Logical Theories of Trees

by

Ruaan Kellerman

under the supervision of

Prof. V. F. Goranko

Prof. M. Möller

School of Mathematics
University of the Witwatersrand
Johannesburg
South Africa

A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy.

2010

Summary

Trees occur naturally in many mathematical settings as important partial orders yet no systematic study of their first-order theories exists. We investigate some of the first-order theories of trees. The two problems which motivate the thesis are (i) the first-order definability of sets within a given tree, and (ii) the first-order definability and axiomatisability of particular classes of trees.

Of particular interest is the correspondence between the first-order theory of a tree and the first-order theory of the class of linear orders which comprise the paths in the tree. For every class $\mathcal{C}$ of linear orders we introduce eight classes of trees collectively called the $\mathcal{C}$-classes of trees, the paths of which are related in various natural ways to the linear orders in $\mathcal{C}$. We completely establish both the set-theoretical relationships between these eight classes of trees as well as the relationships between the first-order theories of these eight classes of trees. We also investigate some of the properties of these first-order theories.

A special case is where the class $\mathcal{C}$ consists of a single ordinal α with $\alpha < \omega^\omega$ since such ordinals are finitely axiomatisable. We obtain the first-order theory of the class of trees where every path is isomorphic to the ordinal α for α any finite ordinal and also for the case where $\alpha = \omega$. The remaining cases are more difficult because of the existence of undefinable paths in the tree. For the cases where $\omega < \alpha < \omega^\omega$ we introduce the notion of an almost α-tree and show that every almost α-tree can be elementarily extended in a natural way to a tree of which every path, definable or undefinable, satisfies the first-order theory of α. We also examine what this elementary extension of the almost α-tree looks like for the case where $\alpha = \omega + \mathbf{1}$.

Throughout the thesis we also investigate various first-order properties and theories of trees and establish some results in this regard.

Acknowledgements

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the NRF.

Furthermore I would like to thank

- Prof. Valentin Goranko, for the immeasurable contribution he has made towards my education and for supervising this study.
- Prof. Manfred Möller, for standing in as supervisor during the latter part of the study.
- The University of the Witwatersrand, for creating an environment conducive to research.
- My family, for their support.

Declaration

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Ruaan Kellerman Date

Contents

Chapter 1

Introduction

Trees occur naturally in many mathematical settings as important partial orders. They are amongst the simplest relational structures which exhibit nontrivial behaviour. Independence results in set theory frequently involve constructions which make use of trees (see e.g. [4, 20]). Curiously some set-theoretical results which seemingly have nothing to do with trees can be rephrased in terms of trees, for example the Suslin conjecture which was originally formulated in terms of linear orders but which was later reformulated very elegantly in terms of trees (see e.g. [20, 29]). The theory of automata makes extensive use of trees (see e.g. [30]). Databases (see e.g. [3, 17, 28]) and formal grammars (see e.g. [1, 2]) are commonly modeled as trees. Trees can be seen as models of theories of temporal logics with the paths in the tree representing different histories (see e.g. [9, 11, 13, 15, 16, 26]).

The logical theories of linear orders are systematically studied in [24]. A similar systematic study of the logical theories of trees has not been done. It is known by Rabin's Tree Theorem that the monadic second-order theory of rooted binary trees with infinite paths is decidable and this result can be extended to other classes of trees (see e.g. [14, 23, 31]). In this thesis we investigate certain first-order theories of trees. We adopt a broad set-theoretical definition of trees. We do not require that trees be finite, rooted, discrete, finitely branching or well-founded.

The two questions which underly the thesis are (i) the first-order definability of sets within a given tree, and (ii) the first-order definability and axiomatisability of particular classes of trees. In this regard some of the important known results include the following:

- The first-order theory of well-founded trees is determined in [5].

- The first-order theory of finite ordered trees is determined in [1]. (An ordered tree is a tree with an order relation imposed on the set of immediate successors of every node.)

- The first-order theory of finitely branching trees is studied in [10, 27]. It is shown in [27] that every tree T has a weakly boundedly branching subtree S with $S \preceq_n T$. Hence the class of trees is complete with respect to the class of weakly boundedly branching trees, i.e. if σ is a first-order sentence satisfiable in any tree then σ is satisfiable in some weakly boundedly branching tree.

Of particular interest is the correspondence between the first-order theory of a tree and the first-order theory of the class of linear orders which comprise the paths in the tree. Hence we introduce, for an arbitrary class $\mathcal{C}$ of linear orders, eight classes of trees, the paths of which are related to the linear orders in $\mathcal{C}$ in various natural ways. These classes of trees are collectively called the $\mathcal{C}$-classes of trees. We completely establish the set-theoretical relationships between these classes and also the relationships between their first-order theories. We present these results in [12]. The idea of classifying trees in terms of how their paths are related to the linear orders in a class of linear orders $\mathcal{C}$ is also considered in [13].

The general problem of studying the $\mathcal{C}$-classes of trees based on knowledge of the first-order theory of the class $\mathcal{C}$ is difficult because the class $\mathcal{C}$ may be an entirely arbitrary class of linear orders. Moreover we treat trees as consisting of a set of nodes with an order relation imposed on those nodes, so we are not able to quantify over undefinable sets of nodes, in particular over undefinable paths, within the tree. Hence even when the class $\mathcal{C}$ is a simple one, for example consisting of a single finitely axiomatisable linear order, it may be difficult to use the first-order theory of $\mathcal{C}$ to establish results about the first-order theory of a tree of which every path is drawn from $\mathcal{C}$.

Still we investigate the problem of axiomatising the first-order theory of the class of trees of which every path is isomorphic with the ordinal α for $\alpha < \omega^\omega$. We completely solve this problem for the cases where α is a finite ordinal and for $\alpha = \omega$. The remaining cases are more difficult because of the existence of undefinable paths in the tree. For the cases where $\omega < \alpha < \omega^\omega$ we introduce the notion of an almost α-tree. We then show that every almost α-tree can be elementarily extended in a natural way to a tree of which every path, definable or undefinable, satisfies the first-order theory of α. These results relating to the axiomatisation of the first-order theory of the class of trees of which every path is isomorphic with the ordinal α will still be submitted for publication.

Along the way we also investigate the first-order properties and theories of trees and establish various results in this regard.

This thesis is structured as follows. In **Chapter 2 (Some preliminaries)** we fix some notation and terminology used often in the text and give an overview of relativisation of formulas and characteristic formulas.

Chapter 3 (General theory of trees) introduces trees from a set-theoretical standpoint and investigates some of their basic properties and behaviour. In Section 3.4 (Condensations) we introduce condensations of trees, a generalisation of the notion of the condensation of a linear order described for example in [24]. Condensations of trees give a natural way to factorise a tree into its constituent bridges.

Chapter 4 (Some important classes of trees) examines, from a set-theoretical perspective, well-founded trees in Section 4.1 (Well-founded trees) and finitely branching trees in Section 4.2 (Finitely branching trees). In Section 4.3 (Trees associated with a class of linear orders) we introduce the eight $\mathcal{C}$-classes of trees determined by a class of linear orders $\mathcal{C}$ in terms of how the paths in those trees are related to the linear orders in $\mathcal{C}$. We completely determine the set-theoretical relationships between these eight classes.

We then shift our focus to first-order definability within trees and the first-order theories of trees. **Chapter 5 (First-order definability and trees)** starts with some brief comments on Ehrenfeucht-Fraïssé games and gives a first-order definition of the class of trees. In Section 5.3 (Nodes) we establish that the expressive power of nodes improves with the height of those nodes and we define neighbourhoods of nodes which allows us to capture properties of trees which are locally true. In Section 5.4 (Paths) we introduce path defining formulas. Singular and emergent paths are also introduced and the notion of an emergent path is further refined into that of internal and peripheral paths. The main result in this section is that within certain trees every parametrically definable path can be defined using a single node lying high up on that path as parameter. The chapter ends with a look at the definability of subtrees and condensations.

Chapter 6 (First-order theories of trees) looks at the first-order theories of certain important classes of trees. In Section 6.1 (Well-founded trees) we describe the construction used in [5] to prove that every definably well-founded tree has a well-founded n-equivalent. In Section 6.2 (Finitely branching trees) we show how it is possible in any tree to remove all but finitely many components extending a stem so that the tree obtained is n-equivalent to the original tree. This result is a special case of the result in [27] that every weakly boundedly branching tree T has a subtree S for which $S \preceq_n T$. In Section 6.3 (Finite trees) we axiomatise the first-order theory of the class of finite trees by adapting the method used in [1] to axiomatise the first-order theory of the class of finite ordered trees. In Section 6.4 (Condensations) we show how the first-order theory of a tree may be determined using the first-order theory of its condensation and the first-order theories of the maximal bridges in the tree. Finally Section 6.5 (The $\mathcal{C}$-classes of trees) completely establishes the relationships between the first-order theories of the various $\mathcal{C}$-classes of trees. We also investigate the general problem of axiomatising the various $\mathcal{C}$-classes of trees using the first-order theory of the class $\mathcal{C}$.

In **Chapter 7 (Axiomatisations of ordinal trees)** we study the problem of axiomatising the first-order theory of the class of trees of which every path is isomorphic with the ordinal α with $\alpha < \omega^\omega$. We begin in Section 7.1 (The first-order theory of the ordinal α with $\alpha < \omega^\omega$) by describing the first-order theory of the ordinal α using an axiom system similar to the one in [24]. In Section 7.2 (Tails of ordinals) we establish some results on tails of ordinals which are used later. In Section 7.4 (Towards first-order theories of α-trees) we determine the first-order theories of the classes of $\mathbf{n}$-trees for every finite ordinal $\mathbf{n}$ as well as the first-order theory of the class of ω-trees. We also introduce the class of almost α-trees and show that every almost α-tree can be elementarily embedded in a pathwise uniformly α-like tree. Finally we examine what this elementary extension of the almost α-tree looks like in Section 7.5 (Almost $(\omega + \mathbf{1})$-trees and their extensions) for the case where $\alpha = \omega + \mathbf{1}$.

Chapter 2

Some preliminaries

We begin by fixing some notation and terminology used frequently in the text. In Section 2.3 (Relativising a formula) we describe how to relativise a formula to a definable subtructure of a structure and in Section 2.4 (Characteristic formulas) we describe characteristic formulas which will allow us to formalise the structure of trees up to n-equivalence.

For further information on linear orders, the reader is referred to the text [24]. For further information on logic or model theory, the reader is referred to [6, 7, 18, 19, 22].

2.1 Notation

Let $\mathfrak{A}$ be a structure and let $\varphi(x_1, \ldots, x_n)$ be a first-order formula. The domain of $\mathfrak{A}$ is denoted as $|\mathfrak{A}|$ or simply as A. Let $c_1, \ldots, c_n \in |\mathfrak{A}|$ and put $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{c} = (c_1, \ldots, c_n)$. Then $\varphi(x_1, \ldots, x_n)$ is also written as $\varphi(\bar{x})$. When evalutating the truth of φ in $\mathfrak{A}$ when the elements c_i are substituted for x_i for every i ($1 \leqslant i \leqslant n$), we also denote the expression $\mathfrak{A} \models \varphi(c_1/x_1, \ldots, c_n/x_n)$ as $\mathfrak{A} \models \varphi(\bar{c}/\bar{x})$. When enriching the signature of $\mathfrak{A}$ with $c_1, \ldots, c_n$ as parameters, we also denote $(\mathfrak{A}; c_1, \ldots, c_n)$ as $(\mathfrak{A}; \bar{c})$ and $(\mathfrak{A}; \bar{c}) \models \varphi(c_1, \ldots, c_n)$ as $(\mathfrak{A}; \bar{c}) \models \varphi(\bar{c})$.

For $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{b} = (b_1, \ldots, b_n)$, the notation $\bar{a}c$ indicates the $(k+1)$-tuple $(a_1, \ldots, a_k, c)$ and $\bar{a}\bar{b}$ indicates the $(k+n)$-tuple $(a_1, \ldots, a_k, b_1, \ldots, b_n)$.

The quantifier rank (the greatest number of nested quantifiers) of a first-order formula φ is denoted as $\mathrm{qr}(\varphi)$. Elementary equivalence of structures $\mathfrak{A}$

and $\mathfrak{B}$ is denoted as $\mathfrak{A} \equiv \mathfrak{B}$, and n-equivalence (equivalence with respect to all sentences of quantifier rank at most n) of $\mathfrak{A}$ and $\mathfrak{B}$ is denoted as $\mathfrak{A} \equiv_n \mathfrak{B}$. The notation $\mathfrak{A} \preceq \mathfrak{B}$ indicates that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ while $\mathfrak{A} \preceq_n \mathfrak{B}$ indicates that

(i) $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, and

(ii) for every formula $\varphi(x, \bar{y})$ where $\bar{y}$ is a k-tuple of variables and with $\mathrm{qr}(\varphi) + k + 1 \leqslant n$, if

$$\mathfrak{B} \models \exists x\, \varphi(x, \bar{a}/\bar{y})$$

for some k-tuple of elements $\bar{a}$ from $\mathfrak{A}$, then there exists $b \in |\mathfrak{A}|$ such that

$$\mathfrak{B} \models \varphi(b/x, \bar{a}/\bar{y}).$$

In particular, if $\mathfrak{A} \preceq_n \mathfrak{B}$ then $\mathfrak{A} \equiv_n \mathfrak{B}$.

For Γ a finite theory, define $\bigwedge \Gamma := \bigwedge\{\gamma : \gamma \in \Gamma\}$. For Γ any theory, $\mathrm{MOD}(\Gamma)$ denotes the class of models of Γ. For $\mathcal{C}$ any class of structures, $\mathrm{TH}(\mathcal{C})$ denotes the first-order theory of $\mathcal{C}$.

Let $\mathcal{A}$ be a class of structures and let $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathrm{TH}(\mathcal{A})$ is called **complete** with respect to $\mathcal{B}$ when, for every sentence σ with $\mathfrak{A} \models \sigma$ for some $\mathfrak{A} \in \mathcal{A}$, there exists $\mathfrak{B} \in \mathcal{B}$ with $\mathfrak{B} \models \sigma$.

For κ a cardinal, a theory Γ is called **κ-categorical** when Γ has precisely one model of cardinality κ, up to isomorphism. A structure $\mathfrak{A}$ is called κ-categorical when $\mathrm{TH}(\mathfrak{A})$ is κ-categorical.

The order type of a finite linear order consisting of n elements is denoted as $\mathbf{n}$. The order types of the linear orders $(\mathbb{N}; <)$, $(\mathbb{Z}; <)$, $(\mathbb{Q}; <)$ and $(\mathbb{R}; <)$ are denoted respectively as ω, ζ, η and λ. ω_1 denotes the order type of any uncountable well-order of which every proper initial segment is countable. For convenience, we will sometimes identify a linear order with its order type. For example, the linear order $(\mathbb{N}; <)$ may be written simply as ω etc. The reverse order of a linear order L is denoted as $L^\star$.

ZF denotes Zermelo-Fraenkel set theory and ZFC denotes Zermelo-Fraenkel set theory with the axiom of choice.

The notation $\mathbb{N}^+$ indicates the set of positive integers.

Define for every positive integer n the sentences

$$\begin{aligned} \lambda_n &:= \exists x_1 \ldots \exists x_n \left(\bigwedge_{i \neq j} x_i \neq x_j \right), \\ \mu_n &:= \neg \lambda_{n+1}. \end{aligned}$$

The sentence λ_n states that there are at least n elements and μ_n states that there are at most n elements. The sentence $\lambda_n \wedge \mu_n$ states that there are precisely n elements.

Using a signature containing the symbols $=$ and $<$, the expressions $x \leqslant y$, $x \nless y$ and $x < z < y$ are abbreviations respectively for $x < y \vee x = y$, $\neg (x < y)$ and $x < z \wedge z < y$. The expression $\exists!$ indicates unique existence. The formula $\exists! x\, \varphi(x)$ may be seen as short for $\exists x\, (\varphi(x) \wedge \forall y\, (\varphi(y) \rightarrow y = x))$.

2.2 Definability without and with parameters

Let $\mathfrak{A}$ be a structure. Let $c \in |\mathfrak{A}|$ be an element, let $R \subseteq |\mathfrak{A}|^n$ be a relation and let $f : |\mathfrak{A}|^n \rightarrow |\mathfrak{A}|$ be a function. Let $\bar{a}$ be a k-tuple of elements from $|\mathfrak{A}|$ and let $\bar{y} = (y_1, \ldots, y_k)$.

(i) A first-order formula $\varphi(x)$ **defines** c in $\mathfrak{A}$ when

$$(\mathfrak{A}; c) \models \forall x\, (\varphi(x) \leftrightarrow x = c)\,.$$

A formula $\varphi(x, \bar{y})$ **defines** c **with parameters** $\bar{a}$ in $\mathfrak{A}$ when $\varphi(x, \bar{a})$ defines c in the structure $(\mathfrak{A}; \bar{a})$.

(ii) A first-order formula $\varphi(x_1, \ldots, x_n)$ **defines** R in $\mathfrak{A}$ when

$$(\mathfrak{A}; R) \models \forall x_1 \ldots \forall x_n\, (\varphi(x_1, \ldots, x_n) \leftrightarrow R(x_1, \ldots, x_n))\,.$$

A formula $\varphi(x_1, \ldots, x_n, \bar{y})$ **defines** R **with parameters** $\bar{a}$ in $\mathfrak{A}$ when $\varphi(x_1, \ldots, x_n, \bar{a})$ defines R in the structure $(\mathfrak{A}; \bar{a})$.

Let $B \subseteq |\mathfrak{A}|$ and define the unary relation R on $|\mathfrak{A}|$ as $R(x)$ if and only if $x \in B$. A formula φ defines the set B (without or with parameters) when φ defines R in $\mathfrak{A}$.

(iii) A first-order formula $\varphi(x_1,\ldots,x_n,x_{n+1})$ **defines** f in $\mathfrak{A}$ when

$$(\mathfrak{A};f)\models\forall x_1\ldots\forall x_n\forall x_{n+1}\left(\varphi(x_1,\ldots,x_n,x_{n+1})\leftrightarrow f(x_1,\ldots,x_n)=x_{x+1}\right).$$

A formula $\varphi(x_1,\ldots,x_n,x_{n+1},\bar{y})$ **defines** f **with parameters** $\bar{a}$ in $\mathfrak{A}$ when $\varphi(x_1,\ldots,x_n,x_{n+1},\bar{a})$ defines f in the structure $(\mathfrak{A};\bar{a})$.

2.3 Relativising a formula

Relativisations give a neat method for imposing first-order properties on definable substructures of a structure. The following definition and results are taken from [24, pp. 259-260].

Let $\mathfrak{A}$ be any structure and let $a_1,\ldots,a_k\in|\mathfrak{A}|$. Fix $\bar{x}=(x_1,\ldots,x_n)$, $\bar{y}=(y_1,\ldots,y_k)$ and $\bar{a}=(a_1,\ldots,a_k)$.

Definition 2.1 ([24]) Let $\varphi(\bar{x})$ and $\theta(u,\bar{y})$ be any first-order formulas. The **relativisation** of φ to θ is denoted as φ^θ (where $\varphi^\theta=\varphi^\theta(\bar{x},\bar{y})$) and is defined as follows:

(i) if φ is atomic then $\varphi^\theta:=\varphi$;

(ii) if $\varphi=\neg\psi$ then $\varphi^\theta:=\neg\left(\psi^\theta\right)$;

(iii) if $\varphi=\psi_1\star\psi_2$ then $\varphi^\theta:=\psi_1^\theta\star\psi_2^\theta$, where $\star$ is any of $\vee$, $\wedge$, $\rightarrow$ or $\leftrightarrow$;

(iv) if $\varphi=\exists x\psi$ then $\varphi^\theta:=\exists x\left(\theta(x,\bar{y})\wedge\psi^\theta\right)$;

(v) if $\varphi=\forall x\psi$ then $\varphi^\theta:=\forall x\left(\theta(x,\bar{y})\rightarrow\psi^\theta\right)$.

Note that if φ is quantifier-free then φ^θ contains the variables $y_1,\ldots,y_k$ vacuously, while if φ contains quantifiers then the variables $y_1,\ldots,y_k$ will appear explicitly in φ^θ.

Remark 2.2 ([24]) For φ and θ any first-order formulas, it can be seen (using structural induction on formulas) that $\mathrm{qr}(\varphi^\theta)=\mathrm{qr}(\varphi)+\mathrm{qr}(\theta)$.

For $\theta(u,\bar{y})$ any first-order formula, define

$$(\mathfrak{A};\bar{a})^\theta:=\{b\in|\mathfrak{A}|:(\mathfrak{A};\bar{a})\models\theta(b/u,\bar{a})\}.$$

Proposition 2.3 ([24]) Let $\varphi(\bar{x})$ and $\theta(u, \bar{y})$ be first-order formulas with the tuples $\bar{x}$ and $\bar{y}$ disjoint. For any $b_1, \ldots, b_n \in \left|(\mathfrak{A}; \bar{a})^\theta\right|$ and with $\bar{b} = (b_1, \ldots, b_n)$,

$$\mathfrak{A} \models \varphi^\theta \left(\bar{b}/\bar{x}, \bar{a}/\bar{y}\right) \Leftrightarrow (\mathfrak{A}; \bar{a})^\theta \models \varphi\left(\bar{b}/\bar{x}\right).$$

Proof By structural induction on formulas. QED

Corollary 2.4 ([24]) Let σ be a first-order sentence and let $\theta(u, \bar{y})$ be a first-order formula. Then

$$\mathfrak{A} \models \sigma^\theta \left(\bar{a}/\bar{y}\right) \Leftrightarrow (\mathfrak{A}; \bar{a})^\theta \models \sigma.$$

Corollary 2.5 ([24]) Let σ be a first-order sentence and let $\theta(u)$ be a first-order formula. Then

$$\mathfrak{A} \models \sigma^\theta \Leftrightarrow \mathfrak{A}^\theta \models \sigma.$$

Example 2.6 ([24]) Consider the formula

$$\theta(u) := \forall v \left(v < u \rightarrow \exists w \left(v < w < u\right)\right).$$

$\theta(u)$ states that u has no immediate predecessor. In the context of well-orders, this means that u is a limit point.

Next consider the sentence

$$\sigma := \forall x \exists y \left(x < y\right).$$

Then

$$\sigma^\theta = \forall x \Big(\forall v \left(v < x \rightarrow \exists w \left(v < w < x\right)\right) \rightarrow \exists y \big(\forall v \left(v < y \rightarrow \exists w \left(v < w < y\right)\right) \wedge x < y\big)\Big).$$

By Corollary 2.5 the sentence σ^θ holds in a well-order $\mathfrak{A} = (A; <)$ if and only if A contains no greatest limit point.

We will make use of the following abbreviations:

θ	φ^θ	$\mathfrak{A}^\theta$
$y_1 < u < y_2$	$\varphi^{(y_1,y_2)}$	$\mathfrak{A}^{(y_1,y_2)}$
$y_1 \leqslant u < y_2$	$\varphi^{[y_1,y_2)}$	$\mathfrak{A}^{[y_1,y_2)}$
$y_1 < u \leqslant y_2$	$\varphi^{(y_1,y_2]}$	$\mathfrak{A}^{(y_1,y_2]}$
$y_1 \leqslant u \leqslant y_2$	$\varphi^{[y_1,y_2]}$	$\mathfrak{A}^{[y_1,y_2]}$

θ	φ^θ	$\mathfrak{A}^\theta$
$u < y_1$	$\varphi^{<y_1}$	$\mathfrak{A}^{<y_1}$
$u \leqslant y_1$	$\varphi^{\leqslant y_1}$	$\mathfrak{A}^{\leqslant y_1}$
$y_1 < u$	$\varphi^{>y_1}$	$\mathfrak{A}^{>y_1}$
$y_1 \leqslant u$	$\varphi^{\geqslant y_1}$	$\mathfrak{A}^{\geqslant y_1}$

2.4 Characteristic formulas

Characteristic formulas give a syntactic formalisation of the Ehrenfeucht-Fraïssé game played on a pair of structures. The following definition and results are taken from [5]. An excellent account of characteristic formulas may also be found in [7].

Fix structures $\mathfrak{A}$ and $\mathfrak{B}$. Let $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{b} = (b_1, \ldots, b_k)$, where $a_1, \ldots, a_k \in |\mathfrak{A}|$ and $b_1, \ldots, b_k \in |\mathfrak{B}|$. Put $\bar{x} = (x_1, \ldots, x_k)$.

Definition 2.7 ([5]) For $n \in \mathbb{N}$ we define the formula $[\![(\mathfrak{A};\bar{a})]\!]^n$ (where $[\![(\mathfrak{A};\bar{a})]\!]^n = [\![(\mathfrak{A};\bar{a})]\!]^n(\bar{x})$) inductively as follows:

(i) $[\![(\mathfrak{A};\bar{a})]\!]^0 := \bigwedge \big\{\varphi(\bar{x}) : \varphi \text{ an atomic or negated atomic formula with } \mathfrak{A} \models \varphi(\bar{a}/\bar{x})\big\};$

(ii) $[\![(\mathfrak{A};\bar{a})]\!]^{m+1} := \bigwedge_{a_{k+1}\in|\mathfrak{A}|} \exists x_{k+1} [\![(\mathfrak{A};\bar{a}a_{k+1})]\!]^m \wedge \forall x_{k+1} \bigvee_{a_{k+1}\in|\mathfrak{A}|} [\![(\mathfrak{A};\bar{a}a_{k+1})]\!]^m.$

The formula $[\![(\mathfrak{A};\bar{a})]\!]^n$ is known as the ***n*-characteristic** of $\bar{a}$ in $\mathfrak{A}$.

Lemma 2.8 ([5]) For $n \in \mathbb{N}$ the following hold:

(i) $\mathfrak{A} \models [\![(\mathfrak{A};\bar{a})]\!]^n\, (\bar{a}/\bar{x})$;

(ii) the formula $[\![(\mathfrak{A};\bar{a})]\!]^n$ has quantifier rank n.

Proof By induction on n. QED

Theorem 2.9 ([5]) For $n \in \mathbb{N}$ the following conditions are equivalent:

(i) $(\mathfrak{A};\bar{a}) \equiv_n \big(\mathfrak{B};\bar{b}\big)$;

(ii) $\mathfrak{B} \models [\![(\mathfrak{A};\bar{a})]\!]^n\, \big(\bar{b}/\bar{x}\big)$;

(iii) the formulas $[\![(\mathfrak{A};\bar{a})]\!]^n$ and $[\![\big(\mathfrak{B};\bar{b}\big)]\!]^n$ are equivalent.

Proof See [5, Theorem 1.6.3]. QED

Corollary 2.10 For $n \in \mathbb{N}$ the following conditions are equivalent:

(i) $\mathfrak{A} \equiv_n \mathfrak{B}$;

(ii) $\mathfrak{B} \models [\![\mathfrak{A}]\!]^n$;

(iii) the sentences $[\![\mathfrak{A}]\!]^n$ and $[\![\mathfrak{B}]\!]^n$ are equivalent.

Proof Immediate from Theorem 2.9. QED

Hence the n-characteristics of empty tuples are canonical objects associated with classes of structures which are n-equivalent.

When working with a finite signature, there will be only finitely many n-characteristics of k-tuples.

Theorem 2.11 ([5]) Let $\{\mathfrak{A}_i\}_{i \in I}$ be a class of structures over the same finite signature. Let $k, n \in \mathbb{N}$ and for every $i \in I$, let $\bar{a}_i$ be a k-tuple of elements from $|\mathfrak{A}_i|$. There are only finitely many non-equivalent formulas in the set $\{[\![(\mathfrak{A}_i; \bar{a}_i)]\!]^n : i \in I\}$.

Proof Use induction on n. QED

The number of n-characteristics of empty tuples over a given finite signature is called the ***n*-characteristic index** of that signature.

Example 2.12 Consider the linear order $\omega := (\mathbb{N}; <)$. Then

$$\begin{aligned}[\![(\omega; (0,0))]\!]^0 \quad = \quad & x_1 = x_1 \wedge x_2 = x_2 \wedge x_1 = x_2 \wedge x_2 = x_1 \wedge \\ & \neg(x_1 < x_1) \wedge \neg(x_2 < x_2) \wedge \neg(x_1 < x_2) \wedge \neg(x_2 < x_1)\end{aligned}$$

and $[\![(\omega; (a_1, a_2))]\!]^0$ is equivalent to $[\![(\omega; (0,0))]\!]^0$ when $a_1 = a_2$. Furthermore

$$\begin{aligned}[\![(\omega; (0,1))]\!]^0 \quad = \quad & x_1 = x_1 \wedge x_2 = x_2 \wedge \neg(x_1 = x_2) \wedge \neg(x_2 = x_1) \wedge \\ & \neg(x_1 < x_1) \wedge \neg(x_2 < x_2) \wedge x_1 < x_2 \wedge \neg(x_2 < x_1)\end{aligned}$$

and $[\![(\omega; (a_1, a_2))]\!]^0$ is equivalent to $[\![(\omega; (0,1))]\!]^0$ when $a_1 < a_2$. Finally

$$\begin{aligned}[\![(\omega; (1,0))]\!]^0 \quad = \quad & x_1 = x_1 \wedge x_2 = x_2 \wedge \neg(x_1 = x_2) \wedge \neg(x_2 = x_1) \wedge \\ & \neg(x_1 < x_1) \wedge \neg(x_2 < x_2) \wedge \neg(x_1 < x_2) \wedge x_2 < x_1\end{aligned}$$

and $[\![(\omega; (a_1, a_2))]\!]^0$ is equivalent to $[\![(\omega; (1,0))]\!]^0$ when $a_2 < a_1$.

Next

$$\llbracket(\omega;0)\rrbracket^1 = \exists x_2 \llbracket(\omega;(0,0))\rrbracket^0 \wedge \exists x_2 \llbracket(\omega;(0,1))\rrbracket^0 \wedge \\ \forall x_2 \left(\llbracket(\omega;(0,0))\rrbracket^0 \vee \llbracket(\omega;(0,1))\rrbracket^0\right)$$

and

$$\llbracket(\omega;1)\rrbracket^1 = \exists x_2 \llbracket(\omega;(1,0))\rrbracket^0 \wedge \exists x_2 \llbracket(\omega;(0,0))\rrbracket^0 \wedge \exists x_2 \llbracket(\omega;(0,1))\rrbracket^0 \wedge \\ \forall x_2 \left(\llbracket(\omega;(1,0))\rrbracket^0 \vee \llbracket(\omega;(0,0))\rrbracket^0 \vee \llbracket(\omega;(0,1))\rrbracket^0\right).$$

The formula $\llbracket(\omega;a_1)\rrbracket^1$ is equivalent to $\llbracket(\omega;1)\rrbracket^1$ when $a_1 > 1$.

Finally we get

$$\llbracket\omega\rrbracket^2 = \exists x_1 \llbracket(\omega;0)\rrbracket^1 \wedge \exists x_1 \llbracket(\omega;1)\rrbracket^1 \wedge \forall x_1 \left(\llbracket(\omega;0)\rrbracket^1 \vee \llbracket(\omega;1)\rrbracket^1\right).$$

By Corollary 2.10, for any structure $(L;<_L)$, $(L;<_L) \equiv_2 \omega$ if and only if $(L;<_L) \models \llbracket\omega\rrbracket^2$.

Chapter 3

General theory of trees

We begin our study of trees by defining some basic notions and we investigate some of the structural properties of trees from a set-theoretical standpoint. The reader may also consult [20, 21, 25, 32] in this regard. In Section 3.4 (Condensations) we introduce condensations of trees obtained by collapsing every maximal bridge in a tree to a single point and we use this notion to show how a tree can be factorised into the product of a simpler tree with a class of linear orders which are associated with the nodes in that tree.

3.1 Definition of a tree

Let A be a non-empty set and let $<$ be a binary relation on A. The structure $(A; <)$ is called an **ordered set**, and when there is no ambiguity, the ordered set $(A; <)$ will sometimes be written simply as A. If $a < b$ we say that a is a **predecessor** of b and that b is a **successor** to a. The qualifier **immediate** indicates that there is no x in A for which $a < x < b$. Thus b is an immediate successor to a when $A \models s(a/x, b/y)$, where

$$s(x, y) := x < y \wedge \neg\exists z\,(x < z < y)\,.$$

If $B, C \subseteq A$ with $x < y$ for all $x \in B$ and $y \in C$ then we write $B < C$, while the notation $a < B$ indicates that $a < y$ for all $y \in B$, etc.

The relation $<$ is **irreflexive** when $x \not< x$ for all $x \in A$, and **transitive** if, for all $x, y, z \in A$, whenever $x < y$ and $y < z$ then $x < z$. When $<$ is both irreflexive and transitive then $(A; <)$ is called a **partial order**.[1]

If $a, b \in A$ are such that $a < b$ or $a = b$ or $b < a$, we say that a and b are **comparable**, and this will be denoted as $a \smile b$. Otherwise, a and b are said to be **incomparable**, and this will be denoted as $a \not\smile b$. The property of two nodes being comparable can be formalised using the first-order formula

$$x \smile y \;:=\; x < y \vee x = y \vee y < x.$$

An ordered set of which all elements are pairwise incomparable is called an **antichain**. From Zorn's Lemma it follows that every antichain can be extended to a maximal antichain.

The ordered set A is **total** when every two elements in A are comparable, and **subtotal** when the set $\{y \in A : y < x\}$ is total for every $x \in A$. A total partial order is called a **linear order**.

A linear order A is called **dense** when for every $x, y \in A$ with $x < y$ there exists $z \in A$ with $x < z < y$. A is called **complete** when every non-empty bounded subset of A has an infimum and a supremum.

A is **connected** when, for every $x, y \in A$, there exists $z \in A$ such that $z \leqslant x$ and $z \leqslant y$. If A is not connected then it is called **disconnected**. A maximal connected subset of A is called a **component** of A.

Definition 3.1 A subtotal partial order $(T; <)$ is called a **forest**. A connected forest is called a **tree**.

In what is to follow, our definitions and results will be phrased mainly within the context of trees. However, many of these definitions and results apply also to forests by observing that every tree is a forest, and every forest is a union of trees.

3.2 Nodes, paths and segments

3.2.1 Nodes and paths

The elements of a tree are called **nodes**. If a tree has a minimal node, then that node is unique and is called the **root** of the tree. A tree containing a

[1] In the literature, partial orders are often defined as being ordered sets which are reflexive, transitive and antisymmetric. In such cases, ordered sets which are irreflexive and transitive are usually called strict partial orders.

root is called a **rooted** tree. A maximal node of a tree, if it exists, is called a **leaf**.

Given a tree T with $a \in T$, define

$$\begin{aligned} a_{>} &:= \{x \in T : x < a\}, \\ a_{\geqslant} &:= \{x \in T : x \leqslant a\}, \\ a_{<} &:= \{x \in T : a < x\}, \\ a_{\leqslant} &:= \{x \in T : a \leqslant x\}. \end{aligned}$$

The sets $a_{>}$ and $a_{\geqslant}$ are linear orderings. The set $a_{\leqslant}$ is a tree, while $a_{<}$ is a forest but not necessarily a tree. The sets $a_{>}$, $a_{\geqslant}$, $a_{<}$ and $a_{\leqslant}$ will also be treated as substructures of T.

A maximal total set of nodes is called a **path**. Using Zorn's Lemma, it is easy to see that every total subset of a tree is contained in a path.

A tree T is called **downwards discrete** when every non-root node in T has an immediate predecessor. T is called **weakly upwards discrete** when every non-leaf node has an immediate successor, and **upwards discrete** when, for every path X in T, every non-leaf node in X has an immediate successor belonging to X. T is called **weakly discrete** when it is both downwards discrete as well as weakly upwards discrete, and **discrete** when it is both downwards discrete and upwards discrete.

For an upwards discrete tree T and for any node $a \in T$, we define the set $S(a)$ as consisting of all the immediate successors to a in T.

3.2.2 Segments

Let T be a tree and let $A \subseteq T$. The set A is called

(i) **downwards convex** when, for every $x \in A$, if $y < x$ then $y \in A$;

(ii) **upwards convex** when, for every $x \in A$, if $x < y$ then $y \in A$;

(iii) **convex** when, for every $x, y \in A$ with $x < y$, if $x < z < y$ then $z \in A$.

A total and convex subset of a tree is called a **segment**. A total and downwards convex subset of a tree is called a **stem**. Hence a stem is simply a downwards convex subset of a path. A subset B of a path A is called a **branch** when, if $x \in B$ and $y \in A$ with $x < y$, then $y \in B$. For every node

$a \in T$, the sets $a_>$ and $a_\geqslant$ are stems. The empty set vacuously constitutes a segment, stem and branch.

Given a tree T and nodes $a, b \in T$, the intervals (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ will be defined as usual, namely $(a, b) := \{x \in T : a < x < b\}$ etc. All these intervals are examples of segments, although not every segment has this form. For example, consider a path of the form $\zeta \cdot 3$ and the segment consisting of the second copy of ζ in that path, or a path of the form η and the segment consisting of all rational numbers x with $\sqrt{2} < x < \sqrt{3}$.

For T a tree and A a non-empty segment in T, we define the following sets:[2]

$$\begin{aligned} A_> &:= \{x \in T : x < A\}, \\ A_< &:= \{x \in T : A < x\}. \end{aligned}$$

Proposition 3.2 Let T be a tree and let A and B be segments in T such that $A \cup B$ is total and $A \cap B = \emptyset$. Let $a \in A$ and $b \in B$ with $a < b$. Then $A < B$.

Proof Put $a_1 := a$ and $b_1 := b$. Suppose to the contrary that there exists $a_2 \in A$ and $b_2 \in B$ with $b_2 < a_2$. Since $A \cup B$ is total then the nodes a_1, a_2, b_1 and b_2 can be related to each other in the following ways:

$$\begin{array}{ll} b_2 < a_2 \leqslant a_1 < b_1 & \text{giving} \quad a_1 \in B, \\ b_2 \leqslant a_1 < a_2 \leqslant b_1 & \text{giving} \quad a_1 \in B, \\ b_2 \leqslant a_1 < b_1 < a_2 & \text{giving} \quad a_1 \in B, \\ a_1 < b_2 < a_2 \leqslant b_1 & \text{giving} \quad b_2 \in A, \\ a_1 < b_2 \leqslant b_1 < a_2 & \text{giving} \quad b_2 \in A, \\ a_1 < b_1 < b_2 < a_2 & \text{giving} \quad b_2 \in A. \end{array}$$

In each case, the assumption that $A \cap B = \emptyset$ is violated. QED

Proposition 3.3 Let T be a tree and let A and B be segments in T with $A \cup B$ total and with the sets $\{x \in T : A < x < B\}$ and $\{x \in T : B < x < A\}$ empty. Then $A \cup B$ is a segment.

[2] The notations $x < A$ and $A < B$ are defined on p. 13.

Proof Let $a, b \in A \cup B$ with $a < b$ and let $a < c < b$. We must show that $c \in A \cup B$. We may assume without loss of generality that $a \in A \backslash B$ and $b \in B \backslash A$. The are two cases to consider.

First, consider the case where $A \cap B \neq \emptyset$, say $d \in A \cap B$. If $b \leqslant d$ then from A being a segment we get $b \in A$, a contradiction. Hence $d < b$. Since T is subtotal then $c \smile d$. Thus we either have $a < c \leqslant d$, in which case $c \in A$, or $d < c < b$, in which case $c \in B$.

Next consider the case where $A \cap B = \emptyset$. From Proposition 3.2 we get $A < B$. Suppose to the contrary that $c \notin A \cup B$.

Let $y \in B$. If $b \leqslant y$ then since T is transitive this gives $c < y$ and so $c \smile y$. If, on the other hand, $y < b$, then again $c \smile y$ since T is subtotal. Now if $y \leqslant c$ then B being a segment gives $c \in B$, a contradiction. Hence $c < y$ and it follows that $c < B$.

Moreover, since T is subtotal and $A < b$ then $x \smile c$ for all $x \in A$. If $c \leqslant x$ then since A is a segment we get $c \in A$, a contradiction. Hence $x < c$ and so $A < c$.

But the set $\{x \in T : A < x < B\}$ is empty. This contradiction shows that $c \in A \cup B$. QED

3.2.3 Bridges and furcations

Definition 3.4 A segment A in a tree T is called a **bridge** when, for every path X in T, either $X \cap A = \emptyset$ or $X \cap A = A$ (i.e. $A \subseteq X$). A segment that is not a bridge is called a **furcation**.

Thus a segment A is a furcation if there is a path B with $\emptyset \neq B \cap A \neq A$. The empty set trivially constitutes a bridge.

Proposition 3.5 Let T be a tree and let A and B be bridges in T with $A \cap B \neq \emptyset$. Then $A \cup B$ is a bridge.

Proof Let C be a path with $A \cap B \subseteq C$. Since A and B are bridges then $A, B \subseteq C$ hence $A \cup B$ is total. By Proposition 3.3 we get that $A \cup B$ is a segment.

Let X be any path with $X \cap (A \cup B) \neq \emptyset$, say $X \cap A \neq \emptyset$. Then $X \cap A = A \supseteq A \cap B$ so that $X \cap B \neq \emptyset$, from which $X \cap B = B$. This gives $X \cap (A \cup B) = X \cap A \cup X \cap B = A \cup B$, hence $A \cup B$ is a bridge. QED

3.3 Subtrees

Definition 3.6 Let $T = (T; <)$ be a tree and let $S \subseteq T$ be such that $(S; <\restriction_S)$ is a tree, where $<\restriction_S$ is the relation $<$ restricted to S. The structure $S = (S; <\restriction_S)$ is called a **subtree** of T.

Let $T = (T; <)$ be a tree and let $S \subseteq T$. Since irreflexivity, transitivity and subtotalness are universal properties, then $S = (S; <\restriction_S)$ satisfies these properties automatically. Hence every subset of a tree forms a forest, and the set S will form a subtree of T if and only if S is connected.

A subtree S of a tree T is called **downwards convex** (respectively **upwards convex**, **convex**) when the set S is downwards convex (respectively upwards convex, convex) in T.

Example 3.7 Let T be a tree.

(a) For every node x, the set $x_\leqslant$ forms a subtree of T, and every rooted subtree of T has the form $x_\leqslant$ for some x.

(b) Let S be an upwards convex subtree of T and define $A := \{x \in T : x < S\}$. Since T is subtotal then A is total, and from the transitivity of T it follows that A is downwards convex. Hence A is a stem in T and $S = A_>$. Every upwards convex subtree of T has the form $X_<$ for some stem X.

Proposition 3.8 Let T be a tree, let S be an upwards convex subtree of T, and let A be a path in T. Then $S \cap A$ is a path in S.

Proof Clearly $S \cap A$ is total. Let $a \in S$ with $a \smile x$ for all $x \in S \cap A$. Suppose there exists $b \in S \cap A$ with $a < b$. Since A is downwards convex then $a \in S \cap A$. Hence suppose $a > S \cap A$ and let $c \in S \cap A$. Since S is upwards convex then $\{x \in S \cap A : x \geqslant c\} = \{x \in A : x \geqslant c\}$, from which it follows that $a > \{x \in A : x \geqslant c\}$ and hence $a > A$, a contradiction with the fact that A is maximal total in T. It follows that $S \cap A$ is maximal total in S, as required. QED

Proposition 3.9 Let T be a tree, let S be an upwards convex subtree of T, and let A be a path in S. Define $B := \{x \in T : x < S\}$. Then $A \cup B$ is a path in T.

Proof Since T is subtotal then B is total, and from the fact that A is total together with the fact that $B < A$, $A \cup B$ is total.

Next let $a \in T$ with $a \smile x$ for all $x \in A \cup B$. If $a \in S$ then $a \in A$ since A is maximal total in S. Hence assume $a \notin S$. Then $a < A$ since S is upwards convex. Let $b \in S$ and $c \in A$. From the connectedness of S, there exists $d \in S$ with $d \leqslant b, c$. Hence $d \in A$ and so $a < d$. Since $d \leqslant b$ then $a < b$. It follows that $a \in B$. Hence $A \cup B$ is maximal total in T, as required. QED

3.4 Condensations

We now introduce condensations of trees, a notion roughly similar to that of condensations of linear orderings as discussed for example in [24].

3.4.1 About maximal bridges

Proposition 3.10 Let T be a tree and let A be a bridge in T. Then A is contained in a unique maximal bridge.

Proof Let $\mathcal{A}$ be a chain of maximal bridges with $A \subseteq X$ for every $X \in \mathcal{A}$ and put $A_0 := \cup\mathcal{A}$. Note that A_0 is total and convex, and hence a segment.

Let B be a path with $B \cap A_0 \neq \emptyset$. Then $B \cap A_1 \neq \emptyset$ for some $A_1 \in \mathcal{A}$ so that $B \cap A_1 = A_1 \supseteq A$. Hence $A \subseteq B$ from which $B \cap X = X$ for all $X \in \mathcal{A}$. This gives

$$B \cap A_0 = B \cap (\cup\mathcal{A}) = \cup\{B \cap X : X \in \mathcal{A}\} = \cup\{X : X \in \mathcal{A}\} = A_0.$$

It follows that A_0 is a bridge. From Zorn's Lemma, A can therefore be extended to a maximal bridge.

Next let C_1 and C_2 be maximal bridges with $A \subseteq C_1, C_2$. By Proposition 3.5 we get that $C_1 \cup C_2$ is a bridge, and by the maximality of C_1 and C_2 this means $C_1 = C_1 \cup C_2 = C_2$. QED

If A and B are maximal bridges in a tree T, then either $A \cap B = \emptyset$ or $A = B$. The set of maximal bridges in T forms a partition of the tree, and the relation of two nodes in T belonging to the same maximal bridge forms an equivalence relation on T.

Definition 3.11 For $a \in T$ the maximal bridge in T containing a will be denoted as $[a]$.

As with other sets of nodes, the notation $[a] < [b]$ will indicate that $x < y$ for all $x \in [a]$ and $y \in [b]$ and $[a] \smile [b]$ will indicate that $[a] < [b]$ or $[a] = [b]$ or $[b] < [a]$ etc.

Proposition 3.12 Let T be a tree and let $a, b \in T$.

(i) If $a < b$ and $[a] \neq [b]$ then $[a] < [b]$.

(ii) If $a \not\smile b$ then $x \not\smile y$ for all $x \in [a]$ and $y \in [b]$.

Proof (i) Let A be a path in T with $a, b \in A$. Since $[a]$ and $[b]$ are bridges then $[a], [b] \subseteq A$ so that $[a] \cup [b]$ is total. Since $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$. By Proposition 3.2 this gives $[a] < [b]$.

(ii) Follows from (i). QED

Corollary 3.13 Let T be a tree and let $a, b \in T$. Then $a \smile b$ if and only if $[a] \smile [b]$.

Proof From Proposition 3.12. QED

Proposition 3.14 Let T be a tree with $a, b \in T$. The following conditions are equivalent:

(i) there exists a bridge B such that $a, b \in B$;

(ii) $[a] = [b]$;

(iii) for every path X in T, $a \in X$ if and only if $b \in X$;

(iv) for every node $x \in T$, $x \smile a$ if and only if $x \smile b$.

Proof (i) $\Leftrightarrow$ (ii) Immediate.

(ii) $\Rightarrow$ (iii) Suppose $[a] = [b]$. Let A be a path with $a \in A$. Then $[a] \subseteq A$ and so $b \in [b] \subseteq A$ which gives $b \in A$. It follows that for every path X, if $a \in X$ then $b \in X$. It can likewise be shown that for every path X, if $b \in X$ then $a \in X$.

(iii) $\Rightarrow$ (ii) Suppose condition (iii) holds. If $a = b$ then the result is immediate, so assume $a \neq b$ and let A be a path with $a \in A$. Then $b \in A$ and so $a \smile b$, say $a < b$. Consider the segment $[a, b]$ and let X be any path with $X \cap [a, b] \neq \emptyset$. Then $a \in X$ and so $b \in X$, from which $[a, b] \subseteq X$. It follows that $[a,b]$ is a bridge with $a,b \in [a,b]$. By Proposition 3.10, $[a,b]$ is contained in a unique maximal bridge, from which it follows that $[a]=[b]$.

(iii) $\Rightarrow$ (iv) Suppose condition (iii) holds. Let $c \in T$ with $c \smile a$ and let A be a path with $c,a \in A$. Then $b \in A$ and so $c \smile b$. Hence for every $x \in T$, if $x \smile a$ then $x \smile b$, and likewise for every $x \in T$, if $x \smile b$ then $x \smile a$.

(iv) $\Rightarrow$ (iii) Suppose condition (iv) holds. Let A be a path with $a \in A$. Since $x \smile a$ for every $x \in A$ then $x \smile b$ for every $x \in A$, from which $b \in A$. Hence for every path X, if $a \in X$ then $b \in X$, and likewise it can be shown that if $b \in X$ then $a \in X$. QED

Hence two nodes x and y belong to the same maximal bridge if and only if they satisfy the formula

$$\beta(x,y) := \forall z\,(z \smile x \leftrightarrow z \smile y)\,. \tag{3.1}$$

The formula β determines an equivalence relation.

3.4.2 Condensations of trees

Definition 3.15 Let $(T;<)$ be a tree. Define $[T] := \{[x] : x \in T\}$ and for $[a],[b] \in [T]$, define the relation $<$ on $[T]$ in the usual way, namely $[a]<[b]$ if and only if $x<y$ for all $x \in [a]$ and $y \in [b]$. The structure $([T];<)$ is called the **condensation** of the tree T.

Thus the condensation of a tree is simply the quotient structure of that tree generated by the relation of membership to the same maximal bridge.

The operator $[\cdot]$ defines a mapping

$$[\cdot] : T \to [T].$$

For $X \subseteq T$, $y \in [T]$ and $Y \subseteq [T]$, we denote

$$\begin{aligned}
[X] &:= \{[x] \in [T] : x \in X\},\\
[y]^{-1} &:= \{x \in T : [x] = y\},\\
[Y]^{-1} &:= \{y \in T : [y] \in Y\}.
\end{aligned}$$

Then $X \subseteq [[X]]^{-1}$ and $[[Y]^{-1}] = Y$ for all $X \subseteq T$ and $Y \subseteq [T]$.

Proposition 3.16 For T any tree, $[T]$ is also a tree.

Proof It is straightforward to check that $[T]$ is irreflexive, transitive, subtotal and connected. QED

Thus we will treat the elements of $[T]$ as nodes within $[T]$.

Example 3.17 Figure 3.1 shows a tree T together with its condensation $[T]$. The bridges A_1 through A_6 are linear orders which may be infinite, and are condensed respectively to the nodes $[A_1]$ through $[A_6]$ in $[T]$, so that $[T]$ is finite.

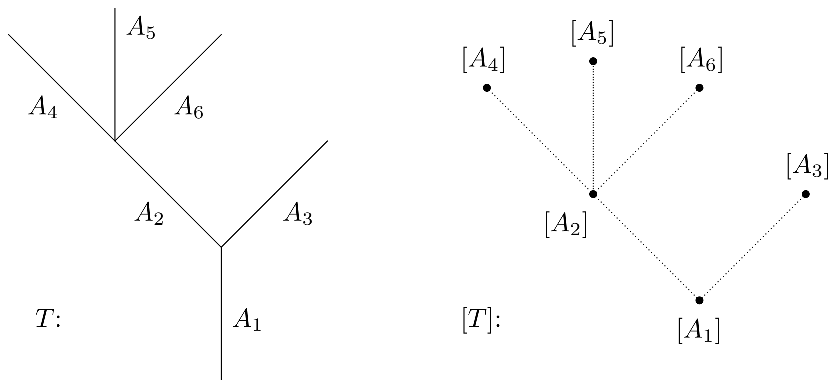

Figure 3.1: The condensation of a tree (see Example 3.17).

3.4.3 Preservation of structure

Proposition 3.18 Let T be a tree and let $A \subseteq T$.

(i) If A is an antichain then $[A]$ is an antichain.

(ii) If A is total then $[A]$ is total.

(iii) If A is maximal total in T then $[A]$ is maximal total in $[T]$.

(iv) If A is convex in T then $[A]$ is convex in $[T]$.

(v) If A is downwards convex in T then $[A]$ is downwards convex in $[T]$.

(vi) If A is upwards convex in $[T]$ then $[A]$ is upwards convex in $[T]$.

Proof (i), (ii): From Corollary 3.13.

(iii): Let A be maximal total in T. Since A is total then $[A]$ is total. Let $[b] \in [T] \backslash [A]$. Then $b \notin A$ so $b \not\sim c$ for some $c \in A$. This gives $[b] \not\sim [c]$ and since $[c] \in [A]$ then it follows that $[A]$ is maximal total in $[T]$.

(iv): Let $[a], [b] \in [A]$ and let $x \in [T]$ with $[a] < x < [b]$. Then $a < [x]^{-1} < b$ and $[x]^{-1} \subseteq A$. This gives $x = [[x]^{-1}] \in [A]$.

(v), (vi): Similar to (iv). QED

Proposition 3.19 Let T be a tree and let $B \subseteq [T]$.

(i) If B is total then $[B]^{-1}$ is total.

(ii) If B is maximal total in $[T]$ then $[B]^{-1}$ is maximal total in T.

(iii) If B is convex in $[T]$ then $[B]^{-1}$ is convex in T

(iv) If B is downwards convex in $[T]$ then $[B]^{-1}$ is downwards convex in T.

(v) If B is upwards convex in $[T]$ then $[B]^{-1}$ is upwards convex in T.

Proof The proof is similar to that of Proposition 3.18. QED

Thus paths, segments and stems are preserved between a tree and its condensation under the mapping $[\cdot]$ and its inverse. The next result shows that branches are likewise preserved.

Proposition 3.20 Let T be a tree.

(i) If A is a branch in T then $[A]$ is a branch in $[T]$.

(ii) If B is a branch in $[T]$ then $[B]^{-1}$ is a branch in T.

Proof (i) Let A be a branch in T. Let B be a path in T with $A \subseteq B$ and with the property that for every $x \in A$, if $y \in B$ with $x < y$, then $y \in A$. Then $[A] \subseteq [B]$ and from Proposition 3.18, $[B]$ is a path in $[T]$. Let $x \in [A]$ and $y \in [B]$ with $x < y$, and let $z \in [x]^{-1}$ and $w \in [y]^{-1}$ with $z \in A$ and $w \in B$. Then $z < w$ so $w \in A$. Hence $y = [w] \in [A]$, and it follows that $[A]$ is a branch in $[T]$.

(ii) The proof is similar to that of (i), but using Proposition 3.19 instead of Proposition 3.18. QED

A path A in a tree T is called **singular** (see also Definition 5.7) if there exists $a \in A$ such that $a_{\leqslant}$ is total. Otherwise A is called **emergent**.

Proposition 3.21 Let T be a tree and let A be a path in T. A is singular if and only if $[A]$ contains a greatest node.

Proof Let A be singular, let $a \in A$ be such that $a_{\leqslant}$ is total, and note that $a_{\leqslant}$ is a bridge. Suppose there exists $[b] \in [A]$ such that $[a] < [b]$. Then $a < b$ which gives $b \in a_{\leqslant} \subseteq [a]$ so $[a] = [b]$, a contradiction. Thus $[a]$ is the greatest node of $[A]$.

Conversely suppose $[A]$ contains a greatest node $[a]$. Obviously $a \in A$. Let $b, c \in a_{\leqslant}$ (i.e. $a \leqslant b, c$). Then $[a] \leqslant [b], [c]$. This gives $[b], [c] = [a]$ hence $b, c \in [a]$. Thus $b \smile c$ and so $a_{\leqslant}$ is total, as required. QED

Thus a path A is emergent if and only if $[A]$ does not contain a greatest node.

A tree T is called **well-founded** (see also Section 4.1) when every non-empty set of nodes from T contains a minimal node.

Corollary 3.22 Let T be a well-founded tree and let A be a path in T.

(i) A is singular if and only if the order type of $[A]$ is a successor ordinal.

(ii) A is emergent if and only if the order type of $[A]$ is a limit ordinal.

Proof From Proposition 3.21. QED

3.4.4 Condensed trees

Definition 3.23 A tree T is called **condensed** when $T \cong [T]$.

Lemma 3.24 Let T be a tree. Then every non-empty bridge in the tree $[T]$ consists of a single node.

Proof Let $[a], [b] \in [T]$ with $[a] \neq [b]$. Then a and b belong to different maximal bridges in T. From Proposition 3.14 we may conclude, without loss of generality, that there exists $c \in T$ such that $c \smile a$ and $c \not\smile b$. By Corollary 3.13 this gives that $[c] \smile [a]$ and $[c] \not\smile [b]$, so that $[a]$ and $[b]$ belong to different maximal bridges in $[T]$. QED

Proposition 3.25 Let T be a tree. The following conditions are equivalent:

(i) T is condensed;

(ii) $T \cong [S]$ for some tree S;

(iii) $[x] = \{x\}$ for every $x \in T$.

Proof (i) $\Rightarrow$ (ii) Let T be condensed. Then $T \cong [T]$.

(ii) $\Rightarrow$ (iii) Let $T \cong [S]$ for some tree S and let $f : T \to [S]$ be an isomorphism. Let $a, b \in T$ with $a \neq b$. Then $f(a) \neq f(b)$ so by Lemma 3.24, $f(a)$ and $f(b)$ belong to different maximal bridges in $[S]$. From Proposition 3.14 we may conclude, without loss of generality, the existence of $c \in [S]$ such that $c \smile f(a)$ and $c \not\smile f(b)$. Since f is an isomorphism then $f^{-1}(c) \smile a$ and $f^{-1}(c) \not\smile b$. Hence $[a] \neq [b]$ and the result follows.

(iii) $\Rightarrow$ (i) Assume that $[x] = \{x\}$ for every $x \in T$, and verify that the map given as $x \mapsto [x]$ defines an isomorphism from T to $[T]$. QED

3.4.5 Products of trees with linear orderings

For sets A and B, the **cartesian product** of A and B is denoted as $A \times B$.

Given partial orders $A := (A; <_A)$ and $B := (B; <_B)$, the **lexicographical product** of A and B is the partial order $A \times_{\text{lex}} B$ (with $A \times_{\text{lex}} B = (A \times B, <_{\text{lex}})$), where for $(x_1, y_1), (x_2, y_2) \in A \times B$,

$$(x_1, y_1) <_{\text{lex}} (x_2, y_2) \;:\Leftrightarrow\; x_1 <_A x_2 \text{ or both } x_1 = x_2 \text{ and } y_1 <_B y_2.$$

Definition 3.26 Let $T = (T; <_T)$ be a tree, let $\mathcal{L} = \{(L_i; <_i) : i \in I\}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. The f**-product** of T with $\mathcal{L}$ is the structure $T \times_f \mathcal{L} = (|T \times_f \mathcal{L}|\,; <)$, defined as follows:

(i) $|T \times_f \mathcal{L}| := \bigcup_{x \in T} (\{x\} \times |f(x)|)$;

(ii) for $(x_1, y_1), (x_2, y_2) \in |T \times_f \mathcal{L}|$,

$$(x_1, y_1) < (x_2, y_2) \;:\Leftrightarrow\; x_1 <_T x_2 \text{ or both } x_1 = x_2 \text{ and } y_1 <_{f(x_1)} y_2.$$

Remark 3.27 Suppose there exists $L_0 \in \mathcal{L}$ such that $f(x) = L_0$ for all $x \in T$. Then $T \times_f \mathcal{L} = T \times_{\text{lex}} L_0$.

Example 3.28 Consider the tree T as depicted in Figure 3.2 and let $\mathcal{L} := \{\omega, \omega^\star\}$, where $\omega^\star$ is the reverse order $(\mathbb{N}; >)$ of ω. Define $f : T \to \mathcal{L}$ as

$$f(x) = \begin{cases} \omega^\star & \text{when } x \in \{a_4, a_6, a_8\}, \\ \omega & \text{otherwise.} \end{cases}$$

For every i, let A_i be the linear order consisting only of the point a_i. The f-product of T with $\mathcal{L}$ is shown in Figure 3.2. The linear orders A, B, C and D are as follows:

$$\begin{array}{lcllcl} A & := & A_1 \times_{\text{lex}} \omega + A_2 \times_{\text{lex}} \omega + A_3 \times_{\text{lex}} \omega & \cong & \omega \cdot 3; \\ B & := & A_4 \times_{\text{lex}} \omega^\star + A_5 \times_{\text{lex}} \omega & \cong & \omega^\star + \omega = \zeta; \\ C & := & A_6 \times_{\text{lex}} \omega^\star + A_7 \times_{\text{lex}} \omega & \cong & \omega^\star + \omega = \zeta; \\ D & := & A_8 \times_{\text{lex}} \omega^\star & \cong & \omega^\star. \end{array}$$

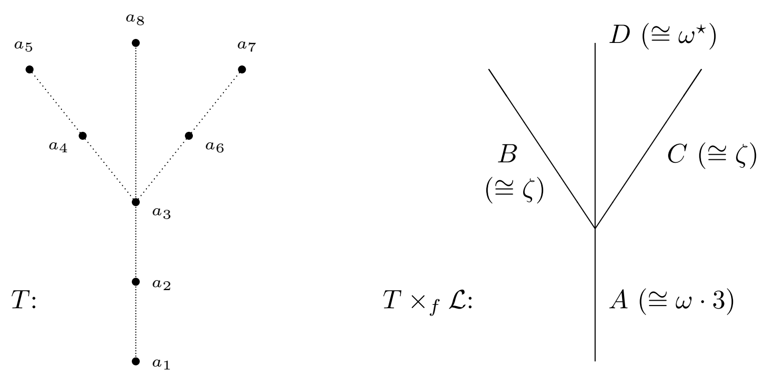

Figure 3.2: The f-product of T with $\mathcal{L}$ (see Example 3.28).

Proposition 3.29 Let T be a tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. Then $T \times_f \mathcal{L}$ is a tree.

Proof Let $(a, b) \in |T \times_f \mathcal{L}|$. Since T and $f(a)$ are both irreflexive then $a \not<_T a$ and $b \not<_{f(a)} b$. Hence $(a, b) \not< (a, b)$. This shows that $T \times_f \mathcal{L}$ is irreflexive.

Next let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in |T \times_f \mathcal{L}|$ with $(a_1, b_1) < (a_2, b_2)$ and $(a_2, b_2) < (a_3, b_3)$. Then either $a_1 <_T a_2$ or both $a_1 = a_2$ and $b_1 <_{f(a_1)} b_2$, and either $a_2 <_T a_3$ or both $a_2 = a_3$ and $b_2 <_{f(a_2)} b_3$. We have the following possibilities:

	$a_2 <_T a_3$	$a_2 = a_3$, $b_2 <_{f(a_2)} b_3$
$a_1 <_T a_2$	$a_1 <_T a_3$	$a_1 <_T a_3$
$a_1 = a_2$, $b_1 <_{f(a_1)} b_2$	$a_1 <_T a_3$	$a_1 = a_3$, $b_1 <_{f(a_1)} b_3$

In each case we get $(a_1, b_1) < (a_3, b_3)$. It follows that $T \times_f \mathcal{L}$ is transitive.

To show that $T \times_f \mathcal{L}$ is subtotal, let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in |T \times_f \mathcal{L}|$ with $(a_1, b_1), (a_2, b_2) < (a_3, b_3)$. Then either $a_1 <_T a_3$ or both $a_1 = a_3$ and $b_1 <_{f(a_1)} b_3$, and either $a_2 <_T a_3$ or both $a_2 = a_3$ and $b_2 <_{f(a_2)} b_3$. We have the following possibilities:

	$a_2 <_T a_3$	$a_2 = a_3$, $b_2 <_{f(a_2)} b_3$
$a_1 <_T a_3$	$a_1 \smile_T a_2$	$a_1 <_T a_2$
$a_1 = a_3$, $b_1 <_{f(a_1)} b_3$	$a_2 <_T a_1$	$a_1 = a_2$, $b_1 \smile_{f(a_1)} b_2$

Again in each case we get $(a_1, b_1) \smile (a_2, b_2)$, and it follows that $T \times_f \mathcal{L}$ is subtotal.

Finally let $(a_1, b_1), (a_2, b_2) \in |T \times_f \mathcal{L}|$. Since T is connected then there exists $a_3 \in T$ such that $a_3 \leqslant_T a_1, a_2$. Let b_3 be any element in $f(a_3)$ such that, if $a_3 = a_1$ then $b_3 \leqslant_{f(a_1)} b_1$, and if $a_3 = a_2$ then $b_3 \leqslant_{f(a_2)} b_2$. Then $(a_3, b_3) \leqslant (a_1, b_1), (a_2, b_2)$. It follows that $T \times_f \mathcal{L}$ is connected. QED

Proposition 3.30 Let T be a tree. Let $[T]$ be the condensation of T and let $\mathcal{L} := \{[x] : x \in T\}$ be the class of linear orders consisting of all the maximal bridges in T. Let $f : [T] \to \mathcal{L}$ be given by $f([x]) = [x]$. Then $T \cong [T] \times_f \mathcal{L}$.

Proof Verify that the map $x \mapsto ([x], x)$ defines an isomorphism from T to $[T] \times_f \mathcal{L}$. QED

Corollary 3.31 Every tree can be expressed in the form $T \times_f \mathcal{L}$ for T a condensed tree, $\mathcal{L}$ a class of linear orders, and $f : T \to \mathcal{L}$ a function.

Proof From Proposition 3.30 and the fact that $[T]$ is condensed for every tree T. QED

Lemma 3.32 Let T be a tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. Let $A \subseteq T$ and define $B := \bigcup_{x \in A} (\{x\} \times f(x))$.

(i) A is total in T if and only if B is total in $T \times_f \mathcal{L}$.

(ii) A is convex (respectively downwards convex, upwards convex) in T if and only if B is convex (respectively downwards convex, upwards convex) in $T \times_f \mathcal{L}$.

It follows that A is a segment (respectively stem, branch) in T if and only if B is a segment (respectively stem, branch) in $T \times_f \mathcal{L}$.

Proof (i) Straightforward.

(ii) Assume A is convex in T. Let $(a_1, b_1), (a_2, b_2) \in B$ and let $(a_3, b_3) \in |T \times_f \mathcal{L}|$ with $(a_1, b_1) < (a_3, b_3) < (a_2, b_2)$. There are four possibilites which are tabulated below:

	$a_3 <_T a_2$	$a_3 = a_2,\ b_3 <_{f(a_3)} b_2$
$a_1 <_T a_3$	$a_3 \in A$	$a_3 \in A$
$a_1 = a_3,\ b_1 <_{f(a_1)} b_3$	$a_3 \in A$	$a_3 \in A$

In each case the fact that $a_3 \in A$ means that $(a_3, b_3) \in B$ so B is convex in $T \times_f \mathcal{L}$.

It is straightforward to see that A is convex in T when B is convex in $T \times_f \mathcal{L}$.

The argument to show that downwards convexivity and upwards convexivity is preserved between A and B is similar. QED

Lemma 3.33 Let T be a tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. Let $A \subseteq T$ and define $B := \bigcup_{x \in A} (\{x\} \times f(x))$.

(i) A is a bridge in T if and only if B is a bridge in $T \times_f \mathcal{L}$.

(ii) A is a maximal bridge in T if and only if B is a maximal bridge in $T \times_f \mathcal{L}$.

(iii) Every maximal bridge in $T \times_f \mathcal{L}$ has the form $\bigcup_{x \in X} (\{x\} \times f(x))$ for some maximal bridge $X \subseteq T$.

Proof (i) We know from Lemma 3.32 that A is a segment in T if and only if B is a segment in $T \times_f \mathcal{L}$.

Assume A is a bridge in T. In order to show that B is a bridge in $T \times_f \mathcal{L}$ it suffices, by Proposition 3.14, to show that for every $(a_1, b_1), (a_2, b_2) \in B$ and $(x, y) \in |T \times_f \mathcal{L}|$, $(x, y) \smile (a_1, b_1)$ if and only if $(x, y) \smile (a_1, b_1)$. Hence let $(a_1, b_1), (a_2, b_2) \in B$, say with $(a_1, b_1) < (a_2, b_2)$, and let $(x, y) \in |T \times_f \mathcal{L}|$.

Suppose $(x, y) \smile (a_1, b_1)$. We consider two cases:

Case 1: $a_1 = a_2$. If $x \neq a_1$ then it is immediate that $(x, y) \smile (a_2, b_2)$, while if $x = a_1$ then it follows that $(x, y) \smile (a_2, b_2)$ from the fact that $f(a_1)$ is a linear order.

Case 2: $a_1 \neq a_2$. Then $a_1 <_T a_2$. If $x = a_1$ then clearly $(x, y) \smile (a_2, b_2)$ while if $x = a_2$ then $(x, y) \smile (a_2, b_2)$ from the fact that $f(a_2)$ is a linear order. If $x \neq a_1, a_2$ then since $x \smile_T a_1$ and A is a bridge in T, $x \smile_T a_2$ giving $(x, y) \smile (a_2, b_2)$.

In each of the above cases we get that $(x, y) \smile (a_2, b_2)$. An identical argument shows that if $(x, y) \smile (a_2, b_2)$ then $(x, y) \smile (a_1, b_1)$. Hence B is a bridge in $T \times_f \mathcal{L}$.

It is straightforward to see that if B is a bridge in $T \times_f \mathcal{L}$ then A is a bridge in T.

(ii) Follows from (i).

(iii) Let $C \subseteq |T \times_f \mathcal{L}|$ be a maximal bridge with $(a, b) \in C$. Since $[a]$ is a maximal bridge in T then from (ii) we know that $\bigcup_{x\in[a]} (\{x\} \times f(x))$ is a maximal bridge in $T \times_f \mathcal{L}$. Since $(a, b) \in \bigcup_{x\in[a]} (\{x\} \times f(x))$ then it follows that $C = \bigcup_{x\in[a]} (\{x\} \times f(x))$. QED

Corollary 3.34 Let T be a condensed tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. For every $x \in T$, the set $\{x\} \times f(x)$ is a maximal bridge in $T \times_f \mathcal{L}$, and every maximal bridge in $T \times_f \mathcal{L}$ has the form $\{x\} \times f(x)$ for some $x \in T$.

Proof From Proposition 3.25 and Lemma 3.33. QED

Proposition 3.35 Let T be a condensed tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. The function $g : [T \times_f \mathcal{L}] \to T$ defined as $g([(x, y)]) = x$ for every $[(x, y)] \in [T \times_f \mathcal{L}]$ is an isomorphism.

Proof Let $[(a_1, b_1)], [(a_2, b_2)] \in [T \times_f \mathcal{L}]$ with $[(a_1, b_1)] \neq [(a_2, b_2)]$. Then $a_1 \neq a_2$ since by Corollary 3.34, $\{a_1\} \times f(a_1)$ and $\{a_2\} \times f(a_2)$ are maximal bridges in $T \times_f \mathcal{L}$. Hence $g([(a_1, b_1)]) = a_1 \neq a_2 = g([(a_2, b_2)])$ and so g is injective.

Next let $a \in T$ and $b \in f(a)$. Then $g([(a, b)]) = a$ hence g is surjective.

Finally, for $[(a_1, b_1)], [(a_2, b_2)] \in [T \times_f \mathcal{L}]$, note that

$$\begin{aligned} g([(a_1, b_1)]) <_T g([(a_2, b_2)]) &\Leftrightarrow a_1 <_T a_2 \\ &\Leftrightarrow (a_1, b_1) < (a_2, b_2) \text{ and } a_1 \neq a_2 \\ &\Leftrightarrow [(a_1, b_1)] < [(a_2, b_2)] \\ &\quad \text{(from Proposition 3.12} \\ &\quad \text{ and Corollary 3.34).} \end{aligned}$$

Hence g is an isomorphism. QED

Chapter 4

Some important classes of trees

In this chapter we investigate some important classes of trees starting with well-founded trees in Section 4.1 (Well-founded trees). We then move to finitely branching trees in Section 4.2 (Finitely branching trees). The notion of finite branching is usually studied in the context of trees which are well-founded and discrete and admits problematic cases when applied to trees which are not well-founded or discrete. We introduce a notion of finite branching which appears to be consistent with the intuitive understanding of finite branching and generalises well to trees which are not well-founded or discrete. In Section 4.3 (Trees associated with a class of linear orders) we introduce eight classes of trees determined by how the paths in those trees relate to the linear orders in some class $\mathcal{C}$ of linear orders and we completely establish the set-theoretical relationships between these eight classes of trees.

4.1 Well-founded trees

Let $(A;<)$ be a partial order. An element $a \in A$ is called the **least element** of A if $a \leqslant x$ for all $x \in A$. The element a is called a **minimal element** in A if there exists no element $x \in A$ for which $x < a$.

Definition 4.1 A linear order L is called **well-ordered** when every non-empty subset of L contains a least element. A tree T is called **well-founded** when every non-empty set of nodes from T contains a minimal node.

Proposition 4.2 Let T be a tree. Then T is well-founded if and only if every path in T is well-ordered.

Proof $\Rightarrow$ Immediate.

$\Leftarrow$ Suppose T is not well-founded. Let $\{a_i\}_{i\in\mathbb{N}}$ be an infinite strictly descending chain in T. Then $\{a_i\}_{i\in\mathbb{N}}$ can be extended to a path A which is not well-ordered. QED

Proposition 4.3 Let T be a well-founded tree. Then T is upwards discrete.

Proof Assume T is well-founded. Let A be a path in T and let $a \in A$ with a not a leaf. Define $B := \{x \in A : x > a\}$. Then B contains a minimal node b and b will be an immediate successor to a in A. Hence T is upwards discrete. QED

Let T be a well-founded tree. For a non-empty segment A in T, let $S(A)$ be the set of minimal nodes in $A_<$. Hence $S(A)$ represents the set of immediate successors to A. The **level** of a node $a \in T$, denoted $l(a)$, is the order type of the set $a_>$. The node a is called a **successor node** when $l(a)$ is a successor ordinal. The node a is called a **limit node** when $l(a)$ is a limit ordinal. The supremum of the set $\{l(x) + \mathbf{1} : x \in T\}$ is called the **height** of T.

The following describes a naming convention for refering to nodes and paths in a well-founded tree T. For every stem $X \subseteq T$, let I_X be a set such that $|S(X)| \leqslant |I_X|$, and let $f_X : S(X) \to I_X$ be an injective function. We assign **labels** $\ell(x)$ to nodes x in T as follows:

(i) The root of T is given the label (0).

(ii) If a is a successor node, let b be the node for which $a \in S(b)$ and suppose b has as its label the sequence $\bar{b}$ of length γ. Then a is assigned as label the sequence $(\bar{b}, f_{a_>}(a))$ of length $\gamma + \mathbf{1}$.

(iii) If a is a limit node then a is assigned as label the sequence γ of length $l(a) + \mathbf{1}$ which is defined as follows. For every node b with $b < a$, $\ell(b)$ is the initial subsequence of γ of length $l(b) + \mathbf{1}$. The last entry in γ is $f_{a_>}(a)$.

We can impose an order $<_L$ on labels by specifying that, for any two labels $\ell(x)$ and $\ell(y)$ obtained from T, $\ell(x) <_L \ell(y)$ when $\ell(x)$ is an initial subsequence of $\ell(y)$. Then if L is the set of all labels obtained from T, the structures $(L; <_L)$ and $(T; <)$ are isomorphic under the function which maps nodes in T to their respective labels in L.

We can assign labels $\ell(X)$ to paths X in T as follows. Let A be a path in T and suppose A has order type α. Then A is assigned as label the sequence γ of length α defined by the property that for every node $b \in A$, $\ell(b)$ is the initial subsequence of γ of length $l(b) + \mathbf{1}$.

Example 4.4 Let T be a binary tree of which every path is isomorphic with ω. Clearly T contains $\aleph_0$ many nodes. For every node x in T, let $I_{x_\geqslant} = \{0, 1\}$ and, if y and z are the immediate successors to x with y located to the left of z as T is depicted in Figure 4.1, define $f_{x_\geqslant}(y) := 0$ and $f_{x_\geqslant}(z) := 1$. This gives a labeling of the nodes in T.

Under this labeling, every path in T has a binary sequence $(0, x_1, x_2, \ldots)$ of length ω as its label, corresponding to the real number with binary representation $(0.x_1x_2\cdots)_2$. Hence T contains $2^{\aleph_0}$ many paths. For $a_i = 0$, note that

$$(0, a_1, \ldots, a_i, 1, 1, 1, \ldots) \neq (0, a_1, \ldots, a_{i-1}, 1, 0, 0, 0, \ldots)$$

but

$$(0.a_1 \cdots a_i 111 \cdots)_2 = (0.a_1 \cdots a_{i-1} 1000 \cdots)_2.$$

Hence the function which maps labels of paths in T to their corresponding binary numbers in the interval $[0, 1]$ is surjective but not injective.

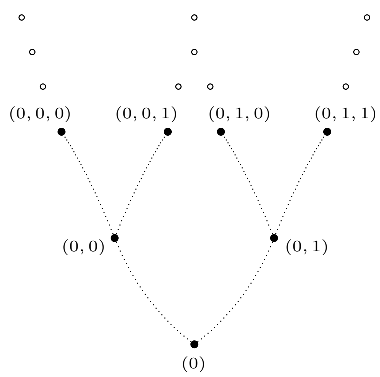

Figure 4.1: A labeling of the tree described in Example 4.4.

4.2 Finitely branching trees

The concept of finite branching of trees is studied in [27] and [10]. The definitions used for finite branching in these two texts are roughly equivalent. Here we will make use of a slightly broader notion of finite branching.

4.2.1 Definition of bounded branching

The notation $X \leqslant_{\mathrm{p}} Y$ indicates that for every $y \in Y$, there exists $x \in X$ such that $x \leqslant y$.

Definition 4.5 For $n \in \mathbb{N}$, a tree T is called **n-branching** from the stem A when, for every antichain X in T with $A < X$, there exists a set L_X in T with $|L_X| \leqslant n$ and such that $A < L_X \leqslant_{\mathrm{p}} X$.

T is called **finitely branching** from A if it is n-branching from A for some n, and **infinitely branching** from A when it is not finitely branching from A.

T is called **n-branching** when it is n-branching from each of its stems, and **boundedly branching** when it is n-branching for some n. T is called **finitely branching** when it is finitely branching from each of its stems.

We can also view a tree as being n-branching, finitely branching or infinitely branching from one of its nodes x by considering whether it is n-branching, finitely branching or infinitely branching from the stem $x_{\geqslant}$.

Clearly if a tree T is n-branching from a stem A, then T will be m-branching from A for all m with $m \geqslant n$. The notion of n-branching also applies to the empty stem, and if A is a path in T then Definition 4.5 yields that T is 0-branching from A.

For $n \in \mathbb{N}$, an upwards discrete tree is called ***n*-ary** when every node in that tree has exactly n immediate successors. A 2-ary tree is called simply a **binary** tree.

Example 4.6 (a) Consider the tree T obtained by taking the order type of the rationals η and at every positive number in η, we attach another copy of η (see Figure 4.2). Then T is infinitely branching from the node 0 located in the copy of η with which we started.

(b) Consider the tree T obtained by taking the order $\omega + \zeta$, and at every node in the copy of ζ, we attach a copy of the order ω (see Figure 4.3). Then T is infinitely branching from the stem consisting of the natural numbers in the copy of ω in the path $\omega + \zeta$.

Proposition 4.7 Let T be a well-founded tree and let A be a stem in T. Then T is n-branching from A if and only if $|S(A)| \leqslant n$.

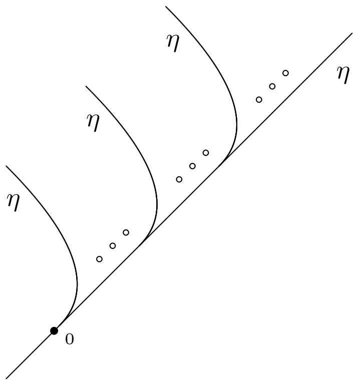

Figure 4.2: Depiction of the non-finitely branching tree described in Example 4.6(a).

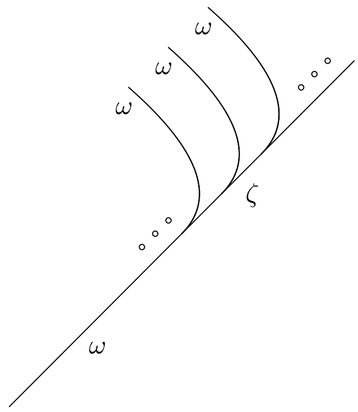

Figure 4.3: Depiction of the non-finitely branching tree described in Example 4.6(b).

Proof Let T be n-branching from A and note that $S(A)$ is an antichain. Hence there exists L with $|L| \leqslant n$ and such that $A < L \leqslant_{\mathrm{p}} S(A)$. It follows that $S(A) \subseteq L$ hence $|S(A)| \leqslant n$.

Conversely assume $|S(A)| \leqslant n$ and let G be an antichain with $A < G$. Let $a \in G$ and define $B := \{x : A < x \leqslant a\}$. Since T is well-founded then B contains a least node b. From the definition of $S(A)$ this gives $b \in S(A)$ and it follows that $A < S(A) \leqslant_{\mathrm{p}} G$. Hence T is n-branching from A. QED

Definition 4.8 Let T be a tree and let $A \subseteq T$ and $B \subseteq T$ with $B \subseteq A$. Then B is said to **span** A when, for every $x \in A$, there exists $y \in B$ such that $y \smile x$.

Proposition 4.9 Let T be a tree and let A be a stem in T. Then T is n-branching from A if and only if there exists $H \subseteq A_<$ with $|H| \leqslant n$ and such that H spans $A_<$.

Proof Suppose T is n-branching from A. Let G be a maximal antichain in $A_<$ and let L be a set of nodes with $|L| \leqslant n$ and such that $A < L \leqslant_{\mathrm{p}} G$. Let $a \in A_<$. Then there exists $b \in G$ such that $b \smile a$, and there exists $c \in L$ such that $c \leq b$. It follows that $c \smile a$ hence L spans $A_<$.

Conversely, suppose H spans $A_<$ for some $H \subseteq A_<$ with $|H| \leqslant n$, and let G be an antichain in $A_<$. Then for every $x \in G$, there exists $b_x \in H$ such that $b_x \smile x$. Take $c_x := \min\{x, b_x\}$ and let $L := \{c_x : x \in G\}$. Then L satisfies the condition $A < L \leqslant_{\mathrm{p}} G$. Moreover $|L| \leqslant n$, for if $|L| > n$ then from the fact that $|H| \leqslant n$, there must exist $x, y \in G$ with $x \neq y$, and $b \in H$ with $b \smile x$ and $b \smile y$, such that $c_x = \min\{x, b\}$ and $c_y = \min\{y, b\}$ but with $c_x \neq c_y$. Since G is an antichain this leads to a contradiction. Hence T is n-branching from A. QED

4.2.2 Other notions of finite branching

We will now introduce the notions of finite branching used in [27] and [10] and show that the notion of finite branching used in this text admits a smaller class of trees than the notions of finite branching used in [27] and [10].

Let T be a tree and let C be a path in T, A a stem in T and B a branch in T with $A, B \subseteq C$. If A and B are such that $A \cup B = C$ and $A \cap B = \emptyset$ then A is called a stem of B and B is called a branch of A.

Two branches which share a stem A are called **siblings** with stem A. A pair of siblings are called **twins** when their intersection is non-empty. A set of siblings $\mathcal{B}$ which share the stem A are called a **litter** of A when $\mathcal{B}$ is maximal with respect to the property that every two siblings in $\mathcal{B}$ are twins.

Remark 4.10 Let T be a tree, let A be a stem in T, and let $\{\mathcal{B}_i\}_{i\in I}$ be the set of all litters of A. Then the set $\{\bigcup \mathcal{B}_i\}_{i\in I}$ forms a partition of the set $A_<$.

Lemma 4.11 Let T be a tree, let A be a stem in T, and let $\mathcal{B}$ be a set of branches of A. Then $\mathcal{B}$ is a litter of A if and only if $\bigcup \mathcal{B}$ is a component of $A_<$.

Proof Immediate. QED

We will make use of the following terminology. Let T be a tree and let A and B be segments in T. If $A < B$ and the set $\{x \in T : A < x < B\}$ is empty then B is said to **extend** A. If $B \subseteq A$ then B is called a **subsegment** of A. If B is a subsegment of A and the set $\{x \in A : x < B\}$ is empty, then B is called an **initial subsegment** of A.

Lemma 4.12 Let T be a tree and let A be a stem in T. If T is n-branching from A then A has at most n litters in T.

Proof Let T be n-branching from the stem A and let $\{\mathcal{B}_i\}_{i\in I}$ be the set of litters of A. Suppose $|I| > n$. For every $i \in I$, choose exactly one node a_i from the set $\bigcup \mathcal{B}_i$ and define $G := \{a_i : i \in I\}$. Then G constitutes an antichain so that there exists L with $|L| \leqslant n$ and $A < L \leqslant_{\mathrm{p}} G$. For every $i \in I$, the set $\bigcup \mathcal{B}_i$ must contain at least one node from L, from which $|L| > n$, a contradiction. Hence $|I| \leqslant n$. QED

Corollary 4.13 Let T be a tree that is n-branching from the stem A and let $\{B_i\}_{i\in I}$ be a set of pairwise disjoint bridges in T which extend A. Then $|I| \leqslant n$.

Proof If B_i and B_j are disjoint bridges which extend A then B_i and B_j are contained within different litters of A. The result hence follows from Lemma 4.12. QED

The converse of Lemma 4.12 fails. For example, the stem A consisting of the non-positive rationals in the starting copy of η in the tree described in Example 4.6(a) has only one litter, but the tree is infinitely branching from A. Likewise the stem consisting of the elements in the copy of ω in the path $\omega + \zeta$ in the tree described in Example 4.6(b) has only one litter, but again the tree is infinitely branching from this stem. The next result gives a more complete relationship between the branching behaviour of a stem and the number of litters of that stem.

Proposition 4.14 Let T be a tree and let A be a stem in T. Then T is n-branching from A if and only if the following two conditions are satisfied:

(i) A has at most n litters in T.

(ii) If B is a segment that extends A then B has an initial subsegment that is a bridge.

Proof Let T be n-branching from the stem A. Condition (i) holds by Lemma 4.12. If B is a bridge then condition (ii) holds immediately, so consider the case where B is a furcation. If B contains a least node a, then the set $\{a\}$ forms an initial subsegment of B that is a bridge. Hence consider the case where B has no least node.

For every path X with $\emptyset \neq X \cap B \subsetneq B$, let $a_X \in X$ with $a_X > X \cap B$. Let G be the set consisting of all these nodes a_X and let G_0 be a maximal antichain in G. Then $A < G_0$ so there exists L with $|L| \leqslant n$ and $A < L \leqslant_{\mathrm{p}} G_0$. For every $x \in L$, let $x^- \in B$ be any node such that $x^- \leqslant x$ and put $b := \min\{x^- : x \in L\}$. Then the set $\{x : A < x < b\}$ is an initial subsegment of B and a bridge.

Next let A be a stem in T and assume that conditions (i) and (ii) hold. Let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be the litters of A and for each i with $1 \leqslant i \leqslant k$, let $b_i \in \bigcup \mathcal{B}_i$ be any node. Then for each i, there exists a bridge B_i which is an initial subsegment of the segment $\{x : A < x \leqslant b_i\}$. For each i, let $a_i \in B_i$ be any node and take $H := \{a_1, \ldots, a_k\}$. Then H spans $A_<$ and since $|H| = k \leqslant n$ then T is n-branching from A by Proposition 4.9. QED

Let T be a tree and let $a, b \in T$. Define the set

$$T_{ab} := \{x \in T : \text{ if } y \in T \text{ with } y \leqslant a, b \text{ then } y < x\}.$$

Hence $T_{ab} = (a_\geqslant \cap b_\geqslant)_<$.

Proposition 4.15 ([10]) Let T be a tree and let n be a positive natural number. The following properties are equivalent:

(i) For every $x, y \in T$, the set T_{xy} has at most n components.

(ii) For every stem X in T, X has at most n litters.

Proof For $x, y \in T$ define the stem $A(x, y) := \{z \in T : z \leqslant x, y\}$. Then $T_{xy} = (A(x, y))_<$ and the result follows from Lemma 4.11. QED

Remark 4.16 Trees are defined in [27] and [10] as being ordered sets which are irreflexive, transitive and subtotal, which coincides with the notion of forest as introduced in this text.

Finite branching is defined in [27] as follows. A tree T is *n-branching* when, for all $x, y \in T$, the set T_{xy} has at most n components. A tree is *finite-branching* when it is n-branching for some n.

Bounded branching is defined in [10] as follows. A tree T is *boundedly branching* if there exists $n \in \mathbb{N}$ such that every stem has at most n litters.

Proposition 4.15 shows that the definition of finite-branching in [27] is equivalent to the definition of boundedly branching in [10].

Lemma 4.12 together with Example 4.6 show that the notion of finitely branching introduced in this text is a refinement of the notion of finite branching used in [27], and of the notion of boundedly branching used in [10]. The trees in Example 4.6 would be regarded as finite-branching using the definition of finite-branching in [27], and boundedly branching using the definition of boundedly branching in [10]. These trees are not finitely branching in the sense that we have defined the notion in this text.

4.2.3 Condensations of finitely branching trees

Lemma 4.17 Let T be a finitely branching tree. Then $[T]$ is well-founded.

Proof Suppose $[T]$ is not well-founded. Then there exists $\{b_i\}_{i\in\mathbb{N}} \subseteq [T]$ with $b_{i+1} < b_i$ for all $i \in \mathbb{N}$. Let $A := \{x \in [T] : x < \{b_i\}_{i\in\mathbb{N}}\}$ and $B := \{x \in [T] : A < x \leqslant b_0\}$. Then $[A]^{-1}$ is a stem in T and $[B]^{-1}$ is a segment in T extending $[A]^{-1}$. But every initial subsegment of $[B]^{-1}$ is a furcation. From Proposition 4.14, it follows that T is infinitely branching from $[A]^{-1}$. QED

Proposition 4.18 Let T be a tree. Then T is n-branching if and only if $[T]$ is well-founded and $|S(X)| \leqslant n$ for every stem X in $[T]$.

Proof Let T be n-branching. Then by Lemma 4.17, $[T]$ is well-founded. Next let A be a stem in $[T]$. Then $\mathcal{B} := \{[x]^{-1} : x \in S(A)\}$ is a set of maximal bridges (hence pairwise disjoint) which extend $[A]^{-1}$ in T, so by Corollary 4.13, $|\mathcal{B}| \leqslant n$ hence $|S(A)| \leqslant n$.

Conversely, assume that $[T]$ is well-founded and that $|S(X)| \leqslant n$ for every stem X in $[T]$. Let A be a stem in T. First consider the case where $A \subsetneq [A]$ and let $a \in [A]\backslash A$. Then the set $\{a\}$ spans $A_<$. Next consider the case where $A = [A]$. Treating $[A]$ as a stem in $[T]$, for every $x \in S([A])$, let a_x be any node in $[x]^{-1}$ and put $B := \{a_x : x \in S([A])\}$. Then B spans $A_<$ and $|B| \leqslant n$. From Proposition 4.9 we get that T is n-branching from A. QED

4.2.4 Branching behaviour and height of a tree

The branching behaviour and height of well-founded trees are related. The reader is referred to [20] for a more detailed explanation of the results that follow. We begin with a well-known result.

König's Lemma: Let T be a well-founded tree of height ω that is finitely branching. Then T contains a path which is isomorphic with ω.

König's Lemma is a theorem in ZFC but not in ZF. König's Lemma can be extended to the following result: if T is a well-founded tree of height ω_1 and having the property that the set $\{x \in T : l(x) = \alpha\}$ is finite for every order type α then T contains a path isomorphic with ω_1.

Let κ be an infinite cardinal. A well-founded tree T is called a **κ-Aronszajn tree** when

(i) T has height κ, and

(ii) for every order type α, $|\{x \in T : l(x) = \alpha\}| < \kappa$, and

(iii) T contains no paths which are isomorphic with κ.

König's Lemma states that there are no $\aleph_0$-Aronszajn trees. The existence of $\aleph_1$-Aronszajn trees is a theorem of ZFC. For $n \geq 2$ the existence of $\aleph_n$-Aronszajn trees involves large cardinal assumptions.

A well-founded tree T of height ω_1 is called a **Kurepa tree** when $|\{x \in T : l(x) = \alpha\}| < \aleph_1$ for every order type α and when T contains at least $\aleph_2$ many paths which are isomorphic with ω_1. The existence of Kurepa trees is undecidable within ZFC.

A well-founded tree T of height ω_1 is called a **Suslin tree** when T does not contain paths which are isomorphic with ω_1 and when T does not contain any uncountable antichains. Hence a Suslin tree is an $\aleph_1$-Aronszajn tree which does not contain any uncountable antichains. The existence of Suslin trees is undecidable within ZFC.

Let R be a complete dense linear order without endpoints and with the property that every set of pairwise disjoint non-empty open intervals in R is countable. If R is not isomorphic with λ (the order type of the reals) then R is called a **Suslin line**. The existence of Suslin trees is equivalent to the existence of Suslin lines.

4.3 Trees associated with a class of linear orders

We now introduce several classes of trees which arise naturally from a class of linear orders. The idea of classifying trees in terms of how their paths are related to some class of linear orders is also considered in [13].

4.3.1 Definition of $\mathcal{C}$-classes of trees

Let α be an order type. A path A in a tree T is called an **α-path** when $A \cong \alpha$. A is called an **α-like path** when $A \equiv \alpha$.

Definition 4.19 Let $\mathcal{C}$ be a class of linear orders. A tree T is called a:

(i) **$\mathcal{C}$-tree** when every path X in T is an α-path for some $\alpha \in \mathcal{C}$ dependent on X;

(ii) **uniformly $\mathcal{C}$-like tree** (U-$\mathcal{C}$-like tree) if $T \equiv S$ for some $\mathcal{C}$-tree S;

(iii) **$\mathcal{C}$-like tree** if, for every $n \in \mathbb{N}$, there is a $\mathcal{C}$-tree S such that $T \equiv_n S$;

(iv) **pathwise uniformly $\mathcal{C}$-like tree** (PU-$\mathcal{C}$-like tree) if, for every path X in T, there exists $\alpha \in \mathcal{C}$ such that $X \equiv \alpha$;

(v) **pathwise $\mathcal{C}$-like tree** (P-$\mathcal{C}$-like tree) if, for every path X in T and for every $n \in \mathbb{N}$, there exists $\alpha \in \mathcal{C}$ such that $X \equiv_n \alpha$;

(vi) **definably $\mathcal{C}$-tree** (D-$\mathcal{C}$-tree) if every parametrically definable path X in T is an α-path for some $\alpha \in \mathcal{C}$ dependent on X;

(vii) **definably uniformly $\mathcal{C}$-like tree** (DU-$\mathcal{C}$-like tree) if, for every parametrically definable path X in T, there exists $\alpha \in \mathcal{C}$ such that $X \equiv \alpha$;

(viii) **definably $\mathcal{C}$-like tree** (D-$\mathcal{C}$-like tree) if, for every parametrically definable path X in T and for every $n \in \mathbb{N}$, there exists $\alpha \in \mathcal{C}$ such that $X \equiv_n \alpha$. Equivalently (since the language of trees has finite signature) if every parametrically definable path in T is a model of the first-order theory of $\mathcal{C}$.

If $\mathcal{C} = \{\alpha\}$ then T is simply called an α-tree, a uniformly α-like tree, etc.

The classes described above are collectively referred to as **$\mathcal{C}$-classes** of trees. We follow with six examples of trees and classes of linear orders which will be used as counterexamples in the proof of Theorem 4.26.

Example 4.20 Let $B_{\omega+\mathbf{1}}$ be the binary tree of which every path is isomorphic to the order type $\omega + \mathbf{1}$. The tree $B_{\omega+\mathbf{1}}$ is uncountable. Let T_1 be any countable elementary substructure of $B_{\omega+\mathbf{1}}$ (the existence of such T_1 follows from the Downward Löwenheim-Skolem Theorem), and let $\mathcal{C}_1 = \{\omega + \mathbf{1}\}$. T_1

will be a binary tree not containing any finite paths, so every path in T_1 will be either an ω-path or an $(\omega + \mathbf{1})$-path. Moreover, T_1 does actually contain both ω-paths and $(\omega + \mathbf{1})$-paths. It follows that T_1 can be seen as the result of having removed an uncountable set of leaves from $B_{\omega+\mathbf{1}}$.

T_1 will contain paths which are not elementarily equivalent with $\omega + \mathbf{1}$. In fact, the ω-paths in T_1 already fail to be 2-equivalent with $\omega + \mathbf{1}$ since they satisfy the sentence $\forall x \exists y\,(x < y)$.

Suppose $\varphi(x, \bar{z})$ defines a path A in T_1 with parameters $\bar{c}$. Then $(T_1, \bar{c}) \models \pi_\varphi(\bar{c})$ and so $(B_{\omega+\mathbf{1}}, \bar{c}) \models \pi_\varphi(\bar{c})$. Hence $\varphi(x, \bar{z})$ defines a path in $B_{\omega+\mathbf{1}}$ with parameters $\bar{c}$ also. Now since every path in $B_{\omega+\mathbf{1}}$ is an $(\omega+\mathbf{1})$-path then we get that $(B_{\omega+\mathbf{1}}, \bar{c}) \models \exists x\,(\mathrm{leaf}(x) \wedge \varphi(x, \bar{c}))$ and so $(T_1, \bar{c}) \models \exists x\,(\mathrm{leaf}(x) \wedge \varphi(x, \bar{c}))$. Hence A will contain a leaf.

Thus every parametrically definably path in T_1 will contain a leaf, and since every path in T_1 containing a leaf is parametrically definable using that leaf as parameter, it follows that the parametrically definable paths in T_1 are precisely its $(\omega + \mathbf{1})$-paths.

Hence T_1 is a uniformly $(\omega+\mathbf{1})$-like tree, and also a definably $(\omega+\mathbf{1})$-tree, but neither an $(\omega + \mathbf{1})$-tree, nor a pathwise $(\omega + \mathbf{1})$-like tree.

Example 4.21 Let T_2 be the tree indicated in Figure 4.4 and let $\mathcal{C}_2 = \{\omega\}$. Each of the two paths in T_2 are parametrically definable and are elementarily equivalent with ω. In any ω-tree, every parametrically definable set contains a minimal node. The set of nodes in T_2 defined by the formula

$$\varphi(x) = \forall y \forall z\,(x < y \wedge x < z \to y \smile z)$$

contains no minimal node. Thus T_2 is a definably uniformly ω-like tree, but not an ω-like tree.
Let σ_1 be the sentence

$$\sigma_1 := \exists u \varphi(u) \to \exists u\,(\varphi(u) \wedge \forall w\,(w < u \to \neg\varphi(w)))$$

stating that the set defined by $\varphi(x)$ contains a minimal node, where $\varphi(x)$ is defined as above. This sentence will be used further.

Example 4.22 Let T_3 be the tree indicated in Figure 4.4 and let $\mathcal{C}_3 = \{\mathbf{n} : n \in \mathbb{N}\}$. Both of the paths in T_3 are parametrically definable. It is known (e.g. [24]) that for every m there exists some sufficiently large n such that $\omega + \omega^\star \equiv_m \mathbf{n}$. However, $\omega + \omega^\star \not\equiv \mathbf{n}$ for every n. In any definably uniformly

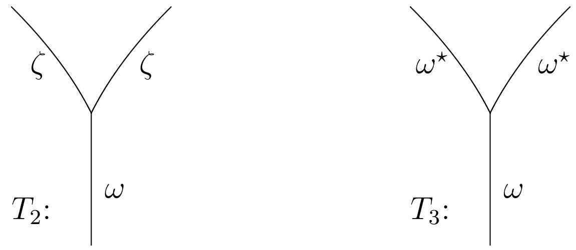

Figure 4.4: The trees T_2 and T_3 described in Example 4.21 and Example 4.22.

$\mathcal{C}_3$-like tree, the set defined by the formula $\varphi(x)$ from Example 4.21 will contain a minimal node. However, the subset of T_3 defined by $\varphi(x)$ does not contain a minimal node. Thus T_3 is a definably $\mathcal{C}_3$-like tree, even a pathwise $\mathcal{C}_3$-like tree, but neither a $\mathcal{C}_3$-like tree nor a definable uniformly $\mathcal{C}_3$-like tree.

Example 4.23 Let T_4 be the linear order $\omega + \omega^\star$ and let $\mathcal{C}_4 = \{\mathbf{n} : n \in \mathbb{N}\}$. Again we note that there exists, for every m, some sufficiently large n such that $\omega + \omega^\star \equiv_m \mathbf{n}$, but that $\omega + \omega^\star \not\equiv \mathbf{n}$ for every n.

Example 4.24 Let T_5 be the binary tree of which every path is an ω-path and take $\mathcal{C}_5 = \{\omega + \mathbf{1}\}$. Note that T_5 contains no parametrically definable paths. Let σ_2 be the sentence

$$\sigma_2 := \forall x \exists y \, (x \leqslant y \wedge \mathrm{leaf}(y)) \, .$$

This sentence will be used further.

Example 4.25 Let T_6 be the linear order $\omega + \zeta$ and take $\mathcal{C}_6 = \{\omega\}$. It is known that $\omega \equiv \omega + \zeta$.

4.3.2 Relationships between $\mathcal{C}$-classes of trees

Theorem 4.26 Let $\mathcal{C}$ be a class of linear orders. The set-theoretical inclusions and non-inclusions which hold between the $\mathcal{C}$-classes of trees are presented in Figure 4.5.

Proof To begin with the inclusions, we will show that the class of $\mathcal{C}$-like trees is contained in the class of D-$\mathcal{C}$-like trees. The argument to show that the class of U-$\mathcal{C}$-like trees is contained in the class of DU-$\mathcal{C}$-like trees is similar. The remaining inclusions are easy to verify.

Let T be a $\mathcal{C}$-like tree and let A be a path in T defined in $(T;\bar{c})$ by the formula $\varphi(x,\bar{c})$ for some tuple of nodes $\bar{c}$ from T. Suppose that A has n-characteristic τ. Then $T \models \pi_\varphi(\bar{c}/\bar{z})$ and $T \models \tau^\varphi(\bar{c}/\bar{z})$ hence $T \models \exists\bar{z}\,(\pi_\varphi(\bar{z}) \wedge \tau^\varphi(\bar{z}))$.[1] Since T is a $\mathcal{C}$-like tree then there exists a $\mathcal{C}$-tree S for which $S \models \exists\bar{z}\,(\pi_\varphi(\bar{z}) \wedge \tau^\varphi(\bar{z}))$. Thus $\varphi(x,\bar{d})$ defines a path B in $(S;\bar{d})$ for some tuple of nodes $\bar{d}$ from S and $B \models \tau$. But B is isomorphic with some linear order C in $\mathcal{C}$ and so $A \equiv_n C$. It follows that T is a D-$\mathcal{C}$-like tree.

As an example of a non-inclusion demonstrated by a counterexample, we show that the class of P-$\mathcal{C}$-like trees is not always included in the class of $\mathcal{C}$-like trees. Note that the tree T_2 from Example 4.21 is a P-$\mathcal{C}_2$-like tree, but not a $\mathcal{C}_2$-like tree, where $\mathcal{C}_2$ is the class of linear orders defined in Example 4.21. This is because every $\mathcal{C}_2$-like tree must satisfy the sentence σ_1 defined in Example 4.21, while T_2 does not satisfy σ_1. Hence the class of P-$\mathcal{C}_2$-like trees is not contained in the class of $\mathcal{C}_2$-like trees.

As an example of a non-inclusion obtained through transitive completion in Figure 4.5, consider the claim that the class of P-$\mathcal{C}$-like trees is not generally a subclass of the class of PU-$\mathcal{C}$-like trees. If, to the contrary, the class of P-$\mathcal{C}$-like trees were a subclass of the class of PU-$\mathcal{C}$-like trees for all classes of linear orders $\mathcal{C}$, then since the class of PU-$\mathcal{C}$-like trees is also a subclass of the class of DU-$\mathcal{C}$-like trees for all classes $\mathcal{C}$, we would get that the class of P-$\mathcal{C}$-like trees is a subclass of the class of DU-$\mathcal{C}$-like trees for all classes $\mathcal{C}$. But this contradicts the fact that the tree T_4 from Example 4.23 is a P-$\mathcal{C}_4$-like tree, where $\mathcal{C}_4$ is defined in Example 4.23, but T_4 is not a DU-$\mathcal{C}_4$-like tree. This establishes the non-inclusion.

The remaining non-inclusions are easily verified. QED

Proposition 4.27 Let $\mathcal{C}$ consist of a single linear order. In addition to the set-theoretical inclusions which have been shown to hold between the $\mathcal{C}$-classes of trees in Theorem 4.26, the following inclusions also hold:

(i) the class of P-$\mathcal{C}$-like trees $\subseteq$ the class of PU-$\mathcal{C}$-like trees;

(ii) the class of D-$\mathcal{C}$-like trees $\subseteq$ the class of DU-$\mathcal{C}$-like trees.

Consequently

(iii) the class of $\mathcal{C}$-like trees $\subseteq$ the class of DU-$\mathcal{C}$-like trees;

[1] The formula π_φ is defined in Section 5.4.1.

(iv) the class of P-$\mathcal{C}$-like trees $\subseteq$ the class of DU-$\mathcal{C}$-like trees.

Proof Routine. QED

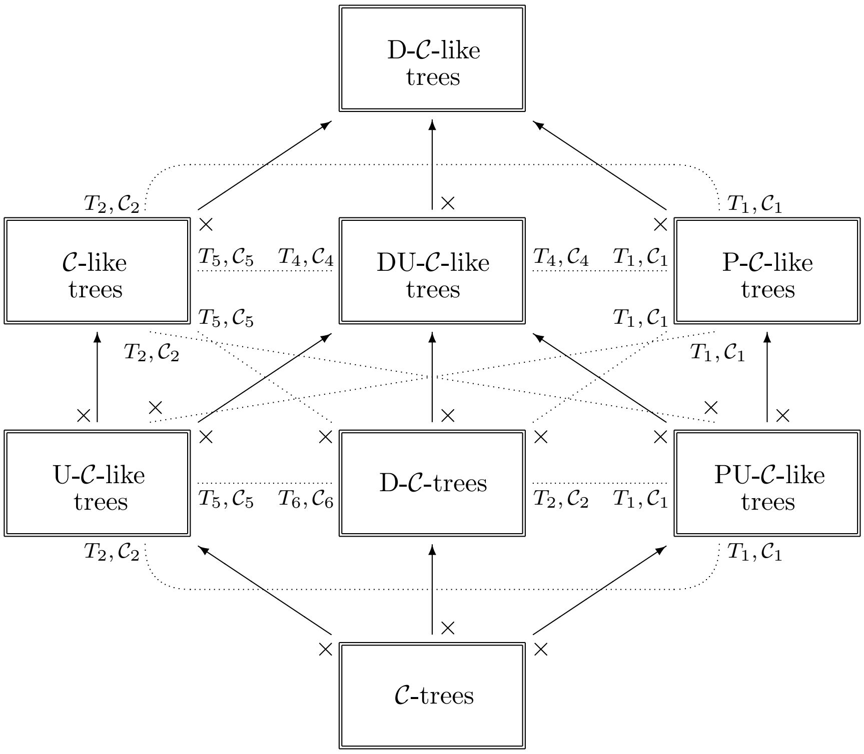

Figure 4.5: Relationships between the $\mathcal{C}$-classes of trees (see Theorem 4.26). Inclusions $X \subseteq Y$ are denoted as $X \to Y$. Non-inclusions are indicated by specifying a counterexample drawn from Examples 4.20 - 4.25 or, when obtained through transitive completion of the diagram, by $\times$. There are no downwards directed inclusions between any of the classes.

Chapter 5

First-order definability and trees

We now shift our focus to the first-order theories of trees. In this chapter we investigate the first-order definability of particular sets of nodes within a tree. In Section 5.3 (Nodes) we show in the relevant trees that the expressive power of nodes increases with the height of those nodes. We also introduce neighbourhoods of nodes which can be useful for imposing properties which are locally true on a tree. In Section 5.4 (Paths) we investigate paths which are parametrically definable and shed some light on some of the reasons why it may happen that a path is not parametrically definable. We also show in the relevant trees that if a path is parametrically definable then it can be defined using a node lying high up on the path. The remainder of the chapter looks at elementary equivalence between trees obtained from one another by substitution of a subtree in Section 5.5 (Subtrees) and by constructions involving condensations in Section 5.6 (Condensations).

5.1 Ehrenfeucht-Fraïssé games on trees

The Ehrenfeucht-Fraïssé game of length n on a pair of structures $\mathfrak{A}$ and $\mathfrak{B}$ will be denoted as $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$. The situation where Player II has a winning strategy for this game will be denoted as $\mathrm{II}_n(\mathfrak{A}, \mathfrak{B})$. For more information on Ehrenfeucht-Fraïssé games the reader is referred to [6]. The paper [33] investigates Ehrenfeucht-Fraïssé games played specifically on trees.

For T a tree and $x \in T$, define the set

$$C(x) := T \setminus x_<$$

consisting of all nodes y in T such that $y \leqslant x$ or y is incomparable with x. The set $C(x)$ will also be treated as a substructure of T.

The following is a well known result in the study of linear orders.

Proposition 5.1 (Splitting Lemma) ([24, Theorem 6.6]) Let L_1 and L_2 be linear orders. Then $\mathrm{II}_{n+1}(L_1, L_2)$ if and only if the following two conditions are satisfied:

(i) for every $a \in L_1$, there exists $b \in L_2$ such that $\mathrm{II}_n(a_>, b_>)$ and $\mathrm{II}_n(a_<, b_<)$;

(ii) for every $b \in L_2$, there exists $a \in L_1$ such that $\mathrm{II}_n(a_>, b_>)$ and $\mathrm{II}_n(a_<, b_<)$.

Proof By induction on n. QED

This result generalises as follows to trees.

Proposition 5.2 Let T_1 and T_2 be trees. Then $\mathrm{II}_{n+1}(T_1, T_2)$ if and only if the following two conditions are satisfied:

(i) for every $a \in T_1$, there exists $b \in T_2$ such that $\mathrm{II}_n((C(a); a), (C(b); b))$ and $\mathrm{II}_n(a_<, b_<)$;

(ii) for every $b \in T_2$, there exists $a \in T_1$ such that $\mathrm{II}_n((C(a); a), (C(b); b))$ and $\mathrm{II}_n(a_<, b_<)$.

Proof Let σ be a winning strategy for Player II for the game $\mathrm{EF}_{n+1}(T_1, T_2)$. Let $a \in T_1$ and suppose the response of Player II in the game $\mathrm{EF}_{n+1}(T_1, T_2)$, using the strategy σ, where Player I chooses the node $a \in T_1$ for his first move, is the node $b \in T_2$. Then $\mathrm{II}_n((T_1; a), (T_2; b))$. In particular, $\mathrm{II}_n((C(a); a), (C(b); b))$ and $\mathrm{II}_n(a_<, b_<)$. This proves condition (i). The proof of condition (ii) is similar.

Next assume that the conditions (i) and (ii) hold. We outline a winning strategy for Player II for the game $\mathrm{EF}_{n+1}(T_1, T_2)$. For his first move, suppose Player I chooses the node $a_1 \in T_1$. According to condition (i), there exists

$b_1 \in T_2$ such that $\mathrm{II}_n((C(a_1); a_1), (C(b_1); b_1))$ and $\mathrm{II}_n((a_1)_<, (b_1)_<)$. Player II then responds by choosing the node $b_1 \in T_2$ for her first move.

Let σ_1 and σ_2 be winning strategies for Player II for the games $\mathrm{EF}_n((C(a_1); a_1), (C(b_1); b_1))$ and $\mathrm{EF}_n((a_1)_<, (b_1)_<)$ respectively. Then Player II plays her remaining n moves according to the strategies σ_1 and σ_2 as follows.

When Player I chooses for his i-th move the node $a_i \in C(a_1) \subseteq T_1$ (respectively $b_i \in C(b_1) \subseteq T_2$) then Player II responds with the node $b_i \in C(b_1) \subseteq T_2$ (respectively $a_i \in C(a_1) \subseteq T_1$) using the strategy σ_1 and based on the nodes that have already been played in $C(a_1)$ and $C(b_1)$.

And when Player I chooses for his i-th move the node $a_i \in (a_1)_< \subseteq T_1$ (respectively $b_i \in (b_1)_< \subseteq T_2$) then Player II responds with the node $b_i \in (b_1)_< \subseteq T_2$ (respectively $a_i \in (a_1)_< \subseteq T_1$) using the strategy σ_2 and based on the nodes that have already been played in $(a_1)_<$ and $(b_1)_<$.

The case where Player I begins the game by choosing the node $b_1 \in T_2$ is handled analogously using condition (ii) instead. QED

5.2 First-order definition of trees

The class of forests can be defined using the following first-order sentences:

$\mathsf{Ir} : \forall x\, (x \not< x);$

$\mathsf{Tr} : \forall x \forall y \forall z\, (x < y \wedge y < z \rightarrow x < z);$

$\mathsf{ST} : \forall x \forall y \forall z\, (y < x \wedge z < x \rightarrow y \smile z).$

Adding the sentence

$\mathsf{Co} : \forall x \forall y \exists z\, (z \leqslant x \wedge z \leqslant y)$

gives a first-order definition of the class of trees. The class of all linear orders can be first-order defined using the sentences Ir and Tr, together with the sentence

$\mathsf{To} : \forall x \forall y\, (x \smile y).$

The set of sentences consisting of Ir, Tr and ST, which define the class of forests, is denoted as $\mathsf{A_F}$, while the set of sentences $\mathsf{A_F} \cup \{\mathsf{Co}\}$, which defines the class of trees, is denoted as $\mathsf{A_T}$. The set of sentences consisting of Ir, Tr and To, which defines the class of linear orders, is denoted as $\mathsf{A_L}$.

5.3 Nodes

5.3.1 Some definable nodes

Roots and leaves can be defined using the respective formulas

$$\begin{aligned} \mathrm{root}(x) &:= \forall y\,(x \leqslant y)\,, \\ \mathrm{leaf}(x) &:= \forall y\,(x \leqslant y \to x = y)\,. \end{aligned}$$

It is known (see [24] and also Section 7.1 below) that for every ordinal α with $\alpha < \omega^\omega$, there exists a first-order sentence Φ_α which axiomatises the first-order theory of α, and $\Phi_\alpha \equiv \Phi_\beta$ if and only if $\alpha = \beta$. Hence the set of nodes in a well-founded tree T having level α with $\alpha < \omega^\omega$ can be defined using the formula

$$\mathrm{level}_\alpha(x) := \Phi_\alpha^{<x}.$$

The next result shows that in well-founded trees T of height less than ω^ω, the ability of nodes to define subsets of T improves with the level of those nodes.

Proposition 5.3 Let T be a well-founded tree of height less than ω^ω. Let A be a set of nodes in T definable using the formula $\varphi(x, \bar{z})$ with parameters $\bar{c}$ from T substituted for $\bar{z}$, where $\bar{z} = (z_1, \ldots, z_k)$ and $\bar{c} = (c_1, \ldots, c_k)$. For every i, let $d_i \in T$ with $c_i \leqslant d_i$. Then there is a formula $\psi(x, \bar{z})$ which defines A with the parameters $\bar{d}$ substituted for $\bar{z}$, where $\bar{d} = (d_1, \ldots, d_k)$.

Proof For every i, suppose c_i has level α_i. Then c_i can be defined in T using the formula $\gamma_i(y, z) := y \leqslant z \wedge \mathrm{level}_{\alpha_i}(y)$ with the parameter d_i substituted for z. Hence take

$$\psi(x, \bar{z}) := \forall y_1 \ldots \forall y_k \left(\bigwedge_{i=1}^{k} \gamma_i(y_i, z_i) \to \varphi(x, y_1, \ldots, y_k) \right).$$

QED

In particular, in well-founded trees of height less than ω^ω satisfying the property

Do : $\forall x \exists y\,(\mathrm{leaf}(y) \wedge x \leqslant y)$

every parametrically definable set of nodes can be defined using leaves as parameters. The sentence Do states that every node is dominated by a leaf.

5.3.2 Neighbourhoods of nodes

Define the formulas

$$\begin{aligned} d_0(x,y) &:= x = y, \\ d_1(x,y) &:= s(x,y) \vee s(y,x), \end{aligned}$$

and for $k \geqslant 2$,

$$d_k(x,y) := \exists z_1 \cdots \exists z_{k-1} \left(\bigwedge_{i \neq j} z_i \neq z_j \wedge d_1(x,z_1) \wedge \bigwedge_{i=1}^{k-2} d_1(z_i,z_{i+1}) \wedge d_1(z_{k-1},y) \right).$$

Let T be a tree and let $a,b \in T$. For $k \geqslant 2$, $T \models d_k(a/x,b/y)$ if and only if a and b can be reached from one another by traversing exactly $k-1$ nodes along the order relation of T. It is easy to see that there is at most one value of k for which $T \models d_k(a/x,b/y)$.

Let F be any set of nodes in T with the property that, for all $u,v \in F$, $T \models d_k(u/x,v/y)$ for some natural number k. For $u,v \in F$ define $\rho_F(u,v)$ to be the unique natural number k for which $T \models d_k(u/x,v/y)$. Then $\rho_F : F^2 \to \mathbb{R}$ forms a metric on the set of nodes F. In particular, if T is an ω-tree then ρ_T forms a metric on the entire set of nodes in T.

For $k \geqslant 0$ define the formula

$$r_k(x,y) := \bigvee_{i=0}^{k} d_k(x,y).$$

For every $a \in T$ we define the **neighbourhood** of a of radius k as the set

$$N_k(a) := \{v \in T : T \models r_k(a/x,v/y)\}.$$

For $a \in T$ and $G := \{v \in T : T \models d_i(a/x,v/y) \text{ for some } i \in \mathbb{N}\}$ it is clear that $N_k(a) = \{v \in T : \rho_G(a,v) \leqslant k\}$.

5.4 Paths

5.4.1 Path-defining formulas

For $k \in \mathbb{N}$ and $\varphi(x,\bar{z})$ any formula with $\bar{z} = (z_1,\ldots,z_k)$, define the formula

$$\pi_\varphi(\bar{z}) := \exists x\, \varphi(x,\bar{z}) \wedge \forall x \forall y\, (\varphi(x,\bar{z}) \wedge \varphi(y,\bar{z}) \to x \smile y) \wedge \forall x \forall y\, (x < y \wedge \varphi(y,\bar{z}) \to \varphi(x,\bar{z})) \wedge \neg \exists x \forall y\, (\varphi(y,\bar{z}) \to y < x)\,. \quad (5.1)$$

If $k = 0$ then π_φ becomes a sentence. Moreover, it is clear that if φ has quantifier rank n then π_φ has quantifier rank $n + 2$.

The formula π_φ formalises the claim that the formula φ defines a path.

Proposition 5.4 Let T be a tree and let $\bar{c}$ be a k-tuple of nodes from T. The formula $\varphi(x, \bar{z})$ (with $\bar{z} = (z_1, \ldots, z_k)$) defines a path in T with parameters $\bar{c}$ substituted for $\bar{z}$ if and only if $T \models \pi_\varphi(\bar{c}/\bar{z})$.

Proof Define $A := \{u \in T : (T; \bar{c}) \models \varphi(u/x, \bar{c})\}$.

If A is a path then it is straightforward to see that each of the four conjuncts in (5.1) hold true in T with $\bar{c}$ substituted for $\bar{z}$.

Next assume that $T \models \pi_\varphi(\bar{c}/\bar{z})$. From the first conjunct in (5.1), A is non-empty. From the second conjunct in (5.1), A is total.

Let $B \subseteq T$ be total and with $A \subseteq B$. Let $b \in B$. By the fourth conjunct in (5.1), there exists $a \in A$ with $b \leqslant a$. The third conjunct in (5.1) then gives that $b \in A$. Hence $A = B$. It follows that A is maximal total, hence A is a path. QED

It follows that if T_1 and T_2 are trees with $T_1 \equiv T_2$ then a formula φ defines a path in T_1 if and only if φ defines a path in T_2.

Proposition 5.5 Let T be a finitely branching tree which is well-founded and in which every node has finite level. Let A be a path in T definable using parameters $\bar{c} = (c_1, \ldots, c_k)$. Then there exists $d \in A$ such that A is definable using only d as parameter.

Proof We first show that the parameter c_k can be replaced with a parameter d_k from A itself. Hence let $\varphi(x, \bar{z})$ define A in T with $\bar{c}$ substituted for $\bar{z}$, where $\bar{z} = (z_1, \ldots, z_k)$. Suppose φ has quantifier rank n and that c_k has level l_k. Let

$$B := \Big\{ u \in T : T \models [\![(T; \bar{c})]\!]^{n+2}(c_1/x_1, \ldots, c_{k-1}/x_{k-1}, u/x_k) \text{ and } u \text{ has level } l_k \Big\}.$$

From the fact that T is finitely branching and that every node in T has finite level it follows that B is finite. B can be defined in T using the formula

$$\xi(x, z_1, \ldots, z_{k-1}) := [\![(T; \bar{c})]\!]^{n+2}(z_1, \ldots, z_{k-1}, x) \wedge \text{level}_{l_k}(x)$$

with parameters $c_1, \ldots, c_{k-1}$ substituted for $z_1, \ldots, z_{k-1}$.

Since $(T;\bar{c}) \models \pi_\varphi(\bar{c})$ and $(T;c_1,\ldots,c_k) \equiv_{n+2} (T;c_1,\ldots,c_{k-1},u)$ for every $u \in B$ then φ defines a path in T with $c_1,\ldots,c_{k-1},u$ substituted for $z_1,\ldots,z_k$. Hence the formula

$$\zeta(x,z_1,\ldots,z_{k-1}) := \exists y\,(\xi(y,z_1,\ldots,z_{k-1}) \wedge \varphi(x,z_1,\ldots,z_{k-1},y))$$

defines a downwards convex subtree T_0 of T with the parameters $c_1,\ldots,c_{k-1}$ substituted for $z_1,\ldots,z_{k-1}$ and where T_0 contains only finitely many paths, amongst which is the path A.

Choose any $d_k \in A$ such that d_k does not belong to any path in T_0 other than the path A. Then A can be defined in T using the formula

$$\chi(x,\bar{z}) := \zeta(x,z_1,\ldots,z_{k-1}) \wedge x \smile z_k$$

with the parameters $c_1,\ldots,c_{k-1},d_k$ substituted for $z_1,\ldots,z_k$. Hence we have succeeded in replacing the parameter c_k with a parameter d_k from A.

Repeating this procedure for the parameters $c_{k-1},\ldots,c_1$, we eventually obtain nodes $d_1,\ldots,d_k \in A$ and a formula $\chi'(x,z_1,\ldots,z_k)$ which defines A in T with the parameters $d_1,\ldots,d_k$ substituted for $z_1,\ldots,z_k$. Suppose without loss of generality that $d_i \leqslant d_1$ for every i ($i \geqslant 2$) and that the level of d_i is m_i. Then d_i can be defined in T using the formula $x_i \leqslant z \wedge \mathrm{level}_{m_i}(x_i)$ with the parameter d_1 substituted for z. It follows that A can be defined in T using the formula

$$\psi(x,z) := \forall z_2 \ldots \forall z_k \left(\bigwedge_{i=2}^{k} \Big(z_i \leqslant z \wedge \mathrm{level}_{m_i}(z_i) \Big) \rightarrow \chi'(x,z,z_2,\ldots,z_k) \right)$$

with d_1 substituted for z. Hence take $d = d_1$. QED

Lemma 5.6 Let T be a tree and let A be a path in T that is not parametrically definable.

(i) For every $a \in A$ and $n \in \mathbb{N}$, there exists $b \in A$ and $c \in T\backslash A$ with $b,c \geqslant a$ and such that $b_\leqslant \equiv_n c_\leqslant$.

(ii) For every $a \in A$ and $n \in \mathbb{N}$, there exists $b \in A$ and $c \in T\backslash A$ with $b,c \geqslant a$ and such that $C(b) \equiv_n C(c)$.

Proof (i) Let $a \in A$ and $n \in \mathbb{N}$ but suppose to the contrary that $u_\leqslant \not\equiv_n v_\leqslant$ for every $u \in A$ and for every $v \in T\backslash A$ with $u,v \geqslant a$. Let $\tau_1,\ldots,\tau_m$ be all n-characteristics of empty tuples over the language of ordered sets. Let $U := \{i : u_{\leqslant} \models \tau_i$ for some $u \in A$ with $u \geqslant a\}$. Then for every u satisfying $u \geqslant a$, we have that $u_{\leqslant} \models \tau_i$ for some $i \in U$ if and only if $u \in A$. But then A can be defined in T using the formula

$$\varphi(x,z) := x < z \vee x \geqslant z \wedge \left(\bigvee_{i \in U} \tau_i^{\geq x} \right)$$

with the parameter a substituted for z, a contradiction.

(ii) Note that for every $d \in T$, the subtree $C(d)$ of T can be defined in T using the formula

$$\theta(w,y) := \neg\,(y < w)$$

with the parameter d substituted for y. The proof is then similar to that of part (i). QED

5.4.2 Singular and emergent paths

Definition 5.7 Let T be a tree and let A be a path in T. A is called **singular** if there exists $a \in A$ such that $a_{\leqslant}$ is total. Otherwise the path A is called **emergent**. If $\mathcal{B}$ is a set of paths in T with $A \notin \mathcal{B}$ and with $A \subseteq \bigcup \mathcal{B}$ then A is said to **emerge** from $\mathcal{B}$.

For a more detailed analysis of singular and emergent paths, the reader is referred to [13].

Example 5.8 Let T be the tree obtained by taking the linear order $A := \omega$ and at each point in A, we adjoin a copy of ω (see Figure 5.1). Thus every path in T is isomorphic with ω. The path A is an emergent path, while every other path in T is singular.

It is immediate that every path containing a leaf must be singular.

Proposition 5.9 Let T be a tree and let A be a singular path in T. Then A is parametrically definable.

Proof Let $a \in A$ such that $a_{\leqslant}$ is total. Then A can be defined in T using the formula

$$\varphi(x,z) := x \smile z$$

with the parameter a substituted for z. QED

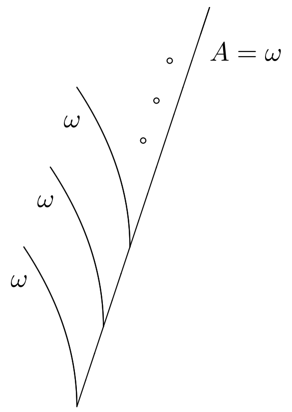

Figure 5.1: Singular and emergent paths (see Example 5.8).

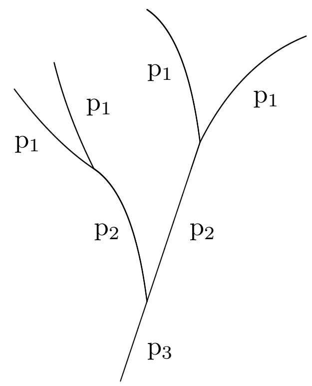

Figure 5.2: The formula p_n (see Example 5.12(a)).

Proposition 5.10 Let T be a tree satisfying the sentence Do. Every path in T which is not parametrically definable emerges from a set of parametrically definable paths from T.

Proof Let A be a path in T which is not parametrically definable. For every $x \in A$ there exists a leaf a_x with $a_x \notin A$. Let $\mathcal{B} := \{(a_x)_{\geqslant} : x \in A\}$. Then $\mathcal{B}$ is a set of parametrically definable paths and A emerges from $\mathcal{B}$. QED

The notion of an emergent path can be further refined as follows. For every $n \in \mathbb{N}^+$, define the formula[1]

$$\mathrm{p}_n(x) := \forall y_1 \dots \forall y_n \left(x \leqslant y_1 \leqslant y_2 \leqslant \dots \leqslant y_n \to \beta(x, y_1) \vee \bigvee_{i=1}^{n-1} \beta(y_i, y_{i+1}) \right).$$

For T a tree and $a \in T$ we have that $T \models \mathrm{p}_n(a/x)$ if and only if any strictly ascending sequence of maximal bridges starting with the maximal bridge $[a]$ consists of at most n maximal bridges. Note that if $T \models \mathrm{p}_n(a/x)$ then $T \models \mathrm{p}_m(a/x)$ for all m with $m \geqslant n$.

Definition 5.11 Let T be a tree and let A be a path in T. Then A is called a **peripheral** path if there exists $a \in A$ and $n \in \mathbb{N}^+$ such that, for every $u \in a_{\leqslant} \backslash A$, we have $T \models \mathrm{p}_n(u/x)$. Otherwise A is called an **internal** path.

[1]Recall that the formula $\beta(x, y)$, defined on p. 21, states that x and y belong to the same maximal bridge.

Intuitively a path A in a tree T is peripheral when for some $n \in \mathbb{N}^+$, nodes high up in A can be reached from the top-end of the tree by traversing at most n maximal bridges along paths which branch off from A.

Example 5.12 (a) The tree depicted in Figure 5.2 shows the maximal bridges where which the formula p_n ($n = 1, 2, 3$) holds for nodes in those bridges.

(b) The emergent path A in the tree depicted in Figure 5.1 is a peripheral path.

(c) For every $n \in \mathbb{N}$, let B_n denote the binary $(\mathbf{n+1})$-tree (up to isomorphism), i.e. B_n is the binary tree of which every path is isomorphic to the linear order $\mathbf{n+1}$. Let T be the tree obtained by taking the linear order $C := \omega$ and at every point n in C, we adjoin the tree B_n, as shown in Figure 5.3. Then the path C is internal.

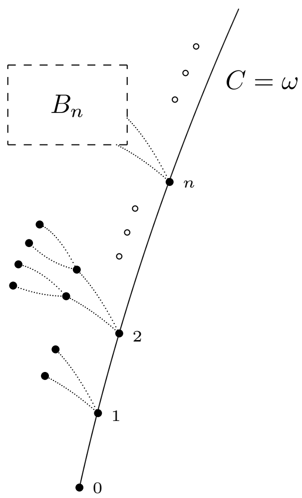

Figure 5.3: An internal path (see Example 5.12(c)).

Note that every singular path is vacuously peripheral.

Lemma 5.13 Let T be a tree and let $n \in \mathbb{N}^+$. The set $T^{\neg \mathrm{p}_n}$ is downwards convex in T.

Proof Let $a \in T^{\neg \mathrm{p}_n}$ and let $b < a$. Hence $T \models \neg \mathrm{p}_n(a/x)$ and since

$$\neg \mathrm{p}_n(x) = \exists y_1 \ldots \exists y_n \Big(x \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n \wedge \neg\beta(x, y_1) \wedge \bigwedge_{i=1}^{n-1} \neg\beta(y_i, y_{i+1}) \Big)$$

then from the transitivity of the relation $<$ we get that $T \models \neg \mathrm{p}_n(b/x)$. Hence $b \in T^{\neg \mathrm{p}_n}$, as required. QED

Lemma 5.14 Let T be a tree and let $n \in \mathbb{N}^+$. Then $T^{\neg \mathrm{p}_n}$ is a subtree of T.

Proof We need to show that $T^{\neg \mathrm{p}_n}$ is connected. Let $a, b \in T^{\neg \mathrm{p}_n}$. From the connectedness of T there exists $c \in T$ such that $c \leqslant a, b$. From Lemma 5.13 we get $c \in T^{\neg \mathrm{p}_n}$, as required. QED

Intuitively the tree $T^{\neg \mathrm{p}_n}$ is the tree that remains when all branches which are contained within singular paths and which consist of at most n distinct maximal bridges have been removed from T.

Peripheral paths are related to singular paths in the following way.

Lemma 5.15 Let T be a tree and let A be an emergent path in T that is also peripheral. There exists $n \in \mathbb{N}^+$ such that A is a singular path in $T^{\neg \mathrm{p}_n}$.

Proof Since A is emergent then it follows that $T \models \neg \mathrm{p}_m(u/x)$ for every $m \in \mathbb{N}^+$ and for all $u \in A$ and so $A \subseteq T^{\neg \mathrm{p}_m}$. From the fact that A is maximal total in T it follows that A will be maximal total in $T^{\neg \mathrm{p}_m}$. Hence A is a path in $T^{\neg \mathrm{p}_m}$ for every $m \in \mathbb{N}^+$.

Since A is peripheral then there exists $a \in A$ and $n \in \mathbb{N}^+$ such that $T \models \mathrm{p}_n(u/x)$ for every $u \in a_{\leqslant} \backslash A$. Hence $u \notin T^{\neg \mathrm{p}_n}$ for every $u \in a_{\leqslant} \backslash A$. It follows that the branch $\{x \in A : x \geqslant a\}$ is total in $T^{\neg \mathrm{p}_n}$. Hence A is a singular path in $T^{\neg \mathrm{p}_n}$. QED

Proposition 5.16 Let T be a tree and let A be a peripheral path in T. Then A is parametrically definable in T.

Proof In the case where A is a singular path we have already demonstrated that A is parametrically definable in T in Proposition 5.9. Hence assume A is an emergent path in T. Then by Lemma 5.15, there exists $n \in \mathbb{N}^+$ such that A is a singular path in $T^{\neg \mathrm{p}_n}$. From Proposition 5.9 we know that there exists $a \in A$ such that the formula $\varphi(x, z) := x \smile z$ defines A in $T^{\neg \mathrm{p}_n}$ with the parameter a substituted for z. Hence the formula $\varphi^{\neg \mathrm{p}_n}(x, z)$ defines A in T with the parameter a substituted for z. QED

5.5 Subtrees

The next result can also be found in [10].

Proposition 5.17 ([10]) Let $T_1 = (|T_1|\,;<_{T_1})$ be a tree and let $\{S_i : i \in I\}$ be a pairwise disjoint set of upwards convex subtrees of T_1. For every $i \in I$, let $U_i = (|U_i|\,;<_{U_i})$ be a tree with $S_i \equiv_n U_i$. Let T_2 be the tree obtained from T_1 by replacing every subtree S_i with the tree U_i. Formally we define $T_2 = (|T_2|\,;<_{T_2})$ as follows:

- $|T_2| := \big(|T_1| \setminus \bigcup_{i\in I} |S_i|\big) \cup \bigcup_{i\in I} |U_i|$, and
- for $x, y \in |T_2|$, $x <_{T_2} y$ if and only if one of the following conditions are satisfied:
  (i) $x, y \in |T_1| \setminus \bigcup_{i\in I} |S_i|$ and $x <_{T_1} y$, or
  (ii) $x, y \in |U_i|$ for some i and $x <_{U_i} y$, or
  (iii) $x \in |T_1| \setminus |S_i|$ and $y \in |U_i|$ for some i, and $x <_{T_1} z$ for some $z \in |S_i|$.

Then $T_1 \equiv_n T_2$. Consequently if $S_i \equiv U_i$ for every $i \in I$ then $T_1 \equiv T_2$.

Proof A winning strategy for Player II for the game $\mathrm{EF}_n(T_1, T_2)$ is as follows. Whenever Player I chooses a node from $|T_1| \setminus \bigcup_{i\in I} |S_i|$ ($\subseteq |T_1|$) or from $|T_2| \setminus \bigcup_{i\in I} |U_i|$ ($\subseteq |T_2|$), then Player II responds by choosing the exact same node from the tree not used by Player I for that move. And whenever Player I chooses a node from $|S_i|$ ($\subseteq |T_1|$) or from $|U_i|$ ($\subseteq |T_2|$) for some i, then Player II selects a node using her winning strategy for the game $\mathrm{EF}_n(S_i, U_i)$, and based on the nodes already played in S_i and U_i. QED

When an upwards convex subtree of tree is replaced with an elementary extension of that subtree, we have the following result.

Proposition 5.18 Let $T_1 = (|T_1|\,;<_{T_1})$ be a tree and let $\{S_i : i \in I\}$ be a pairwise disjoint set of upwards convex subtrees of T_1. For every $i \in I$, let $U_i = (|U_i|\,;<_{U_i})$ be a tree with $S_i \preceq U_i$. Let T_2 be the tree obtained from T_1 by replacing every subtree S_i with the tree U_i as in Proposition 5.17. Then $T_1 \preceq T_2$.

Proof Let $\bar{c}$ be a tuple of nodes in T_1 and let $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ be all those trees from the set $\{S_i : i \in I\}$ which contain nodes from the tuple $\bar{c}$. Suppose without loss of generality that $\bar{c} = \bar{c}_0\bar{c}_1 \cdots \bar{c}_k$ where $\bar{c}_j$ is a tuple of nodes from S_{i_j} and where $\bar{c}_0$ is a tuple of nodes in $|T_1| \setminus \bigcup_{i\in I} |S_i|$. Hence we have that $S_i \equiv U_i$ for all i with $i \neq i_1, \ldots, i_k$, and $(S_{i_j}; \bar{c}_j) \equiv (U_{i_j}; \bar{c}_j)$ for all j with $j = 1, \ldots, k$.

A winning strategy for Player II for the game $\mathrm{EF}_n((T_1; \bar{c}), (T_2; \bar{c}))$ is as follows. Whenever Player I chooses a node from $|T_1| \setminus \bigcup_{i\in I} |S_i|$ ($\subseteq |T_1|$) or from $|T_2| \setminus \bigcup_{i\in I} |U_i|$ ($\subseteq |T_2|$), then Player II responds by choosing the exact same node from the structure not used by Player I for that move. Whenever Player I chooses, for some i with $i \neq i_1, \ldots, i_k$, a node from $|S_i|$ ($\subseteq |T_1|$), respectively from $|U_i|$ ($\subseteq |T_2|$), then Player II selects a node from $|U_i|$, respectively from $|S_i|$, using her winning strategy for the game $\mathrm{EF}_n(S_i, U_i)$, and based on the nodes already played in S_i and U_i. And finally, whenever Player I chooses, for some j with $j = 1, \ldots, k$, a node from $\left|S_{i_j}\right|$ ($\subseteq |T_1|$), respectively from $\left|U_{i_j}\right|$ ($\subseteq |T_2|$), then Player II selects a node from $\left|U_{i_j}\right|$, respectively from $\left|S_{i_j}\right|$, using her winning strategy for the game $\mathrm{EF}_n\big((S_{i_j}; \bar{c}_j), (U_{i_j}; \bar{c}_j)\big)$, and based on the nodes already played in S_{i_j} and U_{i_j}.

Hence $(T_1; \bar{c}) \equiv_n (T_2; \bar{c})$ and it follows that $T_1 \preceq T_2$. QED

Proposition 5.19 Let T be a tree, let $a \in T$ and let A be a path in the tree $a_{\leqslant}$ which is parametrically definable in $a_{\leqslant}$. Then the path $a_{>} + A$ in T is parametrically definable in T.

Proof Suppose the formula $\varphi(x, \bar{z})$ defines the path A in $a_{\leqslant}$ with parameters $\bar{c}$ from $a_{\leqslant}$ substituted for $\bar{z}$. Then the formula

$$\varphi^{\geqslant u}(x, \bar{z}u) \lor x < u$$

defines $a_{>} + A$ in T with the parameters $\bar{c}a$ substituted for $\bar{z}u$. QED

5.6 Condensations

Proposition 5.20 Let T be a tree, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be classes of linear orders, and let $f : T \to \mathcal{L}_1$ and $g : T \to \mathcal{L}_2$ be functions. Let $n \in \mathbb{N}$ and suppose that for every $x \in T$, $f(x) \equiv_n g(x)$. Then $T \times_f \mathcal{L}_1 \equiv_n T \times_g \mathcal{L}_2$. Consequently if $f(x) \equiv g(x)$ for every $x \in T$ then $T \times_f \mathcal{L}_1 \equiv T \times_g \mathcal{L}_2$.

Proof For every $x \in T$, let σ_x be a winning strategy for Player II for the game $\mathrm{EF}_n(f(x), g(x))$. We will describe a winning strategy for Player II for the game $\mathrm{EF}_n(T \times_f \mathcal{L}_1, T \times_g \mathcal{L}_2)$.

Suppose the first $i-1$ moves of the game consist of the nodes $a_1, \ldots, a_{i-1} \in |T \times_f \mathcal{L}_1|$ and $b_1, \ldots, b_{i-1} \in |T \times_g \mathcal{L}_2|$, where for every j $(1 \leqslant j \leqslant i-1)$ the nodes a_j and b_j have the form $a_j = (a_{j,1}, a_{j,2})$ and $b_j = (b_{j,1}, b_{j,2})$, with $a_{j,1}, b_{j,1} \in T$ and with $a_{j,2} \in f(a_{j,1})$ and $b_{j,2} \in g(b_{j,1})$.

Suppose that for his i-th move, Player I chooses the node $a_i \in |T \times_f \mathcal{L}_1|$, where $a_i = (a_{i,1}, a_{i,2})$ with $a_{i,1} \in T$ and $a_{i,2} \in f(a_{i,1})$. Let $a_{j_1,1}, \ldots, a_{j_k,1}$ be all the nodes from amongst $a_{1,1}, \ldots, a_{i-1,1}$ for which $a_{j_1,1}, \ldots, a_{j_k,1} = a_{i,1}$.

Consider the game $\mathrm{EF}_n(f(a_{i,1}), g(a_{i,1}))$. Suppose that the first k moves of the game consist of the elements $a_{j_1,2}, \ldots, a_{j_k,2} \in f(a_{i,1})$ and $b_{j_1,2}, \ldots, b_{j_k,2} \in g(a_{i,1})$. Suppose that, using the strategy $\sigma_{a_{i,1}}$, the response of Player II when Player I chooses for his $(k+1)$-th move the element $a_{i,2} \in f(a_{i,1})$ is that Player II chooses the element $b_{i,2} \in g(a_{i,1})$.

Let $b_{i,1} = a_{i,1}$. For her i-th move of the game $\mathrm{EF}_n(T \times_f \mathcal{L}_1, T \times_g \mathcal{L}_2)$, Player II then chooses the node $b_i \in |T \times_g \mathcal{L}_2|$ where $b_i = (b_{i,1}, b_{i,2})$.

The case where Player I instead chooses some $b_i \in |T \times_g \mathcal{L}_2|$ for his i-th move is similar.

This choice of nodes will result in a win for Player II for the game $\mathrm{EF}_n(T \times_f \mathcal{L}_1, T \times_g \mathcal{L}_2)$ and the result follows. QED

Example 5.21 In part (a) of this example we show that two given trees T_1 and T_2 are elementarily equivalent. In part (b) we define a class of linear orders $\mathcal{L}_1$ together with a function $f : T_1 \to \mathcal{L}_1$ and then show that there is no function $g : T_2 \to \mathcal{L}_1$ for which $T_1 \times_f \mathcal{L}_1 \equiv T_2 \times_g \mathcal{L}_1$. We do this by showing that for every class of linear orders $\mathcal{L}_2$ and for every function $g : T_2 \to \mathcal{L}_2$ such that $T_1 \times_f \mathcal{L}_1 \equiv T_2 \times_g \mathcal{L}_2$, it must be the case that $\mathcal{L}_1 \subsetneq \mathcal{L}_2$.

(a) Consider the trees T_1 and T_2 as depicted in Figure 5.4 and Figure 5.5. The tree T_1 is obtained by taking the linear order $A := \omega$ and at the i-th element c_{i-1} of A we attach the node c_{i-1}^+. T_2 is obtained by taking the linear order $B := \omega + \zeta$ and at every element x in B we attach the node x^+. Let d_{i-1} be the i-th element in the copy of ω in B; then d_{i-1}^+ is the node attached to it.

Then $T_1 \equiv T_2$. The following describes a winning strategy for Player II for the game $\mathrm{EF}_n(T_1, T_2)$ with $n \in \mathbb{N}$.

Let the first $i-1$ moves of the game consist of the nodes $a_1, \ldots, a_{i-1} \in T_1$

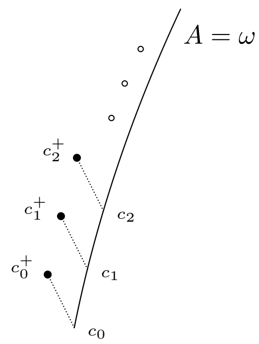

Figure 5.4: The tree T_1 (see Example 5.21(a)).

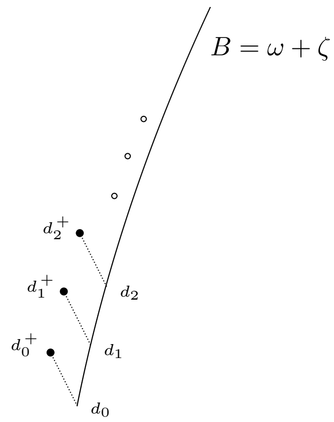

Figure 5.5: The tree T_2 (see Example 5.21(a)).

and $b_1, \ldots, b_{i-1} \in T_2$. For his i-th move, suppose Player I chooses the node $a_i \in T_1$.

For any node x, define x^- as follows: if x has the form $x = y^+$ for some node y then $x^- := y$; otherwise $x^- := x$. In other words, x^- is the greatest node in the path A or in the path B for which $x^- \leqslant x$.

As is well known (see e.g. [24, Corollary 6.12]), $\omega \equiv \omega + \zeta$ and so $A \equiv B$. Thus Player II has a winning strategy for the game $\mathrm{EF}_n(A, B)$. Suppose that the first $i-1$ moves of the game $\mathrm{EF}_n(A, B)$ are $a_1^-, \ldots, a_{i-1}^- \in A$ and $b_1^-, \ldots, b_{i-1}^- \in B$. Using her winning strategy for the game $\mathrm{EF}_n(A, B)$, in response to Player I choosing for his i-th move the node $a_i^- \in A$, let Player II choose the node $b_i^\star \in B$.

If a_i has the form $a_i = x^+$ for some x then Player II's i-th move in the game $\mathrm{EF}_n(T_1, T_2)$ is the node $b_i \in T_2$, where $b_i := (b_i^\star)^+$; otherwise Player II's i-th move is the node $b_i \in T_2$, where $b_i := b_i^\star$.

The case where Player I chooses for his i-th move the node $b_i \in T_2$ is similar.

(b) Let $\mathcal{L}_1 := \{\mathbf{n} : n \in \mathbb{N}^+\}$ and define $f : T_1 \to \mathcal{L}_1$ by specifying that $f(c_0) = \mathbf{2}$, $f(c_k) = \mathbf{1}$ for every $k \in \mathbb{N}^+$, and $f(c_k^+) = \mathbf{k} + \mathbf{1}$ for every $k \in \mathbb{N}$.

The tree $T_1 \times_f \mathcal{L}_1$ is depicted in Figure 5.6. It consists of a path C (isomorphic with ω) with the linear order $\mathbf{n}$ attached to the $(n+1)$-th element of C.

We will now construct a class of linear orders $\mathcal{L}_2$ and a function $g : T_2 \to \mathcal{L}_2$ such that $T_1 \times_f \mathcal{L}_1 \equiv T_2 \times_g \mathcal{L}_2$. In particular, for every such class $\mathcal{L}_2$ we

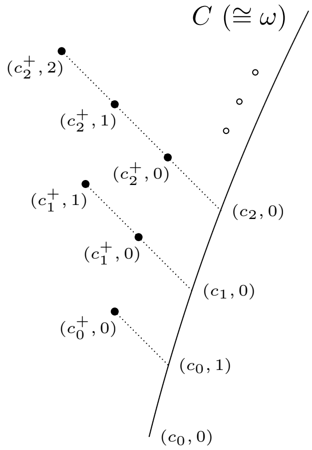

Figure 5.6: The tree $T_1 \times_f \mathcal{L}_1$ (see Example 5.21(b)).

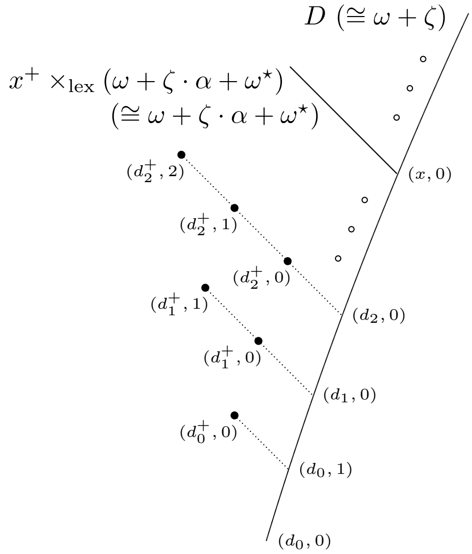

Figure 5.7: The tree $T_2 \times_f \mathcal{L}_2$ (see Example 5.21(b)).

will have $\mathcal{L}_1 \subsetneq \mathcal{L}_2$.

Clearly we will need $\mathbf{1}, \mathbf{2} \in \mathcal{L}_2$ with $g(d_0) = \mathbf{2}$ and $g(x) = \mathbf{1}$ for every $x \in B$ with $x \neq d_0$. Let

$$\varphi(x) := \mathsf{To}^{\geqslant x} \wedge \forall y \left(y < x \rightarrow \neg \mathsf{To}^{\geqslant y}\right).$$

For every $n \geqslant 1$ we have

$$T_1 \times_f \mathcal{L}_1 \models \exists ! x \left(\varphi(x) \wedge (\lambda_n \wedge \mu_n)^{\geqslant x}\right).$$

It follows that we will need $\mathbf{n} \in \mathcal{L}_2$ for every $n \geqslant 3$ and $g(d_k^+) = \mathbf{k+1}$ for every $k \in \mathbb{N}$ if we are to have $T_1 \times_f \mathcal{L}_1 \equiv T_2 \times_g \mathcal{L}_2$, and $g(x^+) \neq \mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^+$ and with x any element in the copy of ζ in B.

As is well known (see e.g. [24, Exercise 6.11]), $\mathbf{k} \equiv_n \omega + \zeta \cdot \alpha + \omega^\star$ for $k \geqslant 2^n - 1$ and for every order type α. Hence let $\mathcal{L}_2$ also contain the class of order types $\{\omega + \zeta \cdot \alpha + \omega^\star : \alpha \in \mathcal{C}\}$ for some class $\mathcal{C}$. Hence $\mathcal{L}_2 := \{\mathbf{n} : n \in \mathbb{N}^+\} \cup \{\omega + \zeta \cdot \alpha + \omega^\star : \alpha \in \mathcal{C}\}$.

For every element x in the copy of ζ in B, let $g(x^+) = \omega + \zeta \cdot \alpha + \omega^\star$ for some order type $\alpha \in \mathcal{C}$ depending on x.

The tree $T_2 \times_g \mathcal{L}_2$ is depicted in Figure 5.7. It consists of a path D, isomorphic with the linear order $\omega + \zeta$, with the linear order $\mathbf{k}$ attached to the $(k+1)$-th element of the copy of ω in D, and with some linear order

$x^+ \times_{\text{lex}} (\omega + \zeta \cdot \alpha + \omega^\star)$ (isomorphic with $\omega + \zeta \cdot \alpha + \omega^\star$) attached to every element x in the copy of ζ in D.

We will now show that $T_1 \times_f \mathcal{L}_1 \equiv T_2 \times_g \mathcal{L}_2$. Fix $n \geqslant 1$ and note that for all $x \in A$ with $x \geqslant c_{2^n-2}$ we have $x^+ \times_{\text{lex}} f(x^+) \equiv_n \omega + \omega^\star$, and for all $x \in B$ with $x \geqslant d_{2^n-2}$ we have $x^+ \times_{\text{lex}} g(x^+) \equiv_n \omega + \omega^\star$. Let S_1 be the tree obtained from $T_1 \times_f \mathcal{L}_1$ by replacing the branch $x^+ \times_{\text{lex}} f(x^+)$ in $T_1 \times_f \mathcal{L}_1$ with the linear order $\omega + \omega^\star$ for every $x \in A$ with $x \geqslant c_{2^n-2}$, and let S_2 be the tree obtained from $T_2 \times_g \mathcal{L}_2$ by replacing the branch $x^+ \times_{\text{lex}} g(x^+)$ in $T_2 \times_g \mathcal{L}_2$ with the linear order $\omega + \omega^\star$ for every $x \in B$ with $x \geqslant d_{2^n-2}$. It then follows from Proposition 5.17 that $T_1 \times_f \mathcal{L}_1 \equiv_n S_1$ and $T_2 \times_g \mathcal{L}_2 \equiv_n S_2$.

Hence S_1 can be seen as consisting of the linear order ω with the linear order $\mathbf{k}$ attached to the $(k+1)$-th element of ω for $k \leqslant 2^n - 2$ and with the linear order $\omega + \omega^\star$ attached to every other element of ω. The tree S_2 can be seen as consisting of the linear order $\omega + \zeta$ with the linear order $\mathbf{k}$ attached to the $(k+1)$-th element of $\omega + \zeta$ for $k \leqslant 2^n - 2$ and with the linear order $\omega + \omega^\star$ attached to every other element of $\omega + \zeta$. By modifying the winning strategy empoyed by Player II for the game $\text{EF}_n(T_1, T_2)$ above, it is easy to see that Player II also has a winning strategy for the game $\text{EF}_n(S_1, S_2)$ and so $S_1 \equiv_n S_2$. Hence $T_1 \times_f \mathcal{L}_1 \equiv_n T_2 \times_g \mathcal{L}_2$ and the result follows.

For $\varphi(x)$ any formula, define the sentence χ_φ as[2]

$$\chi_\varphi := \forall x \exists y \left(\varphi(y) \wedge \beta(x,y)\right) \wedge \forall x \forall y \left(\varphi(x) \wedge \varphi(y) \wedge \beta(x,y) \to x = y\right).$$

The sentence χ_φ states that every maximal bridge contains exactly one node satisfying the formula $\varphi(x)$.

The next result shows that $T^\varphi \cong [T]$ if and only if $T \models \chi_\varphi$.

Proposition 5.22 Let T be a tree and let $\varphi(x)$ be a formula. Consider the function $[\cdot] : T \to [T]$ which maps nodes in T to their maximal bridges in $[T]$ and let $[\cdot]\restriction_{T^\varphi}$ be the restriction of $[\cdot]$ to the set T^φ. Then $[\cdot]\restriction_{T^\varphi}$ is an isomorphism if and only if $T \models \chi_\varphi$.

Proof Straightforward using Proposition 3.12. QED

[2]Recall that the formula $\beta(x,y)$, defined on p. 21, states that x and y belong to the same maximal bridge.

Example 5.23 Let T be a well-founded tree. Every maximal bridge in T contains a unique minimal node hence $[T]$ is isomorphic to the tree formed by these minimal nodes taken from each maximal bridge. It follows from Proposition 3.14 that the condensation of T can be defined up to isomorphism using the formula

$$\varphi(x) := \forall y \, (y < x \rightarrow \exists z \, (z \smile y \wedge z \not< x)) \, .$$

Note that the root of T vacuously satisfies the formula $\varphi(x)$.

Chapter 6

First-order theories of trees

We now look at the first-order theories of some important classes of trees. In Section 6.1 (Well-founded trees) we describe the construction used in [5] to prove that every definably well-founded tree has a well-founded n-equivalent. In Section 6.2 (Finitely branching trees) we show how it is possible in any tree to remove all but finitely many components extending a stem so that the tree obtained is n-equivalent to the original tree. This result is a special case of the result in [27] that every weakly boundedly branching tree T has a subtree S for which $S \preceq_n T$. In Section 6.3 (Finite trees) we axiomatise the first-order theory of the class of finite trees by adapting the method used in [1] to axiomatise the first-order theory of the class of finite ordered trees. In Section 6.4 (Condensations) we show how the first-order theory of a tree may be determined using the first-order theory of its condensation and the first-order theories of the maximal bridges in the tree. Finally Section 6.5 (The $\mathcal{C}$-classes of trees) completely establishes the relationships between the first-order theories of the various $\mathcal{C}$-classes of trees. We also investigate the general problem of axiomatising the various $\mathcal{C}$-classes of trees using the first-order theory of the class $\mathcal{C}$.

6.1 Well-founded trees

A tree T is called **definably well-founded** when every parametrically definable non-empty set of nodes in T contains a minimal element. The property of being definably well-founded can be formalised using the scheme $\mathsf{A_W}$ consisting of the sentences

$$\forall \bar{z}\, (\exists x \varphi(x, \bar{z}) \to \exists x\, (\varphi(x, \bar{z}) \wedge \forall y\, (\varphi(y, \bar{z}) \wedge y \leqslant x \to y = x)))$$

for all $k \in \mathbb{N}$ and for every formula $\varphi(x, \bar{z})$ with $\bar{z} = (z_1, \ldots, z_k)$.

The dual of the property of well-foundedness in a tree states that every non-empty set of nodes in the tree contains a maximal node. The property that every parametrically definable non-empty set of nodes contains a maximal node can be formalised using the scheme $\mathsf{A_{WD}}$, where $\mathsf{A_{WD}}$ consists of the sentences

$$\forall \bar{z}\, (\exists x \varphi(x, \bar{z}) \to \exists x\, (\varphi(x, \bar{z}) \wedge \forall y\, (\varphi(y, \bar{z}) \wedge x \leqslant y \to y = x)))$$

for all $k \in \mathbb{N}$ and for every formula $\varphi(x, \bar{z})$ with $\bar{z} = (z_1, \ldots, z_k)$.

Proposition 6.1 Let T be a tree.

(i) If T satisfies the scheme $\mathsf{A_W}$ (in particular when T is well-founded) then T is upwards discrete.

(ii) If T satisfies the scheme $\mathsf{A_{WD}}$ (in particular when T satisfies the dual of the property of well-foundedness) then T is downwards discrete.

Proof (i) Let T satisfy the scheme $\mathsf{A_W}$. Let A be a path in T and let a be a non-leaf node in A. Then there exists $b \in A$ with $a < b$. Define $\varphi(x, z_1, z_2) := z_1 < x \leqslant z_2$. The formula $\varphi(x, z_1, z_2)$ defines the interval $(a, b]$ in T with the parameters a and b substituted for z_1 and z_2 respectively. From $\mathsf{A_W}$ this interval $(a, b]$ contains a minimal node c which clearly satisfies the condition that $c \in A$ and c is an immediate succesor to a. Hence T is upwards discrete.

(ii) Similar to (i). QED

Proposition 6.2 Let T be a tree.

(i) Suppose every path in T contains a leaf and T is downwards discrete. If T satisfies the scheme $\mathsf{A_W}$ then T satisfies the scheme $\mathsf{A_{WD}}$. If T is well-founded then T satisfies the dual of the property of well-foundedness.

(ii) Suppose T is rooted and upwards discrete. If T satisfies the scheme $\mathsf{A_{WD}}$ then T satisfies the scheme $\mathsf{A_W}$. If T satisfies the dual of the property of well-foundedness then T is well-founded.

Proof (i) Suppose every path in T contains a leaf and T is downwards discrete and let T satisfy the scheme $\mathsf{A_W}$. Let $\bar{c}$ be a k-tuple of nodes in T and let $\varphi(x,\bar{z})$ be a formula with $\bar{z}$ a k-tuple of variables and suppose that the set A defined in T by $\varphi(x,\bar{z})$ with the parameters $\bar{c}$ substituted for $\bar{z}$ is nonempty. In order to show that T satisfies the scheme $\mathsf{A_{WD}}$ we need to show that A contains a maximal node. If A contains a leaf then we are done so assume A does not contain any leaves. Define

$$\psi(x,\bar{z}) := \neg\exists y\,(x \leqslant y \wedge \varphi(y,\bar{z})) \wedge \exists y\,(y < x \wedge \varphi(y,\bar{z}))\,.$$

The set B defined in T by $\psi(x,\bar{z})$ with the parameters $\bar{c}$ substituted for $\bar{z}$ is non-empty since it will contain a leaf and by the scheme $\mathsf{A_W}$ it follows that B contains a minimal node a. Since T is downwards discrete then a has an immediate predecessor b and the node b clearly is a maximal element of A as required.

A similar argument can be used to show that if T is well-founded then T satisfies the dual of the property of well-foundedness.

(ii) Similar to (i). QED

It is worth noting that the property of a tree being downwards discrete can be formalised using the sentence

$$\mathsf{D_1} : \forall x\,(\exists y\,(y < x) \rightarrow \exists y\,(y < x \wedge \forall z\,(\neg\,(y < z < x))))$$

and the property of a tree being upwards discrete can be formalised using the sentence

$$\mathsf{D_2} : \forall x \forall y\,(x < y \rightarrow \exists z\,(x < z \leqslant y \wedge \forall u\,(\neg\,(x < u < z)))).$$

The following result is taken from [5] where a detailed proof can also be found. We will give an outline of the proof.

Theorem 6.3 ([5, Theorem 4.1]) Let T be a definably well-founded tree. For every $n \in \mathbb{N}$ there is a well-founded tree S such that $S \equiv_n T$.

Proof The basic idea is to move up from the root of the tree and systematically replace subtrees which are not well-founded with well-founded n-equivalents.

The first part of the proof consists of showing that if T is a definably well-founded tree then for $a, b \in T$ with $a < b$, there exists $(R;\beta)$ with β

a well-ordered interval in R and with all the components of $R\backslash\beta$ definably well-founded and such that $(a_{\leqslant}\backslash b_{\leqslant}, [a,b)) \equiv_n (R;\beta)$. For let X be the set of nodes b in T with the property that, if $a < b$ then there exists $(R;\beta)$ with β well-ordered and with all the components of $R\backslash\beta$ definably well-founded and with $(a_{\leqslant}\backslash b_{\leqslant}, [a,b)) \equiv_n (R;\beta)$, but suppose $X \neq T$. Let $\tau_1, \ldots, \tau_k$ be the characteristic formulas of rank n over the empty tuple which are satisfied in trees $(R;\beta)$ for which β is well-ordered and for which all the components of $R\backslash\beta$ are definably well-founded. Assume that the sentences τ_i do not contain the variables x and y and for each i, let $\tau_i'(x,y)$ be the formula obtained from τ_i by replacing every instance of the expression $\beta(u)$ with the expression $x \leqslant u < y$. Let

$$\theta(u, y_1, y_2) := y_1 \leqslant u \wedge \neg\,(y_2 \leqslant u).$$

The formula $\theta(u, y_1, y_2)$ defines the set $a_{\leqslant}\backslash b_{\leqslant}$ in $(T; a, b)$ with a substituted for y_1 and b substituted for y_2. Then the formula

$$\varphi(y) := \forall x \left(x < y \rightarrow \bigvee_{i=1}^{k} \left(\tau_i'(x,y)\right)^{\theta(u,x,y)} \right)$$

defines the set X in T. Hence $T\backslash X$ is definable and so contains a minimal node b. Let $a \in T$ with $a < b$ be such that $(a_{\leqslant}\backslash b_{\leqslant}, [a,b))$ does not have an n-equivalent $(R;\beta)$ of the required form. By the minimality of b together with the fact that T is definably well-founded it follows that b does not have an immediate predecessor. Hence let $\{a_\xi\}_{\xi<\alpha} \subseteq [a,b)$ be cofinal in $[a,b)$ with $a_0 = a$ and $a_\xi < a_\zeta$ for $\xi < \zeta$. For every ξ let $(R_\xi;\beta_\xi)$ be a tree with β_ξ well-ordered and with all the components of $R_\xi\backslash\beta_\xi$ definably well-founded and with $((a_\xi)_{\leqslant}\backslash(a_{\xi+1})_{\leqslant}\,;\,[a_\xi, a_{\xi+1})) \equiv_n (R_\xi;\beta_\xi)$. Now take $(R;\beta)$ to be the tree obtained as the union of the trees $(R_\xi;\beta_\xi)$ by glueing the segments β_ξ one after the other in ascending order of the index ξ. From its construction, β will be well-ordered and all the components of $R\backslash\beta$ will be definably well-founded, and it can be seen using an Ehrenfeucht-Fraïssé game that $(a_{\leqslant}\backslash b_{\leqslant}, [a,b)) \equiv_n (R;\beta)$, as required. This completes the first part of the proof.

Using this result, we can next show that if T is a definably well-founded tree and if $b \in T$ then there exists a tree R with $c \in R$ and such that (i) the interval $c_{>}$ is well-ordered, (ii) all the components of $R\backslash c_{>}$ are definably well-founded, and (iii) $(T;b) \equiv_n (R;c)$. This is done by first noting that, being definably well-founded, T will be rooted. Then apply the result of the first part of the proof by taking a as the root of T and replacing $(a_{\leqslant}\backslash b_{\leqslant}, [a,b))$ with an appropriate n-equivalent and take $c = b$ so as to obtain R.

Using induction, we next extend this result to state the following: if T is a definably well-founded tree and if $B \subseteq T$ is finite then there exists a tree R with $C \subseteq R$ and such that (i) the interval $c_>$ is well-ordered for every $c \in C$, (ii) every component of $R\backslash\big(\bigcup_{c\in C} c_>\big)$ is definably well-founded, and (iii) $(T; b)_{b\in B} \equiv_n (R; c)_{c\in C}$.

Finally we construct a sequence of trees $T_0, T_1, \ldots$ and a sequence of sets $A_0, A_1, \ldots$ with $A_i \subseteq T_i$ as follows. Take $T_0 := T$ and $A_0 := \emptyset$. Then given the tree T_i and set A_i for some i, the tree T_{i+1} and set A_{i+1} are obtained as follows. For every component C in $T_i\backslash A_i$ choose $B \subseteq C$ in such a way that, for every $c \in C$ there exists $b \in B$ with $(C; b) \equiv_{n-1} (C; c)$, and with B finite. This can be done since there are only finitely many characteristic formulas of any given rank over any given tuple of variables. Then from the previous result in the proof we know there is a tree C' and a set of nodes $B' \subseteq C'$ such that (i) the interval $b'_>$ is well-ordered for every $b' \in B'$, (ii) every component of $C'\backslash\big(\bigcup_{b'\in B'} b'_>\big)$ is definably well-founded, and (iii) $(C; b)_{b\in B} \equiv_n (C', b')_{b'\in B'}$. Then T_{i+1} is obtained from T_i by replacing every component C in $T_i\backslash A_i$ with the tree C' and A_{i+1} is obtained as the union of the set A_i together with the sets $\bigcup_{b'\in B'} b'_{\geqslant}$ for every set B'.

Then take $S := \bigcup_{i=1}^{\infty} A_i$ where each set A_i is treated as a substructure of the tree T_i. From the way the tree S is constructed it will be well-founded, and it can be seen, using an Ehrenfeucht-Fraïssé game, that $S \equiv_n T$. That this works relies on the fact that every A_{i+1} was chosen to be large enough as to capture all first-order behaviour up to n-equivalence in the structure $T_i\backslash A_i$. QED

Theorem 6.4 ([5]) The first-order theory of the class of well-founded trees can be axiomatised using the theory

$$\mathsf{A_T} \cup \mathsf{A_W}.$$

Proof Immediate from Theorem 6.3. QED

6.2 Finitely branching trees

Motivated by [27], we call a tree T **weakly n-branching** when, for every $x, y \in T$, the set T_{xy} has at most n components, and **weakly boundedly**

branching when T is weakly n-branching for some natural number n. A forest is called **weakly n-branching** when it has at most n components and each of its components is weakly n-branching, and **weakly boundedly branching** when it is weakly n-branching for some n.

Proposition 6.5 Let k be the n-characteristic index of the language with equality and order. Let T be a tree and let $a, b \in T$ be nodes for which $|T_{ab}| > nk$ (in particular, when T_{ab} is infinite). Then it is possible to remove all but nk components from T_{ab} so as to obtain a tree S for which $S \equiv_n T$.

Proof Let $\tau_1, \ldots, \tau_k$ be the characteristic formulas of rank n over the empty tuple. For every q with $1 \leqslant q \leqslant k$, we perform the following construction. Let $\{A_i^q\}_{i \in I_q}$ be the components in T_{ab} for which $A_i^q \models \tau_q$. If $|I_q| > n$ then remove all but any n components in $\{A_i^q\}_{i \in I_q}$ from T.

Once this construction has been done for every q, we are left with at most nk components in T_{ab}. Let S be the tree thus obtained. Then $S \equiv_n T$. To see this consider the Ehrenfeucht-Fraïssé game $\mathrm{EF}_n(T, S)$. We will describe a winning strategy for Player II for this game.

Suppose the first j moves of the game consist of nodes $a_1, \ldots, a_j \in T$ and $b_1, \ldots, b_j \in S$. For every q with $1 \leqslant q \leqslant k$ and for every i with $i \in I_q$, let $(a_i^q)_1, \ldots, (a_i^q)_{j(q,i)}$ be the nodes already played from T for which $(a_i^q)_1, \ldots, (a_i^q)_{j(q,i)} \in A_i^q$, and let $(b_i^q)_1, \ldots, (b_i^q)_{j(q,i)}$ be the corresponding nodes played from S. Assume the game has been played such that for every q and i, there exists $A_{i'}^q$ for which the nodes already played from S and belonging to $A_{i'}^q$ are precisely $(b_i^q)_1, \ldots, (b_i^q)_{j(q,i)} \in A_{i'}^q$.

First consider the case where Player I selects, for his $(j+1)$-th move, a node t from T with $t \notin T_{ab}$. Since $T \backslash T_{ab} = S \backslash T_{ab}$ then the identity map determines a natural correspondence between nodes in $T \backslash T_{ab}$ and $S \backslash T_{ab}$. Player II hence responds for her $(j+1)$-th move by choosing the node from S corresponding to t. Likewise when Player I chooses for his $(j+1)$-th move a node s from S with $s \notin T_{ab}$, then Player II chooses the corresponding node from T.

Next consider the case where Player I chooses, for his $(j+1)$-th move, a node t from T with $t \in T_{ab}$. If $t \in A_i^q$ and if no node has yet been selected from A_i^q by either player for any of their earlier moves, then there exists $A_{i'}^q$ in S from which no nodes have been played yet either. Using her winning strategy for the game $\mathrm{EF}_n(A_i^q, A_{i'}^q)$, Player II then selects a node s from $A_{i'}^q$ in response to the node t chosen by Player I from A_i^q. Likewise when Player

I selects a node s from S with $s \in A^q_{i'}$ for some $A^q_{i'}$ from which no nodes have been played yet.

Finally consider the case where Player I chooses, for his $(j+1)$-th move, a node t from T with $t \in T_{ab}$, where $t \in A^q_i$, and for which the nodes already played from A^q_i are $(a^q_i)_1, \ldots, (a^q_i)_{j(q,i)}$. Player II responds by choosing a node s from $A^q_{i'}$ using her winning strategy for the game $\mathrm{EF}_n(A^q_i, A^q_{i'})$, where the first $j(q,i)$ nodes played for this game are $(a^q_i)_1, \ldots, (a^q_i)_{j(q,i)} \in A^q_i$ and $(b^q_i)_1, \ldots, (b^q_i)_{j(q,i)} \in A^q_{i'}$, and where the $(j(q,i)+1)$-th move of Player I for this game is t. Likewise when Player I selects a node s from S with $s \in A^q_{i'}$ for some $A^q_{i'}$ from which the nodes already played are $(b^q_i)_1, \ldots, (b^q_i)_{j(q,i)}$.

Clearly this strategy constitutes a winning strategy for Player II for the game $\mathrm{EF}_n(T, S)$ hence $S \equiv_n T$. QED

More generally, using this same technique, we can convert a tree T of which only finitely many sets of the form T_{ab} contain more than nk components, into a weakly nk-branching tree S for which $S \equiv_n T$.

In [27], the following similar but substantially stronger result is proved using the notion of nuclearity.

Theorem 6.6 ([27]) Let T be a tree and let $n \in \mathbb{N}$. There exists a weakly boundedly branching tree S with $S \preceq_n T$.

Proof The result is proved in [27, Lemma 2.5] for the class of forests (i.e. for every forest T there is a weakly boundedly branching forest T_1 with $T_1 \preceq_n T$). From the proof of [27, Lemma 2.5], the result applies to trees as well: we need to show that T_1 is connected, so let $a_1, a_2 \in T_1$. Using the notation in the proof of [27, Lemma 2.5], it follows that $a_1, a_2 \in A_m$ for some m. Then for some even j ($j \geqslant m$), there exists $b \in A_{j+1}$ with $b \leqslant a_1, a_2$. Hence $b \in T_1$ and it follows that T_1 is connected, as required. QED

In particular, every tree has a weakly boundedly branching n-equivalent for $n \in \mathbb{N}$.

Remark 6.7 If T is a finite tree then T itself satisfies the condition of being weakly boundedly branching with $T \preceq_n T$. Hence consider the case where T is infinite. From the proof of [27, Lemma 2.5], the tree S mentioned in Theorem 6.6 will be infinite: using the notation in the proof of [27, Lemma 2.5] with $S = T_1$, note that for odd j, we have $b \notin A_j$ so that $A_{j+1} = A_j \cup \{b\} \supsetneq A_j$. Hence $T_1 = \bigcup_{j \in \mathbb{N}} A_j$ will be infinite.

Example 6.8 Consider the tree T described in Example 4.6(a). Let σ be the sentence

$$\exists x \exists y\, (x < y \wedge \forall z \forall u\, (x \leqslant z < u \leqslant y \rightarrow (\exists v\, (z < v < u) \wedge \neg \beta(z,u)))) .$$

The sentence σ states that there exists a dense segment $[x,y]$ with no two distinct nodes from $[x,y]$ belonging to the same bridge. Then $T \models \sigma$ but σ does not hold in any boundedly branching tree. Since $\mathrm{qr}(\sigma) = 5$ then for $n \geqslant 5$, there is no boundedly branching tree S with $S \equiv_n T$. Hence the result of Theorem 6.6 does not hold for the stronger notion of a boundedly branching tree.

Corollary 6.9 ([27]) The first-order theory of the class of trees is complete with respect to the class of weakly boundedly branching trees.

Proof Follows from Theorem 6.6. QED

Theorem 6.10 ([27]) Let T be an $\aleph_0$-categorical tree. The first-order theory of T is decidable.

Proof See [27, Theorem 2.1]. QED

Theorem 6.11 ([27]) The first-order theory of the class of $\aleph_0$-categorical trees is complete with respect to the class of weakly boundedly branching $\aleph_0$-categorical trees.

Proof See [27, Theorem 2.8], where the result is proved for forests (i.e. the first-order theory of the class of $\aleph_0$-categorical forests is complete with respect to the class of weakly boundedly branching $\aleph_0$-categorical forests), but using a slightly broader definition of $\aleph_0$-categoricity (namely that a theory Γ is $\aleph_0$-categorical when Γ has, up to isomorphism, precisely one model of cardinality less than or equal to $\aleph_0$).

As in Theorem 6.6, the result again applies to trees, and using the notation in the proof of [27, Theorem 2.8], in the event that T is infinite, then T_1 will be infinite as well, so that T_1 will be $\aleph_0$-categorical in the sense used in this text. QED

Theorem 6.12 ([27]) Let T be an $\aleph_0$-categorical tree. The first-order theory of T is finitely axiomatizable if and only if T is weakly boundedly branching.

Proof See [27, Theorem 2.2]. QED

6.3 Finite trees

Define the sentence

$\mathsf{Ro}: \exists x \forall y\, (x \leqslant y)$

which states the existence of a root.

In [1] they study trees using a language which includes an order relation on the set of immediate successors of every node. The following result is adapted from there.

Theorem 6.13 (See also [1].) The first-order theory of the class of trees of which all paths are finite can be axiomatised using the theory

$$\{\mathsf{Ir}, \mathsf{Tr}, \mathsf{ST}, \mathsf{Ro}, \mathsf{D}_2\} \cup \mathsf{A}_{\mathsf{WD}}.$$

Proof Clearly every tree of which the paths are all finite satisfies the given theory.

Let T be a model of the theory. We describe, for $n \in \mathbb{N}$, the construction of a tree S from the tree T having only finite paths and such that $S \equiv_n T$.

Let $\tau_1, \ldots, \tau_k$ be all the characteristic formulas of rank n over empty tuples. For every i $(1 \leqslant i \leqslant k)$ let $\varphi_i(x) := \tau_i^{\geqslant x}$.

The first step in the construction of S is as follows. Let $a_{0,0}$ be the root of T (which exists since T satisfies the sentence Ro) and suppose that $(a_{0,0})_{\leqslant} \models \tau_{i_{0,0}}$. Then $T \models \varphi_{i_{0,0}}(a_{0,0}/x)$, so by the scheme A_{WD} there exists a node $b_{0,0} \in T$ maximal with the property that $T \models \varphi_{i_{0,0}}(b_{0,0}/x)$, i.e. $b_{0,0}$ is maximal with the property that $(b_{0,0})_{\leqslant} \models \tau_{i_{0,0}}$. Let T_0 be the tree obtained from T by replacing the subtree $(a_{0,0})_{\leqslant}$ (which in this case equals T itself) with the tree $(b_{0,0})_{\leqslant}$. Then $b_{0,0}$ is maximal in T_0 with the property that $(b_{0,0})_{\leqslant} \models \tau_{i_{0,0}}$, and by Proposition 5.17, $T_0 \equiv_n T$.

For $m \geqslant 1$, the $(m{+}1)$-th step of the construction now proceeds as follows. Suppose we have obtained from the m-th step a tree T_{m-1} such that, for every node $b_{j,z} \in T_{m-1}$ of which the order type of the set $\{x \in T_{m-1} : x \leqslant b_{j,z}\}$ is at most $\mathbf{m}$, if $(b_{j,z})_{\leqslant} \models \tau_{i_{j,z}}$ then the node $b_{j,z}$ is maximal in T_{m-1} with the property that $(b_{j,z})_{\leqslant} \models \tau_{i_{j,z}}$.[1] Let $a_{m,z} \in T_{m-1}$ be a node for which the

[1] The tree T_{m-1} need not be well-founded hence we refrain from formulating the condition that the order type of the set $\{x \in T_{m-1} : x \leqslant b_{j,z}\}$ is at most $\mathbf{m}$ by saying that the node $b_{j,z}$ has level at most $\mathbf{m}$.

order type of the set $\{x \in T_{m-1} : x \leqslant a_{m,z}\}$ is $\mathbf{m+1}$, and suppose that $(a_{m,z})_{\leqslant} \models \tau_{i_{m,z}}$. Then $T \models \varphi_{i_{m,z}}(a_{m,z}/x)$, so by the scheme A_{WD} there exists a node $b_{m,z} \in T$ with $b_{m,z} \geqslant a_{m,z}$ and maximal such that $T \models \varphi_{i_{m,z}}(b_{m,z}/x)$. Hence $b_{m,z}$ is maximal in T with the property that $(b_{m,z})_{\leqslant} \models \tau_{i_{m,z}}$.

Let T_m be the tree obtained from T_{m-1} by replacing every subtree $(a_{m,z})_{\leqslant}$ of T_{m-1} with the tree $(b_{m,z})_{\leqslant}$. Then T_m satisfies the property that for every node $b_{j,z} \in T_m$ of which the order type of the set $\{x \in T_m : x \leqslant b_{j,z}\}$ is at most $\mathbf{m+1}$, if $(b_{j,z})_{\leqslant} \models \tau_{i_{j,z}}$ then the node $b_{j,z}$ is maximal in T_m with the property that $(b_{j,z})_{\leqslant} \models \tau_{i_{j,z}}$. By Proposition 5.17 we also have that $T_m \equiv_n T_{m-1}$.

From the fact that there are only finitely many non-equivalent characteristic formulas of rank n over empty tuples, it follows that the above construction will terminate after finitely many steps to give a tree T_q such that $T_q \equiv_n T$, and T_q will have the property that for every path X in T_q, every node in X satisfies some formula $\varphi_i(x)$, and every formula $\varphi_i(x)$ is satisfied by at most one node from X. Since there are only finitely many formulas $\varphi_i(x)$ then X will be finite. Hence take $S = T_q$. QED

Theorem 6.14 (See also [1].) The first-order theory of the class of finite trees can be axiomatised using the theory

$$\{\mathsf{Ir}, \mathsf{Tr}, \mathsf{ST}, \mathsf{Ro}, \mathsf{D}_2\} \cup \mathsf{A}_{\mathsf{WD}}.$$

Proof Follows from Theorem 6.6 and Theorem 6.13. QED

6.4 Condensations

Theorem 6.15 Let T be a condensed tree, let $\mathcal{L}$ be a class of linear orders, and let $f : T \to \mathcal{L}$ be a function. Suppose we have the following:

- a theory Γ which defines T up to isomorphism;
- for every $L \in \mathcal{L}$, a theory Σ_L which axiomatises the first-order theory of L;
- for every $L \in \mathcal{L}$, a formula $\varphi_L(x)$ such that for every $a \in T$,

$$T \models \varphi_L(a/x) \Leftrightarrow f(a) = L;$$

- a formula $\alpha(x)$ which defines in $T \times_f \mathcal{L}$ a set of nodes consisting of exactly one node from every maximal bridge of $T \times_f \mathcal{L}$.

Then the first-order theory of $T \times_f \mathcal{L}$ can be axiomatised using the theory

$$\{\chi_\alpha\} \bigcup \{\gamma^\alpha : \gamma \in \Gamma\} \bigcup$$
$$\Big\{\forall y \left(\alpha(y) \wedge \varphi_L^\alpha(y) \to \sigma_L^{\beta(x,y)}(y)\right) : \ L \in \mathcal{L} \text{ and } \sigma_L \in \Sigma_L\Big\}.$$

Proof We first show that $T \times_f \mathcal{L}$ satisfies the theory. It is immediate that $T \times_f \mathcal{L} \models \chi_\alpha$. From Proposition 5.22 the function

$$[\cdot]\restriction_{(T \times_f \mathcal{L})^\alpha} : (T \times_f \mathcal{L})^\alpha \to [T \times_f \mathcal{L}]$$

is an isomorphism, and from Proposition 3.35 the function $g : [T \times_f \mathcal{L}] \to T$ given as $g([(x, y)]) = x$ is an isomorphism. Let

$$h := g \circ \left([\cdot]\restriction_{(T \times_f \mathcal{L})^\alpha}\right).$$

Thus $h : (T \times_f \mathcal{L})^\alpha \to T$ with $h((x, y)) = x$ for every $(x, y) \in (T \times_f \mathcal{L})^\alpha$, and h is an isomorphism.

Let $\gamma \in \Gamma$. Since Γ defines T then $T \models \gamma$ hence $(T \times_f \mathcal{L})^\alpha \models \gamma$ so that $T \times_f \mathcal{L} \models \gamma^\alpha$.

Finally let $L \in \mathcal{L}$ and $\sigma_L \in \Sigma_L$ and suppose that $T \times_f \mathcal{L} \models \alpha((a, b)/y)$ and $T \times_f \mathcal{L} \models \varphi_L^\alpha((a, b)/y)$ for some $(a, b) \in |T \times_f \mathcal{L}|$. Then $(a, b) \in (T \times_f \mathcal{L})^\alpha$ so that $(T \times_f \mathcal{L})^\alpha \models \varphi_L((a, b)/y)$ hence $T \models \varphi_L(h((a, b))/y)$. This gives $T \models \varphi_L(a/y)$ so $f(a) = L$.

From Corollary 3.34 we know that $\{a\} \times L$ is a maximal bridge in $T \times_f \mathcal{L}$, and since $(a, b) \in \{a\} \times L$ then $[(a, b)] = \{a\} \times L$. Moreover, $(T \times_f \mathcal{L}; (a, b))^{\beta(x,y)} = [(a, b)]$ and since $\{a\} \times L \models \sigma_L$ then we get $(T \times_f \mathcal{L}; (a, b))^{\beta(x,y)} \models \sigma_L$. This gives $T \times_f \mathcal{L} \models \sigma_L^{\beta(x,y)}((a, b)/y)$, as required. Thus $T \times_f \mathcal{L}$ is a model of the theory.

Next let S be a model of the theory. We need to show that $S \equiv T \times_f \mathcal{L}$. Since $S \models \gamma^\alpha$ for all $\gamma \in \Gamma$ then $S^\alpha \cong T$. Let $g : T \to S^\alpha$ be an isomorphism. Also since $S \models \chi_\alpha$ then the function $[\cdot]\restriction_{S^\alpha} : S^\alpha \to [S]$ is an isomorphism. Hence the function $([\cdot]\restriction_{S^\alpha}) \circ g : T \to [S]$ is an isomorphism. Put $h := ([\cdot]\restriction_{S^\alpha}) \circ g$.

Let $\mathcal{S} = \{[x] : x \in S\}$ and let $\iota : [S] \to \mathcal{S}$ be given by $\iota([x]) = [x]$. From Proposition 3.30 we get that

$$S \cong [S] \times_\iota \mathcal{S} \cong T \times_{\iota \circ h} \mathcal{S}.$$

Let $L \in \mathcal{L}$ and $\sigma_L \in \Sigma_L$ and let $a \in T$ with $f(a) = L$ and $g(a) = b \in S^\alpha \subseteq S$. Since $T \models \varphi_L(a/y)$ then $S^\alpha \models \varphi_L(g(a)/y)$ hence $S \models \varphi_L^\alpha(b/y)$. Also since $b \in S^\alpha$ then $S \models \alpha(b/y)$. From the axioms this gives $S \models \sigma_L^{\beta(x,y)}(b/y)$ and since $(S; b)^{\beta(x,y)} = [b]$ then we get $[b] \models \sigma_L$.

It follows that $(\iota \circ h)(a) = h(a) = [g(a)] = [b] \equiv L = f(a)$ and from Proposition 5.20 we get that $S \cong T \times_{\iota \circ h} \mathcal{S} \equiv T \times_f \mathcal{L}$. QED

6.5 The $\mathcal{C}$-classes of trees

6.5.1 Relationships between FO theories of the $\mathcal{C}$-classes

Theorem 6.16 Let $\mathcal{C}$ be a class of linear orders. The set-theoretical inclusions which hold between the first-order theories of the $\mathcal{C}$-classes of trees are summarised in Figure 6.1.

Proof Let $\sigma \in$ TH(P-$\mathcal{C}$-like trees) say with $\mathrm{qr}(\sigma) = n$. Let T be a $\mathcal{C}$-like tree. Then $T \equiv_n T_0$ for some $\mathcal{C}$-tree T_0. But T_0 is also a P-$\mathcal{C}$-like tree hence $T_0 \models \sigma$ and so $T \models \sigma$. It follows that $\sigma \in$ TH($\mathcal{C}$-like trees) and so TH(P-$\mathcal{C}$-like trees) $\subseteq$ TH($\mathcal{C}$-like trees).

The following inclusions can be proven using a similar argument:

- TH(DU-$\mathcal{C}$-like trees) $\subseteq$ TH($\mathcal{C}$-like trees),
- TH(D-$\mathcal{C}$-trees) $\subseteq$ TH($\mathcal{C}$-like trees),
- TH(PU-$\mathcal{C}$-like trees) $\subseteq$ TH($\mathcal{C}$-like trees),
- TH(U-$\mathcal{C}$-like trees) $\subseteq$ TH($\mathcal{C}$-like trees),
- TH(PU-$\mathcal{C}$-like trees) $\subseteq$ TH(U-$\mathcal{C}$-like trees),
- TH(P-$\mathcal{C}$-like trees) $\subseteq$ TH(U-$\mathcal{C}$-like trees),
- TH(D-$\mathcal{C}$-trees) $\subseteq$ TH(U-$\mathcal{C}$-like trees).

The inclusion TH($\mathcal{C}$-trees) $\subseteq$ TH(U-$\mathcal{C}$-like trees) is immediate.

The remaining inclusions follow from Theorem 4.26 and the accompanying diagram in Figure 4.5.

We briefly discuss the non-inclusions shown in Figure 4.5. Consider for example the non-inclusion TH(DU-$\mathcal{C}_3$-like trees) $\not\subseteq$ TH(P-$\mathcal{C}_3$-like trees). To see this, let T be a DU-$\mathcal{C}_3$-like tree and suppose that $T \models \varphi(a/x)$ for some $a \in T$, where $\varphi(x)$ is as in Example 4.22. Then a belongs to a singular, and hence parametrically definable, path A, with $A \equiv \mathbf{n}$ for some $n \in \mathbb{N}$. Hence A will be finite. It follows that $T \models \sigma_1$, where σ_1 is as in Example 4.21, and so $\sigma_1 \in$ TH(DU-$\mathcal{C}_3$-trees). However, T_3 is a P-$\mathcal{C}_3$-like tree with $T_3 \not\models \sigma_1$, so that $\sigma_1 \notin$ TH(P-$\mathcal{C}_3$-like trees). This serves as a counterexample to establish the non-inclusion TH(DU-$\mathcal{C}_3$-like trees) $\not\subseteq$ TH(P-$\mathcal{C}_3$-like trees).

A similar argument shows that TH(D-$\mathcal{C}_2$-trees) $\not\subseteq$ TH(PU-$\mathcal{C}_2$-like trees).

The non-inclusions which use the class $\mathcal{C}_5$ and the sentence σ_2 from Example 4.24 as counterexample are easily verified.

Finally, the non-inclusions obtained through completion are trivial. For example, TH($\mathcal{C}$-trees) is not generally a subtheory of TH(D-$\mathcal{C}$-trees), for if it were, then using the fact that TH(PU-$\mathcal{C}$-like trees) $\subseteq$ TH($\mathcal{C}$-trees) for all classes $\mathcal{C}$, this would give TH(PU-$\mathcal{C}$-like trees) $\subseteq$ TH(D-$\mathcal{C}$-trees) for all classes $\mathcal{C}$, contradicting the fact that TH(PU-$\mathcal{C}_5$-like trees) $\not\subseteq$ TH(D-$\mathcal{C}_5$-trees).

Likewise the theory TH($\mathcal{C}$-like trees) is not generally a subtheory of the theory TH(D-$\mathcal{C}$-trees), for if it were, then the theories TH(D-$\mathcal{C}$-trees), TH($\mathcal{C}$-like trees) and TH(U-$\mathcal{C}$-like trees) would coincide for all classes $\mathcal{C}$. But this would contradict the fact that TH($\mathcal{C}$-trees) $\subseteq$ TH(U-$\mathcal{C}$-like trees) for all classes $\mathcal{C}$, while there exist classes $\mathcal{C}$ for which TH($\mathcal{C}$-trees) $\not\subseteq$ TH(D-$\mathcal{C}$-trees).

The remaining non-inclusions can be shown by similar reasoning. QED

Proposition 6.17 Let $\mathcal{C}$ consist of a single linear order. In addition to the set-theoretical inclusions which have been shown to hold between the first-order theories of the $\mathcal{C}$-classes of trees in Theorem 6.16, the following inclusions also hold:

(i) TH(DU-$\mathcal{C}$-like trees) $\subseteq$ TH(D-$\mathcal{C}$-like trees);

(ii) TH(DU-$\mathcal{C}$-like trees) $\subseteq$ TH(P-$\mathcal{C}$-like trees);

(iii) TH(PU-$\mathcal{C}$-like trees) $\subseteq$ TH(P-$\mathcal{C}$-like trees).

The remaining non-inclusions stay the same.

Proof Straightforward using Proposition 4.27. QED

6.5.2 Axiomatising the FO theories of the $\mathcal{C}$-classes

We now investigate the first-order theories of some of the $\mathcal{C}$-classes of trees. In its most general form this problem is difficult because $\mathcal{C}$ is an arbitrary class of linear orders. The next example shows that the class of $\mathcal{C}$-trees need not generally be first-order definable.

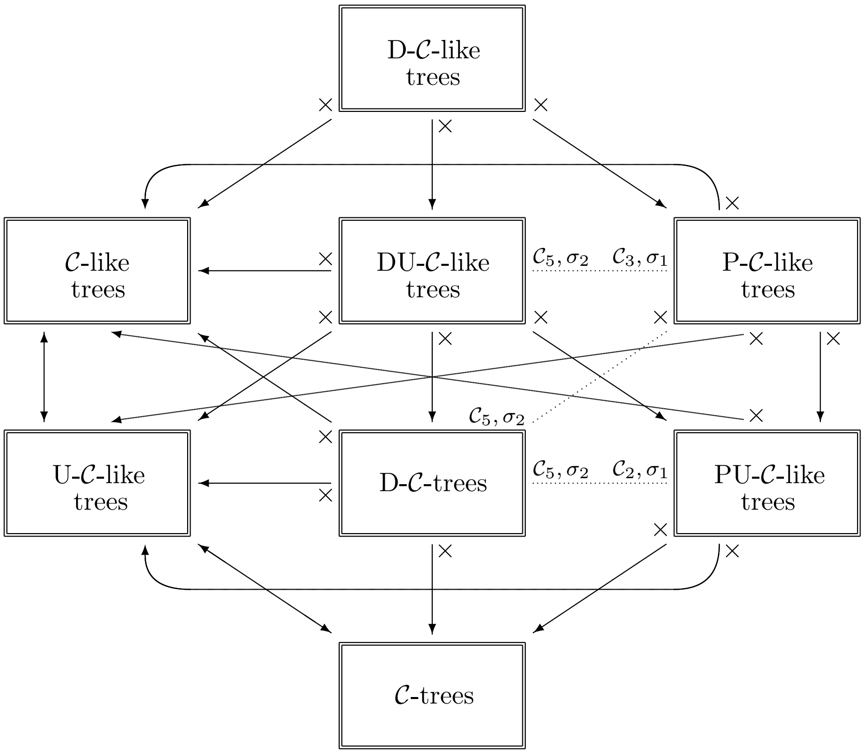

Figure 6.1: Relationships between the first-order theories of the $\mathcal{C}$-classes of trees (see Theorem 6.16). Inclusions $X \subseteq Y$ are denoted as $X \to Y$. Non-inclusions are indicated by specifying a counterexample drawn from Examples 4.20 - 4.25 or, when obtained through transitive completion of the diagram, by the symbol $\times$.

Example 6.18 Let α be an ordinal with $\alpha > \omega$ and let T be a binary α-tree. Thus $|T| \geqslant 2^{\aleph_0}$. From the Downward Löwenheim-Skolem Theorem it follows that T has a countable elementary substructure S. In particular, S will not be an α-tree. Thus the class of α-trees is not first-order definable.

Let Σ be the first-order theory of some class of linear orders. We define the scheme De_Σ as consisting of the sentences

$$\forall \bar{z}\, (\pi_\varphi(\bar{z}) \to \sigma^\varphi(\bar{z}))$$

for every formula $\varphi(x, \bar{z})$ (including formulas $\varphi(x)$ for which the tuple $\bar{z}$ is empty) and for every sentence $\sigma \in \Sigma$. If $\Sigma = \{\sigma\}$ then De_Σ is written simply as De_σ. The scheme De_Σ states that every parametrically definable path satisfies the theory Σ.

Theorem 6.19 Let $\mathcal{C}$ be a class of linear orders axiomatised by the theory Σ. The class of definably $\mathcal{C}$-like trees is precisely the class of models of the theory

$$\mathsf{A_T} \cup \mathsf{De}_\Sigma.$$

Proof Let T be a definably $\mathcal{C}$-like tree. It is immediate that T satisfies $\mathsf{A_T}$. Let $\varphi(x, \bar{z})$ be a formula with $\bar{z}$ a k-tuple of variables ($\bar{z}$ may be empty), let $\bar{c}$ be a k-tuple of nodes in T, and let $T \models \pi_\varphi(\bar{c}/\bar{z})$. Then from Proposition 5.4 there is a path A defined in T by $\varphi(x, \bar{z})$ with the parameters $\bar{c}$ substituted for $\bar{z}$. But $A \models \sigma$ for every $\sigma \in \Sigma$ and $A = (T; \bar{c})^\varphi$ so using Corollary 2.4, $T \models \sigma^\varphi(\bar{c}/\bar{z})$ for every $\sigma \in \Sigma$. It follows that T satisfies the scheme De_Σ.

Next let T be a structure which satisfies the theory $\mathsf{A_T} \cup \mathsf{De}_\Sigma$. Since T satisfies $\mathsf{A_T}$ then T is a tree. Let $\bar{c}$ be a (possibly empty) k-tuple of nodes in T and let A be a path defined in T using the formula $\varphi(x, \bar{z})$ with the parameters $\bar{c}$ substituted for $\bar{z}$, where $\bar{z}$ is a k-tuple of variables. Then $T \models \pi_\varphi(\bar{c}/\bar{z})$ hence $T \models \sigma^\varphi(\bar{c}/\bar{z})$ for every $\sigma \in \Sigma$. But $A = (T; \bar{c})^\varphi$ so from Corollary 2.4, $A \models \sigma$ for every $\sigma \in \Sigma$. Hence $A \in \mathrm{MOD}\,(\mathrm{TH}\,(\mathcal{C}))$ and it follows that T is a definably $\mathcal{C}$-like tree. QED

Corollary 6.20 Let $\mathcal{C}$ be a class of linear orders axiomatised by the theory Σ. Then the class of definably $\mathcal{C}$-trees is precisely the class of models of the theory

$$\mathsf{A_T} \cup \mathsf{De}_\Sigma.$$

Proof By the fact that when $\mathcal{C}$ is axiomatisable then the class of definably $\mathcal{C}$-like trees coincides with the class of definably $\mathcal{C}$-trees. QED

Corollary 6.21 Let $\mathcal{C}$ be a finite class of linear orders axiomatised by the theory Σ. Then the class of definably uniformly $\mathcal{C}$-like trees is precisely the class of models of the theory

$$\mathsf{A_T} \cup \mathsf{De}_\Sigma.$$

Proof If $\mathcal{C}$ is finite then the class of definably $\mathcal{C}$-like trees coincides with the class of definably uniformly $\mathcal{C}$-like trees. QED

Let El denote the scheme consisting of the sentences

$$\forall \bar{z}\Big(\forall x\,(\varphi(x,\bar{z}) \to \exists y\,(x < y \wedge \varphi(y,\bar{z}))) \to \\ \exists x\Big(\mathrm{leaf}(x) \wedge \forall y\big(y < x \to \exists u\,(y < u < x \wedge \varphi(u,\bar{z}))\,\big)\Big)\Big)$$

for every formula $\varphi(x,\bar{z})$ (including formulas $\varphi(x)$ for which the tuple $\bar{z}$ is empty). The scheme El holds in a tree T when T has the property that if A is a cofinal set of nodes in T then there exists a leaf a (an Elder) and a subset B of A with $B \subseteq a_>$ such that B is cofinal in the set $a_>$.

Example 6.22 Let T be the tree constructed by starting with the linear order $A := \omega$ and at each positive element $n \in A$ we attach the n-ary $(\omega + \mathbf{1})$-tree $B^n_{\omega+\mathbf{1}}$. Hence every path in T is an $(\omega + \mathbf{1})$-path except for the path A which is an ω-path. The i-th node of the path A in T has i immediate successors while every node not belonging to A but lying in the subtree $B^j_{\omega+\mathbf{1}}$ has j immediate successors.

We will show that the scheme El holds in T. Let $\varphi(x,\bar{z})$ be a formula with $\bar{z}$ an n-tuple of variables and let $\bar{c}$ be an n-tuple of nodes in T. Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of nodes in T with $a_i < a_i$ when $i < j$ and suppose that $(T;\bar{c}) \models \varphi(a_i/x,\,\bar{c})$ for all a_i. We distinguish two cases.

First consider the case where, for some k, we have that $a_i \notin A$ when $i \geqslant k$. Then the set $\{a_i\}_{i\in\mathbb{N}}$ is contained in an $(\omega + \mathbf{1})$-path B of T, so that there is a leaf in B which is a successor to all of the nodes a_i.

Next consider the case where $\{a_i\}_{i\in\mathbb{N}} \subseteq A$. Suppose that the quantifier rank of $\varphi(x,\bar{z})$ is m. It is easy to see, for example by using an Ehrenfeucht-Fraïssé game, that for some sufficiently large value of p, $(T;\bar{c}\,a_p) \equiv_m (T;\bar{c}\,u)$

for every non-leaf node u with $u \geqslant a_p$. Hence there exists a sequence of nodes $\{b_i\}_{p\leqslant i,\, i\in\mathbb{N}} \subseteq T\backslash A$ with $b_i < b_j$ for $i < j$ and such that $(T;\bar{c}) \models \varphi(b_i/x,\, \bar{c})$ for all b_i. The set $C := \{a_i : 0 \leqslant i \leqslant p-1\} \cup \{b_i : i \geqslant p\}$ is contained in an $(\omega + \mathbf{1})$-path B of T, so that there is a leaf in B which is a successor to all of the nodes in C.

It follows that El holds in T.

For Σ any theory, Le_Σ denotes the scheme consisting of the sentences

$$\forall x\, (\mathrm{leaf}(x) \to \sigma^{\leqslant x})$$

for every $\sigma \in \Sigma$. If $\Sigma = \{\sigma\}$ then Le_Σ is written simply as Le_σ. The scheme Le_Σ states that every path containing a leaf satisfies the theory Σ.

If α is a linear order with a greatest point and α is axiomatised by the theory Σ then the effect of the scheme Le_Σ within an α-tree is to ensure that every parametrically definable path is α-like.

Proposition 6.23 Let α be a linear order containing a greatest element and suppose the sentence σ axiomatises the first-order theory of α. Let T be a definably uniformly α-like tree containing only finitely many paths which are not parametrically definable. Then for every $n \in \mathbb{N}$ there exists a pathwise uniformly α-like tree S such that $S \equiv_n T$.

Proof It suffices to prove the result for large n so let $n \geqslant \mathrm{qr}(\sigma) + 1$. Let $A_1, \ldots, A_k$ be the paths in T which are not parametrically definable and for every i, let $a_i \in A_i$ be such that $a_i \notin A_j$ for all j with $j \neq i$. By Lemma 5.6 there exists, for every i, nodes $b_i \in A_i$ and $c_i \in T\backslash A_i$ with $b_i, c_i \geqslant a_i$ such that $(b_i)_\leqslant \equiv_n (c_i)_\leqslant$.

Let S be the tree obtained by taking the tree T and for every i we replace the subtree $(b_i)_\leqslant$ of T with the tree $S_i := (c_i)_\leqslant$. From the way S is constructed, every path in S will contain a leaf hence every path in S is definable using that leaf as parameter. Define

$$\tau := \forall x \left(\mathrm{leaf}(x) \to \sigma^{\leqslant x}\right).$$

i.e. $\{\tau\} = \mathsf{Le}_\sigma$. Note that $\mathrm{qr}(\tau) = \mathrm{qr}(\sigma) + 1 \leqslant n$. By Proposition 5.17 we get $S \equiv_n T$ and since T satisfies τ then S also satisfies τ. Since every path in S contains a leaf then it follows that every path in S satisfies σ hence S is a pathwise uniformly α-like tree. QED

Proposition 6.24 Let α be a linear order containing a greatest element and let Σ axiomatise the first-order theory of α. Let T be a tree satisfying the schemes El and Le_Σ. Then T satisfies the scheme De_Σ.

Proof Suppose that T satisfies the schemes El and Le_Σ. Let $\varphi(x, \bar{z})$ be a formula with $\bar{z}$ a (possibly empty) k-tuple of variables, and suppose that $T \models \pi_\varphi(\bar{c}/\bar{z})$ for some k-tuple of nodes $\bar{c}$ from T. Then $\varphi(x, \bar{z})$ defines a path A in T with the parameters $\bar{c}$ substituted for $\bar{z}$. Since for every node $u \in A$ we have that $T \models \varphi(u/x,\, \bar{c}/\bar{z})$ then it follows by the scheme El that there exists a leaf $a \in T$ such that $T \models \varphi(u/x,\, \bar{c}/\bar{z})$ for every $u < a$ and clearly $a \in A$. Now from the scheme Le_Σ we get that $T \models \sigma^{\leqslant x}(a/x)$ for every sentence $\sigma \in \Sigma$. Hence $a_\geqslant = A = (T; \bar{c})^\varphi$ satifies σ and so $T \models \sigma^\varphi(\bar{c}/\bar{z})$. It follows that the scheme De_Σ holds in T. QED

The next example shows that the scheme El does not generally follow from the scheme De_Σ. In it we make use of a sentence $\Phi_{\omega+\mathbf{1}}$ which axiomatises the first-order theory of the ordinal $\omega + \mathbf{1}$. This sentence will be fully described in Section 7.1.

Example 6.25 Let T be the tree obtained from the binary ω-tree B_ω by attaching a copy of the linear order $\omega + \mathbf{1}$ to each of its nodes. Every path in T is either an ω-path or an $(\omega + \mathbf{1})$-path. Nodes lying high up on any of the $(\omega + \mathbf{1})$-paths will have only one immediate successor, whereas nodes belonging to any of the ω-paths will have three immediate successors.

The parametrically definable paths in T are precisely the $(\omega + \mathbf{1})$-paths. For let A be an ω-path in T, let $\bar{c}$ be a (possibly empty) k-tuple of nodes in T, and let $\psi(x, \bar{z})$ be a formula with $\bar{z}$ a k-tuple of variables, such that $(T; \bar{c}) \models \psi(u/x,\, \bar{c})$ for all $u \in A$. Let $a \in A$ with $a \geqslant (A \cap (c_i)_\geqslant)$ for all i. Let b be a node in T, different from a, with $b \geqslant (A \cap (c_i)_\geqslant)$ for all i, and with b having the same level as a. In particular, $b \notin A$. From the structure of T it is clear that $(T; \bar{c}\, a) \cong (T; \bar{c}\, b)$. Hence $(T; \bar{c}) \models \psi(b/x,\, \bar{c})$ and it follows that the formula $\psi(x, \bar{z})$ cannot possibly define the path A in T with the parameters $\bar{c}$ substituted for $\bar{z}$. Hence the ω-paths in T are not parametrically definable, whereas the $(\omega + \mathbf{1})$-paths are all definable using a leaf belonging to the path as parameter. Therefore T satisfies the scheme $\mathsf{De}_{\Phi_{\omega+\mathbf{1}}}$.
Now consider the formula

$$\varphi(x) := \exists y_1 \exists y_2 \exists y_3 \left(\bigwedge_{i \neq j} y_i \neq y_j \ \wedge \ \bigwedge_{i=1}^{3} s(x, y_i) \right)$$

which holds for a node $u \in T$ when u has three distinct immediate successors. Hence $T^{\varphi} = B_{\omega}$ and for every node $u \in T$ with $T \models \varphi(u/x)$, there is a node $w \in T$ with $u < w$ and for which $T \models \varphi(w/x)$. However, there is no leaf $d \in T$ for which the set $d_{>}$ contains a cofinal sequence of nodes satisfying the formula $\varphi(x)$. It follows that T does not satisfy the scheme EI.

Chapter 7

Axiomatisations of ordinal trees

In this chapter we study the problem of axiomatising the first-order theory of the class of trees of which every path is isomorphic with the ordinal α with $\alpha < \omega^\omega$. We begin in Section 7.1 (The first-order theory of the ordinal α with $\alpha < \omega^\omega$) by describing the first-order theory of the ordinal α using an axiom system similar to the one in [24]. In Section 7.2 (Tails of ordinals) we establish some important results on tails of ordinals which are used later to establish results on almost α-trees. Section 7.3 (Some general observations) contains some results which are of general use. In Section 7.4 (Towards first-order theories of α-trees) we determine the first-order theories of the classes of $\mathbf{n}$-trees for every finite ordinal $\mathbf{n}$ as well as the first-order theory of the class of ω-trees. We also introduce the class of almost α-trees and show that this class is a proper subclass of the class of definably uniformly α-like trees. We obtain the result that every almost α-tree can be elementarily embedded in a pathwise uniformly α-like tree. Finally we examine what this elementary extension of the almost α-tree looks like in Section 7.5 (Almost $(\omega + \mathbf{1})$-trees and their extensions) for the case where $\alpha = \omega + \mathbf{1}$.

For convenience, we summarise the formulas which will be used frequently in this chapter:

Notation	Formula
λ_n	contains at least n elements
μ_n	contains at most n elements
Tr	transitivity
Co	connectedness
Ro	contains a root
Do	$\forall x \exists y \,(\text{leaf}(y) \wedge x \leqslant y)$
D_1	downwards discreteness
D_2	upwards discreteness
D_2'	weak upwards discreteness of tree / upwards discreteness of linear order
D_2''	$\forall x \exists y \,(x < y)$
$\delta(x)$	defines limit points
N_1	$\forall x \forall y \left((x < y \wedge \delta(x) \wedge \delta(y) \wedge \neg\exists z \,(\delta(z) \wedge x < z < y)) \rightarrow \Phi_\omega^{[x,y)}\right)$
N_2	$\forall x \exists y \exists z \,(\delta(y) \wedge \delta(z) \wedge y \leqslant x < z)$
Φ_α	axiomatises first-order theory of the ordinal α
$\mathsf{A_F}$	defines class of forests
$\mathsf{A_T}$	defines class of trees
$\mathsf{A_L}$	defines class of linear orders
$\mathsf{A_W}$	definable well-foundedness
De_Σ	parametrically definable paths satisfy Σ
Le_Σ	every path containing a leaf satisfies Σ

7.1 The first-order theory of the ordinal α with $\alpha < \omega^\omega$

It is known (see e.g. [24]) that for every ordinal α with $\alpha < \omega^\omega$, the first-order theory of α can be axiomatised using a single sentence Φ_α. We briefly describe this axiomatisation here. The reader is also referred to [24, pp. 253-262] in this regard. The axiom system presented here differs slightly from the one in [24].

7.1.1 Finite α

We first consider finite ordinals.

Proposition 7.1 For every $n \in \mathbb{N}^+$, the ordinal $\mathbf{n}$ can be defined using the sentence

$$\Phi_{\mathbf{n}} := \bigwedge \mathsf{A_L} \wedge \lambda_n \wedge \mu_n.$$

Proof Immediate. QED

Proposition 7.2 For $n = 1, 2$ we have $\mathrm{qr}(\Phi_{\mathbf{n}}) = 3$ and for $n \geqslant 3$ we have $\mathrm{qr}(\Phi_{\mathbf{n}}) = n + 1$.

Proof Since $\mathrm{qr}\left(\bigwedge \mathsf{A_L}\right) = 3$ and $\mathrm{qr}(\lambda_n \wedge \mu_n) = n + 1$. QED

7.1.2 Powers of ω

For α any order type, a linear order A is called **α-like** when $A \equiv \alpha$.

We next investigate the class of ω-like linear orders. Define the sentence

$\mathsf{D}_2' :\ \forall x \exists y\, (x < y \wedge \forall z\, (\neg\, (x < z < y)))$.

The sentence D_2' states that every element has an immediate successor.

Proposition 7.3 ([24]) The first-order theory of the ordinal ω can be axiomatised using the sentence

$$\Phi_\omega := \bigwedge \mathsf{A_L} \wedge \mathsf{Ro} \wedge \mathsf{D_1} \wedge \mathsf{D}_2'.$$

Proof See [24, p. 254]. QED

The class of models of the sentence Φ_ω consists of all linear orders having order type $\omega + \zeta \cdot \alpha$ for α any order type (see [24]). Moreover, $\omega + \zeta \cdot \alpha_1 \equiv \omega + \zeta \cdot \alpha_2$ for all order types α_1 and α_2. In particular, $\omega \equiv \omega + \zeta \cdot \alpha$ for every order type α.

We now turn to powers of ω. For every $n \in \mathbb{N}^+$ define the class $\mathcal{A}(\omega^n)$ of linear orders by induction as follows:

(i) $\mathcal{A}(\omega)$ consists of all linear orders with order type $\omega + \zeta \cdot \alpha$ for α any order type;

(ii) for $n \geq 2$ the class $\mathcal{A}(\omega^n)$ consists of all linear orders of the form

$$\sum_{k\in\omega} W_k + \sum_{i\in I}\left(\sum_{z\in\zeta} W^i_z\right) \tag{7.1}$$

for I any linearly ordered set, and where $W_k, W^i_z \in \mathcal{A}(\omega^{n-1})$ for all k, i and z.

Proposition 7.4 ([24]) Let A be an ordered set and let $n \in \mathbb{N}^+$. Then $A \in \mathcal{A}(\omega^n)$ if and only if $A \equiv \omega^n$.

Proof See [24, Proposition 13.25]. QED

Hence for every $n \in \mathbb{N}^+$ the class $\mathcal{A}(\omega^n)$ consists of the class of ω^n-like linear orders. Since $\omega^0 = \mathbf{1}$ then the class of ω^0-like linear orders consists of those linear orders which have a singleton set as domain. Hence the class of ω^0-like linear orders can be defined by the sentence $\Phi_{\mathbf{1}}$.

Proposition 7.5 ([24]) Let A be an ordered set and let $n \in \mathbb{N}^+$. The following conditions are equivalent:

(i) $A = \sum_{i\in V} V_i$ for some V and $\{V_i\}_{i\in V}$, where V is an ω^{n-1}-like linear order and V_i is a ω-like linear order for every $i \in V$;

(ii) $A = \sum_{i\in W} W_i$ for some W and $\{W_i\}_{i\in W}$, where W is an ω-like linear order and W_i is ω^{n-1}-like linear order for every $i \in W$.

Proof See [24, pp. 257-258]. QED

Define the formula

$$\delta(x) := \forall y\,(y < x \to \exists z\,(y < z < x))\,.$$

The formula $\delta(x)$ defines limit points. A least element, if it exists, will satisfy the formula $\delta(x)$ and hence will be treated as a limit point.

Define the sentences

$\mathsf{N}_1 :\ \forall x \forall y \left((x < y \wedge \delta(x) \wedge \delta(y) \wedge \neg\exists z\,(\delta(z) \wedge x < z < y)) \to \Phi^{[x,y)}_{\omega}\right);$

$\mathsf{N}_2 :\ \forall x \exists y \exists z\,(\delta(y) \wedge \delta(z) \wedge y \leqslant x < z).$

The sentence N_1 states that the interval between every pair of successive limit points is ω-like. The sentence N_2 states that every element is contained in an interval formed by two successive limit points.

Proposition 7.6 (See also [24].) Define inductively

$$\gamma_1 := \mathsf{Ro} \wedge \mathsf{D}_1 \wedge \mathsf{D}_2' \quad \text{and}$$

$$\gamma_n := \gamma_{n-1}^{\delta} \wedge \mathsf{N}_1 \wedge \mathsf{N}_2 \quad \text{for } n \geqslant 2.$$

Then for $n \geqslant 2$ the first-order theory of the ordinal ω^n can be axiomatised using the sentence

$$\Phi_{\omega^n} := \bigwedge \mathsf{A_L} \wedge \gamma_n.$$

Proof It is clear that $\omega^n \models \Phi_{\omega^n}$ for every n with $n \geqslant 2$. Hence we need to show that if $A \models \Phi_{\omega^n}$ then $A \equiv \omega^n$ for every n with $n \geqslant 2$. The proof runs by induction on n. We already know that Φ_ω axiomatises the first-order theory of ω. For $k \geqslant 1$ assume that Φ_{ω^k} axiomatises the first-order theory of ω^k and let A be a structure with $A \models \Phi_{\omega^{k+1}}$.

Since $A \models \bigwedge \mathsf{A_L}$ and the sentences in $\mathsf{A_L}$ express universal properties then $A^\delta \models \bigwedge \mathsf{A_L}$. Also $A \models \gamma_k^\delta$ and so $A^\delta \models \gamma_k$. Hence $A^\delta \models \Phi_{\omega^k}$ and by the induction hypothesis A^δ is ω^k-like.

Thus for every $x \in A^\delta$ there exists $y_x \in A^\delta$ such that y_x is the immediate successor to x in A^δ. Since $A \models \mathsf{N}_1$ then the set $A_x := \{z \in A : x \leqslant z < y_x\}$ is ω-like.

From the fact that $A \models \mathsf{N}_2$ it then follows that A can be written as

$$A = \sum_{x \in A^\delta} A_x$$

where A^δ is ω^k-like and where A_x is ω-like for every $x \in A^\delta$. Using Lemma 7.5 we get that $A \in \mathcal{A}(\omega^{k+1})$. By Proposition 7.4 this gives $A \equiv \omega^{k+1}$ as required. QED

Proposition 7.7 (See also [24].) For $n \geqslant 1$ we have $\mathrm{qr}(\Phi_{\omega^n}) = 2n + 1$.

Proof We use induction on n. First note that $\mathrm{qr}\left(\bigwedge \mathsf{A_L}\right) = 3$, $\mathrm{qr}(\mathsf{Ro}) = 2$, $\mathrm{qr}(\mathsf{D_1}) = 3$ and $\mathrm{qr}(\mathsf{D_2'}) = 3$. Hence $\mathrm{qr}(\Phi_\omega) = 3$. Moreover

$$\mathrm{qr}(\gamma_{m-1}^{\delta}) = \mathrm{qr}(\gamma_{m-1}) + \mathrm{qr}(\delta) = \mathrm{qr}(\gamma_{m-1}) + 2$$

for every $m \geqslant 2$ and since $\mathrm{qr}(\mathsf{N_1}) = 5$ and $\mathrm{qr}(\mathsf{N_2}) = 5$ then $\mathrm{qr}(\Phi_{\omega^2}) = 5$.

Next assume that $\mathrm{qr}(\Phi_{\omega^k}) = 2k+1$ for some k with $k \geqslant 2$. It follows that $\mathrm{qr}(\gamma_k) = 2k+1$, which gives

$$\mathrm{qr}(\Phi_{\omega^{k+1}}) = \mathrm{qr}(\gamma_{k+1}) = \mathrm{qr}(\gamma_k^{\delta}) = 2k+3 = 2(k+1)+1$$

as required. QED

7.1.3 The general case

Finally we consider any ordinal α with $\alpha < \omega^\omega$ and where α is neither finite nor a power of ω. Assume that the Cantor normal form of such α is

$$\alpha = \omega^{n_1} \cdot a_1 + \omega^{n_2} \cdot a_2 + \ldots + \omega^{n_k} \cdot a_k \tag{7.2}$$

where $n_1 > n_2 > \ldots > n_k$ and $a_i \neq 0$ for all i.

Proposition 7.8 ([24]) Let A be a linear order. Then A is α-like if and only if A has the form

$$\left(W_1^{n_1} + \ldots + W_{a_1}^{n_1}\right) + \left(W_1^{n_2} + \ldots + W_{a_2}^{n_2}\right) + \ldots + \left(W_1^{n_k} + \ldots + W_{a_k}^{n_k}\right)$$

where $W_j^{n_i}$ is an ω^{n_i}-like linear order for all n_i and j.

Proof See [24, Theorem 13.26]. QED

Let ψ_1, ψ_2 and ψ_3 be the formulas

$$\begin{aligned}
\psi_1 &:= \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{a_i-1} \left(x_j^{n_i} < x_{j+1}^{n_i}\right) \wedge \bigwedge_{i=1}^{k-1} \left(x_{a_i}^{n_i} < x_1^{n_{i+1}}\right); \\
\psi_2 &:= \forall y \left(\bigvee_{i=1}^{k} \bigvee_{j=1}^{a_i-1} \left(x_j^{n_i} \leqslant y < x_{j+1}^{n_i}\right) \vee \bigvee_{i=1}^{k-1} \left(x_{a_i}^{n_i} \leqslant y < x_1^{n_{i+1}}\right) \vee x_{a_k}^{n_k} \leqslant y \right); \\
\psi_3 &:= \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{a_i-1} \Phi_{\omega^{n_i}}^{\left[x_j^{n_i}, x_{j+1}^{n_i}\right)} \wedge \bigwedge_{i=1}^{k-1} \Phi_{\omega^{n_i}}^{\left[x_{a_i}^{n_i}, x_1^{n_{i+1}}\right)} \wedge \Phi_{\omega^{n_k}}^{\geqslant x_{a_k}^{n_k}}.
\end{aligned}$$

The formulas ψ_1, ψ_2 and ψ_3 have free variables $x_j^{n_i}$, where $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant a_i$. The formula ψ_1 states that the variables $x_j^{n_i}$ can be ordered as

$$x_1^{n_1} < \ldots < x_{a_1}^{n_1} < x_1^{n_2} < \ldots < x_{a_2}^{n_2} < \ldots < x_1^{n_k} < \ldots < x_{a_k}^{n_k}.$$

The formula ψ_2 states that every element lies in one of the intervals as described in the formula which are determined by the variables $x_j^{n_i}$. The formula ψ_3 states that each of the intervals determined by successive variables, as well as the interval at the end determined by the variable $x_{a_k}^{n_k}$, is ω^{n_i}-like for the appropriate value of i.

Proposition 7.9 (See also [24].) Let α be an ordinal of the form described in Equation (7.2). The first-order theory of α can be axiomatised using the sentence

$$\Phi_\alpha := \bigwedge \mathsf{A_L} \wedge \exists \bar{x}^{n_1} \ldots \exists \bar{x}^{n_k} \Big(\psi_1 \wedge \psi_2 \wedge \psi_3 \Big)$$

where $\bar{x}^{n_i} = \left(x_1^{n_i}, \ldots, x_{a_i}^{n_i}\right)$ for every i with $1 \leqslant i \leqslant k$.

Proof From Proposition 7.8 and the way that Φ_α is constructed. QED

Proposition 7.10 (See also [24].) For α of the form described in Equation (7.2) we have

$$\mathrm{qr}(\Phi_\alpha) = 2n_1 + a_1 + \ldots + a_k + 1.$$

Proof Since $\mathrm{qr}(\Phi_{\omega^{m_1}}) > \mathrm{qr}(\Phi_{\omega^{m_2}})$ for $m_1 > m_2$ and since $\mathrm{qr}(\Phi_{\omega^m}^{[x,y)}) = \mathrm{qr}(\Phi_{\omega^m})$ and $\mathrm{qr}(\Phi_{\omega^m}^{\geqslant x}) = \mathrm{qr}(\Phi_{\omega^m})$ for every m we get that

$$\mathrm{qr}(\Phi_\alpha) = a_1 + \ldots + a_k + \mathrm{qr}(\Phi_{\omega^{n_1}})$$

and the result follows by Proposition 7.7. QED

It need not be the case that $\mathrm{qr}(\Phi_\alpha) < \mathrm{qr}(\Phi_\beta)$ for $\alpha < \beta$. As an example, for n a positive integer we get that $\mathrm{qr}(\Phi_{\omega+\mathbf{n}}) = n + 4$, $\mathrm{qr}(\Phi_{\omega \cdot n}) = n + 3$ and $\mathrm{qr}(\Phi_{\omega^2}) = 5$.

Proposition 7.11 Let $\alpha < \omega^\omega$ and let A be an α-like well-order. Then $A \cong \alpha$.

Proof We prove the result using induction on α. The result obviously holds for finite α.

If A is an ω-like well-order then $A \cong \omega + \zeta \cdot \gamma$ for some order type γ and A being well-ordered ensures that $\gamma = \mathbf{0}$. Hence $A \cong \omega$ so the result holds for $\alpha = \omega$.

Next let $m \geqslant 1$ and assume the result holds for $\alpha = \omega^m$. Let A be an ω^{m+1}-like well-order. By Proposition 7.4, A will have the form

$$\sum_{k \in \omega} W_k + \sum_{i \in I} \left(\sum_{z \in \zeta} W_z^i \right)$$

for some linearly ordered set I and where W_k and W_z^i are ω^m-like for all k, i and z. Since A is well-ordered then $I = \emptyset$. Moreover every W_k will be well-ordered so from the induction hypothesis, $W_k \cong \omega^m$ for every k. Hence $A \cong \omega^{m+1}$ and the result holds for $\alpha = \omega^{m+1}$.

Finally assume the result holds for $\alpha = \omega^m$ for every $m \in \mathbb{N}$. Now let A be a β-like well-order, where β has the form

$$\beta = \omega^{n_1} \cdot a_1 + \omega^{n_2} \cdot a_2 + \ldots + \omega^{n_k} \cdot a_k$$

with $n_1 > n_2 > \ldots > n_k$ and $a_i \neq 0$ for every i. By Proposition 7.8, A has the form

$$\left(W_1^{n_1} + \ldots + W_{a_1}^{n_1}\right) + \left(W_1^{n_2} + \ldots + W_{a_2}^{n_2}\right) + \ldots + \left(W_1^{n_k} + \ldots + W_{a_k}^{n_k}\right)$$

where $W_j^{n_i}$ is an ω^{n_i}-like linear order for all n_i and j. Since A is well-ordered then every $W_j^{n_i}$ will also be well-ordered so by the induction hypothesis, $W_j^{n_i} \cong \omega^{n_i}$ for all n_i and j. It follows that $A \cong \beta$ and this completes the proof. QED

7.2 Tails of ordinals

If B is a non-empty upwards convex subset of a linear order A then B is called a **tail** of A. Clearly if β is a tail of the ordinal α then β is also an ordinal with $\beta \leqslant \alpha$ and α can be written as $\alpha = \gamma + \beta$ for some ordinal γ.

Proposition 7.12 Let α be an ordinal with $\alpha < \omega^\omega$ and let β be a tail of α. Then $\mathrm{qr}(\Phi_\beta) \leqslant \mathrm{qr}(\Phi_\alpha)$.

Proof It can be seen by induction on k that if β is a tail of α with $\alpha = \omega^k$ for some k then $\beta = \omega^k$, in which case $\mathrm{qr}(\Phi_\beta) = \mathrm{qr}(\Phi_{\omega^k}) = \mathrm{qr}(\Phi_\alpha)$. Hence assume that α is not a power of ω. Let γ be an ordinal such that $\alpha = \gamma + \beta$ and let the Cantor normal forms of α, β and γ be

$$\begin{aligned}
\alpha &= \omega^{q_1} \cdot a_1 + \omega^{q_2} \cdot a_2 + \ldots + \omega^{q_k} \cdot a_k, \\
\beta &= \omega^{r_1} \cdot b_1 + \omega^{r_2} \cdot b_2 + \ldots + \omega^{r_m} \cdot b_m, \\
\gamma &= \omega^{s_1} \cdot c_1 + \omega^{s_2} \cdot c_2 + \ldots + \omega^{s_n} \cdot c_n,
\end{aligned}$$

where $q_1 > q_2 > \ldots > q_k$ and $r_1 > r_2 > \ldots > r_m$ and $s_1 > s_2 > \ldots > s_n$ with $a_i, b_i, c_i \neq 0$ for all i.

It can be seen by induction that $\omega^{n_1} + \omega^{n_2} = \omega^{n_2}$ when $n_1 < n_2$. From this fact it follows that

$$\gamma + \beta = \sum \left\{ \omega^{s_i} \cdot c_i : 1 \leqslant i \leqslant n \text{ and with } s_i \geqslant r_1 \right\} + \sum_{i=1}^{m} \omega^{r_i} \cdot b_i.$$

Let c_t be the coefficient (possibly 0) of the term containing the power ω^{r_1} in the Cantor normal form expansion of γ. Since the Cantor normal form expansion of α is unique, this gives

$$\begin{aligned}
r_i &= q_{k-m+i} \text{ for } 1 \leqslant i \leqslant m, \text{ and} \\
c_t + b_1 &= a_{k-m+1}, \text{ and} \\
b_i &= a_{k-m+i} \text{ for } 2 \leqslant i \leqslant m.
\end{aligned}$$

Since $\mathrm{qr}(\Phi_{\omega^{r_1}}) = 2r_1 + 1 \leqslant 2q_1 + 1 = \mathrm{qr}(\Phi_{\omega^{q_1}})$ then

$$\begin{aligned}
\mathrm{qr}(\Phi_\beta) &= b_1 + \ldots + b_m + \mathrm{qr}(\Phi_{\omega^{r_1}}) \\
&\leqslant a_1 + \ldots + a_{k-m} + (c_t + b_1) + b_2 + \ldots + b_m + \mathrm{qr}(\Phi_{\omega^{q_1}}) \\
&= a_1 + \ldots + a_k + \mathrm{qr}(\Phi_{\omega^{q_1}}) \\
&= \mathrm{qr}(\Phi_\alpha)
\end{aligned}$$

as required. QED

Lemma 7.13 Let A be an ω^n-like linear order ($n \in \mathbb{N}$) and let B be a tail of A containing a least point. Then B is ω^n-like.

Proof We prove the result using induction on n. The case for $n = 0$ is straightforward, while for $n = 1$ the result follows from the fact that A will have the form $\omega + \zeta \cdot \alpha$ for some order type α.

Assume the result holds for ω^k-like linear orders (where $k \geqslant 1$) and let A be ω^{k+1}-like. From Proposition 7.4, A will have the form

$$A = \sum_{m \in \omega} W_m + \sum_{i \in I} \left(\sum_{z \in \zeta} W_z^i \right)$$

for I some linearly ordered set, and where W_m and W_z^i are ω^k-like for all m, i and z. It follows that B will have one of the forms

$$B = W_q^\star + \sum_{m=q+1}^{\infty} W_m + \sum_{i \in I} \left(\sum_{z \in \zeta} W_z^i \right)$$

for some $q \in \mathbb{N}$ and where $W_q^\star$ is a tail of W_q containing a least point, or

$$B = (W_q^j)^\star + \sum_{z \in \zeta, z > q} W_z^j + \sum_{i \in I, i > j} \left(\sum_{z \in \zeta} W_z^i \right)$$

for some $j \in I$ and $q \in \zeta$ and where $(W_q^j)^\star$ is a tail of W_q^j containing a least point.

By the induction hypothesis, $W_q^\star$ and $(W_q^j)^\star$ will be ω^k-like, so in both of the above cases, B can be written as an ω-like sum of ω^k-like linear orders. Hence B will have the form described in Equation (7.1) for an ω^{k+1}-like linear order, so B is ω^{k+1}-like as required. QED

Proposition 7.14 Let A be a linear order and let B be a tail of A containing a least point. Suppose that A is α-like for some ordinal α with $\alpha < \omega^\omega$. Then there exists an ordinal β with β a tail of α and such that B is β-like.

Proof We have already proved the result in Lemma 7.13 for the case where α is a power of ω, so consider the case where the Cantor normal form of α is

$$\alpha = \omega^{n_1} \cdot a_1 + \omega^{n_2} \cdot a_2 + \ldots + \omega^{n_k} \cdot a_k$$

where $n_1 > n_2 > \ldots > n_k$ and with $a_i \neq 0$ for all i. Since A is α-like then by Proposition 7.8, A can be written in the form

$$A = \sum_{i=1}^{k} \sum_{j=1}^{a_i} W_j^{n_i},$$

where $W_j^{n_i}$ is an ω^{n_i}-like linear order for every n_i and j. Let B be a tail of A containing a least point. It follows that

$$B = (W_m^{n_q})^\star + \sum_{j=m+1}^{a_q} W_j^{n_q} + \sum_{i=q+1}^{k} \sum_{j=1}^{a_i} W_j^{n_i}$$

for some q and m with $1 \leqslant q \leqslant k$ and $1 \leqslant m \leqslant a_q$ and where $(W_m^{n_q})^\star$ is a tail of $W_m^{n_q}$ containing a least point. By Lemma 7.13, $(W_m^{n_q})^\star$ will be ω^{n_q}-like. Take β to be the ordinal

$$\beta = \omega^{n_q} \cdot (a_q - m + 1) + \sum_{i=q+1}^{k} \omega^{n_i} \cdot a_i.$$

Then β is a tail of α and from Proposition 7.8, B is β-like. QED

In particular, every tail having a least point of an ω-like linear order is also ω-like.

Proposition 7.15 Let α be an ordinal with $\alpha < \omega^\omega$. Let A be a linear order and let B_1 and B_2 be linear orders both having least points, and such that $A + B_1 \equiv \alpha$ and $A + B_2 \equiv \alpha$. Then $B_1 \equiv B_2$.

Proof The result clearly holds for finite α. We prove the result for infinite α using induction on α.

First consider the case where $\alpha = \omega$. From Proposition 7.14 it follows that $B_1 \equiv \omega \equiv B_2$.

We next show that if the result holds for an ordinal α then the result also holds for the ordinal $\alpha + \mathbf{1}$. Hence let α be an ordinal for which the result is true. If $A + B_1 \equiv \alpha + \mathbf{1}$ and $A + B_2 \equiv \alpha + \mathbf{1}$ then B_1 and B_2 must both have greatest points. Let B_1' and B_2' be the linear orders obtained respectively from B_1 and B_2 by removing their greatest points. Then we get $A + B_1' \equiv \alpha$ and $A + B_2' \equiv \alpha$ from which $B_1' \equiv B_2'$ by the inductive hypothesis. This in turn gives $B_1 = B_1' + \mathbf{1} \equiv B_2' + \mathbf{1} = B_2$ as required.

Finally let α be a limit ordinal and suppose the result holds for all ordinals ξ with $\xi < \alpha$. We need to show that the result also holds for α. Let the Cantor normal form of α be

$$\alpha = \omega^{n_1} \cdot a_1 + \omega^{n_2} \cdot a_2 + \ldots + \omega^{n_k} \cdot a_k$$

where $n_1 > n_2 > \ldots > n_k$ and $a_i \neq 0$ for all i. Since $A + B_1$ is α-like then by Proposition 7.8 we can write

$$A + B_1 = \left(W_1^{n_1} + \ldots + W_{a_1}^{n_1}\right) + \ldots + \left(W_1^{n_k} + \ldots + W_{a_k}^{n_k}\right)$$

where each of the linear orders $W_j^{n_i}$ is ω^{n_i}-like. Hence we get for some p and q with $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant a_p$ that

$$\begin{aligned} A &= \sum_{i=1}^{p-1} \sum_{j=1}^{a_i} W_j^{n_i} + \sum_{j=1}^{q-1} W_j^{n_p} + \left(W_q^{n_p}\right)^- \\ B_1 &= \left(W_q^{n_p}\right)^\star + \sum_{j=q+1}^{a_p} W_j^{n_p} + \sum_{i=p+1}^{k} \sum_{j=1}^{a_i} W_j^{n_i} \end{aligned}$$

where $\left(W_q^{n_p}\right)^-$ and $\left(W_q^{n_p}\right)^\star$ are linear orders such that $\left(W_q^{n_p}\right)^- + \left(W_q^{n_p}\right)^\star = W_q^{n_p}$ and where $\left(W_q^{n_p}\right)^\star$ has a least point.

Since $A + B_2$ is α-like then using Proposition 7.8 it now follows that we can write

$$B_2 = \left(V_q^{n_p}\right)^\star + \sum_{j=q+1}^{a_p} V_j^{n_p} + \sum_{i=p+1}^{k} \sum_{j=1}^{a_i} V_j^{n_i}$$

where each of the linear orders $V_j^{n_i}$ is ω^{n_i}-like and where $\left(V_q^{n_p}\right)^\star$ is a linear order with a least point such that $\left(W_q^{n_p}\right)^- + \left(V_q^{n_p}\right)^\star$ is ω^{n_p}-like.

Hence $\left(W_q^{n_p}\right)^- + \left(W_q^{n_p}\right)^\star \equiv \omega^{n_p} \equiv \left(W_q^{n_p}\right)^- + \left(V_q^{n_p}\right)^\star$ so by the inductive hypothesis we have $\left(W_q^{n_p}\right)^\star \equiv \left(V_q^{n_p}\right)^\star$ and since $W_j^{n_i} \equiv V_j^{n_i}$ for all i and j then it follows that $B_1 \equiv B_2$ as required. QED

7.3 Some general observations

Proposition 7.16 Let α be a successor ordinal with $\alpha < \omega^\omega$ and let T be a definably uniformly α-like tree. Suppose T contains only finitely many paths which are not parametrically definable. Then for every $n \in \mathbb{N}$ there exists a pathwise uniformly α-like tree S such that $S \equiv_n T$.

Proof Follows from Proposition 6.23 and the fact that the first-order theory of α can be axiomatised by the sentence Φ_α. QED

Proposition 7.17 Let α be an ordinal with $\alpha < \omega^\omega$ and let T be a well-founded pathwise uniformly α-like tree. Then T is an α-tree.

Proof Follows using Proposition 7.11. QED

Proposition 7.18 Let α be an ordinal with $\alpha < \omega^\omega$ and let Σ be a finite theory which defines the class of pathwise uniformly α-like trees. Then the first-order theory of the class of α-trees can be axiomatised using the theory $\Sigma \cup \mathsf{A_W}$.

Proof Let $\mathcal{K}$ be the class of α-trees. We need to show that

$$\mathrm{MOD}(\Sigma \cup \mathsf{A_W}) = \mathrm{MOD}(\mathrm{TH}(\mathcal{K})).$$

Since every α-tree satisfies all of the sentences in the theory $\Sigma \cup \mathsf{A_W}$ then it follows that $\mathrm{MOD}(\mathrm{TH}(\mathcal{K})) \subseteq \mathrm{MOD}(\Sigma \cup \mathsf{A_W})$.

Next let T satisfy the theory $\Sigma \cup \mathsf{A_W}$ and let $\sigma \in \mathrm{TH}(\mathcal{K})$ with $\mathrm{qr}(\sigma) = k$. Assume that all of the sentences in Σ have quantifier rank at most m and let $n := \max\{k, m\}$. By Theorem 6.3 there exists a well-founded tree S with $S \equiv_n T$. Hence S satisfies Σ so S is a pathwise uniformly α-like tree and being well-founded we get that S is an α-tree. This gives $S \models \sigma$ and so $T \models \sigma$. It follows that $T \in \mathrm{MOD}(\mathrm{TH}(\mathcal{K}))$ and this establishes the inclusion $\mathrm{MOD}(\Sigma \cup \mathsf{A_W}) \subseteq \mathrm{MOD}(\mathrm{TH}(\mathcal{K}))$. QED

7.4 Towards first-order theories of α-trees

7.4.1 The finite case

Proposition 7.19 Let n be a positive natural number. The class of $\mathbf{n}$-trees can be defined using the theory

$$\Psi_{\mathbf{n}} := \{\mathsf{Tr}, \mathsf{Co}, \mathsf{Do}\} \cup \mathsf{Le}_{\Phi_{\mathbf{n}}}.$$

Proof It is clear that every $\mathbf{n}$-tree satisfies the theory $\Psi_{\mathbf{n}}$.

Next let T be a structure which satisfies the theory $\Psi_{\mathbf{n}}$.

Let $u \in T$. From Do and $\mathsf{Le}_{\Phi_{\mathbf{n}}}$ there exists a leaf $v \in T$ with $u \leqslant v$ and $T \models \Phi_{\mathbf{n}}^{\leqslant x}(v/x)$. Hence $v_{\geqslant} \models \Phi_{\mathbf{n}}$ so $v_{\geqslant} \cong \mathbf{n}$ and since $u \in v_{\geqslant}$ then $u \not< u$. It follows that $T \models \mathsf{Ir}$.

Next let $u, v, w \in T$ with $v, w < u$. From Do and $\mathsf{Le}_{\Phi_\mathbf{n}}$ there again exists a leaf $z \in T$ with $z_\geqslant \cong \mathbf{n}$ and such that $u \in z_\geqslant$. Since $u \leqslant z$ then from Tr we get $v, w < z$ so $v, w \in z_\geqslant$. Since $\mathbf{n} \models \mathsf{To}$ then $v \smile w$. It follows that $T \models \mathsf{ST}$.

Hence $T \models \bigwedge \mathsf{A_T}$ so T is a tree.

Finally let A be a path in T and let $u \in A$. Again by Do and $\mathsf{Le}_{\Phi_\mathbf{n}}$ there exists a leaf $v \in T$ with $v_\geqslant \cong \mathbf{n}$ and with $u \in v_\geqslant$. Since $u_\geqslant \subseteq v_\geqslant$ then $|u_\geqslant| \leqslant n$. It follows that $|A| \leqslant n$. Hence A contains a greatest element w and w will be a leaf. By $\mathsf{Le}_{\Phi_\mathbf{n}}$ we get that $A = w_\geqslant \cong \mathbf{n}$.

Hence T is an $\mathbf{n}$-tree. QED

7.4.2 The class of ω-trees

Define the sentence

$\mathsf{D}_2'' : \forall x \exists y \, (x < y)$.

Proposition 7.20 The class of pathwise uniformly ω-like trees can be defined using the theory

$$\Psi_\omega := \mathsf{A_F} \cup \{\mathsf{Ro}, \mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_2''\}.$$

Proof If T is a pathwise uniformly ω-like tree then it is immediate that T satisfies the sentences in $\mathsf{A_F}$. That the sentences Ro, D_1, D_2 and D_2'' hold in T can be verified using the fact that every path in T is elementarily equivalent with ω and so will be of the form $\omega + \zeta \cdot \alpha$ for some order type α.

Next let T be a structure which satisfies the theory Ψ_ω. From Ro it follows that $T \models \mathsf{Co}$. Hence $T \models \bigwedge \mathsf{A_T}$ so T is a tree.

Let A be a path in T. We need to show that $A \equiv \omega$. Since A is a linear order then A satisfies all of the sentences in $\mathsf{A_L}$. Since $T \models \mathsf{Ro}$ then T contains a root which will also be a least node in A so $A \models \mathsf{Ro}$.

To show that $A \models \mathsf{D}_1$, let $x \in A$ be any node which has a predecessor belonging to A. Since $T \models \mathsf{D}_1$ then there exists $y \in T$ such that $y < x$, and there is no $z \in T$ with $y < z < x$. Since A is downwards convex this gives $y \in A$. Hence y is an immediate predecessor to x in A, so $A \models \mathsf{D}_1$.

Since $T \models \mathsf{D}_2''$ then T contains no leaves and since A is maximal total in T then it follows that every node in A has a successor from A. Let $x \in A$ be any node and let $y \in A$ be a successor to x. By the fact that $T \models \mathsf{D}_2$ there exists $z \in T$ such that z is an immediate successor to x and $x < z \leqslant y$ which implies $z \in A$. Hence every node in A has an immediate successor from A so $A \models \mathsf{D}_2'$.

Thus $A \models \Phi_\omega$ so $A \equiv \omega$. It follows that the tree T is pathwise uniformly ω-like. QED

Theorem 7.21 The first-order theory of the class of ω-trees can be axiomatised using the theory

$$\Psi_\omega \cup \mathsf{A}_\mathsf{W}.$$

Proof From Proposition 7.20 and Proposition 7.18. QED

7.4.3 The class of almost α-trees

Definition 7.22 Let $A := (A; <_A)$ be a linear order and let $F := (F; <_F)$ be a forest. Then $A + F := (|A + F|; <_{A+F})$ denotes the tree obtained by adding F to the end of A. Formally $A + F$ is defined as follows:

(i) $|A + F| := A \cup F$ and

(ii) $<_{A+F} \;:=\; <_A \,\cup\, <_F \,\cup\, \{(x, y) : x \in A \text{ and } y \in F\}$.

If $T := (T; <_T)$ is a tree and A is a path in T then $T +_A F := (|T +_A F|; <)$ denotes the tree obtained from T by adding the forest F to the end of the path A. Formally $T +_A F$ is defined as follows:

(i) $|T +_A F| := T \cup F$, and

(ii) $< \;:=\; <_T \,\cup\, <_F \,\cup\, \{(x, y) : x \in A \text{ and } y \in F\}$.

Definition 7.23 Let α be an ordinal. A tree T is called an **almost α-tree** when T is definably uniformly α-like and when the following property holds:

A_α : for every path X in T which is not α-like, there exists a forest F such that

(i) the tree $X + F$ is pathwise uniformly α-like, and

(ii) $T \preceq T +_X F$.

If F is a forest satisfying the above two conditions then the tree $T +_X F$ is called an **α-completion** of T with respect to the path X.

Part (ii) of the property A_α is, by the Tarski-Vaught criterion for elementary substructures (see [18, Proposition 4.31]), equivalent to the property that for every tuple $\bar{c}$ in T, if the formula $\varphi(x, \bar{c})$ holds true in the tree $(T +_X F; \bar{c})$ for some element a from $T +_X F$, then $\varphi(x, \bar{c})$ already holds true in $(T; \bar{c})$ for some element b from T.

Example 7.24 Let B_ω be the binary ω-tree and let A be any path in B_ω. Let G be the forest consisting of the set of nodes $\{a, b\}$ with $a \not\prec b$ in G. Let T be the tree obtained from B_ω by adding a copy of the forest G to the end of every path in B_ω other than the path A. The set A remains a path in T and A is not parametrically definable in T whereas every path in T different from A is parametrically definable. It follows that T is definably uniformly $(\omega+\mathbf{1})$-like. Clearly $A+G$ is pathwise uniformly $(\omega+\mathbf{1})$-like and $T \preceq T+_A G$. Hence T is an almost $(\omega + \mathbf{1})$-tree and $T +_A G$ is an $(\omega + \mathbf{1})$-completion of T with respect to the path A.

Let B_ζ be the binary ζ-tree and let B_ζ^+ be the tree obtained from B_ζ by adding a copy of the forest G to the end of every path in B_ζ. The tree $A + B_\zeta^+$ is pathwise uniformly $(\omega + \mathbf{1})$-like and it is easy to see that Player II has a winning strategy for the game $\mathrm{EF}_n\big((T; \bar{c}), \big(T +_A B_\zeta^+; \bar{c}\big)\big)$, where $\bar{c}$ is any tuple of nodes from T, hence $T \preceq T +_A B_\zeta^+$. Therefore $T +_A B_\zeta^+$ is also an $(\omega + \mathbf{1})$-completion of T with respect to the path A. This shows that the choice of forest F for which the tree $T +_A F$ is an $(\omega + \mathbf{1})$-completion of T with respect to the path A, is not unique.

Note that

$$T \models \forall x \exists y \, (y \neq x \wedge \forall z \, (z < x \leftrightarrow z < y)) \, .$$

Hence it is not sufficient to take F to consist of a single node if we require that $T \preceq T +_A F$.

Next we have an example of a tree which is definably uniformly $(\omega+\mathbf{1})$-like but not an almost $(\omega + \mathbf{1})$-tree.

Example 7.25 Let T be the tree from Example 6.25 which we have already shown to be definably uniformly $(\omega + \mathbf{1})$-like. Every non-leaf node in T has either precisely one or precisely three immediate successors and if x is a node with precisely one immediate successor then every non-leaf node y with $x < y$ also has precisely one immediate successor.

Let A be any ω-path in T and suppose F is a forest such that the tree $T +_A F$ is an $(\omega + \mathbf{1})$-completion of T with respect to A. It follows that every path in F must have a stem of which every node in that stem has precisely three immediate successors, for suppose to the contrary that u is a node in F such that x has precisely one immediate successor for every $x \in F$ with $x < u$. Let

$$\varphi(x, z) := x < z \wedge \exists y_1 \exists y_2 \exists y_3 \left(\bigwedge_{i \neq j} y_i \neq y_j \ \wedge \ \bigwedge_{i=1}^{3} s(x, y_i) \right).$$

Then $T +_A F \models \Phi_\omega^\varphi(u/z)$ whereas $T \not\models \exists z \Phi_\omega^\varphi(z)$, a contradiction with the fact that $T \preceq T +_A F$.

For every node x in T having precisely three immediate successors there exists a node y in T also having precisely three immediate successors and with $x < y$. Since $T \preceq T +_A F$ then it follows that F contains a path B such that the set of nodes in the path $A + B$ in the tree $T +_A F$ having precisely three immediate successors has the form $\omega + \zeta \cdot \gamma$ for some order type γ. But the path $A + B$ contains a leaf b and since $\omega + \zeta \cdot \gamma \equiv \omega$ then $T +_A F \models \Phi_\omega^\varphi(b/x)$ whereas $T \not\models \exists z \Phi_\omega^\varphi(z)$, a contradiction with the fact that $T \preceq T +_A F$.

Hence T has no $(\omega + \mathbf{1})$-completion with respect to the path A so T is not an almost $(\omega + \mathbf{1})$-tree.

Let α be an ordinal with $\alpha < \omega^\omega$. By Corollary 6.21 the class of definably uniformly α-like trees is precisely the class of models of the theory

$$\mathsf{A_T} \cup \mathsf{De}_{\Phi_\alpha}.$$

Hence we shift our attention to the property A_α in the study of almost α-trees.

Proposition 7.26 Let α be an ordinal with $\alpha < \omega^\omega$. Let T be an almost α-tree and let b be any node in T. Then the subtree $b_\leqslant$ of T is an almost β-tree for some ordinal β with $b_> + \beta \equiv \alpha$.

Moreover let X be a path in T with $b \in X$ and let Y be the path in $b_\leqslant$ defined as $Y := b_\leqslant \cap X$. If F is a forest such that $T +_X F$ is an α-completion of T with respect to the path X, then $b_\leqslant +_Y F$ is a β-completion of $b_\leqslant$ with respect to the path Y.

Proof We consider two cases: firstly where b lies in a path which is parametrically definable in T, and secondly where none of the paths which contain b are parametrically definable in T.

Hence let A be a parametrically definable path in T with $b \in A$. Then $A \equiv \alpha$. By Proposition 7.14 there exists an ordinal β with $\beta < \alpha$ and such that the stem $\{x \in A : b \leqslant x\}$ is β-like. Clearly $b_> + \beta \equiv \alpha$. We will show that $b_\leqslant$ is an almost β-tree.

$b_\leqslant$ is definably uniformly β-like: Let B be a path defined in $b_\leqslant$ by the formula $\varphi(x, \bar{y})$ with the parameters $\bar{a}$ from $b_\leqslant$ substituted for $\bar{y}$. Then the formula

$$\varphi^{\geqslant z}(x, \bar{y}) \vee x < z$$

defines the path $b_> + B$ in T with $\bar{a}$ substituted for $\bar{y}$ and b substituted for z. Hence $b_> + B \equiv \alpha$ and it follows from Proposition 7.15 that $B \equiv \beta$, as required.

$b_\leqslant$ satisfies A_β: Let B be a path in $b_\leqslant$ that is not β-like. Since $\{x \in A : b \leqslant x\} \equiv \beta$ and $b_> + \{x \in A : b \leqslant x\} \equiv \alpha$, it follows from Proposition 7.15 that the path $b_> + B$ in T is not α-like. Since T satisfies A_α, there exists a forest F such that $(b_> + B) + F$ is a pathwise uniformly α-like tree and $T \preceq T +_{b_> + B} F$.

$B + F$ is a pathwise uniformly β-like tree: Let C be a path in $B + F$. Then $b_> + C$ is a path in $(b_> + B) + F$ so $b_> + C \equiv \alpha$. From Proposition 7.15 it again follows that $C \equiv \beta$, as required.

Finally we show that $b_\leqslant \preceq b_\leqslant +_B F$:

$$\begin{array}{rcl}
\text{Let } (b_\leqslant +_B F; \bar{c}) & \models & \exists x \psi(x, \bar{c}). \\
\text{Then } (T +_{b_> + B} F; \bar{c}b)^{\geqslant b} & \models & \exists x \psi(x, \bar{c}) \ \big(\text{since } (b_\leqslant +_B F; \bar{c}) = \\
 & & \qquad (T +_{b_> + B} F; \bar{c}b)^{\geqslant b}\big) \\
\text{and so } (T +_{b_> + B} F; \bar{c}b) & \models & (\exists x \psi(x, \bar{c}))^{\geqslant b}. \\
\text{This gives } (T +_{b_> + B} F; \bar{c}b) & \models & \exists x \big(x \geqslant b \wedge \psi^{\geqslant b}(x, \bar{c})\big) \\
\text{hence } (T +_{b_> + B} F; \bar{c}b) & \models & \big(x \geqslant b \wedge \psi^{\geqslant b}(x, \bar{c})\big)(d/x) \ \text{ for some } d \in T \\
 & & \qquad \big(\text{since } T \preceq T +_{b_> + B} F\big), \\
\text{i.e. } (T +_{b_> + B} F; \bar{c}bd) & \models & \psi^{\geqslant b}(d, \bar{c}) \text{ and where } d \in b_\leqslant. \\
\text{Then } (T +_{b_> + B} F; \bar{c}bd)^{\geqslant b} & \models & \psi(d, \bar{c}) \\
\text{which gives } (b_\leqslant +_B F; \bar{c}d) & \models & \psi(d, \bar{c}) \ \big(\text{since } (T +_{b_> + B} F; \bar{c}bd)^{\geqslant b} = \\
 & & \qquad (b_\leqslant +_B F; \bar{c}d)\big).
\end{array}$$

$$\text{Hence } (b_{\leqslant} +_B F; \bar{c}) \models \psi(d/x, \bar{c}), \text{ as required.}$$

Hence $b_{\leqslant}$ is almost β-like. This concludes the case where b lies in a path that is parametrically definable in T.

Next consider the case where no path in T containing b is parametrically definable. First suppose that for every path Z in $b_{\leqslant}$ the path $b_{>} + Z$ in T is α-like. Let A be any path in $b_{\leqslant}$. It follows from Proposition 7.14 that $A \equiv \beta$ for some ordinal β with $\beta < \alpha$ and from Proposition 7.15 it follows that $Z \equiv \beta$ for every path Z in $b_{\leqslant}$. Hence $b_{\leqslant}$ is a pathwise uniformly β-like tree and therefore also an almost β-tree. It is clear that $b_{>} + \beta \equiv \alpha$.

Next suppose that for some path B in $b_{\leqslant}$, the path $b_{>} + B$ in T is not α-like. Since T satisfies the property A_α, there exists a forest F such that $(b_{>} + B) + F$ is a pathwise uniformly α-like tree and $T \preceq T +_{b_{>}+B} F$.

Let C be any path in F. The path $b_{>} + B + C$ in T is α-like and so, by Proposition 7.14, $B + C$ is β-like for some ordinal β with $\beta < \alpha$. Moreover $b_{>} + \beta \equiv \alpha$. We will show that $b_{\leqslant}$ is an almost β-tree.

$b_{\leqslant}$ contains no paths which are parametrically definable, for if Z were a path defined in $b_{\leqslant}$ by the formula $\varphi(x, \bar{y})$ with the parameters $\bar{a}$ substituted for $\bar{y}$, then $b_{>} + Z$ would be a path defined in T by the formula

$$\varphi^{\geqslant z}(x, \bar{y}) \vee x < z$$

with $\bar{a}$ substituted for $\bar{y}$ and b substituted for z, a contradiction with the fact that no path in T containing b is parametrically definable. Hence $b_{\leqslant}$ is definably uniformly β-like and it remains to show that $b_{\leqslant}$ satisfies the property A_β.

Let D be a path in $b_{\leqslant}$ with $D \not\equiv \beta$. Since $b_{>}+B+C \equiv \alpha$ and $B+C \equiv \beta$ it follows from Proposition 7.15 that $b_{>}+D \not\equiv \alpha$. Since T satisfies the property A_α, there exists a forest G such that $(b_{>} + D) + G$ is a pathwise uniformly α-like tree and $T \preceq T +_{b_{>}+D} G$.

$D+G$ is pathwise uniformly β-like: if Z is any path in $D+G$ then $b_{>}+Z$ is a path in $(b_{>} + D) + G$ so $b_{>} + Z \equiv \alpha$. Since $b_{>} + B + C \equiv \alpha$ with $B + C \equiv \beta$ it follows by Proposition 7.15 that $Z \equiv \beta$.

$b_{\leqslant} \preceq b_{\leqslant} +_D G$: This can be seen using a similar argument as the one above where it is shown that $b_{\leqslant} \preceq b_{\leqslant} +_B F$.

This completes the proof. QED

Recall that in ZFC every set can be well-ordered. Hence if $\mathcal{A}$ is a set of paths in a tree then we can express $\mathcal{A}$ in the form $\mathcal{A} = \{A_i : i \in \beta\}$ for β

some ordinal. If $\mathcal{A}$ is infinite we may take β to be a limit ordinal. This is because every infinite successor ordinal can be written in the form $\gamma + \mathbf{n}$ for γ a limit ordinal and with $n \in \mathbb{N}^+$, so instead of well-ordering $\mathcal{A}$ as $\gamma + \mathbf{n}$ we can well-order it as $\mathbf{n} + \gamma = \gamma$.

The following result allows us to elementarily extend an almost α-tree to a tree of which every path is elementarily equivalent to the ordinal α.

Theorem 7.27 Let T be an almost α-tree for α an ordinal with $\alpha < \omega^\omega$. For β some limit ordinal, let $\mathcal{A} = \{A_i : i \in \beta\}$ be the set of all paths in T which are not α-like, and for every i ($i \in \beta$), let F_i be a forest such that $T +_{A_i} F_i$ is an α-completion of T with respect to the path A_i. Define

$$\begin{aligned} T_{\mathbf{0}} &:= T, \\ T_\xi &:= \bigcup_{i<\xi} (T +_{A_i} F_i) \quad \text{for} \quad \mathbf{1} \leqslant \xi \leqslant \beta, \end{aligned}$$

i.e. T_ξ is the tree obtained from T by adding for every i ($i < \xi$) the forest F_i to the end of the path A_i. Then T_ξ is an almost α-tree for every ξ ($\xi \leqslant \beta$). Moreover for every i ($\xi \leqslant i < \beta$) the tree $T_\xi +_{A_i} F_i$ is an α-completion of T_ξ with respect to the path A_i.

Proof We use induction on the set of ordinals ξ with $\xi \leqslant \beta$. Note that in order to show that the tree T_ξ satisfies the property A_α, it suffices to show for every i ($\xi \leqslant i < \beta$) that $T_\xi +_{A_i} F_i$ is an α-completion of T_ξ with respect to the path A_i.

By assumption $T_{\mathbf{0}}$ is an almost α-tree and $T_{\mathbf{0}} +_{A_i} F_i$ forms an α-completion of $T_{\mathbf{0}}$ with respect to the path A_i for every i ($\mathbf{0} \leqslant i < \beta$).

Next let γ be an ordinal with $\gamma < \beta$ and suppose that T_γ is a definably uniformly α-like tree and that $T_\gamma +_{A_i} F_i$ forms an α-completion of T_γ with respect to the path A_i for every i ($\gamma \leqslant i < \beta$). We need to show that $T_{\gamma+\mathbf{1}}$ is also a definably uniformly α-like tree and that $T_{\gamma+\mathbf{1}} +_{A_i} F_i$ forms an α-completion of $T_{\gamma+\mathbf{1}}$ with respect to the path A_i for every i ($\gamma + \mathbf{1} \leqslant i < \beta$).

Consider the path A_j in $T_{\gamma+\mathbf{1}}$ for some j ($\gamma + \mathbf{1} \leqslant j < \beta$). Then $A_j + F_j$ is a pathwise uniformly α-like tree. Let a be a node in A_j with $a \notin A_\gamma$. Note that the subtree $a_\leqslant$ of $T_{\gamma+\mathbf{1}}$ is the same as the subtree $a_\leqslant$ of T_γ. Let $Y := \{x \in A_j : x \geqslant a\}$. Since T_γ is an almost α-tree with $T_\gamma +_{A_j} F_j$ an α-completion of T_γ with respect to the path A_j, we have by Proposition 7.26 that the tree $a_\leqslant$ is an almost δ-tree for some ordinal δ with $a_> + \delta \equiv \alpha$

and $a_{\leqslant} +_Y F_j$ is a δ-completion of $a_{\leqslant}$ with respect to the path Y in $a_{\leqslant}$. In particular $a_{\leqslant} \preceq a_{\leqslant} +_Y F_j$. Since $T_{\gamma+\mathbf{1}} +_{A_j} F_j$ can be seen as the result of replacing the subtree $a_{\leqslant}$ of $T_{\gamma+\mathbf{1}}$ with the tree $a_{\leqslant} +_Y F_j$, it follows from Proposition 5.18 that $T_{\gamma+\mathbf{1}} \preceq T_{\gamma+\mathbf{1}} +_{A_j} F_j$. Hence $T_{\gamma+\mathbf{1}} +_{A_j} F_j$ is an α-completion of $T_{\gamma+\mathbf{1}}$ with respect to the path A_j.

Since the class of definably uniformly α-like trees can be defined by the theory $\mathsf{A}_\mathsf{T} \cup \mathsf{De}_{\mathbf{\Phi}_\alpha}$ and since T_γ is definably uniformly α-like and $T_\gamma \preceq T_{\gamma+\mathbf{1}}$, it follows that $T_{\gamma+\mathbf{1}}$ is also definably uniformly α-like. Hence $T_{\gamma+\mathbf{1}}$ is an almost α-tree.

Next consider the case where λ is a limit ordinal with $\lambda \leqslant \beta$. Suppose that for every ξ with $\xi < \lambda$, T_ξ is a definably uniformly α-like tree and $T_\xi +_{A_i} F_i$ forms an α-completion of T_ξ with respect to the path A_i for every i ($\xi \leqslant i < \beta$). We need to show that the tree T_λ is definably uniformly α-like and that $T_\lambda +_{A_i} F_i$ forms an α-completion of T_λ with respect to the path A_i for every i ($\lambda \leqslant i < \beta$).

First note that since the tree T_ξ satisfies the property A_α for every ξ ($\xi < \lambda$), it follows that $\{T_\xi\}_{\xi<\lambda}$ is an elementary chain so

$$T_\tau \preceq \bigcup_{\xi<\lambda} T_\xi = T_\lambda$$

for every τ with $\tau < \lambda$.

It again follows that since the class of definably uniformly α-like trees can be defined using the theory $\mathsf{A}_\mathsf{T} \cup \mathsf{De}_{\mathbf{\Phi}_\alpha}$ and since the tree $T_{\mathbf{0}}$ is definably uniformly α-like then the tree T_λ must also be definably uniformly α-like.

Next consider the path A_j in T_λ for some j ($\lambda \leqslant j < \beta$). We already know that $A_j + F_j$ is a pathwise uniformly α-like tree. Let U be the set of nodes u in T_λ for which (i) $u \notin A_j$, and (ii) the immediate predecessor of u belongs to A_j, and (iii) $u \in A_i + F_i$ for some i with $i < \lambda$. For every ξ ($\xi < \lambda$) and for every $u \in U$, define the trees

$$\begin{aligned} V_u^\xi &:= \{x \in T_\xi : u \leqslant x\} = \{x \in T_\xi +_{A_j} F_j : u \leqslant x\}, \\ W_u &:= \{x \in T_\lambda : u \leqslant x\}. \end{aligned}$$

Now $V_u^\xi \preceq W_u$ for every ξ ($\xi < \lambda$) and $u \in U$. To see this, let $\bar{c}$ be a tuple of nodes in V_u^ξ and suppose that $(W_u; \bar{c}) \models \exists x \varphi(x, \bar{c})$. Then $(T_\lambda; \bar{c}\, u)^{\geqslant u} \models \exists x \varphi(x, \bar{c})$ so $(T_\lambda; \bar{c}\, u) \models \exists x\, (x \geqslant u \wedge \varphi^{\geqslant u}(x, \bar{c}))$. Since $T_\xi \preceq T_\lambda$ this gives $(T_\lambda; \bar{c}\, u) \models \varphi^{\geqslant u}(d/x, \bar{c})$ for some $d \in T_\xi$ with $d \geqslant u$. Hence

$(W_u;\bar{c}) \models \varphi(d/x,\bar{c})$ with $d \in V_u^\xi$ and it follows by the Tarski-Vaught criterion for elementary substructures that $V_u^\xi \preceq W_u$.

Next we show that $T_\xi \preceq T_\lambda +_{A_j} F_j$ for every ξ with $\xi < \lambda$. We know by the inductive hypothesis that $T_\xi \preceq T_\xi +_{A_j} F_j$. The tree $T_\lambda +_{A_j} F_j$ can be obtained from the tree $T_\xi +_{A_j} F_j$ by replacing the subtree V_u^ξ of $T_\xi +_{A_j} F_j$ with the tree W_u for every $u \in U$. By Proposition 5.18 this gives $T_\xi +_{A_j} F_j \preceq T_\lambda +_{A_j} F_j$. Hence we get $T_\xi \preceq T_\lambda +_{A_j} F_j$.

Finally let $\varphi(x,\bar{y})$ be a formula and let $\bar{c}$ be a tuple of nodes in T_λ for which $\left(T_\lambda +_{A_j} F_j;\bar{c}\right) \models \exists x\varphi(x,\bar{c})$. Then $\bar{c}$ is a tuple of nodes in the tree T_τ for some τ ($\tau < \lambda$). Since $T_\tau \preceq T_\lambda +_{A_j} F_j$ we get that $(T_\tau;\bar{c}) \models \varphi(d/x,\bar{c})$ for some $d \in T_\tau$ and since $T_\tau \preceq T_\lambda$ this then gives $(T_\lambda;\bar{c}) \models \varphi(d/x,\bar{c})$ where $d \in T_\lambda$ also. By the Tarski-Vaught criterion for elementary substructures we therefore have that $T_\lambda \preceq T_\lambda +_{A_j} F_j$. Hence $T_\lambda +_{A_j} F_j$ forms an α-completion of T_λ with respect to the path A_j. It follows that T_λ is an almost α-tree.

This completes the induction argument. QED

Corollary 7.28 Let T be an almost α-tree for α an ordinal with $\alpha < \omega^\omega$. There exists a pathwise uniformly α-like tree S for which $T \preceq S$.

Proof Take S to be the tree T_β as in Theorem 7.27. QED

Let α be an ordinal with $\alpha < \omega^\omega$ and suppose we have a theory Δ which axiomatises the class of trees satisfying the property A_α. Then the theory

$$\Delta \cup \mathsf{De}_{\Phi_\alpha}$$

axiomatises the class of pathwise uniformly α-like trees. To see this first note that every pathwise uniformly α-like tree satisfies $\Delta \cup \mathsf{De}_{\Phi_\alpha}$. Next suppose T is a model of the theory $\Delta \cup \mathsf{De}_{\Phi_\alpha}$. It follows that $T \equiv T'$ for some almost α-tree T' and by Corollary 7.28, $T' \equiv S$ for some pathwise uniformly α-like tree S giving $T \equiv S$, as required.

7.5 Almost $(\omega + 1)$-trees and their extensions

Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$. A formula $\varphi(\bar{x})$ is called **existential** when $\varphi(\bar{x})$ has the form $\exists y_1 \ldots \exists y_m\, \psi(\bar{x},\bar{y})$ with $\psi(\bar{x},\bar{y})$ a quantifier-free formula. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and suppose that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. Then $\mathfrak{A}$ is called **existentially closed** in $\mathfrak{B}$ if, for every existential formula $\varphi(\bar{x})$, if $(\mathfrak{B};\bar{b}) \models \varphi(\bar{b})$ for some n-tuple of elements $\bar{b}$ from $|\mathfrak{B}|$, then already $(\mathfrak{A};\bar{a}) \models \varphi(\bar{a})$ for some n-tuple of elements $\bar{a}$ from $|\mathfrak{A}|$.

Every path in an almost $(\omega+\mathbf{1})$-tree is either an ω-like path or an $(\omega+\mathbf{1})$-like path. The easiest way to turn an ω-like path into an $(\omega+\mathbf{1})$-like path is to add a node to the end of the path although the tree so obtained need not be an elementary extension of the original tree. As the next result shows however, an almost $(\omega+\mathbf{1})$-tree will be existentially closed in the tree obtained from it by adding a node to the end of each of its ω-like paths.

Proposition 7.29 Let T be an almost $(\omega+\mathbf{1})$-tree and let T^+ be the tree obtained from T by augmenting every ω-like path in T with a leaf. Then T is existentially closed in T^+.

Proof Let $\varphi(\bar{x})$ be a quantifier-free formula with $\bar{x} = (x_1,\dots,x_k)$ and suppose that $T^+ \models \varphi(\bar{a}/\bar{x})$ for some tuple of nodes $\bar{a} = (a_1,\dots,a_k)$ from T^+. Let $a_{i_1},\dots,a_{i_m}$ be all those nodes from $a_1,\dots,a_k$ which belong to T and let $a_{j_1},\dots,a_{j_n}$ be all those nodes from $a_1,\dots,a_k$ which belong to $T^+\backslash T$. Obviously $a_{j_1},\dots,a_{j_n}$ must all be leaves and for every r the set $A_{j_r} := \{x \in T : x < a_{j_r} \text{ in } T^+\}$ forms an ω-like path in T.

For every r $(1 \leqslant r \leqslant n)$ and for every s for which $a_s \neq a_{j_r}$, let $b_{j_r,s}$ be the least node in T^+ with the property that $b_{j_r,s} \in A_{j_r}$ and $b_{j_r,s} \not< a_s$. Such a node $b_{j_r,s}$ must exist for suppose first that $A_{j_r} \subseteq (a_s)_>$. Now if $s = i_t$ for some t then the set A_{j_r} is not maximal total in T which contradicts the fact that A_{j_r} is a path in T. And if $s = j_t$ for some t then it would mean that in the construction of T^+, the path A_{j_r} in T was augmented with not one but two leaves a_{j_r} and a_{j_t}, again a contradiction. It follows that there exists a node $b_{j_r,s}$ with the property that $b_{j_r,s} \in A_{j_r}$ and $b_{j_r,s} \not< a_s$. That we can find a least such $b_{j_r,s}$ is due to the fact that A_{j_r} is an ω-like path.

Again since A_{j_r} is ω-like then for every r $(1 \leqslant r \leqslant n)$ we can find a least node $b_{j_r} \in A_{j_r}$ for which $b_{j_r} > b_{j_r,s}$ for every s with $a_s \neq a_{j_r}$.

For every i $(1 \leqslant i \leqslant k)$ define

$$c_i := \begin{cases} a_i & \text{when } i = i_t \text{ for some } i_t \ (1 \leqslant t \leqslant m) \\ b_i & \text{when } i = j_t \text{ for some } j_t \ (1 \leqslant t \leqslant n) \end{cases}$$

and let $\bar{c} = (c_1,\dots,c_k)$.

We show that $T \models \varphi(\bar{c}/\bar{x})$ by showing, for all i and j with $1 \leqslant i,j \leqslant k$, that $a_i < a_j$ in T^+ if and only if $c_i < c_j$ in T, and $a_i = a_j$ if and only if $c_i = c_j$.

$a_i < a_j \Leftrightarrow c_i < c_j$: First assume that $a_i < a_j$. The node a_i cannot be amongst $a_{j_1}, \ldots, a_{j_n}$ since then a_i would be a leaf in T^+ and consequently $a_i \not< a_j$. Hence a_i must be a node amongst $a_{i_1}, \ldots, a_{i_m}$. If a_j is also a node amongst $a_{i_1}, \ldots, a_{i_m}$ then we get $c_i = a_i < a_j = c_j$. Hence consider the case where a_j is a node amongst $a_{j_1}, \ldots, a_{j_n}$. Then $a_i \in A_j$ and $b_{j,i} \in A_j$ hence $a_i \smile b_{j,i}$ and it follows that $a_i \leqslant b_{j,i}$. This gives $c_i = a_i \leqslant b_{j,i} < b_j = c_j$.

Next assume that $c_i < c_j$. Observe that the index i cannot be amongst $j_1, \ldots, j_n$ since then we would have that $b_{i,j} < b_i = c_i < c_j$. Then if j is amongst $i_1, \ldots, i_m$ we would further have $c_j = a_j$, while if j is amongst $j_1, \ldots, j_n$ we would have $c_j = b_j < a_j$. In either case this would give $b_{i,j} < a_j$, a contradiction. Now if the indices i and j are both amongst $i_1, \ldots, i_m$ then we get that $a_i = c_i < c_j = a_j$, and if i is amongst $i_1, \ldots, i_m$ and j is amongst $j_1, \ldots, j_n$ then since $c_j \in A_j$ we get $a_i = c_i < c_j < a_j$.

$a_i = a_j \Leftrightarrow c_i = c_j$: Let $a_i = a_j$ with $i \neq j$. Either both i and j must be amongst the indices $i_1, \ldots, i_m$ or both i and j must be amongst the indices $j_1, \ldots, j_n$. If i and j are both amongst $i_1, \ldots, i_m$ then $c_i = a_i = a_j = c_j$. If i and j are both amongst $j_1, \ldots, j_n$ then from the way b_i and b_j are chosen we get $b_i = b_j$ so $c_i = b_i = b_j = c_j$.

Now let $c_i = c_j$ with $i \neq j$. If i is amongst $i_1, \ldots, i_m$ and j is amongst $j_1, \ldots, j_n$ then $b_{j,i} < b_j = c_j = c_i = a_i$ which contradicts the fact that $b_{j,i} \not< a_i$. Hence it cannot be that i is amongst $i_1, \ldots, i_m$ and j is amongst $j_1, \ldots, j_n$. Next if i and j are amongst $i_1, \ldots, i_m$ then $a_i = c_i = c_j = a_j$. Finally consider the case where both i and j are amongst $j_1, \ldots, j_n$. Suppose that $a_i \neq a_j$. Then $b_{i,j} < b_i = c_i = c_j = b_j < a_j$. This contradicts the fact that $b_{i,j} \not< a_j$ and so $a_i = a_j$, as required.

From the fact that the nodes c_i were chosen such that $c_i = a_i$ when $a_i \in T$, it now follows that if $\bar{d}$ is a tuple of nodes from T and if $\psi(\bar{y}, \bar{z})$ is a quantifier-free formula such that $\big(T^+; \bar{d}\big) \models \exists \bar{y}\, \psi(\bar{y}, \bar{d})$ then $\big(T; \bar{d}\big) \models \exists \bar{y}\, \psi(\bar{y}, \bar{d})$. Hence T is existentially closed in T^+. QED

Let T be an almost $(\omega + \mathbf{1})$-tree and let T^+ be the tree obtained from T by augmenting every ω-like path in T with a leaf. Since T is existentially closed in T^+ then there exists a tree S for which $T^+ \subseteq S$ and $T \preceq S$ (see e.g. [22]). As the next two examples show, such a tree S need not be pathwise uniformly $(\omega + \mathbf{1})$-like.

Example 7.30 Let B_ω be the binary ω-tree and let B be the tree obtained from B_ω as follows. We assign labels (recall Section 4.1) to the paths in B_ω

by assigning the label (0) to the root of B_ω and for every node $x \in B_\omega$ we use the set $I_{x_\geqslant} = \{0, 1\}$ and fix $f_{x_\geqslant} : S(x_\geqslant) \to I_{x_\geqslant}$ to be any bijection. Then as in Example 4.4, for every path X in B_ω the label $\ell(X)$ of X corresponds a real number r with $0 \leqslant r \leqslant 1$. The tree B is taken as the result of adding a node to the end of every path X of B_ω of which the label $\ell(X)$ corresponds to a rational number r.

Let W be the tree constructed by taking the linear order ζ and at every point in ζ we attach a copy of the tree B. Thus W is a $\mathcal{C}$-tree, where $\mathcal{C} = \{\zeta, \zeta + \mathbf{1}\}$, in which every non-leaf node has precisely two immediate successors.

Finally we take T to be the tree B. Using an argument similar to the one used in Example 6.25 it can be seen that the parametrically definable paths in T are precisely the $(\omega + \mathbf{1})$-paths hence T is a definably uniformly $(\omega + \mathbf{1})$-like tree. Moreover it is easy to verify using an Ehrenfeucht-Fraïssé game that the tree obtained from T by adding a node to the end of any of its ω-paths is an elementary extension of T. Hence T satisfies the property $A_{\omega+\mathbf{1}}$ and it follows that T is an almost $(\omega + \mathbf{1})$-tree.

Let T^+ be the tree obtained from T by adding a node to the end of every ω-path in T. Then T^+ is an $(\omega + \mathbf{1})$-tree of which every non-leaf node has precisely two immediate successors.

Construct the tree S from T^+ as follows. For every leaf z in $T^+ \backslash T$ we augment the stem $z_>$ with a copy of the tree W by inserting a copy of W next to the leaf z, i.e. $x < w$ for every $x \in z_>$ and $w \in W$ while $z \not< w$ for every $w \in W$. Let S be the tree so obtained. Note that S is a $\mathcal{K}$-tree for $\mathcal{K} = \{\omega + \mathbf{1}, \omega + \zeta, \omega + \zeta + \mathbf{1}\}$. In particular, S is not a pathwise uniformly $(\omega + \mathbf{1})$-tree.

Now $T^+ \subseteq S$ and using an Ehrenfeucht-Fraïssé game it is easy to see that $(T; \bar{c}) \equiv_n (S; \bar{c})$ for every k-tuple $\bar{c}$ in T and for every $n \in \mathbb{N}$. Hence $T \preceq S$.

The next example shows that the tree S need not be an end-extension of the tree T^+.

Example 7.31 Let B be the tree from Example 7.30. Let W be the tree consisting the linear order $Z := \omega^\star$ with a copy of the tree B attached to every point in Z. Hence W is a $\mathcal{C}$-tree, where $\mathcal{C} = \{\zeta, \zeta + \mathbf{1}\}$, and every node in W has precisely two immediate successors, except for the greatest node in Z which has only one immediate successor, and the leaves in W which all have no successor.

As in Example 7.30, let T be the tree B. In Example 7.30 we showed that T is an almost $(\omega + \mathbf{1})$-tree. Let T^+ be the tree obtained from T by adding a node to the end of every ω-path in T.

Construct S as follows. For every leaf z in $T^+ \backslash T$ we (i) insert a copy of the tree W between z and $z_>$ by changing the path $z_> + z$ in T^+ to $z_> + Z + z$ with all the copies of W branching off from the points in Z as usual, and (ii) we adjoin a copy of the tree B to the end of the path $z_> + Z + z$ formed in (i). Take S to be the tree that so results. Hence every non-leaf node in S will have two immediate successors and S is a $\mathcal{K}$-tree with $\mathcal{K} = \{\omega + \mathbf{1}, \omega + \zeta, \omega + \zeta + \mathbf{1}\}$. In particular, S is not a pathwise uniformly $(\omega + \mathbf{1})$-tree

Then $T^+ \subseteq S$ and using an Ehrenfeucht-Fraïssé game, it can be seen that $(T; \bar{c}) \equiv_n (S; \bar{c})$ for every k-tuple $\bar{c}$ in T and for every $n \in \mathbb{N}$. Hence $T \preceq S$.

Remark 7.32 Let T be an almost $(\omega + \mathbf{1})$-tree, let T^+ be the tree obtained from T by adding a node to each of its ω-like paths, and let S be any tree for which $T^+ \subseteq S$ and $T \preceq S$. In the construction of S from T^+ there are certain configurations of nodes which cannot be added to T^+ to obtain S. Assume that S is a proper extension of T^+ and let $s \in S \backslash T^+$. Some restrictions on the location of s relative to the nodes in T include the following:

(i) *The node s cannot lie below the root of T.* This follows from the fact that the formula root(x) defines the root of a tree and $T \preceq S$ hence the root of T is also the root of S.

(ii) *The node s cannot lie above any leaf from T.* This follows from the fact that the formula leaf(x) defines the set of leaves of a tree and $T \preceq S$ hence every leaf in T is also a leaf in S.

(iii) *The node s cannot lie between two nodes from T of which the one node is an immediate successor to the other node.* This is because if $b_1, b_2 \in T$ and b_2 is an immediate successor to b_1 then $(T; b_1, b_2) \models s(b_1, b_2)$ and since $T \preceq S$ then $(S; b_1, b_2) \models s(b_1, b_2)$ so b_2 is an immediate successor to b_1 in S also.

(iv) *If T is finitely branching then s either extends a path from T or s is bounded above by nodes from T.* Suppose to the contrary that $X \not< s$ for every path X from T and that $s \not< x$ for every node $x \in T$. Let $A := \{x \in T : x < s \text{ in } S\}$ and note that A is a stem in T. Suppose T is n-branching from A. It follows from Proposition 4.9 that there exists a set of nodes $H := \{a_1, \ldots, a_n\}$ in the subtree $A_<$ of T such that H spans $A_<$. The set H does not span the subtree $A_<$ of S though since s belongs to the subtree $A_<$ of S but $s \not\smile a_i$ for every i. Let $\bar{a} := (a_1, \ldots, a_n)$. Then $(T; \bar{a})$ satisfies the sentence

$$\forall x \forall y \left(\bigwedge_{i=1}^{n} x < a_i \to \left(x < y \to \bigvee_{i=1}^{n} y \smile a_i \right) \right)$$

but $(S; \bar{a})$ does not. This contradicts the fact that $T \preceq S$.

Chapter 8

Concluding remarks

Trees are important structures occuring in many diverse fields of mathematics and computer science. A systematic study of their first-order theories, along the lines of what [24] does for linear orders, does not exist and much work remains to be done in this regard.

We have defined trees as consisting of a set of nodes with an order relation imposed on those nodes, but we have placed particular emphasis on the structure of the paths within the tree. First-order logic lacks the expressive capability to reliably capture the structure of these paths hence the general problem of studying the first-order theory of a tree based on knowledge of the first-order theory of the class of linear orders which comprise its paths is not an easy one.

We have defined eight classes of trees which arise naturally from a class of linear orders $\mathcal{C}$ in terms of how the paths in those trees are related to the linear orders in the class $\mathcal{C}$ and we have established all the set-theoretical relationships between these eight classes of trees as well as between their first-order theories. We have also investigated the first-order theories of some of these classes of trees based on knowledge of the first-order theory of the class $\mathcal{C}$.

The particular case where $\mathcal{C}$ consists of a single finitely axiomatisable ordinal merits special attention. We have investigated the first-order theory of the class of trees of which every path is isomorphic with the ordinal α for $\alpha < \omega^\omega$. For the case where α is finite or where $\alpha = \omega$ we have determined the first-order theory of this class. For the case where $\omega < \alpha < \omega^\omega$ we have introduced the notion of an almost α-tree and showed that every almost α-tree can be elementarily extended in a natural way to a tree of which every path satisfies the first-order theory of α.

We have also studied some general set-theoretical and logical properties of trees, specifically the broad problem of the axiomatisability of various classes of trees and the definability of various sets of nodes within a tree.

Some directions for further study include the following:

1. The general problem, for any axiomatisable class of linear orders $\mathcal{C}$, of axiomatising the first-order theories of all eight of the classes of $\mathcal{C}$-trees based on the first-order theory of the class $\mathcal{C}$.

2. For an arbitrary class of linear orders $\mathcal{C}$, investigating the transfer of logical properties, such as the decidability of the first-order theory of $\mathcal{C}$, to the first-order theories of the eight $\mathcal{C}$-classes of trees, and vice versa.

3. To axiomatise the first-order theory of the class of α-trees for α an ordinal with $\omega < \alpha < \omega^\omega$.

4. Further study of the definable subsets, notably the definable paths, of trees.

5. Studying trees using extensions of first-order logic which are still generally weaker than monadic second-order logic. Possible such languages include first-order logic with colours (see e.g. [10]), first-order logic with transitive closure (see e.g. [8]), and first-order logic with fixed points (also see e.g. [8]).

List of symbols

Symbol or notation	Page where defined
$\mathfrak{A} \preceq_n \mathfrak{B}$	5
$\mathbf{n}$, ω, ζ, η, λ, ω_1	6
$L^\star$	6
ZF, ZFC	6
$\mathbb{N}^+$	6
λ_n, μ_n	6
φ^θ	8
$(\mathfrak{A};\bar{a})^\theta$	8
$\varphi^{(x,y)}$, $\varphi^{<x}$ etc.	9
$[\![(\mathfrak{A};\bar{a})]\!]^n$	10
$s(x,y)$	13
$X < Y$	13
$x \smile y$	13
$x_>$ etc.	14
$S(x)$	15
(x,y) etc.	15
$X_>$ etc.	16
$[x]$	19
$\beta(x,y)$	21
$[T]$	21
$[X]$, $[y]^{-1}$, $[Y]^{-1}$	21
$X \times Y$	25
$X \times_{\text{lex}} Y$, $(x_1,y_1) <_{\text{lex}} (x_2,y_2)$	25

Bibliography

[1] R. Backofen, J. Rogers and K. Vijay-Shankar, *A First-Order Axiomatization of the Theory of Finite Trees*, ***Journal of Logic, Language and Information***, vol. 4 (1995), pp. 5–39.

[2] P. Blackburn and M. Meyer-Viol, *Linguistics, Logic and Finite Trees*, ***Bulletin of the IGPL***, vol. 2 (1994), pp. 3–29.

[3] D. Comer, *The Ubiquitous B-Tree*, ***Computing Surveys***, vol. 11 (1979), pp. 121-137.

[4] J. Cummings and M. Foreman, *The Tree Property*, ***Advances in Mathematics***, vol. 133 (1998), pp. 1–32.

[5] K. Doets, ***Completeness and Definability: Applications of the Ehrenfeucht Game in Second-Order and Intensional Logic***, Ph.D. thesis, University of Amsterdam, 1987.

[6] K. Doets, ***Basic Model Theory***, CSLI Publications & The European Association for Logic, Language and Information, 1996.

[7] H.-D. Ebbinghaus, J. Flum and W. Thomas, ***Mathematical Logic***, Springer, 1996.

[8] A. Gheerbrant and B. Ten Cate, *Complete Axiomatizations of MSO, FO(TC^1) and FO(LFP^1) on Finite Trees*, ***Proceedings of LFCS 2009*** (S. Artëmov and A. Nerode, editors), Springer, 2009, pp. 180–196.

[9] R. Goldblatt, ***Logics of Time and Computation, 2nd Edition***, CSLI Publications & The European Association for Logic, Language and Information, 1992.

[10] V. Goranko, *Trees and Finite Branching*, ***Proceedings of the 2nd Panhellenic Logic Symposium***, 1999, pp. 91–101.

[11] V. Goranko, *Temporal Logics with Reference Pointers and Computation Tree Logics*, ***Journal of Applied Non-Classical Logics***, vol. 10 (2000), pp. 221–242.

[12] V. Goranko and R. Kellerman, *Classes and Theories of Trees Associated with a Class of Linear Orders*, to appear in ***Logic Journal of the IGPL***.

[13] V. Goranko and A. Zanardo, *From Linear to Branching-Time Temporal Logics: Transfer of Semantics and Definability*, ***Logic Journal of the IGPL***, vol. 15 (2007), pp. 53–76.

[14] Y. Gurevich, *Monadic Second-Order Theories*, ***Model-Theoretical Logics*** (J. Barwise and S. Feferman, editors), Springer, 1985, pp. 479–506.

[15] Y. Gurevich and S. Shelah, *To the Decision Problem for Branching Time Logic*, ***Foundations of Logic and Linguistics: Problems and Their Solutions*** (G. Dorn and P. Weingartner, editors), Plenum, 1985, pp. 181–198.

[16] Y. Gurevich and S. Shelah, *The Decision Problem for Branching Time Logic*, ***The Journal of Symbolic Logic***, vol. 50 (1985), pp. 668–681.

[17] A. Guttman, *R-Trees: A Dynamic Index Structure For Spatial Searching*, ***SIGMOD'84, Proceedings of Annual Meeting*** (B. Yormark, editor), ACM, 1984, pp. 47-57.

[18] S. Hedman, ***A First Course in Logic: An Introduction to Model Theory, Proof Theory, Computability, and Complexity***, Oxford University Press, 2004.

[19] W. Hodges, ***A Shorter Model Theory***, Cambridge University Press, 1997.

[20] T. Jech, *Trees*, ***The Journal Of Symbolic Logic***, vol. 36 (1971), pp. 1–14.

[21] W. Just and M. Weese, ***Discovering Modern Set Theory: I. The Basics***, Graduate Studies in Mathematics, vol. 8, AMS Bookstore, 1996.

[22] B. Poizat, ***A Course in Model Theory: An Introduction to Contemporary Mathematical Logic***, Springer, 2000.

[23] M. Rabin, *Decidability of Second-Order Theories and Automata on Infinite Trees*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[24] J. Rosenstein, ***Linear Orderings***, Academic Press, 1982.

[25] M. Rubin, ***The Reconstruction of Trees from their Automorphism Groups***, Contemporary Mathematics, vol. 151, AMS Bookstore, 1993.

[26] M. Sabbadin and A. Zanardo, *Topological Aspects of Branching-Time Semantics*, ***Studia Logica***, vol. 75 (2003), pp. 271–286.

[27] J. Schmerl, *On $\aleph_0$-Categoricity and the Theory of Trees*, ***Fundamenta Mathematicae***, vol. 94 (1977), pp. 121–128.

[28] M. Stonebraker, *Inclusion of New Types in Relational Database Systems*, ***Proceedings of the Second International Conference on Data Engineering***, IEEE Computer Society, 1986, pp. 262-269.

[29] M. Suslin, *Problème 3*, ***Fundamenta Mathematicae***, vol. 1 (1920), p. 223.

[30] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, S. Tison and M. Tommasi, ***Tree Automata Techniques and Applications***, `http://tata.gforge.inria.fr/`, 2007.

[31] W. Thomas, *Languages, Automata, and Logic*, ***Handbook of Formal Languages, Vol. 3: Beyond Words***, (G. Rozenberg and A. Salomaa, editors), Springer, 1997, pp. 389–455.

[32] S. Todorčević, *Trees and Linearly Ordered Sets*, ***Handbook of Set Theoretic Topology*** (K. Kunen and J.E. Vaughan, editors), North-Holland, 1984, pp. 235–293.

[33] S. Todorčević and J. Väänänen, *Trees and Ehrenfeucht-Fraïssé Games*, ***Annals of Pure and Applied Logic***, vol. 100 (1999), pp. 69–97.

Index