Applications of fuzzy set theory and near vector spaces to functional analysis
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Abstract

We prove an original version of the Hahn-Banach theorem in the fuzzy setting. Convex compact sets occur naturally in set-valued analysis. A question that has not been satisfactorily dealt with in the literature is: What is the relationship between collections of such sets and vector spaces? We thoroughly clarify this situation by making use of Rådström's embedding theorem, leading up to the definition of a near vector space. We then go on to successfully apply these results to provide an original method of proof of Doob's decomposition of submartingales.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Andrew L Pinchuck

This ____________ day of April 2011, at Johannesburg, South Africa.

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Contents

Introduction

We have made a study comprising a number of important related themes in modern functional analysis. This thesis can essentially be divided into two parts. The first part deals with fuzzy Hahn-Banach theorems, while the second focuses on near vector spaces, leading up to applications thereof in Doob's decomposition theorems.

A significant part of this work involves the use of fuzzy sets and the application of fuzzy set theory to classical theories. Over the last few decades, fuzzy set theory has proved to be a useful tool when dealing with inherent uncertainty that is caused by inexact data due to imprecision of human knowledge. We look at such concepts as ordering of fuzzy sets, fuzzy points, mappings and inverse mappings defined on fuzzy sets. We discuss the notion of level sets associated with each fuzzy set, and we present a number of theorems that make use of these level sets which will be used at various points in the thesis. We also explain the typical ways in which fuzzy set theory is used to extend mathematics in the classical setting, which is a re-occuring theme.

In Chapter 3, we discuss and present our original versions of the famous Hahn-Banach theorem in the fuzzy setting. We also present our careful study of previous attempts at fuzzy Hahn-Banach theorems and compare them with our own results. The Hahn-Banach theorem is of central importance in functional analysis and there have been several successful attempts to generalize this theorem over the years.

The central ideas from Chapter 4 were initiated by Samuel's attempt to define a fuzzy vector lattice in [70]. We noticed an error in the relevant paper - in this paper Samuel claimed that the set of fuzzy points is a Riesz space. We showed, by means of a counterexample, that this is definitively not the case. Since Samuel used this mistaken fact to then define a fuzzy Riesz space and to develop a theory of integration in subsequent papers, we attempted to rectify this problem which we have managed to do successfully. The above mentioned error in this first paper, which was carried along in subsequent papers by the same author, was that he assumed that each fuzzy point has an additive inverse with respect to the usual addition of fuzzy sets. This is a similar situation to classical set theory, where certain collections of sets have all of the properties of vector spaces except for the additive inverse property. By examining ways to bridge this problem, we were lead to the work of Rådström. In [68], Rådström proved a theorem that is the key to successfully overcoming the problem that we faced. Using a special case of Rådström's embedding procedure, we developed the concept of a near vector space. We noticed that for this type of embedding to be performed, we required the law of cancellation ($A + C = B + C \Rightarrow A = B$). Thus near vector spaces are spaces that satisfy all of the vector axioms but the one that guarantees the existence of an inverse of each element and in addition satisfy the cancellation law. In our study of such objects, we discovered natural examples arising from certain collections of sets and fuzzy sets. This issue has never been satisfactorily dealt with in the literature and has immediate applications in the area of set-valued analysis. Since compact sets are extremely important and frequently occurring in functional analysis, our results seem to be a significant contribution. We considered three main collections of subsets of a separable Banach space, namely $\mathrm{ck}(X), \mathrm{cwk}(X)$ and $\mathrm{cbf}(X)$, the compact convex sets, the weakly compact sets and the closed and bounded sets respectively since these sets are endowed with certain required properties. After studying the literature on the topic, we noticed that set-valued mappings from a measure space into the collection of compact subsets of a separable Banach space equipped with the Hausdorff metric have been much studied but such mappings into the collection of weakly compact subsets and into closed, bounded subsets have not been adequately considered. By examining these further cases we were able to clarify and extend existing theory. A fruitful discussion about this concept is presented, involving the above mentioned collections of sets, culminating in several original results. Using Rådström's embedding theorem, we can show that these hyperspaces can be embedded in $C(\Omega)$ spaces. We explore these spaces, with a particular interest in applications to probability theory.

Random variables and the related notion of conditional expectations have played a significant role in probability theory, ergodic theory, quantum statistical mechanics and financial mathematics. We therefore present a comprehensive discussion about set-valued random variables.

Set-valued martingales are generalizations of the classical notion of a martingale that first arose in the study of betting strategies in the 1800's. Convergence of martingales is an extremely important topic in probability theory for many applications. Our main focus in the final chapter is Doob's decomposition theorem. Notably, we have found a simpler way of proving Doob's well-known decomposition of set-valued submartingales which we believe to be an important contribution. This is an immediate application of the near vector space ideas developed in Chapter 4.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we lay the foundations for the concepts that will be discussed in this thesis. To successfully engage the material, the reader will need to be familiar with certain aspects of topology, functional analysis and measure theory. We thus present an overview of the important notions as they are used in the theory of vector spaces and Riesz spaces. For a detailed presentation of these results, the reader is referred to [2, 16, 18, 23, 59, 76, 78, 83].

Section 1.2 deals with some essential notions from set theory, topology and vector spaces and Section 1.3 provides a basis for our work involving operator theory. Section 1.4 introduces the basic ideas of measure theory and then the L^p spaces for the scalar case. In section 1.5, we present the vector case of the L^p spaces. We will not always refer to these preliminary results explicitly in order to enhance readability, and it will subsequently be assumed that the reader is familiar with the material of this chapter.

1.2 Normed vector spaces and Riesz spaces

Set theory is fundamental to the topics discussed in this thesis, and thus, we begin by collecting and presenting some well-known concepts and notational conventions in order to make this thesis more self-contained. It must be emphasized that we make no attempt to discuss this elementary material thoroughly. We assume a familiarity with the most well-known and basic mathematical symbols and abbreviations.

We will always be considering sets that are subsets of a universal set U, and we usually adhere to the convention that elements of U will be denoted by small case letters and subsets of U will be denoted by capital letters. If x is an element and A is a set then we denote the fact that x is *an element of* A by $x \in A$. We denote that x is not an element of A by $x \notin A$. The collection of all elements that satisfy a condition $P(x)$ will be written as $\{x \in U : P(x)\}$. If A and B are two sets then we say that A is a *subset* of B, denoted $A \subset B$, if every element of A is also an element of B. We denote by $\emptyset$ the *empty set*, that is, the set that contains no elements. If $A, B \subset U$, then we denote

the *union* by $A \cup B = \{x \in U : x \in A$ or $x \in B\}$,
the *intersection* by $A \cap B = \{x \in U : x \in A$ and $x \in B\}$,
the *complement* by $A^c = \{x \in U : x \notin A\}$, and
the *relative complement* of B with respect to A by $A \backslash B = \{x \in A : x \notin B\}$.

If $\{A_i\}_{i \in I}$ is a collection of sets with $A_i \subset U$, for all $i \in I$, for I an index set; then we denote the union and intersection of $\{A_i\}_{i \in I}$ respectively by

$$\bigcup_{i \in I} A_i = \{x \in U : \exists i \in I, x \in A_i\},$$
$$\bigcap_{i \in I} A_i = \{x \in U : \forall i \in I, x \in A_i\}.$$

As usual, we denote the *natural numbers* by $\mathbb{N}$, the *integers* by $\mathbb{Z}$, the *rational numbers* by $\mathbb{Q}$, the *real numbers* by $\mathbb{R}$ and the *complex numbers* by $\mathbb{C}$. We will denote the *extended real line* by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. To denote intervals of the extended real line, we use the following standard notation for $a, b \in \overline{\mathbb{R}}$, $a \leq b$:

$$[a, b] = \{x \in \overline{\mathbb{R}} : a \leq x \leq b\},$$
$$[a, b) = \{x \in \overline{\mathbb{R}} : a \leq x < b\},$$
$$(a, b] = \{x \in \overline{\mathbb{R}} : a < x \leq b\},$$
$$(a, b) = \{x \in \overline{\mathbb{R}} : a < x < b\}.$$

Throughout the thesis, $\mathbb{R}_+$ will denote the set $[0, \infty)$ of nonnegative real numbers.

1.2.1 Definition
Let X and Y be sets. A *function* (*mapping*) f from X to Y, denoted by $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique element $y \in Y$. We write $y = f(x)$ to denote that f assigns the element $x \in X$ to the element $y \in Y$. We refer to the set X as the *domain*, Y as the *codomain*, and the set $\{f(x) : x \in X\}$ as the *range* of f respectively.

1.2.2 Definition
Let X and Y be sets. A function $f : X \to Y$ is said to be

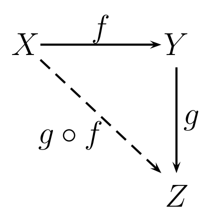
(a) *injective* (or *one-to-one*) if for for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$.

(b) *surjective* (or *onto*) if for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$.

(c) *bijective* if f is both injective and surjective.

1.2.3 Definition
Let $X,\ Y,$ and Z be sets; $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition* of f and g, denoted by $g \circ f$, is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

Diagrammatically:



Let X and Y be sets, $A \subset X$ and f a mapping from X into Y. We denote by $f|_A$, the *restriction of f to A*, defined by

$$\begin{aligned} f|_A : A &\to Y, \\ a &\mapsto f(a), \end{aligned}$$

for each $a \in A$.

Let X and Y be sets, then for a function $f : X \to Y$, there corresponds a function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$, where $f[A] = \{f(x) : x \in A\}$ is called the *direct image* of $A \subset X$; and a function $f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X)$, where $f^{\leftarrow}[B] = \{x \in X : f(x) \in B\}$ is called the *pre-image* of $B \subset Y$.

1.2.4 Definition

(a) Two sets A and B are said to *have the same cardinality*, denoted by $|A| = |B|$, if there exists a bijective function from A onto B. Sets that have the same cardinality are also said to be *equipotent* or *equinumerous*.

(b) A set S is said to be *finite* if $S = \emptyset$ or if there is an $n \in \mathbb{N}$ such that

$$|S| = |\{1, 2, \ldots, n\}|.$$

(c) A set S is said to be *countable* if S is finite or $|S| = |\mathbb{N}|$.

(d) A set S is said to be *uncountable* if S is not countable.

We define for a subset A of a universal set X the *characteristic function* of A, denoted by χ_A, by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We use $\mathcal{P}(X)$ to denote the *power set* of a set X, that is, $\mathcal{P}(X) = \{A : A \subset X\}$.

1.2.5 Definition

A *partition of a set* is a decomposition of the set into subsets such that every element of the set is in *one and only one* of the subsets. We call these subsets the *cells* of the partition.

1.2.6 Theorem

Let S be a nonempty set and let $\sim$ be a relation between elements of S that satisfies the following properties for all $x, y, z \in S$:

(1) $x \sim x$ (*Reflexivity*).

(2) If $x \sim y$, then $y \sim x$ (*Symmetry*).

(3) If $x \sim y$ and $y \sim z$, then $x \sim z$ (*Transitivity*).

Then $\sim$ yields a natural partition of S, where

$$[x] = \{s \in S : s \sim x\}$$

is the cell containing x, for all $x \in S$.

Conversely, each partition of S gives rise to a natural $\sim$ satisfying the reflexive, symmetric and transitive properties if $x \sim y$.

This leads to the following well-known definition.

1.2.7 Definition

A relation $\sim$ on a set S satisfying the reflexive, symmetric and transitive properties described in Theorem 1.2.6 is an *equivalence relation on* S. Each cell $[x]$ in the natural partition given by an equivalence relation is an *equivalence class*.

Since vector spaces are of particular importance throughout this thesis, we present the definitions of some well-known algebraic structures leading to a precise definition of a vector space.

1.2.8 Definition (Binary operation)
A *binary operation* $*$ *on a set* S is a rule that assigns to each ordered pair (x, y) of elements of S some element of S.

1.2.9 Definition
A *group* $(G, *)$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:

(a) For each $x, y, z \in G$, $(x * y) * z = x * (y * z)$ ($*$ is *associative*).

(b) There is an element e in G such that $e * x = x * e = x$, for all $x \in G$. This element e is called the *identity element* for $*$ on G.

(c) For each $x \in G$, there is an element $x' \in G$ with the property that $x * x' = x' * x = e$. The element x' is an *inverse of* x *with respect to the operation* $*$.

A group is a set together with a binary operations but provided that there is no ambiguity, we simply write G to denote the group consisting of G together with the binary operation $*$.

1.2.10 Definition (Abelian group)
A group G is *abelian* if and only if for all $x, y \in G$, $x * y = y * x$ ($*$ is *commutative*).

1.2.11 Definition
A *ring* $(R, +, \cdot)$ is a set R together with two binary operations $+$ and $\cdot$ which we call addition and multiplication, defined on R such that the following axioms are satisfied:

(a) $(R, +)$ is an abelian group.

(b) Multiplication is associative.

(c) For all $x, y, z \in R$, the *left distributive law*, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, and the *right distributive law*, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$, hold.

Again, we can refer to the ring R when we mean $(R, +, \cdot)$, provided this causes no confusion.

1.2.12 Definition
A ring in which the multiplication is commutative is a *commutative ring*. A ring with a multiplicative identity 1 such that $1 \cdot x = x \cdot 1 = x$, for all $x \in R$, is a *ring with unity*.

1.2.13 Definition
Let R be a ring with unity. An element u in R is a *unit of* R if it has a multiplicative inverse in R. If every nonzero element of R is a unit, then R is a *division ring*. A *field* is a commutative division ring.

1.2.14 Definition
Let $\mathbb{F}$ be a field. A *vector space* (*linear space*) *over* $\mathbb{F}$ consists of an abelian group X under addition together with an operation of scalar multiplication of each element of X by each element of $\mathbb{F}$ on the left, such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$, the following conditions are satisfied:

(a) $\alpha \cdot x \in X$.

(b) $\alpha \cdot (\beta x) = (\alpha\beta)x$.

(c) $(\alpha + \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$.

(d) $\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$.

(e) $1 \cdot x = x$.

The elements of X are *vectors*, and the elements of $\mathbb{F}$ are called *scalars*.

$(X, +, \cdot)$ will be denoted simply by X, provided there is no confusion. Throughout this thesis, for every vector space under consideration, the field of scalars $\mathbb{F}$ will be either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. When $\mathbb{F} = \mathbb{R}$, we say that X is a *real vector space*, and when $\mathbb{F} = \mathbb{C}$, we say that X is a *complex vector space*. When we refer to a vector space or a vector space over $\mathbb{F}$, we will mean either a real or a complex vector space.

Let X be a vector space over $\mathbb{F}$, $x \in X$; A and B subsets of X and $\lambda \in \mathbb{F}$. The following notation is standard:

$$\begin{aligned} x + A &:= \{x + a : a \in A\}, \\ A + B &:= \{a + b : a \in A, b \in B\}, \\ \lambda A &:= \{\lambda a : a \in A\}. \end{aligned}$$

1.2.15 Definition
A subset M of a linear space X over $\mathbb{F}$ is called a *linear subspace* of X if

(a) $x + y \in M$, for all $x, y \in M$, and

(b) $\lambda x \in M$, for all $x \in M$ and for all $\lambda \in \mathbb{F}$.

Clearly, a subset M of a vector space X is a linear subspace if an only if $M + M \subset M$ and $\lambda M \subset M$, for all $\lambda \in \mathbb{F}$. Often, we will simply say subspace rather than the more cumbersome 'linear subspace', when it is understood that the type of subspace in question is a linear one.

1.2.16 Definition
Let K be a subset of a linear space X. The *linear hull* of K, denoted by $\text{lin}(K)$ or $\text{span}(K)$, is the intersection of all linear subspaces of X that contain K.

The linear hull of K is also called the *linear subspace of X spanned (generated) by K.*

1.2.17 Definition

(1) A subset $\{x_1, x_2, \ldots, x_n\}$ of a linear space X is said to be *linearly independent* if the equation
$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0,$$
$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, only has the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Otherwise, the set $\{x_1, x_2, \ldots, x_n\}$ is *linearly dependent.*

(2) A subset K of a linear space X is said to be *linearly independent* if every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K is linearly independent.

1.2.18 Definition
If $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent subset of linear space X and
$$X = \text{lin}\{x_1, x_2, \ldots, x_n\},$$
then X is said to have *dimension* n. In this case we say that $\{x_1, x_2, \ldots, x_n\}$ is a *basis* for the linear space X. If a linear space X does not have a finite basis, we say that it is *infinite-dimensional.*

1.2.19 Definition
Let S be a subset of a linear space X. We say that:

(a) S is *convex* if $\lambda x + (1-\lambda)y \in S$, whenever $x, y \in S$ and $\lambda \in [0, 1]$.

(b) S is *balanced* if $\lambda x \in S$, whenever $x \in S$ and $|\lambda| \leq 1$.

(c) S is *absolutely convex* if S is convex and balanced.

(d) S is *absorbing* if for each $x \in X$, there corresponds some $r > 0$ such that $x \in \alpha S$ if $|\alpha| \geq r$.

1.2.20 Definition
Let S be a subset of a linear space X. Then we define the *convex hull* of S, denoted $\text{co}S$, to be the intersection of all convex sets which contain S.

In order to apply the techniques of analysis to vector spaces, it is necessary for the vector space to have a topological structure.

1.2.21 Definition

(a) $\tau \subset \mathcal{P}(X)$ is said to be a *topology* on the set X if it satisfies the following conditions:

(i) $\emptyset \in \tau$ and $X \in \tau$.

(ii) If $\{A_i\}_{i=1}^n \subset \tau$, for $n \in \mathbb{N}$; then $\bigcap_{i=1}^n \in \tau$.

(iii) If $\{A_i\}_{i\in I} \subset \tau$, for I an index set; then $\bigcup_{i\in I} A_i \in \tau$.

(b) If τ is a topology on X, then the pair (X, τ) is called a *topological space*, and the sets in τ are called *open sets* in X. We may leave out the reference to τ if there is no confusion. We say that a set $A \subset X$ is *closed* if its complement is open.

(c) If τ_1 and τ_2 are topologies on X with $\tau_1 \subset \tau_2$, then we say that τ_1 is *weaker* (*coarser*) than τ_2 and that τ_2 is *stronger* (*finer*) than τ_1.

(d) If X, Y are topological spaces and $f : X \to Y$, then f is called *continuous* if A open in Y implies $f^{-1}(A)$ is open in X. f is called *open* if A open in X implies that $f(A)$ is open in Y.

(e) If $\{f_i\}_{i\in I}$ is a family of mappings from a topological space X to a topological space Y, then the *weak topology on X generated by* $\{f_i\}_{i\in I}$, for I an index set, is defined to be the weakest topology on X such that f_i is continuous for each $i \in I$.

(f) We say that a topological space X is *Hausdorff* if each pair of distinct points can be separated by open sets. That is, for each $x, y \in X$ such that $x \neq y$, we have that there exists $U_x, U_y \in \tau$ such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.

1.2.22 Definition

Let A be a nonempty subset of a topological space (X, τ). We define the collection $\tau_A = \{G \cap A : G \in \tau\}$. It is simple to confirm that τ_A is a topology on A. It is called the *relative* (*subspace*) *topology* on A and we say that A is a *topological subspace* of X.

1.2.23 Definition

Let $(X, \tau_1), (Y, \tau_2)$ be topological spaces.

(a) X is said to be *homeomorphic* (or *topologically equivalent*) to Y if there exists a bijective mapping $f : X \to Y$ such that f and f^{-1} are continuous. In this case, the function f is called a *homeomorphism*.

(b) X is said to be *embedded* in Y if X is homeomorphic to a topological subspace of Y. In this case, we say that there exists an *embedding* $g : X \to Y$ and we often write $g : X \hookrightarrow Y$.

1.2.24 Definition

Let (X, τ) be a topological space and $A \subset X$. Then the *interior* of A, denoted A°, is defined as

$$A^\circ = \bigcup \Big\{ B \in \tau : B \subset A \Big\}.$$

This is the largest open set contained in A and we immediately have that A is open if and only if $A = A^\circ$.

1.2.25 Definition
Let (X, τ) be a topological space and $A \subset X$. Then the *closure* of A, denoted by $\overline{A}$ (or cl(A)), is given by

$$\overline{A} = \bigcap \Big\{ B \subset X : B^c \in \tau, A \subset B \Big\}.$$

Therefore, the closure of A is the smallest closed set containing A, and we trivially have that A is closed if and only if $A = \overline{A}$.

1.2.26 Definition
Let (X, τ) be a topological space.

(a) If A is a subset of X, then we say that a set $B \subset A$ is *dense* in A if $A \subset \overline{B}$. We sometimes say that B is τ-dense in A to indicate that B is dense in A with respect to the topology τ.

(b) X is said to be *separable* if there exists a countable set dense in X.

1.2.27 Definition

(a) Let A be a subset of a topological space X. We call $\mathcal{A} = \{G_i \in \tau : i \in I\}$, a collection of open subsets of X for some index set I, an *open cover* for A if $A \subset \cup_{i \in I} G_i$. Furthermore, if a finite subcollection of $\mathcal{A}$ is also an open cover of A, we say that $\mathcal{A}$ is *reducible to a finite cover* (or contains a *finite subcover*).

(b) A subset A of a topological space X is *compact* if every open cover of A is reducible to a finite cover.

1.2.28 Definition
Let (X, τ) be a topological space.

(a) A set $B \subset \mathcal{P}(X)$ is called a *base* for τ if and only if each element of τ is the union of elements of B.

(b) A set $S \subset \mathcal{P}(X)$ is called a *subbase* for τ if and only if the family of all finite intersections of elements of S is a base for τ.

1.2.29 Definition
Let $\{(A_i, \tau_i)\}_{i \in I}$ be a collection of topological spaces for I an index set. For each $k \in I$, we define the *projection* p_k to be the mapping

$$p_k : \prod\nolimits_{i \in I} A_i \to A_i, (a_i) \mapsto a_k.$$

We define the *product topology* on $\prod_{i \in I} A_i$ to be the weak topology generated by all such projections.

It is sometimes more convenient to work with neighbourhoods instead of open sets.

1.2.30 Definition
Let (X, τ) be a topological space and let $x \in X$. A subset N of X is a *neighbourhood* of x if and only if N contains an open set O containing x. The class of neighbourhoods of $x \in X$, denoted $\mathcal{N}_x$, is called the *neighbourhood system* of x.

There are four central facts about the neighbourhood system $\mathcal{N}_x$ of any point $x \in X$ and are thus termed the neighbourhood axioms. It should be noted that these axioms may be used to define a topology on X. We list the neighbourhood axioms below as a proposition and they are simple to prove.

1.2.31 Proposition

(1) $\mathcal{N}_x$ is not empty and x belongs to each member of $\mathcal{N}_x$.

(2) The intersection of any two members of $\mathcal{N}_x$ belongs to $\mathcal{N}_x$.

(3) Every superset of a member of $\mathcal{N}_x$ belongs to $\mathcal{N}_x$.

(4) Each member $N \in \mathcal{N}_x$ is a superset of a member $M \in \mathcal{N}_x$, where M is a neighbourhood of each of its points; i.e., $M \in \mathcal{N}_y$, for every $y \in M$.

1.2.32 Definition
A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space (X, τ) *converges* to a point $x \in X$, or x is the *limit* of the sequence (x_n), denoted by

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty,$$

if and only if for each open set O containing x, there exists $N \in \mathbb{N}$ such that $n \geq N$, implies that $x_n \in O$.

1.2.33 Theorem
Let (X, τ) be a topological space and $A \subset X$. A set $B \subset X$ is dense in A if and only if for each $a \in A$, there exists a sequence (b_n) contained in B such that $\lim\limits_{n\to\infty} b_n = a$.

1.2.34 Definition

(a) Let X be a nonempty set. A *metric* on X is a real function d on $X \times X$ that satisfies:

  (i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$.
  (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$ (Symmetry).
  (iii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (Triangle inequality).

(b) Let X be a set and d a metric on X. Then ordered pair (X, d) is called a *metric space*.

(c) Let (X, d) is a metric space. We define an *open ball* to be a set $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$, for $x \in X$ and $\epsilon \in \mathbb{R}_+$.

If (X, d) is a metric space, then it is a simple matter to show that the collection of open balls $\{B_\epsilon(x) : x \in X, \epsilon \in \mathbb{R}_+\}$ is in fact a base for a topology. This topology is referred to as the *metric topology* on X. It is easy to show that every set equipped with a metric topology is a Hausdorff space.

1.2.35 Definition

(a) Let (x_n) be a sequence in a a metric space (X, d). Then a sequence (x_n) in X is called a *Cauchy sequence* if given $\epsilon > 0$ there exists a natural number $N = N(\epsilon)$ such that

$$d(x_n, x_m) < \epsilon, \text{ for all } n, m \geq N.$$

Equivalently, (x_n) is Cauchy if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

(b) A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges in X. A subset M of X is complete if every Cauchy sequence in M converges in M.

Following [67], if X is a set and $d\colon X \times X \to \mathbb{R}_+$ satisfies the properties (ii) and (iii) in Definition 1.2.34 (a) but with property (i) replaced by:

(i') $d(x, y) \geq 0$, and $d(x, x) = 0$, for all $x, y \in S$,

then d is said to be a *semimetric*, and the ordered pair (X, d) is called a *semimetric space*.

1.2.36 Definition

Let X be a real vector space.

(a) A map $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* if the following conditions are satisfied:

(i) $\|x\| \geq 0$, for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$, for all $x \in X$ and $\alpha \in \mathbb{R}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ (Triangle inequality).

(b) The pair $(X, \|\cdot\|)$ is called a *normed linear space*.

(d) The set $B_X := \{x \in X : \|x\| \leq 1\}$ is called the *closed unit ball* in X.

(e) If $\|\cdot\|$ satisfies properties (ii) and (iii) listed in (a) above but condition (i) is replaced by:

(i') $\|x\| \geq 0$, for all $x \in X$,

then it is called a *seminorm* and we call the pair $(X, \|\cdot\|)$ a *seminormed linear space*.

1.2.37 Theorem

(1) If $(X, \|\cdot\|)$ is a normed linear space, then

$$d(x, y) = \|x - y\|$$

defines a metric on X. Such a metric d is said to be *induced* or *generated* by the norm $\|\cdot\|$. Thus, every normed linear space is a metric space and therefore inherits a natural topological structure as a metric space. Unless otherwise indicated, we will always consider $(X, \|\cdot\|)$ to be a topological space with respect to the induced metric.

(2) If d is a metric on a linear space X satisfying the properties: For all $x, y, z \in X$ and for all $\lambda \in \mathbb{F}$,

$$\begin{aligned} &\text{(i)} \qquad d(x, y) = d(x + z, y + z) \quad \text{(Translation Invariance)} \\ &\text{(ii)} \qquad d(\lambda x, \lambda y) = |\lambda| d(x, y) \quad \text{(Absolute Homogeneity)}, \end{aligned}$$

then

$$\|x\|_d = d(x, 0)$$

defines a norm on X.

1.2.38 Example

Consider the set of real numbers $\mathbb{R}$. It is easy to show that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+, (x, y) \mapsto |x - y|$ is a metric on $\mathbb{R}$. The corresponding induced topology τ_{ord} is referred to as the *usual topology* on $\mathbb{R}$.

1.2.39 Definition

A *linear topology* on a vector space X over $\mathbb{R}$ is a topology such that the two mappings

$$\begin{aligned} + : X \times X \to X, &\quad (x, y) \mapsto x + y, \\ \cdot : \mathbb{R} \times X \to X, &\quad (t, y) \mapsto t \cdot y, \end{aligned}$$

are continuous when $\mathbb{R}$ is equipped with τ_{ord}, and $\mathbb{R} \times X$ and $X \times X$ have the corresponding product topologies.

A vector space X with a linear topology is called a *topological vector space* (*topological linear space*).

1.2.40 Definition

Let (X, τ) be a topological space and A a topological subspace of $(\mathbb{R}, \tau_{\text{ord}})$, where τ_{ord} is once again the usual topology on $\mathbb{R}$. We say that a function $f : X \to A$ is *upper* (resp., *lower*) *semicontinuous* if $\{x \in X : f(x) < \alpha\} \in \tau$ (resp., $\{x \in X : f(x) > \alpha\} \in \tau$), for each $\alpha \in A$.

If $(X, \|\cdot\|)$ is complete with respect to the norm, then $(X, \|\cdot\|)$ is called a *Banach space* and if a Banach space $(X, \|\cdot\|)$ is separable as a topological space, then we will refer to $(X, \|\cdot\|)$ as a *separable Banach space*.

1.2.41 Definition
Let A be a subset of a normed linear space $(X, \|\cdot\|)$. We define the *closed convex hull* of A, denoted $\overline{\text{co}}A$ to be the closure of the convex hull of A.

Clearly, the closed convex hull of A is the smallest closed convex set containing A. and A is closed and convex if and only if $A = \overline{\text{co}}A$.

We say that a subset of a Banach space $(X, \|\cdot\|)$ is *norm compact* if it is compact with respect to the metric topology induced by $\|\cdot\|$.

1.2.42 Theorem (Mazur, see p51, [16])
The closed convex hull of a norm compact subset of a Banach space is norm compact.

1.2.43 Definition
A subset S of a be a normed linear space $(X, \|\cdot\|)$ is *bounded* if $S \subset \overline{B_r(x)} = \{y \in X : \|x - y\| \leq r\}$, for some $x \in X$ and $r > 0$.

1.2.44 Theorem ([78])
Let $(X, \|\cdot\|)$ be a normed linear space. Then every compact set is bounded and complete.

Note that the converse of the theorem above does not hold in general.

1.2.45 Theorem
Let $(X, \|\cdot\|)$ be a Banach space and let S be a subset of X. Then S is complete if and only if S is closed in X.

If it causes no confusion, we will refer to the normed linear space $(X, \|\cdot\|)$ simply as X.

1.2.46 Definition

(a) Let M be linear subspace of a vector space X. For all $x, y \in X$, define

$$x \equiv y (\text{mod } M) \Leftrightarrow x - y \in M.$$

It is easy to verify that $\equiv$ defines an equivalence relation on X.

(b) For $x \in X$, denote by

$$[x] := \{y \in X : x \equiv y (\text{mod } M)\} = \{y \in X : x - y \in M\} = x + M,$$

the *coset* of x with respect to M. The *quotient space* (*factor space*) X/M consists of all equivalence classes $[x]$, $x \in X$.

1.2.47 Proposition ([78])
Let M be a linear subspace of a vector space X over $\mathbb{F}$. For any $x, y \in X$ and $\lambda \in \mathbb{F}$, define the operations

$$[x] + [y] = [x + y] \text{ and } \lambda[x] = [\lambda x].$$

Then X/M is a linear space with respect to these operations.

Note that the operations above are well defined and do not depend on the chosen representants.

We now introduce a norm on a quotient space in a natural way.

1.2.48 Definition
Let M be a closed linear subspace of a normed linear space X over $\mathbb{F}$. For $x \in X$, define

$$\|[x]\| := \inf_{y \in [x]} \|y\|.$$

1.2.49 Proposition
Let M be a closed linear subspace of a normed linear space X over $\mathbb{F}$. The quotient space X/M is a normed linear space with respect to the norm

$$\|[x]\| := \inf_{y \in [x]} \|y\|, \text{ where } [x] \in X/M.$$

1.2.50 Definition
Let S be a nonempty set. A *partial order relation* (*partial ordering*) in S is a relation $\leq$ which satisfies the following properties:

(1) $x \leq x$ for every $x \in S$ (Reflexivity).

(2) $x \leq y$ and $y \leq x \Rightarrow x = y$, for all $x, y \in S$ (Antisymmetry).

(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$, for all $x, y, z \in S$ (Transitivity).

Furthermore, if for any two elements x and y of S one of the relations:

$$x \leq y \text{ or } y \leq x$$

holds, then S is said to be *totally ordered* under $\leq$. A nonempty set S in which there is defined a partial order relation is called a *partially ordered set*. We sometimes say that the ordered pair $(S, \leq)$ is a partially ordered set if there is any confusion as to which partial ordering we are referring to.

1.2.51 Definition
Let S be a subset of a partially ordered set L $(= (L, \leq))$.

(a) An element $u \in L$ is an *upper bound* for S if $x \leq u$, for all $x \in S$.

(b) An element $l \in L$ is an *lower bound* for S if $l \leq x$, for all $x \in S$.

(c) An element $m \in L$, denoted by sup(S), is said to be a *supremum* (*least upper bound*) for S if

  (i) m is an upper bound for S, and

  (ii) for all upper bounds u for S, we have that $m \leq u$.

(d) An element $m \in L$, denoted inf(S), is said to be an *infimum* (*greatest lower bound*) for S if

  (i) m is a lower bound for S, and

  (ii) for all lower bounds l for S, we have that $l \leq m$.

(e) An element $m \in S$ is a *maximal element* (resp., *minimal element*) of S if $s \in S$ such that $m \leq s$ implies that $m = s$ (resp., $s \in S$ such that $s \leq m$ implies that $s = m$), denoted max(S) (resp., min(S)).

1.2.52 Definition
Let $(L_1, \leq_1), (L_2, \leq_2)$ be partially ordered sets. Then we say that a mapping $T : L_1 \to L_2$ is *order preserving* if for any $x, y \in L_1$,

$$x \leq_1 y \Rightarrow T(x) \leq_2 T(y).$$

1.2.53 Definition
Let L $(= (L, \leq))$ be a partially ordered set.

(a) If every subset of L consisting of two elements has a supremum and an infimum, then L is a *lattice*. We also denote $\sup\{x, y\}$ by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$, for all $x, y \in L$.

(b) If L is a lattice, we say that L is *complete*, if each subset $D \subset L$ has a *join* (the supremum): $\bigvee D \in L$. By the duality principle, this is equivalent to the requirement that each $D \subset L$ has a *meet* (the infimum): $\bigwedge D$. If $\{x_i : i \in I\} \subset L$, for I an index set, then we denote the supremum (resp., infimum) of $\{x_i : i \in I\}$ by $\bigvee_{i \in I} x_i$ (resp., $\bigwedge_{i \in I} x_i$).

(c) If L is a lattice, then L is called a *distributive lattice* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for all $x, y, z \in L$.

(d) If L is a lattice such that for all $x \in L$ and $\{y_j\}_{j \in J} \subset L$, for J an index set, we have that $x \wedge (\bigvee_{j \in J} y_j) = \bigvee_{j \in J}(x \wedge y_j)$, then L is said to be *infinitely distributive* (a *frame*).

(e) If L and S are lattices under the same partial ordering and $S \subset L$, then we say that S is a *sublattice* of L.

1.2.54 Definition
Let L $(= (L, \leq))$ be a lattice. A mapping $' : L \to L$ is called an *involution*, if for each $a \in L$, $(a')' = a$ and any lattice with an involution is said to satisfy *the law of double negation*. An involution is said to be *order reversing* if $a \leq b$ implies $b' \leq a'$, for all $a, b \in L$. If L is equipped with an order reversing involution, then we say that L is *de Morgan*.

1.2.55 Definition

(a) A *chain* is a totally ordered subset of a partially ordered set.

(b) A *choice function* is a function f, defined on a collection X of nonempty sets, such that for every set S in X, $f(S)$ is an element of S.

A brief mention of Zorn's lemma and the axiom of choice is appropriate since these statements are implicitly assumed in many of our results.

1.2.56 Lemma (Zorn's lemma)
Every partially ordered set, in which every chain has an upper bound, contains at least one maximal element.

1.2.57 Lemma (Axiom of choice)
For any collection of nonempty sets X, there exists a choice function.

It is well-known that Zorn's lemma is logically equivalent to the axiom of choice. For further reading about the axiom of choice and it's consequences, the reader is referred to [26].

1.2.58 Definition
Let (X, d) be a metric space. If $A \in \mathcal{P}(X)$ and $x \in X$, the *distance* between x and A is defined by

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

If $A, B \in \mathcal{P}(X)$, then the *Hausdorff distance* d_H between A and B is defined by

$$d_H(A, B) = \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A).$$

In the special case where $B = \{0\}$, let

$$\|A\|_H = d_H(A, \{0\}).$$

In general d_H is not a metric but a semimetric and $\| \cdot \|_H$ is not a norm but a seminorm.

We now recall some more terminology from [56, 59, 74, 83].

1.2.59 Definition

(a) A real vector space E is called an *ordered vector space*, if $(E, \leq)$ is a partially ordered set and the vector space structure is compatible with the order structure, i.e., for all $x, y, z \in E$:

(O.1) $x \leq y \;\Rightarrow\; x + z \leq y + z$, and

(O.2) $x \leq y \;\Rightarrow\; \lambda x \leq \lambda y$, for all $\lambda \in \mathbb{R}_+$.

If, in addition, E is a lattice with respect to $\leq$, then E is a *vector lattice*, also known as a *Riesz space*.

(b) If E is a vector lattice, then $E_+ := \{x \in E \colon x \geq 0\}$ denotes the *positive cone* of E. Furthermore, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x \vee (-x)$ denote, respectively, the *positive part, negative part* and *absolute value* of $x \in E$. The following identities hold in E, for all $x, y \in E$:

(I.1) $(x \vee y) + (x \wedge y) = x + y$,

(I.2) $(x - y)^+ = (x \vee y) - y$.

(c) An ordered vector space E is said to have an *order unit* $e \in E_+$, if for each $x \in E$, there exists $K \in \mathbb{R}_+$ such that $-Ke \leq x \leq Ke$.

(d) If E is a Riesz space, then the cone E_+ is said to be *generating* if $E = E_+ - E_+$.

(e) If E is a Riesz space, then we say that E_+ is *Archimedean* if it follows from $y - nx \in E_+$, for all $n \in \mathbb{N}$, with $y \in E_+$ and $x \in E$ that $-x \in E_+$. If E_+ is Archimedean, then E is called an *Archimedean Riesz space*.

If an ordered vector space E has an order unit $e \in E_+$, the *gauge* (also known as the *Minkowski functional*) p_e of the order interval $\{x \in E : -e \leq x \leq e\}$, is defined by

$$p_e(x) := \bigwedge \{K > 0 : -Ke \leq x \leq Ke\}, \text{ for all } x \in E.$$

It is well-known (and easy to verify) that p_e is a seminorm on E. If E is an Archimedean ordered vector space with an order unit $e \in E_+$, then p_e is a norm on E.

We recall from [83], that if E is a vector lattice and $\|\cdot\|$ is a (semi)norm on E, then $\|\cdot\|$ is called a *Riesz (semi)norm* on E provided that:

(i) $0 \leq y \leq x$ in E implies that $\|y\| \leq \|x\|$, and

(ii) $\|\,|x|\,\| = \|x\|$, for all $x \in E$.

1.2.60 Definition ([74])

Let E be a vector lattice.

(a) $A \subset E$ is called *solid* if $x \in A, y \in E$, and $|y| \leq |x|$ implies $y \in A$. A solid vector subspace I of E is called an *ideal* (*lattice ideal*) of E.

(b) Let $B \subset E$, then B is contained in a smallest ideal $I(B)$ of E containing B, called the *ideal generated* by B. The ideal of E generated by a singleton $\{u\}$, for $u \in E$, is called a *principle ideal* and denoted by E_u.

(c) An element $x \geq 0$ of a topological vector lattice E is called a *quasi-interior point* of E_+ (or a *quasi-interior positive element of* E) if the principle ideal E_x is dense in E.

1.3 Operator theory on normed linear spaces

We present some fundamental results from functional analysis. The reader is referred to [83] for a more comprehensive presentation.

1.3.1 Definition
Let X and Y be vector spaces.

(a) We call a mapping $T : X \to Y$ a *linear operator* if we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for each $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Note that we denote $T(x)$ by Tx.

(b) A linear operator $P : X \to X$ is called a *projection* if $P^2 x = P(P(x)) = Px$, for all $x \in X$.

(c) We shall denote by $I_X : X \to X$ the *identity* operator on a vector space X, which is defined by $I_X(x) = x$, for all $x \in X$.

1.3.2 Definition
Let X and Y be vector spaces and $T : X \to Y$ be a linear operator.

(a) We denote the *range* of T by $\mathcal{R}(T) = \{y \in Y : \exists x \in X \text{ such that } Tx = y\}$. $\mathcal{R}(T)$ is a vector subspace of Y.

(b) We denote the *kernel* or *null space* of T by $\mathcal{N}(T) = \{x \in X : Tx = 0\}$. $\mathcal{N}(T)$ is a vector subspace of X.

(c) We define the *rank* of a linear operator to be the dimension of $\mathcal{R}(T)$ as a vector space.

(d) We define the *nullity* of the operator to be the dimension of $\mathcal{N}(T)$ as a vector space.

1.3.3 Definition
Let (X, d_1) and (Y, d_2) denote metric spaces and let $T : X \to Y$ be an operator.

(1) T is said to be an *isomorphism* if T is a bijection.

(2) T is called an *isometry* (or T is *isometric*) if $d_2(Tx, 0) = d_1(x, 0)$, for all $x \in X$.

1.3.4 Definition
Let X and Y denote normed linear spaces and let $T : X \to Y$ denote a linear operator.

(a) $T : X \to Y$ is called *bounded* if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$, for all $x \in X$.

(b) $T : X \to Y$ is called a *metric surjection* if T is surjective and

$$\|y\| = \inf\{\|x\| : x \in X, Tx = y\},$$

for every $y \in Y$. Metric surjections are sometimes referred to as *quotient operators*.

1.3.5 Remark
(1) Note that with this terminology, we are able to give a precise meaning to the statement that one metric space is essentially the same as another - that is, a metric space is said to be *isometrically isomorphic* to another metric space if there exists an isometric isomorphism from the one space onto the other.

(2) In view of the fact that normed linear spaces are metric spaces, we consider two normed linear spaces to be essentially the same as each other (isometrically isomorphic), if there is a linear operator from the one onto the other that is also an isometric isomorphism.

(3) An isomorphism from a vector lattice onto another vector lattice, which is also order preserving, is called a *vector lattice isomorphism*.

(4) It is a well-known fact that a linear operator from one vector space to another is bounded if and only if it is continuous (see e.g. [76]), and we will therefore use these terms interchangeably.

(5) Statement (b) in the previous definition is equivalent to $T : X \to Y$ mapping the open unit ball of X onto the open unit ball of Y. This implies that Y is isometrically isomorphic to the quotient space

$$X/\mathcal{N}(T) = \{x + y : x \in X, y \in \mathcal{N}(T)\}.$$

1.3.6 Theorem (completion)
Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$ which has a subspace $\tilde{W}$ that is, isometric with X and is dense in $\tilde{X}$. This space $\tilde{X}$ is unique except for isometries, that is, if $\hat{X}$ is any complete metric space having a dense subspace $\hat{W}$ isometric with X, then $\hat{X}$ and $\tilde{X}$ are isometric.

1.3.7 Definition
Let X and Y be normed linear spaces.

(a) We shall denote by $L(X, Y)$ the collection of all linear operators from X into Y. If $X = Y$, then we shall write $L(X, X)$ as $L(X)$.

(b) We define addition and scalar multiplication on $L(X, Y)$ as follows. Let $T, S \in L(X, Y)$ and let λ be a scalar.

$$(T + S)x := Tx + Sx, \text{ and}$$
$$(\lambda T)x := \lambda Tx,$$

for all $x \in X$. Note that $L(X, Y)$ is a vector space under these operations.

(c) In the case where $Y = \mathbb{R}$, we shall write $L(X, Y)$ as X^*. The elements of X^* are called *linear functionals* and X^* is called the *algebraic dual* of X.

(d) $X^{**} = (X^*)^*$ is called the *algebraic bidual* of X.

1.3.8 Definition

(a) We define the normed linear space $\mathcal{L}(X, Y)$ by $\mathcal{L}(X, Y) := \{T \in L(X, Y) : T \text{ is bounded}\}$ together with the operator norm $\|\cdot\|$ defined by $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$, for all $T \in \mathcal{L}(X, Y)$. If $X = Y$, then we write $\mathcal{L}(X, Y)$ as $\mathcal{L}(X)$.

(b) In the case where $Y = \mathbb{R}$, we shall write $\mathcal{L}(X, Y)$ as X'. The elements of X' are called *linear functionals* and X' is called the *continuous dual* of X.

(c) We call $X'' = (X')'$ the *continuous bidual* of X.

1.3.9 Theorem (Hahn-Banach)
Let $(X, \|\cdot\|)$ be a normed linear space and f a bounded linear functional on a linear subspace M of X. Then there exists a linear functional F on X, such that $F|_M = f$ and $\|F\| = \|f\|$.

1.3.10 Corollary (Hahn-Banach)

(1) If X is a normed linear space and $x \in X$, then there exists $x' \in X'$ of norm 1 such that $x'(x) = \|x\|$.

(2) If X is a normed linear space, then, for all $x \in X$, we have $\|x\| = \sup\{|x'(x)| : \|x'\| \leq 1, x' \in X'\}$.

(3) If X is a normed linear space and $x'(x) = 0$, for all $x' \in B_{X'}$, then $x = 0$, i.e., $B_{X'}$ separates points in X.

Let X be a normed linear space. Then each element $x \in X$ gives rise to a linear functional F_x in X'' in the following way. We define F_x by

$$F_x(f) = f(x),$$

where $f \in X'$.

F_x is called the functional on X' *induced* by the vector x and we refer to functionals of this type as *induced linear functionals*. The mapping $x \to F_x$ is an isometric isomorphism of X into X'' (see [76], p. 231) and we can thus regard X as a subset of X''.

1.3.11 Definition
Let $(X, \|\cdot\|)$ be a normed linear space.

(a) The topology on X generated by the norm is called the *strong topology* on X.

(b) The topology on X generated by all the linear functionals in X' is called the *weak topology* on X and is denoted by $\sigma(X, X')$.

(c) The weak topology on X' is denoted by $\sigma(X', X'')$.

(d) The weak topology on X' generated by all the induced linear functionals on X' is called the *weak** *topology* on X' and is denoted by $\sigma(X', X)$. Note that $\sigma(X', X)$ is weaker than $\sigma(X', X'')$.

(e) A normed linear space X is called *reflexive* if $X = X''$, in this case the weak and the weak* topologies on X' coincide.

(f) We define the *weakly compact* sets of X to be the subsets of X that are compact with respect to the weak topology on X.

1.3.12 Theorem (Krein-Smulian, [36], p.582)
The closed convex hull of a weakly compact subset of a Banach space is weakly compact.

1.3.13 Definition
Let E, F be ordered vector spaces and let $T : E \to F$ be a linear operator.

(a) T is called *positive* if $Tx \geq 0$ for all $x \in E$, $x \geq 0$; T is called *strictly positive* if $Tx > 0$, for all $x \in E$, $x > 0$.

(b) If E, F are vector lattices, T is called a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$, for all $x, y \in E$.

1.4 Scalar-valued L^p spaces

The classical L^p space is an important concrete example of a Riesz spaces. For various reasons, L^p spaces are studied extensively in functional analysis and measure theory. There are many texts that cover L^p spaces in more detail and the reader is referred to [23] for a more thorough treatment. We will also present the basic notions and notation of measure theory which are necessary for our discussion about L^p spaces.

1.4.1 Definition
Let Ω denote a nonempty set.

(a) $\Sigma \subset \mathcal{P}(\Omega)$ is called a *σ-algebra* on Ω if the following conditions hold:

(i) $\Omega \in \Sigma$.

(ii) $A \in \Sigma$, then $A^c \in \Sigma$.

(iii) $\{A_i\}_{i\in\mathbb{N}} \subset \Sigma$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

(b) If Σ and Σ_0 are σ-algebras on Ω such that $\Sigma_0 \subset \Sigma$, then we say that Σ_0 is a *sub-σ-algebra* of Σ.

(c) If Σ is a σ-algebra on Ω, then the pair (Ω, Σ) is called a *measurable space* and the sets in Σ are called *measurable sets*.

1.4.2 Definition
Let (Ω, Σ) be a measurable space. A (nonnegative, real) *measure* on (Ω, Σ) is a set function
$\mu : \Sigma \to [0, \infty]$ such that:

(i) μ is countably additive (or σ-additive). That is, for a mutually disjoint collection $\{A_i\}_{i\in\mathbb{N}}$, with $A_i \subset \Omega$, for each $i \in \mathbb{N}$, we have $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, and

(ii) $\mu(\emptyset) = 0$.

1.4.3 Definition
Let (Ω, Σ) be a measurable space and μ a measure on (Ω, Σ).

(a) The triple (Ω, Σ, μ) is called a (real) *measure space*.

(b) If $\mu(\Omega) = 1$, then μ is called a *probability measure* and (Ω, Σ, μ) is called a *probability space*. In this case we usually denote μ by P.

(c) If $\mu(\Omega) < \infty$, then we say that the measure space (Ω, Σ, μ) is finite.

(d) If (Ω, Σ, μ) is a measure space, (X, τ) a topological space, then $f : \Omega \to X$ is called *μ-measurable* if U open in X implies $f^{-1}(U)$ is measurable in Ω. We may drop the reference to μ if there is no confusion.

(e) If (Ω, Σ, μ) is a measure space, (X, τ) a topological space and Σ_0 a sub-σ-algebra of Σ, then $f : \Omega \to X$ is said to be *Σ_0-measurable* if U open in X implies $f^{-1}(U) \in \Sigma_0$.

1.4.4 Definition
Let (Ω, τ) be a topological space.

(1) We define the *Borel algebra* $\mathcal{B}(\Omega)$ to be the smallest σ-algebra on Ω that contains all of the elements of τ.

(2) A *Borel measure* is any measure on $\mathcal{B}(\Omega)$.

Throughout the remainder of this section, (Ω, Σ, μ) will denote a real measure space unless otherwise specified. Let $M(\Omega, \Sigma, \mu)$ denote the set of all real-valued, μ-measurable functions on Ω. Then $M(\Omega, \Sigma, \mu)$ becomes a real vector space under pointwise addition and scalar multiplication.

1.4.5 Definition
We define a *null set* to be a set of measure zero, and we identify functions that differ only on a null set, denoted $f \sim g$ if and only if $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. In this case we write $f = g$ *almost everywhere* [μ], often abbreviated to $f = g$ a.e. [μ]. Once again, the reference to μ can be omitted if there is no confusion.

It should be be noted that it can easily be shown that $\sim$ as defined above is an equivalence relation and hence, induces a partition on $M(\Omega, \Sigma, \mu)$.

1.4.6 Definition
Let (Ω, Σ, μ) be a measure space. We define $L^0(\Omega, \Sigma, \mu)$ to be the collection of equivalence classes of a.e. [μ] equivalent functions in $M(\Omega, \Sigma, \mu)$. If there is no confusion, we can simply write $L^0(\mu)$.

$L^0(\Omega, \Sigma, \mu)$ is a real vector space under $[f] + [g] = [f + g]$ and $\alpha[f] = [\alpha f]$, for $[f], [g] \in L^0(\Omega, \Sigma, \mu)$ and α a scalar. $L^0(\Omega, \Sigma, \mu)$ is called the space of μ-measurable functions in $M(\Omega, \Sigma, \mu)$, and we treat its elements as functions rather than equivalence classes.

1.4.7 Definition
Let (Ω, Σ, μ) be a measure space. A function $s : \Omega \to \mathbb{R}$, with $s(\Omega)$ finite is called a *step-function* (*simple function*). If $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ are the values of s and $A_i = s^{-1}(\{\alpha_i\})$, then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$. Note that s is measurable if and only if the A_i's are measurable sets.

We define order on $L^0(\mu)$ pointwise almost everywhere, i.e., $f \leq g \Leftrightarrow f(\omega) \leq g(\omega)$, for all $\omega \in \Omega$ a.e. Then $L^0(\mu)$ is a Riesz space under the lattice operations $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$, for all $f, g \in L^0(\mu)$.

1.4.8 Theorem
Let (Ω, Σ, μ) be a measure space. Let $f : \Omega \to [0, \infty]$ be measurable. Then there exists a nondecreasing sequence (s_n) of measurable step-functions that converge pointwise to f on Ω, and uniformly to f on any set on which f is bounded.

This leads us to the definition of the integral of a nonnegative function on Ω.

1.4.9 Definition
Let (Ω, Σ, μ) be a measure space.

(a) Let $s : \Omega \to [0, \infty)$ be a measurable step-function, then, for $E \in \Sigma$, we define the integral of s with respect to μ to be

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(s^{-1}(\{\alpha_i\}) \cap E),$$

with $s(\Omega) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

(b) If $f : \Omega \to [0, \infty]$ is a measurable function and $E \in \Sigma$, then we define the *Lebesgue integral* of f with respect to μ to be

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu : 0 \leq s \leq f, s \text{ a step-function} \right\}.$$

(c) Let Σ_1 be a sub-σ-algebra of Σ. We define

$$\int_\Omega^{(\Sigma_1)} f d\mu = \sup \left\{ \int_A f d\mu : A \in \Sigma_1 \right\}.$$

The following two theorems are of fundamental importance in measure theory. Note that since $L^0(\mu)$ is a Riesz space, we use the notation of Definition 1.2.59.

1.4.10 Lemma (Fatou's Lemma)
Let (Ω, Σ, μ) be a measure space. If (f_n) is in $L^0(\mu)_+$, then

$$\int_\Omega \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

1.4.11 Theorem (Lebesgue's Monotone Convergence Theorem)
Let (Ω, Σ, μ) be a measure space. If (f_n) is a nondecreasing sequence in $L^0(\mu)_+$ and $f(x) = \lim_{n\to\infty} f_n(x)$, then f is Σ-measurable, and

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

As a consequence of Lebesgue's Monotone Convergence Theorem, this integral is linear. Therefore, if $f : \Omega \to [-\infty, \infty]$ is a measurable function, then $\int_E (f^+ - f^-) d\mu = \int_E f^+ - \int_E f^- d\mu$, which is well defined if at least one of the integrals of f^+ or f^- is finite. Similarly, $\int_E |f| d\mu = \int_E (f^+ + f^-) d\mu = \int_E f^+ d\mu + \int_E f^- d\mu$, which leads us to the definition of integrability.

1.4.12 Definition
Let (Ω, Σ, μ) be a measure space. A function $f : \Omega \to [-\infty, \infty]$ is *integrable* if and only if $\int_\Omega |f| d\mu < \infty$.

1.4.13 Lemma
Let (Ω, Σ, μ) be a measure space and f a nonnegative real-valued measurable function on Ω. Then $\int_X f d\mu = 0$ if and only if $f = 0$ a.e.

We can now define the L^p-spaces.

1.4.14 Definition
Let (Ω, Σ, μ) be a measure space.

(a) Let $1 \leq p < \infty$.

(i) We define the function $\| \cdot \|_p : L^0(\Omega, \Sigma, \mu) \to \mathbb{R}_+$ by

$$\|f\|_p = \left(\int\limits_\Omega |f|^p d\mu \right)^{\frac{1}{p}}.$$

(ii) We define the *Lebesgue space* of p-integrable functions to be

$$L^p(\Omega, \Sigma, \mu) = \{f \in L^0(\Omega, \Sigma, \mu) : \|f\|_p < \infty\}.$$

We shall use the shorthand notation $L^p(\mu)$ provided that this does not lead to confusion.

(b) Let $p = \infty$.

(i) We define $\| \cdot \|_\infty : L^0(\Omega, \Sigma, \mu) \to \mathbb{R}_+$ by

$$\|f\|_\infty = \inf\{M > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > M\}) = 0\}.$$

$\| \cdot \|_\infty$ is called the *essential supremum norm* of f.

(ii) We define the *Lebesgue space* of essentially bounded functions to be

$$L^\infty(\Omega, \Sigma, \mu) = \{f \in L^0(\Omega, \Sigma, \mu) : \|f\|_\infty < \infty\}.$$

Again, we use the notation $L^\infty(\mu)$ for brevity provided no ambiguity is caused.

1.4.15 Note
It should be noted that it is easy to verify, that for $1 \leq p \leq \infty$, $\| \cdot \|_p$ is an example of a Riesz norm on $L_p(\mu)$.

In the case of $p = 1$, we have from the above definition, that $L^p(\mu) = L^1(\mu)$ is just the vector space of integrable functions on Ω. The following theorem describes some important properties of $\|\cdot\|_p$.

1.4.16 Theorem
Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. Choose q such that $p^{-1} + q^{-1} = 1$. Then the following statements hold:

(1) [Hölder's Inequality] For all $f, g \in L^0(\mu)$, we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

(2) [Minkowski's Inequality] For all $f, g \in L^0(\mu)$, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

(3) $\|\cdot\|_p$ defines a norm on $L^p(\mu)$.

(4) $L^p(\mu)$ is complete with respect to $\|\cdot\|_p$.

1.4.17 Theorem
Let (Ω, Σ, μ) be a measure space and let

$$S_p(\mathbb{R}) = \{s \in L^p(\mu) : s \text{ a step-function}\}.$$

Then $S_p(\mathbb{R})$ is dense in $L^p(\mu)$ with respect to $\|\cdot\|_p$, for all $1 \leq p < \infty$.

1.5 Vector-valued L^p spaces

We can generalize the classical L^p-spaces of real-valued functions to functions that take on values in a Banach space. Once again, throughout this section, (Ω, Σ, μ) will denote a measure space with μ a real measure unless otherwise specified. Let X denote a Banach space and $f : \Omega \to X$ denote a function on Ω taking on values in X. The material in this section is taken mainly from [16]. Notice that for $A \subset \Omega$, $x \in X$ and $t \in \Omega$ we have that

$$x\chi_A(t) = \begin{cases} x & \text{if } t \in A \\ 0 & \text{if } t \notin A. \end{cases}$$

1.5.1 Definition
Let (Ω, Σ, μ) be a measure space. We call $s : \Omega \to X$ a *step-function* if there exists $x_1, x_2, ...x_n \in X$ and $A_1, A_2, ..., A_n \in \Sigma$ disjoint such that $s = \sum_{i=1}^{n} x_i \chi_{A_i}$.

1.5.2 Definition

Let (Ω, Σ, μ) be a measure space. We say that $f : \Omega \to X$ is *μ-measurable* if there exists a sequence of step-functions (s_n) such that

$$\lim_{n\to\infty} \|s_n - f\| = 0 \text{ a.e. } [\mu].$$

We denote the space of a.e. $[\mu]$ equivalent measurable functions by $L^0(\Omega, \Sigma, \mu, X)$, and we will just write $L^0(\mu, X)$ provided there is no ambiguity.

1.5.3 Definition

Let (Ω, Σ, μ) be a measure space.

(a) Let $s : \Omega \to X$ be the step-function $s = \sum_{i=1}^{n} x_i \chi_{A_i}$, for $x_1, x_2, ...x_n \in X$ and $A_1, A_2, ..., A_n \in \Sigma$ mutually disjoint, then we define

$$\int_E s d\mu = \sum_{i=1}^{n} x_i \mu(A_i \cap E),$$

for each $E \in \Sigma$.

(b) A measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence (s_n) of X-valued step-functions such that

$$\lim_{n\to\infty} \int_\Omega \|s_n - f\| d\mu = 0.$$

In this case, $\int_E f d\mu$ is defined to be

$$\int_E f d\mu := \lim_{n\to\infty} \int_E s_n d\mu,$$

for each $E \in \Sigma$.

It is known that the above definition of the integral of a Bochner integrable function is well defined and independent of the defining sequence (s_n) (see [16, p.45]).

We present a natural characterization of Bochner integrability that is similar to the scalar case. The proof can be found in [16].

1.5.4 Proposition

Let (Ω, Σ, μ) be a measure space. A measurable function $f : \Omega \to X$ is Bochner integrable if and only if

$$\int_\Omega \|f\| d\mu < \infty.$$

1.5.5 Definition
Let (Ω, Σ, μ) be a measure space and X a Banach space.

(a) Let $1 \leq p < \infty$.

(i) We define the function $\triangle_p : L^0(\Omega, \Sigma, \mu, X) \to \mathbb{R} \cup \{\infty\}$ by

$$\triangle_p(f) = \left(\int_\Omega \|f\|^p d\mu \right)^{\frac{1}{p}}.$$

(ii) We define the *Bochner space* of p-integrable functions to be

$$L^p(\Omega, \Sigma, \mu, X) = \{f \in L^0(\Omega, \Sigma, \mu, X) : \triangle_p(f) < \infty\}.$$

When unambiguous, we shall use the shorthand notation $L^p(\mu, X)$.

(b) Let $p = \infty$.

(i) We define the function $\triangle_\infty : L^0(\Omega, \Sigma, \mu, X) \to \mathbb{R} \cup \{\infty\}$ by

$$\triangle_\infty(f) = \inf\{M > 0 : \mu(\{\omega \in \Omega : \|f(\omega)\| > M\}) = 0\}.$$

(ii) We define the *Bochner space* of essentially bounded functions to be

$$L^\infty(\Omega, \Sigma, \mu, X) = \{f \in L^0(\Omega, \Sigma, \mu, X) : \triangle_\infty(f) < \infty\}.$$

Again we use the shorthand notation $L^\infty(\mu, X)$ provided that no ambiguity is caused.

The Banach space X in the above definition is referred to as the *target space*. We now present some important properties of $\triangle_p$. The proof of the following theorem is similar to the proof for the scalar case.

1.5.6 Theorem ([16])
Let (Ω, Σ, μ) be a measure space.

(1) [Minkowski's Inequality] $\triangle_p(f + g) \leq \triangle_p(f) + \triangle_p(g)$, for all $f, g \in L^p(\mu, X)$.

(2) $\triangle_p$ defines a norm on $L^p(\mu, X)$.

(3) $L^p(\mu, X)$ is complete with respect to $\triangle_p$.

Since the Banach space X has no order structure, $L^p(\mu, X)$ does not inherit a pointwise ordering from X as in the scalar case. The final result for this section is similar to the result in the scalar case.

1.5.7 Theorem ([10])
Let (Ω, Σ, μ) be a measure space and let

$$S_p(X) = \{s \in L^p(\mu, X) : s \text{ a step-function}\},$$

then, for $1 \leq p < \infty$, $S_p(X)$ is dense in $L^p(\mu, X)$ with respect to $\triangle_p$.

PROOF.

Let (Ω, Σ, μ) be a measure space, $1 \leq p < \infty$ and $f \in L^p(\mu, X)$. Then since f is integrable, there exists a sequence (s_n) of step-functions such that

$$\lim_{n\to\infty} \triangle_1(f - s_n) = \lim_{n\to\infty} \int_\Omega \|f - s_n\| d\mu = 0.$$

We need to show that $\lim\limits_{n\to\infty} \triangle_p(f - s_n) = 0$. To that end, assume that $\triangle_p(f - s_n) \not\to 0$ as $n \to \infty$. Then there exists an $\epsilon' > 0$ such that for any $N' \in \mathbb{N}$, there exists $n \geq N'$ such that

$$\triangle_p(f - s_n) = \left(\int_\Omega \|f - s_n\|^p d\mu\right)^{\frac{1}{p}} \geq \epsilon'$$

$\Leftrightarrow \int_\Omega \|f - s_n\|^p d\mu \geq (\epsilon')^p > 0 \Rightarrow \|f - s_n\|^p \neq 0$ a.e., by Lemma 1.4.13. Therefore, $\exists \delta > 0, \exists A \in \Sigma, \mu(A) > 0$ such that for all $\omega \in A$, $\|f(\omega) - s_n(\omega)\|^p \geq \delta$, for all $n \geq N'$,

$$\Leftrightarrow \forall \omega \in A, \|f(\omega) - s_n(\omega)\| \geq \delta^{\frac{1}{p}}$$
$$\Rightarrow \triangle_1(f - s_n) \geq \int_A \|f - s_n\| d\mu \geq \int_A \delta^{\frac{1}{p}} d\mu = \delta^{\frac{1}{p}} \mu(A) > 0 \text{ (again by Lemma 1.4.13)}$$
$$\Rightarrow \triangle_1(f - s_n) \not\to 0 \text{ as } n \to \infty.$$

That is, $\triangle_1(f - s_n) \to 0$ as $n \to \infty$ implies $\triangle_p(f - s_n) \to 0$ as $n \to \infty$ and thus we have $\triangle_p(f - s_n) \to 0$ as $n \to \infty$.

Now, let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned}\triangle_p(s_n) &= \triangle_p(s_n - f + f)\\ &\leq \triangle_p(s_n - f) + \triangle_p(f)\\ &< \epsilon + \infty = \infty.\end{aligned}$$

i.e., $s_n \in S_p(X)$, for all $n \geq N$. We have shown that the sequence $(s_n)_{n \geq N}$ in $S_p(X)$ converges to f, and hence, by Theorem 1.2.33, $S_p(X)$ is dense in $L^p(\mu, X)$.

□

Chapter 2

Fuzzy sets

2.1 Introduction

Set theory forms the foundation of mathematics and every mathematical object can be viewed as a set or a class. Fuzzy set theory was effectively started when Zadeh published his now famous paper [84] in 1965. In terms of practical applications, fuzzy sets have successfully been used in mathematical modelling and control theory when inherent, non-probabilistic uncertainty is involved. That is, uncertainty that arises from insufficient information or imprecise boundaries.

One of the main themes of this thesis is the use of fuzzy set theory to extend recent ideas. Essentially, fuzzy set theory is a generalization of set theory. To highlight the similarity, we present a set calculus of $\mathcal{P}(X)$, and then naturally extend this calculus to fuzzy sets. We thus need to define notions which are analogues of subset, union, intersection and complement in such a way that when the fuzzy sets are in fact crisp sets, then the respective notions reduce to the usual crisp ones.

We are also interested in the concept of α-cuts which can be used to decompose a fuzzy set into a supremum of a sequence of special fuzzy sets, determined by crisp sets. This idea provides the basic mechanism of relating the fuzzy setting to the classical setting.

Since we deal with notions such as compactness and closure, we need to discuss a few topological aspects of fuzzy sets. The material in this introductory chapter is all well-known. For further information with regard to fuzzy sets and fuzzy topology, the reader is referred to [19, 20, 21, 48, 52, 53, 54, 64, 84].

2.1.1 Definition
Let X be a set and I the unit interval $[0,1]$. A *fuzzy set* on X (*fuzzy-subset of* X) is a map from X into I. That is, if A is a fuzzy subset of X, then $A \in I^X$, where I^X denotes the

collection of all maps from X into I.

Let X be a set and $A \subset X$. We naturally associate with A the mapping

$$\chi_A : X \to \{0, 1\},$$

as defined in Section 1.2. In fuzzy set theory, we drop the codomain restriction and use

$$A : X \to [0, 1].$$

In other words, we have generalized classical set theory by rejecting the bivalency requirement, and can consider various degrees of membership associated with an element of a fuzzy set. We will henceforth refer to ordinary sets as *crisp sets* as opposed to fuzzy sets. When we generalize a theory that relies on classical set theory, by replacing the crisp sets with fuzzy sets, we refer to this as *fuzzification*. One of the goals of fuzzification is that by considering the more general fuzzy theory, we can obtain new results in the original setting.

2.2 Order-structure of fuzzy subsets

Let X be a nonempty set and $A \subset X$. Note that $\chi_A \in \{0, 1\}^X = 2^X$, and hence, there is a natural bijection between $\mathcal{P}(X)$ and 2^X. We observe that for $A, B \in \mathcal{P}(X)$:

- $\forall x \in X, \chi_\emptyset(x) = 0,$
- $\forall x \in X, \chi_X(x) = 1,$
- $\forall x \in X, \chi_{A^c}(x) = 1 - \chi_A(x),$
- $\forall x \in X, \chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x),$
- $\forall x \in X, \chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x),$
- $A \subset B \Leftrightarrow \forall x \in X, \chi_A(x) \leq \chi_B(x).$

Now, on I^X, with $A, B \in I^X$, we define correspondingly:

- The *empty fuzzy set* $\chi_\emptyset$ is: $\forall x \in X, \chi_\emptyset(x) = 0,$
- The *whole fuzzy set* χ_X is: $\forall x \in X, \chi_X(x) = 1,$
- $A = B \Leftrightarrow \forall x \in X, A(x) = B(x),$

- $A \le B \Leftrightarrow \forall x \in X, A(x) \le B(x)$,
- $(A \vee B)(x) \equiv A(x) \vee B(x), x \in X$,
- $(A \wedge B)(x) \equiv A(x) \wedge B(x), x \in X$,
- $(\bigvee_{j \in J} A_j)(x) \equiv \bigvee_{j \in J} A_j(x), x \in X$,
- $(\bigwedge_{j \in J} A_j)(x) \equiv \bigwedge_{j \in J} A_j(x), x \in X$,
- $A'(x) \equiv A(x)', x \in X$ (where $'$ is any order reversing involution).

Thus, I^X is naturally equipped with an order structure induced by I, and since I is a complete lattice, de Morgan and a frame, so is I^X. We have the usual order reversing involution: $A'(x) = 1 - A(x), x \in X$.

For a treatment of L-fuzzy sets where I is replaced by a general complete, de Morgan lattice L which is also a frame (see e.g. [20, 64]).

2.3 Fuzzy sets induced by maps

In Zadeh's historical paper [84], he defined fuzzy analogues to the notions of direct image and pre-image as discussed in Section 1.2. For X and Y sets, $f : X \to Y$, $A \in I^X$ and $B \in I^Y$, we define the *direct image* of A, denoted by $f[A]$, and the *pre-image* of B, denoted by $f^{\leftarrow}[B]$, as follows.

For $y \in Y$,

$$f[A](y) = \sup_{f(x)=y} A(x),$$

with the convention that $\sup \emptyset = 0$, and

$$f^{\leftarrow}[B](x) = (B \circ f)(x),$$

for all $x \in X$.

This concept of a direct image above is known as Zadeh's extension principle. It is a simple matter to confirm that both definitions above reduce to the usual crisp ones in the case where A and B are crisp.

2.3.1 Theorem
Let X and Y be sets.

(1) If $A \in I^{X\times Y}$, then

$$\sup_{(x,y)\in X\times Y} A(x,y) = \sup_{x\in X}\sup_{y\in Y} A(x,y),$$
$$\inf_{(x,y)\in X\times Y} A(x,y) = \inf_{x\in X}\inf_{y\in Y} A(x,y),$$
$$\sup_{x\in X}\inf_{y\in Y} A(x,y) \leq \inf_{y\in Y}\sup_{x\in X} A(x,y).$$

(2) If $X, Y \subset I$, then

$$\sup X \wedge \sup Y = \sup_{x\in X}\sup_{y\in Y} x\wedge y,$$
$$\inf X \wedge \inf Y = \inf_{x\in X}\inf_{y\in Y} x\wedge y.$$

(3) If $A, B \in I^X$, then

$$\sup_{x\in X}(A\wedge B)(x) \leq \sup_{x\in X} A(x) \wedge \sup_{x\in X} B(x).$$

(4) If $A \in I^X$ and $E, F \subset X$, then

$$\sup_{x\in E} A(x) \wedge \sup_{y\in F} A(y) = \sup_{x\in E}\sup_{y\in F}(A(x)\wedge A(y)).$$

2.3.2 Theorem

Let X, Y, Z be sets and let $f \in Y^X$, $g \in Z^Y$, $A \in I^X$, $B \in I^Y$ and $C \in I^Z$. Let $(A_j : j \in J) \in (I^X)^J$ and $(B_j : j \in J) \in (I^Y)^J$. Then

(1) $(g\circ f)[A] = g[f[A]]$,

(2) $(g\circ f)^{\leftarrow}[C] = f^{\leftarrow}[g^{\leftarrow}[C]]$,

(3) $f^{\leftarrow}[\bigvee_{j\in J} B_j] = \bigvee_{j\in J} f^{\leftarrow}[B_j]$,

(4) $f^{\leftarrow}[\bigwedge_{j\in J} B_j] = \bigwedge_{j\in J} f^{\leftarrow}[B_j]$,

(5) $f^{\leftarrow}[B'] = (f^{\leftarrow}[B])'$,

(6) $B_1 \leq B_2 \Rightarrow f^{\leftarrow}[B_1] \leq f^{\leftarrow}[B_2]$,

(7) $f[\bigvee_{j\in J} A_j] = \bigvee_{j\in J} f[A_j]$,

(8) $f[\bigwedge_{j\in J} A_j] \leq \bigwedge_{j\in J} f[A_j]$,

(9) $f[A]' \leq f[A']$,

(10) $A_1 \leq A_2 \Rightarrow f[A_1] \leq f[A_2]$,

(11) $f[f^{\leftarrow}[B]] \leq B$, with equality if f is surjective,

(12) $A \leq f^{\leftarrow}[f[A]]$, with equality if f is injective,

(13) $f[f^{\leftarrow}[B] \wedge A] \leq f[A]$, with equality if f is injective.

2.4 Crisp subsets of X associated with a fuzzy set

For a given fuzzy subset of a set X, we associate collections of crisp subsets of X with it.

If $A \in I^X$ and $\alpha \in I$ we define

$$A^\alpha := \{x \in X : A(x) > \alpha\},$$
$$A_\alpha := \{x \in X : A(x) \geq \alpha\}.$$

These crisp sets are referred to as α-*levels* (or α-*cuts*), strong and weak respectively. When a crisp theory is to be fuzzified, very often we expect that if A is a fuzzy set that has a certain fuzzy property, then A_α (or A^α) has the crisp property which is analogous to that particular fuzzy property. We will work mainly with weak cuts in this thesis although in most cases similar results can be obtained by using the strong cuts.

For a fuzzy set A, we call supp$A = A^0$ the *support* of A.

2.4.1 Lemma
Let X be a set and $A, B \in I^X$. Then $A \leq B \Leftrightarrow$ for all $\alpha \in (0,1]$, $A_\alpha \subset B_\alpha$.

We have the following immediate consequence of the previous lemma.

2.4.2 Lemma
Let X be a set and A, B fuzzy sets on X, then

$$A = B \Leftrightarrow \text{ for all } \alpha \in (0,1], A_\alpha = B_\alpha.$$

We will need the following useful characterization of fuzzy sets.

2.4.3 Theorem ([48])
Let X be a set. For a $A \in I^X$ and $x \in X$, we have

$$A(x) = \sup_{\alpha \in (0,1]} \left\{\alpha \chi_{A_\alpha}(x)\right\}.$$

Also, notice that I is separable and therefore, there exists a set $\Lambda = \{\alpha_n\}_{n \in \mathbb{N}} = \mathbb{Q} \cap I$, which is countable and dense in I. We can now state the following theorem.

2.4.4 Theorem
Let X be a set. If $A \in I^X$ and $x \in X$, then

$$A(x) = \sup_{\alpha_n \in \Lambda} \left\{\alpha_n \chi_{A_{\alpha_n}}(x)\right\}$$

where $\Lambda = \{\alpha_n\}_{n \in \mathbb{N}} = \mathbb{Q} \cap I$.

PROOF.

Let X be a set, $A \in I^X$ and $x \in X$, then we have

$$A(x) = \sup_{\alpha \in (0,1]} \alpha \chi_{A_\alpha}(x) = a \in I.$$

Thus,

$$a = \begin{cases} \sup_{\alpha \in (0,1]}\{\alpha : A(x) \geq \alpha\} & \text{if } \exists \alpha \in (0,1] \text{ such that } A(x) \geq \alpha \\ 0 & \text{if } \nexists \alpha \in (0,1] \text{ such that } A(x) \geq \alpha. \end{cases}$$

Let

$$\begin{aligned} b &= \sup_{\alpha_n \in \Lambda} \left\{\alpha_n \chi_{A_{\alpha_n}}(x)\right\} \\ &= \begin{cases} \sup_{\alpha_n \in \Lambda}\{\alpha_n : A(x) \geq \alpha_n\} & \text{if } \exists \alpha_n \in \Lambda \text{ such that } A(x) \geq \alpha_n \\ 0 & \text{if } \nexists \alpha_n \in \Lambda \text{ such that } A(x) \geq \alpha_n. \end{cases} \end{aligned}$$

Clearly, $a \geq b$ since $\Lambda \subset I$, so we need only show that $a \not> b$. Assume that $a > b$, then $a = b + \epsilon$ for some $\epsilon > 0$. Then the set $O = (a - \epsilon, a)$ is an open set on I such that $O \cap \Lambda = \emptyset$, which is a contradiction to the fact that Λ is dense in I.

□

There is an important subset of I^X that we discuss in subsequent chapters.

Let X be a set and let $x \in X$ and $\alpha \in (0,1]$. Then

$$\alpha\chi_{\{x\}} = \begin{cases} \alpha & \text{on } x \\ 0 & \text{elsewhere.} \end{cases}$$

We call $\alpha\chi_{\{x\}}$ a *fuzzy point* with *support* at x and *value* α. We will denote the set of fuzzy points in I^X by $\widetilde{X}$. A fuzzy point can be viewed as a generalization of a point in ordinary set theory.

2.5 Fuzzy topology

Shortly after fuzzy set theory was developed, mathematicians began to fuzzify various areas of classical mathematics. A typical example of such a fuzzification is the study of fuzzy topology. The concept of a fuzzy topological space follows naturally from the corresponding classical notion. Since a crisp topological space can be defined in terms of open sets, we simply replace the crisp open sets with fuzzy open sets, which leads directly to Chang's definition of a fuzzy topological space (see [9]).

2.5.1 Definition
A *fuzzy topology* on a set X is a subset τ of I^X satisfying:

(a) $\chi_\emptyset, \chi_X \in \tau$.

(b) $A, B \in \tau \Rightarrow A \wedge B \in \tau$.

(c) $\forall j \in J, A_j \in \tau \Rightarrow \bigvee_{j\in J} A_j \in \tau$.

The pair (X, τ) is called a *fuzzy topological space* (*fts*) and the members of τ the *fuzzy open sets* of X.

In [51], Lowen defines a subset $\tau \subset I^X$ to be a fuzzy topology on X if (a), (b), (c) hold as well as:

(d) $\forall a \in I, a\chi_X \in \tau$. In this case we speak of a *fuzzy topology in Lowen's sense*.

2.5.2 Examples

(1) The *discrete* fuzzy topology on X: $\tau = I^X$.

(2) The *indiscrete* fuzzy topology in Lowen's sense on X: $\tau = \{\alpha\chi_X : \alpha \in I\}$.

(3) Any ordinary (crisp) topology T on X generates a fuzzy topology on X - simply identify with the open sets, their characteristic functions.

(4) Any ordinary topology T on a set X generates a fuzzy topology $\omega(T)$ in Lowen's sense, where

$$\omega(T) = \{\delta : (X, T) \to I_r \text{ is lower semicontinuous}\},$$

and I_r is the topological space obtained by giving the unit interval I the subspace topology as a subspace of $\mathbb{R}$ equipped with the usual topology τ_{ord} (see [51]).

If τ_1 and τ_2 are fuzzy topologies on a set X, then we say that τ_1 is *smaller* (*coarser*) than τ_2 (or equivalently τ_2 is *bigger* (*finer*) than τ_1) if and only if $\tau_1 \subset \tau_2$.

As in general topology, we define the concepts of a base and subbase.

2.5.3 Definition
Let (X, τ) be a fuzzy topological space.

(a) A set $\mathcal{B} \subset I^X$ is called a *base* for τ if and only if each element of τ is the supremum of members of $\mathcal{B}$.

(b) $\mathcal{S} \subset I^X$ is called a *subbase* for τ if and only if the family of all finite infima of members of $\mathcal{S}$ is a base for τ.

(c) If $\mathcal{P} \subset I^X$, then the fuzzy topology *generated by* $\mathcal{P}$, written $\tau = \langle \mathcal{P} \rangle$, is given by

$$\langle \mathcal{P} \rangle = \Big\{ \bigvee_{J \in K} \bigwedge_{j \in J} A_j : J \text{ finite and each } A_j \in \mathcal{P} \Big\},$$

where K is any index set.

2.5.4 Lemma
Let X be a set and for each $j \in J$, let τ_j be a fuzzy topology on X. Then $\tau = \bigcap_{j \in J} \tau_j$ is a fuzzy topology.

PROOF.

(a) $\chi_\emptyset, \chi_X \in \tau$ trivially.

(b) Let $A, B \in \tau$. Then

$$\begin{aligned} \forall j \in J; A, B \in \tau_j &\Rightarrow \forall j \in J, A \wedge B \in \tau_j \\ &\Leftrightarrow A \wedge B \in \tau. \end{aligned}$$

(c) For each $k \in K$, let $A_k \in \tau$. That is,

$$\begin{aligned} \forall k \in K, \forall j \in J, A_k \in \tau_j &\Rightarrow \forall j \in J, \bigvee_{k \in K} A_k \in \tau_j \\ &\Leftrightarrow \bigvee_{k \in K} A_k \in \tau. \end{aligned}$$

□

2.5.5 Lemma
If we define, for τ_1, τ_2 fuzzy topologies on a set X, and with

$$\tau_1 \vee \tau_2 := \langle \tau_1 \cup \tau_2 \rangle,$$

and

$$\tau_1 \wedge \tau_2 := \tau_1 \cap \tau_2,$$

then the collection of fuzzy topologies on X is a lattice.

2.5.6 Lemma

Let X be a set and $\mathcal{S} \subset I^X$. Then

$$\langle \mathcal{S} \rangle = \bigcap_{\delta \supset \mathcal{S}, \delta \in T} \delta,$$

where $\mathcal{T}$ is the collection of all fuzzy topologies on X.

PROOF.

(1) Let $A \in \langle S \rangle$. Then $A \in \bigvee_{J \in K} \bigwedge_{j \in J} A_j$, where each $J \in K$ is finite and each $A_j \in S$.

Let δ be a fuzzy topology on X such that $\delta \supset S$, then $\forall J \in K, \forall j \in J$, $A_j \in \delta$ and thus,

$$\begin{aligned} \forall J \in K, \bigwedge_{j \in J} A_j \in \delta &\Rightarrow \bigvee_{J \in K} \bigwedge_{j \in J} A_j \in \delta \\ &\Rightarrow A \in \bigcap_{\delta \in T, \delta \supset S} \delta. \end{aligned}$$

(2) $A \in \bigcap_{\delta \in T, \delta \supset S} \delta$ implies that for all δ such that $\delta \supset S$ and $\delta \in T$, we have that $A \in \delta$. Hence, $A \in \langle S \rangle$, as $\langle S \rangle$ is a fuzzy topology which contains S.

□

2.5.7 Definition

Let (X, τ) be a fuzzy topological space and $A \subset X$. Then the *fuzzy interior* A° of A is the join of all members of τ contained in A. i.e.,

$$A^\circ = \bigvee \{B \in I^X : B \in \tau, B \leq A\}.$$

We sometimes write $\text{int}_X(A)$ to denote the closure of A with respect to the fuzzy topological space X. This is the largest open fuzzy set contained in A, and trivially, A is open if and only if $A = A^\circ$.

Due to the fact that we have an order reversing involution $'$ defined on I, we are able to give reasonable definitions of closedness and related notions.

2.5.8 Definition

A fuzzy set A in a fuzzy topological space (X, τ) is τ*-closed* (*fuzzy closed*) if and only if $A' \in \tau$. Once again we may omit reference to τ and 'fuzzy' if there is no confusion.

It then follows trivially, from the definition of a closed fuzzy set, that the collection of closed fuzzy sets $\mathcal{C}$ on a set X satisfies the following properties:

(1) $\chi_\emptyset, \chi_X \in \mathcal{C}$.

(2) If $A, B \in \mathcal{C}$, then $A \vee B \in \mathcal{C}$.

(3) If $\{A_j : j \in J\} \subset \mathcal{C}$, then $\bigwedge_{j \in J} A_j \in \mathcal{C}$.

The concept of a closed set leads naturally to the notion of a closure operator.

2.5.9 Definition
Let (X, τ) be a fuzzy topological space. The *fuzzy closure*, denoted $\overline{A}$ (or cl(A)), of a fuzzy set is the meet of all τ-closed sets which contain A. That is,

$$\overline{A} = \bigwedge \{B \in I^X : B' \in \tau, A \leq B\}.$$

Therefore, $\overline{A}$ is the smallest τ-closed set which contains A and A is closed if and only if $A = \overline{A}$. We sometimes write cl$_X(A)$ to denote the closure of A with respect to the fuzzy topological space X.

It is clear that the closure operator in fuzzy topology has similar properties to its classical analogue, as is illustrated by the following two propositions.

2.5.10 Proposition ([82])
Let (X, τ) be a fuzzy topological space and $A, B \in I^X$. Then the closure operator $^-: I^X \to I^X$ has the following properties:

(1) $\overline{\chi_\emptyset} = \chi_\emptyset$,

(2) $\overline{A \vee B} = \overline{A} \vee \overline{B}$,

(3) $\overline{\overline{A}} = \overline{A}$,

(4) $A \leq \overline{A}$.

2.5.11 Proposition ([82])
Let (X, τ) be a fuzzy topological space and $A \in I^X$. Then

(1) $\overline{A}' = (A')^\circ$,

(2) $(A^\circ)' = \overline{(A')}$.

PROOF.

(1)

$$\begin{aligned}\overline{A}' &= [\bigwedge\{B \in I^X : B' \in \tau, A \leq B\}]' \\ &= \bigvee\{B' \in I^X : B' \in \tau, A \leq B\} \\ &= \bigvee\{B' \in I^X : B' \in \tau, A' \geq B'\} \\ &= (A')^\circ.\end{aligned}$$

(2) Simply replace A with A' in (1).

□

2.6 Continuous Functions

The notion of (fuzzy) continuity was first introduced in [9] by Chang in 1968.

2.6.1 Definition
Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is *(fuzzy) continuous* if and only if $\forall B \in \tau_2, f^{\leftarrow}[B] \in \tau_1$.

When discussing continuity, we will often omit the word 'fuzzy' provided that the context makes it clear which type of continuity is under discussion.

2.6.2 Proposition
Let (X, τ_1), (Y, τ_2) and (Z, τ_3) be fuzzy topological spaces. If $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ are continuous functions, then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is continuous.

2.6.3 Theorem ([82])
Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces and $f : (X, \tau_1) \to (Y, \tau_2)$ a function. Then the following are equivalent:

(1) f is continuous.

(2) For each τ_2-closed B, $f^{\leftarrow}[B]$ is τ_1-closed.

(3) For each $B \in I^Y$, $\overline{f^{\leftarrow}[B]} \leq f^{\leftarrow}[\overline{B}]$.

(4) For each $A \in I^X$, $f[\overline{A}] \leq \overline{f[A]}$.

(5) For each $B \in I^Y$, $f^{\leftarrow}[B^\circ] \leq (f^{\leftarrow}[B])^\circ$.

2.6.4 Theorem ([82])
Let $(X, \tau_1), (Y, \tau_2)$ be fuzzy topological spaces and $\mathcal{T}$ be a subbase of τ_2. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if and only if $\forall A \in \mathcal{T}$, $f^{\leftarrow}[A] \in \tau_1$.

2.6.5 Definition
Let $\{(X_j, \delta_j) : j \in J\}$ be a family of fuzzy topological spaces, for J an index set, and let X be a set without a fuzzy topology. For each $j \in J$, let

$$f_j : X \to (X_j, \delta_j)$$

be a mapping.

Now consider the subbase $S = \{f_j^{\leftarrow}[A_j] : A_j \in \delta_j, j \in J\}$. Let τ_1 be the fuzzy topology generated by S. We call τ_1 the *initial (fuzzy) topology* and it is the smallest fuzzy topology on X such that all mappings f_j will be continuous.

We list the following special cases of initial fuzzy topologies as examples.

2.6.6 Examples

(1) **Product Spaces**

For $X = \prod_{j \in J} X_j$ where (X_j, τ_j) are fuzzy topological spaces and $f_j = p_j$ for $j \in J$ (the projection maps), i.e., $\forall j \in J$, $p_j((x_i : i \in J)) = x_j$.

Now for each $j \in J$, let $A_j \in \tau_j$. For $j_1 \in J$, we have

$$f_{j_1}^{\leftarrow}[A_{j_1}] = \prod_{j \in J} A_j,$$

where $A_j = \chi_X$ for $j \neq j_1$.

Hence, the initial fuzzy topology is the one generated by the subbase $\{f_j^{\leftarrow}[A_j] : j \in J; \;\; \forall j \in J, A_j \in \tau_j\}$. This fuzzy topology is referred to as the *product (fuzzy) topology* on X.

(2) **Subspaces**

Let (X, τ) be a fuzzy topological space with $S \subset X$ and let

$$i_S = \begin{cases} S \to X \\ x \mapsto x. \end{cases}$$

Then $i_S^{\leftarrow}[A] = A|_S$ for $A \in I^X$. The initial fuzzy topology $\tau_S = \{A|_S : A \in \tau\}$ is the *subspace (fuzzy) topology*, that is, the collection of elements of τ restricted to S.

2.6.7 Theorem ([82])
Let (Y, τ_2) be a fuzzy topological space and let $f : (Y, \tau_2) \to (X, \tau_1)$ a mapping. Then f is continuous if and only if $\forall j \in J$, $f_j \circ f$ is continuous.

2.6.8 Lemma
Let (X, τ) be a fuzzy topological space, let $S \subset X$ and let τ_S be the subspace fuzzy topology on S. Then, for $A \in I^X$,

(1) $\text{int}_X(A)|_S \leq \text{int}_S(A|_S)$,

(2) $\text{cl}_S(A|_S) \leq \text{cl}_X(A|_S)$,

where

$\text{int}_X(A)$ is the interior of $A \in I^X$ with respect to τ,
$\text{int}_S(A)$ is the interior of $A \in I^S$ with respect to τ_S,
$\text{cl}_X(A)$ is the closure of $A \in I^X$ with respect to τ, and
$\text{cl}_S(A)$ is the closure of $A \in I^S$ with respect to τ_S.

PROOF.

(1)
$$\text{int}_X(A)|_S = \bigvee\{B|_S : B \in \tau, B \leq A\},$$
and
$$\text{int}_S(A|_S) = \bigvee\{B \in I^S : B \in \tau_S, B \leq A|_S\}.$$
Let $C = \{B \in I^X : B \in \tau, B \leq A\}|_S$ and $D = \{B \in I^S : B \in \tau_A, B \leq A|_S\}$. Then $C = \{B|_S : B \in \tau, B \leq A\}$. We know that:

(i) $B \in I^X \Rightarrow B|_S \in I^S$,
(ii) $B \in \tau \Rightarrow B|_S \in \tau_S$,
(iii) $B \leq A \Rightarrow B|_S \leq A|_S$.

So, $C \subset D$ and therefore, $\text{int}_X(A)|_S \leq \text{int}_S(A|_S)$.

(2)
$$\text{cl}_S(A|_S) = \bigwedge\{B \in I^S : B' \in \tau_S, A|_S \leq B\},$$
and
$$\text{cl}_X(A)|_S = \bigwedge\{B|_S : B' \in \tau, A \leq B\}.$$
Let $E = \{B \in I^S : B' \in \tau_S, A|_S \leq B\}$ and $F = \{B \in I^X : B' \in \tau, A \leq B\}|_S$. Now $F = \{B|_S : B' \in \tau, A \leq B\}$. In addition to (a) and (c) above, we know that

(d) $B' \in \tau \Rightarrow (B|_S)' = B'|_S \in \tau|_S$.

Thus, $F \subset E$ and thus, $\text{cl}_S(A|_S) \leq \text{cl}_X(A)|_S$.

□

The inequalities of the preceding lemma cannot be replaced by equalities as is illustrated in the following example.

2.6.9 Example
Let X be a set and $S \subset X$. Consider the following fuzzy topology τ on X defined as follows.

$$\tau := \{A \in I^X : A \geq a\} \cup \{\chi_\emptyset\},$$

for some $a \in (0, 1]$.

Then $\tau_A = \{B \in I^S : B \geq a\} \bigcup \{\chi_S\}$. Let $b > a$ and choose c such that $0 < c < a$. We define

$$\omega(x) = \begin{cases} b & \text{if } x \in S \\ c & \text{if } x \notin S. \end{cases}$$

Now, $\omega^\circ = \chi_\emptyset$ but $(\omega|_S)^\circ \neq \chi_\emptyset$.

We could similarly construct an example that shows that the inequality of Lemma 2.6.8 (2) cannot be replaced by equality either.

2.7 Fuzzy neighbourhoods spaces

As in classical topology, we can define a fuzzy topology in terms of neighbourhoods instead of open sets as the two concepts are equivalent. The work by Warren [81] is often quoted in this regard, but his requirement that a neighbourhood $n(x)$ of a point x should contain an open set g such that $g(x) = n(x)$, is to ensure a one-to-one correspondence between neighbourhood spaces and topological spaces. We will not consider this requirement (cf. also [22] where a neighbourhood without this condition was called an s-neighbourhood). In order to clarify the relationships between neighbourhoods and open sets, we supply the following.

2.7.1 Definition
Let X be a set. If for every $x \in X$, there exists a nonempty family $\mathcal{N}_x$ of fuzzy subsets of X called *neighbourhoods of* x, satisfying the following axioms:

(a) $U \in \mathcal{N}_x \Rightarrow U(x) > 0$.

(b) $U, V \in \mathcal{N}_x \Rightarrow U \wedge V \in \mathcal{N}_x$.

(c) $U \in \mathcal{N}_x, U \leq V \Rightarrow V \in \mathcal{N}_x$.

(d) If $U \in \mathcal{N}_x$, then there exists a $W \in \mathcal{N}_x$, $W \leq U$ such that if $W(y) > 0$, then $W \in \mathcal{N}_y$.

The family $\mathcal{N} = \{\mathcal{N}_x : x \in X\}$ is called the *neighbourhood system on* X and the ordered pair $(X, \mathcal{N})$ is called a *neighbourhood space*. We simply refer to the neighbourhood space X if this causes no confusion.

2.7.2 Definition

Let $X = (X, \mathcal{N})$ be a fuzzy neighbourhood space.

(a) A nonzero fuzzy subset A on X is *open* if whenever $A(x) > 0$, there exists a $V \in \mathcal{N}_x$ such that $V \leq A$. (Or we say A is open if and only if x is an *interior point* of A, for all x such that $A(x) > 0$). We also take $\chi_\emptyset$ to be open.

(b) A fuzzy subset A on X is *closed* if A' is open.

2.7.3 Proposition

An open set A is a neighbourhood of x if and only if $A(x) > 0$.

PROOF.

Follows from Axiom (c) of neighbourhoods.

□

2.7.4 Theorem

If X is a fuzzy neighbourhood space, then

(1) χ_X and $\chi_\emptyset$ are open.

(2) If A and B are open, then so is $A \wedge B$.

(3) If A_i is open for each $i \in I$, then so is $\bigvee_i A_i$.

PROOF.

(1) Follows from Axiom (c) and Definition 2.7.2.

(2) Follows from Axiom (b) of neighbourhoods.

(3) Follows from Axiom (c) of neighbourhoods.

□

We have shown that by defining open sets as in Definition 2.7.2, each neighbourhood space generates a fuzzy topology. If $(X, \mathcal{N})$ is a neighbourhood space, the collection $\tau_{\mathcal{N}}$ of all the open subsets of X is called *the fuzzy topology* on X (generated by the neighbourhood system $\mathcal{N}$).

2.7.5 Proposition
Let X be a set and $x \in X$. Then every $U \in \mathcal{N}_x$ contains an open set A such that $A(x) > 0$.

PROOF.

Follows from axiom (d) if neighbourhoods.

□

We can now characterize fuzzy continuity on neighbourhood spaces.

2.7.6 Theorem (Characterization of fuzzy continuity)
Let $f : (X, \mathcal{N}_1) \to (Y, \mathcal{N}_2)$ be a map between two fuzzy neighbourhood spaces. Then the following statements are equivalent:

(1) f is fuzzy continuous as a mapping from $(X, \tau_{\mathcal{N}_1})$ to $(Y, \tau_{\mathcal{N}_2})$.

(2) If B is closed in $(Y, \tau_{\mathcal{N}_2})$, then $f^{\leftarrow}[B]$ is closed in $(X, \tau_{\mathcal{N}_1})$.

(3) $f[\mathrm{cl}_X A] \leq \mathrm{cl}_Y(f[A])$, for all $A \in I^X$ (where cl_X is the fuzzy closure in $(X, \tau_{\mathcal{N}_1})$ and cl_Y is the fuzzy closure in $(Y, \tau_{\mathcal{N}_2})$).

(4) For every $x \in X$ and for every neighbourhood $N_{f(x)}$ of $f(x)$ in Y, $f^{\leftarrow}[N_{f(x)}]$ is a neighbourhood of x in X.

(5) For every $x \in X$ and for every neighbourhood $N_{f(x)}$ in Y, there exists a neighbourhood N_x of x in X such that $f[N_x] \leq N_{f(x)}$.

PROOF.

(1) ⇒ (2): Follows from

$$\begin{aligned} f^{\leftarrow}[\chi_Y - A] &= (\chi_Y - A)f \\ &= \chi_X - f^{\leftarrow}[A]f. \end{aligned}$$

(1) $\Rightarrow$ (4): $N_{f(x)}$ contains an $A \in \tau_2$ such that $A(f(x)) > 0$. So,

$$\begin{aligned} f^{\leftarrow}[N_{f(x)}] &= N_{f(x)}f \\ &\geq Af \\ &= f^{\leftarrow}[A]. \end{aligned}$$

By (1), $f^{\leftarrow}[A] \in \tau_1$, and is greater than 0. So by definition, $f^{\leftarrow}[N_{f(x)}]$ is a neighbourhood of x.

(4) $\Rightarrow$ (5): Let $N_x = f^{\leftarrow}[N_{f(x)}]$ of (4). Then

$$\begin{aligned} f[N_x] &= f[f^{\leftarrow}[N_{f(x)}]] \\ &\leq N_{f(x)}. \end{aligned}$$

(5) $\Rightarrow$ (4): We have $f[N_x] \leq N_{f(x)}$. Then

$$\begin{aligned} N_x &\leq f^{\leftarrow}[f[N_x]] \\ &\leq f^{\leftarrow}[N_{f(x)}], \end{aligned}$$

and hence, $f^{\leftarrow}[N_{f(x)}]$ is a neighbourhood on x.

(4) $\Rightarrow$ (1): Since $A \in \tau_2$ is a neighbourhood for each $y \in Y$ for which $A(y) > 0$, $f^{\leftarrow}[A] = Af$ is a neighbourhood for each x, for which $A(f(x)) > 0$. Thus, $f^{\leftarrow}[A] \in \tau_1$.

(2) $\Rightarrow$ (3): $\text{cl}_Y(f[A])$ is closed in Y, and so by (2), $f^{\leftarrow}[\text{cl}_Y f[A]]$ is closed in X. Now,

$$\begin{aligned} A &\leq f^{\leftarrow}[f[A]] \\ &\leq f^{\leftarrow}[\text{cl}_Y f[A]]. \end{aligned}$$

So, $\text{cl}_X A \leq f^{\leftarrow}[\text{cl}_Y f[A]]$ and hence, $f[\text{cl}_X A] \leq \text{cl}_Y f[A]$.

(3) $\Rightarrow$ (2): Let W be closed in Y. Then by (3),

$$\begin{aligned} f[\text{cl}_X f^{\leftarrow}[W]] &\leq \text{cl}_Y[f[f^{\leftarrow}[W]] \\ &\leq \text{cl}_Y W \\ &= W. \end{aligned}$$

Hence, $\text{cl}_X f^{\leftarrow}[W] \leq f^{\leftarrow}[W]$. Therefore, $f^{\leftarrow}[W]$ is closed.

□

Chapter 3

A fuzzy Hahn-Banach theorem

3.1 Introduction

The Hahn-Banach theorem is an elegant and powerful statement which has applications in many disparate branches of mathematics such as game theory, thermodynamics, linear equations and control theory amongst other areas. Above all, it is of extreme importance to the analyst and has been referred to as 'the analyst's form of the axiom of choice' and even 'The crown jewel of functional analysis' (see [47]). While, in fact, the Hahn-Banach theorem is strictly weaker than the axiom of choice - it does encapsulate the spirit of functional analysis, and is indeed one of the most important theorems in this area. Since the theorem was first proved in the 1920's, it has been generalized in many different directions. The reader is referred to [5, 47, 61] for further insight into the history and significance of the Hahn-Banach theorem.

In Chapter 1, we stated a well-known version of the theorem and its most important corollaries. The main idea of this chapter follows from the fact that a norm is an example of convex sublinear functional and we use this fact to obtain original versions of the Hahn-Banach theorem in the fuzzy setting. The central results of this chapter were presented in [35], in which we used a particular form of classical Hahn-Banach Theorem as stated in [2] or [18]:

3.1.1 Theorem (Real case)
Let M be a subspace of the real vector space X and p a sublinear functional on X, i.e., $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$, for all $x, y \in X$ and $\alpha \geq 0$, and f a real-valued linear functional on M such that for all $x \in M$, $f(x) \leq p(x)$. Then there exists a linear functional g on X, extending f (so $f(x) = g(x)$ on M) such that $g(x) \leq p(x)$, for all $x \in X$.

3.1.2 Theorem (Complex case)
Let M be a subspace of the complex vector space X and p a functional on X such that $p(x) \geq 0$, $p(x+y) \leq p(x) + p(y)$, $p(\alpha x) = |\alpha| p(x)$, for all $x, y \in X$ and $\alpha \in \mathbb{C}$, and f is a linear functional on M such that for all $x \in M$, $|f(x)| \leq p(x)$. Then there exists a linear functional g on X, extending f such that $|g(x)| \leq p(x)$, for all $x \in X$.

It should be emphasized that Theorem 1.3.9 immediately follows from the statements above due to the fact that a norm is a sublinear functional. We will use these results to prove counterparts for the fuzzy case. The Axiom of Choice is required for the proof of these two theorems, and so is inherent in our results as well.

3.2 Preliminaries

3.2.1 Operations on fuzzy sets

In order to obtain our fuzzy Hahn-Banach theorem, we need a suitable definition of a fuzzy normed space. In order to do this, we must generalize the concepts of addition and scalar multiplication of sets as well as the notions from Definition 1.2.19 to the fuzzy setting.

We require the following preliminaries. Throughout this section, X will denote a vector space over $\mathbb{F}$, and I the unit interval $I = (I, \leq)$, which is a complete lattice.

We extend the addition and scalar multiplication of crisp sets to the fuzzy setting using Zadeh's extension principle as discussed in Section 2.3.

3.2.1 Definition
Let X be a vector space; $A, B \in I^X$; $t \in \mathbb{F}$ and $x \in X$. Then we define:

(a) [Addition] $(A+B)(x) = \sup_{x_1 + x_2 = x}\{A(x_1) \wedge A(x_2)\}$.

(b) [Scalar multiplication] $t \cdot A(x) = A(\frac{x}{t})$ for $t \neq 0$. If $t = 0$:

$$t \cdot A(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \sup A & \text{if } x = 0. \end{cases}$$

This is indeed the natural way in which to define $t \cdot A$. Let $X := \mathbb{R}$. Let $A = \chi_{[a,b]}$ for

$a, b \in \mathbb{R}, a \leq b$, then for $x \in \mathbb{R}$ and $t \in \mathbb{R}$ ($t > 0$),

$$\begin{aligned} t \cdot A(x) &= \chi_{[a,b]}\Big(\frac{x}{t}\Big) \\ &= \begin{cases} 1 & \text{if } \frac{x}{t} \in [a,b] \\ 0 & \text{if } \frac{x}{t} \notin [a,b]. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} \frac{x}{t} \in [a,b] &\Leftrightarrow a \leq \frac{x}{t} \leq b \\ &\Leftrightarrow ta \leq x \leq tb, \end{aligned}$$

we have

$$\begin{aligned} t \cdot A(x) &= A\Big(\frac{x}{t}\Big) \\ &= \begin{cases} 1 & \text{if } x \in [ta,tb] \\ 0 & \text{if } x \notin [ta,tb]. \end{cases} \end{aligned}$$

That is, the set $[a, b]$ is stretched by a factor of t. For $t < 0$, via a similar argument, we have

$$\begin{aligned} t \cdot A(x) &= \chi_{[a,b]}\Big(\frac{x}{t}\Big) \\ &= \begin{cases} 1 & \text{if } x \in [tb,ta] \\ 0 & \text{if } x \notin [tb,ta]. \end{cases} \end{aligned}$$

If, on the other hand, we have that $t = 0$, then $0 \cdot \chi_{[a,b]} = \chi_{\{0\}}$, i.e., we have the fuzzy point with support 0 and value 1.

Similarly, it can be shown trivially that the definition of addition on fuzzy sets defined above coincides with the crisp definition of addition of sets in the case that the fuzzy sets, in question, are crisp.

3.2.2 Remark

Let X be a vector space. The $\cdot$ for scalar multiplication of a fuzzy set is used simply to avoid confusion with the usual pointwise multiplication of a scalar by a function as in Definition 2.4.4. Definition 3.2.1 follows from defining $A + B$ as the direct image of $f : X \times X \to X$, where f is given by $f(x, y) = x + y$; and defining $t \cdot A$ as the direct image of $g : X \to X$, where g is given by $g(x) = tx$:

$$g[A](y) = \begin{cases} \bigvee\{A(x) : g(x) = y\} & \\ 0 & \text{if } g(x) \neq y, \text{ for all } x \in X \end{cases}$$
$$= \begin{cases} \bigvee\{A(x) : tx = y\} & \\ 0 & \text{if } tx \neq y, \text{ for all } x \in X \end{cases}$$
$$= \begin{cases} A(\frac{y}{t}) & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \text{ and } y \neq 0 \\ \bigvee A(x) & \text{for } t = 0 \text{ and } y = 0. \end{cases}$$

3.2.3 Lemma
Let $s, t \in \mathbb{R}$ and let A, A_1 and A_2 be fuzzy sets on a vector space X. Then

(1) $s \cdot (t \cdot A) = t \cdot (s \cdot A) = (st) \cdot A$, and

(2) $A_1 \leq A_2 \Rightarrow t \cdot A_1 \leq t \cdot A_2$.

PROOF.

(1) If $s, t \neq 0$:

$$s \cdot (t \cdot A)(x) = (t \cdot A)\Big(\frac{x}{s}\Big) = \Big(A\Big(\frac{x}{st}\Big)\Big) = (s \cdot A)\Big(\frac{x}{t}\Big) = t \cdot (s \cdot A)(x).$$

Also,

$$(st) \cdot A(x) = A\Big(\frac{x}{st}\Big).$$

If $s = 0$ and $t \neq 0$:

$$0 \cdot (t \cdot A)(x) = \begin{cases} \sup(t \cdot A) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$
$$= \begin{cases} \sup A & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

As $\sup_{x \in X} A(x) = \sup_{x \in X} A(\frac{x}{t})$ (replace x by tx).

$$
\begin{aligned}
t \cdot (0 \cdot A)(x) &= (0 \cdot A)\Big(\frac{x}{t}\Big) \\
&= \begin{cases} \sup A & \text{if } \frac{x}{t} = 0 \\ 0 & \text{if } \frac{x}{t} \neq 0 \end{cases} \\
&= \begin{cases} \sup A & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(0t) \cdot A(x) &= 0 \cdot A(x) \\
&= \begin{cases} \sup A & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}
\end{aligned}
$$

Obviously, the case where $t = 0$ and $s \neq 0$ is the same as the preceding case.

If $t = s = 0$:

$$
\begin{aligned}
0 \cdot (0 \cdot A)(x) &= \begin{cases} \sup(0 \cdot A) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \\
&= \begin{cases} \sup A & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \\
&= 0 \cdot A(x).
\end{aligned}
$$

(2) Choose $x \in X$. We have that $A_1(x) \leq A_2(x)$. If $t \neq 0$, then

$$
\begin{aligned}
t \cdot A_1(x) &= A_1\Big(\frac{x}{t}\Big) \\
&\leq A_2\Big(\frac{x}{t}\Big) \\
&= t \cdot A_2(x).
\end{aligned}
$$

If $t = 0$ and $x = 0$, then $0 \cdot A_1(0) = \sup A_1$ and $0 \cdot A_2(0) = \sup A_2$. Since we have that $\sup A_1 \leq \sup A_2$, we have that $0 \cdot A_1(0) \leq 0 \cdot A_2(0)$. If $t = 0$ and $x \neq 0$, then $0 \cdot A_1(x) = 0 = 0 \cdot A_2(x)$.

□

3.2.4 Lemma

Let X, Y be real vector spaces and let $f : X \to Y$ be a linear mapping. Let $A, B \in I^X$ and let $k \in \mathbb{R}$. Then

(1) $f[k \cdot A] = k \cdot f[A]$, and

(2) $f[A+B] = f[A] + f[B]$.

PROOF.

Let $y \in Y$.

(1) If $k \neq 0$:

$$f[k \cdot A](y) = \sup_{z:f(z)=y} k \cdot A(z) = \sup_{z:f(z)=y} A\Big(\frac{z}{k}\Big),$$

and

$$k \cdot f[A](y) = k \sup_{z:f(z)=y} A(z) = \sup_{z:f(z)=y} A\Big(\frac{z}{k}\Big).$$

If $k = 0$:

$$\begin{aligned} f[0 \cdot A](y) &= \sup_{z:f(z)=y} 0 \cdot A(z) \\ &= \begin{cases} \sup A & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y, \end{cases} \end{aligned}$$

and

$$\begin{aligned} f[0 \cdot A](y) &= 0 \cdot \sup_{z:f(z)=y} A(z) \\ &= \begin{cases} \sup\{\sup_{z:f(z)=y} A(z)\} & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y \end{cases} \\ &= \begin{cases} \sup A & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y. \end{cases} \end{aligned}$$

(2)

$$
\begin{aligned}
f[A+B](y) &= \sup_{z:f(z)=y} (A+B)(z) \\
&= \sup_{z:f(z)=y} \sup_{z_1+z_2=z} \big\{A(z_1) \wedge B(z_2)\big\} \\
&= \sup_{f(z_1+z_2)=y} \big\{A(z_1) \wedge B(z_2)\big\} \\
&= \sup_{f(z_1)+f(z_2)=y} \big\{A(z_1) \wedge B(z_2)\big\} \quad \text{(since } f \text{ is linear)} \\
&= \sup_{x_1+x_2=y} \Big\{ \sup_{f(z_1)=x_1, f(z_2)=x_2} \big\{A(z_1) \wedge B(z_2)\big\}\Big\} \\
&= \sup_{x_1+x_2=y} \Big\{ \sup_{f(z_1)=x_1} \sup_{f(z_2)=x_2} \big\{A(z_1) \wedge B(z_2)\big\}\Big\} \quad \text{(from Theorem 2.3.1 (1))} \\
&= \sup_{x_1+x_2=y} \Big\{ \sup_{f(z_1)=x_1} \big\{A(z_1)\big\} \wedge \sup_{f(z_2)=x_2} \big\{B(z_2)\big\}\Big\} \\
&= \sup_{x_1+x_2=y} \big\{f[A](z_1) \wedge f[B](z_2)\big\} \\
&= \big(f[A]+f[B]\big)(y).
\end{aligned}
$$

□

3.2.5 Proposition ([30])
Let $A, A_1, ..., A_n$ be fuzzy sets on a vector space X and $r_1, ..., r_n \in \mathbb{R}$, then the following assertions are equivalent:

(1) $r_1 \cdot A_1 + \ldots + r_n \cdot A_n \leq A$.

(2) $\forall x_1, ..., x_n \in X$, we have

$$A(r_1x_1 + \ldots + r_nx_n) \geq \min\{A_1(x_1), \ldots, A_n(x_n)\}.$$

PROOF.

(1) ⇒ (2):

$$
\begin{aligned}
A(r_1x_1 + \ldots + r_nx_n) &\geq (r_1 \cdot A_1 + \ldots + r_n \cdot A_n)(r_1x_1 + \ldots + r_nx_n) \\
&\geq \min\{r_1 \cdot A_1(r_1x_1), \ldots, r_n \cdot A_n(r_nx_n)\} \quad \text{(from Definition 3.2.1 (1))} \\
&\geq \min\{A_1(x_1), \ldots, A_n(x_n)\}. \quad \text{(from Definition 3.2.1 (2))}
\end{aligned}
$$

(2) ⇒ (1): By rearranging the order if necessary, we may assume that $r_i \neq 0$, for $i = 1, \ldots, k$, and $r_i = 0$, for $k < i \leq n$. If $\forall i = 1, \ldots, n$, $r_i \neq 0$, then this method of proof

is still valid. Let $x_1, \ldots, x_k$ be elements of X. From (2), we have that for all $y_1, \ldots, y_{n-k} \in X$,

$$A(r_1x_1 + \ldots + r_kx_k) \geq \min\{A_1(x_1), \ldots, A_{k+1}(y_1), \ldots, A_n(y_{n-k})\}.$$

Since $0 \cdot A_j(0) = \sup_{y \in X} A_j(y)$, we get

$$A(r_1x_1 + \ldots + r_kx_k) \geq \min\{A_1(x_1), \ldots, A_k(x_k), 0 \cdot A_{k+1}(0), \ldots, 0 \cdot A_n(0)\}.$$

Now,

$$(r_1 \cdot A_1 + \ldots + r_n \cdot A_n)(z) = \sup_{x_1+\ldots+x_k=z} \Big\{ \min\{r_1 \cdot A_1(x_1), \ldots, r_n \cdot A_n(x_n)\} \Big\}$$

(from Definition 3.2.1 (1))

$$= \sup_{x_1+\ldots+x_k=z} \Big\{ \min\{r_1 \cdot A_1(x_1), \ldots, r_k \cdot A_k(x_k), 0 \cdot A_{k+1}(0), \ldots, 0 \cdot A_n(0)\Big\}$$

$$= \sup_{x_1+\ldots+x_k=z} \Big\{ \min\{A_1\Big((\frac{1}{r_1})x_1\Big), \ldots, A_k\Big((\frac{1}{r_k})x_k\Big), 0 \cdot A_{k+1}(0), \ldots, 0 \cdot A_n(0)\}\Big\}$$

$$\leq \sup_{x_1+\ldots+x_k=z} A\Big(r_1(\frac{1}{r_1})x_1 + \ldots + r_k(\frac{1}{r_k})x_k\Big) = A(z).$$

□

With the classical definitions of convex, balanced and absorbing in mind, we respectively extend these notions to the fuzzy setting and collect a number of useful associated results.

3.2.2 Convex fuzzy sets

3.2.6 Definition

Let X be a vector space and A a fuzzy subset of X. A is *convex* if

$$A(kx + (1-k)y) \geq A(x) \wedge A(y),$$

whenever $x, y \in X$ and $0 \leq k \leq 1$.

3.2.7 Remark

Let X be a vector space, A a convex and crisp subset of X; $x, y \in A$ and $k \in [0,1]$, then

$$\chi_A(kx + (1-k)y) \geq \chi_A(x) \wedge \chi_A(y).$$

Now, since $\chi_A(x) = \chi_A(y) = 1$, we have

$$kx + (1-k)y \in A.$$

So, A is convex in the classical sense. We thus have that our definition of convexity reduces to the classical notion of convexity in the crisp case.

3.2.8 Proposition

Let A be a fuzzy set on a vector space X, then the following three assertions are equivalent:

(1) A is convex.

(2) $\forall k \in [0,1], k \cdot A + (1-k) \cdot A \leq A$.

(3) $\forall \alpha \in I, A_\alpha$ is convex.

PROOF.

The equivalence of (1) and (2) follows from Proposition 3.2.5, with

$$\begin{aligned} r_1 &:= k, & r_2 &:= 1-k, \\ x_1 &:= x, \quad \text{and} & x_2 &:= y. \end{aligned}$$

(1) $\Rightarrow$ (3): Choose $k \in [0,1]$ and $\alpha \in I$. Let $x, y \in A_\alpha$, then $A(x) \geq \alpha$ and $A(y) \geq \alpha$, and thus,

$$A(x) \wedge A(y) \geq \alpha.$$

So, from the convexity of A, we have

$$A(kx + (1-k)y) \geq A(x) \wedge A(y) \geq \alpha.$$

Thus, $kx + (1-k)y \in A_\alpha$. i.e., A_α is convex.

(3) $\Rightarrow$ (1): Choose $k \in [0,1]$. Let $x, y \in X$ and let $\alpha := A(x) \wedge A(y) \in I$. Then $x, y \in A_\alpha$. By the convexity of A_α, we have

$$kx + (1-k)y \in A_\alpha.$$

Hence,

$$A(kx + (1-k)y) \geq \alpha = A(x) \wedge A(y).$$

□

3.2.9 Proposition

Let X, Y be real vector spaces and let $f : X \to Y$ be a linear map. If A is a convex fuzzy set in X, then $f[A]$ is a convex fuzzy set in Y. Similarly, $f^{\leftarrow}[B]$ is a convex fuzzy set in X, whenever B is a convex fuzzy set in Y.

PROOF.

Let $k \in [0,1]$ and A a convex fuzzy set on a vector space X. Then by Lemma 3.2.4, we have

$$\begin{aligned} f[k \cdot A + (1-k) \cdot A] &= f[k \cdot A] + f[(1-k) \cdot A] \\ &= k \cdot f[A] + (1-k) \cdot f[A]. \end{aligned}$$

By Proposition 3.2.8, we have that $k \cdot A + (1-k) \cdot A \leq A$. Now, by Theorem 2.3.2 (10), we have $f[k \cdot A + (1-k) \cdot A] \leq f[A]$, which implies

$$k \cdot f[A] + (1-k) \cdot f[A] \leq f[A].$$

So, $f[A]$ is convex by Proposition 3.2.8.

Now, assume that B is a convex fuzzy set in Y and let $k \in [0,1]$. Set $M = k \cdot f^{\leftarrow}[B] + (1-k) \cdot f^{\leftarrow}[B]$.

Then

$$\begin{aligned} f(M) &= f[k \cdot f^{\leftarrow}[B] + (1-k) \cdot f^{\leftarrow}[B]] \\ &= f[k \cdot f^{\leftarrow}[B]] + f[(1-k) \cdot f^{\leftarrow}[B]] \quad \text{(by Lemma 3.2.4 (2))} \\ &= k \cdot f[f^{\leftarrow}[B]] + (1-k) \cdot f[f^{\leftarrow}[B]] \quad \text{(by Lemma 3.2.4 (1))} \\ &\leq k \cdot B + (1-k) \cdot B \quad \text{(by Theorem 2.3.2 (11))} \\ &\leq B. \quad \text{(by Proposition 3.2.8)} \end{aligned}$$

Now, by Theorem 2.3.2 (6), we have $f^{\leftarrow}[f[M]] \leq f^{\leftarrow}[B]$ and hence, by Theorem 2.3.2 (12), we have $M \leq f^{\leftarrow}[B]$.

□

3.2.10 Proposition
If A, B are convex fuzzy sets on a vector space X, then $A + B$ is a convex fuzzy set in X.

PROOF.

Let A, B be convex fuzzy sets. Let $x, y \in X$ and choose $k \in [0,1]$. Then

$$(A+B)(kx + (1-k)y) = \bigvee_{z_1 + z_2 = kx + (1-k)y} \{A(z_1) \wedge B(z_2)\}.$$

If $x_1 + x_2 = x$ and $y_1 + y_2 = y$, for $x_1, x_2, y_1, y_2 \in X$, then $(kx_1 + (1-k)y_1) + (kx_2 + (1-k)y_2) = kx + (1-k)y$, and

$$\begin{aligned}(A+B)(kx+(1-k)y) &\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{A(kx_1+(1-k)y_1)\wedge B(kx_2+(1-k)y_2)\Big\}\\ &\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{(A(x_1)\wedge A(y_1))\wedge((B(x_2)\wedge B(y_2))\Big\}\\ &\text{(since } A \text{ and } B \text{ are convex)}\\ &\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{(A(x_1)\wedge B(x_2))\wedge((A(y_1)\wedge B(y_2))\Big\}\\ &\text{(from Theorem 2.3.1 (1))}\\ &\geq \bigvee_{x_1+x_2=x} \Big\{A(x_1)\wedge B(x_2)\}\wedge \bigvee_{y_1+y_2=y}\{A(y_1)\wedge B(y_2)\Big\}\\ &= (A+B)(x)\wedge(A+B)(y).\end{aligned}$$

□

3.2.11 Remark
A convex $\Rightarrow t\cdot A$ convex for $t \neq 0$:

$$\begin{aligned}t\cdot A(kx+(1-k)y) &= A\Big(\frac{1}{t}kx+\frac{1}{t}(1-k)y\Big)\\ &\geq A\Big(\frac{x}{t}\Big)\wedge A\Big(\frac{y}{t}\Big)\\ &= [t\cdot A(x)]\wedge[t\cdot A(y)].\end{aligned}$$

3.2.3 Balanced fuzzy sets

3.2.12 Definition
Let X be a vector space and A a fuzzy subset of X. A is called *balanced* (*circled*) if $A(kx) \geq A(x)$ whenever $x \in X$, $k \in \mathbb{F}$, $|k| \leq 1$.

3.2.13 Remark
Let X be a vector space and $A \in I^X$.

(1) A balanced $\Rightarrow A(-x) = A(x)$, for all $x \in X$.

(2) A balanced $\Rightarrow A(0) = \sup_{x\in X} A(x)$.

(3) Let χ_A be balanced for a crisp set A. Let $x \in X$ and $k \in \mathbb{R}$ such that $|k| \leq 1$. Consider the case where $k \neq 0$. We then have $\chi_A(x) \geq k\cdot\chi_A(x) \geq \chi_A(\frac{x}{k})$.

So,

$$\chi_A\Big(\frac{x}{k}\Big) = 1 \Rightarrow \chi_A(x) = 1.$$

Thus, $\frac{x}{k} \in A \Rightarrow x \in A$, and hence, $x \in k \cdot A \Rightarrow x \in A$. Therefore, $k \cdot A \subset A$, and hence, A is balanced in the classical sense.

(4) If A is balanced, $(\frac{t_1}{t_2}) \cdot A(x) = A(\frac{t_2}{t_1}x) \geq A(x)$ if $|\frac{t_2}{t_1}| \leq 1$. So, $t_1 \cdot A \geq t_2 \cdot A$ if $0 < |t_2| \leq |t_1|$.

(5) A balanced $\Leftrightarrow t \cdot A \leq A$ whenever $|t| \leq 1$.

3.2.14 Proposition
Let A be a fuzzy set on a vector space X, then the following two assertions are equivalent:

(1) A is balanced.

(2) $\forall \alpha \in I$, A_α is balanced.

PROOF.

(1) $\Rightarrow$ (2): Choose $\alpha \in I$. Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and choose $x \in A_\alpha$. Then $k \cdot A \leq A$ by Remark 3.2.13 (5).

(a) If $k \neq 0$, then

$$k \cdot A(x) \leq A(x) \Rightarrow A\left(\frac{x}{k}\right) \leq A(x).$$

So, $A\left(\frac{x}{k}\right) \geq \alpha \Rightarrow A(x) \geq \alpha$, and thus, $\frac{x}{k} \in A_\alpha \Rightarrow x \in A_\alpha$. i.e., $x \in [k \cdot A]_\alpha \Rightarrow x \in A_\alpha$

(b) If $k = 0$ and $x = 0$, then (2) trivially.

$$0 \in A_\alpha \Rightarrow 0 \in 0A_\alpha = \{0\}.$$

(c) If $k = 0$ and $x \neq 0$, then

$$x \in 0 \cdot A_\alpha = \{0\} \Rightarrow x = 0,$$

a contradiction.

From (a), (b) and (c) above, we have that A_α is balanced.

(2) $\Rightarrow$ (1): Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and let $\alpha \in I$. We have that

$$x \in A_\alpha \Rightarrow kx \in A_\alpha.$$

Thus, $A(x) \geq \alpha \Rightarrow A(kx) \geq \alpha$ and hence, $A(x) \leq A(kx)$.

□

3.2.15 Proposition
Let X, Y be real vector spaces and let $f : X \to Y$ be a linear map. If A is a balanced fuzzy set in X, then $f[A]$ is a balanced fuzzy set in Y. Similarly, $f^{\leftarrow}[B]$ is a balanced fuzzy set in X whenever B is a balanced fuzzy set in Y.

PROOF.

Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and let $A \in I^X$ be balanced. We have by Proposition 3.2.14, that $k \cdot A \leq A$. Now, by Theorem 2.3.2 (10), we have $f[k \cdot A] \leq f[A] \Leftrightarrow k \cdot f[A] \leq f[A]$ (from Lemma 3.2.4 (1)). By Proposition 3.2.14, we have that $f[A]$ is balanced in Y.

Let $B \in I^Y$ be balanced and choose $k \in \mathbb{R}$ such that $|k| \leq 1$. By Proposition 3.2.14, we have $k \cdot B \leq B$ and by Theorem 2.3.2 (6), we have $f^{\leftarrow}[k \cdot B] \leq f^{\leftarrow}[B]$. Now, if $k \neq 0$, then for $x \in X$,

$$\begin{aligned}
f^{\leftarrow}[k \cdot B](x) \leq f^{\leftarrow}[B](x) &\Leftrightarrow k \cdot B(f(x)) \leq B(f(x)) \\
&\Leftrightarrow B\Big(\frac{1}{k} f(x)\Big) \leq B(f(x)) \\
&\Leftrightarrow B\Big(f(\frac{x}{k})\Big) \leq B(f(x)) \\
&\Leftrightarrow f^{\leftarrow}[B]\Big(\frac{x}{k}\Big) \leq f^{\leftarrow}[B](x) \\
&\Leftrightarrow k \cdot f^{\leftarrow}[B](x) \leq f^{\leftarrow}[B](x).
\end{aligned}$$

If, on the other hand, $k = 0$, then

$$\begin{aligned}
0 \cdot f^{\leftarrow}[B](x) &= 0 \cdot B(f(x)) \\
&= \begin{cases} \sup B & \text{if } f(x) = 0 \\ 0 & \text{if } f(x) \neq 0. \end{cases}
\end{aligned}$$

We have from Proposition 3.2.25, that $B(0) = \sup B$ and as

$$\begin{aligned}
f^{\leftarrow}[B](x) &= B(f(x)) \\
&= \begin{cases} B(0) & \text{if } f(x) = 0 \\ B(f(x)) & \text{if } f(x) \neq 0, \end{cases}
\end{aligned}$$

we have $0 \cdot f^{\leftarrow}[B] \leq f^{\leftarrow}[B]$.

□

3.2.16 Proposition
If A, B are balanced fuzzy sets on a vector space X, then $A + B$ is a balanced fuzzy set in X.

PROOF.

Let A, B be balanced and choose $k \in \mathbb{R}$ such that $|k| \leq 1$. Choose $x \in X$. Now,

$$\begin{aligned}
(A+B)(x) &= \bigvee_{x_1+x_2=x} \Big\{A(x_1) \wedge B(x_2)\Big\} \\
&\leq \bigvee_{x_1+x_2=x} \Big\{A(kx_1) \wedge B(kx_2)\Big\} \quad \text{(since } A \text{ is balanced)} \\
&\leq \bigvee_{kx_1+kx_2=kx} \Big\{A(kx_1) \wedge B(kx_2)\Big\} \\
&\text{(since } x_1 + x_2 = x \Rightarrow kx_1 + kx_2 = kx, \text{ for } x_1, x_2 \in X) \\
&\leq \bigvee_{z_1+z_2=kx} \Big\{A(z_1) \wedge B(z_2)\Big\} = (A+B)(kx).
\end{aligned}$$

□

3.2.17 Proposition
If $\{A_j\}_{j \in J}$ is a family of convex (resp., balanced) fuzzy sets on a vector space X, for J an index set, then $A = \bigwedge_{j \in J} A_j$ is a convex (resp., balanced) fuzzy set in X.

PROOF.

Let $\alpha \in I$, then

$$A_\alpha = \{x \in X : A(x) \geq \alpha\} = \bigcap_{j \in J} \{x \in X : A_j(x) \geq \alpha\}.$$

Since the intersection of ordinary convex (balanced) subsets of X is convex (balanced), the result follows from Propositions 3.2.8 and 3.2.14.

□

3.2.18 Lemma
Let X be a vector space and A a fuzzy set on X. If A is convex (resp., balanced), then supp(A) is convex (resp., balanced).

PROOF.

(i) Let $A \in I^X$ be convex. Then, by Proposition 3.2.8, we have that for all $\alpha \in I$, A_α is convex. Let $x, y \in$ supp(A) and let $k \in I$. Then $A(x) > 0$ and $A(y) > 0$, and therefore, $x, y \in A_\alpha$, where $\alpha = A(x) \wedge A(y) > 0$. Since A_α is convex we have that $A(kx + (1-k)y) \geq A(x) \wedge A(y) = \alpha$, and thus, we have that $kx + (1-k)y \in A_\alpha \subset \text{supp}(A)$ which means that supp(A) is convex.

(ii) Let $A \in I^X$ be balanced and let k be such that $|k| \leq 1$. We have, by Proposition 3.2.14, that A_α is balanced for each $\alpha \in I$. Let $x \in \text{supp}(A)$, then $A(x) > 0$. Thus $x \in A_\alpha$ where $\alpha = A(x)$. Since A_α is balanced, we have that $kx \in A_\alpha \subset \text{supp}(A)$, and thus, supp($A$) is balanced.

□

3.2.4 Absorbing fuzzy sets

3.2.19 Definition
A fuzzy set A on a vector space X is *absorbing* if $\bigvee_{t>0} t \cdot A = \chi_X$.

3.2.20 Remark
It is clear the that the concept of absorbing in the fuzzy setting is a direct generalization of the classical analogue. Unlike the notions of convexity and balancedness, however, the notion absorbing does not reduce to the classical notion in the following sense. It is possible to have a crisp set A that is not absorbing in the classical sense, yet its characteristic function χ_A is absorbing in the fuzzy sense. This is illustrated by he following example.

3.2.21 Example
Consider the set $\mathbb{R} \times \mathbb{R}$. Let

$$A := \{(0,0)\} \bigcup \{(x,y) \in \mathbb{R} \times \mathbb{R} : 1 \leq x^2 + y^2 \leq 2\}.$$

Now,

$$\bigvee_{t>0} t \cdot \chi_A = 1,$$

but for $x = (1,1)$, $\nexists q \in \mathbb{R}$, $q > 0$ such that $\forall s \in \mathbb{R}$, $|s| < q$, $sx \in A$, and hence, A is not absorbing in the classical sense.

3.2.22 Lemma
Let X be a vector space and let $A \in I^X$ be balanced and absorbing. Then supp(A) is balanced and absorbing.

PROOF.

If $A \in I^X$ is balanced, then we have, by Lemma 3.2.18, that supp(A) is both balanced in the classical sense.

We thus need only show that supp(A) is absorbing. Choose $x \in \text{supp}(A)$. Since A is absorbing, we have that $(\bigvee_{t>0} A)(tx) = 1$. There exists $q \in \mathbb{R}$, $q > 0$ such that $A(qx) > 0$. If this were not the case, then we would have $\bigvee_{t>0} tA(x) = 0$, a contradiction. Choose $s \in \mathbb{R}$ such that $|s| \leq q$. Since $|s| \leq q$, we have that $|\frac{s}{q}| \leq 1$. Because A is balanced, we have that

$$\begin{aligned} A\Big(\frac{s}{q}x\Big) \geq A(x) &\Leftrightarrow A\Big(\frac{s}{q}x\Big) \geq A\Big(\frac{q}{q}x\Big) \\ &\Leftrightarrow q \cdot A(sx) \geq q \cdot A(qx) \\ &\Leftrightarrow \Big(\frac{1}{q}q\Big) \cdot A(sx) \geq \Big(\frac{1}{q}q\Big) \cdot A(qx) \\ &\Leftrightarrow A(sx) \geq A(qx) > 0. \end{aligned}$$

So, $sx \in \text{supp}(A)$, and therefore supp(A) is absorbing.

□

Note that if A absorbing, then $\sup_{t>0} t \cdot A(0) = A(0) = 1$.

3.2.23 Proposition
Let $f : X \to Y$ be a linear map for X, Y real vector spaces, A an absorbing fuzzy set in Y, and $t \in \mathbb{R}$. Then $f^{\leftarrow}[A]$ is an absorbing fuzzy set in X.

PROOF.

Let $x \in X$. Then

$$\begin{aligned} t \cdot f^{\leftarrow}[A](x) &= f^{\leftarrow}[A](\frac{x}{t}) \\ &= A(f(\frac{x}{t})) \\ &= A\big(\frac{1}{t}f(x)\big) \\ &= t \cdot A(f(x)). \end{aligned}$$

So,

$$\bigvee_{t>0} t \cdot f^{\leftarrow}[A](x) = \bigvee_{t>0} t \cdot A(f(x)) = 1,$$

since A is absorbing.

□

3.2.5 Fuzzy vector spaces

The next definition provides us with a concept of a fuzzy vector space.

3.2.24 Definition
Let X be a vector space. Then $A \in I^X$ is called a *fuzzy subspace* of X (*fuzzy vector space* on X) if $\forall a, b \in \mathbb{R}$ and $\forall x, y \in X$,

$$A(ax + by) \geq A(x) \wedge A(y).$$

3.2.25 Proposition
Let A be a fuzzy subspace of a vector space X, then:

(1) $A(0) = \sup_{x \in X} A(x)$.

(2) For each $\alpha \in I$, A_α is a linear subspace of X.

(3) $x \in X, a \neq 0 \Rightarrow A(ax) = A(x)$.

PROOF.

(1)

$$\begin{aligned} x \in X \Rightarrow A(0) &= A(0x + 0x) \\ &\geq A(x) \wedge A(x) \\ &= A(x). \end{aligned}$$

(2) Choose $\alpha \in I$. If $A_\alpha = \emptyset$, then it is a linear subspace of X. If not, then choose $x, y \in A_\alpha$. Then

$A(x) \geq \alpha$ and $A(x) \geq \alpha$.

Since A is a fuzzy subspace, we have $\forall a, b \in \mathbb{R}$,

$$\begin{aligned} A(ax + by) &\geq A(x) \wedge A(y) \\ &\geq \alpha \wedge \alpha \\ &= \alpha. \end{aligned}$$

Hence, $ax + by \in A_\alpha$, and so A_α is a linear subspace.

(3)

$$\begin{aligned} x \in X, a \neq 0 \Rightarrow A(ax) &= A(ax + 0x) \\ &\geq A(x) \wedge A(x) \\ &= A(x). \end{aligned}$$

Now, replace x by ax and a by $\frac{1}{a}$, to get $A(x) \geq A(ax)$. Equality follows.

□

We are now in a position to give a characterization of a fuzzy subspace.

3.2.26 Lemma ([30])
Let A be a fuzzy set on a vector space X. Then the following are equivalent:

(1) A is a fuzzy subspace of X.

(2) $\forall k, m \in \mathbb{R}$, we have $k \cdot A + m \cdot A \leq A$.

(3) The following two conditions hold:

- (a) $A + A \leq A$,
- (b) $\forall t \in \mathbb{R}, t \cdot A \leq A$.

PROOF.

(3) ⇒ (2): Follows trivially, also (1) and (2) are equivalent by Proposition 3.2.5.

(2) ⇒ (3): $A + A = 1 \cdot A + 1 \cdot A \leq A$ and $k \cdot A = k \cdot A + 0 \cdot A \leq A$.

□

3.2.27 Proposition
Let X be a vector space, $u, v \in X$ and A a fuzzy subspace of X such that $A(u) > A(v)$. Then $A(u + v) = A(v)$.

PROOF.

Since $A(u) > A(v)$, we have $A(u + v) \geq A(v)$. Also

$$\begin{aligned} A[(u + v) - u] &= A(v) \\ &\geq A(u + v) \wedge A(u). \end{aligned}$$

Since $A(u) > A(v)$, we have $A(u + v) \leq A(v)$. Consequently, $A(u + v) = A(v)$.

□

3.2.28 Proposition
If A is a fuzzy subspace of a vector space X and $v, w \in X$ with $A(v) \neq A(w)$, then $A(v + w) = A(v) \wedge A(w)$.

PROOF.

Apply Proposition 3.2.27.

□

3.2.29 Proposition ([30])
If A and B are fuzzy subspaces of a vector space X and $k \in \mathbb{R}$, then $k \cdot A$ and $A + B$ are fuzzy subspaces.

PROOF.

(1) We have that for $x, y \in X$ and $a, b \in \mathbb{R}$,

$$A(ax + by) \geq A(x) \wedge A(y).$$

Let $k \in \mathbb{R}$ and assume that $k \neq 0$. Then

$$\begin{aligned} k \cdot A(ax + by) &= A\Big(\frac{1}{k}(ax + by)\Big) \\ &= A\Big(\frac{a}{k}x + \frac{b}{k}y\Big) \\ &\geq A\Big(\frac{x}{k}\Big) \wedge A\Big(\frac{y}{k}\Big) \\ &\geq k \cdot A(x) \wedge k \cdot A(y). \end{aligned}$$

If, on the other hand, we have that $k = 0$, then

$$0 \cdot A(ax + by) = \begin{cases} 0 & \text{if } ax + by \neq 0 \\ \sup A & \text{if } ax + by = 0 \end{cases}$$

If $ax + by = 0$:

$0 \cdot A(ax + by) = \sup A \geq A(x) \wedge A(y)$.

If $ax + by \neq 0$:

We have that $0 \cdot A(ax + by) = 0$. We must show that $(0 \cdot A(x)) \wedge (0 \cdot A(y)) = 0$. Assume that $(0 \cdot A(x)) \wedge (0 \cdot A(y)) \neq 0$. Then

$0 \cdot A(x) > 0$ and $0 \cdot A(y) > 0$. So, $y = x = 0$. A contradiction.

(2)

$$(A + B)(ax + by) = \bigvee_{z_1 + z_2 = ax + by} \big\{A(z_1) \wedge B(z_2)\big\}.$$

Now, if $x_1 + x_2 = x$ and $y_1 + y_2 = y$, for $x_1, x_2, y_1, y_2 \in X$, then

$$(ax_1 + by_1) + (ax_2 + by_2) = ax + by.$$

So,

$$\begin{aligned}
(A+B)(ax+by) &\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{\big(A(ax_1+by_1) \wedge B(ax_2+by_2)\big)\Big\} \\
&\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{\big(A(x_1) \wedge A(y_1)\big) \wedge \big(B(x_2) \wedge B(y_2)\big)\Big\} \\
&\text{(A and B are both fuzzy subspaces)} \\
&= \bigvee_{x_1+x_2=x, y_1+y_2=y} \Big\{\big(A(x_1) \wedge B(x_2)\big) \wedge \big(A(y_1) \wedge B(y_2)\big)\Big\} \\
&= \bigvee_{x_1+x_2=x} \bigvee_{y_1+y_2=y} \Big\{\big(A(x_1) \wedge B(x_2)\big) \wedge \big(A(y_1) \wedge B(y_2)\big)\Big\} \\
&\text{(by Theorem 2.3.1 (1))} \\
&= \Big[\bigvee_{x_1+x_2=x} \Big\{A(x_1) \wedge B(x_2)\Big\}\Big] \bigwedge \Big[\bigvee_{y_1+y_2=y} \Big\{A(y_1) \wedge B(y_2)\Big\}\Big] \\
&= (A+B)(x) \wedge (A+B)(y).
\end{aligned}$$

□

3.2.30 Proposition ([30])
If $(A_j)_{j \in J}$ is a collection of fuzzy subspaces of a vector space X, for J an index set, then $\bigwedge_{j \in J} A_j$ is also a fuzzy subspace of X.

PROOF.

Let $m, k \in \mathbb{R}$ and $x, y \in X$. Then

$$\begin{aligned}
\Big(\bigwedge_{j \in J} A_j\Big)(mx+ky) &= \bigwedge_{j \in J} A_j(mx+ky) \\
&\geq \bigwedge_{j \in J} \big(A_j(x) \wedge A_j(y)\big) \quad \text{(since the A_j's are fuzzy subspaces)} \\
&= \Big(\bigwedge_{j \in J} A_j(x)\Big) \wedge \Big(\bigwedge_{j \in J} A_j(y)\Big).
\end{aligned}$$

□

3.3 Fuzzy topological vector spaces and normed spaces

We now present Katsaras's definition of a fuzzy topological vector space. The material of this section is taken from [30], [31] and [32].

The fuzzy norm and fuzzy seminorm were first formulated by Katsaras in [32], and we now present his motivation and definitions. If p is a seminorm on a vector space X, then the set $V = \{x : p(x) < 1\}$ is convex, balanced, absorbing and the family $\{t \cdot V : t > 0\}$ is a base at zero for a linear topology. Further, p is a norm if and only if $\bigcap_{t>0} t \cdot V = \{0\}$. Conversely, if W is a balanced, convex, absorbing subset of E, then the Minkowski functional p of W,

$$p(x) = \inf\{t > 0 : x \in t \cdot W\}$$

is a seminorm on X. We also have

$$\{x : p(x) < 1\} \subset W \subset \{x : p(x) \leq 1\}.$$

So, we have that the linear topology generated by p coincides with the linear topology which has as a base at zero the family $\{t \cdot W : t > 0\}$. This leads us to the following definition.

If $(X, \|\cdot\|)$ is a normed space and B is the unit ball (open/closed), then B is convex, balanced and absorbing in the classical sense and $\chi_B \in I^X$ has the same properties (as defined above).

Furthermore, if $x \in X, x \neq 0$, then there exists $t > 0$ such that $x \notin t \cdot B$ (i.e., $\chi_B(\frac{x}{t}) = 0$). This condition distinguishes a norm from a seminorm.

3.3.1 Definition ([32])
Let X be a vector space.

(1) A convex, balanced and absorbing $\rho \in I^X$ is called a *fuzzy seminorm* on X. If in addition, $\forall x \neq 0, \inf_{t>0} t \cdot \rho(x) = 0$, ρ is called a *fuzzy norm*.

(2) A *fuzzy seminormed space* is a pair (X, ρ), X a vector space, ρ a fuzzy seminorm on X. A *fuzzy normed space* is a pair (X, ρ), X a vector space and ρ a norm on X.

3.3.2 Example
Consider the function χ_B defined on the vector space $\mathbb{C}$ as follows.

$$\chi_B(z) = \begin{cases} 1 & \text{on } B \\ 0 & \text{off } B, \end{cases}$$

where $B = \{z \in \mathbb{C} : |z| \leq 1\}$.

Then, χ_B is a fuzzy norm on $\mathbb{C}$:

(i) Let $x, y \in B$. If $k \in \{0, 1\}$, then trivially $\chi_B(kx + (1-k)y) = 1$. Let $k \in (0, 1)$, then, for each $\alpha \in (0, 1]$, $[\chi_B]_\alpha = B$, which is a convex set in the classical sense. Thus by 3.2.8, we have that χ_B is fuzzy convex.

(ii) Let $x \in X$ and $\alpha \in \mathbb{C}$ be such that $|\alpha| \leq 1$. Then $|\alpha x| = |\alpha||x| \leq |x|$. Now, it follows that if $|x| \leq 1$, then $\chi_B(x) = 1$ and $\chi_B(\alpha x) = 1$. If, on the other hand, $|x| > 1$, then, $\chi_B(x) = 0$, then clearly $\chi_B(\alpha x) \geq \chi_B(x)$. Thus, χ_B is balanced.

(iii) Let $x \in X$. If $x = 0$, then $\chi_B(x) = 1$. If $x \neq 0$, then, χ_B is absorbing since if $t = |x| \neq 0$, then $|\frac{x}{t}| = 1$ and thus,

$$\bigvee_{s>0} s\chi_B(x) = t\chi_B(x) = 1.$$

(iv) Finally, let $x \in X$, then $\bigwedge_{t>0} t\chi_B(x) = \{0\}$, which means that χ_B is a fuzzy norm.

3.3.3 Definition

Let X be a vector space. Given $x \in X$ and $A \in I^X$, then $x + A \in I^X$ is defined as

$$(x + A)(y) = A(y - x), \forall y \in X.$$

Note that in the definition above, it can easily be shown that $x + A = \chi_{\{x\}} + A$.

3.3.4 Definition

A *linear fuzzy topology* on a vector space X over $\mathbb{R}$ is a fuzzy topology (containing all the constant sets) such that the two mappings

$$+ : X \times X \to X, \quad (x, y) \mapsto x + y,$$
$$\cdot : \mathbb{R} \times X \to X, \quad (t, y) \mapsto t \cdot y,$$

are continuous when $\mathbb{R}$ is equipped with $\omega(\tau_{\text{ord}})$, the fuzzy topology generated (in Lowen's sense, as explained in Example 2.5.2 (4)), by the usual topology on $\mathbb{R}$; and $\mathbb{R} \times X$ and $X \times X$ have the corresponding product fuzzy topologies.

A vector space X with a linear fuzzy topology is called a *fuzzy topological vector space* (*fuzzy topological linear space*).

3.3.5 Definition

A collection $\mathbb{B}$ of fuzzy sets on a vector space X is a *base at zero* for a linear fuzzy topology if the collection

$$\mathcal{N}_0 = \{A \in I^X : \exists B \in \mathbb{B}, A \geq B, A(0) = B(0)\}$$

is a collection of neighbourhoods of zero for a linear fuzzy topology.

For each $x \in X$, we define $\mathcal{N}_x$ the collection of all neighbourhoods of x in the following way.

$$\mathcal{N}_x = \{x + A : A \in \mathcal{N}_0\}.$$

These collections generate a topology by defining the open sets as in Definition 2.7.2.

3.3.6 Theorem ([31])
Let $\mathbb{B}$ be a family of balanced fuzzy sets on a vector space X. Then $\mathbb{B}$ is a base at zero for a linear fuzzy topology if and only if $\mathbb{B}$ satisfies the following conditions:

(1) For each $A \in \mathbb{B}$, $A(0) > 0$.

(2) For each nonzero constant fuzzy set c in X and and $l \in (0, c)$, there exists $A \in \mathbb{B}$ with $A \leq c$ and $A(0) > l$.

(3) If $A_1, A_2 \in \mathbb{B}$ and $l \in (0, A_1(0) \wedge A_2(0))$, then there exists $A \in \mathbb{B}$ with $A \leq A_1 \wedge A_2$ and $A(0) > l$.

(4) If $A \in \mathbb{B}$ and $t \in \mathbb{R}$, $t \neq 0$, then for each $l \in (0, A(0))$, there exists $A_1 \in \mathbb{B}$, with $A_1 \leq t \cdot A$ and $A_1(0) > l$.

(5) Let $A \in \mathbb{B}$ and let $l \in (0, A(0))$. Then there exists $A_1 \in \mathbb{B}$ such that $A_1(0) > l$ and $A_1 + A_1 \leq A$.

(6) Let $A \in \mathbb{B}$ and $x_0 \in X$. If $l \in (0, A(0))$, then there exists a positive number s such that for all $t \in \mathbb{R}$ such that $|t| \leq s$, we have $A(tx_0) > l$.

(7) For each $A \in \mathbb{B}$ there exists a fuzzy set A_1 in X, with $A_1 \leq A$ and $A_1(0) = A(0)$, and such that for each $x_0 \in X$, for which $A_1(x_0) > 0$, and each n such that $0 < n < A_1(x_0)$, there exists $B \in \mathbb{B}$, with $B \leq -x_0 + A_1$ and $B(0) > n$.

3.3.7 Theorem ([32])
If ρ is a fuzzy seminorm on a vector space X, then the family

$$B_\rho = \{l \wedge (t \cdot \rho) : t > 0, l \in (0, 1]\}$$

is a base at zero for a linear fuzzy topology τ_ρ.

Now, by Definition 3.3.5, $\mathcal{N}_0$ the collection of all neighbourhoods of 0 is defined in in the following way.

$$\mathcal{N}_0 = \langle B_\rho \rangle = \{A \in I^X : \exists B \in B_\rho, A \geq B, A(0) = B(0)\}.$$

Hence, we can state the following corollary.

3.3.8 Corollary
A fuzzy seminormed space (and hence a fuzzy normed space) is a fuzzy topological vector space.

3.3.9 Lemma
If (X, ρ) is a fuzzy (semi)normed space and M is a linear subspace of X, then $(M, \rho|_M)$ is a fuzzy (semi)normed subspace of (X, ρ).

PROOF.

Let (X, ρ) be a fuzzy seminormed space, M a linear subspace of X and $x, y \in M$.

(i) Let $k \in [0, 1]$, then,

$$\begin{aligned} \rho|_M(kx + (1-k)y) &= \rho(kx + (1-k)y) \\ &\geq \rho(x) \wedge \rho(y) \\ &= \rho|_M(x) \wedge \rho|_M(y). \end{aligned}$$

That is, $\rho|_M$ is fuzzy convex.

(ii) Now let α be such that $|\alpha| \leq 1$. Then $\rho|_M(\alpha x) = \rho(\alpha x) \geq \rho(x) = \rho|_M(x)$, and hence, $\rho|_M$ is balanced.

(iii) Also, $\bigvee_{t>0} \rho(tx) = 1$, for all $x \in X$, and thus, $\bigvee_{t>0} \rho|_M(tx) = \bigvee_{t>0} \rho(tx) = 1$, for all $x \in M$. i.e., $\rho|_M$ is absorbing.

(iv) Finally, in the case that ρ is a fuzzy norm, we have that $\bigwedge_{t>0} \rho(tz) = 0$, for all $z \in X$, $z \neq 0$, and therefore,

$$\bigwedge_{t>0} \rho|_M(tx) \leq \bigwedge_{t>0} \rho(tx) = 0,$$

for all $x \in M \subset X$, $x \neq 0$. Thus, $\bigwedge_{t>0} \rho|_M(tx) = 0$, for all $x \in M, x \neq 0$, which implies that $\rho|_M$ is a fuzzy norm on M.

□

3.4 Hahn-Banach theorems in the fuzzy setting

Katsaras introduced a meaningful idea of a fuzzy seminorm in [32] and thus, the stage was set for our main results of this chapter. Gil Seob Rhie and In Ah Hwang fuzzified the theorem in [27].

Before we reach the statement and proof of the [27] version of the theorem, it is necessary to establish a few preliminary notions. For this section we will again be considering the vector space X over the field $\mathbb{R}$ of real numbers.

The following definition was yielded by Krishna and Sarma in [38], in their discussion on how to generate a fuzzy vector topology from an ordinary vector topology on a vector space.

3.4.1 Lemma ([38])
If ρ is a fuzzy seminorm on a vector space X (see Definition 3.3.1), then for each $l \in (0,1)$,

$$P_l(x) = \bigwedge \{t > 0 : t \cdot \rho(x) > l\} \quad (\in \mathbb{R}_+)$$

gives an ordinary seminorm on X. This seminorm is called the *induced seminorm*.

3.4.2 Lemma ([27])
Let X be a vector space. The function $P : X \to \mathbb{R}_+$, defined by

$$P(x) = \bigwedge \{P_l(x) : l \in (0,1)\},$$

is a seminorm on X.

3.4.3 Theorem ([27])
Let X be a vector space, ρ_1, ρ_2 fuzzy seminorms on X and let P_l^1, P_l^2 be induced ordinary seminorms, respectively. If $\forall x \in X, \rho_1(x) \leq \rho_2(x)$, then $\forall x \in X, \forall l \in (0,1)$,

$$P_l^1(x) \geq P_l^2(x).$$

3.4.4 Definition (The $*$-property, [27])
Let ρ be a fuzzy seminorm on a vector space X. ρ is said to have the $*$-*property* if for every $x \in X$,

$$\rho(x) = \bigwedge \{\rho(tx) : 0 < t < 1\}.$$

3.4.5 Example
Let $X = \mathbb{R}$ and define the function $\rho : \mathbb{R} \to I$ by

$$\rho(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{if } x \notin [-1,1]. \end{cases}$$

Then it can easily be shown that ρ is a seminorm with the $*$-property.

3.4.6 Lemma ([27])
Let ρ be a fuzzy seminorm on a vector space X with the $*$-property. If $x_0 \in X$ and $\rho(x_0) < l < 1$, then $P_l(x_0) > 1$.

3.4.7 Theorem ([27])
Let ρ_1 and ρ_2 be two fuzzy seminorms on a vector space X and ρ_2 have the $*$-property. If $\forall l \in (0,1), \forall x \in X, P_l^1(x) \geq P_l^2$, then $\forall x \in X, \rho_1(x) \leq \rho_2(x)$.

We are now in a position to state and prove Gil Seob Rhie and In Ah Hwangs' fuzzification of an analytical form of the Hahn-Banach Theorem.

3.4.8 Theorem ([27])
Let X be a vector space over $\mathbb{R}$, let ρ be a fuzzy seminorm on X, and let $M \subset X$ be a linear subspace. If f is a linear functional on M such that $\chi_{B_f} \geq \rho$ on M, then there exists a linear functional g on X such that:

(1) $\forall x \in M, f(x) = g(x)$, and

(2) $\chi_{B_g} \geq \rho$ on X,

where $B_f = \{x \in M : |f(x)| \leq 1\}, B_g = \{x \in X : |g(x)| \leq 1\}$.

PROOF.

Let $\chi_{B_f} = \rho_1$. Then $\forall x \in M, l \in (0,1)$,

$$\begin{aligned} P_l^1(x) &= \bigwedge\{t > 0 : t \cdot \rho_1(x) > l\} \\ &= \bigwedge\{t > 0 : \rho_1(\frac{x}{t}) > l\} \\ &= \bigwedge\{t > 0 : \rho_1(\frac{x}{t}) = 1\} \quad (\text{as } \rho_1 = \chi_{B_f}) \\ &= \bigwedge\{t > 0 : |f(\frac{x}{t})| \leq 1\} \quad (\text{as } \frac{x}{t} \in B_f) \\ &= \bigwedge\{t > 0 : |f(x)| \leq t\} = |f(x)|. \end{aligned}$$

So, by Theorem 3.4.3, $\forall l \in (0,1), \forall x \in M, |f(x)| \leq P_l(x)$, where $\forall x \in X$,

$$P_l(x) = \bigwedge\{t > 0 : t \cdot \rho(x) > l\}.$$

Now by Lemma 3.4.2, we have $\forall x \in M$,

$$|f(x)| \leq P(x) = \bigwedge\{P_l(x) : l \in (0,1)\}.$$

Therefore, by the classical Hahn-Banach Theorem, there exists a linear functional g on X such that:

(1) $\forall x \in M, g(x) = f(x)$, and

(2) $\forall x \in X, |g(x)| \leq P(x)$.

Let $\chi_{B_g} = \rho_2$. Then for all $x \in X, a \in (0,1)$,

$$\begin{aligned} P_a^2(x) &= \bigwedge\{s > 0 : s \cdot \rho_2(x) > a\} \\ &= \bigwedge\{s > 0 : \rho_2(\frac{x}{s}) > a\} \\ &= \bigwedge\{s > 0 : \rho_2(\frac{x}{s}) = 1\} \qquad (\text{since } \rho_2 = \chi_{B_g}) \\ &= \bigwedge\{s > 0 : |g(\frac{x}{s})| \leq 1\} \qquad (\text{since } \frac{x}{s} \in B_g) \\ &= \bigwedge\{s > 0 : |g(x)| \leq s\} = |g(x)|. \end{aligned}$$

Thus, $\forall a \in (0,1), \forall x \in X$,

$$P_a^2(x) \leq P(x) = \bigwedge\{P_l(x) : l \in (0,1)\},$$

and hence, $\forall l \in (0,1), \forall x \in X$,

$$P_l^2(x) \leq P_l(x).$$

Since χ_{B_g} has the $*$-property, $\chi_{B_g} \geq \rho$ by Theorem 3.4.7.

□

The remainder of this section is our original from [35]. We can now define a fuzzy topology on a fuzzy seminormed space (X, ρ) in the following way.

3.4.9 Definition
Let (X, ρ) be a fuzzy seminormed space. The *basic neighbourhoods of* 0 (the zero vector of X) are the fuzzy subsets $t \cdot \rho$, where $t > 0$. A fuzzy subset A is called a *neighbourhood of* 0 if there exists a $t > 0$ such that $t \cdot \rho \leq A$.

The collection of all neighbourhoods of 0, defined above, is denoted by $\mathcal{N}(0)$.

3.4.10 Proposition
Let Let (X, ρ) be a fuzzy seminormed space.

(1) $A(0) > 0$, for all $A \in \mathcal{N}(0)$.

(2) $A \in \mathcal{N}(0)$ and $S \leq B \Rightarrow B \in \mathcal{N}(0)$.

(3) If $A_1, A_2 \in \mathcal{N}(0)$, then $A_1 \wedge A_2 \in \mathcal{N}(0)$.

(4) $t \cdot \rho \in \mathcal{N}(0)$, for each $t > 0$ (in fact for each $t \in \mathbb{F}, t \neq 0$).

(5) If $A \in \mathcal{N}(0)$, then A is absorbing.

(6) If $A \in \mathcal{N}(0)$, then there exists a convex $A^* \in \mathcal{N}(0)$ such that $A^* \leq A$.

PROOF.

(1) $A(0) \geq t \cdot \rho(0) = \rho(0) = \sup_{x \in X} \rho(x) > 0$.

(2) Obvious.

(3) We have $t_1, t_2 > 0$ such that $t_1 \cdot \rho \leq A_1$ and $t_2 \cdot \rho \leq A_2$. Thus,

$$(t_1 \cdot \rho) \wedge (t_2 \cdot \rho) \leq A_1 \wedge A_2.$$

If $t_2 \leq t_1$, then by Remark 3.2.11 (5), $t_2 \cdot \rho \leq A_1 \wedge A_2$, and so $A_1 \wedge A_2 \leq \mathcal{N}(0)$.

(4) Obvious. The remark in parentheses follows for $t \in \mathbb{R}\backslash\{0\}$ since ρ being balanced, $\rho(-x) = \rho(x)$. If $t \in \mathbb{C}\backslash\{0\}$, then by Remark 5, $r \cdot \rho \leq t \cdot \rho$, for $0 < r \leq |t|$, and so $t \cdot \rho \in \mathcal{N}(0)$.

(5) There exists a $t_1 > 0$ such that $t_1 \cdot \rho \leq A$. So, $\sup_{t>0} tt_1 \cdot \rho \leq \sup_{t>0} t \cdot A$, or $1 = \sup_{s>0} s \cdot \rho \leq \sup_{t>0} t \cdot A$. Thus, $\sup t \cdot A = 1$.

(6) We have $t \cdot \rho \leq A$, for a $t > 0$. ρ is convex and hence, $t \cdot \rho$ is convex by Remark 3.2.11.

□

One can define neighbourhoods of an arbitrary point $x \in X$ by translation.

3.4.11 Definition
Let (X, ρ) be a fuzzy seminormed space. Given $x \in X$ and $A \in I^X$, then $x + A \in I^X$ is defined as $(x + A)(y) = A(y - x)$, for all $y \in X$. (cf. $\chi_{[a+x,b+x]}(y) = \chi_{[a,b]}(y - x)$). The *neighbourhoods of* x are defined as the sets of the form $x + A$, where $A \in \mathcal{N}(0)$. The collection of all the neighbourhoods of x is denoted by $\mathcal{N}(x)$.

3.4.12 Theorem
Let (X, ρ) be a fuzzy seminormed space. The family $\mathcal{N}(x)$ defines a neighbourhood system as specified in Definition 2.7.1.

PROOF.

We simply need to show that $\mathcal{N}(x)$ satisfies the four neighourhood axioms of Definition 2.7.1.

(1) $A = x + B$, where $B \in \mathcal{N}(0)$. So, $A(y) = (x + B)(y) = B(y - x)$, and thus, $A(x) = B(0) > 0$, by Proposition 3.4.10 (1).

(2) $A(y) = (x + A_1)(y) = A_1(y - x)$ and $B(y) = A_2(y - x)$, with $A_1, A_2 \in \mathcal{N}(0)$. Thus, $A \wedge B(y) = A_1(y - x) \wedge A_2(y - x) = A_3(y - x)$, where $A_3 \in \mathcal{N}(0)$, by Proposition 3.4.10 (3), which is equal to $(x + A_3)(y)$.

(3) $x + A \leq x + B \Rightarrow B \in \mathcal{N}(0)$, by Proposition 3.4.10 (2). Thus, $x + B \in \mathcal{N}(x)$.

(4) $A = x + B$ and there exists a $t > 0$ such that $t \cdot \rho \leq B$. So, $x + t \cdot \rho \leq A$. If $C(y) = (x+t\cdot\rho)(y) > 0$, then $t\cdot\rho(y-x) > 0$, or $y+t\cdot\rho(x) > 0 \quad (\rho(-x) = \rho(x))$, and is a neighbourhood of y.

□

3.4.13 Theorem
A linear map $T : (X_1, \rho_1) \rightarrow (X_2, \rho_2)$ between two fuzzy seminormed spaces is fuzzy continuous on X_1 if and only if $A \in \mathcal{N}(0)$ in $X_2 \Rightarrow T^{\leftarrow}[A] = A \circ T \in \mathcal{N}(0)$ in X_1.

PROOF.

Let $A \in \mathcal{N}(Tx_0)$ in X_2. So, $A = Tx_0 + B$, where $B \in \mathcal{N}(0)$ in X_2. We immediately have by Theorem 2.7.6 that $T^{\leftarrow}[B] = A \circ T \in \mathcal{N}(0)$ in X_1. Now,

$$\begin{aligned} A \circ T(x) &= (Tx_0 + B)(Tx) \\ &= B(Tx - Tx_0) \\ &= B \circ T(x - x_0) \\ &= C(x - x_0) \\ &= x_0 + C(x), \end{aligned}$$

where $C = B \circ T \in \mathcal{N}(0)$ in X_1 by the assumption. Thus, $A \circ T = T^{\leftarrow}[A]$ is a neighbourhood on x_0 in X_1.

□

With the background notions established, we are able to present our two main results of this section. We consider the real and the complex case separately. It should be mentioned that the basic idea for this work was conceived by J. J. Chadwick at a seminar at Rhodes University in the 1990's.

3.5 The real case

Firstly, notice that $\chi_{[-1,1]}$ on $\mathbb{R}$ is an example of a fuzzy norm on $\mathbb{R}$. For this section we use the following notation to prove the main results.

3.5.1 Definition
Let (X, ρ) be a fuzzy seminormed space where X is a real vector space. Let X' denote the set of all fuzzy continuous linear functionals from (X, ρ) into $(\mathbb{R}, \chi_{[-1,1]})$.

Let $A \in \mathcal{N}(0)$. Since A is absorbing, by Proposition 3.4.10 (5), supp(A) is absorbing, and if A is convex (balanced), so is supp(A) (by Lemma 3.2.22, Proposition 3.2.8 (Proposition 3.2.14)).

3.5.2 Proposition
Let (X, ρ) be a fuzzy seminormed space and $f : X \to \mathbb{R}$ a linear mapping. Then $f \in X'$ if and only if there exists $A \in \mathcal{N}(0)$ in X such that $|f(x)| \leq 1$, for all $x \in$ supp(A).

PROOF.

$\Rightarrow$: $f \in X'$. Now, $\chi_{[-1,1]}$ is a (basic) neighbourhood of 0 in $\mathbb{R}$, and so $A = f^{\leftarrow}[\chi_{[-1,1]}] \in \mathcal{N}(0)$ in X. If $x \in$ supp(A), $A(x) > 0$, i.e., $f^{\leftarrow}[\chi_{[-1,1]}](x) = \chi_{[-1,1]}(f(x)) > 0$, and therefore, $|f(x)| \leq 1$.

$\Leftarrow$: Assume that $|f(x)| \leq 1$, for $x \in$ supp(A), where $A \in \mathcal{N}(0)$. Then $A(x) \leq \chi_{[-1,1]}(f(x))$, for all $x \in X$. Let A be a neighbourhood of 0 in $(\mathbb{R}, \chi_{[-1,1]})$. Choose $t_1 > 0$ such that $t_1\chi_{[-1,1]} \leq A$ and $t_2 > 0$ such that $t_2\rho \leq A$. Then for $x \in X$,

$$\begin{aligned}
(t_1t_2) \cdot \rho(x) &\leq t_1 \cdot A(x) \\
&= A(\frac{x}{t_1}) \\
&\leq \chi_{[-1,1]}(f(\frac{x}{t_1})) \\
&= \chi_{[-1,1]}(\frac{1}{t_1}f(x)) \\
&= t_1 \cdot \chi_{[-1,1]}(f(x)) \\
&\leq A(f(x)) \\
&= f^{\leftarrow}[A](x).
\end{aligned}$$

Hence, $(t_1t_2) \cdot \rho \leq f^{\leftarrow}[A]$, and so $f^{\leftarrow}[A] \in \mathcal{N}(0)$. Thus, f is fuzzy continuous by Theorem 3.4.13.

□

3.5.3 Corollary
Let (X, ρ) be a fuzzy seminormed space, then X' is a vector space.

PROOF.

Let $f, g \in X'$. Choose $A_f, A_g \in \mathcal{N}(0)$ such that $|f(x)| \leq 1$ on supp(A_f), $|g(x)| \leq 1$ on supp(A_g). Put $A = \frac{1}{2} \cdot A_f \wedge \frac{1}{2} \cdot A_g$. So, $A \in \mathcal{N}(0)$, by Proposition 3.4.10. Now,

$$\begin{aligned} x \in \text{supp}(A) &\Rightarrow \frac{1}{2} \cdot A_f(x) \wedge \frac{1}{2} \cdot A_g(x) > 0 \\ &\Rightarrow \frac{1}{2} \cdot A_f(x) > 0, \end{aligned}$$

and $\frac{1}{2} \cdot A_g(x) > 0 \Rightarrow A_f(2x) > 0$ and $A_g(2x) > 0 \Rightarrow |f(2x)| \leq 1$ and $|g(2x)| \leq 1 \Rightarrow |(f+g)(x)| \leq 1$. So, by the preceding Proposition, $f + g \in X'$. Likewise, we can show that if $f \in X'$, r real, then $rf \in X'$:

The case $r = 0$ is trivially true. So, for $r \neq 0$, put $A = \frac{1}{r} \cdot A_f$. Then by Proposition 3.4.10 (4), $A \in \mathcal{N}(0)$. Then

$$\begin{aligned} x \in \text{supp}(A) &\Rightarrow \frac{1}{r} \cdot A_f(x) > 0 \\ &\Rightarrow A_f(rx) > 0 \\ &\Rightarrow |f(rx)| \leq 1 \\ &\Rightarrow |rf(x)| \leq 1 \\ &\Rightarrow rf \in X'. \end{aligned}$$

□

Let (X, ρ) be a fuzzy seminormed space and M a linear subspace of X. The neighbourhoods of 0 in $(M, \rho|_M)$ are $A|_M$, where A is a neighbourhood of 0 in (X, ρ). This can be shown in a similar manner to the proof of Theorem 3.4.12. Note that

$$\text{supp}(A|_M) = \{m \in M : A(m) > 0\} = M \cap \text{supp}(A),$$

where M' has the obvious meaning.

3.5.4 Theorem (Hahn-Banach - real case)
Let (X, ρ) be a fuzzy seminormed space over $\mathbb{R}$, M a linear subspace of X and $f \in M'$. Then there exists a $g \in X'$ such that $g(m) = f(m)$, for $m \in M$.

PROOF.

Since $f \in M'$, there exists a neighbourhood B on 0 in (M, ρ) such that

$$|f(m)| \leq 1, \text{ for all } m \in \text{supp}(B).$$

Now, there exists a neighbourhood A of 0 in X such that $B = A|_M$. By Proposition 3.4.10 (6), there exists a convex $A \in \mathcal{N}(0)$ such that $A \leq A$. So,

$$|f(m)| \leq 1, \text{ for all } m \in \text{supp}(A|_M) = M \cap \text{supp}(A).$$

We can assume A is balanced (or if necessary, replace A by $A \wedge (-A)$ with $-A(x) = A(-x)$ as per Definition 3.2.1). The convexity and absorption properties are retained, the latter because in a frame arbitrary suprema distribute over finite infima. The set supp(A) is convex, balanced and absorbing (by Proposition 3.2.8, Propostion 3.2.14 and Lemma 3.2.22). Thus, the Minkowski functional ('gauge'),

$$p(x) = \inf\{t > 0 : x \in t(\text{supp}(A))\},$$

defines a sublinear functional on X as in Theorem 3.1.1 (see e.g. [33] or [81]).

Now,

$$\begin{aligned} f(m) \leq p(m), \text{ for all } m \in M, p(m) < 1 &\Rightarrow m \in M \cap \text{supp}(A) \\ &\Rightarrow |f(m)| \leq 1. \end{aligned}$$

For any $m \in M$, $k > 0$,

$$p\left(\frac{m}{p(m)+k}\right) = \frac{p(m)}{p(m)+k} < 1.$$

So,

$$\left| f(\frac{p(m)}{p(m)+k}) \right| \leq 1 \text{ or } |f(m)| \leq p(m) + k.$$

Since $k > 0$ is arbitrary, $|f(m)| \leq p(m)$, and hence, $f(m) \leq p(m)$. Apply the classical result Theorem 3.1.1 and we obtain a g defined on X such that $g(x) \leq p(x)$, for all $x \in X$ and $g(m) = f(m)$, for all $m \in M$. If $x \in$ supp(A), then $p(x) \leq 1$, so $g(x) \leq 1$. Also, since supp(A) is balanced, we have

$$g(-x) = -g(x) \leq 1.$$

Thus, $|g(x)| \leq 1$ on supp(A). It follows that $g \in X'$.

$\square$

3.6 The complex case

Consider now a vector space X over $\mathbb{C}$, ρ a fuzzy seminorm on X, with X' the set of all fuzzy continuous linear functionals from (X, ρ) into $(\mathbb{C}, \chi_B)$, where $B = \{z \in \mathbb{C} : |z| \leq 1\}$, i.e.,

$$\chi_B(z) = \begin{cases} 1 & \text{on } B \\ 0 & \text{off } B. \end{cases}$$

This is a fuzzy norm on $\mathbb{C}$ (see Example 3.3.2). Proposition 3.5.2 is also valid for this X' as is Corollary 3.5.3 ($f \in X'$, $k \in \mathbb{C} \Rightarrow kf \in X'$ follows again from Proposition 3.4.10 (4)).

Theorem 3.5.4 then has a counterpart for this case. In the corresponding proof we need A balanced as well. If not, replace it first with $A^* = A \wedge (-A)$. Then for k real and $|k| \leq 1$, $A^*(kx) \geq A^*(x)$. Then put $\tilde{A}(kx) = A^*(|k|x)$ for k complex. Then $\tilde{A}(kx) \geq A^*(x) = \tilde{A}(x)$ (for $|k| \leq 1$).

The convexity and absorption properties are carried over from A to $\tilde{A}$. Then $p(x) = \inf\{t > 0 : x \in t \cdot \text{supp}(A)\}$ defines a sublinear functional on X (see [33] or [78]) of the type as in Theorem 3.1.2, which can be applied to get the following theorem.

3.6.1 Theorem (Hahn-Banach - complex case)
Let (X, ρ) be a fuzzy seminormed space over $\mathbb{C}$, M a linear subspace of X and $f \in M'$. Then there exists a $g \in X'$ such that $g(m) = f(m)$, for $m \in M$.

3.7 Notes and remarks

A comparison between Theorem 3.4.8 and our fuzzy Hahn-Banach theorem is in order. Let (X, ρ) be a fuzzy seminormed space M a linear subspace of X.

The conditions in Theorems 3.5.4 and 3.4.8 are respectively:

A: $f \in M'$, i.e., there exists a $A \in \mathcal{N}_X(0)$ such that for all $x \in \text{supp}(A) \cap M$, $|f(x)| \leq 1$.

B: On M, $\rho \leq \chi_{B_f}$, where $B_f = \{x \in M : |f(x)| \leq 1\}$.

Now, B $\Rightarrow$ A:

The fuzzy seminorm ρ is a neighbourhood on 0. Since $\rho \leq \chi_{B_f}$ on M, for all $x \in \text{supp}(\rho) \cap M$, $|f(x)| \leq 1$. So, $f \in M'$. On the other hand A implies :

There exists $t > 0$ such that $t \cdot \rho \leq A$, and therefore, on M we have $t \cdot \rho \leq \chi_{B_f}$.

So, Theorem 3.5.4 has as the following corollary.

3.7.1 Corollary

If (X, ρ) is a fuzzy seminormed space and M a linear subspace of X such that B holds, then there exists a $g \in X'$ such that for all $x \in M$, $g(x) = f(x)$.

This is exactly the statement of Theorem 3.4.8 in view of the comments above.

Chapter 4

Rådström's embedding theorem

4.1 Introduction

It is a natural question to ask what the relationship between sets and vectors is since certain collections of sets satisfy all but one of the vector space axioms - that is, all but axiom (c) of Definition 1.2.9. In this chapter, we solve the problem of how to embed such a collection of sets into a vector space, which was the central theme in [46]. We firstly look at the motivation for one of our main results and also present the necessary preliminary material that leads to these main results. This research was initiated by the search for a good way of defining a fuzzy Riesz space. We became aware of an incorrect definition of a fuzzy Riesz space, in the literature [70, 71, 72, 73], and realized that the situation was more complicated than it initially appeared. We responded to this error in [44], which lead to the results of this chapter.

Modelled on the classic Daniell integral in the crisp setting, M.S. Samuel considered fuzzy vector lattices to investigate extension properties of fuzzy Daniell integrals in [70, 71, 72, 73]. He considered the vector lattice $\overline{\mathbb{R}}^X = [-\infty, \infty]^X$ of extended real-valued functions on a set X. Using the notion of a *fuzzy vector space* on $\overline{\mathbb{R}}$, as can be found in [30], Samuel introduces the notion of a *fuzzy vector lattice* A on $\overline{\mathbb{R}}$ and the notion of *fuzzy points* of A. Samuel shows that the fuzzy points $\widetilde{A}$ of a fuzzy vector space A on $\overline{\mathbb{R}}$ can be endowed, in a natural way, with an order structure. Extensions are then considered of *fuzzy Daniell integrals*; i.e., linear maps $\tau : \widetilde{A} \to \widetilde{\mathbb{R}}$, with the property that if (x_n) is a decreasing sequence in $\widetilde{A}$ and $\bigwedge_n x_n = 0$, then $\lim_{n\to\infty} \tau(x_n) = 0$. The map τ is extended to the space $\widetilde{A}_u$ consisting of limits of increasing sequences of fuzzy points in $\widetilde{A}$.

He then introduces an *upper Daniell integral* $\overline{\tau}$ and a *lower Daniell integral* $\underline{\tau}$, which agree on $\widetilde{A}_u$. This leads naturally to the notion of τ-integrability of a fuzzy point; a fuzzy point g is τ-*integrable* provided that $\overline{\tau}(g) = \underline{\tau}(g) \in \widetilde{\mathbb{R}}$.

The set of all fuzzy points which are τ-integrable, is denoted by $\widetilde{A}_1$. The main properties of $\widetilde{A}_1$, as claimed in [71, Proposition 3.10] and [72, Proposition 4.5], are that $\widetilde{A}_1$ is again a vector lattice and τ can be uniquely extended to a Daniell integral on $\widetilde{A}_1$.

The extension of a Daniell integral, as proposed in [70, 71, 72, 73], hinges on the premise that if A is a fuzzy vector lattice on a given vector lattice X, then the set of fuzzy points $\widetilde{A}$ of A is a vector lattice. Unfortunately, this is not the case, as the counter example below shows. Consequently, Samuel's Daniell integrals are then not linear and neither are their extensions, contrary to his claims in [70, 71, 72].

We now prove, by means of a counter example, that the central premise that Samuel used in his definition of a fuzzy Riesz space is unsound.

Let X be a crisp set. For any fuzzy set A of X, once again $\widetilde{A}$ will denote the fuzzy points of A. That is, $\widetilde{A} = \{\alpha\chi_{\{x\}} \in \widetilde{X} : \alpha \leq A(x)\}$ as in [70].

If B is a fuzzy vector space, we adhere to the following convention, which guarantees that $\chi_{\{0\}} \in \widetilde{B}$.

Convention: If X is a vector space and B a fuzzy vector space on X, define $B(0) = 1$.

Let X be a vector lattice. Following [70, Definition 1.3], A is is said to be a *fuzzy vector lattice* on X, if A is a fuzzy vector space on X, and for all $x, y \in X$,

$$A(x \wedge y) \geq \min\{A(x), A(y)\} \text{ and } A(x \vee y) \geq \max\{A(x), A(y)\}.$$

In [70], the following theorems are deduced from the natural definitions of addition and scalar multiplication of fuzzy vector spaces, as in Definition 3.2.1.

4.1.1 Theorem (see Note 1.9, [70])
Let X be a vector space and B be a fuzzy vector space on X. Then, for all $a, b \in \mathbb{R}$ and $\alpha\chi_{\{x\}}, \beta\chi_{\{y\}} \in \widetilde{B}$, the following hold:

(1) $a \cdot \alpha\chi_{\{x\}} = \alpha\chi_{\{ax\}}$,

(2) $a \cdot \alpha\chi_{\{x\}} + b \cdot \beta\chi_{\{y\}} = \min\{\alpha, \beta\}\chi_{\{ax+by\}}$.

Let A be a fuzzy vector lattice. As in [70], define an ordering on A as follows. If $\alpha\chi_{\{x\}}, \beta\chi_{\{y\}} \in \widetilde{A}$, define

$$\alpha\chi_{\{x\}} \geq \chi_{\{0\}} \Leftrightarrow x \geq 0,$$

and

$$\alpha\chi_{\{x\}} \geq \beta\chi_{\{y\}} \Leftrightarrow x \geq y \text{ and } \alpha \leq \beta.$$

4.1.2 Theorem (see Theorem 1.7, Result 1.10, [70])
Let A be a fuzzy vector lattice. Then, for all $\alpha\chi_{\{x\}}, \beta\chi_{\{y\}} \in \widetilde{A}$, the following hold:

(1) $\alpha\chi_{\{x\}} \vee \beta\chi_{\{y\}} = \min\{\alpha, \beta\}\chi_{\{x\vee y\}}$,

(2) $\alpha\chi_{\{x\}} \wedge \beta\chi_{\{y\}} = \max\{\alpha, \beta\}\chi_{\{x\wedge y\}}$.

Samuel claims in [70, Result 1.10], that if A is a fuzzy vector lattice, then $\widetilde{A}$ is a vector lattice. Unfortunately this is not the case, as the following example shows.

4.1.3 Example
Let A be a fuzzy vector lattice. We claim that $\widetilde{A}$ is not a vector lattice. In fact, $\widetilde{A}$ is not a group, since the cancellation law

$$\forall x, y, z \in A\,(x + z = y + z \Rightarrow x = y)$$

is not satisfied. This follows from

$$\begin{aligned}\left(\frac{3}{4}\right)\chi_{\{a\}} + \left(\frac{1}{2}\right)\chi_{\{b\}} &= \min\left\{\frac{3}{4}, \frac{1}{2}\right\}\chi_{\{a+b\}} \\ &= \min\left\{\frac{2}{3}, \frac{1}{2}\right\}\chi_{\{a+b\}} \\ &= \left(\frac{2}{3}\right)\chi_{\{a\}} + \left(\frac{1}{2}\right)\chi_{\{b\}},\end{aligned}$$

but $\left(\frac{3}{4}\right)\chi_{\{a\}} \neq \left(\frac{2}{3}\right)\chi_{\{a\}}$, for any $A(a) \geq \frac{3}{4}$ and $A(b) \geq \frac{2}{3}$.

This leads us to our conclusion. Our counter example shows that the set of fuzzy points of fuzzy vector lattices does not form a vector lattice. As a consequence, the error in [70] has an adverse effect on the linearity claims of the fuzzy Daniell integral in [71, 72, 73].

Moreover, it seems that Samuel's definition of a fuzzy vector lattice is inadequate, since it demands $B(x \wedge y) \geq \min\{B(x), B(y)\}$ and $B(x \vee y) \geq \max\{B(x), B(y)\}$. In order to reflect the notion that B is a lattice, the 'max' should possibly be replaced by 'min' in the latter inequality.

4.2 Near vector spaces

In light of the discussion above, the problem with Samuel's definition of a fuzzy vector lattice is that the set of fuzzy points, of a given fuzzy set, is not a vector space. In fact more pertinently, such sets are not even additive groups. This is similar to collections of sets or fuzzy sets. In fact, it is fairly trivial to show that, for a Banach space X, $\mathcal{P}(X)$ satisfies all but one of the requirements of 1.5. The problem is that, for a given set $A \in \mathcal{P}(X)$, an additive inverse of A does not exist in general. The problem that we then undertook to solve was to establish under what conditions such objects can be naturally embedded into vector spaces. Rådström's work in [68] is the key to this problem and we need a special case of his embeddings result.

Rådströms embedding theorem for 'near vector spaces', which are essentially vector spaces without additive inverses, is extended to embeddings of near vector lattices, which are essentially vector lattices without additive inverses, into vector lattices. If the near vector space is endowed with a metric, properties of the metric are considered for which the norm completion of the embedding space is one of the classical L^p Banach spaces.

Rådström proved, in [68], that a 'near vector space', can be embedded into a vector space. He also showed that if the 'near vector space' is endowed with a metric compatible with the addition and multiplication by positive scalars defined on the near vector space, then the embedding space can be normed and the embedding also preserves distance.

Rådström's embedding theorem is cited in papers in many different areas of mathematics, but in particular, in set-valued analysis (cf. [1, 6, 8, 14, 17, 40, 39, 57, 58, 63, 68, 79], which is by no means an exhaustive list).

In set-valued analysis, such 'near vector spaces' arise naturally. Let X be a Banach space and let $\mathcal{P}_0(X)$ be the set of all nonempty subsets of X. There are also two natural operations on $\mathcal{P}_0(X)$, namely addition and scalar multiplication, defined by

$$A + B := \{a + b : a \in A, b \in B\} \text{ and } \lambda A := \{\lambda a : a \in A\},$$

for all $A, B \in \mathcal{P}_0(X)$ and $\lambda \in \mathbb{R}$. It is not always possible to find an additive inverse for a subset A of X. Thus, the set $\mathcal{P}_0(X)$ does not, in general, form a vector space with respect to the above defined addition and scalar multiplication. Certain hyperspaces (i.e., spaces of subsets of $\mathcal{P}_0(X)$) can be embedded in a vector space with preservation of addition and multiplication by positive scalars. Hyperspaces, which are embeddable, must at least obey a cancellation law from the outset for an embedding into a vector space to be possible. Rådström noted in [68, Lemmas 1 and 2], that if A and C are nonempty convex closed subsets of X and B is a nonempty bounded subset of X, then

$$A + B \subset C + B \Rightarrow A \subset C.$$

Consequently, if A, B and C are nonempty convex bounded closed subsets of X, then $A + B = C + B \Rightarrow A = C$.

We define the following hyperspaces:

- $\mathrm{f}(X)$ of nonempty closed subsets of X,

- $\mathrm{bf}(X)$ of nonempty closed bounded subsets of X,
- $\mathrm{cf}(X)$ of nonempty convex closed subsets of X,
- $\mathrm{cbf}(X)$ of nonempty convex bounded closed subsets of X,
- $\mathrm{cwk}(X)$ of nonempty convex weakly compact subsets of X,
- $\mathrm{ck}(X)$ of nonempty convex compact subsets of X,

where X is a Banach space.

Notice that $\mathrm{ck}(X) \subset \mathrm{cwk}(X) \subset \mathrm{cbf}(X) \subset \mathrm{bf}(X) \subset \mathrm{f}(X)$.

It is well-known that the Hausdorff distance d_H is a metric on $\mathrm{bf}(X)$, and that $(\mathrm{bf}(X), d_H)$ is a complete metric space. We may thus speak of the *Hausdorff topology* on $\mathrm{bf}(X)$, where X a Banach space. Furthermore, $\mathrm{cbf}(X)$ is a closed subspace of $\mathrm{bf}(X)$ (cf. [48]).

However, $\mathrm{bf}(X)$ is not closed under $+$, since $A + B$ need not be a closed set (even in the case where $A, B \in \mathrm{bf}(X)$). If we define $\oplus$ on $\mathrm{bf}(X)$ by

$$A \oplus B = \overline{A + B}$$

(where the latter denotes the norm closure of $A + B$ in X), then $\mathrm{bf}(X)$ is closed under $\oplus$.

Set inclusion is a natural ordering on $\mathcal{P}_0(X)$ which is compatible with addition $+$ and multiplication $\cdot$ by positive scalars. It is interesting to note that Rådström's embedding procedure does not take the natural ordering on the hyperspace into account (The ordering on the hyperspace was taken into account in [79] to consider an embedding procedure for a hyperspace of convex compact fuzzy sets). We extend Rådström's embedding for a 'near vector space' and show that a 'near vector lattice', which is essentially a vector lattice without additive inverses, can be embedded into a vector lattice. If the 'near vector lattice' is endowed with a metric, we consider the conditions on the metric required to guarantee that the associated norm on the embedding space is a Riesz norm.

Based on [67], we introduce the notion of an 'order unit' in an 'ordered near vector space'. We show that this notion corresponds to the standard notion in vector lattices on the embedding space. This correspondence also yields a correspondence between the Hausdorff metric on the 'ordered near vector space' and the Minkowski functional of the corresponding order unit of the embedding space. By using Kakutani's (M)-space representation theorem, we give a representation for 'near vector lattices' with order units, endowed with the Hausdorff metric, in terms of $C(\Omega)$-spaces.

We consider properties of the metric on the 'near vector lattice' that connects the embedding space to the classical Banach spaces $L^p(\mu)$ for $1 \leq p < \infty$. If the 'near vector lattice' has an order unit, we consider properties on the endowed metric on the 'near vector lattice' for which it is possible to embed the 'near vector lattice' into an $L^1(\mu)$-space, where (Ω, Σ, μ) is a probability space. Our approach uses Kakutani's (L)-space representation theorem.

We also consider 'near vector lattices' which do not necessarily have order units. For such spaces, we consider properties that the metric on the 'near vector lattice' has to satisfy so as to connect the embedding space to the classical Banach spaces $L^p(\mu)$ for $1 \leq p < \infty$. This approach uses the Kakutani-Bohnenblust representation for L^p-spaces.

The order embedding procedure is then applied to obtain representations of $\mathrm{cbf}(X)$, $\mathrm{cwk}(X)$ and $\mathrm{ck}(X)$.

The reader is referred to [56, 59, 69, 74, 83] for further reading in this area.

We introduce the following terminology.

4.2.1 Definition
Let S be a nonempty set.

(1) Then S is said to be a *near vector space*, provided that: addition $+\colon S \times S \to S$ is defined such that $(S, +)$ is a cancellative commutative semigroup; i.e., for all $x, y, z \in S$:

  (a) $x + z = y + z \Rightarrow x = y$,

  (b) $x + y = y + x$,

  (c) $(x + y) + z = x + (y + z)$,

  and multiplication $\cdot : \mathbb{R}_+ \times S \to S$ by positive scalars is defined such that for all $x, y \in S$ and $\lambda, \delta \in \mathbb{R}_+$:

  (d) $\lambda x + \lambda y = \lambda(x + y)$,

  (e) $(\lambda + \delta)x = \lambda x + \delta x$,

  (f) $(\lambda\delta)x = \lambda(\delta x)$,

  (g) $1x = x$.

(2) If S is a near vector space and $d\colon S \times S \to \mathbb{R}_+$ is a metric on S, then d is said to be an *invariant metric* on S, provided that:

  (h) the two mappings

$$+ : S \times S \to S, \quad (x, y) \mapsto x + y, \text{ and}$$
$$\cdot : \mathbb{R}_+ \times S \to S, \quad (t, y) \mapsto t \cdot x,$$

are continuous when $\mathbb{R}_+$ is equipped with the subspace topology (as a subspace of $\mathbb{R}$ with the usual topology), S is equipped with the metric topology induced by d, and $\mathbb{R}_+ \times S$ and $S \times S$ are equipped with the corresponding product topologies.

(i) $d(\lambda x, \lambda y) = \lambda d(x, y)$, for all $\lambda \in \mathbb{R}_+$ and $x, y \in S$,

(j) $d(x + z, y + z) = d(x, y)$, for all $x, y, z \in S$.

The following is a special case of what Rådström proved in [68, Theorem 1].

4.2.2 Theorem
Let S be a near vector space.

(1) There exist a vector space $R(S)$ and a map $j \colon S \to R(S)$ such that

  (a) j is injective,

  (b) $j(\alpha x + \beta y) = \alpha j(x) + \beta j(y)\ \forall \alpha, \beta \in \mathbb{R}_+$ and $x, y \in S$,

  (c) $R(S) = j(S) - j(S) := \{j(x) - j(y) : x, y \in S\}$.

(2) If $d \colon S \times S \to \mathbb{R}$ is an invariant metric, then there exists a norm $\|\cdot\|_d$ on $R(S)$ such that

$$d(x, y) = \|j(x) - j(y)\|_d, \text{ for all } x, y \in S.$$

Since Rådström's construction of $R(S)$ plays an fundamental role in the remainder of the this work, we include a proof outline of Theorem 4.2.2 for the convenience of the reader.

Consider $S \times S$ and define $\sim$ on $S \times S$ by

$$(x, y) \sim (x_1, y_1) \Leftrightarrow x + y_1 = x_1 + y.$$

Then $\sim$ is an equivalence relation on $S \times S$. Let

$$[x, y] := \{(x_1, y_1) \in S \times S : (x, y) \sim (x_1, y_1)\}.$$

On the quotient $R(S) := (S \times S)/_\sim, = \{[x, y] : (x, y) \in S \times S\}$, define addition by

$$[x, y] + [x_1, y_1] = [x + x_1, y + y_1].$$

Then $R(S)$ is an abelian group with additive identity $[x, x]$ and additive inverse

$$-[x, y] := [y, x], \text{ for any } (x, y) \in S \times S.$$

If multiplication by positive scalars is defined on S with the properties as stated above, define scalar multiplication $\cdot : \mathbb{R} \times R(S) \to R(S)$ by

$$\lambda \cdot [x, y] := \begin{cases} [\lambda x, \lambda y] & \lambda \in \mathbb{R}_+ \\ [-\lambda y, -\lambda x] & -\lambda \in \mathbb{R}_+ . \end{cases}$$

Then $R(S)$ is a vector space.

If d is an invariant metric on S, then $\|\cdot\|_d$, defined by

$$\|\,[x,y]\,\|_d := d(x,y), \text{ for all } [x,y]\in R(S),$$

is a norm on $R(S)$ with the desired property.

The map $j\colon S\to R(S)$, defined by

$$j(x)=[x+z,z], \text{ for all } x\in S,$$

for any $z\in S$, has the desired properties.

The following two results are analogues of those noted by Rådström in [68, Lemma 1].

4.2.3 Lemma
Let X be a Banach space. If A and C are nonempty convex closed subsets of X and B is a nonempty bounded subset of X, then $A\oplus B\subset C\oplus B$ implies $A\subset C$.

Proof.

Let $a\in A$. We show that $a\in C$. If $b_1\in B$, then $a+b_1\in C\oplus B$. Since $C+B$ is dense in $C\oplus B$, we may select $c_1\in C$ and $b_2\in B$ such that $\|a+b_1-(c_1+b_2)\|<\frac{1}{2}$. For the same reason, we may select $c_2\in C$ and $b_3\in B$ such that $\|a+b_2-(c_2+b_3)\|<\frac{1}{2^2}$. Repeat the process and if $b_n\in B$ has been chosen, we select $c_n\in C$ and $b_{n+1}\in B$ such that

$$\|a+b_n-(c_n+b_{n+1})\|<\frac{1}{2^n}.$$

Consequently, for each $n\in\mathbb{N}$,

$$\Big\|na+\sum_{i=1}^{n}b_i-\Big(\sum_{i=1}^{n}c_i+\sum_{i=2}^{n+1}b_i\Big)\Big\|\le\sum_{i=1}^{n}\Big\|a+b_i-(c_i+b_{i+1})\Big\|<\sum_{i=1}^{n}\frac{1}{2^i},$$

i.e.,

$$\Big\|a+\frac{1}{n}b_1-\frac{1}{n}b_{n+1}-\frac{1}{n}\sum_{i=1}^{n}c_n\Big\|<\frac{1}{n}.$$

Let $d_n=\frac{1}{n}\sum_{i=1}^{n}c_n$. Then $d_n\in C$ by the convexity of C. Since

$$\begin{aligned}\|a-d_n\| &\le \Big\|a+\frac{1}{n}b_1-\frac{1}{n}b_{n+1}-d_n\Big\|+\frac{1}{n}\Big\|b_1\Big\|+\frac{1}{n}\Big\|b_{n+1}\Big\|\\ &< \frac{1}{n}+\frac{1}{n}\Big\|b_1\Big\|+\frac{1}{n}\Big\|b_{n+1}\Big\|,\end{aligned}$$

and B is bounded, we get that $\lim_{n\to\infty} \|a - d_n\| = 0$. By the closedness of C, it follows that $a \in C$.

□

4.2.4 Lemma
Let X be a Banach space. Then $(\mathrm{cbf}(X),\oplus)$ satisfies the following cancellation law

(C) If $A, B, C \in \mathrm{cbf}(X)$ and $A \oplus B = C \oplus B$, then $A = C$.

PROOF.

The result follows from the preceding lemma.

□

4.3 Rådström's embedding and order

If S is an ordered near vector space or a near vector lattice, we want to consider Rådström's embedding by taking the given order or lattice structure on S into account. We, therefore, recall some terminology and introduce new terminology.

Recall that a partially ordered set $(P, \leq)$ is called a *join-semilattice* if the least upper bound of x and y, denoted $x \vee y$, exists for all $x, y \in P$.

4.3.1 Definition
Let S be a near vector space.

(a) If $(S, \leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars; i.e.,

  (i) $x \leq y \;\Rightarrow\; x + z \leq y + z$, and
  (ii) $x \leq y \;\Rightarrow\; \alpha x \leq \alpha y$, for all $\alpha \in \mathbb{R}_+$,

  then S is called an *ordered near vector space*.

(b) If S is an ordered near vector space and $(S, \leq)$ is a join-semilattice for which

$$(x \vee y) + z = (x + z) \vee (y + z), \text{ for all } x, y, z \in S,$$

  then S is called a *near vector lattice*.

If S_1 and S_2 are partially ordered sets and both are join-semilattices and $T\colon S_1 \to S_2$ is *join preserving*; i.e., $T(x \vee y) = T(x) \vee T(y)$, for all $x, y \in S_1$, then T is *order preserving*.

4.3.2 Examples

Let X be a Banach space.

(1) By Lemma 4.2.4, $(\mathrm{cbf}(X), \oplus)$ satisfies the cancellation law (C). It is then easily verified, that $(\mathrm{cbf}(X), \oplus, \cdot\,, \subset)$ satisfies the other properties required to be an ordered vector space. Furthermore, $(\mathrm{cbf}(X), \subset)$ is a join-semilattice with join given by

$$K_1 \vee K_2 = \overline{\mathrm{co}}(K_1 \cup K_2)$$

(where the latter denotes the norm closure of the convex hull of $K_1 \cup K_2$). Then, for all $K_1, K_2, K_3 \in \mathrm{cbf}(X)$,

$$(K_1 \vee K_2) \oplus K_3 = (K_1 \oplus K_3) \vee (K_2 \oplus K_3) :$$

It is easy to see that $(K_1 \oplus K_3) \vee (K_2 \oplus K_3) \subset (K_1 \vee K_2) \oplus K_3$. The crux of proving the reverse containment is to note that if $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$, $\sum_{i=1}^n \lambda_i = 1$, $e_1, \ldots, e_n \in K_1 \cup K_2$ and $b \in K_3$, then

$$\Big(\sum_{i=1}^n \lambda_i e_i\Big) + b = \sum_{i=1}^n \lambda_i(e_i + b) \in \mathrm{co}((K_1 \oplus K_3) \cup (K_2 \oplus K_3)).$$

That is, $\mathrm{co}((K_1 \vee K_2) + K_3) \subset \mathrm{co}((K_1 \oplus K_3) \vee (K_2 \oplus K_3))$. Taking closures on both sides yields $(K_1 \vee K_2) \oplus K_3 \subset (K_1 \oplus K_3) \vee (K_2 \oplus K_3)$. Consequently, $\mathrm{cbf}(X)$ is a near vector lattice.

(2) Theorem 1.3.12 states that if A is a nonempty weakly compact subset of X, then $\overline{\mathrm{co}}A \in \mathrm{cwk}(X)$ (see also [16, p.51]). Thus, if $K_1, K_2 \in \mathrm{cwk}(X)$, then $K_1 \vee K_2 \in \mathrm{cwk}(X)$. It also follows, from Theorem 1.3.12, that if $A, B \in \mathrm{cwk}(X)$, then $A \oplus B = A + B \in \mathrm{cwk}(X)$ (see [36]). Therefore, $(\mathrm{cwk}(X), +, \cdot\,)$ is a near vector space with respect to the operations induced by the operations on $\mathrm{cbf}(X)$. Furthermore, $\mathrm{cwk}(X)$ is an ordered near vector space with respect to $\subset$. We also have that $\mathrm{cwk}(X) \subset \mathrm{cbf}(X)$. Consequently, $\mathrm{cwk}(X)$ is a near vector lattice with respect to the operations and order induced by $\mathrm{cbf}(X)$.

(3) It is well-known that if $K_1, K_2 \in \mathrm{ck}(X)$, then $K_1 \oplus K_2 = K_1 + K_2 \in \mathrm{ck}(X)$.

Thus, by Theorem 1.2.42, if $K_1, K_2 \in \mathrm{ck}(X)$, then $K_1 \vee K_2 \in \mathrm{ck}(X)$. It follows easily that $\mathrm{ck}(X)$ is a near vector lattice with respect to the operations and order induced by $\mathrm{cwk}(X)$.

Let S be an ordered near vector space. Define an order $\leq$ on $R(S)$ by

$$[x, y] \leq [x_1, y_1] \Leftrightarrow x + y_1 \leq y + x_1.$$

4.3.3 Theorem
Let S be a near vector lattice. Then $R(S)$ is a vector lattice, with positive cone $R(S)_+ := \{[x, y] : y \leq x\}$, in which the following formulas hold:

(1) $[x, y]^+ = [x \vee y, y]$,

(2) $[x, y]^- = [x \vee y, x]$,

(3) $|\,[x, y]\,| = [2(x \vee y), x + y]$,

(4) $[x, y] \vee [x_1, y_1] = [(x_1 + y) \vee (x + y_1), y + y_1]$,

(5) $[x, y] \wedge [x_1, y_1] = [x + x_1, (x_1 + y) \vee (x + y_1)]$.

PROOF.

It is readily verified that $R(S)$ is an ordered vector space with positive cone $R(S)_+ = \{[x, y] : y \leq x\}$.

To prove (1), note that $[u \vee v, v]$ is an upper bound for $\{[u, v], [z, z]\}$. If $[a, b]$ is also an upper bound for $\{[u, v], [z, z]\}$, then $u + b \leq a + v$ and $b + z \leq a + z$. It follows from $u \vee v + b + z = (u + b + z) \vee (v + b + z) \leq a + v + z$ that $[u \vee v, v] \leq [a + z, b + z] = [a, b]$. Thus, $[u \vee v, v] = [u, v]^+$. This also proves that $R(S)$ is a vector lattice.

The remaining proofs follow from (1) and appropriate application of properties (I.1) and (I.2) from Definition 1.2.59 (b).

□

4.3.4 Corollary
Let S be a near vector lattice. Then the embedding $j \colon S \to R(S)$ is join preserving.

PROOF.

Let $x, y \in S$. Then, for any $z \in S$, it follows from Theorem 4.3.3 (4), that

$$\begin{aligned} j(x) \vee j(y) &= [x + z, z] \vee [y + z, z] \\ &= [(y + z + z) \vee (x + z + z), z + z] \\ &= j(x \vee y). \end{aligned}$$

□

Let S_1 and S_2 be ordered near vector spaces, $t \colon S_1 \to S_2$ and define

$$T([x, y]) = [t(x), t(y)], \text{ for all } x, y \in S_1.$$

If t preserves addition, then it is readily verified that T is a well defined map from $R(S_1)$ to $R(S_2)$. The following result relates properties of t to T.

4.3.5 Theorem
Let S_1 and S_2 be ordered near vector spaces, $t\colon S_1 \to S_2$ addition preserving and $T\colon R(S_1) \to R(S_2)$ as defined above. Then the following statements hold:

(1) If $t(\alpha x + \beta y) = \alpha t(x) + \beta t(y)$, for all $x, y \in S_1$ and $\alpha \in \mathbb{R}_+$, then T is linear.

(2) If t is order preserving, then T is a positive map.

(3) If t is injective, so is T.

Moreover, if S_1 and S_2 are near vector lattices, then

(4) $t(x \vee y) = t(x) \vee t(y)$, for all $x, y \in S_1$, implies that

$$T([x, y] \vee [a, b]) = T([x, y]) \vee T([a, b]), \text{ for all } x, y, a, b \in S_1.$$

PROOF.

(1) Firstly, we notice that because t is addition preserving, we immediately have that $t(0) = 0$ and that $t(-x) = -t(x)$, for all $x \in S_1$. Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in S_1$. We also have that $j(\alpha x + \beta y) = [\alpha x + \beta y, 0]$. Now,

$$\begin{aligned} T([\alpha x + \beta y, 0]) &= [t(\alpha x + \beta y), t(0)] \\ &= [t(\alpha x) + t(\beta y), 0] \\ &= [t(\alpha x), 0] + [t(\beta y), 0]. \end{aligned}$$

If $\alpha \in \mathbb{R}_+$, then $[t(\alpha x), 0] = [\alpha t(x), 0] = \alpha[t(x), 0]$, while on the other hand, if $-\alpha \in \mathbb{R}_+$, then

$$\begin{aligned} [t(\alpha x), 0] &= [t((-\alpha)(-x)), 0] \\ &= [-\alpha t(-x), 0] \\ &= [(-\alpha)(-t(x)), 0] \\ &= [\alpha t(x), 0] \\ &= \alpha[t(x), t(0)]. \end{aligned}$$

Similarly, $[t(\beta y), 0] = \beta[t(y), t(0)]$, which gives us

$$T([\alpha x + \beta y, 0]) = \alpha T([x, 0]) + \beta T([y, 0]).$$

i.e., T is linear.

(2) Let $x, y \in S_1$ such that $[x, y] \geq [0, 0]$. Then $x \geq y$, and hence, $t(x) \geq t(y)$ by the fact that t is order preserving. This implies that

$$T([x, y]) = [t(x), t(y)] \geq [0, 0],$$

and thus, T is a positive map.

(3) Let $x, y, a, b \in S_1$ such that $T([x, y]) = T([a, b])$. Then

$$\begin{aligned}[t(x), t(y)] = [t(a), t(b)] &\Leftrightarrow t(x) + t(b) = t(a) + t(y)\\ &\Leftrightarrow t(x) - t(a) = t(y) - t(b)\\ &\Leftrightarrow t(x - a) = t(y - b)\\ &\Leftrightarrow x - a = y - b \text{ (since } t \text{ is injective)}\\ &\Leftrightarrow x + b = y + a\\ &\Leftrightarrow [x, y] = [a, b].\end{aligned}$$

That is, T is injective.

(4) Assume that $t(x \vee y) = t(x) \vee t(y)$, for all $x, y \in S_1$. Now, let $x, y, a, b \in S_1$. Then,

$$\begin{aligned}T([x, y] \vee [a, b]) &= T([(a + y) \vee (x + b), y + b]) \text{ (by Theorem 4.3.3)}\\ &= [t((a + y) \vee (x + b)), t(y + b)].\end{aligned}$$

On the other hand,

$$\begin{aligned}T([x, y]) \vee T([a, b]) &= [t(x), t(y)] \vee [t(a), t(b)]\\ &= [(t(a) + t(y)) \vee (t(x) + t(b)), t(y) + t(b)] \text{ (by Theorem 4.3.3)}\\ &= [(t(a + y)) \vee (t(x + b)), t(y) + t(b)]\\ &= [t((a + y) \vee (x + b)), t(y + b)].\end{aligned}$$

That is, $T([x, y] \vee [a, b]) = T([x, y]) \vee T([a, b])$.

□

4.3.6 Definition

(a) Let S_2 be an ordered near vector space and S_1 a nonempty subset of S_2. S_1 is said to be a *sub ordered near vector space* of S_2 provided that S_1 is closed under the operations addition, multiplication by positive scalars and join.

(b) If S_1 is a subset of a near vector lattice S_2 and S_1 is a near vector lattice under the same ordering as S_2, then we say that S_1 is a *sub-near vector lattice* of S_2.

4.3.7 Corollary

If S_1 is a sub-near vector lattice of a near vector lattice S_2, then $R(S_1)$ is a vector sublattice of $R(S_2)$.

PROOF.

The proof follows from Theorem 4.3.5.

□

Example. If X is a Banach space, then $\mathrm{ck}(X)$ is a sub-near vector lattice of $\mathrm{cwk}(X)$. The latter is a sub-near vector lattice of $\mathrm{cbf}(X)$. By Corollary 4.3.7, $R(\mathrm{ck}(X))$ is a vector sublattice of $R(\mathrm{cwk}(X))$, which is a vector sublattice of $R(\mathrm{cbf}(X))$.

If S is an ordered near vector space or a near vector lattice which is endowed with a metric d, we consider conditions on d which guarantee that the associated norm $\|\cdot\|_d$ on $R(S)$ is compatible with the order or lattice structure of $R(S)$.

4.3.8 Definition
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ an invariant metric. Then d is said to be a *Riesz metric* on S, provided that:

(a) if $x \le y \le z \in S$, then $d(x,y) \le d(x,z)$, and

(b) if $x, y \in S$, then $d(x,y) = d(2(x \vee y), x+y)$.

4.3.9 Example
Let $S = [0,\infty)$ with the usual ordering of real numbers. Then S is a near vector lattice and the metric d given by $d(x,y) = |x-y|$, for $x, y \in S$, is a Riesz metric on S.

We consider a connection between Riesz metrics on near near vector lattices and Riesz norms on the Rådström spaces that they generate.

4.3.10 Lemma
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ an invariant metric. Then d is a Riesz metric on S if and only if $\|\cdot\|_d$ is a Riesz norm on the vector lattice $R(S)$.

PROOF.

Suppose d is a Riesz metric. Let $a, b, x, y, z \in S$ and $[z,z] \le [x,y] \le [a,b]$. It follows from $y \le x$, $b \le a$ and $x+b \le a+y$, that $y+b \le x+b \le a+y$. Hence, $d(x,y) = d(x+b, y+b) \le d(a+y, y+b) = d(a,b)$. Consequently,

$$\|\,[x,y]\,\|_d \le \|\,[a,b]\,\|_d.$$

Let $x, y \in S$. Since $[2(x \vee y), x+y] = |\,[x,y]\,|$, we get

$$\|\,[x,y]\,\|_d = d(x,y) = d(2(x \vee y), x+y) = \|\,|\,[x,y]\,|\,\|_d.$$

Thus, $\| \cdot \|_d$ is a Riesz norm on $R(S)$.

Conversely, if $\| \cdot \|_d$ is a Riesz norm on $R(S)$, let $x \leq y \leq z$ in S. Then $[u, u] \leq [y, x] \leq [z, x]$. Consequently,

$$d(y, x) = \| [y, x] \|_d \leq \| [z, x] \|_d = d(z, x).$$

Let $x, y \in S$. Since $[2(x \vee y), x + y] = | [x, y] |$, we get

$$d(x, y) = \| [x, y] \|_d = \| | [x, y] | \|_d = d(2(x \vee y), x + y).$$

Thus, d is a Riesz metric on S.

□

If E is a vector lattice, then E_+ is a near vector lattice. We want to compare $R(E_+)$ to E. We first introduce some more terminology.

4.3.11 Definition
Let S be an ordered near vector space [near vector lattice] which also satisfies

(Z) there exists $0 \in S$ such that $x + 0 = x$, for all $x \in S$ and $\lambda 0 = 0$, for all $\lambda \in \mathbb{R}_+$.

Then S is said to be an *ordered near vector space [near vector lattice] with a zero*.

If E is a vector lattice, then E_+ is a near vector lattice with a zero and $x \in E_+$ if and only if $x \geq 0$. We claim that the vector lattice $R(E_+)$ has as positive cone $R(E_+)_+ = \{[x, 0] : x \in E_+\}$. Clearly, $\{[x, 0] : x \in E_+\} \subset R(E_+)_+$. Conversely, if $[x, y] \in R(E_+)_+$, then $y \leq x$. But then $z := x - y \in E_+$ and $y + z = x$. Thus, $(x, y) \sim (z, 0)$, hence, $[x, y] = [z, 0]$.

We recall from [83, p.17], that $(x \vee y) - (x \wedge y) = |x - y|$, for all $x, y \in E$. It follows from (I.1) from Definition 1.2.59 (b), that

(I.3) $2(x \vee y) - (x + y) = |x - y|$, for all $x, y \in E$.

This identity is used in the proof of the following theorem.

4.3.12 Theorem
Let E be a vector lattice. Then $J \colon R(E_+) \to E$, defined by

$$J([x, y]) = x - y, \text{ for all } x, y \in E_+,$$

is a vector lattice isomorphism.

PROOF.

It is readily verified that J is well-defined and injective. To verify that J is surjective, let $x \in E$. Then $x^+, x^- \in E_+$ and $J([x^+, x^-]) = x$. The linearity of J follows easily. To complete the proof, we show that $J(\,|\,[x,y]\,|\,) = |\,J([x,y])\,|$, for all $x, y \in E_+$. This follows from

$$\begin{aligned} J(\,|\,[x,y]\,|\,) &= J\big(2(x \vee y), x+y\big) \\ &= 2(x \vee y) - (x+y) \\ &= |x-y| \\ &= |\,J([x,y])\,|. \end{aligned}$$

□

4.3.13 Corollary
Let E be a Riesz normed vector lattice. Then

(1) $d_{\|\cdot\|} \colon E_+ \times E_+ \to \mathbb{R}_+$, defined by $d_{\|\cdot\|}(x,y) = \|x-y\|$, for all $x, y \in E_+$, is a Riesz metric on E_+, and

(2) the map J, as in Theorem 4.3.12, is a surjective vector lattice and isometric isomorphism.

PROOF.

(a) Let $x, y, z \in E_+$ such that $x \leq y \leq z$. Then $0 \leq y - x \leq z - x$, from which we get that $d_{\|\cdot\|}(x,y) = \|y-x\| \leq \|z-x\| = d_{\|\cdot\|}(z,x)$.

If $x, y \in E_+$, then

$$\begin{aligned} d_{\|\cdot\|}\big(2(x \vee y), x+y\big) &= \|2(x \vee y) - (x+y)\| \\ &= \|\,|x-y|\,\| \\ &= \|x-y\| \\ &= d_{\|\cdot\|}(x,y). \end{aligned}$$

Thus, $d_{\|\cdot\|}$ is a Riesz metric.

(b) For all $x, y \in E_+$, we have

$$\|\,[x,y]\,\|_{d_{\|\cdot\|}} = d_{\|\cdot\|}(x,y) = \|x-y\| = \|J(\,[x,y]\,)\|.$$

Thus, J has the desired properties.

□

4.4 Order units

4.4.1 Definition
A Riesz (semi)norm $\|\cdot\|$ on a vector lattice E is called an M-*(semi)norm* on E, provided that

(M) $\|x \vee y\| = \max\{\|x\|, \|y\|\}$, for all $x, y \in E_+$.

In this case we call $E = (E, \|\cdot\|)$ an M-*normed space*.

Let Ω be a compact Hausdorff space and

$$C(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ is continuous}\}.$$

Endow the latter with pointwise addition, scalar multiplication and order, and norm given by

$$\|f\|_\infty := \bigvee\{|f(t)| : t \in \Omega\}.$$

Then $C(\Omega)$ is a norm complete M-normed vector lattice. The function $\mathbf{1} : \Omega \to \mathbb{R}$, defined by

$$\mathbf{1}(s) = 1, \text{ for all } s \in \Omega,$$

is an order unit of $C(\Omega)$.

4.4.2 Theorem (Kakutani's (M)-space representation theorem, [29])
Let E be any M-normed space with an order unit, then there exists a compact Hausdorff space Ω such that E is isometric and lattice isomorphic to the space $C(\Omega)$ of all continuous real-valued functions $f(\omega)$ defined on Ω.

4.4.3 Lemma
Let S be a near vector lattice and $d : S \times S \to \mathbb{R}_+$ an invariant metric. Then d is a Riesz metric and

(M_d) $d\big((x_1 + y) \vee (x + y_1), y + y_1\big) = \max\{d(x, y), d(x_1, y_1)\}$, for all $x, y, x_1, y_1 \in S$ such that $y \le x$ and $y_1 \le x_1$

if and only if $\|\cdot\|_d$ is an M-norm on $R(S)$.

PROOF.

By Lemma 4.3.10, d is a Riesz metric on S if and only if $\|\cdot\|_d$ is a Riesz norm on $R(S)$. If $x, y, x_1, y_1 \in S$ and $y \le x$ and $y_1 \le x_1$, it follows from

$$[(x_1 + y) \vee (x + y_1), y + y_1] = [x, y] \vee [x_1, y_1],$$

that d has property (M_d) if and only if $\|\cdot\|_d$ has property (M).

□

Let X be a Banach space. We once again denote by X' the continuous dual of X, and by B_X the closed unit ball of X.

Recall that in Definition 1.2.59 (e), an order unit of a Riesz space was defined. Kakutani gave a complete description of Archimedean vector lattices with order units in [29]. If E is an Archimedean vector lattice with an order unit $e \in E_+$, then the Minkowski functional p_e on E is an M-norm on E. By the (M)-space representation Theorem of Kakutani (Theorem 4.4.2), the norm completion $\overline{E}$ of (E, p) is vector lattice and isometrically isomorphic to $C(\Omega)$, where Ω is a compact Hausdorff space. In fact, Ω is the set of extreme points of $H_0 := \{F \in \overline{E}' : \|F\| = 1\}$ and Ω is endowed with the topology induced by the weak* topology $\sigma(\overline{E}', \overline{E})$. The reader is referred to [69] for properties of the weak* topology.

In terms of Rådström completions, we now look for a suitable notion of an order unit of S that yields on $R(S)$ the usual notion of an order unit of a vector lattice. This motivates the following.

4.4.4 Definition
Let S be an ordered near vector space. If $e \in S$ has the property that

(ou) for all $x, y \in S$, there exists $K \in \mathbb{R}_+$ such that $x \leq Ke + y$ and $y \leq Ke + x$,

then e is called an *order unit* of S.

4.4.5 Note
If S is an ordered near vector space with zero, then it is readily verified that e is an order unit of S if and only if for each $x \in S$ there exists $K \in \mathbb{R}_+$ such that $x \leq Ke$ and $0 \leq Ke + x$. Furthermore, $0 \leq e$.

It is well-known that if X is a Banach space, then

- $B_X \in \mathrm{cwk}(X)$ if and only if X is reflexive, and
- $B_X \in \mathrm{ck}(X)$ if and only if X is finite dimensional.

4.4.6 Example
Let X be a Banach space.

(1) Then B_X is an order unit of the near vector lattice $\mathrm{cbf}(X)$.

(2) If X is reflexive, then B_X is an order unit of the near vector lattice $\mathrm{cwk}(X)$.

(3) If X is finite dimensional, then B_X is an order unit of the near vector lattice $\mathrm{ck}(X)$.

If X is a Banach space, the Hausdorff metric d_H on $\mathrm{bf}(X)$ is sometimes given in the following form in the literature (see [6, 48, 67]).

$$d_H(A, C) = \inf\{K > 0 : A \subset KB_X + C \text{ and } C \subset KB_X + A\},$$

for all $A, C \in \mathrm{bf}(X)$. It is easily verified that

$$d_H(A, C) = \inf\{K > 0 : A \subset KB_X \oplus C \text{ and } C \subset KB_X \oplus A\},$$

for all $A, C \in \mathrm{bf}(X)$. This motivates the following.

4.4.7 Definition
Let S be an ordered near vector space and let $e \in S$ be an order unit of S. Define, for all $x, y \in S$,

$$h_e(x, y) = \inf\{K > 0 : x \leq Ke + y \text{ and } y \leq Ke + x\}.$$

4.4.8 Proposition
Let S be an ordered near vector space and let $e \in S$ be an order unit of S. Then h_e, as defined in Definition 4.4.7 above, is a semimetric.

4.4.9 Definition
Let S be an ordered near vector space with an order unit $e \in S$. Then S is said to be *Archimedean*, provided that S satisfies

(A) if $x, y \in S$ and $x \leq \frac{1}{n}e + y$, for all $n \in \mathbb{N}$, then $x \leq y$.

Let S be an Archimedean ordered near vector space and let $e \in S$ be an order unit of S. Then h_e also satisfies

(a′) if $h_e(x, y) = 0$, then $x = y$,

i.e., h_e is a metric on S.

4.4.10 Example
Let X be a Banach space.

(1) Then $\mathrm{cbf}(X)$ is an Archimedean near vector lattice with B_X as an order unit. Moreover, h_{B_X} is the Hausdorff metric d_H on $\mathrm{cbf}(X)$.

(2) If X is not reflexive, then $B_X \notin \mathrm{cwk}(X)$. Thus, B_X is, therefore, not an order unit of the Archimedean ordered near vector lattice $\mathrm{cwk}(X)$.

(3) If X is not finite dimensional, then $B_X \notin \mathrm{ck}(X)$. Thus, B_X is, therefore, not an order unit of the Archimedean ordered near vector lattice $\mathrm{ck}(X)$.

In terms of Rådström completions, the following result shows how the notions on S mesh with the standard notions of vector lattices on $R(S)$.

4.4.11 Lemma
Let S be an ordered near vector space which has an order unit e. Then

(1) $[e+z, z]$ is an order unit of the ordered vector space $R(S)$, for any fixed $z \in S$, and

(2) $R(S)$ is Archimedean if S has a zero and is Archimedean.

PROOF.

(a) Let $[x, y] \in R(S)$. Select $K \in \mathbb{R}_+$ such that $x \leq Ke + y$ and $y \leq Ke + x$. Then $x+Kz \leq Ke+y+Kz$ and $y+Kz \leq Ke+x+Kz$. Consequently, $[x, y] \leq [Ke+Kz, Kz]$ and $-[x, y] = [y, x] \leq [Ke+Kz, Kz]$, showing that $-K[e+z, z] \leq [x, y] \leq K[e+z, z]$. Thus, $[e + z, z]$ is an order unit of $R(S)$.

(b) Let $a, b, u, v, z \in S$ and $[z, z] \leq n[u, v] \leq [a, b]$, for all $n \in \mathbb{N}$. By the preceding part of the proof, there exists $K \in \mathbb{R}_+$ such that $[a, b] \leq K[e, 0]$. Consequently, $[u, v] \leq \frac{K}{n}[e, 0]$, for all $n \in \mathbb{N}$. It follows from $u \leq \frac{K}{n}e + v$, for all $n \in \mathbb{N}$ that $u \leq v$. But then $u = v$, i.e., $[u, v] = [z, z]$. This means that $R(S)$ is Archimedean.

□

4.4.12 Lemma
Let S be an Archimedean near vector lattice with a zero and an order unit e. Then $h_e\colon S \times S \to \mathbb{R}_+$ is an invariant metric on S with the property that the associated norm $\|\cdot\|_{h_e}$ on $R(S)$ is exactly the Minkowski functional $p_{[e,0]}$ of the order unit $[e, 0]$ of the Archimedean vector lattice $R(S)$.

PROOF.

Since $R(S)$ is an Archimedean vector lattice with $[e, 0]$ as an order unit, the Minkowski functional $p_{[e,0]}$ is an M-norm on $R(S)$.

We claim that $h_e(x, y) = p_{[e,0]}([x, y])$, for all $x, y \in S$. If $x, y \in S$ and $K \in \mathbb{R}_+$, then

$$\begin{aligned} x \leq Ke + y \text{ and } y \leq Ke + x &\Leftrightarrow [x, y] \leq K[e, 0] \text{ and } [y, x] \leq K[e, 0] \\ &\Leftrightarrow [x, y] \leq K[e, 0] \text{ and } -[x, y] \leq K[e, 0] \\ &\Leftrightarrow -K[e, 0] \leq [x, y] \leq K[e, 0]. \end{aligned}$$

Thus, $h_e(x, y) = p_{[e,0]}([x, y])$, for all $x, y \in S$, as claimed.

Consequently, h_e is an invariant metric on S and the associated norm $\|\cdot\|_{h_e}$ on $R(S)$ is exactly the Minkowski functional $p_{[e,0]}$ of the order unit $[e, 0]$ on $R(S)$.

□

4.4.13 Theorem
Let S be an Archimedean near vector lattice with a zero and an order unit e. Then the invariant metric $h_e \colon S \times S \to \mathbb{R}_+$ is a Riesz metric on S with property M_d.

PROOF.

Since $p_{[e,0]}$ is an M-norm on $R(S)$, the result follows directly from Lemmas 4.4.12 and 4.4.3.

□

4.4.14 Corollary
Let X be a Banach space. Then the following statements hold:

(1) The Hausdorff metric d_H is a Riesz metric on $\mathrm{cbf}(X)$ with property M_d.

(2) The restriction of the Hausdorff metric d_H to $\mathrm{cwk}(X)$ is a Riesz metric on $\mathrm{cwk}(X)$ with property M_d.

(3) The restriction of the Hausdorff metric d_H to $\mathrm{ck}(X)$ is a Riesz metric on $\mathrm{ck}(X)$ with property M_d.

PROOF.

(1) Since $\mathrm{cbf}(X)$ is an Archimedean near vector lattice with $\{0\}$ as zero and B_X as an order unit, it follows from Theorem 4.4.13 that h_{B_X} is a Riesz metric on $\mathrm{cbf}(X)$ with property M_d. But h_{B_X} equals d_H, as already noted in the preceding example.

(2) The Hausdorff metric d_H on $\mathrm{cbf}(X)$ corresponds to the Minkowski functional $p_{[e,0]}$ on $R(\mathrm{cbf}(X))$. Since $\mathrm{cwk}(X)$ is a sub-near vector lattice of $\mathrm{cbf}(X)$, it follows from Corollary 4.3.7, that $R(\mathrm{cwk}(X))$ is a vector sublattice of $R(\mathrm{cbf}(X))$. Consequently, the restriction of $p_{[e,0]}$ to $R(\mathrm{cwk}(X))$ is an M-norm on $R(\mathrm{cwk}(X))$ and corresponds to the restriction of d_H to $\mathrm{cwk}(X)$. By Lemma 4.4.3, it follows that the restriction of d_H to $\mathrm{cwk}(X)$ is a Riesz metric on $\mathrm{cwk}(X)$ with property M_d.

(3) The proof is almost verbatum the same as that of (2).

□

4.5 A $C(\Omega)$ embedding

We relate $R(S)$ to some of the classical Banach spaces, under appropriate assumptions on d.

The following is one of our main results of this section.

4.5.1 Theorem
Let S be an Archimedean near vector lattice with a zero and an order unit $e \in S$. Then there exist a compact Hausdorff space Ω and a map $K\colon S \to C(\Omega)$ such that:

(1) K is injective.

(2) $K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

(3) $K(S) - K(S)$ is norm dense in $C(\Omega)$.

(4) $h_e(x, y) = \|K(x) - K(y)\|_\infty$, for all $x, y \in S$.

(5) $K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.

(6) $K(e) = \mathbf{1}$.

PROOF.

By Lemma 4.4.12, $h_e\colon S \times S \to \mathbb{R}_+$ is a Riesz metric on S with the property that the associated norm $\|\cdot\|_{h_e}$ on $R(S)$ is exactly the Minkowski functional $p_{[e,0]}$ of the order unit $[e, 0]$ of the Archimedean vector lattice $R(S)$. There exist, by Kakutani's (M)-space representation theorem (Theorem 4.4.2), a compact Hausdorff space Ω and a vector lattice and isometric isomorphism V from the norm completion of $R(S)$ onto $C(\Omega)$ such that $V(e) = \mathbf{1}$. Let $K = V \circ j$, where $j\colon S \to R(S)$ is the natural embedding. By Theorem 4.2.2 and Corollary 4.3.4, it follows that K has the desired properties.

□

4.5.2 Corollary
Let X be a Banach space. Then the following statements hold:

(1) $S = \mathrm{cbf}(X)$ fulfills the requirements of Theorem 4.5.1 with $e = B_X$.

(2) If X is reflexive, then $S = \mathrm{cwk}(X)$ fulfills the requirements of Theorem 4.5.1 with $e = B_X$.

(3) If X is finite dimensional, then $S = \mathrm{ck}(X)$ fulfills the requirements of Theorem 4.5.1 with $e = B_X$.

Our next aim is to derive a version of Theorem 4.5.1 for the case where S does not necessarily have an order unit.

We recall that a completely regular topological space is called a *Stone space* if the closure of every open set is open.

4.5.3 Definition
A Riesz (semi)norm on a vector lattice E is called an *L-(semi)norm*, provided that

$$\|x+y\| = \|x\| + \|y\|, \text{ for all } x, y \in E_+.$$

A vector lattice equipped with an L-norm is said to be an L-normed space.

4.5.4 Theorem (Kakutani's (L)-space representation theorem (see [28, 59, 74]))
Let E be a L-normed Banach lattice with an order unit. Then there corresponds a compact Hausdorff space Ω, a strictly positive Borel measure μ on Ω such that E is isomorphic with $L^1(\mu)$. Furthermore, E possesses a weak order unit u if and only if Ω can be chosen to be compact and such that the isomorphism $E \to L^1(\mu)$ maps E_u onto $L^\infty(\mu)$.

This brings us to our second main result of this section, in which we drop the assumption of an order unit of S, but assume S can be embedded into an ordered near vector lattice with an order unit.

4.5.5 Theorem
Let S_1 be an Archimedean near vector lattice with a zero, denoted by $\mathbf{0}$, and an order unit, denoted by e. If S is a sub-near vector lattice of S_1 such that $\mathbf{0} \in S$, then there exist a compact Hausdorff Stone space Ω and a map $J\colon S \to C(\Omega)$ such that:

(1) J is injective.

(2) $J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

(3) $J(S) - J(S)$ is w^*-dense (i.e., $\sigma(C(\Omega), R(S)')$-dense) in $C(\Omega)$.

(4) $h(x, y) = \|J(x) - J(y)\|_\infty$, for all $x, y \in S$. where h denotes the restriction of $h_e\colon S_1 \times S_1 \to \mathbb{R}_+$ to $S \times S$.

(5) $J(x \vee y) = J(x) \vee J(y)$, for all $x, y \in S$.

PROOF.

By Theorem 4.5.1, $h_e\colon S_1 \times S_1 \to \mathbb{R}_+$ yields the Minkowski functional $p_{[e,0]}$ as the norm $\|\cdot\|_{h_e}$ on the Archimedean vector lattice $R(S_1)$. By Corollary 4.3.5, $R(S)$ is a vector sublattice of $R(S_1)$. Consequently, the restriction $p_{R(S)}$ of $p_{[e,0]}$ to $R(S)$ is an M-norm.

The dual $(R(S)', p'_{R(S)})$ of $(R(S), p_{R(S)})$ is a norm complete L-normed vector lattice with L-norm given by

$$p'_{R(S)}(x') := \sup\{|x'(x)| : p_{R(S)}(x) \leq 1\}, \text{ for all } x' \in R(S)'$$

(see [74, Ch 2]). The bidual $(R(S)'', p''_{R(S)})$ of $(R(S), p_{R(S)})$ is a norm complete M-normed vector lattice with M-norm given by

$$p''_{R(S)}(x'') := \sup\{|x''(x')| : p'_{R(S)}(x') \leq 1\}, \text{ for all } x'' \in R(S)''.$$

Moreover, f'', defined by

$$f''(x') = p^*_{R(S)}(x^{*+}) - p^*_{R(S)}(x^{*-}), \text{ for all } x' \in R(S)',$$

is an order unit of $R(S)''$ (see [74, Ch 2]). By Kakutani's (M)-space representation theorem, there exist a compact Hausdorff space Ω and a vector lattice and isometric isomorphism V from $R(S)''$ onto $C(\Omega)$. Let $J = V \circ \kappa \circ j$, where $j\colon S \to R(S)$ and $\kappa\colon R(S) \to R(S)''$ are the natural embeddings.

By Theorem 4.2.2 and Corollary 4.3.4, and noting that any Banach space X is weak*-dense in its bidual X'', it follows that J has the desired properties.

□

The following complements Corollary 4.5.2 (2) and (3).

4.5.6 Corollary
Let X be a Banach space.

(1) If X is not reflexive, then $S = \mathrm{cwk}(X)$ fullfills the requirements of Theorem 4.5.5, provided that $S_1 = \mathrm{cbf}(X)$, $e = B_X$ and $\mathbf{0} = \{0\}$.

(2) If X is not finite dimensional, then $S = \mathrm{ck}(X)$ fullfills the requirements of Theorem 4.5.5, provided that $S_1 = \mathrm{cbf}(X)$, $e = B_X$ and $\mathbf{0} = \{0\}$.

Let X be a Banach space. The well-known embedding procedure of Hörmander to embed $\mathrm{cbf}(X)$ and $\mathrm{ck}(X)$ into $C(\Omega)$-spaces, as can be found in [48], requires X to be separable. Our embedding of $\mathrm{cbf}(X)$ in Corollary 4.5.2 does not require the separability of X. Combining the ideas in Theorem 4.5.1 with Hörmander's embedding procedure, it is possible to also obtain norm-dense embeddings of $\mathrm{cwk}(X)$ and $\mathrm{ck}(X)$ in $C(\Omega)$-spaces, rather than weak*-dense embeddings, as in Corollary 4.5.6. Furthermore, X is not required to be separable. The details follow.

For every bounded subset C of X and each $x' \in X'$, let

$$s(x', C) := \sup\{x'(x) : x \in C\}.$$

4.5.7 Lemma

Let X be a Banach space. Then, for all nonempty bounded subsets A and C of X and for all $x' \in X'$,

$$s\big(x', \overline{\text{co}}\big(A \cup C\big)\big) = \max\{s(x', A), s(x', C)\}.$$

PROOF.

Direct verification yields that

$$s(x', A \cup C) = \max\{s(x', A), s(x', C)\},$$

for all nonempty bounded subsets A and C of X and for all $x' \in X'$. We claim that for any nonempty bounded subset A of X and for all $x' \in X'$,

$$s(x', \overline{\text{co}}A) = s(x', A).$$

To establish our claim, let $x \in \overline{\text{co}}A$. Then there exist sequences $(x_i), (y_i) \subset A$ and $(\lambda_i) \subset [0, 1]$ such that $\lim_{i\to\infty} \big(\lambda_i x_i + (1 - \lambda_i)y_i\big) = x$. Since $x'\big(\lambda_i x_i + (1 - \lambda_i)y_i\big) \leq s(x', A)$, for all $x' \in X'$, we obtain $x'(x) \leq s(x', A)$, for all $x' \in X'$. Thus,

$$s(x', \overline{\text{co}}A) \leq s(x', A),$$

for all $x' \in X'$. Since the reverse inequality follows readily, our claim is proved. To complete the proof of the lemma, let A and B be nonempty bounded subsets of X and $x' \in X'$. Then, by the preceding parts of the proof,

$$s\big(x', \overline{\text{co}}(A \cup C)\big) = s(x', A \cup C) = \max\{s(x', A), s(x', C)\}.$$

□

4.5.8 Theorem

Let X be a Banach space. Then there exist a compact Hausdorff space Ω and a map $V\colon \text{cwk}(X) \to C(\Omega)$ such that:

(1) $V(\alpha A + \beta C) = \alpha V(A) + \beta V(C)$, for all $A, C \in \text{cwk}(X)$ and $\alpha, \beta \in \mathbb{R}_+$.

(2) $d_H(A, C) = \|V(A) - V(C)\|_\infty$, for all $A, C \in \text{cwk}(X)$.

(3) $V(\text{cwk}(X))$ is norm closed in $C(\Omega)$.

(4) $V\big(\overline{\text{co}}(A \cup C)\big) = \max\{V(A), V(C)\}$, for all $A, C \in \text{cwk}(X)$.

PROOF.

Let $l_\infty(B_{X'})$ be the Banach space of all bounded real-valued functions defined on $B_{X'}$, equipped with the supremum norm $\|\cdot\|_\infty$. Then, by [6, Lemma 1.1], the map $U\colon \text{cwk}(X) \to l_\infty(B_{X'})$, defined by $U(A) = s(\,\cdot\,, A)$, satisfies:

(1) $U(\alpha A + \beta C) = \alpha U(A) + \beta U(C)$, for all $A, C \in \mathrm{cwk}(X)$ and $\alpha, \beta \in \mathbb{R}_+$.

(2) $d_H(A, C) = \|U(A) - U(C)\|_\infty$, for all $A, C \in \mathrm{cwk}(X)$.

(3) $U(\mathrm{cwk}(X))$ is norm closed in $l_\infty(B_{X'})$.

Moreover, by Lemma 4.5.7 and the result of Krein and Smulian (Theorem 1.3.12), U also satisfies

(d) $U\big(\overline{\mathrm{co}}(A \cup C)\big) = \max\{U(A), U(C)\}$, for all $A, C \in \mathrm{cwk}(X)$.

But $l_\infty(B_{X'})$ is a norm complete M-normed vector lattice with an order unit. Thus, by Kakutani's (M)-space representation theorem, there exist a compact Hausdorff space Ω and an isometric and vector lattice isomorphism W from $l_\infty(B_{X'})$ onto $C(\Omega)$. Then $V = W \circ U$ has the desired properties.

□

A similar embedding result as Theorem 4.5.8 can be obtained for $\mathrm{ck}(X)$ (see also [17]).

4.5.9 Theorem
Let X be a Banach space. Then there exist a compact Hausdorff space Ω and a map $V\colon \mathrm{ck}(X) \to C(\Omega)$ such that:

(1) $V(\alpha A + \beta C) = \alpha V(A) + \beta V(C)$, for all $A, C \in \mathrm{ck}(X)$ and $\alpha, \beta \in \mathbb{R}_+$.

(2) $d_H(A, C) = \|V(A) - V(C)\|_\infty$, for all $A, C \in \mathrm{ck}(X)$.

(3) $V(\mathrm{ck}(X))$ is norm closed in $C(\Omega)$.

(4) $V\big(\overline{\mathrm{co}}(A \cup C)\big) = \max\{V(A), V(C)\}$, for all $A, C \in \mathrm{ck}(X)$.

PROOF.

In the proof of Theorem 4.5.8, replace $l_\infty(B_{X'})$ by $C(B_{X'})$ and the Theorem of Krein and Smulian by that of Mazur (Theorem 1.2.42). Furthermore, there is no need here to use Kakutani's (M)-space representation theorem.

□

4.6 An $L^p(\mu)$ embedding

We proceed with our investigation of properties on the metric d on S for which $R(S)$ is a classical Banach space.

A Riesz seminorm on a vector lattice E is called *p-additive* ($1 \le p < \infty$), provided that

$$\|x+y\|^p = \|x\|^p + \|y\|^p, \text{ for all } x, y \in E_+, \text{ for which } x \wedge y = 0.$$

If (Ω, Σ, μ) is any measure space and $1 \le p < \infty$, then $L^p(\mu)$ endowed with $\|\cdot\|_p$ (as defined in Definition 1.4.14) with addition, scalar multiplication and order which is defined pointwise almost everywhere, is an example of a norm complete p-additive Riesz normed vector lattice.

4.6.1 Lemma
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ an invariant metric. Then d is a Riesz metric and

(L_d) $d(x+x_1, y+y_1) = d(x,y) + d(x_1,y_1)$, for all $x, y, x_1, y_1 \in S$ such that $y \le x$ and $y_1 \le x_1$

if and only if $\|\cdot\|_d$ is an L-norm on $R(S)$.

PROOF.

$\Rightarrow$: Let d be a Riesz metric satisfying (L_d). Then,

$$\begin{aligned} \|x+y\|_d &= d(x+y, 0) \\ &= d(x,0) + d(y,0) \text{ (by } (L_d)) \\ &= \|x\|_d + \|y\|_d, \end{aligned}$$

and hence, $\|\cdot\|_d$ is an L-norm on $R(S)$.

$\Leftarrow$: Let $\|\cdot\|_d$ be an L-norm on $R(S)$, then d is a metric by Theorem 1.2.37.
Let $x, y, x_1, y_1 \in S$ such that $y \le x$ and $y_1 \le x_1$. Then,

$$\begin{aligned} d(x+x_1, y+y_1) &= d(x+x_1-y-y_1) \\ &= \|x+x_1-y-y_1\|_d \\ &= \|(x-y)+(x_1-y_1)\|_d \\ &= \|x-y\|_d + \|x_1-y_1\|_d \text{ (since } \|\cdot\|_d \text{ is an } L\text{-norm)} \\ &= d(x-y, 0) + d(x_1-y_1, 0) \\ &= d(x,y) + d(x_1,y_1), \end{aligned}$$

which is condition (L_d).

Now, let $x, y, z \in S$ such that $x \leq y \leq z$. Then

$$\begin{aligned} d(x,z) &= \|z - x\| \\ &= \|z - y + y - x\| \\ &= \|z - y\| + \|y - x\| \text{ (by the fact that } \|\cdot\| \text{ is an } L\text{-norm)} \\ &\geq \|y - x\| \\ &= d(y,x). \end{aligned}$$

Lastly, let $x, y \in S$. If $x \leq y$, then

$$\begin{aligned} d(2(x \vee y), x + y) &= \|2(x \vee y) - (x + y)\| \\ &= \|2x - x - y\| \\ &= \|x - y\| \\ &= d(x,y). \end{aligned}$$

Similar for the case when $y \leq x$, and thus, d is a Riesz metric.

□

4.6.2 Theorem

Let S be an Archimedean near vector lattice with 0 as zero and e as an order unit. If $d\colon S \times S \to \mathbb{R}_+$ is a Riesz metric with property (L_d) and $d(e,0) = 1$, then there exist a probability space (Ω, Σ, μ) and a map $K\colon S \to L^1(\mu)$ such that:

(1) K is injective.

(2) $K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

(3) $K(S) - K(S)$ is norm dense in $L^1(\mu)$.

(4) $d(x,y) = \|K(x) - K(y)\|_1$, for all $x, y \in S$.

(5) $K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.

(6) $K(e) = \mathbf{1}$, where the latter denotes a function which is one almost everywhere.

PROOF.

The norm completion $\overline{R(S)}^d$ of $(R(S), \|\cdot\|_d)$ is a norm complete L-normed vector lattice. In the terminology of [74, Ch.2, §8] this means that $[e,0]$ is a quasi-interior point of $\overline{R(S)}^d$, since $[e,0]$ is an order unit of $R(S)$. By Kakutani's (L)-space representation for such spaces with quasi-interior points, there exist a compact Hausdorff space Ω, a strictly positive regular Borel measure μ on Ω and a vector lattice and isometric isomorphism

$V\colon \overline{R(S)}^d \to L^1(\mu)$ such that $V(R(S)) = L^\infty(\mu) = C(\Omega)$, $V([e,0]) = \mathbf{1}$, and the norm functional $\varphi_{\|\cdot\|_d}$ on $\overline{R(S)}^d$, defined by

$$\varphi_{\|\cdot\|_d}(f) = \|f^+\|_d - \|f^-\|_d, \text{ for all } f \in \overline{R(S)}^d,$$

has the property that

$$\varphi_{\|\cdot\|_\phi}([x,y]) = \int_\Omega V([x,y])\, d\mu, \text{ for all } x, y \in S$$

(see [74, Ch.2, §8]). Then

$$1 = \varphi_{\|\cdot\|_\phi}([e,0]) = \int_\Omega 1\, d\mu = \mu(\Omega),$$

showing that μ is a probability measure.

If we let $K = V \circ j$, where $j\colon S \to R(S)$ is the natural embedding, then, by Theorem 4.2.2 and Corollary 4.3.4, K has the desired properties.

□

The preceding result can be extended to the following theorem.

4.6.3 Theorem
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ a Riesz metric with property (L_d). Then there exist a measure space (Ω, Σ, μ) and a map $K\colon S \to L^1(\mu)$ such that:

(1) K is injective.

(2) $K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

(3) $K(S) - K(S)$ is norm dense in $L^1(\mu)$.

(4) $d(x,y) = \|K(x) - K(y)\|_1$, for all $x, y \in S$.

(5) $K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.

PROOF.

Since $\|\cdot\|_d$ is an L-norm on the vector lattice $R(S)$, it follows from Kakutani's (L)-space representation theorem (Theorem 4.5.4), that there exist a measure space (Ω, Σ, μ) and a vector lattice and isometric isomorphism V from the norm completion of $R(S)$ onto $L^1(\mu)$. Let $K = V \circ j$, where $j\colon S \to R(S)$ is the natural embedding. By Theorem 4.2.2 and Corollary 4.3.4, it follows that K has the desired properties.

□

The preceding result can be extended to cover the cases $1 \leq p < \infty$.

4.6.4 Lemma
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ an invariant metric. Then d is a Riesz metric with the property that

(P_d) $d(x+x_1, y+y_1)^p = d(x,y)^p + d(x_1,y_1)^p$ for $1 < p < \infty$ and for all $x, y, x_1, y_1 \in S$ such that $y \leq x,\ y_1 \leq x_1$ and $x + x_1 = (x_1 + y) \vee (x + y_1)$

if and only if $\|\cdot\|_d$ is a p-additive Riesz norm on $R(S)$.

PROOF.

Note that if $x, x_1, y, y_1 \in S$, then $x + x_1 = (x_1 + y) \vee (x + y_1)$ if and only if

$$[z, z] = [x + x_1, (x_1 + y) \vee (x + y_1)] = [x, y] \wedge [x_1, y_1].$$

The result now follows trivially.

□

4.6.5 Theorem (Kakutani-Bohnenblust representation for L^p-spaces, see [3, 4, 59, 74])
Let X be a vector lattice equiped with a p-additive Riesz norm $\|\cdot\|$. Then there exists a measure space (Ω, Σ, μ) and a vector lattice and isometric isomorphism V from the norm completion of X onto $L^p(\mu)$.

4.6.6 Theorem
Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ a Riesz metric with property (P_d). Then there exist a measure space (Ω, Σ, μ) and a map $K\colon S \to L^p(\mu)$ such that:

(1) K is injective.

(2) $K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

(3) $K(S) - K(S)$ is norm dense in $L^p(\mu)$.

(4) $d(x, y) = \|K(x) - K(y)\|_p$, for all $x, y \in S$.

(5) $K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.

PROOF.

Since $\| \cdot \|_d$ is a p-additive Riesz norm on the vector lattice $R(S)$, there exist, by the theorem of Kakutani-Bohnenblust (Theorem 4.6.5), a measure space (Ω, Σ, μ) and a vector lattice and isometric isomorphism V from the norm completion of $R(S)$ onto $L^p(\mu)$. Let $K = V \circ j$, where $j \colon S \to R(S)$ is the natural embedding. By Theorem 4.2.2 and Corollary 4.3.4, it follows that K has the desired properties.

□

Chapter 5

Generalized random variables

5.1 Random variables

The theory of conditional expectations has been established for Banach space-valued, Bochner-integrable functions. In [65], the central ideas of this chapter were developed. We present the introduction to random variables, which is important not only for the main result of this chapter, but also for the work on martingales in the subsequent chapter.

Throughout the remainder of this thesis we will again be considering functions from a measure space (Ω, Σ, μ) into a separable real Banach space $(X, \|\cdot\|)$.

A *random variable* is a measurable function $f : \Omega \to X$. A random variable can be thought of as an unknown value that may change every time it is inspected. Thus, from a mathematical point of view, we simply regard a random variable as a function mapping the sample space of a random process to X. As in section 1.5, we will again denote by $L^0(\mu, X)$ and $L^p(\mu, X)$, the collections of measurable and p-integrable functions $f : \Omega \to X$ respectively, for $1 \leq p \leq \infty$. Due to the completeness of X we have several characterizations of this class of functions (see [24] Theorem 1.0).

5.1.1 Definition
Let (Ω, Σ, μ) be a measure space, X a separable real Banach space and f an integrable random variable.

(a) We define the *expectation* of a random variable f to be

$$\mathbb{E}(f) = \int_{\Omega} f d\mu.$$

(b) Let Σ_0 a sub-σ-algebra of Σ. Then the *conditional expectation* $\mathbb{E}[f|\Sigma_0]$ of f relative to (with respect to) Σ_0 is a Σ_0-measurable function g such that

$$\int_{\Omega}^{(\Sigma_0)} f d\mu = \int_{\Omega}^{(\Sigma_0)} g d\mu.$$

Uniqueness of the conditional expectation can be shown to be almost sure. That is, different conditional expectations of the same function with respect to a given sub-σ-algebra of Σ will only differ on a null set.

There are several different senses in which random variables can be considered to be equivalent.

5.1.2 Definition
Let (Ω, Σ, μ) be a measure space, X a separable real Banach space, and f and g two random variables.

(a) We say that f and g are *weakly equal* if $\mathbb{E}(f) = \mathbb{E}(g)$.

(b) f and g are *equal in p-th mean* if $\mathbb{E}(\|f - g\|^p) = 0$. This type of equality provides a method of determining the distance between two random variables, namely $d_p(f, g) = \mathbb{E}(\|f - g\|^p)$.

(c) f and g are said to be *equal almost surely* if $f = g$ a.e.

(d) f and g are *equal* if $f(\omega) = g(\omega)$, for all $\omega \in \Omega$.

Note that for all practical purposes in probability theory, almost sure equality is as strong as actual equality. For this reason it is ubiquitous in probability theory literature, and therefore the type of equality that we are most interested in.

In Definition 1.2.32, we defined the general concept of convergence of sequences. There are a number of senses in which sequences of random variables can converge. We list several of the most ubiquitous of these below.

5.1.3 Definition
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. Suppose that (f_n) is a sequence of random variables and that f is a random variable.

(a) If $\lim_{n\to\infty} \mathbb{E}(f_n) = \mathbb{E}(f)$, then we say that (f_n) *converges weakly* to f.

(b) To say that the sequence (f_n) *converges in measure* towards f means

$$\lim_{n\to\infty} \mu(\|f_n - f\|) \geq \epsilon) = 0,$$

for every $\epsilon > 0$. If μ is a probability measure that satisfies the condition above, then we say that (f_n) *converges in probability* towards f.

(c) To say that the (f_n) converges *surely* or *everywhere* or *pointwise* towards f means that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega),$$

for all $\omega \in \Omega$.

(d) The sequence (f_n) converges *almost surely* or *almost everywhere* means that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \text{ a.e.}$$

(e) The sequence (f_n) converges *uniformly* if

$$\lim_{n\to\infty} \|f - f_n\|_\infty = 0,$$

where $\|f\|_\infty := \sup_{\omega\in\Omega} \|f(\omega)\|$ is the supremum norm.

(f) We say that the sequence (f_n) converges *in the* p^{th} *mean* towards f, if $p \geq 1$, $E(\|f_n\|^p) < \infty$, for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \mathbb{E}(\|f_n - f\|^p) = 0.$$

Notice that (e) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a) and also that $r > s \geq 1$, then convergence with respect to $\|\cdot\|_r$ implies convergence with respect to $\|\cdot\|_s$.

5.2 Set-valued random variables

Hiai and Umegaki have generalized random variables and conditional expectations to the set-valued (multivalued) setting in [24]. We also contribute to the theory of set-valued martingales and since martingales are important in probability theory, we feel that there are potential applications to be developed from this work.

Let (Ω, Σ) be a measurable space, X a metric space and $\mathcal{P}_0(X) = \{A \subset X : A \neq \emptyset\}$. A mapping $F : \Omega \to \mathcal{P}_0(X)$ is called a *set-valued mapping*. The set

$$G(F) = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\},$$

is called the *graph* of F, and the set

$$F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\}, \quad A \subset X,$$

the *inverse image* of F.

5.2.1 Definition

Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space. A set-valued mapping $F : \Omega \to \text{f(X)}]$ is called *strongly measurable* if, for each closed subset C of X, $F^{-1}(C) \in \Sigma$. A set-valued mapping $F : \Omega \to \text{f(X)}$ is called *weakly measurable* if, for each open set O of X, $F^{-1}(O) \in \Sigma$. A weakly measurable set-valued mapping is also called a *set-valued random variable* or a *random set*. We denote the collection of weakly measurable random variables by $\mathbf{M}[\Omega, \text{f(X)}]$. If $U \subset \text{f}(X)$, then we denote by $\mathbf{M}[\Omega, U]$, the collection $\{F \in U^{\Omega} : F \in \mathbf{M}[\Omega, \text{f(X)}]\}$.

5.2.2 Remark

Let (Ω, Σ, μ) be a measure space, X a separable Banach space and $F : \Omega \to \text{f(X)}$. Then F is strongly measurable if and only if F is weakly measurable (see [48]). In this work, we require that weak measurability and strong measurability coincide and this justifies the requirement that the Banach space X is separable.

5.2.3 Definition

Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space.

(a) A measurable function $f : \Omega \to X$ is called a *measurable selection* of F if $f(\omega) \in F(\omega)$ a.e. That is, f is a measurable selection of F if $f(\omega) \in F(\omega)$, for all $\omega \in \Omega$ except on a null set. We define the set

$$S_F^p = \{f \in L^p(\mu, X) : f(\omega) \in F(\omega) \text{ a.e.}\}.$$

It is easy to show that S_F^p is a closed subset of $L^p(\mu, X)$ for $1 \leq p \leq \infty$.

(b) If $F \in \mathbf{M}[\Omega, \text{f}(X)]$, then F is called *integrable* provided that $S_F^1 \neq \emptyset$.

(c) If $F \in \mathbf{M}[\Omega, \text{f}(X)]$, then F is called *integrably bounded* provided that there exists $\rho \in L^1(\mu)$ such that $\|x\|_X \leq \rho(\omega)$, for all $x \in F(\omega)$ and for all $\omega \in \Omega$. In this case, $F(\omega) \in \text{f}(X)$ a.e. and $\|F(\omega)\|_H = \sup\{\|x\|_X : x \in F(\omega)\} \leq \rho(\omega)$, for all $\omega \in \Omega$.

Let $\mathcal{L}^1[\Omega, \Sigma, \mu, \text{f}(X)]$ denote the set of all equivalence classes of a.e. equal $F \in \mathbf{M}[\Omega, \text{f}(X)]$, which are integrably bounded. We will simply write $\mathcal{L}^1[\Omega, \text{f}(X)]$ if there is no confusion. If $\Delta\colon \mathcal{L}^1[\Omega, \text{f}(X)] \times \mathcal{L}^1[\Omega, \text{f}(X)] \to \mathbb{R}_+$ is defined by

$$\Delta(F_1, F_2) = \int_{\Omega} d_H\big(F_1(\omega), F_2(\omega)\big) d\mu,$$

where d_H is the Hausdorff metric, then $(\mathcal{L}^1[\Omega, \text{bf}(X)], \Delta)$ is a complete metric space (see [24, 48]).

It now follows that with respect to the Hausdorff topology on $\mathbf{M}[\Omega, \text{bf(X)}]$, that if $(F_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{M}[\Omega, \text{bf(X)}]$, then we have

$$\lim_{n\to\infty} F_n = F$$

if and only if

$$\lim_{n\to\infty} d_H(F_n(\omega), F(\omega)) = 0 \text{ a.e.}$$

5.2.4 Definition
Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space.

(a) Let Σ_0 be a sub-σ-algebra of Σ. If $F : \Omega \to \mathrm{f}(X)$ is Σ_0-measurable, then we define

$$S^1_F(\Sigma_0) = \{f \in L^1(\Omega, \Sigma_0, \mu, X) : f(\omega) \in F(\omega) \text{ a.e.}[\mu|_{\Sigma_0}]\}.$$

(b) Let U be any subset of $\mathrm{f}(X)$. Then we define

$$\mathcal{L}^1[\Omega, U] = \{F \in \mathcal{L}^1[\Omega, \mathrm{f}(X)] : F(\omega) \in U \text{ a.e.}\}.$$

Note that $\mathcal{L}^1[\Omega, U] \subset \mathcal{L}^1[\Omega, \mathrm{f}(X)]$.

5.2.5 Definition
Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space. Let $F \in \mathbf{M}[\Omega, \mathrm{f(X)}]$. We define $\mathcal{E}(F)$, the *expectation* of F, to be

$$\mathcal{E}(F) = \{\mathbb{E}(f) : f \in X^\Omega, \mathbb{E}(\|f\|) < \infty, f(\omega) \in F(\omega) \text{ a.e.}\}.$$

5.2.6 Lemma ([24])
Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space. Let $F \in \mathbf{M}[\Omega, \mathrm{f(X)}]$ and $1 \le p \le \infty$. If S^p_F is nonempty, then there exists a sequence $(f_i)_{i\in\mathbb{N}}$ contained in S^p_F such that $F(\omega) = \mathrm{cl}\{f_i(\omega) : i \in \mathbb{N}\}$, for $\omega \in \Omega$.

5.2.7 Corollary ([24])
Let (Ω, Σ, μ) denote a measure space and X a separable real Banach space. Let $F_1, F_2 \in \mathbf{M}[\Omega, \mathrm{f(X)}]$ and $1 \le p \le \infty$. If $S^p_{F_1} = S^p_{F_2} \neq \emptyset$, then $F_1(\omega) = F_2(\omega)$, for all $\omega \in \Omega$.

A brief discussion of the generalization of conditional expectations is necessary before we reach the main result of this chapter. This material is covered more comprehensively in [24].

Let Σ_0 be a sub-σ-algebra of Σ, then we define

$$S^1_F(\Sigma_0) = \{f \in L^1(\Omega, \Sigma_0, \mu, X) : f(\omega) \in F(\omega) \text{ a.e.}\}, \text{ and}$$

$$\int_\Omega^{(\Sigma_0)} F d\mu = \left\{ \int_\Omega f d\mu : f \in S^1_F(\Sigma_0) \right\}.$$

We define addition $\oplus$, scalar multiplication $\cdot$ and an order relation pointwise on $\mathcal{L}^1[\Omega, \mathrm{f}(X)]$.

5.2.8 Definition
Let (Ω, Σ, μ) be a measure space and let X be a separable real Banach space. If $F, F_1, F_2 \in \mathbf{M}[\Omega, \mathrm{f}(X)]$ and $\lambda \in \mathbb{R}$, define $F_1 \oplus F_2$, λF_1 and $\overline{\mathrm{co}} F_1$, respectively, by

(a)
$$(F_1 \oplus F_2)(\omega) := F_1(\omega) \oplus F_2(\omega),$$
$$(\lambda F)(\omega) := \lambda(F(\omega)), \text{ and}$$
$$(\overline{\mathrm{co}} F)(\omega) := \overline{\mathrm{co}}(F(\omega)),$$

for all $\omega \in \Omega$. Here $\overline{\mathrm{co}}(F(\omega))$ denotes the norm closure in X of the convex hull $\mathrm{co}(F(\omega))$ of $F(\omega)$.

(b) We define a partial order relation on $\mathbf{M}[\Omega, \mathrm{f}(X)]$ as follows:
$$F_1 \leq F_2 \Leftrightarrow F_1(\omega) \subset F_2(\omega) \text{ a.e.}$$

The following theorem provides a natural extension of the notion of a conditional expectation to the set-valued setting.

5.2.9 Theorem ([24])
Let (Ω, Σ, μ) denote a measure space, X a separable real Banach space and Σ_0 a sub-σ-algebra of Σ. Let $F \in \mathcal{L}^1[\Omega, \mathrm{f(X)}]$. Then there exists a unique $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Omega, \Sigma_0, \mu, \mathrm{f(X)}]$ such that
$$S^1_{\mathcal{E}[F|\Sigma_0]}(\Sigma_0) = \mathrm{cl}\{\mathbb{E}[f|\Sigma_0] : f \in S^1_F\},$$
where the closure is taken with respect to the norm Δ on $\mathcal{L}^1[\Omega, \mathrm{f(X)}]$.

5.2.10 Definition
Let (Ω, Σ, μ) denote a measure space, X a separable real Banach space and Σ_0 a sub-σ-algebra of Σ. The unique $\mathcal{E}[F|\Sigma_0]$, as defined in the theorem above, is called the *conditional expectation of F relative to Σ_0.*

It is well known that $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Sigma_0, \mathrm{f}(X)]$, for all $F \in \mathcal{L}^1[\Omega, \mathrm{f}(X)]$ (see [48]).

5.2.11 Theorem (see [24, 48])
Let (Ω, Σ, μ) be a measure space, X a separable real Banach space and Σ_0 a sub-σ-algebra of Σ. If $F \in \mathcal{L}^1[\Omega, \Sigma, \mu, \mathrm{f}(X)]$, then the conditional expectation $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Omega, \Sigma_0, \mu, \mathrm{f}(X)]$ of F with respect to Σ_0 has the following properties:

(E1) If $F_1, F_2 \in \mathcal{L}^1[\Omega, \mathrm{f}(X)]$, then $\mathcal{E}[F_1 \oplus F_2|\Sigma_0] = \mathcal{E}[F_1|\Sigma_0] \oplus \mathcal{E}[F_2|\Sigma_0]$.

(E2) If $F \in \mathcal{L}^1[\Omega, \mathrm{f}(X)]$ and $\lambda \in \mathbb{R}_+$, then $\mathcal{E}[\lambda F|\Sigma_0] = \lambda \mathcal{E}[F|\Sigma_0]$.

(E3) If $F_1, F_2 \in \mathcal{L}^1[\Omega, \mathrm{f}(X)]$, then $F_1 \leq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \leq \mathcal{E}[F_2|\Sigma_0]$.

(E4) If $F \in \mathcal{L}^1[\Omega, \Sigma_0, \mu, \mathrm{f}(X)]$, then $\mathcal{E}[F|\Sigma_0] = F$.

(E5) If Σ_1 and Σ_2 are σ-algebras such that $\Sigma_1 \subset \Sigma_2 \subset \Sigma$ and $F \in \mathcal{L}^1[\Omega, \Sigma_1, \mu, \mathrm{f}(X)]$, then $\mathcal{E}\left[\,\mathcal{E}[F|\Sigma_2]\,|\,\Sigma_1\right] = \mathcal{E}[F|\Sigma_1]$.

5.2.12 Theorem (see [24, 48])
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. A function $F\colon \Omega \to \mathrm{f}(X)$ is Σ-*measurable* provided that there exists a sequence $(f_i)_{n\in\mathbb{N}}$ such that each function $f_i\colon \Omega \to X$ is:

(M1) μ-measurable; (as in Definition 1.4.3 (d))

(M2) a *selection* of F, and

(M3) $F(\omega) = \overline{\{f_i(\omega) : i \in \mathbb{N}\}}$, for all $\omega \in \Omega$, where the closure is the norm closure in X.

Chapter 6

Generalized martingales and submartingales

6.1 Preliminaries

The concept of martingale in probability theory was introduced by Paul Pierre Lévy. Part of the motivation for that work was to show the impossibility of successful betting strategies. A martingale was originally devised as an indexed sequence of random variables with the index representing time. If t is a later time than s, then the idea is that the conditional expected value at time t given the same observations as at time s will be equal to the expected value at time s. The notion of a martingale has proved to be a powerful tool in probability theory and related fields. The advantage of martingale theory is its intrinsic simplicity and intuitive nature and martingale theory is an extremely important application of functional analysis. The range of applications of martingale theory is enhanced by the construction of stochastic integrals and a martingale calculus.

The main results of this section are generalizations of the Doob-Meyer decomposition theorem, which is a theorem in stochastic calculus stating the conditions under which a submartingale may be decomposed in a unique way as the sum of a martingale and a continuous increasing process. It is named after Joseph Leo Doob and Paul-André Meyer and Doob published the original decomposition theorem in 1953 which gives a unique decomposition for certain discrete time martingales. He conjectured a continuous time version of the theorem, and in two publications in 1962 and 1963 Paul-André Meyer proved such a theorem, which became known as the Doob-Meyer decomposition.

The basis of this chapter was developed in [65] and [45]. We present a precise definition of the classical notion of a martingale, followed by the generalization to the set-valued setting, leading up the main results.

We remind the reader of the following standard terminology.

6.1.1 Definition
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space.

(a) A *filtration* on (Ω, Σ, μ) is a sequence of σ-algebras (Σ_n) such that $\Sigma_n \subset \Sigma$ and $\Sigma_n \subset \Sigma_{n+1}$, for all $n \in \mathbb{N}$.

(b) A sequence of set-valued random variables (F_n) is *predictable* with respect to a filtration (Σ_n) if F_n is Σ_n-measurable, for each $n \in \mathbb{N}$.

6.1.2 Definition
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. If (f_n) is a sequence of random variables and (Σ_n) a filtration, then the sequence (f_n, Σ_n) is called a *martingale* if we have for each $n \in \mathbb{N}$,

(a) f_n is Σ_n-measurable and $\mathbb{E}[f_n|\Sigma_n] < \infty$, and

(b) $\mathbb{E}[f_{n+1}|\Sigma_n] = f_n$.

Alternatively, if property (b) is replaced by

(b') $\mathbb{E}[f_{n+1}|\Sigma_n] \geq f_n$ (resp., $\mathbb{E}[f_{n+1}|\Sigma_n] \leq f_n$), then (f_n, Σ_n) is called a *submartingale* (resp., *supermartingale)*.

The following definition asserts that we can equally well define martingales in terms of projections.

6.1.3 Definition
Let S be any nonempty set, (T_i) a commuting sequence (i.e., $T_iT_j = T_jT_i = T_i$, for all $i \leq j$) of projections on S and (f_i) a sequence in S. Then

(a) (f_i, T_i) is a *martingale* in S, provided that $f_i = T_i f_j$, for all $i \leq j$.

If, in addition, $(S, \leq)$ is a partially ordered set and each T_i is order preserving, then

(b) (f_i, T_i) is called a *supermartingale* (resp., *submartingale*) in S, provided that $f_i \in \mathcal{R}(T_i)$, for all i (where $\mathcal{R}(T_i)$ is the range of T_i) and $f_i \leq T_i f_j$ (resp., $f_i \geq T_i f_j$), for all $i \leq j$.

6.2 Doob's decomposition

In mathematics, specifically in stochastic analysis, Doob's martingale convergence theorems are a collection of results on the long-time limits of submartingales and supermartingales.

We concern ourselves with Doob's decomposition of submartingales, which states the conditions under which a submartingale may be decomposed in a unique way as the sum of a martingale and a continuous increasing process. This decomposition was extended from the classical setting of real-valued martingales to set-valued martingales by Daures, Ni and Zhang (see [48, 62]) and also by Shen and Wang (see [75]).

In the set-valued setting, the first problem that one encounters, is that neither the range spaces of the submartingales nor the spaces of submartingales are vector spaces. However, these spaces are near vector spaces; i.e., they have all the properties of a vector space except that elements do not necessarily have additive inverses, (see [46, 68]).

Our aim is to use ideas from measure-free martingale theory (see [11, 12, 13, 41, 77, 80]), together with Rådström's completion of a near vector space, to give an elementary proof for Doob's decomposition of set-valued submartingales.

Our strategy is as follows. After introducing the necessary preliminaries and notation, we consider Doob's decomposition of a submartingale in an ordered vector space. From this, and with the aid of Rådström's completion of a near vector space (see [46]), we obtain a Doob decomposition of a submartingale in an ordered near vector space. We then specialize the ordered near vector space to the appropriate set-valued space of submartingales that are integrable. As special cases, we obtain the Daures, Ni and Zhang result by using the fact that martingales which are integrably bounded are integrable (see [48]). We also derive an analogue of the Doob decomposition of set-valued submartingales, as noted by Shen and Wang.

Let (Ω, Σ, μ) be a measure space. If Σ_0 is a sub σ-algebra of Σ, denote by $L^0(\Omega, \Sigma_0, \mu)$ the set of Σ_0-measurable functions $f : \Omega \to \mathbb{R}$.

We now generalize the concept of a martingale to the set-valued setting in the natural way.

6.2.1 Definition
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. Let (F_n) be a sequence of set-valued random variables and (Σ_n) a filtration, then the sequence (F_n, Σ_n) is called a *set-valued martingale* if we have, for each $n \in \mathbb{N}$,

(a) F_n is Σ_n-measurable and $\mathcal{E}[F_n|\Sigma_n] < \infty$, and

(b) $\mathcal{E}[F_{n+1}|\Sigma_n] = F_n$ a.e.

Alternatively, if property (b) is replaced by

(b') $\mathcal{E}[F_{n+1}|\Sigma_n] \geq F_n$ a.e. (resp., $\mathcal{E}[F_{n+1}|\Sigma_n] \leq F_n$ a.e.),

then (F_n, Σ_n) is called a *set-valued submartingale* (resp., *set-valued supermartingale).*

The following well-known result relates submartingales to martingales.

6.2.2 Theorem (Doob's Decomposition, see [13, 62])
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. If (Σ_i) an increasing sequence of sub σ-algebras of Σ, and (f_i, Σ_i) is a submartingale, then (f_i, Σ_i) has a unique decomposition

$$f_i(\omega) = M_i(\omega) + A_i(\omega) \text{ a.e.},$$

where (M_i, Σ_i) is a set-valued martingale and (A_i) is a predictable (i.e., A_i is Σ_{i-1}-measurable, for all $i \geq 2$), increasing sequence such that:

(a) $A_1(\omega) = 0$ a.e.,

(b) $A_j(\omega) = \sum_{i=1}^{j-1} \Big(\mathbb{E}[f_{i+1}|\Sigma_i](\omega) - f_i(\omega)\Big)$ a.e., for $j \geq 2$,

(c) $M_j(\omega) = f_j(\omega) - A_j(\omega)$ a.e., for all $j \in \mathbb{N}$.

Daures, Ni and Zhang proved an analogue of Doob's decomposition for set-valued submartingales (see [13, 62]). Before we state their result, as can be found in [48], we first recall some terminology from [24, 48].

Let X be a Banach space. There is a canonical addition operation $+$, a canonical scalar multiplication operation $\cdot$ and a canonical subtraction operation $\ominus$ on

$$\mathcal{P}_0(X) := \{A \subset X : A \text{ is nonempty}\},$$

defined, for all $A, B, C \in \mathcal{P}_0(X)$ and $\lambda \in \mathbb{R}$, by

$$A \ominus B := \{x \in X : x + B \subset A\},$$

where $x + B := \{y = x + b : b \in B\}$. It is well-known that the set $\mathcal{P}_0(X)$ does not, in general, form a vector space with respect to the above defined operations.

6.3 Doob's decomposition in an ordered near vector space

We are now in a position to state the Doob decomposition theorem of Daures, Ni and Zhang, as can be found in [48].

6.3.1 Theorem (see [48], p.159)

Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. Let (F_i, Σ_i) be a set-valued submartingale in $\mathcal{L}^1[\Omega, \mathrm{cf}(X)]$; If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$,

(i) $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega)) - s(\,\cdot\,, F_{i-1}(\omega))$, and

(ii) $s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega))$,

are convex functions on X', then (F_i, Σ_i) can be decomposed as

$$F_i(\omega) = M_i(\omega) \oplus A_i(\omega), \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) is a set-valued martingale and (A_i) is a set-valued predictable increasing sequence such that for all $\omega \notin B$:

(a) $A_1(\omega) = 0$,

(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$, for all $j \geq 2$,

(c) $M_1(\omega) = F_1(\omega)$, and

(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j}[F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$, for all $j \geq 2$.

The proof of Theorem 6.3.1, as given in [48], exploits the properties of the the functions $s(\cdot, C)$ where $C \in \mathrm{cf}(X)$. Our aim is to give an elementary proof of Doob's decomposition of a set-valued submartingale.

To achieve our aim, we derive a Doob decomposition theorem for submartingales in ordered near vector spaces as considered in [46]. As a consequence and as an intermediate step, we obtain a Doob decomposition for set-valued submartingales which are integrable. The latter then yields the Daures, Ni and Zhang result (see [13, 62]) as a special case, as integrably bounded functions are integrable (see [48, p.31]). It also yields an analogue of the Doob decomposition of set-valued submartingales, as noted by Shen and Wang (see [75]).

It was noted in [46], that if X is a Banach space, then $(\mathrm{cbf}(X), \oplus, \cdot)$ is a near vector space (see Chapter 4).

If X is a separable Banach space, then, as noted in [42, 43]:

(a) $(\mathbf{M}[\Omega, \mathrm{f}(X)], \oplus, \cdot, \leq)$, and

(b) $(\mathcal{L}^1[\Omega, \mathrm{cbf}(X)], \oplus, \cdot, \leq)$,

are ordered near vector space with with $\{0\}$ as zero. In fact, $(\mathcal{L}^1[\Omega, \mathrm{cbf}(X)], \oplus, \cdot, \leq)$ is a sub ordered near vector space of $(\mathbf{M}[\Omega, \mathrm{f}(X)], \oplus, \cdot, \leq)$.

It is clear that if S is an ordered near vector space with a zero, then there exists a subtraction operation on $R(S)$, but this does not guarantee the existence of a subtraction operation on S under which S is closed.

To overcome this problem, we consider the following.

6.3.2 Definition
Let S be an ordered near vector space with a zero and define $\sqsubseteq$ by

$$y \sqsubseteq x \Leftrightarrow \exists\, z \in S\, [0 \leq z \text{ and } y + z = x].$$

Then, by the cancellation law, z is unique in Definition 6.3.2 and we define

$$x - y := z.$$

Also,

$$x \sqsubseteq y \Rightarrow x \leq y, \text{ for all } x, y \in S$$

and it follows that $\sqsubseteq$ is a partial ordering on S. We call $\sqsubseteq$ the *ordering associated with* $\leq$.

Also note that, for all $x \in S$,

$$0 \sqsubseteq x \Leftrightarrow 0 \leq x;$$

i.e.,

$$S_+ := \{x \in S : 0 \leq x\} = \{x \in S : 0 \sqsubseteq x\}.$$

It is readily verified that $(S, \sqsubseteq)$ is an ordered near vector space with 0 as zero. Furthermore, if we consider the Rådström completion $R(S)$ of $(S, +, \cdot, \sqsubseteq)$, then

$$\begin{aligned} y \sqsubseteq x \quad &\Leftrightarrow \quad \exists\, z \in S \ (0 \leq z \text{ and } [z, 0] = [x, y]) \\ &\Leftrightarrow \quad \exists\, x - y \in S \ (0 \leq x - y \text{ and } [x - y, 0] = [x, y])\,. \end{aligned}$$

Our strategy is now as follows. We first consider Doob's decomposition of a submartingale in an ordered vector space. Then we use this ordered vector space result to obtain a Doob decomposition of a submartingale in an ordered near vector space. We specialize the ordered near vector space to the appropriate set-valued space of submartingales that are integrable and obtain the Daures, Ni and Zhang result as a special case from the latter for integrably bounded martingales.

Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. As was noted in [43], it follows from Theorem 5.2.11 that if (Ω, Σ, μ) a measure space, X a separable real Banach space, (Σ_i) a filtration and if we set

$$T_i(F) = \mathcal{E}[F|\Sigma_i], \text{ for all } F \in \mathcal{L}^1[\Omega, \mathrm{cbf}(X)] \text{ and } i \in \mathbb{N},$$

then (T_i) is a commuting sequence of order preserving projections on the ordered near vector space $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ and the range $\mathcal{R}(T_i)$ of T_i is $\mathcal{L}^1[\Omega, \Sigma_i, \mu, \mathrm{cbf}(X)]$ for each $i \in \mathbb{N}$. Furthermore, if (F_i) is contained in $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ and (Σ_i) is an increasing sequence of sub-σ-algebras of Σ, then (F_i, T_i) is a *martingale* (resp., *submartingale*) in the ordered near vector space $\mathcal{L}^1[\Omega, \mathrm{f}(X)]$ (resp., $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$) in the sense of Definition 6.1.3.

The following result, which is based on a vector lattice version in [41], is the first step in achieving our aim of proving the Daures, Ni and Zang result in an elementary way.

6.3.3 Theorem

Let E be an ordered vector space, (f_i) a sequence in E and $(T_i)_{i\in\mathbb{N}}$ a commuting sequence of positive linear projections on E. If $(f_i, T_i)_{i\in\mathbb{N}}$ is a submartingale such that:

(a) $A_1 = 0$,

(b) $A_j = \sum_{i=1}^{j-1} \left(T_i f_{i+1} - f_i\right)$, for all $j \geq 2$, and

(c) $M_j = f_j - A_j$, for all $j \in \mathbb{N}$,

then the decomposition $f_i = M_i + A_i, i \in \mathbb{N}$, is the unique decomposition of (f_i, T_i) with (M_j, T_j) a martingale in E, $(A_j) \subset E^{\mathbb{N}}$ a positive and increasing sequence and $A_{j+1} \in \mathcal{R}(T_j)$, for all $j \in \mathbb{N}$.

PROOF.

As $A_1 = 0 \in \mathcal{R}(T_j)$, for all $j \in \mathbb{N}$ and $T_i(f_{i+1} - f_i) \in \mathcal{R}(T_i) \subset \mathcal{R}(T_j)$, for all $2 \leq i \leq j$, it follows that $A_j \in \mathcal{R}(T_j)$, for all $j \in \mathbb{N}$. But (f_i, T_i) is a submartingale, so $T_i(f_{i+1} - f_i) \geq f_i - f_i = 0$. Consequently, $A_j \geq 0$ and the sequence (A_j) is increasing.

We verify that $T_i(M_j) = M_i$, for all $i \leq j$. Let $i \leq j$. Then

$$\begin{aligned} T_i(A_j) &= \sum_{k=1}^{i-1} T_i T_k(f_{k+1} - f_k) + \sum_{k=i}^{j-1} T_i T_k(f_{k+1} - f_k) \\ &= \sum_{k=1}^{i-1} T_k(f_{k+1} - f_k) + \sum_{k=i}^{j-1} T_i(f_{k+1} - f_k) \\ &= A_i + T_i(f_j - f_i). \end{aligned}$$

Hence,

$$\begin{aligned} T_i(M_j) &= T_i(f_j) - T_i(A_j) \\ &= T_i(f_j) - [A_i + T_i(f_j - f_i)] \\ &= T_i(f_j) - [A_i + T_i(f_j) - T(f_i)] \\ &= T_i(f_j) - [A_i + f_i - f_i] \\ &= f_i - A_i \\ &= M_i, \end{aligned}$$

proving the first part of the theorem.

Suppose that $f_i = \tilde{M}_i + \tilde{A}_i$, for all $i \in \mathbb{N}$ is another decomposition satisfying the conditions of the theorem. We prove by induction on i, that $A_i = \tilde{A}_i$ and $M_i = \tilde{M}_i$, for all $i \in \mathbb{N}$. As $A_1 = 0 = \tilde{A}_0$ and $M_1 + A_1 = f_1 = \tilde{M}_1 + \tilde{A}_1$, it follows that $M_1 = \tilde{M}_1$. Now suppose that $A_i = \tilde{A}_i$ and $M_i = \tilde{M}_i$. As (M_i, T_i) and $(\tilde{M}_i, T_i)$ are martingales,

$$\begin{aligned} M_i + T_i(A_{i+1}) &= T_i(M_{i+1} + A_{i+1}) \\ &= T_i(f_{i+1}) \\ &= T_i(\tilde{M}_{i+1} + \tilde{A}_{i+1}) \\ &= \tilde{M}_i + T_i(\tilde{A}_{i+1}); \end{aligned}$$

thus, $T_i(A_{i+1}) = T_i(\tilde{A}_{i+1})$. But $A_{i+1}, \tilde{A}_{i+1} \in \mathcal{R}(T_i)$, which yields $A_{i+1} = \tilde{A}_{i+1}$; hence, $M_{i+1} = \tilde{M}_{i+1}$.

□

Let S be an ordered near vector space. A map $T\colon S \to S$ is called $\mathbb{R}_+$-*linear* provided that $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.

Let (f_i, T_i) be a submartingale in an ordered near vector space S, where (T_i) is a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S. Then $([f_i, 0], \hat{T}_i)$ is a submartingale in $R(S)$ and $(\hat{T}_i)$ is a commuting sequence of order preserving linear projections on $R(S)$.

We need the following notion.

6.3.4 Definition
Let S be an ordered near vector space with a zero, $(f_i) \subset S^{\mathbb{N}}$, (T_i) a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S.We call (f_i, T_i) a $\sqsubseteq$-*submartingale* in S if $f_i \in \mathcal{R}(T_i)$, for all i, and $f_j \sqsubseteq T_j(f_i)$, for all $j \leq i$.

6.3.5 Theorem
Let S be an ordered near vector space with a zero, $(f_i) \subset S^{\mathbb{N}}$ and (T_i) a commuting sequence of increasing $\mathbb{R}_+$-linear projections on S. If (f_i, T_i) is a $\sqsubseteq$-submartingale such that:

(a) $A_1 = 0$,

(b) $A_j = \sum_{i=1}^{j-1}[T_i f_{i+1} - f_i, 0]$, for all $j \geq 2$,

(c) $M_1 = [f_1, 0]$, and

(d) $M_j = [f_1, 0] + \sum_{i=1}^{j-1}[f_{i+1}, T_i f_{i+1}]$, for all $j \in \mathbb{N}$,

then the decomposition $[f_i, 0] = M_i + A_i$, for all $i \in \mathbb{N}$, is the unique decomposition of $([f_i, 0], \hat{T}_i)$ with (M_j, T_j) a martingale in $R(S)$, (A_j) a positive and increasing sequence in $R(S)$ and $A_{j+1} \in \mathcal{R}(T_j)$, for all $j \in \mathbb{N}$.

PROOF.

Let (f_i, T_i) be a $\sqsubseteq$-submartingale in S. Then (f_i, T_i) is a submartingale and $T_i f_{i+1} - f_i \in S_+$, for all $i \in \mathbb{N}$. By the preceding theorem, $([f_i, 0], \hat{T}_i)$ has a unique decomposition

$$[f_i, 0] = M_i + A_i, \text{ for all } i \in \mathbb{N}, \tag{6.1}$$

where $A_1 = 0$, and for all $j \geq 2$,

$$A_j = \sum_{i=1}^{j-1}[T_i f_{i+1}, f_i] = \sum_{i=1}^{j-1}[T_i f_{i+1} - f_i, 0], \tag{6.2}$$

and $0 \leq A_i - A_{i-1} = [T_{i-1} f_i, f_{i-1}] = [T_{i-1} f_i - f_{i-1}, 0]$. Moreover, $(M_i, \hat{T}_i)$ is a martingale in $R(S)$ given by $M_1 = [f_1, 0]$, and for all $j \geq 2$,

$$M_j = [f_j, 0] - \sum_{i=1}^{j-1}[T_i f_{i+1}, f_i] = [f_j, 0] + \sum_{i=1}^{j-1}[f_i, T_i f_{i+1}].$$

As the identity $[a, b] = [a, 0] + [0, b]$ holds in $R(S)$, for all $a, b \in S$, it readily follows, for all $j \geq 2$, that

$$\begin{aligned}
[f_j, 0] + \sum_{i=1}^{j-1}[f_i, T_i f_{i+1}] &= [f_j, 0] + [f_{j-1}, T_{j-1} f_j] + \cdots + [f_2, T_2 f_3] + [f_1, T_1 f_2] \\
&= [f_j, T_{j-1} f_j] + [f_{j-1}, T_{j-2} f_{j-1}] + \cdots + [f_2, T_1 f_2] + [f_1, 0] \\
&= [f_1, 0] + \sum_{i=1}^{j-1}[f_{i+1}, T_i f_{i+1}];
\end{aligned}$$

i.e., for all $j \geq 2$,

$$M_j = [f_1, 0] + \sum_{i=1}^{j-1}[f_{i+1}, T_i f_{i+1}].$$

□

6.4 The Daures-Ni-Zhang version of Doob's decomposition

Let X be a Banach space. We first specialize our above discussion on the associated ordering to the ordered near vector space $(\mathrm{cbf}(X), \oplus, \cdot, \subset)$.

The ordering $\sqsubseteq$ on $\mathrm{cbf}(X) \times \mathrm{cbf}(X)$ associated with $\subset$ is given by

$$A \sqsubseteq B \Leftrightarrow \exists C \in \mathrm{cbf}(X)\ (\{0\} \subset C \text{ and } A \oplus C = B).$$

Before we relate, in the definition of $A \sqsubseteq B$, the set C in the equation $A \oplus C = B$ to $B \ominus A$, we first note that it is well-known and also easy to verify (see [48]) that:

- $0 \in A \ominus B \Leftrightarrow B \subset A$, for all $A, B \in \mathcal{P}_0(X)$.
- If A is bounded, then $A \ominus A = \{0\}$.
- If $A \in \mathrm{f}(X)$, then $A \ominus B \in \mathrm{f}(X)$.
- If A is convex, so is $A \ominus B$ provided $A \ominus B \neq \emptyset$.
- If A and B are bounded, then $A \ominus B$ is also bounded provided $A \ominus B \neq \emptyset$.

6.4.1 Theorem
Let X be a Banach space.

(1) If $A, B \in \mathrm{f}(X)$, then there exists $C \in \mathrm{f}(X)$ such that $B \oplus C = A$ if and only if $B \oplus (A \ominus B) = A$.

(2) If $A, B \in \mathrm{cbf}(X)$, then there exists $C \in \mathrm{cbf}(X)$ such that $B \oplus C = A$ if and only if $B \oplus (A \ominus B) = A$. Moreover, in this case, $A \ominus B$ is the unique C satisfying $A = C \oplus B$.

PROOF.

(1) Suppose there exists a closed subset C of X such that $A = B \oplus C$. Since $C + B \subset A$, we get $C \subset \{x \in X : \{x\} + B \subset A\} = A \ominus B$. But then $C + B \subset (A \ominus B) + B$. Consequently, $A = \overline{(C + B)} \subset \overline{(B + (A \ominus B))} = B \oplus (A \ominus B)$. To see that $B \oplus (A \ominus B) \subset A$, notice that it follows readily from the definition of $\ominus$ that $B + (A \ominus B) \subset A$. Hence, $B \oplus (A \ominus B) \subset A$. Thus, $B \oplus (A \ominus B) = A$.

Conversely, let $B \oplus (A \ominus B) = A$. Since A and B are closed, it follows easily that $A \ominus B$ is closed. So, we take $C = A \ominus B$.

(2) Let $A, B \in \mathrm{cbf}(X)$. Then we have that $A \ominus B \in \mathrm{cbf}(X)$. Suppose there exists $C \in \mathrm{cbf}(X)$ such that $B \oplus C = A$. Then $B \oplus (A \ominus B) = A$, by (1). Since $(\mathrm{cbf}(X), \oplus)$ satisfies the cancellation law:

$$(\forall\, U, V \in \mathrm{cbf}(X))\,(U \oplus V = U \oplus W \Rightarrow V = W)\,,$$

we get that $C = A \ominus B$ and that C is unique.

Conversely, let $B \oplus (A \ominus B) = A$. Then $C = A \ominus B$ is the unique member of $\mathrm{cbf}(X)$, which satisfies $B \oplus C = A$.

□

6.4.2 Corollary
Let X be a Banach space and $A, B \in \mathrm{cbf}(X)$. Then the following statements are equivalent:

(1) There exists $C \in \mathrm{cbf}(X)$ such that $B \oplus C = A$.

(2) $B \oplus (A \ominus B) = A$.

(3) $s(\,\cdot\,, A) - s(\,\cdot\,, B)$ is a convex function on X'.

PROOF.

(1)⇔(2) This equivalence was proved in Theorem 6.4.1.

(2)⇔(3) A proof for this equivalence may be found in [48, Lemma 4.7.6].

□

6.4.3 Corollary
Let X be a Banach space. Then, for all $A, B \in \mathrm{cbf}(X)$, the following statements are equivalent:

(1) $B \sqsubseteq A$.

(2) $\{0\} \subset A \ominus B$ and $B \oplus (A \ominus B) = A$.

(3) $B \subset A$ and $s(\,\cdot\,, A) - s(\,\cdot\,, B)$ is a convex function on X'.

PROOF.

It is known that

$$\{0\} \subset A \ominus B \Leftrightarrow B \subset A,$$

(see [48, p.159]). The rest follows from Theorem 6.4.1.

□

We now use Theorem 6.3.5 to obtain the following.

6.4.4 Theorem
Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. Let (F_i, Σ_i) be a set-valued submartingale in $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$. If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$,

(i) $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega)) - s(\,\cdot\,, F_{i-1}(\omega))$, and

(ii) $s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega))$,

are convex functions on X', then (F_i, Σ_i) has a decomposition

$$F_i(\omega) = M_i(\omega) \oplus A_i(\omega), \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) is a set-valued martingale and (A_i) is a set-valued predictable increasing sequence such that, for all $\omega \notin B$:

(a) $A_1(\omega) = 0$,

(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$, for all $j \geq 2$,

(c) $M_1(\omega) = F_1(\omega)$, and

(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right)} + F_1(\omega)$, for all $j \geq 2$.

PROOF.

We want to apply Theorem 6.3.5 to the ordered near vector space $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$. It was noted earlier that $(\mathcal{E}[\,\cdot\,|\Sigma_i])$ is a commuting sequence of increasing $\mathbb{R}_+$-linear projections on $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ such that $\mathcal{R}(\mathcal{E}[\,\cdot\,|\Sigma_i]) = \mathcal{L}^1[\Omega, \Sigma_i, \mu, \mathrm{cbf}(E)]$, for all $i \in \mathbb{N}$. We first verify that (F_i, Σ_i) is a set-valued $\sqsubseteq$-submartingale.

As (F_i, Σ_i) is a set-valued submartingale, it follows from $F_i(\omega) \subset \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ a.e., for all $i \in \mathbb{N}$, that

$$\{0\} \subset \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega) \text{ a.e., for all } i \in \mathbb{N}.$$

Also, $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))$, for all $\omega \notin B$ and all $i \in \mathbb{N}$ means that

$$F_i(\omega) \oplus \Big(\mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\Big) = \mathcal{E}[F_{i+1}|\Sigma_i](\omega),$$

for all $\omega \notin B$ and all $n \in \mathbb{N}$; consequently,

$$F_i(\omega) \sqsubseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \text{ a.e., for all } i \in \mathbb{N}.$$

Hence, (F_i, Σ_i) is a set-valued $\sqsubseteq$-submartingale.

Let $A_1(\omega) = 0$, for all $\omega \notin B$, and for all $j \geq 2$,

$$A_j = \sum_{i=1}^{j-1} \Big[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \{0\}\Big] = \Big[\overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big), \{0\}}\Big],$$

$M_1 = [F_1, 0]$, and for all $j \geq 2$,

$$M_j = [F_1, 0] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big].$$

Then it follows from Theorem 6.3.5, that in the Rådström's completion $R(\mathcal{L}^1[\Omega, \text{cbf}(X)])$ of $\mathcal{L}^1[\Omega, \text{cbf}(X)]$, we have that the submartingale $([F_i, \{0\}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i(\omega), \{0\}(\omega)] = M_i(\omega) + A_i(\omega), \text{ for all } \omega \notin B \text{ and } i \in \mathbb{N},$$

where with $(M_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}^1[\Omega, \text{cbf}(X)])$, $(A_j) \subset \mathcal{L}^1[\Omega, \text{cbf}(X)]$ a positive and increasing sequence and $A_{j+1} \in \mathcal{L}^1[\Omega, \Sigma_j, \mu, \text{cbf}(E)]$, for all $j \in \mathbb{N}$.

From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega))$ is convex, for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \mathcal{E}[F_i|\Sigma_{i-1}](\omega) \oplus \Big(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\Big)$. Hence, in $R(\mathcal{L}^1[\Omega, \text{cbf}(X)])$, it follows that

$$\Big[F_i, \mathcal{E}[F_i|\Sigma_{i-1}]\Big] = \Big[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \{0\}\Big].$$

But then, for all $j \geq 2$,

$$\begin{aligned} M_j &= [F_1, \{0\}] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big] \\ &= [F_1, \{0\}] + \sum_{i=1}^{j-1} \Big[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \{0\}\Big] \\ &= \Big[\overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1}, \{0\}\Big]. \end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)}, \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i] \Big) + F_1}, \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i \oplus A_i, \text{ for all } i \in \mathbb{N},$$

with the desired properties.

□

We are now in a position to prove Theorem 6.3.1, the Doob decomposition as noted by Daures, Ni and Zhang, using Theorem 6.4.4.

Proof of Theorem 6.3.1. As $\mathcal{L}^1[\Omega, \mathrm{cf}(X)]$ is a sub ordered near vector space of $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ (see [43]), we may replace $\mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ by $\mathcal{L}^1[\Omega, \mathrm{cf}(X)]$, as in Corollary 6.4.4, which completes the proof of Theorem 6.3.1. □

6.5 The Shen-Wang version of Doob's decomposition

If $\{0\} \neq E$ is a Banach lattice, then the canonical embedding $E \hookrightarrow \mathrm{cbf}(E)$, given by $x \mapsto \{x\}$, is not order preserving if $\mathrm{cbf}(E)$ is endowed with the ordering of set inclusion. We want to relate the ordering on E to an appropriate ordering on $\mathrm{cbf}(E)$. We therefore consider:

$$\begin{aligned} \mathrm{f}(E_+) : &= \{A \in \mathcal{P}_0(E_+) : A \text{ is closed and bounded}\}, \\ \mathrm{cbf}(E_+) : &= \{A \in \mathrm{f}(E_+) : A \text{ is convex}\}. \end{aligned}$$

For all $F, G \in \mathrm{cbf}(E)$, define

$$F \preceq G \Leftrightarrow \exists H \in \mathrm{cbf}(E_+)\, (F \oplus H = G).$$

Direct verification yields that

(1) if $F \in \mathrm{cbf}(E)$, then $\{0\} \preceq F$ if and only if $0 \leq f$, for all $f \in F$, and

(2) $(\mathrm{cbf}(E), \preceq)$ is a partially ordered set and $(\mathrm{cbf}(E), \oplus, \cdot, \preceq)$ is an ordered near vector space.

It is also clear that the ordering $\sqsubseteq$ associated with $\preceq$ on $\mathrm{cbf}(E)$ coincides with $\preceq$.

We extend the ordering $\preceq$ pointwise to the space. $\mathcal{L}^1[\Omega, \mathrm{cbf}(E)]$. $(\mathcal{L}^1[\Omega, \mathrm{cbf}(E)], \oplus, \cdot, \preceq)$ is an ordered near vector space.

The next result shows that conditional expectations are $\preceq$-preserving.

6.5.1 Lemma
Let E be a Banach lattice, (Ω, Σ, μ) a measure space and Σ_0 a sub σ-algebra of Σ. Then the following holds:

(E3′) If $F_1, F_2 \in \mathcal{L}^1[\Omega, \mathrm{cbf}(E)]$, then $F_1 \preceq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$.

PROOF.

Let $F_1 \preceq F_2$. Select $H \in \mathcal{L}^1[\Omega, \mathrm{cbf}(E)]$ for which $H(\omega) \in \mathrm{cbf}(E_+)$ a.e. and $F_1 \oplus H = F_2$. Then $\mathcal{E}[F_1|\Sigma_0] \oplus \mathcal{E}[H|\Sigma_0] = \mathcal{E}[F_2|\Sigma_0]$. To conclude that $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$, it suffices to show that $\{0\} \preceq \mathcal{E}[H|\Sigma_0]$.

If $h \in L^1(\Omega, \Sigma, \mu)$ such that $h(\omega) \in H(\omega)$ a.e., then, as $H(\omega) \in \mathrm{cbf}(E_+)$ a.e., it follows that $h(\omega) \geq 0$ a.e.; consequently, $\mathbb{E}[h|\Sigma_0](\omega) \geq 0$ a.e. and

$$S_H^1(\Sigma_0) = \{h \in L^1(\Omega, \Sigma_0, \mu) : 0 \leq h(\omega) \in H(\omega) \text{ a.e.}\}.$$

But then $\{0\} \preceq \overline{\{\mathbb{E}[h|\Sigma_0] : h \in S_H^1(\Sigma_0)\}}$. From the definition of $\mathcal{E}[H|\Sigma_0]$, it follows that $\{0\} \preceq \mathcal{E}[H|\Sigma_0]$, and the proof is complete.

$\square$

The following version of Doob's decomposition is similar to a result noted by Shen and Wang (see [75]). Their result differs from the one below mainly in the assumption (1) in Theorem 6.5.2 below. This assumption yields an explicit description of the martingale involved in the decomposition, which they do not obtain in their result.

6.5.2 Theorem
Let E be a Banach lattice, (F_i, Σ_i) be a $\preceq$-submartingale in $\mathcal{L}^1[\Omega, \mathrm{cbf}(E)]$ (alternatively, $\mathcal{L}^1[\Omega, \mathrm{cf}(E)]$). If there exists $B \in \Sigma$ with $\mu(B) = 0$, and for each $i \geq 2$,

$$s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]), \text{ for all } \omega \notin B,$$

is a convex function on X', then there is a decomposition of (F_i, Σ_i) as

$$F_i(\omega) = M_i(\omega) \oplus A_i(\omega), \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) is a set-valued martingale in $\mathcal{L}^1[\Omega, \mathrm{cbf}(E)]$ (alternatively, $\mathcal{L}^1[\Omega, \mathrm{cf}(E)]$) and (A_i) is a set-valued predictable $\preceq$-increasing sequence such that for all $\omega \notin B$:

(a) $A_1(\omega) = 0$,

(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_k](\omega) \ominus F_k(\omega) \right)}$, for all $j \geq 2$,

(c) $M_1(\omega) = F_1(\omega)$, and

(d) $M_j(\omega) = \overline{\Big(\sum_{i=2}^{j} F_k(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega) \Big) + F_1(\omega)}$, for all $j \geq 2$.

Moreover, the decomposition is unique.

PROOF.

The proof is very similar to that of Theorem 6.4.4, although we are considering the ordering $\preceq$ instead of $\subset$. The details follow.

From Lemma 6.5.1, we know that $\mathcal{E}[\,\cdot\,|\Sigma_i]$ is $\preceq$-preserving. Hence, in $R(\mathcal{L}^1[\Omega, \mathrm{cbf}(X)])$ we have that $([F_i, \{0\}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i, \{0\}] = M_i + A_i, \text{ for all} i \in \mathbb{N},$$

where $A_1 = 0$, and for all $j \geq 2$,

$$A_j = \sum_{i=1}^{j-1} \Big[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \{0\} \Big] = \Big[\overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i \Big), \{0\}} \Big],$$

$M_1 = [F_1, 0]$, and for all $j \geq 2$,

$$M_j = [F_1, 0] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i] \Big]$$

with $(M_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}^1[\Omega, \mathrm{cbf}(X)])$, $(A_j) \subset \mathcal{L}^1[\Omega, \mathrm{cbf}(X)]$ a positive and increasing sequence and $A_{j+1} \in \mathcal{L}^1[\Omega, \Sigma_j, \mu, \mathrm{cbf}(E)]$, for all $j \in \mathbb{N}$.

From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega))$, for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \Big(\mathcal{E}[F_i|\Sigma_{i-1}](\omega) \Big) \oplus \Big(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega) \Big)$. Hence, in $R(\mathcal{L}^1[\Omega, \mathrm{cbf}(X)])$, it follows that

$$\Big[F_i, \mathcal{E}[F_i|\Sigma_{i-1}] \Big] = \Big[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \{0\} \Big].$$

But then, for all $j \geq 2$,

$$\begin{aligned} M_j &= [F_1, \{0\}] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i] \Big] \\ &= [F_1, \{0\}] + \sum_{i=1}^{j-1} \Big[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \{0\} \Big] \\ &= \Big[\overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i] \Big) + F_1}, \{0\} \Big] \end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)}, \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1}, \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i \oplus A_i, \text{ for all } i \in \mathbb{N},$$

with the desired properties.

□

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