Self-adjoint Fourth Order Differential Operators With Eigenvalue Parameter Dependent Boundary Conditions

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of MSc in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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This ______________ day of February 2009, at Johannesburg, South Africa.

Abstract

The eigenvalue problem $y^{(4)}(\lambda, x) - (gy')'(\lambda, x) = \lambda^2 y(\lambda, x)$ with boundary conditions

$$y(\lambda, 0) = 0,$$
$$y''(\lambda, 0) = 0,$$
$$y(\lambda, a) = 0,$$
$$y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) = 0,$$

where $g \in C^1[0, a]$ is a real valued function and $\alpha > 0$, has an operator pencil $L(\lambda) = \lambda^2 - i\alpha\lambda K - A$ realization with self-adjoint operators A, M and K. It was shown that the spectrum for the above boundary eigenvalue problem is located in the upper-half plane and on the imaginary axis. This is due to the fact that A, M and K are self-adjoint. We consider the eigenvalue problem $y^{(4)}(\lambda, x) - (gy')'(\lambda, x) = \lambda^2 y(\lambda, x)$ with more general λ-dependent separated boundary conditions $B_j(\lambda)y = 0$ for $j = 1, \cdots, 4$ where $B_j(\lambda)y = y^{[p_j]}(a_j)$ or $B_j(\lambda)y = y^{[p_j]}(a_j) + i\epsilon_j\alpha\lambda y^{[q_j]}(a_j)$, $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$, $\alpha > 0$, $\epsilon_j = -1$ or $\epsilon_j = 1$. We assume that at least one of the $B_1(\lambda)y = 0$, $B_2(\lambda)y = 0$, $B_3(\lambda)y = 0$, $B_4(\lambda)y = 0$ is of the form $y^{[p]}(0) + i\epsilon\alpha\lambda y^{[q]}(0) = 0$ or $y^{[p]}(a) + i\epsilon\alpha\lambda y^{[q]}(a) = 0$ and we investigate classes of boundary conditions for which the corresponding operator A is self-adjoint.

To my late mother Delphine Ouensavi Zinsou and my wife Hermance Kokode Zinsou

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Contents

Chapter 1

Introduction

An important area of Mathematics is the study of differential equations. Differential equations were introduced in the seventeenth century to describe fundamental laws in Physics [4]. Differential equations are used today in Physics, but also in many other areas such as Engineering, Geophysics, Geography, Economics and others. Differential equations are used for optimal design of ships, aircrafts and space shuttles. They are used in climatology for studying climatic changes, to forecast population and economic trends. They also have applications in Hydrodynamics, in Electricity and Mechanics.

One of the tools for studying differential equations is linear operators. Linear operators are mostly studied in Hilbert or Banach spaces where notions such as resolvent sets, spectra, discrete spectra, continuous spectra, eigenvalues and associated eigenfunctions or eigenvector are discussed. The field of differential operators involves topics such as eigenvalues, eigenfunctions, boundary conditions, regularity, normality, minimality, maximality, boundedness, compactness, operator graphs, closable and closed operators, symmetric operators, adjoint operators, self-adjoint operators, operator extensions, operator pencils. We make use of many of these terms in this research. Relevant terms are defined in Chapter 2.

Extensive research on linear operators exists. For example, Kato [8] has discussed perturbation theory of linear operators and Weidmann [19] has studied linear operators in Hilbert spaces. Furthermore Naimark has discussed linear operators in [17] and [18]. Möller and Zettl have presented semi-boundedness ordinary differential operators in [14] and symmetric differential operators and their Friedrichs extension in [15]. Mennicken and Möller have presented their study on non self-adjoint boundary eigenvalue problems in [12]. Other studies on boundary eigenvalue problems have been conducted. We consider in this research an eigenvalue problem with general λ-dependent boundary conditions and investigate classes of boundary conditions which are self-adjoint. Following are summaries of some of the studies describing boundary eigenvalue problems with λ-dependent boundary conditions.

1. Binding, Browne and Watson [1] have considered the Sturm-Liouville equation

$$ly := -y'' + qy = \lambda y \quad \text{on } [0, 1], \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} y(0)\cos\alpha &= y'(0)\sin\alpha, \quad \alpha \text{ in } [0, \pi), &(1.2)\\ f(\lambda) &= \frac{y'}{y}(1) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k}, &(1.3)\end{aligned}$$

where all the coefficients are real and $a \geq 0$, $b_k > 0$ and $c_1 < c_2 < \cdots < c_N$, $N \geq 0$. They have investigated the existence of eigenvalues and the associated oscillation theory of the problem (1.1) - (1.3). They have studied a transformation, with certain eigenvalue-preserving properties, from a problem of the type (1.1) - (1.3) to a 'simpler' one with a new potential $\hat{q}$ in place of q and with $f(\lambda)$ replaced by $F(\lambda)$ where $F(\lambda) = \frac{\mu-\lambda}{f(\lambda)-f(\mu)} - f(\mu)$ and μ is a constant less than c_1. Some of the results they have obtained are:

- The eigenvalues of (1.1) - (1.3) are real, simple and form a sequence $\lambda_0 < \lambda_1 < \cdots$ accumulating only at ∞ and with $\lambda_0 < c_1$.
- If b is decreased and c_k is increased, then each positive $\lambda_j > c_k$ is increased.

- If $a > 0$ is decreased and b_k is increased, then each positive $\lambda_j > c_k$ is increased.

2. Binding, Browne and Watson have considered in [2], again, the Sturm-Liouville equation subject to the boundary conditions (1.1) - (1.3) defined in [1]. Assuming that $a > 0$, they have defined, in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^{N+1}$, the inner product

$$\langle Y, Z\rangle = \int_0^1 y\bar{z} + \sum_{k=1}^{N} \frac{y_k \bar{z}_k}{b_k} + y_{N+1}\bar{z}_{N+1}$$

and have posed the boundary value problem $(1.1)-(1.3)$ by considering the operator

$$LY = \begin{pmatrix} ly \\ c_1 y_1 - b_1 y(1) \\ \vdots \\ c_N y_n - b_N y(1) \\ y'(1) - by(1) - \sum\limits_{k=1}^{N} y_k \end{pmatrix} \quad \text{where} \quad Y = \begin{pmatrix} y \\ y_1 \\ \vdots \\ y_{N+1} \end{pmatrix}$$

and the domain of L is

$$\mathscr{D}(L) = \{Y \in H : y, y' \in AC, y, y' \in L_2(0,1), y(0)\cos\alpha = y'(0)\sin\alpha, y_{N+1} = ay(1)\},$$

where $ly = -y'' + qy$, and AC denotes the space of absolutely continuous functions. They have proved, with the above formulations, that:

- L is self-adjoint on H.
- The eigenvalues of $(1.1)-(1.3)$ coincide with those of L, are real and (algebraically) simple.
- L is bounded below and has compact resolvent on H.
- The eigenfunctions of $(1.1)-(1.3)$ augmented to eigenvectors of L, form an orthonormal basis of H.

3. Kir, Bascanbaz-Tunga and Yanik [9] have considered the operator $L(\lambda)$ generated in the Hilbert space

$$L_2(\mathbb{R}_+, \mathbb{C}^2) := \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \int_0^\infty (|f_1(x)|^2 + |f_2(x)|^2)dx < \infty \right\}$$

by the system

$$\begin{cases} iy_1' + q_1(x)y_2 &= \lambda y_1, \\ -iy_2' + q_2(x)y_1 &= \lambda y_2, \quad x \in \mathbb{R}_+ \end{cases}$$

and the spectral parameter dependent boundary condition

$$(a_1\lambda + b_1)y_2(0) - (a_2\lambda + b_2)y_1(0) = 0,$$

where q_i $i = 1,\ 2,$ are complex - valued functions, λ is a complex parameter, $a_i,\ b_i$ are complex constants, $b_i \neq 0,\ i = 1,\ 2,$ and $|a_1|^2 + |a_2|^2 \neq 0$. They have studied the spectrum of $L(\lambda)$ and have proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities under the conditions $|q_i(x)| \leq ce^{-\varepsilon x} < \infty$, $i = 1,\ 2,\ \varepsilon > 0,\ c > 0$.

4. Hinton [5] has considered the eigenvalue problem:

$$ly = \frac{1}{r}(-(py')' + qy) = \lambda y, \tag{1.4}$$

$$\cos\alpha y(a) + \sin\alpha (py')(a) = 0, \quad \alpha \in [0, \pi), \tag{1.5}$$

$$-[\beta_1 y(b) - \beta_2 (py')(b)] = \lambda[\beta_1 y(b) - \beta_2 (py')(b)], \tag{1.6}$$

where $r,\ p,$ and q are real continuous functions on the interval $[a, b]$ with r and p positive; and the numbers $\beta_1,\ \beta_2,\ \beta_1',$ and β_2' are real. Hinton has remarked that a self-adjoint operator A can be associated with the problem defined in (1.4) – (1.6) if the condition

$$\rho = \beta_1'\beta_2 - \beta_1\beta_2' > 0 \tag{1.7}$$

holds. Hinton has defined in the Hilbert space $H = \mathscr{L}^2(a, b) \oplus \mathbb{C}$ a inner product in H by

$$\langle (F_1, F_2), (G_1, G_2) \rangle = \int_a^b rF_1\bar{G}_1 + \frac{1}{\rho}F_2\bar{G}_2,$$

the operator A acting on H by

$$A(F_1, F_2) = (lF_1, -\beta_1 F_1(b) + \beta_2(p(F_1')(b))),$$

where

$$\mathscr{D}(A) = \left\{ \begin{matrix} (F_1, F_2) \in H : F_1,\ pF_1' \in AC,\ lF_1 \in \mathscr{L}^2(r; a, b), \\ \cos\alpha F_1(a) + \sin\alpha (pF_1')(a) = 0, \quad F_2 = \beta_1' F_1(b) - \beta_2'(pF_1')(b) \end{matrix} \right\},$$

AC is the space of absolutely continuous functions and $\mathscr{L}^2(r; a, b)$ is the complex Hilbert space of Lebesgue measurable functions f satisfying

$$\int_a^b r|f|^2 < \infty.$$

Hinton has remarked that F_1 satisfies (1.4) - (1.6) if and only if $F = (F_1, \beta_1' F_1(b) - \beta_2'(pF_1')(b))$, $F \in \mathscr{D}(A)$ and $AF = \lambda F$. Hinton has observed that A is a self-adjoint operator and bounded below.

5. Marletta, Shkalikov and Tretter [11] have defined a problem of the form:

$$N(y) = \lambda P(y), \quad y = y(x), \quad x \in [0, a],\ a > 0, \tag{1.8}$$

$$B_j^0(y) = \lambda B_j^a(y),\ j = 1, 2, \ldots, n, \tag{1.9}$$

where N and P are ordinary differential expressions of order n and p, respectively, with coefficients, $n > p$, and $B_j^0(y)$, $B_j^a(y)$ are linear forms containing the variables $y^{(k)}(0)$ and $y^{(k)}(a)$ with $k = 0, 1, \ldots, n-1$ in the Hilbert space $L_2(0, a) \times \mathbb{C}^r$, where $r \leq n$ is the number of λ-dependent boundary conditions after a suitable normalization. They have used eigenvalue problems of fourth order differential operators and pencils of fourth order differential operators with eigenvalue parameter in the boundary conditions associated to the above (1.8) - (1.9) problem to investigate the following questions:

- The existence of a function space W densely embedded in $L_2(0, 1)$ such that the system of eigenfunctions of problems associated to (1.8) $-$ (1.9) is complete and minimal simultaneously.
- The existence of a function space W and a linear operator T acting in W such that the eigenvalue and the eigenfunctions of T coincide with those of problems associated with (1.8) $-$ (1.9).

- The construction of pairs (W, T), where W and T are as defined above, (or *linearization pair*) for problems associated to (1.8) – (1.9).

6. Möller and Pivovarchik [13] have proved that the eigenvalues of the eigenvalue problem (1.10) with boundary conditions (1.11) – (1.14) lie in the closed upper half-plane and on the imaginary axis. They have derived a formula for the asymptotic distribution of the eigenvalues and have investigated the spectral properties of the problem mentioned above.

In their study, Möller and Pivovarchik [13] have considered the eigenvalue problem:

$$ly(\lambda, x) := y^{(4)}(\lambda, x) - (gy')'(\lambda, x) = \lambda^2 y(\lambda, x), \tag{1.10}$$

with boundary conditions:

$$\begin{aligned} y(\lambda, 0) &= 0, && (1.11)\\ y''(\lambda, 0) &= 0, && (1.12)\\ y(\lambda, a) &= 0, && (1.13)\\ y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) &= 0, && (1.14) \end{aligned}$$

where $g \in C^1[0, a]$ is a real valued function and $\alpha > 0$. They have stated that the boundary conditions (1.11) – (1.14) are self-adjoint for the differential expression (1.10) and they have derived a statement about the location of the spectrum of the eigenvalue problem (1.10) - (1.14). While Möller and Pivovarckik [13] have been investigating the spectrum of the eigenvalue problem (1.10) with boundary conditions (1.11) – (1.14), the eigenvalue problem (1.10) – (1.14) leads to the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\alpha\lambda K - A, \tag{1.15}$$

where A is the differential operator acting in $L_2(0, a) \oplus \mathbb{C}$ with domain

$$\mathscr{D}(A) = \left\{ Y = \begin{pmatrix} y(x) \\ y'(a) \end{pmatrix} : y \in W_4^2(0, a),\ y(0) = y''(0) = y(a) = 0 \right\},$$

$$\text{given by } AY = \begin{pmatrix} y^{(4)} - (gy')' \\ y''(a) \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

The spectrum of the operator pencil $L(\lambda)$ can be quite arbitrary. However it has been shown, in [13], that the spectrum for (1.10) $-$ (1.14) is located in the upper-half plane and on the imaginary axis. This is due to the fact that A, K and M are self-adjoint and $K \geq 0$.

We, therefore, consider in this study the boundary eigenvalue problem (1.10) with more general λ-dependent boundary conditions and decide to investigate existence of other classes of boundary conditions for this boundary eigenvalue problem for which the corresponding operator A in the corresponding operator pencil defined in (1.15) is self-adjoint. These classes will, therefore, be classes of conditions corresponding to self-adjoint main operator.

The organization of this document is as follows: we give a presentation of basic definitions and properties and also the notion of Sobolev spaces on intervals in Chapter 2. We present in Chapter 3 the proof of the self-adjointness of the differential operator A defined in the boundary eigenvalue problem (1.10) - (1.14). We give in Section 4.2 additional definitions and properties of differential operators, which are used in the remainder of the document. Also in Section 4.3 we give preliminaries that are needed for the study of conditions correponding to a a self-adjoint main operator independent of λ for the differential expression (1.10). Second order self-adjoint differential operators with boundary conditions independent of λ are widely studied and well known. However there exists only a few studies on fourth order differential operators, therefore we study conditions corresponding to a self-adjoint main operator independent of λ for the differential expression (1.10) in Section 4.4. We give characterizations of boundary conditions corresponding to a self-adjoint main operator in Section 4.5. In Section 5.2 we recall definitions and properties of closed symmetric operators. We start the discussion of boundary conditions depending on λ corresponding to a self-adjoint main operator in Section 5.3. We study symmetric operators for boundary conditions dependent of λ for the differential expression (1.10) in Subsection 5.3.1. We develop Theorem 5.23 which characterizes symmetric operators, while we discuss the adjoint operators for boundary conditions dependent on λ

for the differential expression (1.10) in Subsection 5.3.2. We give in this subsection the proof of Theorem 5.41. Theorem 5.41 is derived from Theorem 5.23, for boundary conditions dependent on λ corresponding to a self-adjoint main operator for the differential expression (1.10). We use Theorem 5.41 to characterize classes of self-adjoint boundary conditions dependent on λ for the differential expression (1.10). The document ends with Subsection 5.3.3 where we develop, from Theorem 5.41, Theorem 5.46 which presents a characterization of self-adjoint boundary conditions dependent on λ for the differential expression (1.10).

To obtain the main results of this study, we consider the differential operator associated with the boundary value problem with differential expression (1.10) and boundary conditions $B_j(\lambda)y = 0$ for $j = 1, \cdots, 4$ where $B_j(\lambda)y = y^{[p_j]}(a_j) + i\epsilon_j\alpha\lambda y^{[q_j]}(a_j)$ or $B_j(\lambda)y = y^{[p_j]}(a_j)$, $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$, $\alpha > 0$, $\epsilon_j = -1$ or $\epsilon_j = 1$. We assume that at least one of $B_1(\lambda)y = 0$, $B_2(\lambda)y = 0$, $B_3(\lambda)y = 0$, $B_4(\lambda)y = 0$ is of the form $y^{[p]}(0)+i\epsilon\alpha\lambda y^{[q]}(0) = 0$ or $y^{[p]}(a)+i\epsilon\alpha\lambda y^{[q]}(a) = 0$. The maximal differential operator A_{max} is defined on $L_2(0, a)\times\mathbb{C}^k$ by $A_{max} = \begin{pmatrix} M_{A_0} \\ A_1 \end{pmatrix}$, with domain

$$\mathscr{D}(A_{max}) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix}, y \in W_4^2(0, a) \right\},$$

where $1 \le k \le 4$ is the number of the $B_j(\lambda)y = 0$ depending on λ, $a > 0$, $g \in C^1[0, a]$ is a real value function, $M_{A_0}y = y^{[4]} = y^{(4)} - (gy')'$. There exist two $k \times 8$ matrices V_0 and V_1 such that $V_0Y_R = D_0y$ and $V_1Y_R = A_1y$, see Section 5.3 where Y_R is given in Definition 4.12. D_0y and A_1y are respectively the components independent of λ and the coefficients of λ of k λ-dependent boundary conditions. We, then, define the differential operator $T(U)$ by

$$\mathscr{D}(T(U)) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix}, y \in W_4^2(0, a) \text{ and } UY_R = 0 \right\}$$

and

$$T(U)Y = A_{max}Y \ (Y \in \mathscr{D}(T(U))),$$

where U is the $l \times 8$ matrix defined in (5.16), ($l = 4 - k$), which represents the boundary conditions independent of λ. We say that the boundary conditions $B_j(\lambda)y = 0$, $j = 1, \cdots, 4$, correspond to a self-adjoint main self-adjoint if the differential operator $T(U)$ is self-adjoint.

Since $T(U) \subset A_{max}$ then $A^*_{max} \subset T(U)^*$. If the differential operator $T(U)$ is self-adjoint, that is $T(U)^* = T(U)$, then $A^*_{max} \subset T(U) \subset A_{max}$. Hence a necessary condition for $T(U)$ to be self-adjoint is that A^*_{max} must be symmetric. It is shown that A^*_{max} is symmetric if and only if rank$(W) = 8 - 2k$, see Theorem 5.38, where $W = D + (V_0^* V_1 - V_1^* V_0)$ and D is the matrix defined in (4.21) for $n = 4$. We assume that rank$(W) = 8 - 2k$ where $1 \leq k \leq 4$ and we denote by X the space $(N(W))^\perp$. We consider the matrices W_X and U_X as respectively the restrictions of the matrices W and U to X. We prove that the differential operator $T(U)$ is self-adjoint if and only if $U_X W_X U_X^* = 0$.

From the above result we give explicit characterizations for boundary conditions corresponding to a self-adjoint main operator depending on λ. We denote by P_0 the set of p in $y^{[p]}(0) = 0$ for the λ-independent boundary conditions and by P_a the corresponding set for $y^{[p]}(a) = 0$. Then we prove that the differential operator $T(U)$ is self-adjoint if and only if $p + q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\epsilon_j\lambda y^{[q]}(a_j) = 0$ where $\epsilon_j = 1$ if ($q = 0$ and $a_j = 0$) or ($q = 2$ and $a_j = 0$) or ($q = 1$ and $a_j = a$) or ($q = 3$ and $a_j = a$), $\epsilon_j = -1$ otherwise, $\{0, 3\} \not\subset P_0$, $\{1, 2\} \not\subset P_0$, $\{0, 3\} \not\subset P_a$ and $\{1, 2\} \not\subset P_a$.

An interesting extension to this study will be to investigate the spectral properties of the differential expression (1.10). Möller and Pivovarchik [13] have shown that the spectrum for (1.10) $-$ (1.14) is located in the upper-half plane and on the imaginary axis due to the fact that A, K and M are self-adjoint and $K \geq 0$. Therefore, it will be interesting to investigate the location of the classes of boundary conditions for the differential operator (1.10) for which the corresponding operators A, K and M are self-adjoint and $K \geq 0$.

In the remainder of the document, we call boundary conditions self-adjoint, if the corresponding operator A is self-adjoint.

Chapter 2

Preliminaries

2.1 Introduction

In Section 2.2 we present basic definitions and properties of linear operators. Section 2.3 deals with Sobolev spaces on intervals. We provide definitions and properties of test functions, followed by definitions and properties of distributions in Subsection 2.3.1, while we give definitions and properties of Sobolev spaces in Subsection 2.3.2. These definitions and properties are necessary for the comprehension of the content of this document. They are extensively used to obtain the research results that we present. Other definitions and properties are presented in subsequent chapters, where they are more relevant.

2.2 Definitions

The following definition is taken from [17, page 3].

Definition 2.1. A linear differential expression is an expression of the form:

$$l(y) := p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y$$

where $\frac{1}{p_0(x)},\ p_1(x),\ p_2(x),\ \ldots\ ,p_n(x)$ are continuous functions on a fixed, finite, interval $[a,b]$ and $y \in C^n[a,b]$.

The following definition is taken from [17, page 3].

Definition 2.2. If linear combinations:

$$B_j(y) = \alpha_0^j y(a)+\alpha_1^j y'(a)+\cdots+\alpha_{n-1}^j y^{(n-1)}(a)+\beta_0^j y(b)+\beta_1^j y'(b)+\cdots+\beta_{n-1}^j y^{(n-1)}(b),\ j=1,\ldots,m$$

of the values of the function y and its first $n-1$ successive derivatives at the boundary points a and b of the interval $[a,b]$ have been specified and the conditions $B_j(y) = 0,\ j = 1,\ldots,m$, are imposed on the functions $y \in C^n[a,b]$, these conditions which the functions y must satisfy are called boundary conditions.

The following proposition is taken from [18, page 18].

Proposition 2.3. *If the coefficients $p_k(x),\ k = 0,\ldots,n$ of the differential expression $l(y)$ have continuous derivatives up to the order $(n-k)$ inclusive on the interval $[a,b]$, then there exists a differential expression $l^*(z)$, where $z \in C^{(n)}[a,b]$ such that*

$$\int_a^b l(y)\bar{z}dx = \int_a^b y\overline{l^*(z)}dx + [y,z]_a^b, \tag{2.1}$$

where

1. $$[y,z] = \sum_{k=1}^{n}(y^{[k-1]}\bar{z}^{[2n-k]} - y^{[2n-k]}\bar{z}^{[k-1]}) \tag{2.2}$$

   is Lagrange's form,

2.

$$[y, z]_a^b \tag{2.3}$$

is the difference in the values for the function $[y, z]$*, defined in* (2.2)*, for* $x = b$ *and* $x = a$*,*

3. (2.1) *is said to be Lagrange's identity in integral form.*

The following definition is taken from [17, page 7].

Definition 2.4. The differential expression $l^*(z)$, defined in (2.1) is called the adjoint of the differential expression $l(y)$.

The following definition is taken from [18, page 50].

Definition 2.5. *If* $l = l^*$*, then* $l(y)$ *is said to be self-adjoint.*

The following proposition is taken from [17, page 8].

Proposition 2.6. *Any self-adjoint differential expression with real coefficients is necessarily of even order and has the form*

$$l(y) = (p_0 y^{(\mu)})^{(\mu)} + (p_1 y^{(\mu-1)})^{(\mu-1)} + \cdots + (p_{\mu-1} y')' + p_\mu y$$

where $p_0,\ p_1, \ldots, p_\mu$ *are real-valued functions.*

The following remark is taken from [17, page 9].

Remark 2.7. Let $B_1, \ldots, B_m$ be linearly independent forms in the variables $y_a, y_a', \ldots, y_a^{(n-1)}$, $y_b, y_b', \ldots, y_b^{(n-1)}$, where $y_b^{(j)} = y^{(j)}(b)$ and $y_a^{(j)} = y^{(j)}(a)$; if $m < 2n$, we shall supplement these forms with other forms $B_{m+1}, \ldots, B_{2n}$ to obtain a linearly independent system of $2n$ forms $B_1, B_2, \ldots, B_{2n}$. Since $B_1, B_2, \ldots, B_{2n}$ are linearly independent, the variables $y_a, y_a', \ldots, y_a^{(n-1)}$, $y_b, y_b', \ldots, y_b^{(n-1)}$ can be expressed as linear combinations of the forms $B_1, B_2, \ldots, B_{2n}$.

If we substitute these expressions in the expression (2.3) which occurred in (2.1), then (2.3) becomes a linear, homogeneous form in the variable $B_1, B_2, \ldots, B_{2n}$, and its $2n$ coefficients are themselves linear, homogeneous forms, which we denote by $V_{2n}, V_{2-1}, \ldots, V_1$, in the variables $z_a, z'_a, \ldots, z_a^{(n-1)}, z_b, z'_b, \ldots, z_b^{(n-1)}$. Then (2.1) takes the form

$$\int_a^b l(y)\bar{z}dx = \int_a^b y\overline{l^*(z)}dx + B_1V_{2n} + B_2V_{2n-1} + \cdots + B_{2n}V_1. \tag{2.4}$$

The forms $V_1, V_2, \ldots, V_{2n}$ are linearly independent.

The following definition is taken from [17, page 10].

Definition 2.8. The boundary conditions

$$V_1 = 0, V_2 = 0, \ldots, V_{2n-m} = 0 \tag{2.5}$$

(and all boundary conditions equivalent to them) are said to be the *adjoint* to the original boundary conditions

$$B_1 = 0, B_2 = 0, \ldots, B_m = 0 \tag{2.6}$$

The following definition is taken from [17, page 10].

Definition 2.9. Let L be the operator generated by the expression $l(y)$ and the boundary conditions (2.6). Then the operator generated by $l^*(z)$ and the boundary conditions (2.5) will be denoted by L^* and called the adjoint operator to L.

The following remark is taken from [17, page 10].

Remark 2.10. It follows from (2.4) and the boundary conditions (2.5) and (2.6) that the equation

$$\int_a^b (Ly)\bar{z}dx = \int_a^b y(\overline{L^*z})dx \tag{2.7}$$

holds for the operators L and L^*, for all y in the domain of definition of L and for all z in the domain of definition of L^*.

With the notation

$$(y, z) = \int_a^b y(x)\overline{z(x)}dx,$$

(2.7) becomes

$$(Ly, z) = (y, L^*z). \tag{2.8}$$

The following definition is taken from [17, page 13].

Definition 2.11. A complex number λ is called an *eigenvalue* of an operator $L = L(\lambda)$, generated by a differential expression $l(y) = \lambda y$ and the boundary conditions $B_1(y) = 0, \dots, B_n(y) = 0$, if there exists in the domain of definition $\mathscr{D}(L)$ of the operator L a function y not identically zero such that $Ly = \lambda y$. The function y is called an *eigenfunction*, of the operator L, for the eigenvalue λ .

The following definition is taken from [18, page 17].

Definition 2.12. Let L be an operator defined in a Hilbert space $\mathscr{H}$.

1. A number λ is said to be in the resolvent set of the operator L if the inverse $R_\lambda = (L - \lambda I)^{-1}$ exists and represents a bounded operator defined in the whole space $\mathscr{H}$. The operator R_λ is then called the resolvent of the operator L.
2. All points λ not in the resolvent set, are called points of the spectrum of the operator L. The eigenvalues λ of an operator belong to its spectrum.

The following definition is taken from [10, pages 4, 5].

Definition 2.13. Let $\mathscr{L}(\mathcal{H}, H)$ be the space of linear bounded operators from $\mathcal{H}$ into H, where $\mathcal{H}$ and H are Hilbert spaces and $\mathcal{H}$ is dense in H. An operator *pencil* is a polynomial $\mathcal{A}(\lambda) = \sum\limits_{q=0}^{l} A_{l-q}\lambda^q$ in $\lambda \in \mathbb{C}$ where $A_q \in \mathscr{L}(\mathcal{H}, H)$. If the equation $\mathcal{A}(\lambda_0)y = 0$ has nontrivial solutions, then λ_0 is called an *eigenvalue* of the operator pencil $\mathcal{A}$, and the corresponding nontrivial solutions are called *eigenvectors* related to λ_0.

The following remark is taken from Section 1 of [11, page 894].

Remark 2.14. A problem of the form:

$$N(y) = \lambda P(y), \quad y = y(x), \quad x \in [0,a],\ a > 0, \tag{2.9}$$

$$B_j^0(y) = \lambda B_j^a(y),\ j = 1, 2, \ldots, n, \tag{2.10}$$

where N and P are ordinary differential expressions of order n and p, respectively, with $n > p$, and $B_j^0(y)$, $B_j^a(y)$ are linear forms containing the variables $y^{(k)}(0)$ and $y^{(k)}(a)$ with $k = 0, 1, \ldots, n-1$ can be viewed as a linear pencil in the Hilbert space $L_2(0,a) \times \mathbb{C}^r$, where $r \leq n$ is the number of λ-dependent boundary conditions after a suitable normalization.

The following definition is Definition 2.7.4 of [7, page 157].

Definition 2.15. If the domain $\mathscr{D}(L)$ of a linear operator L is *dense* in a Hilbert space $\mathscr{H}$, then L is said to be densely defined. If L is densely defined and its range is contained in a Hilbert space $\mathscr{K}$, the mapping L^*, the adjoint of L has as domain $D(L^*) = \{z \in \mathscr{K} : \exists\ w \in \mathscr{H}\ \ \langle y, w\rangle = \langle Ly, z\rangle\ \ \forall y \in \mathscr{D}(L)\}$.

The following definitions are taken from [18, page 13].

Definition 2.16. Let $L : \mathscr{H} \to \mathscr{H}$ be a linear operator, where $\mathscr{H}$ is a Hilbert space. Then

- L is said to be Hermitian if for all $y,\ z \in \mathscr{D}(L)$ $\langle Ly, z\rangle = \langle y, Lz\rangle$ holds.
- A Hermitian operator which is densely defined in a Hilbert space $\mathscr{H}$ is called symmetric operator.
- A symmetric operator L in a Hilbert space $\mathscr{H}$ is said to be self-adjoint if $L = L^*$.

The following theorem is Theorem 5.32. of [19, page 115].

Theorem 2.17. *If L is a symmetric operator on the Hilbert space $\mathscr{H}$ such that $\langle y, Ly\rangle \geq \gamma\|y\|^2$ with $\gamma \in \mathbb{R}$ (respectively $\|Ly\| \geq \gamma\|y\|$) for all $y \in \mathscr{D}(L)$, then for each $k \in (-\infty, \gamma)$ (respectively $k \in (-\gamma, \gamma)$), there exists a self-adjoint extension T_k of L such that $\langle y, T_k y\rangle \geq k\|y\|^2$ (respectively $\|T_k y\| \geq k\|y\|$) for all $y \in \mathscr{D}(T_k)$.*

2.3 Sobolev Spaces on Intervals

We assume in this section that a and b are real numbers with $a < b$. Let $1 \leq p \leq \infty$.

2.3.1 Test Functions and Distributions

The following definition is taken from [12, pages 53, 54].

Definition 2.18. Let $I \subset \mathbb{R}$ be an interval.

- $C(I) = C^0(I)$ denotes the space of all continuous functions on I to $\mathbb{C}$. For a positive integer k, $C^k(I)$ denotes the space of k-times continuously differentiable functions on I.
- For $f \in C(I)$ the set $\operatorname{supp} f := \overline{\{x \in I : f(x) \neq 0\}}$ is called the support of f, where the closure is taken with respect to I.
- Let I be open and let $C^\infty(I) := \bigcap\limits_{k=1}^{\infty} C^k(I)$. A function $f \in C^\infty(I)$ is called a test function if its support is a compact subset of I. The space of all test functions on an open interval I is denoted by $C_0^\infty(I)$.

Remark 2.19. Mennicken and Möller [12, page 54] identify $C_0^\infty(I)$ with a subset of $C_0^\infty(\mathbb{R})$ by setting $f = 0$ outside of I for each $f \in C_0^\infty(I)$. And they denote $C_0^\infty(I) = \bigcup\limits_{K \subset I,\ \text{compact}} C_0^\infty(K)$, where $C_0^\infty(K) := \{f \in C_0^\infty(\mathbb{R}) : \operatorname{supp} f \subset K\}$.

The following definition is taken from [12, page 54].

Definition 2.20. Let I be an open interval. A linear functional u on $C_0^\infty(I)$ is called a distribution on I if for each compact set $K \subset I$ there are numbers $k \in \mathbb{N}$ and $C \geq 0$ such that

$$|\langle \phi, u \rangle| \leq C \sum_{j=0,\ x \in K}^{k} \sup |\phi^{(j)}(x)| \qquad (\phi \in C_0^\infty(K)),$$

where $\langle \phi, u \rangle := u(\phi)$. The space of distribution is denoted by $\mathscr{D}'(I)$.

The following definition is taken from [12, page 55].

Definition 2.21. For $u \in \mathscr{D}'(I)$, where I is an open interval, the support of u, denoted $\operatorname{supp} u$, is the set of points $x \in I$ such that for each neighborhood $U \subset I$ of x there exists $\phi \in C_0^\infty(U)$ such that $\langle \phi, u \rangle \neq 0$.

The following definition is taken from [12, page 55].

Definition 2.22. Let I be an open interval and let $u \in \mathscr{D}'(I)$. Then

$$\langle \phi, u' \rangle = -\langle \phi', u \rangle \quad (\phi \in C_0^\infty(I))$$

defines a distribution u' on I, called the derivative in the sense of distributions of u. Recursively for $k = 1, 2, \cdots$

$$u^{(k+1)} := {u^{(k)}}'.$$

The following theorem is Theorem 3.1.4. of [6, page 57].

Theorem 2.23. *If $u \in \mathscr{D}'(X)$ where X is an open interval on $\mathbb{R}$ and if $u' = 0$, then u is a constant.*

2.3.2 Definitions and Properties of Sobolev spaces

The following definition is Definition 2.1.1. of [12, page 55].

Definition 2.24. Let $I \subset \mathbb{R}$ be an open interval, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. The space

$$W_k^p(I) := \{f \in L_p(I) : \ \forall j \in \{1, \dots, k\} \ f^{(j)} \in L_p(I)\}$$

is called a Sobolev space. Here the derivatives $f^{(j)}$ are the derivatives in sense of distributions. For $f \in W_k^p(I)$ we set

$$|f|_{p,k} := \sum_{j=0}^{k} |f^{(j)}|_p.$$

Note that $W_0^p(I) = L_p(I)$ and that $L_2(I)$ is a Hilbert space with respect to the inner product

$$(f, g) = \int_I f(x)\bar{g}(x)dx, \ \ f, g \in L_2(I).$$

The following remark is Remark 2.1.2. of [12, page 55].

Remark 2.25. Let I be an open interval. Let $AC^{loc}(I)$ be the set of functions f on I such that $f|_K$ is absolutely continuous for each compact subinterval K of I. Then for $k > 0$, $W_k^p(I) = \{f \in AC^{loc}(I) : \ \forall j \in \{1, \dots, k-1\} \ f^{(j)} \in AC^{loc}(I) \cap L_p(I), \ f^{(k)} \in L_p(I)\}$.

The following proposition is Proposition 2.1.3. of [12, page 56].

Proposition 2.26. *Let $I \subset \mathbb{R}$ be an open interval, $\gamma \in \bar{I}$ and $g \in L_p(I)$.Set*

$$G(x) := \int_\gamma^x g(t)dt \quad (x \in \bar{I}).$$

Then G is continuous on $\bar{I}$ and $G' = g$ in $\mathscr{D}'(I)$.

The following corollary is Corollary 2.1.4. of [12, page 56].

Corollary 2.27. *Let $k \in \mathbb{N}$ and $u \in \mathscr{D}'(a, b)$ such that $u' \in W_k^p(a, b)$. Then $u \in W_{k+1}^p(a, b)$.*

The following proposition is Proposition 2.1.5. of [12, page 56].

Proposition 2.28. *Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N} \backslash \{0\}$.*

1. *Let $f \in L_p(I)$ and $\gamma \in \bar{I}$. Then $f \in W_k^p(I)$ if and only if there are $g \in W_{k-1}^p(I)$ and $c \in \mathbb{C}$ such that*
$$f(x) = c + \int_\gamma^x g(t)dt \quad (x \in I).$$
In this case, $g = f'$, f has continuous extension to $\bar{I}$, which we also denote by f, and $c = f(\gamma)$.

2. *$W_k^p(I) \subset C^{k-1}(\bar{I})$.*

The following proposition is Proposition 2.1.6. of [12, page 57].

Proposition 2.29. *Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}$. Then $W_k^p(I)$ is a Banach space with respect to the norm $| \quad |_{p,k}$.*

The following proposition is Proposition 2.1.7. of [12, page 57].

Proposition 2.30. *For each $k \in \mathbb{N}$ and $1 \leq p \leq q \leq \infty$ we have*

1. *$W_k^q(a,b) \subset W_k^p(a,b)$,*

2. *$W_{k+1}^p(0,a) \subset C^k[a,b]$,*

3. *$C^k[a,b] \subset W_k^p(a,b)$,*

where the inclusions holds topologically, i.e., the corresponding inclusion maps are continuous.

Chapter 3

The self-adjointness of the differential operator A

3.1 Introduction

We present in this chapter the proof of the self-adjointness of the differential operator A, defined by Möller and Pivovarchick [13]. Definitions and properties that we present in Chapter 2 are extensively used in Section 3.2 where we prove that the operator A is self-adjoint. We prove, first that A is densely defined, next that A is a symmetric operator. We, next, characterize the domain of the adjoint A^* of A. Finally we use the characterization of the domain of A^* and the symmetry of A to prove that A is self-adjoint.

3.2 The operator A

Let A be the operator acting in $L_2(0,a)\oplus\mathbb{C}$ with domain

$$\mathscr{D}(A)=\left\{Y=\begin{pmatrix} y \\ y'(a)\end{pmatrix} : y\in W_4^2(0,a), y(0)=y''(0)=y(a)=0\right\},$$

given by $AY=\begin{pmatrix} y^{(4)}-(gy')' \\ y''(a)\end{pmatrix}$, where $a>0$ and $g\in C^1[0,a]$ is a real valued function.

Proposition 3.1. *A is densely defined.*

Proof. Let $W=\begin{pmatrix} w \\ c\end{pmatrix}\in L_2(0,a)\oplus\mathbb{C}$ be such that $\langle Y,W\rangle=0$ for all $Y\in\mathscr{D}(A)$, where

$$\langle Y,W\rangle=\int_0^a y(x)\bar{w}(x)\,dx+y'(a)\bar{c}.$$

Let $y\in C_0^\infty(0,a)$. Then $y'(a)=0$, and $Y=\begin{pmatrix} y \\ y'(a)\end{pmatrix}\in\mathscr{D}(A)$, so

$$\int_0^a y(x)\bar{w}(x)dx=0 \text{ for all } y\in C_0^\infty(0,a).$$

Thus $w=0$. Let

$$y(x)=x^3(x-a).$$

Then

$$y(0)=y(a)=y''(0)=0.$$

Hence

$$Y=\begin{pmatrix} y \\ y'(a)\end{pmatrix}\in\mathscr{D}(A).$$

Since

$$y'(x)=4x^3-3x^2a,$$

it follows that

$$y'(a) = a^3 \neq 0.$$

Since $w = 0$, then

$$\langle Y, W \rangle = y'(a)\bar{c} = 0;$$

but

$$y'(a) = a^3 \neq 0,$$

thus

$$c = 0,$$

so

$$W = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence

$$\mathscr{D}(A)^{\perp} = \{0\}.$$

Therefore A is densely defined. □

Proposition 3.2. *A is a symmetric operator.*

Proof. Let $Y, Z \in \mathscr{D}(A)$,

$$\begin{aligned} \langle AY, Z \rangle &= \int_0^a [y^{(4)}(x) - (gy')'(x)]\bar{z}(x)dx + y''(a)\bar{z}'(a) \\ &= \int_0^a y^{(4)}(x)\bar{z}(x)dx - \int_0^a (g(x)y'(x))'\bar{z}(x)dx + y''(a)\bar{z}'(a). \end{aligned}$$

Recall that

$$(y^{(4)}, z) = \int_0^a y^{(4)}(x)\bar{z}(x)dx$$

and

$$((gy')', z) = \int_0^a (g(x)y'(x))'\bar{z}(x)dx.$$

Then,

$$\begin{aligned}(y^{(4)}, z) &= [y^{(3)}(x)\bar{z}(x)]_0^a - \int_0^a y^{(3)}(x)\bar{z}'(x)dx \\ &= -[y''(x)\bar{z}'(x)]_0^a + \int_0^a y''(x)\bar{z}''(x)dx \quad (\text{since} \;\; z(a) = z(0) = 0).\end{aligned}$$

But the scalar product

$$\int_0^a y''(x)\bar{z}''(x)dx = (y^{(4)}, z) + [y''(x)\bar{z}'(x)]_0^a$$

is symmetric, so

$$\int_0^a y''(x)\bar{z}''(x)dx = (y, z^{(4)}) + [y'(x)\bar{z}''(x)]_0^a.$$

Thus

$$(y^{(4)}, z) = (y, z^{(4)}) - [y''(x)\bar{z}'(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a.$$

Similarly

$$\begin{aligned}((gy')', z) &= [g(x)y'(x)\bar{z}(x)]_0^a - \int_0^a y'(x)g(x)\bar{z}'(x)dx \\ &= -\int_0^a y'(x)g(x)\bar{z}'(x)dx \quad (\text{since } z(a) = z(0) = 0).\end{aligned}$$

The scalar product

$$\int_0^a y'(x)g(x)\bar{z}'(x)dx = -((gy')', z)$$

is symmetric, so

$$\int_0^a y'(x)g(x)\bar{z}'(x)dx = -(y, (gz')').$$

Hence

$$
\begin{aligned}
\langle AY, Z\rangle &= (y^{(4)}, z) - ((gy')', z) + y''(a)\bar{z}'(a) \\
&= (y, z^{(4)}) - [y''(x)\bar{z}'(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a - (y, (gz')') + y''(a)\bar{z}'(a) \\
&= (y, z^{(4)}) - (y, (gz')') - [y''(x)\bar{z}'(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a + y''(a)\bar{z}'(a) \\
&= (y, z^{(4)} - (gz')') - [y''(x)\bar{z}'(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a + y''(a)\bar{z}'(a) \\
&= (y, z^{(4)} - (gz')') - y''(a)\bar{z}'(a) + y''(0)\bar{z}'(0) + y'(a)\bar{z}''(a) - y'(0)\bar{z}''(0) \\
&+ y''(a)\bar{z}'(a) \\
&= (y, z^{(4)} - (gz')') - y''(a)\bar{z}'(a) + y'(a)\bar{z}''(a) + y''(a)\bar{z}'(a) \\
&\quad \text{(since } y''(0) = z''(0) = 0) \\
&= (y, (z^{(4)} - (gz')') + y'(a)\bar{z}''(a) \\
&= \int_0^a y(x)[\bar{z}^{(4)}(x) - (g(x)\bar{z}'(x))']dx + y'(a)\bar{z}''(a) \\
&= \langle Y, AZ\rangle.
\end{aligned}
$$

Since A is densely defined and $\langle AY, Z\rangle = \langle Y, AZ\rangle$ for all $Y, Z \in \mathscr{D}(A)$, then according to Definition 2.16 A is symmetric. □

Remark 3.3. Let $g \in C^1[0, a]$ be a real valued function. Since the multiplication by g is a continuous linear operator $g\cdot$ from $C^1[0, a]$ into itself, its adjoint $(g\cdot)^*$ from $(C_0^1(0, a))'$ into itself is well-defined. Let $(\cdot, \cdot)_{C_0^1(0,a)}$ be the sesquilinear form on $C_0^1(0, a) \times (C_0^1(0, a))'$. Note that for $f \in L_2(0, a)$ and $\phi \in C_0^1(0, a)$,

$$
\begin{aligned}
(g\phi, f)_{C_0^1(0,a)} &= (g\phi, f) \\
&= (\phi, gf),
\end{aligned}
$$

so that $(g\cdot)^* f = gf$. Hence we write

$$gu = (g\cdot)^* u \tag{3.1}$$

for all $u \in (C_0^1(0, a))'$.

Also note that we have the continuous embeddings with dense ranges

$$C_0^\infty(0,a) \hookrightarrow C_0^1(0,a) \hookrightarrow L_2(0,a),$$

whence

$$L_2(0,a) \hookrightarrow (C_0^1(0,a))' \hookrightarrow \mathscr{D}'(0,a).$$

In particular, $gu \in \mathscr{D}'(0,a)$ for all $u \in (C_0^1(0,a))'$.

Lemma 3.4. *If $z \in L_2(0,a)$ and $g \in C^1[0,a]$, then $gz' \in \mathscr{D}'(0,a)$ and $gz' = (gz)' - g'z$.*

Proof. Since $z \in L_2(0,a) \hookrightarrow (C_0^1(0,a))' \hookrightarrow \mathscr{D}'0,a)$ and $g \in C^1[0,a]$, then according to Remark 3.3

$$gz \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a). \tag{3.2}$$

Since $g \in C^1[0,a]$, then $g' \in C[0,a]$. And as $z \in L_2(0,a)$, then

$$g'z \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a). \tag{3.3}$$

But according to Definition 2.22 (3.2) implies

$$(gz)' \in \mathscr{D}'(0,a). \tag{3.4}$$

Thus (3.3) and (3.4) give

$$(gz)' - g'z \in \mathscr{D}'(0,a). \tag{3.5}$$

According to (3.5), for all $\phi \in C_0^\infty(0,a)$ we have

$$\begin{aligned}
(\phi, (gz)' - g'z)_{C_0^\infty(0,a)} &= (\phi, (gz)')_{C_0^\infty(0,a)} - (\phi, g'z)_{C_0^\infty(0,a)} \\
&= -(\phi', gz)_{C_0^\infty(0,a)} - (g'\phi, z)_{C_0(0,a)} \\
&= -(g\phi', z)_{C_0^1(0,a)} - (g'\phi, z)_{C_0(0,a)} \\
&= -(g\phi' + g'\phi, z)_{C_0(0,a)} \\
&= -((g\phi)', z)_{C_0(0,a)} \\
&= (g\phi, z')_{C_0^1(0,a)} \\
&= (\phi, gz')_{C_0^\infty(0,a)}.
\end{aligned}$$

Since for all $\phi \in C_0^\infty(0,a)$ $(\phi, (gz)' - g'z)_{C_0^\infty(0,a)} = (\phi, gz')_{C_0^\infty(0,a)}$, then

$$g'z = (gz)' - gz'. \tag{3.6}$$

Hence (3.5) and (3.6) give $gz' \in \mathscr{D}'(0,a)$. □

Proposition 3.5. *If* $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*)$, *then* $z \in W_4^2(0,a)$ *and there exists* $c \in \mathbb{C}$ *such that* $A^*Z = \begin{pmatrix} z^{(4)} - (gz')' \\ c \end{pmatrix}$.

Proof. Since $Z \in \mathscr{D}(A^*)$, then according to Definition 2.15, there exists

$$W = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}$$

such that

$$\langle AY, Z \rangle = \langle Y, W \rangle,$$

for all $Y \in \mathscr{D}(A)$.

Let $Y = \begin{pmatrix} y \\ 0 \end{pmatrix} \in C_0^\infty(0,a) \oplus \{0\}$. Since $y \in C_0^\infty(0,a)$, thus $y''(a) = 0$. Then $Y \in \mathscr{D}(A)$ and

$$\begin{aligned} \langle AY, Z \rangle &= (y^{(4)} - (gy')', z) + y''(a)\bar{d} \\ &= (y^{(4)} - (gy')', z) \\ &= (y^{(4)}, z)_{C_0^\infty(0,a)} - ((gy')', z)_{L_2(0,a)}. \end{aligned} \tag{3.7}$$

Since $z \in L_2(0,a)$, then $z \in \mathscr{D}'(0,a)$. Therefore

$$(y^{(4)}, z)_{C_0^\infty(0,a)} = (-1)^4 (y, z^{(4)})_{C_0^\infty(0,a)}.$$

So

$$(y^{(4)}, z)_{C_0^\infty(0,a)} = (y, z^{(4)})_{C_0^\infty(0,a)}.$$

Since $z \in L_2(0,a)$, and $g \in C^1[0,a]$, then according to Lemma 3.4 $gz' \in \mathscr{D}'(0,a)$.

$$\begin{aligned}((gy')', z) &= ((gy')', z)_{C_0(0,a)} \\ &= -(gy', z')_{C_0^1(0,a)} \\ &= -(y', gz')_{C_0^\infty(0,a)}.\end{aligned}$$

Hence

$$\begin{aligned}((gy')', z)_{L_2(0,a)} &= -(y', gz')_{C_0^\infty(0,a)} \\ &= (-1)^2(y, (gz')')_{C_0^\infty(0,a)}.\end{aligned}$$

Thus

$$((gy')', z)_{L_2(0,a)} = (y, (gz')')_{C_0^\infty(0,a)}.$$

Therefore

$$\begin{aligned}\langle AY, Z\rangle &= (y, z^{(4)})_{C_0^\infty(0,a)} - (y, (gz')')_{C_0^\infty(0,a)} \\ &= (y, z^{(4)} - (gz')')_{C_0^\infty(0,a)}.\end{aligned} \tag{3.8}$$

But

$$\begin{aligned}\langle AY, Z\rangle &= \langle Y, W\rangle \\ &= \int_0^a y(x)\bar{w}(x)dx + y'(a)\bar{c}.\end{aligned}$$

Since

$$y \in C_0^\infty(0,a), \qquad y'(a) = 0.$$

Then

$$\begin{aligned}\langle AY, Z\rangle &= \langle Y, W\rangle \\ &= \int_0^a y(x)\bar{w}(x)dx \\ &= (y, w).\end{aligned} \tag{3.9}$$

From (3.8) and (3.9) it follows that

$$\langle AY, Z\rangle = (y, z^{(4)} - (gz')')_{C_0^\infty(0,a)}. \tag{3.10}$$

So (3.7) and (3.10) give

$$(y^{(4)} - (gy')', z) = (y, z^{(4)} - (gz')')_{C_0^\infty(0,a)}. \tag{3.11}$$

Thus

$$(y, z^{(4)} - (gz')')_{C_0^\infty(0,a)} = (y, w)_{C_0^\infty(0,a)}.$$

Then

$$z^{(4)} - (gz')' = w \in L_2(0, a). \tag{3.12}$$

Let $v = z^{(3)} - gz'$. Then according to (3.12)

$$v' = z^{(4)} - (gz')' = w \in L_2(0, a). \tag{3.13}$$

Let

$$u(x) = \int_0^x w(t)dt.$$

Then

$$u \in L_2(0, a). \tag{3.14}$$

Thus (3.14) gives

$$u \in W_1^2(0, a). \tag{3.15}$$

Then $v' - u' = w - u' = 0$. And according to Theorem 2.23 $v - u = c_1$, where c_1 denotes a constant. Thus $v = u + c_1$. Whence (3.15) implies $z^{(3)} - gz' = v \in W_1^2(0, a)$. But by Lemma 3.4, $gz' = (gz)' - g'z$. Thus

$$z^{(3)} - (gz)' + g'z = v \in W_1^2(0, a) \subset L_2(0, a). \tag{3.16}$$

Since $L_2(0, a)$ is a vector space, then (3.3) and (3.16) imply $z^{(3)} - (gz)' \in L_2(0, a)$.

Repeating the above reasoning,

$$z^{(3)} - (gz)' \in L_2(0, a) \Longrightarrow z^{(2)} - gz \in W_1^2(0, a) \subset L_2(0, a).$$

Then

$$\begin{aligned} z^{(2)} - gz \in L_2(0,a) \text{ and } gz \in L_2(0,a) \quad &\Longrightarrow \quad z^{(2)} \in L_2(0,a) \\ &\Longrightarrow \quad z \in W_2^2(0,a). \end{aligned}$$

Since

$$z \in W_2^2(0,a)$$

then,

$$z' \in W_1^2(0,a).$$

And since

$$g \in C^1[0,a],$$

then

$$gz' \in W_1^2(0,a),$$

and thus

$$(gz')' \in W_0^2[0,a] = L_2(0,a).$$

Since

$$z^{(4)} - (gz')' \in L_2(0,a) \quad \text{and} \quad (gz')' \in W_0^2[0,a],$$

then

$$z^{(4)} \in W_0^2[0,a].$$

As

$$z^{(4)} \in W_0^2[0,a],$$

then according to Corollary 2.27

$$\begin{aligned} z^{(3)} &\in W_1^2(0,a), \\ z'' &\in W_2^2(0,a), \\ z' &\in W_3^2(0,a), \\ z &\in W_4^2(0,a). \end{aligned} \tag{3.17}$$

□

Theorem 3.6. *The operator A is self-adjoint.*

Proof. Let $Y \in \mathscr{D}(A)$ and $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*)$. Then

$$\begin{aligned} \langle AY, Z \rangle &= \int_0^a [y^{(4)}(x) - (g(x)y'(x))']\bar{z}(x)dx + y''(a)\bar{d} \\ &= \int_0^a y^{(4)}(x)\bar{z}(x)dx - \int_0^a (g(x)y'(x))'\bar{z}(x)dx + y''(a)\bar{d}. \end{aligned}$$

Since

$$z \in W_4^2(0,a) \quad \text{by} \quad \text{Proposition 3.5},$$

then

$$\begin{aligned} (y^{(4)}, z) &= [y^{(3)}(x)\bar{z}(x)]_0^a - \int_0^a y^{(3)}(x)\bar{z}'(x)dx \\ &= [y^{(3)}(x)\bar{z}(x)]_0^a - [y''(x)\bar{z}'(x)]_0^a + \int_0^a y''(x)\bar{z}''(x)dx. \end{aligned}$$

But the scalar product

$$\int_0^a y''(x)\bar{z}''(x)dx = (y^{(4)}, z) - [y^{(3)}(x)\bar{z}(x)]_0^a + [y''(x)\bar{z}'(x)]_0^a$$

is symmetric, so

$$\int_0^a y''(x)\bar{z}''(x)dx = (y, z^{(4)}) - [y(x)\bar{z}^{(3)}(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a.$$

Then

$$\begin{aligned} (y^{(4)}, z) &= (y, z^{(4)}) + [y^{(3)}(x)\bar{z}(x)]_0^a - [y''(x)\bar{z}'(x)]_0^a - [y(x)\bar{z}^{(3)}(x)]_0^a + [y'(x)\bar{z}''(x)]_0^a \\ &= (y, z^{(4)}) + y^{(3)}(a)\bar{z}(a) - y^{(3)}(0)\bar{z}(0) - y''(a)\bar{z}'(a) + y'(a)\bar{z}''(a) - y'(0)\bar{z}''(0) \\ &= (y, z^{(4)}) + y^{(3)}(a)\bar{z}(a) - y''(a)\bar{z}'(a) + y'(a)\bar{z}''(a) - y^{(3)}(0)\bar{z}(0) - y'(0)\bar{z}''(0) \end{aligned}$$

since

$$y(a) = y''(0) = y(0) = 0.$$

Also,

$$((gy')', z) = [g(x)y'(x)\bar{z}(x)]_0^a - \int_0^a y'(x)g(x)\bar{z}'(x)dx.$$

The scalar product

$$\int_0^a y'(x)g(x)\bar{z}'(x)dx = -((gy')', z) + [g(x)y'(x)\bar{z}(x)]_0^a$$

is symmetric, so

$$\int_0^a y'(x)g(x)\bar{z}'(x)dx = -(y, (gz')') + [g(x)y(x)\bar{z}'(x)]_0^a.$$

Then

$$\begin{aligned} ((gy')', z) &= [g(x)y'(x)\bar{z}(x)]_0^a - [g(x)y(x)\bar{z}'(x)]_0^a + (y, (gz')') \\ &= g(a)y'(a)\bar{z}(a) - g(0)y'(0)\bar{z}(0) + (y, (gz')') \end{aligned}$$

since

$$y(0) = y(a) = 0.$$

Thus

$$\begin{aligned} \langle AY, Z \rangle &= (y^{(4)}, z) - ((gy')', z) + y''(a)\bar{d} \\ &= (y, z^{(4)}) - (y, (gz')') + y^{(3)}(a)\bar{z}(a) - y''(a)\bar{z}'(a) \\ &+ y'(a)\bar{z}''(a) + y''(a)\bar{d} - g(a)y'(a)\bar{z}(a) - y^{(3)}(0)\bar{z}(0) - y'(0)\bar{z}''(0) + g(0)y'(0)\bar{z}(0) \\ &= (y, z^{(4)} - (gz')') + y'(a)\bar{z}''(a) + (y^{(3)}(a) - g(a)y'(a))\bar{z}(a) - y''(a)(\bar{z}'(a) - \bar{d}) \\ &- y^{(3)}(0)\bar{z}(0) + y'(0)g(0)\bar{z}(0) - y'(0)\bar{z}''(0). \end{aligned}$$

But

$$\langle AY, Z \rangle = \langle Y, A^*Z \rangle = \langle Y, W \rangle \quad \text{where} \quad W = \begin{pmatrix} w \\ c \end{pmatrix},$$

and according to Proposition 3.5

$$w = z^{(4)} - (gz')'.$$

So

$$\begin{aligned}\langle AY, Z\rangle &= \langle Y, A^*Z\rangle \\ &= (y, z^{(4)} - (gz')') + y'(a)\bar{c}.\end{aligned}$$

Then

$$\begin{aligned}y'(a)\bar{c} &= y'(a)\bar{z}''(a) + (y^{(3)}(a) - g(a)y'(a))\bar{z}(a) - y''(a)(\bar{z}'(a) - \bar{d}) - y^{(3)}(0)\bar{z}(0) \\ &+ y'(0)g(0)\bar{z}(0) - y'(0)\bar{z}''(0).\end{aligned}$$

So

$$\begin{aligned}0 &= (y^{(3)}(a) - g(a)y'(a))\bar{z}(a) + y'(a)(\bar{z}''(a) - \bar{c}) - y''(a)(\bar{z}'(a) - \bar{d}) \\ &+ (y'(0)g(0) - y^{(3)}(0))\bar{z}(0) - y'(0)\bar{z}''(0).\end{aligned} \tag{3.18}$$

But there exists a polynomial y_1 such that $y_1(0) = y_1''(0) = y_1(a) = 0$, $y_1^{(3)}(a) \neq 0$ and $y_1'(a) = y_1''(a) = y_1'(0) = y_1^{(3)}(0) = 0$. Then $\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_1^{(3)}(a)\bar{z}(a) = 0. \tag{3.19}$$

Since $y_1^{(3)}(a) \neq 0$, then $\bar{z}(a) = 0$, so (3.19) gives

$$z(a) = 0, \tag{3.20}$$

and (3.18) reduces to

$$y'(a)(\bar{z}''(a) - \bar{c}) - y''(a)(\bar{z}'(a) - \bar{d}) + (y'(0)g(0) - y^{(3)}(0))\bar{z}(0) - y'(0)\bar{z}''(0) = 0. \tag{3.21}$$

Also there exists a polynomial y_2 such that $y_2(0) = y_2''(0) = y_2(a) = 0$ and $y_2'(a) \neq 0$, $y_2''(a) = y_2'(0) = y_2^{(3)}(0) = 0$. Then $\begin{pmatrix} y_2 \\ 0 \end{pmatrix} \in \mathscr{D}(A)$ So

$$y_2'(a)(\bar{z}''(a) - \bar{c}) = 0. \tag{3.22}$$

since $y_2'(a) \neq 0$ then (3.22) gives

$$c = z''(a). \tag{3.23}$$

And (3.21) reduces to

$$-y''(a)(\bar{z}'(a) - \bar{d}) + (y'(0)g(0) - y^{(3)}(0))\bar{z}(0) - y'(0)\bar{z}''(0) = 0. \tag{3.24}$$

There exists a polynomial y_3 such that $y_3(0) = y_3''(0) = y_3(a) = 0$, $y_3'(0) = y_3^{(3)}(0) = 0$ and $y_3''(a) \neq 0$. Then $\begin{pmatrix} y_3 \\ 0 \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_3''(a)(\bar{z}'(a) - \bar{d}) = 0. \tag{3.25}$$

Since $y_3''(a) \neq 0$, then (3.25) gives

$$z'(a) = d. \tag{3.26}$$

Then (3.24) reduces to

$$(y'(0)g(0) - y^{(3)}(0))\bar{z}(0) - y'(0)\bar{z}''(0) = 0. \tag{3.27}$$

There exists a polynomial y_4, such that $y_4(0) = y_4''(0) = y_4(a) = 0$, $y_4'(0) = 0$ and $y_4^{(3)}(0) \neq 0$, so $\begin{pmatrix} y_4 \\ 0 \end{pmatrix} \in \mathscr{D}(A)$. Then (3.27) reduces to

$$y_4^{(3)}(0)\bar{z}(0) = 0. \tag{3.28}$$

Since $y_4^{(3)}(0) \neq 0$, then (3.28) gives

$$z(0) = 0, \tag{3.29}$$

and (3.27) reduces to

$$y'(0)\bar{z}''(0) = 0. \tag{3.30}$$

There exists a polynomial y_5, such that $y_5(0) = y_5''(0) = y_5(a) = 0$ and $y_5'(0) \neq 0$. So $\begin{pmatrix} y_5 \\ 0 \end{pmatrix} \in \mathscr{D}(A)$. Since $y_5'(0) \neq 0$, then (3.30) gives

$$z''(0) = 0. \tag{3.31}$$

Hence, from (3.20), (3.23), (3.26), (3.29) and (3.31) we have

$$z(0) = z''(0) = z(a) = 0, \quad c = z''(a) \text{ and } d = z'(a).$$

Thus $Z \in \mathscr{D}(A)$, so $\mathscr{D}(A^*) \subset \mathscr{D}(A)$. Since A is symmetric, see Proposition 3.2, then according to Definition 2.16, it follows that A is self-adjoint. □

Chapter 4

Self-adjoint Boundary Conditions Independent of λ

4.1 Introduction

We present in this chapter characterizations of boundary conditions corresponding to a self-adjoint main operator independent of λ. We present in Section 4.2 maximal and minimal operators and quasi-differential expressions. We give in Section 4.4 a theorem which characterizes boundary conditions corresponding to a self-adjoint main operator independent of λ and in Section 4.5. We give examples of classes of boundary conditions, corresponding to a self-adjoint main operator, independent of λ. The definitions and properties we present in Section 4.2 are given in the general case with matrix coefficients from $M_m(L_{loc}(0, a))$, the set of $m \times m$ matrices with entries from $L_{loc}(0, a)$. However for the study we present in this document, we consider the case $m = 1$.

4.2 Quasi-Differential Expressions and Maximal and Minimal Operators

The following definition is taken from Section 2 [14, page 25].

Definition 4.1. Let $I = (a, b)$ be an interval with $-\infty \leq a < b \leq \infty$ and let $n,\ m$ be postive integers. For a given set $S,\ M_{n,m}(S)$ denotes the set of $n \times m$ matrices with entries in S. If $n = m$, we write also $M_n(S)$, and if $m = 1$, we write S^n.

Let

$$\begin{aligned} Z_{n,m}(I) \ &:= \ \{A = (a_{rs})_{r,s=1}^{n} \in M_n(M_m(L_{loc}(I))), \\ &\quad a_{r,r+1} \ \ \textit{invertible} \ \ \textit{a.e.} \ \ \textit{for} \ \ 1 \leq r \leq n-1, \\ &\quad a_{rs} = 0 \ \textit{for} \ 2 \leq r+1 < s \leq n\}. \end{aligned} \tag{4.1}$$

Let $A \in Z_{n,m}(I)$. We define

$$V_0 := \{y : I \to \mathbb{C}^m, y \ \textit{measurable}\} \tag{4.2}$$

and

$$y^{[0]} := y \ \ (y \in V_0). \tag{4.3}$$

Inductively, for r = 1, . . . , n, we define

$$V_r = \{y \in V_{r-1} : y^{[r-1]} \in (AC_{loc}(I))^m\}, \tag{4.4}$$

$$y^{[r]} = a_{r,r+1}^{-1}(y^{[r-1]'} - \sum_{s=1}^{r} a_{rs}y^{[s-1]}) \ \ (y \in V_r), \tag{4.5}$$

where $a_{n,n+1} := I_m$, the $m \times m$ identity matrix; and $AC_{loc}(I)$ denotes the set of complex valued functions which are absolutely continuous on all compact sub-intervals of I. Finally we set

$$My := i^n y^{[n]} \quad (y \in V_n). \tag{4.6}$$

The expression $M = M_A$ is called the quasi-differential expression associated with A.

The following remark can be found in Section 2 [14, page 26].

Remark 4.2. For V_n we use also the notations $\mathscr{D}(A)$ and $V(M)$.

The following proposition is Proposition 2.2 [14, page 26].

Proposition 4.3. *Let $A \in Z_{n,m}(I)$ and $f \in (L_{loc}(I))^m$. Set* $F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}$.

(i) If $Y \in (AC_{loc}(I))^{nm}$ is a solution of

$$Y' = AY + F, \tag{4.7}$$

then there is a (unique) $y \in \mathscr{D}(A)$ such that

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix} \tag{4.8}$$

and

$$y^{[n]} = f. \tag{4.9}$$

(ii) If $y \in \mathscr{D}(A)$ is a solution of

$$y^{[n]} = f, \tag{4.10}$$

then

$$Y := \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix} \in (AC_{loc}(I))^{nm} \tag{4.11}$$

and

$$Y' = AY + F. \tag{4.12}$$

The following lemma is Lemma 3.1 [14, page 29].

Lemma 4.4. *Let I be bounded. Let A, B be $k \times k$ complex matrix functions on I, such that all components of A and B are in $L(I)$. Let F, G be $k \times m$ matrix functions on I, such that all components of F and G are in $L(I)$. If $Y' = AY + F$ and $Z' = BZ + G$ and $C \in M_k(\mathbb{C})$, then*

$$(Z^*CY)' = Z^*(B^*C + CA)Y + Z^*CF + G^*CY. \tag{4.13}$$

The following corollary is Corollary 3.2 [14, page 30].

Corollary 4.5. *Let the assumptions be as in Lemma 4.4. If, in addition, C is invertible and $B = -C^{-1*}A^*C^*$, then*

$$(Z^*CY)' = Z^*CF + G^*CY. \tag{4.14}$$

The following lemma is Lemma 3.3 [14, page 30].

Lemma 4.6. *Let $A \in Z_{n,m}(I)$ and $C := ((-1)^r \delta_{r,n+1-s} I_m)_{r,s=1}^n$. Let $B := -C^{-1}A^*C$. Then $B \in Z_{n,m}(I)$ and for any $y \in \mathscr{D}(A)$ and $z \in \mathscr{D}(B)$ we have*

$$z^* M_A y - (M_B z)^* y = [y, z]' \tag{4.15}$$

where

$$[y, z] = i^n \sum_{r=0}^{n-1} (-1)^{n+1-r} z_B^{[n-r-1]^*} y_A^{[r]}. \tag{4.16}$$

Remark 4.7. The δ in Lemma 4.6 is the Kronecker delta.

The following definition can be found in Section 4 of [14, page 31].

Definition 4.8. Let $A \in Z_{n,m}(I)$.

1. The maximal operator $T = T_A$ associated with the matrix $A \in Z_{n,m}(I)$ is defined by

$$Ty = M_A y \quad (y \in \mathscr{D}(T)), \quad \text{where}$$
$$\mathscr{D}(T) = \{y \in L_2(I) : y \in \mathscr{D}(A), M_A y \in L_2(I)\}.$$

2. The pre-minimal operator $T_0' = T_{A,0}'$ associated with A is defined by

$$\mathscr{D}(T_0') = \{y \in L_2(I) : y \text{ has support compact}\},$$
$$T_0' y = Ty \quad (y \in \mathscr{D}(T)).$$

3. The closure T_0 of T_0' is a linear operator. It is called the minimal operator associated with A.

Remark 4.9. The minimal operator T_0 is also denoted $T_{A,0}$.

The following proposition can be found in Section 2 [15, page 53].

Proposition 4.10. *If $T_{A,0}$ is symmetric, then $T_{A,0}^* = T_A$ and $T_A^* = T_{A,0}$.*

The following proposition can be found in Section 2 [15, page 53].

Proposition 4.11. *Let $A \in Z_{n,m}(I)$, T_0 be the minimal operator and T_A be the maximal operator.*

1. *T_A is a $2nm$ dimensional extension of T_0.*

2. *T_0 has self-adjoint extension and every self-adjoint extension of T_0 is an nm dimensional extension.*

3. *Every nm dimensional symmetric extension of T_0 is self-adjoint.*

The following definition is from the proof of Lemma 1.1 [15, page 51].

Definition 4.12. Let $A \in Z_{n,m}(I)$. For $y \in \mathscr{D}(T_A)$, $Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}$, $Y_R = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}$.

The following definition is Definition 1.2 [15, page 51]. It is equivalent to Definition 2.2 and is the definition we have used to conduct this study.

Definition 4.13. Let l be an integer with $0 \le l \le 2nm$. Any $l \times 2nm$ matrix $U \in M_{l,2nm}(\mathbb{C})$ with rank l is called a boundary matrix and the equation

$$UY_R = 0 \quad (y \in \mathscr{D}(T_A)) \tag{4.17}$$

is called a boundary condition. For any such U we define an operator $T(U)$ from $L_2(I)$ into itself by

$$\begin{aligned} \mathscr{D}(T(U)) &= \{y \in \mathscr{D}(T_A) : UY_R = 0\} \\ T(U)y &= T_A y \quad (y \in \mathscr{D}(T(U))). \end{aligned} \tag{4.18}$$

When $l = 0$ we have $U = 0$ and $T(U) = T_A$.

The following remark is taken from [15, page 52].

Remark 4.14. From (4.17) and (4.18) it is clear that

$$T_{A,0} \subset T(U) \subset T_A \tag{4.19}$$

The following theorem is Theorem 2.3 [15, page 54].

Theorem 4.15. *Let U be an $l \times 2nm$ matrix with* rank l *where $nm \leq l \leq 2nm$.*

Then the operator $T(U)$ is symmetric if and only if

$$N(U) \subset R(DU^*) \tag{4.20}$$

where $N(U)$ denotes the null space of U, $R(DU^*)$ denotes the range of the matrix DU^* and D is the matrix

$$D = i^n \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \qquad C = ((-1)^r \delta_{r,n+1-s} I_m)_{r,s=1}^n. \tag{4.21}$$

Note that

$$C^{-1} = (-1)^{n+1} C = C^*. \tag{4.22}$$

Remark 4.16. The δ in Theorem 4.15 is the Kronecker delta.

The following proposition can be found in Section 2 [15, page 54].

Proposition 4.17. *$T(U)$ is symmetric if and only if*

$$Z_R^* D Y_R = 0 \quad \text{for all} \quad y, z \in \mathscr{D}(T(U)) \tag{4.23}$$

.

The following proposition is taken from Section 2 [15, page 54].

Proposition 4.18. *The following equivalent statements are also equivalent to Proposition 4.17:*

(i) $c^* D d = 0$ *for all* $c, d \in N(U)$

(ii) $N(U) \perp D(N(U))$

(iii) $D(N(U)) \subset (N(U))^\perp = R(U^*)$

(iv) $N(U) \subset R(D^{-1}U^*) = R(DU^*)$.

The following theorem is Theorem 2.4 [15, page 55].

Theorem 4.19. *Let A, B be $l \times nm$ matrices of complex numbers and let $U = (A : B)$ have* rank l. *Then the operator $T(U)$ is self-adjoint if and only if*

$$l = nm \quad \text{and} \quad ACA^* = BCB^*. \tag{4.24}$$

4.3 Preliminaries

Let A_{max} be the maximal operator defined on $W_4^2(0,a)$ by for $y \in W_4^2(0,a)$, $A_{max}y = y^{(4)} - (gy')'$, where $a > 0$ and $g \in C^1[0,a]$ is a real value function.

For $y \in W_4^2(0,a)$, let

$$y^{[0]} = y \tag{4.25}$$

$$y^{[1]} = y' \tag{4.26}$$

$$y^{[2]} = y'' \tag{4.27}$$

$$y^{[3]} = y^{(3)} - gy' \tag{4.28}$$

$$y^{[4]} = y^{(4)} - (gy')' \tag{4.29}$$

Then

$$y^{[0]'} = y' = y^{[1]} \tag{4.30}$$

$$y^{[1]'} = y'' = y^{[2]} \tag{4.31}$$

$$y^{[2]'} = y^{(3)}. \tag{4.32}$$

But

$$\begin{aligned} y^{(3)} &= y^{[3]} + gy' \\ &= y^{[3]} + gy^{[1]} \end{aligned}$$

So

$$y^{[2]'} = y^{[3]} + gy^{[1]}.$$

$$\begin{aligned} y^{[3]'} &= (y^{(3)} - gy')' \\ &= y^{(4)} - (gy')' \\ &= y^{[4]}. \end{aligned}$$

Therefore

$$y^{[0]'} = y^{[1]} \tag{4.33}$$

$$y^{[1]'} = y^{[2]} \tag{4.34}$$

$$y^{[2]'} = y^{[3]} + gy^{[1]} \tag{4.35}$$

$$y^{[3]'} = y^{[4]} \tag{4.36}$$

So

$$\begin{pmatrix} y^{[0]'} \\ y^{[1]'} \\ y^{[2]'} \\ y^{[3]'} \end{pmatrix} = \begin{pmatrix} y^{[1]} \\ y^{[2]} \\ gy^{[1]} + y^{[3]} \\ y^{[4]} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & g & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^{[4]} \end{pmatrix}.$$

Let

$$\begin{aligned} Z_4(0,a) := \{Q = (q_{rs})_{r,s=1}^4 &\in M_4(L_{loc}(0,a)), \\ q_{r,r+1} \quad &\text{invertible a.e. for} \quad 1 \le r \le 3, \\ q_{r,s} = 0 \quad &\text{for} \quad 2 \le r+1 < s \le 3\}. \end{aligned} \tag{4.37}$$

Let

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & g & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A_0 \in Z_4(0,a).$$

Let C be the matrix defined by

$$C = ((-1)^r \delta_{r,5-s})_{r,s=1}^4,$$

where δ is the Kronecker delta.

Then

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$B(Y,Z) = \langle AY, Z\rangle - \langle Y, AZ\rangle$$

where

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{pmatrix}, \quad Z = \begin{pmatrix} z^{[0]} \\ z^{[1]} \\ z^{[2]} \\ z^{[3]} \end{pmatrix} \quad \text{and} \quad y, z \in \mathscr{D}(A_{max}).$$

Then according to Möller and Zettl [15]

$$B(Y,Z) = Z_R^* D Y_R, \tag{4.38}$$

where

$$D = i^4 \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad Y_R = \begin{pmatrix} Y(0) \\ Y(a) \end{pmatrix} \quad \text{and} \quad Z_R = \begin{pmatrix} Z(0) \\ Z(a) \end{pmatrix}.$$

4.4 The Self-adjoint Boundary Conditions

Let $A_1,\ E$ and B_1 be the following 4×4 matrices:

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix},$$

where $a_{ij},\ b_{kl},\ i,\ k \in \{1,2\},\ j,\ l \in \{1,2,3,4\},$ are complex numbers.

Note that

$$E^* = E,\ E^2 = I \quad \text{and} \quad EB_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} A_1 C A_1^* &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & 0 & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & 0 \\ \overline{a_{13}} & \overline{a_{23}} & 0 & 0 \\ \overline{a_{14}} & \overline{a_{24}} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{14} & -a_{13} & a_{12} & -a_{11} \\ a_{24} & -a_{23} & a_{22} & -a_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & 0 & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & 0 \\ \overline{a_{13}} & \overline{a_{23}} & 0 & 0 \\ \overline{a_{14}} & \overline{a_{24}} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{14}\overline{a_{11}} - a_{13}\overline{a_{12}} + a_{12}\overline{a_{13}} - a_{11}\overline{a_{14}} & a_{14}\overline{a_{21}} - a_{13}\overline{a_{22}} + a_{12}\overline{a_{23}} - a_{11}\overline{a_{24}} & 0 & 0 \\ a_{24}\overline{a_{11}} - a_{23}\overline{a_{12}} + a_{22}\overline{a_{13}} - a_{21}\overline{a_{14}} & a_{24}\overline{a_{21}} - a_{23}\overline{a_{22}} + a_{22}\overline{a_{23}} - a_{21}\overline{a_{24}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Denoting

$$\begin{aligned}
\alpha_{11} &= a_{14}\overline{a_{11}} - a_{13}\overline{a_{12}} + a_{12}\overline{a_{13}} - a_{11}\overline{a_{14}} \\
\alpha_{12} &= a_{14}\overline{a_{21}} - a_{13}\overline{a_{22}} + a_{12}\overline{a_{23}} - a_{11}\overline{a_{24}} \\
\alpha_{21} &= a_{24}\overline{a_{11}} - a_{23}\overline{a_{12}} + a_{22}\overline{a_{13}} - a_{21}\overline{a_{14}} \\
\alpha_{22} &= a_{24}\overline{a_{21}} - a_{23}\overline{a_{22}} + a_{22}\overline{a_{23}} - a_{21}\overline{a_{24}},
\end{aligned}$$

we have

$$A_1CA_1^* = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.39}$$

Defining

$$\begin{aligned}
\beta_{11} &= b_{14}\overline{b_{11}} - b_{13}\overline{b_{12}} + b_{12}\overline{b_{13}} - b_{11}\overline{b_{14}} \\
\beta_{12} &= b_{14}\overline{b_{21}} - b_{13}\overline{b_{22}} + b_{12}\overline{b_{23}} - b_{11}\overline{b_{24}} \\
\beta_{21} &= b_{24}\overline{b_{11}} - b_{23}\overline{b_{12}} + b_{22}\overline{b_{13}} - b_{21}\overline{b_{14}} \\
\beta_{22} &= b_{24}\overline{b_{21}} - b_{23}\overline{b_{22}} + b_{22}\overline{b_{23}} - b_{21}\overline{b_{24}},
\end{aligned}$$

we have accordingly

$$EB_1C(EB_1)^* = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and therefore

$$B_1CB_1^* = E(EB_1CB_1^*E^*)E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_{11} & \beta_{12} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{11} & \beta_{12} \\ 0 & 0 & \beta_{21} & \beta_{22} \end{pmatrix}.$$

Proposition 4.20. *$A_1CA_1^* = B_1CB_1^*$ if and only if $A_1CA_1^* = 0$ and $B_1CB_1^* = 0$.*

Proof.

$$\begin{aligned} A_1CA_1^* = B_1CB_1^* &\iff \alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = 0 \text{ and } \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0 \\ &\iff A_1CA_1^* = 0 \text{ and } B_1CB_1^* = 0. \end{aligned}$$

□

Theorem 4.21. *Let $U = (A_1 : B_1)$ have* rank 4. *Then the operator $T(U)$, defined on $W_4^2(0, a)$ by $\mathscr{D}(T(U)) = \{y \in W_4^2(0, a) : UY_R = 0\}$ and $T(U)y = A_{max}y$ $(y \in \mathscr{D}(T(U)))$, is self-adjoint if and only if $A_1CA_1^* = B_1CB_1^*$.*

Proof. According to Theorem 4.19 $A_1CA_1^* = B_1CB_1^*$ and $\text{rank}(U) = 4$ if and only the operator $T(U)$ is self-adjoint. □

Remark 4.22. U has rank 4 if and only if $(a_{11}, a_{12}, a_{13}, a_{14})$ and $(a_{21}, a_{22}, a_{23}, a_{24})$ are linearly independent and $(b_{11}, b_{12}, b_{13}, b_{14})$ and $(b_{21}, b_{22}, b_{23}, b_{24})$ are linearly independent.

4.5 Examples

4.5.1 Properties

Proposition 4.23. *Let B_1 be the matrix such that $EB_1 = A_1$ where A_1 is the matrix defined in* Section 4.4. *Then $A_1CA_1^* = 0$ if and only if $B_1CB_1^* = 0$.*

Proof.

$$\begin{aligned} B_1CB_1^* = 0 &\iff EB_1CB_1^*E^* = 0 \\ &\iff EB_1C(EB_1)^* = 0 \\ &\iff A_1CA_1^* = 0 \end{aligned}$$

□

Remark 4.24.

$$(A_1CA_1^*)^* = \begin{pmatrix} \overline{\alpha_{11}} & \overline{\alpha_{21}} & 0 & 0 \\ \overline{\alpha_{12}} & \overline{\alpha_{22}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But

$$(A_1CA_1^*)^* = A_1C^*A_1^* = -A_1CA_1^*.$$

So

$$\begin{aligned} \alpha_{11} &= -\overline{\alpha_{11}} \\ \alpha_{12} &= -\overline{\alpha_{21}} \\ \alpha_{21} &= -\overline{\alpha_{12}} \\ \alpha_{22} &= -\overline{\alpha_{22}}. \end{aligned}$$

Proposition 4.25. *Let $\alpha_{ij},\ i,j \in \{1,2\}$, be the entries of matrix $A_1CA_1^*$, see* (4.39). *Then*

1. $\alpha_{11} = ia$ *and* $\alpha_{22} = ib$ *where* $a,\ b \in \mathbb{R}$.

2. $\alpha_{21} = -\overline{\alpha_{12}}$.

3. If there exist $k,\ l \in \mathbb{R}$ *such that* $((a_{14} = ka_{11}$ *and* $a_{24} = ka_{21})$ *or* $(a_{11} = ka_{14}$ *and* $a_{21} = ka_{24}))$ *and* $((a_{13} = la_{12}$ *and* $a_{23} = la_{22})$ *or* $(a_{12} = la_{13}$ *and* $a_{22} = la_{23}))$*, then* $A_1CA_1^* = 0$.

4.5.2 Examples

Let $A_i,\ B_i,\ i \in \{1,2,3,4,5,6\}$ be the 4×4 matrices such that $EB_i = A_i$ where $A_i,\ i \in \{1,2,3,4,5,6\}$ are the following matrices:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As the entries of the matrices $A_i,\ i \in \{1,2,3,4,5,6\}$ are real numbers and $\alpha_{11},\ \alpha_{22}$ are imaginary pure numbers, then $\alpha_{11} = \alpha_{22} = 0$.

1. For $A_1CA_1^*$,

$$\alpha_{12} = 0 \times 0 - 0 \times 1 + 0 \times 0 - 1 \times 0 = 0.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = 0 \text{ then } \alpha_{21} = 0.$$

So

$$A_1CA_1^* = 0 \text{ and according to Proposition 4.23 } B_1CB_1^* = 0.$$

2. For $A_2CA_2^*$,

$$\alpha_{12} = 0 \times 0 - 0 \times 0 + 0 \times 1 - 1 \times 0 = 0.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = 0 \text{ then } \alpha_{21} = 0.$$

So

$$A_2CA_2^* = 0 \text{ and according to Proposition 4.23 } B_2CB_2^* = 0.$$

3. For $A_3CA_3^*$,

$$\alpha_{12} = 0 \times 0 - 0 \times 0 + 0 \times 0 - 1 \times 1 = -1.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = -1 \text{ then } \alpha_{21} = 1.$$

So

$$A_3CA_3^* \neq 0 \text{ and according to Proposition 4.23 } B_3CB_3^* \neq 0.$$

4. For $A_4CA_4^*$,

$$\alpha_{12} = 0 \times 0 - 0 \times 0 + 1 \times 1 - 0 \times 0 = 1.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = 1 \text{ then } \alpha_{21} = -1.$$

So

$$A_4CA_4^* \neq 0 \text{ and according to Proposition 4.23 } B_4CB_4^* \neq 0.$$

5. For $A_5CA_5^*$,

$$\alpha_{12} = 0 \times 0 - 0 \times 0 + 1 \times 0 - 0 \times 0 = 0.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = 0 \text{ then } \alpha_{21} = 0.$$

So

$$A_5CA_5^* = 0 \text{ and according to Proposition 4.23 } B_5CB_5^* = 0.$$

6. For $A_6CA_6^*$,

$$\alpha_{12} = 0 \times 0 - 1 \times 0 + 0 \times 0 - 0 \times 1 = 0.$$

Since

$$\alpha_{21} = -\overline{\alpha_{12}} \text{ and } \alpha_{12} = 0 \text{ then } \alpha_{21} = 0.$$

So

$$A_6CA_6^* = 0 \text{ and according to Proposition 4.23 } B_6CB_6^* = 0.$$

Proposition 4.26. *Let U be the 4×8 matrices $U = (A_i : B_j)$, where for $i, j \in \{1, 2, 3, 4, 5, 6\}$, A_i and B_j are the matrices defined at the beginning of this subsection.*

- *For $i, j \in \{1, 2, 5, 6\}$, $A_iCA_i^* = B_jCB_j^* = 0$ and $\operatorname{rank}(U) = 4$. Then the differential operators $T(U)$ are self-adjoint.*
- *For $i, j \in \{3, 4\}$, $A_iCA_i^* \neq 0$, $B_jCB_j^* \neq 0$. Then for $U = (A_i : B_j)$ where $i \in \{3, 4\}$ and $j \in \{1, 2, 3, 4, 5, 6\}$, $T(U)$ are not self-adjoint. This is also the case if $U = (A_l : B_k)$, where $l \in \{1, 2, 3, 4, 5, 6\}$ and $k \in \{3, 4\}$.*

Chapter 5

Self-adjoint Boundary Conditions Depending On λ

5.1 Introduction

We have presented in the previous chapter boundary conditions corresponding to a self-adjoint main operator independent of the eigenvalue parameter λ. We present in this chapter a characterization theorem (Theorem 5.41) for boundary conditions corresponding to a self-adjoint main operator depending on λ. Boundary conditions depending on the parameter λ are boundary conditions with at least one equation depending on the eigenvalue λ. We consider the differential operator M_{A_0} defined by

$$M_{A_0}y = y^{[4]} = y^{(4)} - (gy')', \tag{5.1}$$

where $a > 0$, $g \in C^1[0, a]$ is a real value function. The boundary conditions considered in this chapter are the following separated boundary conditions

$$B_1(\lambda)y = 0, \tag{5.2}$$

$$B_2(\lambda)y = 0, \tag{5.3}$$

$$B_3(\lambda)y = 0, \tag{5.4}$$

$$B_4(\lambda)y = 0, \tag{5.5}$$

where the $B_j(\lambda)$ are constant or depend on λ linearly, and where at least one of the $B_1(\lambda)$, $B_2(\lambda)$, $B_3(\lambda)$, $B_4(\lambda)$ depends on λ linearly. That is at least one of the equations (5.2) – (5.5) is of the form $y^{[p]}(0) + i\epsilon\alpha\lambda y^{[q]}(0) = 0$ or $y^{[p]}(a) + i\epsilon\alpha\lambda y^{[q]}(a) = 0$, where $\alpha > 0$, $\epsilon = 1$ or $\epsilon = -1$, $0 \le p \le 3$ and $0 \le q \le 3$. Those of the equations (5.2) – (5.5) which do not depend on λ are of the form $y^{[p]}(0) = 0$ or $y^{[p]}(a) = 0$.

Here we assume for simplicity that the boundary conditions have a minimal number of terms. More precisely we assume that either $B_j(\lambda)y = y^{[p_j]}(a_j) + i\epsilon_j\alpha\lambda y^{[q_j]}(a_j)$ or $B_j(\lambda)y = y^{[p_j]}(a_j)$, where $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$. Let $\Theta_1 = \{s \in \{1, 2, 3, 4\} \ : \ B_s(\lambda)y$ depends on $\lambda\}$ and $\Theta_0 = \{1, 2, 3, 4\}\backslash\Theta_1$. We know that $\Theta_1 \neq \emptyset$. Let

$$k := |\Theta_1|. \tag{5.6}$$

From the above assumption on λ-dependence of the boundary conditions it follows that $1 \le k \le 4$. We assume that all the numbers $p_1,\ p_2,\ q_1$ are different if $1 \in \Theta_1$ and all the numbers $p_1,\ p_2,\ q_1,\ q_2$ are different if $2 \in \Theta_1$. Similarly, we assume that $p_3,\ p_4,\ q_3$ are different if $3 \in \Theta_1$ and $p_3,\ p_4,\ q_3\ ,\ q_4$ are different if $4 \in \Theta_1$. Define

$$D_0 y = (\epsilon_j y^{[q_j]}(a_j))_{j\in\Theta_1} \text{ and } A_1 y = (y^{[p_j]}(a_j))_{j\in\Theta_1}. \tag{5.7}$$

Theorem 5.41 is used to investigate classes of boundary conditions corresponding to a self-adjoint main operator with one equation depending on λ, two equations depending on λ, three equations depending on λ and four equations depending on λ respectively. In Section 5.2, we give definitions and properties of closed symmetric operators while in Section 5.3 we present properties for boundary conditions, corresponding to a self-adjoint main operator, depending on λ. We study in this section symmetric operators, adjoint operators and give characterizations of boundary conditions corresponding to a self-adjoint main operator depending on the parameter λ.

5.2 Closed Symmetric Operators

The following theorem is a part of Theorem 4.3 of [19, page 56].

Theorem 5.1. *Let H_1 and H_2 be normed spaces. Let T be an operator from H_1 into H_2.*

We have

$$\begin{aligned} \sup\{||Tf|| : f \in \mathscr{D}(T), ||f|| \le 1\} &= \sup\{||Tf|| : f \in \mathscr{D}(T), ||f|| = 1\} \\ &= \sup\{||Tf|| : f \in \mathscr{D}(T), ||f|| < 1\} \end{aligned}$$

(where the value ∞ is allowed). T is bounded if and only if one of these values is finite; if one is finite, then the others are finite, also, and they are equal to $||T||$.

The following theorem is Theorem 4.15 of [19, page 70].

Theorem 5.2. *Let H_1 and H_2 be Hilbert spaces. A subset G of $H_1 \times H_2$ is the graph of an operator from H_1 into H_2 if and only if G is a subspace possessing the following property: $(0, g) \in G$ implies $g = 0$.*

Each subspace of a graph is a graph.

The following definition can be found in Section 5.1 of [19, page 88].

Definition 5.3. Let H_1 and H_2 be Hilbert spaces. Let T be an operator from H_1 into H_2.

1. T is said to be closed if its graph $G(T) = \{(f, Tf) : f \in \mathscr{D}(T)\}$ is closed in $H_1 \times H_2$.
2. T is said to be closable if $\overline{G(T)}$ is a graph, where $\overline{G(T)}$ is the closure of $G(T)$.

The following proposition can be found in Section 5.1 of [19, page 88].

Proposition 5.4. *Let H_1 and H_2 be Hilbert spaces. Let T be an operator from H_1 into H_2.*

1. *T is closed if and only if the following holds: If (f_n) is a sequence in $\mathscr{D}(T)$ that is convergent in H_1 and the sequence (Tf_n) is convergent in H_2, then we have $\lim f_n \in \mathscr{D}(T)$ and $T(\lim f_n) = \lim Tf_n$.*

2. *T is closable if and only if the following holds: If (f_n) is a sequence in $\mathscr{D}(T)$ such that $f_n \to 0$, and the sequence (Tf_n) in H_2 is convergent, then we have $\lim Tf_n = 0$.*

3. *If T is closable, then*

$$\mathscr{D}(\overline{T}) = \{f \in H_1 : \text{ there exists a sequence } (f_n) \text{ from } \mathscr{D}(T) \text{ such that } f_n \to f, \text{ for which } (Tf_n) \text{ is convergent}\},$$

   $\overline{T}f = \lim Tf_n$ for $f \in \mathscr{D}(\overline{T})$ defines a closed operator $\overline{T}$, called the closure of the operator T.

4. *If T is closed, then $N(T)$ is closed.*

5. *If T is injective, then T is closed if and only if T^{-1} is closed.*

The following theorem is Theorem 5.2. of Section 5.1 of [19, page 89].

Theorem 5.5. *Every bounded operator is closable. A bounded operator T is closed if and only if $\mathscr{D}(T)$ is closed. If T is bounded, then we have $\mathscr{D}(\overline{T}) = \overline{\mathscr{D}(T)}$; The closure $\overline{T}$ is the bounded extension of T onto $\overline{\mathscr{D}(T)}$.*

The following theorem is Theorem 5.3. of Section 5.1 of [19, page 89].

Theorem 5.6. *Let H_1 and H_2 be Hilbert spaces. Let T be a densely defined operator from H_1 into H_2 and T^* be the adjoint operator of the operator T.*

1. *T^* is closed.*

2. *T is closable if and only T^* is densely defined; we then have $\overline{T} = T^{**}$.*

3. If T is closable, then $(\overline{T})^* = T^*$.

The following theorem is a part of Theorem 5.13. of [8, page 234].

Theorem 5.7. *Let X, Y be two Banach spaces and $T \in \mathscr{C}(X, Y)$ where $\mathscr{C}(X, Y)$ is the set of all closed linear operators from X to Y. Let us assume that T is densely defined so that the adjoint operator T^* exists and belongs to $\mathscr{C}(Y^*, X^*)$, where X^* and Y^* are respectively the adjoint spaces of X and Y. Then*

- *$R(T)^{\perp} = N(T^*)$.*
- *$R(T^*)^{\perp} = N(T)$.*
- *If $\dim(X) < \infty$, then $R(T^*) = (N(T))^{\perp}$.*

The following definition can be found in Section 8.1 of [19, page 230].

Definition 5.8. Let T be a closed symmetric operator on a complex Hilbert space H and

$$N_+ = N(i - T^*) = R(-i - T)^{\perp},$$

$$N_- = N(-i - T^*) = R(i - T)^{\perp}.$$

Then N_+ and N_- are called the deficiency spaces of T and the numbers $m_+ = \dim N_+$, $m_- = \dim N_-$ are respectively called the defect index of T and $-i$, the defect index of T and i.

The following theorem is Theorem 8.11. of [19, page 237].

Theorem 5.9. *(The first formula of von Neumann). Let T be a closed symmetric operator on a complex Hilbert space. Then*

$$D(T^*) = D(T) \oplus N_+ \oplus N_-,$$

$$T^*(f_0 + g_+ + g_-) = Tf_0 + ig_+ - ig_- \ for \ f_0 \in D(T), \ g_+ \in N_+, g_- \in N_-.$$

The following theorem is Theorem 8.12. of [19, page 238].

Theorem 5.10. *(The second formula of von Neumann). Let T be a closed symmetric operator on a complex Hilbert space. Then*

(i) S is a closed symmetric extension of T if and only if the following holds:
There are closed subspaces F_+ of N_+ and F_- of N_- and an isometric mapping V of F_+ onto F_- such that

$$D(S) = D(T) + \{g + Vg : g \in F_+\}$$

and

$$\begin{aligned} S(f_0 + g + Vg) &= Tf_0 + ig - iVg \\ &= T^*(f_0 + g + Vg) \ for f_0 \in D(T), \ g \in F_+. \end{aligned}$$

(ii) S is self-adjoint if and only if $F_+ = N_+$ and $F_- = N_-$.

The following theorem is Theorem 8.13. of [19, page 239].

Theorem 5.11. *Let T be a closed symmetric operator on a complex Hilbert space and let S be a closed symmetric extension of T.*

(i) S is an m-dimensional extension if and only if F_+ is m-dimensional.

(ii) If T has defect indices (m, m), then a symmetric extension S of T is self-adjoint if and only if S is an m-dimensional extension of T.

5.3 Self-adjoint Boundary Conditions Depending on λ

Recall that $1 \leq k \leq 4$. Define

$$l = 4 - k. \tag{5.8}$$

Clearly, $0 \leq l \leq 3$.

Let A_{max} be the maximal differential operator associated with the boundary value problem for (5.1) and those boundary conditions from (5.2) - (5.5) which depend on λ. The operator A_{max} is defined on $L_2(0,a) \times \mathbb{C}^k$ by $A_{max} = \begin{pmatrix} M_{A_0} & 0 \\ A_1 & 0 \end{pmatrix}$, with domain

$$\mathscr{D}(A_{max}) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix}, y \in W_4^2(0,a) \right\}.$$

It follows from (5.7) that there are two $k \times 8$ matrices V_0 and V_1 such that $V_0 Y_R = D_0 y$ and $V_1 Y_R = A_1 y$, where D_0 and A_1 are as defined in Section 5.1.

5.3.1 Symmetric Operators

For $Y, Z \in \mathscr{D}(A_{max})$,

$$\langle A_{max} Y, Z \rangle = (M_{A_0} y, z) + (V_1 Y_R, V_0 Z_R)$$

and

$$\langle Y, A_{max} Z \rangle = (y, M_{A_0} z) + (V_0 Y_R, V_1 Z_R),$$

where

$$(V_1 Y_R, V_0 Z_R) = Z_R^* V_0^* V_1 Y_R$$

and

$$(V_0 Y_R, V_1 Z_R) = Z_R^* V_1^* V_0 Y_R.$$

Then

$$\begin{aligned} B(Y,Z) &= \langle A_{max} Y, Z \rangle - \langle Y, A_{max} Z \rangle \\ &= (M_{A_0} y, z) + (V_1 Y_R, V_0 Z_R) - (y, M_{A_0} z) - (V_0 Y_R, V_1 Z_R) \\ &= (M_{A_0} y, z) - (y, M_{A_0} z) + (V_1 Y_R, V_0 Z_R) - (V_0 Y_R, V_1 Z_R) \end{aligned}$$

$$\begin{aligned} &= (M_{A_0}y, z) - (y, M_{A_0}z) + Z_R^* V_0^* V_1 Y_R - Z_R^* V_1^* V_0 Y_R \\ &= (M_{A_0}y, z) - (y, M_{A_0}z) + Z_R^*(V_0^* V_1 - V_1^* V_0)Y_R. \end{aligned}$$

But $(M_{A_0}y, z) - (y, M_{A_0}z) = Z_R^* D Y_R$, see (4.38), where

$$D = i^4 \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad C = ((-1)^r \delta_{r,5-s})_{r,s=1}^4,$$

and δ is the Kronecker delta. So

$$\begin{aligned} B(Y, Z) &= Z_R^* D Y_R + Z_R^*(V_0^* V_1 - V_1^* V_0)Y_R. \\ &= Z_R^*(D + (V_0^* V_1 - V_1^* V_0))Y_R. \end{aligned}$$

Let

$$W = D + (V_0^* V_1 - V_1^* V_0). \tag{5.9}$$

Then $B(Y, Z) = Z_R^* W Y_R$ is the Lagrange identity.

Proposition 5.12. *V_0 and V_1 are two $k \times 8$ matrices of rank k, V_0^* and V_1^* are two $8 \times k$ matrices of* rank *k. Then $V_0^* V_1$ and $V_1^* V_0$ are two 8×8 matrices of* rank *k, $V_0^* V_1 - V_1^* V_0$ is an 8×8 matrix of* rank *$2k$ and the matrix W is an 8×8 matrix of* rank *at least $8 - 2k$.*

Proof. Each non-zero entry of V_0 is coefficient of a coordinate of Y_R which is factor of the eigenvalue λ of a boundary condition, while each non-zero entry of V_1 is a coordinate of Y_R which is not a factor of the eigenvalue λ. Also the equations of the boundary conditions depending on λ are such that $y^{[p]}(0) + i\epsilon\alpha\lambda y^{[q]}(0) = 0$ or $y^{[p]}(a) + i\epsilon\alpha\lambda y^{[q]}(a) = 0$, where $p \neq q$. Therefore each of the positions of non-zero entries of the matrix V_0 is different from the positions of the non-zero entries of the matrix V_1. As the number k is the number of boundary conditions depending on λ, then each of the matrices V_0 and V_1 has exactly k non-zero entries.

Let

$$\Theta_1 = \{\theta_1, \cdots, \theta_k\}, \ \widetilde{p_i} = \begin{cases} p_i + 1 \text{ if } i = 1, 2 \\ p_i + 5 \text{ if } i = 3, 4 \end{cases} \text{ and } \widetilde{q_i} = \begin{cases} q_i + 1 \text{ if } i = 1, 2 \\ q_i + 5 \text{ if } i = 3, 4 \end{cases}. \tag{5.10}$$

Then

$$V_0 = \left(\epsilon_{\theta_i}\delta_{j,\widetilde{q}_{\theta_i}}\right)_{i=1,\cdots,k;\ j=1,\cdots,8} \text{ and } V_1 = \left(\delta_{j,\widetilde{p}_{\theta_i}}\right)_{i=1,\cdots,k;\ j=1,\cdots,8}, \tag{5.11}$$

where δ is the Kronecker symbol.

Hence

$$V_0^* = \left(\epsilon_{\theta_j}\delta_{i,\widetilde{q}_{\theta_j}}\right)_{i=1,\cdots,8;\ j=1,\cdots,k},\ V_1^* = \left(\delta_{i,\widetilde{p}_{\theta_j}}\right)_{i=1,\cdots,8;\ j=1,\cdots,k}, \tag{5.12}$$

and

$$V_0^*V_1 = \left(\sum_{s=1}^{k}\epsilon_{\theta_s}\delta_{i,\widetilde{q}_{\theta_s}}\delta_{j,\widetilde{p}_{\theta_s}}\right)_{i,j=1}^{8} = \sum_{s=1}^{k}(\epsilon_{\theta_s}\delta_{i,\widetilde{q}_{\theta_s}}\delta_{j,\widetilde{p}_{\theta_s}})_{i,j=1}^{8}. \tag{5.13}$$

The non-zero entries of the matrix $V_0^*V_1$ are at positions row index $\widetilde{q}_{\theta_s}$, column index $\widetilde{p}_{\theta_s}$ with $s = 1, \cdots, k$. All the numbers $p_1,\ p_2,\ q_1$ if $1 \in \Theta_1$ are different and all the numbers $p_1,\ p_2,\ q_1,\ q_2$ if $2 \in \Theta_1$ are different. Also all the numbers $p_3,\ p_4,\ q_3$ if $3 \in \Theta_1$ and q_4 if $4 \in \Theta_1$ are different. Thus the numbers $\widetilde{p}_{\theta_s},\ \widetilde{q}_{\theta_s},\ s = 1, \cdots, k$ are different. Therefore there are k different row indices $\widetilde{q}_{\theta_s}$ and k different column indices $\widetilde{p}_{\theta_s}$ for the matrix $V_0^*V_1$. As there are k different row indices $\widetilde{q}_{\theta_s}$ and k different column indices $\widetilde{p}_{\theta_s}$ for the matrix $V_0^*V_1$, then the non-zero entries of the matrix $V_0^*V_1$ are on k different rows and k different columns. Thus $\operatorname{rank} V_0^*V_1 = k$. The matrix

$$V_1^*V_0 = \left(\sum_{s=1}^{k}\epsilon_{\theta_s}\delta_{i,\widetilde{p}_{\theta_s}}\delta_{j,\widetilde{q}_{\theta_s}}\right)_{i,j=1}^{8} = \sum_{s=1}^{k}(\epsilon_{\theta_s}\delta_{i,\widetilde{p}_{\theta_s}}\delta_{j,\widetilde{q}_{\theta_s}})_{i,j=1}^{8}$$

is the transposed of $V_0^*V_1$, so the non-zero entries of the matrix $V_1^*V_0$ are at positions row index $\widetilde{p}_{\theta_s}$, column index $\widetilde{q}_{\theta_s}$ with $s = 1, \cdots, k$ and $\operatorname{rank} V_1^*V_0 = k$.

Clearly,

$$\begin{aligned} V_0^*V_1 - V_1^*V_0 &= \sum_{s=1}^{k}\epsilon_{\theta_s}(\delta_{i,\widetilde{q}_{\theta_s}}\delta_{j,\widetilde{p}_{\theta_s}})_{i,j=1}^{8} - \sum_{s=1}^{k}\epsilon_{\theta_s}(\delta_{i,\widetilde{p}_{\theta_s}}\delta_{j,\widetilde{q}_{\theta_s}})_{i,j=1}^{8} \\ &= \sum_{s=1}^{k}\epsilon_{\theta_s}(\delta_{i,\widetilde{q}_{\theta_s}}\delta_{j,\widetilde{p}_{\theta_s}} - \delta_{i,\widetilde{p}_{\theta_s}}\delta_{j,\widetilde{q}_{\theta_s}})_{i,j=1}^{8}. \end{aligned} \tag{5.14}$$

As the non-zero entries of $V_0^*V_1$ are at the positions row indices $\widetilde{q}_{\theta_s}$, column indices $\widetilde{p}_{\theta_s}$ and the non-zero entries of $V_1^*V_0$ are at the position indices $\widetilde{p}_{\theta_s}$, column indices $\widetilde{q}_{\theta_s}$, with $s = 1, \cdots, k$, then the non-zero entries of the matrix $V_0^*V_1 - V_1^*V_0$ are at positions row index $\widetilde{q}_{\theta_s}$, column index $\widetilde{p}_{\theta_s}$ and row index $\widetilde{p}_{\theta_s}$, column index $\widetilde{q}_{\theta_s}$ with $s = 1, \cdots, k$. So the non-zero entries of $V_0^*V_1 - V_1^*V_0$ are at positions row indices $q_{\theta_s} + 1$, column indices $p_{\theta_s} + 1$, row indices $p_{\theta_s} + 1$, column indices $q_{\theta_s} + 1$ if $\theta_s = 1,\ 2$ and row indices $q_{\theta_s} + 5$, column indices $p_{\theta_s} + 5$, row indices $p_{\theta_s} + 5$, column indices $q_{\theta_s} + 5$ if $\theta_s = 3,\ 4$ with $s = 1, \cdots, k$. But all the numbers $p_1,\ p_2,\ q_1$ if $1 \in \Theta_1$ and q_2 if $2 \in \Theta_1$ are different, also all the numbers $p_3,\ p_4,\ q_3$ if $3 \in \Theta_1$ and q_4 if $4 \in \Theta_1$ are different, see Section 5.1. Thus all the numbers $\widetilde{p}_{\theta_s},\ \widetilde{q}_{\theta_s},\ s = 1, \cdots, k$ are different. Hence the non-zero entries of $V_0^*V_1 - V_1^*V_0$ are on $2k$ different rows and $2k$ different columns. Therefore $\operatorname{rank}(V_0^*V_1 - V_1^*V_0) = 2k$ and

$$8 - 2k \leq \operatorname{rank} W. \tag{5.15}$$

□

Remark 5.13. Let $\Theta_1^{(0)} = \Theta_1 \cap \{1,\ 2\}$ and $\Theta_1^{(a)} = \Theta_1 \cap \{3,\ 4\}$. The matrix $V_0^*V_1 - V_1^*V_0$ can be written

$$V_0^*V_1 - V_1^*V_0 = \begin{pmatrix} V_2 & 0 \\ 0 & V_3 \end{pmatrix},$$

where

$$V_2 = \sum_{\theta_s \in \Theta_1^{(0)}} \epsilon_{\theta_s}(\delta_{i,q_{\theta_s}+1}\delta_{j,p_{\theta_s}+1} - \delta_{i,p_{\theta_s}+1}\delta_{j,q_{\theta_s}+1})$$

and

$$V_3 = \sum_{\theta_s \in \Theta_1^{(a)}} \epsilon_{\theta_s}(\delta_{i,q_{\theta_s}+1}\delta_{j,p_{\theta_s}+1} - \delta_{i,p_{\theta_s}+1}\delta_{j,q_{\theta_s}+1}).$$

Proposition 5.14. $\operatorname{rank} W = 8 - 2k$ *if and only if the following conditions hold:*

1. *for* $\theta_s \in \Theta_1,\ p_{\theta_s} + q_{\theta_s} = 3$,
2. *for* $\theta_s \in \Theta_1^{(0)}$, $\epsilon_{\theta_s} = (-1)^{q_{\theta_s}}$,
3. *for* $\theta_s \in \Theta_1^{(a)}$, $\epsilon_{\theta_s} = (-1)^{q_{\theta_s}+1}$.

Proof. By definition of W $\operatorname{rank} W = 8-2k$ if and only if $\operatorname{rank}(D+(V_0^*V_1 - V_1^*V_0)) = 8-2k$. Since $(V_0^*V_1 - V_1^*V_0)$ has rank $2k$, see Proposition 5.12 and D is invertible and both $V_0^*V_1 - V_1^*V_0$ and D have at most one non-zero element in each row and each column, it follows that $\operatorname{rank} W = 8-2k$ if and only if the $2k$ non-zero entries of $V_0^*V_1 - V_1^*V_0$ cancel $2k$ non-zero entries of D at the corresponding positions row indices $\widetilde{q}_{\theta_s}$, column indices $\widetilde{p}_{\theta_s}$, and row indices $\widetilde{p}_{\theta_s}$ column indices $\widetilde{q}_{\theta_s}$, $\theta_s \in \Theta_1$, see (5.10).

But

$$D = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \text{ and } V_0^*V_1 - V_1^*V_0 = \begin{pmatrix} V_2 & 0 \\ 0 & V_3 \end{pmatrix}.$$

So $\operatorname{rank} W = 8-2k$ if and only if the non-zero entries of V_2 cancel some non-zero entries of C at the corresponding positions row indices $q_{\theta_s}+1$, column indices $p_{\theta_s}+1$, and row indices $p_{\theta_s}+1$ column indices $q_{\theta_s}+1$, $\theta_s \in \Theta_1^{(0)}$ and the non-zero entries of V_3 cancel some non-zero entries of $-C$ at the corresponding positions row indices $q_{\theta_s}+1$, column indices $p_{\theta_s}+1$, and row indices $p_{\theta_s}+1$ column indices $q_{\theta_s}+1$, $\theta_s \in \Theta_1^{(a)}$.

The non-zero entries of C and $-C$ are on rows i and columns j such that $i+j=5$. Then

$$\operatorname{rank} W = 8-2k \quad \text{if and only if} \quad \begin{cases} (p_{\theta_s}+1)+(q_{\theta_s}+1)=5, \\ (-1)^{q_{\theta_s}+1}\delta_{q_{\theta_s}+1,p_{\theta_s}+1} + \epsilon_{\theta_s}\delta_{i,q_{\theta_s}+1}\delta_{j,p_{\theta_s}+1} = 0 \\ (-1)^{p_{\theta_s}+1}\delta_{p_{\theta_s}+1,q_{\theta_s}+1} - \epsilon_{\theta_s}\delta_{i,p_{\theta_s}+1}\delta_{j,q_{\theta_s}+1} = 0 \\ \text{if } \theta_s \in \Theta_1^{(0)} \end{cases}$$

$$\text{and} \quad \begin{cases} (p_{\theta_s}+1)+(q_{\theta_s}+1)=5, \\ -(-1)^{q_{\theta_s}+1}\delta_{q_{\theta_s}+1,p_{\theta_s}+1} + \epsilon_{\theta_s}\delta_{i,q_{\theta_s}+1}\delta_{j,p_{\theta_s}+1} = 0, \\ (-1)^{p_{\theta_s}+1}\delta_{p_{\theta_s}+1,q_{\theta_s}+1} + \epsilon_{\theta_s}\delta_{i,p_{\theta_s}+1}\delta_{j,q_{\theta_s}+1} = 0, \\ \text{if } \theta_s \in \Theta_1^{(a)}. \end{cases}$$

So

$$\operatorname{rank} W = 8-2k \quad \text{if and only if} \quad \begin{cases} p_{\theta_s}+q_{\theta_s}=3,\ \text{if } \theta_s\in\Theta_1, \\ (-1)^{q_{\theta_s}+1}+\epsilon_{\theta_s}=0,\ (-1)^{p_{\theta_s}+1}-\epsilon_{\theta_s}=0,\ \text{if } \theta_s\in\Theta_1^{(0)}, \\ (-1)^{p_{\theta_s}+1}+\epsilon_{\theta_s}=0, (-1)^{q_{\theta_s}}+\epsilon_{\theta_s}=0,\ \text{if } \theta_s\in\Theta_1^{(a)}. \end{cases}$$

$$\text{if and only if} \quad \begin{cases} p_{\theta_s}+q_{\theta_s}=3,\ \text{if } \theta_s\in\Theta_1, \\ \epsilon_{\theta_s}=(-1)^{q_{\theta_s}},\ \epsilon_{\theta_s}=(-1)^{p_{\theta_s}+1}=(-1)^{4-q_{\theta_s}}=(-1)^{q_{\theta_s}} \\ \text{if } \theta_s\in\Theta_1^{(0)}, \\ \epsilon_{\theta_s}=(-1)^{q_{\theta_s}+1}, \epsilon_{\theta_s}=(-1)^{p_{\theta_s}}=(-1)^{3-q_{\theta_s}}=(-1)^{q_{\theta_s}+1} \\ \text{if } \theta_s\in\Theta_1^{(a)}, \end{cases}$$

$$\text{if and only if} \quad \begin{cases} p_{\theta_s}+q_{\theta_s}=3,\ \text{if } \theta_s\in\Theta_1, \\ \epsilon_{\theta_s}=(-1)^{q_{\theta_s}},\ \text{if } \theta_s\in\Theta_1^{(0)}, \\ \epsilon_{\theta_s}=(-1)^{q_{\theta_s}+1},\ \text{if } \theta_s\in\Theta_1^{(a)}. \end{cases}$$

□

Remark 5.15. As

$$D=\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \text{ then } D^*=\begin{pmatrix} C^* & 0 \\ 0 & -C^* \end{pmatrix},$$

but

$$C^*=(-1)^{4+1}C=-C \text{ so } D^*=\begin{pmatrix} -C & 0 \\ 0 & C \end{pmatrix}=-D.$$

Then

$$\begin{aligned} W^* &= (D+(V_0^*V_1-V_1^*V_0))^* \\ &= D^*+(V_0^*V_1)^*-(V_1^*V_0)^* \\ &= -D+V_1^*V_0-V_0^*V_1 \\ &= -(D+(V_0^*V_1-V_1^*V_0))=-W. \end{aligned}$$

Let

$$\Theta_0=\{\sigma_1,\cdots,\sigma_l\},\ \widetilde{p}_{\sigma_i}=\begin{cases} p_{\sigma_i}+1 \text{ if } i=1,2 \\ p_{\sigma_i}+5 \text{ if } i=3,4 \end{cases}$$

Then

$$U = (\delta_{\widetilde{p}_{\sigma_i},j})_{i=1,\cdots,l;\ j=1,\cdots,8}. \tag{5.16}$$

The following system of equations

$$B_i(\lambda)y = 0 \text{ for } i \in \Theta_0 \tag{5.17}$$

can be written $UY_R = 0$, where U is the $l \times 8$ matrix with $l = 4 - k$ defined in (5.16). Let $T(U)$ be the operator defined on $\mathscr{D}(A_{max})$ such that

$$\mathscr{D}(T(U)) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix}, y \in W_4^2(0,a) \text{ and } UY_R = 0 \right\}$$

and

$$T(U)Y = A_{max}Y \ (Y \in \mathscr{D}(T(U))).$$

It follows from Section 5.1 that

$$\mathscr{D}(T(U)) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix},\ y \in W_4^2(0,a),\ y^{[p_j]}(\theta_j) = 0 \text{ for } j \in \Theta_0 \right\} \tag{5.18}$$

and

$$(T(U))Y = \begin{pmatrix} y^{[4]} \\ A_1 y \end{pmatrix} \text{ where } Y \in \mathscr{D}(T(U)). \tag{5.19}$$

The system (5.17) is a system of l linear equations indedependent of λ. The system (5.17) is also linearly independent, since if $y^{[p_i]}(a_i) = 0$ and $y^{[p_j]}(a_j) = 0$ are two equations of the system (5.17), then either $p_i = p_j$ and $a_i \neq a_j$ or $p_i \neq p_j$. Therefore $\operatorname{rank} U = l$.

Remark 5.16. Let $Y \in \mathscr{D}(A_{max})$. Then $Y \in \mathscr{D}(T(U))$ if and only if $Y_R \in N(U)$, where $Y_R = \begin{pmatrix} Y_1(0) \\ Y_1(a) \end{pmatrix}$ and Y_1 is as defined in (4.8).

Proposition 5.17. *The differential operator $T(U)$ is symmetric if and only if for all $Y, Z \in \mathscr{D}(T(U))$, $\quad B(Y,Z) = Z_R^* W Y_R = 0$.*

Proof. The differential operator $T(U)$ is symmetric if and only if $\langle T(U)Y, Z\rangle = \langle Y, T(U)Z\rangle$ for all $Y,\ Z \in \mathscr{D}(T(U))$.

But $\langle T(U)Y, Z\rangle = \langle Y, T(U)Z\rangle$ for all $Y,\ Z \in \mathscr{D}(T(U))$ if and only if $\langle T(U)Y, Z\rangle - \langle Y, T(U)Z\rangle = 0$ for all $Y,\ Z \in \mathscr{D}(T(U))$. Also $B(Y,Z) = Z_R^* W Y_R$. Since $B(Y,Z) = \langle T(U)Y, Z\rangle - \langle Y, T(U)Z\rangle$ for all $Y,\ Z \in \mathscr{D}(T(U))$, then $T(U)$ is symmetric if and only if $B(Y,Z) = 0$ for $Y,\ Z \in \mathscr{D}(T(U))$.

Thus $T(U)$ is symmetric if and only if $Z_R^* W Y_R = 0$ for all $Y,\ Z \in \mathscr{D}(T(U))$. □

Theorem 5.18. *The operator $T(U)$ is symmetric if and only if*

$$W(N(U)) \subset (N(U))^{\perp} = R(U^*).$$

Proof.

$$\begin{aligned} Z_R^* W Y_R = 0 \text{ for } Y_R, Z_R \in N(U) &\iff (WY_R, Z_R) = 0 \text{ for } Y_R, Z_R \in N(U) \\ &\iff Z_R \perp W Y_R \text{ for } Y_R, Z_R \in N(U) \\ &\iff N(U) \perp W(N(U)) \\ &\iff W(N(U)) \subset (N(U))^{\perp}. \end{aligned}$$

But according to Theorem 5.7 $(N(U))^{\perp} = R(U^*)$.

$$\text{So } Z_R^* W Y_R = 0 \text{ for } Y_R, Z_R \in N(U) \iff W(N(U)) \subset (N(U))^{\perp} = R(U^*).$$

□

Corollary 5.19. *If* $\operatorname{rank}(U) = 4 - k$ *and $T(U)$ is symmetric, then* $\operatorname{rank}(W) = 2(4-k)$.

Proof. Let $\operatorname{rank}(W) = h$.

Since

$$\operatorname{rank}(U) = 4 - k$$

then

$$\dim(N(U)) = 8 - (4-k) = 4 + k \tag{5.20}$$

and

$$\dim(N(U))^{\perp} = 8 - (4 + k) = 4 - k. \tag{5.21}$$

Since

$$W(N(U)) \subset N(U)^{\perp},$$

then

$$\dim(W(N(U))) \leq \dim((N(U))^{\perp}) = 4 - k.$$

On the other hand, since $\dim N(W) = 8 - h$, it follows that

$$\dim(W(N(U))) \geq 4 + k - (8 - h) = -4 + k + h.$$

Thus

$$-4 + k + h \leq \dim(W(N(U))) \leq 4 - k. \tag{5.22}$$

Hence

$$h = \operatorname{rank}(W) \leq 2(4 - k). \tag{5.23}$$

It follows from (5.15) and (5.23) that if $\operatorname{rank}(U) = 4 - k$ and $T(U)$ is symmetric, then $\operatorname{rank}(W) = 2(4 - k)$. □

Corollary 5.20. *If* $\operatorname{rank}(U) = 4-k$ *and* $\operatorname{rank}(W) > 2(4-k)$ *then* $T(U)$ *cannot be symmetric.*

Proof. If $\operatorname{rank}(U) = 4 - k$ and $T(U)$ is symmetric then $\operatorname{rank}(W) \leq 2(4 - k)$. Therefore if $\operatorname{rank}(U) = 4 - k$ and $\operatorname{rank}(W) > 2(4 - k)$, then $T(U)$ cannot be symmetric. □

Proposition 5.21. *Let* $\operatorname{rank} W = 8 - 2k$ *and* $\operatorname{rank} U = 4 - k$*, where* $1 \leq k \leq 4$*. Then* $\dim N(W) \leq \dim N(U)$*.*

Proof. $\dim N(U) = 4 + k$, see (5.20) and

$$\dim N(W) = 8 - (8 - 2k) = 2k. \tag{5.24}$$

Since $1 \leq k \leq 4$, then $2k \leq 4 + k$. Hence $\dim N(W) \leq \dim N(U)$. □

Theorem 5.22. *Let* $\operatorname{rank} W = 8 - 2k$ *and* $\operatorname{rank} U = 4 - k$*. Then* $T(U)$ *is symmetric if and only if* $W(N(U)) = R(U^*)$.

Proof. Let $h = 8 - 2k$.

1. If $T(U)$ is symmetric, then according to Theorem 5.18

$$W(N(U)) \subset (N(U))^{\perp} = R(U^*). \tag{5.25}$$

   On the other hand (5.22) gives

$$\dim W(N(U)) = 4 - k. \tag{5.26}$$

   So it follows from (5.21), (5.25) and (5.26) that $W(N(U)) = R(U^*)$.

2. Conversely if $W(N(U)) = R(U^*)$, then $W(N(U)) \subset R(U^*)$. And it follows from Theorem 5.18 that $T(U)$ is symmetric.

□

Recall that W is defined by (5.9) and that U has the representation defined in (5.16)

Theorem 5.23. *Assume that* $\operatorname{rank}(W) = 2(4-k)$ *where* $1 \leq k \leq 4$*,* $X = (N(W))^{\perp}$*,* $W_X = p_X W i_X$ *where* $i_X : X \hookrightarrow \mathbb{C}^8$ *and* $p_X : \mathbb{C}^8 \to X$ *are respectively the canonical injection of* X *into* $\mathbb{C}^8$ *and the orthogonal projection of* $\mathbb{C}^8$ *onto* X*. Let* U *be the matrix of* rank $l = 4-k$ *defined in* (5.16)*, and put* $U_X = U i_X$*. Then the differential operator* $T(U)$ *is symmetric if and only if* $U_X W_X U_X^* = 0$.

Remark 5.24. Let $1 \leq k \leq 3$. Then W_X is the $2(4-k) \times 2(4-k)$ matrix derived from W by removing the $2k$ 0's rows $\widetilde{q}_{\theta_s}, \widetilde{p}_{\theta_s}$ and the $2k$ 0's columns $\widetilde{p}_{\theta_s}, \widetilde{q}_{\theta_s}$, $s = 1, \cdots, k$ of the matrix W. Thus W_X is the matrix obtained from D by removing the corresponding $2k$ rows $\widetilde{q}_{\theta_s}, \widetilde{p}_{\theta_s}$ and $2k$ columns $\widetilde{p}_{\theta_s}, \widetilde{q}_{\theta_s}$, $s = 1, \cdots, k$. Therefore $W_X = p_X D i_X$. And it follows that

$$W_X = C_0 \text{ or } W_X = \begin{pmatrix} C_1 & 0 \\ 0 & -C_2 \end{pmatrix},$$

where

$$C_0 \in \{C_1, C_2\} \text{ and } C_1,\ C_2 \in \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ C \right\}.$$

For $k = 4$, we have $W_X = 0$.

Proposition 5.25. *Let* $X = (N(W))^{\perp}$, *where* $\operatorname{rank} W = 8 - 2k$ *with* $1 \leq k \leq 4$. *Then* $D(X) \subset X$.

Proof. We know that $X = N(W)^{\perp} = \{x \in \mathbb{C}^8 : x_i = 0 \text{ for } i \in \{\widetilde{p}_{\theta_s}, \widetilde{q}_{\theta_s},\ s = 1, \cdots, k\}\}$. For $x \in \mathbb{C}^8$, $(Dx)_i = \pm x_j$, where $i+j = 5$ for $i = 1, \cdots, 4$ and $i+j = 13$ for $i = 5, \cdots, 8$. Now let $x \in X$. If $i = \widetilde{p}_{\theta_s}$ and $j = \widetilde{q}_{\theta_s}$, then $(Dx)_i = \pm x_j = 0$, for $s = 1, \cdots, k$. Similarly $(Dx)_i = 0$ for $i = \widetilde{q}_{\theta_s}$ and $j = \widetilde{p}_{\theta_s}$, $s = 1, \cdots, k$. Therefore $D(X) \subset X$. □

Proposition 5.26. *Let* $1 \leq k \leq 3$ *and* $\operatorname{rank}(W_X) = 8 - 2k$. *Then*

1. $W_X^* = -W_X$.
2. $W_X^* = W_X^{-1}$.

Proof. Let $1 \leq k \leq 3$ and $\operatorname{rank}(W_X) = 8 - 2k$. Then

1. $W_X = p_X D i_X$.

   So
   $$W_X^* = (p_X D i_X)^* = i_X^* D^* p_X^*.$$
   But
   $$i_X^* = p_X,\ p_X^* = i_X \text{ and } D^* = -D.$$
   Thus
   $$W_X^* = p_X(-D)i_X = -p_X D i_X = -W_X.$$

2. Let $\widetilde{W}_X = p_X D^{-1} i_X$. Then

$$\widetilde{W}_X = p_X D^* i_X = i_X^* D^* p_X^* = (p_X D i_X)^* = W_X^*.$$

Let $u \in X$. Then

$$\widetilde{W}_X W_X(u) = (p_X D^{-1} i_X p_X D i_X)(u).$$

Since $u \in X$ and $i_X : X \hookrightarrow \mathbb{C}^8$ is the canonical injection of X into $\mathbb{C}^8$, then $i_X(u) \in i_X(X) \subset \mathbb{C}^8$. As $D(X) \subset X$ see (5.25), then $D(i_X(u)) \in i_X(X)$. But for all $v \in i_X(X)$, $i_X p_X(v) = v$. So $i_X p_X D i_X(u) = D i_X(u)$ for all $u \in X$. Then $i_X p_X D i_X = D i_X$. Hence $\widetilde{W}_X W_X(u) = p_X D^{-1} D i_X(u) = p_X i_X(u)$. As $i_X : X \hookrightarrow \mathbb{C}^8$ is the canonical injection of X into $\mathbb{C}^8$ and $p_X : \mathbb{C}^8 \to X$ is the canonical projection of $\mathbb{C}^8$ onto X, then $p_X(i_X(u)) = u$. Thus $\widetilde{W}_X W_X(u) = u$. Since this is true for all $u \in X$, then $\widetilde{W}_X W_X = I_X$. Hence $W_X^{-1} = \widetilde{W}_X$. As $\widetilde{W}_X = W_X^*$, then $W_X^{-1} = W_X^*$.

□

Proposition 5.27. *$N(U) = N(U_X) \oplus N(W)$, where U is the matrix defined in* (5.16).

Proof. $x \in N(U_X) \iff U i_X x = 0 \iff i_X x \in N(U)$. So

$$N(U) \supset N(U_X) \oplus \{0\}. \tag{5.27}$$

But

$$N(W) = \{x \in \mathbb{C}^8 : x_i = 0 \text{ for } i \notin \{\widetilde{p}_{\theta_s},\ \widetilde{q}_{\theta_s},\ s = 1, \cdots, k\}\} \tag{5.28}$$

and

$$N(U) = \{x \in \mathbb{C}^8 : x_i = 0\ :\ i \in \{\widetilde{p}_{\sigma_j},\ j = 1, \cdots, l\}\}. \tag{5.29}$$

Let $x \in N(W)$ and $i \in \{1, \cdots, 8\}$ such that $i = \widetilde{p}_{\sigma_j}$, where $\sigma_j \in \Theta_0$ for $j \in \{1, \cdots, l\}$, see (5.16). Since $\sigma_j \in \Theta_0$ for $j \in \{1, \cdots, l\}$, then $\sigma_j \notin \Theta_1$ for $j \in \{1, \cdots, l\}$. Thus $i = \widetilde{p}_{\sigma_j} \notin \{\widetilde{p}_{\theta_s},\ \widetilde{q}_{\theta_s},\ s = 1, \cdots, k\}$. As $x \in N(W)$ and $i \notin \{\widetilde{p}_{\theta_s},\ \widetilde{q}_{\theta_s},\ s = 1, \cdots, k\}$, then $x_i = 0$. Thus $x \in N(U)$. Therefore

$$N(W) \subset N(U). \tag{5.30}$$

It follows from (5.27) and (5.30) that

$$N(U) \supset N(U_X) \oplus N(W). \tag{5.31}$$

We know that $\dim N(U) = 4 + k$, see (5.20), $\dim N(W) = 2k$, see (5.24). But

$$\dim N(U_X) = 8 - 2k - l = 4 - k. \tag{5.32}$$

Then

$$\dim N(U_X) + \dim N(W) = 4 + k. \tag{5.33}$$

It follows from (5.20), (5.31) and (5.33) that

$$N(U) = N(U_X) \oplus N(W). \tag{5.34}$$

□

Proposition 5.28. *Let $1 \le k \le 3$ and $l = 4 - k$. Then the differential operator $T(U)$ is symmetric if and only if $W_X(N(U_X)) = R(U_X^*)$.*

Proof. Since $\dim N(W) = 2k$, see (5.24), then

$$\dim X = \dim N(W)^{\perp} = 8 - 2k. \tag{5.35}$$

So

$$\dim N(U_X)^{\perp} = 8 - 2k - (4 - k) = 4 - k. \tag{5.36}$$

We know that $\dim N(U_X) = 4 - k$, see (5.32). Since W_X is invertible, then

$$\dim W_X(N(U_X)) = \dim N(U_X) = 4 - k. \tag{5.37}$$

($\Longrightarrow$) For $y,\ z \in N(U_X)$, we have $(W_X y, z) = (p_X W i_X y, z) = (W i_X y, i_X z)$. For all $y, z \in N(U_X)$, $i_X y,\ i_X z \in N(U)$ and $(W i_X y, i_X z) = 0$, see Theorem 5.18. So

$$z^* W_X y = 0 \tag{5.38}$$

for all $y,\ z \in N(U_X)$. Whence

$$W_X(N(U_X)) \subset N(U_X)^{\perp}. \tag{5.39}$$

And it follows from (5.36), (5.37) and (5.39) that

$$W_X(N(U_X)) = N(U_X)^{\perp}. \tag{5.40}$$

Therefore if $T(U)$ is symmetric, then $W_X(N(U_X)) = N(U_X)^{\perp} = R(U_X^*)$.

$(\Longleftarrow)$

$$\mathbb{C}^8 = N(W)^{\perp} \oplus N(W) = X \oplus N(W).$$

From the definition of W_X, $W = \begin{pmatrix} W_X & 0 \\ 0 & 0 \end{pmatrix}$, where the block decomposition is with respect to $X \oplus N(W)$. By Proposition 5.27,

$$x,\ y \in N(U) \Longrightarrow \begin{cases} x = \begin{pmatrix} u \\ x_W \end{pmatrix} \in X \oplus N(W),\ u \in N(U_X) \\ y = \begin{pmatrix} v \\ y_W \end{pmatrix} \in X \oplus N(W),\ v \in N(U_X). \end{cases}$$

So for $x,\ y \in N(U)$,

$$\begin{aligned} (Wx, y) = y^* W x &= \begin{pmatrix} v^* & y_W^* \end{pmatrix} \begin{pmatrix} W_X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ x_W \end{pmatrix} \\ &= \begin{pmatrix} v^* & y_W^* \end{pmatrix} \begin{pmatrix} W_X u \\ 0 \end{pmatrix} \\ &= v^* W_X u. \end{aligned}$$

Since $u,\ v \in N(U_X)$, then $v^* W_X u = 0$, see (5.38). So $(Wx, y) = 0$. Hence $W(N(U)) \subset N(U)^{\perp} = R(U^*)$. Therefore $T(U)$ is symmetric by Theorem 5.18. □

Proof. (of Theorem 5.23) We have $l = 4 - k$.

1. Let $k = 4$.

   (i) Since $k = 4$, then $l = 0$, $U = 0$ and $W = 0$. So $U_X = 0$ and $W_X = 0$. Therefore $U_X W_X U_X^* = 0$.

   (ii) Let $U_X W_X U_X^* = 0$. Since $k = 4$, then $\operatorname{rank} W = \operatorname{rank} U = 0$ and so $W = 0$ and $U = 0$. Thus $Z_R^* W Y_R = 0$. Therefore, according to Proposition 5.17, $T(U)$ is symmetric.

2. (i) ($\Longrightarrow$) As $1 \leq k \leq 3$ and $l = 4 - k$, then according to Proposition 5.28

$$T(U) \text{ is symmetric if and only if } W_X(N(U_X)) = R(U_X^*).$$

But according to Proposition 5.26

$$W_X^{-1} = W_X^* = -W_X.$$

Then as $l = 4 - k$,

$$T(U) \text{ is symmetric if and only if } W_X^{-1} W_X(N(U_X)) = W_X^{-1} R(U_X^*).$$

So

$$T(U) \text{ is symmetric if and only if } \; N(U_X) = W_X^{-1} R(U_X^*). \tag{5.41}$$

But

$$W_X^{-1} R(U_X) = R(-W_X U_X^*) = R(W_X U_X^*). \tag{5.42}$$

Thus if follows from (5.41) and (5.42) that

$$T(U) \text{ is symmetric if and only if } N(U_X) = R(W_X U_X^*). \tag{5.43}$$

Therefore, if

$$T(U) \text{ is symmetric, then } U_X W_X U_X^* = 0.$$

   (ii) ($\Longleftarrow$) If $U_X W_X U_X^* = 0$, then

$$N(U_X) \supset R(W_X U_X^*). \tag{5.44}$$

$\dim N(U_X)^\perp = 4-k$ see (5.36) and $R(U_X^*) = N(U_X)^\perp$ see Theorem 5.7. Then

$$\dim R(U_X^*) = \dim N(U_X)^\perp = 4-k. \tag{5.45}$$

Since W_X is invertible, then it follows from (5.45) that

$$\dim R(W_X U_X^*) = \dim R(U_X^*) = 4-k. \tag{5.46}$$

But $\dim N(U_X) = 4-k$ see (5.32), thus it follows from (5.44) and (5.46) that $N(U_X) = R(W_X U_X^*)$. Therefore, according (5.43), $T(U)$ is symmetric.

□

5.3.2 Adjoint Operators

The matrix U considered in this subsection is the matrix U defined in (5.16).

Theorem 5.29. *Let A^*_{max} be the adjoint of the maximal operator A_{max}. Let $Z = \begin{pmatrix} z \\ d \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in L_2(0,a)\times\mathbb{C}^k$. Then $Z \in \mathscr{D}(A^*_{max})$ and $A^*_{max}Z = \begin{pmatrix} u \\ v \end{pmatrix}$ if and only if*

1. *$z \in W_4^2(0,a)$, $u = M_{A_0}z$.*

2. *$D^*Z_R + V_1^*d - V_0^*v = 0$.*

Proof. Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$ and $A^*_{max}Z = \begin{pmatrix} u \\ v \end{pmatrix}$.

($\Longrightarrow$)

1.

$$\begin{aligned} A_{max} \; &: \; L_2(0,a)\times\mathbb{C}^k \to L_2(0,a)\times\mathbb{C}^k \\ & W_4^2(0,a)\times\mathbb{C}^k \ni Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix} \to \begin{pmatrix} M_{A_0} y \\ A_1 y \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^*_{max} \; &: \; L_2(0,a)\times\mathbb{C}^k \to L_2(0,a)\times\mathbb{C}^k \\ & Z = \begin{pmatrix} z \\ d \end{pmatrix} \to A^*_{max} Z = \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

where $u \in L_2(0,a)$ and $v \in \mathbb{C}^k$.

Let $Y = \begin{pmatrix} y \\ 0 \end{pmatrix}$, where $y \in C_0^\infty(0,a)$, then $Y \in \mathscr{D}(A_{max})$. Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$. Then

$$\begin{aligned} (M_{A_0} y, z) &= \langle A_{max} Y, Z\rangle \\ &= \langle Y, A^*_{max} Z\rangle \\ &= (y,u)_{C_0^\infty(0,a)}. \end{aligned}$$

Since $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$, then according to Proposition 3.5

$$z \in W_4^2(0,a) \text{ and } u = M_{A_0} z. \tag{5.47}$$

Let $Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix} \in \mathscr{D}(A_{max})$, then

$$\begin{aligned} \langle A_{max} Y, Z\rangle - \langle Y, A^*_{max} Z\rangle &= (M_{A_0} y, z) + (V_1 Y_R, d) - (y, M_{A_0} z) - (V_0 Y_R, v) \\ &= (M_{A_0} y, z) - (y, M_{A_0} z) + (V_1 Y_R, d) - (V_0 Y_R, v) \end{aligned}$$

But $(M_{A_0} y, z) - (y, M_{A_0} z) = Z_R^* D Y_R$, see (4.38). So

$$\begin{aligned} \langle A_{max} Y, Z\rangle - \langle Y, A^*_{max} Z\rangle &= Z_R^* D Y_R + d^* V_1 Y_R - v^* V_0 Y_R \\ &= (Z_R^* D + d^* V_1 - v^* V_0) Y_R \end{aligned}$$

Since A^*_{max} is the adjoint of A_{max}, then $\langle A_{max}Y, Z\rangle - \langle Y, A^*_{max}Z\rangle = 0$ for all $Y \in \mathscr{D}(A_{max})$. So

$$(Z^*_R D + d^*V_1 - v^*V_0)Y_R = 0 \text{ for all } Y \in \mathscr{D}(A_{max}).$$

Then

$$Y^*_R(D^*Z_R + V_1^*d - V_0^*v) = 0 \quad \text{for all } Y \in \mathscr{D}(A_{max}). \tag{5.48}$$

But, for all $c \in \mathbb{C}^8$, there exists a polynomial $y \in W_4^2(0,a)$, such that $Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix} \in \mathscr{D}(A_{max})$ and

$$c = Y_R. \tag{5.49}$$

Thus, it follows from (5.48) and (5.49) that

$$c^*(D^*Z_R + V_1^*d - V_0^*v) = 0 \quad \text{for all } \ c \in \mathbb{C}^8.$$

Therefore

$$D^*Z_R + V_1^*d - V_0^*v = 0. \tag{5.50}$$

($\Longleftarrow$)

Let $z \in W_4^2(0,a)$, $u = M_{A_0}z$ and $D^*Z_R + V_1^*d - V_0^*v = 0$. Since $D^*Z_R + V_1^*d - V_0^*v = 0$, then $Y^*_R(D^*Z_R + V_1^*d - V_0^*v) = 0$ for all $Y \in \mathscr{D}(A_{max})$. So $(Z^*_R D + d^*V_1 - v^*V_0)Y_R = 0$ for all $Y \in \mathscr{D}(A_{max})$.

But

$$\begin{aligned}
(Z^*_R D + d^*V_1 - v^*V_0)Y_R &= Z^*_R D Y_R + d^*V_1Y_R - v^*V_0Y_R \\
&= (M_{A_0}y, z) - (y, M_{A_0}z) + (V_1Y_R, d) - (V_0Y_R, v) \\
&= (M_{A_0}y, z) + (V_1Y_R, d) - (y, M_{A_0}z) - (V_0Y_R, v) \\
&= (M_{A_0}y, z) + (V_1Y_R, d) - (y, u) - (V_0Y_R, v) \\
&= \langle A_{max}Y, Z\rangle - \langle Y, V\rangle
\end{aligned}$$

where

$$V = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since $D^* Z_R + V_1^* d - V_0^* v = 0$, then $\langle A_{max} Y, Z \rangle - \langle Y, V \rangle = 0$. Thus $Z \in \mathscr{D}(A_{max}^*)$ and $V = A_{max}^* Z$. □

Proposition 5.30. *Let V_0 and V_1 be two $k \times 8$ matrices of* rank k*. Then*

1. *The maps*

$$V_0^* \; : \; \mathbb{C}^k \to \mathbb{C}^8$$
$$v \to V_0^* v$$

and

$$V_1^* \; : \; \mathbb{C}^k \to \mathbb{C}^8$$
$$d \to V_1^* d$$

are injective.

2. $R(V_0^*) \cap R(V_1^*) = \{0\}$.

Proof. 1. Since V_0 and V_1 are two $k \times 8$ matrices of rank $k \neq 0$, then V_0^* and V_1^* are two $8 \times k$ matrices of rank $k \neq 0$. The maps

$$V_0^* \; : \; \mathbb{C}^k \to \mathbb{C}^8$$
$$v \to V_0^* v$$

and

$$V_1^* \; : \; \mathbb{C}^k \to \mathbb{C}^8$$
$$d \to V_1^* d$$

are two linear maps. Since V_0^* and V_1^* are linear and $\operatorname{rank} V_0^* = \operatorname{rank} V_1^* = k$, then

$$\dim R(V_0^*) = \dim R(V_1^*) = k. \tag{5.51}$$

Thus V_0^* and V_1^* are injective.

2. Since V_0 and V_1 are two $k \times 8$ matrices of rank k, then V_0^* and V_1^* are two $8 \times k$ matrices of rank k. Thus $\dim(R(V_0^*) + R(V_1^*)) \leq 2k$. But $R(V_0^* V_1 - V_1^* V_0) \subset R(V_0^*) + R(V_1^*)$. So $\dim R(V_0^* V_1 - V_1^* V_0) \leq \dim(R(V_0^*) + R(V_1^*)) \leq 2k$. According to Proposition 5.12, $V_0^* V_1 - V_1^* V_0$ is a 8×8 matrix of rank $2k$, so $\dim R(V_0^* V_1 - V_1^* V_0) = 2k$. Thus $2k \leq \dim(R(V_0^*) + R(V_1^*)) \leq 2k$, then $\dim(R(V_0^*) + R(V_1^*)) = \dim R(V_0^*) + \dim R(V_1^*) = 2k$. Hence $R(V_0^*) \cap R(V_1^*) = \{0\}$.

□

Proposition 5.31. *For all* $v \in \mathbb{C}^k$ *and* $d \in \mathbb{C}^k$ *there is* $z \in W_4^2(0,a)$ *such that* $-DZ_R + V_1^* d - V_0^* v = 0$.

Proof. For all $v \in \mathbb{C}^k$ and $d \in \mathbb{C}^k$, $V_0^* v \in \mathbb{C}^8$ and $V_1^* d \in \mathbb{C}^8$. So $V_0^* v - V_1^* d \in \mathbb{C}^8$. Since D is an 8×8 matrix, then $D(V_0^* v - V_1^* d) \in \mathbb{C}^8$. Thus there exists a polynomial $z \in W_4^2(0,a)$ such that

$$Z_R = D(V_0^* v - V_1^* d). \tag{5.52}$$

Since D is invertible and $D^{-1} = -D$, it follows from (5.52) that

$$-DZ_R + V_1^* d - V_0^* v = 0. \tag{5.53}$$

□

Proposition 5.32. A_{max} *is densely defined.*

Proof. Let $W = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}^k$, such that $\langle Y, W \rangle = 0$, for all $Y \in \mathscr{D}(A_{max})$. Let $y \in C_0^\infty(0,a)$. Then $Y = \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathscr{D}(A_{max})$ and $0 = \langle Y, W \rangle = (y, w)_{C_0^\infty(0,a)}$. So $w = 0$.

Let $c = (c_i)_{i\in\Theta_1}$. There exists a polynomial y_i such that $y_i^{[q_i]}(\theta_i) \neq 0$ and $y_i^{[q_j]}(\theta_j) = 0$ for $j \in \Theta_1,\ j \neq i$, where $0 \leq q_i \leq 3$ and $0 \leq q_j \leq 3$. Then $Y \in \mathscr{D}(A_{max})$ and

$$\begin{aligned}\langle Y, W\rangle &= (V_0 Y_R, c)\\ &= c^* V_0 Y_R\\ &= c_i y_i^{[q_i]}(\theta_i).\end{aligned}$$

Since $\langle Y, W\rangle = 0$, then $c_i y_i^{[q_i]}(\theta_i) = 0$. As $y_i^{[q_i]}(\theta_i) \neq 0$, then $c_i = 0$. Since $c_i = 0$ for all $i \in \Theta_1$, then $c = 0$. As $w = 0$, then $W = \begin{pmatrix} w \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So $\mathscr{D}(A_{max})^{\perp} = \{0\}$. Therefore A_{max} is densely defined. □

Corollary 5.33. *A^*_{max} is a closed operator.*

Proof. Since A_{max} is densely defined, then according to Theorem 5.6 A^*_{max} is a closed operator. □

Proposition 5.34. *A^*_{max} is densely defined.*

Proof. Let $W = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}^k$, such that $\langle Z, W\rangle = 0$ for all $Z \in \mathscr{D}(A^*_{max})$.

Let $z \in C_0^{\infty}(0,a)$. Then $Z = \begin{pmatrix} z \\ 0 \end{pmatrix} \in \mathscr{D}(A^*_{max})$ and $0 = \langle Z, W\rangle = (z, w)_{C_0^{\infty}(0,a)}$. Therefore

$$w = 0. \tag{5.54}$$

Let $d \in \mathbb{C}^k$. Then according to Proposition 5.31, for all $v \in \mathbb{C}^k$, there exists $z \in W_4^2(0,a)$ such that $D^* Z_R + V_1^* d - V_0^* v = 0$ since $D^* = -D$.

In particular, for all $d \in \mathbb{C}^k$ and $v = 0$ there exists a polynomial z such that $\begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$. Then

$$\left\{ d : \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max}) \right\} = \mathbb{C}^k. \tag{5.55}$$

But, since $w = 0$ see (5.54), then

$$0 = \langle Z, W \rangle = (c, d) = c^* d \tag{5.56}$$

for all $d \in \mathbb{C}^k$, such that there exists z and $\begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$.

Whence (5.55) and (5.56) imply

$$c^* d = 0 \text{ for all } d \in \mathbb{C}^k. \tag{5.57}$$

Thus

$$c = 0. \tag{5.58}$$

Then (5.54) and (5.58) give $W = 0$. Therefore A^*_{max} is densely defined. □

Proposition 5.35. *Let* $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^{**}_{max})$, *then* $z \in W^2_4(0, a)$ *and there exists* $c \in \mathbb{C}^k$, *such that* $A^{**}_{max} Z = \begin{pmatrix} M_{A_0} z \\ c \end{pmatrix}$.

Proof. Since $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^{**}_{max})$, then according to Definition 2.15, there exists $W = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0, a) \oplus \mathbb{C}^k$ such that

$$\langle A^*_{max} Y, Z \rangle = \langle Y, W \rangle \text{ for all } Y \in \mathscr{D}(A^*_{max}). \tag{5.59}$$

Let $Y = \begin{pmatrix} y \\ 0 \end{pmatrix} \in C^\infty_0(0, a) \oplus \{0\}$. Then $Y \in \mathscr{D}(A^*_{max})$, so

$$\begin{aligned} \langle A^*_{max} Y, Z \rangle &= (M_{A_0} y, z) \\ &= (y^{(4)} - (g y')', z). \end{aligned} \tag{5.60}$$

Since $z \in L_2(0, a)$, then according to (3.11), (5.60) gives

$$(y^{(4)} - (g y')', z) = (y, z^{(4)} - (g z')')_{C^\infty_0(0,a)}. \tag{5.61}$$

As $Y = \begin{pmatrix} y \\ 0 \end{pmatrix} \in C_0^\infty(0,a)$, then

$$\langle Y, W\rangle = (y, w)_{C_0^\infty(0,a)}. \tag{5.62}$$

Thus (5.61) and (5.62) imply

$$z^{(4)} - (gz')' = w \in L_2(0,a). \tag{5.63}$$

It follows from (3.12), (3.17) and (5.63) that $z \in W_4^2(0,a)$. Thus $w = M_{A_0}z$. □

Proposition 5.36. *The differential operator A_{max} is closed.*

Proof. Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A_{max}^{**})$, such that $A_{max}^{**}Z = \begin{pmatrix} p \\ q \end{pmatrix}$. Then according to Proposition 5.35, $z \in W_4^2(0,a)$ and $p = M_{A_0}z$.

Let $Y = \begin{pmatrix} y \\ c \end{pmatrix} \in \mathscr{D}(A_{max}^{*})$ such that $A_{max}^{*}Y = \begin{pmatrix} u \\ v \end{pmatrix}$, then, according to Theorem 5.29,

$$u = M_{A_0}y \text{ and } D^*Y_R + V_1^*c - V_0^*v = 0. \tag{5.64}$$

So

$$\begin{aligned} \langle A_{max}^{*}Y, Z\rangle - \langle Y, A_{max}^{**}Z\rangle &= (M_{A_0}y, z) + (v, d) - (y, M_{A_0}z) - (c, q) \\ &= (M_{A_0}y, z) - (y, M_{A_0}z) + (v, d) - (c, q). \end{aligned} \tag{5.65}$$

But $(M_{A_0}y, z) - (y, M_{A_0}z) = Z_R^*DY_R$, see (4.38). Thus (5.65) gives

$$\langle A_{max}^{*}Y, Z\rangle - \langle Y, A_{max}^{**}Z\rangle = Z_R^*DY_R + d^*v - q^*c. \tag{5.66}$$

Since A_{max}^{**} is the adjoint of A_{max}^{*}, then

$$\langle A_{max}^{*}Y, Z\rangle - \langle Y, A_{max}^{**}Z\rangle = 0.$$

Thus

$$Z_R^*DY_R + d^*v - q^*c = 0. \tag{5.67}$$

But (5.64) gives

$$Z_R^* D^* Y_R + Z_R^* V_1^* c - Z_R^* V_0^* v = 0 \iff -Z_R^* D Y_R + Z_R^* V_1^* c - Z_R^* V_0^* v = 0. \tag{5.68}$$

Then (5.67) and (5.68) imply

$$(Z_R^* V_1^* - q^*)c - (Z_R^* V_0^* - d^*)v = 0. \tag{5.69}$$

for all $c,\ v \in \mathbb{C}^k$.

For $v = 0$, (5.69) gives

$$(Z_R^* V_1^* - q^*)c = 0 \tag{5.70}$$

for all $c \in \mathbb{C}^k$. Thus, it follows from (5.70) that

$$q = V_1 Z_R. \tag{5.71}$$

For $c = 0$, (5.69) gives

$$(Z_R^* V_0^* - d^*)v = 0 \tag{5.72}$$

for all $v \in \mathbb{C}^k$. Hence (5.72) gives

$$d = V_0 Z_R \tag{5.73}$$

It follows from (5.71) and (5.73) that

$$Z \in \mathscr{D}(A_{max}) \text{ and } A_{max}^{**} Z = A_{max} Z. \tag{5.74}$$

Since (5.74) holds for all $Z \in \mathscr{D}(A_{max}^{**})$, then

$$A_{max}^{**} \subset A_{max}. \tag{5.75}$$

Since A_{max}^* is densely defined (see Propositon 5.34) and A_{max} is densely defined (see Proposition 5.32), then according to Theorem 5.6 A_{max}^{**} is an extension of A_{max}. So

$$A_{max} \subset A_{max}^{**}. \tag{5.76}$$

Thus (5.75) and (5.76) give

$$A_{max} = A^{**}_{max}. \tag{5.77}$$

Since A^*_{max} is densely defined (see Proposition 5.34), then according to Theorem 5.6 A^{**}_{max} is closed. Therefore (5.77) imply that A_{max} is closed. □

Proposition 5.37. *Assume that* $\mathscr{D}(A^*_{max}) \subset \mathscr{D}(A_{max})$. *Then* $\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A^*_{max}) = 8-2k$.

Proof. The map

$$\begin{aligned} V_0^* \;:\; & \mathbb{C}^k \to \mathbb{C}^8 \\ & v \to V_0^* v \end{aligned}$$

is injective (see Proposition 5.30).

$R(V_0^*) \oplus R(V_0^*)^\perp = \mathbb{C}^8$. So $\dim R(V_0^*) + \dim R(V_0^*)^\perp = \dim \mathbb{C}^8 = 8$. Hence $\operatorname{codim} R(V_0^*) = \dim R(V_0^*)^\perp = 8 - \dim R(V_0^*) = 8 - k$.

According to Proposition 5.31, for all $v \in \mathbb{C}^k$ and $d \in \mathbb{C}^k$, there is $z \in W_4^2(0,a)$ such that $-DZ_R + V_1^* d - V_0^* v = 0$. So for all $d \in \mathbb{C}^k$ and for all $v \in \mathbb{C}^k$, there exists $z \in W_4^2(0,a)$, such that $V_0^* v = -DZ_R + V_1^* d$. Let h be the map such that

$$\begin{aligned} h \;:\; & \mathscr{D}(A^*_{max}) \longrightarrow \mathbb{C}^8 \\ & Z = \begin{pmatrix} z \\ d \end{pmatrix} \to -DZ_R + V_1^* d. \end{aligned}$$

Since D is invertible, then the map

$$\begin{aligned} & \mathscr{D}(A^*_{max}) \longrightarrow \mathbb{C}^8 \\ & Z = \begin{pmatrix} z \\ d \end{pmatrix} \to -DZ_R \end{aligned}$$

is surjective. Therefore h is surjective.

Whence

$$\operatorname{codim}_{W_4^2(0,a)\times\mathbb{C}^k} \mathscr{D}(A^*_{max}) = \operatorname{codim}_{\mathbb{C}^8} R(V_0^*) = 8 - k. \tag{5.78}$$

We recall that $\mathscr{D}(A_{max}) = \left\{ Y = \begin{pmatrix} y \\ D_0 y \end{pmatrix}, y \in W_4^2(0,a) \right\}$ where $D_0 y = V_0 Y_R$, with V_0 a $k \times 8$ matrix and Y_R a 8×1 matrix defined by $Y_R = \begin{pmatrix} Y(0) \\ Y(a) \end{pmatrix}$, with

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{pmatrix}.$$

So $D_0 y$ is a $k \times 1$ matrix defined only by y, where $y \in W_4^2(0,a)$. Thus $\mathscr{D}(A_{max}) \oplus (\{0\} \times \mathbb{C}^k) = W_4^2(0,a) \times \mathbb{C}^k$. Therefore

$$\operatorname{codim}_{W_4^2(0,a)\times\mathbb{C}^k} \mathscr{D}(A_{max}) = \dim(\{0\} \times \mathbb{C}^k) = k. \tag{5.79}$$

Then (5.78) and (5.79) give $\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A^*_{max}) = 8 - k - k = 8 - 2k$. □

Theorem 5.38. *Let A^*_{max} be the adjoint of the maximal operator A_{max} and $W = (D + (V_0^* V_1 - V_1^* V_0))$. Then the followings are equivalent*

1. *A^*_{max} is symmetric,*
2. *$\operatorname{rank} W = 8 - 2k$,*
3. *$D(N(W)) \supset R(D - W)$.*

Proof. 1. ($1 \Longrightarrow 2$) Let A^*_{max} be symmetric.

Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A_{max})$. Then according to Theorem5.29

$$Z \in \mathscr{D}(A^*_{max}) \text{ if and only if } \; D^* Z_R + V_1^* d - V_0^* v = 0,$$

where $A^*_{max}Z = \begin{pmatrix} u \\ v \end{pmatrix}$. Since A^*_{max} is symmetric, then $\mathscr{D}(A^*_{max}) \subset \mathscr{D}(A_{max})$ and $A^*_{max}Z = A_{max}Z$. So $d = V_0Z_R$ and $v = V_1Z_R$. Then

$$\begin{aligned} Z \in \mathscr{D}(A^*_{max}) &\iff D^*Z_R + V_1^*V_0Z_R - V_0^*V_1Z_R = 0 \\ &\iff (-D + V_1^*V_0 - V_0^*V_1)Z_R = 0 \\ &\iff -WZ_R = 0 \\ &\iff Z_R \in N(W). \end{aligned}$$

But according to Theorem 5.7 $N(W) = R(W)^\perp$. As $R(W)^\perp \oplus R(W) = \mathbb{C}^8$, then $N(W) \oplus R(W) = \mathbb{C}^8$. As the map

$$\begin{aligned} \mathscr{D}(A_{max}) &\longrightarrow \mathbb{C}^8 \\ Z &\longrightarrow Z_R \end{aligned}$$

is surjective then $\operatorname{codim}_{\mathbb{C}^8} N(W) = \operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A^*_{max}) = 8 - 2k$, see Proposition 5.37. But $\operatorname{codim}_{\mathbb{C}^8} N(W) = \operatorname{rank} W$. Therefore $\operatorname{rank} W = 8 - 2k$. Whence if A^*_{max} is symmetric, then $\operatorname{rank} W = 8 - 2k$.

2. $(2 \Longrightarrow 3)$ Let $\operatorname{rank} W = 8 - 2k$.

   Let $B = D - W = -(V_0^*V_1 - V_1^*V_0)$. Write

$$B : R(B) \oplus R(B)^\perp \to R(B) \oplus R(B)^\perp,$$

   where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

   and

$$\mathbb{C}^8 = R(B) \oplus R(B)^\perp.$$

   According to Theorem 5.7 $R(B)^\perp = N(B^*)$. But $B^* = -(V_0^*V_1 - V_1^*V_0)^* = -(V_0^*V_1 - V_1^*V_0) = -B$. So $R(B)^\perp = N(-B) = N(B)$. Let $\begin{pmatrix} u, v \end{pmatrix} \in R(B) \oplus R(B)^\perp$. Then

$B\begin{pmatrix}0\\v\end{pmatrix}=\begin{pmatrix}B_{12}v\\B_{22}v\end{pmatrix}$. As $v\in N(B)$, then $B_{12}v=0$ and $B_{22}v=0$, thus $B_{12}=0$ and $B_{22}=0$. So $B=\begin{pmatrix}B_{11}&0\\B_{21}&0\end{pmatrix}$. Therefore $B\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}B_{11}u\\B_{21}u\end{pmatrix}$. But $B\begin{pmatrix}u\\v\end{pmatrix}\in R(B)$, so $B_{21}u\in R(B)\cap R(B)^{\perp}=\{0\}$. Thus $B_{21}u=0$. As $u\in R(B)$ was arbitrary, then $B_{21}=0$. Therefore $B=\begin{pmatrix}B_{11}&0\\0&0\end{pmatrix}$.

Let $D=\begin{pmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{pmatrix}$. Then

$$\begin{aligned}WD^{-1}&=\left(\begin{pmatrix}I&0\\0&I\end{pmatrix}+\begin{pmatrix}B_{11}&0\\0&0\end{pmatrix}\begin{pmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{pmatrix}\right)\\&=\begin{pmatrix}I+B_{11}D_{11}&B_{11}D_{12}\\0&I\end{pmatrix},\end{aligned}$$

where I is the identity matrix in the corresponding spaces.

As $\operatorname{rank}W=8-2k$ and D is invertible, then $\operatorname{rank}WD^{-1}=8-2k$. Since $\operatorname{rank}B=2k$, then $\dim R(B)^{\perp}=8-2k$. Thus $\operatorname{rank}I_{R(B)^{\perp}}=8-2k$. As $\operatorname{rank}WD^{-1}=8-2k$ and $\operatorname{rank}I_{R(B)^{\perp}}=8-2k$, then $I+B_{11}D_{11}=0$. So $WD^{-1}=\begin{pmatrix}0&B_{11}D_{12}\\0&I\end{pmatrix}$. Thus

$$\begin{aligned}WD^{-1}(D-W)=WD^{-1}B&=\begin{pmatrix}0&B_{11}D_{12}\\0&I\end{pmatrix}\begin{pmatrix}B_{11}&0\\0&0\end{pmatrix}\\&=\begin{pmatrix}0&0\\0&0\end{pmatrix}.\end{aligned}$$

Therefore $WD^{-1}(D-W)=0$.

But

$$\begin{aligned} WD^{-1}(D-W)=0 &\iff N(W)\supset R(D^{-1}(D-W)) \\ &\iff N(W)\supset D^{-1}R(D-W) \\ &\iff D(N(W))\supset R(D-W). \end{aligned}$$

Hence

$$D(N(W))\supset R(D-W).$$

3. $(3 \implies 1)$

Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*_{max})$ such $A^*_{max}Z = \begin{pmatrix} u \\ v \end{pmatrix}$. Then according to Theorem 5.29, $z \in W_4^2(0,a)$, $u = M_{A_0}z$ and $D^*Z_R + V_1^*d - V_0^*v = 0$. So

$$DZ_R = V_1^*d - V_0^*v \text{ since } D^* = -D. \tag{5.80}$$

Let

$$\begin{aligned} \hat{B} \ : \ & \mathbb{C}^k \oplus \mathbb{C}^k \to \mathbb{C}^8 \\ & \begin{pmatrix} x \\ y \end{pmatrix} \to \hat{B}\begin{pmatrix} x \\ y \end{pmatrix} = V_1^*x - V_0^*y. \end{aligned}$$

Since $\operatorname{rank} V_0^* = \operatorname{rank} V_1^* = k$, then

$$\dim R(\hat{B}) \le 2k. \tag{5.81}$$

Let $w \in \mathbb{C}^8$, $x = V_0w$ and $y = V_1w$. Then $Bw = V_1^*V_0w - V_0^*V_1w = V_1^*x - V_0^*y \in R(\hat{B})$. So

$$R(B)\subset R(\hat{B}). \tag{5.82}$$

As $\dim R(B) = 2k$, it follows from (5.81) and (5.82) that $R(B) = R(\hat{B})$. Since $d,\ v \in \mathbb{C}^k$, then

$$V_0^*v - V_1^*d \in R(B). \tag{5.83}$$

Whence (5.80) implies that

$$Z_R = D(V_0^* v - V_1^* d) \in DR(B). \tag{5.84}$$

Since $D(N(W)) \supset R(B)$, then $N(W) \supset D^{-1}R(B) = -DR(B)$. Thus

$$N(W) \supset DR(B). \tag{5.85}$$

Whence (5.84) and (5.85) imply that $WZ_R = 0$. So $DZ_R + V_0^* V_1 Z_R - V_1^* V_0 Z_R = 0$. Then

$$DZ_R = V_1^* V_0 Z_R - V_0^* V_1 Z_R. \tag{5.86}$$

It follows from (5.80) and (5.86) that $V_1^* d - V_0^* v = V_1^* V_0 Z_R - V_0^* V_1 Z_R$. So

$$V_1^*(d - V_0 Z_R) = V_0^*(v - V_1 Z_R) \in R(V_1^*) \cap R(V_0^*) = \{0\} \tag{5.87}$$

see (2) of Proposition 5.30. Thus

$$V_1^*(d - V_1 Z_R) = 0 \text{ and } V_0^*(v - V_1 Z_R) = 0. \tag{5.88}$$

As V_1^* and V_0^* are injective see (1) of Proposition 5.30, then

$$d = V_0 Z_R \text{ and } v = V_1 Z_R. \tag{5.89}$$

Since $z \in W_4^2(0,a)$ and $u = M_{A_0} z$, then $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A_{max})$ and $A_{max}^* Z = A_{max} Z$. Therefore A_{max}^* is symmetric.

□

Corollary 5.39. *Let A_{max}^* be the adjoint of the maximal operator A_{max}. If* $\operatorname{rank} W = 8 - 2k$, *then $A_{max}^* = A_{min}$ and $A_{min}^* = A_{max}$ where A_{min} denotes the minimal operator.*

Proof. If $\operatorname{rank} W = 8 - 2k$, then according to Theorem 5.38 A_{max}^* is symmetric. Since A_{max}^* is symmetric, then according to Proposition 4.10 $A_{max}^* = A_{min}$ and $A_{max} = A_{min}^*$. □

Corollary 5.40. *Let* $\operatorname{rank} W = 8 - 2k$ *and* U *be the matrix defined in* (5.16). *Then the differential operator* $T(U)$ *is closed.*

Proof. A^*_{max} is a closed operator, see Proposition 5.36. A_{max} is a closed operator, see Corollary 5.33. Since $\operatorname{rank} W = 8-2k$, then A^*_{max} is symmetric see Theorem 5.38. Then $A^*_{max} \subset T(U) \subset A_{max}$ see (4.19) and Corollary 5.39. Thus $\operatorname{codim}_{\mathscr{D}(T(U))} \mathscr{D}(A^*_{max}) \leq \operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A^*_{max})$. But $\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A^*_{max}) = 8-2k$, see Proposition 5.37. So $\operatorname{codim}_{\mathscr{D}(T(U))} \mathscr{D}(A^*_{max}) \leq 8-2k$. Then $\operatorname{codim}_{\mathscr{D}(T(U))} \mathscr{D}(A^*_{max}) < \infty$. Since A^*_{max} is a closed operator and $\operatorname{codim}_{\mathscr{D}(T(U))} \mathscr{D}(A^*_{max}) < \infty$, then $T(U)$ is a closed operator. □

Theorem 5.41. *Assume that* $\operatorname{rank}(W) = 2(4-k)$ *where* $1 \leq k \leq 4$, $X = (N(W))^{\perp}$, $W_X = p_X W i_X$ *where* $i_X : X \hookrightarrow \mathbb{C}^8$ *and* $p_X : \mathbb{C}^8 \to X$ *are respectively the canonical injection of* X *into* $\mathbb{C}^8$ *and the orthogonal projection of* $\mathbb{C}^8$ *onto* X. *Let* U *be the matrix of* rank l *defined in* (5.16), $U_X = U i_X$. *Then the differential operator* $T(U)$ *is self-adjoint if and only if and* $U_X W_X U_X^* = 0$.

Proof.

1. Let $k = 4$. Then $l = 0$ and therefore $T(U)$ is self-adjoint, see (1) of the proof of Theorem 5.23.

2. Let $1 \leq k \leq 3$. Since $\operatorname{rank}(W) = 2(4-k)$, then $A^*_{max} = A_{min}$, see Corollary 5.39. Thus the differential operator $T(U)$ is a closed operator with $A_{min} \subset T(U) \subset A_{max}$, see Corollary 5.40.

   As $\operatorname{rank} W = 8 - 2k$, then

$$\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A_{min}) = 8 - 2k \tag{5.90}$$

   see Proposition 5.37.

   $(\Longrightarrow)$

   If $T(U)$ is self-adjoint, then $T(U)$ is an $l = 4-k$ dimensional extension of A_{min}, see (2) of Proposition 4.11. Also $T(U)$ is symmetric, so $U_X W_X U_X^* = 0$ see Theorem 5.23. Therefore, if $T(U)$ is self-adjoint, then $U_X W_X U_X^* = 0$.

($\Longleftarrow$) If $U_X W_X U_X^* = 0$, then, according to Theorem 5.23, $T(U)$ is symmetric.

Note that for all $Y \in \mathscr{D}(A_{max})$, $Y \in \mathscr{D}(T(U))$ if and only if $Y_R \in N(U)$, where $Y_R = \begin{pmatrix} Y(0) \\ Y(a) \end{pmatrix} \in \mathbb{C}^8$, see Remark 5.16 . But $\dim N(U) = 4 + k$, see (5.20). So $\operatorname{codim}_{\mathbb{C}^8} N(U) = 8 - (4 + k) = 4 - k$. As the map

$$\mathscr{D}(A_{max}) \longrightarrow \mathbb{C}^8$$
$$Z \longrightarrow Z_R$$

is surjective, then

$$\begin{aligned} \operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(T(U)) &= \operatorname{codim}_{\mathbb{C}^8} N(U) \\ &= 4 - k. \end{aligned}$$

Hence it follows from (5.90) that $\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(T(U)) = 4 - k = \frac{\operatorname{codim}_{\mathscr{D}(A_{max})} \mathscr{D}(A_{min})}{2}$. Whence if $l = 4 - k$ and $U_X W_X U_X^* = 0$, then $T(U)$ is an $l = 4 - k$ symmetric extension of A_{min}. Therefore according to (3) of Proposition 4.11 $T(U)$ is self-adjoint.

$\square$

5.3.3 Characterization of self-adjoint boundary conditions depending on λ

Remark 5.42. It follows from (1) of the proof of Theorem 5.41 that if $l = 0$, then $T(U)$ is self-adjoint.

Remark 5.43. Let $\Psi_0 = \{1, \cdots, 8\} \backslash \{\widetilde{p}_i, \widetilde{q}_i \ : i \in \Theta_1\}$. For $i = 1, \cdots, l$, $\hat{p}_{\sigma_i}$ is the position of $\widetilde{p}_{\sigma_i}$ in Ψ_0 if Ψ_0 is ordered in increasing order. Then $U_X = (\delta_{\hat{p}_{\sigma_i}, j})$ where $i = 1, \cdots, l$ and $j = 1, \cdots, 2l$. Thus

$$U_X^* = (\delta_{i, \hat{p}_{\sigma_j}}) \text{ where } i = 1, \cdots, 2l \text{ and } j = 1, \cdots, l \tag{5.91}$$

Recall that

$$W_X = \sum_{s=1}^{l} (\epsilon_{\sigma_s} \delta_{i,r_s} \delta_{t_s,j} - \epsilon_{\sigma_s} \delta_{i,t_s} \delta_{r_s,j})_{i,j=1}^{8-2k}, \tag{5.92}$$

where ϵ_{σ_s} is as defined in Proposition 5.14.

Then

$$\begin{aligned} U_X W_X U_X^* &= \sum_{i=1}^{2l} \sum_{j=1}^{2l} \sum_{s=1}^{l} (\epsilon_{\sigma_s} (\delta_{\hat{p}_{\sigma_u},i} (\delta_{i,r_s} \delta_{t_s,j} - \delta_{i,t_s} \delta_{r_s,j}) \delta_{j,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} \\ &= (\sum_{i=1}^{2l} \sum_{j=1}^{2l} \sum_{s=1}^{l} \epsilon_{\sigma_s} \delta_{\hat{p}_{\sigma_u},i} \delta_{i,r_s} \delta_{t_s,j} \delta_{j,\hat{p}_{\sigma_v}} - \sum_{i=1}^{2l} \sum_{j=1}^{2l} \sum_{s=1}^{l} \epsilon_{\sigma_s} \delta_{\hat{p}_{\sigma_u},i} \delta_{i,t_s} \delta_{r_s,j} \delta_{j,\hat{p}_{\sigma_v}})_{u,v=1}^{l} \\ &= (\sum_{s=1}^{l} \epsilon_{\sigma_s} \delta_{\hat{p}_{\sigma_u},r_s} \delta_{t_s,\hat{p}_{\sigma_v}} - \sum_{s=1}^{l} \epsilon_{\sigma_s} \delta_{\hat{p}_{\sigma_u},t_s} \delta_{r_s,\hat{p}_{\sigma_v}})_{u,v=1}^{l} \\ &= \sum_{s=1}^{l} (\epsilon_{\sigma_s} (\delta_{\hat{p}_{\sigma_u},r_s} \delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s} \delta_{r_s,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} \end{aligned} \tag{5.93}$$

since

$$\sum_{i=1}^{2l} \delta_{\hat{p}_{\sigma_u},i} \delta_{i,r_s} = \delta_{\hat{p}_{\sigma_u},r_s} \text{ and } \sum_{j=1}^{2l} \delta_{r_s,j} \delta_{j,\hat{p}_{\sigma_v}} = \delta_{r_s,\hat{p}_{\sigma_v}}.$$

Corollary 5.44. *Assume that* $\operatorname{rank}(W) = 2(4-k)$ *where* $1 \le k \le 3$, $X = (N(W))^{\perp}$, $W_X = p_X W i_X$ *where* $i_X : X \hookrightarrow \mathbb{C}^8$ *and* $p_X : \mathbb{C}^8 \to X$ *are respectively the canonical injection of* X *into* $\mathbb{C}^8$ *and the orthogonal projection of* $\mathbb{C}^8$ *onto* X*. Let* U *be the matrix of* rank l *defined in* (5.16), $U_X = U i_X$ *. Then the differential operator* $T(U)$ *is self-adjoint if and only if*

$$\sum_{s=1}^{l} (\epsilon_{\sigma_s} (\delta_{\hat{p}_{\sigma_u},r_s} \delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s} \delta_{r_s,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} = 0. \tag{5.94}$$

Proof. It follows from (5.93) and Theorem 5.41 that $T(U)$ is self-adjoint if and only if

$$\sum_{s=1}^{l} (\epsilon_{\sigma_s} (\delta_{\hat{p}_{\sigma_u},r_s} \delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s} \delta_{r_s,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} = 0$$

. □

Corollary 5.45. *Assume that* rank$(W) = 2(4-k)$ *where* $1 \leq k \leq 3$, $X = (N(W))^{\perp}$, $W_X = p_X W i_X$ *where* $i_X : X \hookrightarrow \mathbb{C}^8$ *and* $p_X : \mathbb{C}^8 \to X$ *are respectively the canonical injection of* X *into* $\mathbb{C}^8$ *and the orthogonal projection of* $\mathbb{C}^8$ *onto* X*. Let* U *be the matrix of* rank l *defined in* (5.16)*,* $U_X = U i_X$ *. Then the differential operator* $T(U)$ *is self-adjoint if and only if for all* $i = 1, \cdots, l$

$$\#(\{\hat{p}_{\sigma_j} : j = 1, \cdots, l\} \cap \{r_i, t_i\}) = 1. \tag{5.95}$$

Proof. By assumption, rank $U = l$, see (5.16).

($\Longrightarrow$) If $\sum_{s=1}^{l}(\epsilon_{\sigma_s}(\delta_{\hat{p}_{\sigma_u},r_s}\delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s}\delta_{r_s,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} = 0$, then $\delta_{\hat{p}_{\sigma_u},r_s}\delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s}\delta_{r_s,\hat{p}_{\sigma_v}} = 0$ for all $s = 1, \cdots, l$, $u = 1, \cdots, l$ and $v = 1, \cdots, l$, where the $2l$ numbers r_s, t_s $(s = 1, \cdots, l)$ are pairwise different. So if $\hat{p}_{\sigma_u} = r_s$, then $\hat{p}_{\sigma_v} \neq t_s$. Thus at most one of the number $\hat{p}_{\sigma_j}$, $j = 1, \cdots, l$ is in $\{r_i, t_i\}$, $i = 1, \cdots, l$. Since there are l number of sets $\{r_i, t_i\}$, $i = 1, \cdots, l$ and l number of $\hat{p}_{\sigma_j}$, $j = 1, \cdots, l$, then at leat one number $\hat{p}_{\sigma_j}$, $j = 1, \cdots, l$ is in $\{r_i, t_i\}$, $i = 1, \cdots, l$. Hence $\#(\{\hat{p}_{\sigma_j} : j = 1, \cdots, l\} \cap \{r_i, t_i\}) = 1$.

($\Longleftarrow$) If for all $i = 1, \cdots, l$ $\#(\{\hat{p}_{\sigma_j} : j = 1, \cdots, l\} \cap \{r_i, t_i\}) = 1$, then $\delta_{\hat{p}_{\sigma_j},r_i}\delta_{t_i,\hat{p}_{\sigma_j}} - \delta_{\hat{p}_{\sigma_j},t_i}\delta_{r_i,\hat{p}_{\sigma_j}} = 0$ for all $i = 1, \cdots, l$ and $j = 1, \cdots, l$. So $\delta_{\hat{p}_{\sigma_u},r_i}\delta_{t_i,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_i}\delta_{r_i,\hat{p}_{\sigma_v}} = 0$ for all $i = 1, \cdots, l$, $u = 1, \cdots, l$ and $v = 1, \cdots, l$. Whence

$$\sum_{s=1}^{l}(\epsilon_{\sigma_s}(\delta_{\hat{p}_{\sigma_u},r_s}\delta_{t_s,\hat{p}_{\sigma_v}} - \delta_{\hat{p}_{\sigma_u},t_s}\delta_{r_s,\hat{p}_{\sigma_v}}))_{u,v=1}^{l} = 0.$$

Therefore it follows from Corollary 5.45 that $T(U)$ is self-adjoint if and only if $l = 4-k$ and $\#(\{\hat{p}_{\sigma_j} : j = 1, \cdots, l\} \cap \{r_i, t_i\}) = 1$. □

Theorem 5.46. *Assume that* rank$(W) = 2(4-k)$ *where* $1 \leq k \leq 3$, $X = (N(W))^{\perp}$, $W_X = p_X W i_X$ *where* $i_X : X \hookrightarrow \mathbb{C}^8$ *and* $p_X : \mathbb{C}^8 \to X$ *are respectively the canonical injection of* X *into* $\mathbb{C}^8$ *and the orthogonal projection of* $\mathbb{C}^8$ *onto* X*. Let* U *be the matrix of* rank $l = 4-k$ *defined in* (5.16)*,* $U_X = U i_X$ *.*

1. *If* $l = 1$*, then* $T(U)$ *is self-adjoint.*

2. If $2 \le l \le 3$, then $T(U)$ is self-adjoint if and only if the following are satisfied:

(i) $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{0,3\}) \le 1$ *and* $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{1,2\}) \le 1$

(ii) $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{0,3\}) \le 1$ *and* $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{1,2\}) \le 1.$

Proof. We know that $l = 4 - k$ with $1 \le k \le 3$. Also note that

$$\{0,1,2,3\} \backslash \{p_{\theta_s}, q_{\theta_s}, s \in \Theta_1^{(0)}\} =: \{r_{\theta_s}, t_{\theta_s} : s \in \Theta_0^{(0)}, r_{\theta_s} + t_{\theta_s} = 3\} \tag{5.96}$$

and

$$\{0,1,2,3\} \backslash \{p_{\theta_s}, q_{\theta_s}, s \in \Theta_1^{(a)}\} =: \{r_{\theta_s}, t_{\theta_s} : s \in \Theta_0^{(a)}, r_{\theta_s} + t_{\theta_s} = 3\}. \tag{5.97}$$

It follows from (5.95), (5.96) and (5.97) that

$$\#(\{p_{\sigma_j} : j = 1, \cdots, l\} \cap \{1,2\}) \le 1 \text{ and } \#(\{p_{\sigma_j} : j = 1, \cdots, l\} \cap \{0,3\}) \le 1. \tag{5.98}$$

1. If $l = 1$, then $p_{\sigma_1} = 0$ or $p_{\sigma_1} = 1$ or $p_{\sigma_1} = 2$ or $p_{\sigma_1} = 3$. So $\#(p_{\sigma_1} \cap \{0,3\}) = 1$ or $\#(p_{\sigma_1} \cap \{1,2\}) = 1$. Whence if $l = 1$, then according to Corollary 5.45, $T(U)$ is self-adjoint.

2. If $l = 2$, then $\#\Theta_0^{(0)} = 2$ or $\#\Theta_0^{(a)} = 2$ or $\#\Theta_0^{(0)} = 1$ and $\#\Theta_0^{(a)} = 1$.

   (2i) If $\#\Theta_0^{(0)} = 2$, then it follows from Corollary 5.45 that if $l = 2$, then $T(U)$ is self-adjoint if and only if $\{P_i : i \in \Theta_0^{(0)}\} \neq \{0,3\}$ and $\{P_i : i \in \Theta_0^{(0)}\} \neq \{1,2\}$. Therefore, if $\#\Theta_0^{(0)} = 2$ and $l = 2$, then $T(U)$ is self-adjoint if and only if $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{0,3\}) \le 1$ and $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{1,2\}) \le 1$.

   (2ii) If $\#\Theta_0^{(a)} = 2$, then it follows from Corollary 5.45 that if $l = 2$, then $T(U)$is self-adjoint if and only if $\{P_i : i \in \Theta_0^{(a)}\} \neq \{0,3\}$ and $\{P_i : i \in \Theta_0^{(a)}\} \neq \{1,2\}$. Therefore, if $\#\Theta_0^{(a)} = 2$ and $l = 2$, then $T(U)$ is self-adjoint if and only if $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{0,3\}) \le 1$ and $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{1,2\}) \le 1$.

   (2iii) If $\#\Theta_0^{(0)} = 1$ and $\#\Theta_0^{(a)} = 1$, then $\{P_i : i \in \Theta_0^{(0)}\} \cap \{1,2\} \le 1$ and $\{P_i : i \in \Theta_0^{(0)}\} \cap \{0,3\} \le 1$, $\{P_i : i \in \Theta_0^{(a)}\} \cap \{1,2\} \le 1$ and $\{P_i : i \in \Theta_0^{(a)}\} \cap \{0,3\} \le 1$.

So either $\{P_i : i \in \Theta_0^{(0)}\} \cap \{1,2\} = 1$ or $\{P_i : i \in \Theta_0^{(0)}\} \cap \{0,3\} = 1$ and either $\{P_i : i \in \Theta_0^{(a)}\} \cap \{1,2\} = 1$ or $\{P_i : i \in \Theta_0^{(a)}\} \cap \{0,3\} = 1$. Thus it follows from Corollary 5.45 that if $l = 2$, $\#\Theta_0^{(0)} = 1$ and $\#\Theta_0^{(a)} = 1$, then $T(U)$ is self-adjoint.

3. If $l = 3$, then either $\#\Theta_0^{(0)} = 2$ and $\#\Theta_0^{(a)} = 1$ or $\#\Theta_0^{(0)} = 1$ and $\#\Theta_0^{(a)} = 2$.

   (3i) If $\#\Theta_0^{(0)} = 2$ and $\#\Theta_0^{(a)} = 1$, then it follows from (1) and ($2i$) that $T(U)$ is self-adjoint if and only if $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{0,3\}) \leq 1$ and $\#(\{p_{\sigma_j} : j \in \Theta_0^{(0)}\} \cap \{1,2\}) \leq 1$.

   (3ii) If $\#\Theta_0^{(0)} = 1$ and $\#\Theta_0^{(a)} = 2$, then it follows from (1) and ($2ii$) that $T(U)$ is self-adjoint if and only if $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{0,3\}) \leq 1$ and $\#(\{p_{\sigma_j} : j \in \Theta_0^{(a)}\} \cap \{1,2\}) \leq 1$.

□

Corollary 5.47. *Let U be the matrix of* rank l *defined in* (5.16) *and denote by P_0 the set of p in $y^{[p]}(0) = 0$ for the λ-independent boundary conditions and by P_a the corresponding set for $y^{[p]}(a) = 0$. Then the differential operator $T(U)$ associated with this boundary value problem is self-adjoint if and only if $p+q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\epsilon_j\lambda y^{[q]}(a_j) = 0$ where $\epsilon_j = 1$ if ($q = 0$ and $a_j = 0$) or ($q = 2$ and $a_j = 0$) or ($q = 1$ and $a_j = a$) or ($q = 3$ and $a_j = a$), $\epsilon_j = -1$ otherwise, $\{0,3\} \not\subset P_0$, $\{1,2\} \not\subset P_0$, $\{0,3\} \not\subset P_a$ and $\{1,2\} \not\subset P_a$.*

Proof. $p + q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\epsilon_j\lambda y^{[q]}(a_j) = 0$ where $\epsilon_j = 1$ if ($q = 0$ and $a_j = 0$) or ($q = 2$ and $a_j = 0$) or ($q = 1$ and $a_j = a$) or ($q = 3$ and $a_j = a$), $\epsilon_j = -1$ otherwise, if and only if $\text{rank}(W) = 8 - 2k$, see Proposition 5.14.

The above is a necessary condition for $T(U)$ to be self-adjoint. Under this condition, we have the following two cases:

1. $If l = 0$, then $\#(\{0,3\} \cap P_0) = 0$, $\#(\{1,2\} \cap P_0) = 0$, $\#(\{0,3\} \cap P_a) = 0$ and $\#(\{1,2\} \cap P_a) = 0$ and the differential operator $T(U)$ is self-adjoint, see Remark 5.42.

2. If $1 \leq l \leq 3$, then it follows from Theorem 5.46 that the differential operator $T(U)$ is self-adjoint if and only if $\#(\{0,3\} \cap P_0) \leq 1$, $\#(\{1,2\} \cap P_0) \leq 1$, $\#(\{0,3\} \cap P_a) \leq 1$ and $\#(\{1,2\} \cap P_a) \leq 1$.

Now the statement of this corollary easily follows. □

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