PRICING MODELS FOR INFLATION LINKED DERIVATIVES IN AN ILLIQUID MARKET

by

Thibaut Zafack Takadong

Programme in Advanced Mathematics of Finance
School of Computational and Applied Mathematics

University of the Witwatersrand

A dissertation submitted for the degree of

Master of Science

September 2008

ABSTRACT

Recent financial crises have highlighted the sensitivity and vulnerability of financial markets to inflation, which reduces the value of money and affects the net returns of financial instruments. In response to this, investors who are concerned with maintaining their investment's purchasing power rather than its market value are resorting to inflation linked (IL) products to hedge their inflation risk. Consequently, the inflation market has been rapidly growing for the last decade and has further great potential growth worldwide. It is highly probable that inflation linked derivatives will eventually be as common as conventional products. Another cause of the inflation market boost is the growing extension of the time frame of financial transactions, which has generated an increase in inflation expectation; since 1980 the average time to maturity of long-dated transactions went from one decade to three decades. This is, in part, due to the ageing population in the developed world. This research investigates some alternative models in order to improve the match between model prices and observed prices in the American and South African inflation markets. It takes into account the relative illiquidity of IL products. The main tools used are Lévy distributions, macroeconomic factors, no-arbitrage and pricing kernel models. Lévy processes can replicate the behaviour of the return innovations of a wide range of financial securities. Adding a stochastic time change to the Lévy process randomises the market clock, thus generating stochastic volatilities, higher stochastic return moments and eventually stochastic skewness. These are observed stylised facts most conventional models do not achieve. Moreover, in contrast to the hidden factor approach, each Lévy process component and its stochastic time change can readily be assigned an economic meaning. This explicit economic mapping facilitates the interpretation of current models and provides a more intuitive approach to building new models that capture other observed behaviours. Finally, Lévy processes also provide tractable formulas for derivative pricing and market estimations. In general, inflation is a consequence of macroeconomic factors. Exogenous dynamics of the most significant of these factors are used to deduce the endogenous inflation dynamics in some of the considered models. In these cases, the calibration of the pricing kernel models requires little historical data on IL derivatives. In fact, the required macroeconomic historical data is easily available because of the current national and international legislation.

Key words: market illiquidity, inflation linked products, Lévy processes, pricing kernel, macroeconomic factors.

ACKNOWLEDGMENTS

TO MY DAD AND MY MOM.

There will not have been a calibration to the South African market without the assistance of Shameer Sukha, Simona Levet and Nicolette Roussos. Each of them was very helpful and provided some sample data for the calibration.

Many thanks to David Taylor, coordinator of the Program of Advanced Mathematics of Finance at Wits, for admitting me in the program and for the great courses (both theoretical and practical) he makes sure are always available. I was impressed by the quality of the curriculum and learned a lot from them.

I will also like to thank my “colleagues” in the laboratory at Wits. They created an ideal environment for research and took the time to proof read each draft of the current manuscript. Numerous mistakes were thus corrected and the multiple discussion we had were always fruitful. Any mistake left is entirely due to me.

Last but not least, I can not thank enough my supervisor Raouf Ghomrasni for all the hard work he put in and for all the proof reading, critics and pushing he kept doing.

DECLARATION

This is to certify that

(i) the thesis comprises only my original work toward the MSc, except where indicated in the Preface,

(ii) due acknowledgement has been made in the text to all other material used.

I hereby certify that this thesis was independently written by me. No material was used other than that referred to. Sources directly quoted and ideas used, including figures, tables, sketches, drawings and photos, have been correctly denoted. Those not otherwise indicated belong to the author.

Contents

List of Figures

List of Tables

Preface

For the last decades major economies worldwide have been experiencing a constantly increasing inflation rate. Combined with historically low interest rates and high money growth (more than 18% per year for China [74]), these place the Central Banks far behind the current inflation/interest rate curve. Moreover, if Central Banks fight inflation by raising interest rates, their currency will strengthen, but they will lose market shares. The fact that inflation is rising and should keep increasing steadily for the coming years, if not decades; has led to the introduction of a new type of financial instrument.

Instead of preserving the investment's (nominal) value, these instruments guarantee its purchasing power throughout the years at a certain threshold. These securities are termed inflation linked (IL) or inflation indexed (II) derivatives and have their payoff linked to a price index, i.e. the prices of goods and services.

Furthermore, IL securities are often more profitable than their corresponding nominal (i.e. conventional) counterparts. This is because inflation expectation is mostly non-negative (especially for long maturities e.g. 10, 20 years and more) due to fluctuations in supply and demand. For example[1] not long ago a "normal" car had no air-conditioner, nor CD player. Nowadays, because "most" recent cars have both air conditioning and CD (and even MP3) players they are more expensive; that is inflation. In the meantime, wages did not necessary follow; thus to buy a car people take credit. Since future money gets used today, the amount of money available today is "virtually" increased. But, from the time value of money, R100 moved back to today is not "really" worth R100 today. Hence disrupting the equilibrium between overall value of money in circulation and overall value of "goods" being produced. To restore this equilibrium, the intrinsic value (purchasing power) of money needs to be devaluated. This imbalance gets propagated in other sectors like oil, food, etc.

[1] These are simplified examples, an extensive coverage can be found in Economics books and on the web [105].

Assuming the features of a car didn't change over time, again because of technological evolution in the car production system, over time, more cars get produced during a given period of time. Hence supply "exceeds" demand, to restore the equilibrium between supply and demand, the price of cars should go down. If this happened, the car industry will not get rewarded for their achievements. Thus they won't be aiming for constant improvement. In the previous example setting to maintain the equilibrium between supply and demand, the price of new cars should be identical to old ones' (without air conditioning and player) and the same problem is faced. Governments generally implement schemes to maintain the prices of services and goods "almost" constant, thus contributing to inflation.

The major drawback of the inflation market is its relative illiquidity just as the interest rate market at its beginning. And similarly, to the latter market, the inflation market should experience a fast growth in the coming years; eventually rejoining the interest rate market among the key components of the financial world.

Mathematical models in Finance literature and those used by investors are mainly based on Brownian motion although it is known that real-life financial data provides a different statistical behaviour than that implied by these models [8, 27, 109, 55, 9]. Recent empirical studies [48, 35] have proven that Lévy processes are better distributions than the normal distribution for models in Finance in general and for returns distribution in particular, because of their accuracy and flexibility. Moreover, Lévy processes do not only improve the fit of the distribution, but they give a more realistic and intuitive picture of the price movements at the microstructure level [46]. This study investigates how general Lévy processes can be applied to the inflation market's models.

The emphasis in this study lies on the application of Lévy processes in inflation models, particularly the pricing of IL bonds, swaps, caps and floors. In this process, both no-arbitrage and pricing kernel models are considered. In pricing kernel models, the main question is the modelling of the stochastic process governing the prices of contingent claims. An alternative approach to compensating for market illiquidity is through macroeconomic factors specification, which is not exclusive of previous approaches.

Here is a brief overview of each chapter. Chapter 1 briefly surveys some aspects of inflation and existing inflation derivative models. Chapter 2 reviews generalised hyperbolic distributions and Lévy processes' characteristics, with focus on how Lévy processes can be used to generalise the classical structural approach due to Merton [83]. Chapter 3 generalises the Heath Jarrow Morton

(HJM) approach proposed by Jarrow and Yildirim [71] for IL products' pricing and later extended by Hinnerich [63] for marked point processes. Moreover, the new framework is used to price IL swaps, caps and floors.

Chapters 4 and 5 introduce two pricing kernel frameworks. The former framework built by Hughston and Macrina [68] is a stochastic monetary economy structure to price IL securities. While the latter chapter does a reverse engineering [8] of the nominal and real pricing kernels from bond prices (IL and nominal) and inflation. The latter model is an improvement on the previous model because it does not use the agent's utility functions, thus avoiding the widely documented discrepancies between representative agent theories and observed asset prices [61].

Chapter 6 starts via an empirical study of the inflation market's data both for the South African and the American markets. Note that the former market is in a developing country, i.e. more illiquid than the latter market which is in a developed country. The study looks at the Lévy distributions fit against the normal distribution fit. It comes out that there is always at least a Lévy distribution that performs better than the normal distribution. Moreover, Lévy distributions have more degrees of freedom than their Gaussian counterpart thanks to their increased number of parameters, which make them more flexible and robust for calibration purposes. In the second part, Chapter 6 provides calibration tools used in the other chapters for option prices' calibration.

Chapter 1

Inflation Modelling

As of December 2003 there had been eleven issuances of Treasury Inflation Protected Securities (TIPS) by the US treasury. TIPS are meant to preserve the purchasing power of investors instead of the nominal value of their investment. Since their first appearance in the 18th century, it is only during the last decade that they have become more and more popular. Almost non-existent in 2001, the inflation market grew to about €50bn notional through the broker market in 2004, doubling its value in 2003 [72].

Inflation is a persistent increase in the price of products and services; it is synonym to a persistent decline in the purchase power of money. The opposite price's movement, a decrease in the price of products is called deflation. Although deflation might seem desired, inflation and deflation conflict with the Central Bank objective to stabilize prices through their monetary policy.

Definition 1.1. *Inflation (resp. deflation) is defined as the relative increase (resp. decrease) of the level of the consumer price index over a period of time.*

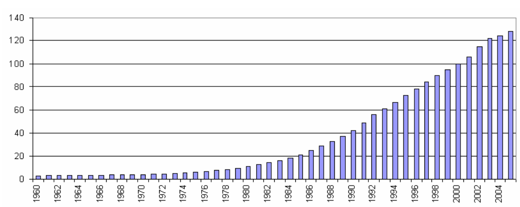
The generalized and constant rise in the prices of goods has generated a growing interest for products whose value is linked to a price index and thus "indirectly" to inflation. These instruments are referred to as inflation linked (IL) or inflation indexed (II) derivatives. They are meant to preserve an investor's purchasing power at a certain level throughout the years. This is achieved by linking their pay-off to the growth rate of prices.

The present chapter gives an overview of the inflation market and existing frameworks for IL derivatives pricing. Section 1.1 describes the inflation market and its main players. Section 1.2 reviews the most commonly traded IL instruments and their main features. And finally, Section 1.3 presents some existing frameworks for the pricing of IL products.

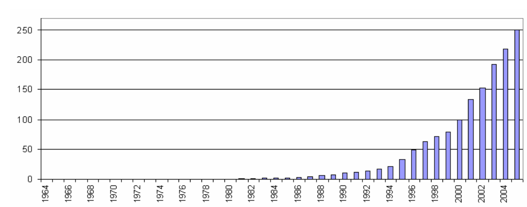
1.1 Inflation Market

Inflation is a measure of the variation of the price of a predefined basket of goods and services. Obviously, the composition of this reference basket greatly impacts the value of the inflation. According to the basket's components and their respective weight, a variety of inflation indexes has been defined. Examples of inflation indexes are the Consumer Price Index (CPI), the Retail Price Index (RPI), the Euro-zone Harmonised Index of Consumer Prices (Euro-HICP) and the Gross Domestic Product (GDP).

Usually inflation is not high enough to be noticed over a short period of time. Nevertheless a straightforward computation shows that if over 30 years we have an averaged inflation of 1% per annum, then R100 at initial time will have the purchasing power of R74 and only R48 if inflation averages 2.5%. Taking a look at the South African CPI (+11.7% y/y in May 2008 [98]) and the US CPI (+0.8% in May 2008 [106]), there is reason to be concerned by inflation especially in South Africa (SA).

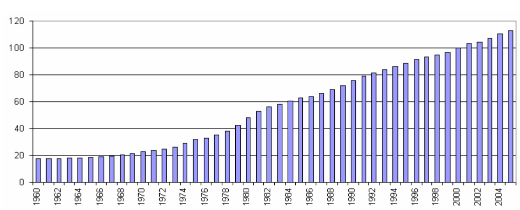


(a) SA annual CPI (2000 = 100) 1960 − 2005

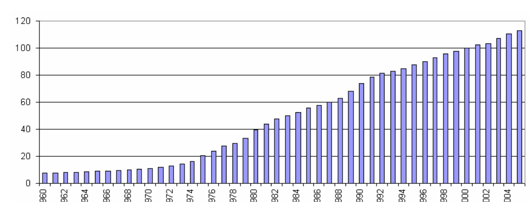


(b) Ghana annual CPI (2000 = 100) 1964 − 2005

Figure 1.1 Evolution of the annual South African and Ghanian CPI [104].



(a) US annual CPI (2000 = 100) 1960 − 2005



(b) UK annual CPI (2000 = 100) 1960 − 2005

Figure 1.2 Evolution of the annual UK and US CPI [104].

Figures 1.1 and 1.2 present the CPI evolution over about forty years for some developed (UK, USA) and emerging (Ghana, SA) countries. Note that these CPI are all normalised at 100 in 2000. As was expected, the lowest growth rate of 557% in 45 years is in a developed country, US; and the highest growth rate of 4048287% in 41 years is observed in an emerging country, Ghana. The Ghanian inflation is more than 7000 times the American inflation over a longer period of time. Moreover, the inflation rate of 1444% over 45 years for UK is still almost 3000 times smaller than that of Ghana. This is without including the high inflation rate observed during the last years which should impact more emerging countries because they do not have in place the structure to efficiently perform inflation rate targeting. South Africa whose economy can be said to be in-between that of a developed country and an emerging country has an inflation rate of 4152% in 45 years which is only about three times that of UK. However, looking at the estimations in the previous paragraph, there is still reason to be worried. These statistics suggest that developed countries should hedge their inflation risk and emerging countries must hedge their inflation risk.

A financial product exists and persists because of the supply and demand. This implies two corresponding groups of players, respectively payers/issuers and receivers/investors. Governments and private corporations constitute the main IL products issuers [38]; while the main investors in IL derivatives are pension funds and retail investors.

1.1.1 IL Products Issuers

Inflation indexed products issuers should have some IL liabilities whose risk they want to share through IL products. Some of the countries issuing IL securities had high inflation prior to this initiative (Mexico and Brazil with respectively 114.8% and 69.2% during the 1950s and 1960s hyperinflation period), but that was not the case for most.

Government

A government might issue IL products for several reasons. Firstly, a government can influence inflation (by reducing public cost through public debt's interest rate or premium) and thus benefit from issuing IL bonds. Secondly, IL bonds are adjusted to inflation whereas the conventional nominal bonds bear the risk of loosing value in real terms over time due to inflation. Another way to see this is through the fact that the conventional nominal interest rate on government's bonds is similar to the real interest rate plus inflation. This relationship is referred to as the Fisher hypothesis[1].

[1] See Equation 1.1

For instance, suppose that a government can issue either conventional nominal bonds with yield 8% or IL bonds with a real yield of 3% with the same maturity. The features of these two bonds imply that the market expectation of inflation over the lifetime of the bonds is 5%. However, if the realised inflation turns out to be 3%, then the government will just have a debt of 6% to repay with IL bonds as opposed to a "fixed" 8% with conventional bonds. In the event that inflation turns out higher than what had been expected, conventional bond issuance would of course have been the cheaper alternative.

Secondly, given that a government can influence inflation, issuance of inflation-indexed securities is a proof of its determination to fight inflation. In case of inflation, investors can transfer their losses through the purchased IL bonds. The involved risks taken by the government shows its firm intention to dampen inflation. Besides, the government's performance in controlling inflation can be gauged through IL products which provide a direct measure of real interest rate necessary to some decision makers. Prior to the issuance of IL securities in the US there was no direct means to study real interest rates [32].

Private Corporations

Private sector entities elect to issue indexed rather than nominal debt mainly for identical reasons as governments. Corporate treasurers judging that the expected inflation (as priced by the market) is too high will consider the issuance of inflation-indexed bonds more attractive. Moreover, the diversification of the company's debt portfolio implied by IL derivatives issuance and the improved risk-return characteristics is also very appealing [38].

The main non-governmental issuers of IL products are insurance companies. Due to the increasing risk of inflation and diminishing pension[2] payments, insurance companies have started selling IL products to take over some if not all of the inflation risk from their customers [111].

1.1.2 IL Products Investors

With the global aging of the world's population, pension funds and the saving system they represent is becoming capital for the economy. The key variable for any pension plan beneficiary is not the nominal amount of the pension payment, but the purchasing power guaranteed thereof.

In a standard contribution pension plan, the plan member bears a considerable risk due to inflation.

[2] See next section 1.1.2.

A distinct feature of pension funds, when compared to other financial institutions such as banks for example, is the very long maturity of their liabilities [49]. The typical duration of pension fund liabilities currently lies over a period of 30 years or more during which the pension benefit acquiring power might diminish. In fact, many plan members may not be aware that the benefits they will obtain from a classical, non-IL pension plan may not be sufficient to carry their expenses in the future, as price levels may have increased due to inflation. A simple calculation shows that an annual inflation rate of 1.5% over 30 years will reduce the real value of R$100,000$ then to R$63,546$. It therefore makes sense to link pension products to inflation.

At an individual level, IL products or structured products can also be used by agents in the market to hedge the risk due to inflation.

1.2 Inflation Products

The most commonly traded inflation linked securities are bonds, swaps, caps and floors. This section reviews the main attributes of these instruments. A more detailed covering of IL securities can be found in [72, 38].

To lighten the text, the nominal currency in this section will be the South African ZAR or R.

1.2.1 Inflation Linked Bonds

A conventional (nominal) bond $P_n(t,T)$ represents the value at time t of an instrument that pays R1 at maturity T. The corresponding IL bond $P_{IL}(t,T)$ represents the value (in ZAR) at time t of an instrument that pays R$I(T)$ at maturity T. If the IL bond's pay-off is measured in unit of $I(t)$ at time t, then it also pays 1 at T. When ignoring the units, its pay-off is similar to that of the nominal bond.

When an IL bond's value is divided by the price index, the corresponding real bond's value is obtained:

$$P_r(t,T) = \frac{P_{IL}(t,T)}{I(t)},$$

where the unit of a real bond, $P_r(t,T)$, is goods and services.

Though real bond's value can be deduced from IL bond's (nominal) value, only nominal and IL bonds are effectively traded on the market. Moreover, although IL bonds are quoted on the market in term of real yield, real bonds only exist by construction and are abstract.

If $r_n(t)$ is the nominal interest rate, then under the risk neutral measure $\mathbb{Q}$

$$P_n(t,T) \quad = \quad \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_n(s)ds\right)\right].$$

Similarly, if $r_r(t)$ is the real interest rate, then under $\mathbb{Q}$

$$P_r(t,T) \quad = \quad \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_r(s)ds\right)\right],$$

and the inflation linked bond is defined by

$$P_{IL}(t,T) \quad = \quad P_r(t,T)I(t).$$

A first order approximation of the relationship between the interest rates, under the nominal risk neutral measure, is given by the following equation known as the Fisher equation[3] [86]:

$$r_n(t,T) \quad = \quad r_r(t,T) + i_t^e(t,T), \tag{1.1}$$

where $r_n(t,T)$ (resp. $r_r(t,T)$) is the nominal (resp. real) interest rate for the time interval $[t,T]$ and $i_t^e(t,T)$ is the expected inflation rate over $[t,T]$ at time t.

The difference between the nominal and inflation yields is referred to as the inflation breakeven rate or inflation compensation. From the Fisher equation, the breakeven rate is a "good" approximation of the expected inflation. However, the relationship between nominal and real yield is more complicated and better estimated using the expectation hypothesis according to which

$$r_n(t) \quad = \quad r_r(t) + i_t^e(t,T) + Pr_{IL}(t),$$

where $Pr_{IL}(t)$ is the inflation risk premium. It is the additional return IL issuers need to pay on nominal bonds compared with IL bonds and depends on the volatility of inflation (higher volatility leads to higher premium) and risk-averseness of investors (the more risk-averse the higher the premium) [38].

Similarly to the interest rate market whose participants' pool is expanded beyond traders in interest rate through other interest rate derivatives (swaps, caps, floors, etc); IL derivatives have added flexibility to the inflation market and given new opportunities to investors.

[3] The full relationship between the nominal and real interest rates is restated in Equation 5.6 with a detailed derivation.

1.2.2 Zero-Coupon Inflation Swaps

Zero-coupon inflation swaps are considered the building block of the inflation market because of their simplicity, their transparency and the new investment opportunities they generate [38]. A zero-coupon inflation swap can be used to convert a nominal bond into a corresponding IL bond or to preserve the purchasing power of its notional with respect to a given inflation index.

By locking in a zero-coupon inflation swap, the participants agree to exchange the change in the inflation index over a period $[t, T]$ against a specified compounded interest rate. If t is the contract signature date (i.e. I_t is known at the signature of the contract), then the swap is spot starting. If t is instead a future date (i.e. I_t not yet known), then the swap is forward starting.

Let N denote the notional of the swap. There is no cash flow initially. At maturity T, the payer pays the net increase in inflation over the swap's life $N\left[\frac{I_T}{I_t} - 1\right]$. The receiver pays the fixed amount corresponding to a predefined annual compound rate b, $N\left[(1+b)^{T-t} - 1\right]$. The rate b is referred to as the breakeven swap rate and quoted in the market.

1.2.3 Year-on-Year Swap

The year-on-year (y-o-y) inflation swap is a variant of the zero-coupon swap with multiple payments (typically annually) over the term of the contract. Let $[T_{i-1}, T_i]$ denote a sub-period of an y-o-y inflation swap with notional N. At T_i, the swap payer pays $N\left[\frac{I_T}{I_t} - 1\right]$ and receives $N\frac{b}{p}$ where b is a pre-agreed zero coupon rate and p its annual periodicity.

1.2.4 Inflation Caps, Floors and Swaptions

Inflation caps, floors and swaptions are inflation volatility products. Inflation caps and floors are mainly use to set boundaries of an investment's pay-off (i.e. limit losses or benefit). For example along with an IL bond, an IL floor on the notional is usually bought as protection against eventual deflation.

An inflation cap (resp. floor) is a collection of caplets (resp. floorlets) each of which is a call (resp. put) on a zero-coupon swap. A caplet (resp. floorlet) pays the difference with respect to a (compounded) strike in case inflation turns out to be higher (resp. lower) than this pre-specified strike. A caplet written over the period $[t, T]$ with notional N and strike K pays a maturity

$$\text{Pay-off} \quad = \quad N \max\left[\alpha\left(\frac{I_T}{I_t} - 1 - K\right), 0\right],$$

where $\alpha = 1$ for a caplet and $\alpha = -1$ for a floorlet.

Just as inflation swaps, IL caps and floors can be spot starting or forward starting.

A swaption is an option to enter into a forward starting inflation swap (zero-coupon, year-on-year, etc) at a pre-specified coupon.

1.3 Inflation Models

Inflation has an analogy both with interest rates and Foreign Exchange (FOREX) [23, 67]. Given the diversity of methods existing to model these two, some "good" approaches were just tailored to IL securities. Most of the IL pricing frameworks use either the foreign currency analogy or the pricing kernel. These methodologies are respectively similar to the short-rate and the pricing kernel for interest rate derivatives. This section starts by a brief review of common interest rate models which are later implicitly used in the foreign currency analogy and pricing kernel frameworks.

1.3.1 Interest Rate Approach

Because of its similarities to the interest rate (defined as a percentage increment to an index) inflation models were first tailored to interest rate models. This subsection is entirely based on the paper by Fischer Black [18].

At the beginning of inflation theory, it was either modeled as a normal, a lognormal or a square root process. Using a normal process, the volatility of the change in the interest rate does not depend on the rate, though it may depend on time. When a lognormal distribution is used, the volatility of the fractional change in the interest rate does not depend on the rate. And with the square root process, the ratio of the variance of the change in the interest rate to the rate does not depend on the rate, so the volatility of the change in the rate is proportional to the square root of the rate at a given time. Of course, mean reversion can be inserted in any of the previous models.

The normal process implies that as the rate goes toward zero, the interest rate volatility does not decline. This contradicts the observed fact that volatility seems to decline with the rate, but it could be considered as an acceptable flaw. Apart from this deficiency, in a normal distribution the nominal interest rate has a non-zero probability of being negative. Though inflation can be negative, the nominal rate is always non-negative. After all, people can hold currency: they would rather keep currency under their mattresses than hold instruments bearing negative interest rates.

The lognormal process assumes that the nominal rate is non-negative, especially that it is always non-zero. However, in the 1930s the US nominal interest rate fell to zero and there are other such historical cases. Furthermore, a lognormal distribution implies that as the rate approaches zero the volatility falls very rapidly. Whereas, from market observations when the volatility falls, it does not seem to fall this rapidly.

The square root process is the most complex of all three and is halfway between the normal and the lognormal. The short rate will be non-zero if the mean reversion is quite strong or the short rate drift is large enough. However, when none of these conditions is satisfied, it is possible for the rate to become zero, we then have to decide whether zero is a reflecting barrier or an absorbing barrier. Assuming that zero is a reflecting barrier implies that the rate will “bounce” at zero while if zero is an absorbing barrier, we must assign a probability for the rate becoming positive again; thus having more complexity. Of the three alternatives, the absorbing barrier seems the most realistic [18].

1.3.2 Foreign Exchange Approach

The Foreign Exchange (FOREX) approach to modelling inflation is the most used nowadays. It is based on the foreign currency analogy in which real and nominal rates are assimilated to currencies in respectively the foreign and domestic economies, and the CPI is similar to the exchange rate [23]. The reference framework to price IL securities is due to Jarrow and Yildirim [71]. The following subsections present this model and extension by Mercurio [81] and Belgrade-Benhamou-Koehler [14].

Jarrow and Yildirim Model

The most quoted foreign currency analogy implementation is due to Jarrow and Yildirim [71] and is based on a Heath-Jarrow-Morton (HJM) model. In analogy with the HJM model of foreign currency they build a three-factor, arbitrage-free term structure model by modeling the dynamics of the real and nominal instantaneous forward interest rates and the inflation. The underlying sources of randomness are allowed to be correlated and the instantaneous forward rates are fitted to the market data.

Under the real world filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the Jarrow and Yildirim model is described by:

$$
\begin{aligned}
df_n(t,T) &= \alpha_n(t,T)dt + \varsigma_n(t,T)dW_n \\
df_r(t,T) &= \alpha_r(t,T)dt + \varsigma_r(t,T)dW_r \\
dI(t) &= I(t)\mu(t)dt + \sigma_I I(t)dW_I
\end{aligned}
$$

with $I(0) = I_0 > 0$, and

$$
f_x(0,T) = f_x^M(0,T), \quad x \in \{n, r\}
$$

where

i. $f(t,T)$ represents an instantaneous forward rate with maturity T at t and $I(t)$ represents the inflation rate at time t;

ii. The Brownian motions (W_k), with $k = n, r, I$ standing respectively for nominal, real and inflation, have correlation $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$;

iii. α_n, α_r, μ are adapted processes;

iv. ς_n, ς_r are deterministic functions;

v. σ_I is a positive constant;

vi. $f_n^M(0,T)$, $f_r^M(0,T)$ are the nominal and real instantaneous forward rates observed in the market at time 0 for maturity T respectively.

Hence Jarrow and Yildirim assumed both nominal and real (instantaneous) rates to be normally distributed under their respective risk neutral measures. Then using the no-arbitrage principle and taking forward rate volatilities of the form[4] $\sigma_k(t,T) = \sigma_k e^{-a_k(T-t)}$ with $k = n, r$, they proved that the real and nominal rates are Ornstein-Uhlenbeck[5] processes under the nominal measure $\mathbb{Q}_n$, and that the inflation index $I(t)$, at each time t, is lognormally distributed under $\mathbb{Q}_n$.
Since the real and nominal rates evolve following a Gaussian distribution, closed form solutions can be computed. However, this model has several drawbacks such as the difficulty to estimate parameters from market data and the possibility of interest rate becoming negative.

[4]For statistical reasons.

[5]i.e. of the form $dr_t = -\theta(r_t - \mu)dt + \sigma dW_t$; where θ, μ, σ are parameters and W_t denotes a Brownian motion.

Mercurio Market Model

Mercurio [81] proposed two variants of market models as an alternative to JY [71] and equivalent to Belgrade et al. [14] for pricing year on year inflation indexed swaps (YYIIS). Resorting to the lognormal LIBOR model, the first market model considers the nominal and the real rates to follow a lognormal distribution and the forward inflation index to follow geometric Brownian motion. The YYIIS price is a function of the nominal and real forward rates' (instantaneous) volatilities and their correlations, for each cash flows payment time; the correlations between real forward rates and forward inflation indices, again for each cash flows payment time. This YYIIS pricing formula is more complicated both in terms of input parameters and in terms of the calculations involved than the JY one. Nevertheless, it can be solved using numerical integration and, as is typical in a market model, the input parameters can be determined more easily than in the JY approach. But this new formula still depends on the volatility of real rates which may be hard to estimate. Given this deficiency, Mercurio developed a second market model to overcome this estimation issue [81].

He obtained a pricing formula for YYIIS combining the advantage of a fully-analytical formula with that of a market-model approach which does not depend on the real rates volatility anymore. The price of YYIIS depends on the (instantaneous) volatilities of the forward inflation indexes and their correlations, the (instantaneous) volatilities of nominal forward rates and the instantaneous correlations between forward inflation indices and nominal forward rates. The drawback of this model is that it is based in a rough approximation for long maturities, especially when the correlations between forward rates and inflation are non zero; the formula is exact when these correlations are zero [81].

Mercurio has shown that these three models produced similar results when calibrating with market data although they differed when away-from-the-money[6] instruments are considered [81].

A consolidated practice in all developed options markets is to include some kind of smile effect when simultaneously pricing caps with different strikes; to achieve this Mercurio and Moreni [82] as in Heston [62] introduce stochastic volatility in the forward CPI dynamics under the spot LIBOR measure. The cap prices obtained are a good approximation of the model's price.

[6] Not at-the-money, i.e. either in-the-money or out-of-the-money.

Belgrade-Benhamou-Koehler Model

The Belgrade-Benhamou-Koehler (BBK) [14] model was designed precisely to solve, using the no-arbitrage principle, the two major disadvantages of the JY model: the lack of link between zero coupon bonds and year-on-year swaps and the non-observable parameters. This model has two main objectives; to be simple (i.e. to have only few parameters) and to be robust (i.e. to replicate market prices). The main assumption made by BBK is that the market model for inflation considers forward inflation index returns as a diffusion with deterministic volatility structure. Under the risk neutral probability measure $\mathbb{Q}$, this index follows geometric Brownian motion with deterministic drift and volatility. In their paper, they consider three different functional forms of volatility (constant, exponentially decaying and adjusted exponentially). They present a method to parametrize the volatility structure to include the market data of caps/floors. They also perform a convexity adjustment of the inflation swaps derived from the difference of martingale measures between the numerator and the denominator. Given that it is not possible to estimate implicit correlations from the market data, they suggest some boundary conditions which for certain model hypotheses (for example constant volatility structure) give unrealistic results. This model is more suited in markets where there is enough information from zero-coupon and year-on-year swaps. It is important to be aware that to derive the model some approximations were done in the process so the solution is not exact. Another drawback of this model is that it is computationally intensive.

1.3.3 Macro-finance models

Because the JY framework is based on the HJM model, it can only perform cross-sectional fitting and thus can not estimate the inflation risk premium. However, this limitation can be overcome by using a macro-finance model of the term structure. Such a model is characterised by the fact that it uses macroeconomic factors to improve the coherence between the model output and the observed term structure on the market. Such models form a subclass of the affine term structure models (i.e. tractability and closed form solutions for asset pricing under certain restrictions). A macro-finance model can, in general, estimate both the correlation between the real and nominal interest rates and the risk premium “endogenously”. These models differ by the complexity used to include the macroeconomic factors in the conventional short rate models. This subsection provides a brief overview of their general properties based on Piazzesi [6].

In a no-arbitrage framework holding a zero coupon bond over a certain period of time $[t, T]$ is equivalent to the return of an average risk free short term rate during the same period under the risk neutral measure $\mathbb{Q}$:

$$P(t,T) \quad = \quad \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right], \tag{1.2}$$

where $P(t,T)$ is the price of a zero coupon bond of maturity T at any time $t \in [0,T]$. The two main components of this model are:

i. the change of measure from the real world $\mathbb{P}$, where the input data is measured and the risk neutral measure $\mathbb{Q}$, where the pricing is actually done because of the properties it has and;

ii. the short rate dynamics.

For an affine model the short rate is of the form

$$r(t) \quad = \quad R(x_t) \quad \text{, with} \quad x_t \in D \subset \mathbb{R}^n,$$

where $R(x)$ is affine and x_t is an affine diffusion process under $\mathbb{Q}$ and the solution of a stochastic differential equation of the form

$$dx_t \quad = \quad \mu(x_t)dt + \sigma(x_t)dW_t \quad \text{where } W_t \text{ is a standard Brownian motion.}$$

Under some regularity conditions, the corresponding guessed closed form solution for pricing zero coupon bonds is affine in the state variables and of the form:

$$P(t,T) \quad = \quad \exp\left[A(t,T) - B(t,T)x_t\right],$$

with some restrictions on $A(t,T)$ and $B(t,T)$. Solving this system of equations gives the short rates dynamics. The change of measure is obtained through the pricing kernel and hence the model is complete.

The pricing kernel or stochastic discount factor $\pi = (\pi_t)_{t\geq 0}$ is defined by

$$P(t,T) \quad = \quad \frac{\mathbb{E}[\pi_T]}{\pi_t} \quad \forall\ t \in [0,T].$$

With the short rate model attributes presented, let us discuss the macroeconomic side of the model. The short rate's dynamics is modelled because it drives the entire yield curve through Equation (1.2). From a macroeconomic point of view the short rate can be governed by the Taylor rule [101, 5]. The Taylor rule gives the interest rate change a central bank should make in response to a divergence in inflation or economic growth. The less complex Fisher equation can also be used to introduce the macroeconomic factors [4, 30]. Making the assumption that agents maximise their utility is also part of the macroeconomic model and translates the reactions to the central bank behavior, the inflation gap (difference between the realized inflation and the expected inflation), etc. The specificity of each micro-finance model is observable through the way the macroeconomic variables are inserted in the term structure model.

1.3.4 Stochastic monetary economy models

Similarly to the micro-finance models described in the previous subsection, the "stochastic monetary economy models" proposed by Hughston and Macrina [68] use macroeconomic factors and the pricing kernel to price IL securities. However, the latter framework does not assume linearity[7] of the macro-finance models; instead, it assumes a positive "nominal" interest rate and the underlying pricing kernel that was advocated for by Flesaker and Hughston (FH) [54].
Assuming N_t is the conventional numéraire, the corresponding pricing kernel is given by $\pi_t = \dfrac{\rho_t}{N_t}$ in the real world probability measure[8], where $(\rho_t)_{t\geq 0}$ denotes the Radon-Nikodym density martingale transforming the real world measure into the risk neutral measure. The latter equation implies that under the real probability measure the asset price process *multiplied with the pricing kernel process* is a martingale. The process $(\pi_t)_{t\geq 0}$ is a decreasing and positive supermartingale (i.e. $\pi_t \geq \pi_{t+h}$ with $h > 0$) thus ensures interest rate positivity.
The IL framework built by Hughston and Macrina is based on the assumption that inflation is a purely monetary phenomenon. Thus the influence of fluctuations in wages, supply and demand, foreign exchange and employment, etc. on inflation is not treated directly, but is rather reflected in the change of the rates of consumption and money supply, and the liquidity benefit of money supply.
In a discrete time[9] model, let the nominal money supply, the aggregate consumption and the nominal liquidity benefit be denoted respectively by $(\{M_i\}_{i\geq 0})$, $(\{k_i\}_{i\geq 1})$ and $(\{\lambda_i\}_{i\geq 0})$. At time t_i, the real benefit (in units of goods and services) provided by the money supply is defined by [68]

$$l_i = \frac{\lambda_i M_i}{C_i} \quad \text{for } i \geq 0.$$

Considering a wealth function of the form

$$W = \mathbb{E}\left[\sum_{n=0}^{N} \pi_n (C_n k_n + \lambda_n M_n)\right],$$

[7] i.e. that the models form a subclass of affine term structure models.

[8] This formula is fulling derived in Chapter 4

[9] The formulas in continuous time have also been derived and are similar to those obtained in discrete time.

where $U(\cdot,\cdot)$ is a bivariate utility function, the CPI ($\{C_i\}_{i\geq 0}$), pricing kernel ($\{\pi_i\}_{i\geq 0}$) and fair price of IL instruments ($H = \{H_i\}_{i\geq 0}$) are determined by maximising a consumer investor's function of the form

$$J = \mathbb{E}\left[\sum_{n=0}^{N} e^{-\gamma t_n} U(k_n, l_n)\right].$$

Example 1. *(i) Considering a log-separable utility function of the form*

$$U(x,y) = A\ln(x) + B\ln(y),$$

where A and B are two non-negative constants; the pricing kernel, the CPI and the pricing formula for an IL security are respectively

$$\begin{aligned} C_n &= \frac{A}{B}\frac{\lambda_n M_n}{k_n}; \\ \pi_n &= \frac{Be^{-\gamma t_n}}{\mu\lambda_n M_n}; \\ H_0 &= \lambda_0 M_0 e^{-\gamma t_j}\mathbb{E}\left[\frac{H_j}{\lambda_j M_j}\right]. \end{aligned}$$

(ii) Considering $p, q \in]-\infty, 1]\backslash\{0\}$, two non-negative constants A and B, and a separable power utility function of the form

$$U(x,y) = \frac{A}{p}x^p + \frac{B}{q}y^q,$$

gives

$$\begin{aligned} C_n &= \left(\frac{A}{B}\right)^{1-q}\frac{\lambda_n M_n}{k_n^{(1-q)/(1-p)}}; \\ \pi_n &= \frac{B^{\frac{1}{1-q}}}{A^{\frac{q}{1-q}}}\frac{k_n^{\frac{q}{1-q}(1-p)}}{\mu\lambda_n M_n}; \\ H_0 &= \frac{\lambda_0 M_0}{k_0^{q(1-p)/(1-q)}}e^{-\gamma t_j}\mathbb{E}\left[\frac{H_j k_j^{q(1-p)/(1-q)}}{\lambda_j M_j}\right]. \end{aligned}$$

Note that the formulas obtained are not directly functions of any IL derivative's price on the market. Therefore, this pricing methodology could be a solution to pricing IL products with inflation market illiquidity. The performances of this framework are further investigated in Chapter 4.

1.3.5 Calibration

The most commonly used data for calibration is that of US market because of its quality (the Federal Reserve publishes constant maturity US treasury bond yield data) and its time span, which is longer than for most of other countries. Even though there are not zero coupon bond rates available on the market, they can be deduced from market data. The common technique is through a bootstrap and an interpolation of coupon bearing bonds (and eventually swaps) often ignoring the seasonality; however this manipulation can introduce some measurement errors.

There is no standard estimation method since estimating both yield and macro data is dependent on the number of parameters assumed in the model. Nevertheless in order to simplify the calculations, researchers usually impose all restrictions on the parameters before the estimation process; this can reduce the number of parameters to compute (e.g. symmetry in a matrix).

Apart from US data, this study also uses South African data to evaluate the performances of the models both in a developed economy and a developing economy. All the models are based on Lévy processes (see Chapter 2) to improve the fit. The parameter estimation is mainly done with the maximum likelihood method. Chapter 6 presents in detail the calibration process.

Chapter 2

The Lévy Process Framework

Lévy processes are basically processes with stationary and independent increments. They are an excellent tool to model distributions in mathematical finance for four main reasons. First, they are the simplest class of processes with jumps. The latter become more obvious the smaller the time step considered between market observations. Second, they are part of both semimartingales and Markov processes with an additional robust mathematical structure. Third, some important processes like Brownian motion, Poisson process, stable and self-decomposable processes are special cases of Lévy processes. Finally, they have been successfully applied to mathematical finance, Physics and other fields both for research and practical usage [7, 70].

This chapter starts by reviewing elements of Lévy processes in Section 2.1. The remaining sections are devoted to more advanced topics for option pricing. Sections 2.2 and 2.3 present the Itô formula for Lévy processes, the Girsanov change of measure and other tools which will be used later on. Section 2.4 describes the General Hyperbolic (GH) distribution and other subclasses considered for the calibration. Finally, Section 2.5 examines option valuation using the Fast Fourier transform [26]. A more detailed presentation, both mathematical and practical, can be found in [70, 88, 35].

2.1 Lévy Processes

The following basic assumption is made throughout this thesis.

Assumption 1. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *with* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ *be a filtered probability space satisfying the usual conditions, that is:*

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ *is complete.*

(ii) All the null sets of $\mathcal{F}$ *are contained in* $\mathcal{F}_0$*, i.e. all impossible events are known beforehand.*

(iii) $\mathbb{F}$ *is a right continuous filtration:*

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \text{ are } \sigma\text{-algebra for } s, t \in \mathbb{R}_+, s \leq t, \text{ and } \mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \text{ for all } t \geq 0.$$

Furthermore, assumption is made that

$$\mathcal{F} \quad = \quad \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$$

This allows to specify a change of probability measure from $\mathbb{P}$ *to* $\mathbb{Q}$ *by giving the density process* $(Z_t)_{t\geq 0}$*, where* $Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$.

The following definition of Lévy processes is from Applebaum (2004).

Definition 2.1. *An adapted stochastic process* $X = (X_t)_{t\geq 0}$ *on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *taking values on* $\mathbb{R}^d$ *such that* $X_0 = 0$ *is called a Lévy process if:*

(i) X *has increments independent of the past, i.e.* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for* $0 \leq s < t < \infty$.

(ii) X *has stationary increments, i.e. the distribution of* $X_{s+t} - X_s$ *does not depend on* s *or equivalently* $X_{s+t} - X_s \stackrel{d}{=} X_t$ *where* $\stackrel{d}{=}$ *stands for the equality in distribution.*

(iii) X_t *is continuous in probability or stochastically continuous, i.e.*

$$\forall\ t \geq 0,\ \forall\ \varepsilon > 0\ \lim_{s\to t} \mathbb{P}\left[|X_t - X_s| > \varepsilon\right] = 0.$$

If the process X satisfies all the previous conditions, then it can be shown (See Theorem 30 in [93]) that there exists a *transformation* $Y = (Y_t)_{t\geq 0}$ of $X = (X_t)_{t\geq 0}$ (i.e. $\mathbb{P}(Y_t \neq X_t) = 0$ for all $t \geq 0$) with the following property

(iv) For almost every $\omega \in \Omega$, the function $t \longmapsto X(t, w)$ is càdlàg (from the French "continue à droite, limite à gauche") that is everywhere right-continuous with left limit.

This transformation is again a Lévy process. Because this transformation is always possible, the latter condition is generally included among the characteristics of a Lévy process.

Definition 2.2. 1. *Processes meeting only conditions* (i) *and* (ii) *are called processes with stationary independent increments (PIIS) [70].*

2. *Processes meeting conditions (i), (iii) and (iv) are called time-inhomogeneous Lévy processes.*

Example 2. *(i) A standard Brownian motion in $\mathbb{R}^d$ is a Lévy process.*

(ii) The Poisson process $(N_t)_{t\geq 0}$ with intensity $\lambda > 0$ is a Lévy process with values in $\mathbb{N}\cup\{0\}$ such that

$$\mathbb{P}(N_t = n) \quad = \quad \frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

Definition 2.3. *A probability distribution X is said to be infinitely divisible if for any positive integer n, there exists n independent and identically distributed (i.i.d.) random variables $Y_i, i = 1, 2, \cdots, n$, such that $Y_1 + Y_2 + \cdots + Y_n$ has distribution X.*

If $(X_t)_{t\geq 0}$ is a Lévy process, the distribution of X_t, for any $t > 0$ is infinitely divisible. Hence X_t can be decomposed into n i.i.d. parts each having the same distribution with appropriately scaled parameters.

The characteristic function of a Lévy process X is of the form

$$\mathbb{E}[e^{iz\cdot X_t}] \quad = \quad e^{t\psi_X(z)}, \quad z \in \mathbb{R}^d$$

where $\psi_X(\cdot) : \mathbb{R}^d \to \mathbb{R}$ is the corresponding characteristic exponent. Since the log-characteristic function is linear in t and X_t is infinitely divisible, the distribution of X_t is fully determined by the distribution of X_1.

Definition 2.4. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. The jump size at time $t \geq 0$ is defined by*

$$\Delta X_t \quad = \quad X_t - X_{t-}.$$

Considering the family of Borel sets $\mathcal{B}(\mathbb{R}^d)$, the Poisson random measure $\mu : \mathbb{R}^+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is defined for every $U \in \mathcal{B}(\mathbb{R}^d)$ whose closure does not contain 0 by

$$\mu(t, U) \quad = \quad \mu(t, U, \omega) = \sum_{s:0<s\leq t} \chi_U(\Delta X_s),$$

where the indicator or characteristic function with respect to U, χ_U is defined by

$$\chi_U(x) \quad = \quad \begin{cases} 1, & x \in U \\ 0, & x \notin U \end{cases}$$

with $U \subset \mathbb{R}^d$. In other words, $\mu(t, U)$ is the number of jumps of size $\Delta X_s \in U$ which occur before or at time t. Henceforth, the randomness parameter ω will be omitted among functions' parameters as in the previous equation.

Definition 2.5. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$, then the measure $\nu : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\nu(A) \quad = \quad \mathbb{E}\left[\#\{t \in [0,1]\} : \Delta X_t \neq 0, \Delta X_t \in A\right], A \in \mathcal{B}(\mathbb{R}^d)$$

*is called Lévy measure. It might be interpreted as the average number of jumps per unit time of the underlying processes (See [35] Equation (*3.10*)).*

An infinite-activity process is one with an infinite number of jumps in any finite time interval. For an infinite-activity process, $\nu(A)$ remains finite for A such that $0 \notin A$. An infinite-activity process, with measure ν, has a finite number of jumps on $\mathbb{R}^d \setminus \{0\}$, but may have an infinite number of jumps with measure zero. The sum of jumps becomes an infinite series. The integrability conditions imposed in the next theorem, guarantee the convergence of this infinite series.

Theorem 2.6 (The Itô-Lévy decomposition)**.** *If X is a Lévy process on $\mathbb{R}^d$, then for every $t \geq 0$, X_t has the decomposition:*

$$dX_t \quad = \quad \alpha dt + dW_t + \int_{|z|\geq R} z\mu(dt,dz) + \int_{|z|<R} z(\mu - \pi)(dt,dz) \tag{2.1}$$

with

- *the constant $R \in [0,\infty]$. In financial literature, $R = 1$ is commonly used for simplification.*
- *the vector $\alpha \in \mathbb{R}^d$.*
- *the random factor $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion with covariance matrix c.*
- *the Poisson random measure μ on $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$ has compensator $\pi(dt,dz) = \nu(dz)dt$ such as $\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\nu(dz) < \infty$.*

The previous theorem splits a Lévy process into respectively a deterministic component (predictable part), a pure Brownian motion (random part), a Poisson integral (the large jumps) and a compensated Poisson integral (the small jumps). The first two parts constitute a Brownian motion with drift which is the continuous part of the Lévy process. Similarly, the last two parts form the discontinuous part of the process.

Note that the Brownian motion and Poisson measure are independent.

The triplet (c, ν, α) is called the Lévy characteristics triplet (or for short Lévy triplet) of X. By Corollary $II.4.19$ in [70], Lévy process are semimartingales and their characteristics are deterministic.

The previous decomposition of Lévy processes can be extended to the case when its parameters are time and space dependent in the form

$$dX_t = \alpha(t)dt + \beta(t)dW_t + \int_{\mathbb{R}} \gamma(t,z)\bar{\mu}(dt,dz) \tag{2.2}$$

where

$$\bar{\mu}(dt,dz) = \begin{cases} (\mu-\pi)(dt,dz), & |z| < R \\ \mu(dt,dz), & |z| \geq R \end{cases}$$

for some constant $R \in [0,\infty]$. Such processes are referred to as Itô-Lévy processes.

2.2 Itô Formula for Lévy Processes

The dynamics of a sufficiently smooth function of Lévy processes can be deduced from those of the underlying processes through Itô's formula. This section reviews the Itô formula for Itô-Lévy processes and other closely related results that are of great use afterwards.

Theorem 2.7 (The one-dimensional Itô formula [88])**.** *Let $(X_t)_{t\geq 0}$ be a real valued Itô-Lévy process of the form*

$$dX_t = \alpha(t)dt + \beta(t)dW_t + \int_{\mathbb{R}} \gamma(t,z)\bar{\mu}(dt,dz) \tag{2.3}$$

where

$$\bar{\mu}(dt,dz) = \begin{cases} \mu(dt,dz) - \nu(dz)dt, & |z| < R \\ \mu(dt,dz), & |z| \geq R \end{cases}$$

for some constant $R \in [0,\infty]$.

Let $f \in \mathcal{C}^{1,2}(\mathbb{R}^2)$ and $Y = (Y_t)_{t\geq 0}$ such that $Y_t = f(t,X_t)$, then the dynamics of the Itô-Lévy process Y_t are given by

$$\begin{aligned} dY_t = {} & \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)\left[\alpha(t)dt + \beta(t)dW_t\right] + \frac{1}{2}\beta^2(t)\frac{\partial^2 f}{\partial x^2}(t,X_t)dt \\ & + \int_{|z|<R} \left\{ f\left(t,X_{t-}+\gamma(t,z)\right) - f\left(t,X_{t-}\right) - \frac{\partial f}{\partial x}\left(t,X_{t-}\right)\gamma(t,z) \right\} \pi(dt,dz) \\ & + \int_{\mathbb{R}} \left\{ f\left(t,X_{t-}+\gamma(t,z)\right) - f\left(t,X_{t-}\right)\right\} \bar{\mu}(dt,dz). \end{aligned}$$

The following corollaries give some applications of Itô's formula for some simple functions. These functions are largely encountered in financial Mathematics as will be seen later on.

Corollary 2.8. *Let $X = (X_t)_{t\geq 0}$ denote a strictly positive real valued Itô-Lévy process with dynamics given by*

$$dX_t \quad = \quad \alpha(t)dt + \beta(t)dW_t + \int_{\mathbb{R}} \gamma(t,z)\bar{\mu}(dt,dz).$$

The Lévy process $Y = (Y_t)_{t\geq 0}$ defined by $Y_t = \dfrac{1}{X_t}$ for $t \geq 0$ has dynamics

$$\begin{aligned} dY_t \quad = \quad & -Y_t^2\left[\alpha(t) - \beta^2(t)Y_t\right]dt - Y_t^2\beta(t)dW_t + \int_{|z|<R}\left\{\frac{Y_{t-}}{1+\gamma(t,z)Y_{t-}}\right. \\ & \left. -Y_{t-} + Y_{t-}^2\gamma(t,z)\right\}\pi(dt,dz) + \int_{\mathbb{R}}\left\{\frac{Y_{t-}}{1+Y_{t-}\gamma(t,z)} - Y_{t-}\right\}\bar{\mu}(dt,dz). \end{aligned}$$

Proof. Consider $f(t,X_t) = \dfrac{1}{X_t}$, then $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial x} = -\dfrac{1}{X_t^2}$ and $\dfrac{\partial^2 f}{\partial^2 x} = \dfrac{2}{X_t^3}$. By the one-dimensional Itô formula (Theorem 2.7)

$$\begin{aligned} dY_t \quad = \quad & -\frac{1}{X_t^2}\left[\alpha(t)dt + \beta(t)dW_t\right] + \frac{1}{2}\beta^2(t)\frac{2}{X_t^3}dt + \int_{\mathbb{R}}\left\{f\left(t, X_{t-} + \gamma(t,z)\right) - f\left(t, X_{t-}\right)\right\}\bar{\mu}(dt,dz) \\ & + \int_{|z|<R}\left\{f\left(t, X_{t-} + \gamma(t,z)\right) - f\left(t, X_{t-}\right) + \frac{1}{X_{t-}^2}\gamma(t,z)\right\}\pi(dt,dz) \\ = \quad & -\frac{1}{X_t^2}\left[\alpha(t)dt + \beta(t)dW_t\right] + \beta^2(t)\frac{1}{X_t^3}dt + \int_{|z|<R}\left\{\frac{1}{X_{t-}+\gamma(t,z)} - \frac{1}{X_{t-}} + \frac{1}{X_{t-}^2}\gamma(t,z)\right\}\pi(dt,dz) \\ & + \int_{\mathbb{R}}\left\{\frac{1}{X_{t-}+\gamma(t,z)} - \frac{1}{X_{t-}}\right\}\bar{\mu}(dt,dz) \\ = \quad & -\frac{1}{X_t^2}\left[\alpha(t) - \beta^2(t)\frac{1}{X_t}\right]dt - \frac{1}{X_t^2}\beta(t)dW_t + \int_{|z|<R}\left\{\frac{1}{X_{t-}+\gamma(t,z)} - \frac{1}{X_{t-}} + \frac{1}{X_{t-}^2}\gamma(t,z)\right\}\pi(dt,dz) \\ & + \int_{\mathbb{R}}\left\{\frac{1}{X_{t-}+\gamma(t,z)} - \frac{1}{X_{t-}}\right\}\bar{\mu}(dt,dz) \\ = \quad & -Y_t^2\left[\alpha(t) - \beta^2(t)Y_t\right]dt - Y_t^2\beta(t)dW_t + \int_{|z|<R}\left\{\frac{Y_{t-}}{1+\gamma(t,z)Y_{t-}} - Y_{t-} + Y_{t-}^2\gamma(t,z)\right\}\pi(dt,dz) \\ & + \int_{\mathbb{R}}\left\{\frac{Y_{t-}}{1+Y_{t-}\gamma(t,z)} - Y_{t-}\right\}\bar{\mu}(dt,dz). \end{aligned}$$

■

Corollary 2.9. *Let $X = (X_t)_{t\geq 0}$ denote an Itô-Lévy process with dynamics given by*

$$\frac{dX_t}{X_{t-}} \quad = \quad \alpha(t)dt + \beta(t)dW_t + \int_{\mathbb{R}} \gamma(t,z)\bar{\mu}(dt,dz).$$

Now consider the process $Y = (Y_t)_{t\geq 0}$ defined by $Y_t = \dfrac{1}{X_t}$ for $t \geq 0$, then the dynamics of Y_t are

$$\frac{dY_t}{Y_{t-}} \quad = \quad \left[-\alpha(t) + \beta^2(t)\right]dt + \int_{|z|<R}\frac{\gamma^2(t,z)}{1+\gamma(t,z)}\pi(dt,dz) - \beta(t)dW_t - \int_{\mathbb{R}}\frac{\gamma(t,z)}{1+\gamma(t,z)}\bar{\mu}(dt,dz).$$

Proof. Consider $f(t,X_t) = \dfrac{1}{X_t}$, then $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial x} = -\dfrac{1}{X_t^2}$ and $\dfrac{\partial^2 f}{\partial^2 x} = \dfrac{2}{X_t^3}$. By the one-dimensional Itô formula

$$\begin{aligned}
dY_t &= -\frac{X_t}{X_t^2}\left[\alpha(t)dt + \beta(t)dW_t\right] + \frac{1}{2}X_t^2\beta^2(t)\frac{2}{X_t^3}dt \\
&\quad + \int_{|z|<R}\left\{f\left(t,(1+\gamma(t,z))X_{t-}\right) - f\left(t,X_{t-}\right) + \frac{1}{X_{t-}^2}X_t\gamma(t,z)\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}}\left\{f\left(t,(1+\gamma(t,z))X_{t-}\right) - f\left(t,X_{t-}\right)\right\}\bar{\mu}(dt,dz)
\end{aligned}$$

$$\begin{aligned}
dY_t &= -\frac{X_t}{X_t^2}\left[\alpha(t)dt + \beta(t)dW_t\right] + \frac{1}{2}X_t^2\beta^2(t)\frac{2}{X_t^3}dt + \int_{|z|<R}\left\{\frac{1}{X_{t-}}\frac{1}{1+\gamma(t,z)}\right. \\
&\quad \left. -\frac{1}{X_{t-}} + \frac{1}{X_{t-}^2}X_t\gamma(t,z)\right\}\pi(dt,dz) + \int_{\mathbb{R}}\left\{\frac{1}{X_{t-}}\frac{1}{1+\gamma(t,z)} - \frac{1}{X_{t-}}\right\}\bar{\mu}(dt,dz) \\
&= -\frac{1}{X_t}\left[\alpha(t)dt + \beta(t)dW_t\right] + \beta^2(t)\frac{1}{X_t}dt + \int_{|z|<R}\left\{\frac{1}{X_{t-}}\frac{1}{1+\gamma(t,z)}\right. \\
&\quad \left. -\frac{1}{X_{t-}} + \frac{1}{X_t}\gamma(t,z)\right\}\pi(dt,dz) + \int_{\mathbb{R}}\left\{\frac{1}{X_{t-}}\frac{1}{1+\gamma(t,z)} - \frac{1}{X_{t-}}\right\}\bar{\mu}(dt,dz).
\end{aligned}$$

Thus

$$\begin{aligned}
dY_t &= -\frac{1}{X_t}\left[\alpha(t)dt + \beta(t)dW_t\right] + \beta^2(t)\frac{1}{X_t}dt + \frac{1}{X_t}\int_{|z|<R}\left\{\frac{1}{1+\gamma(t,z)} - 1 + \gamma(t,z)\right\}\pi(dt,dz) \\
&\quad + \frac{1}{X_t}\int_{\mathbb{R}}\left\{\frac{1}{1+\gamma(t,z)} - 1\right\}\bar{\mu}(dt,dz).
\end{aligned}$$

From $Y_t = \dfrac{1}{X_t}$,

$$\begin{aligned}
\frac{dY_t}{Y_{t-}} &= -\left[\alpha(t)dt + \beta(t)dW_t\right] + \beta^2(t)dt + \int_{|z|<R}\left\{\frac{1}{1+\gamma(t,z)} - 1 + \gamma(t,z)\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}}\left\{\frac{1}{1+\gamma(t,z)} - 1\right\}\bar{\mu}(dt,dz)
\end{aligned}$$

$$\begin{aligned}
\frac{dY_t}{Y_{t-}} &= -\left[\alpha(t)dt + \beta(t)dW_t\right] + \beta^2(t)dt + \int_{|z|<R}\frac{\gamma^2(t,z)}{1+\gamma(t,z)}\pi(dt,dz) - \int_{\mathbb{R}}\frac{\gamma(t,z)}{1+\gamma(t,z)}\bar{\mu}(dt,dz) \\
&= \left[-\alpha(t) + \beta^2(t)\right]dt - \beta(t)dW_t + \int_{|z|<R}\frac{\gamma^2(t,z)}{1+\gamma(t,z)}\pi(dt,dz) - \int_{\mathbb{R}}\frac{\gamma(t,z)}{1+\gamma(t,z)}\bar{\mu}(dt,dz).
\end{aligned}$$

■

Corollary 2.10. *Let $X = (X_t)_{t\geq 0}$ denote an Itô-Lévy process with dynamics given by*

$$\frac{dX_t}{X_{t-}} = \alpha(t)dt + \beta(t)dW_t + \int_{\mathbb{R}}\gamma(t,z)\bar{\mu}(dt,dz).$$

Now consider the process $Y = (Y_t)_{t\geq 0}$ defined by $Y_t = \ln X_t$ for $t \geq 0$, then the dynamics of Y_t are

$$\begin{aligned} dY_t &= \left[\alpha(t) - \frac{1}{2}\beta^2(t)\right] dt + \int_{|z|<R} \{\ln[1+\gamma(t,z)] - \gamma(t,z)\}\, \pi(dt,dz) \\ &\quad + \beta(t) dW_t + \int_{\mathbb{R}} \ln[1+\gamma(t,z)] \bar{\mu}(dt,dz). \end{aligned}$$

Proof. Consider $f(t, X_t) = \ln X_t$, then $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial x} = \dfrac{1}{X_t}$ and $\dfrac{\partial^2 f}{\partial^2 x} = -\dfrac{1}{X_t^2}$. By the one-dimensional Itô formula

$$\begin{aligned} dY_t &= \frac{X_{t-}}{X_t}\left[\alpha(t)dt + \beta(t)dW_t\right] - \frac{1}{2}\beta^2(t)\frac{X_{t-}^2}{X_t^2}dt \\ &\quad + \int_{|z|<R} \left\{ \ln\left[X_{t-} + X_{t-}\gamma(t,z)\right] - \ln X_{t-} - \frac{1}{X_t} X_{t-}\gamma(t,z) \right\} \pi(dt,dz) \\ &\quad + \int_{\mathbb{R}} \left\{ \ln\left[X_{t-} + X_{t-}\gamma(t,z)\right] - \ln X_{t-} \right\} \bar{\mu}(dt,dz) \\ &= \alpha(t)dt + \beta(t)dW_t - \frac{1}{2}\beta^2(t)dt + \int_{|z|<R} \left\{ \ln\left[1+\gamma(t,z)\right] - \gamma(t,z) \right\} \pi(dt,dz) \\ &\quad + \int_{\mathbb{R}} \ln\left[1+\gamma(t,z)\right] \bar{\mu}(dt,dz) \end{aligned}$$

since X is càdlàg. Thus

$$\begin{aligned} dY_t &= \left[\alpha(t) - \frac{1}{2}\beta^2(t)\right] dt + \int_{|z|<R} \left\{ \ln\left[1+\gamma(t,z)\right] - \gamma(t,z) \right\} \pi(dt,dz) \\ &\quad + \beta(t)dW_t + \int_{\mathbb{R}} \ln\left[1+\gamma(t,z)\right] \bar{\mu}(dt,dz). \end{aligned}$$

∎

Next follows a generalisation of Itô's formula which allows the derivative function to depend on multiple processes.

Theorem 2.11 (The multi-dimensional Itô formula). *Let $X = (X_t)_{t\geq 0} \subset \mathbb{R}^n$ be an n-dimensional Itô-Lévy process of the form:*

$$dX_t = \alpha(t,\omega)dt + \beta(t,\omega)dW_t + \int_{\mathbb{R}^l} \gamma(t,z,\omega)\bar{\mu}(dt,dz),$$

where $\alpha : [0,T] \times \Omega \to \mathbb{R}^n$, $\beta : [0,T] \times \Omega \to \mathbb{R}^{n\times m}$ and $\gamma : [0,T] \times \mathbb{R}^l \times \Omega \to \mathbb{R}^{n\times l}$ are adapted processes such that the integrals exist. Here W_t is an m-dimensional Brownian motion and

$$\begin{aligned} \bar{\mu}(dt,dz)^\tau &= \left[\bar{\mu}_1(dt,dz_1), \cdots, \bar{\mu}_l(dt,dz_l)\right] \\ &= \left[\mu_1(dt,dz_1) - \chi_{|z_1|<R_1}\pi_1(dt,dz_1), \cdots, \mu_l(dt,dz_l) - \chi_{|z_l|<R_l}\pi_l(dt,dz_l)\right] \end{aligned}$$

where $(\mu_j)_{1\leq j\leq l}$ are independent Poisson random measures with respective compensator $(\pi_j)_{1\leq j\leq l}$ coming from l independent (1 dimensional) Lévy processes and $(\cdot)^\tau$ denotes the transpose.

Note that each column $\gamma^{(k)}$ of the $n \times l$ matrix $\gamma = [\gamma_{ij}]$ depends on z only through the k^{th} coordinate z_k, i.e

$$\gamma^{(k)}(t,z) \quad = \quad \gamma^{(k)}(t,z_k), \quad \textit{with } z = (z_1, \cdots, z_l) \in \mathbb{R}^l.$$

In particular, for $i = 1, ..., n$,

$$dX_i(t) \quad = \quad \alpha_i(t)dt + \sum_{j=1}^{m} \beta_{ij}(t)dW_t^j + \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{ij}(t,z_j)\bar{\mu}_j(dt,dz_j).$$

Let $f \in \mathcal{C}^{1,2}([0,T], \mathbb{R}^n)$ and $Y = (Y_t)_{t\geq 0}$ such that $Y_t = f(t, X_t)$, then

$$\begin{aligned} dY_t \quad = \quad & \frac{\partial f}{\partial t}(t,X_t)\,dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t,X_t)\left[\alpha_i(t)dt + \beta_i(t)dW_t\right] + \frac{1}{2}\sum_{i,j=1}^{n} (\beta\beta^{\tau})_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(t,X_t)\,dt \\ & + \sum_{k=1}^{l} \int_{|z_k|<R_k} \left\{ f\left(t, X_{t-} + \gamma^{(k)}(t,z)\right) - f(t,X_{t-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t,X_{t-})\,\gamma_i^{(k)}(t,z) \right\} \pi_k(dt,dz_k) \\ & + \sum_{k=1}^{l} \int_{\mathbb{R}} \left\{ f\left(t, X_{t-} + \gamma^{(k)}(t,z)\right) - f(t,X_{t-}) \right\} \bar{\mu}(dt,dz) \end{aligned}$$

where $\gamma_i^{(k)} = \gamma_{ik}$ is the i^{th} component of $\gamma^{(k)} \in \mathbb{R}^n$ which is the k^{th} column of the $n \times l$ matrix $\gamma = [\gamma_{ij}]$.

The next corollaries apply the multidimensional Itô formula to some particularly useful functions.

Corollary 2.12. *Let*

$$dX_i(t) = \alpha_i(t)dt + \beta_i(t)dW_t + \int_{\mathbb{R}} \gamma_i(t,z)\bar{\mu}(dt,dz); \quad i = 1, 2$$

be two Itô-Lévy processes. Then the process $(Z_t)_{t\geq 0}$ defined by $Z_t = X_1(t)X_2(t)$ has $\mathbb{P}$-dynamics:

$$\begin{aligned} dZ_t \quad = \quad & \left[\alpha_1(t)X_2(t) + \alpha_2(t)X_1(t) + \beta_1(t)\beta_2(t) + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\nu(dz)\right] dt \\ & + \left[\beta_1(t)X_2(t) + \beta_2(t)X_1(t)\right] dW_t \\ & + \int_{\mathbb{R}} \left[\gamma_2(t,z)X_1(t-) + \gamma_1(t,z)X_2(t-) + \gamma_1(t,z)\gamma_2(t,z)\right] \bar{\mu}(dt,dz). \end{aligned}$$

Proof. Applying Theorem 2.11 with $f(t, X_1, X_2) = X_1(t)X_2(t)$

$$\begin{aligned}
dZ_t &= X_2(t)\left[\alpha_1(t)dt + \beta_1(t)dW_t\right] + X_1(t)\left[\alpha_2(t)dt + \beta_2(t)dW_t\right] + \frac{1}{2}[\beta_1(t)\beta_2(t) + \beta_2(t)\beta_1(t)]dt \\
&+ \int_{\mathbb{R}} \{[X_1(t-) + \gamma_1(t,z)]\,[X_2(t-) + \gamma_2(t,z)] - X_1(t-)X_2(t-) \\
&-X_2(t-)\gamma_1(t,z) - X_1(t-)\gamma_2(t,z)\}\,\pi(dt,dz) \\
&+ \int_{\mathbb{R}} \{[X_1(t-) + \gamma_1(t,z)]\,[X_2(t-) + \gamma_2(t,z)] - X_1(t-)X_2(t-)\}\,\bar{\mu}(dt,dz) \\
&= \left[\alpha_1(t)X_2(t) + \alpha_2(t)X_1(t) + \beta_1(t)\beta_2(t) + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\nu(dz)\right] dt \\
&+ [\beta_1(t)X_2(t) + \beta_2(t)X_1(t)]\,dW_t \\
&+ \int_{\mathbb{R}} [\gamma_2(t,z)X_1(t-) + \gamma_1(t,z)X_2(t-) + \gamma_1(t,z)\gamma_2(t,z)]\,\bar{\mu}(dt,dz).
\end{aligned}$$

■

Corollary 2.13. *Let*

$$\frac{dX_i(t)}{X_i(t-)} = \alpha_i(t)dt + \beta_i(t)dW_t + \int_{\mathbb{R}} \gamma_i(t,z)\bar{\mu}(dt,dz); \quad i = 1, 2$$

be two Itô-Lévy processes under the statistical probability measure $\mathbb{P}$*. Now consider the process* $Z = (Z_t)_{t\geq 0}$ *defined by* $Z_t = X_1(t)X_2(t)$*, then its* $\mathbb{P}$*-dynamics are:*

$$\begin{aligned}
\frac{dZ_t}{Z_{t-}} &= [\alpha_1(t) + \alpha_2(t) + \beta_1(t)\beta_2(t)]\,dt + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\pi(dt,dz) + [\beta_1(t) + \beta_2(t)]\,dW_t \\
&+ \int_{\mathbb{R}} [\gamma_1(t,z) + \gamma_2(t,z) + \gamma_1(t,z)\gamma_2(t,z)]\,(\mu - \pi)(dt,dz).
\end{aligned}$$

Proof. Applying Theorem 2.11 with $f(t, X_1, X_2) = X_1(t)X_2(t)$

$$\begin{aligned}
dZ_t &= X_2(t)\left[X_1(t)\alpha_1(t)dt + X_1(t)\beta_1(t)dW_t\right] + X_1(t)\left[X_2(t)\alpha_2(t)dt + X_2(t)\beta_2(t)dW_t\right] \\
&+\frac{1}{2}[\beta_1(t)X_1(t)\beta_2(t)X_2(t) + \beta_2(t)X_2(t)\beta_1(t)X_1(t)]dt \\
&+ \int_{\mathbb{R}} \{[X_1(t-) + X_1(t)\gamma_1(t,z)]\,[X_2(t-) + X_2(t)\gamma_2(t,z)] - X_1(t-)X_2(t-) \\
&-X_2(t-)X_1(t)\gamma_1(t,z) - X_1(t-)X_2(t)\gamma_2(t,z)\}\,\pi(dt,dz) \\
&+ \int_{\mathbb{R}} \{[X_1(t-) + X_1(t)\gamma_1(t,z)]\,[X_2(t-) + X_2(t)\gamma_2(t,z)] - X_1(t-)X_2(t-)\}\,\bar{\mu}(dt,dz).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{dZ_t}{Z_{t-}} &= [\alpha_1(t) + \alpha_2(t) + \beta_1(t)\beta_2(t)]\,dt + [\beta_1(t) + \beta_2(t)]\,dW_t + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\pi(dt,dz) \\
&+ \int_{\mathbb{R}} [\gamma_1(t,z) + \gamma_2(t,z) + \gamma_1(t,z)\gamma_2(t,z)]\,\bar{\mu}(dt,dz).
\end{aligned}$$

■

Corollary 2.14. *Let*

$$dX_i(t) = \alpha_i(t)dt + \beta_i(t)dW_t + \int_{\mathbb{R}} \gamma_i(t,z)\bar{\mu}(dt,dz); \quad i = 1,2$$

define two Itô-Lévy processes under the probability measure $\mathbb{P}$ *with* X_2 *strictly positive. Then the process* $(Z_t)_{t\geq 0}$ *defined by* $Z_t = \dfrac{X_1(t)}{X_2(t)}$ *has* $\mathbb{P}$*-dynamics:*

$$\begin{aligned} dZ_t &= \left[\frac{\alpha_1(t)}{X_2(t)} + \alpha_Y(t)X_1(t) - \frac{\beta_1(t)\beta_2(t)}{X_2^2(t)} + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz)\right] dt \\ &+ \left[\frac{\beta_1(t)}{X_2(t)} - \beta_2(t)\frac{X_1(t)}{X_2^2(t)}\right] dW_t + \int_{\mathbb{R}} \left[\gamma_Y(t,z)X_1(t-) + \frac{\gamma_1(t,z)}{X_2(t-)} + \gamma_1(t,z)\gamma_Y(t,z)\right] \bar{\mu}(dt,dz), \end{aligned}$$

where

$$\begin{aligned} \alpha_Y(t) &= -\frac{1}{X_2^2(t)}\left[\alpha_2(t) - \beta_2^2(t)\frac{1}{X_2(t)}\right] \\ &+ \int_{|z|<R} \left[\frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)} + \frac{1}{X_2^2(t-)}\gamma_2(t,z)\right] \nu(dz); \\ \gamma_Y(t,z) &= \frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)}. \end{aligned}$$

Proof. By Corollary 2.8, considering $Y_t = \dfrac{1}{X_2(t)}$

$$\begin{aligned} dY_t &= \left\{-\frac{1}{X_2^2(t)}\left[\alpha_2(t) - \beta_2^2(t)\frac{1}{X_2(t)}\right]\right. \\ &+ \left.\int_{|z|<R} \left[\frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)} + \frac{1}{X_2^2(t-)}\gamma_2(t,z)\right] \nu(dz)\right\} dt \\ &- \frac{1}{X_2^2(t)}\beta_2(t)dW_t + \int_{\mathbb{R}} \left[\frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)}\right] \bar{\mu}(dt,dz). \end{aligned}$$

Applying Corollary 2.12 with $Z_t = X_1(t)Y_t$

$$\begin{aligned} dZ_t &= \left[\frac{\alpha_1(t)}{X_2(t)} + \alpha_Y(t)X_1(t) - \frac{\beta_1(t)\beta_2(t)}{X_2^2(t)} + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz)\right] dt \\ &+ \left[\frac{\beta_1(t)}{X_2(t)} - \beta_2(t)\frac{X_1(t)}{X_2^2(t)}\right] dW_t \\ &+ \int_{\mathbb{R}} \left[\gamma_Y(t,z)X_1(t-) + \frac{\gamma_1(t,z)}{X_2(t-)} + \gamma_1(t,z)\gamma_Y(t,z)\right] \bar{\mu}(dt,dz), \end{aligned}$$

where

$$\begin{aligned} \alpha_Y(t) &= -\frac{1}{X_2^2(t)}\left[\alpha_2(t) - \beta_2^2(t)\frac{1}{X_2(t)}\right] \\ &+ \int_{|z|<R} \left[\frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)} + \frac{1}{X_2^2(t-)}\gamma_2(t,z)\right] \nu(dz); \\ \gamma_Y(t,z) &= \frac{1}{X_2(t-)+\gamma_2(t,z)} - \frac{1}{X_2(t-)}. \end{aligned}$$

■

Corollary 2.15. *Let*

$$\frac{dX_i(t)}{X_i(t-)} = \alpha_i(t)dt + \beta_i(t)dW_t + \int_{\mathbb{R}} \gamma_i(t,z)\bar{\mu}(dt,dz); \quad i = 1,2$$

be two Itô-Lévy processes. Then the process $(Z_t)_{t\geq 0}$ defined by $Z_t = \dfrac{X_1(t)}{X_2(t)}$ has $\mathbb{P}$-dynamics:

$$\begin{aligned}
\frac{dZ_t}{Z_{t-}} &= \left[\alpha_1(t) - \alpha_2(t) + \beta_2^2(t) + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) - \beta_1(t)\beta_2(t)\right] dt \\
&+ \int_{\mathbb{R}} \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) + [\beta_1(t) - \beta_2(t)]\, dW_t \\
&+ \int_{\mathbb{R}} \left[\gamma_1(t,z) + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\right]\bar{\mu}(dt,dz).
\end{aligned}$$

Proof. By Corollary 2.9, considering $Y_t = \dfrac{1}{X_2(t)}$

$$\begin{aligned}
\frac{dY_t}{Y_t} &= \left[-\alpha_2(t) + \beta_2^2(t)\right] dt + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) - \beta_2(t)dW_t - \int_{\mathbb{R}} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\bar{\mu}(dt,dz) \\
&= \left[-\alpha_2(t) + \beta_2^2(t) + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz)\right] dt - \beta_2(t)dW_t - \int_{\mathbb{R}} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\bar{\mu}(dt,dz).
\end{aligned}$$

Applying Corollary 2.13 with $Z(t) = X_1(t)Y(t)$

$$\begin{aligned}
\frac{dZ_t}{Z_{t-}} &= \left[\alpha_1(t) - \alpha_2(t) + \beta_2^2(t) + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) - \beta_1(t)\beta_2(t)\right] dt \\
&+ \int_{\mathbb{R}} \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) + [\beta_1(t) - \beta_2(t)]\, dW_t \\
&+ \int_{\mathbb{R}} \left[\gamma_1(t,z) + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\right]\bar{\mu}(dt,dz).
\end{aligned}$$

■

Derivative securities are priced in the risk neutral probability measure, however the market prices are valued in the statistical probability measure. The following Girsanov Theorem is used to move from one measure to another.

Theorem 2.16 (Girsanov change of measure)**.** *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let X_1 and X_2 be two real-valued adapted processes under the probability measure $\mathbb{P}$, with dynamics*

$$\frac{dX_i(t)}{X_i(t-)} = \beta_i(t)dW_t + \int_{\mathbb{R}} \gamma_i(t,z)\bar{\mu}(dt,dz), \text{ for } i = 1,2;$$

where W_t is a $\mathbb{P}$-Brownian motion, $\pi(dt,dz)$ is the compensator of the random measure of jumps μ and the standard integrability conditions are verified. If $\gamma_i(t,z)$ is deterministic for $i = 1,2$ and

$\gamma_2(t,z) > -1$, then the process $Y = \dfrac{X_1}{X_2}$ is a local martingale and has dynamics

$$\frac{dY_t}{Y_{t-}} = -[\beta_2(t) - \beta_1(t)]d\widetilde{W}_t - \int_{\mathbb{R}} \frac{\gamma_2(t,z) - \gamma_1(t,z)}{1+\gamma_2(t,z)}(\mu - \widetilde{\pi})(dt,dz)$$

under an equivalent measure $\widetilde{\mathbb{P}} \simeq \mathbb{P}$, where $\widetilde{W}_t$ is a $\widetilde{\mathbb{P}}$-Brownian motion given by

$$d\widetilde{W}_t = dW_t - \beta_2(t)dt,$$

and $\widetilde{\pi}(dt,dz)$ is the $\widetilde{\mathbb{P}}$-compensator of μ given by

$$\widetilde{\pi}(dt,dz) = \theta(t,z)\pi(dt,dz)$$

with $\theta(t,z) = 1 + \gamma_2(t,z)$.

The function $\theta(t,z)$ is a non-negative solution to the equation

$$\int_{\mathbb{R}} \frac{\gamma_2^2(t,z) - \gamma_1(t,z)\gamma_2(t,z) - \gamma_1(t,z) + \gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) + \int_{\mathbb{R}} \frac{\gamma_1(t,z) - \gamma_2(t,z)}{1+\gamma_2(t,z)} f(t,z)\pi(dt,dz) = 0.$$

Proof. By applying Corollaries 2.9 and 2.13

$$\begin{aligned}\frac{dY_t}{Y_{t-}} &= [\beta_2^2(t) - \beta_1(t)\beta_2(t)]dt + \int_{\mathbb{R}} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) - \int_{\mathbb{R}} \frac{\gamma_1(t,z)\gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) \\ &\quad -[\beta_2(t) - \beta_1(t)]dW_t - \int_{\mathbb{R}} \frac{\gamma_2(t,z) - \gamma_1(t,z)}{1+\gamma_2(t,z)}(\mu - \pi)(dt,dz). \end{aligned} \tag{2.4}$$

Since the continuous and discontinuous (jump) parts of a semimartingale do not interact [70], they can be studied independently. The continuous part of the process Y is

$$[\beta_2^2(t) - \beta_1(t)\beta_2(t)]dt - [\beta_2(t) - \beta_1(t)]dW_t = -[\beta_2(t) - \beta_1(t)][dW_t - \beta_2(t)dt].$$

This implies that $d\widetilde{W}_t = dW_t - \beta_2(t)dt$ [87, 15].

The discontinuous part of Y is

$$\int_{\mathbb{R}} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) - \int_{\mathbb{R}} \frac{\gamma_1(t,z)\gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) - \int_{\mathbb{R}} \frac{\gamma_2(t,z) - \gamma_1(t,z)}{1+\gamma_2(t,z)}(\mu - \pi)(dt,dz).$$

Following Øksendal and Sulem [88], the last expression can be written as:

$$\begin{aligned}&\int_{\mathbb{R}} \frac{\gamma_1(t,z) - \gamma_2(t,z)}{1+\gamma_2(t,z)}(\mu - \tilde{\pi})(dt,dz) + \int_{\mathbb{R}} \frac{\gamma_2^2(t,z) - \gamma_1(t,z)\gamma_2(t,z) - \gamma_1(t,z) + \gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) \\ &+ \int_{\mathbb{R}} \frac{\gamma_1(t,z) - \gamma_2(t,z)}{1+\gamma_2(t,z)} f(t,z)\pi(dt,dz)\end{aligned}$$

where $\widetilde{\pi}(dt,dz) = \theta(t,z)\pi(t,z)$ for some $\theta \geq 0$ is the compensator of μ under some new probability measure $\widetilde{\mathbb{P}} \simeq \mathbb{P}$. It is obvious from the last equation that for the process Y to be a martingale under the new measure, it is required that

$$\int_{\mathbb{R}} \frac{\gamma_2^2(t,z) - \gamma_1(t,z)\gamma_2(t,z) - \gamma_1(t,z) + \gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) + \int_{\mathbb{R}} \frac{\gamma_1(t,z) - \gamma_2(t,z)}{1+\gamma_2(t,z)} f(t,z)\pi(dt,dz) = 0.$$

The solution to this equation is $\theta(t,z) = 1 + \gamma_2(t,z)$ [87]. ■

2.3 Some Useful Results

In the 1930s, de Finetti and Kolmogorov obtained an explicit and simple expression which describes an infinitely divisible distribution in terms of its characteristic function. This formula is known as the Lévy-Khinchin formula.

Theorem 2.17 (Lévy-Khinchin Formula). *(i) Let $(X_t)_{t>0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (c, ν, α). Then $\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\nu(dz) < \infty$ and*

$$\mathbb{E}[e^{iz \cdot X_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d$$

where

$$\psi(z) = i\alpha \cdot z - \frac{1}{2} z \cdot cz + \int_{\mathbb{R}^d} \left(e^{iz \cdot x} - 1 - iz \cdot x\chi_{|x|\le 1}\right) \nu(dx). \tag{2.5}$$

(ii) Conversely, if c is a symmetric positive matrix, $\alpha \in \mathbb{R}^d$ and ν is a positive measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies $\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\nu(dz) < \infty$, then there exists a Lévy process on $\mathbb{R}^d$ (unique in law) whose characteristic function is given by Equation 2.5.

Note that the characteristic function is the Fourier transform of the probability density function $f(x)$.

Definition 2.18. *The Fourier transform of a function $f \in L^1(\mathbb{R})$ is the function $\hat{f} = \mathcal{F}(f)$ defined by*

$$\hat{f}(z) = \int_{\mathbb{R}} e^{ixz} f(x) dx.$$

Example 3. *Let us consider two simple cases:*

1. *X follows a Brownian motion; its dynamics are given by $dX_t = \alpha dt + \sigma dW_t$ and its Lévy characteristics are $(\sigma^2, 0, \alpha)$. Therefore*

$$\begin{aligned} \psi_X(z) &= i\alpha z - \frac{1}{2}\sigma^2 z^2 \\ \mathbb{E}[e^{izX_t}] &= \exp\left[t\left(i\alpha z - \frac{1}{2}\sigma^2 z^2\right)\right]. \end{aligned}$$

2. *X is a Poisson process of intensity λ; its Lévy characteristics are $(0, \lambda, 0)$. Therefore*

$$\begin{aligned} \psi_X(z) &= \int_{\mathbb{R}} \left(e^{iz \cdot x} - 1\right) \nu(dx) \\ &= \lambda\left(e^{iz \cdot x} - 1\right). \end{aligned}$$

Since ν counts the average number of jumps which is λ. Hence,

$$\mathbb{E}[e^{izX_t}] \quad = \quad \exp\left[\lambda\left(e^{iz\cdot x}-1\right)\right].$$

Theorem 2.19 (Exponential moments [35])**.** *Let $(X_t)_{t>0}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet (c,ν,α) and let $u\in\mathbb{R}$. The exponential moment $\mathbb{E}[e^{uX_t}]$ is finite for some t or, equivalently, for all $t>0$ if and only if $\int_{|x|>1} e^{ux}\nu(dx)<\infty$. In this case*

$$\mathbb{E}[e^{uX_t}] \quad = \quad e^{t\psi_X(-iu)}.$$

where ψ_X is the characteristic exponent of X given by Equation 2.5.

For a purely jump process, its exponential moment can be computed analytically. The next theorems give these moments for a Poisson integral in the cases when it is compensated or not.

Theorem 2.20. *Let A be bounded from below. Then,*

(i) for each $t\geq 0$, $\int_A f(x)\mu(t,dx)$ has compound Poisson distribution such that, for each $u\in\mathbb{R}^d$,

$$\mathbb{E}\left(\exp\left[i\left\langle u,\int_A f(x)\mu(t,dx)\right\rangle\right]\right) \quad = \quad \exp\left[t\int_A (e^{i(u,x)}-1)\pi_f(dx)\right]$$

(ii)

$$\mathbb{E}\left(\exp\left[i\left\langle u,\int_A f(x)(\mu-\pi)(t,dx)\right\rangle\right]\right) \quad = \quad \exp\left\{t\int_A\left[e^{i(u,x)}-1-i(u,x)\right]\pi_f(dx)\right\}$$

where $\pi_f=\pi\circ f^{-1}$ and for $x,y\in\mathbb{R}^d$ such as $x=(x_1,x_2,\cdots,x_d)$ and $y=(y_1,y_2,\cdots,y_d)$, $\langle x,y\rangle=\sum_{i=1}^{d}x_iy_i$.

Proof. See Theorem 2.3.8(1) and Equation (2.9) in Applebaum's book [7]. ∎

Corollary 2.21. *Working in $\mathbb{R}$ and taking $u=-i$, we have*

$$\begin{aligned}\mathbb{E}\left(\exp\left[\int_A f(x)\mu(t,dx)\right]\right) &= \exp\left[t\int_A (e^x-1)\pi_f(dx)\right]\\ \mathbb{E}\left(\exp\left[\int_A f(x)(\mu-\pi)(t,dx)\right]\right) &= \exp\left\{t\int_A\left[e^x-1-x\right]\pi_f(dx)\right\}\end{aligned}$$

where $\pi_f=\pi\circ f^{-1}$.

The previous expectations can be simplified further when $\gamma(t,z)$ is deterministic as shown in the next theorem.

Theorem 2.22. *If $\gamma(t,z)$ is deterministic, the exponential moment of a Poisson integral and a compensated Poisson integral are respectively*

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}}\gamma(t,z)(\mu-\pi)(dt,dz)\right)\right] &= \exp\left\{\int_0^t\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1-\gamma(t,z)\right]\nu(dz)dt\right\};\\ \mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}}\gamma(t,z)\mu(dt,dz)\right)\right] &= \exp\left\{\int_0^t\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1\right]\nu(dz)dt\right\}.\end{aligned}$$

Proof. This proof is based on Exercice 1.6 in Øksendal and Sulem's book [88]. Using Corollary 2.10, the equation

$$\frac{dX_t}{X_{t-}} = \int_{\mathbb{R}}(e^{\gamma(t,z)}-1)(\mu-\pi)(dt,dz); \quad X_0 = 1 \tag{2.6}$$

has solution

$$\begin{aligned}X_T &= \exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)\mu(dt,dz) - \int_0^T\int_{\mathbb{R}}(e^{\gamma(t,z)}-1)\nu(dz)dt\right\}\\ &= \exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)(\mu-\pi)(dt,dz) - \int_0^T\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1-\gamma(t,z)\right]\nu(dz)dt\right\}.\end{aligned}$$

Assuming that

$$\int_0^T\int_{\mathbb{R}}(e^{\gamma(t,z)}-1)^2\nu(dz)dt < \infty,$$

from Equation (2.6), X is a martingale, i.e. $\mathbb{E}[X_t]=1$, hence

$$\mathbb{E}\left[\exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)(\mu-\pi)(dt,dz)\right\}\right] = \exp\left\{\int_0^T\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1-\gamma(t,z)\right]\nu(dz)dt\right\}.$$

Note that

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)(\mu-\pi)(dt,dz)\right\}\right] &= \mathbb{E}\left[\exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)\mu(dt,dz)\right\}\right]\\ &\quad \mathbb{E}\left[\exp\left\{-\int_0^T\int_{\mathbb{R}}\gamma(t,z)\nu(dz)dt\right\}\right]\\ &= \mathbb{E}\left[\exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)\mu(dt,dz)\right\}\right]\\ &\quad \exp\left\{-\int_0^T\int_{\mathbb{R}}\gamma(t,z)\nu(dz)dt\right\},\end{aligned}$$

thus

$$\mathbb{E}\left[\exp\left\{\int_0^T\int_{\mathbb{R}}\gamma(t,z)\mu(dt,dz)\right\}\right] = \exp\left\{\int_0^T\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1\right]\nu(dz)dt\right\}$$

■

The next two results will be used when pricing IL securities using the martingale method in Chapter 3. The convexity correction allows to compute the expected value of the product of two martingales.

Theorem 2.23. *(Convexity correction) Let X_1 and X_2 be two $\mathbb{P}$-martingales with dynamics given by:*

$$\frac{dX_i(t)}{X_i(t-)} = \beta_i(t)dW(t) + \int_{\mathbb{R}} \gamma_i(t,z)(\mu - \pi)(dt, dz), \qquad \text{for } i = 1, 2,$$

where $\beta_i(t)$ and $\gamma_i(t)$ are deterministic and $\int_0^T \int_{\mathbb{R}} \gamma_i^2(t,z)\pi(dt,dz) < \infty$ for $i = 1, 2$. Then for all $t \in [0, T]$

$$\mathbb{E}_t^{\mathbb{P}}[X_1(T)X_2(T)] = X_1(t)X_2(t) \exp\left[\int_t^T \beta_1(u)\beta_2^\tau(u)du + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\pi(dt,dz)\right] \quad (2.7)$$

where $(\cdot)^\tau$ denotes the transpose.

Remark The exponential factor in Equation (2.7) is sometimes referred to the as convexity correction term.

Proof. This proof is from [63] where it was proved in the case of finite jumps. The proof when considering infinite jumps is similar.

Considering $Z_t = X_1(t)X_2(t)$, by Corollary 2.13

$$\begin{aligned}\frac{dZ_t}{Z_{t-}} &= \beta_1(t)\beta_2(t)dt + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_2(t,z)\pi(dt,dz) + [\beta_1(t) + \beta_2(t)]\, dW_t \\ &+ \int_{\mathbb{R}} [\gamma_1(t,z) + \gamma_2(t,z) + \gamma_1(t,z)\gamma_2(t,z)]\, (\mu - \pi)(dt,dz).\end{aligned}$$

Since dW and $(\mu - \pi)$ are $\mathbb{P}$-martingales, for every $t \leq T$,

$$\mathbb{E}_t^{\mathbb{P}}[Z_T] = Z_t + \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T Z_u A_u du\right]$$

where

$$A_u = \alpha_1(u) + \alpha_2(u) + \beta_1(u)\beta_2(u) + \int_{\mathbb{R}} \gamma_1(u,z)\gamma_2(u,z)\nu(dz)$$

is an integrable function of u.

Since A is a non stochastic process which is integrable, the expectation can be moved within the integral sign. Assuming that $C_s = \mathbb{E}_t^{\mathbb{P}}[Z_s]$ with $s \geq t$ yields

$$C_s = Z_t + \int_t^s A_u C_u du.$$

Taking the derivative of this equation gives the the following ordinary differential equation

$$\begin{cases} \dot{C}_s = C_s A_s \\ C_t = Y_t \end{cases}$$

The solution to this differential equation is given by

$$C_T = Y_t \exp\left[\int_t^T A_u du\right].$$

■

The Bayes theorem relates the conditional and marginal probabilities of two random events.

Theorem 2.24 (Bayes theorem). *Let $X = (X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$ and with Radon-Nikodym derivative*

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} \quad on \quad \mathcal{F}.$$

Let $\mathcal{G}$ be a σ-algebra with $\mathcal{G} \subseteq \mathcal{F}$, then

$$\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}] = \frac{\mathbb{E}^{\mathbb{P}}[\lambda \cdot X \mid \mathcal{G}]}{\mathbb{E}^{\mathbb{P}}[\lambda \mid \mathcal{G}]}, \quad \mathbb{Q} - a.s.$$

2.4 Examples of Lévy Processes

Infinitely divisible distributions have been used for modelling financial data as early as 1980 [95] to incorporate skewness and excess kurtosis. Examples of such distributions are the Variance Gamma (VG), the Normal Inverse Gausssian (NIG), the Generalized Hyperbolic (GH) model, the GH skew Student's t distribution and the Hyperbolic model. The VG distribution was introduced by Madan and Seneta [77, 78] to model stock returns in the late 1980s. In 1995, Eberlein and Keller [41] used the Hyperbolic distribution and Barndorff-Nielsen [11] proposed the NIG Lévy process. All the previous models were brought together as special cases of the GH model, which was developed by Eberlein and co-workers in a series of papers [47, 42, 91]. This section presents these selected Lévy processes and other useful results. A more detailed coverage of Lévy processes can be found in [91, 95].

Note that throughout this section, the GH model refers to the univariable Generalized Hyperbolic distribution.

2.4.1 The Generalized Hyperbolic Distribution

The name "Generalized Hyperbolic" is due to the fact that the GH distribution log-density is hyperbolic while the Gaussian distribution log-density is a parabola. In 1977, Barndorff-Nielsen [10] introduced the Generalized Hyperbolic distribution to model grain size distributions of wind blown sand. He was looking for a satisfactory explanation to some empirical laws in geology; later on, this distribution was used in financial Mathematics [41, 91].

Definition 2.25 (Generalized Hyperbolic distribution). *A univariate GH distribution is defined by the following Lebesgue density*

$$gh(x;\lambda,\alpha,\beta,\delta,\mu) = a(\lambda,\alpha,\beta,\delta)[\delta^2+(x-\mu)^2]^{\frac{1}{2}(\lambda-\frac{1}{2})}K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)} \tag{2.8}$$

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})} \tag{2.9}$$

where $x\in\mathbb{R}$ and K_λ is a modified Bessel function of the third kind (See Figure 2.1)

$$K_\lambda(z) = \frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left[-\frac{z}{2}\left(y+\frac{1}{y}\right)\right]dy \text{ for } z>0. \tag{2.10}$$

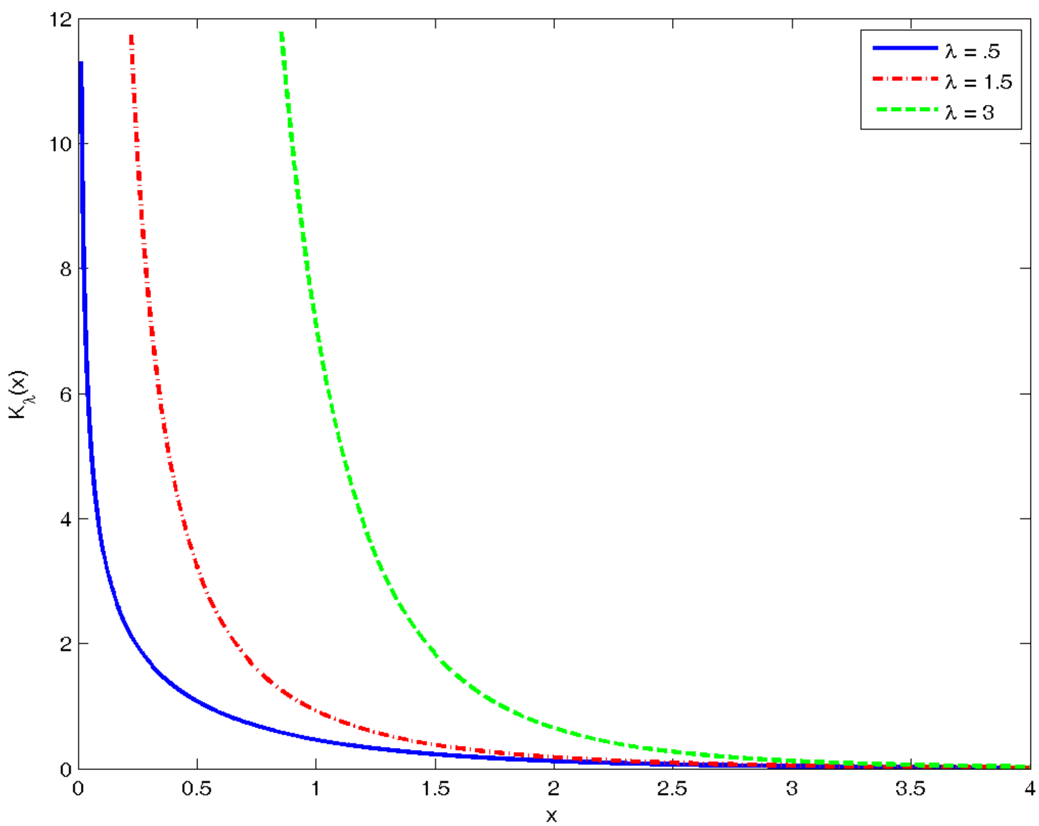


Figure 2.1 Modified Bessel function of the third kind.

The domain of variation of the parameters is $\mu \in \mathbb{R}$ *and*

$$\delta \geq 0, |\beta| < \alpha \quad if \quad \lambda > 0;$$
$$\delta > 0, |\beta| < \alpha \quad if \quad \lambda = 0;$$
$$\delta > 0, |\beta| \leq \alpha \quad if \quad \lambda < 0.$$

The parameters μ, δ, β and α affect respectively the location, the scale, the skewness and the kurtosis.

Proposition 2.26 (Mean and Variance)**.** *The mean and variance of a generalized hyperbolic distributed random variate* X *are given by [91]*

$$\begin{aligned}
\mathbb{E}[X] &= \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}; \\
Var[X] &= \delta^2\left\{\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{\beta^2}{\alpha^2-\beta^2}\left[\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left(\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}\right)^2\right]\right\},
\end{aligned}$$

where $\zeta = \delta\sqrt{\alpha^2-\beta^2}$.

Proposition 2.27. *The characteristic function of the generalized hyperbolic distribution is given by*

$$\varphi_{GH}(u) = e^{i\mu u}\left[\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right]^{\frac{\lambda}{2}}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$$

The GH can also be seen as a normal variance-mean mixture in the form

$$gh(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^\infty \mathcal{N}(x;\mu+\beta w, w)\cdot gig(w;\lambda,\delta^2,\alpha^2-\beta^2)dw$$

where $\mathcal{N}(\cdot)$ is the normal density function and $gig(\cdot)$ the density function of a generalized inverse Gaussian(GIG).

Definition 2.28 (Generalized Inverse Gaussian distribution)**.** *A univariate GIG distribution is defined by the following Lebesgue density*

$$gig(x;\lambda,\chi,\psi) = \frac{\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\psi\chi}\right)}x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\frac{\chi}{x}+\psi x\right)\right], \text{ for } x > 0,$$

where K_λ *is a modified Bessel function of the third kind and* $\lambda \in \mathbb{R}$ *and* $\chi, \psi \in \mathbb{R}_+$.

Remark The normal distribution is obtained from the GH distribution by considering the following limit case: $\delta \to \infty$ and $\dfrac{\delta}{\alpha} \to \sigma^2$.

Although the GH distribution is highly flexible, it is seldom used in practical applications. This might be due to the fact that even for very large sample, it is hard to determine which subclass is the most appropriate [91]. Instead, specific subclasses have been applied in various situations for parameter estimation. The following subsections review some of these subclasses.

2.4.2 The Hyperbolic Distribution

A univariate hyperbolic (HYP) distribution is obtained from a GH distribution for $\lambda = 1$.

Definition 2.29 (Hyperbolic distribution)**.** *A univariate HYP distribution is defined by the following Lebesgue density*

$$hyp(x;\alpha,\beta,\delta,\mu) \quad = \quad \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha K_1(\delta\sqrt{\alpha^2-\beta^2})}\exp\left[-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right],$$

where $x,\mu \in \mathbb{R}, 0 \leq \delta$ *and* $|\beta| < \alpha$.

The mean and variance of an HYP distribution can easily be computed from that of the GH distribution.

2.4.3 The Normal Inverse Gaussian Distribution

The name "Normal Inverse Gaussian" (NIG) stems from the fact that the NIG distribution can be represented as a mixture of a Generalized Inverse Gaussian with a Normal distribution. A univariate NIG distribution is obtained from a GH distribution for $\lambda = -\dfrac{1}{2}$.

Definition 2.30 (Normal Inverse Gaussian distribution)**.** *A univariate NIG distribution is defined by the following Lebesgue density*

$$nig(x;\alpha,\beta,\delta,\mu) \quad = \quad \frac{\alpha\delta}{\pi}\exp\left[\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right]\frac{K_1[\alpha\sqrt{\delta^2+(x-\mu)^2}]}{\sqrt{\delta^2+(x-\mu)^2}},$$

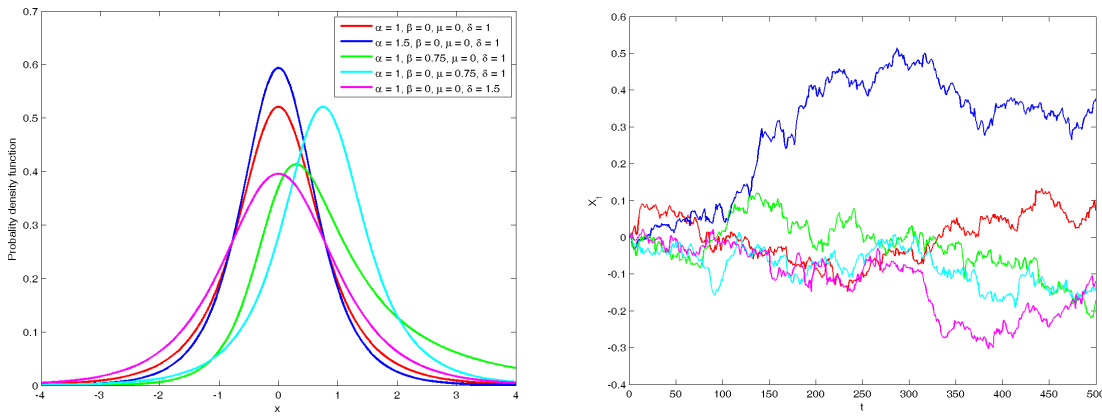
where $x,\mu \in \mathbb{R}, 0 \leq \delta$ *and* $0 \leq |\beta| \leq \alpha$.

The NIG distribution is a lot easier to handle than the HYP distribution because it has a parameter additivity property similar to that of the the normal distribution [92]. If $(X_i)_{1\leq i\leq n}$ are independent NIG random variables with common parameters α and β but having individual parameters μ_i and δ_i, then $\displaystyle\sum_{i=1}^{n} X_i$ is NIG distributed with parameters $\left(\alpha,\beta,\displaystyle\sum_{i=1}^{n}\mu_i,\sum_{i=1}^{n}\delta_i\right)$. Furthermore, if $X \sim nig(\alpha,\beta,\delta,\mu)$ and $Y = aX+b$, then

$$Y \quad \sim \quad nig\left(\frac{\alpha}{|a|},\frac{\beta}{a},|a|\delta,a\mu+b\right).$$

The characteristic function of the NIG distribution is given by

$$\varphi_{NIG}(u) \quad = \quad \exp\left\{\delta[\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iu)^2}]+iu\mu\right\}.$$



(a) Probability density function

(b) Sample paths for $\alpha = 100$, $\beta = 1$, $\mu = 0$ and $\delta = 0.01$

Figure 2.2 Probability density function and sample path for NIG.

2.4.4 The Variance Gamma Distribution

The Variance Gamma (VG) process can be expressed as the difference between two independent Gamma processes [95]. The Gamma process $X^{(Gamma)} = \left\{X_t^{(Gamma)}\right\}_{t\geq 0}$ starts at zero and has independent and stationary increments. The increments are Gamma distributed, i.e. $X_t^{(Gamma)}$ is $Gamma(at;b)$ distributed. So if $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are two Gamma processes, a VG density function can be expressed in the following way,

$$f_{VG}(x) \quad = \quad f_{X+(-Y)}(x) = \int_{-\infty}^{\infty} f_X(x+s)f_Y(s)ds$$

where f_X and f_Y are Gamma density functions. The Gamma density function is given by

$$f_G(x;a,b) \quad = \quad \frac{b^a}{\Gamma(a)}x^{a-1}\exp(-xb), \quad x > 0.$$

The previous method is commonly used when simulating VG paths (See Figures 2.4(a) and 2.4(b)). Alternatively, the representation of a VG process as a Brownian motion subordinated by a Gamma process can also be used. A subordinated Lévy process is a time changing process for which the time changes according to another "increasing" Lévy process. The latter process is referred to as the subordinator. In this situation, a VG process has three parameters: σ, θ and ν which are respectively the volatility of the underlying Brownian motion, the drift of the Brownian motion and the variance of the subordinator.

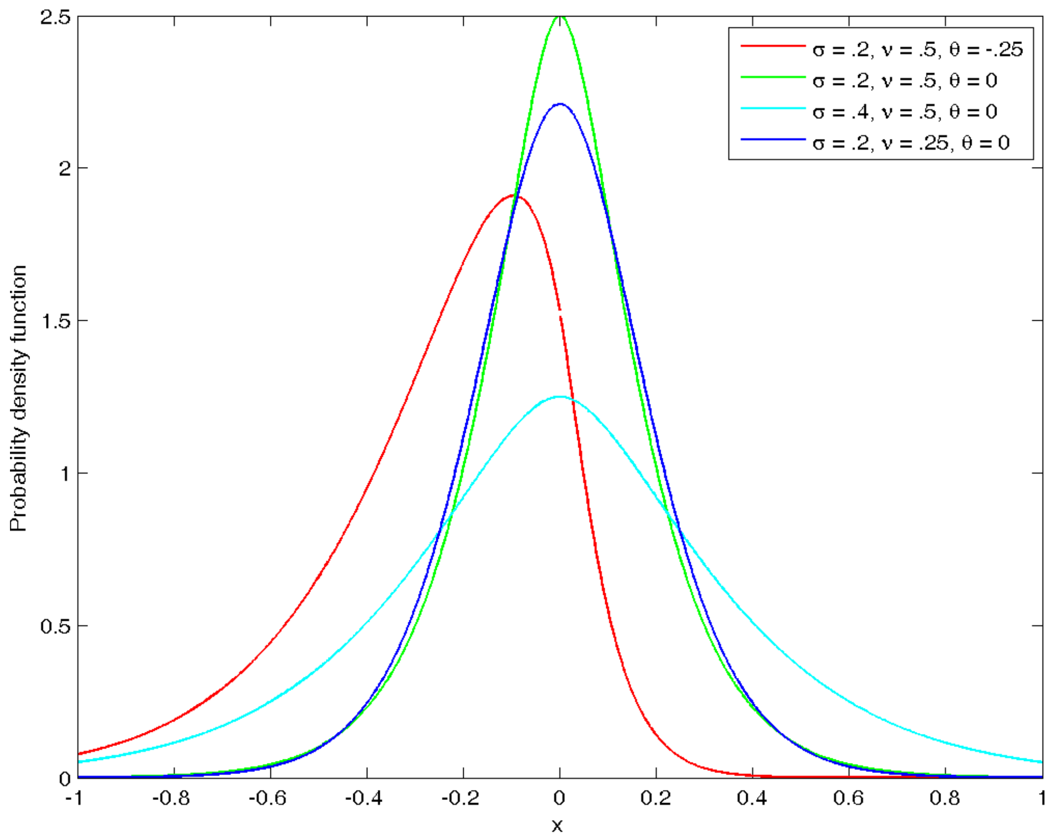


Figure 2.3 Probability density function of some VG processes.

Senata [96] introduced another approach whereby the probability density function of a VG distribution with parameters $(\theta, \sigma, \nu, \mu)$ is

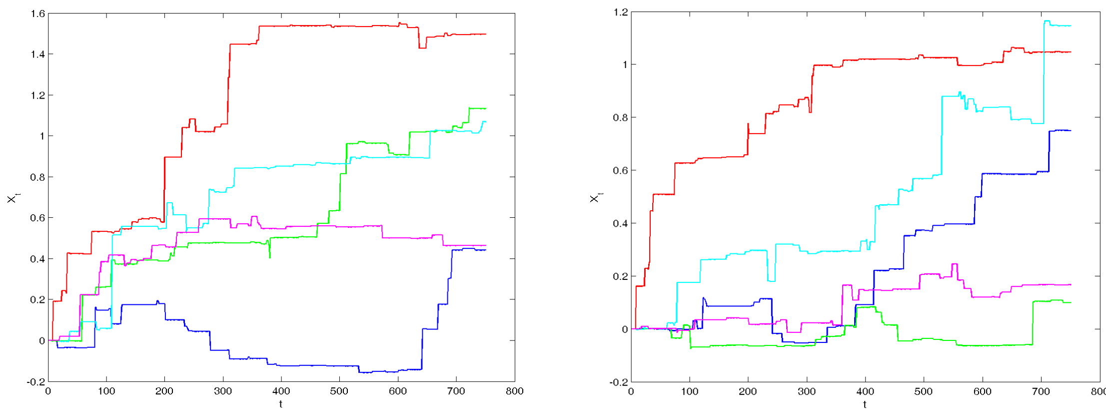
$$vg(x;\theta,\sigma,\nu,\mu) \quad = \quad \frac{2\exp\left[\dfrac{(x-\mu)\,\theta}{\sigma^2}\right]}{\sigma\sqrt{2\pi}\nu^{\frac{1}{\nu}}\Gamma(\frac{1}{\nu})}\left[\frac{(x-\mu)^2}{\theta^2+2\sigma^2/\nu}\right]^{\frac{1}{2\nu}-\frac{1}{4}} \times K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{|x-\mu|}{\sigma^2}\sqrt{\theta^2+2\sigma^2/\nu}\right),$$

and its characteristic function is

$$\varphi_{VG}(x;\theta,\sigma,\nu,\mu) \quad = \quad e^{i\mu x}\left(1-i\theta\nu x+\frac{\sigma^2\nu}{2}x^2\right)^{-1/\nu}.$$

2.4.5 The GH Skew Student's t Distribution

The GH skew Student's t-distribution is ideal for financial modelling. It is not only almost as analytically tractable as the NIG distribution, but its parameter estimation using the maximum



(a) Subordinated VG (b) Difference of 2 Gamma

Figure 2.4 Sample paths of a VG process with $\sigma = 0.2$, $\nu = 0.5$ and $\theta = 0.25$.

likelihood method is quite straightforward [39]. Moreover, the GH skew Student's t-distribution is the only subclass of the GH distribution for which one tail has polynomial behaviour while the other has exponential behaviour. This generalisation of the usual Student's t distribution is obtained from Equation (2.8) by letting $\lambda = -\frac{\nu}{2}$, $\nu > 0$ and $\alpha \to |\beta| > 0$. Its probability density function is [1]

$$\begin{aligned} f_{St}(x;\beta,\delta,\mu,\nu) &= \frac{2^{\frac{1-\nu}{2}}\delta^{\nu}|\beta|^{\frac{\nu+1}{2}}\exp\left[\beta(x-\mu)\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}\left[\sqrt{\delta^2+(x-\mu)^2}\right]^{\frac{\nu+1}{2}}} K_{\frac{\nu+1}{2}}\left[\beta^2\sqrt{\delta^2+(x-\mu)^2}\right], \quad \beta \neq 0; \\ f_{St}(x;\beta,\delta,\mu,\nu) &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\delta\Gamma\left(\frac{\nu}{2}\right)}\left[1+\frac{(x-\mu)^2}{\delta^2}\right]^{-\frac{\nu+1}{2}}, \quad \beta = 0. \end{aligned}$$

The mean and the variance of the skewed Student's t distributed random variate X are

$$\begin{aligned} \mathbb{E}(X) &= \mu + \frac{\beta\delta^2}{\nu-2}, \\ Var(X) &= \frac{2\beta^2\delta^4}{(\nu-2)^2(\nu-4)} + \frac{\delta^2}{\nu-2}. \end{aligned}$$

The mean is finite only when $\nu > 2$ and the same is true for the variance when[1] $\nu > 4$.

2.5 Option Pricing Using the Fast Fourier Transform

Under the assumption that prices follow a Lévy distribution or an exponential Lévy distribution, option pricing using the fast Fourier transform is performed in two steps. First, the Fourier transform of the contingent claim is computed, subsequently the Fourier inverse method gives the option price.

[1]See [21] for a detailed derivation and coverage of the mean and variance for all possible values of ν.

This methodology was first proposed by Carr and Madan [26] to price equity derivatives driven by variance gamma processes, but it has general applicability. The current section presents the valuation of European call-like option since IL caplets, floorlets and swaptions (priced later) can be viewed as particular European call options.

Let r_t denote the market interest rate and $R_t = \ln r_t$. This section values an European call with underlying r_t and strike k. Throughout this section, the characteristic function of r_t

$$\Phi_T(u) \quad = \quad \mathbb{E}[\exp(iur_t)]$$

is considered known analytically. Since the returns' distribution is easily deduced from the market's observation (See Chapter 6), this is not too far fetched. The previous characteristic function is also defined by

$$\Phi_T(u) \quad = \quad \int_{-\infty}^{\infty} e^{iuR} q_T(R) dR, \quad \forall\, u \in \mathbb{R}$$

where $q_T(\cdot)$ is the density of R_t under the risk neutral probability. The European call's value is

$$\begin{aligned} c_T(k) &= p_n(0,T)\mathbb{E}_{\mathbb{Q}}[(r_t - k)^+] \\ &= p_n(0,T) \int_k^{\infty} \left(e^R - e^K\right) q_T(R) dR. \end{aligned}$$

However, the c_T function is not square integrable in K, i.e. c_T does not decay as $K \to -\infty$ (or, i.e. $k \to 0$), thus its Fourier transform does not exist. Following Carr and Madan [26], the modified call price is

$$C_T(K) \quad = \quad \exp(\alpha k) c_T(k),$$

with $\alpha > 0$ chosen such that $C_T(K)$ is integrable in $-\infty$.

The Fourier transform of $C_T(K)$ is

$$\Psi_T(v) \quad = \quad \int_{-\infty}^{+\infty} e^{ivK} C_T(K) dK. \tag{2.11}$$

The fact that

$$C_T(K) \quad \underset{K \to -\infty}{\approx} \quad r_0 \exp(\alpha K),$$

ensures the integrability of the square of $C_T(K)$ at $-\infty$. However, this might accentuate the problem at $+\infty$. For the moment, the assumption is made that $\Psi(0)$ is defined and $C_T(K)$ is integrable at $+\infty$. The latter point will be taken care of in the paragraph containing Equation 2.12.

The Fourier inversion formula gives

$$\begin{aligned} C_T(K) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivK}\Psi_T(v)dv; \\ c_T(K) &= \frac{1}{2\pi}\exp(-\alpha K)\int_{-\infty}^{+\infty} e^{-ivK}\Psi_T(v)dv. \end{aligned}$$

The price $c_T(K)$ is real, therefore $\forall K \in \mathbb{R}$,

$$\Im\left(\int_{-\infty}^{+\infty} e^{-ivK}\Psi_T(v)dv\right) = 0.$$

Let $a(v)$ and $b(v)$ denote respectively the real and imaginary parts of $\Psi_T(v)$. They are defined by

$$\begin{aligned} a &: v \longrightarrow \int_{-\infty}^{+\infty} \cos(vK)c_T(K)dK \\ b &: v \longrightarrow \int_{-\infty}^{+\infty} \sin(vK)c_T(K)dK \end{aligned}$$

a is even and b is odd. Thus, $\forall v \in \mathbb{R}$,

$$\Psi(-v) = a(v) - ib(v).$$

Let A and B be the functions defined for all $K \in \mathbb{R}$ by:

$$\begin{aligned} A(K) &= \int_{-\infty}^{0} e^{-ivK}\Psi_T(v)dv \\ B(K) &= 2\pi\exp(\alpha K)c_T(K) - A(K) \\ &= \int_{0}^{+\infty} e^{-ivK}\Psi_T(v)dv. \end{aligned}$$

With the change of variable $v \to -v$,

$$\begin{aligned} A(K) &= \int_{+\infty}^{0} -e^{ivK}\Psi_T(-v)dv \\ &= \int_{0}^{+\infty} \{\cos(vK)a(v) + \sin(vK)b(v) + i\left[\sin(vK)a(v) - \cos(vK)b(v)\right]\}\,dv. \end{aligned}$$

Comparing the last equation with:

$$\begin{aligned} B(K) &= \int_{0}^{+\infty} e^{-ivK}\Psi_T(v)dv \\ &= \int_{0}^{+\infty} \{\cos(vK)a(v) + \sin(vK)b(v) - i\left[\sin(vK)a(v) - \cos(vK)b(v)\right]\}\,dv, \end{aligned}$$

notice that

$$\begin{aligned} \Re[A(K)] &= \Re[B(K)]; \\ \Im[A(K)] &= -\Im[B(K)]. \end{aligned}$$

Hence

$$2\pi \exp(\alpha K) c_T(K) \quad = \quad 2\Re[B(K)],$$

and

$$c_T(K) \quad = \quad \frac{\exp(-\alpha K)}{\pi} \Re \left[\int_0^{+\infty} e^{-ivK} \Psi_T(v) dv \right].$$

To get the call price as a function of the characteristic function Φ_T, the first step is expressing Ψ_T as function of Φ_T. From Equation (2.11),

$$\Psi_T(v) \quad = \quad p_n(0,T) \int_{-\infty}^{+\infty} \int_K^{+\infty} e^{\alpha K} e^{ivK} \left(e^R - e^K\right) q_T(R) dR dK.$$

The integration domain is defined by the upper half plane defined by $R = K$. With the use of the Fubini theorem,

$$\begin{aligned}
\Psi_T(v) &= p_n(0,T) \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{R} e^{\alpha K + ivK + R} - e^{\alpha K + ivK + K} \right) q_T(R) dR dK \\
&= p_n(0,T) \int_{-\infty}^{+\infty} q_T(R) \left[\frac{e^{\alpha K + ivK + R}}{\alpha + iv} - \frac{e^{\alpha K + ivK + K}}{\alpha + iv + 1} \right]_{-\infty}^{R} dR \\
&= p_n(0,T) \int_{-\infty}^{+\infty} q_T(R) \left(\frac{e^{\alpha K + ivK + R}}{\alpha + iv} - \frac{e^{\alpha K + ivK + K}}{\alpha + iv + 1} \right) dR \\
&= p_n(0,T) \int_{-\infty}^{+\infty} q_T(R) \left[\frac{e^{(\alpha + iv + 1)R}}{(\alpha + iv)(\alpha + iv + 1)} \right] dR \\
&= \frac{p_n(0,T) \Phi_T[v - i(1+\alpha)]}{\alpha^2 + \alpha - v^2 + iv(2\alpha + 1)}.
\end{aligned}$$

The integrability condition at $+\infty$ on α which was $\Psi_T(0) < \infty$ becomes $\Phi_T[0 - i(1+\alpha)] < \infty$, then:

$$\int_{-\infty}^{\infty} q_T(R) e^{(1+\alpha)R} dR < +\infty, \tag{2.12}$$

i.e. $\mathbb{E}_{\mathbb{Q}}\left[r_T^{\alpha+1}\right] < +\infty$.

Hence

$$c_T(K) \quad = \quad \frac{p_n(0,T) e^{-\alpha K}}{\pi} \Re \left\{ \int_0^{+\infty} \frac{e^{-ivK} \Phi_T[v - i(1+\alpha)]}{\alpha^2 + \alpha - v^2 + iv(2\alpha + 1)} \right\}.$$

Notice that for $\alpha = 0$, i.e. non modified price of the call, there is a valuation problem under the integral sign in zero. The choice of a value of α is important for the convergence speed. Carr and Madan [26] suggest close to 0.25 and Schoutens [95] 0.75 to price stock options, while Wu [110] proposes 1 for currency and interest rate options.

Section 6.2 describes how to use Fast Fourrier Transform (FFT) to discretise and implement this pricing scheme.

Chapter 3

Heath-Jarrow-Morton Model

In order to improve the match between model generated and market observed inflation linked (IL) securities prices, this chapter assumes that the consumer price index's log return, nominal and real forward rates follow Lévy processes. This is an extension of the work of Hinnerich [63] where the probability measure had only finite jump processes contrary to infinite jump processes that are used in this chapter. Pricing formulas for swaps, swaptions, caps and floors are derived. Finally, an example of calibration to market data with numerical details is performed.

Here is a summary of the content of this chapter. The first section is related to Björk, Di Masi, Kabanov and Runggaldier [16], where a general semimartingale approach is used for modelling of the inflation linked market in the Lévy setting. Having introduced some basic assumptions, the models for nominal and IL bond prices are specified through the dynamics of inflation, domestic and real instantaneous forward rates. We derive expressions for the real spot and forward inflation rates and consider the problems of existence of nominal risk neutral martingale measures respectively. As a by product we obtain HJM-type conditions on the coefficients for the IL market. Finally, we investigate the question of absence of arbitrage in the international bond market.

The second section is motivated by Eberlein and Özkan [46], where a Lévy Libor model based on a time-inhomogeneous Lévy processes has been introduced. After presentation of several technical results concerning the properties of the driving time-inhomogeneous Lévy process, we translate a few models from the semimartingale setting in the first section to the current Lévy setting. In particular, we specify the models for domestic and foreign instantaneous forward rates, bond prices and foreign spot and forward exchange rates. This allows us to proceed with the specification of the dynamics for domestic and foreign forward processes, followed by the models for domestic and foreign forward Libor rates. Finally, we consider the relationship between domestic and foreign fixed income markets in the discrete-tenor framework.

3.1 The Extended HJM Model

This section extends the HJM model to Itô-Lévy processes. Using the martingale approach, dynamics are derived for nominal bonds, inflation linked bonds, real bonds and inflation under the risk neutral measure. Contrary to previous work [71, 81, 82, 14], no initial assumption is made that the foreign currency analogy holds. Instead, the foreign currency analogy is a result.

Assumption 1. *The probability space carries both an n-dimensional Wiener process $W^{\mathbb{P}}$ and a Poisson random measure $\mu(dt, dz)$ over $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\pi^{\mathbb{P}}(dt, dz) = \nu^{\mathbb{P}}(dz)dt$. The probability space's filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated both by $W^{\mathbb{P}}$ and μ (i.e. $\mathcal{F}_t = \mathcal{F}_t^{W^{\mathbb{P}}} \vee \mathcal{F}_t^{\mu}$) which are independent. The Lévy measure $\nu^{\mathbb{P}}$ is on $\mathbb{R}$ and satisfies:*

(i) $\nu^{\mathbb{P}}(0) = 0$;

(ii) $\displaystyle\int_0^T \int_{\mathbb{R}} (z^2 \wedge 1)\nu^{\mathbb{P}}(dz)dt < \infty.$

For a "smooth" yield curve to be deductible from the market bonds' prices, the next initial assumption is needed where IP stands for inflation protected.

Assumption 2. *There exists a (nominal) market for T-bonds and T-IP-bonds for all maturities $T > 0$. Furthermore, for every fixed t, the nominal bond $p_n(t, T)$ and the inflation linked bond $p_{IP}(t, T)$ are differentiable with respect to the maturity T.*

The corresponding real bond is defined by

$$p_n(t, T) = \frac{p_{IL}(t, T)}{I(t)}.$$

Instantaneous forward rates, contracted at time t are defined by

$$f_i(t, T) = -\frac{\partial \ln p_i(t, T)}{\partial T} \quad \text{for} \quad i = r, n.$$

For $i = n$ (resp. $i = r$), the forward rate is a nominal (resp. real) instantaneous forward rate. From these forward rates, the instantaneous interest rates are deduced by

$$r^i(t) = f_i(t, t) \quad \text{for} \quad i = r, n.$$

The money market accounts are given by

$$B_i(t) = e^{\int_0^t r^i(s)ds} \quad \text{for} \quad i = r, n.$$

For $i = n$, $B_n(t)$ is the nominal money market account at time t measured in dollars; while $B_r(t)$ is the real money market account at time t measured in CPI basket.

Similarly to Jarrow and Yildirim, the next assumption first gives specifications for the dynamics of the consumer price index, the nominal and real forward rates in the statistical probability measure.

Assumption 3. *Under the objective probability measure $\mathbb{P}$, the dynamics of f_r and f_n for every fixed $T > 0$ and the dynamics of I are given by:*

$$\begin{aligned} df_i(t,T) &= \alpha^i(t,T)dt + \beta^i(t,T)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^i(t,z,T)\bar{\mu}(dt,dz) \quad i = r, n \\ dI(t) &= I(t-)a^I(t)dt + I(t-)b^I(t)dW_t^{\mathbb{P}} + I(t-)\int_{\mathbb{R}} c^I(t,z)\bar{\mu}(dt,dz), \end{aligned} \tag{3.1}$$

with

$$\bar{\mu}(dt,dz) = \begin{cases} \mu(dt,dz) - \nu(dz)dt, & |z| < R \\ \mu(dt,dz), & |z| \geq R \end{cases}$$

where $\alpha^i(t,T)$, $\beta^i(t,T)$, $\gamma^i(t,z,T)$, $a^I(t)$, $b^I(t)$ and $c^I(t,z)$ are adapted processes with

$$\int_0^T \int_t^T |\alpha^i(u,s)|dsdu < \infty, \int_0^T \int_t^T |\beta^i(u,s)|^2 dsdu < \infty,$$

for all finite t and $T \geq t$; $\gamma^i(t,z,T) : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ is a real valued function satisfying

$$\int_0^T \int_{\mathbb{R}} \int_t^T |\gamma^i(u,z,s)|^2 ds\pi(du,dz) < \infty,$$

for finite t and $T \geq t$. These conditions guarantee integrability of the coefficients and are satisfied if the coefficients are bounded for t and T from a bounded set and $\pi([0,t] \times \mathbb{R}) < \infty$ for finite t. Additionally, $\alpha(t,T)$, $\beta(t,T)$ and $\gamma(t,x,T)$ equal zero for $T < t$.
The real world Brownian motion $W^{\mathbb{P}}$ will sometime be noted W in short form.

Assumption 4. *The market is arbitrage free.*

In the current economy, the investor maintains his real value holdings in the form of real bonds and the real money market account. In nominal currency these are respectively represented by $P_{IP}(t,T) = I(t)P_r(t,T)$ and $I(t)B_r(t)$. Let $B_{IP}(t)$ denote the nominal value of the real money bank account, i.e. $B_{IP}(t) = I(t)B_r(t)$. Assumption 4 is equivalent to the existence of a (not necessary unique) nominal risk neutral probability measure $\mathbb{Q}^n$. The probability measure $\mathbb{Q}^n$ is such that $\dfrac{P_n(t,T)}{B_n(t)}$, $\dfrac{P_{IP}(t,T)}{B_n(t)}$ and $\dfrac{I(t)B_r(t)}{B_n(t)}$ are $\mathbb{Q}^n$-martingales [3, 71].

Proposition 3.1. *If $f_n(t,T)$, $f_r(t,T)$ and $I(t)$ satisfy the Assumption 3 then $I(t)$, $p_n(t,T)$, $p_{IP}(t,T)$ and $p_r(t,T)$ will under the nominal martingale measure $\mathbb{Q}^n$ satisfy:*

$$\frac{dI(t)}{I(t-)} = \left[r_n(t) - r_r(t) + \int_{|z|\geq R} c^I(t,z)\nu(dz)\right]dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz); \tag{3.2}$$

$$\frac{dp_n(t,T)}{p_n(t,T)} = r_n(t)dt + \sigma^n(t,T)dW_t + \int_{\mathbb{R}} \delta^n(t,z,T)\tilde{\mu}(dt,dz); \tag{3.3}$$

$$\frac{dp_{IP}(t,T)}{p_{IP}(t-,T)} = r_n(t)dt + \sigma^{IP}(t,T)dW_t + \int_{\mathbb{R}} \delta^{IP}(t,z,T)\tilde{\mu}(dt,dz); \tag{3.4}$$

$$\frac{dp_r(t,T)}{p_r(t-,T)} = a^r(t,T)dt + \sigma^r(t,T)dW_t + \int_{\mathbb{R}} \delta^r(t,z,T)\tilde{\mu}(dt,dz), \tag{3.5}$$

where

$$\begin{aligned}
\sigma^i(t,T) &= -\int_t^T \beta^i(t,u)du, \quad \textit{for } i = n,r;\\
\delta^i(t,z,T) &= \exp\left[D^i(t,z,T)\right] - 1, \quad \textit{for } i = n,r;\\
\sigma^{IP}(t,T) &= \left[S^r(t,T) + b^I(t)\right];\\
a^r(t,T) &= r_r(t) - S^r(t,T)b^I(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\\
&\quad - \int_{\mathbb{R}} \left\{\exp\left[D^r(t,z,T)\right]c^I(t,z) - \frac{c^I(t,z)}{1+c^I(t,z)}\right\}\nu(dz);
\end{aligned}$$

$$\begin{aligned}
\delta^{IP}(t,z,T) &= \exp\left[D^r(t,z,T)\right]\left[1+c^I(t,z)\right] - 1;\\
S^i(t,T) &= -\int_t^T \beta^i(t,u)du;\\
D^i(t,z,T) &= -\int_t^T \gamma^i(t,z,u)du.
\end{aligned}$$

Proof. The process p_i for $i = r,n$ is defined by

$$p_i(t,T) = \exp\left[X(t,T)\right] \quad, i.e. \quad \ln p_i(t,T) = X(t,T),$$

with $X(t,T) = -\int_t^T f_i(t,u)du$.

Integrating Equation 3.1 with respect to t on $[0,t]$ gives

$$f_i(t,u) = f_i(0,u) + \int_0^t \alpha^i(s,u)ds + \int_0^t \beta^i(s,u)dW_s + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z,u)\bar{\mu}(ds,dz).$$

In particular, for the instantaneous interest rate $r_i(t) = f_i(t,t)$

$$r_i(t) = f_i(0,t) + \int_0^t \alpha^i(s,t)ds + \int_0^t \beta^i(s,t)dW_s + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z,t)\bar{\mu}(ds,dz).$$

Splitting the integrals, then using Fubini theorem, we get

$$
\begin{aligned}
X(t,T) &= -\int_t^T f_i(0,u)du - \int_0^t \int_t^T \alpha^i(s,u)duds - \int_0^t \int_t^T \beta^i(s,u)dudW_s \\
&\quad - \int_0^t \int_t^T \int_{\mathbb{R}} \gamma^i(s,z,u)du\bar{\mu}(ds,dz) \\
&= -\int_0^T f_i(0,u)du - \int_0^t \int_s^T \alpha^i(s,u)duds - \int_0^t \int_s^T \beta^i(s,u)dudW_s \\
&\quad - \int_0^t \int_s^T \int_{\mathbb{R}} \gamma^i(s,z,u)du\bar{\mu}(ds,dz) + \int_0^t f_i(0,u)du + \int_0^t \int_s^t \alpha^i(s,u)duds \\
&\quad + \int_0^t \int_s^t \beta^i(s,u)dudW_s + \int_0^t \int_s^t \int_{\mathbb{R}} \gamma^i(s,z,u)du\bar{\mu}(ds,dz) \\
&= X(0,T) - \int_0^t \int_s^T \alpha^i(s,u)duds - \int_0^t \int_s^T \beta^i(s,u)dudW_s \\
&\quad - \int_0^t \int_s^T \int_{\mathbb{R}} \gamma^i(s,z,u)du\bar{\mu}(ds,dz) + \int_0^t f_i(0,u)du + \int_0^t \int_0^u \alpha^i(s,u)dsdu \\
&\quad + \int_0^t \int_0^u \beta^i(s,u)dW_s du + \int_0^t \int_0^u \int_{\mathbb{R}} \gamma^i(s,z,u)\bar{\mu}(ds,dz)du
\end{aligned}
$$

Noticing that the four last terms are equal to $\int_0^t r_i(s)ds$, we end up with

$$
\begin{aligned}
X(t,T) &= X(0,T) + \int_0^t r_i(s)ds - \int_0^t \int_s^T \alpha^i(s,u)duds - \int_0^t \int_s^T \beta^i(s,u)dudW_s \\
&\quad - \int_0^t \int_s^T \int_{\mathbb{R}} \gamma^i(s,z,u)du\bar{\mu}(ds,dz)
\end{aligned}
$$

Let us define $A^i(t,T) = -\int_t^T \alpha^i(t,u)du$, $S^i(t,T) = -\int_t^T \beta^i(t,u)du$ and $D^i(t,z,T) = -\int_t^T \gamma^i(t,z,u)du$, if differentiating the previous equation, we have

$$
dX(t,T) = \left[r_i(t) + A^i(t,T)\right]dt + S^i(t,T)dW_t + \int_{\mathbb{R}} D^i(t,z,T)\bar{\mu}(dt,dz).
$$

Using the one-dimensional Itô formula with $p_i(t,T) = f\left[X(t,T)\right] = \exp\left[X(t,T)\right]$, $\alpha(t,T) = r_i(t) + A^i(t,T)$, $\beta(t,T) = S^i(t,T)$ and $\gamma(t,z,T) = D^i(t,z,T)$

$$
\begin{aligned}
dp_i(t,T) &= p_i(t,T)\left[\alpha(t,T)dt + \beta(t,T)dW_t\right] + \frac{1}{2}\beta^2(t,T)p_i(t,T)dt \\
&\quad + \int_{|z|<R} \left\{p_i(t-,T)\exp\left[\gamma(t,z,T)\right] - p_i(t-,T) - p_i(t-,T)\gamma(t,z,T)\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}} \left\{p_i(t-,T)\exp\left[\gamma(t,z,T)\right] - p_i(t-,T)\right\}\bar{\mu}(dt,dz).
\end{aligned}
$$

Thus

$$\begin{aligned}
\frac{dp_i(t,T)}{p_i(t-,T)} &= \left[\alpha(t,T)+\frac{1}{2}\beta^2(t,T)\right]dt+\beta(t,T)dW_t+\int_{|z|<R}\{\exp\left[\gamma(t,z,T)\right] \\
&\quad -\gamma(t,z,T)-1\}\,\pi(dt,dz)+\int_{\mathbb{R}}\{\exp\left[\gamma(t,z,T)\right]-1\}\,\bar{\mu}(dt,dz) \\
&= \left[r_i(t)+A^i(t,T)+\frac{1}{2}\left\|S^i(t,T)\right\|^2\right]dt+S^i(t,T)dW_t+\int_{|z|<R}\left\{\exp\left[D^i(t,z,T)\right]\right. \\
&\quad \left.-D^i(t,z,T)-1\right\}\pi(dt,dz)+\int_{\mathbb{R}}\left\{\exp\left[D^i(t,z,T)\right]-1\right\}\bar{\mu}(dt,dz) \\
&= \left[r_i(t)+A^i(t,T)+\frac{1}{2}\left\|S^i(t,T)\right\|^2\right]dt+S^i(t,T)dW_t+\int_{|z|<R}\left\{\exp\left[D^i(t,z,T)\right]\right. \\
&\quad \left.-D^i(t,z,T)-1\right\}\nu(dz)dt+\int_{\mathbb{R}}\left\{\exp\left[D^i(t,z,T)\right]-1\right\}\bar{\mu}(dt,dz) \qquad (3.6)
\end{aligned}$$

with $\pi(dt,dz)=\nu(dz)dt$.

Combining the $\mathbb{P}$-dynamics of $p_r(t,T)$ and $I(t)$, the Corollary 2.13 applied to $p_{IP}(t,T)=p_r(t,T)I(t)$ with

$$\begin{aligned}
\alpha_1(t,T) &= \left[r_r(t)+A^r(t,T)+\frac{1}{2}\left\|S^r(t,T)\right\|^2\right]+\int_{|z|<R}\{\exp\left[D^r(t,z,T)\right]-D^r(t,z,T)-1\}\,\nu(dz); \\
\beta_1(t,T) &= S^r(t,T); \\
\gamma_1(t,z,T) &= \{\exp\left[D^r(t,z,T)\right]-1\}; \\
\alpha_2(t) &= a^I(t);\quad \beta_2(t) = b^I(t);\quad \gamma_2(t,z) = c^I(t,z);
\end{aligned}$$

gives

$$\begin{aligned}
\frac{dp_{IP}(t,T)}{p_{IP}(t,T)} &= \left[\alpha_1(t,T)+\alpha_2(t)+\beta_1(t,T)\beta_2(t)\right]dt+\int_{\mathbb{R}}\gamma_1(t,z,T)\gamma_2(t,z)\pi(dt,dz) \\
&\quad +\left[\beta_1(t,T)+\beta_2(t)\right]dW_t+\int_{\mathbb{R}}\left[\gamma_1(t,z,T)+\gamma_2(t,z)+\gamma_1(t,z,T)\gamma_2(t,z)\right](\mu-\pi)(dt,dz) \\
&= \left\{\left[r_r(t)+A^r(t,T)+\frac{1}{2}\left\|S^r(t,T)\right\|^2\right]+\int_{|z|<R}\{\exp\left[D^r(t,z,T)\right]-D^r(t,z,T)-1\}\,\nu(dz)\right. \\
&\quad \left.+a^I(t)+S^r(t,T)b^I(t)\right\}dt+\int_{\mathbb{R}}\{\exp\left[D^r(t,z,T)\right]-1\}\,c^I(t,z)\pi(dt,dz) \\
&\quad +\left[S^r(t,T)+b^I(t,T)\right]dW_t++\int_{\mathbb{R}}\left\{\left[1+c^I(t,z)\right]\exp\left[D^r(t,z,T)\right]-1\right\}(\mu-\pi)(dt,dz).
\end{aligned}$$

Next, we would like to change measure from $\mathbb{P}$ to the equivalent (nominal) martingale measure $\mathbb{Q}^n$. By the Girsanov change of measure, we know there is a $\mathbb{P}$-adapted process h_t and a $\mathbb{P}$-predictable process $\rho(t,z)\leq -1\quad \forall z\in\mathbb{R}$ such that $dL_t=h_tL_tdW^{\mathbb{P}}+\int_{\mathbb{R}}\rho(t,z)\tilde{\mu}^{\mathbb{P}}(dt,dz)$ where $L_T=\dfrac{d\mathbb{Q}^n}{d\mathbb{P}}$ on $\mathcal{F}_T$ so that $dW_t^{\mathbb{P}}=h_tdt+dW_t$ and $\pi_t(dz)=\pi_t^{\mathbb{P}}(1+\rho(t,z))$. Here W denotes a $\mathbb{Q}^n$-Brownian motion and $\pi_t(dt,dz)$ it the compensator of μ under the $\mathbb{Q}^n$-measure. Furthermore $\tilde{\mu}(dt,dz)$ denotes the compensated Poisson random measure under $\mathbb{Q}^n$, i.e.

$$\begin{aligned}\tilde{\mu}(dt,dz) &= \mu(dt,dz)-\pi(dt,dz)\\ &= \bar{\mu}(dt,dz)-\pi_t^{\mathbb{P}}(dz)\rho(t,z).\end{aligned}$$

Hence the dynamics of $I(t)$, $p_n(t,T)$, $p_r(t,T)$ and $p_{IP}(t,T)$ under $\mathbb{Q}^n$ are given by:

$$\frac{dI(t)}{I(t-)} = \left[a^I(t)+b^I(t)h_t+\int_{\mathbb{R}}c^I(t,z)\rho(t,z)\nu^{\mathbb{P}}(dz)\right]dt+b^I(t)dW_t+\int_{\mathbb{R}}c^I(t,z)\tilde{\mu}(dt,dz); \tag{3.7}$$

$$\frac{dp_i(t,T)}{p_i(t,T)} = a^i(t,T)dt+\sigma^i(t,T)dW_t+\int_{\mathbb{R}}\delta^i(t,z,T)\tilde{\mu}(dt,dz), \quad i=n,r; \tag{3.8}$$

$$\frac{dp_{IP}(t,T)}{p_{IP}(t,T)} = a^{IP}(t,T)dt+\sigma^{IP}(t,T)dW_t+\int_{\mathbb{R}}\delta^{IP}(t,z,T)\tilde{\mu}(dt,dz); \tag{3.9}$$

where

$$\begin{aligned}a^i(t,T) &= r_i(t)+A^i(t,T)+\frac{1}{2}\left\|S^i(t,T)\right\|^2+h_tS^i(t,T)+\int_{|z|<R}\left\{\exp\left[D^i(t,z,T)\right]\right.\\ &\quad \left.-D^i(t,z,T)-1\right\}\nu(dz)+\int_{\mathbb{R}}\left\{\exp\left[D^i(t,z,T)\right]-1\right\}\rho(t,z)\nu^{\mathbb{P}}(dz); \qquad (3.10)\\ \sigma^i(t,T) &= S^i(t,T);\\ \delta^i(t,z,T) &= \exp\left[D^i(t,z,T)\right]-1;\\ a^{IP}(t,T) &= r_r(t)+A^r(t,T)+\frac{1}{2}\|S^r(t,T)\|^2+\int_{|z|<R}\{\exp\left[D^r(t,z,T)\right]-D^r(t,z,T)-1\}\nu(dz)\\ &\quad +a^I(t)+S^r(t,T)b^I(t)+\int_{\mathbb{R}}\{\exp\left[D^r(t,z,T)\right]-1\}c^I(t,z)\nu(dz)\\ &\quad +\int_{\mathbb{R}}\left\{[1+c^I(t,z)]\exp\left[D^r(t,z,T)\right]-1\right\}\rho(t,z)\nu^{\mathbb{P}}(dz)+h_t\left[S^r(t,T)+b^I(t)\right]; \qquad (3.11)\\ \sigma^{IP}(t,T) &= \left[S^r(t,T)+b^I(t)\right];\\ \delta^{IP}(t,z,T) &= \exp\left[D^r(t,z,T)\right][1+c^I(t,z)]-1.\end{aligned}$$

From Assumption 4, $\dfrac{P_n(t,T)}{B_n(t)}$ and $\dfrac{P_{IP}(t,T)}{B_n(t)}$ are $\mathbb{Q}^n$-martingales, hence the drift of $p_n(t,T)$ and $p_{IP}(t,T)$ must equal the nominal short rate, that is $a^n(t,T)=a^{IP}(t)=r_n(t)$. This, together with Equations (3.8) and (3.9), implies Equations (3.3) and (3.4), respectively.

If we insert the condition $a^n(t,T) = a^{IP}(t,T) = r_n(t)$ into the drift Equations (3.10) and (3.11)

$$\begin{aligned}
r_n(t) &= r_n(t) + A^n(t,T) + \frac{1}{2}\left\|S^n(t,T)\right\|^2 + h_t S^n(t,T) + \int_{|z|<R} \{\exp\left[D^n(t,z,T)\right] \\
&\quad -D^n(t,z,T) - 1\}\,\nu(dz) + \int_{\mathbb{R}} \{\exp\left[D^n(t,z,T)\right] - 1\}\,\rho(t,z)\nu^{\mathbb{P}}(dz) \\
&= r_r(t) + A^r(t,T) + \frac{1}{2}\left\|S^r(t,T)\right\|^2 + \int_{|z|<R} \{\exp\left[D^r(t,z,T)\right] \\
&\quad -D^r(t,z,T) - 1\}\,\nu(dz) + a^I(t) + S^r(t,T)b^I(t) + \int_{\mathbb{R}} \{\exp\left[D^r(t,z,T)\right] - 1\}\,c^I(t,z)\nu(dz) \\
&\quad +h_t\left[S^r(t,T) + b^I(t)\right] + \int_{\mathbb{R}} \left\{\left[1 + c^I(t,z)\right]\exp\left[D^r(t,z,T)\right] - 1\right\}\rho(t,z)\nu^{\mathbb{P}}(dz).
\end{aligned}$$

Since these equations must hold for all T, we get Equations (3.12), (3.13), (3.14) and (3.16).

$$\begin{aligned}
A^n(t,T) &= -\frac{1}{2}\left\|S^n(t,T)\right\|^2 - h_t S^n(t,T) - \int_{|z|<R} \{\exp\left[D^n(t,z,T)\right] - D^n(t,z,T)\}\,\nu(dz) \\
&\quad - \int_{\mathbb{R}} \exp\left[D^n(t,z,T)\right]\rho(t,z)\nu^{\mathbb{P}}(dz); \qquad (3.12) \\
\int_{|z|<R} \nu(dz) &= -\int_{\mathbb{R}} \rho(t,z)\nu^{\mathbb{P}}(dz) \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
r_n(t) &= r_r(t) - \int_{|z|<R} \nu(dz) + a^I(t) - \int_{\mathbb{R}} c^I(t,z)\nu(dz) + h_t b^I(t) - \int_{\mathbb{R}} \rho(t,z)\nu^{\mathbb{P}}(dz); \\
&= r_r(t) + a^I(t) - \int_{\mathbb{R}} c^I(t,z)\nu(dz) + h_t b^I(t) \qquad (3.14)
\end{aligned}$$

from Equation (3.13);

$$\begin{aligned}
a^I(t) &= r_n(t) - r_r(t) + \int_{\mathbb{R}} c^I(t,z)\nu(dz) - h_t b^I(t); \qquad (3.15) \\
A^r(t,T) &= -\frac{1}{2}\left\|S^r(t,T)\right\|^2 - \int_{|z|<R} \{\exp\left[D^r(t,z,T)\right] - D^r(t,z,T)\}\,\nu(dz) - S^r(t,T)b^I(t) - h_t S^r(t,T) \\
&\quad - \int_{\mathbb{R}} \exp\left[D^r(t,z,T)\right]c^I(t,z)\nu(dz) - \int_{\mathbb{R}} \left[1 + c^I(t,z)\right]\exp\left[D^r(t,z,T)\right]\rho(t,z)\nu^{\mathbb{P}}(dz). \qquad (3.16)
\end{aligned}$$

Inserting Equation (3.15) into Equation (3.7)

$$\begin{aligned}
\frac{dI(t)}{I(t-)} &= \left[r_n(t) - r_r(t) + \int_{\mathbb{R}} c^I(t,z)\nu(dz) + \int_{\mathbb{R}} c^I(t,z)\rho(t,z)\nu^{\mathbb{P}}(dz)\right]dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz) \\
&= \left[r_n(t) - r_r(t) + \int_{|z|\geq R} c^I(t,z)\nu(dz)\right]dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz) \qquad (3.17)
\end{aligned}$$

because of Equation (3.13). Furthermore, $\int_{|z|\geq R} c^I(t,z)\nu(dz)$ is finite by definition. Hence Equation (3.2) is proved.

Let us now find the dynamics of p_r under $\mathbb{Q}^n$; by definition $p_r(t,T) = \dfrac{p_{IP}(t,T)}{I(t)}$. Considering $Y_t = \dfrac{1}{I(t)}$, Equation (3.17) and Corollary 2.9 give

$$
\begin{aligned}
\frac{dY_t}{Y_t} &= \left\{-r_n(t) + r_r(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \left[b^I(t)\right]^2\right\} dt + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\pi(dt,dz) \\
&\quad -b^I(t)dW_t - \int_{\mathbb{R}} \frac{c^I(t,z)}{1+c^I(t,z)}\tilde{\mu}(dt,dz) \\
&= \left\{-r_n(t) + r_r(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \left[b^I(t)\right]^2 + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\right\} dt \\
&\quad -b^I(t)dW_t - \int_{\mathbb{R}} \frac{c^I(t,z)}{1+c^I(t,z)}\tilde{\mu}(dt,dz)
\end{aligned}
$$

Applying Corollary 2.13 to $p_r(t,T) = p_{IP}(t,T)Y_t$

$$
\begin{aligned}
\frac{dp_r(t,T)}{p_r(t-,T)} &= \left[r_n(t) - r_n(t) + r_r(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \left[b^I(t)\right]^2 + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\right. \\
&\quad \left. -\sigma^{IP}(t,T)b^I(t)\right] dt - \int_{\mathbb{R}} \delta^{IP}(t,z,T)\frac{c^I(t,z)}{1+c^I(t,z)}\pi(dt,dz) + \left[\sigma^{IP}(t,T)t - b^I(t)\right] dW_t \\
&\quad + \int_{\mathbb{R}} \left[\delta^{IP}(t,z,T) - \frac{c^I(t,z)}{1+c^I(t,z)} - \delta^{IP}(t,z,T)\frac{c^I(t,z)}{1+c^I(t,z)}\right]\tilde{\mu}(dt,dz) \\
&= \left[r_r(t) + \left[b^I(t)\right]^2 - \sigma^{IP}(t,T)b^I(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\right. \\
&\quad \left. - \int_{\mathbb{R}} \frac{\delta^{IP}(t,z,T)c^I(t,z)}{1+c^I(t,z)}\nu(dz)\right] dt + \left[\sigma^{IP}(t,T) - b^I(t)\right] dW_t \\
&\quad + \int_{\mathbb{R}} \left[\delta^{IP}(t,z,T) - \frac{c^I(t,z)}{1+c^I(t,z)} - \frac{\delta^{IP}(t,z,T)c^I(t,z)}{1+c^I(t,z)}\right]\tilde{\mu}(dt,dz)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dp_r(t,T)}{p_r(t-,T)} &= \left[r_r(t) + \left[b^I(t)\right]^2 - \left[S^r(t,T) + b^I(t)\right]b^I(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\right. \\
&\quad \left. - \int_{\mathbb{R}} \left\{\exp\left[D^r(t,z,T)\right][1+c^I(t,z)] - 1\right\}\frac{c^I(t,z)}{1+c^I(t,z)}\nu(dz)\right] dt + \left[S^r(t,T) + b^I(t) - b^I(t)\right] dW_t \\
&\quad + \int_{\mathbb{R}} \left[\exp\left[D^r(t,z,T)\right][1+c^I(t,z)] - 1 - \frac{c^I(t,z)}{1+c^I(t,z)}\right. \\
&\quad \left. - \left\{\exp\left[D^r(t,z,T)\right][1+c^I(t,z)] - 1\right\}\frac{c^I(t,z)}{1+c^I(t,z)}\right]\tilde{\mu}(dt,dz) \\
\frac{dp_r(t,T)}{p_r(t-,T)} &= \left[r_r(t) - S^r(t,T)b^I(t) - \int_{|z|\geq R} c^I(t,z)\nu(dz) + \int_{|z|<R} \frac{\left[c^I(t,z)\right]^2}{1+c^I(t,z)}\nu(dz)\right. \\
&\quad \left. - \int_{\mathbb{R}} \left\{\exp\left[D^r(t,z,T)\right]c^I(t,z) - \frac{c^I(t,z)}{1+c^I(t,z)}\right\}\nu(dz)\right] dt \\
&\quad + S^r(t,T)dW_t + \int_{\mathbb{R}} \left\{\exp\left[D^r(t,z,T)\right] - 1\right\}\tilde{\mu}(dt,dz)
\end{aligned}
$$

which proves Equation (3.5). ■

By construction in the previous proof, $\dfrac{P_n(t,T)}{B_n(t)}$ and $\dfrac{P_{IP}(t,T)}{B_n(t)}$ are $\mathbb{Q}^n$-martingales. For $\mathbb{Q}^n$ to be the sought nominal risk neutral probability measure, $\dfrac{I(t)B_r(t)}{B_n(t)}$ also has to be a $\mathbb{Q}^n$-martingale. The next Lemma states a necessary condition for such a probability measure to exist.

Lemma 3.2. *For the probability measure $\mathbb{Q}^n$ to be a nominal risk neutral probability measure, the integral $\int_{|z|\geq R} c^I(t,z)\nu(dz)$ has to be zero.*

Proof. The definition of $B_r(t)$ and the $\mathbb{Q}^n$-dynamics of $I(t)$ given in Equation (3.2) yield

$$\frac{dB_{IP}(t)}{B_{IP}(t-)} = \left[r_n(t) + \int_{|z|\geq R} c^I(t,z)\nu(dz)\right]dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz).$$

However B_{IP} if a $\mathbb{Q}^n$-martingale is and only if the drift term in the previous equation is $r_n(t)$. ∎

This necessary condition for the existence of a nominal risk neutral probability measure is not part of the generalisation of the previously obtained conditions in more specific settings (See Hinnerich [63], Corollary 2.1 and Jarrow and Yildrim [71], Proposition 2). This additional condition is a restriction only on the inflation's jumps. The fact that this condition is on a jump component was predictable from the work of Jarrow and Yildrim [71], under some drift and volatility restrictions, a unique equivalent risk neutral measure always exists under the normality assumption. However, after introducing finite jumps in the probability measure, Hinnerich did not get this additional condition. A sufficient condition for the previous condition to be met is the next assumption which is commonly used with Lévy processes in Finance in different versions [43, 100, 45]. Before stating the assumption, the general work setting needs to be redefined.

The nominal probability measure is endowed with a canonical filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where the driving process $L = (L_t)_{t\geq 0}$ is a time inhomogeneous Lévy process, i.e. a process with independent increments and absolutely continuous characteristics (PIIAC). The law of the process L_t with characteristic triplet (c,ν,α) is given by the Lévy-Khinchin Formula (See Equation (2.5)) with the standard integrability conditions, where

$$b = \int_0^t b_s ds, \quad c = \int_0^t c_s ds, \quad \nu(z) = \int_0^t \nu_s ds,$$

with the integrals being componentwise. Further assumption is made that

$$\int_0^T \left(|\alpha_s| + \|c_s\| + \int_{\mathbb{R}^d} (|z^2| \wedge 1)\nu(dz)\right) ds < \infty \quad \text{, for } T > 0,$$

where $|\cdot|$ is the norm corresponding to the Euclidian scalar product on $\mathbb{R}^d$ and $\|\cdot\|$ denotes any norm on the $d \times d$ matrices. The following additional moment assumption is made

Assumption 5. *There are constant M, $\varepsilon > 0$, such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$*

$$\int_0^T \int_{|z|>R} \exp\langle u, z\rangle \, \nu(dz)ds < \infty \quad , \text{ for } T > 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidian scalar product on $\mathbb{R}^d$ and R is a positive constant generally taken equal to one.

The previous assumption is equivalent to $\mathbb{E}[\exp\langle u, L_t\rangle] < \infty$ for $t \in [0, T]$ and $u \in \mathbb{R}^d$. This is a natural assumption especially when using the Fast Fourier transform or Laplace transform for option pricing. Recall that for these methods, the characteristic function is supposedly initially known and needs to be finite for obvious reasons. Furthermore, in the HJM framework, the underlying processes are always exponentials of stochastic integrals with respect to the driving processes L. In order to allow the pricing of derivatives these underlying processes have to be martingales under the nominal risk neutral measure and, therefore, *a priori* have to have finite expectations, which is exactly the previous assumption.

In particular, under the previous assumption, the variable L_t itself has finite expectation and consequently a truncation is not needed. In fact, now L is not only a semimartingale, but a special semimartingale and Assumption 3 can be relaxed. Under the objective probability measure $\mathbb{P}$, the dynamics of f_i (for every fixed $T > 0$ and $i = n, r$) and the dynamics of I are given by:

$$\begin{aligned} df_i(t,T) &= \alpha^i(t,T)dt + \beta^i(t,T)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^i(t,z,T)(\mu(dt,dz) - \nu(dz)dt) \quad i = r, n \\ dI(t) &= I(t-)a^I(t)dt + I(t)b^I(t-)dW_t^{\mathbb{P}} + I(t-)\int_{\mathbb{R}} c^I(t,z)[\mu(dt,dz) - \nu(dz)dt], \end{aligned}$$

with the standard integrability conditions. The canonical representation of the process L is

$$L_t = \int_0^t \alpha_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} z(\mu - \pi)(ds, dz),$$

where $\sqrt{c_s}$ is a measurable version of the square root of c_s.

Henceforth, Assumption 5 is supposed verified.

For a nominal risk-free probability measure to exist, additional restrictions on the drift terms in Assumption 3 and some non-degeneracy conditions upon the volatilities are needed.

Corollary 3.3. *The drift conditions that have to be satisfied in order for the market to be free of arbitrage are:*

$$\begin{aligned} \alpha^n(t,T) &= \beta^n(t,T)\left[\int_t^T \beta(t,u)du - h_t\right] - \int_{|z|<R} \delta^n(t,z,T)\gamma^n(t,z,T)\nu(dz) \\ &\quad - \int_{\mathbb{R}} [\delta^n(t,z,T) + 1]\, \gamma^n(t,z,T)\rho(t,z)\nu^{\mathbb{P}}(dz); \end{aligned}$$

$$\begin{aligned}
\alpha^r(t,T) &= \beta^r(t,T)\left[\int_t^T \beta^r(t,u)du - h_t - b^I(t)\right] - \int_{|z|<R} \delta^r(t,z,T)\gamma^r(t,z,T)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \left[\delta^r(t,z,T)+1\right]\gamma^r(t,z,)c^I(t,z)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \left[1+c^I(t,z)\right]\left[\delta^r(t,z,T)+1\right]\gamma^r(t,z,)\rho(t,z)\nu^{\mathbb{P}}(dz); \\
\int_{|z|<R} \nu(dz) &= -\int_{\mathbb{R}} \rho(t,z)\nu^{\mathbb{P}}(dz); \\
a^I(t) &= r_n(t) - r_r(t) + \int_{\mathbb{R}} c^I(t,z)\nu(dz) - h_t b^I(t).
\end{aligned}$$

Hinnerich's Corollary 2.1 in [63] and Jarrow and Yildrim's Proposition 2 in [71] are just particular cases of this corollary.

Proof. We use Equations (3.12), (3.14), (3.13) and (3.16) and take the T-derivative of the first two equations.

$$\begin{aligned}
A^n(t,T) &= -\frac{1}{2}\left\|S^n(t,T)\right\|^2 - h_t S^n(t,T) - \int_{|z|<R} \left\{\exp\left[D^n(t,z,T)\right]\right. \\
&\quad \left. -D^n(t,z,T)\right\}\nu(dz) - \int_{\mathbb{R}} \exp\left[D^n(t,z,T)\right]\rho(t,z)\nu^{\mathbb{P}}(dz); \\
-\alpha^n(t,T) &= -\beta^n(t,T)\int_t^T \beta(t,u)du + h_t\beta^n(t,T) - \int_{|z|<R} \left\{-\gamma^n(t,z,T)\exp\left[D^n(t,z,T)\right]\right. \\
&\quad \left. +\gamma^n(t,z,T)\right\}\nu(dz) + \int_{\mathbb{R}} \exp\left[D^n(t,z,T)\right]\gamma^n(t,z,T)\rho(t,z)\nu^{\mathbb{P}}(dz); \\
\alpha^n(t,T) &= \beta^n(t,T)\left[\int_t^T \beta(t,u)du - h_t\right] + \int_{|z|<R} \left\{1-\exp\left[D^n(t,z,T)\right]\right\}\gamma^n(t,z,T)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \exp\left[D^n(t,z,T)\right]\gamma^n(t,z,T)\rho(t,z)\nu^{\mathbb{P}}(dz) \\
&= \beta^n(t,T)\left[\int_t^T \beta(t,u)du - h_t\right] - \int_{|z|<R} \delta^n(t,z,T)\gamma^n(t,z,T)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \left[\delta^n(t,z,T)+1\right]\gamma^n(t,z,T)\rho(t,z)\nu^{\mathbb{P}}(dz); \\
A^r(t,T) &= -\frac{1}{2}\left\|S^r(t,T)\right\|^2 - S^r(t,T)b^I(t) - h_t S^r(t,T) - \int_{|z|<R} \left\{\exp\left[D^r(t,z,T)\right]\right. \\
&\quad \left. -D^r(t,z,T)\right\}\nu(dz) - \int_{\mathbb{R}} \exp\left[D^r(t,z,T)\right]c^I(t,z)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \left[1+c^I(t,z)\right]\exp\left[D^r(t,z,T)\right]\rho(t,z)\nu^{\mathbb{P}}(dz); \\
-\alpha^r(t,T) &= -\beta^r(t,T)\left[\int_t^T \beta^r(t,u)du - h_t - b^I(t)\right] - \int_{|z|<R} \left\{-\gamma^r(t,z,T)\exp\left[D^r(t,z,T)\right]\right. \\
&\quad \left. +\gamma^r(t,z,T)\right\}\nu(dz) + \int_{\mathbb{R}} \exp\left[D^r(t,z,T)\right]\gamma^r(t,z,)c^I(t,z)\nu(dz) \\
&\quad + \int_{\mathbb{R}} \left[1+c^I(t,z)\right]\exp\left[D^r(t,z,T)\right]\gamma^r(t,z,)\rho(t,z)\nu^{\mathbb{P}}(dz);
\end{aligned}$$

$$
\begin{aligned}
\alpha^r(t,T) &= \beta^r(t,T)\left[\int_t^T \beta^r(t,u)du - h_t - b^I(t)\right] + \int_{|z|<R} \{1-\exp\left[D^r(t,z,T)\right]\}\,\gamma^r(t,z,T)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \exp\left[D^r(t,z,T)\right]\gamma^r(t,z,)c^I(t,z)\nu(dz) \\
&\quad - \int_{\mathbb{R}} [1+c^I(t,z)]\exp\left[D^r(t,z,T)\right]\gamma^r(t,z,)\rho(t,z)\nu^{\mathbb{P}}(dz); \\
&= \beta^r(t,T)\left[\int_t^T \beta^r(t,u)du - h_t - b^I(t)\right] - \int_{|z|<R} \delta^r(t,z,T)\gamma^r(t,z,T)\nu(dz) \\
&\quad - \int_{\mathbb{R}} \left[\delta^r(t,z,T)+1\right]\gamma^r(t,z,)c^I(t,z)\nu(dz) \\
&\quad - \int_{\mathbb{R}} [1+c^I(t,z)]\left[\delta^r(t,z,T)+1\right]\gamma^r(t,z,)\rho(t,z)\nu^{\mathbb{P}}(dz); \\
\int_{|z|<R} \nu(dz) &= -\int_{\mathbb{R}} \rho(t,z)\nu^{\mathbb{P}}(dz); \\
a^I(t) &= r_n(t) - r_r(t) + \int_{\mathbb{R}} c^I(t,z)\nu(dz) - h_t b^I(t).
\end{aligned}
$$

■

Under Assumption 5, the forward rates' dynamics can be rewritten in the form

$$
df_i(t,T) = \alpha^i(t,T)dt - \sigma^i(t,T)dL_t \quad i=r,n \quad (0\le t\le T), \tag{3.18}
$$

where α^i and σ^i satisfy the usual integrability conditions (See [47, 44]).

In this setting, the stochastic differential equation (SDE) (3.6) can be rewritten [89] as

$$
\frac{dp_i(t,T)}{p_i(t-,T)} = [r_i(t) - A^i(t,T)]dt + \Sigma^i(t,T)dL_t,
$$

where $A^i(t,T) = \int_{t\wedge T}^T \alpha^i(t,u)du$ and $\Sigma^i(t,T) = \int_{t\wedge T}^T \sigma^i(t,u)du$.

Therefore

$$
\begin{aligned}
p_i(t,T) &= p_i(0,T)\exp\left\{\int_0^t [r_i(s) - A^i(s,T)]ds + \int_0^t \Sigma^i(s,T)dL_s\right\} \\
&= p_i(0,T)B_i(t)\exp\left\{-\int_0^t A^i(s,T)ds + \int_0^t \Sigma^i(s,T)dL_s\right\}
\end{aligned} \tag{3.19}
$$

By setting $T=t$, the risk free savings account can be written as

$$
B_i(t) = \frac{1}{p_i(0,t)}\exp\left[\int_0^t A^i(s,t)ds - \int_0^t \Sigma^i(s,t)dL_s\right]. \tag{3.20}
$$

Using the two previous equations, the bond prices can be rewritten as

$$
\begin{aligned}
p_i(t,T) &= \frac{p_i(0,T)}{p_i(0,t)}\exp\left\{\int_0^t \left[A^i(s,t) - A^i(s,T)\right]ds + \int_0^t \left[\Sigma^i(s,T) - \Sigma^i(s,t)\right]dL_s\right\} \\
&= \frac{p_i(0,T)}{p_i(0,t)}\exp\left\{-\int_0^t A^i(s,t,T)ds + \int_0^t \Sigma^i(s,t,T)dL_s\right\},
\end{aligned} \tag{3.21}
$$

where

$$\begin{aligned} A^i(s,t,T) &= A^i(s,T) - A^i(s,t); \\ \Sigma^i(s,t,T) &= \Sigma^i(s,T) - \Sigma^i(s,t). \end{aligned}$$

In the risk neutral measure, the nominal money market account is the numeraire, hence discounted bond prices are martingales. The bond prices take the form [44]

$$p_i(t,T) = p_i(0,T) \exp\left\{ ds \int_0^t [r(s) - \theta_s^i(\sigma^i(s,T))]ds + \int_0^t \Sigma^i(s,T) dL_s \right\}, \tag{3.22}$$

where θ is the Laplace cumulant of L_1, given by

$$\theta(z) = \varphi_1(-iz).$$

Comparison with Equation 3.19, yields

$$A^i(s,T) = \theta_s^i(\Sigma^i(s,T)). \tag{3.23}$$

Note that the latter drift condition is only "part" of the conditions in Corollary 3.3 for a nominal risk neutral probability measure to exist. The following sections assume that the latter conditions are in force.

The next propositions and corollaries present some needed probability measures, change of probability measure and associated properties that will be used later for IL options' pricing.

Proposition 3.4. *Let $\mathbb{Q}^{IP}$ denote the probability measure defined by*

$$\frac{d\mathbb{Q}^{IP}}{d\mathbb{Q}^n} = Z_T \quad \text{on } \mathcal{F}_T$$

where

$$Z_t = \frac{B_{IP}(t)}{B_n(t)} \frac{B_n(0)}{B_{IP}(0)},$$

then $\mathbb{Q}^{IP}$ is a martingale measure for the numeraire B_{IP}.

The expression $\frac{B_{IP}(t)}{B_n(t)}$ is the discounted nominal value of the real money bank account.

Proof. Let Π be a stochastic process such that $\dfrac{\Pi(t)}{B_n(t)}$ is a $\mathbb{Q}^n$-martingale, i.e. so that Π is an arbitrage free price process. We have to show that the process $\dfrac{\Pi(t)}{B_{IP}(t)}$ is a $\mathbb{Q}^{IP}$-martingale. Let $s \leq t$, then Bayes formula gives that

$$\begin{aligned} \mathbb{E}_s^{IP}\left[\frac{\Pi(t)}{B_{IP}(t)}\right] &= \frac{\mathbb{E}_s^{\mathbb{Q}}\left[Z_t \frac{\Pi(t)}{B_{IP}(t)}\right]}{Z_s} = \frac{\mathbb{E}_s^{\mathbb{Q}}\left[\frac{B_{IP}(t)}{B_n(t)} \frac{\Pi(t)}{B_{IP}(t)}\right] \frac{B_n(0)}{B_{IP}(0)}}{Z_s} \\ &= \mathbb{E}_s^{\mathbb{Q}}\left[\frac{\Pi(t)}{B_n(t)}\right] \frac{B_n(s)}{B_{IP}(s)} = \frac{\Pi(s)}{B_n(s)} \frac{B_n(s)}{B_{IP}(s)} = \frac{\Pi(s)}{B_{IP}(s)} \end{aligned}$$

■

Corollary 3.5. *Define* $\mathbb{Q}^{T-IP}$ *by*

$$\frac{d\mathbb{Q}^{T-IP}}{d\mathbb{Q}^n} = Z_T \quad \text{on } \mathcal{F}_T$$

where

$$Z_t = \frac{p_{IP}(t,T)}{B_n(t)} \frac{B_n(0)}{p_{IP}(0,T)}$$

then $\mathbb{Q}^{T-IP}$ *is a martingale measure for the numeraire* $p_{IP}(t,T)$.

The expression $\frac{p_{IP}(t,T)}{B_n(t)}$ is the price at time t of the discounted real zero-coupon bond denominated in nominal terms.

If one exchanges $B_{IP}(t)$ for $p_{IP}(t,T)$ in the proof of Proposition 3.4 the corollary follows.

Proposition 3.6. *Let* Π_n *denote an arbitrage free price in the nominal economy. Define the process* Π_r *by* $\Pi_r(t) = \dfrac{\Pi_n(t)}{I(t)}$. *Define* $\mathbb{Q}^r$ *by*

$$\frac{d\mathbb{Q}^r}{d\mathbb{Q}^n} = Z_T \quad \text{on } \mathcal{F}_T$$

where

$$Z_t = \frac{B_r(t)I(t)}{B_n(t)} \frac{B_n(0)}{B_r(0)I(0)}.$$

Then $\mathbb{Q}^r$ *is a martingale measure for the numeraire* $B_r(t)$ *and*

$$\frac{\Pi_r(t)}{B_r(t)} \quad \text{is a } \mathbb{Q}^r\text{-martingale.}$$

Proof. From Proposition 3.4 it follows that $\dfrac{\Pi_n(t)}{B_{IP}(t)}$ is a $\mathbb{Q}^{IP}$-martingale and that $\mathbb{Q}^r$ is equal to $\mathbb{Q}^{IP}$. Since

$$\frac{\Pi_r(t)}{B_r(t)} = \frac{\Pi_r(t)I(t)}{B_r(t)I(t)} = \frac{\Pi_n(t)}{B_{IP}(t)}$$

it follows that $\frac{\Pi_r(t)}{B_r(t)}$ is a $\mathbb{Q}^r$-martingale. ■

Corollary 3.7. *Define* $\mathbb{Q}^{T,r}$ *by*

$$\frac{d\mathbb{Q}^{T,r}}{d\mathbb{Q}^n} = Z_T \quad \text{on } \mathcal{F}_T$$

where

$$Z_t = \frac{p_r(t,T)I(t)}{B_n(t)} \frac{B_n(0)}{p_r(0,T)I(0)}$$

then $\mathbb{Q}^{T,r}$ is a martingale measure for the numeraire $p_r(t,T)$ and

$$\frac{\Pi_r(t)}{p_r(t)} \text{ is a } \mathbb{Q}^{T,r}\text{-martingale.}$$

Corollary 3.8.

$$\frac{p_r(t,T)}{B_r(t)} \text{ is a } \mathbb{Q}^{r}\text{-martingale}$$

$$\frac{p_r(t,S)}{p_r(t,T)} \text{ is a } \mathbb{Q}^{T,r}\text{-martingale.}$$

Assumption 6. *We assume that $\beta^n, \beta^r, \gamma^n, \gamma^r, c^I, \nu^{\mathbb{P}}$ are deterministic. Under this assumption, for 1-dimensionnal processes, there is a single risk neutral measure and thus a unique fair price for IL derivatives just as in the Black-Scholes pricing theory [47].*

3.2 Inflation Linked Swaps

This section focuses on the pricing of the most commonly traded inflation linked (IL) swaps. Zero coupons and year-on-year IL swaps are priced in the previously built HJM framework. Let $T_0, T_1, \cdots, T_M$ denote a fixed set of increasing times and τ_i be defined by

$$\tau_i \quad = \quad T_i - T_{i-1}, \quad \text{for} \quad i = 1, \cdots, M.$$

A typical swap starts at time T_0 with payments occurring at time $T_1, T_2, \cdots, T_M$. On each payment date, Party A pays Party B the inflation rate over a predefined period while Party B pays Party A a fixed rate. The inflation rate is computed as the percentage increase of the level of the price index over a period of time. In the previous description, Party A has entered a receiver swap (i.e. he receives a fixed amount) while Party B has entered a payer swap.

In what follows to lighten the formulas, $\Pi[t, \cdot]$ is used to denote the price in nominal currency (e.g. rands, dollars), of the payoff $(\cdot)$.

3.2.1 Zero Coupon Inflation Indexed Swap

Mercurio [81] showed that the fair price of a ZCIIS is model independent using martingale methods. This fact can also be proved using a replicating argument as proved by Hinnerich [63]. Both proofs are provided in this subsection, this result will be used afterwards to price year-on-year swaps.

A ZCIIS over the time interval $[T_0, T]$ has only one payment at time T without any intermediary payments. If $Z_0(T, K)$ denotes the corresponding payer ZCIIS with swap rate K and nominal N, then a fixed amount of

$$N\left[(1+K)^{T-T_0}-1\right]$$

is paid out at time T and a floating amount of

$$N\left[\frac{I(T)}{I(T_0)}-1\right]$$

is received at time T. Henceforth, the nominal will be taken to be equal to one for simplification. If $Z_0(t,T,K)$ denotes the price of $Z_0(T,K)$ at time t, then the its payoff is

$$Z_0(T,T,K) \quad = \quad \frac{I(T)}{I(T_0)}-(1+K)^{T-T_0}$$

and

$$Z_0(t,T,K) \quad = \quad \mathbb{E}_n\left\{\exp\left(-\int_t^T r_n(s)ds\right)\left[\frac{I(T)}{I(T_0)}-(1+K)^{T-T_0}\right]\Big|_{\mathcal{F}_t}\right\}$$

for $t\in[T_0,T]$, where $\mathcal{F}_t$ is the corresponding filtration. In particular

$$\begin{aligned} Z_0(T_0,T,K) &= \Pi\left[T_0,\frac{I(T)}{I(T_0)}-(1+K)^{T-T_0}\right] \\ &= \Pi\left[T_0,\frac{I(T)}{I(T_0)}\right]-\Pi\left[T_0,(1+K)^{T-T_0}\right]. \end{aligned} \tag{3.24}$$

The inflation linked leg of the ZCIIS is

$$\begin{aligned} \Pi\left[T_0,\frac{I(T)}{I(T_0)}\right] &= \frac{p_n(T_0,T)}{I(T_0)}\mathbb{E}^{T,n}_{T_0}[I(T)] \\ &= \frac{p_n(T_0,T)}{I(T_0)}\mathbb{E}^{T,n}_{T_0}\left[\frac{I(T)p_r(T,T)}{p_n(T,T)}\right]=p_r(T_0,T) \end{aligned}$$

since $p_r(T,T)=p_n(T,T)=1$ and

$$\frac{I(t)p_r(t,T)}{p_n(t,T)} \quad = \quad \frac{p_{IP}(t,T)}{p_n(t,T)}$$

is a $\mathbb{Q}^{T,n}$-martingale.

The fixed leg of the ZCIIS is

$$\Pi\left[T_0,(1+K)^{T-T_0}\right] \quad = \quad p_n(T_0,T)(1+K)^{T-T_0}.$$

Hence fair price of the payer ZCIIS becomes

$$Z_0(T_0,T,K) \quad = \quad p_r(T_0,T)-p_n(T_0,T)(1+K)^{T-T_0}.$$

This price is not based on any assumption about the interest rate behaviour or assets' dynamics, but only on a no-arbitrage argument. The result was first stated in [81]. Its next simple replicating argument counterpart was stated in [63].

To replicate the floating leg of the swap

(i) At time T_0 buy $\frac{1}{I(T_0)}$ IL bonds with maturity date T.

(ii) At time T the dollar value of $\frac{1}{I(T_0)}$ CPI units will be received, that is $\frac{I(T)}{I(T_0)}$.

(iii) The price at time T_0 of $\frac{1}{I(T_0)}$ IL bonds is $\frac{1}{I(T_0)} I(T_0) p_r(T_0, T) = p_r(T_0, T)$.

3.2.2 Year-on-Year Inflation Indexed Swaps

Contrary to the ZCIIS, the year-on-year IL swap's fair price is model independent. In this subsection, the year-on-year IL swap (YYIIS) is priced in the Lévy setting specified in Section 3.1.

A YYIIS has multiple payments dates. Let $Y_m^M(K)$ denote a payer YYIIS that starts at time T_m with payment dates at $T_{m+1}, T_{m+2}, \cdots, T_M$. For each period $[T_i, T_{i+1}]$ for $i = m, \cdots, M-1$ a fixed amount of

$$\tau_{i+1} K$$

is paid out at time T_{i+1}. For the same period a floating amount of

$$\tau_{i+1}[X_{i+1} - 1]$$

where

$$X_{i+1} = \frac{I(T_{i+1})}{I(T_i)}$$

is received at time T_{i+1}.

Let $Y_m^M(t, K)$ denote the price of a $Y_m^M(K)$ at time t where $t \leq T_m$, then

$$\begin{aligned} Y_m^M(t, K) &= \sum_{i=m}^{M-1} \Pi[t, \tau_{i+1}(X_{i+1} - 1)] - \sum_{i=m}^{M-1} \Pi[t, \tau_{i+1} K] \\ &= \sum_{i=m}^{M-1} \Pi[t, \tau_{i+1} X_{i+1}] - (K+1) \sum_{i=m}^{M-1} \tau_{i+1} p_n(t, T_{i+1}), \end{aligned} \tag{3.25}$$

by standard no-arbitrage pricing theory. Therefore the pricing of Y_m^M comes back to the computation of $\sum_{i=m}^{M-1} [t, \tau_{i+1} X_{i+1}]$, which is achieved through the forward swap rate. The forward swap rate of a

YYIIS is the value of swap rate for which the fair price of the swap is zero. Let $R_m^M(t)$ denote the forward swap rate for the swap $Y_m^M(K)$. By definition, $Y_m^M[t, R_m^M(t)] = 0$ and so $R_m^M(t)$ is given by:

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1} \Pi[t, \tau_{i+1}(X_{i+1} - 1)] - \sum_{i=m}^{M-1} \tau_{i+1} p_n(t, T_{i+1})}{\sum_{i=m}^{M-1} \tau_{i+1} p_n(t, T_{i+1})}. \tag{3.26}$$

Now the computation of the model-dependent expression $\sum_{i=m}^{M-1} \Pi[t, \tau_{i+1} X_{i+1}]$ will provide explicit formulas for both the swap price and the forward swap rate.

For $m = 1$ and $t < T_2$, the first term of the summation is

$$\begin{aligned} \Pi[t, \tau_2 X_2] &= p_n(t, T_2) \mathbb{E}_t^{T_2,n} \left[\tau_2 \frac{I(T_2)}{I(T_1)} \right] \\ &= p_n(t, T_2) \tau_2 \mathbb{E}_t^{T_2,n} \left[\frac{1}{I(T_1)} \mathbb{E}_{T_1}^{T_2,n} [I(T2)] \right] \\ &= p_n(t, T_2) \tau_2 \mathbb{E}_t^{T_2,n} \left[\frac{1}{I(T_1)} \mathbb{E}_{T_1}^{T_2,n} \left[\frac{I(T_2) p_r(T_2, T_2)}{p_n(T_2, T_2)} \right] \right] \\ &= p_n(t, T_2) \tau_2 \mathbb{E}_t^{T_2,n} \left[\frac{p_r(T_1, T_2)}{p_n(T_1, T_2)} \right]. \end{aligned} \tag{3.27}$$

Because the numéraire of the expectation in Equation (3.27) is $p_n(T_1, T_2)$, the $\mathbb{Q}^{T_1,n}$-forward measure is more appropriate for its valuation. The change of measure is done with the Bayes formula and the Radon-Nikodým derivative $Z_t^{T_2,n/T_1,n}$ that satisfies for every $t \in [0, T_2]$

$$Z_t^{T_2,n/T_1,n} = \left. \frac{dQ^{T_2,n}}{dQ^{T_1,n}} \right|_t = \frac{p_n(t, T_2)}{p_n(t, T_1)} \frac{p_n(0, T_1)}{p_n(0, T_2)}.$$

Hence

$$\begin{aligned} \mathbb{E}_t^{T_2,n} \left[\frac{p_r(T_1, T_2)}{p_n(T_1, T_2)} \right] &= \frac{\mathbb{E}_t^{T_1,n} \left[\frac{p_r(T_1,T_2)}{p_n(T_1,T_2)} Z_{T_1}^{T_2,n/T_1,n} \right]}{Z_t^{T_2,n/T_1,n}} \\ &= \mathbb{E}_t^{T_1,n} \left[\frac{p_r(T_1, T_2)}{p_n(T_1, T_2)} \frac{p_n(T_1, T_2)}{p_n(T_1, T_1)} \right] \frac{p_n(t, T_1)}{p_n(t, T_2)} \\ &= \frac{p_n(t, T_1)}{p_n(t, T_2)} \mathbb{E}_t^{T_1,n} [p_r(T_1, T_2)] \end{aligned} \tag{3.28}$$

The combination of Equations (3.27) and (3.28) yields

$$\Pi[t, \tau_2 X_2] = \tau_2 p_n(t, T_1) \mathbb{E}_t^{T_1,n} [p_r(T_1, T_2)]. \tag{3.29}$$

So far, no model assumption has been used. However, the expectation in Equation (3.29) is model dependent. Mercurio use a diffusion model to computed it in an environment without jumps [81] while Hinnerich [63] used the martingale approach in a jump diffusion setting. The latter approach will also be used in the Lévy setting.

Once more, the Bayes formula is used, but this time to change measure from $\mathbb{Q}^{T_1,n}$ to $\mathbb{Q}^{T_1,r}$. The expected value in equation (3.29) can thus be rewritten as

$$\mathbb{E}_t^{T_1,n}[p_r(T_1,T_2)] \quad = \quad \frac{\mathbb{E}_t^{T_1,r}\left[\frac{p_r(T_1,T_2)}{p_r(T_1,T_1)} Z_{T_1}^{T_1,n/T_1,r}\right]}{Z_t^{T_1,n/T_1,r}}, \tag{3.30}$$

where

$$Z_t^{T_1,n/T_1,r} \quad = \quad \left.\frac{dQ^{T_1,n}}{dQ^{T_1,r}}\right|_t = \frac{p_n(t,T_1)}{p_r(t,T_1)I(t)} \frac{p_r(0,T_1)I(0)}{p_n(0,T_1)}.$$

Under $\mathbb{Q}^n$ we have the following dynamics

$$\begin{aligned}
\frac{dI(t)}{I(t-)} &= [r_n(t) - r_r(t)]\,dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz); \\
\frac{dp_n(t,T_1)}{p_n(t-,T_1)} &= r_n(t)dt + \sigma^n(t,T_1)dW_t + \int_{\mathbb{R}} \delta^n(t,z,T_1)\tilde{\mu}(dt,dz); \\
\frac{dp_r(t,T_1)}{p_r(t-,T_1)} &= a^r(t,T)dt + \sigma^r(t,T_1)dW_t + \int_{\mathbb{R}} \delta^r(t,z,T_1)\tilde{\mu}(dt,dz)
\end{aligned}$$

Using Corollary 2.13 with $Z(t) = p_r(t,T_1)I(t)$

$$\begin{aligned}
\frac{dZ(t)}{Z(t-)} &= \Big[a^r(t,T) + r_n(t) - r_r(t) + \sigma^r(t,T_1)b^I(t) \\
&\quad + \int_{\mathbb{R}} \delta^r(t,z,T_1)c^I(t,z)\nu(dz)\Big]\,dt + \left[\sigma^r(t,T_1) + b^I(t)\right]dW_t \\
&\quad + \int_{\mathbb{R}} \left[\delta^r(t,z,T_1) + c^I(t,z) + \delta^r(t,z,T_1)c^I(t,z)\right]\tilde{\mu}(dt,dz)
\end{aligned}$$

By Corollary 2.9 with $Y(t) = \dfrac{1}{Z(t)}$

$$\frac{dY(t)}{Y(t)} \quad = \quad \alpha(t)dt - \beta(t)dW_t - \int_{\mathbb{R}} \frac{\gamma(t,z)}{1+\gamma(t,z)}\tilde{\mu}(dt,dz)$$

where

$$\begin{aligned}
\alpha(t) &= -a^r(t,T) - r_n(t) + r_r(t) - \sigma^r(t,T_1)b^I(t) \\
&\quad - \int_{\mathbb{R}} \delta^r(t,z,T_1)c^I(t,z)\nu(dz) + \beta^2(t) + \int_{|z|<R} \frac{\gamma^2(t,z)}{1+\gamma(t,z)}\nu(dz) \\
\beta(t) &= \sigma^r(t,T_1) + b^I(t) \\
\gamma(t,z) &= \delta^r(t,z,T_1) + c^I(t,z) + \delta^r(t,z,T_1)c^I(t,z)
\end{aligned}$$

Applying Corollary 2.13 to $X(t) = p_n(t,T_1)Y(t)$

$$
\begin{aligned}
\frac{dX(t)}{X(t-)} &= [r_n(t) + \alpha(t) - \sigma^n(t,T_1)\beta(t)]\,dt - \int_{\mathbb{R}} \delta^n(t,z,T_1)\frac{\gamma(t,z)}{1+\gamma(t,z)}\pi(dt,dz) \\
&\quad + [\sigma^n(t,T_1) - \beta(t)]\,dW_t \\
&\quad + \int_{\mathbb{R}} \left[\delta^n(t,z,T_1) - \frac{\gamma(t,z)}{1+\gamma(t,z)} - \delta^n(t,z,T_1)\frac{\gamma(t,z)}{1+\gamma(t,z)}\right]\tilde{\mu}(dt,dz) \\
&= [r_n(t) + \alpha(t) - \sigma^n(t,T_1)\beta(t)]\,dt - \int_{\mathbb{R}} \delta^n(t,z,T_1)\frac{\gamma(t,z)}{1+\gamma(t,z)}\pi(dt,dz) \\
&\quad + [\sigma^n(t,T_1) - \beta(t)]\,dW_t + \int_{\mathbb{R}} \frac{\delta^n(t,z,T_1) - \gamma(t,z)}{1+\gamma(t,z)}\tilde{\mu}(dt,dz)
\end{aligned}
$$

Since $\frac{dX(t)}{X(t-)} = \frac{dZ_t^{T_1,n/T_1,r}}{Z_{t-}^{T_1,n/T_1,r}}$, $Z_t^{T_1,n/T_1,r}$ is a martingale under $\mathbb{Q}^{T_1,r}$ and a change of measure does not change either the volatility or the jump component; the dynamics of $Z_t^{T_1,n/T_1,r}$ under $\mathbb{Q}^{T_1,r}$ are given by:

$$
\begin{aligned}
\frac{dZ_t^{T_1,n/T_1,r}}{Z_{t-}^{T_1,n/T_1,r}} &= \left[\sigma^n(t,T_1) - \sigma^r(t,T_1) - b^I(t)\right] dW^{T_1,r}(t) \\
&\quad + \int_{\mathbb{R}} \frac{\delta^n(t,z,T_1) - \left[\delta^r(t,z,T_1) + c^I(t,z) + \delta^r(t,z,T_1)c^I(t,z)\right]}{1 + \delta^r(t,z,T_1) + c^I(t,z) + \delta^r(t,z,T_1)c^I(t,z)}\tilde{\mu}^{T_1,r}(dt,dz)
\end{aligned}
$$

Since both $\frac{p_r(t,T_2)}{p_r(t,T_1)}$ and $Z_t^{T_1,n/T_1,r}$ are $\mathbb{Q}^{T_1,r}$-martingales, Theorem 2.23 and Equation (3.30)give that

$$
\mathbb{E}_t^{T_1,n}[p_r(T_1,T_2] = \frac{p_r(t,T_2)e^{C(t,T_1,T_2)}}{p_r(t,T_1)}
$$

where

$$
C(t,T_1,T_2) = \int_t^{T_1} \left\{ \left[\sigma_s^{n,1} - \sigma_s^{r,1} - b_s^I\right]\left[\sigma_s^{r,2} - \sigma_s^{r,1}\right] + \int_{\mathbb{R}} \Delta_s^{1,2}\pi^{T_1,r}(ds,dz) \right\} ds
$$

with the notation $\sigma_s^{i,j} = \sigma^i(s,T_j)$, $\delta_s^{i,j} = \delta^i(s,T_j)$, $b_s^I = b^I(s)$, $c_t^I = c^I(t,z)$ and

$$
\Delta_s^{1,2} = \frac{\delta_t^{r,2} - \delta_t^{r,1}}{1 + \delta_t^{r,1}} \frac{\delta_t^{n,1} - \left[\delta_t^{r,1} + c_t^I + \delta_t^{r,1}c_t^I\right]}{1 + \delta_t^{r,1} + c_t^I + \delta_t^{r,1}c_t^I}.
$$

Inserting this into Equation (3.29) gives that

$$
\Pi[t, X_{T_2}] = \tau_2 \frac{p_n(t,T_1)p_r(t,T_2)e^{C(t,T_1,T_2)}}{p_r(t,T_1)}
$$

Changing back to the general case with $1 = i$ and $2 = i+1$ we have that

$$
\Pi[t, X_{T_{i+1}}] = \tau_{i+1} \frac{p_n(t,T_i)p_r(t,T_{i+1})e^{C(t,T_i,T_{i+1})}}{p_r(t,T_i)}
$$

Hence the pricing Equation (3.25) is found to be

$$Y_m^M(t,K) = \sum_{i=1}^{M-1} \tau_{i+1} \frac{p_n(t,T_i)p_r(t,T_{i+1})e^{C(t,T_i,T_{i+1})}}{p_r(t,T_i)} - (K+1)\sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1}) \quad (3.31)$$

and the forward swap rate (3.26) is found to be

$$R_m^M(t) = \frac{\displaystyle\sum_{i=1}^{M-1} \tau_{i+1} \frac{p_n(t,T_i)p_r(t,T_{i+1})e^{C(t,T_i,T_{i+1})}}{p_r(t,T_i)} - \sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1})}{\displaystyle\sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1})}$$

For calibration purposes, it is more convenient to rewrite these formulas in function of the IL bond using the relation $p_{IP}(t,T) = p_r(t,T)$. The new formulas are

$$\begin{aligned} Y_m^M(t,K) &= \sum_{i=1}^{M-1} \tau_{i+1} \frac{p_n(t,T_i)p_{IP}(t,T_{i+1})e^{C(t,T_i,T_{i+1})}}{p_{IP}(t,T_i)} - (K+1)\sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1}) \\ R_m^M(t) &= \frac{\displaystyle\sum_{i=1}^{M-1} \tau_{i+1} \frac{p_n(t,T_i)p_{IP}(t,T_{i+1})e^{C(t,T_i,T_{i+1})}}{p_{IP}(t,T_i)} - \sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1})}{\displaystyle\sum_{i=1}^{M-1} \tau_{i+1} p_n(t,T_{i+1})} \end{aligned}$$

3.3 Inflation Linked Caplets/Floorlets

An inflation indexed (II) caplet (resp. floorlet) written at time t over the period $[T_{i-1}, T_i]$ (i.e. with maturity T_i) is a call (resp. put) on the inflation rate $X_i = \dfrac{I(T_i)}{I(T_{i-1})}$ implied by an CPI index. The payoff at time T_i of such an option with strike k is

$$N\tau_i \left[\omega(X_i - 1 - k)\right]^+ \quad (3.32)$$

where N is the contract nominal, τ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

The no-arbitrage price at time t of the i-th caplet is

$$\begin{aligned} CFlet_i(t,k) &= N\tau_i \mathbb{E}_{\mathbb{Q}} \left\{ \frac{B_n(t)}{B_n(T_i)} \left[\omega\left(X_i - 1 - k\right)\right]^+ \Big|_{\mathcal{F}_t} \right\} \\ &= N\tau_i p_n(t,T_i)\mathbb{E}_i \left\{ \left[\omega\left(X_i - K\right)\right]^+ \Big|_{\mathcal{F}_t} \right\}, \end{aligned} \quad (3.33)$$

where $K = k+1$ and $\mathbb{E}_i$ is the short form of $\mathbb{E}_{T_i}^{\mathbb{Q}^n}$, which is the conditional expectation with respect to the nominal risk neutral forward measure at time T_i.

The next sections investigate different paths for computing this price.

3.3.1 Lognormally Distributed CPI

Throughout this section the next assumption is supposed true.

Assumption 7. *The CPI is only driven by a Brownian motion without any jumps.*

The previous assumption leads to the Jarrow-Yildrim framework where the CPI is lognormal under $\mathbb{Q}^n$. The pricing formula given by Equation (3.33) becomes a simple Black and Scholes (BS) formula. Just as with standard BS pricing the next theorem will be useful.

Theorem 3.9. *Let X be a random variable that is lognormally distributed, and denote by M and V the mean and standard deviation of $Y = \ln(X)$. Then*

$$\mathbb{E}\left\{[\omega(X-K)]^+\right\} \;=\; \omega e^{M+\frac{1}{2}V^2}\Phi\left(\omega\frac{M-\ln K+V^2}{V}\right)-\omega K\Phi\left(\omega\frac{M-\ln K}{V}\right),$$

for each $K > 0$, $\omega \in \{-1, 1\}$, where $\mathbb{E}$ denotes expectation with respect to X's distribution and Φ denotes the cumulative standard normal distribution function.

Under Assumption 7 (i.e. $c^I(t,z) = 0$), and by Itô's formula

$$d\ln I(t) \;=\; \left[r_n(s)-r_r(s)-\frac{1}{2}(b^I(t))^2\right]dt+b^I(t)dWt$$

and the CPI is of the form

$$I(T) \;=\; I(t)\exp\left\{\int_t^T\left[r_n(s)-r_r(s)-\frac{1}{2}(b^I(s))^2\right]ds+\int_t^T b^I(s)dW_s\right\},$$

where $t < T$. Therefore, $\ln\left.\frac{I(T)}{I(t)}\right|_{\mathcal{F}_t}$ and $\ln\left.\frac{I(T_i)}{I(T_{i-1})}\right|_{\mathcal{F}_t}$ are lognormal under $Q^{T_i,n}$. From Theorem 3.9, if X_i is a lognormal random variable with mean $\mathbb{E}(X) = m$ and standard deviation of the logarithm distribution $Std[ln(X)] = v$, then

$$\mathbb{E}\left\{[\omega(X_i-K)]^+\right\} \;=\; \omega m\Phi\left(\omega\frac{\ln\frac{m}{K}+\frac{1}{2}v^2}{v}\right)-\omega K\Phi\left(\omega\frac{\ln\frac{m}{K}-\frac{1}{2}v^2}{v}\right), \tag{3.34}$$

with $K = 1 + k$. The conditional expectation of $\frac{I(T_i)}{I(T_{i-1})}$ is obtained from Equation (3.31)

$$\mathbb{E}_t^{T_i,n}[X_i] \;=\; \frac{p_n(t,T_{i-1})}{p_n(t,T_i)}\frac{p_r(t,T_i)}{p_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}$$

The variable v is given by the following corollary.

Corollary 3.10. *The variance of the logarithm of the ratio $\frac{I(T_i)}{I(T_{i-1})}$ under the (nominal) risk neutral measure is given by*

$$Var_t^{T_i,n}[\ln X_i] \;=\; V^2(t,T_{i-1},T_i)$$

where

$$
\begin{aligned}
V^2(t,T_{i-1},T_i) &= \frac{\sigma_n^2}{2a_n^3}\left[1-e^{-a_n(T_i-T_{i-1})}\right]^2\left[1-e^{-2a_n(T_{i-1}-t)}\right]+(b^I)^2(T_i-T_{i-1})\\
&+\frac{\sigma_r^2}{2a_r^3}\left[1-e^{-a_r(T_i-T_{i-1})}\right]^2\left[1-e^{-2a_r(T_{i-1}-t)}\right]\\
&-2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_na_r(a_n+a_r)}\left[1-e^{-a_n(T_i-T_{i-1})}\right]\left[1-e^{-a_r(T_i-T_{i-1})}\right]\left[1-e^{-(a_n+a_r)(T_{i-1}-t)}\right]\\
&+\frac{\sigma_n^2}{a_n^2}\left[T_i-T_{i-1}+\frac{2}{a_n}e^{-a_n(T_i-T_{i-1})}-\frac{1}{2a_n}e^{-2a_n(T_i-T_{i-1})}-\frac{3}{2a_n}\right]\\
&+\frac{\sigma_r^2}{a_r^2}\left[T_i-T_{i-1}+\frac{2}{a_r}e^{-a_r(T_i-T_{i-1})}-\frac{1}{2a_r}e^{-2a_r(T_i-T_{i-1})}-\frac{3}{2a_r}\right]\\
&-2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_na_r}\left[T_i-T_{i-1}-\frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}-\frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}\right.\\
&\left.-\frac{1-e^{-(a_n+a_r)(T_i-T_{i-1})}}{a_n+a_r}\right]+2\rho_{n,I}\frac{\sigma_n\sigma_I}{a_n}\left[T_i-T_{i-1}-\frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}\right]\\
&-2\rho_{r,I}\frac{\sigma_r\sigma_I}{a_r}\left[T_i-T_{i-1}-\frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}\right]
\end{aligned}
$$

Proof. See [81, 23]

■

Hence, by Equation (3.34)

$$
\begin{aligned}
\mathbb{E}_t^{T_{i+1},n}[X_{i+1}] &= \omega m\Phi\left(\omega\frac{\ln\frac{m}{K}+\frac{1}{2}v^2}{v}\right)-\omega K\Phi\left(\omega\frac{\ln\frac{m}{K}-\frac{1}{2}v^2}{v}\right)\\
&= \omega\frac{p_n(t,T_i)}{p_n(t,T_{i+1})}\frac{p_r(t,T_{i+1})}{p_r(t,T_i)}e^{C(t,T_i,T_{i+1})}\Phi\left[\omega\frac{\ln\left(\frac{p_n(t,T_i)}{Kp_n(t,T_{i+1})}\frac{p_r(t,T_{i+1})}{p_r(t,T_i)}e^{C(t,T_i,T_{i+1})}\right)+\frac{1}{2}v^2}{v}\right]\\
&-\omega K\Phi\left[\omega\frac{\ln\left(\frac{p_n(t,T_i)}{Kp_n(t,T_{i+1})}\frac{p_r(t,T_{i+1})}{p_r(t,T_i)}e^{C(t,T_i,T_{i+1})}\right)-\frac{1}{2}v^2}{v}\right]\\
&= \omega\frac{p_n(t,T_i)}{p_n(t,T_{i+1})}\frac{p_r(t,T_{i+1})}{p_r(t,T_i)}e^{C(t,T_i,T_{i+1})}\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_r(t,T_{i+1})}{Kp_n(t,T_{i+1})p_r(t,T_i)}+C(t,T_i,T_{i+1})+\frac{1}{2}v^2}{v}\right]\\
&-\omega K\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_r(t,T_{i+1})}{Kp_n(t,T_{i+1})p_r(t,T_i)}+C(t,T_i,T_{i+1})-\frac{1}{2}v^2}{v}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
CFlet(T_{i+1}) &= \tau_{i+1}p_n(t,T_{i+1})\left\{\omega\frac{p_n(t,T_i)}{p_n(t,T_{i+1})}\frac{p_r(t,T_{i+1})}{p_r(t,T_i)}e^{C(t,T_i,T_{i+1})}\right.\\
&\cdot\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_r(t,T_{i+1})}{Kp_n(t,T_{i+1})p_r(t,T_i)}+C(t,T_i,T_{i+1})+\frac{1}{2}v^2}{v}\right]\\
&\left.-\omega K\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_r(t,T_{i+1})}{Kp_n(t,T_{i+1})p_r(t,T_i)}+C(t,T_i,T_{i+1})-\frac{1}{2}v^2}{v}\right]\right\}.
\end{aligned}
$$

Using the equality $p_{IP}(t,T) = I(t)P_r(t,T)$, the caplet/floorlet can be rewritten has

$$\begin{aligned} CFlet(T_{i+1}) &= \tau_{i+1}p_n(t,T_{i+1})\left\{\omega\frac{p_n(t,T_i)}{p_n(t,T_{i+1})}\frac{p_{IP}(t,T_{i+1})}{p_{IP}(t,T_i)}e^{C(t,T_i,T_{i+1})}\right. \\ &\cdot\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_{IP}(t,T_{i+1})}{Kp_n(t,T_{i+1})p_{IP}(t,T_i)}+C(t,T_i,T_{i+1})+\frac{1}{2}v^2}{v}\right] \\ &\left.-\omega K\Phi\left[\omega\frac{\ln\frac{p_n(t,T_i)p_{IP}(t,T_{i+1})}{Kp_n(t,T_{i+1})p_{IP}(t,T_i)}+C(t,T_i,T_{i+1})-\frac{1}{2}v^2}{v}\right]\right\}. \end{aligned}$$

3.3.2 Pricing with the Bilateral Laplace Transform

To price caplets and floorlets under the assumption of exponential Lévy distribution, the methodology proposed by Eberlein and Kluge in [44] will be followed. First of all, recall that the forward rate dynamics in Assumption (3) can be rewritten as

$$df_i(t,T) = \alpha^i(t,T)dt + \sigma^i(t,T)dL_t \quad i = r,n \quad (0 \le t \le T), \tag{3.35}$$

with some common integrability conditions. A cap (resp. floor) is a series of call (resp. put) options on subsequent variable rates. These single options are called caplets (resp. floorlets). Each caplet (resp. floorlet) is equivalent to a put (resp. call) option on the inflation rate. Thus, deriving suitable formulas for calls and puts on the inflation rate immediately gives formulas for caps and floors.

As described previously, the discounted bond price process $p_i(\cdot,T)$ for $i = n, r$ are martingales with respect to the measure $\mathbb{Q}$ and the corresponding filtration for each T. However, this is not the case for the inflation process and consequently the inflation rate. This difficulty can be avoided through a change of probability measure. Moreover, because the "unwanted" term is the real interest rate that can be evaluated from market data, the calibration process will still be possible. In this new probability measure, which will be denoted by $\mathbb{Q}^{j,r}$, the pricing of a caplet can be achieved by taking the conditional expectation of the discounted payoff. The time-t value of a caplet/floorlet with strike k over the period $[T_{j-1},T_j]$ is given by

$$CFlet_j(t;k) = N\tau_j p_n(t,T_j)\mathbb{E}_j\left\{\left[\omega\left(X_j-K\right)\right]^+\Big|_{\mathcal{F}_t}\right\} \quad (t \le T_j)$$

where $K = k+1$. For simplification, henceforth $w = 1$, $N = 1$ and $\tau_j = 1$. The caplet fair price can be rewritten as

$$CFlet_j(t;K) = p_n(t,T_j)\mathbb{E}_j\left[\left(X_j-K\right)^+\Big|_{\mathcal{F}_t}\right] \quad (t \le T_j).$$

Having a closer look at the inflation's dynamics, through Corollary 2.10 and Equation (3.2)

$$\begin{aligned}
\frac{dI(t)}{I(t-)} &= [r_n(t) - r_r(t)]\,dt + b^I(t)dW_t + \int_{\mathbb{R}} c^I(t,z)\tilde{\mu}(dt,dz);\\
d\ln I_t &= \left[r_n(t) - r_r(t) - \frac{1}{2}(b^I(t))^2\right]dt + \int_{|z|<R}\left\{\ln[1+c^I(t,z)] - c^I(t,z)\right\}\pi(dt,dz)\\
&\quad + b^I(t)dW_t + \int_{\mathbb{R}} \ln[1+c^I(t,z)]\bar{\mu}(dt,dz)\\
\ln\frac{I_T}{I_t} &= \int_{s=t}^{s=T}\left\{\left[r_n(s) - r_r(s) - \frac{1}{2}(b^I(s))^2\right]ds + \int_{|z|<R}\left\{\ln[1+c^I(s,z)] - c^I(s,z)\right\}\pi(ds,dz)\right\}\\
&\quad + \int_{s=t}^{s=T}\left\{b^I(s)dW_s + \int_{\mathbb{R}}\ln[1+c^I(s,z)]\bar{\mu}(ds,dz)\right\}.
\end{aligned}$$

Under Assumption 5, for the inflation process to be a $\mathbb{Q}$-martingale, it has to be multiplied by[1] $\exp\left[\int_0^t r_r(s)ds\right]$ which is deterministic. The caplet price is now

$$CFlet_j(t;K) = p_n(t,T_j)f_j\mathbb{E}_{j,r}\left[\left(X_j^r - Kf_j^{-1}\right)^+\Big|_{\mathcal{F}_t}\right] \quad (t \le T_j),$$

where

$$\begin{aligned}
X_j^r &= X_j f_j^{-1};\\
f_j &= \exp\left[-\int_{T_{j-1}}^{T_j} r_r(s)ds\right]
\end{aligned}$$

and the expectation is under $\mathbb{Q}^{j,r}$ and not the initial nominal risk forward probability measure.

A straightforward approach is to derive the joint (conditional) distribution of the random variables $p_n(t,T)$ and X_j. Although this can easily be done analytically [48], the numerical evaluation of the resulting expression is extremely time consuming. Instead, the change-of-numeraire technique is used here to circumvent the calculation of the joint probability law, that is as previously mentioned, calculations are not conducted in the spot martingale measure $\mathbb{Q}$, but in the "modified" forward martingale measure for the settlement day T_{i-1} $\mathbb{Q}^{j,r}$ (see Geman, El Karoui, and Rochet (1995) for details). More precisely, the measure $\mathbb{Q}^{j,r}$, equivalent to $\mathbb{Q}$, is defined by its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{j,r}}{d\mathbb{Q}} = \frac{1}{B_n(t)X_j\exp\left[\int_{T_{j-1}}^{T_j} r_r(s)ds\right]}.$$

The previous expression can be rewritten as

$$\frac{d\mathbb{Q}_t}{d\mathbb{Q}} = \exp\left[-\int_0^t A^{IP}(s,t)ds + \int_0^t \Sigma^{IP}(s,t)dL_s^{IP}\right]$$

[1] Eventually scaled by a multiplicative constant.

and, when restricted to the σ-field $\mathcal{F}_s$,

$$\left.\frac{d\mathbb{Q}_t}{d\mathbb{Q}}\right|_{\mathcal{F}_s} = \exp\left[-\int_0^s A^{IP}(u,t)du + \int_0^s \Sigma^{IP}(u,t)dL_u^{IP}\right].$$

Equation (3.22) leads to

$$CFlet_j(t;K) = p_n(t,T_j)\mathbb{E}_j\left\{\left.[D\exp(Y)-K]^+\right|_{\mathcal{F}_t}\right\} \quad (t \le T_j).$$

where

$$D = \frac{p_n(t,T_i)}{p_n(t,T_{i-1})}\exp\left\{\int_t^{T_{i-1}}\left[A^{IP}(s,T_{i-1}) - A^{IP}(s,T_i)\right]ds - \int_{T_{j-1}}^{T_j} r_r(s)ds\right\}$$

is deterministic and

$$Y = \int_0^{T_{i-1}}\left[\Sigma^{IP}(s,T_i) - \Sigma^{IP}(s,T_{i-1})\right]dL_s$$

is $\mathcal{F}_t$-measurable. Notice that the expectation is now in the forward risk neutral probability measure. To calculate the option price, the distribution of Y under the measure $\mathbb{Q}_t$ is required; and it can be estimated. If this distribution is represented by the Lebesgue-density φ in $\mathbb{R}$, then

$$CFlet_j(t;K) = p_n(t,T_j)\int_{\mathbb{R}}(De^y - K)^+\varphi(y)dy.$$

To get a numerical estimate of the caplet price, either the Laplace transform method or the Fourier transform method can be used. The Laplace technique was developed in [94] and used to derive exact pricing formulas for pricing caps, floors and swaptions in [44]. The similar and rather simpler technique was previously proposed by Carr and Madan [26] using Fourier transforms. First, the option price is expressed as a convolution. The Laplace transform of this convolution equals the product of the Laplace transforms of the convolution factors. These factors are easy to calculate in this case. Then, to get the price of the option, an inverse Laplace transformation will be performed.

Theorem 3.11. *Let M_t^Y denote the moment generating function of the random variable Y with respect to the measure $\mathbb{Q}_t$. If R is chosen $R < -1$ such that $M_t^Y(-R) < \infty$, then*

$$CFlet_j(t;K) = \frac{1}{2\pi}Kp_n(t,T_j)e^{R\xi}\int_{-\infty}^{\infty}\frac{e^{iu\xi}M_t^Y(-R-iu)}{(R+iu)(R+1+iu)}du$$

with

$$\xi = \ln\frac{p_n(t,T_{i-1})}{p_n(t,T_i)} - \int_t^{T_{i-1}}\left\{A^{IP}(s,T_{i-1}) - A^{IP}(s,T_i)\right\}ds + \ln K.$$

Proof. See [44] Theorem 12. ∎

Theorem 3.12. *Considering $R > 0$ such that $M_t^Y(-R) < \infty$, the price of a floorlet with strike K and exercise period $[T_{i-1}, T_i]$*

$$CFlet_j(t;K)|_{\omega=-1} = \frac{1}{2\pi} K p_n(t,T_j) e^{R\xi} \int_{-\infty}^{\infty} \frac{e^{iu\xi} M_t^Y(-R-iu)}{(R+iu)(R+1+iu)} du$$

with

$$\xi = \ln \frac{p_n(t,T_{i-1})}{p_n(t,T_i)} - \int_t^{T_{i-1}} \left\{ A^{IP}(s,T_{i-1}) - A^{IP}(s,T_i) \right\} ds + \ln K.$$

Proof. See [44] Corollary 14. ∎

The formulas to price the caplet and floorlet are similar except for the values for R. The accuracy of the numerically estimated security price relies on the right choice of R, which has already been discussed in Section 2.5.

3.4 Conclusion

This chapter has presented a non-trivial generalisation of the work by Hinnerich on pricing inflation linked securities. It started with the extension of the HJM framework to Lévy processes, with the underlying proof of the foreign exchange analogy. Afterwards, some inflation linked derivative pricing formulas were derived in the new framework. Some calibration to market data is presented in Chapter 6, where both South African and American market data are used. These results reinforce the fact that Lévy distributions are more appropriate than the conventional normal (or log normal) distribution, providing an improved accuracy and a more straightforward intuition when building and tuning models. However, the main issue encountered was the lack of data necessary for the calibration. This prevented the calibration process to be completed for some of the IL securities; but results of this calibration will be provided in upcoming work.

Chapter 4

Stochastic Monetary Economy Models

Virtually all asset pricing models are special cases of the fundamental equation [53]

$$P_t = \mathbb{E}_t[m_{t+1}(P_{t+1} + D_{t+1})], \tag{4.1}$$

where P_t is the value of an asset at time t, D_{t+1} is the amount of any dividends, interest or other cashflows received at time $t+1$ and m_{t+1} is the stochastic discount factor (SDF) between time t and time $t+1$. Equation (4.1) implies that the price process is a martingale under an appropriate measure.

If m_{t+1} is a strictly positive random variable, equation (4.1) becomes equivalent to the no-arbitrage principle, which states that all portfolios of assets with non-negative payoffs and positive probability of positive payoffs, must have positive prices. While the no-arbitrage principle places restrictions on m_{t+1}, other work explores the implications of equilibrium models for the SDF based on the investor' s utility optimization. A typical consumer/investor's utility optimization involves the Bellman equation:

$$J(W_t, s_t) \equiv \max \mathbb{E}_t \left[U(C_t, \cdot) + J(W_{t+1}, s_{t+1}) \right],$$

where $U(C_t, \cdot)$ is the utility of consumption expenditures at time t, and $J(\cdot, \cdot)$ is the indirect utility of wealth [53].

In the case that an asset pays dividends on a continuous basis, the pricing formula is given by

[66, 68]:

$$M_t \quad = \quad \frac{1}{\pi_t}\mathbb{E}_t\left[\pi_T S_T + \int_t^T \pi_s D_s ds\right] \qquad 0 \leq t \leq T,$$

where $(M_t)_{t\geq 0}$ is a martingale, $(\pi_t)_{t\geq 0}$ is the pricing kernel process, $(S_t)_{t\geq 0}$ is the value of the dividend-paying asset, and $(D_t)_{t\geq 0}$ is the dividend process.

The stochastic monetary economy models built by Hughston and Macrina assume a positive nominal interest rate that was advocated for by Flesaker and Hughston (FH) [54]. Flesaker and Hughston were among the first to propose an entirely new approach to interest-rate modelling resulting in concrete models that are not part of the short-rate world. Instead of modelling instantaneous forward rates they model pricing kernels (also known as state-price densities or pricing operators). Assuming N_t is the conventional numéraire, the corresponding pricing kernel is given by $\pi_t = \dfrac{\rho_t}{N_t}$ in the real world probability measure, where $(\rho_t)_{t\geq 0}$ denotes the Radon-Nikodym density martingale transforming the real world measure into the risk neutral measure. The latter equation implies that under the real probability measure the asset price process *multiplied with the pricing kernel process* is a martingale. The process $(\pi_t)_{t\geq 0}$ is a decreasing and positive supermartingale (i.e. $\pi_t \geq \pi_{t+h}$ with $h > 0$) thus ensures interest rate positivity.

Proof. Consider a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{P}$ is the real world measure with equivalent risk neutral measure $\mathbb{Q}$.

With the standard numéraire approach the arbitrage-free price of a European contingent claim, $(Y_t)_{t\geq 0}$, paying Y_T at its maturity T is given by:

$$Y_t \quad = \quad B_t\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{Y_T}{B_T}\right|_{\mathcal{F}_t}\right], \tag{4.2}$$

where the numéraire is the money market account defined by:

$$B_t \quad = \quad \exp\left[\int_0^t r_s ds\right],$$

with r_s denoting the short rate at time s.

The absence of arbitrage in a financial market is equivalent to the existence of a pricing kernel $(\pi_t)_{t\geq 0}$. In fact, Equation (4.2) can be expressed in terms of a pricing kernel under the real world measure as shown in the next paragraphs.

Let $(\rho_t)_{t\geq 0}$ denote the Radon-Nikodym density martingale transforming the real world measure into the risk neutral measure, i.e.

$$\rho_t \quad = \quad \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Application of Bayes' formula shows that:

$$\begin{aligned} Y_t &= B_t \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{Y_T}{B_T}\right|_{\mathcal{F}_t}\right] \\ &= B_t \frac{\mathbb{E}^{\mathbb{P}}\left[\left.\frac{Y_T}{B_T}\rho_T\right|_{\mathcal{F}_t}\right]}{\rho_t} \\ &:= \frac{\mathbb{E}^{\mathbb{P}}\left[\pi_T Y_T|_{\mathcal{F}_t}\right]}{\pi_t}, \end{aligned}$$

where the pricing kernel is defined to be of form

$$\pi_t = \frac{\rho_t}{B_t} = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \exp\left[-\int_0^t r_s ds\right].$$

■

The IL framework proposed by Hughston and Macrina is based on the assumption that inflation is a purely monetary phenomenon. Thus the influence of fluctuations in wages, supply and demand, foreign exchange and employment, etc. on inflation is not treated directly, but is rather reflected in the change of the rates of consumption and money supply, and the liquidity benefit of money supply. In a discrete time[1] model, let the nominal money supply, the aggregate consumption and the nominal liquidity benefit be denoted respectively by $(\{M_i\}_{i\geq 0})$, $(\{k_i\}_{i\geq 1})$ and $(\{\lambda_i\}_{i\geq 0})$. At time t_i, the real benefit (in units of goods and services) provided by the money supply is

$$l_i = \frac{\lambda_i M_i}{C_i} \quad \text{for } i \geq 0.$$

Given that

$$\begin{aligned} J &= \mathbb{E}\left[\sum_{n=0}^{N} e^{-rt_n} U(k_n, l_n)\right], \\ W &= \mathbb{E}\left[\sum_{n=0}^{N} \pi_n (C_n k_n + \lambda_n M_n)\right], \end{aligned}$$

where $U(\cdot,\cdot)$ is a bivariate utility function. Two examples of utility functions are considered below: the log-separable utility function and the separable power utility function.

Note that the formulas obtained are not directly functions of any IL derivative's price; therefore this novel framework could be a solution to pricing IL products despite the fact that they are illiquid. Sections 4.1 (resp. 4.2) studies the performances of this framework when the macroeconomic factors are Lévy (resp. exponential Lévy) processes.

[1] The formulas in continuous time have also been derived and are similar to those obtained in discrete time.

4.1 Lévy process distribution

Given the high flexibility of Lévy distributions [7], it is reasonable to assume that the nominal money supply, the aggregate consumption and the nominal liquidity benefit can be reproduced using Lévy processes (Assumption 8). Under the previous assumption, this section deduces the dynamics of the CPI, the pricing kernel and IL securities.

The next assumption holds throughout this section.

Assumption 8. *Under the objective probability measure $\mathbb{P}$, the dynamics of $(M_t)_{t\geq 0}$, $(k_t)_{t\geq 0}$ and $(\lambda_t)_{t\geq 0}$ are given by:*

$$\begin{aligned} dM_t &= \alpha_M(t)dt + \beta_M(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_M(t,z)\bar{\mu}(dt,dz); \\ dk_t &= \alpha_k(t)dt + \beta_k(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_k(t,z)\bar{\mu}(dt,dz); \\ d\lambda_t &= \alpha_\lambda(t)dt + \beta_\lambda(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_\lambda(t,z)\bar{\mu}(dt,dz), \end{aligned}$$

with

$$\bar{\mu}(dt,dz) = \begin{cases} (\mu-\pi)(dt,dz), & |z| < R \\ \mu(dt,dz), & |z| \geq R \end{cases}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, $a_i(t)$, $b_i(t)$ and $c_i(t)$ are adapted processes with

$$\int_0^t |\alpha_i(s)|ds < \infty, \int_0^t |\beta_i(s)|^2 ds < \infty,$$

for all finite t; $\gamma_i(t,z) : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a real valued function satisfying

$$\int_0^t \int_{\mathbb{R}} |\gamma_i(s,z)|^2 \pi(ds,dz) < \infty,$$

for finite t. These conditions guarantee integrability of the coefficients and are satisfied if the coefficients are bounded for t from a bounded set and $\pi([0,t]\times\mathbb{R}) < \infty$ for finite t.

The real world Brownian motion $W^{\mathbb{P}}$ will be denoted by W when there is no ambiguity.

Section 4.1.1 (resp. 4.1.2) further assumes that the agent utility function is a log-separable (resp. separable power) utility function; then computes the IL pricing formulas and explicit formulas for the CPI and pricing kernel.

4.1.1 Log-separable utility function

Given two non-negative constants A and B, a log-separable utility function is of the form

$$U(x,y) = A\ln(x) + B\ln(y).$$

In the current pricing framework [68], the pricing kernel, the CPI and the no-arbitrage value of a contingent claim H_t, are respectively

$$\begin{aligned}
C_n &= \frac{A}{B}\frac{\lambda_n M_n}{k_n};\\
\pi_n &= \frac{Be^{-rt_n}}{\kappa\lambda_n M_n};\\
H_0 &= \lambda_0 M_0 e^{-rt_j}\mathbb{E}\left[\frac{H_j}{\lambda_j M_j}\right]
\end{aligned}$$

where κ is an introduced Lagrange multiplier.

The following propositions compute the dynamics of the CPI and pricing kernel, and an explicit formula for the value of the contingent.

Proposition 4.1. *The dynamics of the CPI and pricing kernel are given by*

$$\begin{aligned}
\frac{dC_t}{A/B} &= \alpha_C(t)dt + \beta_C(t)dW_t + \int_{\mathbb{R}}\gamma_C(t,z)\bar{\mu}(dt,dz);\\
\frac{d\pi_t}{B/\kappa} &= \alpha_\pi(t)dt + \beta_\pi(t)dW_t + \int_{\mathbb{R}}\gamma_\pi(t,z)\bar{\mu}(dt,dz),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_C(t) &= \frac{1}{k_t}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\right]\\
&\quad +\alpha_Y(t)\lambda_t M_t - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{\beta_k(t)}{k_t^2}\\
&\quad +\int_{\mathbb{R}}\left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\gamma_Y(t,z)\nu(dz);\\
\beta_C(t) &= \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{1}{k_t} - \beta_k(t)\frac{\lambda_t M_t}{k_t^2};\\
\gamma_C(t,z) &= \gamma_Y(t,z)\lambda_{t-}M_{t-} + \left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{1}{k_{t-}}\\
&\quad +\left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right])\gamma_Y(t,z);
\end{aligned}$$

$$\begin{aligned}
\alpha_\pi(t) &= \left\{-\frac{r}{\lambda_t M_t} - \frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left(\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right)^2\frac{1}{\lambda_t M_t}\right]\right.\\
&\quad \left. +\int_{|z|<R}\left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}} + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz)\right\}e^{-rt};\\
\beta_\pi(t) &= -\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{e^{-rt}}{\lambda_t^2 M_t^2};\\
\gamma_\pi(t,z) &= \left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}\right]e^{-rt};
\end{aligned}$$

$$
\begin{aligned}
\alpha_Y(t) &= -\frac{1}{k_t^2}\left[\alpha_k(t) - \beta_k^2(t)\frac{1}{k_t}\right] + \int_{|z|<R}\left[\frac{1}{k_{t-}+\gamma_k(t,z)} - \frac{1}{k_{t-}} + \frac{1}{k_{t-}^2}\gamma_k(t,z)\right]\nu(dz);\\
\gamma_Y(t,z) &= \frac{1}{k_{t-}+\gamma_k(t,z)} - \frac{1}{k_{t-}};\\
\alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\
\gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}
$$

Proof. Applying Corollary 2.12 with $Z_t = \lambda_t M_t$

$$
\begin{aligned}
dZ_t &= \left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\right]dt + \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]dW_t\\
&\quad + \int_{\mathbb{R}}\left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\bar{\mu}(dt,dz).
\end{aligned}
$$

Then applying Corollary 2.14

$$
\frac{dC_t}{A/B} = \alpha_C(t)dt + \beta_C(t)dW_t + \int_{\mathbb{R}}\gamma_C(t,z)\bar{\mu}(dt,dz),
$$

where

$$
\begin{aligned}
\alpha_C(t) &= \frac{1}{k_t}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\right]\\
&\quad +\alpha_Y(t)\lambda_t M_t - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{\beta_k(t)}{k_t^2}\\
&\quad + \int_{\mathbb{R}}\left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\gamma_Y(t,z)\nu(dz);\\
\beta_C(t) &= \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{1}{k_t} - \beta_k(t)\frac{\lambda_t M_t}{k_t^2};\\
\gamma_C(t,z) &= \gamma_Y(t,z)\lambda_{t-}M_{t-} + \left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{1}{k_{t-}}\\
&\quad + \left[\gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z)\right]\gamma_Y(t,z);\\
\alpha_Y(t) &= -\frac{1}{k_t^2}\left[\alpha_k(t) - \beta_k^2(t)\frac{1}{k_t}\right] + \int_{|z|<R}\left[\frac{1}{k_{t-}+\gamma_k(t,z)} - \frac{1}{k_{t-}} + \frac{1}{k_{t-}^2}\gamma_k(t,z)\right]\nu(dz);\\
\gamma_Y(t,z) &= \frac{1}{k_{t-}+\gamma_k(t,z)} - \frac{1}{k_{t-}}.
\end{aligned}
$$

The deterministic process defined by $X_t = e^{-rt}$, i.e. $\ln X_t = -rt$ has dynamics given by

$$
dX_t = -re^{-rt}dt.
$$

Again applying Corollary 2.14

$$
\begin{aligned}
\frac{d\pi_t}{B/\kappa} &= \left[\frac{-re^{-rt}}{\lambda_t M_t} + \alpha_X(t)e^{-rt}\right]dt - \beta_2(t)\frac{e^{-rt}}{\lambda_t^2 M_t^2}dW_t + \int_{\mathbb{R}}\gamma_X(t,z)e^{-rt}\bar{\mu}(dt,dz)\\
&= \left[-\frac{r}{\lambda_t M_t} + \alpha_X(t)\right]e^{-rt}dt - \beta_2(t)\frac{e^{-rt}}{\lambda_t^2 M_t^2}dW_t + \int_{\mathbb{R}}\gamma_X(t,z)e^{-rt}\bar{\mu}(dt,dz),
\end{aligned}
$$

where

$$\begin{aligned}
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \beta_2^2(t)\frac{1}{\lambda_t M_t}\right] \\
&\quad + \int_{|z|<R}\left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}} + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz); \\
\gamma_X(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}; \\
\alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\beta_2(t) &= \beta_\lambda(t)M_t + \beta_M(t)\lambda_t; \\
\gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

■

Proposition 4.2. *The fair price at $t = 0$ of a contingent claim H_t is*

$$\begin{aligned}
H_0 &= e^{-rt}\left[1 + \lambda_0 M_0 \int_0^t \alpha_X(s)ds\right]\mathbb{E}[H_t] + \lambda_0 M_0 e^{-rt}\mathbb{E}\left[H_t \int_0^t \beta_X(s)dW_s\right] \\
&\quad + \lambda_0 M_0 e^{-rt}\mathbb{E}\left[H_t \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right],
\end{aligned}$$

where

$$\begin{aligned}
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_Y(t) - (\beta_\lambda(t)M_t + \beta_M(t)\lambda_t)^2\frac{1}{\lambda_t M_t}\right] \\
&\quad + \int_{|z|<R}\left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}} + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_Y(t,z)\right]\nu(dz); \\
\beta_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]; \\
\gamma_X(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}; \\
\alpha_Y(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_Y(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

Proof. Using the dynamics of Z_t (see previous proof) in Corollary 2.8 with $X_t = \frac{1}{\lambda_t M_t}$

$$dX_t = \alpha_X(t)dt + \beta_X(t)dW_t + \int_{\mathbb{R}}\gamma_X(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_Y(t) - (\beta_\lambda(t)M_t + \beta_M(t)\lambda_t)^2 \frac{1}{\lambda_t M_t}\right] \\
&\quad + \int_{|z|<R}\left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}} + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_Y(t,z)\right]\nu(dz); \\
\beta_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]; \\
\gamma_X(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}; \\
\alpha_Y(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_Y(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

Hence

$$X_t = X_0 + \int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)$$

and

$$\mathbb{E}[X_tH_t] = \mathbb{E}\left[\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right)H_t\right] + \mathbb{E}[X_0H_t].$$

Therefore

$$H_0 = e^{-rt}\mathbb{E}[H_t] + \lambda_0 M_0 e^{-rt}\mathbb{E}\left[H_t\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right)\right].$$

■

If, additionally, the pricing kernel and the contingent claim are assumed to be independent, then the expression of the fair price can easily be compared to the standard no-arbitrage pricing formula as shown in the next proposition.

Proposition 4.3. *If a further assumption is made that* $\frac{1}{\lambda_t M_t}$ *and* H_t *are independent (i.e. the pricing kernel is independent of the contingent claim), the fair price at* $t = 0$ *of the contingent claim becomes*

$$H_0 = e^{-rt}\left[1 + \lambda_0 M_0\left(\int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right)\right]\mathbb{E}[H_t].$$

where

$$
\begin{aligned}
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_Y(t) - (\beta_\lambda(t)M_t + \beta_M(t)\lambda_t)^2 \frac{1}{\lambda_t M_t}\right] \\
&\quad + \int_{|z|<R}\left[\frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}} + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_Y(t,z)\right]\nu(dz); \\
\beta_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]; \\
\gamma_X(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_Y(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}; \\
\alpha_Y(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_Y(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}
$$

The term $e^{-rt}\left[1 + \lambda_0 M_0\left(\int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right)\right]$ acts as the conventional discount factor e^{-rt}. The additional term

$$
A_t = e^{-rt}\lambda_0 M_0\left(\int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right)
$$

can be thought of as due to the inflation linkage of the underlying securities. Note, however, that contrary to the standard pricing formula, the expectation is not computed in the risk neutral probability measure, but in the statistical probability measure.

Proof. If X_t and H_t are independent, then

$$
\mathbb{E}[X_t H_t] = \mathbb{E}[X_t]\mathbb{E}[H_t], \text{ with}
$$

$$
\begin{aligned}
\mathbb{E}[X_t] &= X_0 + \int_0^t \alpha_X(s)ds + \mathbb{E}\left[\int_0^t \beta_X(s)dW_s\right] + \mathbb{E}\left[\int_0^t\int_{\mathbb{R}}\gamma_X(s,z)\bar{\mu}(ds,dz)\right] \\
&= X_0 + \int_0^t \alpha_X(s)ds + \mathbb{E}\left[\int_0^t\int_{|z|\geq R}\gamma_X(s,z)\mu(ds,dz)\right]
\end{aligned}
$$

since

$$
\mathbb{E}\left[\int_0^t \beta_X(s)dW_s\right] = \mathbb{E}\left[\int_0^t\int_{|z|<R}\gamma_X(s,z)(\mu-\pi)(ds,dz)\right] = 0.
$$

Hence

$$
\mathbb{E}[X_t] = X_0 + \int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz).
$$

Therefore

$$
H_0 = e^{-rt}\left[1 + \lambda_0 M_0\left(\int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right)\right]\mathbb{E}[H_t].
$$

■

4.1.2 Separable power utility function

Given two non-negative constants A and B, a separable power utility function has the form

$$U(x,y) \quad = \quad \frac{A}{p}x^p + \frac{B}{q}y^q,$$

with $p, q \in]-\infty, 1]\backslash\{0\}$.

Assuming that the previous utility function is the agent's utility function in the market, the CPI, the pricing kernel and the fair value of a contingent claim H_t are respectively [68]

$$\begin{aligned}
C_n &= \left(\frac{A}{B}\right)^{1-q} \frac{\lambda_n M_n}{k_n^{(1-q)/(1-p)}};\\
\pi_n &= \frac{B^{\frac{1}{1-q}}}{A^{\frac{q}{1-q}}} \frac{k_n^{\frac{q}{1-q}(1-p)}}{\kappa \lambda_n M_n};\\
H_0 &= \frac{\lambda_0 M_0}{k_0^{q(1-p)/(1-q)}} e^{-\gamma t_j} \mathbb{E}\left[\frac{H_j k_j^{q(1-p)/(1-q)}}{\lambda_j M_j}\right],
\end{aligned}$$

where κ is a Lagrange multiplier.

Proposition 4.4. *The dynamics of the CPI and the pricing kernel are given by*

$$\begin{aligned}
\frac{dC_t}{(A/B)^{1-q}} &= \left\{\frac{\alpha_1(t)}{k_t^a} + \alpha_Y(t)\lambda_t M_t - \frac{a k_t^{a-1}\beta_k(t)}{k_t^{2a}}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\right.\\
&\quad \left. + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz)\right\} dt + \left[\frac{\beta_\lambda(t)M_t + \beta_M(t)\lambda_t}{k_t^a} - a k_t^{a-1}\beta_k(t)\frac{\lambda_t M_t}{k_t^{2a}}\right] dW_t\\
&\quad + \int_{\mathbb{R}} \left[\gamma_Y(t,z)\lambda_{t-}M_{t-} + \frac{\gamma_1(t,z)}{k_{t-}^a} + \gamma_1(t,z)\gamma_Y(t,z)\right] \bar{\mu}(dt,dz);\\
\frac{d\pi_t}{D} &= \left[\frac{\alpha_2(t)}{\lambda_t M_t} + \alpha_X(t)k_t^b - \frac{b k_t^{b-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right] + \int_{\mathbb{R}} \gamma_2(t,z)\gamma_X(t,z)\nu(dz)\right] dt\\
&\quad + \left[\frac{b k_t^{b-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^b}{\lambda_t^2 M_t^2}\right] dW_t\\
&\quad + \int_{\mathbb{R}} \left[\gamma_X(t,z)k_{t-}^b + \frac{\gamma_2(t,z)}{\lambda_{t-}M_{t-}} + \gamma_2(t,z)\gamma_X(t,z)\right] \bar{\mu}(dt,dz),
\end{aligned}$$

where

$$
\begin{aligned}
a &= \frac{1-q}{1-p}; \quad D = \frac{B^{\frac{1}{1-q}}}{\kappa A^{\frac{q}{1-q}}}; \quad b = \frac{q(1-p)}{1-q};\\
\alpha_1(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\
\gamma_1(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z);\\
\gamma_2(t,z) &= [k_{t-} + \gamma_k(t,z)]^a - k_{t-}^a;\\
\alpha_2(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2} + \int_{|z|<R} \left\{\gamma_2(t,z) - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz);\\
\gamma_Y(t,z) &= \frac{1}{k_{t-}^a + \gamma_2(t,z)} - \frac{1}{k_{t-}^a};\\
\alpha_Y(t) &= -\frac{1}{k_t^{2a}}\left[\alpha_2(t) - \left[ak_t^{a-1}\beta_k(t)\right]^2\frac{1}{k_t^a}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{k_{t-}^{2a}}\gamma_2(t,z)\right]\nu(dz);\\
\gamma_X(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_1(t,z)} - \frac{1}{\lambda_{t-}M_{t-}};\\
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_1(t) - (\beta_\lambda(t)M_t + \beta_M(t)\lambda_t)^2\frac{1}{\lambda_t M_t}\right] + \int_{|z|<R}\left[\gamma_X(t,z) + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_1(t,z)\right]\nu(dz).
\end{aligned}
$$

Proof. By the one-dimensional Itô formula (Theorem 2.7), considering

$$Y_t(a) \quad = \quad f(t,k_t) = k_t^a$$

with $a \in \mathbb{R}$.

Because $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial k_t} = ak_t^{a-1}$ and $\frac{\partial^2 f}{\partial^2 k_t} = a(a-1)k_t^{a-2}$,

$$
\begin{aligned}
dY_t &= ak_t^{a-1}\left[\alpha_k(t)dt + \beta_k(t)dW_t\right] + \frac{1}{2}a(a-1)\left[\beta_k(t)\right]^2 k_t^{a-2}dt\\
&\quad + \int_{|z|<R}\left\{[k_{t-} + \gamma_k(t,z)]^a - k_{t-}^a - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\pi(dt,dz)\\
&\quad + \int_{\mathbb{R}}\left\{[k_{t-} + \gamma_k(t,z)]^a - k_{t-}^a\right\}\bar{\mu}(dt,dz)\\
&= \left[ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2}\right]dt + ak_t^{a-1}\beta_k(t)dW_t\\
&\quad + \int_{|z|<R}\left\{[k_{t-} + \gamma_k(t,z)]^a - k_{t-}^a - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\pi(dt,dz)\\
&\quad + \int_{\mathbb{R}}\left\{[k_{t-} + \gamma_k(t,z)]^a - k_{t-}^a\right\}\bar{\mu}(dt,dz).
\end{aligned}
$$

Applying Corollary 2.14 with $X_t = \dfrac{Z_t}{k_t^a}$, $a = \dfrac{1-q}{1-p}$ and the dynamics of Z_t given in the proof of

Proposition 4.1, gives

$$\begin{aligned}\frac{dC_t}{(A/B)^{1-q}} &= \left\{\frac{\alpha_1(t)}{k_t^a} + \alpha_Y(t)\lambda_t M_t - \frac{ak_t^{a-1}\beta_k(t)}{k_t^{2a}}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\right.\\ &\quad \left. + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz)\right\} dt + \left[\frac{\beta_\lambda(t)M_t + \beta_M(t)\lambda_t}{k_t^a} - ak_t^{a-1}\beta_k(t)\frac{\lambda_t M_t}{k_t^{2a}}\right] dW_t\\ &\quad + \int_{\mathbb{R}}\left[\gamma_Y(t,z)\lambda_{t-}M_{t-} + \frac{\gamma_1(t,z)}{k_{t-}^a} + \gamma_1(t,z)\gamma_Y(t,z)\right]\bar{\mu}(dt,dz),\end{aligned}$$

where

$$\begin{aligned}\alpha_1(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\ \gamma_1(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z);\\ \gamma_2(t,z) &= \left[k_{t-} + \gamma_k(t,z)\right]^a - k_{t-}^a;\\ \alpha_2(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2} + \int_{|z|<R}\left\{\gamma_2(t,z) - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz);\\ \gamma_Y(t,z) &= \frac{1}{k_{t-}^a + \gamma_2(t,z)} - \frac{1}{k_{t-}^a};\\ \alpha_Y(t) &= -\frac{1}{k_t^{2a}}\left[\alpha_2(t) - \left(ak_t^{a-1}\beta_k(t)\right)^2\frac{1}{k_t^a}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{k_{t-}^{2a}}\gamma_2(t,z)\right]\nu(dz).\end{aligned}$$

Applying again Corollary 2.14 with $\pi_t = D\dfrac{k_t^a}{Z_t}$, $D = \dfrac{B^{\frac{1}{1-q}}}{\kappa A^{\frac{q}{1-q}}}$ and $a = \frac{q(1-p)}{1-q}$, yields

$$\begin{aligned}\frac{d\pi_t}{D} &= \left[\frac{\alpha_1(t)}{\lambda_t M_t} + \alpha_Y(t)k_t^a - \frac{ak_t^{a-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right] + \int_{\mathbb{R}}\gamma_1(t,z)\gamma_Y(t,z)\nu(dz)\right] dt\\ &\quad + \left[\frac{ak_t^{a-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^a}{\lambda_t^2 M_t^2}\right] dW_t\\ &\quad + \int_{\mathbb{R}}\left[\gamma_Y(t,z)k_{t-}^a + \frac{\gamma_1(t,z)}{\lambda_{t-}M_{t-}} + \gamma_1(t,z)\gamma_Y(t,z)\right]\bar{\mu}(dt,dz),\end{aligned}$$

where

$$\begin{aligned}\gamma_1(t,z) &= \left[k_{t-} + \gamma_k(t,z)\right]^a - k_{t-}^a;\\ \alpha_1(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2} + \int_{|z|<R}\left\{\gamma_1(t,z) - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz);\\ \alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\ \gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z);\\ \gamma_Y(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}};\\ \alpha_Y(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2\frac{1}{\lambda_t M_t}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz).\end{aligned}$$

■

Proposition 4.5. *The fair price at $t = 0$ of a contingent claim H_t is given by*

$$\begin{aligned} H_0 &= e^{-rt}\mathbb{E}[H_t]\left(1 + \frac{\lambda_0 M_0}{k_0^a}\int_0^t \alpha_X(s)ds\right) + \frac{\lambda_0 M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t \beta_X(s)dW_s\right] \\ &\quad + \frac{\lambda_0 M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right], \end{aligned}$$

where

$$\begin{aligned} a &= \frac{q(1-p)}{1-q}; \\ \alpha_X(t) &= \frac{\alpha_1(t)}{\lambda_t M_t} + \alpha_Y(t)k_t^a - \frac{ak_t^{a-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right] + \int_{\mathbb{R}}\gamma_1(t,z)\gamma_Y(t,z)\nu(dz); \\ \beta_X(t) &= \frac{ak_t^{a-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^a}{\lambda_t^2 M_t^2}; \\ \gamma_X(t,z) &= \gamma_Y(t,z)k_{t-}^a + \frac{\gamma_1(t,z)}{\lambda_{t-}M_{t-}} + \gamma_1(t,z)\gamma_Y(t,z); \\ \gamma_Y(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}}; \\ \alpha_Y(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2\frac{1}{\lambda_t M_t}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz); \\ \alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda\beta_M + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\ \gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z); \\ \gamma_1(t,z) &= \left[k_{t-} + \gamma_k(t,z)\right]^a - k_{t-}^a; \\ \alpha_1(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2} + \int_{|z|<R}\left\{\gamma_1(t,z) - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz). \end{aligned}$$

Proof. Similarly to π_t in the previous proof, considering $a = \frac{q(1-p)}{1-q}$ and $X_t = \frac{k_t^a}{\lambda_t M_t}$, its dynamics are given by

$$dX_t = \alpha_X(t)dt + \beta_X(t)dW_t + \int_{\mathbb{R}}\gamma_X(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}
\alpha_X(t) &= \frac{\alpha_1(t)}{\lambda_t M_t} + \alpha_Y(t)k_t^a - \frac{ak_t^{a-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right] + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz);\\
\beta_X(t) &= \frac{ak_t^{a-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^a}{\lambda_t^2 M_t^2};\\
\gamma_X(t,z) &= \gamma_Y(t,z)k_{t-}^a + \frac{\gamma_1(t,z)}{\lambda_{t-}M_{t-}} + \gamma_1(t,z)\gamma_Y(t,z);\\
\alpha_Y(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2 \frac{1}{\lambda_t M_t}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz);\\
\gamma_Y(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}};\\
\alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda\beta_M + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\
\gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z)M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z);\\
\alpha_1(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2} + \int_{|z|<R}\left\{\gamma_1(t,z) - ak_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz);\\
\gamma_1(t,z) &= \left[k_{t-} + \gamma_k(t,z)\right]^a - k_{t-}^a.
\end{aligned}$$

A reasoning similar to the one used in the proof of Proposition 4.2 yields

$$\mathbb{E}[X_tH_t] = \mathbb{E}\left[\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right)H_t\right] + \mathbb{E}[X_0H_t].$$

Therefore

$$\begin{aligned}
H_0 &= e^{-rt}\mathbb{E}[H_t] + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\left(\int_0^t\alpha_X(s)ds + \int_0^t\beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right)\right]\\
&= e^{-rt}\mathbb{E}[H_t] + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\alpha_X(s)ds\right] + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\beta_X(s)dW_s\right]\\
&\quad + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right]\\
&= e^{-rt}\mathbb{E}[H_t]\left(1 + \frac{\lambda_0M_0}{k_0^a}\int_0^t\alpha_X(s)ds\right) + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\beta_X(s)dW_s\right]\\
&\quad + \frac{\lambda_0M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t\int_{\mathbb{R}}\gamma_X(s)\bar{\mu}(ds,dz)\right].
\end{aligned}$$

■

An additional independence assumption similar to that made in Proposition 4.3 yields a similar result.

Proposition 4.6. *If an additional assumption is made that* $\frac{k_t^a}{\lambda_tM_t}$ *and* H_t *are independent (i.e. the pricing kernel is independent of the contingent claim), the fair price at* $t = 0$ *of the contingent claim becomes*

$$H_0 = e^{-rt}\left[1 + \frac{\lambda_0M_0}{k_0^a}\left(\int_0^t\alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right)\right]\mathbb{E}[H_t].$$

where

$$
\begin{aligned}
a &= \frac{q(1-p)}{1-q};\\
\alpha_X(t) &= \frac{\alpha_1(t)}{\lambda_t M_t} + \alpha_Y(t) k_t^a - \frac{a k_t^{a-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t) M_t + \beta_M(t)\lambda_t\right] + \int_{\mathbb{R}} \gamma_1(t,z)\gamma_Y(t,z)\nu(dz);\\
\beta_X(t) &= \frac{a k_t^{a-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t) M_t + \beta_M(t)\lambda_t\right]\frac{k_t^a}{\lambda_t^2 M_t^2};\\
\gamma_X(t,z) &= \gamma_Y(t,z) k_{t-}^a + \frac{\gamma_1(t,z)}{\lambda_{t-}M_{t-}} + \gamma_1(t,z)\gamma_Y(t,z);\\
\alpha_Y(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2 \frac{1}{\lambda_t M_t}\right] + \int_{|z|<R}\left[\gamma_Y(t,z) + \frac{1}{\lambda_{t-}^2 M_{t-}^2}\gamma_2(t,z)\right]\nu(dz);\\
\gamma_Y(t,z) &= \frac{1}{\lambda_{t-}M_{t-} + \gamma_2(t,z)} - \frac{1}{\lambda_{t-}M_{t-}};\\
\alpha_2(t) &= \alpha_\lambda(t) M_t + \alpha_M(t)\lambda_t + \beta_\lambda\beta_M + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz);\\
\gamma_2(t,z) &= \gamma_M(t,z)\lambda_{t-} + \gamma_\lambda(t,z) M_{t-} + \gamma_\lambda(t,z)\gamma_M(t,z);\\
\alpha_1(t) &= a k_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) k_t^{a-2} + \int_{|z|<R}\left\{\gamma_1(t,z) - a k_{t-}^{a-1}\gamma_k(t,z)\right\}\nu(dz);\\
\gamma_1(t,z) &= \left[k_{t-} + \gamma_k(t,z)\right]^a - k_{t-}^a.
\end{aligned}
$$

The additional term due to the inflation linkage is now

$$
A_t = e^{-rt}\frac{\lambda_0 M_0}{k_0^a}\left(\int_0^t \alpha_X(s)ds + \int_0^t\int_{|z|\geq R}\gamma_X(s,z)\nu(ds,dz)\right).
$$

The fair price of a contingent claim H_t is of the form

$$
H_0 = e^{-rt}\left(1 + A_t\right)\mathbb{E}[H_t],
$$

where A_t is function of the agent's utility function considered.

Proof. Similar to that of Proposition 4.3. ■

4.1.3 Arithmetic Brownian distribution

Working in an environment without jumps and assuming that the nominal money supply, aggregate consumption and nominal liquidity benefit are normally distributed, i.e. Assumption 8 with $\mu = \pi = \gamma_M = \gamma_\lambda = \gamma_k = \gamma = 0$ gives the following propositions, which are just particular cases of the previous propositions.

Proposition 4.7. *Considering a log-separable utility function, the dynamics of C_t, π_t and the fair* *price at $t = 0$ of a contingent claim H_t are*

$$
\begin{aligned}
\frac{dC_t}{A/B} &= \alpha_C(t)dt + \beta_C(t)dW_t;\\
\frac{d\pi_t}{B/\mu} &= \alpha_\pi(t)dt + \beta_\pi(t)dW_t;\\
H_0 &= e^{-rt}\left(1 + \lambda_0 M_0 \int_0^t \alpha_X(s)ds\right)\mathbb{E}[H_t] + \lambda_0 M_0 e^{-rt}\mathbb{E}\left[H_t \int_0^t \beta_X(s)dW_s\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_C(t) &= \frac{1}{k_t}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t)\right] + \alpha_Y(t)\lambda_t M_t - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{\beta_k(t)}{k_t^2};\\
\beta_C(t) &= \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{1}{k_t} - \beta_k(t)\frac{\lambda_t M_t}{k_t^2};\\
\alpha_\pi(t) &= \left\{-\frac{r}{\lambda_t M_t} - \frac{1}{\lambda_t^2 M_t^2}\left[\alpha_2(t) - \left(\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right)^2 \frac{1}{\lambda_t M_t}\right]\right\} e^{-rt};\\
\beta_\pi(t) &= -\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{e^{-rt}}{\lambda_t^2 M_t^2};\\
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) - \left(\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right)^2\frac{1}{\lambda_t M_t}\right];\\
\beta_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right];\\
\alpha_Y(t) &= -\frac{1}{k_t^2}\left[\alpha_k(t) - \beta_k^2(t)\frac{1}{k_t}\right];\\
\alpha_2(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t);\\
\beta_2(t) &= \beta_\lambda(t)M_t + \beta_M(t)\lambda_t.
\end{aligned}
$$

Proposition 4.8. *Considering a log-separable utility function and assuming that the pricing kernel is independent of the contingent claim, the fair price at $t = 0$ becomes*

$$
H_0 = e^{-rt}\left(1 + \lambda_0 M_0 \int_0^t \alpha_X(s)ds\right)\mathbb{E}[H_t].
$$

where

$$
\alpha_X(t) = -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) - \left(\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right)^2\frac{1}{\lambda_t M_t}\right].
$$

Proposition 4.9. *Considering a separable power utility function, the dynamics of C_t, π_t and the fair price at t = 0 of a contingent claim H_t are given by*

$$\begin{aligned}
\frac{dC_t}{(A/B)^{1-q}} &= \left\{\frac{\alpha_1(t)}{k_t^a} + \alpha_Y(t)\lambda_t M_t - \frac{ak_t^{a-1}\beta_k(t)}{k_t^{2a}}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\right\} \\
&\quad dt + \left[\frac{\beta_\lambda(t)M_t + \beta_M(t)\lambda_t}{k_t^a} - ak_t^{a-1}\beta_k(t)\frac{\lambda_t M_t}{k_t^{2a}}\right] dW_t; \\
\frac{d\pi_t}{D} &= \left[\frac{\alpha_2(t)}{\lambda_t M_t} + \alpha_X(t)k_t^b - \frac{bk_t^{b-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\right] dt \\
&\quad + \left[\frac{bk_t^{b-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^b}{\lambda_t^2 M_t^2}\right] dW_t; \\
H_0 &= e^{-rt}\left(1 + \frac{\lambda_0 M_0}{k_0^a}\int_0^t \alpha_Z(s)ds\right)\mathbb{E}[H_t] + \frac{\lambda_0 M_0}{k_0^a}e^{-rt}\mathbb{E}\left[H_t\int_0^t \beta_Z(s)dW_s\right],
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{1-q}{1-p}; \quad D = \frac{B^{\frac{1}{1-q}}}{\kappa A^{\frac{q}{1-q}}}; \quad b = \frac{q(1-p)}{1-q}; \\
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_1(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2 \frac{1}{\lambda_t M_t}\right]; \\
\alpha_Y(t) &= -\frac{1}{k_t^{2a}}\left[\alpha_2(t) - \left[ak_t^{a-1}\beta_k(t)\right]^2 \frac{1}{k_t^a}\right]; \\
\alpha_Z(t) &= \frac{1}{\lambda_t M_t}\left[bk_t^{b-1}\alpha_k(t) + \frac{1}{2}b(b-1)\beta_k^2(t)k_t^{b-2}\right] - \frac{bk_t^{b-1}\beta_k(t)}{\lambda_t^2 M_t^2}\left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right] \\
&\quad - \frac{k_t^b}{\lambda_t^2 M_t^2}\left[\alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2\frac{1}{\lambda_t M_t}\right]; \\
\beta_Z(t) &= \frac{bk_t^{b-1}\beta_k(t)}{\lambda_t M_t} - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]\frac{k_t^b}{\lambda_t^2 M_t^2}; \\
\alpha_2(t) &= ak_t^{a-1}\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t)k_t^{a-2}; \\
\alpha_1(t) &= \alpha_\lambda(t)M_t + \alpha_M(t)\lambda_t + \beta_\lambda(t)\beta_M(t).
\end{aligned}$$

Proposition 4.10. *Considering a separable separable power utility function and assuming that the pricing kernel is independent of the contingent claim, the fair price at t = 0 becomes*

$$H_0 = e^{-rt}\left(1 + \frac{\lambda_0 M_0}{k_0^a}\int_0^t \alpha_X(s)ds\right)\mathbb{E}[H_t].$$

where

$$\begin{aligned}
a &= \frac{1-q}{1-p}; \\
\alpha_X(t) &= -\frac{1}{\lambda_t^2 M_t^2}\left[\alpha_1(t) - \left[\beta_\lambda(t)M_t + \beta_M(t)\lambda_t\right]^2\frac{1}{\lambda_t M_t}\right].
\end{aligned}$$

4.2 Exponential Lévy distribution

This section is similar to Section 4.1, but instead of following Lévy processes, the macroeconomic factors are assumed to be exponential Lévy processes. Since the nominal money supply, the aggregate consumption and the nominal liquidity benefit are always positive, this assumption is appropriate. Moreover, formulas obtained under the assumption of exponential Lévy distribution are generally highly tractable.

The following assumption is made throughout this section.

Assumption 9. *Under the objective probability measure $\mathbb{P}$, the dynamics of M_t, k_t and λ_t for every $t > 0$ are given by:*

$$\begin{aligned}
\frac{dM_t}{M_t} &= \alpha_M(t)dt + \beta_M(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_M(t,z)\bar{\mu}(dt,dz); \\
\frac{dk_t}{k_t} &= \alpha_k(t)dt + \beta_k(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_k(t,z)\bar{\mu}(dt,dz); \\
\frac{d\lambda_t}{\lambda_t} &= \alpha_\lambda(t)dt + \beta_\lambda(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma_\lambda(t,z)\bar{\mu}(dt,dz),
\end{aligned}$$

with the standard integrability conditions (See Assumption 8).

4.2.1 Log-separable utility function

Recall that if the agent utility function is a log-separable utility function, then the pricing kernel, the CPI and the fair price of a contingent claim are given by

$$\begin{aligned}
C_t &= \frac{A}{B}\frac{\lambda_t M_t}{k_t}; \\
\pi_t &= \frac{Be^{-rt}}{\kappa\lambda_t M_t}; \\
H_0 &= \lambda_0 M_0 e^{-rt}\mathbb{E}\left[\frac{H_t}{\lambda_t M_t}\right],
\end{aligned}$$

where A and B are two non-negative constants that define the utility function.

Proposition 4.11. *The dynamics of CPI and the pricing kernel are*

$$\begin{aligned}
\frac{dC_t}{C_{t-}} &= \Big\{\alpha_\lambda(t)+\alpha_M(t)-\alpha_k(t)+\beta_\lambda(t)\beta_M(t)+\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\\
&\quad+\beta_k^2(t)+\int_{|z|<R}\frac{\gamma_k^2(t,z)}{1+\gamma_k(t,z)}\nu(dz)-\left[\beta_\lambda(t)+\beta_M(t)\right]\beta_k(t)\Big\}\,dt\\
&\quad+\int_{\mathbb{R}}\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\pi(dt,dz)\\
&\quad+\left[\beta_\lambda(t)+\beta_M(t)-\beta_k(t)\right]dW_t\\
&\quad+\int_{\mathbb{R}}\Big\{\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)+\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\\
&\quad+\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\Big\}\,\bar{\mu}(dt,dz);\\
\frac{d\pi_t}{\pi_{t-}} &= \Big[-r-\alpha_\lambda(t)-\alpha_M(t)-\beta_\lambda(t)\beta_M(t)-\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)+\left[\beta_\lambda(t)+\beta_M(t)\right]^2\\
&\quad+\int_{|z|<R}\frac{\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]^2}{1+\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}\nu(dz)\Big]\,dt-\left[\beta_\lambda(t)+\beta_M(t)\right]dW_t\\
&\quad+\int_{\mathbb{R}}\left[\frac{\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}{1+\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}\right]\bar{\mu}(dt,dz).
\end{aligned}$$

Proof. Applying Corollary 2.13 with $Z_t=\lambda_t M_t$

$$\begin{aligned}
\frac{dZ_t}{Z_{t-}} &= \left[\alpha_\lambda(t)+\alpha_M(t)+\beta_\lambda(t)\beta_M(t)+\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\right]dt+\left[\beta_\lambda(t)+\beta_M(t)\right]dW_t\\
&\quad+\int_{\mathbb{R}}\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]\bar{\mu}(dt,dz).
\end{aligned}$$

Then applying Corollary 2.15

$$\begin{aligned}
\frac{dC_t}{C_{t-}} &= \Big\{\alpha_\lambda(t)+\alpha_M(t)+\beta_\lambda(t)\beta_M(t)+\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)-\alpha_k(t)\\
&\quad+\beta_k^2(t)+\int_{|z|<R}\frac{\gamma_k^2(t,z)}{1+\gamma_k(t,z)}\nu(dz)-\left[\beta_\lambda(t)+\beta_M(t)\right]\beta_k(t)\Big\}\,dt\\
&\quad+\int_{\mathbb{R}}\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\pi(dt,dz)\\
&\quad+\left[\beta_\lambda(t)+\beta_M(t)-\beta_k(t)\right]dW_t\\
&\quad+\int_{\mathbb{R}}\Big\{\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)+\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\\
&\quad+\left[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)\right]\frac{\gamma_k(t,z)}{1+\gamma_k(t,z)}\Big\}\,\bar{\mu}(dt,dz).
\end{aligned}$$

The deterministic process $X_t=e^{-rt}$ has dynamics

$$\frac{dX_t}{X_{t-}} = -rdt.$$

Applying again Corollary 2.15

$$\begin{aligned}\frac{d\pi_t}{\pi_{t-}} &= \Bigg[-r-\alpha_\lambda(t)-\alpha_M(t)-\beta_\lambda(t)\beta_M(t)-\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)+[\beta_\lambda(t)+\beta_M(t)]^2\\&+\int_{|z|<R}\frac{[\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)]^2}{1+\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}\nu(dz)\Bigg]dt-[\beta_\lambda(t)+\beta_M(t)]\,dW_t\\&+\int_{\mathbb{R}}\left[\frac{\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}{1+\gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z)}\right]\bar{\mu}(dt,dz).\end{aligned}$$

■

Proposition 4.12. *The fair price at $t=0$ of a contingent claim H_t is given by*

$$H_0 = e^{-rt}\mathbb{E}\left[H_t\exp\left(\int_0^t\alpha_X(s)ds+\int_0^t\beta_X(s)dW_s+\int_0^t\int_{\mathbb{R}}\gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right],$$

where

$$\begin{aligned}\alpha_X(t) &= -\alpha_\lambda(t)-\alpha_M(t)-\beta_\lambda(t)\beta_M(t)-\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)+[\beta_\lambda(t)+\beta_M(t)]^2\\&+\int_{|z|<R}\frac{\Gamma^2(t,z)}{1+\Gamma(t,z)}\nu(dz);\\\beta_X(t) &= -\beta_\lambda(t)-\beta_M(t);\\\gamma_X(t,z) &= -\frac{\Gamma(t,z)}{1+\Gamma(t,z)};\\\Gamma(t,z) &= \gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z).\end{aligned}$$

Proof. Using the dynamics of Z_t (see previous proof) in Corollary 2.9 with $X_t=\dfrac{1}{\lambda_t M_t}$ yields

$$\begin{aligned}\frac{dX_t}{X_{t-}} &= \Bigg\{-\alpha_\lambda(t)-\alpha_M(t)-\beta_\lambda(t)\beta_M(t)-\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)+[\beta_\lambda(t)+\beta_M(t)]^2\\&+\int_{|z|<R}\frac{\Gamma^2(t,z)}{1+\Gamma(t,z)}\nu(dz)\Bigg\}dt-[\beta_\lambda(t)+\beta_M(t)]\,dW_t-\int_{\mathbb{R}}\frac{\Gamma(t,z)}{1+\Gamma(t,z)}\bar{\mu}(dt,dz)\\&= \alpha_X(t)dt+\beta_X(t)dW_t+\int_{\mathbb{R}}\gamma_X(t,z)\bar{\mu}(dt,dz)\end{aligned}$$

where

$$\begin{aligned}\alpha_X(t) &= -\alpha_\lambda(t)-\alpha_M(t)-\beta_\lambda(t)\beta_M(t)-\int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz)\\&+[\beta_\lambda(t)+\beta_M(t)]^2+\int_{|z|<R}\frac{\Gamma^2(t,z)}{1+\Gamma(t,z)}\nu(dz);\\\beta_X(t) &= -\beta_\lambda(t)-\beta_M(t);\\\gamma_X(t,z) &= -\frac{\Gamma(t,z)}{1+\Gamma(t,z)};\\\Gamma(t,z) &= \gamma_\lambda(t,z)+\gamma_M(t,z)+\gamma_\lambda(t,z)\gamma_M(t,z).\end{aligned}$$

Hence

$$X_t = X_0 \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)$$

and

$$\mathbb{E}[X_t H_t] = \mathbb{E}\left[\frac{H_t}{\lambda_0 M_0} \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right].$$

Therefore

$$H_0 = e^{-rt}\mathbb{E}\left[H_t \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right].$$

■

Proposition 4.13. *Further, assuming that* $\frac{1}{\lambda_t M_t}$ *and* H_t *are independent (i.e. the pricing kernel is independent of the priced security), the fair price at* $t = 0$ *of the contingent claim becomes*

$$H_0 = e^{-rt} \exp\left\{\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma_X}(dz)du\right\} \mathbb{E}[H_t],$$

where

$$\begin{aligned}
\alpha_X(t) &= -\alpha_\lambda(t) - \alpha_M(t) - \beta_\lambda(t)\beta_M(t) - \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz) + [\beta_\lambda(t) + \beta_M(t)]^2 \\
&\quad + \int_{|z|<R} \frac{\Gamma^2(t,z)}{1+\Gamma(t,z)}\nu(dz); \\
\beta_X(t) &= -\beta_\lambda(t) - \beta_M(t); \\
\gamma_X(t,z) &= -\frac{\Gamma(t,z)}{1+\Gamma(t,z)}; \\
\Gamma(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

Recall that the standard no-arbitrage price of a derivative is $e^{-rt}\mathbb{E}[H_t]$, the term

$$D_t = \exp\left\{\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma_X}(dz)du\right\}$$

in the previous equation can be interpreted as a correction factor due to inflation linkage. Note that the standard expectation is computed under the risk neutral probability measure while here it is computed under the statistical (i.e. real world) probability measure.

Proof. If X_t and H_t are independent, then

$$\mathbb{E}[X_t H_t] = \mathbb{E}[X_t]\mathbb{E}[H_t], \text{ with}$$

$$\begin{aligned}\mathbb{E}[X_t] &= X_0 \exp\left(\int_0^t \alpha_X(s)ds\right)\mathbb{E}\left[\exp\left(\int_0^t \beta_X(s)dW_s + \int_0^t\int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right] \\ &= X_0 \exp\left(\int_0^t \alpha_X(s)ds\right)\mathbb{E}\left[\exp\left(\int_0^t \beta_X(s)dW_s\right)\right]\mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right]\end{aligned}$$

since dW_t and $\bar{\mu}(dt,dz)$ are independent.

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}} \gamma_X(u,z)\bar{\mu}(du,dz)\right)\right] &= \mathbb{E}\left[\exp\left(\int_0^t\int_{|z|<R} \gamma_X(u,z)(\mu-\pi)(du,dz)\right.\right. \\ &\quad \left.\left. + \int_0^t\int_{|z|\geq R} \gamma_X(u,z)\mu(du,dz)\right)\right] \\ &= \mathbb{E}\left[\exp\left(\int_0^t\int_{|z|<R} \gamma_X(u,z)(\mu-\pi)(du,dz)\right)\right] \\ &\quad \mathbb{E}\left[\exp\left(\int_0^t\int_{|z|\geq R} \gamma_X(u,z)\mu(du,dz)\right)\right]\end{aligned}$$

since $\{|z|<R\}$ and $\{|z|\geq R\}$ are disjoint.

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^t\int_{\mathbb{R}} \gamma_X(u,z)\bar{\mu}(du,dz)\right)\right] &= \exp\left\{\int_0^t\int_{|z|<R} \left[e^z - 1 - z\right]\pi_{\gamma_X}(dz)du\right\} \\ &\quad \exp\left[\int_0^t\int_{|z|\geq R} (e^z-1)\pi_{\gamma_X}(dz)du\right] \\ &= \exp\left\{\int_0^t\int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right]\pi_{\gamma_X}(dz)du\right\}\end{aligned}$$

by Corollary 2.21 with $\pi_{\gamma_X} = \pi \circ (\gamma_X)^{-1}$.

Recall that $dW_t \sim \mathcal{N}(0,dt)$, hence $\int_0^t \beta_X(u)dW_u \sim \mathcal{N}\left(0, \int_0^t \beta_X^2(u)du\right)$. Let $\langle\cdot,\cdot\rangle$ denote the Euclidean distance, i.e. for $x,y\in\mathbb{R}^d$ such as $x=(x_1,x_2,\cdots,x_d)$ and $y=(y_1,y_2,\cdots,y_d)$, $\langle x,y\rangle = \sum_{i=1}^d x_i y_i$. Let $\psi_{\mathcal{N}(m,V)}$ denote the log-characteristic function of a normal distributed process with mean m and variance V, defined by

$$\psi_{\mathcal{N}(m,V)}(u) = i\langle m,u\rangle - \frac{1}{2}\langle u, Vu\rangle$$

with $m,u\in\mathbb{R}^d$. In particular

$$\psi_{\mathcal{N}(m,V)}(u) = imu - \frac{1}{2}Vu^2 \text{ with } m,u\in\mathbb{R}.$$

Using Theorem 2.19

$$\mathbb{E}\left[\exp\left(\int_0^t \beta_X(u)dW_u\right)\right] = \exp\left[-\frac{1}{2}\int_0^t \beta_X^2(u)du\right].$$

Combining these expectations yields

$$E[X_t] = X_0 \exp\left\{\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t\int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right]\pi_{\gamma_X}(dz)du\right\}.$$

Therefore

$$H_0 = e^{-rt}\exp\left\{\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t\int_{\mathbb{R}}\left[e^z - 1 - z\chi_{|z|<R}\right]\pi_{\gamma_X}(dz)du\right\}\mathbb{E}[H_t].$$

■

For simplicity, the parameter estimation will always assume $\gamma(t,z)$ to be deterministic. In this case, the previous expression becomes more tractable as shown in the next proposition.

Proposition 4.14. *If $\gamma(t,z)$ is deterministic and $\frac{1}{\lambda_t M_t}$ and H_t are independent, the fair price at $t=0$ of the contingent claim is*

$$H_0 = e^{-rt}\exp\left\{\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t\int_{\mathbb{R}}\left[e^{\gamma_X(t,z)} - 1 - \gamma_X(t,z)\chi_{|z|<R}\right]\pi(dz)du\right\}\mathbb{E}[H_t],$$

where the coefficients are given in Proposition 4.13.

4.2.2 Power utility function

Recall that if the agent utility function is a separable power utility function, the CPI, the pricing kernel and the fair price of an IL security are respectively

$$\begin{aligned}
C_t &= \left(\frac{A}{B}\right)^{1-q}\frac{\lambda_t M_t}{k_t^{(1-q)/(1-p)}};\\
\pi_t &= \frac{B^{\frac{1}{1-q}}}{A^{\frac{q}{1-q}}}\frac{k_t^{\frac{q}{1-q}(1-p)}}{\kappa\lambda_t M_t};\\
H_0 &= \frac{\lambda_0 M_0}{k_0^{q(1-p)/(1-q)}}e^{-rt}\mathbb{E}\left[\frac{H_t k_t^{q(1-p)/(1-q)}}{\lambda_t M_t}\right],
\end{aligned}$$

where A and B are two non-negative constants; $p,q\in]-\infty,1]\backslash\{0\}$ defining the utility function. The next proposition computes the dynamics of the CPI and the pricing kernel assuming that the macroeconomic factors are exponential Lévy processes.

Proposition 4.15. *The dynamics of C_t and π_t are given by*

$$\begin{aligned}
\frac{dC_t}{C_{t-}} &= \alpha_C(t)dt + \beta_C(t)dW_t + \int_{\mathbb{R}}\gamma_C(t,z)\bar{\mu}(dt,dz);\\
\frac{d\pi_t}{\pi_{t-}} &= \alpha_\pi(t)dt + \beta_\pi(t)dW_t + \int_{\mathbb{R}}\gamma_\pi(t,z)\bar{\mu}(dt,dz),
\end{aligned}$$

where

$$
\begin{aligned}
\alpha_C(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz) + a^2\beta_k(t)^2 \\
&\quad -a\alpha_k(t) - \frac{1}{2}a(a-1)\beta_k^2(t) - \int_{|z|<R} \{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\}\,\nu(dz) \\
&\quad + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) - a\left[\beta_\lambda(t) + \beta_M(t)\right]\beta_k(t) + \int_{\mathbb{R}} \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\
\beta_C(t) &= \beta_\lambda(t) + \beta_M(t) - a\beta_k(t); \\
\gamma_C(t,z) &= \gamma_2(t,z) + \frac{\gamma_1(t,z)}{1+\gamma_1(t,z)} + \gamma_2(t,z)\frac{\gamma_1(t,z)}{1+\gamma_1(t,z)}; \\
\alpha_\pi(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) \\
&\quad -a\beta_k(t)\left[\beta_\lambda(t) + \beta_M(t)\right] + \int_{\mathbb{R}} \{[1+\gamma_k(t,z)]^a - 1\}\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\
\beta_\pi(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\gamma_\pi(t,z) &= [1+\gamma_k(t,z)]^a - 1 + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \{[1+\gamma_k(t,z)]^a - 1\}\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}; \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R} \{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\}\,\nu(dz); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_1(t,z) &= [1+\gamma_k(t,z)]^a - 1; \\
\gamma_2(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}
$$

Proof. Applying the one-dimensional Itô formula (Theorem 2.7) with $Y_t(a) = f(t,k_t) = k_t^a$ for $a \in \mathbb{R}$. We have $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial k_t} = ak_t^{a-1}$ and $\dfrac{\partial^2 f}{\partial^2 k_t} = a(a-1)k_t^{a-2}$,

$$
\begin{aligned}
dY_t &= ak_t^{a-1}\left[\alpha_k(t)k_{t-}dt + \beta_k(t)k_{t-}dW_t\right] + \frac{1}{2}a(a-1)\left[\beta_k(t)k_{t-}\right]^2 k_t^{a-2}dt \\
&\quad + \int_{|z|<R} \left\{[k_{t-} + \gamma_k(t,z)k_{t-}]^a - k_{t-}^a - ak_{t-}^{a-1}\gamma_k(t,z)k_{t-}\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}} \left\{[k_{t-} + \gamma_k(t,z)k_{t-}]^a - k_{t-}^a\right\}\bar{\mu}(dt,dz) \\
&= a\left[\alpha_k(t)dt + \beta_k(t)dW_t\right] + \frac{1}{2}a(a-1)\beta_k^2(t)dt + \int_{|z|<R} \{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\}\,\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}} \{[1+\gamma_k(t,z)]^a - 1\}\,\bar{\mu}(dt,dz) \\
&= a\left[\alpha_k(t) + \frac{1}{2}(a-1)\beta_k^2(t)\right]dt + a\beta_k(t)dW_t + \int_{|z|<R} \{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\}\,\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} \{[1+\gamma_k(t,z)]^a - 1\}\,\bar{\mu}(dt,dz).
\end{aligned}
$$

Hence

$$\begin{aligned}\frac{dY_t}{Y_{t-}} &= \left\{a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R}\left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz)\right\}dt \\ &\quad + a\beta_k(t)dW_t + \int_{\mathbb{R}}\left\{[1+\gamma_k(t,z)]^a - 1\right\}\bar{\mu}(dt,dz).\end{aligned}$$

Applying Corollary 2.15 with $X_t = \dfrac{Z_t}{k_t^a}$ and $a = \dfrac{1-q}{1-p}$ (Z_t from the proof of Proposition 4.11)

$$\begin{aligned}\frac{dC_t}{C_{t-}} &= \Big\{\alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz) + a^2\beta_k(t)^2 \\ &\quad - a\alpha_k(t) - \frac{1}{2}a(a-1)\beta_k^2(t) - \int_{|z|<R}\left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz) \\ &\quad + \int_{|z|<R}\frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) - a\left[\beta_\lambda(t) + \beta_M(t)\right]\beta_k(t)\Big\}dt \\ &\quad + \int_{\mathbb{R}}\gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\pi(dt,dz) + \left[\beta_\lambda(t) + \beta_M(t) - a\beta_k(t)\right]dW_t \\ &\quad + \int_{\mathbb{R}}\left\{\gamma_1(t,z) + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \gamma_1(t,z)\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\right\}\bar{\mu}(dt,dz)\end{aligned}$$

where

$$\begin{aligned}\gamma_1(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z); \\ \gamma_2(t,z) &= [1+\gamma_k(t,z)]^a - 1.\end{aligned}$$

Again applying Corollary 2.15 with $\pi_t = \dfrac{B^{\frac{1}{1-q}}}{\kappa A^{\frac{q}{1-q}}}\dfrac{k_t^a}{Z_t}$ and $a = \frac{q(1-p)}{1-q}$ yields

$$\frac{d\pi_t}{\pi_{t-}} = \alpha_\pi(t)dt + \beta_\pi(t)dW_t + \int_{\mathbb{R}}\gamma_\pi(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}\alpha_\pi(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 + \int_{|z|<R}\frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) \\ &\quad - a\beta_k(t)\left[\beta_\lambda(t) + \beta_M(t)\right] + \int_{\mathbb{R}}\left\{[1+\gamma_k(t,z)]^a - 1\right\}\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\ \beta_\pi(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\ \gamma_\pi(t,z) &= [1+\gamma_k(t,z)]^a - 1 + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \left\{[1+\gamma_k(t,z)]^a - 1\right\}\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}; \\ \alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R}\left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz); \\ \alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}}\gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\ \gamma_2(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).\end{aligned}$$

∎

The next proposition computes the fair price of an Il derivative under the current assumptions.

Proposition 4.16. *The fair price at $t = 0$ of a contingent claim H_t is*

$$H_0 = e^{-rt}\mathbb{E}\left[H_t \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right],$$

where

$$\begin{aligned}
\alpha_X(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) \\
&\quad -a\beta_k(t)\left[\beta_\lambda(t) + \beta_M(t)\right] + \int_{\mathbb{R}} \left\{[1+\gamma_k(t,z)]^a - 1\right\} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\
\beta_X(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\gamma_X(t,z) &= [1+\gamma_k(t,z)]^a - 1 + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \left\{[1+\gamma_k(t,z)]^a - 1\right\} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}; \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R} \left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_2(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

Proof. Similarly to π_t in the previous proof, the process defined by $X_t = \dfrac{k_t^a}{\lambda_t M_t}$ with $a = \dfrac{q(1-p)}{1-q}$ has dynamics

$$\frac{dX_t}{X_{t-}} = \alpha_X(t)dt + \beta_X(t)dW_t + \int_{\mathbb{R}} \gamma_X(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}
\alpha_X(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) \\
&\quad -a\beta_k(t)\left[\beta_\lambda(t) + \beta_M(t)\right] + \int_{\mathbb{R}} \left\{[1+\gamma_k(t,z)]^a - 1\right\} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\
\beta_X(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\gamma_X(t,z) &= [1+\gamma_k(t,z)]^a - 1 + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \left\{[1+\gamma_k(t,z)]^a - 1\right\} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}; \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R} \left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_2(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

Thus

$$H_0 = e^{-rt}\mathbb{E}\left[H_t \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s + \int_0^t \int_{\mathbb{R}} \gamma_X(s,z)\bar{\mu}(ds,dz)\right)\right].$$

■

When the pricing kernel is independent of the derivative security, a mapping can be done between IL securities and nominal securities as shown in the next proposition.

Proposition 4.17. *Further, assuming that* $\frac{k_t^a}{\lambda_t M_t}$ *and* H_t *are independent (i.e. the pricing kernel is independent of the contingent claim), the fair price at* $t = 0$ *of a contingent claim* H_t *is*

$$H_0 = e^{-rt} \exp\left\{ \int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma_X}(dz)du \right\} \mathbb{E}[H_t],$$

where

$$\begin{aligned}
\alpha_X(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 + \int_{|z|<R} \frac{\gamma_2^2(t,z)}{1+\gamma_2(t,z)}\nu(dz) \\
&\quad - a\beta_k(t)\left[\beta_\lambda(t) + \beta_M(t)\right] + \int_{\mathbb{R}} \left\{[1+\gamma_k(t,z)]^a - 1\right\} \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}\nu(dz); \\
\beta_X(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\gamma_X(t,z) &= [1+\gamma_k(t,z)]^a - 1 + \frac{\gamma_2(t,z)}{1+\gamma_2(t,z)} + \left\{[1+\gamma_k(t,z)]^a - 1\right\}\frac{\gamma_2(t,z)}{1+\gamma_2(t,z)}; \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t) + \int_{|z|<R} \left\{[1+\gamma_k(t,z)]^a - 1 - a\gamma_k(t,z)\right\}\nu(dz); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) + \int_{\mathbb{R}} \gamma_\lambda(t,z)\gamma_M(t,z)\nu(dz); \\
\gamma_2(t,z) &= \gamma_\lambda(t,z) + \gamma_M(t,z) + \gamma_\lambda(t,z)\gamma_M(t,z).
\end{aligned}$$

The correction factor due to inflation linkage is

$$D_t = \exp\left\{ \int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma_X}(dz)du \right\}.$$

Note that the correction factor has the same form as with the log-separable utility function and can readily be written as a function of $N_t = \dfrac{1}{\pi_t}$ which is the standard numéraire in Finance.

Proof. Similar to that of Proposition 4.13. ■

Proposition 4.18. *If* $\gamma(t,z)$ *is deterministic and* $\frac{k_t^a}{\lambda_t M_t}$ *and* H_t *are independent, the fair price at* $t = 0$ *of a contingent claim* H_t *is*

$$H_0 = e^{-rt} \exp\left\{ \int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du + \int_0^t \int_{\mathbb{R}} \left[e^{\gamma_X(t,z)} - 1 - \gamma_X(t,z)\chi_{|z|<R}\right] \pi(dz)du \right\} \mathbb{E}[H_t],$$

where the coefficients are given in Proposition 4.17.

4.2.3 Geometric Brownian distribution

Working in an environment without jumps and assuming that the nominal money supply, aggregate consumption and nominal liquidity benefit follow a geometric distribution, i.e. Assumption 9 with $\mu = \pi = \gamma_M = \gamma_\lambda = \gamma_k = \gamma = 0$ gives the following results.

Proposition 4.19. *Considering a log-separable utility function, the dynamics of C_t, π_t and the fair price at $t = 0$ of a contingent claim H_t are*

$$\begin{aligned}
\frac{dC_t}{C_{t-}} &= \left\{\alpha_\lambda(t) + \alpha_M(t) - \alpha_k(t) + \beta_\lambda(t)\beta_M(t) + \beta_k^2(t) - \left[\beta_\lambda(t) + \beta_M(t)\right]\beta_k(t)\right\} dt \\
&\quad + \left[\beta_\lambda(t) + \beta_M(t) - \beta_k(t)\right] dW_t; \\
\frac{d\pi_t}{\pi_{t-}} &= \left\{-r - \alpha_\lambda(t) - \alpha_M(t) - \beta_\lambda(t)\beta_M(t) + \left[\beta_\lambda(t) + \beta_M(t)\right]^2\right\} dt \\
&\quad - \left[\beta_\lambda(t) + \beta_M(t)\right] dW_t; \\
H_0 &= e^{-rt}\mathbb{E}\left[H_t \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\alpha_X(t) &= -\alpha_\lambda(t) - \alpha_M(t) - \beta_\lambda(t)\beta_M(t) + \left[\beta_\lambda(t) + \beta_M(t)\right]^2; \\
\beta_X(t) &= -\beta_\lambda(t) - \beta_M(t).
\end{aligned}$$

Proposition 4.20. *Considering a log-separable utility function and assuming that the pricing kernel is independent of the contingent claims, the fair price at $t = 0$ becomes*

$$H_0 = e^{-rt}\exp\left(\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du\right)\mathbb{E}[H_t],$$

where

$$\begin{aligned}
\alpha_X(t) &= -\alpha_\lambda(t) - \alpha_M(t) - \beta_\lambda(t)\beta_M(t) + \left[\beta_\lambda(t) + \beta_M(t)\right]^2; \\
\beta_X(t) &= -\beta_\lambda(t) - \beta_M(t).
\end{aligned}$$

Proposition 4.21. *Considering a separable power utility function, the dynamics of C_t, π_t and the fair price at $t = 0$ of a contingent claim H_t are*

$$\begin{aligned}
\frac{dC_t}{C_{t-}} &= \alpha_C(t)dt + \beta_C(t)dW_t; \\
\frac{d\pi_t}{\pi_{t-}} &= \alpha_\pi(t)dt + \beta_\pi(t)dW_t; \\
H_0 &= e^{-rt}\mathbb{E}\left[H_t \exp\left(\int_0^t \alpha_X(s)ds + \int_0^t \beta_X(s)dW_s\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\alpha_C(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t) - a\alpha_k(t) - \frac{1}{2}a(a-1)\beta_k^2(t) - a\left[\beta_\lambda(t) + \beta_M(t)\right]\beta_k(t); \\
\beta_C(t) &= \beta_\lambda(t) + \beta_M(t) - a\beta_k(t);
\end{aligned}$$

$$\begin{aligned}
\alpha_X(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 - a\beta_k(t)\,[\beta_\lambda(t) + \beta_M(t)]\,; \\
\beta_X(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\alpha_\pi(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 - a\beta_k(t)\,[\beta_\lambda(t) + \beta_M(t)]\,; \\
\beta_\pi(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t).
\end{aligned}$$

Proposition 4.22. *Considering a separable power utility function and assuming that the pricing kernel is independent of the contingent claims the fair price at $t = 0$ becomes*

$$H_0 = e^{-rt}\exp\left(\int_0^t \alpha_X(u)du - \frac{1}{2}\int_0^t \beta_X^2(u)du\right)\mathbb{E}[H_t]$$

where

$$\begin{aligned}
\alpha_X(t) &= \alpha_1(t) - \alpha_2(t) + [\beta_\lambda(t) + \beta_M(t)]^2 - a\beta_k(t)\,[\beta_\lambda(t) + \beta_M(t)]\,; \\
\beta_X(t) &= a\beta_k(t) - \beta_\lambda(t) - \beta_M(t); \\
\alpha_1(t) &= a\alpha_k(t) + \frac{1}{2}a(a-1)\beta_k^2(t); \\
\alpha_2(t) &= \alpha_\lambda(t) + \alpha_M(t) + \beta_\lambda(t)\beta_M(t).
\end{aligned}$$

4.3 Conclusion

In this chapter, pricing formulas for IL securities were successfully derived. As initially expected, these formulas are mainly function of the selected macroeconomic factors and less function of the observed market prices. One of the main advantage of this approach is that there is a unique pricing formula for "every" IL derivatives. This fair price is not a function of the actual security being priced and in theory might be applied even to customised and exotic IL securities.

The calibration process has been started in Chapter 6. However, it was not pushed as far as planned because of the unavailability of the necessary market data. The preliminary study conducted justifies our choice of Lévy distributions as the driving factor and source of randomness of all processes.

Chapter 5

Reverse Engineering

In this chapter, the real and nominal pricing kernels are modelled without the use of utility functions. In 1994, Backus and Zin [8] developed the methodology of "Reverse engineering the yield curve" with application to the nominal yield curve. In 2003, Craig and Haubrich [36] conducted an implementation with the real term structure. Both studies were made in discrete time with underlying AR and ARMA processes; whereas the framework built here is in continuous time and its underlying processes follow Lévy and exponential Lévy distributions. The pricing kernels' dynamics are deduced from the market observed yield curves and the inflation. The obtained estimates of the nominal and real pricing kernels (See Section 5.5) reflect their relatively complex dynamics and more importantly the existing interaction between nominal economy and real economy. This provides a new perspective on how the market yield curves reveals nominal and real influences.

5.1 Inflation Breakeven Rate

Despite the fact that the approach presented in this chapter can be applied to any contingent claim, the current study only focuses on bonds (nominal and inflation linked). This sections briefly reviews some standard concepts related to bonds in general and others specific to inflation linked (IL) bonds. Among the latter is the less known inflation breakeven rate or inflation compensation, which is the difference between the yield of nominal bonds and the real yield obtained from IL bonds with same maturity. It has been widely used as a proxy of inflation expectation because of the Fischer hypothesis that is recalled later.

From now on, the subscript/exponent n (resp. r and IP) stands for nominal (resp. real and inflation protected).

For a "smooth" yield curve to be deductible from the market bonds, the following initial assumption is made.

Assumption 10. *There exists a market for nominal and IL bonds for all maturities $T > 0$ or at least for $T^* > T > 0$ with T^* fixed. Furthermore, for every fixed $t \in [0, T]$, $p_n(t, T)$ and $p_{IP}(t, T)$ are differentiable with respect to the maturity T.*

The instantaneous forward rates, contracted at time t are defined by

$$f_i(t,T) \quad = \quad -\frac{\partial \ln p_i(t,T)}{\partial T} \quad \text{for} \quad i = r, n;$$

and the instantaneous interest rates

$$r^i(t) \quad = \quad f_i(t,t) \quad \text{for} \quad i = r, n.$$

The continuously compounded yields y_i for $i = n, r$ are defined by

$$p_i(t,T) \quad = \quad \exp\left[-y_i(t,T) \cdot (T-t)\right].$$

Therefore

$$y_i(t,T) \quad = \quad -\frac{\ln p_i(t,T)}{T-t}.$$

The continuously compounded nominal yield can also be computed from the real bonds by [57]

$$y_n(t,T) \quad = \quad -\frac{1}{T-t}\ln\frac{p_r(t,T)I(t)}{I(T)} = -\frac{1}{T-t}\ln\frac{p_{IP}(t,T)}{I(T)}.$$

The previous equation can be decomposed as follow

$$\begin{aligned} y_n(t,T) \quad &= \quad -\frac{1}{T-t}\left[\ln p_r(t,T) + \ln\frac{I(t)}{I(T)}\right] \\ &= \quad y_r(t,T) + \frac{1}{T-t}\ln\frac{I(T)}{I(t)}. \end{aligned}$$

In the last equation, all the components except the $I(T)$ are known at time t; thus the last term is random at time t and its expected value will be used instead. The expression

$$i^e(t,T) \quad = \quad \frac{1}{T-t}\mathbb{E}_t\left[\ln\frac{I(T)}{I(t)}\right]$$

is the standard expectation of the average inflation between t and T. Hence, the Fisher hypothesis is retrieved

$$y_n(t,T) \quad = \quad y_r(t,T) + i^e(t,T).$$

Another relationship between the nominal and real yields can be computed using the nominal and real pricing kernels. Let $\pi^i = (\pi^i(t))_{t\geq 0}$ with $i = n, r$ denote the pricing kernels; the shorthand notation π^i_t for $\pi^i(t)$ will also be used.

Considering $t < T$, the real pricing kernel is defined by [68, 37]

$$p_r(t,T) \quad = \quad \frac{\mathbb{E}_t\left[\pi^r_T p_r(T,T)\right]}{\pi^r_t},$$

where $\mathbb{E}_t$ denotes the expectation conditional on the information available at time t.

Considering $\delta > 0$ and $s = t + \delta$ such that $t < s < T$, then

$$p_r(t,T) \quad = \quad \frac{\mathbb{E}_t\left[\pi^r_s p_r(s,T)\right]}{\pi^r_t}. \tag{5.1}$$

However, real bond prices are not directly observable on the market. The only way to ensure purchasing power is by taking a position in inflation protected securities (here IL bonds). At time t, the real and inflation protected bonds with maturity T are related by

$$p_r(t,T) \quad = \quad \frac{p_{IP}(t,T)}{I(t)},$$

where $I(t)$ is the inflation at time t. Substituting $p_r(t,T)$ in equation (5.1), gives

$$\begin{aligned} p_{IP}(t,T) &= \frac{\mathbb{E}_t\left[\pi^r_s p_{IP}(s,T)\dfrac{I_t}{I_s}\right]}{\pi^r_t} \\ &= \frac{\mathbb{E}_t\left[\pi^r_s p_{IP}(s,T)\dfrac{1}{G_s}\right]}{\pi^r_t}, \end{aligned} \tag{5.2}$$

where $G_s = \dfrac{I_s}{I_{s-\delta}}$ is the gross inflation return over $[s-\delta, s]$.

Since the nominal pricing kernel π^n is also given by

$$p_{IP}(t,T) \quad = \quad \frac{\mathbb{E}_t\left[\pi^n_s p_{IP}(s,T)\right]}{\pi^n_t}, \tag{5.3}$$

because $p_{IP}(t,T)$ is a nominal contingent claim. Identification between the last two equations breaks the nominal pricing kernel into the real pricing kernel and another component[1] due to inflation. The

[1]The term inflation is not directly used because of Equation (5.9) that will be used instead in our pricing framework.

obtained no-arbitrage relationship between the pricing kernels is

$$\pi_s^n = \frac{\pi_s^r}{I_s} \tag{5.4}$$

This relationship is model independent (i.e. satisfied without any assumption on the dynamics of the pricing kernels and inflation) and has been stated in [37]. Combined with the inflation dynamics, Equation (5.1) enables the partition of the nominal pricing kernel into real and nominal components which can both be estimated.

A relationship similar to Equation (5.1) exists between the nominal bonds and the nominal pricing kernel

$$p_n(t,T) = \frac{\mathbb{E}_t\left[\pi_s^n p_n(s,T)\right]}{\pi_t^n}. \tag{5.5}$$

Using Equation (5.4), the previous expression can be rewritten has

$$p_n(t,T) = \mathbb{E}_t\left[\frac{\pi_s^r}{\pi_t^r}\frac{1}{G_s}p_n(s,T)\right].$$

Taking $s = T$ yields

$$\begin{aligned} p_n(t,s) &= \mathbb{E}_t\left[\frac{\pi_s^r}{\pi_t^r}\frac{1}{G_s}\right] \\ &= \mathbb{E}_t\left[\frac{\pi_s^r}{\pi_t^r}\right]\mathbb{E}_t\left[\frac{1}{G_s}\right] + Cov\left[\frac{\pi_s^r}{\pi_t^r}, \frac{1}{G_s}\right] \\ &= p_r(t,s)\mathbb{E}_t\left[\frac{1}{G_s}\right] + Cov\left[\frac{\pi_s^r}{\pi_t^r}, \frac{1}{G_s}\right] \end{aligned}$$

by Equation (5.1). Taking the logarithm of the last equality, then taking the negation of its division by the time to maturity gives

$$y_n(t,T) = y_r(t,T) + i^e(t,T) + p^I(t,T), \tag{5.6}$$

where $i^e(t,T)$ and $p^I(t,T)$ are respectively the expected inflation and the inflation risk premium over $[t,T]$. The inflation risk premium is generally decomposed in two components: the Jensen's effect and the covariance effect [37] which are respectively defined by

$$J(t,T) = -\frac{\ln \mathbb{E}_t\left[G_s^{-1}\right] - \mathbb{E}_t\left[\ln G_s^{-1}\right]}{T-t}$$

and

$$c(t,T) = -\frac{1}{T-t}\ln\left(1 + \frac{Cov\left[\frac{\pi_s^r}{\pi_t^r}, G_s^{-1}\right]}{\mathbb{E}_t\left[G_s^{-1}\right]}\right).$$

If the inflation premium is zero, then the Fisher Hypothesis is recovered. However, recall from the Subsection 1.1.1 that the existence of a "non-zero" inflation risk premium in nominal bonds was one of the reasons for the issuance of IL bonds [38]. This suggest that the Fisher Hypothesis should in "general" mis-estimate the expected inflation because it ignores the covariance between the real pricing kernel and inflation. Nevertheless, the improvement is not without disadvantage; the pricing kernel approach requires models for (i.e. assumptions on) the pricing kernels and the CPI, while the Fisher hypothesis gives a simple (no models for yields) means to estimated inflation expectation.

The next section presents an assumption on the "information flow", which simplifies the computation of formulas and the calibration of the pricing kernel approach. Afterwards Sections 5.4 and 5.3 each makes assumptions on the dynamics of the pricing kernels and inflation for data fitting.

5.2 Theoretical framework

The existence of a pricing kernel is synonymous to the well known no-arbitrage principle. In fact, given a pricing kernel the price of any bond or derivative security can be computed. Here the reverse operation is accomplished: from bond prices, the nominal and real pricing kernels are inferred.

Following Backus et al. [8], to simplify the current framework and the fitting process, the next assumption will be made on the filtration.

Assumption 11. *In Equations (5.1) and (5.3), the pricing kernels π^i for $i = n, r$ are contained in $\mathbb{E}_t$, i.e. π_t^r is considered to be part of the information known at time t that is represented by $\mathcal{F}_t$ with $\mathbb{E}_t[\cdot] = \mathbb{E}_{\mathcal{F}_t}[\cdot]$.*

Note that derivative pricing is generally conducted at $t = 0$, where $\pi_0^i = 1$ for $i = n, r$. In which case the previous assumption is satisfied, therefore this property has been extended to forward pricing, i.e. pricing a security forward in time. Another interpretation of the previous assumption is that instead of modelling the pricing kernels, the change in the pricing kernel (in these equations between times t and s) is modelled. If $t = 0$, because $\pi_0^i = 1$ for $i = n, r$, this comes back to modelling the pricing kernels. Equation (5.1) becomes

$$p_r(t,T) \quad = \quad \mathbb{E}_t\left[\pi_s^r p_r(s,T)\right]$$

or similarly to [36]

$$\mathbb{E}_t\left[\pi_s^r R(t;s,T)\right] \quad = \quad 1, \tag{5.7}$$

where $R(t;s,T) = \dfrac{p_r(s,T)}{p_r(t,T)}$ is the gross real return over the period $[t,s]$. Inserting the IL bonds gives

$$p_{IP}(t,T) \quad = \quad \mathbb{E}_t\left[\pi_s^r p_{IP}(s,T)\frac{I(t)}{I(s)}\right].$$

Equation (5.3) is now

$$p_{IP}(t,T) \quad = \quad \mathbb{E}_t\left[\pi_s^n p_{IP}(s,T)\right]. \tag{5.8}$$

Identification of these two expressions of the IL bonds yields the new no-arbitrage relationship between the pricing kernels

$$\pi_s^n \quad = \quad \frac{\pi_s^r}{G_s}, \tag{5.9}$$

which is still model independent.

The reverse engineering approach [8] then proceeds as follows. It first specifies processes for π_t^r and G_t (or I_t); and then uses those to price the term structure, i.e. derive the yield on zero-coupon bonds of different maturities. This contrasts with the consumption based view (see examples in Chapter 4), in which the asset pricing equation takes the form

$$p_i(t,T) = \mathbb{E}_t\left[B\frac{U(C_s)}{U(C_t)}p_i(s,T)\right]$$

for a time separable utility function $U(\cdot)$. In this case, the stochastic process for consumption and the form of the utility function determine the pricing kernel.

Secondly, the time series and cross section properties implied by the theory are then matched with the data to derive the parameters of the underlying process and to deduce the pricing kernel and inflation. Once deduced, the two halves of the pricing kernel can function as a metric for assessing asset pricing theories and as an engine for pricing securities.

The general asset pricing condition (5.7) becomes a theory of bond pricing once the pricing kernel π^r and the gross inflation rate $G = (G_t)_{t\geq 0}$ are characterized.

The gross inflation return can be approximated as a function of the logarithm inflation rate in the following way:

$$\begin{aligned} G_s &= \frac{I_s}{I_{s-\delta}} = \frac{I_s - I_{s-\delta}}{I_{s-\delta}} + 1 \approx \ln\left(\frac{I_s - I_{s-\delta}}{I_{s-\delta}} - 1\right) + 1 \\ &= \ln\left(\frac{I_s}{I_{s-\delta}}\right) + 1. \end{aligned}$$

The approximation holds because $\frac{I_s - I_{s-\delta}}{I_{s-\delta}}$ is assumed "small enough". Note that the last line expresses the gross return in terms of the absolute return on inflation that will be denoted by r_s^I. So instead of studying the gross returns process G_s, the differenced log-price process r_s^I is studied. Recall that taking the log returns of a data set is a standard procedure from time-series analysis, which transforms a non stationary sequence to one that is plausibly modelled as stationary [24]. Sections 5.3 and 5.4 follow Craig et al. [36] by making assumptions on the distribution (of the logarithm) of the inflation gross return. If the gross return has an exponential Lévy growth (Section 5.3), then inflation has an exponential of exponential of Lévy growth. This is rather unrealistic especially given the "good" results other works [81, 82, 14, 71] obtained when pricing IL securities under the hypothesis that inflation had an exponential growth. Furthermore, this fast growth assumption might explain the fact that Craig et al. had good results on short maturities, but worsening results the longer the maturity considered. The more "realistic" hypothesis of inflation exponential growth is used in Section 5.4.

5.3 Exponential Lévy Gross return

The inflation gross return is a positive process, thus a reasonable choice of distribution is an exponential Lévy distribution. The following assumption extends the work of Craig et al. [36] by assuming the logarithm of inflation gross return follows a Lévy process instead of an AR(ARMA) process. Note that this makes the inflation an exponential of exponential of Lévy process.

Assumption 12. *Under the objective probability measure $\mathbb{P}$ the inflation gross return G and the real pricing kernel π^r follow exponential Lévy processes. Their dynamics are given by*

$$\begin{aligned}\frac{d\pi_t^r}{\pi_{t-}^r} &= \alpha^r(t)dt + \beta^r(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^r(t,z)\bar{\mu}(dt,dz)\\ \frac{dG_t}{G_{t-}} &= \alpha^G(t)dt + \beta^G(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^G(t,z)\bar{\mu}(dt,dz).\end{aligned}$$

with

$$\bar{\mu}(dt,dz) = \begin{cases} (\mu-\pi)(dt,dz), & |z| < R \\ \mu(dt,dz), & |z| \geq R \end{cases}$$

where the coefficients $\alpha^i(t)$, $\beta^i(t)$ and $\gamma^i(t,z)$ are adapted processes with

$$\int_0^t |\alpha^i(s)|ds < \infty \quad \text{and} \quad \int_0^t |\beta^i(s)|^2 ds < \infty$$

for all finite t; $\gamma^i(t,z) : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is a real valued function satisfying*

$$\int_0^t \int_{\mathbb{R}} |\gamma^i(s,z)|^2 \pi(ds,dz) < \infty,$$

for finite t. These conditions guarantee integrability of the coefficients and are satisfied if the coefficients are bounded for t from a bounded set and $\pi([0,t]\times\mathbb{R}) < \infty$ *for finite t.*

The following propositions compute the dynamics and analytic formulas for the pricing kernels and forward rates.

Proposition 5.1. *Under Assumption 12, the dynamics of the nominal pricing kernel are given by*

$$\frac{d\pi_t^n}{\pi_{t-}^n} = \alpha^n(t)dt + \beta^n(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^n(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}
\alpha^n(t) &= \alpha^r(t) - \alpha^G(t) + [\beta^G(t)]^2 + \int_{|z|<R} \frac{[\gamma^G(t,z)]^2}{1+\gamma^G(t,z)}\nu(dz) - \beta^r(t)\beta^G(t) \\
&\quad + \int_{\mathbb{R}} \gamma^r(t,z)\frac{\gamma^G(t,z)}{1+\gamma^G(t,z)}\nu(dz); \\
\beta^n(t) &= \beta^r(t) - \beta^G(t); \\
\gamma^n(t,z) &= \gamma^r(t,z) + \frac{\gamma^G(t,z)}{1+\gamma^G(t,z)} + \gamma^r(t,z)\frac{\gamma^G(t,z)}{1+\gamma^G(t,z)}.
\end{aligned}$$

Proof. Direct application of Corollary 2.15.

■

Under Assumption 11, the nominal and real bonds are given in terms of their corresponding pricing kernels by

$$p_i(t,T) = \mathbb{E}_t\left[\pi_s^i p_i(s,T)\right] \text{ for } i = n, r.$$

For a maturity $T = s$, i.e. $p_n(s,T) = 1$

$$p_i(t,T) = \mathbb{E}_t\left[\pi_T^i\right] \text{ for } i = n, r.$$

Hence, the time t price of a bond (nominal or real) is the expected value of its corresponding pricing kernel at the bond's maturity. Without Assumption 11, this expectation would have been multiplied by the pricing kernel at time t to get the bond price. This simplified relationship is used in the next proposition to compute the nominal and real bonds prices.

Proposition 5.2. *At time t, the nominal and real bonds with maturity T are worth*

$$\begin{aligned} p_i(t,T) &= \exp\left\{\int_0^T \alpha^i(u)du - \frac{1}{2}\int_t^T [\beta^i(u)]^2 du + \int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)du \right. \\ &\quad \left. + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz)\right\} \text{ for } i = n, r, \end{aligned}$$

where $\pi_{\gamma^i} = \pi \circ (\gamma^i)^{-1}$.

The dynamics of $p_i(t,T)$ *are*

$$\frac{dp_i(t,T)}{p_i(t-,T)} = a^i(t)dt + b^i(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} c^i(t,z)\bar{\mu}(dt,dz) \text{ for } i = n, r,$$

where

$$\begin{aligned} a^i(t) &= [\beta^i(t)]^2 - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz) + \int_{|z|<R} \left\{\exp[\gamma^i(t,z)] - 1 - \gamma^i(t,z)\right\} \nu(dz); \\ b^i(t) &= \beta^i(t); \quad c^i(t,z) = \exp[\gamma^i(t,z)] - 1; \end{aligned}$$

with constraints

$$\int_0^t \alpha^n(u)du + \int_0^t \beta^n(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^n(s,z)\bar{\mu}(ds,dz) = 0 \quad \forall t \in \mathbb{R}.$$

If the coefficient $\gamma^i(t,z)$ is deterministic, then

$$a^i(t) = [\beta^i(t)]^2 - \int_{\mathbb{R}} \left[e^{\gamma^i(t,z)} - 1 - \gamma^i(t,z)\chi_{|z|<R}\right] \pi(dz) + \int_{|z|<R} \left\{\exp[\gamma^i(t,z)] - 1 - \gamma^i(t,z)\right\} \nu(dz);$$

Proof. For $i = n, r$

$$\begin{aligned} \mathbb{E}_t[\pi_T^i] &= \pi_t^i \exp\left(\int_t^T \alpha^i(s)ds\right) \mathbb{E}_t\left[\exp\left(\int_t^T \beta^i(s)dW_s + \int_t^T \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz)\right)\right] \\ &= \pi_t^i \exp\left(\int_t^T \alpha^i(s)ds\right) \mathbb{E}_t\left[\exp\left(\int_t^T \beta^i(s)dW_s\right)\right] \mathbb{E}_t\left[\exp\left(\int_t^T \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz)\right)\right] \end{aligned}$$

since dW_t and $\bar{\mu}(dt,dz)$ are independent.

$$\begin{aligned} \mathbb{E}_t\left[\exp\left(\int_t^T \int_{\mathbb{R}} \gamma^i(u,z)\bar{\mu}(du,dz)\right)\right] &= \mathbb{E}_t\left[\exp\left(\int_t^T \int_{|z|<R} \gamma^i(u,z)(\mu-\pi)(du,dz) \right.\right. \\ &\quad \left.\left. + \int_t^T \int_{|z|\geq R} \gamma^i(u,z)\mu(du,dz)\right)\right] \\ &= \mathbb{E}_t\left[\exp\left(\int_t^T \int_{|z|<R} \gamma^i(u,z)(\mu-\pi)(du,dz)\right)\right] \\ &\quad \times \mathbb{E}_t\left[\exp\left(\int_t^T \int_{|z|\geq R} \gamma^i(u,z)\mu(du,dz)\right)\right] \end{aligned}$$

since $\{|z| < R\}$ and $\{|z| \geq R\}$ are disjoint. Using Corollary 2.21 for the first equality

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\int_t^T \int_{\mathbb{R}} \gamma^i(u,z)\bar{\mu}(du,dz)\right)\right] &= \exp\left\{\int_t^T \int_{|z|<R} [e^z - 1 - z]\, \pi_{\gamma^n}(dz)du\right\} \\
&\quad \times \exp\left[\int_t^T \int_{|z|\geq R} (e^z - 1)\pi_{\gamma^n}(dz)du\right] \\
&= \exp\left\{\int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^n}(dz)du\right\}
\end{aligned}$$

with $\pi_{\gamma^i} = \pi \circ (\gamma^i)^{-1}$.

Recall that $dW_t \sim \mathcal{N}(0, dt)$, hence $\displaystyle\int_t^T \beta^i(u)dW_u \sim \mathcal{N}\left(0, \int_t^T [\beta^i(u)]^2 du\right)$. Let $\langle \cdot, \cdot \rangle$ be the inner product, i.e. if $x, y \in \mathbb{R}^d$ such as $x = (x_1, x_2, \cdots, x_d)$ and $y = (y_1, y_2, \cdots, y_d)$, then $\langle x, y\rangle = \displaystyle\sum_{i=1}^d x_i y_i$.

Let $\psi_{\mathcal{N}(m,V)}$ denote the log-characteristic function of a normal distributed process with mean m and variance V, defined by

$$\psi_{\mathcal{N}(m,V)}(u) = i\,\langle m, u\rangle - \frac{1}{2}\langle u, Vu\rangle \text{ with } m, u \in \mathbb{R}^d.$$

In particular

$$\psi_{\mathcal{N}(m,V)}(u) = imu - \frac{1}{2}Vu^2 \text{ with } m, u \in \mathbb{R}.$$

Using Theorem 2.19

$$\mathbb{E}_t\left[\exp\left(\int_t^T \beta^i(u)dW_u\right)\right] = \exp\left\{-\frac{1}{2}\int_t^T [\beta^i(u)]^2 du\right\}.$$

Combining these expectations yields

$$\mathbb{E}_t[\pi_T^i] = \pi_t^i \exp\left\{\int_t^T \alpha^i(u)du - \frac{1}{2}\int_t^T [\beta^i(u)]^2 du + \int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)du\right\}.$$

Therefore

$$\begin{aligned}
p_i(t,T) &= \exp\left\{\int_t^T \alpha^i(u)du - \frac{1}{2}\int_t^T [\beta^i(u)]^2 du + \int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)du\right\} \\
&\quad \times \exp\left[\int_0^t \alpha^i(s)ds + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz)\right] \\
&= \exp\left\{\int_0^T \alpha^i(u)du - \frac{1}{2}\int_t^T [\beta^i(u)]^2 du + \int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)du\right. \\
&\quad \left. + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz)\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}\ln p_i(t,T) &= \int_0^T \alpha^i(u)du - \frac{1}{2}\int_t^T [\beta^i(u)]^2 du + \int_t^T \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)du \\ &\quad + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz).\end{aligned}$$

Denoting $\ln p_i(t,T)$ by $X(t,T)$

$$dX(t,T) = \frac{1}{2}[\beta^i(t)]^2 dt - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)dt + \beta^i(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^i(t,z)\bar{\mu}(dt,dz),$$

with constraints

$$X(t,t) = \int_0^t \alpha^i(u)du + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz).$$

But, by definition of nominal and real bonds, $\ln p_i(t,t) = 0$; therefore, the constraint becomes

$$\int_0^t \alpha^i(u)du + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz) = 0.$$

Using $p_i(t,T) = f[X(t,T)] = \exp[X(t,T)]$ in the one-dimensional Itô formula (Theorem 2.7), we have $\dfrac{\partial f}{\partial t} = 0$ and $\dfrac{\partial f}{\partial x} = \dfrac{\partial^2 f}{\partial^2 x} = p_n(t,T)$. Thus

$$\begin{aligned}dp_i(t,T) &= p_i(t,T)\left\{\frac{1}{2}[\beta^i(t)]^2 dt - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)dt + \beta^i(t)dW_t^{\mathbb{P}} + \frac{1}{2}[\beta^i(t)]^2 dt\right\} \\ &\quad + \int_{|z|<R} \left\{p_i(t-,T)\exp[\gamma^i(t,z)] - p_i(t-,T) - p_i(t-,T)\gamma^i(t,z)\right\}\pi(dt,dz) \\ &\quad + \int_{\mathbb{R}} \left\{p_i(t-,T)\exp[\gamma^i(t,z)] - p_i(t-,T)\right\}\bar{\mu}(dt,dz).\end{aligned}$$

The dynamics of p_i are given by

$$\begin{aligned}\frac{dp_i(t,T)}{p_i(t-,T)} &= \left\{\frac{1}{2}[\beta^i(t)]^2 dt - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz)dt + \beta^i(t)dW_t^{\mathbb{P}} + \frac{1}{2}[\beta^i(t)]^2 dt\right\} \\ &\quad + \int_{|z|<R} \left\{\exp[\gamma^i(t,z)] - 1 - \gamma^i(t,z)\right\}\pi(dt,dz) + \int_{\mathbb{R}} \left\{\exp[\gamma^i(t,z)] - 1\right\}\bar{\mu}(dt,dz) \\ &= \left\{[\beta^i(t)]^2 - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz) + \beta^i(t)dW_t^{\mathbb{P}} \right. \\ &\quad \left. + \int_{|z|<R} \left\{\exp[\gamma^i(t,z)] - 1 - \gamma^i(t,z)\right\}\nu(dz)\right\} dt + \int_{\mathbb{R}} \left\{\exp[\gamma^i(t,z)] - 1\right\}\bar{\mu}(dt,dz),\end{aligned}$$

with

$$\int_0^t \alpha^i(u)du + \int_0^t \beta^i(s)dW_s^{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} \gamma^i(s,z)\bar{\mu}(ds,dz) = 0.$$

If $\gamma^i(t,z)$ is assumed deterministic, then Theorem 2.22 is used. ∎

Proposition 5.3. *The nominal and real forward rate are*

$$f_i(t,T) = -\alpha^i(T) + \frac{1}{2}[\beta^i(T)]^2 - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz) \text{ for } i = n, r.$$

If γ^i is assumed deterministic, then

$$f_i(t,T) = -\alpha^i(T) + \frac{1}{2}[\beta^i(T)]^2 - \int_{\mathbb{R}} \left[e^{\gamma^i(t,z)} - 1 - \gamma^i(t,z)\chi_{|z|<R}\right] \pi(dz).$$

Proof. For $i = r, n$, the instantaneous forward rates contracted at time t are given by

$$\begin{aligned} f_i(t,T) &= -\frac{\partial \ln p_i(t,T)}{\partial T} \\ &= -\alpha^i(T) + \frac{1}{2}[\beta^i(T)]^2 - \int_{\mathbb{R}} \left[e^z - 1 - z\chi_{|z|<R}\right] \pi_{\gamma^i}(dz). \end{aligned}$$

If γ^i is deterministic, then Theorem 2.22 is again used. ∎

5.4 Exponential Lévy Inflation

Many studies [81, 82, 14, 71] have used the assumption that inflation is lognormal, which implies an exponential distribution for IL securities[2] instead of the previously assumed exponential of exponential distribution. This suggests that although the CPI is fast growing, it does grow at an exponential of exponential pace. The former is more realistic especially given that "most" can still afford daily expenses. The already successful lognormal distribution is here extended to an exponential Lévy process for the the inflation. By similitude, the pricing kernel is also assumed to be an exponential Lévy process. The dynamics of inflation's gross return will be implied by those of inflation as is shown in the next assumption.

Assumption 13. *Under the objective probability measure $\mathbb{P}$ the inflation I_t and the real pricing kernel follow exponential Lévy processes. Their dynamics are given by*

$$\begin{aligned} \frac{d\pi_t^r}{\pi_{t-}^r} &= \alpha^r(t)dt + \beta^r(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^r(t,z)\bar{\mu}(dt,dz); \\ \frac{dI_t}{I_{t-}} &= \alpha^I(t)dt + \beta^I(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^I(t,z)\bar{\mu}(dt,dz). \end{aligned}$$

with the standard integrability conditions imposed on the coefficients.

The log formulation is useful because yields and interest rates are easier to work with than bond prices, and it allows us to exploit some property of exponential Lévy distributions. Furthermore, in

[2]See Chapter 3 for example.

this form, π^r and G are not independent, but they do not depend directly on one another. Splitting inflation into nominal and real parts involves more than merely using more complicated distributions; it requires explicit consideration of the interactions between real and nominal rates.

The nominal pricing kernel is given by

$$\pi_t^n = \frac{\pi_t^r}{r_t^I + 1}.$$

This, in conjunction with equation (5.7), prices assets. It describes how the pricing kernel evolves over time, or equivalently, how the discount rate depends on both real and nominal shocks.

Proposition 5.4. *The dynamics of the nominal pricing kernel are given by*

$$\frac{d\pi_t^n}{\pi_{t-}^n} = \alpha^n(t)dt + \beta^n(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^n(t,z)\bar{\mu}(dt,dz),$$

where

$$\begin{aligned}
\alpha^n(t) &= \alpha^r(t) - Y_t\left[\alpha^I(t) - [\beta^I(t)]^2 Y_t\right] - \beta^r(t)Y_t\beta^I(t) \\
&\quad + \int_{|z|<R} \left\{\frac{1}{1+\gamma^I(t,z)Y_{t-}} - 1 + Y_{t-}\gamma^I(t,z)\right\}\nu(dz) \\
&\quad + \int_{\mathbb{R}} \gamma^r(t,z)\left(\frac{1}{1+Y_{t-}\gamma^I(t,z)} - 1\right)\nu(dz); \\
\beta^n(t) &= \beta^r(t) - Y_t\beta^I(t); \\
\gamma^n(t,z) &= \frac{\gamma^r(t,z)+1}{1+Y_{t-}\gamma^I(t,z)} - 1; \\
Y_t &= \frac{1}{r_t^I+1} = \frac{1}{G_t}.
\end{aligned}$$

Proof.

Assuming that δ is small enough, from the inflation dynamics (Assumption 13), the absolute inflation return

$$r_t^I = \ln\left(\frac{I_t}{I_{t-\delta}}\right)$$

has dynamics

$$dr_t^I = \alpha^I(t)dt + \beta^I(t)dW_t^{\mathbb{P}} + \int_{\mathbb{R}} \gamma^I(t,z)\bar{\mu}(dt,dz).$$

Applying the one-dimensional Itô formula (Theorem 2.7) with $Y_t = f(t, r_t^I) = \dfrac{1}{r_t^I + 1}$ whose derivatives are $\dfrac{\partial f}{\partial t} = 0$, $\dfrac{\partial f}{\partial r_t^I} = -\dfrac{1}{(r_t^I+1)^2}$ and $\dfrac{\partial^2 f}{\partial^2 r_t^I} = \dfrac{2}{(r_t^I+1)^3}$; we have

$$\begin{aligned}
dY_t &= -\frac{1}{(r_t^I+1)^2}\left[\alpha^I(t)dt + \beta^I(t)dW_t\right] + \frac{1}{2}[\beta^I(t)]^2\frac{2}{(r_t^I+1)^3}dt \\
&\quad + \int_{|z|<R}\left\{\frac{1}{r_{t-}^I+\gamma^I(t,z)+1} - \frac{1}{r_{t-}^I+1} + \frac{\gamma^I(t,z)}{(r_{t-}^I+1)^2}\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}}\left\{\frac{1}{r_{t-}^I+\gamma^I(t,z)+1} - \frac{1}{r_{t-}^I+1}\right\}\bar{\mu}(dt,dz) \\
&= -\frac{1}{(r_t^I+1)^2}\left[\alpha^I(t) - [\beta^I(t)]^2\frac{1}{r_t^I+1}\right]dt - \frac{1}{(r_t^I+1)^2}\beta(t)dW_t \\
&\quad + \int_{|z|<R}\left\{\frac{1}{r_{t-}^I+\gamma^I(t,z)+1} - \frac{1}{r_{t-}^I+1} + \frac{\gamma^I(t,z)}{(r_{t-}^I+1)^2}\right\}\pi(dt,dz) \\
&\quad + \int_{\mathbb{R}}\left\{\frac{1}{r_{t-}^I+\gamma^I(t,z)+1} - \frac{1}{r_{t-}^I+1}\right\}\bar{\mu}(dt,dz) \\
&= -Y_t^2\left[\alpha^I(t) - [\beta^I(t)]^2 Y_t\right]dt - Y_t^2\beta^I(t)dW_t \\
&\quad + \int_{|z|<R}\left\{\frac{Y_{t-}}{1+\gamma^I(t,z)Y_{t-}} - Y_{t-} + Y_{t-}^2\gamma^I(t,z)\right\}\pi(dt,dz) + \int_{\mathbb{R}}\left\{\frac{Y_{t-}}{1+Y_{t-}\gamma^I(t,z)} - Y_{t-}\right\}\bar{\mu}(dt,dz).
\end{aligned}$$

Now applying the multidimensional Itô formula (Theorem 2.11) with $f(t,\pi_t^r,Y_t) = \pi_t^r Y_t$

$$\begin{aligned}
d\pi_t^n &= \Big[\pi_{t-}^r\alpha^r(t)Y_t - Y_t^2\left[\alpha^I(t) - [\beta^I(t)]^2Y_t\right]\pi_t^r - \pi_{t-}^r\beta^r(t)Y_t^2\beta^I(t) \\
&\quad + \int_{|z|<R}\left\{\frac{Y_{t-}}{1+\gamma^I(t,z)Y_{t-}} - Y_{t-} + Y_{t-}^2\gamma^I(t,z)\right\}\nu(dz)\pi_t^r \\
&\quad + \int_{\mathbb{R}}\pi_{t-}^r\gamma^r(t,z)\left(\frac{Y_{t-}}{1+Y_{t-}\gamma^I(t,z)} - Y_{t-}\right)\nu(dz)\Big]dt \\
&\quad + \left[\pi_{t-}^r\beta^r(t)Y_t - Y_t^2\beta^I(t)\pi_t^r\right]dW_t \\
&\quad + \int_{\mathbb{R}}\Big[\frac{Y_{t-}}{1+Y_{t-}\gamma^I(t,z)}\pi_{t-}^r - Y_{t-}\pi_{t-}^r + \pi_{t-}^r\gamma^r(t,z)Y_{t-} \\
&\quad + \pi_{t-}^r\gamma^r(t,z)\frac{Y_{t-}}{1+Y_{t-}\gamma^I(t,z)} - \pi_{t-}^r\gamma^r(t,z)Y_{t-}\Big]\bar{\mu}(dt,dz)
\end{aligned}$$

$$\begin{aligned}
d\pi_t^n &= \Big[\pi_{t-}^n\alpha^r(t) - Y_t\left[\alpha^I(t) - [\beta^I(t)]^2Y_t\right]\pi_t^n - \pi_{t-}^n\beta^r(t)Y_t\beta^I(t) \\
&\quad + \int_{|z|<R}\left\{\frac{\pi_{t-}^n}{1+\gamma^I(t,z)Y_{t-}} - \pi_{t-}^n + \pi_t^nY_{t-}\gamma^I(t,z)\right\}\nu(dz) \\
&\quad + \int_{\mathbb{R}}\gamma^r(t,z)\left(\frac{\pi_{t-}^n}{1+Y_{t-}\gamma^I(t,z)} - \pi_{t-}^n\right)\nu(dz)\Big]dt \\
&\quad + \left[\pi_{t-}^n\beta^r(t) - Y_t\beta^I(t)\pi_t^n\right]dW_t \\
&\quad + \int_{\mathbb{R}}\left[\frac{\pi_{t-}^n}{1+Y_{t-}\gamma^I(t,z)} - \pi_{t-}^n + \gamma^r(t,z)\frac{\pi_{t-}^n}{1+Y_{t-}\gamma^I(t,z)}\right]\bar{\mu}(dt,dz).
\end{aligned}$$

■

From here on, a reasoning similar to that of Section 5.3 gives similar formulas for the IL bonds, nominal and real forward instantaneous forward rates.

5.5 Parameter estimation

At time t, the zero coupon bond price $p_i(t,T)$ for $i = n, r$ is known for a maturity T. To get the entire yield curve, forward rates over the period $[T, T^*]$ are needed, where T^* is the maturity up to which the yield curve is required. This comes back to estimating the unique driving process in the nominal and real yield curves just as in Subsection 6.1.8. The methodology of parameter estimation will not be repeated here.

Work in progress aims at first obtaining the historical real yield curve data for the South African market. Afterward, the calibration and interpretation of results will be conducted.

Chapter 6

Empirical Study and Calibration

Chapters 3, 4 and 5 each presented a different framework to price inflation linked (IL) derivatives taking into account the inflation market's illiquidity. This chapter now performs parameter estimation from market data for each of the previous models. The chapter is divided into two main sections. The first section conducts an empirical study of historical data to highlight the shortcomings of the normality assumption and the appropriateness of Lévy distributions. The second section deals with the actual pricing of derivative securities (swaps, caps and floors).

6.1 Empirical study

Let $(X_i)_{i\in\mathbb{N}}$ denote an observed macroeconomic factor and Δt be the fixed time step between the observations. Depending on the process being considered, Δt will be a day, a week or even a month. The observation X_i represents the value taken by the factor X at time $i\Delta t$ and $p_X(x_i)$ is the probability that X has the value x_i.

6.1.1 Data

The financial data used for parameter estimation is from both the Johannesburg Stocks Exchange (JSE) securities exchange and the New York Stock Exchange (NYSE). The former is an emerging (African) market while the latter is a developed (American) market. Thus the overall performances of the models in both types of markets can be tested. A detailed description of the data is provided in the Subsections 6.1.9 and 6.1.10.

The statistical study is carried out on the log returns of the macroeconomical factors for two main reasons. Firstly, financially, the log return corresponds to the continuously compounded return of the factor. It is dimensionless and assumed smooth "enough". Secondly, numerically, this is done to constrain return values to be positive. Others arguments for this approach are the extensive evidence of returns stationarity in the literature [34]; plus the direct transferability of return independence and identical distribution to its logarithm. Considering a discrete process (X_i), with $i = 1, 2, \cdots$, the corresponding log return process r_i, with $i = 1, 2, \cdots$, is defined by:

$$r_i \quad = \quad \frac{1}{\Delta t} \ln \left(\frac{X_{i+1}}{X_i} \right).$$

The time interval Δt is generally constant and equal to one time step and is thus ignored.

6.1.2 Statistics

This section briefly reviews some common descriptive statistics that will be used. A more detailed coverage can be found in Hamilton (1994). For illustration, the South African (SA) monthly Consumer Price Index (CPI) data between January 1965 and March 2008 (Figure 6.4(b)) is used. Monthly SA CPI between January 1960 and December 1964 were mostly constant over periods of at least six months and thus ignored because they do not provide any "useful" information for the study.

Definition 6.1 (Mean)**.** *The (sample) mean or first moment of X is defined by*

$$\mathbb{E}[X] \quad = \quad \sum_i x_i p_X(x_i),$$

where $(x_i)_{i \in I}$ is the set of attainable values by X with $I \subset \mathbb{N}$.

From here on, μ represents the mean of the distribution X.

Example 4. *The sample mean of South African (SA) Consumer Price Index (CPI) monthly log return between January 1965 and March 2008 is 0.74% which is rather low given that it is monthly and South Africa's target for monthly CPIX*[1] *is about $0.8666\% - 1.7321\%$, i.e. $3\% - 6\%$ annually.*

Definition 6.2 (Variance)**.** *The (sample) variance or central second moment of X is defined by*

$$Var(X) = \mathbb{E}[(X - \mu)^2] \quad = \quad \sum_i (x_i - \mu)^2 p_X(x_i),$$

[1] See Subsection 6.1.9 for a definition.

where $(x_i)_{i\in I}$ *is the set of attainable values by* X *with* $I \subset \mathbb{N}$.

The variance is characterized by

$$Var(X) \quad = \quad \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

where $\mathbb{E}[X^2] = \sum_i x_i^2 p_X(x_i)$ *is the second non-central moment of* X.

The variance and its square root, the standard deviation or volatility, are measures of the uncertainty of the return of a specific factor. In the market, a period of relatively low (resp. high) risk is generally followed by periods of relatively low (resp. high) risk. This phenomenon is referred to as volatility clustering [59].

From here on, σ represents the standard deviation of the distribution X.

Example 5. *The standard deviation of SA CPI log returns over the period from January* 1965 *to March* 2008 *is* 0.70% *which is only* 4 *basis points (bp) less than its mean. This can be explained by the rather low average and the high volatility of the earliest CPI (Figure 6.4(b)). Unfortunately the small amount of data (all recorded SA CPI) prevent the exclusion of the earliest data sample.*

Definition 6.3 (Skewness)**.** *The (sample) skewness is a measure of the asymmetry of a distribution with respect to (in short w.r.t.) its mean. It is defined by*

$$S(X) \quad = \quad \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3}.$$

Note that for a normal distribution the skewness is zero. A distribution with positive skewness (right skewed) has a fatter right tail, i.e. it is more likely to have its values above the mean value than below. Likewise, a distribution with negative skewness (left skewed) has a fatter left tail, i.e. it is more likely to have its values below the mean value than above.

Example 6. *The skewness of the SA CPI log returns is* 0.9163*. Therefore, the CPI is asymmetric (i.e. non-normal) and has mass concentrated on the right.*

Definition 6.4 (Kurtosis)**.** *The (sample) Kurtosis is a measure of the peakedness (i.e. tail behavior) of a distribution. It is defined by*

$$K(X) \quad = \quad \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4}.$$

A normal distribution has a kurtosis of 3. A distribution with kurtosis greater than 3 is said to have “fat tails” or to be leptokurtic, i.e. it is more peaked than a Gaussian around the mean. A distribution with kurtosis less than 3 is said to be platykurtic.

Some programs return the excess kurtosis which is the kurtosis minus 3 instead of the kurtosis.

Example 7. *The kurtosis of the SA CPI log returns is* $4.5877 > 3$*. Hence the log return innovations' density function is more peaked than the normal density function.*

A financial time series can be viewed as a sequence of random observations. This random sequence, or stochastic process, may exhibit some degree of correlation from one observation to the next. This correlation structure can be used to predict future values of the process based on past observations. Exploiting the correlation structure, if any, allows the decomposition of the time series into a deterministic component (i.e. the forecast), and a random component (i.e. the error, or uncertainty, associated with the forecast). Autocorrelation and partial autocorrelation are important tools for studying stationary time series such as simple autoregressive (AR) models, moving-average (MA) models, autoregressive moving-average (ARMA) models and seasonal models [59].

Definition 6.5 (Autocorrelation)**.** *The autocorrelation function (ACF) is a measure of cross-dependence of a distribution with itself given a time lag. It is useful to find repeating patterns in a distribution. The* j^{th} *(sample) autocorrelation of the distribution* X *is defined by*

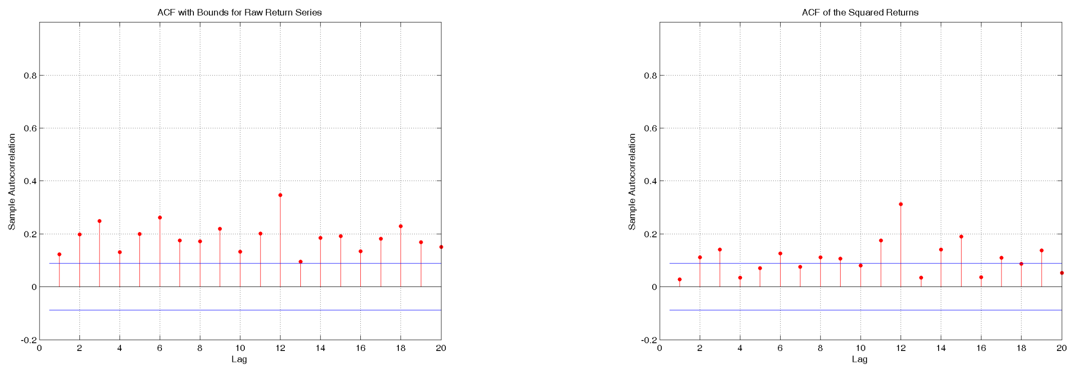
$$\rho_j = \frac{Cov(X_i, X_{i+j})}{\sqrt{Var(X_i)Var(X_{i+j})}},$$

where

$$Cov(X, Y) = \mathbb{E}[(X - \mu_x)(Y - \mu_y)]$$

represents the (sample) covariance of the distributions X *and* Y*; with* μ_x *(resp.* μ_y*) the average of* X *(resp.* Y*).*

Example 8. *Figures 6.1 show that the South African CPI log returns and its square are slightly autocorrelated. The highest correlation (at lag* 12*) is due to annual seasonality of the CPI. This translates to the fact that during festive periods (Christmas, end of year, etc) prices tend to increase because of the higher demand; and (most of) agriculture follow an annual cycle and thus the prices of related goods is seasonal, this annual seasonality is not surprising. Moreover, it is present in most (if not all) CPIs worldwide. Notice the similar distribution of the spikes (of log returns) for the subsets* $\{1, 2, 3\}$*,* $\{4, 5, 6\}$ *and* $\{10, 11, 12\}$*. The spikes at lag* 1*,* 4 *and* 10 *(resp.* 2*,* 5 *and* 11*) are almost (resp. perfectly) identical. The spike at lag* 12 *is surely greater than those at lags* 3 *and* 6 *just because of the annual seasonality. Strangely enough, log returns are more correlated than their square this might be due to the small size of data.*



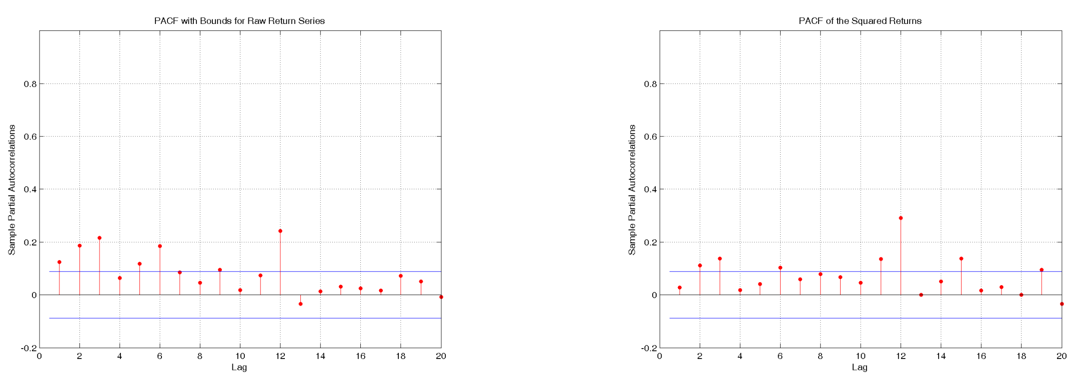
(a) Log returns (b) Squared returns

Figure 6.1 Monthly SA CPI correlograms.

Definition 6.6 (Partial autocorrelation)**.** *The partial autocorrelation function (PACF) is a measure of the conditional cross-dependence of a distribution with itself given a time lag. The PACF removes the effect of shorter lag autocorrelation from the correlation estimate at longer lags. The* m^{th} *(sample) partial autocorrelation of the distribution* X *is defined by*

$$\varphi_{m,m} = \frac{\rho_m - \sum_{j=1}^{m-1} \varphi_{m-1,j}\rho_{m-1}}{1 - \sum_{j=1}^{m-1} \varphi_{m-1,j}\rho_j},$$

where ρ_j *is the autocorrelation with a time lag of* j*.*



(a) Log returns (b) Squared returns

Figure 6.2 Monthly SA CPI partial correlograms.

Example 9. *In Figure 6.2, as expected the monthly SA CPI partial correlograms' spikes are below that of the correlograms. The log returns autocorrelation spikes are only maintained at lags* 1, 2, 3, 5 *and* 6*; meaning that the remaining higher-order autocorrelations are due to these initial autocorrelations. Hence, when forecasting monthly CPI, it is not "necessary" to use data beyond a year from the prediction date. This is a common and successful practise in the South African market. However, this should only give "good" results for a year or less forecast. For a longer period forecast and a more accurate forecasting framework, Lévy distributions should be used. Recall that the normal distribution is a particular case of a Lévy distribution, a generalisation of the standard forecast will thus be obtained by using Lévy processes.*

The autocorrelations at lags 1 *and* 5 *are "quite" small and can be ignored. Between the raw squared returns ACF and PACF, the only significant autocorrelation maintained is the one at lag* 12*. This corresponds to a year periodicity, i.e. the CPI seasonality. The non-existence of a high autocorrelation in the squared innovations might be due to data sparsity (granularity and size). This issue is "solved" later by increasing the data size (See Subsection 6.1.6).*

6.1.3 Hypothesis Tests

A hypothesis test is a procedure used to check if a certain criterion is satisfied by a given sample distribution. This study conducts two families of hypothesis tests:

(i) Normality tests: Jarque-Bera test, Kolmogorov-Smirnov test, the Pearson's Chi-squared test, Normal Probability Plots and Quantile-Quantile Plots;

(ii) Heteroscedasticity tests: Ljung-Box-Pierce Q-test and Engle's ARCH test.

Normality tests investigate if a sample data comes from a normal distribution. While heteroscedasticity (i.e. ARCH/GARCH effects) tests investigate if the sample data's variance is non-constant (i.e. time varying). Heteroscedasticity tests are commonly used to quantify the correlation in a sample data. Here is a brief description of the selected hypothesis tests.

Jarque-Bera test

The Jarque-Bera (JB) test examines whether a specific distribution is normal or not. The JB-value is calculated as:

$$JB(X) \quad = \quad \frac{n-l}{6}\left\{S^2(X) + \frac{[K(X)-3]^2}{4}\right\},$$

where n is the number of observations, $S(\cdot)$ the skewness function, $K(\cdot)$ the kurtosis function and l is the number of estimated parameters. The intuition behind this test is that the larger the JB-value is, the lower the probability is that the given series is drawn from a normal distribution. For large sample size, the test statistic of the Jarque-Bera test is χ^2-distributed with 2 degrees of freedom under the null hypothesis that the series is normally distributed.

Example 10. *The Jarque-Bera test of the monthly SA CPI log returns yields $h = 1$, $p = 10^{-3}$. In fact the p-value is less than 10^{-3}, which is the smallest value returned by the Matlab function* jbtest. *The p-value is below the default significance level of 5%, and the test rejects the null hypothesis that the distribution is normal.*

Kolmogorov-Smirnov test

The Kolmogorov-Smirnov (KS) test is a goodness of fit test used to determine whether two underlying one-dimensional probability distributions differ, or whether an underlying probability distribution differs from a hypothesized distribution, in either case based on finite samples.
The one-sample KS test compares the empirical distribution function with the cumulative distribution function specified by the null hypothesis. The main applications are testing goodness of fit with the normal and uniform distributions. For normality testing, minor improvements made by Lilliefors lead to the Lilliefors test. This test is sensitive to differences in both location and shape of the empirical cumulative distribution functions of the two distributions.
The Anderson-Darling (AD) [99] is another modification of the KS test. It gives more weight to the tails than does the KS test. Contrary to the KS test, the AD test's critical values are functions of the specific distribution being tested. This implies a more sensitive test, but critical values have to be computed for each distribution.

Example 11. *The Kolmogorov-Smirnov test of monthly SA CPI log returns yields $h = 1$, $p = 4.5267 \cdot 10^{-112} \approx 0$, $k = 0.4970$ and $c = 0.0595$ where k (resp. c) is the test statistic, i.e. the maximum difference between the cumulative distribution functions (resp. the cutoff value for determining if k is significant). Since $h = 1$, the test rejects the null hypothesis that the values come from a normal distribution at the 5% significance level. A look at the Kolmogorov-Smirnov test in Figure 6.3 confirms the unsuitability of the normal distribution.*
Henceforth, p-value as small as the previous will be assimilated to 0.
The Lilliefors test returns $h = 1$ and $p = 0$. This test also rejects the normality of the monthly SA

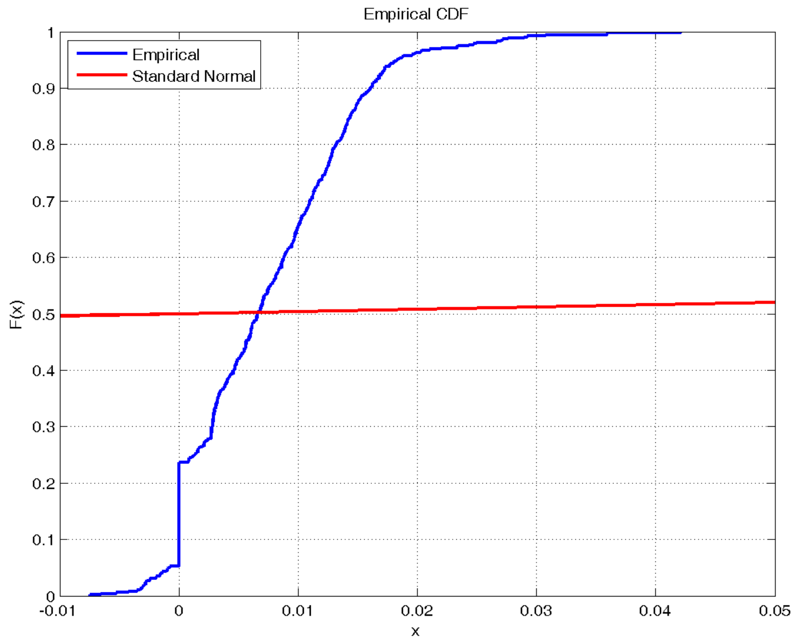


Figure 6.3 SA CPI raw returns Kolmogorov-Smirnov test.

CPI log returns. The AD test also reject the normality of the monthly SA CPI log returns.

The Chi-squared Test

The χ^2 test has as null hypothesis that the provided data comes from a specified distribution with unknown parameters. If the considered distribution is a normal distribution, then the null hypothesis is that the data sample is from a normal distribution with unknown mean and standard variance, which are estimated from the sample data. The test counts the number of sample points falling into certain intervals (referred to as bins) and compares these counts with the expected number in these intervals under the null hypothesis. The χ^2 statistic is given by

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i},$$

where O_i and E_i are respectively the observed and expected counts.

Normal Probability Plot

A normal probability plot (See Figure 6.11(a)) is a useful graph for assessing that a data sample comes from a normal distribution. Many statistical procedures make the assumption that the underlying distribution of the data is normal, so this plot can provide some assurance that the assumption of normality is not being violated, or provide an early warning of a problem.

Quantile-Quantile Plot

A quantile-quantile plot (See Figure 6.11(b)) is useful for determining whether two samples come from the same distribution (the distribution can be normal or not).
Even though the parameters and sample sizes are different, the straight line relationship shows that the two samples come from a similar kind of distribution. The set consisting of numerous pluses is the quantiles of each sample. By default the number of pluses is the number of data values in the smaller sample. The solid line joins the 25^{th} and 75^{th} percentiles of the samples. The dashed line extends the solid line to the extent of the sample.

Ljung-Box-Pierce Q-test

The Ljung-Box-Pierce (LBP) Q-test is performed to test jointly whether several autocorrelations of data series are significant or not. The LBP Q-value is calculated by:

$$Q_k = n(n+2)\sum_{i=1}^{k}\frac{\rho_i^2}{n-i},$$

where n is the sample size, k is the number of lags and ρ_i the i^{th} autocorrelation. If Q_k is large then the probability that the process has uncorrelated data decreases. The null hypothesis for the test is that there exists no correlation and under that hypothesis, Q_k is χ^2-distributed with k degrees of freedom.

Example 12. *The Ljung-Box-Pierce Q-test estimate of the autocorrelation present in the raw and squared SA CPI returns when tested for up to* 10, 15, *and* 20 *lags at* 0.05 *level of significance gives*

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat**	**Crit**
10	1	0	193.7546	18.3070	0	0.7479	6.7599	18.3070
15	1	0	321.7936	24.9958	1	0	65.5773	24.9958
20	1	0	405.1546	31.4104	1	0	69.8175	31.4104

Table 6.1 Ljung-Box-Pierce Q-test for SA CPI raw and squared returns.

The column "Stat" (resp. "Crit") is the vector of Q-statistics for each lag (resp. the vector of critical values of the χ^2 distribution for comparison with the corresponding element of "Stat").
The correlation in the squared of raw returns translates the existing volatility clustering that will be captured by the $GARCH(1,1)$ filter presented in Section 6.1.5.

Engle's ARCH Tests

It is fairly easy to test whether the residuals from a regression have conditional heteroskedasticity or not. The test is based on Ordinary Least Squares (OLS) regression, where the OLS residuals $\hat{u}_t$ from the regression are saved. The process $\hat{u}_t^2$ is thereafter regressed on a constant and its own m-lagged values. This is done for all samples $t = 1, 2, \cdots, n$. This regression has a corresponding R^2-value. The distribution nR^2 is then asymptotically χ^2-distributed with m degrees of freedom under the null hypothesis that $\hat{u}_t$ is i.i.d. $\mathcal{N}(0, \sigma^2)$ [50].
This ARCH-test can also be performed as a test for GARCH-effects. The ARCH-test for lag $(p+q)$ is locally equivalent to a test for GARCH effects with lags $(p; q)$. (MathWorks 2007)
The null hypothesis, H_0, is that no ARCH effects exist.

Example 13. *The Engle's ARCH test (Table 6.2) confirms the existence of autocorrelation and GARCH effects only for time lags of* 15 *and* 20*.*

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat**	**Crit**
10	0	0.8166	5.9839	18.3070	0	1	0.5681	18.3070
15	1	0	56.4319	24.9958	1	0	48.8665	24.9958
20	1	0	59.2076	31.4104	1	0.0004	48.4818	31.4104

Table 6.2 Engle's ARCH test results for SA CPI raw and squared returns.

6.1.4 Goodness of Fit

This section fits a wide variety of distributions to the empirical distribution of a data sample. To evaluate the performance of each distribution, multiple goodness of fit assessment measures are also used. The latter can be divided in two majors groups; visual assessment and quantitative assessment.

The visual assessment indicators are the normal probability plots and quantile-quantile plots that have already been introduced.
Similarly, most of the quantitative assessment indicators have already been presented. These are Kolmogorov-Smirnov test, the Pearson's Chi-squared test, which are not only restricted to the test of normality. The log-likelihood estimate is a common measure generally associated to the maximum likelihood estimator which is described in Section 6.2.1.
The Akaike information criterion (AIC) that comes with the *ghyp* package of R is also used. These goodness-of-fit test will be used after the parameter estimation for the Lévy distributions in Subsection 6.1.7.

6.1.5 Data Filtering

The hypothesis of independent price returns is extremely important in financial modelling. So is the time varying volatility which can be reproduced by Lévy models. To reinforce the observed volatility clustering, a GARCH filter first captures the persistence in the volatility. Moreover, McNeil et al [80] argued that a GARCH$(1,1)$ model with Student t innovations is enough to remove the dependence in return series. This approach is used here to render the return series "more" independent and identically distributed (i.i.d.). In this study, GARCH$(1,1)$ filters with normal and student-t innovations are considered.

GARCH model

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) is a generalization of the ordinary ARCH-model. The model structure was introduced by Bollerslev [22]. The generalization with respect to ARCH model is similar to the extension of an AR(p) to an ARMA(p,q).
The intuitive introduction to GARCH models presented here is similar to that done by John Hull in [69]. GARCH models are generally used to reproduce and forecast volatility and correlation. As mentioned previously, the standard deviation σ_t or its square is a convenient measure of risk. The continuously compounded interest rate y_t of the asset price represented by X_t is defined by

$$y_i = \ln\left(\frac{X_i}{X_{i-1}}\right).$$

In this section σ_i denotes the standard deviation of the rate y_i at time $i\Delta t$. An estimate of σ_i using the m most recent observations is

$$\sigma_i^2 = \frac{1}{m-1}\sum_{j=1}^{m}(y_{i-j}-\bar{y})^2, \tag{6.1}$$

where $\bar{y}$ denotes the average of y_i for $i \in [i-m, i-1]$:

$$\bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_{i-j}.$$

From Equation (6.1), the following approximations are made

(i) the rate y_i is defined as the percentage change between time $(i-1)\Delta t$ and $i\Delta t$:

$$y_i = \frac{X_i - X_{i-1}}{X_{i-1}}; \tag{6.2}$$

(ii) the average $\bar{y}$ is considered to be zero;

(iii) the denominator $m-1$ is replaced by m.

These changes simplify the variance formula to

$$\sigma_i^2 = \frac{1}{m}\sum_{j=1}^{m} y_{i-j}^2,$$

where y_i is given by Equation (6.2). Moreover, they do not affect the variance estimates much.

However, the previous equation gives the same weight to all the used observations; because of volatility clustering it is more reasonable to give higher weights to recent observations. This yields

$$\sigma_i^2 = \sum_{j=1}^{m} \alpha_j y_{i-j}^2,$$

with $\displaystyle\sum_{j=1}^{m}\alpha_j = 1$. The weights are all positive and $\alpha_j < \alpha_k$ for $j > k$ translates the fact that less weight is given to older observations.

Under the further assumption that there is a long-run average rate y_L which should be given a weight,

$$\sigma_i^2 = \gamma y_L + \sum_{j=1}^{m} \alpha_j y_{i-j}^2 \quad \text{or}$$

$$\sigma_i^2 = \omega + \sum_{j=1}^{m} \alpha_j y_{i-j}^2,$$

with $\omega = \gamma y_L$ and $\gamma + \displaystyle\sum_{j=1}^{m}\alpha_j = 1$.

The latter model is known as an ARCH(m) model. A GARCH(m, n) model extends the ARCH(m) model, by assuming that σ_i is not only a function of the long-run average rate and the last m observed rates, but also of the last n variances. The model is defined by

$$\begin{aligned}\sigma_i^2 &= \gamma y_L + \sum_{j=1}^{m} \alpha_j y_{i-j}^2 + \sum_{j=1}^{n} \beta_j \sigma_{i-j}^2 \quad \text{or}\\ \sigma_i^2 &= \omega + \sum_{j=1}^{m} \alpha_j y_{i-j}^2 + \sum_{j=1}^{n} \beta_j \sigma_{i-j}^2,\end{aligned}$$

with $\omega = \gamma y_L$ and $\gamma + \sum_{j=1}^{m} \alpha_j + \sum_{j=1}^{n} \beta_j = 1$.

GARCH $(1, 1)$

In the case where $m = n = 1$, the model reduces to

$$\begin{aligned}\sigma_i^2 &= \gamma y_L + \alpha y_{i-1}^2 + \beta \sigma_{i-1}^2 \quad \text{or}\\ \sigma_i^2 &= \omega + \alpha y_{i-1}^2 + \beta \sigma_{i-1}^2,\end{aligned}$$

with $\omega = \gamma y_L$ and $\gamma + \alpha + \beta = 1$.

When estimating the parameters, ω, α and β are first evaluated and γ deduced as $1 - \alpha - \beta$. A stable GARCH$(1, 1)$ model requires $\alpha + \beta < 1$ for γ to be positive. Note that this model is mean reverting since it assumes that the variance rate is always pulled back to the long-run average.

Parameter estimation

The GARCH$(1, 1)$ filter is not directly applied to the log returns, but to the intermediate distribution

$$\hat{r}_i = \frac{r_i - \bar{r}}{\sqrt{Var(r)}}. \tag{6.3}$$

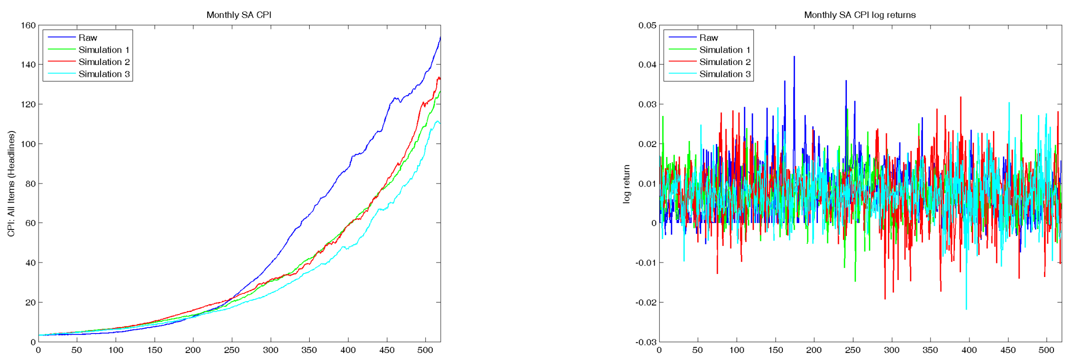
This distribution has an average of zero which agrees with the approximation (ii) made when building the GARCH model. After the GARCH$(1, 1)$ parameters for $\hat{r}_i$ are estimated, the filtered interest rate is obtained from the model generated $\hat{r}_i$ through Equation (6.3).

Example 14. *The presence of heteroscedasticity (GARCH effects), shown in the previous analysis, indicates that GARCH modelling is appropriate. The Matlab function* garchfit *is used with Student t innovations to estimate the $GARCH(1, 1)$ parameters. After* 19 *iterations, the monthly SA CPI from January* 1965 *to* 2008 *gives the following parameters: $C = -0.095175$, $K = 0.010126$, $GARCH(1) =$*

0.94397 *and* $ARCH(1) = 0.045577$*. Hence, the constant conditional mean/*$GARCH(1,1)$ *conditional variance model that best fits the observed data is*

$$\begin{aligned}\hat{r}_t &= -0.095175 + \varepsilon_t \\ \hat{\sigma}_t^2 &= 0.010126 + 0.94397 \cdot \hat{\sigma}_{t-1}^2 + 0.045577 \cdot \varepsilon_{t-1}^2,\end{aligned}$$

where $\hat{\sigma}_t$ *is the standard deviation of* $\hat{r}_t$ *and* ε_t *represents the student* t *innovations.*

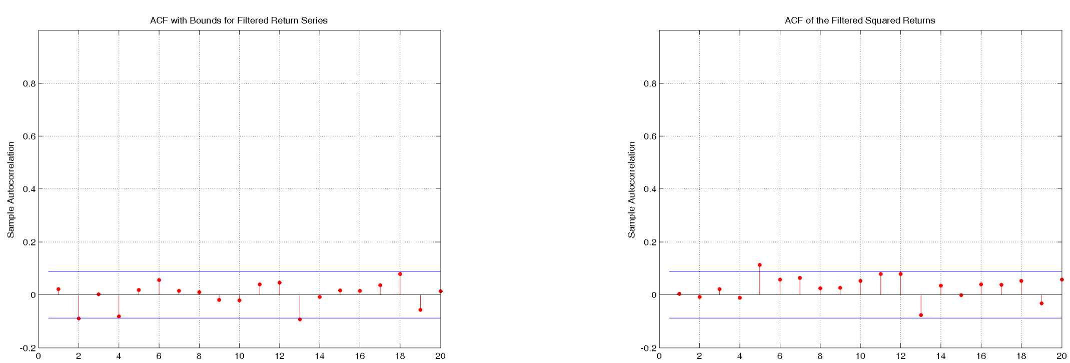


(a) CPI

(b) Log returns

Figure 6.4 Filtered vs raw SA CPI data series.

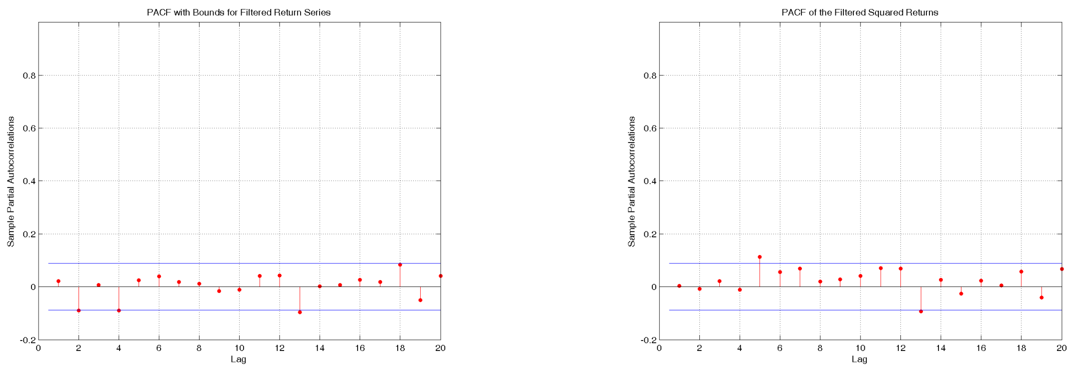
Figures 6.4 give plots of the raw vs filtered simulations for respectively the SA CPI and its log return. The filtered data has kept the general behaviour of the initial data without the unwanted trend at the beginning of the log returns.



(a) Log returns

(b) Squared returns

Figure 6.5 Filtered monthly SA CPI correlograms.



(a) Log returns

(b) Squared returns

Figure 6.6 Filtered monthly SA CPI partial correlograms.

Figures 6.5 and 6.6 show that the ACF and PACF of both the filtered log return series of SA CPI and its square have little serial correlation, thus the GARCH filter is “good”. Recall that by definition the increments of Lévy distributions are independent, i.e. not correlated. Therefore, this filtered series is better suited for Lévy distributions’ parameter estimation than the initial series.

6.1.6 Increasing the data size

A “good” empirical study, generally requires a large data sample for many reasons. Firstly, the parameter estimation for most of the models used in this empirical study needs such a dataset. For example, it is advised on the Willmot forum to have at least 700 to 800 data points in the empirical series for a $GARCH(1,1)$ model fitting. Secondly, the bigger the sample data, the smoother the QQ-plots and density plots obtained, which will be used to assess the distributions’ goodness of fit. Thirdly, because there are a lot of data points, there is no need to run multiple simulations as is commonly done with Monte Carlo simulations. However, for most of our South African data, the available dataset has less than 700 elements. To remedy to this, two alternatives are considered:

(i) Linear interpolation is used to get a daily dataset from the monthly dataset. This is a common practice when dealing with CPI or CPIX, therefore the obtained results are still relevant.

(ii) The GARCH$(1,1)$ filter is used to increase the size of the data. This is done in two steps: first parameters of the GARCH model are estimated; then when generating the filtered data, a bigger dataset is generated.

With the two approaches, there is no added information in the filtered dataset. The first method will only use the most recent information (CPI over the last 4 years for example), thus ignoring old data whose behaviour is less likely to be related to today's data behaviour. This is particularly true for SA who is having high CPI hikes nowadays against almost constant CPI in 1965. In general, the bigger the time span of the data, the more different the initial and final sub-data's behaviour are. One inconvenience of this approach, is that the change of granularity of data through linear interpolation might generate more (positive) correlation.

The second approach focuses on maintaining general volatility behaviour of the dataset. But given the small size of the data, the GARCH model used to increase the data's size is not that "well" fitted to the initial data. Therefore, the forecast (i.e. added data points) might tamper with the results. However, because the primary goal of IL securities is to protect against inflation risk, it seems reasonable to give priority to reproducing the inflation's volatility.

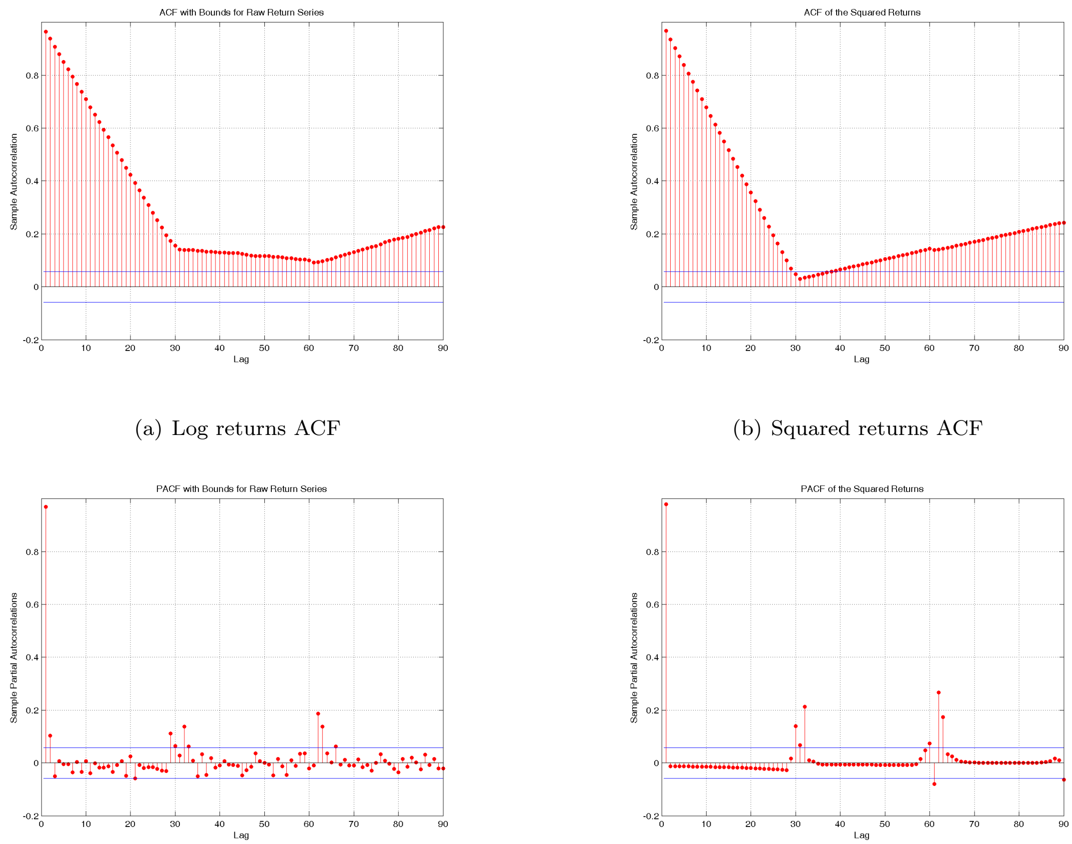
Daily South African CPI

The descriptive statistics of the daily SA CPI between January 2005 (in fact the 31^{st} December 2004 for interpolation purpose) and March 2008 are provided in Table 6.8. In total the data has 1187 data points. The SA CPI went from being more peaked than the normal density function for the monthly dataset to less peaked than the normal density function. The daily dataset is also more symmetric than its monthly counterpart. These changes suggest that assuming normal distribution should give better results with the daily CPI as compared to monthly CPI.

As expected the only major change in the autocorrelation's spikes compared to the monthly SA CPI is the appearance of spikes at lags $1 - 30$. These are due to the linear interpolation performed in between the monthly (i.e. 30 days on average) CPI to get the daily CPI. The spikes for the monthly lags should not have change much compared to Figures 6.1 and 6.2.

For daily CPI, the GARCH filter also successfully reduces data autocorrelation (See Figure 6.8).

The flat levels observed for the raw log returns in Figure 6.9(b) are due to the linear interpolation (i.e. almost constant return over a month). The filtered simulations have a "quasi" zero volatility; this is more obvious when looking at the simulated CPI (Figure 6.9(a)). The simulated daily SA CPI are "perfectly" superposed and linear, thus the CPI is "fully" deterministic which is not wanted in the model. Taking a bigger sample size (back up to 2001, i.e. 2648 data points) did not solve the problem of linearity and non-zero volatility; however increasing the granularity of the data (weekly



(a) Log returns ACF

(b) Squared returns ACF

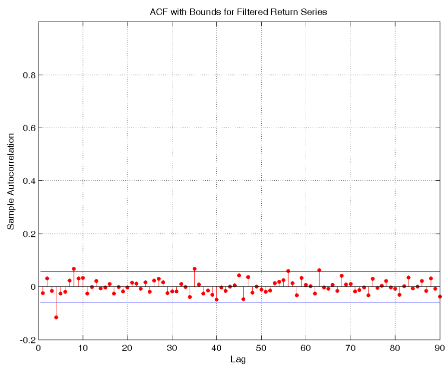
(c) Log returns PACF

(d) Squared returns PACF

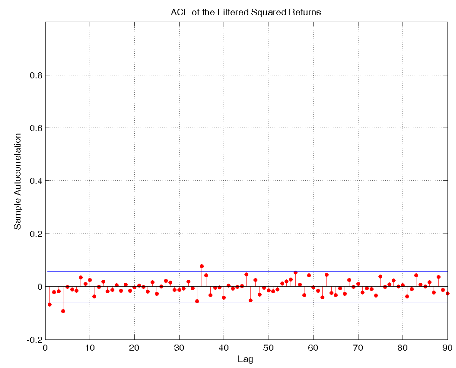
Figure 6.7 Daily SA CPI (partial) correlograms.

instead of daily) might reduce the effect of the linear interpolation. Taking weekly data should preserve some of the volatility and increase the data size following common market practise.

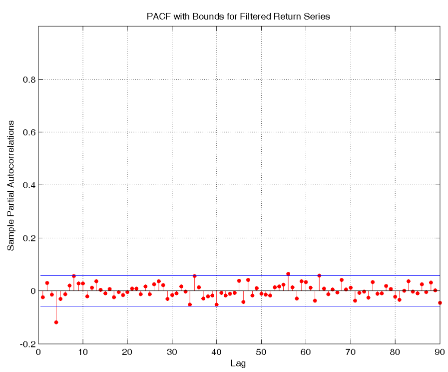
The previous GARCH filter is using normal innovations instead of student-t innovations. As can be seen in Figure 6.10, when using student-t innovations, the log returns are "almost" constant, i.e. zero volatility. The fact that the daily CPI is deterministic is more true with student-t innovations than with normal innovations. In the former case, it suffices to estimate the "constant" log return to be able to forecast CPI. However, the observed log return of the CPI on the market is not constant.



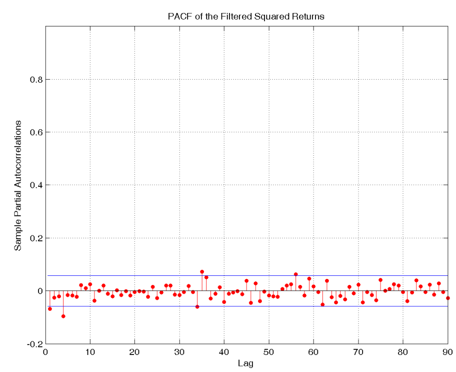
(a) Log returns ACF



(b) Squared returns ACF

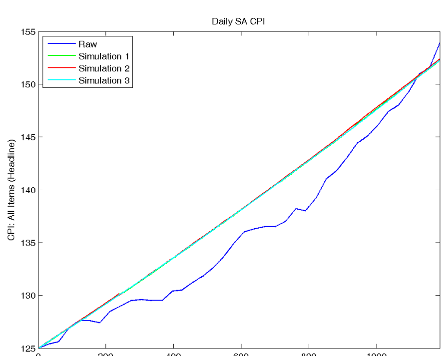


(c) Log returns PACF

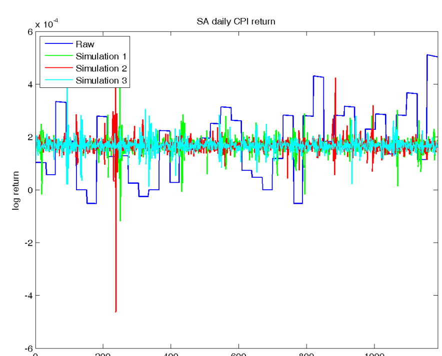


(d) Squared returns PACF

Figure 6.8 Daily filtered SA CPI (partial) correlograms.

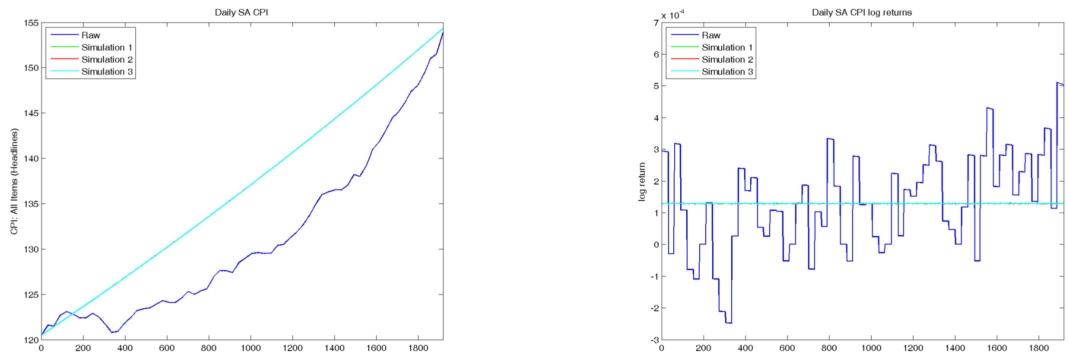


(a) Raw vs Filtered CPI



(b) Raw vs Filtered CPI log return

Figure 6.9 Daily raw vs filtered SA CPI (normal innovations).



(a) Raw vs Filtered CPI

(b) Raw vs Filtered CPI log return

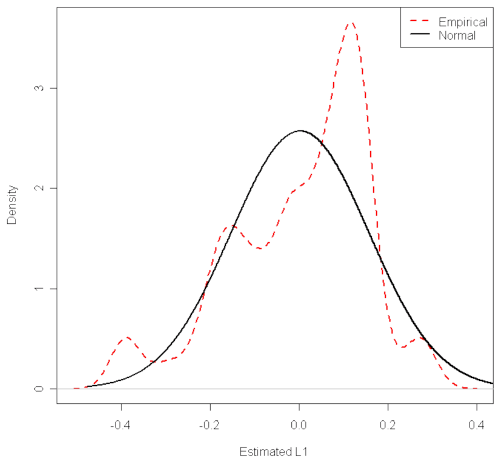
Figure 6.10 Daily raw vs filtered SA CPI (student-t innovqtions).

6.1.7 Lévy Distributions' Parameter Estimation

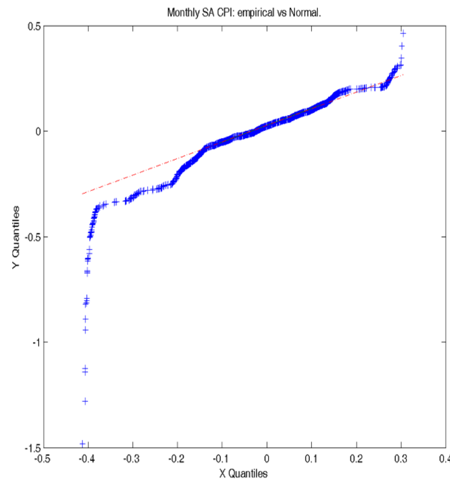
The calibration assumes that the Lévy characteristic triplet is not time dependent, i.e. $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are constants over time. Under this assumption, for 1-dimensional processes, there is a single risk neutral measure and thus a unique fair price for IL derivatives just as in the Black-Scholes pricing theory [47]. The parameter estimation of all Lévy distributions was performed using the maximum likelihood parameter estimation under R with the package *ghyp*. Parameters are estimated both for the daily and the monthly South African consumer price index.

Monthly SA CPI

Figure 6.11 shows the empirical against the normal density functions of monthly SA CPI log returns with the corresponding QQ plot. The sample normal data is generated from a normal distribution with same mean and standard deviation as the empirical distribution. The plots indicates that the normality assumption is "highly" questionable for monthly SA CPI log returns. The normal distribution does not reproduce the peakedness of the market around its mean (Figure 6.11(a)), neither does it match the upper and lower tails behaviour (Figure 6.11(b)). The fit with the lower tail is worse than that with the upper tail and the peaks of the normality plots are not vertically aligned. If normal innovations were used in the GARCH filter instead, then a better fit to the empirical data is obtained (see Appendix A.1.1). In summary, the empirical density function is taller than the corresponding normal density function, its peak is more to the left; however their support is "quite" similar.

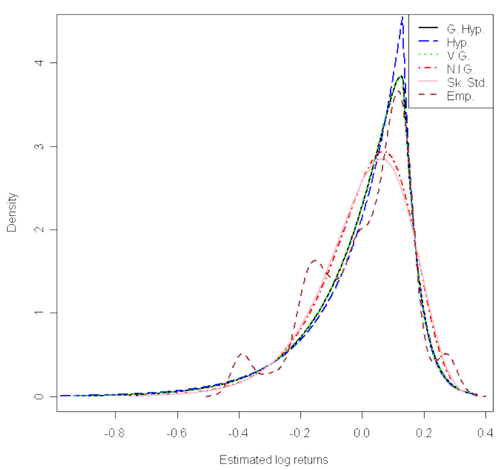


(a) Probability plot

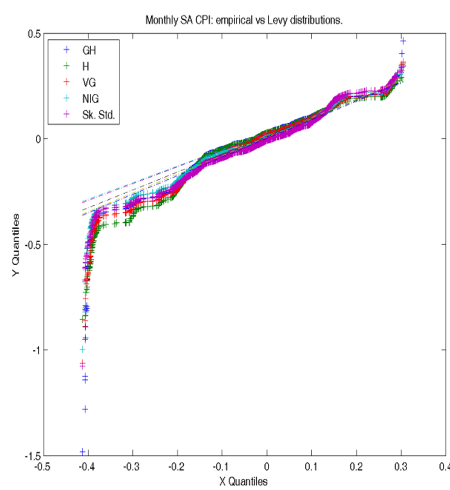


(b) QQ plot

Figure 6.11 Monthly SA CPI probability and QQ plots: Empirical vs normal.



(a) Probability plot



(b) QQ plot

Figure 6.12 Monthly SA CPI probability plots: Empirical vs Lévy.

The normal density and QQ plots' fit with Lévy distributions (Figures 6.12 and A.3) is in general better than under the normality assumption. Despite the fact that the parameter estimation did not converge for the GH and Student-t distributions, all the Lévy distributions did better than the normal distribution is reproducing the empirical peakedness. The VG distribution has the best performances in matching the empirical peak. However, it is not the best in reproducing the empirical tail behaviour. With respect to the latter, the best fit is under the assumption of NIG distribution. To accurately model the monthly SA CPI, the appropriate distribution in this case is either the VG or the NIG distribution according to what is believed to be more important.

The Lévy distributions parameters for the SA CPI log returns are given in Table 6.3. Notice that all the distributions agree on the "high" asymmetry (i.e. non-normality) since beta is non-zero. They also agree on the value of the location parameter μ. Recall that the Student-t distribution calibration did not converge, the corresponding results can thus be ignored.

Model	α	β	$\delta(\times 10^5)$	μ	λ	**LLH**
NIG	32.01031	-24.13984	0.22725	22724.94	-0.5	773.7516
H	13.86566	-8.11529	3.31747	0.13085	1	803.5996
GH	15.04463	-8.36209	18.18795	0.13220	1.21520	806.6979
VG	15.04654	-8.35547	0	0.13224	1.21616	806.7002
Sk.Std.	1347687	-1347687	46.93758	0.70584	-22.10530	771.4530

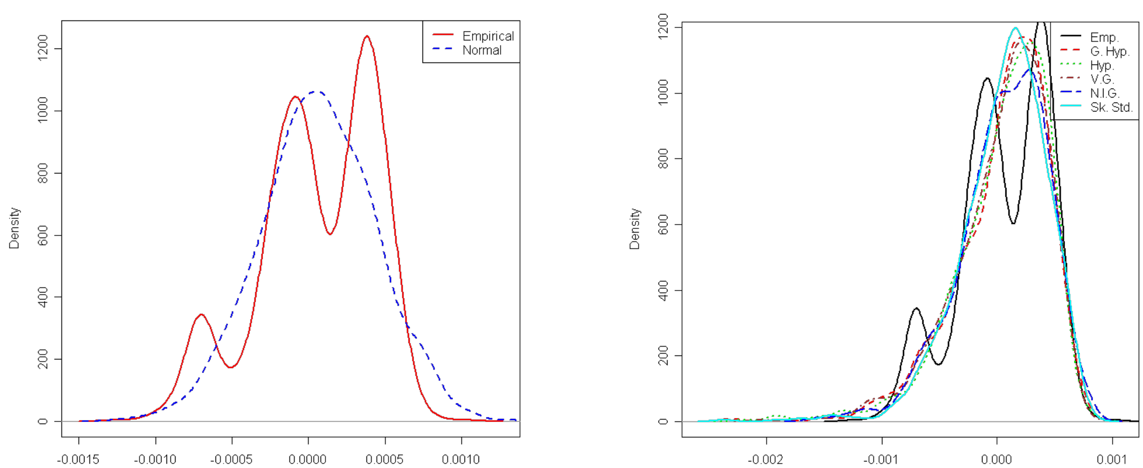
Table 6.3 Monthly SA CPI Lévy distributions' estimated parameters.

The AIC model selection returns the VG model as having the best fit to the empirical data among all distributions (normal included). Unfortunately the KS test and Chi-squared goodness of fit test available under Matlab (resp. *kstest* and *chi2gof*) do not allow a two-samples goodness of fit test. The corresponding functions under R (resp. *ks.test* and *chisq.test*) do perform a two samples plot, but they keep returning a warning when performing the test. The *Wafo* [108] library for Matlab was used instead for the Chi-squared goodness of fit test which is better suited than the KS test in this settings. The results of the test are presented in Table 6.9. All the distributions have the same p-value; the test was not decisive. Recall that the larger the p-value, the better the fit. However, the smallest test value is obtained with the hyperbolic distribution, not the VG. This is not that relevant since both distributions have the same p-value.

Daily SA CPI

Because of the linearity of filtered daily SA CPI and their non-zero volatility, some will judge the Lévy distributions' parameter estimation useless. In fact, if the filtered SA CPI is linear, then a "good" characteristic of its evolution is its slope. However, the parameter estimation might provide a better insight into the evolution process.

Since the probability density function of the filtered daily SA CPI between 2005 and 2008 is not "well behaved" (see the red and solid line in Figure 6.13(a)), the daily SA CPI is extended from 2001 to 2008. Contrary to the filtered daily SA CPI, the raw daily CPI is volatile, therefore it might also be useful to estimate the Lévy distributions' parameter estimation for the raw data.

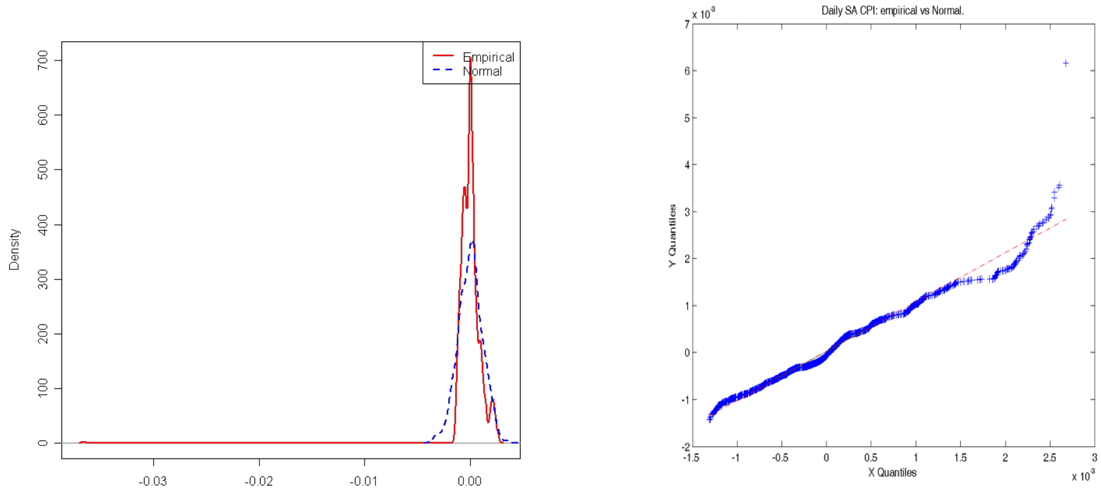


(a) Empirical vs Normal

(b) Empirical vs Lévy

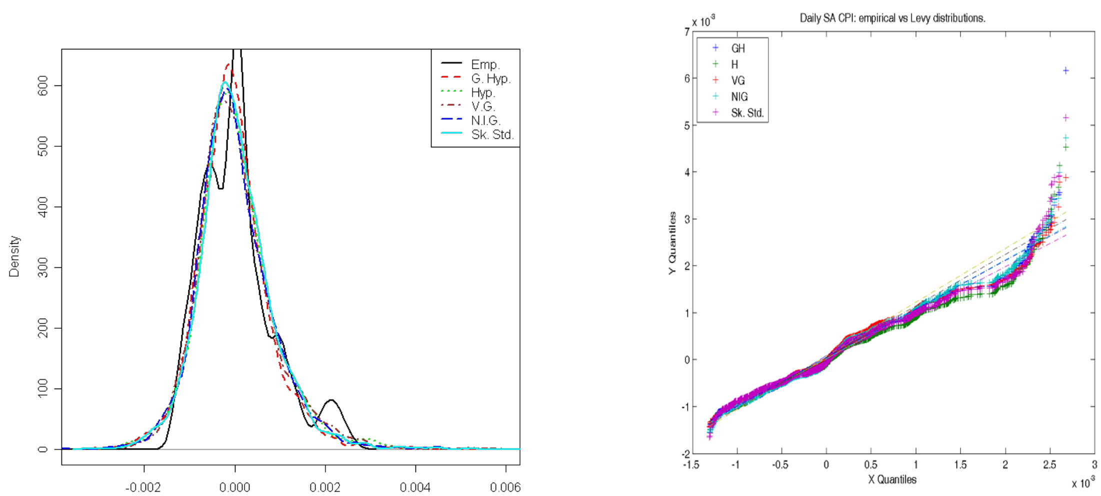
Figure 6.13 Filtered daily SA CPI 2005 – 2008 probability plots.

Even when the sample data is not well behaved as with the daily CPI, the fit with Lévy distributions proves to be better than that achieved under the normality assumption. When performing model calibration, monthly CPI will be preferred to daily CPI because of their previous behaviour. The daily CPI will be obtained by interpolation as is the convention in the market.



(a) Probability plot (b) QQ plot

Figure 6.14 Filtered daily SA CPI probability and QQ plots: Empirical vs normal.



(a) Probability plot (b) QQ plot

Figure 6.15 Filtered daily SA CPI probability plots: Empirical vs Lévy.

6.1.8 Forward rates

The parameter estimation of the term structure is substantially more difficult than in the case of macroeconomic factors. The difficulty is due to the fact that a number of different assets (in theory an infinite number) are driven by only "one" process. Therefore, to extract the parameters of the (unique) driving process from the assets (in this case log returns of "zero coupon" bond prices or discount factors) is not straightforward.

For the real world study, the approach for Lévy forward rate model proposed by Eberlein and Wolfgang [43] is used to deduce the unique Lévy driving process from market zero coupon bonds. The methodology is first presented before providing the results obtained.

The initial assumptions and notations used in this subsection were introduced in Section 3.1. Considering the logarithm of the ratio between the bond price and its forward price on the day before, i.e.

$$LR_i(t,T) = \ln \frac{p_i(t+1;t+T)}{p_i(t,t+1,t+T)},$$

for $i = n, r$, where the subscript n (resp. r) stands for nominal (resp. real). The forward price of $p_i(t+1;t+T)$ at time t is

$$p_i(t,t+1,t+T) = \frac{p_i(t,t+T)}{p_i(t,t+1)}.$$

The variable $LR_i(t-1,t)$ denotes the daily log return. Using Equation (3.21)

$$\ln p_i(t,T) = \ln p_i(0,T) - \ln p_i(0,t) - \int_0^t A^i(s,t,T)ds + \int_0^t \Sigma^i(s,t,T)dL_s.$$

Therefore

$$\begin{aligned}
LR_i(t,T) &= \ln p_i(t+1,t+T) - \ln p_i(t,t+T) + \ln p_i(t,t+1) \\
&= \ln p_i(0,t+T) - \ln p_i(0,t+1) - \int_0^{t+1} A^i(s,t+1,t+T)ds + \int_0^{t+1} \Sigma^i(s,t+1,t+T)dL_s \\
&\quad - \ln p_i(0,t+T) + \ln p_i(0,t) + \int_0^t A^i(s,t,t+T)ds - \int_0^t \Sigma^i(s,t,t+T)dL_s \\
&\quad + \ln p_i(0,t+1) - \ln p_i(0,t) - \int_0^t A^i(s,t,t+1)ds + \int_0^t \Sigma^i(s,t,t+1)dL_s \\
&= - \int_0^{t+1} A^i(s,t+1,t+T)ds + \int_0^t A^i(s,t,t+T)ds - \int_0^t A^i(s,t,t+1)ds \\
&\quad + \int_0^{t+1} \Sigma^i(s,t+1,t+T)dL_s - \int_0^t \Sigma^i(s,t,t+T)dL_s + \int_0^t \Sigma^i(s,t,t+1)dL_s
\end{aligned}$$

$$\begin{aligned}
LR_i(t,T) &= -\int_0^{t+1} A^i(s,t+T)ds + \int_0^{t+1} A^i(s,t+1)ds + \int_0^t A^i(s,t+T)ds \\
&\quad - \int_0^t A^i(s,t)ds - \int_0^t A^i(s,t+1)ds + \int_0^t A^i(s,t)ds \\
&\quad + \int_0^{t+1} \Sigma^i(s,t+T)dL_s - \int_0^{t+1} \Sigma^i(s,t+1)dL_s - \int_0^t \Sigma^i(s,t+T)dL_s \\
&\quad + \int_0^t \Sigma^i(s,t)dL_s + \int_0^t \Sigma^i(s,t+1)dL_s - \int_0^t \Sigma^i(s,t)dL_s \\
&= -\int_0^{t+1} A^i(s,t+T)ds + \int_0^{t+1} A^i(s,t+1)ds + \int_0^t A^i(s,t+T)ds - \int_0^t A^i(s,t+1)ds \\
&\quad + \int_0^{t+1} \Sigma^i(s,t+T)dL_s - \int_0^{t+1} \Sigma^i(s,t+1)dL_s - \int_0^t \Sigma^i(s,t+T)dL_s + \int_0^t \Sigma^i(s,t+1)dL_s \\
&= -\int_t^{t+1} A^i(s,t+T)ds + \int_t^{t+1} A^i(s,t+1)ds + \int_t^{t+1} \Sigma^i(s,t+T)dL_s - \int_t^{t+1} \Sigma^i(s,t+1)dL_s \\
&= -\int_t^{t+1} A^i(s,t+1,t+T)ds + \int_t^{t+1} \Sigma^i(s,t+1,t+T)dL_s.
\end{aligned}$$

The next "stationarity" assumptions allows to get rid of the cumbersome integrals.

Assumption 14. *(i) The volatility structure is stationary, i.e.* $\Sigma^i(s,T)$ *depends only on* $(T-s)$ *for* $s < T$.

(ii) Similarly, the drift term satisfy some stationarity condition, namely

$$A(s,T) = A(0,T-s) \quad \textit{for } s < T.$$

Note that the second assumptions follows from the first assumption and Equation (3.23). It yields

$$-\int_t^{t+1} A^i(s,t+1,t+T)ds = -\int_0^1 A^i(s,1,T)ds := f(T),$$

where ":=" means "denoted by" and is used to define the function f which is independent of t. For the second integral, let's consider the Ho-Lee volatility structure, i.e. $\Sigma^i(s,T) = \sigma^i \times (T-s)$ for constants σ^i, which will be set equal to one henceforth without loss of generality. The second integral becomes

$$\begin{aligned}
\int_t^{t+1} \Sigma^i(s,t+1,t+T)dL_s &= \int_t^{t+1} \Sigma^i(s,t+T)dL_s - \int_t^{t+1} \Sigma^i(s,t+1)dL_s \\
&= (t+T-s)\int_t^{t+1} dL_s - (t+1-s)\int_t^{t+1} dL_s \\
&= (T-1)\int_t^{t+1} dL_s = (T-1)(L_{t+1}-L_t)
\end{aligned}$$

Hence

$$LR_i(t,T) = f(T) + (T-1)Y_{t+1} \tag{6.4}$$

where $Y_{t+1} = L_{t+1} - L_t \sim L_1$ is $\mathcal{F}_{t+1}$ measurable and does not depend on T.

Let $\mathbb{D}$ (resp. $\mathbb{T}$) denote the set of days (resp. set of bonds' maturity in years) for which data is available. Considering $d \in \mathbb{D}$ and $n \in \mathbb{T}$, the daily log returns are determined by

$$LR(d, d+n) \quad = \quad \ln B(d+1, d+n) + \ln \frac{B(d, d+1)}{B(d, d+n)}.$$

Since $B(d, d+1)$ and $B(d+1, d+n)$ are not provided (in the initial discount factors), the negative of the logarithm of the bond prices is interpolated with a cubic spline to get those. That is because bond prices decrease exponentially with the time to maturity, thus linear interpolation will generate errors. On the other hand, the negative of the logarithm of the bond prices is linear in the time to maturity for constant interest rates, therefore linear interpolation will introduce less error. The transformation, then the interpolation, are conducted for each considered day (i.e. $d \in \mathbb{D}$) separately.

For the South African nominal yield curve, daily market coupon bearing bonds between the 31^{st} July 2000 and the 30^{th} May 2008 (i.e. 1953 trading days) were initially used for computation. Hermite polynomials were applied on the interest rates to get zero coupon interest rates with maturities ranging from one to thirty years in steps of one year. This dataset was generously provided by Nicolette Roussos from Standard Bank, South Africa.

Zero coupon interest rates at maturities 1, 3, 6 and 9 month(s) were also provided, but not used for the calibration process. However, the discount factor of one for the zero year maturity was included for the calibration. The short term (i.e. less than a year) interest rates were surely computed from the money market (i.e. JIBAR and other) which is quite different from the bonds market. Furthermore, the concern here, inflation, is more in the long term than in the short term. Figure 6.16(a) (resp. 6.16(b)) gives the South African nominal yield curve (resp. interpolated negative log bond price) on the 30^{th} May 2008.

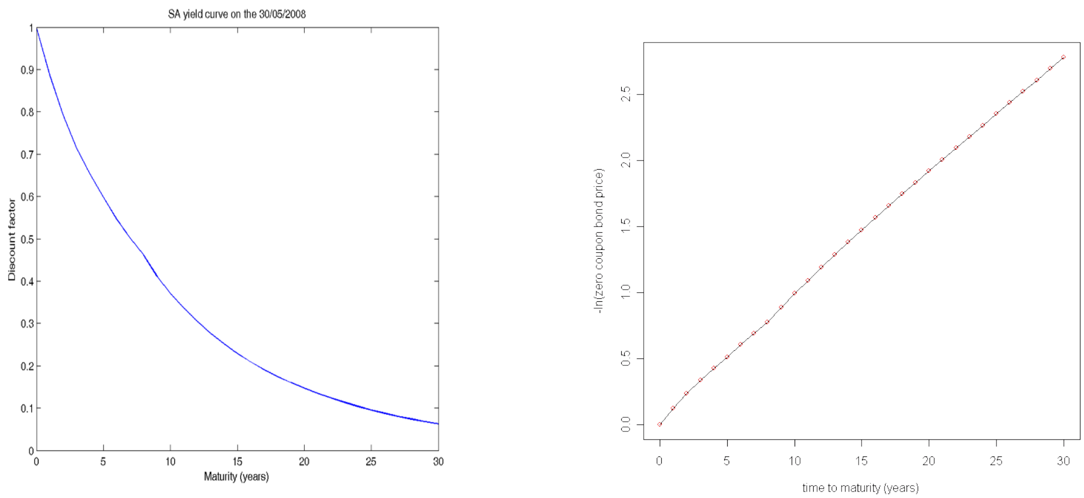
Taking the expectation of Equation (6.4) gives

$$\mathbb{E}[LR_i(t,T)] \quad = \quad f(T),$$

since $\mathbb{E}[L_1] = 0$. Therefore

$$LR_i(t,T) - \mathbb{E}[LR_i(t,T)] \quad = \quad (T-1)Y_{t+1}.$$

Recall that Y_{t+1} for $t \in \mathbb{D}$ are independent and equal to L_1 in distribution, thus the last equation means that the centred log returns are affine linear in T.



(a) Yield curve (b) interpolated "-ln"

Figure 6.16 Nominal yield and interpolated "-ln" on the 30^{th} May 2008.

For a fixed $n \in \mathbb{T}$, the sample value y_{d+1} corresponding to Y_{d+1} should be computed as

$$y_{d+1} = \frac{LR(d, d+n) - \overline{x}_n}{n-1},$$

where

$$\overline{x}_n = \frac{1}{|\mathbb{D}|} \sum_{\mathbb{D}} LR(d, d+n).$$

However, since the centred log returns are not exactly linear in n (See Figure 6.17), the sample values y_{d+1} will depend on n. This is not the case of L_1 which does not depend on the bonds' maturity. Thus, a different approach is used: a linear regression through the origin and with the points $[n-1, LR_i(d, d+n) - \overline{x}_n]$ for $n \in \mathbb{T}$ is performed. The value of y_{d+1} is the gradient of the straight line. Recall that in the expression $n-1$, n is in years while 1 is in days. Figure 6.18 gives the estimated values of L_1 between the 31^{st} July 2000 and the 29^{th} May 2008.

The linear regression is conducted under Matlab with the function *polyfit*. For the 29^{th} May 2008, the linear regression of the centred empirical daily log return return the following model:

$$y = -0.002458x + 0.003171.$$

The linear regression's slope estimation is performed for each of trading days (except the last), then the gradient are used for the parameter estimation of L_1. The procedure is similar to that of the macroeconomic factors where the gradients replace the log returns.

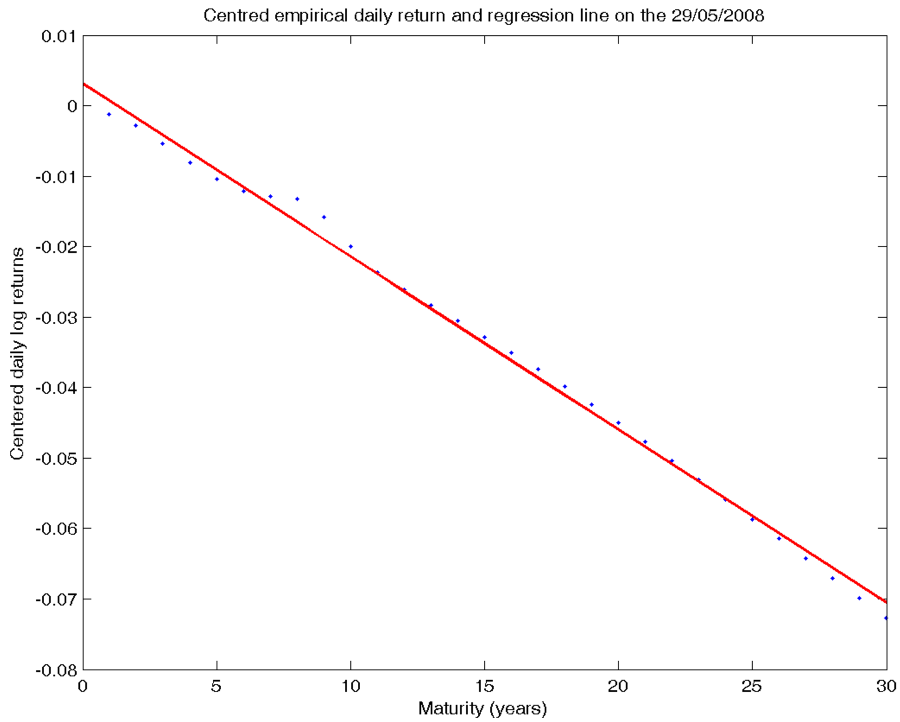


Figure 6.17 SA centred empirical daily return and regression line 29^{th} May 2008.

Model	α	β	δ $(\times 10^5)$	$\mu(\times 10^5)$	λ	**LLH**
NIG	621.6958	24.04922	64.19035	-2.46877	-0.5	10974.58
H	1499.659	35.21145	9.70121	-3.19967	1	10934.69
GH	25.52232	19.69284	102.9604	-2.32664	-1.46109	10994.66
VG	1388.358	36.26896	0	-3.36557	0.90868	10934.82
Sk.Std.	19.03043	19.03043	103.2756	-2.32732	-1.46824	10994.69

Table 6.4 Estimated parameters for empirical L_1 for SA nominal forward rate.

The log likelihood estimate and the AIC return the student-t as the distribution (normal included) having the best fit with the sample data.

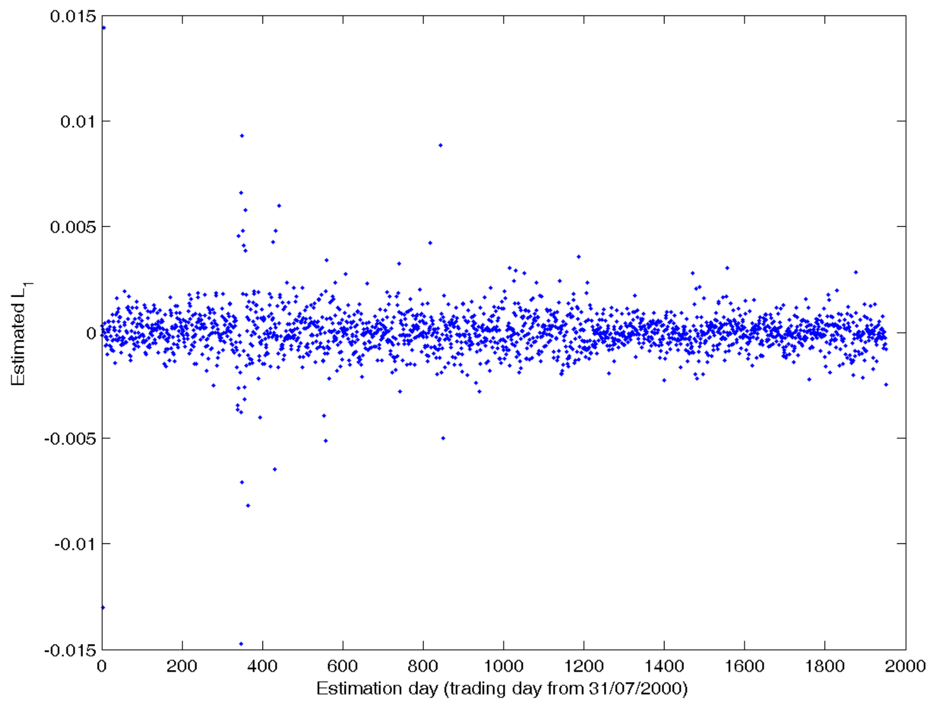
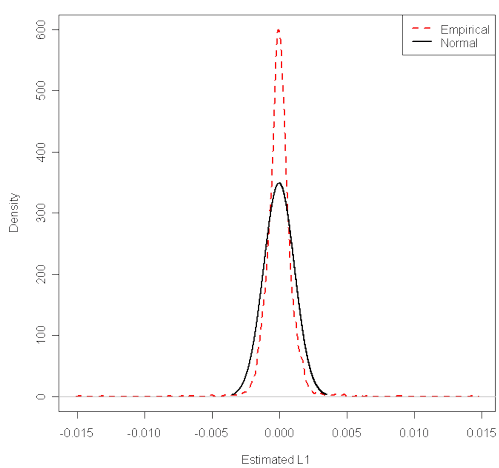
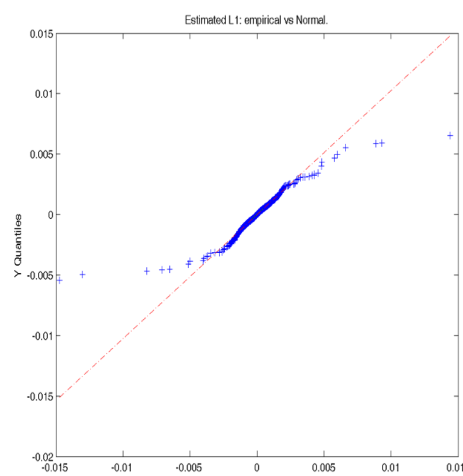


Figure 6.18 SA estimated L_1 between the 31^{st} July 2000 and the 29^{th} May 2008.

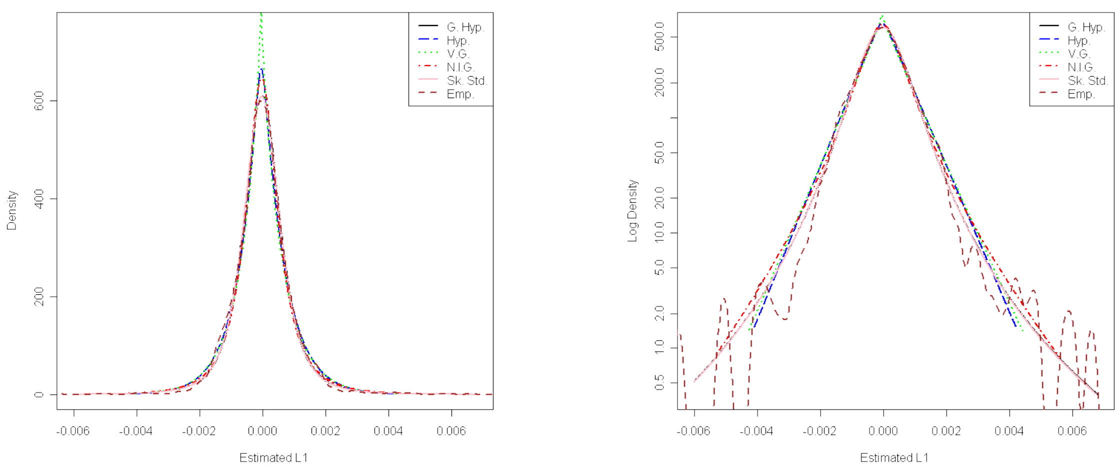


(a) Probability plot



(b) QQ Plot

Figure 6.19 Estimated L_1: Empirical vs normal.



(a) Density (b) Log Density

Figure 6.20 Estimated L_1 probability plots: Empirical vs Lévy.

6.1.9 South African data

Monthly and daily South African consumer price index have already been studied in details and thus will not be mentioned again. The other South African data investigated are the CPIX and the money supply aggregates. The SA CPIX is the SA CPI without interest rates on mortgage bonds; it is generally used interchangeably with the SA CPI. The money supply aggregates have had different roles in monetary policy as their reliability as guides has changed. They are mainly indicators of the monetary structure and flow of a given country. Here is a brief description of the main money supply aggregates in South Africa:

1. M0: Deposit of banks, mutual banks with the South African Reserve Bank (SARB) and notes and coins outside the SARB and SA mint.

2. M1A: Coins and banknotes in circulation outside the monetary sector, cheque and transmission deposit with banking institutions and post office savings bank.

3. M1: M1A plus other demand deposit with banking institutions.

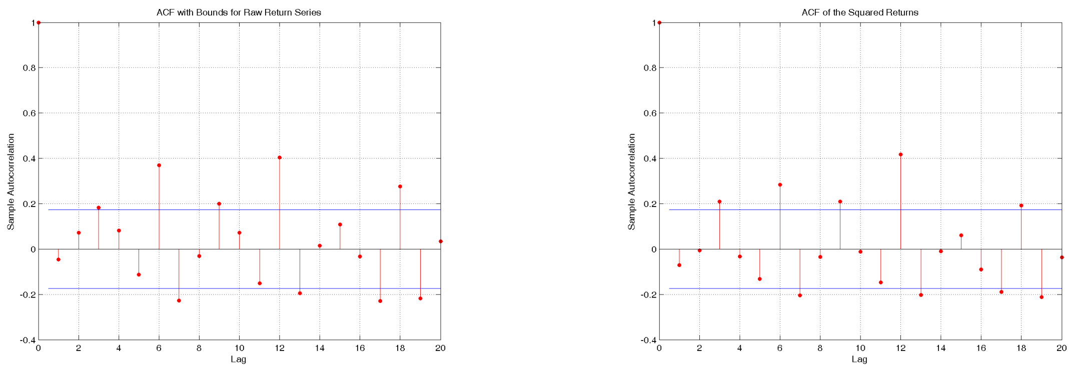
4. M2: M1 plus other short term deposits, and all medium term deposits (including savings deposits) with the monetary banking institutions.

5. M3: M2 plus all long term deposit with monetary banking institutions.

The following subsections present each of these data series more in details. Most of the corresponding plots and parameter estimates are provided in Appendix A.

Consumer Price Index (CPIX)

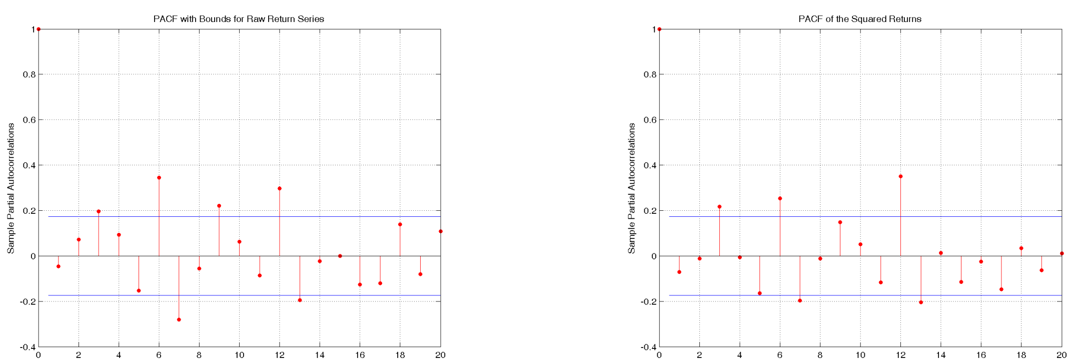
The monthly South African Consumer Price Index Metropolitan and urban areas excluding interest rates on mortgage bonds (CPIX) time series data is obtained from the South African Reserve Bank [97]. The data is from January 1997 to February 2008 normalised at 100 in 2000, which is 134 data points. That is not enough data points for a "good" statistical study; the GARCH filter will be used to increase the number of data points. The data series code is $KBP7113J$ and its unit R millions. In Figure 6.21, the only positive autocorrelations in the CPIX are at three months intervals, most before the 12^{th} month. Given that the spikes at lags 3 and 9 are "quite" small, there might be a semi annual cycle in the CPIX instead of a clear annual seasonality as for the SA CPI. This possibility is reinforced by Figure 6.22(a), where the spike at lag 6 is higher than that at lag 12. This suggest that



(a) Log returns (b) Squared returns

Figure 6.21 Monthly SA CPIX (1997-2008) correlograms.

most of the previous 12 months' lag autocorrelation was due to the 6 months' lag autocorrelation. However, this remark is not true for the squared log returns where the PACF spike at lag 12 is higher than that at lag 6.



(a) Log returns (b) Squared returns

Figure 6.22 Monthly SA CPIX (1997-2008) partial correlograms.

The LBP Q-test identifies GARCH effects only in the raw returns and not in their square. However, the Engle's ARCH test finds GARCH effects neither in the log returns nor their square. It is the reverse LBP Q-test's results (i.e. no GARCH effects in log returns and some GARCH effects in their square) that is more appropriate for a GARCH model calibration. The Engle's ARCH test output just confirms the fact that the sample's volatility does not vary "much" with time. Nevertheless, Lévy distributions' parameter estimation will be performed to compare their fit with that of the

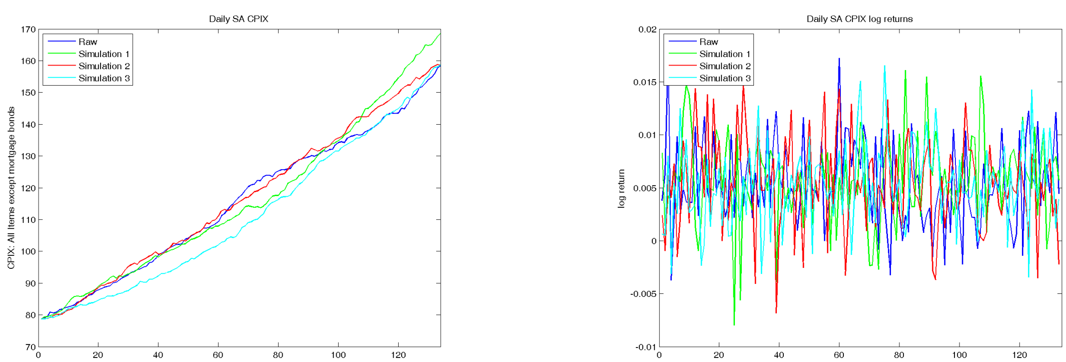
	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	41.8128	18.3070	0	0.8504	5.5649	18.3070
15	1	0	76.9280	24.9958	0	0.8661	9.2125	24.9958
20	1	0	104.7414	31.4104	0	0.8752	13.0499	31.4104

Table 6.5 Ljung-Box-Pierce Q-test for SA Monthly CPIX (1997-2008) raw and squared returns.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	0	0.7774	6.4354	18.3070	0	0.9992	1.4237	18.3070
15	0	0.8397	9.6740	24.9958	0	1.0000	1.8261	24.9958
20	0	0.8456	13.6963	31.4104	0	0.9991	5.8234	31.4104

Table 6.6 Engle's ARCH test results for SA Monthly CPIX (1997-2008) raw and squared returns.

normal distribution and the inflation index in the South African settings will always be the CPI and not the CPIX.

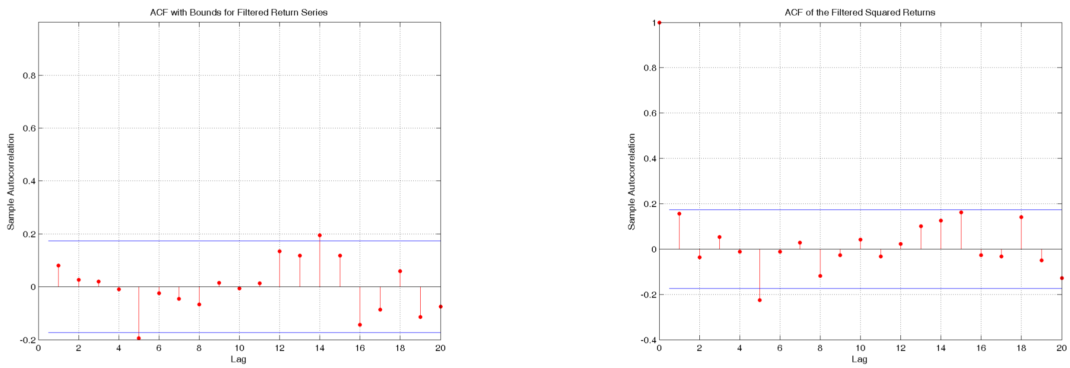


(a) CPIX

(b) CPIX Log returns

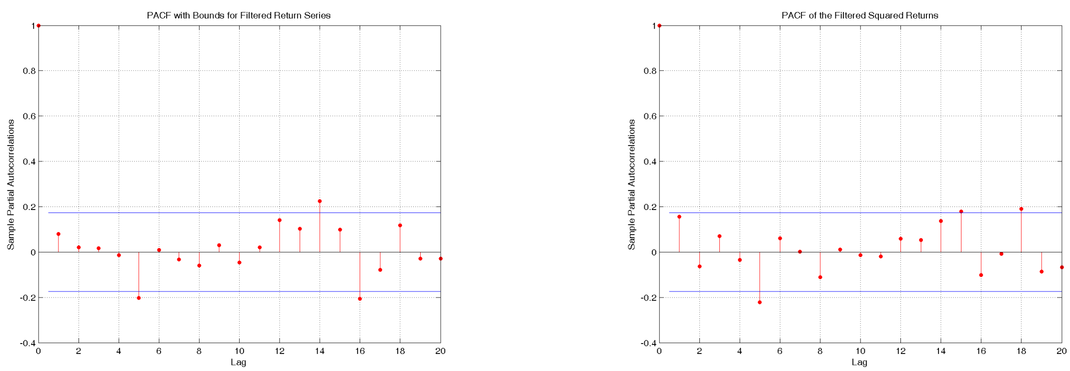
Figure 6.23 Filtered vs raw monthly SA CPIX (1997-2008) data series.

Figures 6.24 and 6.25 show that despite the small size of the sample, the GARCH filter has reduced its autocorrelation.



(a) Log returns (b) Squared returns

Figure 6.24 Filtered monthly SA CPIX (1997-2008) correlograms.



(a) Log returns (b) Squared returns

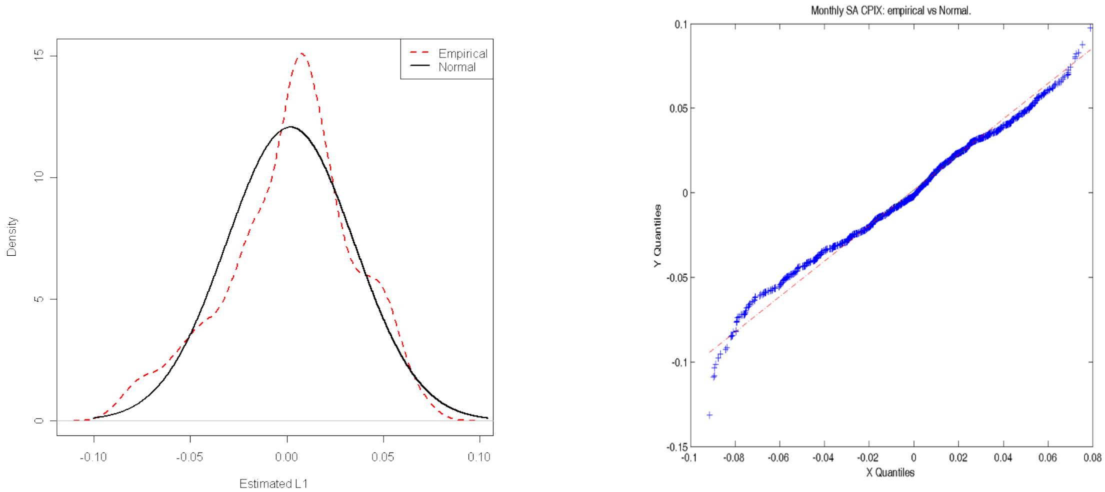
Figure 6.25 Filtered monthly SA CPIX (1997-2008) partial correlograms.

For the Lévy distributions' parameter estimation, the GARCH filter is used to multiply the sample size by 10.

Figures 6.26 and 6.27 give the normality plots and QQ-plots with the empirical distribution. Unfortunately, none of the parameter estimation for Lévy distributions did converge. However, the fit with the Lévy distributions is still better than that with the normal distribution.

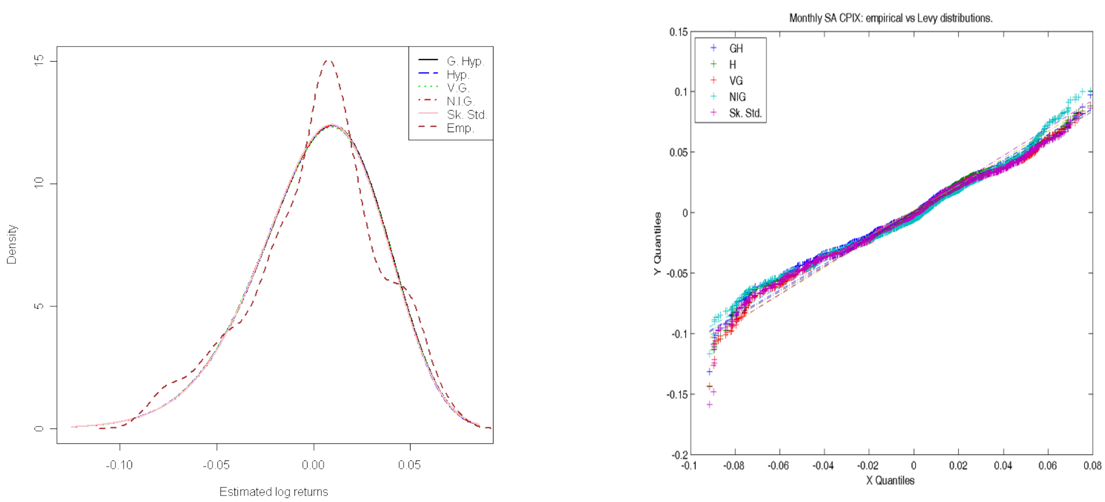
Money Supply aggregate M1A

The monthly South African Monetary aggregate M1(A) time series data is obtained from the South African Reserve Bank [97]. The data is from March 1979 to December 2007 that is 346 data points overall. That is not enough data points for a "good" statistical study; the GARCH filter will be used



(a) Probability plot (b) QQ plot

Figure 6.26 Filtered monthly SA CPIX probability and QQ plots: Empirical vs normal.



(a) Probability plot (b) QQ plot

Figure 6.27 Filtered monthly SA CPIX probability plots: Empirical vs Lévy.

Model	α	β	δ	μ	λ	**LLH**
NIG	4303.523	-4206.992	0.04419	0.20696	-0.5	2683.354
H	797.9752	-704.9338	0.08325	0.16626	1	2683.190
GH	1128.630	-1033.692	0.07289	0.17756	1.32246	2683.241
VG	1383.623	-1254.324	0	0.13672	18.34853	2683.819
Sk.Std.	3812.483	-3812.483	0.09803	0.26698	-70.0834	2682.768

Table 6.7 Estimated parameters for monthly SA CPIX log returns.

to triple the number of data points. The data series code is $KBP1374M$ and its unit R millions.

Money Supply aggregate M1, M2 and M3

The monthly South African Monetary aggregates M1 (resp. M2, M3) time series data is obtained from the South African Reserve Bank [97]. The data is from March 1965 to December 2007 that is 514 data points overall. The GARCH filter will be used to double the number of data points. The data series code is $KBP1373M$ (resp. $KBP1372M$, $KBP1370M$) and its unit R millions.

Series	$\mu(\%)$	$\sigma(\%)$	**Skew.**	**Kurt.**	**Min.(%)**	**Max.(%)**
M1A	1.36	4.14	-0.1212	2.8675	-10.16	12.85
M1	1.21	3.40	-0.0826	3.4912	-9.07	13.28
M2	1.20	1.65	0.0645	3.2218	-3.41	6.44
M3	1.14	1.23	0.0815	3.4580	-2.74	5.44
CPI(Monthly)	0.74	0.70	0.9198	4.6009	-0.74	4.21
CPI(Daily)	0.017537	0.013374	0.2031	2.4956	-0.0052327	0.051089
CPIX	0.53	0.40	0.3912	3.1446	-0.37	1.75

Table 6.8 Descriptive statistics of S.A. Data series log returns.

Table 6.9 Chi squared Pearson's test.

	Normal		GH		H		VG		NIG		Sk.t	
	p	Test	p	Test	P	Test	P	Test	P	Test	P	Test
CPI	0	345.3730	0	211.4566	0	192.0096	0	231.0064	0	295.7781	0	320.7814

6.1.10 United State of America data

The American data studied is similar to the South African data seen in the previous subsection. However, although the US money aggregates are named identically to their South African counterpart, they are not exactly identical. The following details their principal components [102]:

1. M0: The total of all physical currency, plus accounts at the central bank that can be exchanged for physical currency.
2. M1: M0 minus those portions of M0 held as reserves or vault cash plus the amount in demand accounts (“checking” or “current” accounts).
3. M2: M1 plus most savings accounts, money market accounts, and small denomination time deposits (certificates of deposit of under $100,000).
4. M3: M2 plus all other CDs (large time deposits, institutional money market mutual fund balances), deposits of eurodollars and repurchase agreements.

The CPIX is particular to South Africa, therefore there is only one potential inflation index for US. Table 6.10 (resp. 6.11) contains descriptive statistics (resp. hypothesis tests) of all our considered data samples. For a brief overview of these descriptive statistics and hypothesis tests see Subsection 6.1.2.

Series	$\mu(\%)$	$\sigma(\%)$	**Skew.**	**Kurt.**	**Min.(%)**	**Max.(%)**
CPI (186 yrs)	0.22	1.45	1.7607	70.1835	−16.83	19.72
CPI (70 yrs)	0.32	0.47	2.2667	24.2856	−1.40	5.72
M1	0.082719	2.08	−0.4315	4.1987	−9.22	10.81
M1 (Adj.)	0.085279	0.62	1.6958	63.5971	−6.84	10.14
M2	0.11	0.59	0.1760	3.2690	−2.56	3.02
M2 (Adj.)	0.11	0.19	2.4551	55.2031	−1.47	3.22
M3	0.12	0.43	0.0537	3.1615	−1.19	2.18
M3 (Adj.)	0.12	0.19	1.3564	18.6892	−0.88	2.31

Table 6.10 Descriptive statistics of USA Data series log returns.

In Table 6.10, the kurtosis is always larger than three, which would have been the kurtosis if the sample data series were taken from normal distributions. This behaviour is commonly observed in the market [8, 27, 109, 55, 9]. In particular the money supply aggregates's kurtosis increases considerably when adjusting it for seasonality. However, the maximum-likelihood estimators (see Subsection 6.2.1) used for Lévy distributions' parameter estimation assumes that the data points are independent identically distributed, i.e. no autocorrelation. Therefore the seasonally adjusted data series, which are less normal, are more appropriate than their non-adjusted counterpart for the parameter estimation.

The non-adjusted money aggregate M1 is the only one with a negative skewness. Its general behaviour might differ from that of the other money aggregates, therefore it will not be used later when modelling the money aggregate as a macroeconomic factor.

Series	**JB**		**K-S**		**Lill.**		**AD**	
	H	p	H	p	H	p	H	p
CPI (186 yrs)	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}
CPI (70 yrs)	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}
M1	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}
M1 (Adj.)	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}
M2	1	0.0054	1	0	1	10^{-3}	1	10^{-3}
M2 (Adj.)	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}
M3	0	0.3464	1	0	0	0.5000	0	5%
M3 (Adj.)	1	10^{-3}	1	0	1	10^{-3}	1	10^{-3}

Table 6.11 Hypothesis Tests of US Data series log returns.

Money Supply aggregate M1 and M2 (Adjusted and unadjusted)

The weekly American Monetary aggregates M1, M2 (both adjusted and unadjusted) time series data is obtained from the Federal Reserve System [103]. The data is from the 5^{th} January 1981 to the 21^{st} April 2008 that is 1425 data points overall. The data was obtained on the 1^{st} May 2008 and is measured in billions of US dollars.

Some of the results obtained with theses data series are presented in Appendix B.

Money Supply aggregate M3

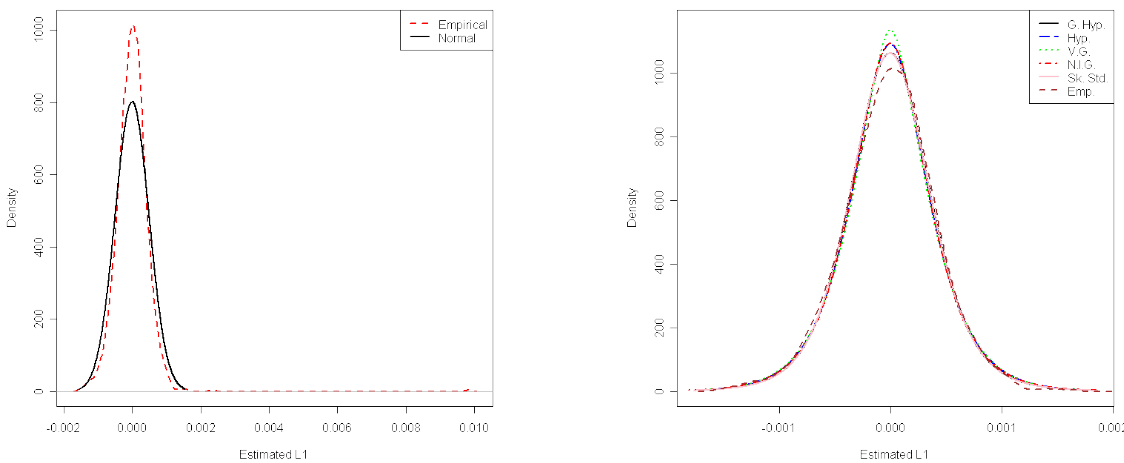
The weekly seasonally adjusted and unadjusted American Monetary aggregates M3 time series data were obtained from the Federal Reserve System [103]. The data is from the 5^{th} January 1981 to the 13^{tH} March 2006 that is 1315 data points overall. The data is for the 1^{st} May 2008 and measured in billions of US dollars.

The Federal Reserve ceased publishing M3 statistics in March 2006, claiming that M3 did not appear to convey additional information about economic activity compared to M2, had not been used in determining economic policy, and that the costs to collect M3 data outweighed the benefits. Some of the data used to calculate M3 are still collected and published on a regular basis.

The results obtained with all the previous US macroeconomic factors confirms the better fit of Lévy distributions compare to that of the conventional normal distribution. Only results obtained with the real and nominal US yield curves are presented in this subsection.

Nominal and Real Yields

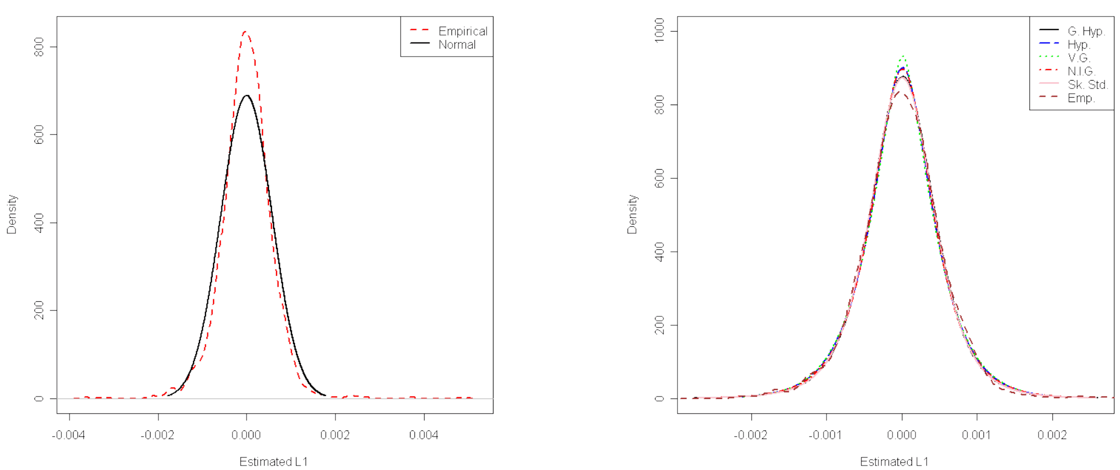
The nominal and real daily historical yield curves were downloaded from the U.S. treasury website [107] from January 2003 to September 2008. The sample data has 1430 data points, with only the trading days considered. The fit with the Lévy distributions are still better than that under normality assumption (Figures 6.29 and 6.29). In both cases the best fit according to the AIC is obtained with the Student-t distribution. This is also the case when using the log-likelihood estimate. Further calibration results are provided in Appendix B.3.



(a) Empirical vs Normal

(b) Empirical vs Lévy

Figure 6.28 Nominal yield curve: Empirical vs model.



(a) Empirical vs Normal

(b) Empirical vs Lévy

Figure 6.29 Real yield curve: Empirical vs model.

Model	α	β	$\delta(\cdot 10^5)$	μ	λ	**LLH**
NIG	2728.92	24.64427	53.96559	$-4.83289 \cdot 10^{-6}$	-0.5	9056.900
H	3938.503	26.19406	34.55043	$-5.20755 \cdot 10^{-6}$	1	9050.542
GH	73.83758	70.64118	91.40627	$-1.40600 \cdot 10^{-5}$	-3.19922	9072.226
VG	4437.195	3.61602	0	$-6.79980 \cdot 10^{-7}$	35	9046.039
Sk.Std.	67.62844	67.62844	91.2076	$-1.36021 \cdot 10^{-5}$	-3.18674	9072.230

Table 6.12 Estimated parameters for US nominal forward rate.

Model	α	β	$\delta(\cdot 10^5)$	$\mu(\cdot 10^5)$	λ	**LLH**
NIG	1890.162	-51.50919	60.56153	1.65484	-0.5	8725.085
H	2909.199	-54.34723	32.85109	1.68966	1	8720.764
GH	52.25032	-44.33243	93.34401	1.46063	-2.33771	8729.895
VG	3304.674	-45.19182	0	1.42259	1.72004	8717.872
Sk.Std.	39.62266	-39.62266	94.60124	1.28412	-2.38186	8729.890

Table 6.13 Estimated parameters for US real forward rate.

Table 6.14 Kolmogorov-Smirnov test.

$(\times 10^3)$	Normal		GH		H		VG		NIG		Sk.t	
	p	D	p	D	P	D	P	D	P	D	P	D
US Nom.	2.423	68.5	43.45	51.7	693.6	26.6	537.6	30.1	840.9	23.1	86.63	46.9
US Real	43.45	51.7	866.4	22.4	840.9	23.1	240.8	38.5	240.8	38.5	813.7	23.8

6.2 Option pricing

After the statistical study which was comparing the fit with the empirical data of normal distribution against that of Lévy distributions, this section reviews some calibration tools for option pricing. It begins by the maximum-likelihood parameter estimation method that was used in the previous section without a specific description. Afterward, the discretisation (i.e. numerical implementation) of the Fast Fourier Transform (FFT) is presented.

6.2.1 Maximum-Likelihood Estimator

Contrary to the normal distribution for which the parameters (sample's average and variance) are easily computed, there is no formula to estimated a Lévy distribution parameters. The maximum likelihood estimator (MLE) method described in this subsection can be used for Lévy distributions' parameter estimation. It is a common method in statistic for curve fitting and parameter estimation. Considering independent and identically distributed (i.i.d.) observations $x_1, x_2, \cdots, x_n$, the likelihood function of parameter θ is

$$lik(\theta) = f(x_1, x_2, \cdots, x_n|\theta),$$

where f is the frequency function. If the distribution is discrete, the likelihood function gives the probability of observing the given data as function of the parameter θ. With maximum likelihood estimator (MLE) a maximisation of the probability is performed. Since $x_1, x_2, \cdots, x_n$ are assumed i.i.d. and the natural logarithm is a monotonic function a maximisation is conducted on the log likelihood function

$$l(\theta) = \sum_{1}^{n} \ln[f(x_i|\theta)].$$

The MLE also have good theoretical properties such as being asymptotically efficient according to Cramer-Rao Inequality. Of course, this parameter estimation can also be used for a normal distribution. In fact, this case yields unbiased estimations of μ and σ^2.

For the GH distribution, the log likelihood function is

$$\begin{aligned} l(\theta) &= \ln a(\lambda, \alpha, \beta, \delta) + \left(\frac{\lambda}{2} - \frac{1}{4}\right) \sum_{i=1}^{n} \ln[\delta^2 + (x-\mu)^2] \\ &\quad + \sum_{i=1}^{n} \left[\ln K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x_i-\mu)^2}) + \beta(x_i-\mu)\right], \end{aligned}$$

with a defined in Section 2.4.1. Simpler expressions are obtained for hyperbolic and NIG distributions by taking respectively $\lambda = 1$ and $\lambda = -\frac{1}{2}$.

6.2.2 Discretisation of the FFT

Recall from Section 2.5 that the formula to be numerically evaluated is

$$c_T(K) = \frac{\exp(-\alpha K)}{\pi} \Re\left[\int_0^{+\infty} e^{-ivK}\Psi_T(v)dv\right].$$

First an approximation of ∞ is made. Let η represents the integration step and $N \in \mathbb{N}$ be a "large" enough number, the previous equation can be approximated by

$$c_T(K) \approx \frac{\exp(-\alpha K)}{\pi} \Re\left[\int_0^{N\eta} e^{-ivK}\Psi_T(v)dv\right].$$

The discretisation of the integral can be done using the Simpson's weighting rule [26], the midpoint rule, the trapeze method or any other common integral discretisation scheme. The trapeze method will be used here. The call fair price is now

$$c_T(K) \approx \frac{\exp(-\alpha K)}{\pi} \Re\left[\sum_{j=0}^{N} e^{-iv_jK}\Psi_T(v_j)\eta\theta_j\right],$$

where

$$\theta_j = \begin{cases} 0.5 & \text{for } j = 0, N \\ 1 & \text{otherwise.} \end{cases}$$

The Fast Fourrier Transform (FFT) returns the call price for $N + 1$ strike price with a regular interval. The FFT parameters (initial strike and strike step) are chosen such that the strike of the options to be priced are in the range of the estimated. An interpolation will also eventually be used to get the wanted option price(s).

6.3 Conclusion

An empirical study of the sample data from the South African and American markets was performed in this chapter. The results for the developing and developed markets all agree on the fact that market data is non-normal (and non-lognormal). This agrees with well documented stylised facts highlighting the non-normality of markets.

It is shown here that a calibration using Lévy distributions and specifically Generalised Hyperbolic, Hyperbolic, Variance Gamma, Normal Inverse Gaussian and Student-t prove to give better results in "every" case. Moreover, the calibration cost with Lévy distributions might eventually be "less" expensive than under normality assumption. A number of test of normality and goodness of fit test are also used to quantify how inappropriate the normal assumption is and how well each distribution performed. In most of the cases, the best fit is obtained with the Student-t distribution.

Appendix A

Empirical Study SA

This Appendix is divided in two sections which gives parameters estimated and other results obtained. Section A.1 (resp. A.2) presents results obtained from a GARCH filter with normal (resp. student t) innovations.

A.1 Normal innovations

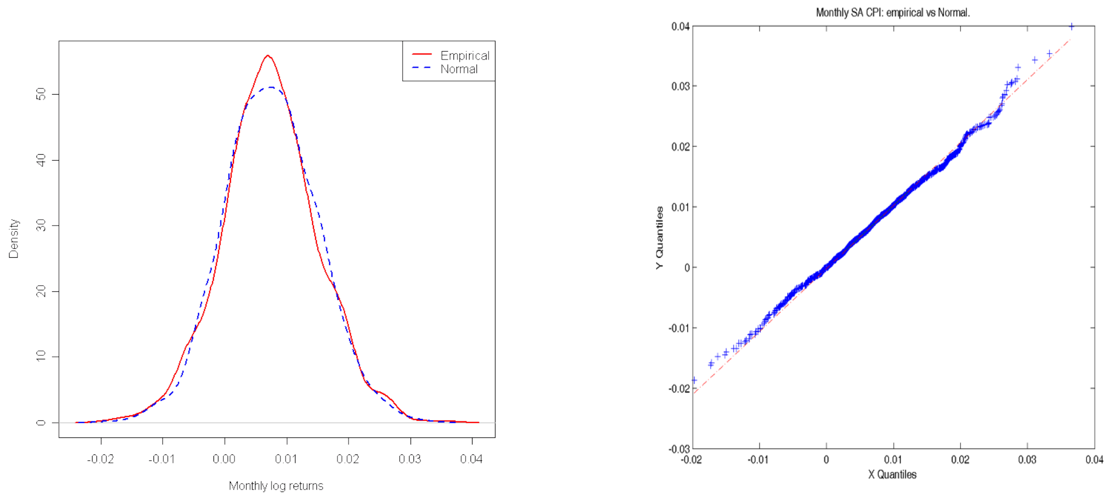
In this section a GARCH filter with normal innovations was used. Recall that the filter is meant to reduce the autocorrelation in the sample data.

A.1.1 Monthly SA CPI

When using normal innovations, the fit with the lower tail is better than that with the upper tail (Figure A.1(b)), i.e. the normality assumption will have more difficulties predicting high inflation increases than low increases. But, an investor is more concerned about high inflation rates than low rates, these “forecasting” performances are contrary to what is needed. In summary, the empirical density function is taller than the corresponding normal density function, however their support and skin’s shape are “quite” similar.

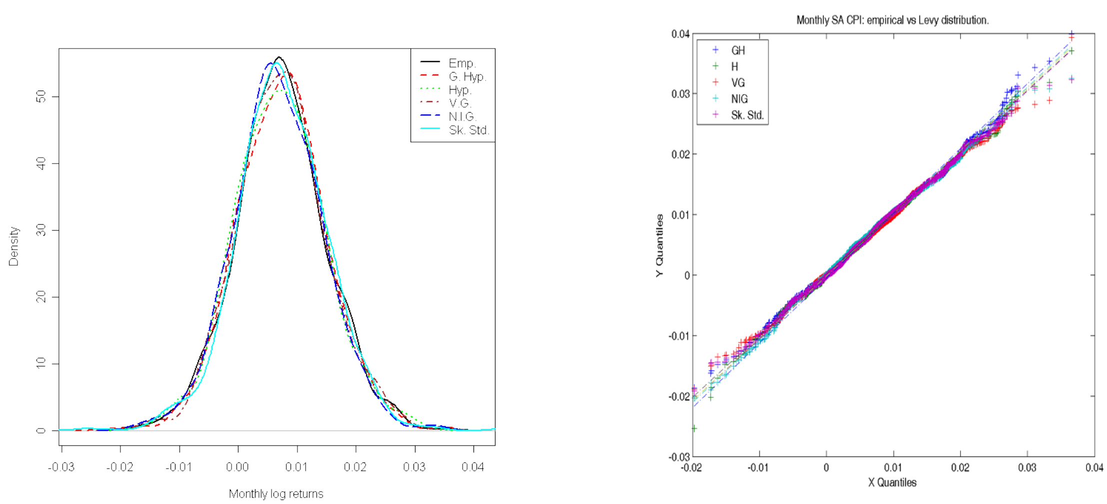
A.1.2 Consumer Price Index (CPIX)

Figures A.6 and A.7 show that despite the the small size of the sample, the GARCH filter has reduced its autocorrelation.



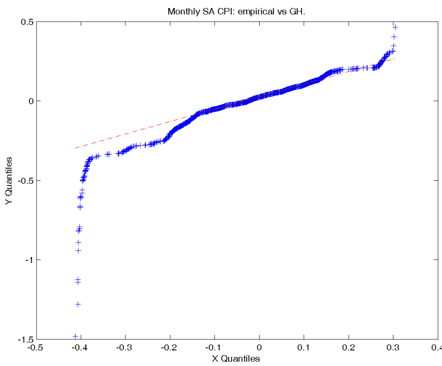
(a) Probability plot (b) QQ plot

Figure A.1 Monthly SA CPI probability and QQ plots: Empirical vs normal.

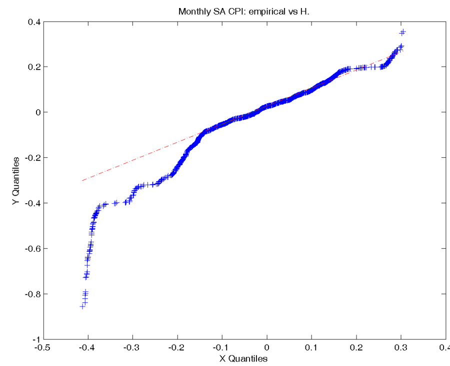


(a) Probability plot (b) QQ plot

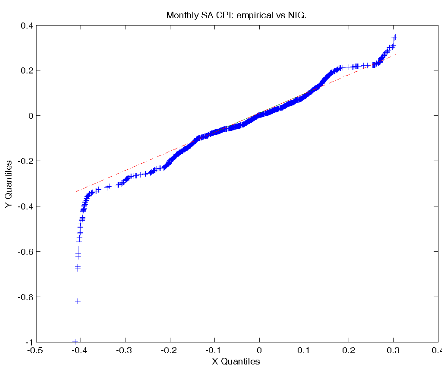
Figure A.2 Monthly SA CPI probability plots: Empirical vs Lévy.



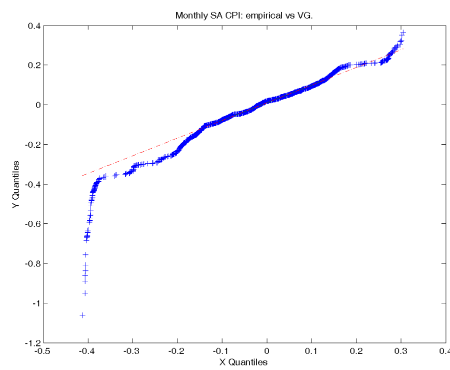
(a) GH



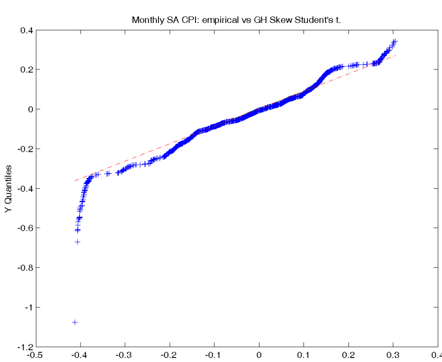
(b) H



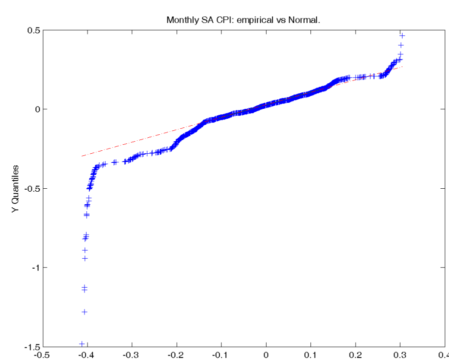
(c) NIG



(d) VG

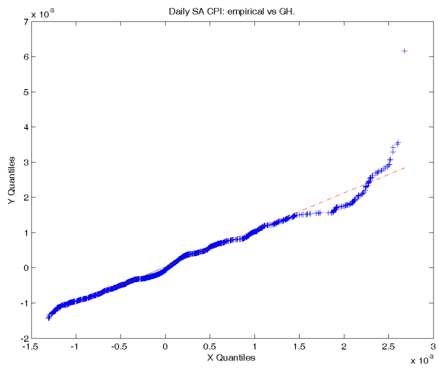


(e) Skw. Std.

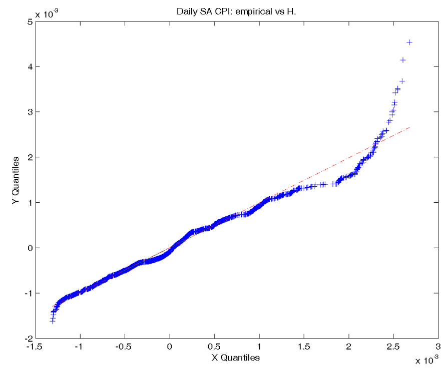


(f) Normal

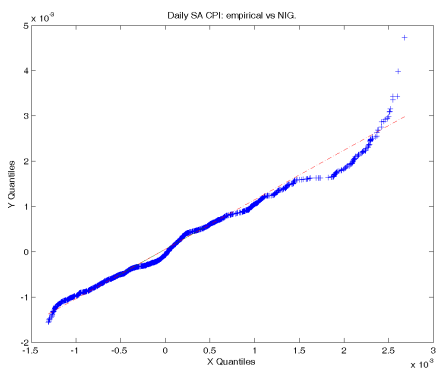
Figure A.3 Monthly SA CPI QQ plots (normality assumption).



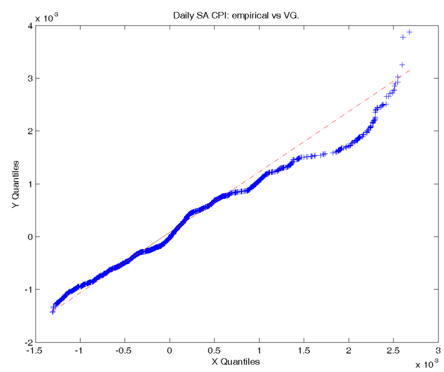
(a) GH



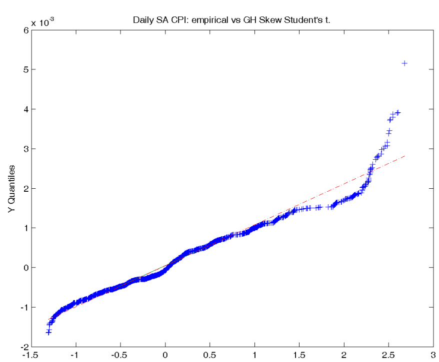
(b) H



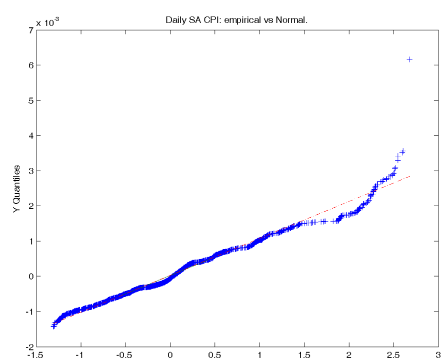
(c) NIG



(d) VG

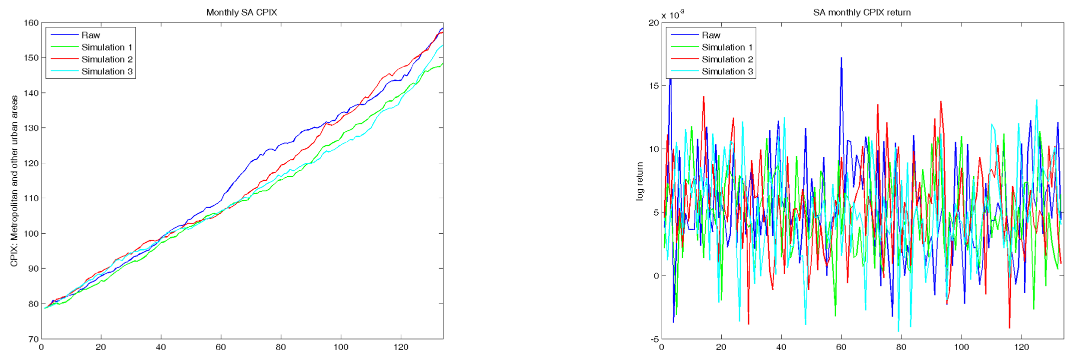


(e) Skw. Std.



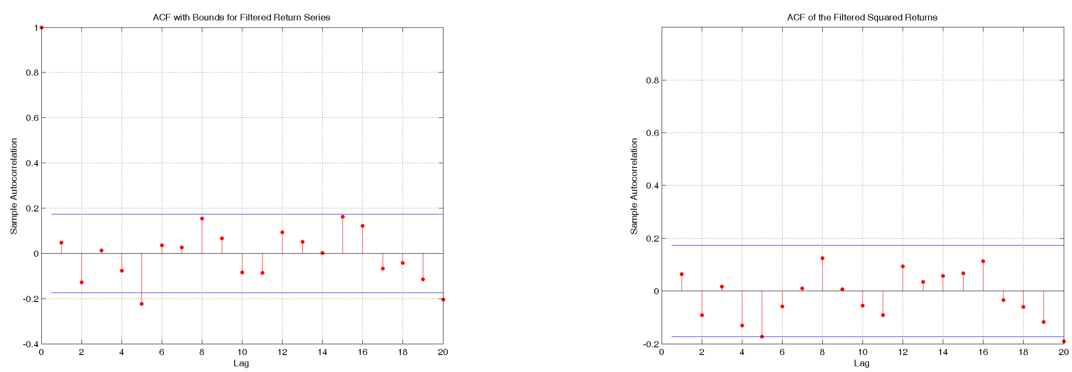
(f) Normal

Figure A.4 Filtered daily SA CPI QQ plots.



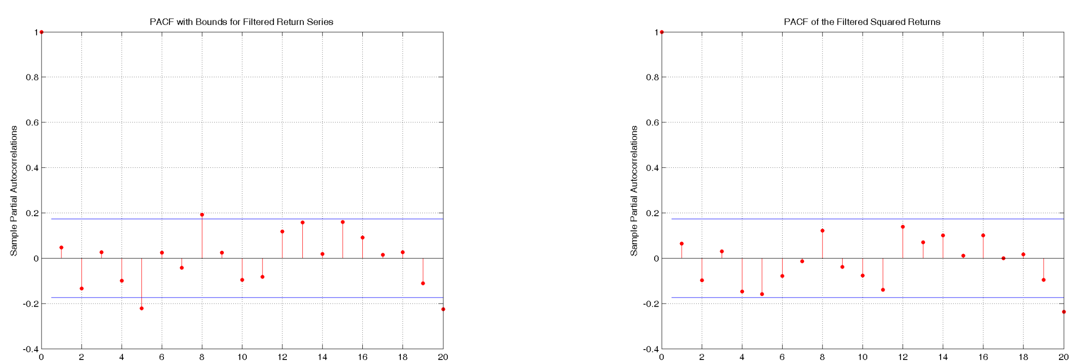
(a) CPIX (b) CPIX Log returns

Figure A.5 Filtered vs raw monthly SA CPIX (1997-2008) data series.



(a) Log returns (b) Squared returns

Figure A.6 Filtered monthly SA CPIX (1997-2008) correlograms.



(a) Log returns (b) Squared returns

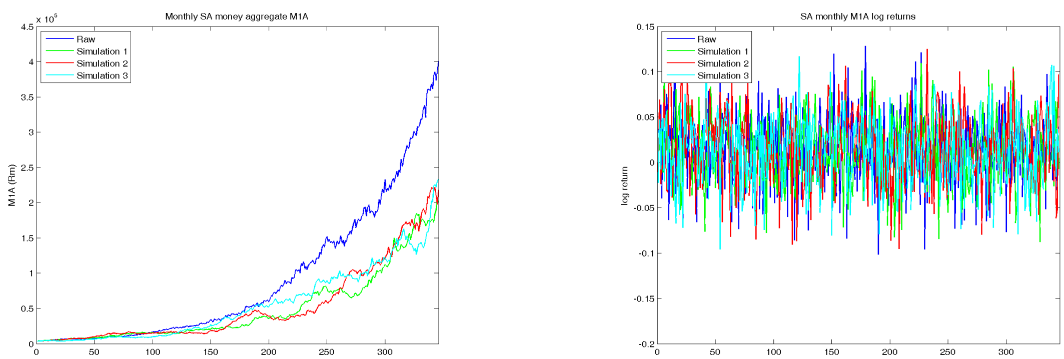
Figure A.7 Filtered monthly SA CPIX (1997-2008) partial correlograms.

Model	α	β	δ	μ	λ
NIG	342.72583	18.73317	0.020182	0.0062122	-0.5
H	369.70173	18.44541	0.017584	0.0062295	1
GH	465.85525	19.08309	$1.90 \cdot 10^{-5}$	0.0061911	6.38998
VG	465.93679	19.09688	0	0.0061914	6.39349
Sk.Std.	18.12631	18.12631	0.00423256	0.0062488	0.053276

Table A.1 Monthly SA CPI Lévy distributions' estimated parameters.

For the Lévy distributions' parameters estimation, the GARCH filter is used to multiply the sample size by 10.

A.1.3 Money Supply aggregate M1A



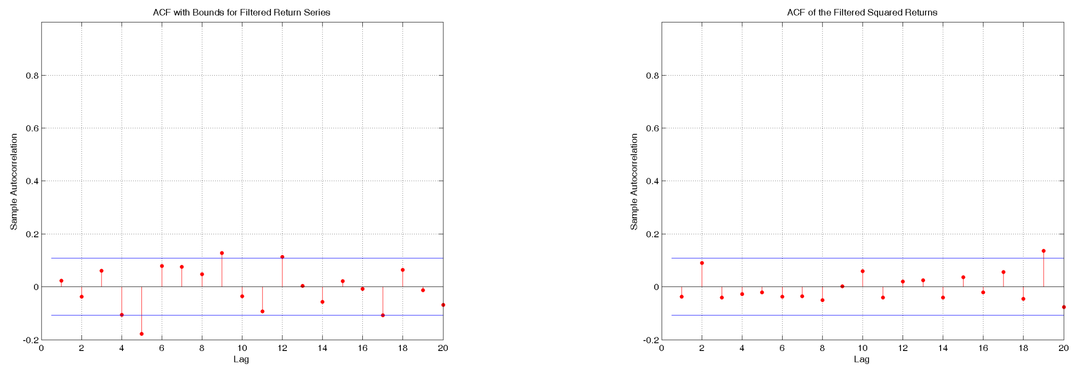
(a) M1A

(b) M1A Log returns

Figure A.8 Filtered vs raw monthly SA M1A (1979-2007) data series.

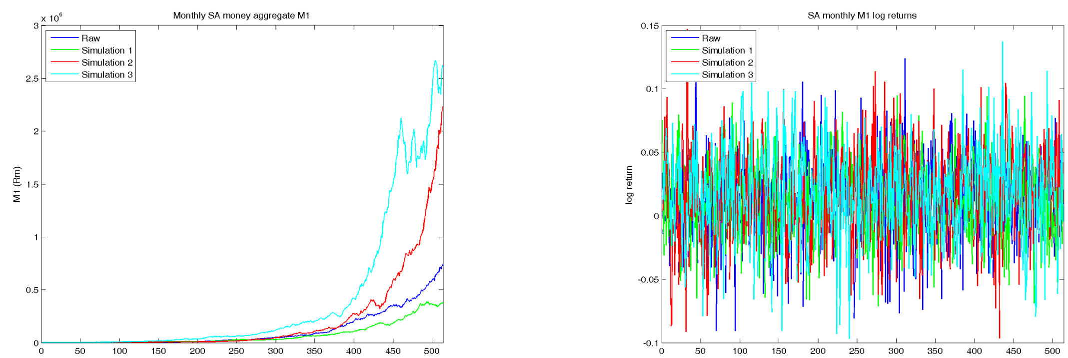
A.1.4 Money Supply aggregate M1

A.1.5 Money Supply aggregate M2



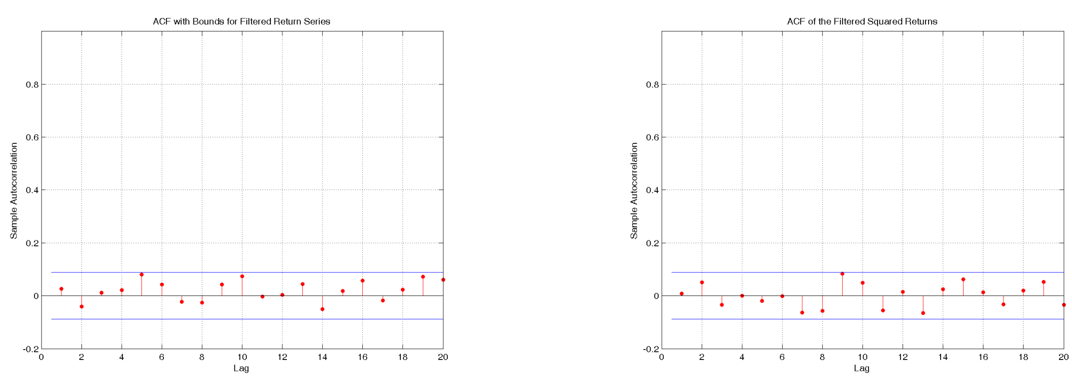
(a) Log returns (b) Squared returns

Figure A.9 Filtered monthly SA M1A (1979-2007) correlograms.



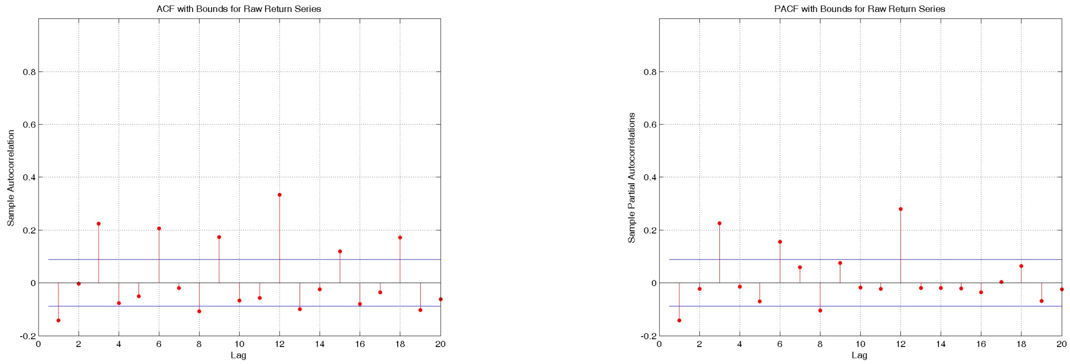
(a) M1 (b) M1 Log returns

Figure A.10 Filtered vs raw monthly SA M1 (1965-2007) data series.



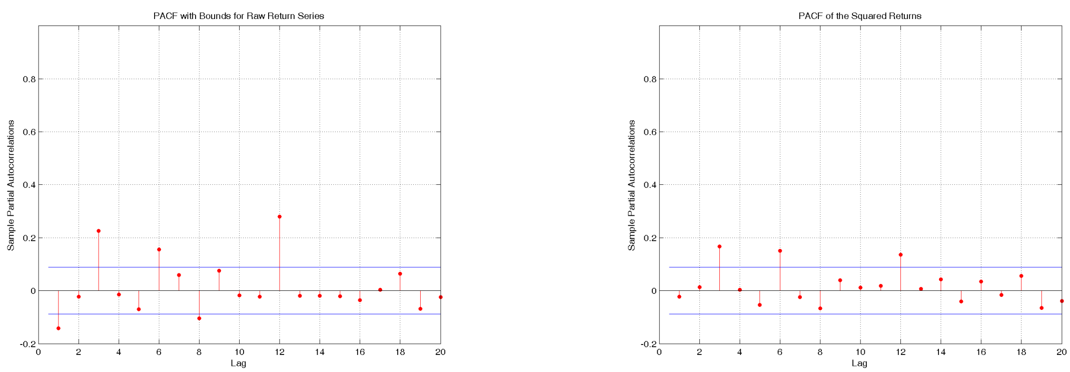
(a) Log returns (b) Squared returns

Figure A.11 Filtered monthly SA M1 (1965-2007) correlograms.



(a) Log returns (b) Squared returns

Figure A.12 Monthly SA Money Supply M2 (1965-2007) correlograms.



(a) Log returns (b) Squared returns

Figure A.13 Monthly SA Money Supply M2 (1965-2007) partial correlograms.

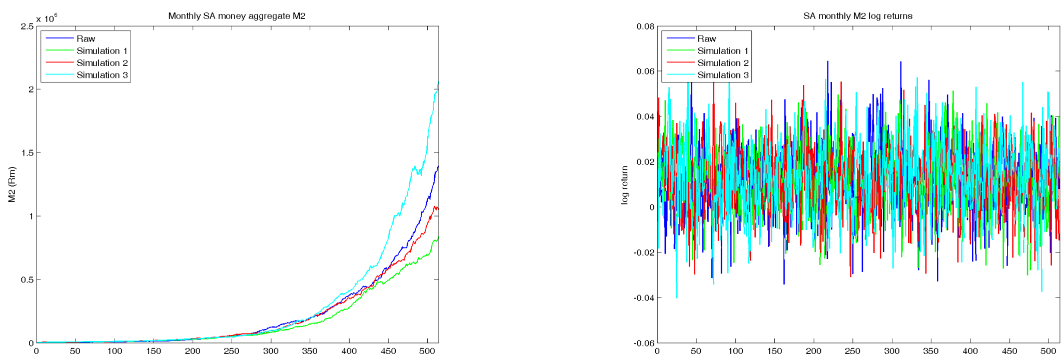
	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	87.7421	18.3070	0	0.6379	7.9071	18.3070
15	1	0	161.2922	24.9958	0	0.1300	21.2190	24.9958
20	1	0	188.6595	31.4104	0	0.1950	25.1705	31.4104

Table A.2 Ljung-Box-Pierce Q-test for SA Monthly M2 (1965-2007) raw and squared returns.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	0	0.6974	7.2942	18.3070	0	0.7489	6.7487	18.3070
15	0	0.2219	18.8214	24.9958	0	0.6849	11.9225	24.9958
20	0	0.3202	22.3815	31.4104	0	0.8147	14.3047	31.4104

Table A.3 Engle's ARCH test results for SA M2 (1965-2007) raw and squared returns.

The LBP Q-test identifies GARCH effects only in the raw returns and not in their square. However, the Engle's ARCH test finds GARCH effects neither in the log returns nor their square. The Engle's ARCH test output confirms the fact that the sample's volatility does not vary "much" with time. Nevertheless, Lévy distributions' parameters estimation will be performed to compare their fit with that of the normal distribution.

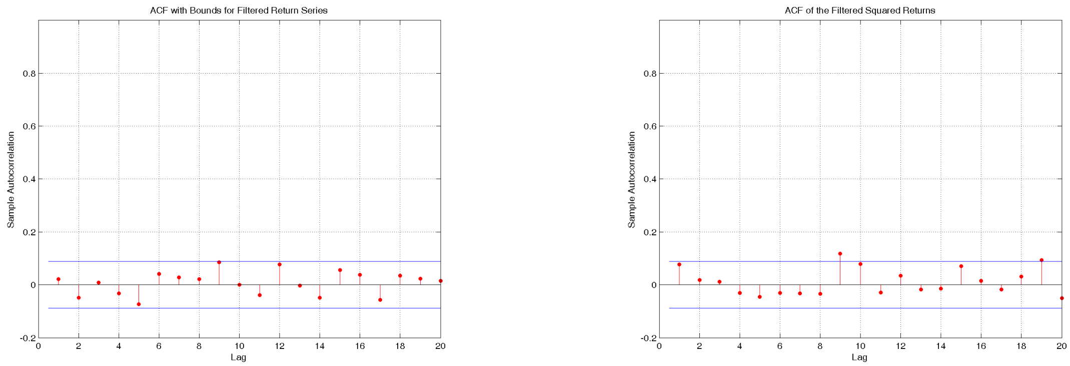


(a) M2

(b) M2 Log returns

Figure A.14 Filtered vs raw monthly SA M2 (1965-2007) data series.

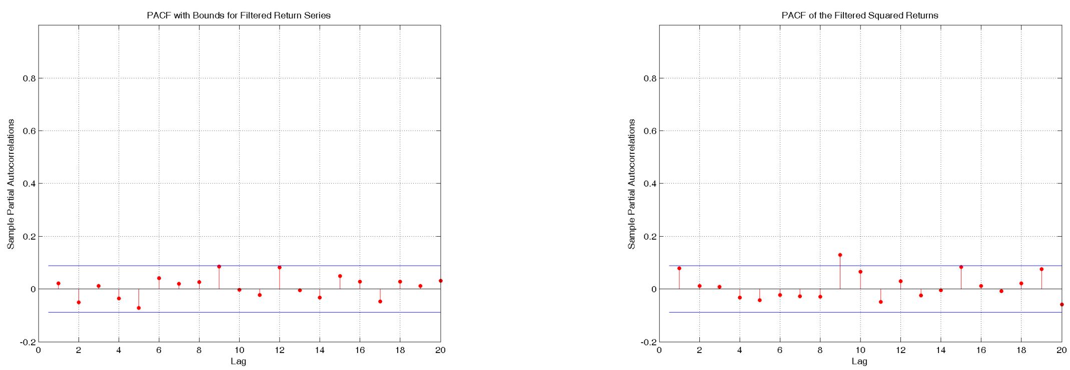
Money Supply aggregate M3



(a) Log returns

(b) Squared returns

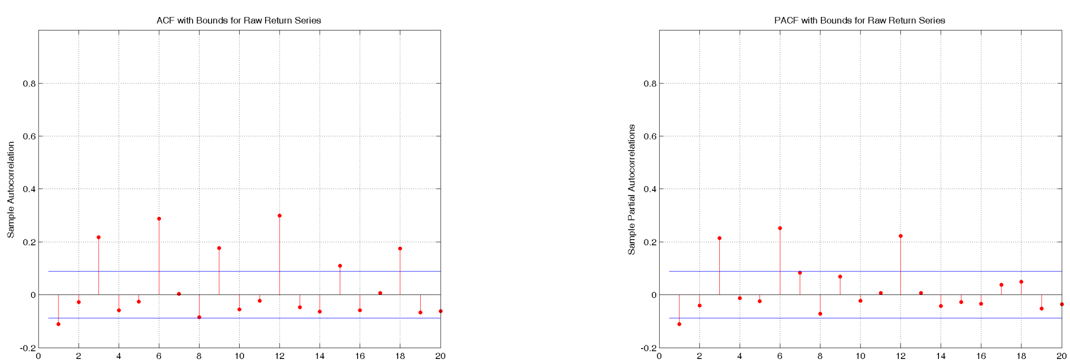
Figure A.15 Filtered monthly SA M2 (1965-2007) correlograms.



(a) Log returns

(b) Squared returns

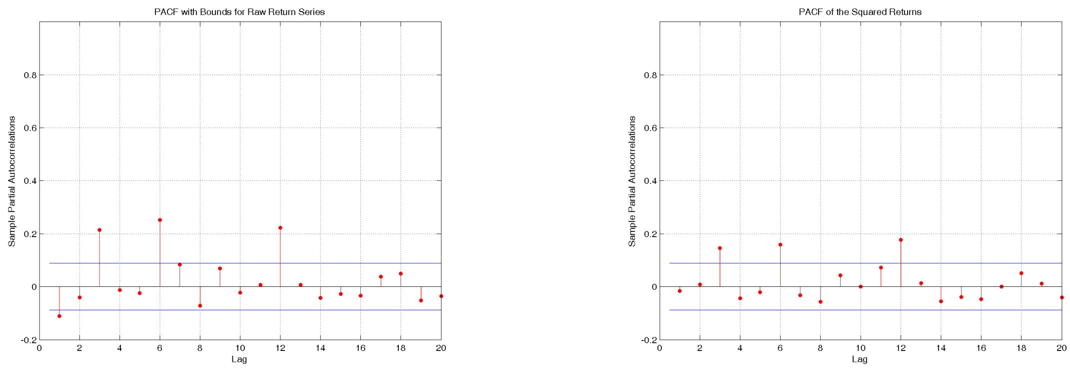
Figure A.16 Filtered monthly SA M2 (1965-2007) partial correlograms.



(a) Log returns

(b) Squared returns

Figure A.17 Monthly SA Money Supply M3 (1965-2007) correlograms.



(a) Log returns (b) Squared returns

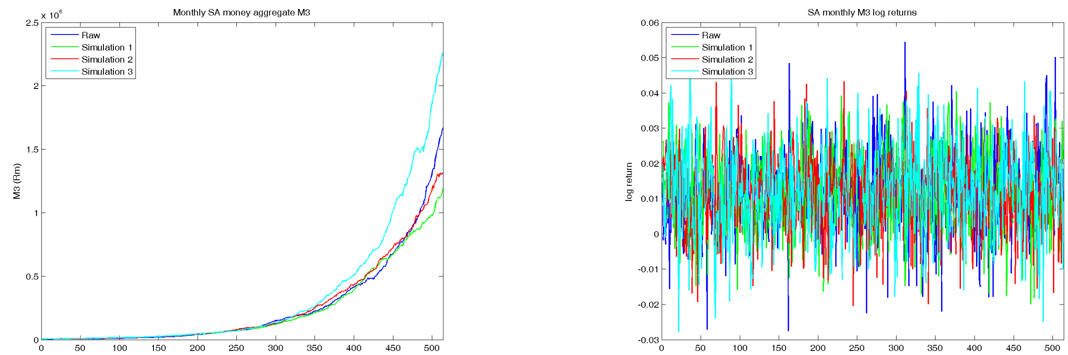
Figure A.18 Monthly SA Money Supply M3 (1965-2007) partial correlograms.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	98.2356	18.3070	0	0.3576	10.99907	18.3070
15	1	0	155.7858	24.9958	0	0.0545	24.6732	24.9958
20	1	0	178.4955	31.4104	1	0.0466	31.7032	31.4104

Table A.4 Ljung-Box-Pierce Q-test for SA Monthly M3 (1965-2007) raw and squared returns.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	0	0.3916	10.5742	18.3070	0	0.4477	9.9184	18.3070
15	0	0.1650	20.1843	24.9958	0	0.6191	12.7825	24.9958
20	0	0.1324	27.1015	31.4104	0	0.8047	14.4929	31.4104

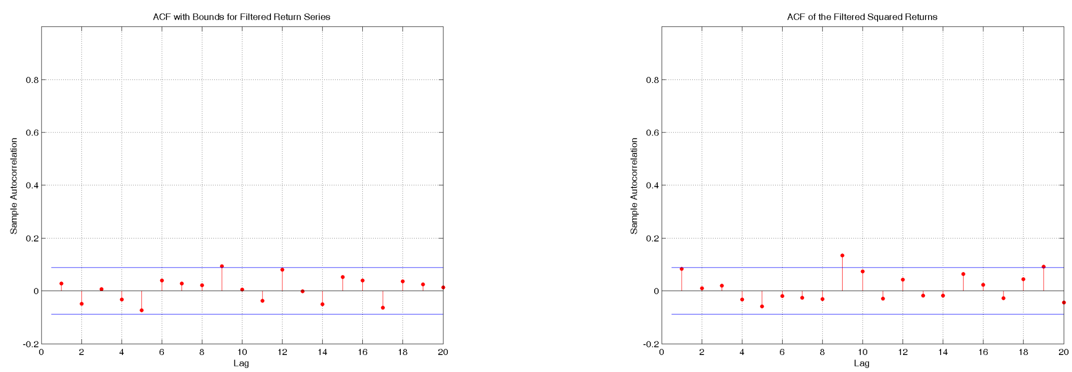
Table A.5 Engle's ARCH test results for SA M3 (1965-2007) raw and squared returns.



(a) M3

(b) M3 Log returns

Figure A.19 Filtered vs raw monthly SA M3 (1965-2007) data series.



(a) Log returns

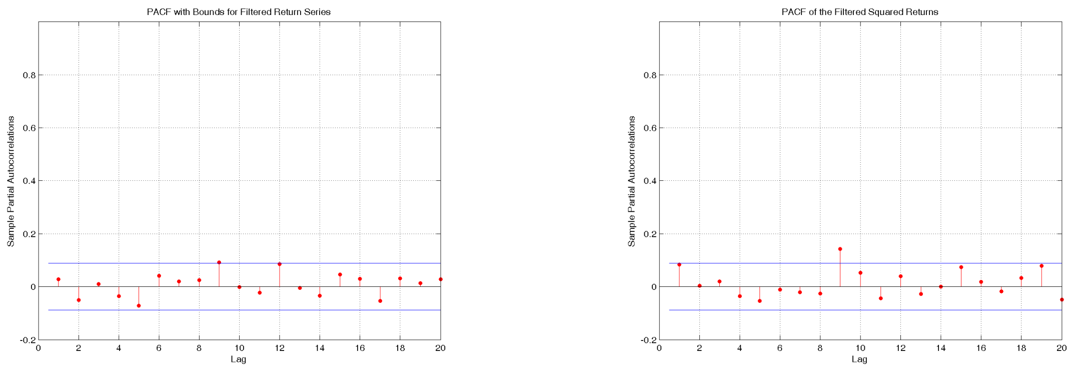
(b) Squared returns

Figure A.20 Filtered monthly SA M3 (1965-2007) correlograms.

A.2 Student t innovations

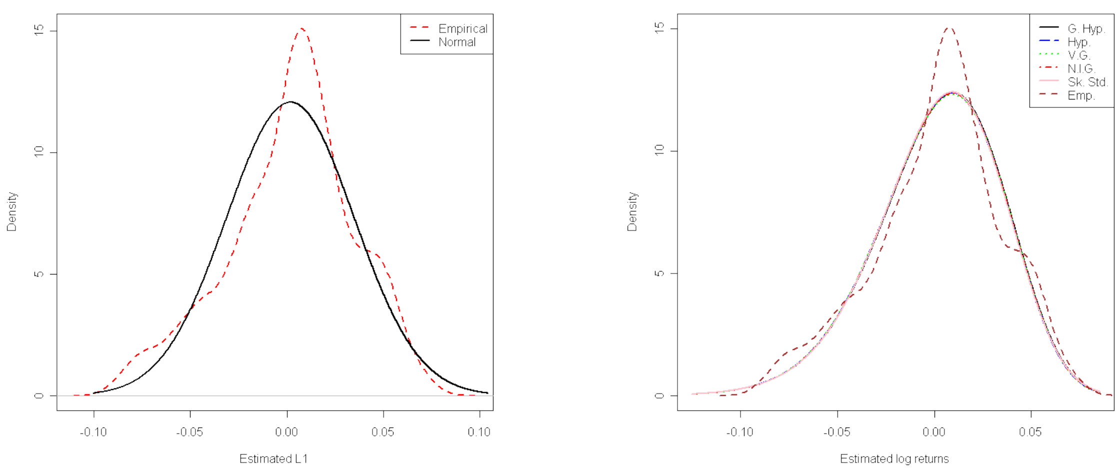
A.2.1 SA CPIX

A.3 Forward estimates



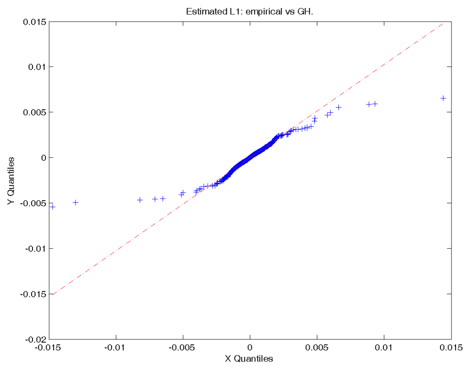
(a) Log returns (b) Squared returns

Figure A.21 Filtered monthly SA M3 (1965-2007) partial correlograms.

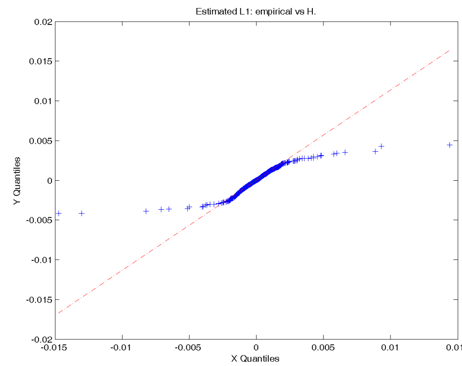


(a) Empirical vs Normal (b) Empirical vs Lévy

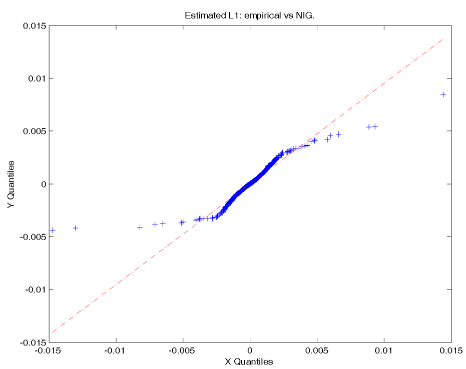
Figure A.22 Probability plots: Empirical vs model.



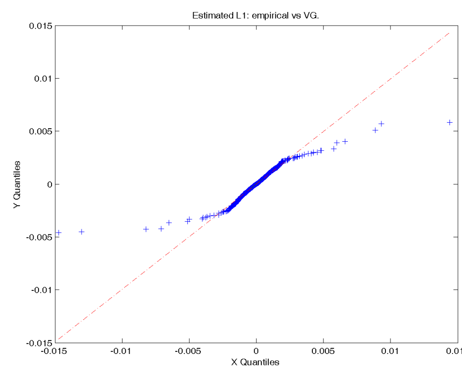
(a) GH



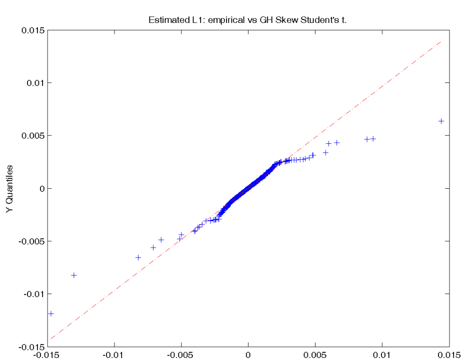
(b) H



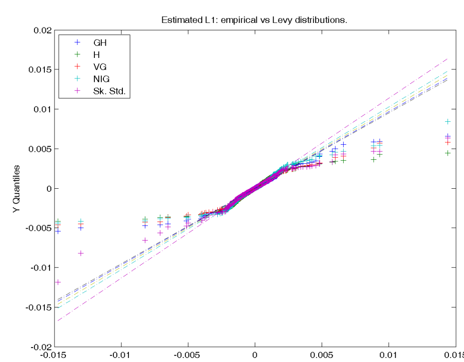
(c) NIG



(d) VG



(e) Skw. Std.



(f) Normal

Figure A.23 Estimated L_1 QQ plots.

Appendix B

Empirical Study US

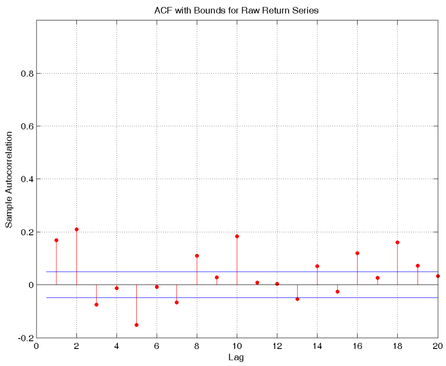
B.1 Normal innovations

B.1.1 Consumer Price Index

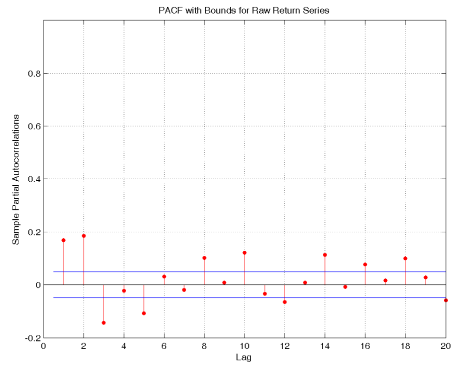
The monthly United States of America Consumer Price Index is taken from the Bureau of Labor Statistics' (BLS) website [106]. The data is sampled from the 31^{rst} December 1821 to the 30^{th} November 2007. The entire historical data covers a period of 186 years corresponding to 1649 observation points. The empirical study is first conducted on the entire data, then on the most recent half. The latter coincide with the period going from the 31^{rst} January 1937 to the 30^{th} November 2007, with 850 observations covering 70 years. The sample size is big enough for the filtering using the $GARCH(1,1)$ to give "good" results and 70 years is big enough to cover the investment of a particularly long lived client of a pension fund.

Figures B.2 and B.1 show that there is no annual seasonality (i.e. high spike at lag 12). This is quite surprising, this might reflect the "success" of inflation targeting in US.

Figure B.6 (resp. B.7) shows the empirical density and log density functions of monthly log returns of US CPI (1821-2007). Each graph also has the normal (resp. Lévy) probability density and log density functions with parameters evaluated from the sample data and presented in Tables B.3 and 6.10. The plots indicate that the monthly US CPI is non-normal and the Lévy distributions are more suited for the calibration. The empirical density and log density functions are more peaked than the corresponding normal distribution, but Lévy distributions reproduce "fairly" well the peakedness. The normal distribution's density function also has fatter tails than its empirical

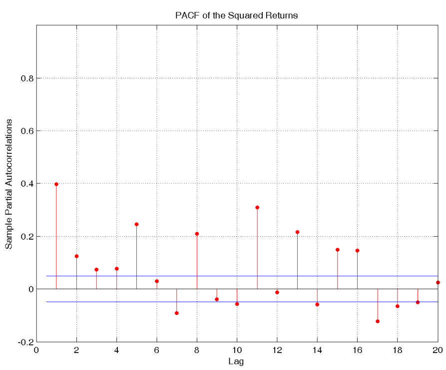


(a) Log returns

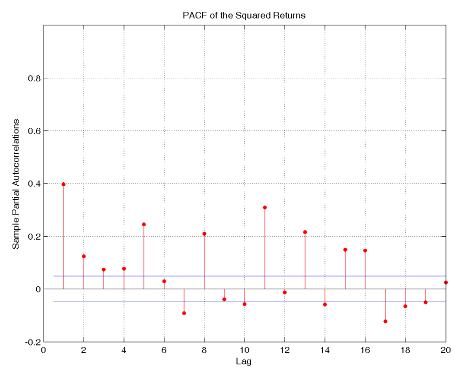


(b) Squared returns

Figure B.1 Monthly USA CPI (1821-2007) correlograms.

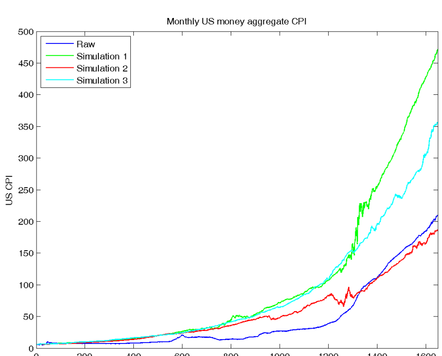


(a) Log returns

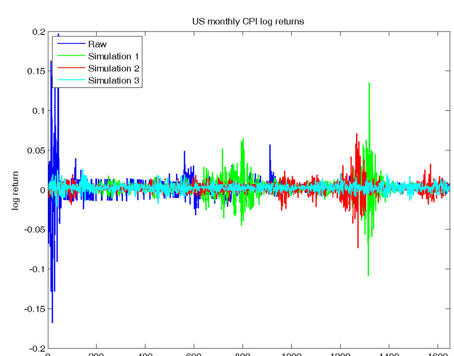


(b) Squared returns

Figure B.2 Monthly USA CPI (1821-2007) partial correlograms.



(a) CPI



(b) Log returns

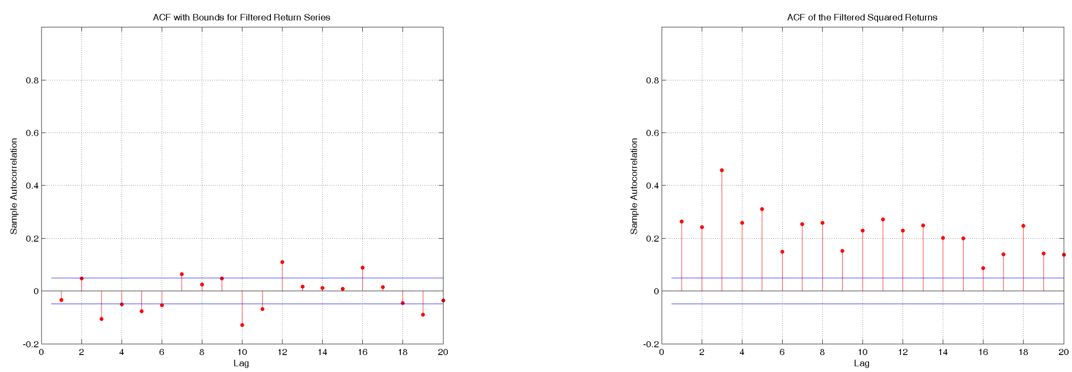
Figure B.3 Filtered vs raw monthly USA CPI (1821-2007) data series .

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	303.2123	18.3070	1	0	964.3	18.3
15	1	0	324.1402	24.9958	1	0	1570.2	25.0
20	1	0	400.5362	31.4104	1	0	1854.6	31.4

Table B.1 Ljung-Box-Pierce Q-test for USA CPI (1821-2007) raw and squared returns.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	453.6761	18.3070	1	0	112.9745	18.3070
15	1	0	609.4648	24.9958	1	0	280.9518	24.9958
20	1	0	813.3873	31.4104	1	0	383.9034	31.4104

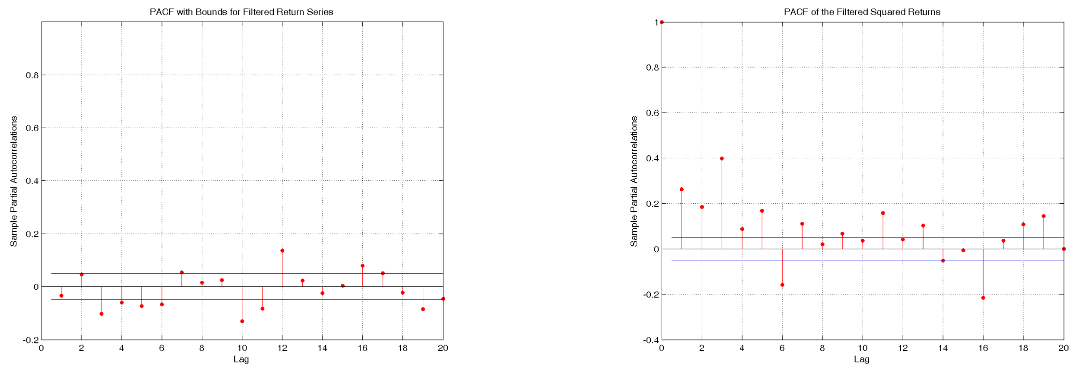
Table B.2 Engle's ARCH test results for USA CPI (1821-2007) raw and squared returns.



(a) Log returns (b) Squared returns

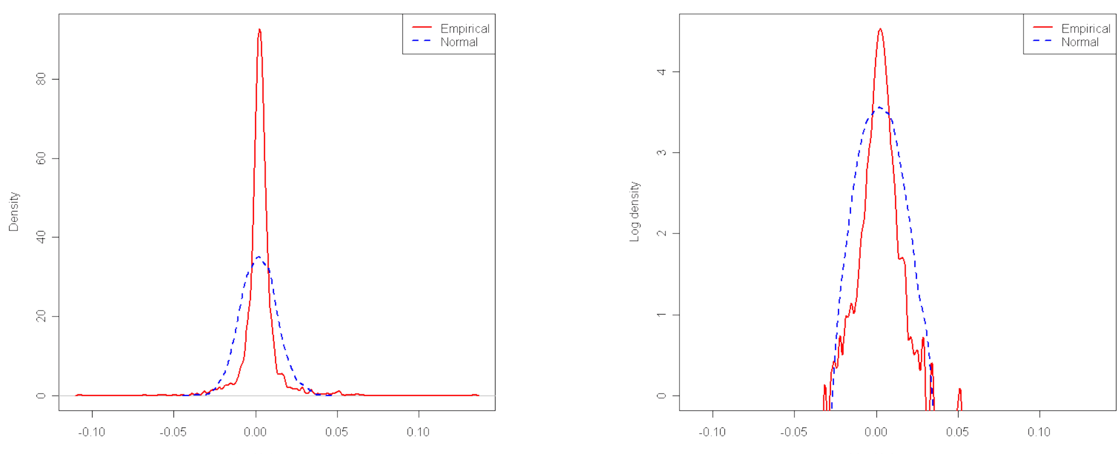
Figure B.4 Filtered monthly USA CPI (1821-2007) correlograms.

counterpart; with a wider support. While Lévy distribution have a support "almost" identical to that of the empirical density function. However, they do not perform so well in reproducing the tail behaviour of the empirical sample. This is more obvious when looking at the QQ plots (Figures B.8 and B.9). In other words, the empirical density and log density functions are taller, skinnier, but with a smaller support than their normal counterpart. While the Lévy distributions' density and log density functions have the same general structure as the corresponding empirical function, with



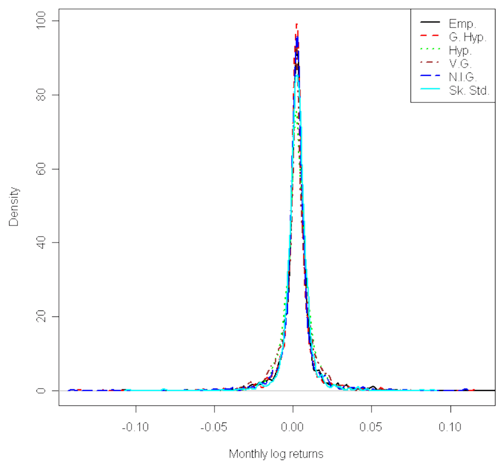
(a) Log returns (b) Squared returns

Figure B.5 Filtered monthly USA CPI (1821-2007) partial correlograms.

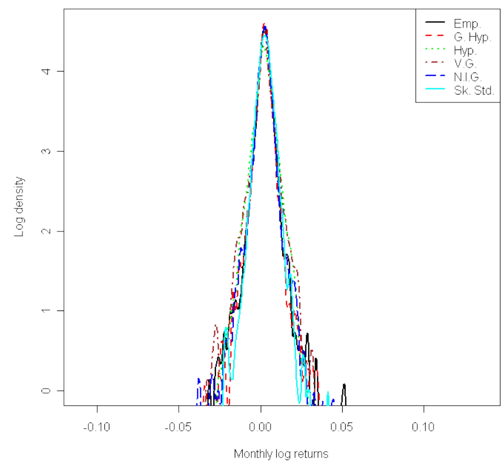


(a) Density (b) Log Density

Figure B.6 Monthly USA CPI (1821-2007) probability plots: Empirical vs normal.



(a) Density



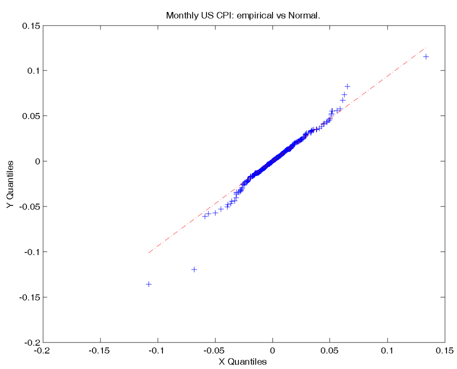
(b) Log Density

Figure B.7 Monthly USA CPI (1821-2007) probability plots: Empirical vs Lévy.

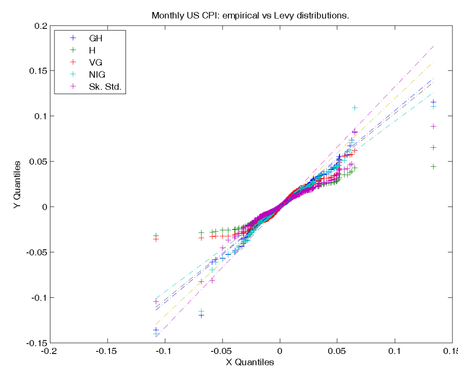
some mismatches on the tails.

Model	α	β	δ	μ	λ	**LLH**
NIG	28.00664	1.153026	0.00339	0.00248	-0.5	5805.90
H	169.93788	3.39996	$1.77 \cdot 10^{-6}$	0.00238	1	5669.98
GH	23.66755	1.06193	0.00362	0.00248	-0.56210	5805.17
VG	114.86472	2.85505	0	0.00239	0.54532	5726.15
Sk.Std.	0.09226	0.09226	N/A	0.00249	0.49989	5788.29

Table B.3 Estimated parameters for USA CPI (1821-2007).

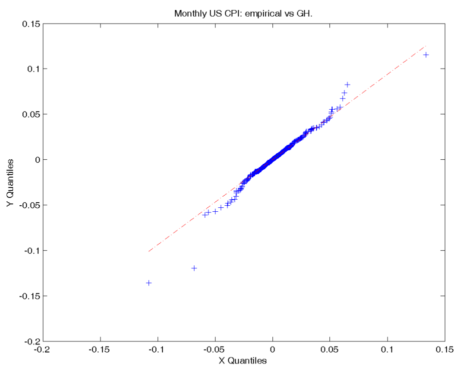


(a) Normal

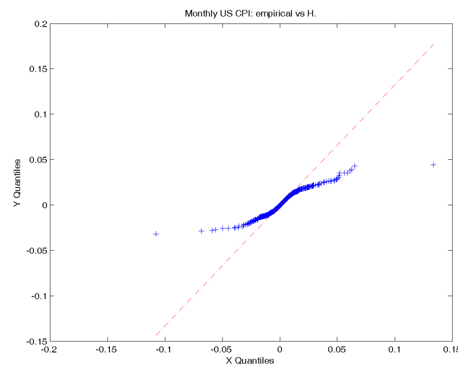


(b) Lévy

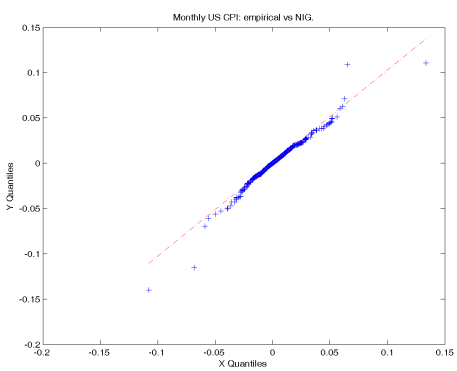
Figure B.8 Monthly US CPI QQ plots (individually).



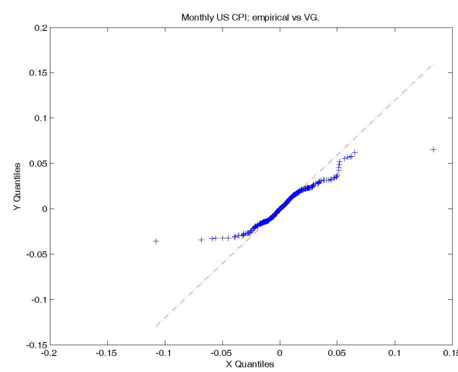
(a) GH



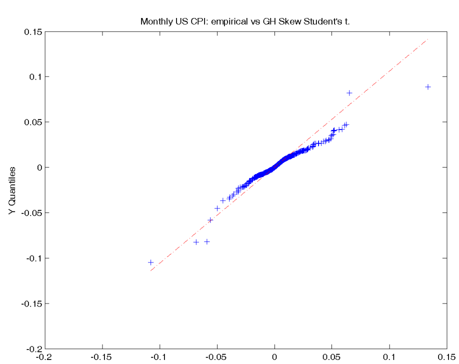
(b) H



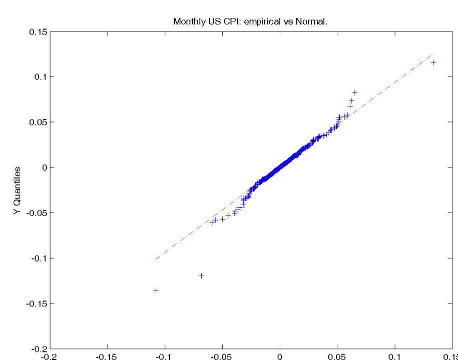
(c) NIG



(d) VG



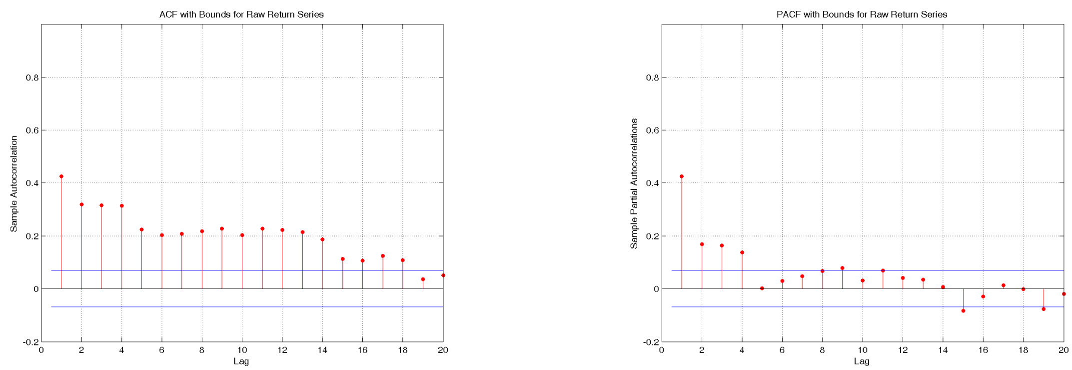
(e) Skw. Std.



(f) Normal

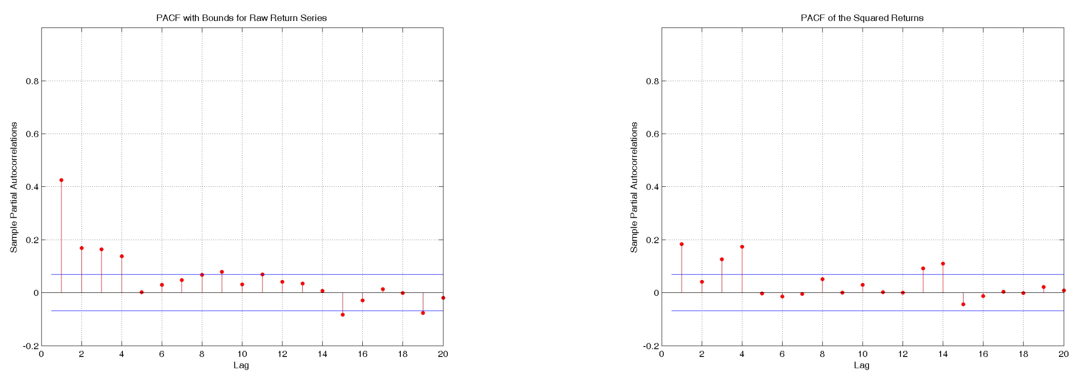
Figure B.9 Monthly US CPI QQ plots.

B.1.2 Consumer Price Index (End half)



(a) Log returns (b) Squared returns

Figure B.10 Monthly USA CPI (1937-2007) correlograms.



(a) Log returns (b) Squared returns

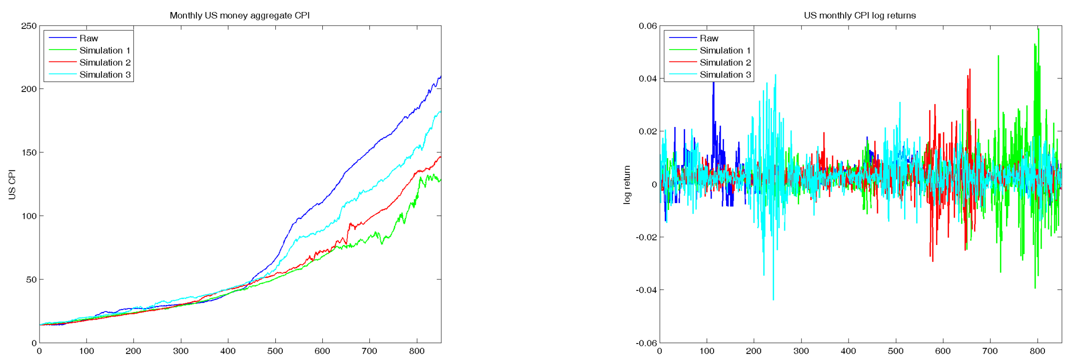
Figure B.11 Monthly USA CPI (1937-2007) partial correlograms.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	647.9843	18.3070	1	0	55.6300	18.3070
15	1	0	816.7679	24.9958	1	0	72.3584	24.9958
20	1	0	853.6211	31.4104	1	0	78.5209	31.4104

Table B.4 Ljung-Box-Pierce Q-test for USA CPI (1937-2007) raw and squared returns.

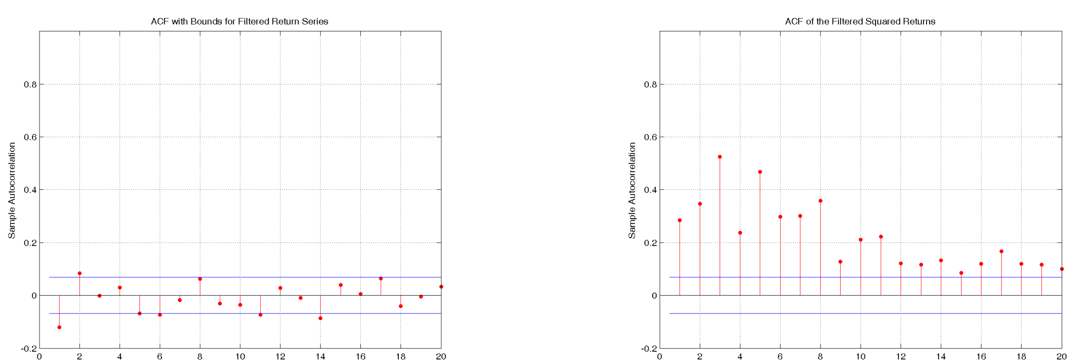
	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	40.6426	18.3070	0	1	0.4588	18.3070
15	1	0	50.7111	24.9958	0	1	0.6084	24.9958
20	1	0.0001	51.7990	31.4104	0	1	0.6089	31.4104

Table B.5 Engle's ARCH test results for USA CPI (1937-2007) raw and squared returns.



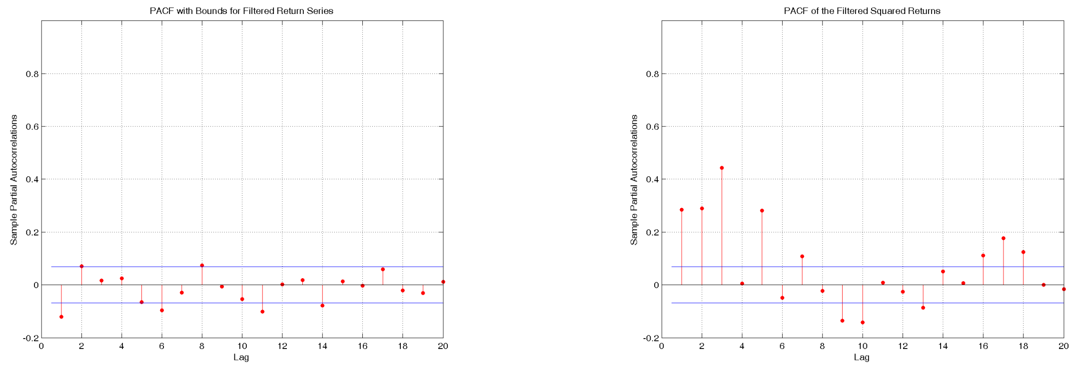
(a) CPI (b) Log returns

Figure B.12 Filtered vs raw monthly USA CPI (1937-2007) data series .



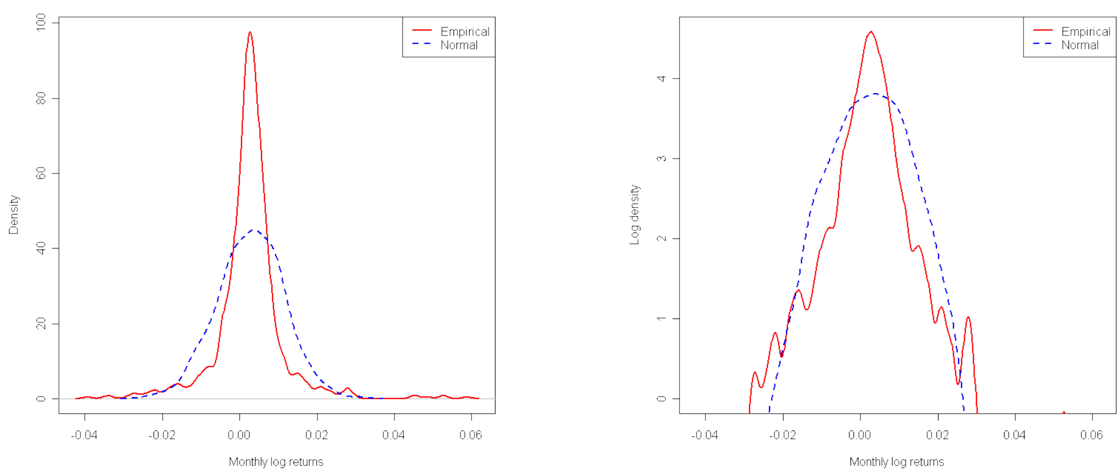
(a) Log returns (b) Squared returns

Figure B.13 Filtered monthly USA CPI (1937-2007) correlograms.



(a) Log returns (b) Squared returns

Figure B.14 Filtered monthly USA CPI (1937-2007) partial correlograms.

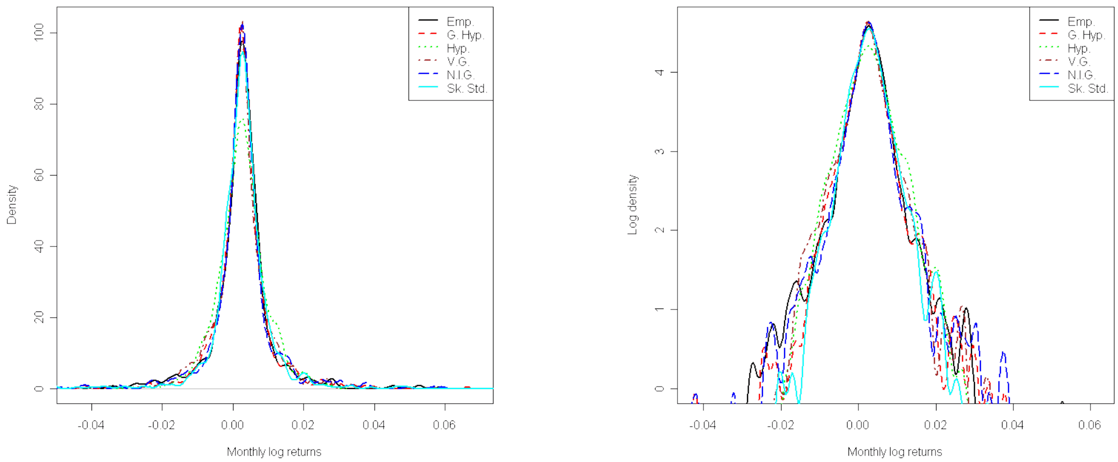


(a) Density (b) Log Density

Figure B.15 Monthly USA CPI (1937-2007) probability plots: Empirical vs normal.

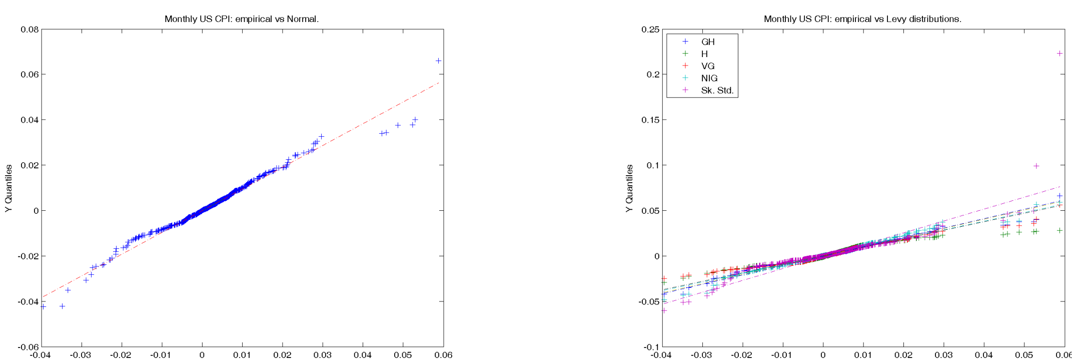
Model	α	β	δ	μ	λ	**LLH**
NIG	42.17912	-2.20823	0.00344	0.00278	-0.5	3055.83
H	190.76526	-1.44290	$1.24 \cdot 10^{-5}$	0.00268	1	3020.98
GH	55.97538	-2.35718	0.00287	0.00279	-0.32058	3056.39
VG	137.34979	0.10904	0	0.00264	0.60524	3038.41
Sk.Std.	0.51414	-0.51414	N/A	0.00272	0.49952	3047.13

Table B.6 Estimated parameters for USA CPI (1937-2007).



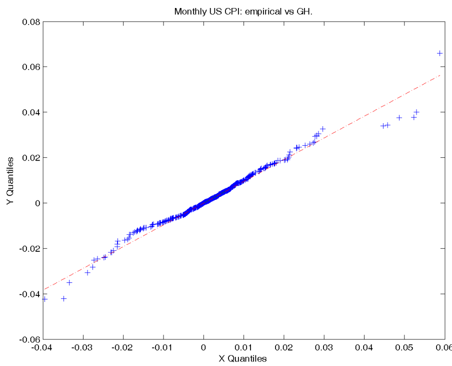
(a) Density (b) Log Density

Figure B.16 Monthly USA CPI (1937-2007) probability plots: Empirical vs Lévy.

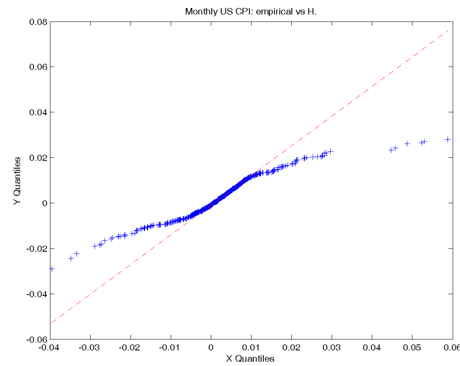


(a) Normal (b) Lévy

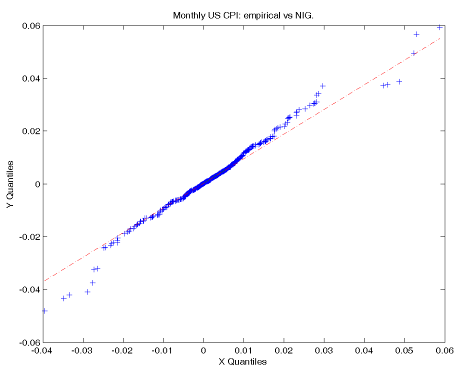
Figure B.17 Monthly US CPI QQ plots.



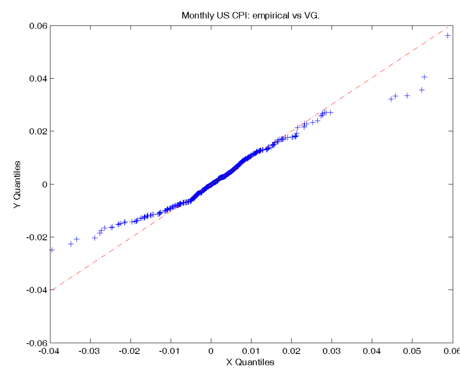
(a) GH



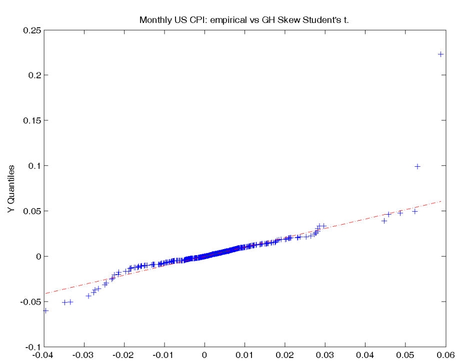
(b) H



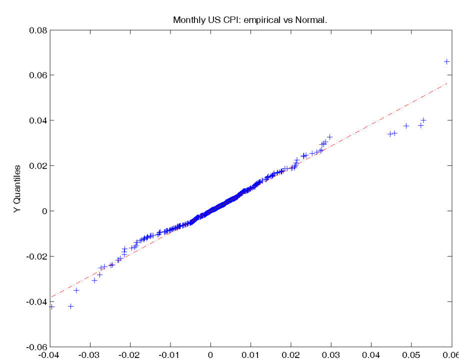
(c) NIG



(d) VG



(e) Skw. Std.

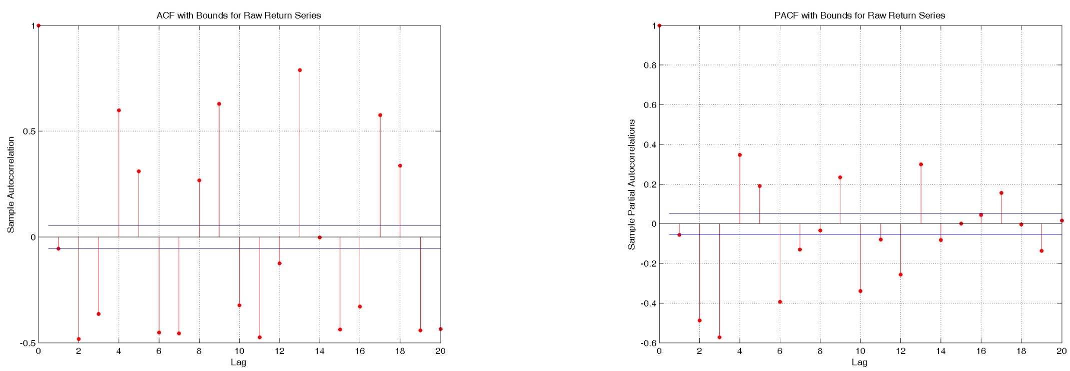


(f) Normal

Figure B.18 Monthly US CPI QQ plots (individually).

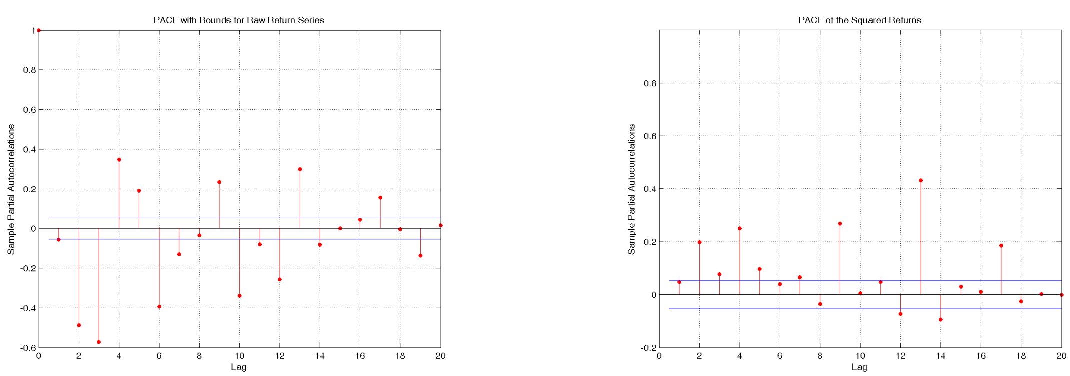
B.1.3 Money Supply aggregate M1

The weekly American Monetary aggregates M1 time series data is obtained from the Federal Reserve System [103]. The data is from the 5^{th} January 1981 to the 21^{st} April 2008 that is 1425 data points overall. That is more than enough data points for a "good" statistical study. The data is for the 1^{st} May 2008 and measured in billions of US dollars.



(a) Log returns (b) Squared returns

Figure B.19 Weekly USA Money Supply M1 (1981-2008) correlograms.



(a) Log returns (b) Squared returns

Figure B.20 Weekly USA Money Supply M1 (1981-2008) partial correlograms.

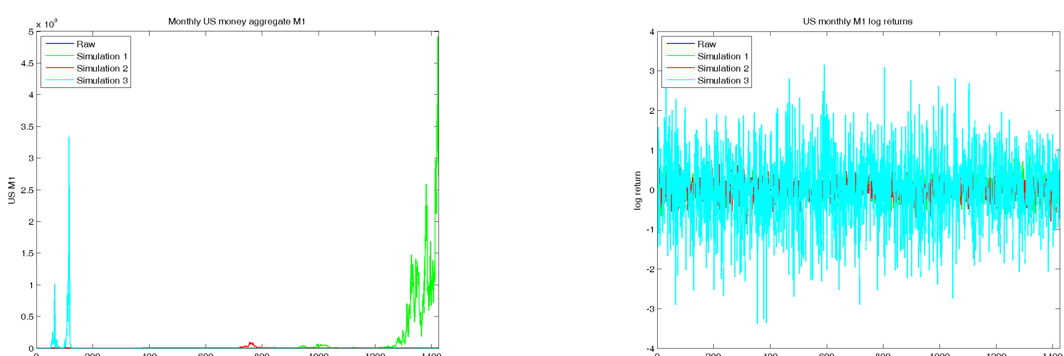
The current estimated GARCH parameters might generate a highly volatile filtered sample data (Figure B.21). This might be due to the high volatility of the money aggregate M1. Notice that in Table 6.10, M1 has the highest volatility which is about three time that of the next most volatile money aggregate. Therefore, for stability reasons, the other two aggregates M2 and M3 will be

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	2583.8	18.3070	1	0	416.0	18.3070
15	1	0	4099.5	24.9958	1	0	892.9	24.9958
20	1	0	5456.4	31.4104	1	0	1096.8	31.4104

Table B.7 Ljung-Box-Pierce Q-test for USA Weekly M1 (1981-2008) raw and squared returns.

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	265.9485	18.3070	1	0.0010	29.5679	18.3070
15	1	0	498.9969	24.9958	1	0	69.2088	24.9958
20	1	0	530.3411	31.4104	1	0	79.2994	31.4104

Table B.8 Engle's ARCH test results for USA M1 (1981-2008) raw and squared returns.

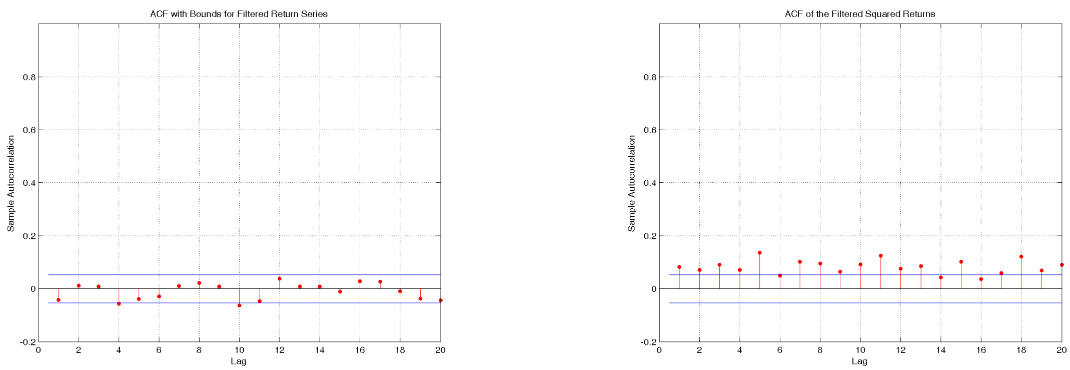


(a) M1

(b) M1 Log returns

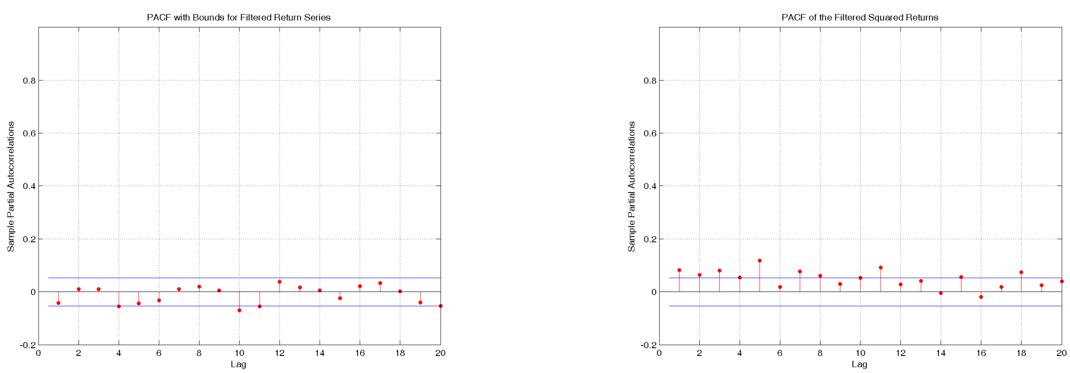
Figure B.21 Filtered vs raw weekly USA M1 (1981-2008) data series.

preferred to M1 for our calibrations. In Table 6.10, M1 is also the only sample with negative skewness, this might also be due to its high volatility.



(a) Log returns (b) Squared returns

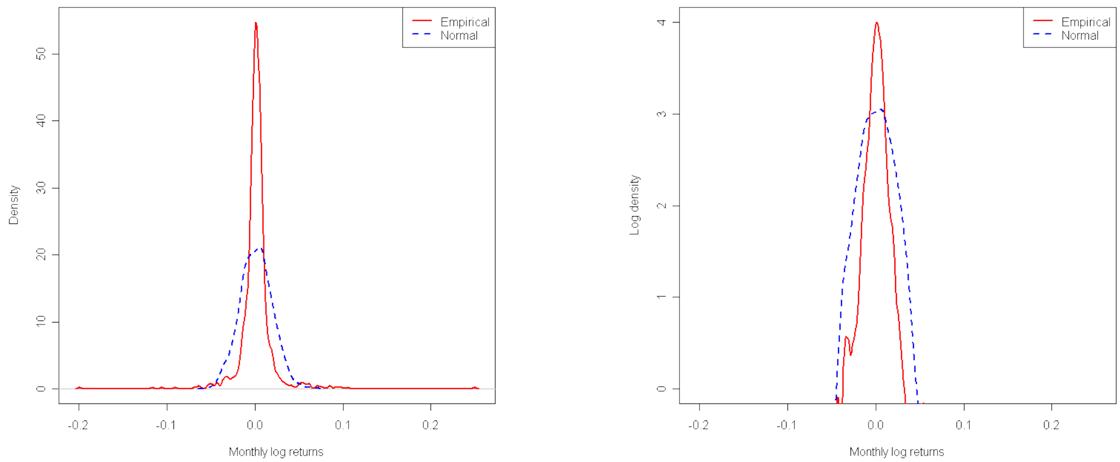
Figure B.22 Filtered weekly USA M1 (1981-2008) correlograms.



(a) Log returns (b) Squared returns

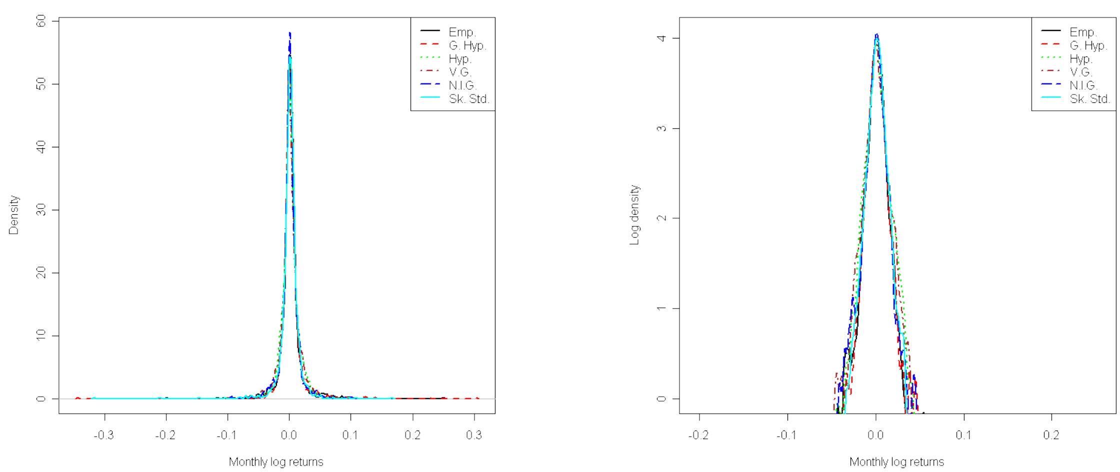
Figure B.23 Filtered weekly USA M1 (1981-2008) partial correlograms.

Figures B.22 and B.23 show that the GARCH filter successfully reduced the autocorrelation in our sample data. However, there is still a “slight” positive correlation in the squared returns. This might be what is sometime translated by a highly volatile filtered sample data.



(a) Density (b) Log Density

Figure B.24 Weekly USA M1 (1981-2008) probability plots: Empirical vs normal.

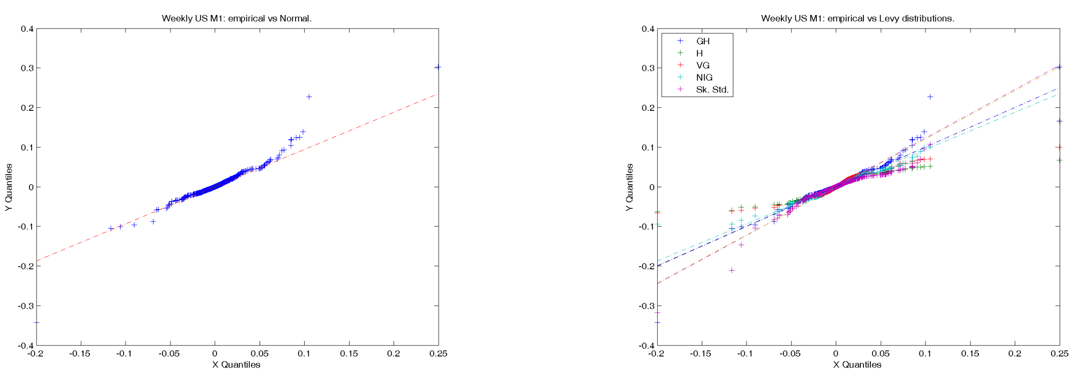


(a) Density (b) Log Density

Figure B.25 Weekly USA M1 (1981-2008) probability plots: Empirical vs Lévy.

Model	α	β	δ	μ	λ	**LLH**
NIG	18.71749	0.86203	0.00567	0.00121	-0.5	
H	105.69644	2.12448	$6.61 \cdot 10^{-6}$	0.00108	1	4222.08
GH	10.33677	0.79936	0.00682	0.00120	-0.69369	4340.19
VG	70.50968	4.08735	0	0.00041	0.55041	4261.61
Sk.Std.	0.02403	0.02403	N/A	0.00122	0.49993	4332.68

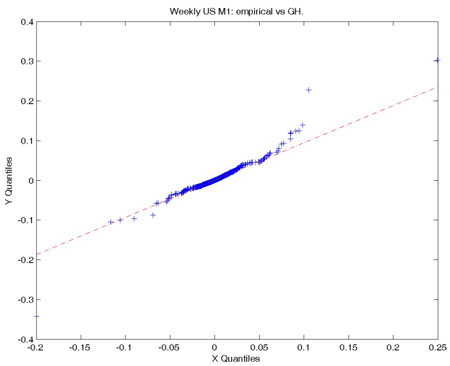
Table B.9 Estimated parameters for weekly USA M1 (1981-2008).



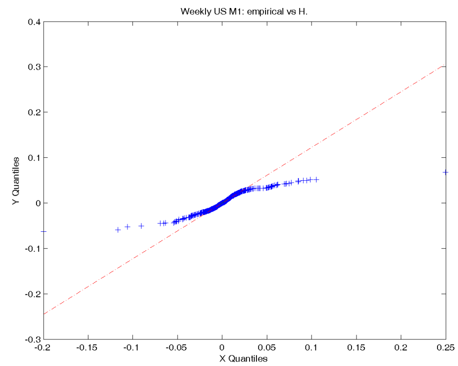
(a) Normal

(b) Lévy

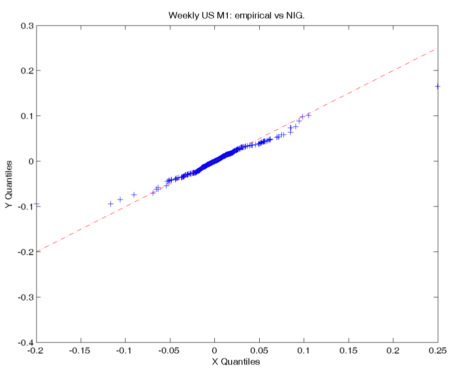
Figure B.26 Weekly US M1 QQ plots: Empirical vs normal.



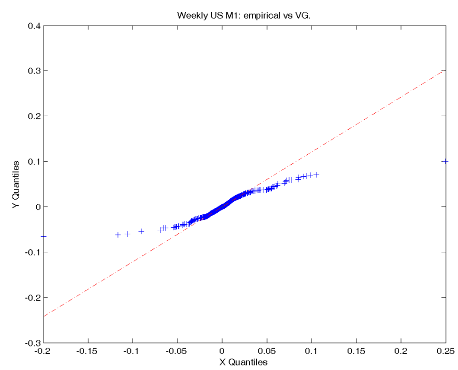
(a) GH



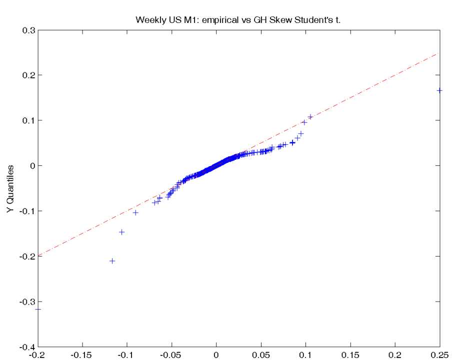
(b) H



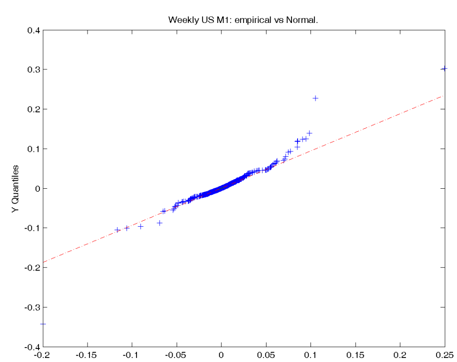
(c) NIG



(d) VG



(e) Skw. Std.

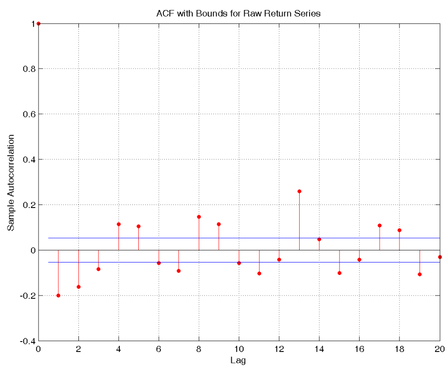


(f) Normal

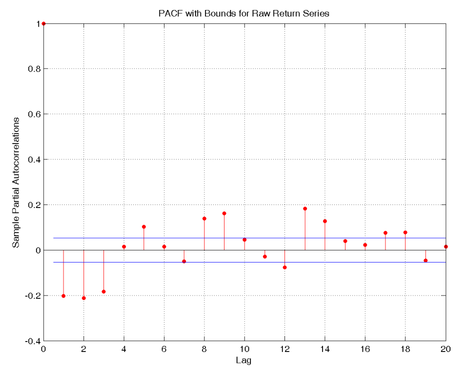
Figure B.27 Weekly US M1 QQ plots.

Money Supply aggregate M1: Seasonally adjusted

The weekly seasonally adjusted American Monetary aggregates M1 time series data is obtained from the Federal Reserve System [103]. The data is from the 5^{th} January 1981 to the 21^{st} April 2008 that is 1425 data points overall. That is more than enough data points for a "good" statistical study. The data is for the 1^{st} May 2008 and measured in billions of US dollars.

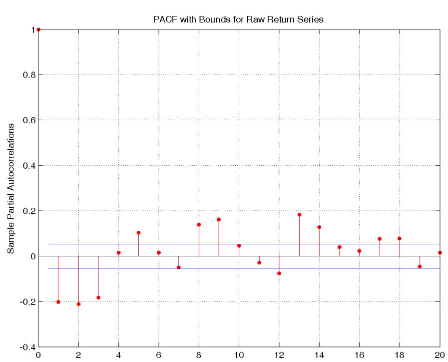


(a) Log returns

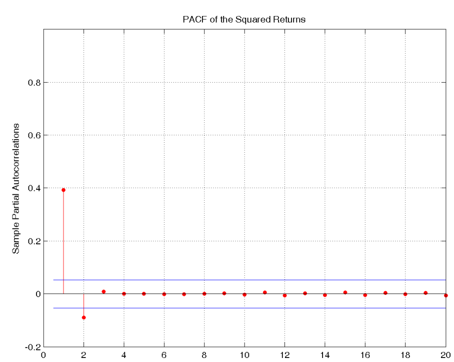


(b) Squared returns

Figure B.28 Seasonally adjusted weekly USA Money Supply M1 (1981-2008) correlograms.



(a) Log returns



(b) Squared returns

Figure B.29 Seasonally adjusted weekly USA Money Supply M1 (1981-2008) partial correlograms.

Figures B.31 and B.32 show that the GARCH filter was not that successful this time in reducing the sample autocorrelation.

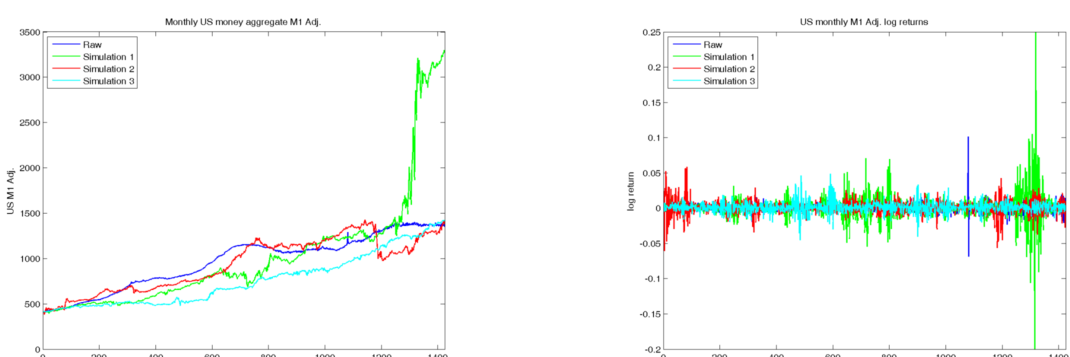
The normal distribution perform better with this sample (see Figure B.33) than with the previous

	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	210.6978	18.3070	1	0	242.8464	18.3070
15	1	0	343.8959	24.9958	1	0	242.8661	24.9958
20	1	0	392.7793	31.4104	1	0	242.9056	31.4104

Table B.10 Ljung-Box-Pierce Q-test for seasonally adjusted USA Weekly M2 (1981-2008) raw and squared returns.

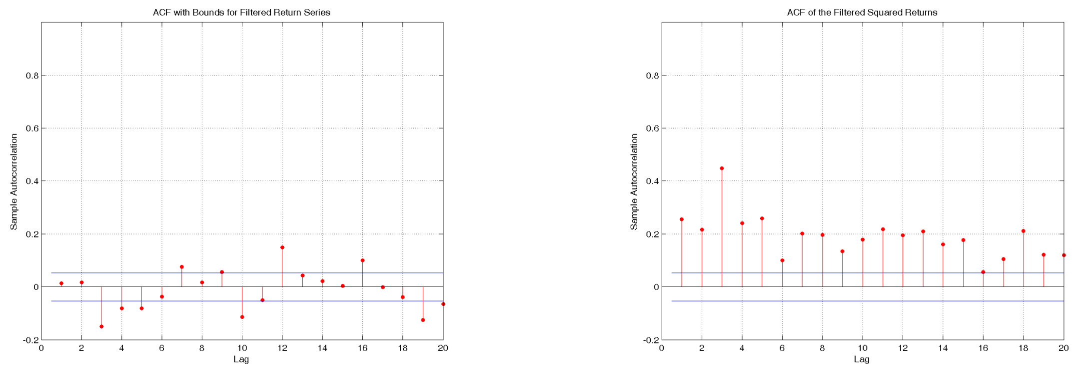
	Raw return				Squared Raw return			
Lag	**H**	**p**	**Stat**	**Crit**	**H**	**p**	**Stat.**	**Crit.**
10	1	0	243.5764	18.3070	1	0	67.6032	18.3070
15	1	0	242.7911	24.9958	1	0	67.3666	24.9958
20	1	0	242.0485	31.4104	1	0	67.1300	31.4104

Table B.11 Engle's ARCH test results for seasonally adjusted USA M1 (1981-2008) raw and squared returns.



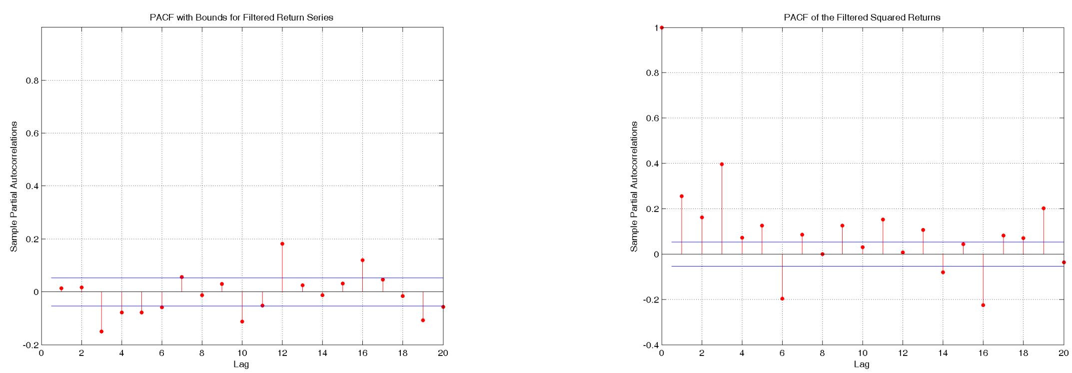
(a) M1 Adj. (b) M1 Adj. Log returns

Figure B.30 Filtered vs raw weekly USA M1 Adj. (1981-2008) data series.



(a) Log returns (b) Squared returns

Figure B.31 Filtered weekly USA M1 Adj. (1981-2008) correlograms.

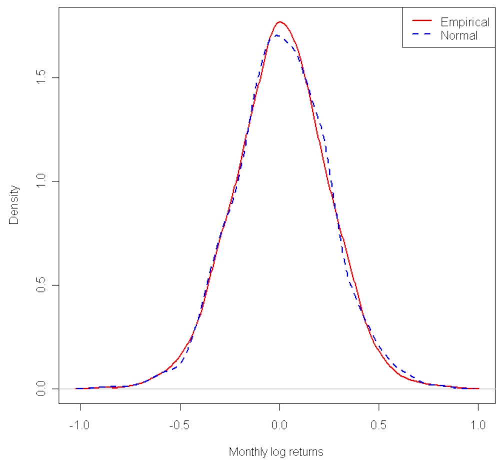


(a) Log returns (b) Squared returns

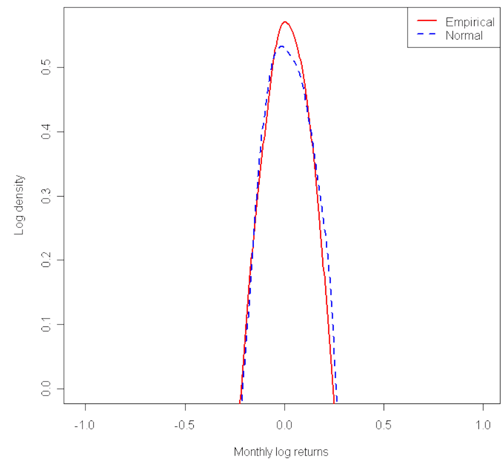
Figure B.32 Filtered weekly USA M1 Adj. (1981-2008) partial correlograms.

samples. For the first time, the fit under normality assumption is better than with one of the Lévy distribution (see Figure B.34) that is the GH distribution.

The QQ plot in Figure B.35(a) confirms the fact that the match under normality assumption is “good”.

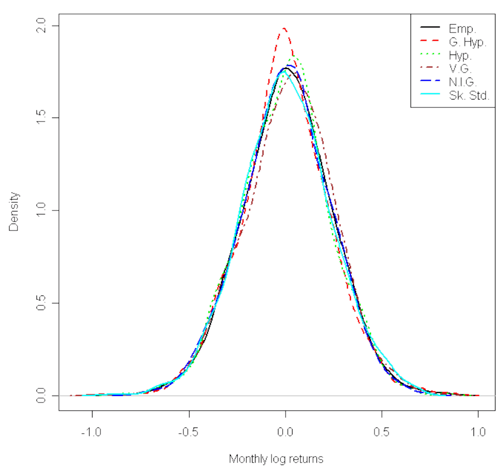


(a) Density

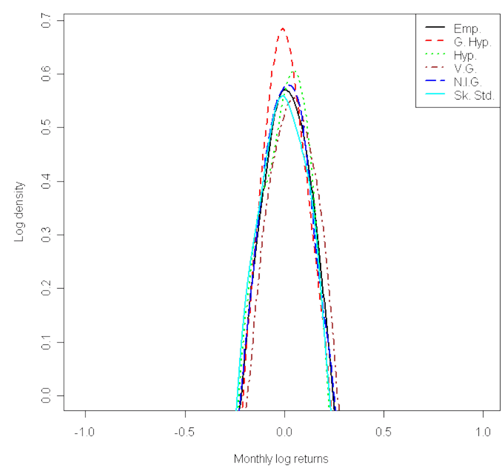


(b) Log Density

Figure B.33 Weekly USA Adj. M1 (1981-2008) probability plots: Empirical vs normal.



(a) Density

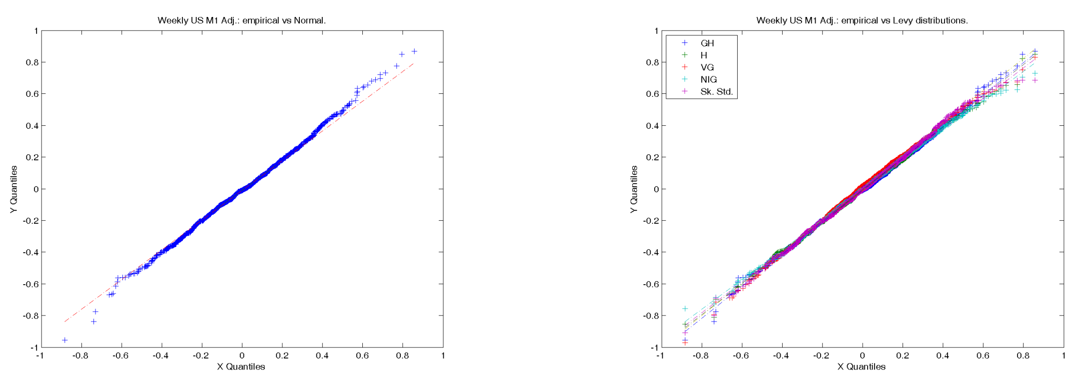


(b) Log Density

Figure B.34 Weekly USA Adj. M1 (1981-2008) probability plots: Empirical vs Lévy.

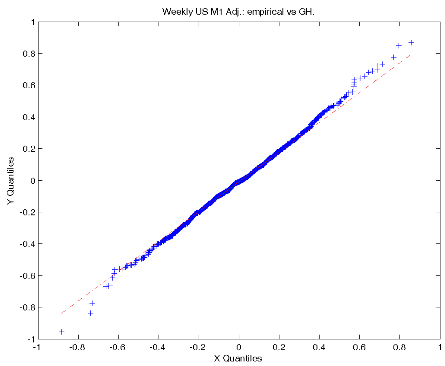
Model	α	β	δ	μ	λ	**LLH**
NIG	13.87633	−0.34898	0.75585	0.03012	N/A	52.65
H	14.76969	−0.35021	0.70075	0.03019	N/A	52.65
GH	19.80023	−0.36612	0.00371	0.03105	10.67644	52.66
VG	19.8000	−0.36612	N/A	0.03105	0.04067	52.66
Sk.Std.	0.32700	−0.32700	N/A	0.02892	0.04067	52.62

Table B.12 Estimated parameters for USA Adj. M1 (1981-2008).

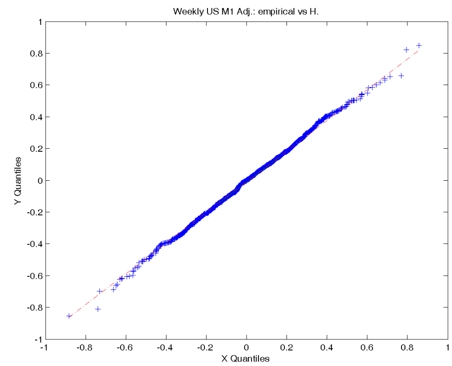


(a) Normal (b) Lévy

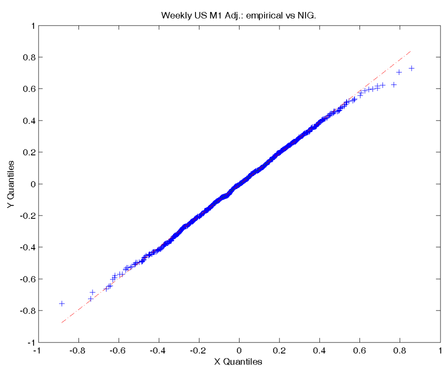
Figure B.35 Weekly US M1 Adj. QQ plots: Empirical vs normal.



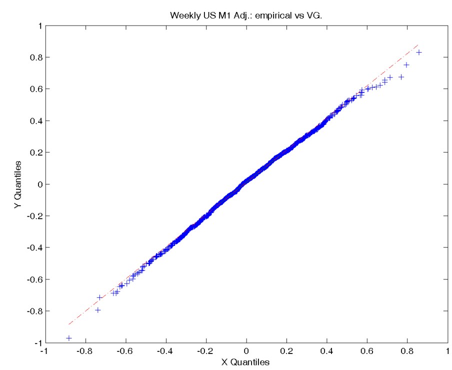
(a) GH



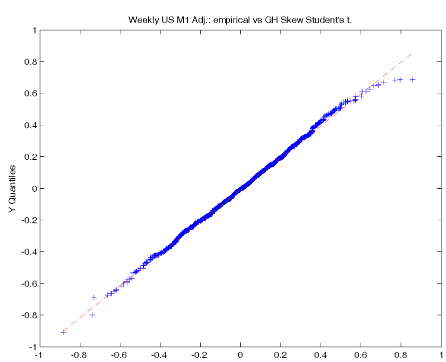
(b) H



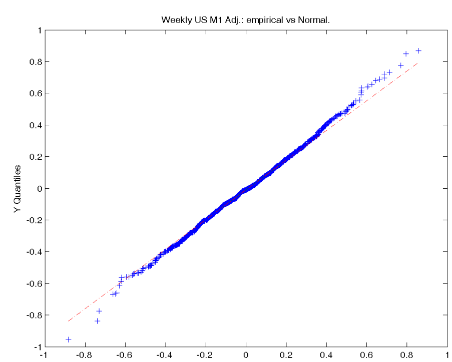
(c) NIG



(d) VG



(e) Skw. Std.

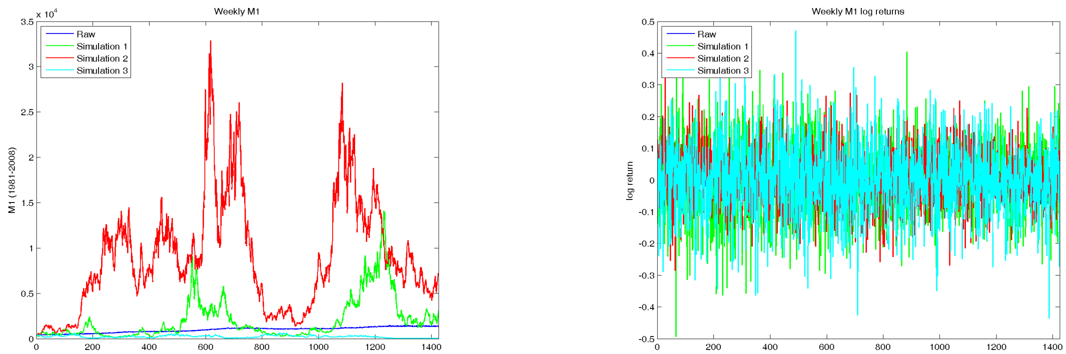


(f) Normal

Figure B.36 Weekly US M1 Adj. QQ plots.

B.2 Student t innovations

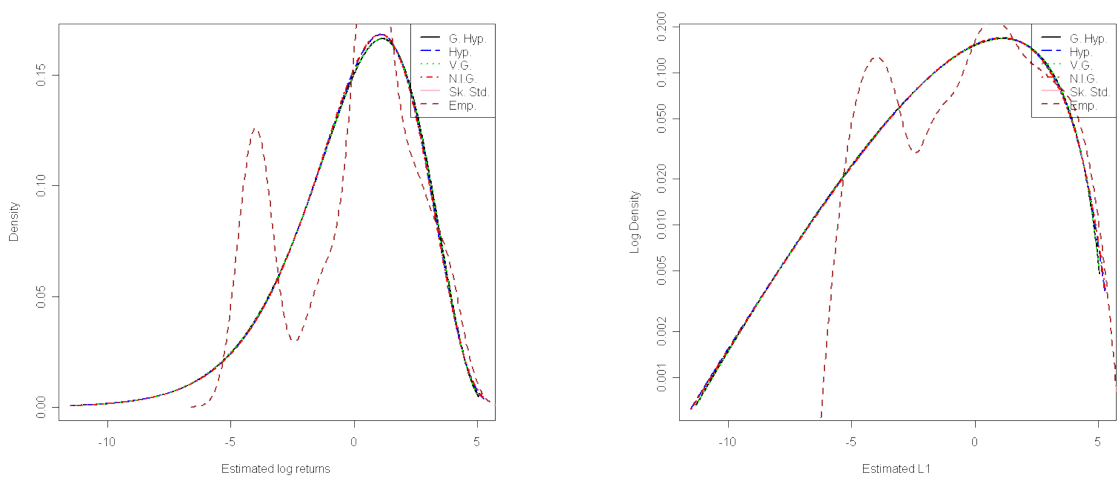
Money Supply aggregate M1



(a) M1

(b) M1 Log returns

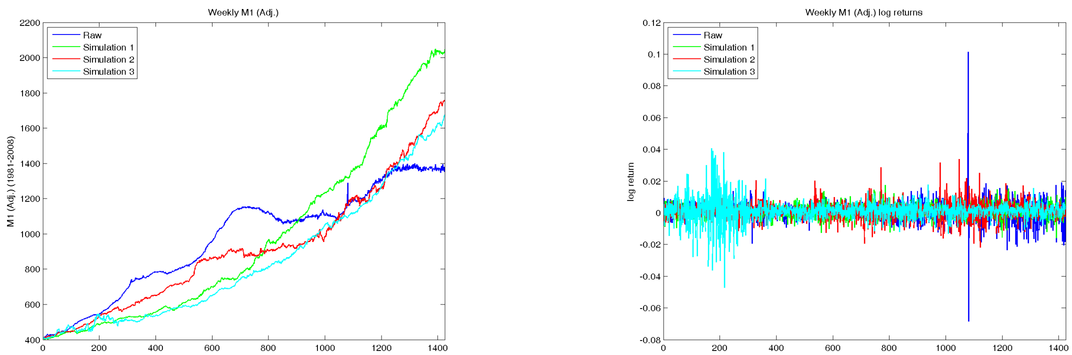
Figure B.37 Filtered vs raw weekly USA M1 (1981-2008) data series.



(a) Density

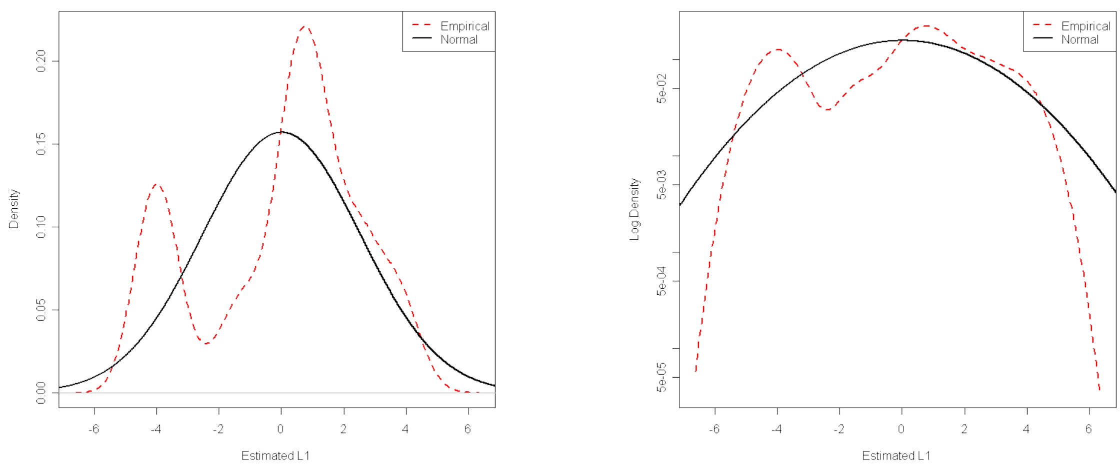
(b) Log Density

Figure B.38 Weekly USA M1 (1981-2008) probability plots: Empirical vs Lévy.



(a) M1 Adj. (b) M1 Adj. Log returns

Figure B.39 Filtered vs raw weekly USA M1 Adj. (1981-2008) data series.



(a) Density (b) Log Density

Figure B.40 Weekly USA Adj. M1 (1981-2008) probability plots: Empirical vs normal.

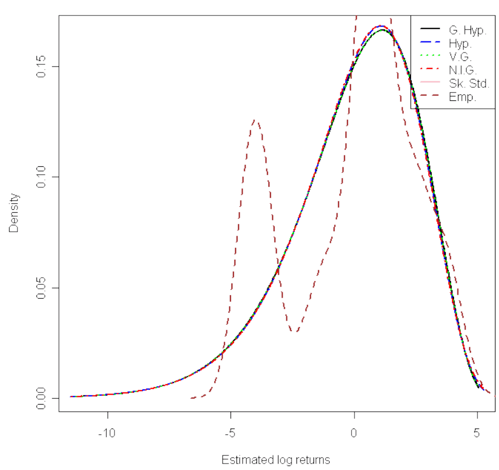
Money Supply aggregate M1: Seasonally adjusted

Money Supply aggregate M2

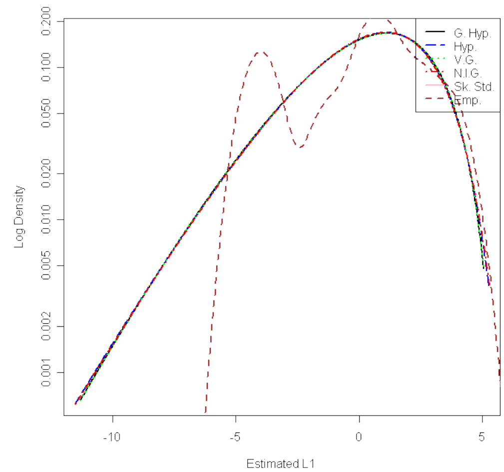
B.3 Yield Curves

Nominal Yield Curve

The nominal and real daily historical yield curves were downloaded from the U.S. treasury website [107] from January 2003 to September 2008. The sample data has 1430 data points, with only the

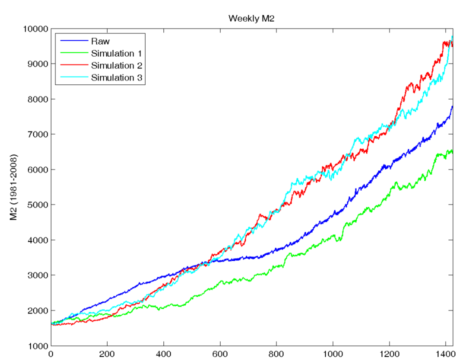


(a) Density

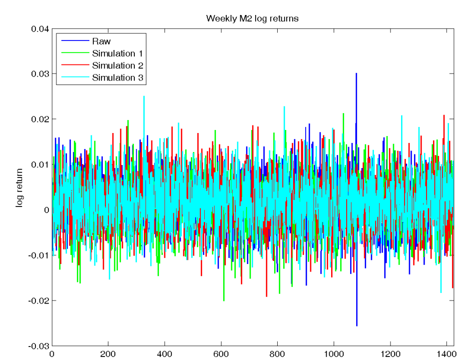


(b) Log Density

Figure B.41 Weekly USA Adj. M1 (1981-2008) probability plots: Empirical vs Lévy.



(a) M2

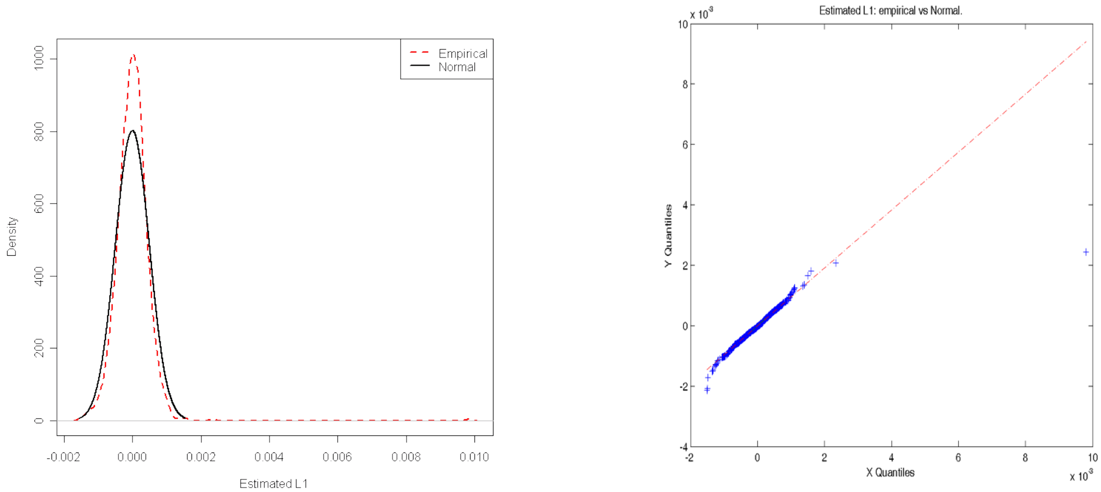


(b) M2 Log returns

Figure B.42 Filtered vs raw weekly USA M2 (1981-2008) data series.

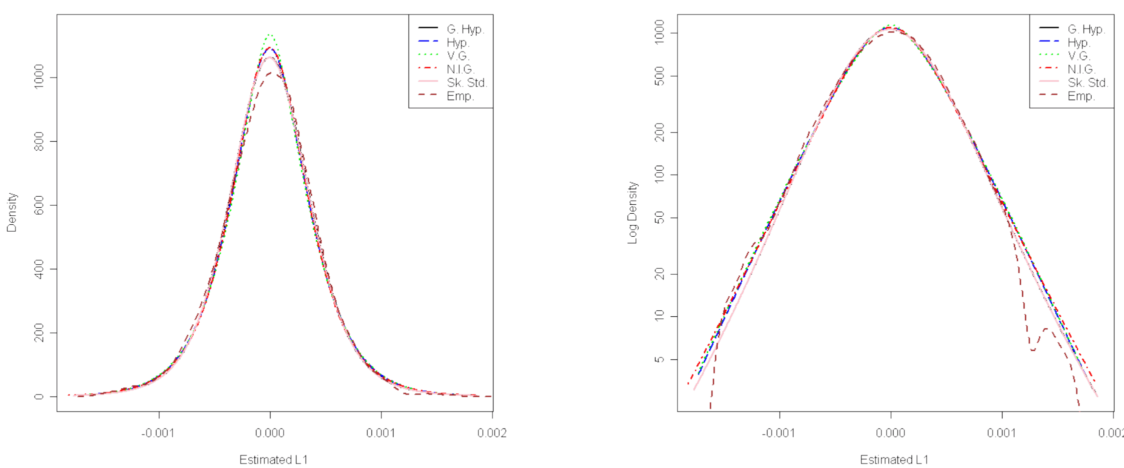
trading days considered.

AIC best fit Student-t.



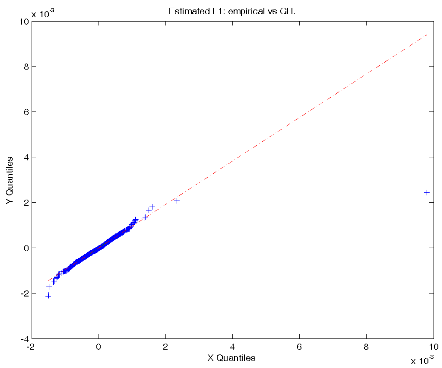
(a) Probability plot (b) QQ Plot

Figure B.43 Estimated L_1: Empirical vs normal.

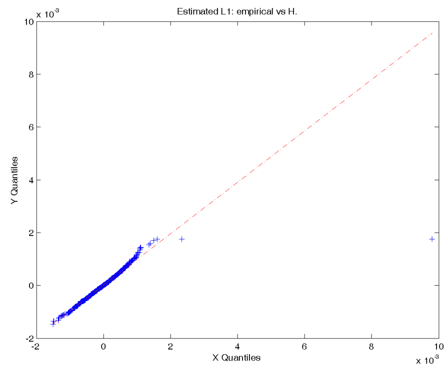


(a) Density (b) Log Density

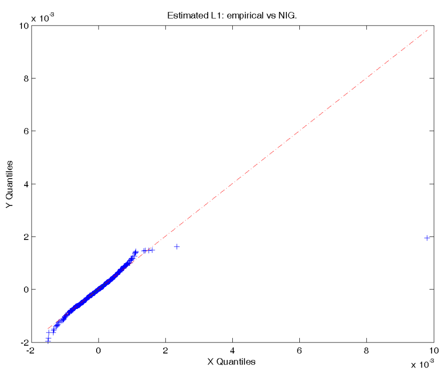
Figure B.44 Estimated L_1 probability plots: Empirical vs Lévy.



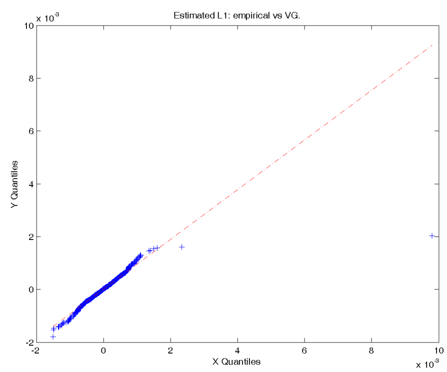
(a) GH



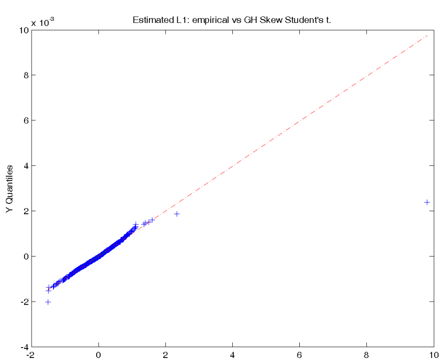
(b) H



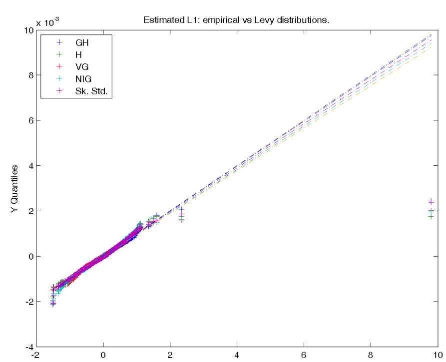
(c) NIG



(d) VG



(e) Skw. Std.



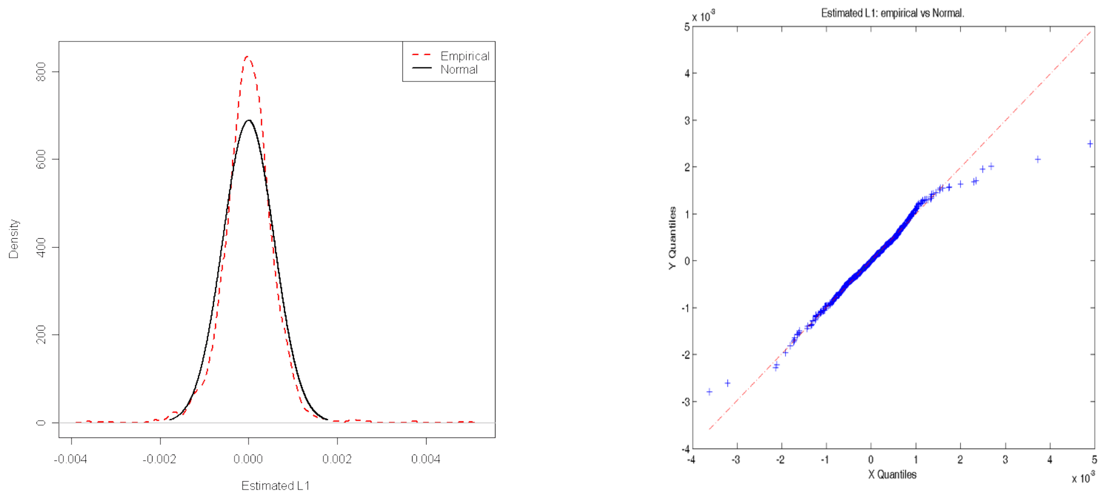
(f) Normal

Figure B.45 Estimated L_1 QQ plots.

Model	α	β	$\delta(\cdot 10^5)$	μ	λ	**LLH**
NIG	2728.92	24.64427	53.96559	$-4.83289 \cdot 10^{-6}$	-0.5	9056.900
H	3938.503	26.19406	34.55043	$-5.20755 \cdot 10^{-6}$	1	9050.542
GH	73.83758	70.64118	91.40627	$-1.40600 \cdot 10^{-5}$	-3.19922	9072.226
VG	4437.195	3.61602	0	$-6.79980 \cdot 10^{-7}$	35	9046.039
Sk.Std.	67.62844	67.62844	91.2076	$-1.36021 \cdot 10^{-5}$	-3.18674	9072.230

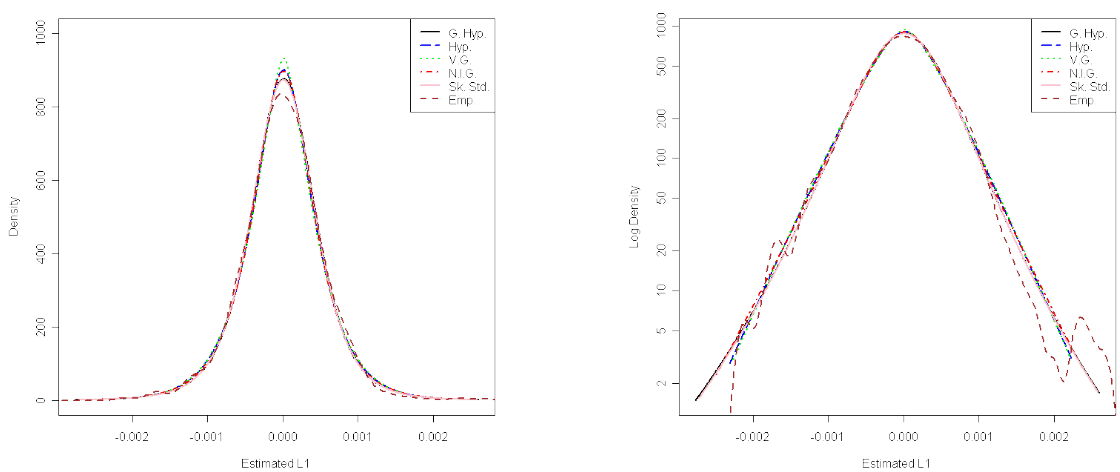
Table B.13 Estimated parameters for empirical L_1 for SA nominal forward rate.

Real Yield Curve



(a) Probability plot (b) QQ Plot

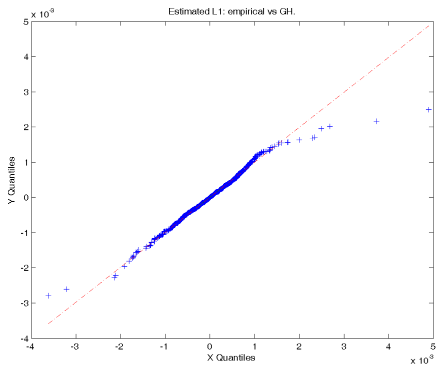
Figure B.46 Estimated L_1: Empirical vs normal.



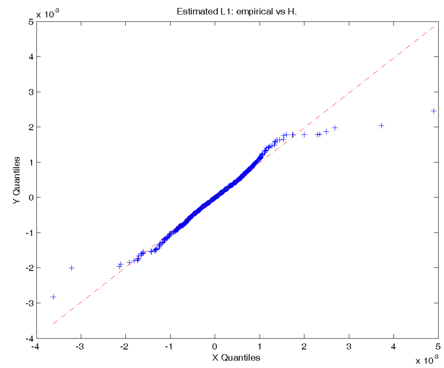
(a) Density (b) Log Density

Figure B.47 Estimated L_1 probability plots: Empirical vs Lévy.

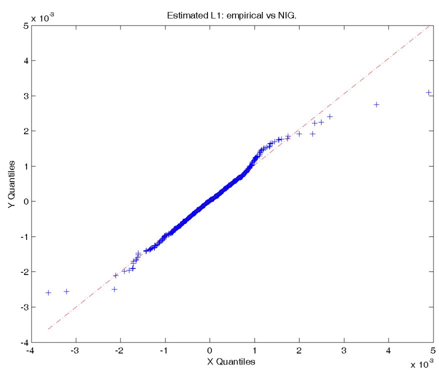
AIC best fit Student-t.



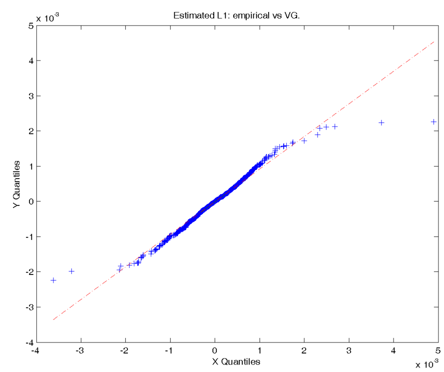
(a) GH



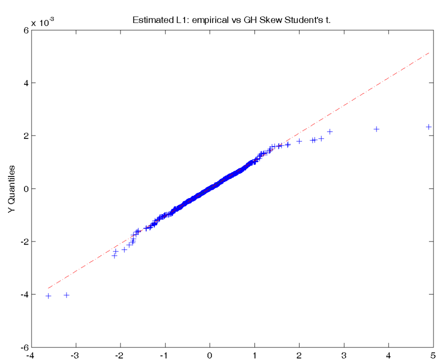
(b) H



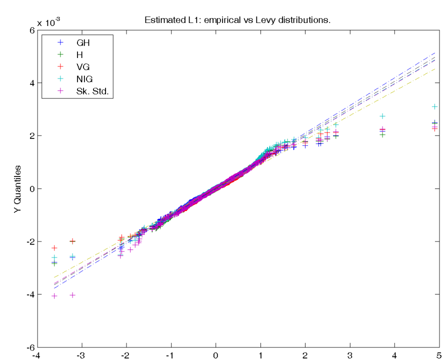
(c) NIG



(d) VG



(e) Skw. Std.



(f) Normal

Figure B.48 US real curve: Estimated L_1 QQ plots.

Model	α	β	$\delta(\cdot 10^5)$	$\mu(\cdot 10^5)$	λ	**LLH**
NIG	1890.162	-51.50919	60.56153	1.65484	-0.5	8725.085
H	2909.199	-54.34723	32.85109	1.68966	1	8720.764
GH	52.25032	-44.33243	93.34401	1.46063	-2.33771	8729.895
VG	3304.674	-45.19182	0	1.42259	1.72004	8717.872
Sk.Std.	39.62266	-39.62266	94.60124	1.28412	-2.38186	8729.890

Table B.14 Estimated parameters for empirical L_1 for US real forward rate.

Bibliography

[1] K. Aas and I. H. Haff. The Generalized Hyperbolic Skew Student's t-Distribution. *Journal of Financial Econometrics*, 4(2):275–309, 2006.

[2] N. Alexander. The elusive quest for fiscal space. Technical report, The Bretton Woods Project, April 2007.

[3] K. I. Amin and R. A. Jarrow. Pricing foreign currency options under stochastic interest rates. *Journal of International Money and Finance*, 10(3):310–329, September 1991.

[4] A. Ang, G. Bekaert, and M. Wei. The term structure of real rates and expected inflation. Working Paper 12930, National Bureau of Economic Research, February 2007.

[5] A. Ang, S. Dong, and M. Piazzesi. No-Arbitrage Taylor Rules. Nber working papers, National Bureau of Economic Research, Inc, Sept. 2007.

[6] A. Ang and M. Piazzesi. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics*, 50(4):745–787, May 2003.

[7] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Number 93 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[8] D. Backus, S. Foresi, K. Li, and L. Wu. Accounting for biases in Black-Scholes. Working paper series, August 2004.

[9] G. Bakshi, N. Kapadia, and D. Madan. Stock return characteristics, skew laws, and the differential pricing of individual equity options. *Review of Financial Studies*, 16(1), 2003.

[10] O. E. Barndorff-Nielsen. Exponentially Decreasing Distributions for the Logarithm of Particle Size. *Proceedings of the Royal Society of London*, 353(1674):401–419, Mar. 1977.

[11] O. E. Barndorff-Nielsen. Normal inverse Gaussian distributions and the modeling of stock returns. *Scandinavian Journal of Statistics*, 24(300), March 1997.

[12] T. Beletski. *Inflation-Linked Products and Optimal Investment with Macro Derivatives.* PhD thesis, Fachbereich Mathematik der Technischen Universität Kaiserslautern, 2006.

[13] T. Beletski and R. Korn. *Advances in Risk Management (Edited by Greg N. Gregoriou)*, chapter 9, pages 170–190. Finance and Capital Markets. Palgrave-MacMillan, 2006. Chapter title: Optimal Investment with Investment Inflation-Linked Products.

[14] N. Belgrade, E. Benhamou, and E. Koehler. A market model for inflation. Cahiers de la Maison des Sciences Economiques, Université Panthéon-Sorbonne (Paris 1), 2004.

[15] T. Björk. *Arbitrage Theory in Continuous Time.* Oxford University Press, Second edition, 2004.

[16] T. Björk, G. Di Masi, Y. Kabanov, and W. Runggaldier. Towards a general theory of bond markets. *Finance and Stochastics*, 1(2):141–174, 1997.

[17] T. Björk, Y. Kabanov, and W. Runggaldier. Bond market structure in the presence of marked point processes. *Mathematical Finance*, 7(2):211–239, 1997.

[18] F. Black. Interest rates as options. *Journal of Finance*, 50(5):1371–76, December 1995.

[19] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 1973.

[20] P. Blaesild. The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data. *Biometrika*, 68(1):251–263, April 1981.

[21] P. Blaesild. Generalized hyperbolic and generalized inverse Gaussian distributions. Technical report, University of Århus, 1999. Unpublished.

[22] T. Bollerslev. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3):307–327, April 1986.

[23] D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*. Number 133 in Finance. Springer, September 2006.

[24] P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods*. Series in Statistics. Springer, Second edition, September 1998.

[25] P. Carr and H. Geman. The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75(2):305–332, April 2002.

[26] P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2:61–73, March 1999.

[27] P. Carr and L. Wu. The finite moment log stable process and option pricing. *Journal of Finance*, 58(2):753–778, April 2003.

[28] P. Carr and L. Wu. Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71(1):113–141, January 2004.

[29] P. Carr and L. Wu. Stochastic skew in currency options. *Journal of Financial Economics*, 86(1):213–247, October 2007.

[30] R.-R. Chen, B. Liu, and X. Cheng. Inflation, Fisher Equation, and The Term Structure of Inflation Risk Premia: Theory and Evidence from TIPS. Working paper, Rutgers Business School- New Brunswick, September 2005.

[31] M. Chétouane and M. Marcussen. Can Lifecycle Plans Weather the Pension Storm? - A Micro - and Macroeconomic Perspective. Preliminary version, European Forecasting Research Association for the Macro-Economy (EUROFRAME), May 2007.

[32] Q. C. Chu and D. N. Pittman. *Treasury inflation-indexed Securities*. Springer, 2006.

[33] Q. C. Chu, D. N. Pittman, and J.-H. Chen. A Tale of two inflation regimes: evidence from maturing TIPS prices. In *Annual Meeting of the Financial Management Association International*, September 2007.

[34] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1, 2001.

[35] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Financial Mathematics Series. Chapman & Hall/CRC, 2003.

[36] B. R. Craig and J. G. Haubrich. Pricing kernels, inflation, and the term structure of interest rates. Working paper 03-08, September 2003.

[37] S. D'Amico, D. H. Kim, and M. Wei. Tips from TIPS: the informational content of Treasury Inflation-Protected Security prices. Technical report, 2008.

[38] M. Deacon, A. Derry, and D. Mirfendereski, editors. *Inflation-Indexed Securities Bonds, Swaps and Other Derivatives*. Wiley, Second edition, January 2004.

[39] A. P. Dempster, N. M. Laird, and D. Rubin. Maximum likelihood from incomplete data using the EM algorithm. *Journal of the Royal Statistical Society*, 39, 1977.

[40] M. Dodgson and D. Kainth. Inflation-Linked Derivatives. Risk training course, Royal Bank of Scotland Quantitative Research Centre, September 2006.

[41] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[42] E. Eberlein, U. Keller, and K. Prause. New Insights into Smile, Mispricing, and Value at Risk: The Hyperbolic Model. *Journal of Business*, 71(3):371–405, July 1998.

[43] E. Eberlein and W. Kluge. Calibration of Lévy term structure models. Fu, Michael C. (ed.) et al., Advances in mathematical finance. Papers presented at the 'Mathematical finance conference in honor of the 60th birthday of Dilip B. Madan', College Park, MD, USA, September 29 – October 1, 2006. Boston, MA: Birkhäuser. Applied and Numerical Harmonic Analysis, 147-172 (2007).

[44] E. Eberlein and W. Kluge. Exact Pricing Formulae for Caps and Swaptions in a Lévy Term Structure Model. *Journal of Computational Finance*, 9(2), 2006.

[45] E. Eberlein and N. Koval. A cross-currency Lévy market model. *Quantitative Finance*, 6(6):465–480, December 2006.

[46] E. Eberlein and F. Özkan. Time consistency of Lévy models. *Quantitative Finance*, 93(1):40–50, 2003.

[47] E. Eberlein and K. Prause. The generalized hyperbolic model: Financial derivatives and risk measures. In *Mathematical Finance Bachelier Congress 2000, Geman*, pages 245–267. Springer, 2002.

[48] E. Eberlein and S. Raible. Term structure models driven by general Lévy processes. *Mathematical Finance*, 9(1):31–53, 1999.

[49] J. P. Engel, H. M. Kat, and T. P. Kocken. Strategic Interest Rate Hedges Or How Derivatives Can Help Solve the Pension Fund Crisis Part II. Working Paper 0024, September 2005. Cass Business School Research Paper.

[50] R. F. Engle. Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, 50(4):987–1007, July 1982.

[51] E. F. Fama. The behavior of stock-market prices. *The Journal of Business*, 38(1):34–105, 1965.

[52] E. F. Fama. Short-term interest rates as predictors of inflation. *American Economic Review*, 65(3):269–82, June 1975.

[53] W. E. Ferson. Asset pricing models. *Encyclopedia of Finance*, pages 364–375, 2006.

[54] B. Flesaker and L. Hughston. Positive interest. *Risk Magazine*, 9:46–49, 1996.

[55] S. Foresi and L. Wu. Crash-O-Phobia: A Domestic Fear or a Worldwide Concern? *Journal of Derivatives*, Winter 2005.

[56] C. J. Fuhrer. Intrinsic and inherited inflation persistence. MPRA Paper, University Library of Munich, Germany, June 2006.

[57] R. S. Gürkaynak, B. Sack, and J. H. Wright. The TIPS yield curve and inflation compensation. Technical report, 2008.

[58] R. W. Hafer and S. E. Hein. Forecasting inflation using interest-rate and time-series models: Some international evidence. *Journal of Business*, 63(1):1–17, January 1990.

[59] J. D. Hamilton. *Time Series Analysis*. Princeton University Press, January 1994.

[60] B. E. Hansen. Autoregressive conditional density estimation. *International Economic Review*, 35(3):705–30, August 1994.

[61] L. P. Hansen and R. Jagannathan. Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99(2):225–62, April 1991.

[62] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–43, 1993.

[63] M. Hinnerich. Inflation-indexed swaps and swaptions. *Journal of Banking & Finance*, 32(11):2293–2306, November 2008.

[64] P. Honohan. The accidental tax: Inflation and the financial sector. *New York: Oxford University Press*. In Taxation of Financial Intermediation: Theory and Practice for Emerging Economies.

[65] G. Huberman and G. W. Schwert. Information aggregation, inflation, and the pricing of indexed bonds. *Journal of Political Economy*, 93(1):92–114, February 1985.

[66] L. Hughston and A. Rafailidis. A chaotic approach to interest rate modelling. *Finance and Stochastics*, 9(1):43–65, January 2005.

[67] L. P. Hughston. Inflation derivatives. *Department of Mathematics King's College London*, 1998.

[68] L. P. Hughston and A. Macrina. *Information, inflation, and interest.* 2008.

[69] J. C. Hull. *Options, Futures, & other Derivatives*, chapter 17, pages 372–391. Prentice Hall, Fifth edition, 2005.

[70] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Number 288. Springer, Second edition, December 2003.

[71] R. Jarrow and Y. Yildirim. Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM Model. *Journal of Financial and Quantitative Analysis*, 38(2), June 2003.

[72] J. Kerkhof. Inflation derivatives explained: Markets, products, and pricing. Fixed income-quantitative research, Lehman Brothers, July 2005.

[73] V. Konlack. Pricing Equity Swaps in an Economy Driven by Geometric Itô-Lévy Processes. December 2007.

[74] C. Laird. First Inflation then Deflation? - Financial Crash, February 2007. Available at: http://www.financialsense.com/fsu/editorials/laird/2007/0222.html.

[75] A. Lioui and P. Poncet. General equilibrium pricing of CPI derivatives. *Journal of Banking & Finance*, 29(5):1265–1294, May 2005.

[76] A. Macrina. *An Information-Based Framework for Asset Pricing: X-Factor Theory and its Applications*. PhD thesis, Department of Mathematics, King's College London, 2006.

[77] D. B. Madan and E. Seneta. Chebyshev polynomial approximations and characteristic function estimation. *Journal of the Royal Statistical Society*, 49(2):163–169, 1987.

[78] D. B. Madan and E. Seneta. The Variance Gamma (VG) Model for Share Market Returns. *Journal of Business*, 63(4):511–24, October 1990.

[79] L. Malvaez. Valuation of inflation-indexed derivatives with three factor model. Master's thesis, Keble College, University of Oxford, 2005.

[80] A. J. Mcneil, R. Frey, and P. Embrechts. *Quantitative risk management: Concepts, Techniques and Tools*, chapter 4. Princeton Series in Finance. Princeton University Press, Princeton, NJ, 2005.

[81] F. Mercurio. Pricing inflation-indexed derivatives. *Quant. Finance*, 5(3):289–302, 2005.

[82] F. Mercurio and N. Moreni. Pricing inflation-indexed options with stochastic volatility. August 2005.

[83] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1-2), 1976.

[84] F. S. Mishkin. A multi-country study of the information in the shorter maturity term structure about future inflation. *Journal of International Money and Finance*, 10(1):2–22, March 1991.

[85] H. Mo and L. Wu. International capital asset pricing: Evidence from options. *Journal of Empirical Finance*, 14(4):465–498, September 2007.

[86] N. Moreni. *Monte Carlo Methods and Option Pricing*. PhD thesis, University Paris 6 and University of Pavia, December 2005.

[87] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer, Second edition, January.

[88] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions (Universitext)*. Springer, Second edition, June 2007.

[89] F. Özkan. *Lévy processes in credit risk and market models*. PhD thesis, Freiburg im Breisgau: Univ. Freiburg, Mathematische Fakultät, 153 p., 2002.

[90] M. Piazzesi. Affine term structure models. March 2004. Contribution to the Handbook of Financial Econometrics (Edited by Yacine Ait-Sahalia and Lars Peter Hansen).

[91] K. Prause. *The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures*. PhD thesis, Freiburg i. Br.: Univ. Freiburg i. Br., Mathematische Fakultät, 160 p. , 1999.

[92] M. Predota. On European and Asian option pricing in the generalized hyperbolic model. *Euro. Jnl of Applied Mathematics*, 16(1):111–144, 2005.

[93] P. E. Protter. *Stochastic Integration and Differential Equations*. Number 21 in Stochastic Modelling and Applied Probability. Springer, Second edition, May 2005.

[94] S. Raible. *Lévy processes in finance: Theory, numerics, and empirical facts*. PhD thesis, Freiburg im Breisgau: Univ. Freiburg, Mathematische Fakultät, viii, 161 p., 2000.

[95] W. Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley Series in Probability and Statistics. Wiley, May 2003.

[96] E. Seneta. Fitting the variance-gamma model to financial data. *Journal of Applied Probability*, 41:177–187, 2004.

[97] South African Reserve Bank. Website: www.reservebank.co.za.

[98] Statistics South Africa. Website: www.statssa.gov.za.

[99] M. A. Stephens. EDF Statistics for Goodness of Fit and Some Comparisons. *Journal of the American Statistical Association*, 69(347):730–737, September 1974.

[100] S. Svoboda-Greenwood. *Volatility Specifications in the LIBOR Market Model*. PhD thesis, University of Oxford, July 2008.

[101] J. B. Taylor. Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series.*

[102] The Federal Reserve Board. The federal reserve system: Purposes and functions. Technical report, 2005. Available from http://www.federalreserve.gov/pf/pf.htm.

[103] The Federal Reserve Board, The Central Bank of United State. Website: www.federalreserve.gov.

[104] Université de Sherbrooke: World Perspective. Website: worldperspective.usherbrooke.ca.

[105] Unknown. Understanding inflation. Website: www.futurecasts.com. Article available at: http://www.futurecasts.com/Understanding%20Inflation.html.

[106] U.S. Department of Labor: Bureau of Labor Statistics. Website for CPI data: www.bls.gov/cpi.

[107] U.S. Treasury. Website: www.treas.gov.

[108] WAFO-group. *WAFO - A Matlab Toolbox for Analysis of Random Waves and Loads - A Tutorial*. Math. Stat., Center for Math. Sci., Lund Univ., Lund, Sweden, 2000. ISBN XXXX.

[109] L. Wu. Dampened power law: Reconciling the tail behavior of financial security returns. *Journal of Business*, 79(3):1445–1474, May 2006.

[110] L. Wu. Modeling Financial Security Returns Using Lévy Processes. Working paper series, City University of New York, CUNY Baruch College - Zicklin School of Business, Januarry 2006.

[111] A. Zhang, R. Korn, and C.-O. Ewald. Optimal management and inflation protection for defined contribution pension plans. *Blätter der DGVFM*, 28(2):239–258, 2007.