6, 8]). When the smart materials are added into the elastic structures, the Young's modulus, the mass density and the damping coefficients are changed accordingly. This passive method, on the one hand, makes the distributed control practically implementable but on the other hand, brings some new mathematical challenges which attract an increasing research interests, see for instance [33, 34, 40, 48, 85, 88, 92, 96, 102, 103] for beam equations, and [18, 30, 43, 44, 52, 65, 74, 82, 106] for wave equations. For the controllability study of this kind of systems, we refer to [56, 57, 61, 63, 66, 68, 104]. The research of spectral analysis can be found in [4, 43, 81, 84, 98, 101]. The results of exponential stability by bounded viscous damping can be found in [9, 15, 30, 73, 106]. The exponential stability of one-dimensional wave and Euler-Bernoulli beam equations with Kelvin-Voigt damping is discussed in [71, 72], and [74] for multi-dimensional case. Other studies can be found in [1, 13, 14, 37, 38, 41, 45, 46, 47, 50, 53, 54, 64, 75, 89, 94, 97, 100].

Among these works, two types of unbounded viscoelastic damping, Kelvin-Voigt damping and Boltzmann damping, are specially important. These kinds of passive control can now be accomplished as active vibration control through piezoelectric actuator/sensor ([86]). The Kelvin-Voigt damping models of linear viscoelasticity assume that the instantaneous stress depends on the instantaneous strain and the strain rate linearly. The Boltzmann models of linear viscoelasticity assume that the instantaneous stress depends on the instantaneous strain and the entire history of strain rate linearly. For the Boltzmann models, we refer to [7, 28, 69, 72, 82, 103] and the references therein. Basically, there are two types of Boltzmann integrals. One is with the infinite entire memory ([7, 69, 72, 75, 103]), and another is with finite memory ([5, 29, 82]). The Kelvin-Voigt damping that is a special case of Boltzmann damping where the relaxation function is a delta function and is stronger than the Boltzmann damping. Some results of Kelvin-Voigt damping can be found in [16, 43, 70, 71, 74, 81, 98, 101, 102].

However, for these works aforementioned, only partial properties of the vibration frequencies are presented. This is an unfortunate situation since it has been shown for many other elastic systems in [33, 34, 76, 90] that for a vibrating system, the vibration frequencies could determine all dynamic behaviors of the system. The reason for occurrence of this situation is that for a viscoelastic system, the resolvent of system operator is not compact anymore, which is in sharp contrast with that discussed in [33, 34, 76, 90]. Nevertheless, a viscoelastic system with constant coefficients still shows the validity of Riesz basis property due to the fact that the continuous spectrum is the limit set of point spectrum ([55]). But for the viscoelastic systems with variable coefficients, the situation is quite different even in one-dimensional cases, see for instance [59, 60, 81, 98].

In this thesis, we study the wave and Euler-Bernoulli beam equations with Boltzmann type or Kelvin-Voigt damping. The spectrum for these systems are analyzed, and the Riesz basis properties are investigated. Partial results have been published in my publications during the candidate ([42, 43, 51, 101]).

1.2 Preliminaries

A linear time-invariant infinite-dimensional system on a Hilbert Space $\mathcal{H}$ can be represented as

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1.2.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on $\mathcal{H}$.

In this section, we introduce some basic concepts about the linear operators in Hilbert space. The strong continuous semigroups for linear operators (C_0-semigroup in short) and Riesz basis are also briefly introduced.

1.2.1 Basic results of the linear operators

Firstly, we recall some definitions (see [2, 23, 31, 32], or check on wikipedia) about the spectrum and eigenvector of the linear operators.

Let $\mathcal{A}$ be a linear operator in a Hilbert space $\mathcal{H}$ with the domain $D(\mathcal{A})$. Then the adjoint operator of $\mathcal{A}$ is denoted by $\mathcal{A}^*$. $\sigma(\mathcal{A})$, $\sigma_p(\mathcal{A})$, $\sigma_r(\mathcal{A})$, $\sigma_c(\mathcal{A})$, $\rho(\mathcal{A})$ will denote respectively the spectrum of $\mathcal{A}$, the point spectrum, the residual spectrum, the continuous spectrum and the resolvent set of $\mathcal{A}$. For any $\lambda \in \rho(\mathcal{A})$,

$$R(\lambda, \mathcal{A}) := (\lambda I - \mathcal{A})^{-1} \quad (I \text{ is the identity operator})$$

is called the resolvent operator of $\mathcal{A}$.

Definition 1.2.1. Let $\mathcal{A}$ be a closed linear operator in a Hilbert space. The set of complex numbers λ is called the **essential spectrum** of $\mathcal{A}$, and is denoted by $\sigma_{ess}(\mathcal{A})$, if one of the following three conditions is satisfied:

(i). $\mathcal{R}(\lambda I - \mathcal{A})$, the range of $\lambda I - \mathcal{A}$, is not closed.

(ii). $\dim \mathcal{N}(\lambda I - \mathcal{A}) = \infty$, here $\mathcal{N}(\lambda I - \mathcal{A})$ denotes the null space of $\lambda I - \mathcal{A}$.

(iii). $\dim(\mathcal{R}(\lambda I - \mathcal{A}))^\perp = \infty$, here $(\mathcal{R}(\lambda I - \mathcal{A}))^\perp$ is the orthogonal complement space of range $\mathcal{R}(\lambda I - \mathcal{A})$ of $\lambda I - \mathcal{A}$.

Notice that if $\mathcal{A}$ is densely defined, then (iii) of Definition 1.2.1 can be replaced by $\dim \mathcal{N}(\overline{\lambda} I - \mathcal{A}^*) = \infty$.

The following Proposition 1.2.1 is a direct consequence of Corollary 4.4 of [31, p.378].

Proposition 1.2.1. *Let T be a closed linear operator in a Hilbert space and G a compact operator. Then*

$$\sigma_{ess}(T) = \sigma_{ess}(T + G).$$

Definition 1.2.2. A nonzero element Φ in a Hilbert space $\mathcal{H}$ is called a **generalized eigenvector** of a closed linear operator $\mathcal{A}$, corresponding to an eigenvalue λ of $\mathcal{A}$, if there is a nonnegative integer n such that

$$(\lambda I - \mathcal{A})^n \Phi \neq 0 \quad \text{and} \quad (\lambda I - \mathcal{A})^{n+1} \Phi = 0.$$

If $n = 0$, then Φ is an ordinary eigenvector. The root subspace of $\mathcal{A}$ corresponding to λ is defined as

$$\mathcal{N}_\lambda(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{N}((\lambda I - \mathcal{A})^n).$$

The dimension $m_{\lambda,a}$ of $\mathcal{N}_\lambda(\mathcal{A})$ is called the **algebraic multiplicity** of λ. When $m_{\lambda,a} = 1$, we say λ is **algebraically simple**. The collection of all eigenvectors of $\mathcal{A}$ corresponding to an eigenvalue λ is just $\mathcal{N}(\lambda I - \mathcal{A})$ and it is called the **geometric eigenspace** of λ. The dimension $m_{\lambda,g}$ of $\mathcal{N}(\lambda I - \mathcal{A})$ is called the **geometric multiplicity** of λ. When $m_{\lambda,g} = 1$, we say λ is **geometrically simple**.

Now we introduce C_0-semigroup of linear operators (see [21, 25, 26, 32, 62, 79]).

Definition 1.2.3. Let X be a Banach space. A one parameter family $T(t),\ 0 \le t < \infty$, of bounded linear operators from X into X is a **semigroup of bounded linear operators** on X if

(i) $T(0) = I$;

(ii) $T(t + s) = T(t)T(s)$ for every $t, s \ge 0$. (the semigroup property).

A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The linear operator $\mathcal{A}$ defined by

$$\mathcal{A}x = \lim_{t\to 0} \frac{T(t)x - x}{t},\ \forall x \in D(\mathcal{A})$$

and

$$D(\mathcal{A}) = \left\{ x \in X |\lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

is the **infinitesimal generator of the semigroup** $T(t)$.

Definition 1.2.4. Let X be a Banach space. A semigroup $T(t), 0 \le t < \infty$, of bounded linear operators on X is a **strongly continuous semigroup of bounded linear operators** if

$$\lim_{t\to 0} T(t)x = x,\ \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a $\mathbf{C_0}$**-semigroup**.

For a C_0-semigroup, we have following Theorem 1.2.1.

Theorem 1.2.1. *Let $T(t)$ be a C_0-semigroup. Then there exist constants ω and $M \ge 1$ such that*

$$\|T(t)\| \le Me^{\omega t} \ \ \forall 0 \le t < \infty.$$

In Theorem 1.2.1, if $\omega = 0$, then $T(t)$ is called **uniformly bounded** and if moreover $M = 1$ it is called a C_0**-semigroup of contractions**.

The next Theorem, **Hille-Yosida Theorem** is very important in theory of semigroups of linear operators. It is a characterization of generators of arbitrary strongly continuous semigroups.

Theorem 1.2.2. *Let $\mathcal{A}$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}, M \geq 1$ be constants. Then the following properties are equivalent.*

(i) $\mathcal{A}$ generates a C_0-semigroup $T(t), t \geq 0$ satisfying

$$\|T(t)\| \leq Me^{\omega t}, \ \forall t \geq 0;$$

(ii) $\mathcal{A}$ is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A})$ and

$$\|(\lambda - \omega)^n R(\lambda, \mathcal{A})^n\| \leq M, \quad \forall n \in \mathbb{N};$$

(iii) $\mathcal{A}$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$ one has $\lambda \in \rho(\mathcal{A})$ and

$$\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

For contraction semigroups, the Hille-Yosida characterization is especially simple (case $M = 1, \omega = 0$).

Corollary 1.2.1. *An operator $\mathcal{A}$ is the generator of a C_0-contraction semigroup if and only if it is closed, densely defined, and $\rho(\mathcal{A})$ contains $\mathbb{R}^+$ and for every $\lambda > 0$,*

$$\|\lambda R(\lambda, \mathcal{A})\| \leq 1.$$

Definition 1.2.5. Let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Hilbert space $\mathcal{H}$. Consider

$$\omega(\mathcal{A}) := \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\|,$$

the growth exponent bound of $T(t)$, and

$$s(\mathcal{A}) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{A})\},$$

the spectral bound of the operator $\mathcal{A}$. If $\omega(\mathcal{A}) = s(\mathcal{A})$, we say that the **spectrum determined growth condition** holds.

Remark 1.2.1. From the Hille-Yosida Theorem, we know that $s(\mathcal{A}) \leq \omega(\mathcal{A})$ for any infinitesimal generator of strongly continuous semigroups. However, in general, $s(\mathcal{A}) \geq \omega(\mathcal{A})$ is not true. A counterexample can be found in [76, 105].

Next, we give characterization of C_0-semigroup of contractions, namely, the **Lumer-Phillips Theorem** (Theorem 1.2.4). Before doing so, some definitions are needed.

Let X be a Banach space and let X^* be its dual. We denote the value $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$. For every $x \in X$ we define duality set $F(x) \subset X^*$ by

$$F(x) = \left\{ x^* \in X^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

From the Hahn-Banach Theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$.

Definition 1.2.6. A linear operator $\mathcal{A}$ on a Banach space X is dissipative if for every $x \in D(\mathcal{A})$ there is an $x^* \in F(x)$ such that

$$\mathrm{Re}\langle Ax, x^* \rangle \leq 0.$$

A useful characterization of dissipative operators is given next.

Theorem 1.2.3. *A linear operator $\mathcal{A}$ is dissipative if and only if*

$$\|(\lambda I - \mathcal{A})x\| \geq \lambda \|x\|, \quad \forall x \in D(\mathcal{A}), \lambda > 0.$$

Corollary 1.2.2. *Let X be a Hilbert space. Then a linear operator $\mathcal{A}$ is dissipative if and only if*

$$\mathrm{Re}\langle \mathcal{A}x, x \rangle \leq 0, \quad \forall x \in D(\mathcal{A}).$$

Theorem 1.2.4. *(**Lumer-Phillips Theorem**) Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Banach space X.*

(i) If $\mathcal{A}$ is dissipative and there is a $\lambda_0 > 0$ such that the range, $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.

(ii) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X, then $\mathcal{R}(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$ and $\mathcal{A}$ is dissipative. Moreover,

$$\mathrm{Re}\langle \mathcal{A}x, x^* \rangle \leq 0, \quad \forall x \in D(\mathcal{A}), x^* \in F(x).$$

Corollary 1.2.3. *Let $\mathcal{A}$ be a closed and densely defined linear operator on a Banach space X. If both $\mathcal{A}$ and A^* are dissipative, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.*

We define the exponential stability for a linear infinite-dimensional system (see [21, 76]).

Definition 1.2.7. Suppose that the infinitesimal generator $\mathcal{A}$ of system (1.2.1) generates a strongly continuous semigroup of linear operators $T(t)$ on Hilbert space $\mathcal{H}$. $T(t)$ is said to be exponentially stable if there are constants $\alpha > 0$ and $M > 0$ such that

$$\|T(t)\| \leq Me^{-\alpha t}, \quad \text{for } t > 0.$$

The minimal such of α is called the **decay rate** for the semigroup $T(t)$.

In systems theory, the exponential stability is the most desirable type of stability because if a semigroup associated with an infinite-dimensional system is exponentially stable, then the system is automatically input-output stable and robustly stable. The following necessary and sufficient condition for the exponential stability of a C_0-semigroup is due to Huang (see [58]):

Theorem 1.2.5. *Let $T(t)$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. Then $T(t)$ is exponentially stable if and only if*

$$\sup \{\mathrm{Re}\lambda : \ \lambda \in \sigma(\mathcal{A})\} < 0$$

and

$$\sup_{\mathrm{Re}\lambda \geq 0} \|(\lambda I - \mathcal{A})^{-1}\| < \infty$$

hold.

From Definition 1.2.5, we have another sufficient condition for the exponential stability: Assume the semigroup generalized by the system satisfies the spectrum determined growth condition and $s(\mathcal{A}) < 0$, then the semigroup is exponentially stable. However, the verification of spectrum determined growth condition represents a challenge problem in infinite-dimensional system theory.

1.2.2 Riesz basis

We give the definition of Riesz basis first, which can be found in [21].

Definition 1.2.8. A sequence $\{\Phi_n : n \geq 1\}$ in a Hilbert space $\mathcal{H}$ is said to be a **Riesz basis** for $\mathcal{H}$ if the following two conditions hold:

1.

$$\overline{\mathrm{span}_{n\geq 1}\{\Phi_n\}} = \mathcal{H};$$

2. There exist positive constants C_1 and C_2 such that for arbitrary $N \in \mathbb{N}$ and arbitrary scalars $\alpha_n, n = 1, 2, \cdots, N$, we have

$$C_1 \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \Phi_n \right\|^2 \leq C_2 \sum_{n=1}^{N} |\alpha_n|^2.$$

Riesz basis is equivalent to an orthonormal basis (see [99]).

Proposition 1.2.2. *A sequence $\{\Phi_n : n \geq 1\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis if and only if there is a linear bounded and boundedly invertible operator $\mathcal{T}$ on $\mathcal{H}$ such that*

$$\mathcal{T}\Phi_n = e_n, \quad n \geq 1,$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

The following proposition reveals the importance of Riesz basis property in infinite-dimensional systems (see [21, 87]).

Proposition 1.2.3. *Let $\mathcal{A}$ be a densely defined closed linear operator in Hilbert space $\mathcal{H}$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be its eigenvalues with finite multiplicities and let $\{\Phi_{n,j} | a \leq j \leq m_{\lambda_n,a}\}$ be a family of generalized eigenvectors of $\mathcal{A}$ corresponding to the eigenvalue λ_n, that form a basis for the finite-dimensional space $\mathcal{N}_{\lambda_n}(A)$. Suppose that λ_n is separated and simple for sufficient large n, i.e.,*

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n| \leq |\lambda_{n+1}| \leq \cdots, \quad \lambda_j \neq \lambda_k, \ \textit{if } j \neq k,$$

and for large enough $N \in \mathbb{N}$,

$$m_{\lambda_n,a} = 1, \quad \textit{for } n \geq N.$$

Suppose

$$\{\Phi_{n,j} | 1 \leq j \leq m_{\lambda_n,a}\}_{n=1}^{N-1} \cup \{\Phi_n\}_{n=N}^{\infty}$$

forms a Riesz basis in $\mathcal{H}$.

1. If $\{\Psi_{n,j} | 1 \leq j \leq m_{\overline{\lambda}_n,a}\}$ is the set of generalized eigenvectors of $\mathcal{A}^*$ corresponding to the eigenvalue $\overline{\lambda}_n$, that forms a basis for the finite-dimensional space $\mathcal{N}_{\overline{\lambda}_n}(\mathcal{A}^*)$, then we can normalize $\{\Psi_{n,j} | 1 \leq j \leq m_{\overline{\lambda}_n,a}\}$ so that $\{\Phi_{n,j} | 1 \leq j \leq m_{\lambda_n,a}\}$ and $\{\Psi_{n,j} | 1 \leq j \leq m_{\overline{\lambda}_n,a}\}$ are biorthogonal. In general, the generalized eigenvectors $\{\Phi_{n,j} | 1 \leq j \leq m_{\lambda_n,a}\}$ and $\{\Psi_{n,j} | 1 \leq j \leq m_{\overline{\lambda}_n,a}\}$ can be constructed from the following procedures:

$$\mathcal{A}\Phi_{n,j} = \lambda_n \Phi_{n,j} + \Phi_{n,j+1}, \quad j = 1, 2, \cdots, m_{\lambda_n,a} - 1,$$

$$\mathcal{A}\Phi_{n,m_{\lambda_n,a}} = \lambda_n \Phi_{n,m_{\lambda_n,a}},$$

$$\mathcal{A}^*\Psi_{n,j} = \overline{\lambda}_n \Psi_{n,j} + \Psi_{n,j-1}, \quad j = 2, \cdots, m_{\lambda_n,a},$$

$$\mathcal{A}^*\Psi_{n,1} = \overline{\lambda}_n \Psi_{n,1},$$

and obtain

$$\langle \Phi_{n,s}, \Psi_{n,k} \rangle = \delta_{sk} = \begin{cases} 1, & \text{if } s = k, \\ 0, & \text{if } s \neq k. \end{cases}$$

2. Every $x \in \mathcal{H}$ can be represented uniquely by

$$x = \sum_{n=1}^{N-1} \sum_{j=1}^{m_{\lambda_n},a} \langle x, \Psi_{n,j} \rangle \Phi_{n,j} + \sum_{n=N}^{\infty} \langle x, \Psi_n \rangle \Phi_n$$

and there exist positive constants C_1 and C_2 such that

$$C_1 \left(\sum_{n=1}^{N-1} \sum_{j=1}^{m_{\lambda_n},a} |\langle x, \Psi_{n,j} \rangle|^2 + \sum_{n=N}^{\infty} |\langle x, \Psi_n \rangle|^2 \right) \leq ||x||^2$$
$$\leq C_2 \left(\sum_{n=1}^{N-1} \sum_{j=1}^{m_{\lambda_n},a} |\langle x, \Psi_{n,j} \rangle|^2 + \sum_{n=N}^{\infty} |\langle x, \Psi_n \rangle|^2 \right).$$

3. Under $\{\Phi_{n,j}\}$ and $\{\Psi_{n,j}\}$, the operator $\mathcal{A}$ has the form

$$\mathcal{A}x = \sum_{n=1}^{N-1} \left(\lambda_n \langle x, \Psi_{n,m_{\lambda_n,a}} \rangle \Phi_{n,m_{\lambda_n,a}} + \sum_{j=1}^{m_{\lambda_n,a}-1} \langle x, \Psi_{n,j} \rangle (\lambda_n \Phi_{n,j} + \Phi_{n,j+1}) \right)$$
$$+ \sum_{n=N}^{\infty} \lambda_n \langle x, \Psi_n \rangle \Phi_n$$

for every $x \in D(\mathcal{A})$, and

$$D(\mathcal{A}) = \left\{ x \in \mathcal{H} \Big| \sum_{n=N}^{\infty} |\lambda_n|^2 |\langle x, \Psi_n \rangle|^2 < \infty \right\}.$$

4. Operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ in $\mathcal{H}$ if and only if

$$\sup_{n \geq 1} \mathrm{Re}\lambda_n < \infty.$$

In that case, the semigroup $T(t)$ is given by

$$T(t) = \sum_{n=1}^{N-1} e^{\lambda_n t} \sum_{j=1}^{m_{\lambda_n,a}} \langle \cdot, \Psi_{n,j} \rangle \sum_{s=1}^{j} \frac{t^{j-s}}{(j-s)!} \Phi_{n,s} + \sum_{n=N}^{\infty} e^{\lambda_n t} \langle \cdot, \Psi_n \rangle \Phi_n.$$

5. If $\sup_{n \geq 1} \mathrm{Re}\lambda_n < \infty$, then the spectrum determined growth condition holds: $\omega(\mathcal{A}) = s(\mathcal{A})$.

In infinite-dimensional control theory, Riesz basis property is one of the most wanted properties, particular for elastic vibrating systems for which the Riesz basis property is significant both theoretically and practically. Usually, the Riesz basis property will lead to the establishment such as the spectrum determined growth condition, and the exponential stability of the system. However, verification of the Riesz basis is extremely difficult since the associated system operator is non-self-adjoint. Here are some useful methods on the verification of the Riesz basis property:

(i) Let $\mathcal{A}$ be a closed linear operator in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is a self-adjoint and resolvent compact operator, then its eigenvectors form an orthonormal basis for $\mathcal{H}$ and hence a Riesz basis.

(ii) The Riesz basis can be established via some classical perturbation results for discrete operators as illustrated in [24] and [67].

(iii) For a system that has its generalized eigenvectors asymptotically close to the set of linear combinations of non-harmonic exponentials, the property can be established through the method of non-harmonic exponentials (see [3] and [99]).

(iv) Classical Bari's Theorem (see, for example, [9, 19, 20, 35, 36, 39, 49, 83, 95]): if $\{\Phi_n\}_{n=1}^{\infty}$ is a Riesz basis for a Hilbert space $\mathcal{H}$, and $\{\Psi_n\}_{n=1}^{\infty}$, an ω-linearly independent sequence in $\mathcal{H}$, is quadratically close to $\{\Phi_n\}_{n=1}^{\infty}$ in the sense that

$$\sum_{n=1}^{\infty} \|\Phi_n - \Psi_n\|^2 < \infty,$$

then $\{\Psi_n\}_{n=1}^{\infty}$ is also a Riesz basis itself for $\mathcal{H}$.

The following Lemma (see [80]) will be used later.

Lemma 1.2.1. *Let $\{\phi_n\}_1^{\infty}$ be a Riesz basis for a Hilbert space $\mathcal{H}$. Suppose there are $N_0 \geq 1$ and an ω-linearly independent sequence $\{\psi_n\}_{N_0}^{\infty}$ such that*

$$\sum_{n=N_0}^{\infty} \|\psi_n - \phi_n\|^2 < \infty.$$

Then $\{\psi_n\}_{N_0}^{\infty}$ forms a Riesz basis for the subspace spanned by itself.

1.3 Organization of the thesis

This thesis consists of three parts.

Chapter 2 is devoted to the spectral analysis of a one-dimensional wave equation with clamped boundary conditions and internal Kelvin-Voigt damping. The spectrum of the system operator is discussed. The asymptotic expression for eigenvalues is presented.

In Chapter 3, we examine an Euler-Bernoulli beam equation with Kelvin-Voigt damping, which was initiated in [98]. The spectral property of the equation is considered. The asymptotic behavior of eigenvalues is also presented.

In Chapter 4, we study the one-dimensional wave equation with Boltzmann damping. Two different Boltzmann integrals that represent the memory of materials are considered. The spectral properties for both cases are thoroughly analyzed. Riesz basis property for the equation with memory counted from the vibrating starting moment is described.

Chapter 2

Spectral Analysis of a One-Dimensional Wave Equation with Internal Kelvin-Voigt Damping

2.1 Introduction

In this chapter, we shall consider a one-dimensional wave equation with clamped boundary conditions and internal Kelvin-Voigt damping. It is shown that the spectrum of the system operator is composed of two parts: point spectrum and continuous spectrum. The point spectrum consists of isolated eigenvalues of finite algebraic multiplicity, and the continuous spectrum that is identical to the essential spectrum is an interval on the left real axis. The asymptotic behavior of eigenvalues is presented.

We shall give an outline of the Chapter: In next section, Section 2.2, we formulate the problem into an abstract evolution equation in the state space. Section 2.3 is devoted to the analysis of essential spectrum and continuous spectrum of the system operator, see Theorem 2.3.2 and 2.3.3. The asymptotic expression for eigenvalues is presented in Section 2.4, see Theorem 2.4.1.

2.2 System operator setup

The system that we are concerned with is the following wave equation with Kelvin-Voigt damping and clamped boundary conditions:

$$\begin{cases} \rho(x)y_{tt}(x,t) - (a(x)y_x(x,t) + b(x)y_{xt}(x,t))' = 0, \ 0 < x < 1, \ t > 0, \\ y(0,t) = y(1,t) = 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \end{cases} \tag{2.2.1}$$

where the continuous function $b(\cdot) \geq 0$ is the damping function, and the continuous functions $\rho(\cdot), a(\cdot) > 0$ are system parameter functions in spacial variable. In this thesis, we use prime "$\prime$" to represent the derivative with respect to x. The system energy is

$$E(t) = \frac{1}{2} \int_0^1 \left[a(x)|y_x(x,t)|^2 + \rho(x)|y_t(x,t)|^2\right] dx. \tag{2.2.2}$$

For any positive continuous function ρ, set $L^2_\rho = L^2(0,1)$ with norm

$$\|f\|^2_{L^2_\rho} = \int_0^1 \rho(x)|f(x)|^2 dx$$

and $V = H^1_0(0,1)$, the first order Sobolev space with zero boundary values. We consider the system (2.2.1) in the energy state Hilbert space $\mathcal{H} = V \times L^2_\rho$ with the inner product:

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \int_0^1 [a(x)f_1'(x)\overline{f_2'(x)} + \rho(x)g_1(x)\overline{g_2(x)}]dx, \quad \forall\, (f_i, g_i) \in \mathcal{H}, \ i = 1, 2. \tag{2.2.3}$$

Define the system operator $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ as

$$\begin{cases} \mathcal{A}(f,g) = \left(g, \ \dfrac{1}{\rho}(af' + bg')'\right), \\ D(\mathcal{A}) = \left\{(f,g) \in H^1_0(0,1) \times H^1_0(0,1) | \ af' + bg' \in H^1(0,1)\right\}. \end{cases} \tag{2.2.4}$$

Then (2.2.1) can be formulated into an abstract evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} Y(t) = \mathcal{A}Y(t), \\ Y(0) = Y_0, \end{cases} \tag{2.2.5}$$

where $Y(t) = (y(\cdot,t), y_t(\cdot,t))$ is the state variable and $Y_0 = (y_0(\cdot), y_1(\cdot))$ is the initial value.

The following Lemma 2.2.1 is straightforward.

Lemma 2.2.1. *Let $\mathcal{A}$ be defined by (2.2.4). Then its adjoint $\mathcal{A}^*$ has the following form:*

$$\begin{cases} \mathcal{A}^*(f,g) = \left(-g, \ -\dfrac{1}{\rho}(af' - bg')'\right), \\ D(\mathcal{A}^*) = \left\{(f,g) \in V \times V | \ af' - bg' \in H^1(0,1)\right\}. \end{cases} \tag{2.2.6}$$

Proposition 2.2.1. *Let $\mathcal{A}$ and $\mathcal{A}^*$ be given by (2.2.4) and (2.2.6) respectively. Then $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, and hence $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. For any $(f, g) \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}(f,g),(f,g)\rangle &= \left\langle (g, \frac{1}{\rho}(af'+bg')'),\ (f,g)\right\rangle \\
&= \int_0^1 \left[a(x)g'(x)\overline{f'(x)} + (a(x)f'(x)+b(x)g'(x))'\overline{g(x)}\right] dx \\
&= \int_0^1 \left[a(x)g'(x)\overline{f'(x)} - a(x)f'(x)\overline{g'(x)}\right] dx - \int_0^1 b(x)|g'(x)|^2 dx,
\end{aligned}$$

and hence

$$\mathrm{Re}\langle \mathcal{A}(f,g),(f,g)\rangle = -\int_0^1 b(x)|g'(x)|^2 dx \le 0.$$

Similarly for any $(u, v) \in D(\mathcal{A}^*)$,

$$\begin{aligned}
\langle \mathcal{A}^*(u,v),(u,v)\rangle &= \left\langle (-v, -\frac{1}{\rho}(au'-bv')'),\ (u,v)\right\rangle \\
&= \int_0^1 \left[-a(x)v'(x)\overline{u'(x)} - (a(x)u'(x)-b(x)v'(x))'\overline{v(x)}\right] dx \\
&= \int_0^1 \left[-a(x)v'(x)\overline{u'(x)} + a(x)u'(x)\overline{v'(x)}\right] dx - \int_0^1 b(x)|v'(x)|^2 dx,
\end{aligned}$$

and hence

$$\mathrm{Re}\langle \mathcal{A}^*(u,v),(u,v)\rangle = -\int_0^1 b(x)|v'(x)|^2 dx \le 0.$$

Therefore, both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. By the Lumer-Phillips Theorem, $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. ∎

2.3 Essential and continuous spectrum

In this section, we consider the spectrum of $\mathcal{A}$. First, let us formulate the eigenvalue problem. Suppose $\mathcal{A}(f, g) = \lambda(f, g)$ with $(f, g) \in D(\mathcal{A})$ and $(f, g) \neq 0$. Then $g = \lambda f$ and $f \in H_0^1(0, 1)$ satisfies

$$\begin{cases} ((a(x)+\lambda b(x))f'(x))' = \lambda^2 \rho(x) f(x), \\ f(0) = f(1) = 0. \end{cases} \tag{2.3.1}$$

The Theorem 2.3.1 following shows that the set $\sigma_r(\mathcal{A})$ is empty.

Theorem 2.3.1. $\sigma_r(\mathcal{A}) = \emptyset$.

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$ if and only if $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$, it suffices to show that

$$\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*).$$

Suppose $\mathcal{A}^*(f,g) = \lambda(f,g)$ for some $(f,g) \in D(\mathcal{A}^*)$ and $(f,g) \neq 0$. Then $g = -\lambda f$ and f satisfies

$$\begin{cases} (a(x)f'(x) + \lambda b(x)f'(x))' = \lambda^2 \rho(x) f(x), \\ f(0) = f(1) = 0. \end{cases} \tag{2.3.2}$$

It is seen that (2.3.2) is the same with (2.3.1). Hence, $\lambda \in \sigma_p(\mathcal{A}^*)$ if and only if $\lambda \in \sigma_p(\mathcal{A})$. Since the eigenvalues of $\mathcal{A}^*$ are symmetric with real axis, we have $\sigma_r(\mathcal{A}) = \emptyset$. ■

Proposition 2.3.1. *Let $\mathcal{A}$ be defined by (2.2.4). Then $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is given by*

$$\mathcal{A}^{-1}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} g_1(x) - \displaystyle\int_0^x \frac{b(\tau)}{a(\tau)} f'(\tau)d\tau + \frac{a_1(x)}{a_1(1)}\left[\int_0^1 \frac{b(\tau)}{a(\tau)} f'(\tau)d\tau - g_1(1)\right] \\ f(x) \end{pmatrix}, \tag{2.3.3}$$

where

$$\begin{cases} g_1(x) = \displaystyle\int_0^x \frac{1}{a(\tau)}\left[\int_0^\tau \rho(s)g(s)ds\right]d\tau, \\ a_1(x) = \displaystyle\int_0^x \frac{1}{a(\tau)}d\tau. \end{cases} \tag{2.3.4}$$

Proof. Let $(f,g) \in \mathcal{H}$. By $\mathcal{A}(\phi,\psi) = (f,g)$, we have

$$\begin{cases} \psi(x) = f(x), \\ \dfrac{1}{\rho(x)}(a(x)\phi'(x) + b(x)\psi'(x))' = g(x). \end{cases} \tag{2.3.5}$$

These together with the boundary conditions show that

$$\begin{cases} (a(x)\phi'(x) + b(x)f'(x))' = \rho(x)g(x), \\ \phi(0) = \phi(1) = 0. \end{cases}$$

A direct computation gives

$$\phi(x) = g_1(x) - \int_0^x \frac{b(\tau)}{a(\tau)} f'(\tau)d\tau + C_1 a_1(x),$$

where $g_1(x)$, $a_1(x)$ are given by (2.3.4). Using the boundary condition $\phi(1) = 0$, it gives

$$C_1 = \frac{1}{a_1(1)}\left[\int_0^1 \frac{b(\tau)}{a(\tau)} f'(\tau)d\tau - g_1(1)\right].$$

Therefore

$$\phi(x) = g_1(x) - \int_0^x \frac{b(\tau)}{a(\tau)} f'(\tau) d\tau + \frac{a_1(x)}{a_1(1)} \left[\int_0^1 \frac{b(\tau)}{a(\tau)} f'(\tau) d\tau - g_1(1) \right].$$

This together with (2.3.5) gives (2.3.3). The proof is complete. ■

Next, we consider the essential spectrum of $\mathcal{A}$. Define a bounded linear operator $\mathcal{D} : V \to L_a^2$ by

$$(\mathcal{D}f)(x) = f'(x), \ \forall\, f \in V = H_0^1(0,1). \tag{2.3.6}$$

The following Lemma 2.3.1 is straightforward.

Lemma 2.3.1. *Let $\mathcal{D}$ be defined by (2.3.6). Then the following assertions hold:*

(i) For any $\phi \in V$, $\|\mathcal{D}\phi\|_{L_a^2} = \|\phi\|_V$.

(ii) The range of $\mathcal{D}$,

$$\mathcal{R}(\mathcal{D}) = \left\{ f \in L_a^2 \middle| \ \left\langle f(\cdot), \frac{1}{a(\cdot)} \right\rangle_{L_a^2} = 0 \right\} \tag{2.3.7}$$

is a closed subspace of L_a^2.

(iii) $\mathcal{D}^{-1}$ is a bounded linear operator from $\mathcal{R}(\mathcal{D})$ onto V given by

$$\mathcal{D}^{-1} f(x) = \int_0^x f(\tau) d\tau, \quad \forall\, f \in \mathcal{R}(\mathcal{D}). \tag{2.3.8}$$

Let $\mathcal{H}_1 = \mathcal{R}(\mathcal{D}) \times L_\rho^2$ with the same inner product defined by (2.2.3). Define the linear operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}_1$ by

$$\mathcal{T}(\phi, \psi) = (\mathcal{D}\phi, \psi) = (\phi', \psi), \ \forall\, (\phi, \psi) \in \mathcal{H}. \tag{2.3.9}$$

Then, it is easy to see that

$$\mathcal{T}^{-1}(f, g) = (\mathcal{D}^{-1} f, g), \quad \forall\, (f, g) \in \mathcal{H}_1$$

and

$$\|\mathcal{T}(\phi, \psi)\|_{\mathcal{H}_1}^2 = \int_0^1 \left[a(x)|\phi'(x)|^2 + \rho(x)|\psi(x)|^2 \right] dx = \|(\phi, \psi)\|_{\mathcal{H}}^2, \quad \forall\, (\phi, \psi) \in \mathcal{H}.$$

Define a linear operator $\widetilde{\mathcal{A}} : D(\widetilde{\mathcal{A}}) (\subset \mathcal{H}_1) \to \mathcal{H}_1$ by

$$\widetilde{\mathcal{A}} = \mathcal{T}\mathcal{A}\mathcal{T}^{-1}. \tag{2.3.10}$$

Then $\tilde{\mathcal{A}}$ is explicitly given by

$$\begin{cases} \widetilde{\mathcal{A}}(\phi, \psi) = \left(\psi', \ \dfrac{1}{\rho}(a\phi + b\psi')' \right), \\ D(\widetilde{\mathcal{A}}) = \left\{ (\phi, \psi) \in \mathcal{R}(\mathcal{D}) \times H_0^1(0,1) |\ a\phi + b\psi' \in H^1(0,1) \right\}. \end{cases} \tag{2.3.11}$$

By (2.3.10), we have Lemma 2.3.2 below.

Lemma 2.3.2. $\sigma(\mathcal{A}) = \sigma(\widetilde{\mathcal{A}})$.

Proposition 2.3.2. *Let $\widetilde{\mathcal{A}}$ be defined by (2.3.11). Then $\widetilde{\mathcal{A}}^{-1}$ exists and has the following expression:*

$$\widetilde{\mathcal{A}}^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{P}\begin{pmatrix} f \\ g \end{pmatrix} + \mathcal{Q}\begin{pmatrix} f \\ g \end{pmatrix}, \quad \forall\, (f,g) \in \mathcal{H}_1, \tag{2.3.12}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are bounded operators on $\mathcal{H}_1$ and have the following expressions respectively: for each $(f,g) \in \mathcal{H}_1$,

$$\mathcal{P}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} \dfrac{1}{a(x)}\displaystyle\int_0^x \rho(s)g(s)ds - \dfrac{g_1(1)}{a_1(1)}\dfrac{1}{a(x)} \\ \displaystyle\int_0^x f(\tau)d\tau \end{pmatrix} \tag{2.3.13}$$

and

$$\mathcal{Q}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} -\dfrac{b(x)}{a(x)}f(x) + \dfrac{1}{a_1(1)}\dfrac{1}{a(x)}\displaystyle\int_0^1 \dfrac{b(\tau)}{a(\tau)}f(\tau)d\tau \\ 0 \end{pmatrix}. \tag{2.3.14}$$

Moreover,

(i) $\mathcal{P}$ is compact and skew-adjoint on $\mathcal{H}_1$;

(ii) $\mathcal{Q}$ is self-adjoint on $\mathcal{H}_1$, and its essential spectrum is given by

$$\sigma_{ess}(\mathcal{Q}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}. \tag{2.3.15}$$

Proof. Since $\widetilde{\mathcal{A}} = \mathcal{T}\mathcal{A}\mathcal{T}^{-1}$, $\widetilde{\mathcal{A}}^{-1}$ exists and $\widetilde{\mathcal{A}}^{-1} = \mathcal{T}\mathcal{A}^{-1}\mathcal{T}^{-1}$. For any $(f,g) \in \mathcal{H}_1$,

$$\begin{aligned}
\widetilde{\mathcal{A}}^{-1}\begin{pmatrix} f \\ g \end{pmatrix}(x) &= \mathcal{T}\mathcal{A}^{-1}\mathcal{T}^{-1}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \mathcal{T}\mathcal{A}^{-1}\begin{pmatrix} \displaystyle\int_0^x f(\tau)d\tau \\ g(x) \end{pmatrix} \\
&= \mathcal{T}\begin{pmatrix} g_1(x) - \displaystyle\int_0^x \dfrac{b(\tau)}{a(\tau)}f(\tau)d\tau + \dfrac{a_1(x)}{a_1(1)}\left[\displaystyle\int_0^1 \dfrac{b(\tau)}{a(\tau)}f(\tau)d\tau - g_1(1)\right] \\ \displaystyle\int_0^x f(\tau)d\tau \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{a(x)}\displaystyle\int_0^x \rho(s)g(s)ds - \dfrac{b(x)}{a(x)}f(x) + \dfrac{1}{a_1(1)}\dfrac{1}{a(x)}\left[\displaystyle\int_0^1 \dfrac{b(\tau)}{a(\tau)}f(\tau)d\tau - g_1(1)\right] \\ \displaystyle\int_0^x f(\tau)d\tau \end{pmatrix} \\
&= \mathcal{P}\begin{pmatrix} f \\ g \end{pmatrix}(x) + \mathcal{Q}\begin{pmatrix} f \\ g \end{pmatrix}(x),
\end{aligned}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are defined by (2.3.13) and (2.3.14) respectively.

Notice that when $b(\cdot) \equiv 0$, $\mathcal{A}$ is skew-adjoint and is of compact resolvent. So is for $\widetilde{\mathcal{A}}$ when $b(\cdot) \equiv 0$ and in this case, $\widetilde{\mathcal{A}}^{-1} = \mathcal{P}$. Hence, $\mathcal{P}$ is compact and skew-adjoint on $\mathcal{H}_1$. (i) is thus proved.

Next we prove (ii). We prove $\mathcal{Q}$ is self-adjoint first. Actually, for any $(f, g),\ (u, v) \in \mathcal{H}_1$, by (2.3.7), $f, u \in \mathcal{R}(\mathcal{D})$,

$$\int_0^1 f(x)dx = \int_0^1 u(x)dx = 0,$$

and hence

$$\begin{aligned}
\langle \mathcal{Q}(f,g),(u,v)\rangle_{\mathcal{H}_1} &= \int_0^1 a(x)\left[-\frac{b(x)}{a(x)}f(x) + \frac{1}{a_1(1)}\frac{1}{a(x)}\int_0^1 \frac{b(\tau)}{a(\tau)}f(\tau)d\tau\right]\overline{u(x)}dx \\
&= -\int_0^1 b(x)f(x)\overline{u(x)}dx \\
&= \langle (f,g), \mathcal{Q}(u,v)\rangle_{\mathcal{H}_1},
\end{aligned}$$

which shows that $\mathcal{Q}$ is self-adjoint on $\mathcal{H}_1$.

Now we show

$$\sigma(\mathcal{Q}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}. \tag{2.3.16}$$

Let $\lambda \in \mathbb{C}$. For any $(u, v) \in \mathcal{H}_1$, consider the equation

$$(\lambda I - \mathcal{Q})(f, g) = (u, v),$$

which is equivalent to

$$\lambda g(x) = v(x)$$

and f satisfies

$$\lambda f(x) + \frac{b(x)}{a(x)}f(x) - \frac{1}{a_1(1)}\frac{1}{a(x)}\int_0^1 \frac{b(\tau)}{a(\tau)}f(\tau)d\tau = u(x). \tag{2.3.17}$$

Since $\langle u(\cdot),\ \frac{1}{a(\cdot)}\rangle_{L^2_a} = 0$, if (2.3.17) admits a solution, integrating both sides of (2.3.17) over $[0, 1]$ shows that it must have

$$\langle \lambda f(\cdot),\ \frac{1}{a(\cdot)}\rangle_{L^2_a} = 0.$$

When $\lambda \neq 0$ and $\lambda + \dfrac{b(x)}{a(x)} \neq 0$ for any $x \in [0, 1]$,

$$g(x) = \lambda^{-1}v(x)$$

and it follows from (2.3.17) that

$$f(x) = \frac{a(x)}{\lambda a(x) + b(x)}\left[u(x) + \frac{1}{a_1(1)}\frac{1}{a(x)}\int_0^1 \frac{b(\tau)}{a(\tau)}f(\tau)d\tau\right]. \tag{2.3.18}$$

A direct computation gives

$$\int_0^1 \frac{b(\tau)}{a(\tau)} f(\tau)d\tau = \frac{1}{\lambda} a_1(1) \left(\int_0^1 \frac{1}{\lambda a(\tau)+b(\tau)} d\tau \right)^{-1} \int_0^1 \frac{b(\tau)u(\tau)}{\lambda a(\tau)+b(\tau)} d\tau.$$

Hence,

$$\begin{aligned} f(x) &= \frac{1}{\lambda a(x)+b(x)} \Big[a(x)u(x) \\ &\quad + \frac{1}{\lambda} \int_0^1 \frac{b(\tau)u(\tau)}{\lambda a(\tau)+b(\tau)} d\tau \left(\int_0^1 \frac{1}{\lambda a(\tau)+b(\tau)} d\tau \right)^{-1} \Big]. \end{aligned} \tag{2.3.19}$$

So

$$(f,g) \in \mathcal{H}_1.$$

Therefore $\lambda \in \rho(\mathcal{Q})$, which implies that

$$\sigma(\mathcal{Q}) = \mathbb{C} \backslash \rho(\mathcal{Q}) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}. \tag{2.3.20}$$

Moreover, when $\lambda = 0$, since for each $(f,g) \in \mathcal{H}_1$,

$$\begin{aligned} &\left[(\lambda I - \mathcal{Q}) \begin{pmatrix} f \\ g \end{pmatrix} \right](x) \\ &= \begin{pmatrix} \dfrac{\lambda a(x)+b(x)}{a(x)} f(x) - \dfrac{1}{a_1(1)} \dfrac{1}{a(x)} \displaystyle\int_0^1 \frac{b(\tau)}{a(\tau)} f(\tau)d\tau \\ \lambda g(x) \end{pmatrix}, \end{aligned} \tag{2.3.21}$$

it has $\{0\} \times L^2_\rho \subset \mathcal{N}(\mathcal{Q})$. Hence $\dim \mathcal{N}(\mathcal{Q}) = \infty$ and by Definition 1.2.1,

$$0 \in \sigma_{ess}(\mathcal{Q}). \tag{2.3.22}$$

If $\lambda \neq 0$ and $\lambda a(\xi) + b(\xi) = 0$ for some $\xi \in [0,1]$, we claim that

$$\mathcal{R}(\lambda I - \mathcal{Q}) \neq \mathcal{H}_1.$$

In fact, define

$$E_\lambda = \{x \in [0,1] | \lambda a(x) + b(x) = 0\}.$$

If the measure of E_λ is nonzero and (2.3.17) admits a solution, it must have

$$u(x) = C/a(x) \text{ in } E_\lambda \text{ for some constant } C.$$

Obviously, such a function cannot represent all functions of $\mathcal{R}(\mathcal{D})$ on E_λ, that is

$$\mathcal{R}(\lambda I - \mathcal{Q}) \neq \mathcal{H}_1.$$

Now suppose that the measure of E_λ is zero and (2.3.17) has solution $f \in \mathcal{R}(\mathcal{D})$ for any $u \in \mathcal{R}(\mathcal{D})$. Then f must be of the form (2.3.18). Take special $u \in \mathcal{R}(\mathcal{D})$ in (2.3.18) as following

$$u(x) = \begin{cases} \dfrac{1}{\sqrt[3]{x-\xi}}, & x \in E_1, \\ \dfrac{-1}{1-\mathrm{mes}(E_1)} \displaystyle\int_{E_1} \dfrac{1}{\sqrt[3]{x-\xi}} dx, & x \in [0,1] \backslash E_1, \end{cases}$$

where $E_1 \subset [0,1]$ is a given small closed interval containing ξ, and $\mathrm{mes}(E_1)$ is the measure of E_1, $0 < \mathrm{mes}(E_1) < 1$. Obviously, for this special u, there exists a closed interval $E_2 \subset E_1$, $\xi \in E_2$, such that the corresponding solution f satisfies

$$\left| a(x)u(x) + \frac{1}{a_1(1)} \int_0^1 \frac{b(\tau)}{a(\tau)} f(\tau) d\tau \right| > 1, \ x \in E_2,$$

and hence by (2.3.18),

$$\left\| \frac{1}{\lambda a + b} \right\|_{L^2(E_2)} \leq \|f\|_{L^2(E_2)} < \infty,$$

which means that

$$\frac{1}{\lambda a + b} \in L^2(E_2).$$

This fact together with (2.3.18) shows that

$$\frac{a\widetilde{u}}{\lambda a + b} \in L^2(E_2), \ \forall\, \widetilde{u} \in L^2(E_2).$$

Define the multiplication operator $\mathcal{F} : L^2(E_2) \to L^2(E_2)$ by

$$(\mathcal{F}\widetilde{u})(x) = \frac{a(x)}{\lambda a(x) + b(x)} \widetilde{u}(x), \ \forall\, \widetilde{u} \in L^2(E_2). \tag{2.3.23}$$

Then $\mathcal{F}$ is a closed operator on $L^2(E_2)$. In fact, for any sequence $\{\widetilde{u}_n\} \subset L^2(E_2)$, if

$$\|\widetilde{u}_n - \widetilde{u}\|_{L^2(E_2)} \to 0, \ \|\mathcal{F}\widetilde{u}_n - \widehat{u}\|_{L^2(E_2)} \to 0,$$

for some $\widetilde{u}, \widehat{u} \in L^2(E_2)$, then there exist subsequences $\{\widetilde{u}_{n_k}\}$ and $\{\mathcal{F}\widetilde{u}_{n_k}\}$ converge to $\widetilde{u}$ and $\widehat{u}$ almost everywhere for $x \in E_2$, respectively. Therefore, by definition (2.3.23), we have

$$(\mathcal{F}\widetilde{u})(x) = \frac{a(x)}{\lambda a(x) + b(x)} \widetilde{u}(x) = \widehat{u}(x), \ x \in E_2 \text{ a.e..}$$

Hence $\mathcal{F}$ is closed on $L^2(E_2)$. By the Closed Graph Theorem, $\mathcal{F}$ is bounded on $L^2(E_2)$, which implies that

$$\frac{a}{\lambda a + b} \in L^\infty(E_2).$$

This contradicts to $\lambda a(\xi) + b(\xi) = 0$ and continuity of a, b. Hence

$$\mathcal{R}(\lambda I - \mathcal{Q}) \neq \mathcal{H}_1.$$

Therefore

$$\{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\} \subseteq \sigma(\mathcal{Q}). \tag{2.3.24}$$

Combining (2.3.20), (2.3.22) and (2.3.24) gives (2.3.16).

Finally, we show (2.3.15). Since $\sigma_{ess}(\mathcal{Q}) \subseteq \sigma(\mathcal{Q})$, we only need to show that

$$\sigma(\mathcal{Q}) \subseteq \sigma_{ess}(\mathcal{Q}).$$

Let

$$\begin{cases} m &= \min\limits_{0\le x\le 1} \{\lambda | \ \lambda a(x) + b(x) = 0\}, \\ M &= \max\limits_{0\le x\le 1} \{\lambda | \ \lambda a(x) + b(x) = 0\}. \end{cases}$$

By (2.3.16) and (2.3.22), it suffices to show that

$$[m, M] \subset \sigma_{ess}(\mathcal{Q}).$$

There are two cases:

Case I: $m = M$**.** In this case, $b(x)/a(x) = -m$ is a constant. It follows from (2.3.21) that

$$(mI - \mathcal{Q})(f, g) = (0, mg).$$

Hence,

$$\mathcal{R}(\mathcal{D}) \times \{0\} \subset \mathcal{N}(mI - \mathcal{Q}),$$

which means by Definition 1.2.1 that

$$\lambda = m \in \sigma_{ess}(\mathcal{Q}).$$

Case II: $m < M$**.** In this case, λ can be taken as any point of interval $[m, M]$ by the continuity of $b(x)/a(x)$. So by (2.3.16),

$$[m, M] \subseteq \sigma(\mathcal{Q}).$$

Since $\mathcal{Q}$ is self-adjoint, $[m, M] \subseteq \sigma_{ess}(\mathcal{Q})$ follows from Theorem 5 of [23, p.1395] which says that for a self-adjoint operator, all non-isolated spectrum must be essential spectrum (note that in [23], the essential spectrum of a closed operator is defined as only those that (i) of our Definition 1.2.1 is satisfied). ■

With these preparations, we could summarize the properties of $\sigma_{ess}(\mathcal{A})$ as Theorem 2.3.2 following.

Theorem 2.3.2. *Let $\mathcal{A}$ be defined by (2.2.4). Then the following assertions hold.*

(i) The essential spectrum of operator $\mathcal{A}$ is given by

$$\sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid a(\xi) + \lambda b(\xi) = 0 \textit{ for some } \xi \in [0,1]\}. \tag{2.3.25}$$

(ii) $\sigma(\mathcal{A})\backslash\sigma_{ess}(\mathcal{A})$ consists of at most countable isolated eigenvalues of finite algebraic multiplicity.

Proof. Suppose (i) is valid. Then $\sigma(\mathcal{A})\backslash\sigma_{ess}(\mathcal{A})$ is an open connected subset of $\mathbb{C}\backslash\sigma_{ess}(\mathcal{A})$, (ii) is then a direct consequence of Theorem 2.1 of [31, p.373]. So only proof of (i) is needed. Since $\widetilde{\mathcal{A}}^{-1} = \mathcal{P} + \mathcal{Q}$ and $\mathcal{P}$ is compact, it follows that

$$\sigma_{ess}(\widetilde{\mathcal{A}}^{-1}) = \sigma_{ess}(\mathcal{Q}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}.$$

Since $\lambda \in \sigma_{ess}(\widetilde{\mathcal{A}}^{-1})$ if and only if $\lambda^{-1} \in \sigma_{ess}(\widetilde{\mathcal{A}})$, we have

$$\sigma_{ess}(\widetilde{\mathcal{A}}) = \{\lambda \in \mathbb{C} \mid a(\xi) + \lambda b(\xi) = 0 \text{ for some } \xi \in [0,1]\}.$$

The desired result then follows directly through the relation (2.3.10). ■

Next, we consider the continuous spectrum for the system (2.2.1).

Lemma 2.3.3. *Let $\mathcal{A}$ be defined by (2.2.4) and the following conditions are satisfied:*

$$\begin{cases} a(x),\ b(x) \textit{ and } \rho(x) \textit{ are analytic in } [0,1]; \\ \forall\, \lambda \in \mathbb{R}, \dfrac{(x-\xi)^2}{a(x) + \lambda b(x)} \textit{ is analytic in a neighboorhood of any } \xi \in [0,1]. \end{cases} \tag{2.3.26}$$

Then the set of the continuous spectrum of $\mathcal{A}$ satisfies

$$\sigma_c(\mathcal{A}) = \sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C} | \ a(\xi) + \lambda b(\xi) = 0 \textit{ for some } \xi \in [0,1]\}.$$

Proof. Suppose that $a(\xi) + \lambda b(\xi) = 0$ for some $\xi \in [0,1], \lambda \in \mathbb{R}$. If $\lambda \in \sigma_p(\mathcal{A})$, then there exists a nonzero $f \in H_0^1(0,1)$ satisfying the characteristic equation (2.3.1). The proof will be accomplished if we can show that $f \equiv 0$ because $\sigma_r(\mathcal{A}) = \emptyset$ claimed by Theorem 2.3.1. This will be divided into three steps:

Step 1: We claim that in a neighborhood of ξ,

$$f(x) = C_\xi \left(1 + \sum_{n=1}^{\infty} a_n (x-\xi)^n\right) \tag{2.3.27}$$

or there exists a $r_1 > 0$ such that

$$f(x) = D_\xi (x-\xi)^{r_1} \left[1 + \sum_{n=1}^{\infty} b_n (x-\xi)^n\right], \tag{2.3.28}$$

where C_ξ and D_ξ are constants and each series in equation (2.3.27) and (2.3.28) converges uniformly in a neighborhood of ξ and defines a function that is analytic at $x = \xi$.

It follows from (2.3.26) that ξ is the regular singular point of the first equation in (2.3.1). Using Theorem 4.4 of [12, p.192], (2.3.1) must admit a Frobenius series solution in a neighborhood of ξ. The procedure is as follows. By (2.3.26), assume that

$$a(x) + \lambda b(x) = (x - \xi)^k \varphi(x),$$

where $k = 1$ or 2 and φ is analytic in $[0, 1]$, $\varphi(\xi) \neq 0$. Thus,

$$\frac{(a(x) + \lambda b(x))'}{a(x) + \lambda b(x)} = \frac{k}{x - \xi} + \frac{\varphi'(x)}{\varphi(x)}.$$

Let

$$\begin{cases} p_0 &= \lim\limits_{x \to \xi} (x - \xi) \dfrac{(a(x) + \lambda b(x))'}{a(x) + \lambda b(x)}, \\ q_0 &= \lim\limits_{x \to \xi} (x - \xi)^2 \dfrac{-\lambda^2 \rho(x)}{a(x) + \lambda b(x)}. \end{cases}$$

The indicial equation of (2.3.1) is (see e.g., Theorem 4.4 of [12, p.192])

$$F(r) = r(r - 1) + p_0 r + q_0 = 0.$$

A simple calculation shows that $p_0 = 1, q_0 = 0$ when ξ is the first order zero point of $a + \lambda b$, and $p_0 = 2, q_0 \neq 0$ while ξ is the second order zero point of $a + \lambda b$.

Since f is required to be continuous, when ξ is the first order zero point of $a + \lambda b$, $F(r) = 0$ has only zero solution and hence f is of the form (2.3.27). While ξ is the second order zero point of $a + \lambda b$, let r_1, r_2 be the roots of $F(r) = 0$:

$$r_{1,2} = \frac{-1 \pm \sqrt{1 - 4q_0}}{2}.$$

If r_1 is a nonreal number, then $\mathrm{Re}(r_1) = -\frac{1}{2}$. In this case, f must be identical to zero in a neighborhood of ξ and $D_\xi = 0$ in (2.3.28). Otherwise, we may suppose $r_1 > 0 > r_2$. Since f is continuous in $[0, 1]$, it must be of the form (2.3.28).

Step 2: We claim that there is a sequence $\{\gamma_n\}_{n=1}^{\infty} \subset [0, 1]$, $\gamma_i \neq \gamma_j$ for any $i \neq j$, $i, j = 1, 2, \cdots$, such that

$$f(\gamma_n) = 0, \quad n = 1, 2, \cdots$$

and

$$\lim_{n \to \infty} \gamma_n = \xi. \tag{2.3.29}$$

To do this, it suffices to show that for any $[x_1, x_2] \subset [0, 1]$, if $\xi \in [x_1, x_2]$ and $f(x_1) = f(x_2) = 0$, then there exists a $\gamma \in (x_1, x_2)$ such that $f(\gamma) = 0$. In fact, if there exists a second order zero point γ of $a + \lambda b$ in (x_1, x_2), it follows from **Step 1** that

$$f(\gamma) = 0.$$

If there exists no second order zero point of $a + \lambda b$ in (x_1, x_2), by **Step 1**, f must be analytic in (x_1, x_2). By Rolle's Theorem, it follows that there exists an $\eta \in (x_1, x_2)$ such that

$$f'(\eta) = 0.$$

If $\xi = \eta$, then by (2.3.1) and $(a + \lambda b)(\xi) = 0$, it has

$$f(\eta) = 0.$$

In this case, we take $\gamma = \eta$. If $\xi \neq \eta$, then we have

$$[(a + \lambda b)f'](\xi) = [(a + \lambda b)f'](\eta) = 0.$$

By using Rolle's Theorem again, there exists a γ between ξ and η such that

$$[(a + \lambda b)f']'(\gamma) = 0,$$

which yields $f(\gamma) = 0$ from (2.3.1).

Step 3: It follows from **Step 1** and **Step 2** that there is a neighborhood $\mathcal{O}_\xi$ of ξ such that

$$f \equiv 0 \text{ in } \mathcal{O}_\xi.$$

Since f is identical to zero in a neighborhood of any regular singular point ξ, f must be identical to zero everywhere by the uniqueness theorem of the regular ordinary differential equations. The proof is complete. ■

Theorem 2.3.3. *Let $\mathcal{A}$ be defined by (2.2.4) and $a(x)$, $b(x)$, $\rho(x)$ are analytic in $[0, 1]$. Then*

$$\sigma_c(\mathcal{A}) = \sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C} |\ a(\xi) + \lambda b(\xi) = 0 \textit{ for some } \xi \in [0, 1]\}.$$

Proof. Suppose $a(\xi) + \lambda b(\xi) = 0$ for some $\xi \in [0, 1]$ and $\lambda \in \mathbb{C}$. If $a \equiv -\lambda b$, it is trivially that the solution f of equation (2.3.1) must be identical to zero. By Lemma 2.3.3, we may assume that

$$a(x) + \lambda b(x) = (x - \xi)^m \varphi(x),$$

where $m > 2$ is a positive integer and φ is analytic in $[0,1]$, $\varphi(\xi) \neq 0$. We show that

$$f \equiv 0.$$

This case corresponds the irregular singular point for equation (2.3.1). The proof will be divided into three steps:

Step 1: We claim that

$$f(\xi) = 0.$$

In fact, we can rewrite (2.3.1) as

$$f''(x) + \frac{1}{x-\xi}\left[m + \frac{\varphi'(x)}{\varphi(x)}(x-\xi)\right] f'(x) + \frac{1}{(x-\xi)^m}\frac{-\lambda^2\rho(x)}{\varphi(x)} f(x) = 0 \tag{2.3.30}$$

with the boundary conditions:

$$f(0) = f(1) = 0.$$

By the analyticity of φ and ρ, we may assume that

$$\begin{cases} c(x) = \dfrac{1}{x-\xi}\left[m + \dfrac{\varphi'(x)}{\varphi(x)}(x-\xi)\right] = \dfrac{1}{x-\xi}\left[m + \displaystyle\sum_{i=1}^{\infty} h_i (x-\xi)^i\right], \\ d(x) = \dfrac{1}{(x-\xi)^m}\dfrac{-\lambda^2\rho(x)}{\varphi(x)} = \dfrac{1}{(x-\xi)^m}\left[l_0 + \displaystyle\sum_{i=1}^{\infty} l_i (x-\xi)^i\right], \end{cases}$$

where the two series on the right side above are the Taylor series and by assumption $l_0 \neq 0$. We only need to discuss the case of $x \geq \xi$ since the case of $x \leq \xi$ can be treated similarly. Let $x - \xi = t^2$. Then (2.3.30) is equivalent to

$$y''(t) + C(t)y'(t) + D(t)y(t) = 0, \tag{2.3.31}$$

where

$$C(t) = 2tc(\xi + t^2) - \frac{1}{t} = \frac{2m-1}{t} + 2[h_1 t + h_2 t^3 + \cdots + h_n t^{2n+1} + \cdots]$$

and

$$D(t) = 4t^2 d(\xi + t^2) = \frac{4}{t^{2m-2}}[l_0 + l_1 t^2 + l_2 t^4 + \cdots + l_n t^{2n} + \cdots].$$

Then $f(\xi) = 0$ is equivalent to

$$y(0) = 0.$$

We choose $k = m - 2$ and let

$$c_0 = t^{k+1} C(t) \mid_{t=0} = 0$$

and

$$d_0 = t^{2k+2}D(t) \mid_{t=0} = 4l_0.$$

Then the solutions of (2.3.31) are of the form (see, e.g., [17, p.224]).

$$y(t) = e^{F(t)}Y(t), \tag{2.3.32}$$

where

$$F(t) = \frac{A_k}{t^k} + \frac{A_{k-1}}{t^{k-1}} + \cdots + \frac{A_1}{t} \tag{2.3.33}$$

and

$$Y(t) = \sum_{n=0}^{\infty} a_n t^{n+s}, a_0 \neq 0 \tag{2.3.34}$$

is a Frobenius series. Substitute (2.3.32) into (2.3.31) to obtain the differential equation satisfied by Y:

$$Y''(t) + [C(t) + 2F'(t)]Y'(t) + \left[D(t) + C(t)F'(t) + [F'(t)]^2 + F''(t)\right] Y(t) = 0. \tag{2.3.35}$$

Choose the constants A_n, $n = 1, 2, \cdots, k$ to eliminate the most singular terms in the coefficient of Y in (2.3.35) to get, after a calculation, that

$$\begin{cases} A_k &= \dfrac{c_0 \pm \sqrt{c_0^2 - 4d_0}}{2k} = \pm\dfrac{2}{k}\sqrt{-l_0} \neq 0, \\ A_{k-1} &= A_{k-3} = \cdots = 0. \end{cases}$$

There are two cases:

Case I: $l_0 > 0$. In this case,

$$\mathrm{Re}A_k = 0.$$

By a simple calculation, we find that

$$\mathrm{Re}A_n = 0, \quad n = 1, 2, \cdots, k.$$

By equation (6.53) of [17, p.226],

$$\mathrm{Re}s = -\frac{-(2m-1)kA_k + k(k+1)A_k}{-2kA_k} = -\frac{m}{2} < 0.$$

Let

$$\begin{cases} F(t) &= i\tau(t), \\ Y(t) &= t^s[u(t) + iv(t)], \\ s &= \alpha + i\beta, \end{cases}$$

where $\alpha = -\dfrac{m}{2}, \beta, \tau(t), u(t), v(t) \in \mathbb{R}$ and u, v are analytic at $t = 0$, $u^2(0) + v^2(0) \neq 0$. Then

$$\begin{aligned} y(t) &= [\cos\tau(t) + i\sin\tau(t)]t^{\alpha}[\cos(\beta\ln t) + i\sin(\beta\ln t)][u(t) + iv(t)] \\ &= t^{\alpha}[\cos(\tau(t) + \beta\ln t) + i\sin(\tau(t) + \beta\ln t)][u(t) + iv(t)] \\ &= t^{\alpha}[u(t)\cos(\tau(t) + \beta\ln t) - v(t)\sin(\tau(t) + \beta\ln t)] \\ &\quad +it^{\alpha}[v(t)\cos(\tau(t) + \beta\ln t) + u(t)\sin(\tau(t) + \beta\ln t)]. \end{aligned}$$

Let $z(t) = \tau(t) + \beta\ln t$. Since $A_k \neq 0$, $\lim_{t\to 0^+} z(t) = \infty$, this together with the continuity of $z(t)$ enables us to easily show that $[u(t)\cos z(t) - v(t)\sin z(t)]$ and $[v(t)\cos z(t) + u(t)\sin z(t)]$ are linearly independent. Therefore the general solution of (2.3.31) is of the form

$$y(t) = t^{\alpha}\left\{b_1[u(t)\cos z(t) - v(t)\sin z(t)] + b_2[v(t)\cos z(t) + u(t)\sin z(t)]\right\}, \tag{2.3.36}$$

where b_1 and b_2 are real constants. Since $f(x)$ is continuous at $x = \xi$, so is for $y(t)$ at $t = 0$. Since $\alpha < 0$, it must have

$$\lim_{t\to 0^+}\{b_1[u(t)\cos z(t) - v(t)\sin z(t)] + b_2[v(t)\cos z(t) + u(t)\sin z(t)]\} = 0. \tag{2.3.37}$$

Since $\lim_{t\to 0^+} z(t) = \infty$, one can find two positive sequences $\{t_{n_1}\}$ and $\{t_{n_2}\}$ such that

$$\begin{aligned} &\lim_{n_1\to\infty} t_{n_1} = 0, \quad \lim_{n_2\to\infty} t_{n_2} = 0, \\ &\cos z(t_{n_1}) = 1, \ \sin z(t_{n_1}) = 0, \\ &\cos z(t_{n_2}) = 0, \ \sin z(t_{n_2}) = 1. \end{aligned}$$

This together with (2.3.37) gives

$$\begin{cases} b_1u(0) + b_2v(0) = \lim\limits_{n_1\to\infty}[b_1u(t_{n_1}) + b_2v(t_{n_1})] = 0, \\ -b_1v(0) + b_2u(0) = \lim\limits_{n_2\to\infty}[-b_1v(t_{n_2}) + b_2u(t_{n_2})] = 0. \end{cases}$$

Since $u^2(0) + v^2(0) \neq 0$, it has

$$b_1 = b_2 = 0.$$

By (2.3.36),

$$y(t) \equiv 0. \tag{2.3.38}$$

Case II: $l_0 < 0$. In this case,

$$A_k \in \mathbb{R}.$$

By the similar calculation as **Case I**, we have

$$A_n \in \mathbb{R},\ n = 1, 2, \cdots, k$$

and the general solution of (2.3.31) is of the form

$$y(t) = c_1 e^{F(t)} Y_1(t) + c_2 e^{-F(t)} Y_2(t),$$

where c_1 and c_2 are constants, $F(t)$ is of the form (2.3.33), and $Y_i(t)$, $i = 1, 2$ is of the form (2.3.34). Now, we may assume without loss of generality that

$$A_k = -\frac{2}{k}\sqrt{-l_0} < 0.$$

Then by the continuity of y and $\lim_{t \to 0^+} e^{-F(t)} Y_2(t) = \infty$, it must have

$$y(t) = c_1 e^{F(t)} Y_1(t) \tag{2.3.39}$$

and hence

$$y(0) = 0.$$

Step 2: We claim that there is a sequence $\{\gamma_n\}_{n=1}^{\infty} \subset [0, 1]$, $\gamma_i \neq \gamma_j$ for any $i \neq j, i, j = 1, 2, \cdots$, such that $f(\gamma_n) = 0$ for $n = 1, 2, \cdots$, and

$$\lim_{n \to \infty} \gamma_n = \xi. \tag{2.3.40}$$

To do this, it suffices to show that for any $[x_1, x_2] \subset [0, 1]$, if $\xi \in [x_1, x_2]$ and $f(x_1) = f(x_2) = 0$, then there exists a $\gamma \in (x_1, x_2)$ such that $f(\gamma) = 0$. In fact, if there exists a zero point γ of $a + \lambda b$ in (x_1, x_2) whose order is greater than one, it follows from **Step 1** and the proof of Lemma 2.3.3 that

$$f(\gamma) = 0.$$

Otherwise, by the proof of Lemma 2.3.3, f is analytic at any first order zero of $a + \lambda b$ in (x_1, x_2), and $\xi = x_1$ or $\xi = x_2$. But since the solution y of (2.3.31) is of (2.3.38) or (2.3.39), it is differentiable in a neighborhood of $t = 0$ except $t = 0$. So the solution f of (2.3.1) is differentiable in a neighborhood of $x = \xi$ except $x = \xi$. In any case, f is differentiable in (x_1, x_2). By Rolle's Theorem, there exists an $\eta \in (x_1, x_2)$ such that

$$f'(\eta) = 0.$$

Clearly $\xi \neq \eta$. Since

$$[(a + \lambda b) f'](\xi) = [(a + \lambda b) f'](\eta) = 0,$$

using Rolle's Theorem again, there exists a γ between ξ and η such that

$$[(a+\lambda b)f']'(\gamma) = 0,$$

which yields $f(\gamma) = 0$ from (2.3.1).

Step 3: By **Step 2**, the solution y of (2.3.31) has infinitely many zero points approaching zero. Since y is of (2.3.38) or (2.3.39), $y \equiv 0$ in a neighborhood of $t = 0$. Equivalently, there is a neighborhood $\mathcal{O}_\xi$ of ξ such that

$$f \equiv 0 \text{ in } \mathcal{O}_\xi.$$

Since there are at most finite number of singular points ξ, f must be identical to zero everywhere by the uniqueness of the solution for regular linear ordinary differential equations. The proof is complete. ■

If there is no analyticity, Theorem 2.3.3 is not true anymore. This is suggested by many studies on Sturm-Liouville problem. The following is a counter-example.

Example 2.3.1. Let

$$-p(x) = a(x) + \lambda b(x) = -x^{1/3}.$$

Then

$$1/p \in L^1(0,1).$$

According to Theorem 0 of [27], there are countable number of positive μ such that the Sturm-Liouville problem:

$$\begin{cases} -(p(x)f'(x))' = \mu f(x), \; x \in (0,1), \\ f(0) = f(1) = 0 \end{cases} \tag{2.3.41}$$

admits nonzero absolutely continuous solutions f. Take specially $\mu > 0$ for such a μ. Then we may choose

$$\begin{cases} a(x) = \sqrt{\mu} + 2x^{1/3}, \\ b(x) = 1 + \dfrac{3}{\sqrt{\mu}}x^{1/3}, \\ \lambda = -\sqrt{\mu}, \\ \rho(x) = 1. \end{cases}$$

equation (2.3.1) is now having a nonzero absolutely continuous solution f. Suppose $(pf')(0) = c$. We show that $f \in H_0^1(0,1)$ or equivalently $f' \in L^2(0,1)$, which hence serves as a counter-example. Indeed, set

$$\begin{cases} \alpha = \dfrac{1}{3}, \\ A(x) = \begin{pmatrix} 0 & x^{-\alpha} \\ -\mu & 0 \end{pmatrix}, \\ y(x) = \begin{pmatrix} f(x) \\ p(x)f'(x) \end{pmatrix}, \\ y_0 = y(0) = \begin{pmatrix} 0 \\ c \end{pmatrix}. \end{cases}$$

Then (2.3.41) is rewritten as

$$\frac{dy}{dx} = A(x)y(x).$$

According to Theorem 1 of [[78], Section 16], the above equation is equivalent to the following integral equation

$$y(x) = y_0 + \int_0^x A(\xi)y(\xi)d\xi$$

for which the solution can be represented uniformly in $[0,1]$ as

$$y(x) = \lim_{n\to\infty} y_n(x),$$

where

$$\begin{aligned} y_{n+1}(x) &= \begin{pmatrix} f_{n+1}(x) \\ h_{n+1}(x) \end{pmatrix} \\ &= y_0 + \int_0^x A(\xi)y_n(\xi)d\xi, \quad n = 0, 1, 2, \cdots. \end{aligned}$$

A direct computation shows that

$$\begin{cases} f_{2n+1}(x) = f_{2n+2}(x) = \dfrac{c}{1-\alpha}x^{1-\alpha} + \displaystyle\sum_{k=1}^{n} a_k x^{k+(k+1)(1-\alpha)}, \\ h_{2n}(x) = h_{2n+1}(x) = x^{\alpha} f'_{2n+1}(x), \end{cases} \qquad n = 1, 2, \cdots,$$

where

$$a_k = \frac{(-1)^k c\mu^k}{(1-\alpha)(2-\alpha)(3-2\alpha)(4-2\alpha)(5-3\alpha)\cdots(2k-k\alpha)(2k+1-(k+1)\alpha)}.$$

Therefore,

$$f(x) = \frac{c}{1-\alpha}x^{1-\alpha} + \sum_{k=1}^{\infty} a_k x^{k+(k+1)(1-\alpha)}.$$

The above series is absolutely uniformly convergent since

$$|f(x)| \leq |c|/(1-\alpha) + \sum_{k=1}^{\infty} |a_k|$$

and

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k-1}|} = \lim_{k\to\infty} \frac{\mu}{(2k-k\alpha)(2k+1-(k+1)\alpha)} = 0.$$

Now,

$$f'(x) = cx^{-\alpha} + \sum_{k=1}^{\infty} a_k[2k+1-(k+1)\alpha]x^{k-1+(k+1)(1-\alpha)}.$$

The above series is also absolutely uniformly convergent since

$$\left|\sum_{k=1}^{\infty} a_k[2k+1-(k+1)\alpha]x^{k-1+(k+1)(1-\alpha)}\right| \leq \sum_{k=1}^{\infty} |a_k[2k+1-(k+1)\alpha]|$$

and

$$\lim_{k\to\infty} \frac{|a_k[2k+1-(k+1)\alpha]|}{|a_{k-1}[2k-1-k\alpha]|} = \lim_{k\to\infty} \frac{\mu}{(2k-k\alpha)(2k-1-k\alpha)} = 0.$$

This shows that

$$f'(x) = cx^{-\alpha} + g(x),$$

where g is a continuous function. Hence

$$f' \in L^2(0,1).$$

2.4 Asymptotic behavior of eigenvalues

In this section, we consider the asymptotic behavior of eigenvalues for the system (2.2.1). To do this, we assume further that

$$a(x), b(x) \in C^1[0,1] \text{ and } a(x), b(x) > 0 \text{ for all } x \in [0,1]. \tag{2.4.1}$$

Suppose that λ is an eigenvalue with large modulus. Then

$$a(x) + \lambda b(x) \neq 0 \text{ for any } x \in [0,1] \tag{2.4.2}$$

and we rewrite the characteristic equation (2.3.1) as

$$\begin{cases} [a(x)+\lambda b(x)]f''(x) + [a'(x)+\lambda b'(x)]f'(x) = \lambda^2 \rho(x) f(x), \\ f(0) = f(1) = 0. \end{cases} \tag{2.4.3}$$

By (2.4.3), it is apparently seen that λ must be geometrically simple.

The following Lemma 2.4.1 is immediate.

Lemma 2.4.1. *Let* $\lambda \in \mathbb{C}$*. Then as* $|\lambda| \to \infty$*, it has*

$$\begin{aligned} \frac{1}{a(x)+\lambda b(x)} &= \frac{1}{\lambda b(x)} \cdot \frac{1}{1+\frac{a(x)}{\lambda b(x)}} \\ &= \frac{1}{\lambda b(x)} - \frac{a(x)}{\lambda^2 b^2(x)} + \frac{a^2(x)}{\lambda^3 b^3(x)} + \mathcal{O}(|\lambda|^{-4}) \end{aligned} \tag{2.4.4}$$

and

$$[a(x)+\lambda b(x)]f''(x) + [a'(x)+\lambda b'(x)]f'(x) = \lambda^2 \rho(x) f(x) \tag{2.4.5}$$

has the following asymptotic expression:

$$\begin{aligned} & f''(x) \\ & + \left[\frac{1}{\lambda b(x)} - \frac{a(x)}{\lambda^2 b^2(x)} + \frac{a^2(x)}{\lambda^3 b^3(x)} + \mathcal{O}(|\lambda|^{-4})\right] [a'(x)+\lambda b'(x)] f'(x) \\ & -\lambda \left[1 - \frac{a(x)}{\lambda b(x)} + \frac{a^2(x)}{\lambda^2 b^2(x)} + \mathcal{O}(|\lambda|^{-3})\right] \rho_0^2(x) f(x) = 0. \end{aligned} \tag{2.4.6}$$

where

$$\rho_0(x) = \sqrt{\frac{\rho(x)}{b(x)}}. \tag{2.4.7}$$

In order to find the asymptotic fundamental solutions of (2.4.6), we introduce the following space-scaling transformation:

$$\begin{cases} \phi(z) = f(x), \\ z = \dfrac{1}{h} \displaystyle\int_0^x \rho_0(\tau) d\tau, \\ h = \displaystyle\int_0^1 \rho_0(\tau) d\tau. \end{cases} \tag{2.4.8}$$

Under this transformation, (2.4.6) becomes

$$\begin{aligned} & \phi''(z) + h \left\{ \frac{1}{\rho_0(x)} \left[\frac{1}{\lambda b(x)} - \frac{a(x)}{\lambda^2 b^2(x)} + \frac{a^2(x)}{\lambda^3 b^3(x)} + \mathcal{O}(|\lambda|^{-4})\right] [a'(x)+\lambda b'(x)] \right. \\ & \left. + \frac{\rho_0'(x)}{\rho_0^2(x)} \right\} \phi'(z) - \lambda h^2 \left[1 - \frac{a(x)}{\lambda b(x)} + \frac{a^2(x)}{\lambda^2 b^2(x)} + \mathcal{O}(|\lambda|^{-3})\right] \phi(z) = 0 \end{aligned} \tag{2.4.9}$$

with the boundary conditions:

$$\phi(0) = \phi(1) = 0. \tag{2.4.10}$$

Proposition 2.4.1. *The equation (2.4.6) with boundary condition (2.4.3) is equivalent to (2.4.9) and (2.4.10). That is,* (λ, f)*,* $f \neq 0$*, satisfies (2.4.6) and boundary conditions (2.4.3) if and only if* (λ, ϕ)*,* $\phi \neq 0$*, satisfies (2.4.9) and (2.4.10).*

Now we consider (2.4.9) and (2.4.10). Since the eigenvalues are symmetric about the real axis and $\mathrm{Re}\lambda \leq 0$ for any $\lambda \in \sigma(\mathcal{A})$, we only consider those eigenvalues λ with

$$\frac{\pi}{2} \leq \arg\lambda \leq \pi.$$

Let $\lambda = \mu^2$. Then as $\frac{\pi}{2} \leq \arg\lambda \leq \pi$, we consider μ locating on the following sector:

$$\mathcal{S} = \left\{ \mu \in \mathbb{C} \mid \frac{\pi}{4} \leq \arg\mu \leq \frac{\pi}{2} \right\}. \tag{2.4.11}$$

Lemma 2.4.2. *Suppose $\lambda = \mu^2 \neq 0$. Then for $z \in [0,1]$ and $\mu \in \mathcal{S}$,*

$$e^{\mu h z},\ e^{-\mu h z} \tag{2.4.12}$$

are linearly independent fundamental solutions of

$$\phi''(z) - \mu^2 h^2 \phi(z) = 0,$$

and for $|\mu|$ large enough, (2.4.9) has the following asymptotic fundamental solutions:

$$\begin{cases} \phi_1(z) = e^{\mu h z}\left[\phi_{10}(z) + \phi_{11}(z)\mu^{-1} + \mathcal{O}(\mu^{-2})\right], \\ \phi_2(z) = e^{-\mu h z}\left[\phi_{20}(z) + \phi_{21}(z)\mu^{-1} + \mathcal{O}(\mu^{-2})\right], \end{cases} \tag{2.4.13}$$

where

$$\begin{cases} \phi_{10}(z) = \phi_{20}(z) = exp\left\{ -\frac{1}{2}\int_0^z \rho_1(x)\rho_0(x)dx \right\}, \\ \phi_{11}(z) = -\frac{1}{2}\int_0^z exp\left\{ -\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx \right\}\rho_2(\zeta)d\zeta, \\ \phi_{21}(z) = -\phi_{11}(z) = \frac{1}{2}\int_0^z exp\left\{ -\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx \right\}\rho_2(\zeta)d\zeta \end{cases} \tag{2.4.14}$$

and

$$\begin{cases} \rho_1(x) = \frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}, \\ \rho_2(z) = \frac{1}{h}\phi_{10}''(z) + \left(\frac{\rho_0'(x(z))}{\rho_0^2(x(z))} + \frac{1}{\rho_0(x(z))}\frac{b'(x(z))}{b(x(z))} \right)\phi_{10}'(z) \\ \qquad\qquad + h\frac{a(x(z))}{b(x(z))}\phi_{10}(z). \end{cases} \tag{2.4.15}$$

Proof. The first claim is trivial. We only need to show that (2.4.13) are the asymptotic fundamental solutions of (2.4.9). This can be done along the same way of [11] and [[78], Section 4]. Here we present briefly a simple calculation to (2.4.13)-(2.4.15).

Let

$$\begin{cases} \widetilde{\phi}_1(z,\mu) = e^{\mu h z}\left[\phi_{10}(z) + \phi_{11}(z)\mu^{-1}\right], \\ \widetilde{\phi}_2(z,\mu) = e^{-\mu h z}\left[\phi_{20}(z) + \phi_{21}(z)\mu^{-1}\right], \end{cases} \tag{2.4.16}$$

where $\phi_{ki}(z), k = 1, 2,\ i = 0, 1$ are some functions to be determined, and

$$\begin{aligned}
&D(\phi) \\
= \quad & \phi''(z) + h\left\{\frac{1}{\rho_0(x)}\left[\frac{1}{\mu^2 b(x)} - \frac{a(x)}{\mu^4 b^2(x)} + \frac{a^2(x)}{\mu^6 b^3(x)} + \mathcal{O}(|\mu|^{-8})\right][a'(x) + \mu^2 b'(x)]\right. \\
& \left. + \frac{\rho_0'(x)}{\rho_0^2(x)}\right\}\phi'(z) - \mu^2 h^2\left[1 - \frac{a(x)}{\mu^2 b(x)} + \frac{a^2(x)}{\mu^4 b^2(x)} + \mathcal{O}(|\mu|^{-6})\right]\phi(z).
\end{aligned}$$

Substitute $\widetilde{\phi}_1(z, \mu)$ into $D(\phi)$ to yield

$$\begin{aligned}
& e^{-\mu h z} D(\widetilde{\phi}_1(z, \mu)) \\
= \quad & \mu^2 h^2\left[\phi_{10}(z) + \phi_{11}(z)\mu^{-1}\right] + 2\mu h\left[\phi_{10}'(z) + \phi_{11}'(z)\mu^{-1}\right] + \left[\phi_{10}''(z) + \phi_{11}''(z)\mu^{-1}\right] \\
& -\mu^2 h^2\left[1 - \frac{a(x)}{\mu^2 b(x)} + \frac{a^2(x)}{\mu^4 b^2(x)} + \mathcal{O}(|\mu|^{-6})\right]\left[\phi_{10}(z) + \phi_{11}(z)\mu^{-1}\right] \\
& +h\left\{\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\left[\frac{1}{\mu^2 b(x)} - \frac{a(x)}{\mu^4 b^2(x)} + \frac{a^2(x)}{\mu^6 b^3(x)} + \mathcal{O}(|\mu|^{-8})\right][a'(x) + \mu^2 b'(x)]\right\} \\
& \times\left\{\mu h\left[\phi_{10}(z) + \phi_{11}(z)\mu^{-1}\right] + \left[\phi_{10}'(z) + \phi_{11}'(z)\mu^{-1}\right]\right\} \\
= \quad & \mu\left[2h\phi_{10}'(z) + h^2\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{10}(z)\right] \\
& +\left[2h\phi_{11}'(z) + h^2\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{11}(z)\right. \\
& \quad \left. +\phi_{10}''(z) + h^2\frac{a(x)}{b(x)}\phi_{10}(z) + h\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{10}'(z)\right] + \mathcal{O}(\mu^{-1}).
\end{aligned}$$

Letting the coefficients of μ^1 and μ^0 be zero gives

$$2\phi_{10}'(z) + h\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{10}(z) = 0$$

and

$$\begin{aligned}
2h\phi_{11}'(z)\ & +h^2\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{11}(z) \\
& +\phi_{10}''(z) + h^2\frac{a(x)}{b(x)}\phi_{10}(z) + h\left(\frac{\rho_0'(x)}{\rho_0^2(x)} + \frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)}\right)\phi_{10}'(z) = 0.
\end{aligned}$$

Use the conditions $\phi_{10}(0) = 1$ and $\phi_{11}(0) = 0$ to obtain

$$\phi_{10}(z) = exp\left\{-\frac{1}{2}h\int_0^z \rho_1(x(\tau))d\tau\right\}$$

and

$$\phi_{11}(z) = -\frac{1}{2}\int_0^z exp\left\{-\frac{1}{2}h\int_0^{z-\zeta}\rho_1(x(\tau))d\tau\right\}\rho_2(\zeta)d\zeta,$$

where $\rho_1(x)$ and $\rho_2(z)$ are given by (2.4.15). From (2.4.8), we have

$$\frac{dz}{dx} = \frac{1}{h}\rho_0(x),$$

and so

$$\frac{dx}{dz} = \frac{h}{\rho_0(x)}.$$

Hence

$$\begin{aligned}\phi_{10}(z) &= exp\left\{-\frac{1}{2}\int_0^z \rho_1(x(\tau))\rho_0(x(\tau))\frac{dx}{d\tau}d\tau\right\}\\ &= exp\left\{-\frac{1}{2}\int_0^z \rho_1(x)\rho_0(x)dx\right\}\end{aligned}$$

and

$$\phi_{11}(z) = -\frac{1}{2}\int_0^z exp\left\{-\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx\right\}\rho_2(\zeta)d\zeta.$$

Same arguments also give $\phi_{20}(z)$ and $\phi_{21}(z)$ as in (2.4.14) and (2.4.15). The proof is complete. ■

Let $\lambda = \mu^2$ with large modulus and $\mu \in \mathcal{S}$ defined by (2.4.11). Let ϕ be a solution of (2.4.9) and (2.4.10). There are constants c_1 and c_2 such that

$$\phi(z) = c_1\phi_1(z) + c_2\phi_2(z), \tag{2.4.17}$$

where $\phi_1(z)$ and $\phi_2(z)$ are fundamental solutions given by (2.4.13)-(2.4.15). By using the boundary conditions of (2.4.10), we have

$$\Delta(\mu)[c_1, c_2]^\top = 0, \tag{2.4.18}$$

where

$$\Delta(\mu) = \begin{bmatrix} 1 & 1 \\ \phi_1(1) & \phi_2(1) \end{bmatrix}. \tag{2.4.19}$$

Hence, $\phi(z)$ has a nontrivial solution if and only if

$$\det(\Delta(\mu)) = 0.$$

That is, $\mu \in \mathcal{S}$ satisfies the characteristic equation:

$$\begin{aligned}\det(\Delta(\mu)) &= \phi_2(1) - \phi_1(1)\\ &= e^{-\mu h}\left[\phi_{20}(1) + \phi_{21}(1)\mu^{-1} + \mathcal{O}(\mu^{-2})\right]\\ &\qquad -e^{\mu h}\left[\phi_{10}(1) + \phi_{11}(1)\mu^{-1} + \mathcal{O}(\mu^{-2})\right] \qquad (2.4.20)\\ &= \phi_{10}(1)\left\{e^{-\mu h} - e^{\mu h} + k_0\mu^{-1}\left[e^{-\mu h} + e^{\mu h}\right]\right\} + \mathcal{O}(\mu^{-2})\\ &= 0,\end{aligned}$$

where k_0 is a constant satisfying

$$k_0 = -\frac{\phi_{11}(1)}{\phi_{10}(1)} = \frac{1}{2}\frac{\int_0^1 exp\left\{-\frac{1}{2}\int_0^{1-\zeta}\rho_1(x)\rho_0(x)dx\right\}\rho_2(\zeta)d\zeta}{exp\left\{-\frac{1}{2}\int_0^1\rho_1(x)\rho_0(x)dx\right\}}. \tag{2.4.21}$$

Lemma 2.4.3. *Let $\Delta(\mu)$ be given by (2.4.19). Then the characteristic determinant $\det(\Delta(\mu))$ has the following asymptotic expression:*

$$\frac{1}{\phi_{10}(1)}\det(\Delta(\mu)) = e^{-\mu h} - e^{\mu h} + k_0\mu^{-1}\left[e^{-\mu h} + e^{\mu h}\right] + \mathcal{O}(\mu^{-2}), \tag{2.4.22}$$

where k_0 is given by (2.4.21).

Theorem 2.4.1. *Let $\lambda = \mu^2$ with large modulus and $\mu \in \mathcal{S}$ defined by (2.4.11). Then λ, which must be geometrically simple as indicated in the beginning of the section, has the following asymptotic form:*

$$\lambda_n = -\frac{n^2\pi^2}{h^2} + 2\frac{k_0}{h} + \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots, \tag{2.4.23}$$

where k_0 is given by (2.4.21) and h is given by (2.4.8).

Proof. Since in sector $\mathcal{S}$, $\det(\Delta(\mu))$ has the asymptotic form given by (2.4.22), it follows from $\det(\Delta(\mu)) = 0$ that

$$e^{-\mu h} - e^{\mu h} + k_0\mu^{-1}\left[e^{-\mu h} + e^{\mu h}\right] + \mathcal{O}(\mu^{-2}) = 0, \tag{2.4.24}$$

which can also be rewritten as

$$1 - e^{-2\mu h} + \mathcal{O}(\mu^{-1}) = 0. \tag{2.4.25}$$

Since in sector $\mathcal{S}$, the solutions of $1 - e^{-2\mu h} = 0$ are given by

$$\widetilde{\mu}_n = \frac{n\pi i}{h}, \quad n = 0, 1, 2, \ldots,$$

it follows from Rouché's Theorem that the solutions of equation (2.4.25) have the form of

$$\mu_n = \widetilde{\mu}_n + \alpha_n = \frac{n\pi i}{h} + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}),\ n = N, N+1, N+2, \ldots,$$

where N is a sufficiently large positive integer. Substitute μ_n into (2.4.24) and use the fact that $e^{2\widetilde{\mu}_n h} = 1$ to obtain

$$1 - e^{2\alpha_n h} + k_0\mu_n^{-1}\left[1 + e^{2\alpha_n h}\right] + \mathcal{O}(\mu^{-2}) = 0. \tag{2.4.26}$$

Expand the exponential function in (2.4.26) according to its Taylor series, to give

$$2\alpha_n h = 2k_0\widetilde{\mu}_n^{-1} + \mathcal{O}(\mu_n^{-2}).$$

Hence we obtain that

$$\alpha_n = \frac{k_0}{h}\widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-2}),$$

and

$$\mu_n = \widetilde{\mu}_n + \alpha_n = \frac{n\pi i}{h} + \frac{k_0}{h}\widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-2}), \quad n = N, N+1, \ldots.$$

Due to $\lambda_n = \mu_n^2$, we get eventually

$$\begin{aligned}\lambda_n &= \left(\frac{n\pi i}{h}\right)^2 + 2\frac{k_0}{h} + \mathcal{O}(n^{-1}) \\ &= -\frac{n^2\pi^2}{h^2} + 2\frac{k_0}{h} + \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots.\end{aligned}$$

The proof is complete. ■

Theorem 2.4.1 gives the asymptotic expression of high eigenfrequencies. To end this section, we indicate that the high eigenfrequencies are actually real, which is stronger than that claimed by Theorem 1 of [81] under the assumption $b > 0$.

Proposition 2.4.2. *Suppose $b > 0$. Let $\mathcal{A}$ be defined by (2.2.4) and*

$$\Lambda_0 = \{\lambda \in \sigma(\mathcal{A}) |\ Im\lambda \neq 0\}. \tag{2.4.27}$$

Then Λ_0 is a bounded set of $\mathbb{C}$. Moreover, there is no spectrum on the imaginary axis and hence $\mathrm{Re}\lambda \leq -\alpha$ for some $\alpha > 0$ for all $\lambda \in \sigma(\mathcal{A})$.

Proof. By Theorems 2.3.1 and 2.3.2, $\Lambda_0 \subset \sigma_p(\mathcal{A})$. For any $\lambda = \tau + i\omega \in \Lambda_0$, we may take $(f, \lambda f), f \neq 0$ to be an eigenfunction corresponding to λ. Multiply the first equation of (2.3.1) by $\overline{f_n}$ and then integrate over $[0,1]$ with respect to x, to obtain, after separating real part and imaginary part, that

$$\begin{cases} (\tau^2 - \omega^2)\displaystyle\int_0^1 \rho(x)|f(x)|^2 dx + \int_0^1 [a(x) + \tau b(x)]|f'(x)|^2 dx = 0, \\ 2\tau\omega \displaystyle\int_0^1 \rho(x)|f(x)|^2 dx + \omega \int_0^1 b(x)|f'(x)|^2 dx = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} |\lambda|^2 \displaystyle\int_0^1 \rho(x)|f(x)|^2 dx = \int_0^1 a(x)|f'(x)|^2 dx, \\ -2\mathrm{Re}\lambda \displaystyle\int_0^1 \rho(x)|f(x)|^2 dx = \int_0^1 b(x)|f'(x)|^2 dx. \end{cases} \tag{2.4.28}$$

Thus $\mathrm{Re}\lambda \neq 0$, and

$$|\lambda| \leq \frac{|\lambda|^2}{|\mathrm{Re}\lambda|} = 2\frac{\displaystyle\int_0^1 a(x)|f'(x)|^2 dx}{\displaystyle\int_0^1 b(x)|f'(x)|^2 dx} \leq 2 \max_{0\leq x\leq 1} \frac{a(x)}{b(x)}. \tag{2.4.29}$$

So Λ_0 is a bounded set of $\mathbb{C}$ and there is no eigenvalue on the imaginary axis. These together with (i) of Theorem 2.3.2 show that $\mathrm{Re}\lambda \leq -\alpha$ for some $\alpha > 0$ for all $\lambda \in \sigma(\mathcal{A})$. The proof is complete. ■

It is notice that we only get the asymptotic expression for larger eigenvalues. For constant case that both a and b are constant, there is a sequence of finite eigenvalues that approach to continuous spectrum. However, for variable a, b, it becomes complicated that needs further investigations.

Chapter 3

On the Spectrum of an Euler-Bernoulli Beam Equation with Kelvin-Voigt Damping

3.1 Introduction

In this Chapter, we shall generalize the results of [43] to an Euler-Bernoulli beam equation with clamped boundary conditions and internal Kelvin-Voigt damping, which was initiated in [98]. The spectral property of the equation is considered. We show rigorously that the essential spectrum of the system operator is identified to be an interval on the left real axis. Moreover, under some assumptions on the coefficients, we show that the essential spectrum also contains continuous spectrum only, and the point spectrum consists of isolated eigenvalues of finite algebraic multiplicity. The asymptotic behavior of eigenvalues is presented.

The chapter is organized as follows. Firstly, in next section, Section 3.2, we formulate the problem into an abstract evolution equation in the state space. Secondly, in Section 3.3, we show that the system operator has no residual spectrum, see Theorem 3.3.1. The continuous spectrum is discussed under the analyticity of coefficient functions, see Theorem 3.3.3. Finally, in Section 3.4, we develop the asymptotic property of the eigenvalues when the damping is global, see Theorem 3.4.1.

3.2 System operator setup

The system that we are concerned with is the following Euler-Bernoulli beam equation clamped at two boundaries with internal Kelvin-Voigt damping:

$$\begin{cases} \rho(x)y_{tt}(x,t) + (a(x)y_{xx}(x,t) + b(x)y_{xxt}(x,t))_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = y(1,t) = y_x(1,t) = 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), \end{cases} \tag{3.2.1}$$

where the continuous function $b(\cdot) \geq 0$ is the damping function, and the continuous functions $\rho(\cdot), a(\cdot) > 0$ are system parameter functions in spacial variable. The system energy is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[a(x)|y_{xx}(x,t)|^2 + \rho(x)|y_t(x,t)|^2\right] dx. \tag{3.2.2}$$

Let $H_0^2(0,1)$ be the usual Sobolev space equipped with the inner product:

$$\langle f, g\rangle := \int_0^1 a(x)f''(x)\overline{g''(x)}dx, \quad \forall\ f, g \in H_0^2(0,1).$$

For any positive continuous function ρ, denote by $L^2_\rho(0,1) = L^2(0,1)$ with norm

$$\|f\|^2_{L^2_\rho(0,1)} = \int_0^1 \rho(x)|f(x)|^2 dx.$$

We consider system (3.2.1) in the energy state Hilbert space

$$\mathcal{H} = H_0^2(0,1) \times L^2_\rho(0,1)$$

with the inner product:

$$\begin{gathered} \langle (f_1, g_1), (f_2, g_2)\rangle = \int_0^1 [a(x)f_1''(x)\overline{f_2''(x)} + \rho(x)g_1(x)\overline{g_2(x)}]dx, \\ \forall\ (f_i, g_i) \in \mathcal{H},\ i = 1, 2. \end{gathered} \tag{3.2.3}$$

Define the system operator $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ as follows:

$$\begin{cases} \mathcal{A}(f, g) = \left(g,\ -\frac{1}{\rho}(af'' + bg'')''\right), \\ D(\mathcal{A}) = \left\{(f, g) \in H_0^2(0,1) \times H_0^2(0,1) | af'' + bg'' \in H^2(0,1)\right\}. \end{cases} \tag{3.2.4}$$

With the operator $\mathcal{A}$ at hand, we can write system (3.2.1) into an evolutionary equation in $\mathcal{H}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \mathcal{A}Y(t), \\ Y(0) = Y_0, \end{cases} \tag{3.2.5}$$

where $Y(t) = (y(\cdot,t), y_t(\cdot,t))$ is the state variable and $Y_0 = (y_0(\cdot), y_1(\cdot))$ is the initial value.

The following Lemma 3.2.1 is straightforward.

Lemma 3.2.1. *Let $\mathcal{A}$ be defined by (3.2.4). Then its adjoint $\mathcal{A}^*$ has the following form:*

$$\begin{cases} \mathcal{A}^*(f,g) = \left(-g,\ \dfrac{1}{\rho}(af'' - bg'')''\right), \\ D(\mathcal{A}^*) = \left\{(f,g) \in H_0^2(0,1) \times H_0^2(0,1) |\ af'' - bg'' \in H^2(0,1)\right\}. \end{cases} \tag{3.2.6}$$

Proposition 3.2.1. *Let $\mathcal{A}$ and $\mathcal{A}^*$ be given by (3.2.4) and (3.2.6) respectively. Then $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, and hence $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. For any $(f,g) \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}(f,g),(f,g)\rangle &= \left\langle (g,\ -\frac{1}{\rho}(af'' + bg'')''),\ (f,g) \right\rangle \\ &= \int_0^1 \left[a(x)g''(x)\overline{f''(x)} - (a(x)f''(x) + b(x)g''(x))''\overline{g(x)} \right] dx \\ &= \int_0^1 a(x)g''(x)\overline{f''(x)}dx + \int_0^1 (a(x)f''(x) + b(x)g''(x))'\overline{g'(x)}dx \\ &= \int_0^1 \left[a(x)g''(x)\overline{f''(x)} - a(x)f''(x)\overline{g''(x)} \right] dx - \int_0^1 b(x)|g''(x)|^2 dx, \end{aligned}$$

and hence

$$\mathrm{Re}\langle \mathcal{A}(f,g),(f,g)\rangle = -\int_0^1 b(x)|g''(x)|^2 dx \le 0.$$

Similarly for any $(u,v) \in D(\mathcal{A}^*)$,

$$\begin{aligned} \langle \mathcal{A}^*(u,v),(u,v)\rangle &= \left\langle (-v,\ \frac{1}{\rho}(au'' - bv'')''),\ (u,v) \right\rangle \\ &= \int_0^1 \left[-a(x)v''(x)\overline{u''(x)} + (a(x)u''(x) - b(x)v''(x))''\overline{v(x)} \right] dx \\ &= -\int_0^1 a(x)v''(x)\overline{u''(x)}dx - \int_0^1 (a(x)u''(x) - b(x)v''(x))'\overline{v'(x)}dx \\ &= \int_0^1 \left[-a(x)v''(x)\overline{u''(x)} + a(x)u''(x)\overline{v''(x)} \right] dx - \int_0^1 b(x)|v''(x)|^2 dx, \end{aligned}$$

and hence

$$\mathrm{Re}\langle \mathcal{A}^*(u,v),(u,v)\rangle = -\int_0^1 b(x)|v''(x)|^2 dx \le 0.$$

Therefore, both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. By the Lumer-Phillips Theorem, $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. ∎

3.3 Essential and continuous spectrum

In this section, we consider the spectrum of $\mathcal{A}$. First, let us formulate the eigenvalue problem. Suppose that $\mathcal{A}(f,g) = \lambda(f,g)$ with $(f,g) \in D(\mathcal{A})$ and $(f,g) \neq 0$. Then $g = \lambda f$

and $f \in H_0^2(0,1)$ satisfies

$$\begin{cases} \lambda^2\rho(x)f(x) + ((a(x)+\lambda b(x))f''(x))'' = 0, \\ f(0) = f(1) = f'(0) = f'(1) = 0. \end{cases} \tag{3.3.1}$$

The Theorem 3.3.1 following shows that the set of residual spectrum of $\mathcal{A}$ is empty.

Theorem 3.3.1. $\sigma_r(\mathcal{A}) = \emptyset$.

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$ implies $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$, it suffices to show that

$$\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*).$$

Suppose that $\mathcal{A}^*(f,g) = \lambda(f,g)$ for some $(f,g) \in D(\mathcal{A}^*)$ and $(f,g) \neq 0$. Then $g = -\lambda f$ and f satisfies

$$\begin{cases} \lambda^2\rho(x)f(x) + (a(x)f''(x) + \lambda b(x)f''(x))'' = 0, \\ f(0) = f(1) = f'(0) = f'(1) = 0. \end{cases} \tag{3.3.2}$$

It is seen that (3.3.2) is the same as (3.3.1). Hence, $\lambda \in \sigma_p(\mathcal{A}^*)$ if and only if $\lambda \in \sigma_p(\mathcal{A})$, and consequently $\sigma_r(\mathcal{A}) = \emptyset$. ■

The following Theorem 3.3.2 about the essential spectrum of $\mathcal{A}$ is from [98].

Theorem 3.3.2. *Let $\mathcal{A}$ be defined by (3.2.4). Then following assertions hold:*

(i) The essential spectrum of operator $\mathcal{A}$ is given by

$$\sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid a(\xi) + \lambda b(\xi) = 0 \textit{ for some } \xi \in [0,1]\}. \tag{3.3.3}$$

(ii) $\sigma(\mathcal{A})\backslash\sigma_{ess}(\mathcal{A})$ consists of all isolated eigenvalues of finite multiplicity.

Proof. The (ii) of Theorem 3.3.2 was claimed in [98] without proof. Here we give a simple explanation. Suppose that (i) is valid. Then $\sigma(\mathcal{A})\backslash\sigma_{ess}(\mathcal{A})$ is an open connected subset of $\mathbb{C}\backslash\sigma_{ess}(\mathcal{A})$, (ii) is then a direct consequence of Theorem 2.1 of [31, p.373].

(i) of Theorem 3.3.2 was also claimed in [98] but the proof there is incomplete. Actually, in [98], the authors defined a bounded operator $\mathcal{B}$ on $\mathcal{H}_1$ as following:

$$\mathcal{B}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} -\dfrac{b(x)}{a(x)}f(x) + \dfrac{xG_1(f)+G_2(f)}{a(x)} \\ 0 \end{pmatrix}, \ \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}_1, \tag{3.3.4}$$

where

$$\mathcal{H}_1 = L_E^2(0,1) \times L_\rho^2(0,1),$$

$$L_E^2(0,1) := \left\{ f \in L_a^2(0,1) \,\middle|\, \langle f, \tfrac{x}{a}\rangle_{L_a^2} = 0,\ \langle f, \tfrac{1}{a}\rangle_{L_a^2} = 0 \right\}$$

and

$$\begin{aligned} G_1(f) &= \frac{\int_0^1 \frac{b(\tau)}{a(\tau)} f(\tau)d\tau \int_0^1 \frac{1-\tau}{a(\tau)}d\tau - \int_0^1 (1-\tau)\frac{b(\tau)}{a(\tau)} f(\tau)d\tau \int_0^1 \frac{1}{a(\tau)}d\tau}{\int_0^1 \frac{\tau^2}{a(\tau)}d\tau \int_0^1 \frac{1}{a(\tau)}d\tau - \left[\int_0^1 \frac{\tau}{a(\tau)}d\tau\right]^2}, \\ G_2(f) &= -\frac{\int_0^1 \frac{b(\tau)}{a(\tau)} f(\tau)d\tau \int_0^1 \frac{(1-\tau)\tau}{a(\tau)}d\tau - \int_0^1 (1-\tau)\frac{b(\tau)}{a(\tau)} f(\tau)d\tau \int_0^1 \frac{\tau}{a(\tau)}d\tau}{\int_0^1 \frac{\tau^2}{a(\tau)}d\tau \int_0^1 \frac{1}{a(\tau)}d\tau - \left[\int_0^1 \frac{\tau}{a(\tau)}d\tau\right]^2}. \end{aligned}$$

It was proved in [98] that $\mathcal{B}$ is self-adjoint and its essential spectrum is given by

$$\sigma_{ess}(\mathcal{B}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}. \tag{3.3.5}$$

However, in the proof of

$$\sigma_{ess}(\mathcal{B}) \supset \{\lambda \in \mathbb{C}\backslash\{0\} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\} \tag{3.3.6}$$

in [98], the authors claimed unfoundedly that when $\lambda \neq 0$ and $\lambda a(\xi) + b(\xi) = 0$ for some $\xi \in [0,1]$, the rang of operator $\lambda I - \mathcal{B}$ satisfies

$$\mathcal{R}(\lambda I - \mathcal{B}) \supset \left\{ y \in L^2(0,1) \,\middle|\, y(\xi) = \frac{c_1\xi + c_2}{a(\xi)}, c_1, c_2 \in \mathbb{C} \right\}.$$

Here we give a correct proof for (3.3.6). First, we show that

$$\sigma(\mathcal{B}) \supset \{\lambda \in \mathbb{C}\backslash\{0\} \mid \lambda a(\xi) + b(\xi) = 0 \text{ for some } \xi \in [0,1]\}. \tag{3.3.7}$$

Let $\lambda \in \mathbb{C}\backslash\{0\}$ and $\lambda a(\xi) + b(\xi) = 0$ for some $\xi \in [0,1]$. Define

$$E_\lambda = \{x \in [0,1] | \lambda a(x) + b(x) = 0\}.$$

For any $(u,v) \in \mathcal{H}_1$, consider the equation

$$(\lambda I - \mathcal{B})(f,g) = (u,v),$$

which is equivalent to

$$\lambda g(x) = v(x)$$

and f satisfying

$$\lambda f(x) + \frac{b(x)}{a(x)} f(x) - \frac{xG_1(f) + G_2(f)}{a(x)} = u(x). \tag{3.3.8}$$

If the measure of E_λ is nonzero and (3.3.8) admits a solution, it must have

$$u(x) = \frac{C_1 x + C_2}{a(x)} \text{ in } E_\lambda \text{ for some constants } C_1, C_2.$$

Obviously, such functions cannot represent all functions of $L^2_E(0,1)$ on E_λ, that is

$$\mathcal{R}(\lambda I - \mathcal{B}) \neq \mathcal{H}_1.$$

So

$$\lambda \in \sigma(\mathcal{B}).$$

Now suppose that the measure of E_λ is zero and (3.3.8) has solution $f \in L^2_E(0,1)$ for any $u \in L^2_E(0,1)$. In this case, it follows from (3.3.8) that f must be of the form:

$$f(x) = \frac{1}{\lambda a(x) + b(x)}[a(x)u(x) + xG_1(f) + G_2(f)], \ \forall\, x \in [0,1]\backslash E_\lambda. \tag{3.3.9}$$

Take special $u \in L^2_E(0,1)$ in (3.3.9):

$$u(x) = \begin{cases} \dfrac{1}{\sqrt[3]{x-\xi}}, & x \in E_1, \\ \\ c_1 + c_2 x, & x \in E_0 = [0,1]\backslash E_1, \end{cases}$$

where c_1, c_2 are constants to be chosen such that $u \in L^2_E(0,1)$, $E_1 \subset [0,1]$ is a given small closed interval containing ξ, and $\mathrm{mes}(E_1)$ is the measure of E_1, $0 < \mathrm{mes}(E_1) < 1$. A simple computation shows that the sufficient condition for the existence of c_1, c_2 is

$$\left[\int_{E_0} x dx\right]^2 - \int_{E_0} x^2 dx \int_{E_0} 1 dx \neq 0,$$

which is obviously true. Now for this special u, choose a closed interval $E_2 \subset E_1$, $\xi \in E_2$, such that the associated f with this u through (3.3.9) satisfies

$$|a(x)u(x) + xG_1(f) + G_2(f)| > 1, \ x \in E_2,$$

and hence by (3.3.9),

$$\left\| \frac{1}{\lambda a + b} \right\|_{L^2(E_2)} \leq \|f\|_{L^2(E_2)} < \infty,$$

which means that

$$\frac{1}{\lambda a + b} \in L^2(E_2).$$

This fact together with (3.3.9) shows that

$$\frac{a\widetilde{u}}{\lambda a + b} \in L^2(E_2), \ \forall\, \widetilde{u} \in L^2(E_2).$$

Define the multiplication operator $\mathcal{F} : L^2(E_2) \to L^2(E_2)$ by

$$(\mathcal{F}\widetilde{u})(x) = \frac{a(x)}{\lambda a(x) + b(x)}\widetilde{u}(x), \ \forall\, \widetilde{u} \in L^2(E_2).$$

Then $\mathcal{F}$ is a closed operator on $L^2(E_2)$. By the Closed Graph Theorem, $\mathcal{F}$ is bounded on $L^2(E_2)$, which implies that

$$\frac{a}{\lambda a + b} \in L^\infty(E_2).$$

This contradicts to $\lambda a(\xi) + b(\xi) = 0$ and continuity of a, b. Hence

$$\mathcal{R}(\lambda I - \mathcal{B}) \neq \mathcal{H}_1.$$

Therefore

$$\lambda \in \sigma(\mathcal{B})$$

and (3.3.7) holds.

Next, we show (3.3.6). Let

$$\begin{cases} m &= \min\limits_{0\le x\le 1} \{\lambda |\ \lambda a(x) + b(x) = 0\}, \\ M &= \max\limits_{0\le x\le 1} \{\lambda |\ \lambda a(x) + b(x) = 0\}. \end{cases}$$

It suffices to show that

$$[m, M] \subset \sigma_{ess}(\mathcal{B}).$$

There are two cases:

Case I: $m = M$**.** In this case, $b(x)/a(x) = -m$ is a constant, and a simple computation shows that

$$G_1(f) = G_2(f) = 0,\ \forall\ f \in L^2_E(0, 1).$$

By the definition of $\mathcal{B}$,

$$\begin{aligned} (mI - \mathcal{B})(f, g)(x) &= \left(-\frac{xG_1(f) + G_2(f)}{a(x)}, mg(x)\right) \\ &= (0, mg(x)). \end{aligned}$$

Hence,

$$L^2_E(0, 1) \times \{0\} \subset \mathcal{N}(mI - \mathcal{B}),$$

which means, by Definition 1.2.1, that

$$\lambda = m \in \sigma_{ess}(\mathcal{B}).$$

Case II: $m < M$**.** In this case, λ can be taken as any point of interval $[m, M]$ by the continuity of $b(x)/a(x)$. So by (3.3.7), $[m, M] \subseteq \sigma(\mathcal{B})$. Since $\mathcal{B}$ is self-adjoint, $[m, M] \subseteq \sigma_{ess}(\mathcal{B})$ follows from Theorem 5 of [23, p.1395] which says that for a self-adjoint operator, all non-isolated spectrum must be essential spectrum (note that in [23, p.1393], the essential spectrum of a closed operator is defined as only those that (i) of our Definition 1.2.1 is satisfied). ■

Next, we consider the continuous spectrum of $\mathcal{A}$. To do this, we need additional conditions of the following

$$\begin{cases} a(x),\ b(x) \text{ and } \rho(x) \text{ are analytic in } [0,1]; \\ \forall\ \lambda \in \mathbb{R} \text{ and } \xi \in [0,1], \dfrac{(x-\xi)^4}{a(x)+\lambda b(x)} \text{ is analytic in a neighboorhood of } \xi. \end{cases} \tag{3.3.10}$$

Remark 3.3.1. For analytic functions a, b, it can be easily shown that at the nonzero point of b, the second condition of (3.3.10) is equivalent to the following conditions:

$$\begin{cases} a'(x) - \dfrac{a(x)}{b(x)}b'(x) \neq 0 \\ \text{or } a''(x) - \dfrac{a(x)}{b(x)}b''(x) \neq 0 \\ \text{or } a'''(x) - \dfrac{a(x)}{b(x)}b'''(x) \neq 0 \\ \text{or } a^{(4)}(x) - \dfrac{a(x)}{b(x)}b^{(4)}(x) \neq 0 \end{cases} \quad \text{at the point } x \in [0,1] \text{ where } b(x) \neq 0. \tag{3.3.11}$$

Actually, for any $\xi \in [0,1]$ with $b(\xi) \neq 0$, find $\lambda \in \mathbb{R}$ such that $a(\xi) + \lambda b(\xi) = 0$. So

$$\lambda = -a(\xi)/b(\xi).$$

Let

$$a(x) + \lambda b(x) = (x-\xi)^k \varphi(x)$$

with $\varphi(\xi) \neq 0$. By (3.3.10), $k \in \{1,2,3,4\}$. If $k = 1$,

$$(a + \lambda b)'(\xi) = \varphi(\xi) \neq 0,$$

which is equivalent to

$$a'(\xi) - \frac{a(\xi)}{b(\xi)}b'(\xi) = \varphi(\xi) \neq 0.$$

This is the first case of (3.3.11). When $k \in \{2,3,4\}$, the similar arguments lead to the other cases of (3.3.11).

The condition (3.3.11) is easier to check for given functions a, b. For instance, the functions $b(x) = x$ and any function a with $a'' \neq 0$ satisfy (3.3.11).

Theorem 3.3.3. *Let $\mathcal{A}$ be defined by (3.2.4) and (3.3.10) holds. Then the set of continuous spectrum of $\mathcal{A}$ satisfies*

$$\sigma_c(\mathcal{A}) = \sigma_{ess}(\mathcal{A}) = \{\lambda \in \mathbb{C} |\ a(\xi) + \lambda b(\xi) = 0 \textit{ for some } \xi \in [0,1]\}.$$

Proof. Suppose that $a(\xi) + \lambda b(\xi) = 0$ for some $\xi \in [0,1], \lambda \in \mathbb{R}$. If $\lambda \in \sigma_p(\mathcal{A})$, then there is a nonzero $f \in H_0^2(0,1)$ satisfying the characteristic equation (3.3.1). The proof will be accomplished if we can show that $f \equiv 0$ because $\sigma_r(\mathcal{A}) = \emptyset$ claimed by Theorem 3.3.1. This will be split into three steps:

Step 1: We claim that the solution f of the first equation in (3.3.1) that is rewritten specifically in following

$$\lambda^2\rho(x)f(x) + ((a(x) + \lambda b(x))f''(x))'' = 0 \tag{3.3.12}$$

is either analytic in a neighborhood of ξ with $f(\xi) = 0$, or

$$f(\xi) = f'(\xi) = 0.$$

We consider f in a neighborhood of ξ. By (3.3.10), we assume that

$$a(x) + \lambda b(x) = (x - \xi)^k \varphi(x),$$

where $k \in \{1, 2, 3, 4\}$ and φ is analytic in $[0, 1]$, $\varphi(\xi) \neq 0$. Then ξ is the regular singular point of the equation (3.3.12) ([10, p.62]). Set

$$\begin{aligned}
P_0(x) &= \frac{\lambda^2\rho(x)}{a(x) + \lambda b(x)} = \frac{1}{(x-\xi)^k}\left[c + \sum_{i=1}^{\infty} p_{0,i}(x-\xi)^i\right], \\
P_2(x) &= \frac{a''(x) + \lambda b''(x)}{a(x) + \lambda b(x)} = \frac{1}{(x-\xi)^2}\left[k(k-1) + \sum_{i=1}^{\infty} p_{2,i}(x-\xi)^i\right], \\
P_3(x) &= \frac{2(a'(x) + \lambda b'(x))}{a(x) + \lambda b(x)} = \frac{1}{(x-\xi)}\left[2k + \sum_{i=1}^{\infty} p_{3,i}(x-\xi)^i\right],
\end{aligned}$$

where the three series on the right side above are the Taylor series and by assumption

$$c = \frac{\lambda^2\rho(\xi)}{\varphi(\xi)} \neq 0.$$

The indicial equation of (3.3.12) is (see e.g., [10, p.76])

$$F(r) = r(r-1)(r-2)(r-3) + 2kr(r-1)(r-2) + k(k-1)r(r-1) + \widetilde{c} = 0, \tag{3.3.13}$$

where

$$\widetilde{c} = \begin{cases} 0, & k < 4, \\ c, & k = 4. \end{cases}$$

If the four roots r do not differ by integers, then there will be four linearly independent solutions of equation (3.3.12) of the form (see e.g., [10, p.63,76]):

$$f(x) = (x - \xi)^r A(x). \tag{3.3.14}$$

Otherwise, the form of the solution of equation (3.3.12) must be generalized to

$$f(x) = (x-\xi)^r A(x) \ln|x-\xi| + C(x)(x-\xi)^{r_2}, \tag{3.3.15}$$

or

$$f(x) = (x-\xi)^r A_2(x)[\ln|x-\xi|]^2 + (x-\xi)^{r_2} A_1(x) \ln|x-\xi| + (x-\xi)^{r_3} A_0(x), \tag{3.3.16}$$

or

$$f(x) = (x-\xi)^\alpha \sum_{j=0}^{3} [\ln|x-\xi|]^j B_j(x). \tag{3.3.17}$$

Here r, r_2, r_3, α are roots of (3.3.13), $A(x), A_i(x), B_j(x), C(x)$ are analytic functions at ξ that have Taylor series whose radius of convergence are at least as large as the distance to the nearest singular point of the coefficient functions in the equation (3.3.12). A direct calculation shows that

$$F(r) = \begin{cases} r(r-1)^2(r-2), & k=1, \\ r^2(r-1)^2, & k=2, \\ r^2(r-1)(r+1), & k=3, \\ r(r-1)(r+1)(r+2)+c, & k=4. \end{cases}$$

We have now two cases:

Case I: $k<4$**.** In this case, we claim that f must be analytic in a neighborhood of ξ and

$$f(\xi) = 0.$$

We only give a proof for the case of $k=1$ since other cases can be treated similarly. Now, the roots of (3.3.13) are $r = 0, 1, 1, 2$. By (3.3.14), (3.3.15), (3.3.16) and (3.3.17), in a neighborhood of ξ, the solution of (3.3.12) must be

$$\begin{aligned} f(x) \;=\;& C_1(x-\xi)^2 A_1(x) + C_2\left[(x-\xi)^2 A_1(x)\ln|x-\xi| + (x-\xi)A_2(x)\right] \\ & + C_3\left[(x-\xi)^2 A_1(x)[\ln|x-\xi|]^2 + (x-\xi)A_2(x)\ln|x-\xi| + (x-\xi)A_3(x)\right] \\ & + C_4\left[(x-\xi)^2 A_1(x)[\ln|x-\xi|]^3 + (x-\xi)A_2(x)[\ln|x-\xi|]^2 \right. \\ & \left. + (x-\xi)A_3(x)\ln|x-\xi| + A_4(x)\right], \end{aligned}$$

where $A_i(x)$, $i = 1, 2, 3, 4$ are analytic at ξ, $A_i(\xi) \neq 0$, and have Taylor series whose radius of convergence are at least as large as the distance to the nearest singular point of the coefficient functions in the equation (3.3.12). Then

$$\begin{aligned}
f'(x) \;=\;& C_4\left[2(x-\xi)A_1(x)+(x-\xi)^2A_1'(x)\right][\ln|x-\xi|]^3 \\
&+C_4\left[3(x-\xi)A_1(x)+A_2(x)+(x-\xi)A_2'(x)\right][\ln|x-\xi|]^2 \\
&\quad +C_3\left[2(x-\xi)A_1(x)+(x-\xi)^2A_1'(x)\right][\ln|x-\xi|]^2 \\
&+C_4\left[2A_2(x)+A_3(x)+(x-\xi)A_3'(x)\right]\ln|x-\xi| \\
&\quad +C_3\left[2(x-\xi)A_1(x)+A_2(x)+(x-\xi)A_2'(x)\right]\ln|x-\xi| \\
&\quad +C_2\left[2(x-\xi)A_1(x)+(x-\xi)^2A_1'(x)\right]\ln|x-\xi| \\
&+C_4\left[A_3(x)+A_4'(x)\right] \\
&\quad +C_3\left[A_2(x)+A_3(x)+(x-\xi)A_3'(x)\right] \\
&\quad +C_2\left[(x-\xi)A_1(x)+A_2(x)+(x-\xi)A_2'(x)\right] \\
&\quad +C_1\left[2(x-\xi)A_1(x)+(x-\xi)^2A_1'(x)\right].
\end{aligned}$$

Since f' is continuous, it must have

$$C_3 = C_4 = 0.$$

Thus,

$$f(x) = C_1(x-\xi)^2A_1(x) + C_2\left[(x-\xi)^2A_1(x)\ln|x-\xi| + (x-\xi)A_2(x)\right]$$

and

$$f''(x) = [2C_2A_1(x) + (x-\xi)B_1(x)]\ln|x-\xi| + B_2(x),$$

where

$$\begin{aligned}
B_1(x) \;=\;& 4A_1'(x) + (x-\xi)A_1''(x), \\
B_2(x) \;=\;& C_1\left[2A_1(x) + 4(x-\xi)A_1'(x) + (x-\xi)^2A_1''(x)\right] \\
&+C_2\left[3A_1(x) + 2(x-\xi)A_1'(x) + 2A_2'(x) + (x-\xi)A_2''(x)\right].
\end{aligned}$$

So,

$$\begin{aligned}
&(a(x)+\lambda b(x))f''(x) \\
&\quad = \left[2C_2(x-\xi)A_1(x) + (x-\xi)^2B_1(x)\right]\varphi(x)\ln|x-\xi| + B_2(x)(x-\xi)\varphi(x).
\end{aligned}$$

By the continuity of $((a+\lambda b)f'')'$, we have

$$C_2 = 0.$$

Hence,

$$f(x) = C_1(x-\xi)^2 A_1(x),$$

which implies that f must be analytic in a neighborhood of ξ and

$$f(\xi) = 0.$$

Case II: $k = 4$**.** In this case, $0, 1$ are not roots of (3.3.13). By the continuity of f and f', it follows from (3.3.14), (3.3.15), (3.3.16), and (3.3.17) that

$$f(\xi) = f'(\xi) = 0.$$

Step 2: We claim that for any $[\alpha_1, \alpha_2] \subset [0,1]$ with $\xi \in [\alpha_1, \alpha_2]$, if

$$f(\alpha_1) = f(\alpha_2) = f'(\alpha_1) = f'(\alpha_2) = 0$$

and f is analytic in (α_1, α_2), then there are infinitely many $\widetilde{x}_i \in (\alpha_1, \alpha_2)$ such that

$$f(\widetilde{x}_i) = 0,\ \widetilde{x}_i \neq \widetilde{x}_j,\ i \neq j,\ i, j = 1, 2, \cdots . \tag{3.3.18}$$

In fact, since there are at most finitely many regular singular points of (3.3.12) in $[0, 1]$, by the boundary conditions and **Step 1**, such interval $[\alpha_1, \alpha_2]$ exists. We will use the mathematical induction to prove the result.

First, we claim that there is an $\hat{x} \in (\alpha_1, \alpha_2)$ such that

$$f(\hat{x}) = 0.$$

In fact, by $f(\alpha_1) = f(\alpha_2) = 0$ and Rolle's Theorem, there is an $\eta \in (\alpha_1, \alpha_2)$ such that $f'(\eta) = 0$. By $f'(\alpha_1) = f'(\alpha_2) = 0$ and Rolle's Theorem, there are $\gamma_1, \gamma_2 \in (\alpha_1, \alpha_2)$ such that $\gamma_1 < \gamma_2$ and

$$f''(\gamma_1) = f''(\gamma_2) = 0.$$

If $\xi = \gamma_1$ or $\xi = \gamma_2$, take $\hat{x} = \xi$. Then

$$f(\hat{x}) = 0.$$

If $\xi \neq \gamma_1$ and $\xi \neq \gamma_2$, suppose that $\xi < \gamma_1 < \gamma_2$ without loss of generality. By

$$((a + \lambda b)f'')(\xi) = ((a + \lambda b)f'')(\gamma_1) = ((a + \lambda b)f'')(\gamma_2) = 0,$$

there are $\xi_1 \in (\xi, \gamma_1),\ \xi_2 \in (\gamma_1, \gamma_2)$ such that

$$((a + \lambda b)f'')'(\xi_i) = 0,\ i = 1, 2.$$

So there is an $\hat{x}_1 \in (\xi_1, \xi_2)$ such that

$$((a+\lambda b)f'')''(\hat{x}_1) = 0,$$

which, together with (3.3.1) implies that

$$f(\hat{x}_1) = 0.$$

Next, we assume $\xi \neq \alpha_1$ (otherwise, we may assume $\xi \neq \alpha_2$). Suppose generally that

$$f(x_i) = 0,\ i = 1, 2, \cdots, n,\ \alpha_1 < x_1 < x_2 < \cdots < x_n < \alpha_2.$$

We will show that there are $n+1$ number of different zeros $\{\overline{x}_{i1}\}_{i=1}^{n+1}$ of f in (α_1, α_2). Set $x_0 = \alpha_1,\ x_{n+1} = \alpha_2$. Assume $\xi \in (x_j, x_{j+1}]$ for some $j \in \{0, 1, \cdots, n\}$. By Rolle's Theorem, for each $i \in \{1, 2, \cdots, n+1\}$, there is an $x_{i1} \in (x_{i-1}, x_i)$ such that

$$f'(x_{i1}) = 0.$$

Obviously,

$$\alpha_1 < x_{11} < x_{21} < \cdots < x_{n1} < x_{(n+1)1} < \alpha_2.$$

Set $x_{01} = \alpha_1$ and $x_{(n+2)1} = \alpha_2$. Then $\xi \in (x_{j1}, x_{(j+2)1})$ or $\xi = \alpha_2$. By Rolle's Theorem again, for each $i \in \{1, 2, \cdots, n+2\}$, there is an $x_{i2} \in (x_{(i-1)1}, x_{i1})$ such that

$$f''(x_{i2}) = 0. \tag{3.3.19}$$

Obviously,

$$\alpha_1 < x_{12} < x_{22} < \cdots < x_{(n+1)2} < x_{(n+2)2} < \alpha_2.$$

Set $x_{(n+3)2} = \alpha_2$. Then $\xi \in (x_{j2}, x_{(j+3)2})$ or $\xi = \alpha_2$. Now we have two cases:

Case I: $\xi = x_{(j+1)2}$ or $\xi = x_{(j+2)2}$. In this case, by $(a+\lambda b)(\xi) = 0$, we have

$$((a+\lambda b)f'')'(\xi) = 0. \tag{3.3.20}$$

Moreover, from (3.3.19), we have

$$((a+\lambda b)f'')(x_{i2}) = 0,\ i = 1, 2 \cdots, n+2.$$

By Rolle's Theorem, for each $i \in \{1, 2, \cdots, n+1\}$, there is an $x_{i3} \in (x_{i2}, x_{(i+1)2})$ such that

$$((a+\lambda b)f'')'(x_{i3}) = 0.$$

This together with (3.3.20) shows that there is a sequence $\{\overline{x}_i\}_{i=1}^{n+2} \subset (\alpha_1, \alpha_2)$ such that

$$((a+\lambda b)f'')'(\overline{x}_i) = 0,\ i = 1, 2, \cdots, n+2,\ \alpha_1 < \overline{x}_1 < \overline{x}_2 < \cdots < \overline{x}_{n+2} < \alpha_2. \tag{3.3.21}$$

Case II: $\xi \neq x_{(j+1)2}$ **and** $\xi \neq x_{(j+2)2}$**.** In this case,

$$((a+\lambda b)f'')(\xi) = ((a+\lambda b)f'')(x_{i2}) = 0,\ i = 1, 2, \cdots, n+2.$$

By Rolle's Theorem, we can get (3.3.21) also for some different $\{\overline{x}_i\}_{i=1}^{n+2} \subset (\alpha_1, \alpha_2)$.

Since (3.3.21), by Rolle's Theorem, for each $i \in \{1, 2, \cdots, n+1\}$, we can find an $\overline{x}_{i1} \in (\overline{x}_i, \overline{x}_{i+1})$ such that

$$((a+\lambda b)f'')''(\overline{x}_{i1}) = 0.$$

This is equivalent to, by (3.3.1), that

$$f(\overline{x}_{i1}) = 0,\ i = 1, 2, \cdots, n+1,\ \alpha_1 < \overline{x}_{11} < \overline{x}_{21} < \cdots < \overline{x}_{(n+1)1} < \alpha_2.$$

By mathematical induction, we get (3.3.18) eventually.

Step 3: Let τ be an accumulation point of $\{\widetilde{x}_i\}$ satisfy (3.3.18). If τ is a zero of $a+\lambda b$, by (3.3.14), (3.3.15), (3.3.16), and (3.3.17), there is a neighborhood $\mathcal{O}_\tau$ of τ such that

$$f \equiv 0 \text{ in } \mathcal{O}_\tau. \tag{3.3.22}$$

If τ is not a zero of $a+\lambda b$, then it must be an ordinary point of (3.3.12). By the uniqueness theorem of the regular ordinary differential equations, we get (3.3.22) again. Same arguments as **Step 2**, it follows that there is a neighborhood $\mathcal{O}_\xi$ of ξ such that

$$f \equiv 0 \text{ in } \mathcal{O}_\xi.$$

Since f is identical to zero in a neighborhood of any singular point ξ, f must be identical to zero everywhere by the uniqueness theorem of the regular ordinary differential equations. The proof is complete. ■

Remark 3.3.2. For one-dimensional wave equation with internal Kelvin-Voigt damping, we showed that Theorem 3.3.3 is still true for irregular singularity but it is not true without analytic assumption on coefficient functions. These are also expected for the system (3.2.1), which however, needs nontrivial investigations.

3.4 Asymptotic behavior of eigenvalues

In this section, we consider the asymptotic behavior of eigenvalues of $\mathcal{A}$. To do this, we assume further that

$$\rho(x), a(x), b(x) \in C^3[0,1] \text{ and } \rho(x), a(x), b(x) > 0 \text{ for all } x \in [0,1]. \tag{3.4.1}$$

Suppose that λ is an eigenvalue with large modulus. Then

$$a(x)+\lambda b(x)\neq 0 \text{ for any } x\in[0,1] \tag{3.4.2}$$

and we rewrite the characteristic equation (3.3.1) as

$$\begin{cases} (a(x)+\lambda b(x))f^{(4)}(x)+2(a'(x)+\lambda b'(x))f'''(x) \\ \qquad +(a''(x)+\lambda b''(x))f''(x)+\lambda^2\rho(x)f(x)=0, \\ f(0)=f(1)=f'(0)=f'(1)=0. \end{cases} \tag{3.4.3}$$

The following Lemma 3.4.1 is direct.

Lemma 3.4.1. *Let $\lambda\in\mathbb{C}$. Then as $|\lambda|\to\infty$, the first equation of (3.4.3) has the following asymptotic expression:*

$$\begin{aligned} &f^{(4)}(x)+2(a'(x)+\lambda b'(x))\left[\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right]f'''(x) \\ &\quad +(a''(x)+\lambda b''(x))\left[\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right]f''(x) \\ &\quad +\lambda\left[1-\frac{a(x)}{\lambda b(x)}+\frac{a^2(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right]\rho_0^4(x)f(x)=0, \end{aligned} \tag{3.4.4}$$

where

$$\rho_0(x)=\left[\frac{\rho(x)}{b(x)}\right]^{1/4}. \tag{3.4.5}$$

In order to find the asymptotic fundamental solutions of (3.4.3), we introduce the following space-scaling transformation:

$$\varphi(z)=f(x),\ z=\frac{1}{h}\int_0^x\rho_0(\tau)d\tau,\ h=\int_0^1\rho_0(\tau)d\tau. \tag{3.4.6}$$

Under this transformation, (3.4.4) becomes

$$\begin{aligned} &\varphi^{(4)}(z)+\frac{h}{\rho_0^4(x)}\left[2\rho_0^3(x)(a'(x)+\lambda b'(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\right. \\ &\qquad\left.+6\rho_0^2(x)\rho_0'(x)\right]\varphi'''(z) \\ &+\frac{h^2}{\rho_0^4(x)}\left[3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)\right. \\ &\qquad +6\rho_0(x)\rho_0'(x)(a'(x)+\lambda b'(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right) \\ &\qquad\left.+\rho_0^2(x)(a''(x)+\lambda b''(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\right]\varphi''(z) \\ &+\frac{h^3}{\rho_0^4(x)}\left[\rho_0'''(x)+2\rho_0''(x)(a'(x)+\lambda b'(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\right. \\ &\qquad\left.+\rho_0'(x)(a''(x)+\lambda b''(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\right]\varphi'(z) \\ &+\lambda h^4\left[1-\frac{a(x)}{\lambda b(x)}+\frac{a^2(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right]\varphi(z)=0 \end{aligned} \tag{3.4.7}$$

with the boundary conditions:

$$\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0. \tag{3.4.8}$$

With this transformation, we have the following Proposition 3.4.1.

Proposition 3.4.1. *The equation (3.4.4) with boundary condition (3.4.3) is equivalent to (3.4.7) and (3.4.8). That is,* (λ, f), $f \neq 0$, *satisfies (3.4.4) and boundary condition (3.4.3) if and only if* (λ, φ), $\varphi \neq 0$, *satisfies (3.4.7) and (3.4.8).*

Now we consider (3.4.7) with the boundary condition (3.4.8). Since the eigenvalues are symmetric about the real axis and $\mathrm{Re}\lambda \leq 0$ for any $\lambda \in \sigma(\mathcal{A})$, we only consider those eigenvalues λ with $\frac{\pi}{2} \leq \arg\lambda \leq \pi$. Let $\lambda = \mu^4$. Since $\frac{\pi}{2} \leq \arg\lambda \leq \pi$, we consider μ located on the following sector:

$$\mathcal{S} = \left\{ \mu \in \mathbb{C} \mid \frac{\pi}{8} \leq \arg\mu \leq \frac{\pi}{4} \right\}. \tag{3.4.9}$$

Let

$$\omega_1 = e^{\frac{3}{4}\pi i},\ \omega_2 = e^{\frac{5}{4}\pi i},\ \omega_3 = -\omega_2,\ \omega_4 = -\omega_1. \tag{3.4.10}$$

Then we have

$$\mathrm{Re}(\mu\omega_1) \leq \mathrm{Re}(\mu\omega_2) \leq \mathrm{Re}(\mu\omega_3) \leq \mathrm{Re}(\mu\omega_4) \tag{3.4.11}$$

and

$$\begin{cases} \mathrm{Re}(\mu\omega_1) = |\mu| \cos(\arg\mu + \frac{3}{4}\pi) = -|\mu| \sin(\arg\mu + \frac{\pi}{4}) \leq -\frac{\sqrt{2}}{2}|\mu| < 0, \\ \mathrm{Re}(\mu\omega_2) = |\mu| \cos(\arg\mu + \frac{5}{4}\pi) \leq 0. \end{cases} \tag{3.4.12}$$

The following Lemma 3.4.2 gives the form of the asymptotic fundamental solutions of (3.4.7).

Lemma 3.4.2. *Suppose* $\lambda = \mu^4 \neq 0$. *Then for* $z \in [0, 1]$ *and* $\mu \in \mathcal{S}$,

$$e^{\mu h \omega_k z},\ k = 1, 2, 3, 4,$$

are linearly independent fundamental solutions of

$$\varphi^{(4)}(z) + \mu^4 h^4 \varphi(z) = 0,$$

and when $|\mu|$ *large enough, (3.4.7) has the following asymptotic fundamental solutions: for* $k = 1, 2, 3, 4$,

$$\varphi_k(z) = e^{\mu h \omega_k z} \left[\varphi_0(z) + \frac{1}{\omega_k}\varphi_1(z)\mu^{-1} + \frac{1}{\omega_k^2}\varphi_2(z)\mu^{-2} + \frac{1}{\omega_k^3}\varphi_3(z)\mu^{-3} + \mathcal{O}(\mu^{-4}) \right], \tag{3.4.13}$$

where

$$\begin{cases} \varphi_0(z) &= \exp\left\{-\frac{h}{2}\int_0^z \rho_1(x(\tau))d\tau\right\} = \exp\left\{-\frac{1}{2}\int_0^z \rho_1(x)\rho_0(x)dx\right\}, \\ \varphi_1(z) &= -\frac{1}{4}\int_0^z \exp\left\{-\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx\right\}\rho_2(\zeta)d\zeta, \\ \varphi_2(z) &= -\frac{1}{4}\int_0^z \exp\left\{-\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx\right\}\rho_3(\zeta)d\zeta, \\ \varphi_3(z) &= -\frac{1}{4}\int_0^z \exp\left\{-\frac{1}{2}\int_0^{z-\zeta} \rho_1(x)\rho_0(x)dx\right\}\rho_4(\zeta)d\zeta, \end{cases} \tag{3.4.14}$$

and

$$\begin{cases} \rho_1(x) = \dfrac{3\rho_0'(x)}{\rho_0^2(x)} + \dfrac{1}{\rho_0(x)}\dfrac{b'(x)}{b(x)}, \\ \rho_2(z) = \dfrac{6}{h}\varphi_0''(z) + \left(\dfrac{18\rho_0'(x(z))}{\rho_0^2(x(z))} + \dfrac{6}{\rho_0(x(z))}\dfrac{b'(x(z))}{b(x(z))}\right)\varphi_0'(z) \\ \qquad + \dfrac{h\varphi_0(z)}{\rho_0^4(x(z))}\left(\rho_0^2(x(z))\dfrac{b''(x(z))}{b(x(z))} + 6\rho_0(x(z))\rho_0'(x(z))\dfrac{b'(x(z))}{b(x(z))}\right. \\ \qquad\qquad \left. + 4\rho_0(x(z))\rho_0''(x(z)) + 3(\rho_0'(x(z)))^2\right), \\ \rho_3(z) = \dfrac{1}{h^2}\left[6h\varphi_1''(z) + 4\varphi_0'''(z)\right. \\ \qquad + h\left(\dfrac{6\rho_0'(x(z))}{\rho_0^2(x(z))} + \dfrac{2}{\rho_0(x(z))}\dfrac{b'(x(z))}{b(x(z))}\right)(3h\varphi_1'(z) + 3\varphi_0''(z)) \\ \qquad + \dfrac{h^2}{\rho_0^4(x(z))}\left(\rho_0^2(x(z))\dfrac{b''(x(z))}{b(x(z))} + 6\rho_0(x(z))\rho_0'(x(z))\dfrac{b'(x(z))}{b(x(z))}\right. \\ \qquad\qquad \left. + 4\rho_0(x(z))\rho_0''(x(z)) + 3(\rho_0'(x(z)))^2\right)(h\varphi_1(z) + 2\varphi_0'(z)) \\ \qquad \left. + \dfrac{h^3\varphi_0(z)}{\rho_0^4(x(z))}\left(\rho_0'''(x(z)) + 2\rho_0''(x(z))\dfrac{b'(x(z))}{b(x(z))} + \rho_0'(x(z))\dfrac{b''(x(z))}{b(x(z))}\right)\right], \\ \rho_4(z) = \dfrac{1}{h^3}\left[6h^2\varphi_2''(z) + 4h\varphi_1'''(z) + \varphi_0^{(4)}(z) - h^4\dfrac{a(x(z))}{b(x(z))}\varphi_0(z)\right. \\ \qquad + h\left(\dfrac{6\rho_0'(x(z))}{\rho_0^2(x(z))} + \dfrac{2}{\rho_0(x(z))}\dfrac{b'(x(z))}{b(x(z))}\right)\left(3h^2\varphi_2'(z) + 3h\varphi_1''(z) + \varphi_0'''(z)\right) \\ \qquad + \dfrac{h^2}{\rho_0^4(x(z))}\left(\rho_0^2(x(z))\dfrac{b''(x(z))}{b(x(z))} + 6\rho_0(x(z))\rho_0'(x(z))\dfrac{b'(x(z))}{b(x(z))}\right. \\ \qquad \left. + 4\rho_0(x(z))\rho_0''(x(z)) + 3(\rho_0'(x(z)))^2\right)(h^2\varphi_2(z) + 2h\varphi_1'(z) + \varphi_0''(z)) \\ \qquad + \dfrac{h^3(h\varphi_1(z) + \varphi_0'(z))}{\rho_0^4(x(z))}\left(\rho_0'''(x(z)) + 2\rho_0''(x(z))\dfrac{b'(x(z))}{b(x(z))}\right. \\ \qquad\qquad \left.\left. + \rho_0'(x(z))\dfrac{b''(x(z))}{b(x(z))}\right)\right]. \end{cases} \tag{3.4.15}$$

Proof. The first claim is trivial. We only need to show that (3.4.13) are the asymptotic fundamental solutions of (3.4.7). This can be done along the same way of [11] and [[78], Section 4]. Here we present briefly a simple calculation to (3.4.13), (3.4.14) and (3.4.15).

Let

$$\widetilde{\varphi}_k(z,\mu) := e^{\mu h\omega_k z}\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right],$$

where $\varphi_{kj}(z), j=0,1,2,3$ are some functions to be determined, and

$$\begin{aligned}
D(\varphi) &:= \varphi^{(4)}(z)\\
&+\frac{h}{\rho_0^4(x)}\left[6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)(a'(x)+\lambda b'(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\right]\varphi'''(z)\\
&+\frac{h^2}{\rho_0^4(x)}\Big[3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)\\
&\qquad+6\rho_0(x)\rho_0'(x)(a'(x)+\lambda b'(x))\left(\tfrac{1}{\lambda b(x)}-\tfrac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\\
&\qquad+\rho_0^2(x)(a''(x)+\lambda b''(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\Big]\varphi''(z)\\
&+\frac{h^3}{\rho_0^4(x)}\Big[\rho_0'''(x)+2\rho_0''(x)(a'(x)+\lambda b'(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\\
&\qquad+\rho_0'(x)(a''(x)+\lambda b''(x))\left(\frac{1}{\lambda b(x)}-\frac{a(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right)\Big]\varphi'(z)\\
&+\lambda h^4\left[1-\frac{a(x)}{\lambda b(x)}+\frac{a^2(x)}{\lambda^2b^2(x)}+\mathcal{O}(|\lambda|^{-3})\right]\varphi(z).
\end{aligned}$$

Substitute $\widetilde{\varphi}_k(z,\mu), k=1,2,3,4$, into $D(\varphi)$ respectively, to yield

$$\begin{aligned}
&e^{-\mu h\omega_k z}D(\widetilde{\varphi}_k(z,\mu))\\
=\;&\mu^4h^4\omega_k^4\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right]\\
&+4\mu^3h^3\omega_k^3\left[\varphi_{k0}'(z)+\varphi_{k1}'(z)\mu^{-1}+\varphi_{k2}'(z)\mu^{-2}+\varphi_{k3}'(z)\mu^{-3}\right]\\
&+6\mu^2h^2\omega_k^2\left[\varphi_{k0}''(z)+\varphi_{k1}''(z)\mu^{-1}+\varphi_{k2}''(z)\mu^{-2}+\varphi_{k3}''(z)\mu^{-3}\right]\\
&+4\mu h\omega_k\left[\varphi_{k0}'''(z)+\varphi_{k1}'''(z)\mu^{-1}+\varphi_{k2}'''(z)\mu^{-2}+\varphi_{k3}'''(z)\mu^{-3}\right]\\
&+\left[\varphi_{k0}^{(4)}(z)+\varphi_{k1}^{(4)}(z)\mu^{-1}+\varphi_{k2}^{(4)}(z)\mu^{-2}+\varphi_{k3}^{(4)}(z)\mu^{-3}\right]\\
&+\frac{h}{\rho_0^4(x)}\Big[2\rho_0^3(x)(a'(x)+\mu^4b'(x))\left(\frac{1}{\mu^4b(x)}-\frac{a(x)}{\mu^8b^2(x)}+\mathcal{O}(|\mu|^{-12})\right)\\
&\quad+6\rho_0^2(x)\rho_0'(x)\Big]\cdot\Big\{\mu^3h^3\omega_k^3\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right]\\
&\qquad+3\mu^2h^2\omega_k^2\left[\varphi_{k0}'(z)+\varphi_{k1}'(z)\mu^{-1}+\varphi_{k2}'(z)\mu^{-2}+\varphi_{k3}'(z)\mu^{-3}\right]\\
&\qquad+3\mu h\omega_k\left[\varphi_{k0}''(z)+\varphi_{k1}''(z)\mu^{-1}+\varphi_{k2}''(z)\mu^{-2}+\varphi_{k3}''(z)\mu^{-3}\right]\\
&\qquad+\left[\varphi_{k0}'''(z)+\varphi_{k1}'''(z)\mu^{-1}+\varphi_{k2}'''(z)\mu^{-2}+\varphi_{k3}'''(z)\mu^{-3}\right]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h^2}{\rho_0^4(x)}\left[3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)\right.\\
&\qquad+6\rho_0(x)\rho_0'(x)(a'(x)+\mu^4 b'(x))\left(\frac{1}{\mu^4 b(x)}-\frac{a(x)}{\mu^8 b^2(x)}+\mathcal{O}(|\mu|^{-12})\right)\\
&\qquad\left.+\rho_0^2(x)(a''(x)+\mu^4 b''(x))\left(\frac{1}{\mu^4 b(x)}-\frac{a(x)}{\mu^8 b^2(x)}+\mathcal{O}(|\mu|^{-12})\right)\right]\cdot\\
&\left\{\mu^2h^2\omega_k^2\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right]\right.\\
&\qquad+2\mu h\omega_k\left[\varphi_{k0}'(z)+\varphi_{k1}'(z)\mu^{-1}+\varphi_{k2}'(z)\mu^{-2}+\varphi_{k3}'(z)\mu^{-3}\right]\\
&\qquad\left.+\left[\varphi_{k0}''(z)+\varphi_{k1}''(z)\mu^{-1}+\varphi_{k2}''(z)\mu^{-2}+\varphi_{k3}''(z)\mu^{-3}\right]\right\}\\
&+\frac{h^3}{\rho_0^4(x)}\left[\rho_0'''(x)+2\rho_0''(x)(a'(x)+\mu^4 b'(x))\left(\frac{1}{\mu^4 b(x)}-\frac{a(x)}{\mu^8 b^2(x)}+\mathcal{O}(|\mu|^{-12})\right)\right.\\
&\qquad\left.+\rho_0'(x)(a''(x)+\mu^4 b''(x))\left(\frac{1}{\mu^4 b(x)}-\frac{a(x)}{\mu^8 b^2(x)}+\mathcal{O}(|\mu|^{-12})\right)\right]\cdot\\
&\left\{\mu h\omega_k\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right]\right.\\
&\qquad\left.+\left[\varphi_{k0}'(z)+\varphi_{k1}'(z)\mu^{-1}+\varphi_{k2}'(z)\mu^{-2}+\varphi_{k3}'(z)\mu^{-3}\right]\right\}\\
&+\mu^4h^4\left[1-\frac{a(x)}{\mu^4 b(x)}+\frac{a^2(x)}{\mu^8 b^2(x)}+\mathcal{O}(|\mu|^{-12})\right]\left[\varphi_{k0}(z)+\varphi_{k1}(z)\mu^{-1}\right.\\
&\qquad\left.+\varphi_{k2}(z)\mu^{-2}+\varphi_{k3}(z)\mu^{-3}\right]\\
=\;&\mu^3\left[4h^3\omega_k^3\varphi_{k0}'(z)+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)h^3\omega_k^3\varphi_{k0}(z)\right]\\
&+\mu^2\left[4h^3\omega_k^3\varphi_{k1}'(z)+6h^2\omega_k^2\varphi_{k0}''(z)\right.\\
&+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\left(h^3\omega_k^3\varphi_{k1}(z)+3h^2\omega_k^2\varphi_{k0}'(z)\right)\\
&\left.+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)h^2\omega_k^2\varphi_{k0}(z)\right]\\
&+\mu\left[4h^3\omega_k^3\varphi_{k2}'(z)+6h^2\omega_k^2\varphi_{k1}''(z)+4h\omega_k\varphi_{k0}'''(z)\right.\\
&+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\left(h^3\omega_k^3\varphi_{k2}(z)+3h^2\omega_k^2\varphi_{k1}'(z)+3h\omega_k\varphi_{k0}''(z)\right)\\
&+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)\cdot\\
&\qquad\left(h^2\omega_k^2\varphi_{k1}(z)+2h\omega_k\varphi_{k0}'(z)\right)\\
&\left.+\frac{h^3}{\rho_0^4(x)}\left(\rho_0'''(x)+2\rho_0''(x)\frac{b'(x)}{b(x)}+\rho_0'(x)\frac{b''(x)}{b(x)}\right)h\omega_k\varphi_{k0}(z)\right]\\
&+\left[4h^3\omega_k^3\varphi_{k3}'(z)+6h^2\omega_k^2\varphi_{k2}''(z)+4h\omega_k\varphi_{k1}'''(z)+\varphi_{k0}^{(4)}(z)-h^4\frac{a(x)}{b(x)}\varphi_{k0}(z)\right.\\
&+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\cdot\\
&\qquad\left(h^3\omega_k^3\varphi_{k3}(z)+3h^2\omega_k^2\varphi_{k2}'(z)+3h\omega_k\varphi_{k1}''(z)+\varphi_{k0}'''(z)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)\cdot\\
&\qquad(h^2\omega_k^2\varphi_{k2}(z)+2h\omega_k\varphi_{k1}'(z)+\varphi_{k0}''(z))\\
&+\ \frac{h^3}{\rho_0^4(x)}\left[\rho_0'''(x)+2\rho_0''(x)\frac{b'(x)}{b(x)}+\rho_0'(x)\frac{b''(x)}{b(x)}\right](h\omega_k\varphi_{k1}(z)+\varphi_{k0}'(z))\Bigg]\\
&+\mathcal{O}(|\mu|^{-1}).
\end{aligned}
$$

Setting the coefficients of $\mu^j, j=0,1,2,3$ be zero gives

$$
4h^3\omega_k^3\varphi_{k0}'(z)+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)h^3\omega_k^3\varphi_{k0}(z)=0,
$$

$$
\begin{aligned}
&4h^3\omega_k^3\varphi_{k1}'(z)+6h^2\omega_k^2\varphi_{k0}''(z)\\
&\quad+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\left(h^3\omega_k^3\varphi_{k1}(z)+3h^2\omega_k^2\varphi_{k0}'(z)\right)\\
&\quad+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)h^2\omega_k^2\varphi_{k0}(z)=0,
\end{aligned}
$$

$$
\begin{aligned}
&4h^3\omega_k^3\varphi_{k2}'(z)+6h^2\omega_k^2\varphi_{k1}''(z)+4h\omega_k\varphi_{k0}'''(z)\\
&\quad+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\left(h^3\omega_k^3\varphi_{k2}(z)+3h^2\omega_k^2\varphi_{k1}'(z)+3h\omega_k\varphi_{k0}''(z)\right)\\
&\quad+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)\cdot\\
&\qquad(h^2\omega_k^2\varphi_{k1}(z)+2h\omega_k\varphi_{k0}'(z))\\
&\quad+\frac{h^3}{\rho_0^4(x)}\left(\rho_0'''(x)+2\rho_0''(x)\frac{b'(x)}{b(x)}+\rho_0'(x)\frac{b''(x)}{b(x)}\right)h\omega_k\varphi_{k0}(z)=0
\end{aligned}
$$

and

$$
\begin{aligned}
&4h^3\omega_k^3\varphi_{k3}'(z)+6h^2\omega_k^2\varphi_{k2}''(z)+4h\omega_k\varphi_{k1}'''(z)+\varphi_{k0}^{(4)}(z)-h^4\frac{a(x)}{b(x)}\varphi_{k0}(z)\\
&\quad+\frac{h}{\rho_0^4(x)}\left(6\rho_0^2(x)\rho_0'(x)+2\rho_0^3(x)\frac{b'(x)}{b(x)}\right)\cdot\\
&\qquad\left(h^3\omega_k^3\varphi_{k3}(z)+3h^2\omega_k^2\varphi_{k2}'(z)+3h\omega_k\varphi_{k1}''(z)+\varphi_{k0}'''(z)\right)\\
&\quad+\frac{h^2}{\rho_0^4(x)}\left(3(\rho_0'(x))^2+4\rho_0(x)\rho_0''(x)+6\rho_0(x)\rho_0'(x)\frac{b'(x)}{b(x)}+\rho_0^2(x)\frac{b''(x)}{b(x)}\right)\cdot\\
&\qquad(h^2\omega_k^2\varphi_{k2}(z)+2h\omega_k\varphi_{k1}'(z)+\varphi_{k0}''(z))\\
&\quad+\frac{h^3}{\rho_0^4(x)}\left[\rho_0'''(x)+2\rho_0''(x)\frac{b'(x)}{b(x)}+\rho_0'(x)\frac{b''(x)}{b(x)}\right](h\omega_k\varphi_{k1}(z)+\varphi_{k0}'(z))=0.
\end{aligned}
$$

Use the conditions $\varphi_{k0}(0)=1$ and $\varphi_{ki}(0)=0, i=1,2,3$ to obtain

$$\begin{aligned}
\varphi_{k0}(z) &= \exp\left\{-\frac{h}{2}\int_0^z \rho_1(x(\tau))d\tau\right\},\\
\varphi_{k1}(z) &= \frac{1}{\omega_k}\left[-\frac{1}{4}\int_0^z \exp\left\{-\frac{h}{2}\int_0^{z-\zeta}\rho_1(x(\tau))d\tau\right\}\rho_{k2}(\zeta)d\zeta\right],\\
\varphi_{k2}(z) &= \frac{1}{\omega_k^2}\left[-\frac{1}{4}\int_0^z \exp\left\{-\frac{h}{2}\int_0^{z-\zeta}\rho_1(x(\tau))d\tau\right\}\rho_{k3}(\zeta)d\zeta\right],\\
\varphi_{k3}(z) &= \frac{1}{\omega_k^3}\left[-\frac{1}{4}\int_0^z \exp\left\{-\frac{h}{2}\int_0^{z-\zeta}\rho_1(x(\tau))d\tau\right\}\rho_{k4}(\zeta)d\zeta\right],
\end{aligned}$$

where

$$\begin{aligned}
\rho_1(x) &= \frac{3\rho_0'(x)}{\rho_0^2(x)}+\frac{1}{\rho_0(x)}\frac{b'(x)}{b(x)},\\
\rho_{k2}(z) &= \frac{6}{h}\varphi_{k0}''(z)+\left(\frac{18\rho_0'(x(z))}{\rho_0^2(x(z))}+\frac{6}{\rho_0(x(z))}\frac{b'(x(z))}{b(x(z))}\right)\varphi_{k0}'(z)\\
&\quad+\frac{h\varphi_{k0}(z)}{\rho_0^4(x(z))}\left(3(\rho_0'(x(z)))^2+4\rho_0(x(z))\rho_0''(x(z))+6\rho_0(x(z))\rho_0'(x(z))\frac{b'(x(z))}{b(x(z))}\right.\\
&\qquad\left.+\,\rho_0^2(x(z))\frac{b''(x(z))}{b(x(z))}\right),\\
\rho_{k3}(z) &= \frac{1}{h^2}\left[6h\omega_k\varphi_{k1}''(z)+4\varphi_{k0}'''(z)\right.\\
&\quad+h\left(\frac{6\rho_0'(x(z))}{\rho_0^2(x(z))}+\frac{2}{\rho_0(x(z))}\frac{b'(x(z))}{b(x(z))}\right)(3h\omega_k\varphi_{k1}'(z)+3\varphi_{k0}''(z))\\
&\quad+\frac{h^2}{\rho_0^4(x(z))}\left(3(\rho_0'(x(z)))^2+4\rho_0(x(z))\rho_0''(x(z))+6\rho_0(x(z))\rho_0'(x(z))\frac{b'(x(z))}{b(x(z))}\right.\\
&\qquad\left.+\,\rho_0^2(x(z))\frac{b''(x(z))}{b(x(z))}\right)(h\omega_k\varphi_{k1}(z)+2\varphi_{k0}'(z))\\
&\quad\left.+\,\frac{h^3\varphi_{k0}(z)}{\rho_0^4(x(z))}\left(\rho_0'''(x(z))+2\rho_0''(x(z))\frac{b'(x(z))}{b(x(z))}+\rho_0'(x(z))\frac{b''(x(z))}{b(x(z))}\right)\right],\\
\rho_{k4}(z) &= \frac{1}{h^3}\left[6h^2\omega_k^2\varphi_{k2}''(z)+4h\omega_k\varphi_{k1}'''(z)+\varphi_{k0}^{(4)}(z)-h^4\frac{a(x(z))}{b(x(z))}\varphi_{k0}(z)\right.\\
&\quad+h\left(\frac{6\rho_0'(x(z))}{\rho_0^2(x(z))}+\frac{2}{\rho_0(x(z))}\frac{b'(x(z))}{b(x(z))}\right)\left(3h^2\omega_k^2\varphi_{k2}'(z)+3h\omega_k\varphi_{k1}''(z)+\varphi_{k0}'''(z)\right)\\
&\quad+\frac{h^2}{\rho_0^4(x(z))}\left(3(\rho_0'(x(z)))^2+4\rho_0(x(z))\rho_0''(x(z))+6\rho_0(x(z))\rho_0'(x(z))\frac{b'(x(z))}{b(x(z))}\right.\\
&\qquad\left.+\,\rho_0^2(x(z))\frac{b''(x(z))}{b(x(z))}\right)(h^2\omega_k^2\varphi_{k2}(z)+2h\omega_k\varphi_{k1}'(z)+\varphi_{k0}''(z))\\
&\quad+\frac{h^3(h\omega_k\varphi_{k1}(z)+\varphi_{k0}'(z))}{\rho_0^4(x(z))}\left(\rho_0'''(x(z))+2\rho_0''(x(z))\frac{b'(x(z))}{b(x(z))}\right.\\
&\qquad\left.\left.+\,\rho_0'(x(z))\frac{b''(x(z))}{b(x(z))}\right)\right].
\end{aligned}$$

From (3.4.6), we have

$$\frac{dx}{dz} = \frac{h}{\rho_0(x)}.$$

Hence

$$\begin{aligned}\int_0^z \rho_1(x(\tau))d\tau &= \frac{1}{h}\int_0^z \rho_1(x(\tau))\rho_0(x(\tau))\frac{dx}{d\tau}d\tau \\ &= \frac{1}{h}\int_0^z \rho_1(x)\rho_0(x)dx.\end{aligned}$$

Therefore

$$\varphi_{kj}(z) = \frac{1}{\omega_k^j}\varphi_j(z),\ j = 0, 1, 2, 3,$$

where $\varphi_j(z)$, $j = 0, 1, 2, 3$ are given by (3.4.14) and (3.4.15). The proof is complete. ■

Next, we consider the asymptotic behavior of eigenvalues λ of $\mathcal{A}$. Let $\lambda = \mu^4$ and $\mu \in \mathcal{S}$ defined by (3.4.9). Let φ be a solution of (3.4.7). There are constants c_k, $k = 1, 2, 3, 4$, such that

$$\varphi(z) = c_1\varphi_1(z) + c_2\varphi_2(z) + c_3\varphi_3(z) + c_4\varphi_4(z), \tag{3.4.16}$$

where $\varphi_k(z)$, $k = 1, 2, 3, 4$ are fundamental solutions given by (3.4.13), (3.4.14), (3.4.15), and (3.4.5). By the boundary conditions of (3.4.8), we have

$$\Delta(\mu)[c_1, c_2, c_3, c_4]^\top = 0,$$

where

$$\Delta(\mu) = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi_1'(0) & \varphi_2'(0) & \varphi_3'(0) & \varphi_4'(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \\ \varphi_1'(1) & \varphi_2'(1) & \varphi_3'(1) & \varphi_4'(1) \end{bmatrix}. \tag{3.4.17}$$

Hence, $\varphi(z)$ has a nontrivial solution if and only if

$$\det(\Delta(\mu)) = 0.$$

Set

$$\begin{aligned}\Phi_k(1) = \omega_k\varphi_0(1) + [\varphi_1(1) + \frac{1}{h}\varphi_0'(1)]\mu^{-1} \\ + \frac{1}{\omega_k}[\varphi_2(1) + \frac{1}{h}\varphi_1'(1)]\mu^{-2} + \frac{1}{\omega_k^2}[\varphi_3(1) + \frac{1}{h}\varphi_2'(1)]\mu^{-3}, \quad k = 1, 2, 3, 4\end{aligned}$$

and

$$[a]_4 = a + \mathcal{O}(\mu^{-4}), \quad \forall a \in \mathbb{C}.$$

Then $\mu \in \mathcal{S}$ satisfies the characteristic equation

$$\det(\Delta(\mu)) = \det\left[\Delta_1(\mu),\ \Delta_2(\mu),\ \Delta_3(\mu),\ \Delta_4(\mu)\right] = 0,$$

where

$$\Delta_k(\mu) := \begin{bmatrix} [1]_4 \\ \mu h\left[\omega_k + \frac{1}{h}\varphi_0'(0)\mu^{-1} + \frac{1}{h}\frac{1}{\omega_k}\varphi_1'(0)\mu^{-2} + \frac{1}{h}\frac{1}{\omega_k^2}\varphi_2'(0)\mu^{-3}\right]_4 \\ e^{\mu h\omega_k}\left[\varphi_0(1) + \frac{1}{\omega_k}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_k^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_k^3}\varphi_3(1)\mu^{-3}\right]_4 \\ \mu h e^{\mu h\omega_k}[\Phi_k(1)]_4 \end{bmatrix},\ k = 1, 2, 3, 4.$$

Now we compute the high eigenvalues. A simple computation gives that

$$\det(\Delta(\mu)) = \mu^2 h^2 e^{\mu h(\omega_3+\omega_4)}\left\{\det(\widetilde{\Delta}(\mu)) + \mathcal{O}(\mu^{-4})\right\},$$

where

$$\widetilde{\Delta}(\mu) = \left[\widetilde{\Delta}_1(\mu),\ \widetilde{\Delta}_2(\mu),\ \widetilde{\Delta}_3(\mu),\ \widetilde{\Delta}_4(\mu)\ \right],$$

$$\widetilde{\Delta}_1(\mu) := \begin{bmatrix} 1 \\ \omega_1 + \frac{1}{h}\varphi_0'(0)\mu^{-1} + \frac{1}{h}\frac{1}{\omega_1}\varphi_1'(0)\mu^{-2} + \frac{1}{h}\frac{1}{\omega_1^2}\varphi_2'(0)\mu^{-3} \\ 0 \\ 0 \end{bmatrix},$$

$$\widetilde{\Delta}_2(\mu) := \begin{bmatrix} 1 \\ \omega_2 + \frac{1}{h}\varphi_0'(0)\mu^{-1} + \frac{1}{h}\frac{1}{\omega_2}\varphi_1'(0)\mu^{-2} + \frac{1}{h}\frac{1}{\omega_2^2}\varphi_2'(0)\mu^{-3} \\ e^{\mu h\omega_2}\left[\varphi_0(1) + \frac{1}{\omega_2}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_2^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_2^3}\varphi_3(1)\mu^{-3}\right] \\ e^{\mu h\omega_2}\Phi_2(1) \end{bmatrix},$$

$$\widetilde{\Delta}_3(\mu) := \begin{bmatrix} e^{\mu h\omega_2} \\ e^{\mu h\omega_2}\left[\omega_3 + \frac{1}{h}\varphi_0'(0)\mu^{-1} + \frac{1}{h}\frac{1}{\omega_3}\varphi_1'(0)\mu^{-2} + \frac{1}{h}\frac{1}{\omega_3^2}\varphi_2'(0)\mu^{-3}\right] \\ \varphi_0(1) + \frac{1}{\omega_3}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_3^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_3^3}\varphi_3(1)\mu^{-3} \\ \Phi_3(1) \end{bmatrix},$$

$$\widetilde{\Delta}_4(\mu) := \begin{bmatrix} 0 \\ 0 \\ \varphi_0(1) + \frac{1}{\omega_4}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_4^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_4^3}\varphi_3(1)\mu^{-3} \\ \Phi_4(1) \end{bmatrix}.$$

Notice that

$$\det(\widetilde{\Delta}(\mu)) = \det\left[\widehat{\Delta}_1(\mu),\ \widehat{\Delta}_2(\mu),\ \widehat{\Delta}_3(\mu)\right],$$

where

$$\widehat{\Delta}_1(\mu) := \begin{bmatrix} (\omega_2 - \omega_1) + \frac{1}{h}\varphi_1'(0)[\frac{1}{\omega_2} - \frac{1}{\omega_1}]\mu^{-2} + \frac{1}{h}\varphi_2'(0)[\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}]\mu^{-3} \\ e^{\mu h \omega_2}\left[\varphi_0(1) + \frac{1}{\omega_2}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_2^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_2^3}\varphi_3(1)\mu^{-3}\right] \\ e^{\mu h \omega_2}\Phi_2(1) \end{bmatrix},$$

$$\widehat{\Delta}_2(\mu) := \begin{bmatrix} e^{\mu h \omega_2}\left[-(\omega_2 + \omega_1) + \frac{1}{h}\varphi_1'(0)[\frac{1}{\omega_3} - \frac{1}{\omega_1}]\mu^{-2} + \frac{1}{h}\varphi_2'(0)[\frac{1}{\omega_3^2} - \frac{1}{\omega_1^2}]\mu^{-3}\right] \\ \varphi_0(1) + \frac{1}{\omega_3}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_3^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_3^3}\varphi_3(1)\mu^{-3} \\ \Phi_3(1) \end{bmatrix}$$

and

$$\widehat{\Delta}_3(\mu) := \begin{bmatrix} 0 \\ \varphi_0(1) + \frac{1}{\omega_4}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_4^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_4^3}\varphi_3(1)\mu^{-3} \\ \Phi_4(1) \end{bmatrix}.$$

Thus,

$$\begin{aligned}
&\det(\widetilde{\Delta}(\mu)) \\
&= \left[(\omega_2 - \omega_1) + \frac{1}{h}\varphi_1'(0)[\frac{1}{\omega_2} - \frac{1}{\omega_1}]\mu^{-2} + \frac{1}{h}\varphi_2'(0)[\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}]\mu^{-3}\right] \cdot \\
&\quad \left[(\varphi_0(1) + \frac{1}{\omega_3}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_3^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_3^3}\varphi_3(1)\mu^{-3})\Phi_4(1)\right. \\
&\qquad \left. - (\varphi_0(1) + \tfrac{1}{\omega_4}\varphi_1(1)\mu^{-1} + \tfrac{1}{\omega_4^2}\varphi_2(1)\mu^{-2} + \tfrac{1}{\omega_4^3}\varphi_3(1)\mu^{-3})\Phi_3(1)\right] \\
&- e^{2\mu h \omega_2}\left[-(\omega_2 + \omega_1) + \frac{1}{h}\varphi_1'(0)[\frac{1}{\omega_3} - \frac{1}{\omega_1}]\mu^{-2} + \frac{1}{h}\varphi_2'(0)[\frac{1}{\omega_3^2} - \frac{1}{\omega_1^2}]\mu^{-3}\right] \cdot \\
&\quad \left[(\varphi_0(1) + \frac{1}{\omega_2}\varphi_1(1)\mu^{-1} + \frac{1}{\omega_2^2}\varphi_2(1)\mu^{-2} + \frac{1}{\omega_2^3}\varphi_3(1)\mu^{-3})\Phi_4(1)\right. \\
&\qquad \left. - (\varphi_0(1) + \tfrac{1}{\omega_4}\varphi_1(1)\mu^{-1} + \tfrac{1}{\omega_4^2}\varphi_2(1)\mu^{-2} + \tfrac{1}{\omega_4^3}\varphi_3(1)\mu^{-3})\Phi_2(1)\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\omega_2-\omega_1)\left[1-\frac{1}{h}\varphi_1'(0)\mu^{-2}-\frac{1}{h}\varphi_2'(0)(\omega_2+\omega_1)\mu^{-3}\right]\cdot \\
&\qquad \left[(\varphi_0(1)+\frac{1}{\omega_3}\varphi_1(1)\mu^{-1}+\frac{1}{\omega_3^2}\varphi_2(1)\mu^{-2}+\frac{1}{\omega_3^3}\varphi_3(1)\mu^{-3})\Phi_4(1)\right. \\
&\qquad\quad \left.-(\varphi_0(1)+\frac{1}{\omega_4}\varphi_1(1)\mu^{-1}+\frac{1}{\omega_4^2}\varphi_2(1)\mu^{-2}+\frac{1}{\omega_4^3}\varphi_3(1)\mu^{-3})\Phi_3(1)\right] \\
&\quad +(\omega_2+\omega_1)e^{2\mu h\omega_2}\left[1+\frac{1}{h}\varphi_1'(0)\mu^{-2}+\frac{1}{h}\varphi_2'(0)(\omega_2-\omega_1)\mu^{-3}\right]\cdot \\
&\qquad \left[(\varphi_0(1)+\frac{1}{\omega_2}\varphi_1(1)\mu^{-1}+\frac{1}{\omega_2^2}\varphi_2(1)\mu^{-2}+\frac{1}{\omega_2^3}\varphi_3(1)\mu^{-3})\Phi_4(1)\right. \\
&\qquad\quad \left.-(\varphi_0(1)+\frac{1}{\omega_4}\varphi_1(1)\mu^{-1}+\frac{1}{\omega_4^2}\varphi_2(1)\mu^{-2}+\frac{1}{\omega_4^3}\varphi_3(1)\mu^{-3})\Phi_2(1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (\omega_2-\omega_1)^2\left[\varphi_0^2(1)-(\omega_2+\omega_1)\varphi_0(1)\varphi_1(1)\mu^{-1}\right. \\
&\qquad +\left[-\frac{1}{h}\varphi_0^2(1)\varphi_1'(0)-\varphi_0(1)[\varphi_2(1)+\frac{1}{h}\varphi_1'(1)]\right. \\
&\qquad\qquad \left.+\varphi_1(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]+\varphi_0(1)\varphi_2(1)\right]\mu^{-2} \\
&\qquad +(\omega_2+\omega_1)\left[\frac{1}{h}\varphi_0(1)\varphi_1(1)\varphi_1'(0)-\frac{1}{h}\varphi_0^2(1)\varphi_2'(0)\right. \\
&\qquad\qquad \left.\left.+\varphi_0(1)[\varphi_3(1)+\frac{1}{h}\varphi_2'(1)]-\varphi_2(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]\right]\mu^{-3}\right] \\
&\quad -(\omega_2+\omega_1)^2e^{2\mu h\omega_2}\left[\varphi_0^2(1)-(\omega_2-\omega_1)\varphi_0(1)\varphi_1(1)\mu^{-1}\right. \\
&\qquad +\left[\frac{1}{h}\varphi_0^2(1)\varphi_1'(0)+\varphi_0(1)[\varphi_2(1)+\frac{1}{h}\varphi_1'(1)]\right. \\
&\qquad\qquad \left.-\varphi_1(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]-\varphi_0(1)\varphi_2(1)\right]\mu^{-2} \\
&\qquad -(\omega_2-\omega_1)\left[\frac{1}{h}\varphi_0(1)\varphi_1(1)\varphi_1'(0)-\frac{1}{h}\varphi_0^2(1)\varphi_2'(0)\right. \\
&\qquad\qquad \left.\left.+\varphi_0(1)[\varphi_3(1)+\frac{1}{h}\varphi_2'(1)]-\varphi_2(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]\right]\mu^{-3}\right].
\end{aligned}
$$

By the facts

$$\omega_2-\omega_1=-\sqrt{2}i,\ \omega_2+\omega_1=-\sqrt{2},$$

we get the following Lemma 3.4.3.

Lemma 3.4.3. *Let $\Delta(\mu)$ be given by (3.4.17). Then the characteristic determinant* $\det(\Delta(\mu))$ *has the following asymptotic expression:*

$$
\begin{aligned}
\det(\Delta(\mu)) = -2\varphi_0^2(1)\mu^2h^2e^{\sqrt{2}\mu h}\left\{[1+\sqrt{2}A_0\mu^{-1}-A_1\mu^{-2}-\sqrt{2}A_2\mu^{-3}]\right. \\
\left.+e^{2\mu h\omega_2}[1+\sqrt{2}iA_0\mu^{-1}+A_1\mu^{-2}+\sqrt{2}iA_2\mu^{-3}]+\mathcal{O}(\mu^{-4})\right\},
\end{aligned}
\tag{3.4.18}
$$

where

$$
\begin{aligned}
A_0 &= \frac{\varphi_1(1)}{\varphi_0(1)},\\
A_1 &= \frac{1}{\varphi_0^2(1)}\left\{\frac{1}{h}\varphi_0^2(1)\varphi_1'(0)+\varphi_0(1)[\varphi_2(1)+\frac{1}{h}\varphi_1'(1)]\right.\\
&\qquad\left.-\varphi_1(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]-\varphi_0(1)\varphi_2(1)\right\}, \qquad (3.4.19)\\
A_2 &= \frac{1}{\varphi_0^2(1)}\left\{\frac{1}{h}\varphi_0(1)\varphi_1(1)\varphi_1'(0)-\frac{1}{h}\varphi_0^2(1)\varphi_2'(0)\right.\\
&\qquad\left.+\varphi_0(1)[\varphi_3(1)+\frac{1}{h}\varphi_2'(1)]-\varphi_2(1)[\varphi_1(1)+\frac{1}{h}\varphi_0'(1)]\right\}.
\end{aligned}
$$

Theorem 3.4.1. *Let $\lambda=\mu^4$ satisfy (3.4.3) with $\mu\in\mathcal{S}$ defined by (3.4.9). Then λ has the following asymptotic expansion:*

$$
\begin{aligned}
\lambda_n &= \left(\widetilde{\mu}_n+k_0\widetilde{\mu}_n^{-1}+k_1\widetilde{\mu}_n^{-2}+k_2\widetilde{\mu}_n^{-3}+\mathcal{O}(n^{-4})\right)^4\\
&= -\frac{1}{h^4}\left(n+\frac{1}{2}\right)^4\pi^4-\frac{1}{h^3}(2n+1)^2\pi^2A_0+\frac{2}{h^2}(2n+1)\pi(A_0^2+A_1) \qquad (3.4.20)\\
&\quad+\frac{4}{h}A_2-\frac{4}{h}A_0A_1-\frac{2}{h^2}A_0^2-\frac{8}{3h}A_0^3+\mathcal{O}(n^{-1}),\ n\to\infty,
\end{aligned}
$$

where

$$
\begin{cases}
\widetilde{\mu}_n = -\dfrac{(2n+1)\pi i}{2h\omega_2},\\
k_0 = \dfrac{\sqrt{2}(1-i)}{2h\omega_2}A_0,\\
k_1 = -\dfrac{1}{h\omega_2}(A_0^2+A_1),\\
k_2 = -\dfrac{\sqrt{2}(1+i)}{2h\omega_2}A_2+\dfrac{1}{2h\omega_2}\left[\sqrt{2}(1+i)A_0A_1+\dfrac{2i}{h\omega_2}A_0^2+\dfrac{2\sqrt{2}(1+i)}{3}A_0^3\right]
\end{cases}
\qquad (3.4.21)
$$

and A_k, $k=0,1,2$ are given by (3.4.19), (3.4.5), (3.4.14), and (3.4.15).

Proof. Since in sector $\mathcal{S}$, $\det(\Delta(\mu))$ has the asymptotic form given by (3.4.18), $\det(\Delta(\mu))=0$ yields

$$
\begin{aligned}
&[1+\sqrt{2}A_0\mu^{-1}-A_1\mu^{-2}-\sqrt{2}A_2\mu^{-3}]\\
&+e^{2\mu h\omega_2}[1+\sqrt{2}iA_0\mu^{-1}+A_1\mu^{-2}+\sqrt{2}iA_2\mu^{-3}]+\mathcal{O}(\mu^{-4})=0,
\end{aligned}
\qquad (3.4.22)
$$

which can also be rewritten as

$$[1+\sqrt{2}A_0\mu^{-1}]+e^{2\mu h\omega_2}[1+\sqrt{2}iA_0\mu^{-1}]+\mathcal{O}(\mu^{-2})=0 \qquad (3.4.23)$$

or

$$1+e^{2\mu h\omega_2}+\mathcal{O}(\mu^{-1})=0. \qquad (3.4.24)$$

Since in sector $\mathcal{S}$, the solutions of $1 + e^{2\mu h\omega_2} = 0$ are given by

$$\widetilde{\mu}_n = -\frac{(2n+1)\pi i}{2h\omega_2}, \quad n = 0, 1, 2, \ldots,$$

it follows from Rouché's Theorem that the solutions of equation (3.4.24) have the form of

$$\widehat{\mu}_n = \widetilde{\mu}_n + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}),\ n \to \infty.$$

Substitute $\widehat{\mu}_n$ into (3.4.23) and use the fact $e^{2\widetilde{\mu}_n h\omega_2} = -1$ to obtain

$$[1 + \sqrt{2}A_0\widehat{\mu}_n^{-1}] - e^{2\alpha_n h\omega_2}[1 + \sqrt{2}iA_0\widehat{\mu}_n^{-1}] + \mathcal{O}(\widehat{\mu}_n^{-2}) = 0. \tag{3.4.25}$$

Since

$$\widehat{\mu}_n^{-1} = (\widetilde{\mu}_n + \alpha_n)^{-1} = \widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-3}),$$

expand the exponential function in (3.4.25) according to its Taylor series, to give

$$2\alpha_n h\omega_2 = \sqrt{2}A_0\widetilde{\mu}_n^{-1} - \sqrt{2}iA_0\widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-2}).$$

Hence

$$\alpha_n = k_0\widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-2})$$

and

$$\widehat{\mu}_n = \widetilde{\mu}_n + k_0\widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-2}),\ n \to \infty.$$

By Rouché's Theorem again, the solutions of equation (3.4.23) have the form of

$$\breve{\mu}_n = \widetilde{\mu}_n + k_0\widetilde{\mu}_n^{-1} + \beta_n, \quad \beta_n = \mathcal{O}(n^{-2}),\ n \to \infty.$$

Substitute $\breve{\mu}_n$ into (3.4.26) below

$$[1 + \sqrt{2}A_0\mu^{-1} - A_1\mu^{-2}] + e^{2\mu h\omega_2}[1 + \sqrt{2}iA_0\mu^{-1} + A_1\mu^{-2}] + \mathcal{O}(\mu^{-3}) = 0 \tag{3.4.26}$$

and use the fact

$$e^{2\widetilde{\mu}_n h\omega_2} = -1,\ \breve{\mu}_n^{-1} = \widetilde{\mu}_n^{-1} + \mathcal{O}(n^{-3})$$

to obtain

$$\begin{aligned}&[1 + \sqrt{2}A_0\widetilde{\mu}_n^{-1} - A_1\widetilde{\mu}_n^{-2}] \\ &\qquad - \exp\left\{\sqrt{2}(1-i)A_0\widetilde{\mu}_n^{-1} + 2h\omega_2\beta_n\right\}[1 + \sqrt{2}iA_0\widetilde{\mu}_n^{-1} + A_1\widetilde{\mu}_n^{-2}] + \mathcal{O}(n^{-3}) = 0.\end{aligned}$$

Similarly, we have

$$\beta_n = k_1\widetilde{\mu}_n^{-2} + \mathcal{O}(n^{-3})$$

and

$$\breve{\mu}_n = \widetilde{\mu}_n + k_0\widetilde{\mu}_n^{-1} + k_1\widetilde{\mu}_n^{-2} + \mathcal{O}(n^{-3}),\ n \to \infty.$$

By Rouché's Theorem again, the solutions of equation (3.4.26) have the form of

$$\mu_n = \widetilde{\mu}_n + k_0\widetilde{\mu}_n^{-1} + k_1\widetilde{\mu}_n^{-2} + \gamma_n, \quad \gamma_n = \mathcal{O}(n^{-3}),\ n \to \infty.$$

Substitute μ_n into (3.4.22), and notice

$$\mu_n^{-1} = \widetilde{\mu}_n^{-1} - k_0\widetilde{\mu}_n^{-3} + \mathcal{O}(n^{-4}),$$

to obtain that

$$\begin{aligned}
&\left[-2h\omega_2 k_0 + \sqrt{2}A_0(1-i)\right]\widetilde{\mu}_n^{-1} \\
&- \left[2h\omega_2 k_1 + 2h\omega_2 k_0\sqrt{2}iA_0 + 2A_1 + 2h^2\omega_2^2 k_0^2\right]\widetilde{\mu}_n^{-2} \\
&- \left\{2h\omega_2 k_2 + \sqrt{2}[A_0 k_0 + A_2] + \sqrt{2}i[-A_0 k_0 + A_2] + 2h\omega_2 k_0 A_1\right. \\
&\qquad \left. + \sqrt{2}iA_0[2h\omega_2 k_1 + 2h^2\omega_2^2 k_0^2] + 4h^2\omega_2^2 k_0 k_1 + \tfrac{4}{3}h^3\omega_2^3 k_0^3\right\}\widetilde{\mu}_n^{-3} \\
&+ \mathcal{O}(n^{-4}) = 0.
\end{aligned}$$

So, we have

$$\gamma_n = k_2\widetilde{\mu}_n^{-3} + \mathcal{O}(n^{-4})$$

and

$$\mu_n = \widetilde{\mu}_n + k_0\widetilde{\mu}_n^{-1} + k_1\widetilde{\mu}_n^{-2} + k_2\widetilde{\mu}_n^{-3} + \mathcal{O}(n^{-4}),\ n \to \infty.$$

We then get (3.4.20) eventually by $\lambda_n = \mu_n^4$. The proof is complete. ∎

Theorem 3.4.1 is about the asymptotic expression for high eigenfrequencies. To end this section, we indicate that the high eigenfrequencies are actually real.

Proposition 3.4.2. *Suppose that $b > 0$. Let $\mathcal{A}$ be defined by (3.2.4) and*

$$\Lambda_0 = \{\lambda \in \sigma(\mathcal{A}) |\ Im\lambda \neq 0\}. \tag{3.4.27}$$

Then Λ_0 is a bounded set of $\mathbb{C}$. Moreover, there is no spectrum on the imaginary axis and hence $\mathrm{Re}\lambda \le -\alpha$ *for some $\alpha > 0$ for all $\lambda \in \sigma(\mathcal{A})$.*

Proof. By Theorems 3.3.1 and 3.3.2, $\Lambda_0 \subset \sigma_p(\mathcal{A})$. For any $\lambda = \tau + i\omega \in \Lambda_0$, we may take $(f, \lambda f), f \neq 0$ to be an eigenfunction corresponding to λ. Multiply the first equation of

(3.3.1) by $\overline{f}$ and then integrate over $[0,1]$ with respect to x, to obtain, after separating real part and imaginary part, that

$$\begin{cases} (\tau^2-\omega^2)\displaystyle\int_0^1 \rho(x)|f(x)|^2dx + \int_0^1 [a(x)+\tau b(x)]|f''(x)|^2dx = 0, \\ 2\tau\omega \displaystyle\int_0^1 \rho(x)|f(x)|^2dx + \omega\int_0^1 b(x)|f''(x)|^2dx = 0, \end{cases}$$

which are equivalent to

$$\begin{cases} |\lambda|^2 \displaystyle\int_0^1 \rho(x)|f(x)|^2dx = \int_0^1 a(x)|f''(x)|^2dx, \\ -2\mathrm{Re}\lambda \displaystyle\int_0^1 \rho(x)|f(x)|^2dx = \int_0^1 b(x)|f''(x)|^2dx. \end{cases} \tag{3.4.28}$$

Thus $\mathrm{Re}\lambda \neq 0$, and

$$|\lambda| \leq \frac{|\lambda|^2}{|\mathrm{Re}\lambda|} = 2\frac{\displaystyle\int_0^1 a(x)|f''(x)|^2dx}{\displaystyle\int_0^1 b(x)|f''(x)|^2dx} \leq 2\max_{0\leq x\leq 1}\frac{a(x)}{b(x)}. \tag{3.4.29}$$

So Λ_0 is a bounded set of $\mathbb{C}$ and there is no eigenvalue on the imaginary axis. These together with (i) of Theorem 3.3.2 show that $\mathrm{Re}\lambda \leq -\alpha$ for some $\alpha > 0$ for all $\lambda \in \sigma(\mathcal{A})$. The proof is complete. ■

Chapter 4

On Spectrum and Riesz Basis Property for One-Dimensional Wave Equation with Boltzmann Damping

4.1 Introduction

In this Chapter, we study the one-dimensional wave equation with Boltzmann damping. Two different Boltzmann integrals that represent the memory of materials are considered. The spectral properties for both cases are thoroughly analyzed. It is found that when the memory of system is counted from the infinity, the spectrum of system contains a left half complex plane, which is a sharp contrast to most results in elastic vibration systems that the vibrating dynamics can be considered from the vibration frequency point of view. This suggests us to investigate the system with memory counted from the vibrating starting moment. In the later case, it is shown that the spectrum of system determines completely the dynamic behavior of the vibration: There is a sequence of generalized eigenfunctions of the system, which forms a Riesz basis for the state space. As the consequences, the spectrum-determined growth condition and exponential stability are concluded.

Particular interest is on the difference between two different types of Boltzmann integrals for the dynamics of vibrating systems. We use the one-dimensional wave equation with Boltzmann model of the viscoelasticity for expository demonstration. It is assumed that the instantaneous stress depends on the instantaneous strain and history of strain rate linearly. When the history is entire, that is, the memory is counted from $-\infty$ to t, then the stress σ at time t and position x is ([72]):

$$
\begin{aligned}
\sigma(x,t) &= \int_{-\infty}^{t} \eta(x,t-s)\varepsilon_t(x,s)ds \qquad (\varepsilon(x,-\infty)=0) \\
&= \eta(x,0)\varepsilon(x,t) + \int_{-\infty}^{t} \eta_t(x,t-s)\varepsilon(x,s)ds \\
&= \eta(x,0)\varepsilon(x,t) + \int_{0}^{\infty} \eta_s(x,s)\varepsilon(x,t-s)ds \\
&= \eta(x,\infty)\varepsilon(x,t) - \int_{0}^{\infty} \eta_s(x,s)[\varepsilon(x,t)-\varepsilon(x,t-s)]ds \\
&= a(x)\varepsilon(x,t) - b(x)\int_{0}^{\infty} g_s(s)[\varepsilon(x,t)-\varepsilon(x,t-s)]ds,
\end{aligned} \tag{4.1.1}
$$

while the memory is finite, that is, the memory is counted from the vibration starting moment 0 to t, the stress is:

$$
\begin{aligned}
\sigma(x,t) &= \int_{0}^{t} \eta(x,t-s)\varepsilon_t(x,s)ds \qquad (\varepsilon(x,0)=0) \\
&= \eta(x,0)\varepsilon(x,t) + \int_{0}^{t} \eta_t(x,t-s)\varepsilon(x,s)ds \\
&= [a(x)+b(x)g(0)]\varepsilon(x,t) + \int_{0}^{t} b(x)g_t(t-s)\varepsilon(x,s)ds,
\end{aligned} \tag{4.1.2}
$$

where we take the relaxation function in the form of ([72])

$$
\eta(x,s) = a(x) + b(x)g(s), \quad g(\infty)=0. \tag{4.1.3}
$$

So, the corresponding governing equation to infinite memory is ([72]):

$$
\begin{cases}
u_{tt}(x,t) = \left(a(x)u_x(x,t) - b(x)\displaystyle\int_{0}^{\infty} g_s(s)[u_x(x,t)-u_x(x,t-s)]ds\right)_x, \\
u(0,t) = u(1,t) = 0, \quad t>0,\ 0<x<1, \\
u(x,t) = u_0(x,t),\ u_t(x,t) = u_1(x,t), \quad t\le 0,\ 0<x<1,
\end{cases} \tag{4.1.4}
$$

and the equation to finite memory is ([82]):

$$
\begin{cases}
u_{tt}(x,t) = \left(a(x)u_x(x,t) + b(x)\displaystyle\int_{0}^{t} g_t(t-s)u_x(x,s)ds\right)_x, \ x\in(0,1),\ t>0, \\
u(0,t) = u(1,t) = 0, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),
\end{cases} \tag{4.1.5}
$$

where in (4.1.5), we replace $a(x)+b(x)g(0)$ by $a(x)$ for the sake of simplicity.

In order to compare the models (4.1.4) and (4.1.5) qualitatively, we take the kernel simply as the finite sum of exponential polynomials, and both a and b are positive constant functions:

$$\begin{cases} g(s) = \sum_{j=1}^{N} a_j e^{-b_j s}, \ 0 < a_j, b_j \in \mathbb{R}, 1 \leq N \in \mathbb{N}, \\ \\ a(x) \equiv a > 0, b(x) \equiv b > 0, \end{cases} \tag{4.1.6}$$

where it is assumed that

$$0 < b_1 < b_2 < \cdots < b_N. \tag{4.1.7}$$

It is noted that since we replace $a + bg(0)$ by a in (4.1.5) and $a > 0$ in modeling (4.1.2), it is natural to assume in (4.1.6) that

$$a - bg(0) = a - b\sum_{j=1}^{N} a_j > 0. \tag{4.1.8}$$

The system (4.1.4) has been formulated into an abstract evolution equation in [72] based on the idea of [22]. In next section, Section 4.2, the spectral analysis for this system with kernel (4.1.6) is thoroughly performed. The asymptotic distribution of eigenvalues is presented. It is shown that the spectrum of the system operator contains a half complex plane, which is an unexpected result for an elastic vibrating system, see Theorem 4.2.2.

Section 4.3 is devoted to the analysis of system (4.1.5), (4.1.6). We adapt the methods used in [91] for the heat equation with finite memory. The spectral analysis for the system operator that is not of compact resolvent shows that there is a sequence of generalized eigenfunctions of the system operator, which forms a Riesz basis for the state space, see Theorem 4.3.5. This is a sharp contrast with the heat equation with memory discussed in [91], but coincides, in reflecting the dynamic behavior of system via the vibrating frequencies, with those presented in [33, 34] where the system operators are of compact resolvent. Consequently, the spectrum-determined growth condition as well as the exponential stability of the system is concluded.

4.2 Infinite memory

In this section, we analyze the spectrum of system (4.1.4) with kernel (4.1.6). Special attention would be paid to the distribution of the spectrum on the complex plane and the asymptotic behavior of the eigenvalues.

4.2.1 System operator setup

The following general formulation comes from [72] for general kernel satisfying

(g1) $g \in C^2(0,\infty) \cap C[0,\infty)$, and $g_s \in L^1(0,\infty)$;

(g2) $g > 0, g_s < 0, g_{ss} > 0$ on $(0,\infty)$;

(g3) $-kg_s \le g_{ss} \le -Kg_s$ on $(0,\infty)$ for some $k, K > 0$;

(g4) $g(\infty) = 0$.

It is easily seen that the special kernel (4.1.6) satisfies the above four conditions. Let

$$y(x,t,s) = u(x,t) - u(x,t-s),\ v = u_t.$$

Then

$$y_t = u_t - y_s,$$

and

$$y(\cdot,\cdot,0) = 0. \tag{4.2.1}$$

The energy of the system (4.1.4) is given by

$$E(t) = \frac{1}{2}\int_0^1 (a|u_x(x,t)|^2 + |u_t(x,t)|^2)dx + \frac{1}{2}\int_0^\infty |g_s(s)| \int_0^1 b|y_x(x,t,s)|^2 dxds. \tag{4.2.2}$$

Let $W = H_0^1(0,1)$ with the inner product:

$$\langle w_1, w_2 \rangle = b\int_0^1 w_1'(x)\overline{w_2'(x)}dx, \quad \forall\ w_1, w_2 \in W. \tag{4.2.3}$$

Define the energy state Hilbert space

$$\mathcal{H} = V \times H \times Y, \tag{4.2.4}$$

where

$$\begin{aligned} V &= H_0^1(0,1), & \|u\|_V^2 &= a\int_0^1 |u'(x)|^2 dx, \\ H &= L^2(0,1), & \|v\|_H^2 &= \int_0^1 |v(x)|^2 dx, \\ Y &= L^2((0,\infty);W), & \|y\|_Y^2 &= \int_0^\infty |g_s(s)|\|y\|_W^2 ds. \end{aligned} \tag{4.2.5}$$

Define the system operator $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ as

$$\begin{cases} \mathcal{A}z = \left(v,\ \left(au' - b\int_0^\infty g_s(s)y'(\cdot,s)ds\right)',\ v - y_s\right), \forall\ z = (u,v,y) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ z \in \mathcal{H} \;\middle|\; \begin{array}{l} v \in V, y_s \in Y, y(\cdot,0) = 0, \\ au' - b\int_0^\infty g_s(s)y'(\cdot,s)ds \in H^1(0,1) \end{array} \right\}. \end{cases} \tag{4.2.6}$$

Then system (4.1.4) can be formulated as an abstract evolution equation in $\mathcal{H}$ ([72]):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = \mathcal{A} z(t), \\ z(0) = z_0, \end{cases} \tag{4.2.7}$$

where $z(t) = (u(\cdot,t), u_t(\cdot,t), y(\cdot,t,\cdot))$, $z_0(x) = (u_0(x,0), u_1(x,0), u_0(x,0) - u_0(x,-s))$ is the state variable and the initial value, respectively.

The Proposition 4.2.1 below justifies the adjoint operator of $\mathcal{A}$.

Proposition 4.2.1. *Let $\mathcal{A}$ be defined by (4.2.6). Then its adjoint $\mathcal{A}^*$ has the following form:*

$$\begin{cases} \mathcal{A}^* z = \left(-v, \ -\left(au' - b\displaystyle\int_0^\infty g_s(s) y'(\cdot,s) ds \right)', -(v - y_s - \dfrac{g_{ss}(s)}{g_s(s)} y) \right), \\ D(\mathcal{A}^*) = \left\{ z = (u,v,y) \in \mathcal{H} \ \middle| \ \begin{array}{l} u, v \in V, y, y_s \in Y, y(\cdot,0) = 0, \\ au' - b\int_0^\infty g_s(s) y'(\cdot,s) ds \in H^1(0,1) \end{array} \right\}. \end{cases} \tag{4.2.8}$$

The next Lemma 4.2.1 comes from Lemma 2.1 in [69].

Lemma 4.2.1. *Suppose that $y \in Y$, $\mathrm{Re}\lambda > -\frac{k}{2}$, g satisfies conditions (g1)-g(4),*

$$h(s) = \int_0^s e^{-\lambda(s-\tau)} y(\tau) d\tau.$$

Then

(i)

$$h \in Y \cap C([0,\infty), W), \ \ h_s \in Y$$

and

$$\|h\|_Y^2 \le \frac{1}{\delta}(2\mathrm{Re}\lambda + k - \delta)^{-1} \|y\|_Y^2 \ \ \textit{for } \delta \in (0, 2\mathrm{Re}\lambda + k); \tag{4.2.9}$$

(ii)

$$\mathrm{Re} \int_0^\infty g_s(s) \langle h_s(s), h(s) \rangle_W ds \le -\frac{k}{2} \|h\|_Y^2.$$

It was explained shortly in [72] that $\mathcal{A}$ is invertible and generates a C_0-semigroup. Here, we give a simple proof.

Proposition 4.2.2. *Let $\mathcal{A}$ be defined by (4.2.6). Then $\mathcal{A}^{-1}$ is given by*

$$\mathcal{A}^{-1} \begin{pmatrix} u \\ v \\ y \end{pmatrix} (x,s) = \begin{pmatrix} h_1(x) + h_2(x) - [h_1(1) + h_2(1)]x \\ u(x) \\ u(x)s - \displaystyle\int_0^s y(x,\zeta) d\zeta \end{pmatrix}, \tag{4.2.10}$$

where

$$\begin{cases} h_1(x) = \dfrac{b}{a}u(x)\displaystyle\int_0^\infty sg_s(s)ds - \dfrac{b}{a}\int_0^x \left[\int_0^\infty g_s(s)\int_0^s y_x(\tau,\zeta)d\zeta ds\right] d\tau, \\ h_2(x) = \dfrac{1}{a}\displaystyle\int_0^x \left[\int_0^\tau v(\zeta)d\zeta\right] d\tau. \end{cases} \tag{4.2.11}$$

And hence $0 \in \rho(\mathcal{A})$*. Moreover,* $\mathcal{A}$ *is dissipative, and thus* $\mathcal{A}$ *generates a* C_0*-semigroup of constructions* $e^{\mathcal{A}t}$ *on* $\mathcal{H}$*.*

Proof. Let $(u, v, y) \in \mathcal{H}$. By $\mathcal{A}(\widetilde{u}, \widetilde{v}, \widetilde{y}) = (u, v, y)$, it has

$$\begin{cases} \widetilde{v}(x) = u(x), \\ \left(a\widetilde{u}_x(x) - b\displaystyle\int_0^\infty g_s(s)\widetilde{y}_x(x,s)ds\right)_x = v(x), \\ \widetilde{v}(x) - \widetilde{y}_s(x,s) = y(x,s). \end{cases}$$

This together with the boundary conditions shows that

$$\widetilde{v} = u,\ \widetilde{y} = us - \int_0^s y(\cdot,\zeta)d\zeta$$

and

$$\begin{cases} \left(a\widetilde{u}'(x) - b\displaystyle\int_0^\infty g_s(s)\left[u'(x)s - \int_0^s y'(x,\zeta)d\zeta\right] ds\right)' = v(x), \\ \widetilde{u}(0) = \widetilde{u}(1) = 0. \end{cases}$$

A direct computation gives

$$\widetilde{u}(x) = h_1(x) + h_2(x) + \frac{C}{a}x,$$

where $h_1(x)$, $h_2(x)$ are given by (4.2.11), and C is a constant to be determined. Using the boundary condition $\widetilde{u}(1) = 0$ gives

$$C = -a[h_1(1) + h_2(1)].$$

Therefore

$$\widetilde{u}(x) = h_1(x) + h_2(x) - [h_1(1) + h_2(1)]x.$$

By Lemma 4.2.1, it has

$$\widetilde{y} \in Y.$$

And hence,

$$(\widetilde{u}, \widetilde{v}, \widetilde{y}) \in D(\mathcal{A}),$$

(4.2.10) holds.

For each $z = (u, v, y) \in D(\mathcal{A})$, it has

$$\begin{aligned}
\langle \mathcal{A}z, z\rangle &= \langle (v, (au' - b\int_0^\infty g_s(s)y'(\cdot, s)ds)', v - y_s)), (u, v, y)\rangle \\
&= \int_0^1 av'(x)\overline{u'(x)}dx + \int_0^1 \left(au'(x) - b\int_0^\infty g_s(s)y'(x,s)ds\right)' \overline{v(x)}dx \\
&\quad + \int_0^\infty |g_s(s)| \int_0^1 b(v(x) - y_s(x,s))'\overline{y'(x,s)}dxds \\
&= \int_0^1 a\left(v'(x)\overline{u'(x)} - u'(x)\overline{v'(x)}\right)dx \\
&\quad + b\int_0^1\int_0^\infty g_s(s)\left(y'(x,s)\overline{v'(x)} - v'(x)\overline{y'(x,s)}\right)dsdx \\
&\quad + b\int_0^1\int_0^\infty g_s(s)y_s'(x,s)\overline{y'(x,s)}dsdx.
\end{aligned}$$

By Lemma 4.2.1, we have

$$\mathrm{Re}\langle \mathcal{A}z, z\rangle \leq -\frac{k}{2}\|y\|_Y^2 \leq 0.$$

Therefore, $\mathcal{A}$ is dissipative. By the Lumer-Phillips Theorem, $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. The proof is complete. ■

4.2.2 Spectral analysis for system operator

In this subsection, we analyze the spectrum of $\mathcal{A}$ with the kernel (4.1.6). First, consider the eigenvalue problem. Suppose $\mathcal{A}z = \lambda z$ for $0 \neq \lambda \in \mathbb{C}$ and $0 \neq z = (u, v, y) \in D(\mathcal{A})$. Then

$$\begin{cases} v(x) = \lambda u(x), \\ \left(au'(x) - b\int_0^\infty g_s(s)y'(x,s)ds\right)' = \lambda v(x), \\ v(x) - y_s(x,s) = \lambda y(x,s), \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.12}$$

From the third equation of (4.2.12) and $y(\cdot, 0) = 0$, we have

$$y(x,s) = \frac{1}{\lambda}(1 - e^{-\lambda s})v(x). \tag{4.2.13}$$

We claim that v can not be identical to a constant. Actually, if this is the case, it follows from (4.2.12) that

$$(u, v, y) = 0.$$

Hence, for any $\mathrm{Re}\lambda \leq -\frac{b_1}{2}$,

$$y \notin Y.$$

Therefore,

$$\sigma_p(\mathcal{A}) \subset D_1 = \{\lambda \in \mathbb{C} \mid -\frac{b_1}{2} < \mathrm{Re}\lambda < 0\}. \tag{4.2.14}$$

By this fact, we always assume that $\lambda \in D_1$ when we mention the eigenvalues of $\mathcal{A}$ in what follows. Collecting these facts just mentioned, we find, from (4.2.12) and (4.2.13), that $\lambda \in \sigma_p(\mathcal{A})$ if and only if $(\lambda, u), u \neq 0$, satisfies

$$\begin{cases} \left(a + b\sum_{j=1}^{N} a_j - b\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j}\right) u''(x) - \lambda^2 u(x) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.15}$$

Lemma 4.2.2. *Let $\mathcal{A}$ be defined by (4.2.6) and*

$$p(\lambda) = a + b\sum_{j=1}^{N} a_j - b\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j}. \tag{4.2.16}$$

Then there exists a unique solution $\lambda_c \in \{\lambda | -b_1 < \mathrm{Re}\lambda < 0\}$ to $p(\lambda) = 0$. Moreover, λ_c is real, and

$$\lambda_c \notin \sigma_p(\mathcal{A}). \tag{4.2.17}$$

Proof. Obviously, for any $j = 1, 2, \cdots, N$, $\lambda = -b_j$ is not the zero point of $p(\lambda)$. Thus, $p(\lambda) = 0$ is equivalent to $\widetilde{p}(\lambda) = 0$, where

$$\begin{aligned} \widetilde{p}(\lambda) &= p(\lambda) \prod_{j=1}^{N} (\lambda + b_j) \\ &= \left(a + b\sum_{j=1}^{N} a_j\right) \prod_{j=1}^{N} (\lambda + b_j) - b\sum_{j=1}^{N} a_j b_j \prod_{k=1, k\neq j}^{N} (\lambda + b_k). \end{aligned}$$

However, $\widetilde{p}(\lambda)$ is an N-th order polynomial, and hence there are at most N number of zeros for $p(\lambda)$. Now we find these zeros. Notice that $p(\lambda)$ is continues in $\left(\cup_{j=1}^{N-1}(-b_{j+1}, -b_j)\right) \cup (-b_1, \infty)$, and

$$\lim_{\lambda \to -b_j^-} p(\lambda) = +\infty, \quad \lim_{\lambda \to -b_j^+} p(\lambda) = -\infty, \quad p(0) > 0, \quad j = 1, 2, \cdots, N.$$

It follows that there exists a solution to $p(\lambda) = 0$ in $(-b_{j+1}, -b_j)$, $j = 0, 1, 2, \cdots, N-1$, here we set $b_0 = 0$. Moreover, when $\lambda_c > -\frac{b_1}{2}$ and $p(\lambda) = 0$, it follows from (4.2.15) that $u \equiv 0$. This together with (4.2.12) gives

$$(u, v, y) = 0.$$

Hence (4.2.17) is valid. The proof is complete. ∎

By Lemma 4.2.2, the eigenvalue problem (4.2.15) can be written as

$$\begin{cases} u''(x) = \dfrac{\lambda^2}{p(\lambda)} u(x), \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.18}$$

The nonzero solution of (4.2.18) is found to be

$$u(x) = e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}x} - e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}x}, \tag{4.2.19}$$

where λ satisfies

$$e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}} - e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}} = 0. \tag{4.2.20}$$

That is

$$e^{2\sqrt{\frac{\lambda^2}{p(\lambda)}}} = 1,$$

or

$$\frac{\lambda^2}{p(\lambda)} = -n^2\pi^2, \ n = 1, 2, \cdots . \tag{4.2.21}$$

Substituting (4.2.21) into (4.2.19) gives the corresponding eigenfunction

$$(u(x), \lambda u(x), (1 - e^{-\lambda s})u(x)),$$

where

$$u(x) = \sin n\pi x, \tag{4.2.22}$$

for some $n \in \mathbb{N}^+$.

Set

$$\widetilde{a} = a + b\sum_{j=1}^{N} a_j. \tag{4.2.23}$$

When $|\lambda|$ is large enough, since

$$\begin{aligned}
\frac{\lambda^2}{p(\lambda)} &= \frac{\lambda^2}{\widetilde{a}} \cdot \frac{1}{1 - \frac{b}{\widetilde{a}}\sum_{j=1}^{N}\frac{a_j b_j}{\lambda + b_j}} \\
&= \frac{\lambda^2}{\widetilde{a}}\left(1 + \frac{b}{\widetilde{a}}\sum_{j=1}^{N}\frac{a_j b_j}{\lambda + b_j} + \frac{b^2}{\widetilde{a}^2}\left(\sum_{j=1}^{N}\frac{a_j b_j}{\lambda + b_j}\right)^2\right) + \mathcal{O}(|\lambda|^{-1}) \\
&= \frac{1}{\widetilde{a}}\left(\lambda^2 + \frac{b}{\widetilde{a}}\sum_{j=1}^{N}\frac{a_j b_j \lambda}{1 + \frac{b_j}{\lambda}} + \frac{b^2}{\widetilde{a}^2}\left(\sum_{j=1}^{N}\frac{a_j b_j}{1 + \frac{b_j}{\lambda}}\right)^2\right) + \mathcal{O}(|\lambda|^{-1}) \\
&= \frac{1}{\widetilde{a}}\left(\lambda^2 + \frac{b}{\widetilde{a}}\sum_{j=1}^{N} a_j b_j \lambda - \frac{b}{\widetilde{a}}\sum_{j=1}^{N} a_j b_j^2 + \frac{b^2}{\widetilde{a}^2}\left(\sum_{j=1}^{N} a_j b_j\right)^2\right) + \mathcal{O}(|\lambda|^{-1}),
\end{aligned}$$

we obtain that

$$\lambda^2 + \frac{b}{\widetilde{a}}\sum_{j=1}^{N} a_j b_j \lambda - \frac{b}{\widetilde{a}}\sum_{j=1}^{N} a_j b_j^2 + \frac{b^2}{\widetilde{a}^2}\left(\sum_{j=1}^{N} a_j b_j\right)^2 + \widetilde{a} n^2\pi^2 + \mathcal{O}(|\lambda|^{-1}) = 0.$$

Thus, the eigenvalues of $\mathcal{A}$ are found to be

$$\lambda_n = -\frac{b}{2\widetilde{a}}\sum_{j=1}^{N} a_j b_j \pm i\sqrt{\widetilde{a}}n\pi + \mathcal{O}(n^{-1}), \quad n \to \infty.$$

When $\lambda \to \lambda_c$, $\mu = \lambda - \lambda_c \to 0$. Since

$$\begin{aligned} p(\lambda) &= a + b\sum_{j=1}^{N} a_j - b\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j} \\ &= a + b\sum_{j=1}^{N} a_j - b\sum_{j=1}^{N} \frac{a_j b_j}{\lambda_c + b_j}\frac{1}{1 + \frac{(\lambda-\lambda_c)}{\lambda_c + b_j}} \\ &= \mu b\sum_{j=1}^{N} a_j b_j \left[\frac{1}{(\lambda_c + b_j)^2} - \frac{\mu}{(\lambda_c + b_j)^3} + \mathcal{O}(\mu^2)\right], \end{aligned}$$

it has

$$\begin{aligned} \frac{\lambda^2}{p(\lambda)} &= \frac{\lambda_c^2 + 2\lambda_c\mu + \mu^2}{p(\lambda)} \\ &= \frac{1}{\mu}\frac{\lambda_c^2}{\sum_{j=1}^{N}\frac{ba_j b_j}{(\lambda_c+b_j)^2}} \cdot \left(1 + \frac{2}{\lambda_c}\mu + \frac{1}{\lambda_c^2}\mu^2\right) \cdot \left(1 - \frac{\sum_{j=1}^{N}\frac{a_j b_j}{(\lambda_c+b_j)^3}}{\sum_{j=1}^{N}\frac{a_j b_j}{(\lambda_c+b_j)^2}}\mu + \mathcal{O}(\mu^2)\right)^{-1} \\ &= \frac{1}{\mu}\frac{\lambda_c^2}{\sum_{j=1}^{N}\frac{ba_j b_j}{(\lambda_c+b_j)^2}} \cdot \left(1 + \frac{2}{\lambda_c}\mu + \frac{1}{\lambda_c^2}\mu^2\right) \cdot \left(1 + \frac{\sum_{j=1}^{N}\frac{a_j b_j}{(\lambda_c+b_j)^3}}{\sum_{j=1}^{N}\frac{a_j b_j}{(\lambda_c+b_j)^2}}\mu\right) + \mathcal{O}(\mu) \\ &= \frac{1}{\mu}\frac{\lambda_c^2}{\Delta}\left[1 + \left(\frac{2}{\lambda_c} + \frac{\Delta_1}{\Delta}\right)\mu\right] + \mathcal{O}(\mu), \end{aligned}$$

where

$$\begin{cases} \Delta = \displaystyle\sum_{j=1}^{N} \frac{ba_j b_j}{(\lambda_c + b_j)^2}, \\ \Delta_1 = \displaystyle\sum_{j=1}^{N} \frac{a_j b_j}{(\lambda_c + b_j)^3}. \end{cases} \tag{4.2.24}$$

This together with (4.2.21) yields

$$\frac{1}{\mu}\frac{\lambda_c^2}{\Delta}\left[1 + \left(\frac{2}{\lambda_c} + \frac{\Delta_1}{\Delta}\right)\mu\right] + \mathcal{O}(\mu) = -n^2\pi^2, \ n \to \infty.$$

Thus

$$\mu_n = -\frac{1}{n^2\pi^2}\frac{\lambda_c^2}{\Delta} + \mathcal{O}(n^{-3}), \quad n \to \infty$$

or

$$\lambda_n = \lambda_c - \frac{1}{n^2\pi^2}\frac{\lambda_c^2}{\Delta} + \mathcal{O}(n^{-3}), \quad n \to \infty.$$

We summarize these results as Theorem 4.2.1 following.

Theorem 4.2.1. *Let $\mathcal{A}$ be defined by (4.2.6). Then the eigenvalues of $\mathcal{A}$ must be located inside of D_1 that is given by (4.2.14). The eigenfunction corresponding to λ is*

$$(u(x), \lambda u(x), (1 - e^{-\lambda s})u(x))$$

with

$$u(x) = \sin n\pi x, \tag{4.2.25}$$

for some $n \in \mathbb{N}^+$. More precisely,

(i). When $\lambda_c > -\frac{b_1}{2}$, where λ_c is given in Lemma 4.2.2, there is a sequence of eigenvalues $\{\lambda_n\}$ of $\mathcal{A}$, which have the following asymptotic expression:

$$\lambda_n = \lambda_c - \frac{1}{n^2\pi^2}\frac{\lambda_c^2}{\Delta} + \mathcal{O}(n^{-3}), \quad n \to \infty, \tag{4.2.26}$$

where Δ is given by (4.2.24). Furthermore, the corresponding eigenfunctions

$$(u_n(x), \lambda_n u_n(x), (1 - e^{-\lambda_n s})u_n(x))$$

are of the form:

$$u_n(x) = \sin n\pi x, \quad n \to \infty. \tag{4.2.27}$$

(ii). When $|\lambda| \to \infty$ and

$$-\frac{b}{2\widetilde{a}}\sum_{j=1}^{N} a_j b_j > -\frac{b_1}{2},$$

the eigenvalues of $\mathcal{A}$ have the following asymptotic expressions:

$$\lambda_n = -\frac{b}{2\widetilde{a}}\sum_{j=1}^{N} a_j b_j \pm i\sqrt{\widetilde{a}}n\pi + \mathcal{O}(n^{-1}), \quad n \to \infty, \tag{4.2.28}$$

where $\widetilde{a}$ is given by (4.2.23). In particular,

$$\mathrm{Re}\lambda_n \to -\frac{b}{2\widetilde{a}}\sum_{j=1}^{N} a_j b_j < 0, \quad n \to \infty, \tag{4.2.29}$$

that is, $\mathrm{Re}\lambda = -\frac{b}{2\widetilde{a}}\sum_{j=1}^{N} a_j b_j$ is the asymptote of the eigenvalues specified by (4.2.28). Furthermore, the corresponding eigenfunctions

$$(u_n(x), \lambda_n u_n(x), (1 - e^{-\lambda_n s})u_n(x))$$

satisfy (4.2.27).

Now we characterize the spectrum of $\mathcal{A}$.

Theorem 4.2.2. *Let $\mathcal{A}$ be defined by (4.2.6), and λ_c be given in Lemma 4.2.2. Then*

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \{\lambda_c\} \cup \{\lambda |\ \mathrm{Re}\lambda \leq -\frac{b_1}{2}\}. \tag{4.2.30}$$

Proof. Let $\lambda \notin \sigma_p(\mathcal{A})$. If $\lambda = 0$, by Proposition 4.2.2, $\lambda \in \rho(\mathcal{A})$. So we need only consider the case of $\lambda \neq 0$. For any $\widetilde{z} = (\widetilde{u}, \widetilde{v}, \widetilde{y}) \in \mathcal{H}$. Solve $(\lambda I - \mathcal{A})z = \widetilde{z}$ for $z = (u, v, y)$, that is,

$$\begin{cases} \lambda u(x) - v(x) = \widetilde{u}(x), \\ \lambda v(x) - \left(au'(x) - b\displaystyle\int_0^\infty g_s(s)y'(x,s)ds\right)' = \widetilde{v}(x), \\ \lambda y(x,s) - (v(x) - y_s(x,s)) = \widetilde{y}(x,s), \\ u(0) = u(1) = 0, \end{cases} \tag{4.2.31}$$

to get

$$\begin{cases} v(x) = \lambda u(x) - \widetilde{u}(x), \\ y(x,s) = \dfrac{1}{\lambda}(1 - e^{-\lambda s})v(x) + e^{-\lambda s}\displaystyle\int_0^s e^{\lambda\tau}\widetilde{y}(x,\tau)d\tau \end{cases} \tag{4.2.32}$$

and

$$\begin{cases} \left(au'(x) - b\displaystyle\int_0^\infty g_s(s)y'(x,s)ds\right)' - \lambda^2 u(x) + \lambda\widetilde{u}(x) + \widetilde{v}(x) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.33}$$

There are three cases:

Case I: $\mathrm{Re}\lambda \leq -\frac{b_1}{2}$**.** We claim that

$$\lambda \in \sigma(\mathcal{A}).$$

In fact, take

$$\widetilde{z} = (\widetilde{u}, \widetilde{v}, \widetilde{y}) = (0, \widetilde{v}, 0), \quad \forall \widetilde{v} \in H,\ \widetilde{v} \neq 0.$$

It follows from (4.2.32) and (4.2.33) that

$$\begin{cases} v(x) = \lambda u(x), \\ y(x,s) = (1 - e^{-\lambda s})u(x), \\ \left(au'(x) - b\displaystyle\int_0^\infty g_s(s)(1 - e^{-\lambda s})u'(x)ds\right)' - \lambda^2 u(x) + \widetilde{v}(x) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.34}$$

If (4.2.34) admits a solution, it must have

$$y \in Y.$$

This together with $\mathrm{Re}\lambda \leq -\frac{b_1}{2}$ shows that $u' \equiv 0$. Thus, $u \equiv 0$. So

$$\widetilde{v} \equiv 0.$$

This is a contradiction. Therefore, there is no solution to equation (4.2.34), which means that

$$\lambda \in \sigma(\mathcal{A}).$$

Case II: $\mathrm{Re}\lambda > -\frac{b_1}{2}$ **and** $\lambda \neq \lambda_c$**.** We show that

$$\lambda \in \rho(\mathcal{A}).$$

By Lemma 4.2.1, it has

$$y \in Y.$$

(4.2.33) is equivalent to

$$\begin{cases} \eta'(x) - \lambda^2 u(x) + \lambda \widetilde{u}(x) + \widetilde{v}(x) = 0, \\ \eta(x) = p(\lambda) u'(x) + \dfrac{1}{\lambda}(a - p(\lambda))\widetilde{u}'(x) \\ \qquad\qquad -b \displaystyle\int_0^\infty g_s(s) \left[\int_0^s e^{-\lambda(s-\tau)} \widetilde{y}'(x,\tau) d\tau \right] ds, \\ u(0) = u(1) = 0. \end{cases} \tag{4.2.35}$$

We write above equation as the following first order system of differential equations

$$\begin{cases} \dfrac{d}{dx} \begin{pmatrix} u(x) \\ \eta(x) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{p(\lambda)} \\ \lambda^2 & 0 \end{pmatrix} \begin{pmatrix} u(x) \\ \eta(x) \end{pmatrix} + \begin{pmatrix} \frac{1}{p(\lambda)} U(x) \\ -\lambda \widetilde{u}(x) - \widetilde{v}(x) \end{pmatrix}, \\ u(0) = u(1) = 0, \end{cases} \tag{4.2.36}$$

where

$$U(x) = -\frac{1}{\lambda}(a - p(\lambda))\widetilde{u}'(x) + b \int_0^\infty g_s(s) \left[\int_0^s e^{-\lambda(s-\tau)} \widetilde{y}'(x,\tau) d\tau \right] ds. \tag{4.2.37}$$

Let

$$A(\lambda) = \begin{pmatrix} 0 & \frac{1}{p(\lambda)} \\ \lambda^2 & 0 \end{pmatrix}.$$

Then

$$e^{A(\lambda)x} = \begin{pmatrix} a_{11}(\lambda,x) & a_{12}(\lambda,x) \\ a_{21}(\lambda,x) & a_{22}(\lambda,x) \end{pmatrix},$$

where

$$\begin{cases} a_{11}(\lambda,x) = \cosh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \\ a_{21}(\lambda,x) = \lambda\sqrt{p(\lambda)}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \end{cases} \quad \begin{cases} a_{12}(\lambda,x) = \frac{1}{\lambda\sqrt{p(\lambda)}}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \\ a_{22}(\lambda,x) = \cosh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right). \end{cases}$$

The general solution of (4.2.36) is given by

$$\begin{pmatrix} u(x) \\ \eta(x) \end{pmatrix} = e^{A(\lambda)x}\begin{pmatrix} u(0) \\ \eta(0) \end{pmatrix} - \int_0^x e^{A(\lambda)(x-\gamma)}\begin{pmatrix} \frac{1}{p(\lambda)}U(\gamma) \\ -\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma) \end{pmatrix} d\gamma.$$

By $u(0)=0$, it has,

$$\begin{aligned} u(x) &= a_{12}(\lambda,x)\eta(0) \\ &\quad - \int_0^x \left[\frac{1}{p(\lambda)}a_{11}(\lambda,x-\gamma)U(\gamma) + a_{12}(\lambda,x-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right] d\gamma \end{aligned} \tag{4.2.38}$$

and

$$\begin{aligned} \eta(x) &= a_{22}(\lambda,x)\eta(0) \\ &\quad - \int_0^x \left[\frac{1}{p(\lambda)}a_{21}(\lambda,x-\gamma)U(\gamma) + a_{22}(\lambda,x-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right] d\gamma. \end{aligned} \tag{4.2.39}$$

Since $\lambda \notin \sigma_p(\mathcal{A})$, it follows from (4.2.20) that

$$a_{12}(\lambda,1) = \frac{1}{\lambda\sqrt{p(\lambda)}}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}\right) \neq 0.$$

By the boundary condition $u(1)=0$, it has

$$\begin{aligned} &\eta(0) \\ &= \frac{1}{a_{12}(\lambda,1)}\int_0^1 \left[\frac{1}{p(\lambda)}a_{11}(\lambda,1-\gamma)U(\gamma) + a_{12}(\lambda,1-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right] d\gamma. \end{aligned} \tag{4.2.40}$$

Hence u is uniquely determined by (4.2.38). By the second equation of (4.2.35) and (4.2.39), it has

$$u' \in L^2(0,1).$$

This together with (4.2.32) shows that $(\lambda I - \mathcal{A})^{-1}$ exists and is bounded, or

$$\lambda \in \rho(\mathcal{A}).$$

Case III: $\lambda = \lambda_c > -\frac{b_1}{2}$. In this case, it follows from (4.2.33) that

$$\begin{cases} u(x) = \dfrac{1}{\lambda^2}\left[\lambda\widetilde{u}(x) + \widetilde{v}(x) - U'(x)\right], \\ u(0) = u(1) = 0, \end{cases} \tag{4.2.41}$$

where U is given by (4.2.37). Since $\widetilde{u} \in H_0^1(0,1)$, (4.2.41) means that (4.2.31) admits a solution if and only if U is differentiable and

$$\widetilde{v}(0) - U'(0) = \widetilde{v}(1) - U'(1) = 0.$$

Thus

$$\lambda \notin \rho(\mathcal{A}).$$

Combing all these cases completes the proof. ■

4.3 Finite memory

In this section, we turn to the system (4.1.5) with kernel (4.1.6). We analyze the spectrum of the system operator first, and then prove the Riesz basis property for the system. The idea comes from [91] but the result is different, particularly for the basis property.

4.3.1 System operator setup

In what follows, we always assume (4.1.8). Set

$$h_j(x,t) = a_j b_j \int_0^t e^{-b_j(t-s)} u_x(x,s)ds, \ j = 1, 2, \cdots, N. \tag{4.3.1}$$

Then it has

$$\begin{cases} (h_j)_t(x,t) = a_j b_j u_x(x,t) - b_j h_j(x,t), \\ (h_j)_x(x,t) = a_j b_j \displaystyle\int_0^t e^{-b_j(t-s)} u_{xx}(x,s)ds, \\ h_j(x,0) = 0. \end{cases} \tag{4.3.2}$$

Thus we can rewrite the system (4.1.5),(4.1.6) as

$$\begin{cases} u_{tt}(x,t) = \left(a u_x(x,t) - b\displaystyle\sum_{j=1}^N h_j(x,t)\right)_x, \ x \in (0,1), \ t > 0, \\ (h_j)_t(x,t) = a_j b_j u_x(x,t) - b_j h_j(x,t), \ j = 1, 2, \cdots, N, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ h_j(x,0) = 0, \ j = 1, 2, \cdots, N. \end{cases} \tag{4.3.3}$$

The system energy is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[a|u_x(x,t)|^2 + |u_t(x,t)|^2 + \sum_{j=1}^{N} |h_j(x,t)|^2 \right] dx. \tag{4.3.4}$$

We consider system (4.3.3) in the energy state Hilbert space

$$\mathcal{H} = H_0^1(0,1) \times (L^2(0,1))^{N+1}$$

with the inner product:

$$\begin{aligned} &\langle (u, v, h_1, \cdots, h_N), (\widetilde{u}, \widetilde{v}, \widetilde{h_1}, \cdots, \widetilde{h_N}) \rangle \\ &= \int_0^1 au'(x)\overline{\widetilde{u}'(x)}dx + \int_0^1 v(x)\overline{\widetilde{v}(x)}dx + \sum_{j=1}^{N} \int_0^1 h_j(x)\overline{\widetilde{h_j}(x)}dx, \\ &\qquad \forall\ (u, v, h_1, \cdots, h_N), (\widetilde{u}, \widetilde{v}, \widetilde{h_1}, \cdots, \widetilde{h_N}) \in \mathcal{H}. \end{aligned} \tag{4.3.5}$$

Define the system operator $\mathcal{B} : D(\mathcal{B})(\subset \mathcal{H}) \to \mathcal{H}$ as

$$\begin{cases} \mathcal{B} \begin{pmatrix} u \\ v \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^{\top} = \begin{pmatrix} v \\ (au' - b\displaystyle\sum_{j=1}^{N} h_j)' \\ a_1 b_1 u' - b_1 h_1 \\ \vdots \\ a_N b_N u' - b_N h_N \end{pmatrix}^{\top}, \\ D(\mathcal{B}) = \left\{ \begin{pmatrix} u \\ v \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^{\top} \middle| \begin{array}{l} u, v \in H_0^1(0,1), \\ h_j \in L^2(0,1),\ j = 1, \cdots, N, \\ au' - b\displaystyle\sum_{j=1}^{N} h_j \in H^1(0,1) \end{array} \right\}. \end{cases} \tag{4.3.6}$$

Then (4.3.3) can be formulated into an abstract evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} U(t) = \mathcal{B} U(t), \\ U(0) = U_0, \end{cases} \tag{4.3.7}$$

where

$$\begin{cases} U(t) = (u(\cdot,t), u_t(\cdot,t), h_1(\cdot,t), \cdots, h_N(\cdot,t)), \\ U_0 = (u_0(\cdot), u_1(\cdot), 0, \cdots, 0) \end{cases}$$

is the state variable and the initial value, respectively.

The following Lemma 4.3.1 shows that $\mathcal{B}$ is invertible.

Lemma 4.3.1. *Let $\mathcal{B}$ be defined by (4.3.6). Then $0 \in \rho(\mathcal{B})$.*

Proof. Let $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{h_1}, \cdots, \widetilde{h_N}) \in \mathcal{H}$. Solve $\mathcal{B}U = \widetilde{U}$ for $U = (u, v, h_1, \cdots, h_N)$, that is

$$\begin{cases} v(x) = \widetilde{u}(x), \\ \left(au'(x) - b\sum_{j=1}^{N} h_j(x)\right)' = \widetilde{v}(x), \\ a_j b_j u'(x) - b_j h_j(x) = \widetilde{h}_j(x), \quad j = 1, 2, \cdots, N, \\ u(0) = u(1) = 0, \end{cases} \tag{4.3.8}$$

to give

$$\begin{cases} v(x) = \widetilde{u}(x), \\ h_j(x) = a_j u'(x) - \dfrac{1}{b_j}\widetilde{h}_j(x), \; j = 1, 2, \cdots, N \end{cases} \tag{4.3.9}$$

and

$$\left(a - b\sum_{j=1}^{N} a_j\right) u'(x) + b\sum_{j=1}^{N} \frac{1}{b_j}\widetilde{h}_j(x) = \int_0^x \widetilde{v}(\tau)d\tau + C_1, \tag{4.3.10}$$

where C_1 is a constant to be determined. By the boundary condition $u(0) = 0$, it follows from (4.3.10) that

$$u(x) = -\frac{b}{A}\int_0^x \sum_{j=1}^{N} \frac{1}{b_j}\widetilde{h}_j(\tau)d\tau + \frac{1}{A}\int_0^x \int_0^s \widetilde{v}(\tau)d\tau ds + \frac{C_1}{A}x, \tag{4.3.11}$$

where

$$A = a - b\sum_{j=1}^{N} a_j.$$

Using the other boundary condition $u(1) = 0$, it gives

$$C_1 = b\int_0^1 \sum_{j=1}^{N} \frac{1}{b_j}\widetilde{h}_j(x)dx - \int_0^1 \int_0^s \widetilde{v}(\tau)d\tau ds. \tag{4.3.12}$$

This together with (4.3.9) and (4.3.11) gives the unique solution $U \in D(\mathcal{B})$ to equation (4.3.8). Hence $\mathcal{B}^{-1}$ exists and is bounded, or

$$0 \in \rho(\mathcal{B}).$$

■

4.3.2 Spectrum of system operator

In this subsection, we consider the spectrum of $\mathcal{B}$. As in previous section, we first consider the eigenvalue problem. Suppose

$$\mathcal{B}U = \lambda U, \quad \lambda \in \mathbb{C}, \ 0 \neq U = (u, v, h_1, \cdots, h_N) \in D(\mathcal{B}),$$

that is,

$$\begin{cases} v(x) = \lambda u(x), \\ \left(au'(x) - b\sum_{j=1}^{N} h_j(x) \right)' = \lambda v(x), \\ a_j b_j u'(x) - b_j h_j(x) = \lambda h_j(x), \quad j = 1, 2, \cdots, N, \\ u(0) = u(1) = 0. \end{cases} \tag{4.3.13}$$

Proposition 4.3.1. *Let $\mathcal{B}$ be defined by (4.3.6). Then $\lambda = -b_j$, $j = 1, 2, \cdots, N$ are eigenvalues of $\mathcal{B}$, which corresponding to eigenfunctions e_{j+2}, $j = 1, 2, \cdots, N$ respectively, where e_j is a constant function whose element is the j-th element of the canonical basis of $\mathbb{R}^{N+2}$. Moreover, each of these eigenvalues is algebraically simple.*

Proof. We only give the proof for $\lambda = -b_1$ because other cases can be treated similarly. Let $\lambda = -b_1$ and $U = (u, v, h_1, \cdots, h_N) \in D(\mathcal{B})$. Since $\lambda = -b_1$, (4.3.13) becomes

$$\begin{cases} v(x) = -b_1 u(x) \\ \left(au'(x) - b\sum_{j=1}^{N} h_j(x) \right)' = -b_1 v(x), \\ a_1 b_1 u'(x) = 0, \\ (b_j - b_1) h_j(x) = a_j b_j u'(x), \quad j = 2, \cdots, N, \\ u(0) = u(1) = 0. \end{cases} \tag{4.3.14}$$

This together with (4.1.7) yields

$$u(x) = v(x) = h_j(x) = 0, \ j = 2, \cdots, N. \tag{4.3.15}$$

By the second equation of (4.3.14), it has

$$h_1'(x) = 0.$$

Therefor, e_3 is an eigenfunction of $\mathcal{B}$ corresponding to $-b_1$. Further computation of

$$(b_1 I + \mathcal{B})U_1 = -e_3,$$

where $U_1 = (\widetilde{u}, \widetilde{v}, \widetilde{h_1}, \cdots, \widetilde{h_N}) \in D(\mathcal{B})$, gives

$$\begin{cases} \widetilde{v}(x) = -b_1 \widetilde{u}(x), \\ \left(a\widetilde{u}'(x) - b \displaystyle\sum_{j=1}^{N} \widetilde{h_j}(x) \right)' = -b_1 \widetilde{v}(x), \\ a_1 b_1 \widetilde{u}'(x) = -1, \\ (b_j - b_1)\widetilde{h_j}(x) = a_j b_j \widetilde{u}'(x), \quad j = 2, \cdots, N, \\ \widetilde{u}(0) = \widetilde{u}(1) = 0. \end{cases} \tag{4.3.16}$$

(4.3.16) has no solution since otherwise, $\widetilde{u}$ satisfies

$$a_1 b_1 \widetilde{u}'(x) = -1, \quad \widetilde{u}(0) = \widetilde{u}(1) = 0,$$

which is impossible. This shows that $-b_1$ is algebraically simple. ■

When $\lambda \neq -b_j$, $j = 1, 2, \cdots, N$, it follows from (4.3.13) that

$$\begin{cases} v(x) = \lambda u(x), \\ h_j(x) = \dfrac{a_j b_j}{\lambda + b_j} u'(x), \quad j = 1, 2, \cdots, N \end{cases} \tag{4.3.17}$$

and u satisfies

$$\begin{cases} \left(a - b \displaystyle\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j} \right) u''(x) = \lambda^2 u(x), \\ u(0) = u(1) = 0. \end{cases} \tag{4.3.18}$$

The following Lemma 4.3.2 is straightforward.

Lemma 4.3.2. *Let $\mathcal{B}$ be defined by (4.3.6) and*

$$\Lambda = \left\{ \lambda \in \mathbb{C} \;\middle|\; a - b \sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j} = 0 \right\}. \tag{4.3.19}$$

Then

$$\Lambda \cap \sigma_p(\mathcal{B}) = \emptyset. \tag{4.3.20}$$

Lemma 4.3.3. *Let $\mathcal{B}$ be defined by (4.3.6). Λ is given by (4.3.19). Then*

$$\Lambda = \{\lambda_{c1}, \lambda_{c2}, \cdots, \lambda_{cN}\}, \tag{4.3.21}$$

where $\lambda_{c1} \in (-b_1, 0)$, and $\lambda_{ck} \in (-b_k, -b_{k-1}), k = 2, \cdots, N$.

Proof. Since $-b_j \notin \Lambda$, $j = 1, 2, \cdots, N$, $p(\lambda) = 0$ is equivalent to $q(\lambda) = 0$, where

$$\begin{cases} p(\lambda) = a - b\displaystyle\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j}, \\ q(\lambda) = p(\lambda)\displaystyle\prod_{j=1}^{N}(\lambda + b_j). \end{cases} \tag{4.3.22}$$

However, $q(\lambda)$ is an N-th order polynomial, and hence there are at most N number of zeros for $p(\lambda)$. Now we find all these zeros.

Since $p(\lambda)$ is continues in $(-b_1, \infty) \cup (\cup_{j=1}^{N-1}(-b_{j+1}, -b_j))$, by the fact

$$\lim_{\lambda \to -b_1^+} p(\lambda) = -\infty$$

and (4.1.8), we see that there exists a solution to $p(\lambda) = 0$ in $(-b_1, 0)$. For any $j = 1, 2, \cdots, N-1$, it has

$$\lim_{\lambda \to -b_{j+1}^+} p(\lambda) = -\infty, \quad \lim_{\lambda \to -b_j^-} p(\lambda) = +\infty.$$

Therefore, there exists a solution to $p(\lambda) = 0$ in $(-b_{j+1}, -b_j)$. The proof is complete. ■

By Lemma 4.3.2, the eigenvalue problem (4.3.18) is equivalent to the following problem:

$$\begin{cases} u''(x) = \dfrac{\lambda^2}{p(\lambda)} u(x), \\ u(0) = u(1) = 0, \end{cases} \tag{4.3.23}$$

where $p(\lambda)$ is given by (4.3.22). Hence

$$u(x) = e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}x} - e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}x}. \tag{4.3.24}$$

By the boundary condition $u(1) = 0$, (4.3.23) has non-trivial solution if and only if

$$e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}} - e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}} = 0. \tag{4.3.25}$$

That is

$$e^{2\sqrt{\frac{\lambda^2}{p(\lambda)}}} = 1,$$

which is equivalent to

$$\frac{\lambda^2}{p(\lambda)} = -n^2\pi^2, \; n = 1, 2, \cdots. \tag{4.3.26}$$

Substituting (4.3.26) into (4.3.24), we obtain the eigenfunction

$$\left(u(x), \lambda u(x), \frac{a_1 b_1}{\lambda + b_1} u'(x), \cdots, \frac{a_N b_N}{\lambda + b_N} u'(x)\right)$$

corresponding to λ, where

$$u(x) = \sin n\pi x, \tag{4.3.27}$$

for some $n \in \mathbb{N}^+$.

When $|\lambda|$ is large enough, since

$$\begin{aligned}
\frac{\lambda^2}{p(\lambda)} &= \frac{\lambda^2}{a} \cdot \frac{1}{1 - \frac{b}{a}\sum_{j=1}^N \frac{a_j b_j}{\lambda + b_j}} \\
&= \frac{\lambda^2}{a}\left(1 + \frac{b}{a}\sum_{j=1}^N \frac{a_j b_j}{\lambda + b_j} + \frac{b^2}{a^2}\left(\sum_{j=1}^N \frac{a_j b_j}{\lambda + b_j}\right)^2\right) + \mathcal{O}(|\lambda|^{-1}) \\
&= \frac{1}{a}\left(\lambda^2 + \frac{b}{a}\sum_{j=1}^N \frac{a_j b_j \lambda}{1 + \frac{b_j}{\lambda}} + \frac{b^2}{a^2}\left(\sum_{j=1}^N \frac{a_j b_j}{1 + \frac{b_j}{\lambda}}\right)^2\right) + \mathcal{O}(|\lambda|^{-1}) \\
&= \frac{1}{a}\left(\lambda^2 + \frac{b}{a}\sum_{j=1}^N a_j b_j \lambda - \frac{b}{a}\sum_{j=1}^N a_j b_j^2 + \frac{b^2}{a^2}\left(\sum_{j=1}^N a_j b_j\right)^2\right) + \mathcal{O}(|\lambda|^{-1}),
\end{aligned}$$

it has

$$\lambda^2 + \frac{b}{a}\sum_{j=1}^N a_j b_j \lambda - \frac{b}{a}\sum_{j=1}^N a_j b_j^2 + \frac{b^2}{a^2}\left(\sum_{j=1}^N a_j b_j\right)^2 + an^2\pi^2 + \mathcal{O}(|\lambda|^{-1}) = 0.$$

Thus, the eigenvalues of $\mathcal{B}$ in this case are found to be

$$\lambda_n = -\frac{b}{2a}\sum_{j=1}^N a_j b_j \pm i\sqrt{a}n\pi + \mathcal{O}(n^{-1}), \quad n \to \infty.$$

For any $\lambda_c \in \Lambda$, when $\lambda \to \lambda_c$, $\mu = \lambda - \lambda_c \to 0$. Notice that

$$p(\lambda) = \mu b \sum_{j=1}^N a_j b_j \left[\frac{1}{(\lambda_c + b_j)^2} - \frac{\mu}{(\lambda_c + b_j)^3} + \mathcal{O}(\mu^2)\right].$$

We have

$$\frac{\lambda^2}{p(\lambda)} = \frac{1}{\mu}\frac{\lambda_c^2}{\Delta}\left[1 + \left(\frac{2}{\lambda_c} + \frac{\widetilde{\Delta}}{\Delta}\right)\mu\right] + \mathcal{O}(\mu),$$

where

$$\begin{cases}
\Delta = \displaystyle\sum_{j=1}^N \frac{b a_j b_j}{(\lambda_c + b_j)^2}, \\
\widetilde{\Delta} = \displaystyle\sum_{j=1}^N \frac{a_j b_j}{(\lambda_c + b_j)^3}.
\end{cases}$$

This together with (4.3.26) yields

$$\frac{1}{\mu}\frac{\lambda_c^2}{\Delta}\left[1 + \left(\frac{2}{\lambda_c} + \frac{\widetilde{\Delta}}{\Delta}\right)\mu\right] + \mathcal{O}(\mu) = -n^2\pi^2, \ n \to \infty.$$

Thus,

$$\mu_n = -\frac{1}{n^2\pi^2}\frac{\lambda_c^2}{\Delta} + \mathcal{O}(n^{-3}), \quad n \to \infty.$$

Hence, the eigenvalues of $\mathcal{B}$ in this case are given by

$$\lambda_n = \lambda_c - \frac{1}{n^2\pi^2}\frac{\lambda_c^2}{\Delta} + \mathcal{O}(n^{-3}), \quad n \to \infty.$$

We summarize these results as Proposition 4.3.2 following.

Proposition 4.3.2. *Let $\mathcal{B}$ be defined by (4.3.6), λ be an eigenvalue of $\mathcal{B}$, satisfying*

$$\lambda \neq -b_j, \quad j = 1, 2, \cdots, N.$$

Then the eigenfunction corresponding to λ is of the form

$$\left(u(x), \lambda u(x), \frac{a_1 b_1}{\lambda + b_1}u'(x), \cdots, \frac{a_N b_N}{\lambda + b_N}u'(x)\right),$$

where

$$u(x) = \sin n\pi x, \tag{4.3.28}$$

for some $n \in \mathbb{N}^+$. Furthermore,

(i). For any $1 \le k \le N$, there is a sequence of eigenvalues $\{\lambda_{nk}\}$ of $\mathcal{B}$, which have the following asymptotic expressions:

$$\lambda_{nk} = \lambda_{ck} - \frac{1}{n^2\pi^2}\frac{\lambda_{ck}^2}{\Delta_k} + \mathcal{O}(n^{-3}), \quad n \to \infty, \tag{4.3.29}$$

where

$$\Delta_k = \sum_{j=1}^{N} \frac{b a_j b_j}{(\lambda_{ck} + b_j)^2}. \tag{4.3.30}$$

The corresponding eigenfunctions

$$\left(u_n(x), \lambda u_n(x), \frac{a_1 b_1}{\lambda + b_1}u_n'(x), \cdots, \frac{a_N b_N}{\lambda + b_N}u_n'(x)\right)$$

satisfy

$$u_n(x) = \frac{1}{n\pi}\sin n\pi x, \quad n \to \infty. \tag{4.3.31}$$

(ii). When $|\lambda| \to \infty$, the eigenvalues $\{\lambda_{n0}, \overline{\lambda_{n0}}\}$ of $\mathcal{B}$ have the following asymptotic expressions:

$$\lambda_{n0} = -\frac{b}{2a}\sum_{j=1}^{N} a_j b_j + i\sqrt{a}n\pi + \mathcal{O}(n^{-1}), \quad n \to \infty, \tag{4.3.32}$$

where $\overline{\lambda_{n0}}$ denotes the complex conjugate of λ_{n0}. In particular,

$$\mathrm{Re}\lambda_{n0} \to -\frac{b}{2a}\sum_{j=1}^{N} a_j b_j < 0, \quad n \to \infty, \tag{4.3.33}$$

that is, $\mathrm{Re}\lambda = -\frac{b}{2a}\sum_{j=1}^{N} a_j b_j$ *is the asymptote of the eigenvalues* λ_{n0} *given by (4.3.32). Furthermore, the corresponding eigenfunctions*

$$\left(u_n(x), \lambda u_n(x), \frac{a_1 b_1}{\lambda + b_1} u_n'(x), \cdots, \frac{a_N b_N}{\lambda + b_N} u_n'(x)\right)$$

satisfy (4.3.31).

Combing Proposition 4.3.1 and 4.3.2, we obtain the following Theorem 4.3.1.

Theorem 4.3.1. *Let* $\mathcal{B}$ *be defined by (4.3.6). Then*

(i). $\mathcal{B}$ *has the eigenvalues*

$$\{-b_j, j = 1, 2, \cdots, N\} \cup \{\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}, \lambda_{n0}, \overline{\lambda_{n0}},\ n \in \mathbb{N}^+\}, \tag{4.3.34}$$

where $\lambda_{nk}, k = 1, 2, \cdots, N$ *and* λ_{n0} *have the asymptotic expressions (4.3.29) and (4.3.32), respectively.*

(ii). The eigenfunction corresponding to $-b_j$ *is* e_{j+2} *for any* $j = 1, 2, \cdots, N$.

(iii). The eigenfunctions corresponding to λ_{nk}, $k = 1, 2, \cdots, N$, *are given by*

$$\begin{aligned} U_{nk}(x) = &\left(\frac{1}{n\pi}\sin n\pi x, 0, \frac{a_1 b_1}{\lambda_{nk} + b_1}\cos n\pi x, \cdots, \frac{a_N b_N}{\lambda_{nk} + b_N}\cos n\pi x\right) \\ &+ (0, 1, \cdots, 1)\mathcal{O}(n^{-1}), \quad n \to \infty. \end{aligned} \tag{4.3.35}$$

(iv). The eigenfunctions corresponding to λ_{n0} *and* $\overline{\lambda_{n0}}$, *are given by*

$$U_{n0}(x) = \left(\frac{1}{n\pi}\sin n\pi x, i\sqrt{a}\sin n\pi x, 0, \cdots, 0\right) + (0, 1, \cdots, 1)\mathcal{O}(n^{-1}),\ n \to \infty \tag{4.3.36}$$

and

$$\overline{U_{n0}}(x) = \left(\frac{1}{n\pi}\sin n\pi x, -i\sqrt{a}\sin n\pi x, 0, \cdots, 0\right) + (0, 1, \cdots, 1)\mathcal{O}(n^{-1}),\ n \to \infty \tag{4.3.37}$$

respectively.

Concerning about $\sigma(\mathcal{B})$, we have the following Theorem 4.3.2.

Theorem 4.3.2. *Let* $\mathcal{B}$ *be defined by (4.3.6).* Λ *is given by (4.3.19). Then*

$$\sigma(\mathcal{B}) = \Lambda \cup \sigma_p(\mathcal{B}). \tag{4.3.38}$$

Proof. Let $\lambda \notin \sigma_p(\mathcal{B})$. For any $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{h_1}, \cdots, \widetilde{h_N}) \in \mathcal{H}$. Solve

$$(\lambda I - \mathcal{B})U = \widetilde{U}$$

for $U = (u, v, h_1, \cdots, h_N)$, that is,

$$\begin{cases} \lambda u(x) - v(x) = \widetilde{u}(x), \\ \lambda v(x) - \left(au'(x) - b\sum_{j=1}^{N} h_j(x) \right)' = \widetilde{v}(x), \\ \lambda h_j(x) - (a_j b_j u'(x) - b_j h_j(x)) = \widetilde{h}_j(x),\ j = 1, 2, \cdots, N, \\ u(0) = u(1) = 0, \end{cases} \tag{4.3.39}$$

to get

$$\begin{cases} v(x) = \lambda u(x) - \widetilde{u}(x), \\ h_j(x) = \dfrac{1}{\lambda + b_j}(a_j b_j u'(x) + \widetilde{h}_j(x)),\ j = 1, 2, \cdots, N \end{cases} \tag{4.3.40}$$

and

$$\begin{cases} \theta'(x) = \lambda^2 u(x) - \lambda \widetilde{u}(x) - \widetilde{v}(x), \\ \theta(x) = p(\lambda) u'(x) - \sum_{j=1}^{N} \dfrac{b}{\lambda + b_j} \widetilde{h}_j(x), \\ u(0) = u(1) = 0, \end{cases} \tag{4.3.41}$$

where $p(\lambda)$ is given by (4.3.22). There are two cases:

Case I: $\lambda \notin \Lambda$. In this case, $p(\lambda) \neq 0$. Since by Lemma 4.3.1, $0 \in \rho(\mathcal{B})$, we only need consider the case of $\lambda \neq 0$. Now, we can rewrite (4.3.41) as the following first order system of differential equations:

$$\begin{cases} \dfrac{d}{dx} \begin{pmatrix} u(x) \\ \theta(x) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{p(\lambda)} \\ \lambda^2 & 0 \end{pmatrix} \begin{pmatrix} u(x) \\ \theta(x) \end{pmatrix} + \begin{pmatrix} \frac{1}{p(\lambda)} V(x) \\ -\lambda \widetilde{u}(x) - \widetilde{v}(x) \end{pmatrix}, \\ u(0) = u(1) = 0, \end{cases} \tag{4.3.42}$$

where

$$V(x) = \sum_{j=1}^{N} \frac{b}{\lambda + b_j} \widetilde{h}_j(x). \tag{4.3.43}$$

Let

$$A(\lambda) = \begin{pmatrix} 0 & \frac{1}{p(\lambda)} \\ \lambda^2 & 0 \end{pmatrix}.$$

Then

$$e^{A(\lambda)x} = \begin{pmatrix} a_{11}(\lambda, x) & a_{12}(\lambda, x) \\ a_{21}(\lambda, x) & a_{22}(\lambda, x) \end{pmatrix},$$

where

$$\begin{cases} a_{11}(\lambda,x)=\cosh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \\ a_{21}(\lambda,x)=\lambda\sqrt{p(\lambda)}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \end{cases} \qquad \begin{cases} a_{12}(\lambda,x)=\frac{1}{\lambda\sqrt{p(\lambda)}}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right), \\ a_{22}(\lambda,x)=\cosh\left(\frac{\lambda}{\sqrt{p(\lambda)}}x\right). \end{cases}$$

The general solution of (4.3.42) is given by

$$\begin{pmatrix} u(x) \\ \theta(x) \end{pmatrix} = e^{A(\lambda)x}\begin{pmatrix} u(0) \\ \theta(0) \end{pmatrix} - \int_0^x e^{A(\lambda)(x-\gamma)}\begin{pmatrix} \frac{1}{p(\lambda)}V(\gamma) \\ -\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma) \end{pmatrix} d\gamma.$$

By $u(0)=0$, it has,

$$\begin{aligned} u(x) &= a_{12}(\lambda,x)\theta(0) \\ &\quad -\int_0^x \left[\frac{1}{p(\lambda)}a_{11}(\lambda,x-\gamma)V(\gamma)+a_{12}(\lambda,x-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right]d\gamma \end{aligned} \tag{4.3.44}$$

and

$$\begin{aligned} \theta(x) &= a_{22}(\lambda,x)\theta(0) \\ &\quad -\int_0^x \left[\frac{1}{p(\lambda)}a_{21}(\lambda,x-\gamma)V(\gamma)+a_{22}(\lambda,x-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right]d\gamma. \end{aligned} \tag{4.3.45}$$

Since $\lambda\notin\sigma_p(\mathcal{B})$, by (4.3.25),

$$a_{12}(\lambda,1)=\frac{1}{\lambda\sqrt{p(\lambda)}}\sinh\left(\frac{\lambda}{\sqrt{p(\lambda)}}\right)\neq 0.$$

By the boundary condition $u(1)=0$, it has

$$\begin{aligned} &\theta(0) \\ &= \frac{1}{a_{12}(\lambda,1)}\int_0^1 \left[\frac{1}{p(\lambda)}a_{11}(\lambda,1-\gamma)V(\gamma)+a_{12}(\lambda,1-\gamma)(-\lambda\widetilde{u}(\gamma)-\widetilde{v}(\gamma))\right]d\gamma. \end{aligned} \tag{4.3.46}$$

Hence u is uniquely determined by (4.3.44). By the second equation of (4.3.41) and (4.3.45), we know that

$$u'\in L^2(0,1).$$

This together with (4.3.40) shows that $(\lambda I-\mathcal{B})^{-1}$ exists and is bounded, or

$$\lambda\in\rho(\mathcal{B}).$$

Case II: $\lambda\in\Lambda$. In this case, $\lambda\neq 0$. By (4.3.41),

$$\begin{cases} u(x)=\frac{1}{\lambda^2}\left(\lambda\widetilde{u}(x)+\widetilde{v}(x)-V'(x)\right), \\ u(0)=u(1)=0, \end{cases} \tag{4.3.47}$$

where V is given by (4.3.43). Since $\widetilde{u} \in H_0^1(0,1)$, (4.3.47) means that (4.3.39) admits a solution if and only if V is differentiable, and

$$\widetilde{v}(0) - V'(0) = \widetilde{v}(1) - V'(1) = 0.$$

Thus $\lambda \notin \rho(\mathcal{B})$.

The result follows by combining of these two cases. ■

In order to investigate the residual and continuous spectrum of $\mathcal{B}$, we need the adjoint operator $\mathcal{B}^*$.

Lemma 4.3.4. *Let $\mathcal{B}$ be defined by (4.3.6). Then*

$$\mathcal{B}^* \begin{pmatrix} u \\ v \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^\top = \begin{pmatrix} -v + \frac{1}{a}\sum_{j=1}^N a_j b_j \int_0^x h_j(\tau)d\tau \\ -au'' \\ bv' - b_1 h_1 \\ \vdots \\ bv' - b_N h_N \end{pmatrix}^\top, \tag{4.3.48}$$

with

$$D(\mathcal{B}^*) = \left\{ \begin{pmatrix} u \\ v \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^\top \middle| \begin{array}{l} u, v, \sum_{j=1}^N a_j b_j \int_0^x h_j(\tau)d\tau \in H_0^1(0,1), \\ u'', h_j \in L^2(0,1),\ j = 1, \cdots, N. \end{array} \right\}. \tag{4.3.49}$$

Theorem 4.3.3. *Let $\mathcal{B}$ be defined by (4.3.6). Then*

$$\sigma_r(\mathcal{B}) = \emptyset, \quad \sigma_c(\mathcal{B}) = \Lambda, \tag{4.3.50}$$

where $\sigma_r(\mathcal{B})$ and $\sigma_c(\mathcal{B})$ denotes the set of residual and continuous spectrum of $\mathcal{B}$, respectively.

Proof. By Lemma 4.3.2 and Theorem 4.3.2, we only need to prove

$$\Lambda \cap \sigma_r(\mathcal{B}) = \emptyset.$$

Since $\lambda \in \sigma_r(\mathcal{B})$ implies $\overline{\lambda} \in \sigma_p(\mathcal{B}^*)$, it suffices to show that

$$\Lambda \cap \sigma_p(\mathcal{B}^*) = \emptyset.$$

Suppose that $\mathcal{B}^* U = \lambda U$ for $\lambda \in \mathbb{C}$ and $0 \neq U = (u, v, h_1, \cdots, h_N) \in D(\mathcal{B}^*)$. Then

$$\begin{cases} -v(x) + \dfrac{1}{a}\displaystyle\sum_{j=1}^{N} a_j b_j \int_0^x h_j(\tau)d\tau = \lambda u(x), \\ -au''(x) = \lambda v(x), \\ bv'(x) - b_j h_j(x) = \lambda h_j(x), \quad j = 1, 2, \cdots, N, \\ v(0) = v(1) = 0. \end{cases}$$

When $\lambda \neq -b_j$, $j = 1, 2, \cdots, N$, v satisfies

$$\begin{cases} \left(a - b \displaystyle\sum_{j=1}^{N} \frac{a_j b_j}{\lambda + b_j} \right) v''(x) = \lambda^2 v(x), \\ v(0) = v(1) = 0. \end{cases} \tag{4.3.51}$$

For any $\lambda \in \Lambda$, it has $v = 0$. This implies that $U = 0$. Therefore,

$$\lambda \notin \sigma_p(\mathcal{B}^*).$$

So, $\Lambda \cap \sigma_p(\mathcal{B}^*) = \emptyset$. The proof is complete. ■

4.3.3 Riesz basis property

Now, we study the Riesz basis property for system (4.3.3). To this purpose, we need the following Theorem 4.3.4, which was proved in [55] (The similar result can be found in [93]). Since [55] is not published, we attach its brief proof as Appendix.

Theorem 4.3.4. *Let A be a densely defined closed linear operator in a Hilbert space H with isolated eigenvalues $\{\lambda_i\}_1^\infty$ and $\sigma_r(A) = \emptyset$. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. Suppose that there are $N_0 \geq 1$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N_0}^\infty$ of A such that*

$$\sum_{n=N_0}^{\infty} \|\psi_n - \phi_n\|_H^2 < \infty. \tag{4.3.52}$$

Then there exist $M (\geq N_0)$ number of generalized eigenvectors $\{\psi_{n_0}\}_1^M$ such that $\{\psi_{n_0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H.

Theorem 4.3.5. *Let $\mathcal{B}$ be defined by (4.3.6). Then*

(i). There is a sequence of generalized eigenfunctions of $\mathcal{B}$, which forms a Riesz basis for the state space $\mathcal{H}$.

(ii). All eigenvalues except finitely many are algebraically simple.

(iii). $\mathcal{B}$ generates a C_0-semigroup $e^{\mathcal{B}t}$ on $\mathcal{H}$.

Therefore, for the semigroup $e^{\mathcal{B}t}$, the Spectrum-determined growth condition holds: $\omega(\mathcal{B}) = s(\mathcal{B})$.

Proof. Since from Theorem 4.3.1, all eigenvalues are located in some left half complex plane, the other parts follow directly from (i) and (ii). So we only need to prove (i) and (ii). For any $n \in \mathbb{N}^+$, set

$$V_{n0} = \left(\frac{1}{n\pi}\sin n\pi x, i\sqrt{a}\sin n\pi x, 0, \cdots, 0\right), \tag{4.3.53}$$

$$\begin{cases} \varphi_{n0} = \left(\sqrt{a}\cos n\pi x, i\sqrt{a}\sin n\pi x, 0, \cdots, 0\right) + (0,1,1,\cdots,1)\mathcal{O}(n^{-1}), \\ \varphi_{nk} = \left(\sqrt{a}, 0, \dfrac{a_1 b_1}{\lambda_{nk}+b_1}, \cdots, \dfrac{a_N b_N}{\lambda_{nk}+b_N}\right)\cos n\pi x + (0,1,1,\cdots,1)\mathcal{O}(n^{-1}), \\ \qquad\qquad k = 1, 2, \cdots, N. \end{cases} \tag{4.3.54}$$

Define the reference sequence:

$$\begin{cases} \psi_{n0} = \left(\sqrt{a}\cos n\pi x, i\sqrt{a}\sin n\pi x, 0, \cdots, 0\right), \\ \psi_{nk} = \left(0, 0, \dfrac{a_1 b_1}{\lambda_{nk}+b_1}, \cdots, \dfrac{a_N b_N}{\lambda_{nk}+b_N}\right)\cos n\pi x, \ \ k = 1, 2, \cdots, N. \end{cases} \tag{4.3.55}$$

Since $b_j \neq b_k$, $\lambda_{nj} \neq \lambda_{nk}$, $1 \le j < k \le N$, a direct computation shows that

$$\det \begin{pmatrix} \frac{a_1b_1}{\lambda_{n1}+b_1} & \frac{a_1b_1}{\lambda_{n2}+b_1} & \cdots & \frac{a_1b_1}{\lambda_{nN}+b_1} \\ \frac{a_2b_2}{\lambda_{n1}+b_2} & \frac{a_2b_2}{\lambda_{n2}+b_2} & \cdots & \frac{a_2b_2}{\lambda_{nN}+b_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_Nb_N}{\lambda_{n1}+b_N} & \frac{a_Nb_N}{\lambda_{n2}+b_N} & \cdots & \frac{a_Nb_N}{\lambda_{nN}+b_N} \end{pmatrix} \neq 0.$$

Hence,

$$\{\psi_{n0}, \overline{\psi_{n0}}, \psi_{n1}, \psi_{n2}, \cdots, \psi_{nN}\}_1^\infty \tag{4.3.56}$$

forms a Riesz basis for $\mathcal{H}_1 = (L^2(0,1))^{N+2}$. By (4.3.54), (4.3.55) and Theorem 4.3.1, there exists an $N_0 \in \mathbb{N}^+$, such that,

$$
\begin{aligned}
&\sum_{n=N_0}^{\infty}\left[\|U_{n0}-V_{n0}\|_{\mathcal{H}}^2+\|\overline{U_{n0}}-\overline{V_{n0}}\|_{\mathcal{H}}^2+\sum_{k=1}^{N}\left\|U_{nk}-\frac{U_{n0}+\overline{U_{n0}}}{2}-\psi_{nk}\right\|_{\mathcal{H}}^2\right]\\
&=\sum_{n=N_0}^{\infty}\left[\|\varphi_{n0}-\psi_{n0}\|_{\mathcal{H}_1}^2+\|\overline{\varphi_{n0}}-\overline{\psi_{n0}}\|_{\mathcal{H}_1}^2+\sum_{k=1}^{N}\left\|\varphi_{nk}-\frac{\varphi_{n0}+\overline{\varphi_{n0}}}{2}-\psi_{nk}\right\|_{\mathcal{H}_1}^2\right] \quad (4.3.57)\\
&<\infty.
\end{aligned}
$$

By Theorem 4.3.4, (i) and hence (ii) hold true. The proof is complete. ∎

Combing Theorem 4.3.1, 4.3.2 and 4.3.5, we conclude the exponential stability of system (4.3.3).

Theorem 4.3.6. *System (4.3.3) is exponentially stable, that is,*

$$E(t) \le Me^{-\omega t}E(0), \tag{4.3.58}$$

for some $M, \omega > 0$, where $E(t)$ is given by (4.3.4).

Appendix

Proof of Theorem 4.3.4

For the sake of coherent understanding, we first define the **spectral projection** corresponding to the operator A which is available in most of textbooks of functional analysis.

Definition *(Spectral Projection): Let A be a closed operator and $\gamma \subset \sigma(A)$ be a compact subset of $\mathbb{C}$ which is open and closed in $\sigma(A)$. A subset with these properties will be called a compact spectral set. With the compact spectral set γ, we can construct a closed Jordan curve Γ, which is oriented in the customary positive sense of complex variable theory, and it bounds a finite domain containing every point of γ and no point of $\sigma(A)\backslash\gamma$. The spectral projection on γ is now defined as*

$$E(\gamma) := \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - A)^{-1}d\lambda. \qquad (Ap1)$$

When $\gamma = \{\lambda\}$, where λ is an isolated eigenvalue of A, we shall denote

$$E(\lambda, A) := E(\{\lambda\}). \qquad (Ap2)$$

The following basic properties of spectral projection can be found in chapter XVIII of [24].

Proposition A1 *The spectral $E(\cdot)$ defined by (Ap1) has the following properties:*

(i). $E(\gamma)$ is a projection (not necessarily self-adjoint).

(ii). $E(\emptyset) = 0$.

(iii) $E(\gamma_1)E(\gamma_2) = E(\gamma_1 \cap \gamma_2)$.

(iv). If λ is an isolated eigenvalue, with finite multiplicity, then $E(\lambda, A)$ is the projection on the space of generalized eigenvectors of A corresponding to λ, that is, the subspace spanned by all those ϕ satisfying $(\lambda I - A)^n\phi = 0$ for some positive integer n.

Next, we give two lemmas.

Lemma A1. *Let A be a linear operator in a Hilbert space H with isolated eigenvalues and residual spectrum $\{\lambda_i\}_1^\infty$, $\rho(A) \neq \emptyset$. Let*

$$\sigma_\infty = \{x \mid E(\lambda_i, A)x = 0, i \geq 1\}. \tag{Ap3}$$

Then σ_∞ is either 0 or infinite dimensional.

Proof. Suppose first that A is bounded and $0 < \dim \sigma_\infty < \infty$. Since σ_∞ is invariant subspace of A, that is, $A\sigma_\infty \subset \sigma_\infty$, A has at least one eigenvector $x_\infty \in \sigma_\infty$ such that $Ax_\infty = \eta x_\infty$ for some constant η. So $\eta = \lambda_i$ for some i, and hence,

$$x_\infty = E(\lambda_i, A)x_\infty = 0,$$

which is a contradiction. So (Ap3) holds true.

If A is unbounded. Take $\lambda_0 \in \rho(A)$ such that $\mid \lambda_0 - \lambda_i \mid \geq \varepsilon > 0$ for all $i \geq 1$. Let $T = (\lambda_0 I - A)^{-1}$, $\mu_i = (\lambda_0 - \lambda_i)^{-1}$, $i = 1, 2, \cdots$. Then it is well-known that

$$\lambda_i \in \sigma_p(A) \text{ if and only if } \mu_i \in \sigma_p(T), \lambda_i \in \sigma_r(A) \text{ if and only if } \mu_i \in \sigma_r(T)$$

and

$$E(\lambda_i, A) = E(\mu_i, T), \quad \text{for all } i \geq 1.$$

Hence

$$\sigma_\infty = \{x \mid E(\mu_i, T)x = 0, \mu_i \in \sigma_p(T) \cup \sigma_r(T)\}.$$

Since T is bounded, σ_∞ is either 0 or infinite dimensional. ■

Lemma A2. *Let A be a densely defined closed operator in a Hilbert space H with isolated eigenvalues $\{\lambda_i\}_1^\infty$. Then*

$$H = \mathrm{sp}(A) \oplus \sigma_\infty^*, \tag{Ap4}$$

where $\mathrm{sp}(A)$ denote the closed linear span of all generalized eigenfunctions of A, and

$$\sigma_\infty^* = \{x \mid E(\overline{\lambda}_i, A^*)x = 0, \lambda_i \in \sigma_p(A)\}.$$

Proof. By a well-known fact $\sigma(A^*) = \{\overline{\lambda} \mid \lambda \in \sigma(A)\}$, $\overline{\lambda}_i$ is an isolated spectral point of A^* and so $E(\overline{\lambda}_i, A^*)$ makes sense. For any $f \in E(\lambda_i, A)H$, $g^* \in \sigma_\infty^*$, we have $E(\lambda_i, A)f = f$. And hence

$$\langle f, g^* \rangle = \langle E(\lambda_i, A)f, g^* \rangle = \langle f, E(\overline{\lambda}_i, A^*)g^* \rangle = 0.$$

So $\mathrm{sp}(A) \subset (\sigma_\infty^*)^\perp$. If $f \notin \mathrm{sp}(A)$, then there exists a functional g^* such that

$$\langle f, g^* \rangle = 1, \langle h, g^* \rangle = 0, \text{ for all } h \in \mathrm{sp}(A).$$

For any $w \in H$, it follows from $E(\lambda_i, A)w \in \mathrm{sp}(A)$ that

$$\langle w, E(\overline{\lambda}_i, A^*)g^* \rangle = \langle E(\lambda_i, A)w, g^* \rangle = 0.$$

By the arbitrary of w, it has $E(\overline{\lambda}_i, A^*)g^* = 0$. That is $g^* \in \sigma_\infty^*$. Hence $f \notin (\sigma_\infty^*)^\perp$. Therefore, $\mathrm{sp}(A) = (\sigma_\infty^*)^\perp$, proving (Ap4). ∎

Proof of Theorem 4.3.4. Condition (4.3.52) implies that there exists an $M \geq N_0$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H. In particular, $(\mathrm{sp}(A))^\perp$ is finite dimensional. This together with (Ap4) shows that σ_∞^* is finite dimensional. It is known that $\overline{\lambda} \in \sigma_p(A^*) \cup \sigma_r(A^*)$ if and only if $\lambda \in \sigma_p(A) \cup \sigma_r(A)$. By our assumption, $\sigma_p(A^*) \cup \sigma_r(A^*) = \{\overline{\lambda}_i\}_1^\infty$. By Lemma A1, it follows that $\sigma_\infty^* = \{0\}$. Therefore,

$$\mathrm{sp}(A) = H. \tag{Ap5}$$

Suppose that $\{\psi_\alpha\} \cup \{\psi_n\}_M^\infty$ is the "maximal" ω-linearly independent set of generalized eigenvector of A, that is, $\{\psi_\alpha\} \cup \{\psi_n\}_M^\infty$ is an ω-linearly independent set and if adding another extra generalized eigenvector of A to $\{\psi_\alpha\} \cup \{\psi_n\}_M^\infty$, the extended set is not ω-linearly independent anymore. By Lemma 1.2.1, $\{\psi_\alpha\} \cup \{\psi_n\}_M^\infty$ forms a Riesz basis for the subspace spanned by itself, which is the whole space as we just proved.

Since a proper subset of a Riesz basis can not be a Riesz basis, it follows from condition (4.3.52) and Bari's Theorem (see Section 2 of [32] on p.309) that the number of $\{\psi_\alpha\}$ is just M. The proof is complete. ∎

Bibliography

[1] G. Amendola, M. Fabrizio, J. M. Golden and B. Lazzari, Free energies and asymptotic behaviour for incompressible viscoelastic fluids, *Appl. Anal.*, 88(2009), No.6, 789–805.

[2] S. Axler, *Linear Algebra Done Right*, Springer-Verlag, New York, 2ed Edition, 1997.

[3] S. A. Avdonin and S. A. Ivanov, *Families of exponentials*, Cambridge University Press, Cambridge, 1995.

[4] A. V. Balakrishnan, M. A. Shubov and C. A. Peterson, Spectral analysis of coupled Euler-Bernoulli and Timoshenko beam model, *Z. Angew. Math. Mech.*, 84(2004), 291-313.

[5] H. T. Banks, J. B. Hood and N. G. Medhin, A molecular based model for polymer viscoelasticity: intra- and inter-molecular variability, *Appl. Math. Model.*, 32(2008), No.12, 2753-2767.

[6] H. T. Banks, K. Ito and Y. Wang, Well posedness for damped second order systems with unbounded input operators, *Differential and Integral Eqs.*, 8(1995), 587-606.

[7] H. T. Banks, G. A. Pinter, L. K. Potter, B. C. Munoz and L. C. Yanyo, Estimation and control related issues in smart material structures and fluids, *The 4th International Conference on Optimization: Techniques and Applications*, Perth, Australia, (1998), 19-34.

[8] H. T. Banks, R. C. Smith, D. E. Brown, R. J. Silcox and V. L. Metcalf, Experimental confirmation of a PDF-based approach to design of feedback control, *SIAM J. Control Optim.*, 35(1997), 1263-1296.

[9] A. Benaddi and B. Rao, Energy decay rate of wave equations with indefinite damping, *J. Diff. Equas.*, 161(2000), 337-357.

[10] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill Book Co., New York, 1978.

[11] G. D. Birkhoff, On the asymptotic character of the solutions of certain linear differential equations containing a parameter, *Trans. Am. Math. Soc.*, 9(1908), 219-231.

[12] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, New York, 3rd Edition, 1977.

[13] S. G. Chai and B. Z. Guo, Feedthrough operator for linear elasticity system with boundary control and observation, *SIAM J. Control Optim.*, 48(2010), 3708-3734.

[14] S. G. Chai and B. Z. Guo, Analyticity of a thermoelastic plate with variable coefficients, *J. Math. Anal. Appl.*, 354(2009), 330-338.

[15] G. Chen, S. A. Fulling, F. J. Narcowich and S. Sun, Exponential decay of energy of evolution equations with locally distributed damping, *SIAM J. Control Optim.*, 51(1991), 266-301.

[16] S. P. Chen, K. S. Liu and Z. Liu, Spectrum and stability for elastic systems with global or local Kelvin-Voigt damping, *SIAM J. Appl. Math.*, 59(1999), 651–668.

[17] H. Cheng, *Advanced Analytic Methods in Applied Mathematics, Science, and Engineering*, Luban Press, 2007.

[18] B. Chentouf and J. M. Wang, A Riesz basis methodology for proportional and integral output regulation of a one-dimensional diffusive-wave equation, *SIAM J. Control Optim.*, 47(2008), 2275-2302.

[19] S. Cox and E. Zuazua, The rate at which energy decays in a string damped at one end, *Indiana Univ. Math. J.*, 44(1995), 545-573.

[20] S. Cox and E. Zuazua, The rate at which energy decays in a damped string, *Comm. Partial Differential Equations*, 19(1994), 213-243.

[21] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[22] C. M. Dafermos, An abstract Volterra equation with application to linear viscoelasticity, *J. Diff. Equas.*, 7(1970), 554-569.

[23] N. Dunford and J. T. Schwartz, *Linear Operators. Part II. Spectral Theory*, Wiley-Interscience, New York, 1967.

[24] N. Dunford and J. T. Schwartz, *Linear Operators. Part III. Spectral Operators*, Wiley-Interscience, New York, 1971.

[25] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[26] K. J. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Springer-Verlag, New York, 2006.

[27] W. N. Everitt, M. K. Kwong and A. Zettl, Oscillation of eigenfunctions of weighted regular Sturm-Liouville problems, *J. London Math. Soc.*, 27(2)(1983), 106-120.

[28] M. Fabrizio and B. Lazzari, On the existence and asymptotic stability of solutions for linearly viscoelastic solids, *Arch. Rat. Mech. Anal.*, 116(1991), 139-152.

[29] M. Fabrizio and S. Polidoro, Asymptotic decay for some differential systems with fading memory, *Appl. Anal.*, 81(2002), 1245-1264.

[30] P. Freitas and E. Zuazua, Stability results for the wave equation with indefinite damping, *J. Diff. Equas.*, 132(1996), 338-352.

[31] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of Linear Operators*, Vol. I, Birkhäuser, Basel, 1990.

[32] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, Rhode Island, Vol. 18, AMS Providence, 1969.

[33] B. Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass, *SIAM J. Control Optim.*, 39(2001), 1736-1747.

[34] B. Z. Guo, Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equations with variable coefficients, *SIAM J. Control Optim.*, 40(2002), 1905-1923.

[35] B. Z. Guo, Basis property of a Rayleigh beam with boundary stabilization, *J. Optim. Theory Appl.*, 112(2002), 529-547.

[36] B. Z. Guo, On the boundary control of a hybrid system with variable coefficients, *J. Optim. Theory Appl.*, 114(2002), 373-395.

[37] B. Z. Guo, Further result for a one-dimensional linear thermoelastic equation with Dirichlet-Dirichlet boundary conditions, *Australia and New Zealand Industrial and Applied Mathematics*, 43(2002), part 3, 449-462.

[38] B. Z. Guo and F. F. Jin, Arbitrary decay rate for two connected strings with joint anti-damping by boundary output feedback, *Automatica*, 46(2010), 1203-1209.

[39] B. Z. Guo and J. M. Wang, Riesz basis generation of abstract second-order partial differential equation systems with general non-separated boundary conditions, *Numer. Funct. Anal. Optim.*, 27(2006), 291–328.

[40] B. Z. Guo, J. M. Wang and K. Y. Yang, Dynamic stabilization of an Euler-Bernoulli beam under boundary control and non-collocated observation, *Systems Control Lett.*, 57(2008), 740-749.

[41] B. Z. Guo, J. M. Wang and S. P. Yung, On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam, *Systems Control Lett.*, 54(2005), 557-574.

[42] B. Z. Guo, J. M. Wang and G. D. Zhang, Frequency analysis of a wave equation with Kelvin-Voigt damping, *48th IEEE Conference on Decision and Control*, Shanghai, P.R. China, (2009), 4471-4476.

[43] B. Z. Guo, J. M. Wang and G. D. Zhang, Spectral analysis of a wave equation with Kelvin-Voigt damping, *Z. Angew. Math. Mech.*, 90(2010), 323-342.

[44] B. Z. Guo and C. Z. Xu, The stabilization of a one-dimensional wave equation by boundary feedback with noncollocated observation, *IEEE Trans. Automat. Control*, 52(2007), 371-377.

[45] B. Z. Guo and C. Z. Xu, On the spectrum-determined growth condition of a vibration cable with a tip mass, *IEEE Trans. Automat. Control*, 45(2000), 89-93.

[46] B. Z. Guo and G. Q. Xu, Expansion of solution in terms of generalized eigenfunctions for a hyperbolic system with static boundary condition, *J. Funct. Anal.*, 231(2006), 245-268.

[47] B. Z. Guo and G. Q. Xu, Riesz bases and exact controllability of C_0-groups with one-dimensional input operators, *Systems Control Lett.*, 52(2004), 221-232.

[48] B. Z. Guo and K. Y. Yang, Dynamic stabilization of an Euler-Bernoulli beam equation with time delay in boundary observation, *Automatica*, 45(2009), 1468-1475.

[49] B. Z. Guo and R. Yu, The Riesz basis property of discrete operators and application to a Euler-Bernoulli beam equation with linear feedback control, *IMA J. of Math. Control and Information*, 18(2001), 241-251.

[50] B. Z. Guo and S. P. Yung, Asymptotic behavior of the eigenfrequency of a one-dimensional linear thermoelastic system, *J. Math. Anal. Appl.*, 213(1997), 406-421.

[51] B. Z. Guo and G. D. Zhang, On spectrum and Riesz basis property for one-dimensional wave equation with Boltzmann damping, submitted.

[52] B. Z. Guo and X. Zhang, The regularity of the wave equation with partial Dirichlet control and colocated observation, *SIAM J. Control Optim.*, 44(2005), 1598-1613.

[53] B. Z. Guo and Z. X. Zhang, Well-posedness of systems of linear elasticity with Dirichlet boundary control and observation, *SIAM J. Control Optim.*, 48(2009), 2139-2167.

[54] B. Z. Guo and W. D. Zhu, On the energy decay of two coupled strings through a joint damper, *J. Sound and Vibration*, 203(1997), 447-455.

[55] B. Z. Guo and H. Zwart, Riesz Spectral System, *Preprint, University of Twente, the Netherlands*, 2001.

[56] A. Haraux, Lacunary series and semi-internal control of the vibrations of a rectangular plate, *J. Math. Pures Apply.(9)*, 68(1989), 457-465.

[57] L. F. Ho, Exact controllability of the one-dimensional wave equation with locally distributed control, *SIAM J. Control Optim.*, 28(1990), 733-748.

[58] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differential Equations*, 1(1985), 43-56.

[59] B. Jacob and C. Trunk, Location of the spectrum of operator matrices which are associated to second order systems, *Operators and Matrices*, 1(1)(2007), 45-60.

[60] B. Jacob, C. Trunk and M. Winklmeier, Analyticity and Riesz basis property of semigroups associated to damped vibrations, *Journal of Evolution Equations*, 8(2008), 263-281.

[61] S. Jaffard, Exact internal control of the vibrations of a rectangular plate, *Portugal. Math.*, 47(1990), 423-429.

[62] S. Kantorovitz, *Topics in Operator Semigroups*, Birkhäuser, Boston, 2010.

[63] V. Komornik, On the exact internal controllability of a Petrowsky system, *J. Math. Pures Appl.(9).*, 71(1992), 331-342.

[64] M. Krstic, B. Z. Guo, A. Balogh and A. Smyshlyaev, Control of a tip-force destabilized shear beam by observer-based boundary feedback, *SIAM J. Control Optim.*, 47(2008), 553-574.

[65] M. Krstic, B. Z. Guo, A. Balogh and A. Smyshlyaev, Output-feedback stabilization of an unstable wave equation, *Automatica J. IFAC*, 44(2008), 63-74.

[66] J. Lagnese, Control of wave processes with distributed controls supported on a subregion, *SIAM J. Control Optim.*, 21(1983), 68-85.

[67] B. R. Li, Perturbation Theory for a Class of Linear Operators and Its Applications, *Acta Math. Sinica (in Chinese).*, 21(1978), 206-222

[68] K. S. Liu, Locally distributed control and damping for conservative systems, *SIAM J. Control Optim.*, 35(1997), 1574-1590.

[69] K. S. Liu and Z. Liu, On the type of C_0-semigroup associated with the abstract linear viscoelastic system, *Z. Angew. Math. Phys.*, 47(1996), 1-15.

[70] K. S. Liu and Z. Liu, Analyticity and differentiability of semigroups associated with elastic systems with damping and gyroscopic forces, *J. Diff. Equas.*, 141(1997), 340-355.

[71] K. S. Liu and Z. Liu, Exponential decay of energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping, *SIAM J. Control Optim.*, 36(1998), 1086-1098.

[72] K. S. Liu and Z. Liu, Exponential decay of energy of vibrating strings with local viscoelasticity, *Z. Angew. Math. Phys.*, 53(2002), 265-280.

[73] K. S. Liu, Z. Liu and B. Rao, Exponential stability of an abstract nondissipative linear system, *SIAM J. Control Optim.*, 40(2001), 149-165.

[74] K. S. Liu and B. Rao, Exponential stability for wave equations with local Kelvin-Voigt damping, *C. R. Math. Acad. Sci. Paris*, 339(2004), 769-774.

[75] Z. Liu, S. M. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, *Quart. Appl. Math.*, 54(1996), 21-31.

[76] Z. H. Luo, B. Z. Guo and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London, 1999.

[77] L. Meirovitch and M. K. Kwak, Dynamics and control of spacecraft with retargeting flexible antennas, *J. Guidance Control Dynam.*, 13(1990), 241-248.

[78] M. A. Naimark, *Linear Differential Operators*, Vol. I, II, Ungar, New York, 1967, 1968.

[79] A. Pazy, *Semigroups of Linear Operator and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[80] B. P. Rao, Optimal energy decay rate in a damped Rayleigh beam, *Contemporary Mathematics*, RI, Providence, 209(1997), 221-229.

[81] M. Renardy, On localized Kelvin-Voigt damping, *Z. Angew. Math. Mech.*, 84(2004), 280-283.

[82] J. E. M. Rivera and H. P. Oquendo, The transmission problem of viscoelastic waves, *Acta Appl. Math.*, 62(2000), 1-21.

[83] M. A. Shubov, Riesz basis property of root vectors of non-self-adjoint operators generated by aircraft wing model in subsonic airflow, *Math. Methods Appl. Sci.*, 23(2000), 1585-1615.

[84] M. A. Shubov, Asymptotic and spectral analysis of the spatially nonhomogeneous Timoshenko beam model, *Math. Nachr.*, 241(2002), 125-162.

[85] A. Smyshlyaev, B. Z. Guo and M. Krstic, Arbitrary decay rate for Euler-Bernoulli beam by backstepping boundary feedback, *IEEE Trans. Automat. Control*, 54(2009), 1134-1140.

[86] H. S. Tzou and J. H. Ding, Optimal control of precision paraboloidal shell structronic systems, *J. Sound and Vibration*, 276(2004), 273-291.

[87] J. M. Wang, *Riesz Basis Property of Some Infinite-dimensional Control Problems and Its Applications*, Ph.D. Thesis, University of Hong Kong, 2004.

[88] J. M. Wang and B. Z. Guo, Analyticity and dynamic behavior of a three-layer damped sandwich beam, *J. Optim. Theory Appl.*, 137(2008), 675-689.

[89] J. M. Wang and B. Z. Guo, Riesz basis and stabilization for the flexible structure of a symmetric tree-shaped beam network, *Math. Methods Appl. Sci.*, 31(2008), 289-314.

[90] J. M. Wang, B. Z. Guo and B. Chentouf, Boundary feedback stabilization of a three-layer sandwich beam: Riesz basis approach, *ESAIM: Control, Optimization and Calculus of Variations*, 12(2006), 12-34.

[91] J. M. Wang, B. Z. Guo and M. Y. Fu, Dynamic behavior of a heat equation with memory, *Math. Methods Appl. Sci.*, 32(2009), 1287-1310.

[92] J. M. Wang, B. Z. Guo and K. Y. Yang, Stability analysis for an Euler-Bernoulli beam under local internal control and boundary observation, *J. Control Theory Appl.*, 6(2008), 341-350.

[93] G. Q. Xu, *Boundary Feedback Control of Elastic Beams*, Ph.D. Thesis, Institute of Mathematical Science and System Science, Academia Sinica, 2000.

[94] G. Q. Xu and B. Z. Guo, Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation, *SIAM J. Control Optim.*, 42(2003), 966-984.

[95] G. Q. Xu and D. X. Feng, The Riesz basis property of a Timoshenko beam with boundary feedback and application, *IMA J. Appl. Math.*, 67(2002), 357-370.

[96] G. Q. Xu, D. X. Feng and S. P. Yung, Riesz basis property of the generalized eigenvector system of a Timoshenko beam, *IMA J. Math. Control Inform.*, 21(2004), 65-83.

[97] G. Q. Xu, D. Y. Liu and Y. Q. Liu, Abstract second order hyperbolic system and applications to controlled network of strings, *SIAM J. Control Optim.*, 47(2008), 1762-1784.

[98] G. Q. Xu and N. E. Mastorakis, Spectrum of an operator arising elastic system with local K-V damping, *Z. Angew. Math. Mech.*, 88(2008), 483-496.

[99] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

[100] C. G. Zhang, H. L. Zhao and K. S. Liu, Exponential stabilization of a nonhomogeneous Timoshenko beam with the coupled control of locally distributed feedback and boundary feedback, *Chinese Ann. Math. Ser. A*, 24(6)(2003), 757-764 (in Chinese).

[101] G. D. Zhang and B. Z. Guo, On the Spectrum of Euler-Bernoulli Beam Equation with Kelvin-Voigt Damping, *J. Math. Anal. Appl.*, 374(2011), 210-229.

[102] H. L. Zhao, K. S. Liu and C. G. Zhang, Stability for the Timoshenko beam system with local Kelvin-Voigt damping, *Acta Math. Sin (Engl.Ser.)*, 21(3)(2005), 655-666.

[103] H. L. Zhao, K. S. Liu and Z. Liu, A note on the exponential decay of energy of a Euler-Bernoulli beam with local viscoelasticity, *J. Elasticity*, 74(2004), 175-183.

[104] Q. Zhou, Exact internal controllability of Maxwell's equations, *Japan J. Indust. Appl. Math.*, 14(1997), 245-256.

[105] J. Zabczyk, A note on C_0-semigroups, *Bull. Acad. Polon. Sci. Sér. Sci. Math.,* 23(1975), 895–898.

[106] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Comm. Partial Diff. Equas.*, 15(1990), 205-235.