Conservation Laws and Their Associated Symmetries for Stochastic Differential Equations

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Abstract

The modelling power of Itô integrals has a far reaching impact on a spectrum of diverse fields. For example, in mathematics of finance, its use has given insights into the relationship between call options and their non-deterministic underlying stock prices; in the study of blood clotting dynamics, its utility has helped provide an understanding of the behaviour of platelets in the blood stream; and in the investigation of experimental psychology, it has been used to build random fluctuations into deterministic models which model the dynamics of repetitive movements in humans.

Finding the quadrature for these integrals using continuous groups or Lie groups has to take families of time indexed random variables, known as Wiener processes, into consideration. Adaptations of Sophus Lie's work to stochastic ordinary differential equations (SODEs) have been done by Gaeta and Quintero [1], Wafo Soh and Mahomed [2], Ünal [3], Meleshko et al. [4], Fredericks and Mahomed [5], and Fredericks and Mahomed [6]. The seminal work [1] was extended in Gaeta [7]; the differential methodology of [2] and [3] were reconciled in [5]; and the integral methodology of [4] was corrected and reconciled in [5] via [6].

Symmetries of SODEs are analysed. This work focuses on maintaining the properties of the Weiner processes after the application of infinitesimal transformations. The determining equations for first-order SODEs are derived in an Itô calculus context. These determining equations are non-stochastic.

Many methods of deriving Lie point-symmetries for Itô SODEs have surfaced. In the Itô calculus context both the formal and intuitive understanding of how to construct these symmetries has led to seemingly disparate results. The impact of Lie point-symmetries on the stock market, population growth and weather SODE models, for example, will not be understood until these different results are reconciled as has been attempted here.

Extending the symmetry generator to include the infinitesimal transformation of the Wiener process for Itô stochastic differential equations (SDEs), has successfully been done in this thesis. The impact of this work leads to an intuitive understanding of the random time change formulae in the context of Lie point symmetries without having to consult much of the intense Itô calculus theory needed to derive it formerly (see Øksendal [8, 9]). Symmetries of nth-order SODEs are studied. The determining equations of these SODEs are derived in an Itô calculus context. These determining equations are not stochastic in nature. SODEs of this nature are normally used to model nature (e.g. earthquakes) or for testing the safety and reliability of models in construction engineering when looking at the impact of random perturbations.

The symmetries of high-order multi-dimensional SODEs are found using form invariance arguments on both the instantaneous drift and diffusion properties of the SODEs. We then apply this to a generalised approximation analysis algorithm. The determining equations of SODEs are derived in an Itö calculus context.

A methodology for constructing conserved quantities with Lie symmetry infinitesimals in an Itô integral context is pursued as well. The basis of this construction relies on Lie bracket relations on both the instantaneous drift and diffusion operators.

Declaration

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

E Fredericks.

Dedication

To my loving parents and friends

Acknowledgements

I am deeply indebted to my thesis supervisor Professor FM Mahomed for having introduced me to the world of Lie Symmetry and its manifold applications to both stochastic and ordinary differential equation. His encouragement and guidance throughout the course of my research work, especially during the final stages, is greatly appreciated. I also wish to register my gratitude for his critical appraisal of the orgininal manuspcript.

I thank Professor B A Watson of the School of Mathematics at the University of the Witwatersrand, for making me aware of the measure theoretic implications of Lie group transformations on stochastic ordinary differential equations.

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Chapter 1

Introduction and Seminal Works

In 1999 a paper was written by Guiseppe Gaeta and Niurka R. Quintero [1], which discussed relations between the symmetries of the Fokker-Planck (FP) equation and its corresponding stochastic (ordinary) differential equation (SDE): The symmetries were found, in the usual way, from the FP equation; these symmetries were then checked against certain conditions; those that met the conditions were symmetries of the corresponding SDE. However, the assumption of projectability is assumed.

Two years later, a paper was released by Celestine Wafo Soh and Fazal M. Mahomed [2], which explained how to derive these Lie point symmtries without referring to the corresponding FP equation. Their methodology was able to incorporate higher order SDEs, like the governing equation for the response of a mass-spring oscillator to a white noise random excitation. The assumption of projectibility was not needed in their derivation. Their result was for a more generalized case.

Almost a year later a paper was published by Gazanfer Ünal [3]; the paper claimed that the determining equations it used for finding symmetries for first order SDE's were far removed from the simplified version of [2] as [2] precluded an extra condition in its derivation, which was introduced in [3].

It is true that there is an error of notation that was carried through in the derivation for the general case in [2], but it must be said that its determining equations that was used to solve for the symmetries of first-order SDEs is correct and by carrying the derivation further, gives the same determining equations as in [3]. In fact it will be shown in this work that Ünal's [3] extra condition is unnecessary and is automatically satisfied.

In this thesis, we also investigate the more formal approach of Meleshko et al. [4]; reconciling it with the most recent findings. The method followed by [4] is incomplete in the sense that the Itô formula was neglected in the transformation of the time index variable, which leads to transforms that under the Itô's formula are not form invariant in terms of the original SODEs.

Extending the work of Gaeta [7] to both first and nth order SODEs is the next step of our investigation and involves not only point transformations but generalized transformations as well. The result of this analysis is then applied to approximate SODEs, which was first investigated by Ibragimov et al. [10] and conserved quantities in an Itô context, which was first studied in Ünal [3].

The outline of the thesis is as follows. In 1.1 Preliminaries, we firstly introduce preliminary results that underlie our research. This is followed by an in depth view into the relationship between the FP equations and its associated SODEs. Specifically, the calculations of [1] are done in more detail and the *ansatz* is justified using as an aid the work of Mahomed and Momoniat [11].

Chapter 2 seeks to reconcile the works of Wafo Soh and Mahomed [2] and Ünal [3], by using the instantaneous drift and diffusion of the Wiener process. The finite Lie point transformation is also analyzed in conjunction with the Itô's formula, which had not been done to date.

Completing the transformation methodology of Meleshko et al. [4] in Chapter 3, reconciles all findings thus far. Examples are used to establish the consequence of precluding Itô's formula in the time index transformation.

We then begin Chapter 4, in attempt to extend the results of Gaeta [7] by using form invariant arguments on the moments of the Wiener process which underlies SODEs. The symmetry operator is also extended as a result. Examples from [7] are then used to establish differences and similarities between our philosophy and that of [7].

Chapter 5 extends these results for generalized symmetries for nth order SODEs, thus extending on the work previously done by [2]. As a result of the philosophy of form invariance for the moments of the transformed Wiener process, new insights into the prolongation formulae needed to find the the prolonged spatial infinitesimals become apparent. The use of the examples establish the differences between what was done in [2] and in the thesis thus far.

The next two chapters are merely applications of all the recent findings. Chapter 6 extends the approximate symmetry analysis for first order SODEs of [10] to nth order SODEs.

Chapter 7 reconciles the work of Ünal [3] with the latest results and in addition finds a new method for construction of conserved quantities. An example from [3] confirms this.

We provide a conclusion, which places all findings into perspective.

1.1 Preliminaries

In order to work with SDEs we first have to familiarize ourselves with how we associate events ω belonging to a sample space Ω with a probability measure $\mathbb{P}$. We apply the probability measure specifically to a system of subsets of Ω, which we denote by $\mathcal{F}$. This σ-algebra $\mathcal{F}$ contains the complement and countable union of any of its arbitrary members, which we call open sets (refer to [12] for summarized definitions concerning measure theory). We then form a *natural filtration* by forming an indexed family of σ-algebras $\mathcal{F}_t$, where t is a time-index, to which the sample paths of our processes are *adapted* (see [13]). The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that we have introduced allows us to proceed with the introduction of the randomess which drives the SDE, namely the Wiener process. The Wiener process is a family of random variables indexed, for our purposes, by time t, which belongs to the interval I, which can be taken to be the positive real line. This process is a mathematical tool used to formalize the physical phenomena of Brownian motion; its sample paths, which are obtained by focusing on a fixed realisation of particular event $\omega \in \Omega$ and following their families of random variables through time, are almost surely continuous and are almost surely nowhere differentiable in the usual sense (there are many books that explain these concepts, e.g. [14] and [15]). We represent it as a function $W(t, \omega)$ which does the following:

$$(t, \omega) \in I \times \Omega \longrightarrow W(t, \omega) \in \mathbb{R}.$$

The ω in the argument of our function is an arbitrary event and is thus suppressed throughout the paper. This process $W(t)$ also has the following characteristics:

- at time zero with probability one, $W(0) = 0$;
- for any strictly increasing sequence of indexed times $\{t_i\} \subset I$, the random variables $W(t_{i+1}) - W(t_i)$ are independent;
- for times $s < t$, $W(t) - W(s)$ is normally distributed with a zero mean and a variance of $t - s$;
- the covariance between two scalar processes at different times $\mathbb{E}\left(W(s)W(t)\right)$ is just the minimum between the two different times $min(t, s)$.

The conditions which were used in deriving the determining equations in [2] and [3] were based upon what is known as the *Itô's multiplication table* - simple mneumonics based on *Itô isometry*, see [9]

	$dW(t,\,\omega)^{(i)}$	$dW(t,\,\omega)^{(j)}$	dt
$dW(t,\,\omega)^{(i)}$	dt	0	0
$dW(t,\,\omega)^{(j)}$	0	dt	0
dt	0	0	0

Here $dW(t,\,\omega)^{(i)}$ and $dW(t,\,\omega)^{(j)}$ are two independent standard Wiener processes; $i, j = 1, \ldots, N$.

The derivative, in the distribution sense, of the Brownian motion is called *white noise* and is represented as $dW(t,\,\omega)\,dt$. One of the earliest descriptions of *white noise* was given in [16]: "Inside the plane ... we hear all frequencies added together at once, producing a noise which is to sound what white light is to light."

In order to construct the *Itô Integral* next, we need to define the *mean square norm*. The *mean square norm* of our defined random variable $\mathbf{W}_t$ is defined by

$$\|\mathbf{W}(t,\,\omega)\| = \sqrt{\mathbb{E}\left(|\mathbf{W}(t,\,\omega)|^2\right)} = \sqrt{\int_\Omega |\mathbf{W}(t,\,\omega)|^2 d\mathbb{P}},$$

where $\mathbb{E}$ and $\mathbb{P}$ are the expectation operator and probability law respectively with

$$|\mathbf{W}(t,\,\omega)| = \sqrt{\sum_{i=1}^{N} (W(t,\,\omega))^2}.$$

Now let $\mathbf{X}(t,\,\omega)$ be a stochastic process such that $\|\mathbf{X}(t,\,\omega)\| < \infty$ for all $t \in [0,T]$, $T > 0$. Then we denote the class of such stochastic processes by $\mathcal{L}^2$.

Unlike the *Riemann Integral* whose approximations converge in $\mathbb{R}$, the *Itô Integral* will be approximated by a sequence of random variables converging in $\mathcal{L}^2$. Also the choice of values that the function being integrated takes, along the discretized temporal axis, in the integral approximation, affects the limit of such approximations: in an interval of $[t_i, t_{i+1})$ the choice of $s_i \in [t_i, t_{i+1})$ for the function being integrated in this interval is crucial, as the choice of t_i would give rise to what is known as the *Itô Integral* and the choice of $\frac{1}{2}(t_i + t_{i+1})$ would give rise to the *Stratonovich Integral*.

We now consider the mesh

$$0 = t_1 < t_2 < \cdots < t_n = T,$$

of $[0,T]$ and let $\Pi_n = \max_{1 \leq k \leq n-1}(t_{k+1} - t_k)$. The following random variable is now formed:

$$\mathbf{Y}_n = \sum_{k=1}^{n-1} \mathbf{X}_{t_k}\left[\mathbf{W}_{t_{k+1}} - \mathbf{W}_{t_k}\right]. \tag{1.1}$$

Thus we have as a consequence that

$$Y_n^{(i)} = \sum_{k=1}^{n-1} X_{t_k}^{(i)}\left[W_{t_{k+1}}^{(i)} - W_{t_k}^{(i)}\right]. \tag{1.2}$$

We now show that our newly formed random variable is finite under the mean square norm:

$$\begin{aligned}
\|Y_n\| &= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} X_{t_i}^{(k)}\left[W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right]\right)^2\right)} \\
&= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} X_{t_i}^{(k)}\left[W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right] X_{t_j}^{(k)}\left[W_{t_{j+1}}^{(k)} - W_{t_j}^{(k)}\right]\right)} \\
&= \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \mathbb{E}\left(X_{t_i}^{(k)}\left[W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right] X_{t_j}^{(k)}\left[W_{t_{j+1}}^{(k)} - W_{t_j}^{(k)}\right]\right)} \\
&= \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1} \mathbb{E}\left(\left(X_{t_i}^{(k)}\right)^2\left[W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right]^2\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\mathbb{E}\left(X_{t_i}^{(k)}\right)^2[t_{i+1}-t_i]} \left(\text{as } \mathbb{E}\left(\left[W_{t_{i+1}}^{(k)}-W_{t_i}^{(k)}\right]\right)\mathbb{E}\left(\left[W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right]\right)=0;\ i\neq j\right)\\
&\leq \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\mathbb{E}\left(X_{t_i}^{(k)}\right)^2\Pi_n}\\
&= \sqrt{\Pi_n}\left[\sum_{i=1}^{n-1}\sum_{k=1}^{N}\mathbb{E}\left(X_{t_i}^{(k)}\right)^2\right]^{\frac{1}{2}}\\
&= \sqrt{\Pi_n}\left[\sum_{i=1}^{n-1}\mathbb{E}\left(\sum_{k=1}^{N}\left(X_{t_i}^{(k)}\right)^2\right)\right]^{\frac{1}{2}}\\
&= \sqrt{\Pi_n\sum_{i=1}^{n-1}\|\mathbf{X}_{t_i}\|^2}\\
&\leq \infty.
\end{aligned}
$$

As $\|\mathbf{X}_t\|$ is finite for all $t\in[0,T]$, it belongs to the class $\mathcal{L}_T^2$. If there is a random variable Y such that

$$\lim_{\substack{n\to+\infty,\\ \Pi_n\to 0}}\|Y_n-Y\|=0,$$

Y is called the *Itô integral* of $\mathbf{X}_t$ and is denoted by $\int_0^a \mathbf{X}_t\, d\mathbf{W}_t$. We illustrate with the following example:

Example 1.1. The squared Wiener processes, $\mathbf{W}_t^2$ belongs to the class $\mathcal{L}_T^2$ since

$$
\begin{aligned}
\|\mathbf{W}_t^2\| &= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(W_t^{(k)}\right)^4\right)} && (1.3)\\
&= \sqrt{\sum_{k=1}^{N}\mathbb{E}\left(\left(W_t^{(k)}\right)^4\right)}\\
&= \sqrt{3\sum_{k=1}^{N}t^2}\\
&= \sqrt{3Nt^2}\ \textit{is thus finite.} && (1.4)
\end{aligned}
$$

In order to find its Itô integral we form the random variable[1].

$$
\begin{aligned}
\mathbf{Y}_n &= \sum_{i=1}^{n}\mathbf{W}_{t_i}^2\left[\mathbf{W}_{t_{i+1}}-\mathbf{W}_{t_i}\right]\ \textit{which is finite under the mean-square norm} && (1.5)\\
&= \frac{1}{3}\sum_{i=1}^{n}\left(\mathbf{W}_{t_{i+1}}^3-\mathbf{W}_{t_i}^3\right)-\sum_{i=1}^{n}\mathbf{W}_{t_i}\left(\mathbf{W}_{t_{i+1}}-\mathbf{W}_{t_i}\right)^2-\frac{1}{3}\sum_{i=1}^{n}\left(\mathbf{W}_{t_{i+1}}-\mathbf{W}_{t_i}\right)^3\\
&= \frac{1}{3}\mathbf{W}_T^3-\mathbf{Y}_{n_1}-\mathbf{Y}_{n_2}-\frac{1}{3}\mathbf{Y}_{n_3},
\end{aligned}
$$

[1] Use the identity $a^2(b-a)=\frac{1}{3}(b^3-a^3)-a(b-a)^2-\frac{1}{3}(b-a)^3$

where

$$\mathbf{Y}_{n_1} = \sum_{i=1}^{n} \mathbf{W}_{t_i} \left(t_{i+1} - t_i\right) \tag{1.6}$$

$$\mathbf{Y}_{n_2} = \sum_{i=1}^{n} \mathbf{W}_{t_i} \left[\left(\mathbf{W}_{t_{i+1}} - \mathbf{W}_{t_i}\right)^2 - \left(t_{i+1} - t_i\right)\right] \tag{1.7}$$

$$\mathbf{Y}_{n_3} = \frac{1}{3} \sum_{i=1}^{n} \left(\mathbf{W}_{t_{i+1}} - \mathbf{W}_{t_i}\right)^3. \tag{1.8}$$

If we choose $\mathbf{Y}_1 = \int_0^T \mathbf{W}_t \, dt$, then

$$\|\mathbf{Y}_{n_1} - \mathbf{Y}_1\| = \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} W_{t_i}^{(k)} \left(t_{i+1} - t_i\right) - \int_0^T W_t^{(k)} \, dt\right)^2\right)} \tag{1.9}$$

$$= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \left(W_{t_i}^{(k)} - W_t^{(k)}\right) \, dt\right)^2\right)} \tag{1.10}$$

$$= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \left(W_{t_i}^{(k)} - W_t^{(k)}\right) \, dt\right)\left(\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} \left(W_{t_j}^{(k)} - W_t^{(k)}\right) \, dt\right)\right)}$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left(W_{t_i}^{(k)} - W_t^{(k)}\right) \, dt\right)\left(\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}\left(W_{t_j}^{(k)} - W_t^{(k)}\right) \, dt\right)}$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left(W_t^{(k)} - W_{t_i}^{(k)}\right)^2 \, dt\right)}$$

$$\left(as \int_{t_i}^{t_{i+1}} \mathbb{E}\left(W_{t_i}^{(k)} - W_t^{(k)}\right) dt \int_{t_j}^{t_{j+1}} \mathbb{E}\left(W_{t_j}^{(k)} - W_t^{(k)}\right) dt \;=\; 0;\; i \neq j\right)$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \left(t - t_i\right) \, dt\right)} \tag{1.11}$$

$$= \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\left(\left.\frac{\left(t - t_i\right)^2}{2}\right|_{t_i}^{t_{i+1}}\right.}$$

$$= \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\left(\frac{\left(t_{i+1} - t_i\right)^2}{2}\right)}$$

$$\leq \sqrt{\sum_{k=1}^{N}\sum_{i=1}^{n-1}\left(\frac{\left(\Pi_n\right)^2}{2}\right)}$$

$$\longrightarrow \; 0 \;\; as \;\; n \to \infty; \Pi_n \to 0. \tag{1.12}$$

Thus $\mathbf{Y}_1$ is the Itô integral of $\mathbf{Y}_{n_1}$. As for the third component $\mathbf{Y}_{n_2}$ we get

$$\|\mathbf{Y}_{n_2}\| = \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n} W_{t_i}^{(k)}\left[\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right) - (t_{i+1}-t_i)\right]\right)^2\right)} \tag{1.13}$$

$$= \left(\mathbb{E}\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{n} W_{t_i}^{(k)}\left[\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^2 - (t_{i+1}-t_i)\right]\right) \times \ldots\right.\right.$$

$$\left.\left.\left(\sum_{j=1}^{n} W_{t_j}^{(k)}\left[\left(W_{t_{j+1}}^{(k)} - W_{t_j}^{(k)}\right)^2 - (t_{j+1}-t_j)\right]\right)\right)\right)^{\frac{1}{2}} \tag{1.14}$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n}\mathbb{E}\left(W_{t_i}^{(k)}\right)^2\mathbb{E}\left[\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^2 - (t_{i+1}-t_i)\right]^2\right)} \tag{1.15}$$

$$\left(as\ \mathbb{E}\left(W_{t_i}^{(k)}\right)\mathbb{E}\left[\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^2 - (t_{i+1}-t_i)\right] \times \cdots\right.$$

$$\left.\mathbb{E}\left(W_{t_j}^{(k)}\right)\mathbb{E}\left[\left(W_{t_{j+1}}^{(k)} - W_{t_j}^{(k)}\right)^2 - (t_{j+1}-t_j)\right] = 0; i \neq j\right)$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n} t_i\mathbb{E}\left[\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^4 - 2\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^2(t_{i+1}-t_i) - (t_{i+1}-t_i)^2\right]\right)}$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n} t_i\left(\mathbb{E}\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^4 - 2\mathbb{E}\left(W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}\right)^2(t_{i+1}-t_i) + (t_{i+1}-t_i)^2\right)\right)}$$

$$= \sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{n} t_i\left(3(t_{i+1}-t_i)^2 - 2(t_{i+1}-t_i)^2 + (t_{i+1}-t_i)^2\right)\right)}$$

$$= \sqrt{2\sum_{k=1}^{N}\sum_{i=1}^{n} t_i(t_{i+1}-t_i)^2}$$

$$\leq \sqrt{2\sum_{k=1}^{N}\sum_{i=1}^{n} t_i(\Pi_n)^2} \tag{1.16}$$

$$\longrightarrow 0\ as\ n \to \infty; \Pi_n \to 0. \tag{1.17}$$

If we look carefully at the second expectation in (1.15), we see that we have actually found justification for the mneumonic '$(d\mathbf{W}_t)^2 = dt$' in the $\mathcal{L}_T^2$ space, i.e.

$$\mathbb{E}\left[\left(\int_0^T \mathbf{W}_s d\mathbf{W}_s\right)^2\right] = \mathbb{E}\left[\int_0^T \mathbf{W}_s^2 dt\right],$$

which is a particular case of what is also known as Itô's Isometry.

Finally the last of the components $\mathbf{Y}_{n_3}$ gives us

$$\|\mathbf{Y}_{n_3}\| = \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\frac{1}{3}\sum_{i=1}^{n}\left(W_{t_{i+1}}^{(k)}-W_{t_i}^{(k)}\right)^3\right)^2\right)} \tag{1.18}$$

$$= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\frac{1}{3}\sum_{i=1}^{n}\left(W_{t_{i+1}}^{(k)}-W_{t_i}^{(k)}\right)^3\right)\left(\frac{1}{3}\sum_{j=1}^{n}\left(W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right)^3\right)\right)}$$

$$= \sqrt{\mathbb{E}\left(\sum_{k=1}^{N}\left(\frac{1}{9}\sum_{j=1}^{n}\left(W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right)^6\right)\right)} \tag{1.19}$$

$$\left(\text{as } \mathbb{E}\left(W_{t_{i+1}}^{(k)}-W_{t_i}^{(k)}\right)^3\mathbb{E}\left(W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right)^3 = 0; i\neq j\right)$$

$$= \sqrt{\left(\sum_{k=1}^{N}\left(\frac{1}{9}\sum_{j=1}^{n}\mathbb{E}\left(W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right)^6\right)\right)}$$

$$= \sqrt{\left(\sum_{k=1}^{N}\left(\frac{1}{9}\sum_{j=1}^{n}\mathbb{E}\left(W_{t_{j+1}}^{(k)}-W_{t_j}^{(k)}\right)^6\right)\right)}$$

$$= \sqrt{\left(\sum_{k=1}^{N}\left(\frac{1}{9}\sum_{j=1}^{n}15\left(t_{j+1}-t_j\right)^3\right)\right)}$$

$$\leq \sqrt{\left(\sum_{k=1}^{N}\left(2\sum_{j=1}^{n}\left(\Pi_n\right)^3\right)\right)}$$

$$\longrightarrow 0, \text{ as } n\to\infty; \Pi_n\to 0.$$

Thus we have found that

$$\int_0^T \mathbf{W}_s^2 d\mathbf{W}_s = \frac{1}{3}\mathbf{W}_T^3 - \int_0^T \mathbf{W}_t\, dt. \tag{1.20}$$

An *Itô process* is a stochastic process $\mathbf{X}_t$ defined formally as a stochastic integral equation

$$\mathbf{X}_t = \mathbf{X}_{t_0} + \int_{t_0}^{t}\mathbf{f}\left(s,\mathbf{X}_s\right)ds + \int_{t_0}^{t}\mathbf{G}\left(s,\mathbf{X}_s\right)d\mathbf{W}_s \tag{1.21}$$

or it can be intuitively presented as a stochastic differential equation

$$d\mathbf{X}_t = \mathbf{f}\left(t,\mathbf{X}_t\right)dt + \mathbf{G}\left(t,\mathbf{X}_t\right)d\mathbf{W}_t, \tag{1.22}$$

where $t_0, t\in I, \mathbf{f}$ is an N vector - valued function, $\mathbf{G}$ is an $N\times M$ matrix-valued function, $\mathbf{W}_t$ is an M-dimensional Wiener process and the second integral is the *Itô integral.* For the existence and uniqueness of a temporally-continuous solution, besides the assumption that $\mathbf{X}_t$ belongs to $\mathcal{L}_T^2$, we also assume that

$$|\mathbf{f}| + |\mathbf{G}| \leq C\left(1+|x|\right), \text{ for some constant } C, \tag{1.23}$$

where $|\mathbf{G}| = \sum_{i=1}^{N}\sum_{j=1}^{M}|G_{ij}|^2$ and the drift and diffusion coefficients are Lipschitz continuous

$$|\mathbf{f}(t,\mathbf{x})-\mathbf{f}(t,\mathbf{y})| + |\mathbf{G}(t,\mathbf{x})-\mathbf{G}(t,\mathbf{y})| \leq D|\mathbf{x}-\mathbf{y}|, \text{ for some constant } D. \tag{1.24}$$

We now state the three main theorems that make our analysis possible.

Theorem 1.1 (*Itô's Formula*, [9])**.**

If $\mathbf{X}(t)$, an N-dimensional vector, is an Itô process,

$$d\mathbf{X}(t) = \mathbf{f}dt + \mathbf{G}\, d\mathbf{W}(t), \tag{1.25}$$

where $\mathbf{f} = \mathbf{f}(t,\ \mathbf{X}(t))$ and $\mathbf{G} = \mathbf{G}(t,\ \mathbf{X}(t))$ are N-dimensional drift vector coefficient and diffusion matrix coefficient of dimension $N \times M$, respectively; then for an arbitrary application $\mathbf{F} : I \times \mathbb{R}^N \to \mathbb{R}^M$, which is twice differentiable in the spatial coordinates, $\mathbf{F}(t, .) \in \mathcal{C}^2\left(\mathbb{R}^N, \mathbb{R}^M\right)$ and only differentiable with respect to time once, $\mathbf{F}(., \mathbf{x}) \in \mathcal{C}^1\left(I, \mathbb{R}^M\right)$ for all $(s, \mathbf{y}) \in I \times \mathbb{R}^{\mathcal{N}}$, an Itô process $\mathbf{F}\left(t, \mathbf{X}(t)\right)$ exists and is written in component form as

$$\begin{aligned} dF_j\left(t, \mathbf{X}(t)\right) &= \left.\frac{\partial F_j\left(t, \mathbf{x}\right)}{\partial t}\right|_{(t,\mathbf{X}(t))} dt + \left.\frac{\partial F_j\left(t, \mathbf{x}\right)}{\partial x_i}\right|_{(t,\mathbf{X}(t))} dX_i(t) \\ &\quad + \frac{1}{2}\left.\frac{\partial^2 F_j\left(t, \mathbf{x}\right)}{\partial x_i \partial x_m}\right|_{(t,\mathbf{X}(t))} dX_i(t)\, dX_m(t), \text{ for } j = 1, ..., N. \end{aligned}$$

The evaluation of each of the partial derivatives in the right-hand side is made at $(t, \mathbf{X}(t))$ and we simply write as

$$dF_j\left(t, \mathbf{X}(t)\right) = \frac{\partial F_j}{\partial t}\, dt + \frac{\partial F_j}{\partial x_i}\, dX_i(t) + \frac{1}{2}\, \frac{\partial^2 F_j}{\partial x_i \partial x_m}\, dX_i(t)\, dX_m(t). \tag{1.26}$$

It should be kept in mind that though $\mathbf{X}(t)$ is indexed by time; it is by its random nature independent of time. The repeated index summation convention is assumed throughout this work. The terms $d\mathbf{X}_i(t)$ and $d\mathbf{X}_i(t)\, d\mathbf{X}_m(t)$ are evaluated using (1.25) and the Itô multiplication table to get

$$dF_j\left(t, \mathbf{X}(t)\right) = \Gamma(F_j)\left(t, \mathbf{X}(t)\right) dt + Y^l(F_j)\left(t, \mathbf{X}(t)\right) dW_l(t), \tag{1.27}$$

where

$$\Gamma(F_j) = \frac{\partial F_j}{\partial t} + f_i\, \frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k\, G_m^k\, \frac{\partial^2 F_j}{\partial x_i \partial x_m}\ , \tag{1.28}$$

$$Y^l(F_j) = G_i^l\, \frac{\partial F_j}{\partial x_i}, \text{ for each } l = 1, ..., M. \tag{1.29}$$

For the existence and uniqueness of a temporally-continuous solution, besides the assumption that $\mathbf{X}(t)$ belongs to $\mathcal{L}^2$ for an interval $[0, T]$, we also assume that the instantaneous mean and diffusion coefficients of (1.25) are Lipschitz continuous (see [15], chap. 7). We give an example to illustrate how Itô's theorem could be applied to find the integral of a function of the Wiener process. From this example one notices how the Newtonian calculus differs from the Itô calculus. We check our tedious calculation from the example 1.1 done earlier (see equation (1.20)).

Example 1.2.

The Wiener process $\mathbf{W}^2(t)$ is an Itô process. We apply the Itô's formula (1.26) to $\mathbf{W}^3(t)$ to find the integral of the process $\mathbf{W}^2(t)$. We therefore obtain

$$\begin{aligned} d\left(\mathbf{W}^3(t)\right) &= 3\,\mathbf{W}^2(t)\, d\mathbf{W}(t) + \frac{1}{2}\, 6\,\mathbf{W}(t)\ \left(d\mathbf{W}(t)\right)^2 \\ &= 3\,\mathbf{W}^2(t)\, d\mathbf{W}(t) + 3\,\mathbf{W}(t)\, dt. \end{aligned} \tag{1.30}$$

Hence we now integrate to arrive at

$$\mathbf{W}^3(T) - \mathbf{W}^3(0) = 3\int_0^T \mathbf{W}^2(t)\, d\mathbf{W}(t) + \int_0^T 3\mathbf{W}(t)\, dt.$$

Dividing through by 3 and rearranging terms this simplifies to

$$\int_0^T \mathbf{W}^2(t)\, d\mathbf{W}(t) = \frac{1}{3}\,\mathbf{W}(T)^3 - \int_0^T \mathbf{W}(t)\, dt, \tag{1.31}$$

which is the same as before, viz. (1.20). One easily identifies the extra term $-\int_0^T \mathbf{W}(t)dt$, as Itô's correction term. This adjusts the answer we would have gotten had we used basic Newtonian calculus methods. Since the calculus governing Wiener processes is not as straightforward as Newtonian calculus, it is the case that the transformation of a Wiener process into another Wiener process would have to be contended with. This brings us to the following theorem.

Theorem 1.2 (*Random Time Change for Itô Integrals,* [8])**.**

Let $c(t,\ \omega)$ be the measureable time change rate which is related to our time change scalar stochastic process $\beta(t,\ \omega)$, by the following equation

$$\beta(t,\ \omega)\ =\ \int^{t} c(s,\ \omega)\, ds \tag{1.32}$$

and $\alpha(t,\ \omega)$ be a scalar stochastic process satisfying

- $\alpha(0,\ \omega) = 0$,
- $d\alpha(t,\ \omega)/dt = 1/c(\alpha(t),\ \omega) \geq 0$, for almost all positive time and almost all $\omega \in \Omega$,
- $\beta(t,\ \omega)$ and $\alpha(t,\ \omega)$ are left and right inverses of each other respectively, $\alpha\left(\beta(t,\omega),\ \omega\right) = \beta\left(\alpha(t,\omega),\ \omega\right) = t$ for all $(t,\omega) \in I \times \Omega$.

Then, under the (random) time change $\bar{t} = \beta(t,\ \omega)$, the Wiener process $\mathbf{W}(\alpha(t),\ \omega)$ is mapped to another Wiener process $\overline{\mathbf{W}}(t,\ \omega)$ according to the relation

$$\sqrt{\frac{d\alpha(t)}{dt}}\, d\overline{\mathbf{W}}(t) =\ d\mathbf{W}(\alpha(t)), \tag{1.33}$$

where we have suppressed ω in the expression above. This can then be expressed as

$$d\overline{\mathbf{W}}(\beta(t)) =\ \sqrt{c(t)}\, d\mathbf{W}(t), \tag{1.34}$$

by using the inverse relation between $\alpha(t)$ and $\beta(t)$ in conjunction with (1.32).

Example 1.3 (An Example of Random Time Change from Øksendal [9])**.**

The example begins with

$$dY(t,\omega) = \frac{1}{|B(t,\omega)|}\sigma(Y(t,\omega))dB(t,\omega) + \frac{1}{|B(t,\omega)|^2}b(Y(t,\omega))dt. \tag{1.35}$$

Now perform the following time change: define

$$Z(t,\omega) = Y(\alpha(t,\omega),\omega). \tag{1.36}$$

Thus we have that

$$\begin{aligned} dY(\alpha(t,\omega),\omega) = {} & \frac{1}{|B(\alpha(t,\omega),\omega)|}\sigma(Y(\alpha(t,\omega),\omega))dB(\alpha(t,\omega),\omega) \\ & + \frac{1}{|B(\alpha(t,\omega),\omega)|^2}b(Y(\alpha(t,\omega),\omega))d\alpha(t,\omega), \end{aligned} \tag{1.37}$$

where

$$\alpha(t,\omega) = \beta^{-1}(t,\omega), \quad \beta(t,\omega) = \int_0^t c(s,\omega)\,ds \;\text{ and }\; c(t,\omega) = \frac{1}{|B|^2}, \tag{1.38}$$

which means that

$$d\tilde{B}(t,\omega) = \frac{1}{|B(\alpha(t,\omega),\omega)|} dB(\alpha(t,\omega),\omega) \tag{1.39}$$

and

$$dt = \frac{d\alpha(t,\omega)}{|B(\alpha(t,\omega),\omega)|^2} \tag{1.40}$$

whence

$$dZ(t,\omega) = \sigma(Z(t,\omega))d\tilde{B}(t,\omega) + b(Z(t,\omega))dt \tag{1.41}$$

when using the random time change formula for Itô integrals.

(1.42)

Theorem 1.3 (*Feynman-Kac Theorem* [9])**.** Define

$$\begin{aligned} \nu(t,\mathbf{x}) &= \mathbb{E}^{t,\mathbf{x}} h(\mathbf{X}_T), \quad 0 \le t \le T, \\ &= \int h(\mathbf{y})\, p(T-t,\mathbf{x},\mathbf{y}) d\mathbf{y}, \end{aligned}$$

where

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t)\, dt + \mathbf{G}(\mathbf{X}_t) d\mathbf{W}_t$$

and $p(\tau,\mathbf{x},\mathbf{y})$ is the transition density function.

Then

$$\frac{\partial \nu}{\partial t} + A_{ij}(t,\mathbf{x})\frac{\partial^2 \nu}{\partial x_i \partial x_j} + B_i(t,\mathbf{x})\frac{\partial \nu}{\partial x_i} + C(t,\mathbf{x})\nu = 0 \tag{1.43}$$

and

$$\nu(T,\mathbf{x}) = h(\mathbf{x}),$$

where

$$A_{ij} = -\frac{1}{2}(\mathbf{G}\mathbf{G}^T)_{ij} \text{ ; its components are not to be all zero,} \tag{1.44}$$

$$B_i = a_i - 2\frac{\partial A_{ik}}{\partial x_k}, \tag{1.45}$$

$$C = \left(\frac{\partial a_i}{\partial x_i}\right) - \frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}. \tag{1.46}$$

The linear partial differential equation (PDE)(1.43) is known as the **Kolmogrov's Backward equation** (KBE). However, throughtout this chapter, we shall be referring to the **Fokker-Plank equation** (FP), which is usually written as

$$L_{FP}(p) = \frac{\partial p(t,\mathbf{x})}{\partial t} + A_{ij}\frac{\partial^2 p(t,\mathbf{x})}{\partial x_i \partial x_j} + B_i\frac{\partial p(t,\mathbf{x})}{\partial x_i} + Cp(t,\mathbf{x}) = 0, \tag{1.47}$$

where $p(t,\mathbf{x})$ is the density function, satisfying

$$\int_{-\infty}^{\infty} p(t,\mathbf{x}) dx_1 \dots dx_N = 1.. \tag{1.48}$$

Please note that the probabilistic equivalence between the FP and Itô equations does not imply uniqueness between the two; it is possible for different Itô equations to share the same FP equation.

Example 1.4.

$$d\mathbf{x}_1 = \mathbf{f}dt + \mathbf{G}d\mathbf{W}_t \tag{1.49}$$

$$d\mathbf{x}_2 = \mathbf{f}dt + \tilde{\mathbf{G}}d\mathbf{W}_t \tag{1.50}$$

where $\mathbf{G}$ is the identity matrix and $\tilde{\mathbf{G}}$ is the identity matrix multiplied by an orthogonal matrix

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}. \tag{1.51}$$

In this example both processes share the same constant $\mathbf{f}$ and thus the FP equations for both are the same

$$\frac{\partial u}{\partial t} - \frac{1}{2}A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + f_i\frac{\partial u}{\partial x_i} = 0 \tag{1.52}$$

where A_{ij} is the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.53}$$

Example 1.5. Let us look at the geometric brownian motion model used for stocks in the financial market,

$$dS(t) = rS(t)dt + \sigma S(t)dW_t, \tag{1.54}$$

with initial condition

$$S(t) = x, \tag{1.55}$$

where r is the fixed risk-free interest rate, σ is the fixed volatility and $S(t)$ is the stock's price. Let us now consider the financial instrument known as a European call option, which has term of $\tau = T - t$ and a strike of K. Then we let $\nu(t, S(t))$ be the expected payoff of the option at maturity T, given that we have all the relevant financial information needed up until time t. This conditional expectation on the future payoff is not dependent on time and when we arrrive at maturity T, ν is simply just value of the payoff. That is, the expected future payoff is unaffected by the amount of knowledge we have or accumulate with time towards the expiry date T.

The expected value of a contingent claim, $\max\{0, s(T) - K\}$, at time t is the discounted value of the expected payout in the future,

$$u(t,x) = e^{r\tau}\mathbb{E}^{t,x}\left[\max\{0, s(T) - K\}\right] \tag{1.56}$$

$$= e^{-r\tau}\nu(t,x) \text{ by definition.} \tag{1.57}$$

at some initial time t where $S(t) = x$. Now if we work out the FP equation w.r.t. ν and then replace ν with $u(t,x)e^{r\tau}$ we get the Black-Scholes PDE.

$$-ru(t,x) + \frac{\partial u(t,x)}{\partial t} + rx\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u(t,x)}{\partial x^2} = 0. \tag{1.58}$$

The terminal condition as was mentioned earlier is

$$\nu(T,x) = \max\{0, x - K\}. \tag{1.59}$$

Notice if we had chosen $h(x) = \max\{0, K - x\}$ (a put option instead), we would have arrived at the same PDE, but for a different function $u(t,x)$.

1.2 Lie Point Symmetries and SDEs

1.2.1 Establishing the *ansatz* used in work [2]

A relationship between the symmetries of the FP equation and those related to the SDE, via what is known as the normalization condition (1.48) was established in the paper of Gaeta and Quintero [1].

Given the SDE

$$d\mathbf{x}_t = \mathbf{f}dt + \mathbf{G}d\mathbf{W}_t \tag{1.60}$$

The operator

$$H_0 = \tau(t, \mathbf{x}, u)\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x}, u)\frac{\partial}{\partial x_j},\ j = 1, \dots, N \tag{1.61}$$

will be used in the transformation of the diffusion and drift coeffecients of (1.60). But before we proceed with the transformation of the spatial and temporal tranformations, we will use [11] to establish the *ansatz* for the operator H that [1] used in their paper, i.e.

$$H_0 = \tau(t)\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x})\frac{\partial}{\partial x_j},\ j = 1, \dots, N. \tag{1.62}$$

We first introduce the Lie characteristic function which is given as

$$Q = \eta(t, \mathbf{x}, u) - \frac{\partial u}{\partial t}\tau(t, \mathbf{x}, u) - \frac{\partial u}{\partial x_i}\xi_i(t, \mathbf{x}, u), \tag{1.63}$$

where

$$H = H_0 + \eta(t, \mathbf{x}, u)\frac{\partial}{\partial u}, \tag{1.64}$$

is the Lie point transformation generator. Thus the functions τ, ξ_i and η can be given in terms of Q

$$\tau = -\frac{\partial Q}{\partial u_{(t)}}, \tag{1.65}$$

$$\xi_i = -\frac{\partial Q}{\partial u_{(i)}}, \tag{1.66}$$

$$\eta = Q - \frac{\partial u}{\partial t}\frac{\partial Q}{\partial u_{(t)}} - \frac{\partial u}{\partial x_i}\frac{\partial Q}{\partial u_{(i)}}, \tag{1.67}$$

where

$$u_{(t)} = \frac{\partial u}{\partial t}, \tag{1.68}$$

$$u_{(i)} = \frac{\partial u}{\partial x_i}, \tag{1.69}$$

$$u_{(ik)} = \frac{\partial^2 u}{\partial x_i \partial x_k}\left(= \frac{\partial^2 u}{\partial x_i \partial x_k}\right), \tag{1.70}$$

and thus in general

$$u_{(i_{1\to N})} = \frac{\partial^N u}{\partial x_{i_1} \dots \partial x_{i_N}}\left(= \frac{\partial^N u}{\partial x_{i_1} \dots \partial x_{i_N}}\right), \tag{1.71}$$

where

$$i_{1\to N} = i_1 i_2 \dots i_N. \tag{1.72}$$

In order to find the *ansazt* for H, we need to apply the second prolongation of H to the second-order evolution PDE (1.47) (we replace p by u in what follows). The second prolongation of H is given by,

$$H^{[2]} = H + \zeta_t \frac{\partial}{\partial u_{(t)}} + \zeta_i \frac{\partial}{\partial u_{(i)}} + \zeta_{it} \frac{\partial}{\partial u_{(it)}} + \zeta_{ik} \frac{\partial}{\partial u_{(ik)}}, \tag{1.73}$$

where

$$\zeta_t = \frac{\partial Q}{\partial t} + u_{(t)} \frac{\partial Q}{\partial u}, \tag{1.74}$$
$$\zeta_i = \frac{\partial Q}{\partial x_i} + u_{(i)} \frac{\partial Q}{\partial u}, \tag{1.75}$$
$$\tag{1.76}$$

and since there is only one dependent variable u,

$$\begin{aligned}
\zeta_{ik} &= D_{(i)} D_{(k)} \left(Q\right), \qquad (1.77)\\
&= \left(\frac{\partial}{\partial x_i} + u_{(i)} \frac{\partial}{\partial u} + u_{(il)} \frac{\partial}{\partial u_{(l)}} + \ldots\right) \left(\frac{\partial}{\partial x_k} + u_{(k)} \frac{\partial}{\partial u} + u_{(kj)} \frac{\partial}{\partial u_{(j)}} + \ldots\right) \left(Q\right),\\
&= \frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)} \frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)} \frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(it)} \frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + \ldots\\
&+ u_{(k)} \frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u} + u_{(ik)} \frac{\partial Q}{\partial u} + u_{(il)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u_{(l)}} + u_{(it)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \ldots\\
&+ u_{(kj)} \frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(kt)} \frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)} u_{(kj)} \frac{\partial^2 Q}{\partial u \partial u_{(j)}} + u_{(i)} u_{(kt)} \frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \ldots\\
&+ u_{(il)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(j)}} + u_{(ij)} u_{(kl)} \frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(l)}} + u_{(ij)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(j)}} + u_{(it)} u_{(kt)} \frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(t)}} + \ldots\\
&+ u_{(il)} u_{(kt)} \frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(t)}} + u_{(it)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(j)}}; \qquad (1.78)
\end{aligned}$$

we do not need ζ_{it}, since our FP equation does not contain $u_{(it)}$. Applying $H^{[2]}$ on $\left(u_{(t)} - F\right)$ at $u_{(t)} = F$, i.e.,

$$H^{[2]} \left(u_{(t)} - F\right)\big|_{u_{(t)}=F} = 0, \tag{1.79}$$

where

$$F = -A_{ik} \frac{\partial^2 u(t, \mathbf{x})}{\partial x_i \partial x_k} - B_i \frac{\partial u}{\partial x_i} - C u(t, \mathbf{x}), \tag{1.80}$$

gives

$$\begin{aligned}
&- \frac{\partial Q}{\partial t} - u_{(t)} \frac{\partial Q}{\partial u} + A_{ik} \left(\frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)} \frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)} \frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(it)} \frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + \ldots\right.\\
&+ u_{(k)} \frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u} + u_{(ik)} \frac{\partial Q}{\partial u} + u_{(il)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u_{(l)}} + u_{(it)} u_{(k)} \frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \ldots\\
&+ u_{(kj)} \frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(kt)} \frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)} u_{(kj)} \frac{\partial^2 Q}{\partial u \partial u_{(j)}} + u_{(i)} u_{(kt)} \frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \ldots\\
&+ u_{(il)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(j)}} + u_{(ij)} u_{(kl)} \frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(l)}} + u_{(ij)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(j)}} + u_{(it)} u_{(kt)} \frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(t)}} + \ldots\\
&+ \left. u_{(il)} u_{(kt)} \frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(t)}} + u_{(it)} u_{(kj)} \frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(j)}}\right) + B_i \left(\frac{\partial Q}{\partial x_i} + u_{(i)} \frac{\partial Q}{\partial u}\right) + \ldots\\
&+ C \left(Q - u_{(t)} \frac{\partial Q}{\partial u_{(t)}} - u_{(j)} \frac{\partial Q}{\partial u_{(j)}}\right) = 0, \text{ on } u_{(t)} = F. \qquad (1.81)
\end{aligned}$$

Substitution of $u_{(t)}$ for F yields

$$
\begin{aligned}
- & \quad \frac{\partial Q}{\partial t} - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u} + \dots \\
+ & \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)}\frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(it)}\frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + \dots\right. \\
+ & \quad u_{(k)}\frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u} + u_{(ik)}\frac{\partial Q}{\partial u} + u_{(il)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(l)}} + u_{(it)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & \quad u_{(kj)}\frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(kt)}\frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)}u_{(kj)}\frac{\partial^2 Q}{\partial u \partial u_{(j)}} + u_{(i)}u_{(kt)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & \quad u_{(il)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(j)}} + u_{(ij)}u_{(kl)}\frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(l)}} + u_{(ij)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(j)}} + u_{(it)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(t)}} + \dots \\
+ & \quad \left. u_{(il)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(t)}} + u_{(it)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(j)}}\right) + B_i\left(\frac{\partial Q}{\partial x_i} + u_{(i)}\frac{\partial Q}{\partial u}\right) + \dots \\
+ & \quad C\left(Q - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u_{(t)}} - u_{(j)}\frac{\partial Q}{\partial u_{(j)}}\right) = 0. \qquad (1.82)
\end{aligned}
$$

Separating out (1.82) by derivatives of u mixed in time and space results in

$$
\begin{aligned}
& \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + u_{(kj)}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(j)}}\right)u_{(it)} + \dots \\
+ & \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + u_{(il)}\frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(t)}}\right)u_{(kt)} + \dots \\
+ & \quad A_{ik}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(t)}}u_{(it)}u_{(kt)} + \dots \\
- & \quad \frac{\partial Q}{\partial t} + \left(A_{ik}u_{(ik)} + B_i u_{(i)} + Cu\right)\frac{\partial Q}{\partial u} + \dots \\
+ & \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)}\frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(k)}\frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u} + \dots\right. \\
+ & \quad u_{(ik)}\frac{\partial Q}{\partial u} + u_{(il)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(l)}} + u_{(kj)}\frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(i)}u_{(kj)}\frac{\partial^2 Q}{\partial u \partial u_{(j)}} + u_{(il)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(j)}} + \dots \\
+ & \quad \left. u_{(ij)}u_{(kl)}\frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(l)}} + u_{(ij)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(j)} \partial u_{(j)}}\right) + B_i\left(\frac{\partial Q}{\partial x_i} + u_{(i)}\frac{\partial Q}{\partial u}\right) + \dots \\
+ & \quad C\left(Q + \left(A_{ik}u_{(ik)} + B_i u_{(i)} + Cu\right)\frac{\partial Q}{\partial u_{(t)}} - u_{(j)}\frac{\partial Q}{\partial u_{(j)}}\right) = 0. \qquad (1.83)
\end{aligned}
$$

By considering only the first three terms of (1.83) we get

$$
u_{(it)}: \qquad A_{ik}\left(\frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + u_{(kj)}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(j)}}\right) = 0, \qquad (1.84)
$$

$$
u_{(kt)}: \qquad A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + u_{(il)}\frac{\partial^2 Q}{\partial u_{(l)} \partial u_{(t)}}\right) = 0, \qquad (1.85)
$$

$$
u_{(it)}u_{(kt)}: \qquad A_{ik}\frac{\partial^2 Q}{\partial u_{(t)} \partial u_{(t)}} = 0. \qquad (1.86)
$$

From (1.86) we see that Q is linear in $u_{(t)}$

$$
Q = \alpha_1(t,\mathbf{x},u)u_{(t)} + \alpha_2(t,\mathbf{x},u), \qquad (1.87)
$$

where $\alpha_1(t,\mathbf{x},u)$ and $\alpha_2(t,\mathbf{x},u)$ are arbitrary functions of t and $\mathbf{x}$. By substituting (1.87) into (1.84) and (1.85); separating by $u_{(k)}$, $u_{(i)}$, $u_{(kj)}$ and $u_{(il)}$ gives

$$
Q = \alpha_1(t)u_{(t)} + \alpha_2(t,\mathbf{x},u). \qquad (1.88)
$$

Separating out (1.82) with respect to $u_{(il)}u_{(kj)}$, $u_{(ij)}u_{(kl)}$ and $u_{(ij)}u_{(kj)}$, gives

$$\begin{aligned}
& A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(l)}\partial u_{(j)}}\right)u_{(il)}u_{(kj)} + A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(l)}}\right)u_{(ij)}u_{(kl)} + A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(j)}}\right)u_{(ij)}u_{(kj)} + \dots \\
- & \frac{\partial Q}{\partial t} - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u} + \dots \\
+ & A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)}\frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(it)}\frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + \dots\right. \\
+ & u_{(k)}\frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u} + u_{(ik)}\frac{\partial Q}{\partial u} + u_{(il)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(l)}} + u_{(it)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & u_{(kj)}\frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(kt)}\frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + u_{(i)}u_{(kj)}\frac{\partial^2 Q}{\partial u \partial u_{(j)}} + u_{(i)}u_{(kt)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & \left. u_{(it)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(t)}\partial u_{(t)}} + u_{(il)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(l)}\partial u_{(t)}} + u_{(it)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(t)}\partial u_{(j)}}\right) + B_i\left(\frac{\partial Q}{\partial x_i} + u_{(i)}\frac{\partial Q}{\partial u}\right) + \dots \\
+ & C\left(Q - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u_{(t)}} - u_{(j)}\frac{\partial Q}{\partial u_{(j)}}\right) = 0. \qquad (1.89)
\end{aligned}$$

Again considering only the first three terms gives rise to

$$u_{(il)}u_{(kj)}: \qquad A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(l)}\partial u_{(j)}}\right) = 0, \qquad (1.90)$$

$$u_{(ij)}u_{(kl)}: \qquad A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(l)}}\right) = 0, \qquad (1.91)$$

$$u_{(ij)}u_{(kj)}: \qquad A_{ik}\left(\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(j)}}\right) = 0. \qquad (1.92)$$

From (1.92) we see that $\alpha_2(t,\mathbf{x},u)$ is linear in $u_{(j)}$, i.e. we have

$$Q = \alpha_1(t)u_{(t)} + \alpha_j(t,\mathbf{x},u)u_{(j)} + \alpha_4(t,\mathbf{x},u). \qquad (1.93)$$

Finally separating out (1.82) with respect to $u_{(il)}u_{(k)}$ and $u_{(i)}u_{(kj)}$, gives

$$\begin{aligned}
& A_{ik}u_{(i)}u_{(kj)}\frac{\partial^2 Q}{\partial u \partial u_{(j)}} + A_{ik}u_{(il)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(l)}} + \dots \\
- & \frac{\partial Q}{\partial t} - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u} + \dots \\
+ & A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial x_k} + u_{(i)}\frac{\partial^2 Q}{\partial u \partial x_k} + u_{(il)}\frac{\partial^2 Q}{\partial x_k \partial u_{(l)}} + u_{(it)}\frac{\partial^2 Q}{\partial x_k \partial u_{(t)}} + \dots\right. \\
+ & u_{(k)}\frac{\partial^2 Q}{\partial u \partial x_i} + u_{(i)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u} + u_{(ik)}\frac{\partial Q}{\partial u} + +u_{(it)}u_{(k)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & u_{(kj)}\frac{\partial^2 Q}{\partial x_i \partial u_{(j)}} + u_{(kt)}\frac{\partial^2 Q}{\partial x_i \partial u_{(t)}} + +u_{(i)}u_{(kt)}\frac{\partial^2 Q}{\partial u \partial u_{(t)}} + \dots \\
+ & u_{(il)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(l)}\partial u_{(j)}} + u_{(ij)}u_{(kl)}\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(l)}} + u_{(ij)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(j)}\partial u_{(j)}} + u_{(it)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(t)}\partial u_{(t)}} + \dots \\
+ & \left. u_{(il)}u_{(kt)}\frac{\partial^2 Q}{\partial u_{(l)}\partial u_{(t)}} + u_{(it)}u_{(kj)}\frac{\partial^2 Q}{\partial u_{(t)}\partial u_{(j)}}\right) + B_i\left(\frac{\partial Q}{\partial x_i} + u_{(i)}\frac{\partial Q}{\partial u}\right) + \dots \\
+ & C\left(Q - \left(-A_{ik}u_{(ik)} - B_i u_{(i)} - Cu(t,\mathbf{x})\right)\frac{\partial Q}{\partial u_{(t)}} - u_{(j)}\frac{\partial Q}{\partial u_{(j)}}\right) = 0. \qquad (1.94)
\end{aligned}$$

This time looking at only the first two terms provide

$$u_{(il)}u_{(k)}: \qquad A_{ik}\frac{\partial^2 Q}{\partial u \partial u_{(j)}} = 0, \tag{1.95}$$

$$u_{(i)}u_{(kj)}: \qquad A_{ik}\frac{\partial^2 Q}{\partial u \partial u_{(l)}} = 0. \tag{1.96}$$

From the above, we see that $\alpha_j(t, \mathbf{x}, u) = \alpha_j(t, \mathbf{x})$, whence

$$Q = \alpha_1(t)u_{(t)} + \alpha_j(t, \mathbf{x})u_{(j)} + \alpha_4(t, \mathbf{x}, u). \tag{1.97}$$

Thus we have established the *ansatz* which [1] used, *viz.*

$$H = \tau(t)\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x})\frac{\partial}{\partial x_j} + \eta(t, \mathbf{x}, u)\frac{\partial}{\partial u}. \tag{1.98}$$

Determining Equations Associated with the Transformation of the Variables

The transformation of the spatial, temporal and the Wiener variables are

$$d\overline{X} = dX + \epsilon d\xi + \mathcal{O}(\epsilon^2) \tag{1.99}$$

$$d\bar{t} = dt + \epsilon d\tau + \mathcal{O}(\epsilon^2) \tag{1.100}$$

and by using the Random Time Change relation (1.34) we get

$$d\overline{W}_{\bar{t}}^{(l)} = dW_t^{(l)}\left(1 + \frac{\epsilon}{2}\frac{d\tau}{dt}\right) + \mathcal{O}(\epsilon^2), \tag{1.101}$$

where, by using Itô's formula (1.27)

$$d\xi = \left(\frac{\partial \xi}{\partial t} + f_i\,\frac{\partial \xi}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_i^k\, G_j^k\,\frac{\partial^2 \xi}{\partial x_i \partial x_j}\right)dt + G_i^j\,\frac{\partial \xi}{\partial x_i}\,dW^j(t) \tag{1.102}$$

$$d\tau = \left(\frac{\partial \tau}{\partial t} + f_i\,\frac{\partial \tau}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_i^k\, G_j^k\,\frac{\partial^2 \tau}{\partial x_i \partial x_j}\right)dt + G_i^j\,\frac{\partial \tau}{\partial x_i}\,dW^j(t). \tag{1.103}$$

However, throughout their paper [1], they made τ a function of time only, thus

$$d\tau = \frac{\partial \tau}{\partial t}dt, \text{ with the indices } i, j = 1, \ldots N. \tag{1.104}$$

Thus, as a consequence of having a projectable symmetry operator H, i.e. with $\tau(t)$

$$d\overline{X} = dX + \epsilon\left(\left(\frac{\partial \xi}{\partial t} + f_i\,\frac{\partial \xi}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_i^k\, G_j^k\,\frac{\partial^2 \xi}{\partial x_i \partial x_j}\right)dt + G_i^j\,\frac{\partial \xi}{\partial x_i}\,dW^j(t)\right) + \mathcal{O}(\epsilon^2) \tag{1.105}$$

$$d\bar{t} = dt\left(1 + \epsilon\frac{\partial \tau}{\partial t}\right) + \mathcal{O}(\epsilon^2) \tag{1.106}$$

$$d\overline{W}_{\bar{t}}^{(l)} = dW_t^{(l)}\left(1 + \frac{\epsilon}{2}\frac{\partial \tau}{\partial t}\right) + \mathcal{O}(\epsilon^2). \tag{1.107}$$

The transformation of $\mathbf{f}$ and $\mathbf{G}$ under our infinitesimal generator H is

$$\begin{aligned} f_i(\bar{t},\bar{\mathbf{x}}) &= f_i(t,\mathbf{x}) + \epsilon H f_i(t,\mathbf{x}) + \mathcal{O}(\epsilon^2) && (1.108)\\ &= f_i + \epsilon\left(\tau(t)\frac{\partial f_i}{\partial t} + \xi_j(t,\mathbf{x})\frac{\partial f_i}{\partial x_j}\right) + \mathcal{O}(\epsilon^2), && (1.109)\\ G^i_k(\bar{t},\bar{\mathbf{x}}) &= G^i_k(t,\mathbf{x}) + \epsilon H G^i_k(t,\mathbf{x}) + \mathcal{O}(\epsilon^2) && (1.110)\\ &= G^i_k + \epsilon\left(\tau(t)\frac{\partial G^i_k}{\partial t} + \xi_j(t,\mathbf{x})\frac{\partial G^i_k}{\partial x_j}\right) + \mathcal{O}(\epsilon^2). && (1.111) \end{aligned}$$

The purpose of the transformation was to leave $d\bar{\mathbf{x}}$ form invariant, i.e.

$$d\overline{X}_l = f_l(\bar{t},\overline{\mathbf{X}})d\bar{t} + G^l_k(\bar{t},\overline{\mathbf{X}})d\overline{W}^k_{\bar{t}}. \tag{1.112}$$

Multiplying out the above components gives

$$\begin{aligned} d\overline{X}_l &= f_l(t,\mathbf{X})dt + G^l_k(t,\mathbf{X})dW^k_t + \epsilon\left(\left(f_l\frac{\partial \tau}{\partial t} + \tau(t)\frac{\partial f_l}{\partial t} + \xi_j(t,\mathbf{X})\frac{\partial f_l}{\partial x_j}\right)dt + \dots\right.\\ &+ \left.\left(\tau(t)\frac{\partial G^i_k}{\partial t} + \xi_j(t,\mathbf{X})\frac{\partial G^i_k}{\partial x_j} + \frac{1}{2}G^i_k\frac{\partial \tau}{\partial t}\right)dW^k_t\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{1.113}$$

Thus by comparing the terms that follow the ϵ in (1.106) and (1.113) we have that

$$f_l\frac{\partial \tau}{\partial t} + \tau\frac{\partial f_l}{\partial t} + \xi_j\frac{\partial f_l}{\partial x_j} - \left(\frac{\partial \xi_l}{\partial t} + f_i\,\frac{\partial \xi_l}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G^k_i\,G^k_j\,\frac{\partial^2 \xi_l}{\partial x_i \partial x_j}\right) = 0, \tag{1.114}$$

$$\text{and}\quad \tau(t)\frac{\partial G^i_k}{\partial t} + \xi_j(t,\mathbf{x})\frac{\partial G^i_k}{\partial x_j} - G^i_k\frac{\partial \xi_l}{\partial x_i} + \frac{1}{2}G^i_k\frac{\partial \tau}{\partial t} = 0. \tag{1.115}$$

The determining equations (1.186) and (1.187) were then used later in the determining equations of the FP equation (1.47). Here are a few examples taken from [1] and re-done in this context.

Example 1.6. Consider

$$dx = \sigma dW_t. \tag{1.116}$$

The determining equations are

$$-\sigma^2\frac{\partial^2 \xi}{\partial x^2} - 2\frac{\partial \xi}{\partial t} = 0 \tag{1.117}$$

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0 \tag{1.118}$$

Solving gives

$$\tau = 2C_1 t + C_2 \tag{1.119}$$

$$\xi = C_1 x + C_3. \tag{1.120}$$

Example 1.7. Now we turn to

$$dx = dt + x dW_t. \tag{1.121}$$

Its determining equations are

$$\dot{\tau} - \frac{\partial \xi}{\partial x} - \frac{\partial \xi}{\partial t} = 0 \tag{1.122}$$

$$-\frac{1}{2}\frac{\partial^2 \xi}{\partial x^2} = 0 \tag{1.123}$$

$$\xi = 0 \tag{1.124}$$

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0 \tag{1.125}$$

after having separated out coefficients with respect to explicit powers of x. Solving results in

$$\tau = C_1 \tag{1.126}$$

$$\xi = 0. \tag{1.127}$$

Example 1.8. We study

$$dx = xdt + dW_t. \tag{1.128}$$

Its determining equations are

$$\xi - \frac{\partial \xi}{\partial t} + x\frac{\partial \xi}{\partial x} = 0 \tag{1.129}$$

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0. \tag{1.130}$$

Solving yields

$$\tau = C_1 e^{2t} + C_3 \tag{1.131}$$

$$\xi = e^t \left(C_1 e^t x + C_2 \right). \tag{1.132}$$

Example 1.9. Finally, consider

$$dx = gdt + \sqrt{D}dW_t. \tag{1.133}$$

The determining equations are

$$g\frac{\partial \xi}{\partial x} - \frac{\partial \xi}{\partial t} = 0 \tag{1.134}$$

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0. \tag{1.135}$$

Solving gives rise to

$$\tau = C_1 2t + C_3 \tag{1.136}$$

$$\xi = C_1(gt + x) + C_2. \tag{1.137}$$

Remark. These results confirm the findings of Gaeta and Quintero [1], even though we are purely using the random time change formula given by Øksendal [9].

Determining Equations Associated with the Fokker-Plank Equation.

In order to find the determining equations of (1.47) we need the second prolongation of the symmetry operator H,

$$H^{[2]} = H + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_{(t)}} + \zeta_i\frac{\partial}{\partial u_{(i)}} + \zeta_{ik}\frac{\partial}{\partial u_{(ik)}}, \tag{1.138}$$

where

$$u_{(t)} = \frac{\partial u}{\partial t}, \tag{1.139}$$

$$u_{(i)} = \frac{\partial u}{\partial x_i}\left(= \frac{\partial u}{\partial x_i}\right), \tag{1.140}$$

$$u_{(ik)} = \frac{\partial^2 u}{\partial x_i \partial x_k}\left(= \frac{\partial^2 u}{\partial x_i \partial x_k}\right) \tag{1.141}$$

and thus in general

$$u_{(i_{1\to N})} = \frac{\partial^N u}{\partial x_{i_1}\dots\partial x_{i_N}}\left(= \frac{\partial^N u}{\partial x_{i_1}\dots\partial x_{i_N}}\right), \tag{1.142}$$

where

$$i_{1\to N} = i_1 i_2 \dots i_N. \tag{1.143}$$

The remaining extended infinitesimals are

$$\zeta_t = D_{(t)}(\eta) - u_{(t)}D_{(t)}(\tau) - u_{(j)}D_{(t)}(\xi_j), \tag{1.144}$$

$$\zeta_i = D_{(i)}(\eta) - u_{(t)}D_{(i)}(\tau) - u_{(j)}D_{(i)}(\xi_j), \tag{1.145}$$

$$\zeta_{ik} = D_{(k)}(\zeta_i) - u_{(it)}D_{(k)}(\tau) - u_{(ij)}D_{(k)}(\xi_j). \tag{1.146}$$

The D operator is the *total derivative* operator defined as

$$D_{(t)} = \frac{\partial}{\partial t} + u_{(t)}\frac{\partial}{\partial u} + u_{(tj)}\frac{\partial}{\partial u_{(j)}} + \dots + u_{(ti_{1\to N})}\frac{\partial}{\partial u_{(i_{1\to N})}} + \dots, \tag{1.147}$$

$$D_{(i)} = \frac{\partial}{\partial x_i} + u_{(i)}\frac{\partial}{\partial u} + u_{(ij)}\frac{\partial}{\partial u_{(j)}} + \dots + u_{(ii_{1\to N})}\frac{\partial}{\partial u_{(i_{1\to N})}} + \dots \tag{1.148}$$

$$\left(= \frac{\partial}{\partial x_i} + u_{(i)}\frac{\partial}{\partial u} + u_{(ij)}\frac{\partial}{\partial u_{(j)}} + \dots + u_{(ii_{1\to N})}\frac{\partial}{\partial u_{(i_{1\to N})}} + \dots\right), \tag{1.149}$$

where $i_{1\to N} = i_1 i_2 \dots i_N$.

Since our symmetry operator was chosen to projectable, i.e. $\tau = \tau(t)$, we have as a result

$$\zeta_t = \frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial u}{\partial t}, \tag{1.150}$$

$$\zeta_i = \frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}, \tag{1.151}$$

$$\begin{aligned}\zeta_{ik} = {} & \frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} + \dots \\ & + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}.\end{aligned} \tag{1.152}$$

An application of our second prolongation on the FP equation should be zero in order for the infinitesimal generator H to be a symmetry of (1.47). That is

$$H^{[2]}\left(L_{FP}(u)\right) = 0, \text{ (on } L_{FP}(u) = 0). \tag{1.153}$$

This gives

$$\begin{aligned}
& \frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial u}{\partial t} + A_{ik}\left(\frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i} + \dots\right.\\
+ \quad & \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} + \dots\\
- \quad & \left.\frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}.\right) + B_i\left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}\right) + \dots\\
+ \quad & C\eta + \left(\xi_j \frac{\partial A_{ik}}{\partial x_j} + \tau \frac{\partial A_{ik}}{\partial t}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} + \left(\xi_j \frac{\partial B_i}{\partial x_j} + \tau \frac{\partial B_i}{\partial t}\right)\frac{\partial u}{\partial x_i} + \dots\\
+ \quad & \left(\xi_j \frac{\partial C}{\partial x_j} + \tau \frac{\partial C}{\partial t}\right) u \;=\; 0, \text{ (on (1.47))}.
\end{aligned} \tag{1.154}$$

We now substitute for $\partial u/\partial t$, i.e.

$$\frac{\partial u}{\partial t} = -A_{ij}\frac{\partial^2 u(t,\mathbf{x})}{\partial x_i \partial x_j} - B_i \frac{\partial u}{\partial x_i} - Cu(t,\mathbf{x}), \tag{1.155}$$

in the above. This yields

$$\begin{aligned}
& \frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\left(-A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - B_i \frac{\partial u}{\partial x_i} - Cu\right) + \dots\\
+ \quad & A_{ik}\left(\frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k} + \dots\right.\\
- \quad & \left.\frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) + B_i\left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}\right) + \dots\\
+ \quad & C\eta + \left(\xi_j \frac{\partial A_{ik}}{\partial x_j} + \tau \frac{\partial A_{ik}}{\partial t}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} + \left(\xi_j \frac{\partial B_i}{\partial x_j} + \tau \frac{\partial B_i}{\partial t}\right)\frac{\partial u}{\partial x_i} + \dots\\
+ \quad & \left(\xi_j \frac{\partial C}{\partial x_j} + \tau \frac{\partial C}{\partial t}\right) u \;=\; 0, \text{ (on (1.47))}.
\end{aligned} \tag{1.156}$$

We now collect the terms into four groups, which are coefficients of $\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}$, $\frac{\partial^2 u}{\partial x_i \partial x_k}$, $\frac{\partial u}{\partial x_i}$ and 1:

$$\begin{aligned}
& \left(A_{ik}\frac{\partial^2 \eta}{\partial u^2}\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + \left(\tau \frac{\partial A_{ik}}{\partial t} + \frac{\partial \tau}{\partial t}\tau A_{ik} + \xi_r \frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} + \dots\\
+ \quad & \left(\tau \frac{\partial B_i}{\partial t} + \frac{\partial \tau}{\partial t}\tau B_i + \xi_r \frac{\partial B_i}{\partial x_r} - B_r \frac{\partial \xi_i}{\partial x_r} - \frac{\partial \xi_i}{\partial t} + A_{ik}\frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right) + \dots\right.\\
+ \quad & \left.A_{ri}\frac{\partial}{\partial x_r}\left(\frac{\partial \eta}{\partial u}\right) - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}\right)\frac{\partial u}{\partial x_i} + \left(\frac{\partial \eta}{\partial t} - \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right) Cu + \dots\right.\\
+ \quad & \left.A_{ik}\frac{\partial^2 \eta}{\partial x_i \partial x_k} + B_i \frac{\partial \eta}{\partial x_i} + C\eta + \left(\xi_r \frac{\partial C}{\partial x_r} + \tau \frac{\partial C}{\partial t}\right) u\right) = 0.
\end{aligned} \tag{1.157}$$

Each of the four groups are equal to zero; we have an over-determined system of equations known as determining equations

$$A_{ik}(t,\mathbf{x})\frac{\partial^2\eta(t,\mathbf{x},u(t,\mathbf{x}))}{\partial u^2} = 0, \tag{1.158}$$

$$\tau(t)\frac{\partial A_{ik}}{\partial t} + \frac{\partial \tau}{\partial t}\tau A_{ik} + \xi(t,\mathbf{x})_r\frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r} = 0, \tag{1.159}$$

$$\begin{aligned}&\tau\frac{\partial B_i(t,\mathbf{x})}{\partial t} + \frac{\partial \tau}{\partial t}\tau B_i + \xi_r\frac{\partial B_i}{\partial x_r} - B_r\frac{\partial \xi_i}{\partial x_r} - \frac{\partial \xi_i}{\partial t} + A_{ik}\frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right) + \ldots\\ &+A_{ri}\frac{\partial}{\partial x_r}\left(\frac{\partial \eta}{\partial u}\right) - A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k} = 0,\end{aligned} \tag{1.160}$$

$$\begin{aligned}&\frac{\partial \eta}{\partial t} - \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)C(t,\mathbf{x})u + A_{ik}\frac{\partial^2\eta}{\partial x_i\partial x_k} + B_i\frac{\partial \eta}{\partial x_i} + C\eta + \ldots\\ &+\left(\xi_r\frac{\partial C}{\partial x_r} + \tau\frac{\partial C}{\partial t}\right)u = 0.\end{aligned} \tag{1.161}$$

Knowing that the A_{ik}'s components are not all zero, we see that

$$\frac{\partial^2\eta}{\partial u^2} = 0, \tag{1.162}$$

which means that η is linear in u

$$\eta = \alpha_1(t,\mathbf{x}) + \alpha_2(t,\mathbf{x})u. \tag{1.163}$$

Substituting for η in (1.160) we get

$$\begin{aligned}&\tau\frac{\partial B_i(t,\mathbf{x})}{\partial t} + \frac{\partial \tau}{\partial t}\tau B_i + \xi_r\frac{\partial B_i}{\partial x_r} - B_r\frac{\partial \xi_i}{\partial x_r} - \frac{\partial \xi_i}{\partial t} + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k} + \ldots\\ &+A_{ri}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_r} - A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k} = 0.\end{aligned} \tag{1.164}$$

(1.165)

After insisting for η, the last of the four determining equations splits into two separate equations: one as a coefficient of u and the other a coefficient of 1:

$$\begin{aligned}&\left(\frac{\partial \alpha_2(t,\mathbf{x})}{\partial t} + \frac{\partial \tau}{\partial t}C + A_{ik}\frac{\partial^2\alpha_2(t,\mathbf{x})}{\partial x_i\partial x_k} + B_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + \ldots\right.\\ +\quad &\left.\xi_r\frac{\partial C}{\partial x_r} + \tau\frac{\partial C}{\partial t}\right)u \qquad (1.166)\\ +\quad &\left(\frac{\partial \alpha_1(t,\mathbf{x})}{\partial t} + A_{ik}\frac{\partial^2\alpha_1(t,\mathbf{x})}{\partial x_i\partial x_k} + B_i\frac{\partial \alpha_1(t,\mathbf{x})}{\partial x_i} + C\alpha_1(t,\mathbf{x})\right) = 0. \qquad (1.167)\end{aligned}$$

The two groups are each equal to zero, since the u is explicit throughout to get

$$\begin{aligned}&\frac{\partial \alpha_2(t,\mathbf{x})}{\partial t} + \frac{\partial \tau}{\partial t}C + A_{ik}\frac{\partial^2\alpha_2(t,\mathbf{x})}{\partial x_i\partial x_k} + B_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + \ldots\\ &+\xi_r\frac{\partial C}{\partial x_r} + \tau\frac{\partial C}{\partial t} = 0\end{aligned} \tag{1.168}$$

$$\frac{\partial \alpha_1(t,\mathbf{x})}{\partial t} + A_{ik}\frac{\partial^2\alpha_1(t,\mathbf{x})}{\partial x_i\partial x_k} + B_i\frac{\partial \alpha_1(t,\mathbf{x})}{\partial x_i} + C\alpha_1(t,\mathbf{x}) = 0 \tag{1.169}$$

Equation (1.169) is just the FP equation for $\alpha_1(t,\mathbf{x})$. Thus it is left alone from here on as this gives the infinite number of solutions symmetries. As for the coefficients A_{ik}, B_i and C, we can eliminate two of the three so as to simplify the determining equations further. The relations (1.45) and (1.46) will now be used in (1.159), (1.164) and (1.168) to achieve this and give

$$\frac{\partial(\tau A_{ik})}{\partial t} + \left(\xi_r \frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0 \tag{1.170}$$

$$\begin{aligned}&\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - 2\left(\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right) + \ldots\right.\\ &\left. + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - \frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r} + \xi_r\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k}\right) = 0\end{aligned} \tag{1.171}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha_2(t,\mathbf{x}) + \tau\left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right)\right) + f_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k} + \ldots\\ &+2\frac{\partial A_{ik}}{\partial x_k}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + \xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r} + \xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_k \partial x_r} = 0\end{aligned} \tag{1.172}$$

Keeping in mind the determining equations associated with the SDEs, (1.186) and (1.187), we now manipulate the determining systems above. We multiply (1.170) by 2

$$2\frac{\partial(\tau A_{ik})}{\partial t} + \left(2\xi_r\frac{\partial A_{ik}}{\partial x_r} - 2A_{ir}\frac{\partial \xi_k}{\partial x_r} - 2A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0, \tag{1.173}$$

differentiate with respect to x_k

$$\begin{aligned}&2\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right) + \left(2\xi_r\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k} + 2\frac{\partial \xi_r}{\partial x_k}\frac{\partial A_{ik}}{\partial x_r} - 2\frac{\partial A_{ir}}{\partial x_k}\frac{\partial \xi_k}{\partial x_r} - 2A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + \ldots\right.\\ - \quad &\left. 2\frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r} - 2A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}\right) = 0,\end{aligned} \tag{1.174}$$

and using the repeated index summation convention, we sum over all k; add the resulting equation (1.174) to (1.171), to arrive at

$$\begin{aligned}&2\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right) + \left(2\xi_r\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k} + 2\frac{\partial \xi_r}{\partial x_k}\frac{\partial A_{ik}}{\partial x_r} - 2\frac{\partial A_{ir}}{\partial x_k}\frac{\partial \xi_k}{\partial x_r} - 2A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + \ldots\right.\\ - \quad &\left. 2\frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r} - 2A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}\right) + \frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} + \ldots\\ &-2\left(\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right) + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - \frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r} + \xi_r\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k}\right),\end{aligned} \tag{1.175}$$

which simplifies to

$$\begin{aligned}&\left(-2A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k}\right) + \frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} + \ldots\\ &-\xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - 2\left(A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right) = 0.\end{aligned} \tag{1.176}$$

Rewriting the above we arrive at

$$\begin{aligned}&\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} + \ldots\\ &-2\left(A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right) = 0.\end{aligned} \tag{1.177}$$

Next we differentiate (1.170) w.r.t. x_k and x_i

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\tau\frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right)+\dots\\
&+\left(\frac{\partial \xi_r}{\partial x_i}\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k}+\xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_r \partial x_k}+\frac{\partial^2 \xi_r}{\partial x_i \partial x_k}\frac{\partial A_{ik}}{\partial x_r}+\frac{\partial \xi_r}{\partial x_k}\frac{\partial^2 A_{ik}}{\partial x_i \partial x_r}+\right.\\
&-\frac{\partial^2 A_{ir}}{\partial x_i \partial x_k}\frac{\partial \xi_k}{\partial x_r}-\frac{\partial A_{ir}}{\partial x_k}\frac{\partial^2 \xi_k}{\partial x_i \partial x_r}-\frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k}-A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k}\dots\\
&\left.-\frac{\partial^2 A_{rk}}{\partial x_i \partial x_k}\frac{\partial \xi_i}{\partial x_r}-\frac{\partial A_{rk}}{\partial x_k}\frac{\partial^2 \xi_i}{\partial x_i \partial x_r}-\frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}-A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}\right)=0.
\end{aligned}
\tag{1.178}
$$

Then differentiate equation (1.177) with repsect to x_i to deduce

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial \xi_i}{\partial x_i}-\tau\frac{\partial f_i}{\partial x_i}\right)+\frac{\partial f_r}{\partial x_i}\frac{\partial \xi_i}{\partial x_r}+f_r\frac{\partial^2 \xi_i}{\partial x_i \partial x_r}+\dots\\
&-\frac{\partial \xi_r}{\partial x_i}\frac{\partial f_i}{\partial x_r}-\xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r}-\frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}-A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}+\dots\\
&-2\left(\frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k}+A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k}+\frac{\partial A_{ik}}{\partial x_i}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}+A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k}\right)=0
\end{aligned}
\tag{1.179}
$$

We now add and subtract (1.179) and (1.178), respectively from (1.172) to get,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_2(t,\mathbf{x})+\tau\left(\frac{\partial f_i}{\partial x_i}+\frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right)\right)+f_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i}+A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k}+\dots\\
&+2\frac{\partial A_{ik}}{\partial x_k}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i}+\xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r}+\xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_k \partial x_r}+\dots\\
&-\frac{\partial}{\partial t}\left(\tau\frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right)+\dots\\
&-\left(\frac{\partial \xi_r}{\partial x_i}\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k}+\xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_r \partial x_k}+\frac{\partial^2 \xi_r}{\partial x_i \partial x_k}\frac{\partial A_{ik}}{\partial x_r}+\frac{\partial \xi_r}{\partial x_k}\frac{\partial^2 A_{ik}}{\partial x_i \partial x_r}+\right.\\
&-\frac{\partial^2 A_{ir}}{\partial x_i \partial x_k}\frac{\partial \xi_k}{\partial x_r}-\frac{\partial A_{ir}}{\partial x_k}\frac{\partial^2 \xi_k}{\partial x_i \partial x_r}-\frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k}-A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k}\dots\\
&\left.-\frac{\partial^2 A_{rk}}{\partial x_i \partial x_k}\frac{\partial \xi_i}{\partial x_r}-\frac{\partial A_{rk}}{\partial x_k}\frac{\partial^2 \xi_i}{\partial x_i \partial x_r}-\frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}-A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}\right)+\dots\\
&\frac{\partial}{\partial t}\left(\frac{\partial \xi_i}{\partial x_i}-\tau\frac{\partial f_i}{\partial x_i}\right)+\frac{\partial f_r}{\partial x_i}\frac{\partial \xi_i}{\partial x_r}+f_r\frac{\partial^2 \xi_i}{\partial x_i \partial x_r}+\dots\\
&-\frac{\partial \xi_r}{\partial x_i}\frac{\partial f_i}{\partial x_r}-\xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r}-\frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k}-A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}+\dots\\
&-2\left(\frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k}+A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k}+\frac{\partial A_{ik}}{\partial x_i}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}+A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k}\right)=0.
\end{aligned}
\tag{1.180}
$$

This simplifies drastically to

$$
\frac{\partial \alpha_2(t,\mathbf{x})}{\partial t}+\frac{\partial}{\partial t}\frac{\partial \xi_i}{\partial x_i}+f_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i}+f_r\frac{\partial^2 \xi_i}{\partial x_i \partial x_r}-A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}-A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k}=0.
\tag{1.181}
$$

Recollecting terms, we arrive at [1]'s result

$$
\left(\frac{\partial}{\partial t}+f_i\frac{\partial}{\partial x_i}-A_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right)\left(\alpha_2(t,\mathbf{x})+\frac{\partial \xi_r}{\partial x_r}\right)=0.
\tag{1.182}
$$

Thus, giving a new system of determining equations

$$\frac{\partial(\tau A_{ik})}{\partial t}+\left(\xi_r\frac{\partial A_{ik}}{\partial x_r}-A_{ir}\frac{\partial \xi_k}{\partial x_r}-A_{rk}\frac{\partial \xi_i}{\partial x_r}\right)=0 \tag{1.183}$$

$$\begin{aligned}&\frac{\partial(\xi_i-\tau f_i)}{\partial t}+f_r\frac{\partial \xi_i}{\partial x_r}-\xi_r\frac{\partial f_i}{\partial x_r}-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}+\ldots\\ &-2\left(A_{ir}\frac{\partial^2\xi_k}{\partial x_r\partial x_k}+A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right)=0.\end{aligned} \tag{1.184}$$

$$\left(\frac{\partial}{\partial t}+f_i\frac{\partial}{\partial x_i}-A_{ik}\frac{\partial^2}{\partial x_i\partial x_k}\right)\left(\alpha_2(t,\mathbf{x})+\frac{\partial\xi_r}{\partial x_r}\right)=0. \tag{1.185}$$

We now compare these results to the determining equations (1.186) and (1.187)

$$f_l\frac{\partial \tau}{\partial t}+\tau\frac{\partial f_l}{\partial t}+\xi_j\frac{\partial f_l}{\partial x_j}-\left(\frac{\partial \xi_l}{\partial t}+f_i\,\frac{\partial \xi_l}{\partial x_i}+\frac{1}{2}\sum_{k=1}^{M}G_i^k\,G_j^k\,\frac{\partial^2\xi_l}{\partial x_i\partial x_j}\right)=0 \tag{1.186}$$

and

$$\tau(t)\frac{\partial G_k^i}{\partial t}+\xi_j(t,\mathbf{x})\frac{\partial G_k^i}{\partial x_j}-G_k^i\frac{\partial \xi_l}{\partial x_i}+\frac{1}{2}G_k^i\frac{\partial\tau}{\partial t}=0. \tag{1.187}$$

Rewriting (1.183), (1.184) and (1.185) w.r.t. $\mathbf{G(t,x)}$ gives

$$\begin{aligned}&G_i^j\left(\tau\frac{\partial G_j^k}{\partial t}+\frac{1}{2}G_j^k\frac{\partial\tau}{\partial t}+\xi_r\frac{\partial G_j^k}{\partial x_r}-G_j^r\frac{\partial\xi_k}{\partial x_r}\right)+\ldots\\ &+G_j^k\left(\tau\frac{\partial G_i^j}{\partial t}+\frac{1}{2}G_i^j\frac{\partial\tau}{\partial t}+\xi_r\frac{\partial G_i^j}{\partial x_r}-G_r^j\frac{\partial\xi_k}{\partial x_r}\right)=0,\end{aligned} \tag{1.188}$$

$$\begin{aligned}&-\left(f_i\frac{\partial\tau}{\partial t}+\tau\frac{\partial f_i}{\partial t}+\xi_r\frac{\partial f_i}{\partial x_r}-\left(\frac{\partial\xi_i}{\partial t}+f_j\,\frac{\partial\xi_i}{\partial x_j}+\frac{1}{2}\sum_{k=1}^{M}G_j^k\,G_r^k\,\frac{\partial^2\xi_i}{\partial x_j\partial x_r}\right)\right)+\ldots\\ &+\sum_{j=1}^{M}G_i^jG_r^j\frac{\partial^2\xi_k}{\partial x_r\partial x_k}+\sum_{j=1}^{M}G_i^jG_k^j\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_k}=0\end{aligned} \tag{1.189}$$

$$\left(\frac{\partial}{\partial t}+f_i\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{j=1}^{M}G_i^jG_k^j\frac{\partial^2}{\partial x_i\partial x_k}\right)\left(\alpha_2(t,\mathbf{x})+\frac{\partial\xi_r}{\partial x_r}\right)=0. \tag{1.190}$$

We find that (1.188) is a satisfied as a direct result of (1.187). However, the converse is not true. Also from (1.189), (1.186) forces

$$\sum_{j=1}^{M}G_i^jG_k^j\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_k}=-\sum_{j=1}^{M}G_i^jG_r^j\frac{\partial^2\xi_k}{\partial x_r\partial x_k}, \tag{1.191}$$

which when we integrate w.r.t. x_k gives

$$\alpha_2(t,\mathbf{x})=-\frac{\partial\xi_r}{\partial x_r}+\beta_1,\ \text{where } \beta_1 \text{ is arbitrary constant.} \tag{1.192}$$

Looking now at the normalization condition (1.48), under the transformation the probability density function $p(t,\mathbf{x})$ undergoes the following

$$\bar{t} = t+\epsilon\tau(t), \tag{1.193}$$

$$\overline{x_i} = x_i+\epsilon\xi_i(t,\mathbf{x}),\ \text{and} \tag{1.194}$$

$$\overline{p} = p+\epsilon\eta(t,\mathbf{x},p), \tag{1.195}$$

The volume element $dx_1\ldots dx_N$ undergoes the following for each i, by use of (1.194)

$$d\overline{x}_i=dx_i+\epsilon\frac{\partial\xi_i}{\partial x_j}dx_j. \tag{1.196}$$

Any element dx_i which is raised to a power higher than one, becomes so small that it becomes negligible; with this in mind we see that

$$d\overline{x}_1 \dots d\overline{x}_N = \left(1 + \epsilon \frac{\partial \xi_j}{\partial x_j}\right) dx_1 \dots dx_N. \tag{1.197}$$

Thus at first order in ϵ we have that

$$\int_{-\infty}^{\infty} p(\overline{t}, \overline{\mathbf{x}}) d\overline{x}_1 \dots d\overline{x}_N = \int_{-\infty}^{\infty} \left(p(t, \mathbf{x}) + \epsilon \left(\alpha_1(t, \mathbf{x}) + \left(\alpha_2(t, \mathbf{x}) + \frac{\partial \xi_j}{\partial x_j}\right) p\right)\right) dx_1 \dots dx_N \tag{1.198}$$

which we separate out to get

$$\begin{aligned} \int_{-\infty}^{\infty} p(\overline{t}, \overline{\mathbf{x}}) d\overline{x}_1 \dots d\overline{x}_N &= \int_{-\infty}^{\infty} p(t, \mathbf{x}) dx_1 \dots dx_N + \dots \\ \epsilon \int_{-\infty}^{\infty} &\left(\alpha_1(t, \mathbf{x}) + \left(\alpha_2(t, \mathbf{x}) + \frac{\partial \xi_j}{\partial x_j}\right) p\right) dx_1 \dots dx_N \end{aligned} \tag{1.199}$$

Thus what [1] showed was that $\alpha_1(t, \mathbf{x})$ should be zero under integration, i.e.

$$\int_{-\infty}^{\infty} \alpha_1(t, \mathbf{x}) dx_1 \dots dx_N = 0, \tag{1.200}$$

in order for the normalization condition (1.48) to be satisified, i.e. in order for term following the ϵ to vanish. This then as a result forced

$$\alpha_2(t, \mathbf{x}) = -\frac{\partial \xi_r}{\partial x_r} \tag{1.201}$$

which is the condition we found earlier (1.192), with the arbitrary constant β_1 put to zero.

Therefore with these conditions (1.200) and (1.201) the complete probabilistic equivalence between the Itô and FP equations, which we saw in the Feynman-Kac theorem, is maintained. Thus the symmetry operator H_0 would transform the Itô equation into another; maintaining all the same probabilistic properties.

In summary it was shown that H could only be a symmetry operator for Itô SDE if (1.187) was satisified. The FP symmetry operator $H_0 + \eta\, \partial/\partial u$, where $\eta = \alpha_1(t, \mathbf{x}) + \alpha_2(t, \mathbf{x})u$, with $\alpha_1(t, \mathbf{x})$ being a solution to the FP equation and satisfying (1.200); $\alpha_2(t, \mathbf{x})$ satisfying (1.192), can now be found. Here are the FP equivalent equations from the previous examples.

Example 1.10. The FP equation

$$\frac{\partial u}{\partial t} = \sigma \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \tag{1.202}$$

has the determining equations

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0 \tag{1.203}$$

$$\frac{\partial \xi}{\partial t} + \frac{3}{2}\sigma^2 \frac{\partial^2 \xi}{\partial x^2} + \sigma^2 \frac{\partial \alpha_2(t, x)}{\partial x} = 0 \tag{1.204}$$

$$\frac{\partial \alpha_2(t, x)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \alpha_2(t, x)}{\partial x^2} + \frac{\partial^2 \xi}{\partial t \partial x} + \frac{1}{2}\sigma^2 \frac{\partial^3 \xi}{\partial x^3} = 0, \tag{1.205}$$

which solves as

$$\tau = C_0 + C_1 2t + C_2 t^2 \tag{1.206}$$

$$\xi = C_1 x + C_2 x t + C_3 + C_4 \sigma^2 t \tag{1.207}$$

$$\phi = \left(\frac{1}{2}\left(\frac{x^2}{\sigma^2} + t\right) u C_2 + C_4 x u + C_5\right) u + \alpha(t, x). \tag{1.208}$$

Example 1.11. The FP equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x^2 \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \quad (1.209)$$

results in the determining equations

$$\xi = 0 \quad (1.210)$$

$$\dot{\tau} = 0 \quad (1.211)$$

$$\frac{\partial \alpha_2(t,x)}{\partial x} = 0 \quad (1.212)$$

$$\frac{\partial \alpha_2(t,x)}{\partial t} = 0. \quad (1.213)$$

$$(1.214)$$

Solving this yields

$$\tau = C_0 \quad (1.215)$$

$$\xi = 0 \quad (1.216)$$

$$\phi = C_1 u + \alpha(t,x). \quad (1.217)$$

Example 1.12. The FP equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \quad (1.218)$$

has the associated determining equations

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0 \quad (1.219)$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial \alpha_2(t,x)}{\partial x} - x\frac{\partial \xi}{\partial x} - \xi = 0 \quad (1.220)$$

$$\frac{\partial \alpha_2(t,x)}{\partial t} + x\frac{\partial \alpha_2(t,x)}{\partial x} + \frac{1}{2}\frac{\partial^2 \alpha_2(t,x)}{\partial x^2} + \frac{\partial^2 \xi}{\partial t \partial x} = 0, \quad (1.221)$$

which gives rise to

$$\tau = C_3 e_{2t} - \frac{C_0}{2} e_{-2t} + C_5 \quad (1.222)$$

$$\xi = \left(C_3 e_{2t} + \frac{C_0}{2} e_{-2t} \right) x + C_4 e_t + \frac{C_1}{2} e_{-t} \quad (1.223)$$

$$\phi = \left(C_0 e_{-2t} \frac{x^2}{2} + C_1 e_{-t} x + C_2 - C_3 e_{2t} \right) u + \alpha(t,x) \quad (1.224)$$

Example 1.13. The FP equation

$$\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial x} = \frac{1}{2} D \frac{\partial^2 u}{\partial x^2} \quad (1.225)$$

is associated to the determining equations

$$\frac{1}{2}\dot{\tau} - \frac{\partial \xi}{\partial x} = 0 \quad (1.226)$$

$$\frac{\partial \xi}{\partial t} + D\frac{\partial \alpha_2(t,x)}{\partial x} - g\frac{\partial \xi}{\partial x} = 0 \quad (1.227)$$

$$\frac{\partial \alpha_2(t,x)}{\partial t} + g\frac{\partial \alpha_2(t,x)}{\partial x} + \frac{1}{2}D\frac{\partial^2 \alpha_2(t,x)}{\partial x^2} + \frac{\partial^2 \xi}{\partial t \partial x} = 0. \quad (1.228)$$

This furnishes

$$\tau = -DC_0t^2 + 2C_3t + C_5 \tag{1.229}$$

$$\xi = (C_3 - DC_0t)\,x + (gC_3 - DC_1)\,t + C_4 \tag{1.230}$$

$$\phi = \left(C_0\frac{x^2}{2} - gC_0tx + C_1x + g^2\frac{C_0}{2}t^2 - \left(gC_1 - \frac{D}{2}C_0\right)t + C_2\right)u + \alpha(t,x). \tag{1.231}$$

Remark. Each of the examples above corresponded with the examples from the previous section. The algebras of the first order SODEs with one-dimensional Wiener processes form a sub-algebra of the original Lie point algebras associated with the FP equation.

1.3 Conclusion

The question of whether or not the Lie algebra of a first order SODEs, with multi-dimensional Wiener processes, will form a sub-algebra of the associated algebra related to FP equations needs to be investigated.

So far the *ansatz* that the temporal infinitesimal must be projective for both the FP and the related SODEs ensures that the symmetries generated from the SODEs is a sub-algebra of the one generated by the associated FP equation.

The question that now arises is whether or not the Lie point transformations of the SODEs are applicable if the infinitesimals are not projective. This leads us to the works of Wafo Soh and Mahomed [2], Ünal [3] and Fredericks and Mahomed [5] where no recourse is made to the FP equation. As a result, the *ansatz* of [1] is not assumed.

Chapter 2

Symmetries of First-Order Stochastic Ordinary Differential Equations Revisited

Symmetries of first order SODEs are analysed. This work focuses on maintaining the properties of the Weiner processes after the application of infinitesimal transformations. The determining equations for first-order SODEs are derived in an Itô calculus context. These determining equations are non-stochastic.

2.1 Introduction

Two years after the seminal work by Gaeta and Quintero [1] which brought to the fore the relations between the symmetries of the Fokker-Planck (FP) equation and its corresponding Itô stochastic (ordinary) differential equation (SDE), a paper by Wafo Soh and Mahomed [2] explained how to derive these Lie point symmetries without referring to the corresponding FP equations and without using these FP dependent symmetries to transform the Itô SDE into a different one as had been done in [7]. This novelty in methodology was able to incorporate higher order SDEs, for instance the governing equation for the response of a mass-spring oscillator to a random excitation induced by white noise, which we discuss in a later chapter.

However, this methodology neglected to apply the invariance principle to the underlying properties which drives the non-deterministic characteristic of Itô SDEs, namely the Wiener process' properties. This implies that the instantaneous mean of the transformed Wiener processes is not zero under expectation as it aught to be. As a result of this oversight, the determining equations related to the invariance of the diffusion coefficient of the SDEs are non-deterministic; there is a white noise term which survives the transformation. What is interesting is the fact that this non-determinstic white noise term does not appear in the determining equations for higher order SDEs.

Ünal [3] uncovered the reason for the non-determinstic nature of the determining equations associated with the diffusion coefficient of the Itô SDEs by applying an invariance principle to the Itô multiplication table (1.1). This removes the white noise term which appears in [2]. The determining equations [3] obtained for finding symmetries of first-order SDEs were superficially not in agreement with the version of [2] as it precluded an extra condition given in his derivation (see [3]).

This chapter is aimed at reconciling these two seminal works for first order SDEs: in the following section we derive the determining equations that are needed to solve for the symmetries. We closely follow the methodology of [2] in this regard:

- Apply infinitesimal transformations on the spatial, temporal and Wiener process variables.
- Apply infinitesimal transformations on the drift and diffusion coefficients of the SDEs.
- Induce an invariance transformation argument on the transformed SDEs in differential form.
- Show that the symmetry operator constructed from the infinitesimals is a symmetry of the Itô SDEs provided that the determining equations are satisfied.

However, we extend the derivation of [2] further and arrive at an alternative form of the same determining equations that were constructed by Ünal [3]. We also seek to ensure that the finite transformations can be recovered from the infinitesimal transformations; this leads to another condition on the temporal infinitesimal which neither works have considered. Thus this route not only leads to the same extra condition that was found in [3], it also yields another important condition on the temporal symmetry variable τ which ensures that the transformed Wiener differential still behaves like a standard Wiener process. We thus, in the third section, review the steps given in [3]; deriving these determining equations and comparing them to the ones found in the previous section mentioned above. We conclude with the same example used in [3] to provide evidence that we have reconciled the works of both [2] and [3]. We in fact show that Ünal's extra condition is a direct consequence of our extension using the properties of the Wiener process.

2.2 Derivation of the Determining Equations

Due to Itô Isometry, the finite quadratic variation of the Wiener process, i.e. $(dW(t))^2$ has a mean-squared value of dt (see Øksendal [9]), the Newton-Leibniz chain rule in differential form, which we need to apply to establish invariance arguments on our spatial, temporal and Wiener variables, has to be adjusted. This change leads to the Itô Formula and the Random Time Change Formula, which we stated earlier.
Consider an Itô process

$$d\mathbf{X}(t) = \mathbf{f}(t,\ \mathbf{X}(t))dt + \mathbf{G}(t,\ \mathbf{X}(t))\, d\mathbf{W}(t), \tag{2.1}$$

where $\mathbf{f}(t,\ \mathbf{x})$ is a vector of N dimension, which is the same as the dimension of the process $\mathbf{X}(t)$ and $\mathbf{G}(t,\ \mathbf{x})$ is an $N \times M$-matrix. These functions are evaluated at $\mathbf{X}(t)$ in the system of Itô processes above. The Lie Point Theorem symmetry approach for ODEs requires spatial and temporal infinitesimals $\xi_j(t,x)$ and $\tau(t,x)$, in its analysis. In the SODEs framework these entities are functionally based on the spatial stochastic process, $\mathbf{X}(t)$ and using Itô's formula (1.27), we have that the jth spatial infinitesimal, for $j = 1, ..., N$ and temporal infinitesimal are themselves solutions to Itô processes given in component form, respectively, as

$$d\xi_j(t,\ \mathbf{X}(t)) \quad = \quad \Gamma(\xi_j)\,(t, \mathbf{X}(t))\, dt + Y^l(\xi_j)\,(t, \mathbf{X}(t))\, dW_l(t) \tag{2.2}$$

and

$$d\tau(t,\ \mathbf{X}(t)) \quad = \quad \Gamma(\tau)\,(t, \mathbf{X}(t))\, dt + Y^l(\tau)\,(t, \mathbf{X}(t))\, dW_l(t), \tag{2.3}$$

where $\Gamma(\xi_j)$, $Y^l(\xi_j)$, $\Gamma(\tau)$ and $Y^l(\tau)$ are the drift and diffusion coefficients of our spatial and temporal infinitesimals, respectively and defined using (1.28) and (1.29). The Lie Point Theorem (see Wafo Soh and Mahomed [2]), as in [2] uses determining equations to furnish symmetries which would enable the transformation of a solution of the equation to another. These determining equations are in fact $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ equations derived from form invariant ODE point transformation analysis. The resultant higher order equations of this form invariant analysis, are functionally dependent on the solution of these equations. We perform a similar point transformation of (2.1)'s spatial, temporal and the Wiener variables

$$\begin{aligned} \overline{X}_j(t) &= e^{\epsilon H}(X_j(t)) \\ &= \int^t \Gamma(e^{\epsilon H}(X_j(s)))\, ds + \int^t Y(e^{\epsilon H}(X_j(s)))\, dW(s), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \bar{t} &= e^{\epsilon H}(t) \\ &= \int^t \Gamma(e^{\epsilon H}(s))\, ds + \int^t Y(e^{\epsilon H}(s))\, dW(s) \end{aligned} \tag{2.5}$$

and

$$d\overline{W}_l(\bar{t}) = \sqrt{\frac{d\left(e^{\epsilon H}(t)\right)}{dt}} dW_l(t), \text{ for each } l = 1, ..., M \tag{2.6}$$

using the random time change formula and Itô's formula; where H is the symmetry generator

$$H = \tau(t, \mathbf{x})\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x})\frac{\partial}{\partial x_j}, \tag{2.7}$$

with the spatial and temporal infinitesimals $\xi_j(t, \mathbf{x})$ and $\tau(t, \mathbf{x})$, respectively. The point transformations of the drift and diffusion coefficients are given by

$$f_j(\bar{t}, \overline{\mathbf{x}}) = e^{\epsilon H}(f_j(t,\, \mathbf{x})) \tag{2.8}$$

and

$$g_i^k(\bar{t}, \overline{\mathbf{x}}) = e^{\epsilon H}(g_i^k(t,\, \mathbf{x})), \tag{2.9}$$

for each $i,\, j = 1, ..., N$ and $k = 1, ..., N$. The transformations (2.4), (2.5), (2.6), (2.8) and (2.9) are used in conjunction with Itô's formula to form an invariant version of the original SODEs (2.1)

$$d\overline{\mathbf{X}}(\bar{t}) = \mathbf{f}(\bar{t},\ \overline{\mathbf{X}}(\bar{t}))d\bar{t} + \mathbf{G}(\bar{t},\ \overline{\mathbf{X}}(\bar{t}))\ d\overline{\mathbf{W}}(\bar{t}). \tag{2.10}$$

The transformed time index should be invariant in terms of its instantaneous drift and diffusion coefficients, which implies

$$\mathbb{E}\left[d\bar{t}(t,\omega)\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = d\bar{t}(t,\omega), \tag{2.11}$$

since this is trivially so for the original differential time index, dt. By using (2.5), we have

$$\int^t \mathbb{E}\left[\Gamma(e^{\epsilon H}(s))\middle|\, \mathbf{W}(s) = \mathbf{w},\ \mathbf{X}(s) = \mathbf{x}\right]\, ds = \int^t \Gamma(e^{\epsilon H}(s))\, ds + \int^t Y(e^{\epsilon H}(s))\, dW(s), \tag{2.12}$$

which forces

$$Y^l(e^{\epsilon H}(t)) = 0 \text{ for each } l = 1, ..., M \tag{2.13}$$

which gives the finite transformation version of the infinitesimal condition that Ünal [3] derived at $\mathcal{O}(\epsilon)$ using a form invariant argument on the Itô multiplication table. We also have that the instantaneous drift of the time index must be constant, i.e.

$$\Gamma(e^{\epsilon H}(t)) = Constant. \tag{2.14}$$

Similarly, the transformed standard Wiener differential process, $d\overline{\mathbf{W}}(\bar{t})$, should be invariant in terms of the existence of an instantaneous mean and variance which implies that the following should still hold, *viz.*

$$\mathbb{P}\left[\left|d\overline{W}_l(\bar{t})\right| > \epsilon\,\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = 0 \text{ for all } \epsilon > 0, \tag{2.15}$$

$$\mathbb{E}\left[d\overline{W}_l(\bar{t})\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = 0, \tag{2.16}$$

$$\mathbb{E}\left[d\overline{W}_l(\bar{t})\, d\overline{W}_m(\bar{t})\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = d\bar{t}\ \delta_l^m. \tag{2.17}$$

Expanding (2.16)

$$\mathbb{E}\left[\sqrt{\frac{\Gamma(e^{\epsilon H}(t))\, dt\, +\, Y^k(e^{\epsilon H}(t))\, dW_k(t)}{dt}} dW_l(t)\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = 0 \tag{2.18}$$

by using (2.5) in conjunction with (2.6) we note that the condition (2.13) allows the invariance argument (2.16) to be satisified

$$\mathbb{E}\left[d\overline{W}_l(\bar{t})\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] = \mathbb{E}\left[\sqrt{\Gamma(e^{\epsilon H}(t))} dW_l(t)\middle|\, \mathbf{W}(t) = \mathbf{w},\ \mathbf{X}(t) = \mathbf{x}\right] \tag{2.19}$$

$$= 0. \tag{2.20}$$

Thus we have the following theorem.

Theorem 2.1 (Lie Point Symmetry Rate of Time Change Formula).

The rate of time change for the random time change formula under the Lie point symmetry approach is the temporal instantaneous drift

$$\bar{t} = \int^{t} \Gamma(e^{\epsilon H}(s))(s,\ \mathbf{X}(s))ds. \tag{2.21}$$

As a result (2.6) becomes

$$d\overline{W}_l(\bar{t}) = \sqrt{\Gamma(e^{\epsilon H}(t))}dW_l(t), \text{ for each } l = 1, ..., M. \tag{2.22}$$

Since the temporal instantaneous drift is measureable as a result of Itô's formula, the random time change formula still holds for this application. Expanding the drift term $\mathbf{f}(\bar{t},\ \overline{\mathbf{X}}(\bar{t}))d\bar{t}$ on the right hand side of (2.10) with simple algebra gives

$$\Bigg\{ \mathbf{f}(t,\ \mathbf{X}(t)) + \epsilon\,(\Gamma(H(t)) + H)\ \mathbf{f}(t,\ \mathbf{X}(t)) + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \Bigg((\Gamma(H(t)) + H)^k\ \mathbf{f}(t,\ \mathbf{X}(t)) + \sum_{j=0}^{k-2} \binom{k}{k-j} \mathbf{f}(t,\ \mathbf{X}(t))\, H^j(t) \left(\Gamma(H^{k-j}(t)) - [\Gamma(H(t))]^{k-j} \right) \Bigg) \Bigg\}\ dt. \tag{2.23}$$

In order to use the Lie Point Theorem in the SODEs context we require that all terms of order higher than $\mathcal{O}(\epsilon)$ be functionally dependent on terms of order $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$. As a result of this dependency, higher order terms can be ignored completely and justifies the methods of [2] and [3]. This dependency, however, forces the following condition

$$e^{\epsilon\,\Gamma(H(t))}(t,\ \mathbf{X}(t)) = \Gamma\left(e^{\epsilon\,H}(t)(t,\ \mathbf{X}(t))\right) \tag{2.24}$$

and the resultant relationship, by separation of coefficients of ϵ, between the drift components of the left and right hand side of (2.10) can be expressed as

$$\Gamma(H^k(\mathbf{x}))(t,\ \mathbf{X}(t)) = (\Gamma(H(t)) + H)^k\ f(t,\ \mathbf{X}(t)), \tag{2.25}$$

for $k = 1, 2, 3, \ldots$. Thus for $k = 1$ we have our first determining equation

$$\Gamma(H(\mathbf{x})) = (\Gamma(H(t)) + H)\ f(t,\ \mathbf{X}(t)) \tag{2.26}$$

which partially solves for the spatial and temporal infinitesimals. By using the determining equation (2.26) in (2.25) for the remaining higher order equations, a direct functional dependency between the two is established by the following

$$\Gamma(H^k(\mathbf{x})) = (\Gamma(H(t)) + H)^{k-1}\ \Gamma(H(\mathbf{x})), \text{ for } k = 2, 3, 4, \ldots. \tag{2.27}$$

We thus have our next theorem,

Theorem 2.2 (Constant Temporal Infinitesimal Instantaneous Drift).

The instantaneous drift of the temporal infinitesimal τ has to be constant, i.e. $\Gamma(\tau)$ has to be constant, in order for the condition (2.24) to be satisfied.
Proof: by merely looking at the second term of (2.24) we have

$$(\Gamma(\tau))^2 = \Gamma(H(\tau)) \tag{2.28}$$

$$= \Gamma\left(H\left(\int^t \Gamma(\tau(s,\ \omega))ds\right)\right) \text{ (note } d\tau = \Gamma(\tau)\,dt) \tag{2.29}$$

$$= \Gamma\left(\tau\,\Gamma(\tau) + \xi_j \int^t \frac{\partial}{\partial x_j}\left(\Gamma(\tau(s,\ \omega))\right)ds\right). \tag{2.30}$$

After applying the Γ operator into the brackets, we have the following resulting equation after canceling out terms

$$\Gamma^2(\tau) + \Gamma(\xi_j) \int^t \frac{\partial}{\partial x_j} (\Gamma(\tau))\, ds + \xi_j \int^t \Gamma \left(\frac{\partial}{\partial x_j} (\Gamma(\tau)) \right) ds = 0. \tag{2.31}$$

This equation, without loss of generality, can only be satisfied if $\Gamma(\tau)$ is constant. But, we have shown that this is true if the transformed differential time index obeys condition (2.11). Before deriving the remaining determining equation, we first note that (2.22) can be written as

$$d\overline{W}_l(\bar{t}) = e^{\frac{\epsilon\Gamma(H(t))}{2}} dW_l(t), \text{ for each } l = 1, ..., M \tag{2.32}$$

as a result of (2.24). If we expand the diffusion component $\mathbf{G}(\bar{t},\ \overline{\mathbf{X}}(\bar{t}))\ d\overline{\mathbf{W}}(\bar{t})$ of (2.10) and then compare these components on both sides of (2.10) by separation of coefficients of ϵ, we get the following

$$Y^l(H(\mathbf{x}))(t,\ \mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H \right) G^l(t,\ \mathbf{X}(t)) \tag{2.33}$$

$$Y^l(H^k(\mathbf{x}))(t,\ \mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H \right)^{k-1} Y^l(H(\mathbf{x})), \text{ for } k = 2, 3, \ldots \tag{2.34}$$

for each $l = 1, ..., M$, where (2.33) is the last determining equation needed to solve for the infinitesimals. The functional dependency of higher order equations on zero and first order ones is satisfied in (2.34). All that remains to be shown is that the determining equations are unique to their SODEs from which they are derived. If we are given the determining equations (2.26) and (2.33) the canonical symmetry that is immediately applicable is the time scaling symmetry $H = \partial/\partial t$. From this we see that the drift and diffusion coefficients have to be functions of the spatial variable only in order to satisfy (2.26) and (2.33). Thus the SODEs associated with this particular symmetry is given by

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))dt + \mathbf{G}(\mathbf{X}(t))\ d\mathbf{W}(t). \tag{2.35}$$

Thus we have proved the following theorem which was partially proved in Wafo Soh and Mahomed [2].

Theorem 2.3. *Lie Point Theorem for SODEs*

The Itô SODEs

$$d\mathbf{X}(t) = \mathbf{f}(t,\ \mathbf{X}(t))dt + \mathbf{G}(t,\ \mathbf{X}(t))\ d\mathbf{W}(t) \tag{2.36}$$

has the following determining equations and conditions that have to hold in order to transform a solution of (2.36) to that of another solution using Lie point symmetry methods evaluated at $(t,\ \mathbf{X}(t))$

$$\Gamma(H(x)) = (\Gamma(H(t)) + H)\ f \tag{2.37}$$

$$Y^l(H(x)) = \left(\frac{\Gamma(H(t))}{2} + H \right) G^l \tag{2.38}$$

$$e^{\epsilon\, \Gamma(H(t))} = \Gamma\left(e^{\epsilon\, H}(t)\right) \tag{2.39}$$

and

$$Y^l(e^{\epsilon H}(t)) = 0, \tag{2.40}$$

for each $l = 1, ..., M$.

To establish a comparison between these results and those of [2] we resort to the definition of the first prolongation of an infinitesimal generator for non-stochastic ODEs

$$H_{[1]} = H + \xi_j^{[1]} \frac{\partial}{\partial \dot{x}_j}, \tag{2.41}$$

where

$$\dot{x}_j = \frac{dx_j}{dt} \tag{2.42}$$

$$= D_t x_j \tag{2.43}$$

$$\xi_j^{[1]} = D_t(\xi_j) - \dot{x}_j D_t(\tau) \tag{2.44}$$

$$= \frac{\partial \xi_j}{\partial t} + \dot{x}_i \frac{\partial \xi_j}{\partial x_i} - \dot{x}_j \left(\frac{\partial \tau}{\partial t} - \dot{x}_i \frac{\partial \tau}{\partial x_i} \right), \tag{2.45}$$

with the total time derivative D_t given as

$$D_t = \frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x_i} + \ddot{x}_i \frac{\partial}{\partial \dot{x}_i} + \dots \quad . \tag{2.46}$$

Applying the first prolongation on $(\dot{x}_j - f_j)$ at $\dot{\mathbf{x}} = \mathbf{f}$, can be represented as

$$H_{[1]}(\dot{x}_j - f_j)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} = \xi_j^{[1]} - H(f_j). \tag{2.47}$$

Using (2.45) we find that (2.47) in conjunction with the second-order derivative terms of the instantaneous spatial and temporal drifts constitute the whole of (2.26) and we can express this as

$$\left((H_{[1]}(\dot{x}_j - f_j)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_p^k \left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_p} - f_j \frac{\partial^2 \tau}{\partial x_i \partial x_p} \right) \right)(t, \mathbf{X}(t)) = 0. \tag{2.48}$$

If we now consider (2.38), there is no *white noise* term, $dW_l(t)/dt$, as was the case in the previous attempt by [2] since $Y^l(\tau) = 0$.

Theorem 2.4 (Infinitesimal Symmetries of the Itô equation)**.**

A vector field $H = \tau(t, \mathbf{x})\partial/\partial t + \xi_j(t, \mathbf{x})\partial/\partial x_j$ is a symmetry of (2.1) if and only if (2.37), (2.38), (2.39) and (2.40) are satisfied.

Proof: If H is an infinitesimal symmetry of (2.1), we merely follow the above derivation to $\mathcal{O}(\epsilon)$ and arrive at the result. If conversely (2.37), (2.38), (2.39) and (2.40) are satisfied, without loss of generality we can use the conanical variables approach to assume that $H = \partial/\partial t$. Thus the equations (2.37), (2.38), (2.39) and (2.40) will lead to the drift and diffusion coefficients of (2.1) being functions of the spatial variables only, i.e. the Itô equation will be invariant under H.

2.3 Ünal's Extra Condition

Ünal [3] commented that the Itô multiplication table for the transformed variables must be applicable, i.e.

$$d\overline{W}_l(\bar{t})\, d\overline{W}_m(\bar{t}) = \delta_l^m\, d\bar{t}, \tag{2.49}$$

$$d\overline{W}_i(\bar{t})\, d\bar{t} = 0, \tag{2.50}$$

$$d\bar{t}\, d\bar{t} = 0 \tag{2.51}$$

for each i, l and $m = 1, ..., M$ and derived his DEs from this standpoint. Recently [4] stated that no strict proof had been done in the past to verify that the transformed Wiener processes using the random time change formula would still satisfy the properties of a Wiener process. All that the random time change formula requires for it to be applicable to SODEs, is the measurability of the rate of time change, which Itô's formula preserves. The spatial process $\mathbf{X}(t)$ is measurable at the onset, so all functions of this stochastic process will be measurable too. The strict proof has been done in [9] and [8]; the consequences of these properties on the symmetry infinitesimals were investigated in [3]. Using the results (2.21) and (2.32), we find

$$d\overline{W}_l(\bar{t})\, d\overline{W}_m(\bar{t}) = e^{\epsilon\frac{\Gamma(\tau)}{2} + \epsilon\frac{\Gamma(\tau)}{2}} dW_l(t) dW_m(t) \delta_l^m = \delta_l^m\, e^{\epsilon\Gamma(\tau)} dt = \delta_l^m\, \overline{dt} \tag{2.52}$$

$$d\overline{W}_l(\bar{t}) d\bar{t} = e^{\frac{3}{2}\epsilon\Gamma(\tau)} dW(t) dt = 0 \tag{2.53}$$

and

$$d\bar{t}d\bar{t} = 0 \text{ are automatically satisfied.} \tag{2.54}$$

for each i, l and $m = 1, ..., M$. Thus our application of the Lie Point Theorem for SODEs is consistent with the criteria set by Ünal [3].

2.4 Examples

We use the same example as Ünal [3] to show that the symmetries, which we arrive at using (2.26) and (2.33), are the same as what was found in [3]. This is our first example

Example 2.1 (Brownian motion on a circle)**.**

Let $\mathbf{X}(t)$ be an Itô process

$$d\mathbf{X}(t) = \mathbf{f}\ dt + \mathbf{G}\ dW(t), \tag{2.55}$$

where $\mathbf{f}$ is the vector

$$\begin{pmatrix} -\frac{1}{2}X_1(t) \\ -\frac{1}{2}X_2(t) \end{pmatrix} \tag{2.56}$$

and $\mathbf{G}$ the vector

$$\begin{pmatrix} -X_2(t) \\ X_1(t) \end{pmatrix}. \tag{2.57}$$

Thus from [2]'s corrected version of the determining equations (2.48) and (2.33), we have for $j = 1$:

$$H^{[1]}\left(\dot{x}_1 + \frac{1}{2}x_1\right)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} + \frac{1}{2}G_i^1 G_p^1 \left(\frac{\partial^2 \xi_1}{\partial x_i \partial x_p} + \frac{1}{2}x_1 \frac{\partial^2 \tau}{\partial x_i \partial x_p}\right) = 0, \tag{2.58}$$

$$-\xi_2 - G_i^1\left(\frac{\partial \xi_1}{\partial x_i}\right) - \frac{1}{2}x_2\left(\frac{\partial \tau}{\partial t} + f_i\,\frac{\partial \tau}{\partial x_i} + \frac{1}{2}\sum_{l=1}^{M} G_i^l\, G_p^l\, \frac{\partial^2 \tau}{\partial x_i \partial x_p}\right) = 0 \tag{2.59}$$

and for $j = 2$:

$$H^{[1]}\left(\dot{x}_2 + \frac{1}{2}x_2\right)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} + \frac{1}{2}G_i^1 G_p^1 \left(\frac{\partial^2 \xi_2}{\partial x_i \partial x_p} + \frac{1}{2}x_2 \frac{\partial^2 \tau}{\partial x_i \partial x_p}\right) = 0 \tag{2.60}$$

$$\xi_1 - G_i^1\left(\frac{\partial \xi_2}{\partial x_i}\right) + \frac{1}{2}x_1\left(\frac{\partial \tau}{\partial t} + f_i\,\frac{\partial \tau}{\partial x_i} + \frac{1}{2}\sum_{l=1}^{M} G_i^l\, G_p^l\, \frac{\partial^2 \tau}{\partial x_i \partial x_p}\right) = 0. \tag{2.61}$$

The prolongations of the spatial infinitesimals are given for j equal to 1 and 2 respectively as

$$\begin{aligned} \xi_1^{[1]} &= \frac{\partial \xi_1}{\partial t} + \dot{x}_i \frac{\partial \xi_1}{\partial x_i} - \dot{x}_1\left(\frac{\partial \tau}{\partial t} - \dot{x}_i \frac{\partial \tau}{\partial x_i}\right) \\ &= \frac{\partial \xi_1}{\partial t} + f_i \frac{\partial \xi_1}{\partial x_i} + \frac{1}{2}x_1\left(\frac{\partial \tau}{\partial t} - f_i \frac{\partial \tau}{\partial x_i}\right), \end{aligned} \tag{2.62}$$

$$\begin{aligned} \xi_2^{[1]} &= \frac{\partial \xi_2}{\partial t} + \dot{x}_i \frac{\partial \xi_2}{\partial x_i} - \dot{x}_2\left(\frac{\partial \tau}{\partial t} - \dot{x}_i \frac{\partial \tau}{\partial x_i}\right) \\ &= \frac{\partial \xi_1}{\partial t} + f_i \frac{\partial \xi_1}{\partial x_i} + \frac{1}{2}x_2\left(\frac{\partial \tau}{\partial t} - f_i \frac{\partial \tau}{\partial x_i}\right) \end{aligned} \tag{2.63}$$

as we are evaluating at $\dot{x}_i = f_i$ in both cases of j. Substituting the above into the refurbished determining equations of Wafo Soh and Mahomed [2], i.e. equations (2.48) and (2.33), we find the following once we have multiplied equations (2.58) and (2.60) by a factor of two:

$$-\xi_2 + x_2\frac{\partial \xi_2}{\partial x_2} - x_1^2\frac{\partial^2\xi_2}{\partial x_2^2} + x_1\frac{\partial \xi_2}{\partial x_1} + 2x_1x_2\frac{\partial^2\xi_2}{\partial x_1\partial x_2} - x_2^2\frac{\partial^2\xi_2}{\partial x_1^2} = 0 \tag{2.64}$$

$$\xi_1 - x_1\frac{\partial \xi_2}{\partial x_2} + x_2\frac{\partial \xi_2}{\partial x_1} = 0 \tag{2.65}$$

$$-\xi_1 + x_2\frac{\partial \xi_1}{\partial x_2} - x_1^2\frac{\partial^2\xi_1}{\partial x_2^2} + x_1\frac{\partial \xi_1}{\partial x_1} + 2x_1x_2\frac{\partial^2\xi_1}{\partial x_1\partial x_2} - x_2^2\frac{\partial^2\xi_1}{\partial x_1^2} = 0 \tag{2.66}$$

$$\xi_2 - x_1\frac{\partial \xi_1}{\partial x_2} + x_2\frac{\partial \xi_1}{\partial x_1} = 0 \tag{2.67}$$

The final determining equation now needed is the extra condition (2.13) which reconciles both papers, *viz.*

$$-x_1\frac{\partial \tau}{\partial x_2} + x_2\frac{\partial \tau}{\partial x_1} = 0, \tag{2.68}$$

where the evaluation at $(t,\ \mathbf{X}(t))$ has not taken place. Solving these deterministic equations give

$$\tau(t, \mathbf{X}(t)) = C_0F_0\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right), \tag{2.69}$$

$$\xi_1(t, \mathbf{X}(t)) = C_1F_1\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)X(t)_1 + C_2F_2\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)X(t)_2 \tag{2.70}$$

and

$$\xi_2(t, \mathbf{X}(t)) = C_1F_1\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)X(t)_2 - C_2F_2\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)X(t)_1 \tag{2.71}$$

which are the same results that Ünal [3] had found. The condition (Lie point SODEs condition) is satisfied, since $\Gamma(\tau) = 0$ and $H(\tau) = \tau\,\Gamma(\tau) = 0$. To demonstrate that a solution of one SODEs is transformed to that of another, we choose a simple example where $F_1\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right) = F_2\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right) = 1$. Thus we have the following resulting symmetry generators

$$H_0 = F_0(\frac{x_2^2 + x_1^2}{2})\frac{\partial}{\partial t}, \tag{2.72}$$

$$H_1 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \tag{2.73}$$

and

$$H_2 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}. \tag{2.74}$$

The point transformations associated with (2.72) are

$$\overline{x}_1(\overline{t}) = x_1 \tag{2.75}$$

$$\overline{x}_2(\overline{t}) = x_2 \tag{2.76}$$

and

$$\overline{t} = t + F_0(\frac{x_2^2 + x_1^2}{2})\epsilon. \tag{2.77}$$

The point transformations associated with (2.73) are

$$\overline{x}_1(\overline{t}) = x_1e^{\epsilon} \tag{2.78}$$

$$\overline{x}_2(\overline{t}) = x_2e^{\epsilon} \tag{2.79}$$

and

$$\bar{t} \,=\, t. \tag{2.80}$$

The point transformation associated with (2.74) are

$$\overline{x}_1(\bar{t}) \,=\, x_1\,\cos(\epsilon)\,+\,x_2\,\sin(\epsilon) \tag{2.81}$$
$$\overline{x}_2(\bar{t}) \,=\, -x_1\,\sin(\epsilon)\,+\,x_2\,\cos(\epsilon) \tag{2.82}$$

and

$$\bar{t} \,=\, t. \tag{2.83}$$

The point transformations associated with (2.72) and (2.73) trivially verify form invariance when Itô's formula is applied. This is especially for H_0 where the temporal infinitesimal is zero under both the Γ and Y^1 operators. Applying Itô's formula to (2.81) and (2.82) gives the following

$$\begin{aligned}
d\overline{X}_1(\bar{t}) \,&=\, dX_1(t)\,\cos(\epsilon)\,+\,dX_2(t)\,\sin(\epsilon)\\
&= \left(\frac{-X_1(t)}{2}\,\cos(\epsilon) + \frac{-X_2(t)}{2}\,\sin(\epsilon)\right)d\bar{t}\,+\,(-X_2(t)\,\cos(\epsilon) - X_1(t)\,\sin(\epsilon))\,d\overline{W}(\bar{t})\\
&= \left(e^{\epsilon H_2}(f_1(X_1(t)))\right)d\bar{t} + \left(e^{\epsilon H_2}(G_1(X_2(t))))\right)d\overline{W}(\bar{t}) && (2.84)\\
&= f_1(\bar{t},\,\overline{\mathbf{X}}(\bar{t}))d\bar{t}\,+\,G_1(\bar{t},\,\overline{\mathbf{X}}(\bar{t}))d\overline{W}(\bar{t}) && (2.85)\\
d\overline{X}_2(\bar{t}) \,&=\, -dX_1(t)\,\sin(\epsilon)\,+\,dX_2(t)\,\cos(\epsilon)\\
&= \left(\frac{-X_2(t)}{2}\,\cos(\epsilon) + \frac{X_1(t)}{2}\,\sin(\epsilon)\right)d\bar{t}\,+\,(X_1(t)\,\cos(\epsilon) + \epsilon X_2(t)\,\sin(\epsilon))\,d\overline{W}(\bar{t})\\
&= \left(e^{\epsilon H_2}(f_2(X_2(t)))\right)d\bar{t} + \left(e^{\epsilon H_2}(G_2(X_1(t)))\right)d\overline{W}(\bar{t}) && (2.86)\\
&= f_2(\bar{t},\,\overline{\mathbf{X}}(\bar{t}))d\bar{t}\,+\,G_2(\bar{t},\,\overline{\mathbf{X}}(\bar{t}))d\overline{W}(\bar{t}) && (2.87)
\end{aligned}$$

which demonstrates form invariance.

Our following examples follows from Gaeta and Quintero [1]. This is to see if the temporal infinitesimal satisfies (2.39).

Example 2.2 (Constant noise)**.**

This one-dimensional example has a constant diffusion term σ

$$dX(t) = \sigma\,dW(t), \tag{2.88}$$

with the determining equations simply given as

$$\frac{\partial \xi}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial \xi}{\partial x^2} = 0 \tag{2.89}$$
$$\frac{\partial \xi}{\partial x} = \frac{\frac{\partial \tau}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial \tau}{\partial x^2}}{2} \tag{2.90}$$
$$\sigma\frac{\partial \tau}{\partial x} = 0. \tag{2.91}$$

From (2.91) we see that the temporal infinitesimal is projective. Thus we have

$$\frac{\partial \xi}{\partial x} = \frac{\dot{a}(t)}{2}, \tag{2.92}$$

where $\tau(t) = a(t)$; $a(t)$ is an arbitrary function of time. We can then solve for the spatial variable in (2.90) to deduce

$$\xi(t,x) = \frac{\dot{a}(t)}{2}\,x + b(t), \tag{2.93}$$

with $b(t)$ being an arbitrary function of the temporal variable. Substituting the above result into (2.89) produces the following

$$\frac{\ddot{a}(t)}{2}\,x + \dot{b}(t) = 0. \tag{2.94}$$

By using comparison of coefficients, we are able to solve for both the spatial and temporal infinitesimals, respectively

$$\xi(x) = \frac{c_1}{2}\,x + c_2 \tag{2.95}$$

and

$$\tau(t) = c_1\,t + c_3. \tag{2.96}$$

Our final condition (2.39) on substitution is satisfied. We demonstrate this by just looking at the second term of the expansion of both sides of the condition. The right-hand side yields

$$\left(\Gamma\left(c_1\,t + c_3\right)\right)^2 = \left(c_1\right)^2 \tag{2.97}$$

while the remaining side gives

$$\Gamma\left(H\left(\tau\right)\right) = \left(c_1\right)^2. \tag{2.98}$$

Hence to verify that we can recover the finite transformations with only having access to the infinitesimal transformations, we consider three cases. In the first case we have

$$\frac{d\overline{x}}{d\epsilon} = \frac{\overline{x}}{2}, \quad \overline{x}\bigg|_{\epsilon=0} = x, \tag{2.99}$$

$$\frac{d\overline{t}}{d\epsilon} = \overline{t}, \quad \overline{t}\bigg|_{\epsilon=0} = t, \tag{2.100}$$

which results in

$$\overline{x} = xe^{\frac{\epsilon}{2}} \tag{2.101}$$

and

$$\overline{t} = te^{\epsilon}. \tag{2.102}$$

The random time change formula gives

$$\begin{aligned} d\overline{W}(\overline{t}) &= \sqrt{\frac{d\overline{t}}{dt}}dW(t) && (2.103)\\ &= e^{\frac{\epsilon}{2}}dW(t). && (2.104)\end{aligned}$$

Applying Itô's formula to the spatial transformed process we have

$$\begin{aligned} d\overline{X}(\overline{t}) &= dX(t)e^{\frac{\epsilon}{2}} && (2.105)\\ &= \sigma\; e^{\frac{\epsilon}{2}}dW(t), \\ &= e^{\epsilon H}(\sigma)\, d\overline{W}(\overline{t}), \\ &= \overline{\sigma}\, d\overline{W}(\overline{t}), && (2.106)\end{aligned}$$

since σ is constant. The remaining two cases are trivial.

(2.107)

Example 2.3.

We now consider

$$dX(t) = x\,dt + dW(t). \tag{2.108}$$

The determining equations are

$$\frac{\partial \xi}{\partial t} + x\,\frac{\partial \xi}{\partial x} + \frac{1}{2}\,\frac{\partial \xi}{\partial x^2} = x\,\left(\frac{\partial \tau}{\partial t} + x\,\frac{\partial \tau}{\partial x} + \frac{1}{2}\,\frac{\partial \tau}{\partial x^2}\right) + \xi \tag{2.109}$$

$$\frac{\partial \xi}{\partial x} = \frac{\frac{\partial \tau}{\partial t} + x\,\frac{\partial \tau}{\partial x} + \frac{1}{2}\sigma^2\,\frac{\partial \tau}{\partial x^2}}{2} \tag{2.110}$$

$$\frac{\partial \tau}{\partial x} = 0. \tag{2.111}$$

From (2.111) we see that the temporal infinitesimal is again projective. Thus we have

$$\frac{\partial \xi}{\partial x} = \frac{\dot{a}(t)}{2}, \tag{2.112}$$

where $\tau(t) = a(t)$; $a(t)$ is an arbitrary function of time. We can then solve for the spatial variable in (2.110) to get

$$\xi(t,x) = \frac{\dot{a}(t)}{2}\,x + b(t), \tag{2.113}$$

with $b(t)$ being an arbitrary function of the temporal variable. Substituting the above result into (2.109) produces the following

$$\frac{\ddot{a}(t)}{2}\,x + \dot{b}(t) + x\,\frac{\dot{a}(t)}{2} = x\,\dot{a}(t) + \frac{\dot{a}(t)}{2}\,x + b(t). \tag{2.114}$$

By using comparison of coefficients, we are able to solve for both the spatial and temporal infinitesimals, respectively

$$\xi(x) = c_1\,e^{2t}\,x + c_2\,e^{t} \tag{2.115}$$

and

$$\tau(t) = c_1\,e^{2t} + c_3. \tag{2.116}$$

The right-hand side of our final condition (2.39) gives

$$\left(\Gamma\left(c_1\,e^{2t} + c_3\right)\right)^2 = (2\,c_1\,e^{2t})^2 \tag{2.117}$$

while the remaining side gives

$$\Gamma\left(H\left(\tau\right)\right) = \left(8\left(c_1\right)^2 e^{4t} + 4\,c_1\,c_3\,e^{2t}\right). \tag{2.118}$$

Therefore only for $c_1 = 0$ will the condition (2.39) be satisfied. Thus the only spatial and temporal infinitesimals that apply are

$$\xi(x) = c_2\,e^{t} \tag{2.119}$$

and

$$\tau(t) = c_3. \tag{2.120}$$

Recovering the finite transformation which leaves the SDEs form invariant, with only having access to the infinitesimal transformations, will be verified by only considering the case associated with c_2, as the remaining case is trivial. We have

$$\frac{d\overline{x}}{d\epsilon} = e^{\overline{t}}, \tag{2.121}$$

$$\frac{d\overline{t}}{d\epsilon} = 0, \tag{2.122}$$

giving

$$\overline{x} = x + \epsilon e^{t} \tag{2.123}$$

since

$$\overline{t} = t. \tag{2.124}$$

The random time change formula leaves the Wiener process unaltered

$$d\overline{W}(\overline{t}) = dW(t). \tag{2.125}$$

Applying Itô's formula to the spatial transformed process we have

$$\begin{aligned} d\overline{X}(\overline{t}) &= dX(t) + \epsilon e^{t} dt \\ &= x\, dt + dW(t) + \epsilon e^{t} dt, \\ &= e^{\epsilon H}(x)\, d\overline{t} \, + \, d\overline{W}(\overline{t}), \end{aligned} \tag{2.126}$$

where $e^{\epsilon H}(x)$ is evaluated at $X(t)$, yielding

$$d\overline{X}(\overline{t}) = \overline{X}(\overline{t})\, d\overline{t} \, + \, d\overline{W}(\overline{t}), \tag{2.127}$$

Example 2.4.

We now look at the system

$$dX_1(t) = \frac{a_1}{X_1}\, dt + dW_1(t), \tag{2.128}$$

$$dX_2(t) = a_2\, dt + dW_2(t), \tag{2.129}$$

which means that the diffusion coefficient matrix $\mathbf{G}$, is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.130}$$

The determining equations here are simply given by

$$\begin{aligned}
\frac{\partial \xi_1}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_1}{\partial x_1} + a_2\frac{\partial \xi_1}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_2^2} &= \frac{a_1}{x_1}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau}{\partial x_1} + a_2\frac{\partial \tau}{\partial x_2} + \frac{1}{2}\frac{\partial \tau}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau}{\partial x_2^2}\right) \\
&\quad + \xi_1\left(-\frac{a_1}{x_1^2}\right) \qquad (2.131) \\
\frac{\partial \xi_2}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_2}{\partial x_1} + a_2\frac{\partial \xi_2}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_2^2} &= a_2\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau}{\partial x_1} + a_2\frac{\partial \tau}{\partial x_2} + \frac{1}{2}\frac{\partial \tau}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau}{\partial x_2^2}\right) \qquad (2.132) \\
G_1^1\frac{\partial \xi_1}{\partial x_1} + G_2^1\frac{\partial \xi_1}{\partial x_2} &= G_1^1\Gamma(\tau) + H(G_1^1) \\
\frac{\partial \xi_1}{\partial x_1} &= \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau}{\partial x_1} + a_2\frac{\partial \tau}{\partial x_2} + \frac{1}{2}\frac{\partial \tau}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau}{\partial x_2^2}\right) \qquad (2.133) \\
G_1^2\frac{\partial \xi_1}{\partial x_1} + G_2^2\frac{\partial \xi_1}{\partial x_2} &= G_1^2\Gamma(\tau) + H(G_1^2) \\
\frac{\partial \xi_1}{\partial x_2} &= 0 \qquad (2.134) \\
G_1^1\frac{\partial \xi_2}{\partial x_1} + G_2^1\frac{\partial \xi_2}{\partial x_2} &= G_2^1\Gamma(\tau) + H(G_2^1) \\
\frac{\partial \xi_2}{\partial x_1} &= 0 \qquad (2.135) \\
G_1^2\frac{\partial \xi_2}{\partial x_1} + G_2^2\frac{\partial \xi_2}{\partial x_2} &= \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau}{\partial x_1} + a_2\frac{\partial \tau}{\partial x_2} + \frac{1}{2}\frac{\partial \tau}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau}{\partial x_2^2}\right) \\
\frac{\partial \xi_2}{\partial x_2} &= \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau}{\partial x_1} + a_2\frac{\partial \tau}{\partial x_2} + \frac{1}{2}\frac{\partial \tau}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau}{\partial x_2^2}\right) \qquad (2.136) \\
G_1^1\frac{\partial \tau}{\partial x_1} + G_2^1\frac{\partial \tau}{\partial x_2} &= 0,
\end{aligned}$$

which imply that

$$\begin{aligned}
\frac{\partial \tau}{\partial x_1} &= 0. \qquad (2.137) \\
G_1^2\frac{\partial \tau}{\partial x_1} + G_2^2\frac{\partial \tau}{\partial x_2} &= 0,
\end{aligned}$$

whence

$$\frac{\partial \tau}{\partial x_2} = 0. \qquad (2.138)$$

From (2.138) we have

$$\tau(t,x) = a(t), \qquad (2.139)$$

where $a(t)$ is an arbitrary function of time. The right-hand side of our final condition (2.39) gives

$$\left(\Gamma\left(a(t)\right)\right)^2 = \left(\dot{a}(t)\right)^2 \qquad (2.140)$$

while the remaining side provides

$$\Gamma\left(H\left(\tau\right)\right) = \left(\left(\dot{a}(t)\right)^2 + a(t)\ddot{a}(t)\right). \qquad (2.141)$$

Thus temporal infinitesimal is forced to be linear with respect to time

$$\tau(t,x) = c_1\, t + c_2. \qquad (2.142)$$

As a consequence, the determining equations (2.131, 2.132), (2.133, 2.134), (2.135) and (2.136) become

$$\frac{\partial \xi_1}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_1}{\partial x_1} + a_2\frac{\partial \xi_1}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_2^2} = \frac{a_1}{x_1}c_1 + \xi_1\left(-\frac{a_1}{x_1^2}\right) \tag{2.143}$$

$$\frac{\partial \xi_2}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_2}{\partial x_1} + a_2\frac{\partial \xi_2}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_2^2} = a_2\,c_1 \tag{2.144}$$

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1}{2}c_1 \tag{2.145}$$

$$\frac{\partial \xi_1}{\partial x_2} = 0 \tag{2.146}$$

$$\frac{\partial \xi_2}{\partial x_1} = 0 \tag{2.147}$$

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}c_1. \tag{2.148}$$

From (2.145, 2.146) and (2.147, 2.147) we observe

$$\xi_1 = \frac{1}{2}c_1\,x_1 + F_1(t) \tag{2.149}$$

and

$$\xi_2 = \frac{1}{2}c_1\,x_2 + F_2(t). \tag{2.150}$$

The expression for ξ_1, in (2.149) reduces (2.143) to

$$\dot{F}_1(t) + \frac{a_1\,c_1}{2\,x_1} = \frac{a_1\,c_1}{x_1} - \left(\frac{1}{2}c_1\,x_1 + F_1(t)\right)\frac{a_1}{x_1^2}. \tag{2.151}$$

By comparison of coefficients we have that $F_1(t)$ must be zero, i.e.

$$\xi_1 = \frac{1}{2}c_1\,x_1. \tag{2.152}$$

For the remaining spatial infinitesimal we have that (2.150) simplifies (2.144) to

$$\dot{F}_3(t) + a_2\frac{1}{2}c_1 = a_2\,c_1. \tag{2.153}$$

The function $F_3(t)$ is forced to be linear with respect to time

$$F_3(t) = \frac{a_2\,c_1}{2}t + c_4. \tag{2.154}$$

In summarizing the results, we have

$$\xi_1 = \frac{1}{2}c_1\,x_1, \tag{2.155}$$

$$\xi_2 = \frac{1}{2}c_1\,x_2 + \frac{a_2\,c_1}{2}t + c_4 \tag{2.156}$$

and the temporal infinitesimal is

$$\tau(t) = c_1\,t + c_3. \tag{2.157}$$

To verify that the finite transformations can be recovered from only knowing the infinitesimal transforms, while still maintaining form invariance, we only consider the second case for which

$$\frac{d\overline{x}_1}{d\epsilon} = \frac{\overline{x}_1}{2}, \tag{2.158}$$

$$\frac{d\overline{x}_2}{d\epsilon} = \frac{\overline{x}_2 + a_2\,\overline{t}}{2} \tag{2.159}$$

and

$$\frac{d\bar{t}}{d\epsilon} = \bar{t}. \tag{2.160}$$

Thus we have that

$$\overline{x}_1 = x_1\, e^{\frac{\epsilon}{2}}, \tag{2.161}$$
$$\bar{t} = t\, e^{\epsilon} \tag{2.162}$$

and

$$\overline{x}_2 = (x_2 - a_2\, t\,)\, e^{\frac{\epsilon}{2}} + a_2\, t\, e^{\epsilon} \tag{2.163}$$

The random time change formula produces

$$d\overline{W}_i(\bar{t}) = \sqrt{\frac{d\bar{t}}{dt}}\, dW_i(t) \tag{2.164}$$
$$= e^{\frac{\epsilon}{2}}\, dW_i(t), \;\; i \,=\, 1,\, 2. \tag{2.165}$$

Applying Itô's formula to the spatial transformed process we obtain

$$\begin{aligned}
d\overline{X}_1(t) &= dX_1(t)\, e^{\frac{\epsilon}{2}} && (2.166)\\
&= \frac{a_1}{X_1}\, e^{\frac{\epsilon}{2}}\, dt \,+\, e^{\frac{\epsilon}{2}}\, dW_1(t), \\
&= \frac{a_1}{X_1\, e^{\frac{\epsilon}{2}}}\, e^{\epsilon}\, dt \,+\, e^{\frac{\epsilon}{2}}\, dW_1(t), \\
&= \frac{a_1}{\overline{X}_1(t)}\, d\bar{t} \,+\, d\overline{W}_1(\bar{t}). && (2.167)\\
d\overline{X}_2(t) &= dX_2(t)\, e^{\frac{\epsilon}{2}} - a_2\, dt\, e^{\frac{\epsilon}{2}} + a_2\, dt\, e^{\epsilon} && (2.168)\\
&= a_2\, e^{\frac{\epsilon}{2}}\, dt \,+\, e^{\frac{\epsilon}{2}}\, dW_2(t) \,-\, a_2\, dt\, e^{\frac{\epsilon}{2}} \,+\, a_2\, dt\, e^{\frac{\epsilon}{2}}, \\
&= a_2\, dt\, e^{\epsilon} \,+\, e^{\frac{\epsilon}{2}}\, dW_2(t), \\
&= a_2\, d\bar{t} \,+\, d\overline{W}_2(\bar{t}). && (2.169)
\end{aligned}$$

Example 2.5.

Finally we consider the system

$$dX_1(t) = X_2\, dt, \tag{2.170}$$
$$dX_2(t) = -k^2\, X_2\, dt + \sqrt{2\, k^2} dW(t), \tag{2.171}$$

which gives rise to the determining equations

$$\frac{\partial \xi_1}{\partial t} + x_2\, \frac{\partial \xi_1}{\partial x_1} - k^2\, x_2\, \frac{\partial \xi_1}{\partial x_2} + k^2\, \frac{\partial^2 \xi_1}{\partial x_2^2} = x_2\, \left(\frac{\partial \tau}{\partial t} + x_2\, \frac{\partial \tau}{\partial x_1} - k^2\, x_2\, \frac{\partial \tau}{\partial x_2} + k^2\, \frac{\partial^2 \tau}{\partial x_2^2}\right) + \xi_2 \tag{2.172}$$

$$\frac{\partial \xi_2}{\partial t} + x_2\, \frac{\partial \xi_2}{\partial x_1} - k^2\, x_2\, \frac{\partial \xi_2}{\partial x_2} + k^2\, \frac{\partial^2 \xi_2}{\partial x_2^2} = -k^2 x_2\, \left(\frac{\partial \tau}{\partial t} + x_2\, \frac{\partial \tau}{\partial x_1} - k^2\, x_2\, \frac{\partial \tau}{\partial x_2} + k^2\, \frac{\partial^2 \tau}{\partial x_2^2}\right) - k^2 \xi_2 \tag{2.173}$$

$$\frac{\partial \xi_1}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\, \frac{\partial \tau}{\partial x_1} - k^2\, x_2\, \frac{\partial \tau}{\partial x_2} + k^2 \frac{1}{2}\frac{\partial^2 \tau}{\partial x_2^2}\right) \tag{2.174}$$

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\, \frac{\partial \tau}{\partial x_1} - k^2\, x_2\, \frac{\partial \tau}{\partial x_2} + k^2 \frac{1}{2}\frac{\partial^2 \tau}{\partial x_2^2}\right) \tag{2.175}$$

$$\frac{\partial \tau}{\partial x_2} = 0. \tag{2.176}$$

Equation (2.176) easily gives

$$\tau(t,x) = F\left(x_1,t\right), \tag{2.177}$$

where F is an arbitrary function. We then apply condition (2.39) to the newly found temporal infinitesimal to furnish what the arbitrary function $F\left(x_1,t\right)$ should be. The right-hand side of our final condition (2.39) yields

$$\left(\Gamma\left(F\left(x_1,t\right)\right)\right)^2 = \left(\frac{\partial F\left(x_1,t\right)}{\partial t} + x_2\,\frac{\partial F\left(x_1,t\right)}{\partial x_1}\right)^2 \tag{2.178}$$

while the remaining side results in

$$\Gamma\left(H\left(\tau\right)\right) = \Gamma\left(F\,\frac{\partial F\left(x_1,t\right)}{\partial t} + \xi_1\,x_2\,\frac{\partial F\left(x_1,t\right)}{\partial x_1}\right). \tag{2.179}$$

By comparison of coefficients we have that $F\left(x_1,t\right) = c_1$ satisfies the condition (2.39). Thus the temporal infinitesimal is forced to be

$$\tau(t,x) = c_1, \tag{2.180}$$

where c_1 is an arbitrary constant. Hence the determining equations (2.172), (2.173), (2.174) and (2.174) become

$$\frac{\partial \xi_1}{\partial t} + x_2\,\frac{\partial \xi_1}{\partial x_1} - k^2\,x_2\,\frac{\partial \xi_1}{\partial x_2} + k^2\,\frac{\partial^2 \xi_1}{\partial x_2^2} = \xi_2 \tag{2.181}$$

$$\frac{\partial \xi_2}{\partial t} + x_2\,\frac{\partial \xi_2}{\partial x_1} - k^2\,x_2\,\frac{\partial \xi_2}{\partial x_2} + k^2\,\frac{\partial^2 \xi_2}{\partial x_2^2} = -k^2\xi_2 \tag{2.182}$$

$$\frac{\partial \xi_1}{\partial x_2} = 0 \tag{2.183}$$

$$\frac{\partial \xi_2}{\partial x_2} = 0. \tag{2.184}$$

From (2.183) and (2.184) we observe

$$\frac{\partial^2 \xi_1}{\partial x_2^2} = 0 \tag{2.185}$$

$$\frac{\partial^2 \xi_2}{\partial x_2^2} = 0, \tag{2.186}$$

which follows by using simple differentiation. As a result of (2.183), (2.184), (2.185) and (2.186) the equations (2.181) and (2.182) can be simplified to

$$\frac{\partial \xi_1(x_1,t)}{\partial t} + x_2\,\frac{\partial \xi_1(x_1,t)}{\partial x_1} = \xi_2(x_1,t) \tag{2.187}$$

and

$$\frac{\partial \xi_2(x_1,t)}{\partial t} + x_2\,\frac{\partial \xi_2(x_1,t)}{\partial x_1} = -k^2\xi_2(x_1,t). \tag{2.188}$$

Comparison by coefficient yields

$$\frac{\partial \xi_1(x_1,t)}{\partial t} - \xi_2(x_1,t) = 0, \tag{2.189}$$

$$\frac{\partial \xi_1(x_1,t)}{\partial x_1} = 0, \tag{2.190}$$

$$\frac{\partial \xi_2(x_1,t)}{\partial t} + k^2\xi_2(x_1,t) = 0 \tag{2.191}$$

and

$$\frac{\partial \xi_2(x_1,t)}{\partial x_1} = 0. \tag{2.192}$$

Therefore, we have that both spatial infinitesimals are functions of time only

$$\xi_2(t) = c_2\, e^{-k^2\, t} \tag{2.193}$$

and

$$\xi_1(t) = c_3 - \frac{c_2}{k^2}\, e^{-k^2\, t}. \tag{2.194}$$

Verification of maintaining form invariance after furnishing the finite transformations from the infinitesimal transformations is trivial for this example.

Remark. The temporal infinitesimal was projective for all the examples taken from [1]. But we saw from the example 2.1, that this is not always the case.

2.5 Concluding Remarks

It has been shown that by taking special care that the transformed Wiener variable is still a standard Wiener process, overlooked in the pioneering work [2] for the Itô process

$$d\mathbf{X}(t) = \mathbf{f}dt + \mathbf{G}\; d\mathbf{W}(t),$$

leads to the same results as that of [3] meaning that no recourse to the Itô's multiplication table for the transformed variables is necessary to find the extra condition (2.13).

This work allows us to investigate the symmetries of SODEs without recourse to the FP equations; precluding the assumption that the symmetry H of the SDEs had to be projectable, i.e. $\tau = \tau(t)$. This work has successfully reconciled the works of [2] and [3]. We have also found a new condition, i.e. $\Gamma(\tau) = Constant$, which allows us to use Lie point symmetry in the SODEs context.

We are able to construct the finite transformations with only having the infinitesimal transformations. As a result, we sometimes have symmetries which are not projectable and hence not belonging to the Lie algebra associated with the FP equations.

Chapter 3

A Formal Approach of Handling Lie Point Symmetries of Scalar First-Order Itô Stochastic Ordinary Differential Equations

Various methods of deriving Lie point symmetries for Itô SODEs has surfaced in the recent past. In the Itô calculus context both the formal and intuitive understanding of how to construct these symmetries has led to seemingly disparate results. The impact of Lie point symmetries on the stock market, population growth and weather SODE models, for example, will not be understood properly until these varying results are reconciled as has been attempted here.

3.1 Introduction

Adjusting Lie's work for ordinary differential equations (ODEs) to Itô SODEs was first investigated by Gaeta and Quintero [1]; extended by Wafo Soh and Mahomed [2]; refined by Ünal [3]; and reconciled by Fredericks and Mahomed [5]. The purpose of the work of Meleshko et al. [4] was to find a formal approach of finding the determining equations needed to obtain the Lie point infinitesimals for scalar SDEs. This work, however, does not make use of the Itô formula in conjunction with the random time change formula. As a result, a conditioning on the temporal infinitesimal, which [3] had constructed with the Itô multiplication table for the transformed Wiener and time index variables, is not accounted. However, unlike the determining equations derived in [2], the integro-differential determining equations of [4] is non-stochastic.

This dissimilarity between the determining equations of [5] and [4] is superficial. We will endeavor to unearth the conditioning on the temporal infinitesimal, which has been precluded from [4]. Without this conditioning, the determining equations are impossible to solve, without having to consider cases. We also extend the work of [4] to multidimensional Wiener processes, M-dimensional Wiener processes.

Another point which has only been addressed by [5], is the recoverability of the finite transformations from the infinitesimal transformations. We will try to re-derive this condition from the methodology followed by [4]. Although the recoverability of the finite transformations from infinitesimal ones is claimed in [4], it is not proven. The SDE examples used by [4] to establish these claims are either trivial or precluding much of the needed steps; in some cases the symmetries are not even a subset of the symmetries found for the related FP equations, which has been demonstrated in [1]. We will redo most of the critical examples, which appear in [4], in conjunction with our seminal condition to validate the recoverability of the finite transformations from the infinitesimal ones.

We commence with Lie point transformations. After applying the Itô formula to the transformations, a comparison between the formal Itô integral representation of the transformed time index variable and the random time change formula defined by [4], will be made. This collation should furnish the needed condition which ensures that the transformed Wiener process' properties remain invariant under the transformation. These invariant properties of the transformed Wiener process is maintained by the way in which [4] defines the rate of time change; we will show that this rate of time change is the instantaneous drift of the transformed time index, which is a consequence of Itô's formula.

Re-deriving the seminal condition, which ensures that the finite transformations are redeemable from the infinitesimal transformations, will be made by following the Itô integral methodology used in [4]. We end with examples.

3.2 Transformation

With the application of a Lie point transformation on the spatial and temporal variables, we intend to transform our probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration is generated by the σ-algebras $\mathcal{F}_t$, to the probability space $(\Omega, \mathcal{F}, \overline{\mathbb{P}})$, where the filtration is generated by the σ-algebras $\mathcal{F}_{\bar{t}}$. Notice that the density function which characterizes the probabilistic characteristics of the Wiener process, is also transformed and hence gives rise to a transformed probability measure $\overline{\mathbb{P}}$. Please note that the Lie point transformation takes place in the Banach space; the Itô formula then takes this transformation to the probability space. As per Lie's work we begin with a one-paramter group of transformation of the time index and an N-dimensional spatial random process, respectively

$$\bar{t} = \theta(t, \mathbf{X}(t,\omega), \epsilon) \quad \overline{\mathbf{X}}(\bar{t}, \omega) = \boldsymbol{\varphi}(t, \mathbf{X}(t,\omega), \epsilon), \tag{3.1}$$

where we link the group transformations to the continuous parameters or infinitesimals via the following relations

$$\frac{\partial \theta}{\partial \epsilon} = \tau(\theta, \boldsymbol{\varphi}) \quad \frac{\partial \varphi_j}{\partial \epsilon} = \xi_j(\theta, \boldsymbol{\varphi}) \text{ for } j \,=\, 1,\, N \tag{3.2}$$

which have the identity boundary condition at $\epsilon \,=\, 0$, i.e.

$$\bar{t}\Big|_{\epsilon\,=\,0} = t, \quad \text{and } \overline{\mathbf{X}}(\bar{t}, \omega)\Big|_{\epsilon\,=\,0} = \mathbf{X}(t,\omega). \tag{3.3}$$

The spatial and temporal infinitesimals form the symmetry operator

$$H = \tau \, \frac{\partial}{\partial t} \,+\, \xi_j \, \frac{\partial}{\partial x_j}, \tag{3.4}$$

which is applied to functions on the Banach space.

(3.5)

3.3 Review of the Work of Meleshko et al. [4]

Using Itô's formula on the temporal and spatial group transformations $\theta(t, X(t,\omega), \epsilon)$ and $\boldsymbol{\varphi}(t, \mathbf{X}(t,\omega), \epsilon)$ respectively, gives

$$\begin{aligned} \theta(t, \mathbf{X}(t,\omega), \epsilon) = \theta(0, \mathbf{X}(0,\omega), \epsilon) &+ \int_0^t \Gamma(\theta)(s, \mathbf{X}(s,\omega), \epsilon)\, ds \\ &+ \int_0^t Y(\theta)(s, \mathbf{X}(s,\omega), \epsilon)\, d\mathbf{W}(s) \end{aligned} \tag{3.6}$$

and

$$\varphi(t,\mathbf{X}(t,\omega),\epsilon) = \varphi(0,\mathbf{X}(0,\omega),\epsilon) + \int_0^t \Gamma(\varphi)(s,\mathbf{X}(s,\omega),\epsilon)\,ds + \int_0^t Y(\varphi)(s,\mathbf{X}(s,\omega),\epsilon)\,d\mathbf{W}(s). \tag{3.7}$$

Since random processes which can be viewed as families of random variables, which are indexed by time, are non-deterministic, we use the random time change formula from Øksendal [8] to perform a random time change

$$\bar{t}(t,\,\mathbf{X}(t,\omega),\,\epsilon) = \int_0^t \eta^2(s,\,\mathbf{X}(s,\omega),\,\epsilon)\,ds, \tag{3.8}$$

where [4] defined the time change rate $\eta^2(t,\,\mathbf{X}(t,\omega),\,\epsilon)$ to be continuously differentiable and adapted to the families of σ-algebras generating the filtration $\mathcal{F}$. By juxtaposing (3.6) and (3.8) we see that the instantaneous drift of the temporal group transformation is the rate of time change, i.e.

$$\int_0^t \eta^2(s,\,\mathbf{X}(s,\omega),\,\epsilon)\,ds = \theta(0,\mathbf{X}(0,\omega),\epsilon) + \int_0^t \Gamma(\theta)(s,\mathbf{X}(s,\omega),\epsilon)\,ds, \tag{3.9}$$

the initial value of the temporal group transformation at time index naught, is then forced to be zero. Since our group transformation of the time index is continuously differentiable, the Itô formula ensures that it is also adapted to the family of σ-algebras that generate the filtration of our original probability space. The instantaneous diffusion component of the temporal group transformation is forced to be zero in order for the comparison above in (3.9) to hold, i.e.

$$\int_0^t Y(\theta)(s,\mathbf{X}(s,\omega),\epsilon)\,dW(s) = 0. \tag{3.10}$$

This is the finite transformation version of the infinitesimal condition that Ünal [3] had derived using an invariance argument on the Itô multiplication table for the transformed Wiener process. Thus the transformed Wiener process, which [4] defines as

$$\overline{\mathbf{W}}(\beta(t,\omega),\omega) = \int^t \eta(s,\,\mathbf{X}(s,\omega),\,\epsilon)\,d\mathbf{W}(s,\omega), \tag{3.11}$$

reads as

$$\overline{\mathbf{W}}(\beta(t,\omega),\omega) = \int^t \sqrt{\Gamma\,(\theta)\,(s,\,\mathbf{X}(s,\omega),\,\epsilon)}\,d\mathbf{W}(s,\omega). \tag{3.12}$$

Form invariance arguments on the integrands require

$$\overline{\mathbf{X}}(\beta(t,\omega),\omega) = \overline{\mathbf{X}}(0,\omega) + \int_0^t \mathbf{f}(\beta(s,\omega),\overline{\mathbf{X}}(\beta(s,\omega),\omega))\,d\beta(s,\omega) + \int_0^t \mathbf{G}(\beta(s,\omega),\overline{\mathbf{X}}(\beta(s,\omega),\omega))\,d\overline{\mathbf{W}}(\beta(s,\omega)) \tag{3.13}$$

which is superficially different to what [4] had, where the form invariance argument was applied to the interval of integration and not to the time indices which followed the transformed spatial random process along the time interval. This can also be represented as

$$\overline{\mathbf{X}}(\beta(t,\omega),\omega) = \overline{\mathbf{X}}(0,\omega) + \int_0^t \mathbf{f}(\beta(s,\omega),\overline{\mathbf{X}}(\beta(s,\omega),\omega))\,\eta^2(s,\,\mathbf{X}(s,\omega),\,\epsilon)\,ds + \int_0^t \mathbf{G}(\beta(s,\omega),\overline{\mathbf{X}}(\beta(s,\omega),\omega))\,\eta(s,\,\mathbf{X}(s,\omega),\,\epsilon)\,d\mathbf{W}(s), \tag{3.14}$$

where we used the relation $d\beta(s,\omega)/ds = \eta^2(s,\, \mathbf{X}(s,\omega),\, \epsilon)$ and $d\mathbf{W}(\beta(s,\omega))/d\mathbf{W}(s) = \eta(s,\, \mathbf{X}(s,\omega),\, \epsilon)$ from (3.8) and (3.11) respectively. We re-write the transformed spatial process above in terms of the symmetry operator

$$\overline{\mathbf{X}}(\beta(t,\omega),\omega) = \overline{\mathbf{X}}(0,\omega) + \int_0^t e^{\epsilon H}(\mathbf{f})\,(s,\, \mathbf{X}(s,\omega))\,\Gamma(e^{\epsilon H}(s))(s,\, \mathbf{X}(s,\omega))\,ds + \int_0^t e^{\epsilon H}(\mathbf{G})(s,\, \mathbf{X}(s,\omega))\,\sqrt{\Gamma(e^{\epsilon H}(s))}(s,\, \mathbf{X}(s,\omega))\,d\mathbf{W}(s). \tag{3.15}$$

By looking at the Riemann integral, we see that

$$\int_0^t e^{\epsilon H}(\mathbf{f})\,(s,\, \mathbf{X}(s,\omega))\,\Gamma(e^{\epsilon H}(s))(s,\, \mathbf{X}(s,\omega))\,ds = \int_0^t \mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \epsilon \int_0^t (\Gamma(H(t)) \,+\, H)\;\mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds$$
$$+ \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \int_0^t \left((\Gamma(H(t)) \,+\, H)^k\, \mathbf{f}(s,\, \mathbf{X}(s,\omega)) + \sum_{j=0}^{k-2} \begin{pmatrix} k \\ k\,-\,j \end{pmatrix} \mathbf{f}(s,\, \mathbf{X}(s,\omega))\; H^j(t)\, \left(\Gamma(H^{k-j}(t)) \,-\, [\Gamma(H(t))]^{k-j} \right) \right) ds. \tag{3.16}$$

Recovery of the spatial finite transformations from the infinitesimal ones can only be achieved if the higher order terms are dependent on the first order term. This is the same argument that was used in the previous chapter. Thus, we arrive at the following condition

$$\Big(\Gamma(H(t))\Big)^k (t,\, \mathbf{X}(t)) \,=\, \Gamma\Big(H^k(t)\Big)(t,\, \mathbf{X}(t)), \text{ for } k \,\in\, \mathbb{N}, \tag{3.17}$$

from which we can conclude the condition (2.24). Thus (3.16) becomes

$$\int_0^t e^{\epsilon H}(\mathbf{f})\,(s,\, \mathbf{X}(s,\omega))\,\Gamma(e^{\epsilon H}(s))(s,\, \mathbf{X}(s,\omega))\,ds$$
$$= \int_0^t \mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \epsilon \int_0^t (\Gamma(H(t)) \,+\, H)\;\mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \int_0^t \Big((\Gamma(H(t)) \,+\, H)^k\, \mathbf{f}(s,\, \mathbf{X}(s,\omega)) \Big)\, ds. \tag{3.18}$$

Comparing the drift components of (3.7) and (3.18) yields

$$\int_0^t \Gamma(\boldsymbol{\varphi})(s, \mathbf{X}(s,\omega), \epsilon)\,ds \,=\, \int_0^t \mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \epsilon \int_0^t (\Gamma(H(t)) \,+\, H)\;\mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \int_0^t \Big((\Gamma(H(t)) \,+\, H)^k\, \mathbf{f}(s,\, \mathbf{X}(s,\omega)) \Big)\, ds. \tag{3.19}$$

Equation (3.7) can also be written in terms of the symmetry operator,

$$\int_0^t \Gamma(\boldsymbol{\varphi})(s, \mathbf{X}(s,\omega), \epsilon)\,ds \,=\, \int_0^t \mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \epsilon \int_0^t \Gamma\,(H(x))\;(s,\, \mathbf{X}(s,\omega))\,ds + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \int_0^t \Big(\Gamma\,\Big(H^k(x)\Big)\,(s,\, \mathbf{X}(s,\omega))\,ds \Big)\, ds. \tag{3.20}$$

The collation of (3.19) and (3.20) recovers the same determining equations that were found in the previous chapter. By differentiating (3.19) and (3.20) by ϵ at $\epsilon = 0$ and then equating their results establishes these determining equations. Differentiating (3.19) and (3.20) by ϵ, respectively gives

$$\int_0^t (\Gamma(H(t)) \,+\, H)\;\mathbf{f}(s,\, \mathbf{X}(s,\omega))\,ds + \sum_{k=2}^{\infty} \frac{k\,\epsilon^{k-1}}{k!} \int_0^t \Big((\Gamma(H(t)) \,+\, H)^k\, \mathbf{f}(s,\, \mathbf{X}(s,\omega)) \Big)\, ds \tag{3.21}$$

and

$$\int_0^t \Gamma\left(H(x)\right)\,(s,\ \mathbf{X}(s,\omega))\,ds + \sum_{k=2}^{\infty}\frac{k\,\epsilon^{k-1}}{k!}\int_0^t \left(\Gamma\left(H^k(x)\right)(s,\ \mathbf{X}(s,\omega))\,ds\right)ds. \tag{3.22}$$

Therefore equating at $\epsilon = 0$ gives our first system of determining equations

$$\int_0^t \left(\Gamma(H(t))\ +\ H\right)\mathbf{f}(s,\ \mathbf{X}(s,\omega))\,ds\ =\int_0^t \Gamma\left(H(x)\right)\,(s,\ \mathbf{X}(s,\omega))\,ds. \tag{3.23}$$

The drift coefficient of the temporal group transformation re-written with respect to the symmetry operator gives

$$\int_0^t \Gamma(\theta)(s,\ \mathbf{X}(s,\omega))\,ds = \int_0^t \Gamma\left(e^{\epsilon\,H}(s)\right)(s,\ \mathbf{X}(s,\omega))\,ds. \tag{3.24}$$

As a consequence of (3.17), the instantaneous drift coefficient of the temporal group transformation has the following simplication,

$$\int_0^t \Gamma(\theta)(s,\ \mathbf{X}(s,\omega))\,ds = \int_0^t e^{\epsilon\,\Gamma(H(s))}(s,\ \mathbf{X}(s,\omega))\,ds. \tag{3.25}$$

The random time change formula (3.12) as a result, abridges to

$$\overline{\mathbf{W}}(\beta(t,\omega),\omega) = \int^t e^{\epsilon\,\frac{\Gamma(H(s))}{2}}(s,\ \mathbf{X}(s,\omega))\,d\mathbf{W}(s,\omega), \tag{3.26}$$

which adjusts the diffusion component of (3.15) to

$$\int_0^t e^{\epsilon H}(\mathbf{G})(s,\ \mathbf{X}(s,\omega))\,e^{\epsilon\,\frac{\Gamma(H(s))}{2}}(s,\ \mathbf{X}(s,\omega))\,d\mathbf{W}(s). \tag{3.27}$$

Writing the diffusion component of (3.7) in terms of the symmetry operator, while simultaneously expanding the terms of (3.27) essentially gives the following equality

$$\int_0^t \mathbf{G}(s,\ \mathbf{X}(s,\omega))\,dW(s) + \epsilon\int_0^t Y\left(H(x)\right)\,(s,\ \mathbf{X}(s,\omega))\,dW(s) + \sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\int_0^t\left(Y\left(H^k(x)\right)(s,\ \mathbf{X}(s,\omega))\right)dW(s)$$

$$=$$

$$\int_0^t \mathbf{G}(s,\ \mathbf{X}(s,\omega))\,dW(s) + \epsilon\int_0^t\left(\frac{\Gamma(H(t))}{2}\ +\ H\right)\mathbf{G}(s,\ \mathbf{X}(s,\omega))\,dW(s)$$

$$+\sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\int_0^t\left((\frac{\Gamma(H(t))}{2}\ +\ H)^k\,\mathbf{G}(s,\ \mathbf{X}(s,\omega))\right)dW(s). \tag{3.28}$$

Differentiating with respect to ϵ results in

$$\int_0^t Y\left(H(x)\right)\,(s,\ \mathbf{X}(s,\omega))\,dW(s) + \sum_{k=2}^{\infty}\frac{k\,\epsilon^{k-1}}{k!}\int_0^t\left(Y\left(H^k(x)\right)(s,\ \mathbf{X}(s,\omega))\right)dW(s) \qquad =$$

$$\int_0^t\left(\frac{\Gamma(H(t))}{2}\ +\ H\right)\mathbf{G}(s,\ \mathbf{X}(s,\omega))\,dW(s)\ +\sum_{k=2}^{\infty}\frac{k\,\epsilon^{k-1}}{k!}\int_0^t\left((\frac{\Gamma(H(t))}{2}\ +\ H)^k\,\mathbf{G}(s,\ \mathbf{X}(s,\omega))\right)dW(s). \tag{3.29}$$

At $\epsilon = 0$ we arrive at the remaining determining equation

$$\int_0^t Y\left(H(x)\right)\,(s,\ \mathbf{X}(s,\omega))\,dW(s) = \int_0^t \left(\frac{\Gamma(H(t))}{2}\ +\ H\right)\mathbf{G}(s,\ \mathbf{X}(s,\omega))\,dW(s). \tag{3.30}$$

Expanding (3.23) and (3.30) respectively gives

$$\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial t}\ +\sum_{l=1}^{N} f_l(t,\,\mathbf{x})\,\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_l}\ +\sum_{p=1}^{M} G_r^p G_s^p(t,\,\mathbf{x})\frac{\partial^2 \xi_j(t,\,\mathbf{x})}{\partial x_r \partial x_s} = \tau(t,\,\mathbf{x})\,\frac{\partial f_j(t,\,\mathbf{x})}{\partial t}\ +\sum_{p=1}^{N} \xi_p(t,\,\mathbf{x})\,\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_p} + f_j(t,\,\mathbf{x})\,\Gamma(\tau(t,\,\mathbf{x})) \tag{3.31}$$

and

$$\sum_{p=1}^{M} G_p^s(t,\,\mathbf{x})\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_p} = \tau(t,\,\mathbf{x})\,\frac{\partial G_j^s(t,\,\mathbf{x})}{\partial t}\ +\sum_{p=1}^{N} \xi_p(t,\,\mathbf{x})\,\frac{\partial G_j^s(t,\,\mathbf{x})}{\partial x_p}\ +\ G_j^s(t,\,\mathbf{x})\,\frac{\Gamma(\tau(t,\,\mathbf{x}))}{2}, \tag{3.32}$$

for $j = 1, N$ and $s = 1, M$. This can then be written with integral terms to give the exact integro-differential equations of [4]

$$\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial t}\ +\sum_{p=1}^{N} f_l(t,\,\mathbf{x})\,\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_l}\ +\sum_{p=1}^{M} G_r^p G_s^p(t,\,\mathbf{x})\frac{\partial^2 \xi_j(t,\,\mathbf{x})}{\partial x_r \partial x_s}\ =\ \frac{\partial f_j(t,\,\mathbf{x})}{\partial t}\int^t \Gamma(\tau(s,\,\mathbf{x}))ds + \sum_{p=1}^{N} \xi_p(t,\,\mathbf{x})\,\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_p} + f_j(t,\,\mathbf{x})\,\Gamma(\tau(t,\,\mathbf{x})) \tag{3.33}$$

and

$$\sum_{p=1}^{M} G_p^s(t,\,\mathbf{x})\frac{\partial \xi_j(t,\,\mathbf{x})}{\partial x_p}\ =\ \frac{\partial G_j^s(t,\,\mathbf{x})}{\partial t}\int^t \Gamma(\tau(s,\,\mathbf{x}))ds + \sum_{p=1}^{N} \xi_p(t,\,\mathbf{x})\,\frac{\partial G_j^s(t,\,\mathbf{x})}{\partial x_p} + G_j^s(t,\,\mathbf{x})\,\frac{\Gamma(\tau(t,\,\mathbf{x}))}{2}. \tag{3.34}$$

With the additional conditions (3.10) and (3.17) we review a few of the examples from Meleshko et al. [4].

3.4 Examples

3.4.1 Example 1

Consider

$$dX(t,\omega) = \mu\,X(t,\omega)\,dt + \sigma\,X(t,\omega)\,dW(t,\omega) \tag{3.35}$$

with the initital condition $X(0,\omega) = x_0$. The determining equations are

$$\Gamma(\xi(t,x)) = H(\mu\,x) + \mu\,x\,\Gamma(\tau(t)),$$

or equivalently

$$\frac{\partial \xi(t,x)}{\partial t} + \mu\, x \frac{\partial \xi(t,x)}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \xi(t,x)}{\partial x^2} = \xi(t,x)\,\mu + \mu\, x \frac{d\tau(t)}{d\,t} \tag{3.36}$$

and

$$Y(\xi(t,x)) = \xi(t,x)\,\sigma + \frac{\sigma\, x}{2}\frac{d\tau(t)}{dt}$$

which is

$$x\frac{\partial \xi(t,x)}{\partial\, x} = \xi(t,x) + \frac{x}{2}\frac{d\tau(t)}{dt} \tag{3.37}$$

since the extra condition forces the temporal infinitesimal, τ, to be a function of time only. By substituting for $\xi(t,x)$ from (3.37), (3.36) becomes

$$\frac{\partial \xi(t,x)}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \xi(t,x)}{\partial x^2} = \frac{\mu\, x}{2}\frac{d\tau(t)}{d\,t}. \tag{3.38}$$

Multiplying (3.37) by x and then differentiating by x gives

$$\begin{aligned} x^2\frac{\partial^2 \xi(t,x)}{\partial\, x^2} &= -\,x\,\frac{\partial \xi(t,x)}{\partial\, x} + \xi(t,x) + x\frac{d\tau(t)}{dt} \\ &= \frac{x}{2}\frac{d\tau(t)}{dt} \end{aligned} \tag{3.39}$$

after using the result (3.37) again. Thus (3.38) becomes

$$\frac{\partial \xi(t,x)}{\partial t} = \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x \frac{d\tau(t)}{dt} \tag{3.40}$$

which means that

$$\xi(t,x) = \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\, \tau(t) + a_1(x), \tag{3.41}$$

where $a_1(x)$ is an arbitrary function of x. Equation (3.39) now becomes

$$a_1''(x) = \frac{\dot{\tau}(t)}{2x} \tag{3.42}$$

which in turn yields

$$a_1(x) = \frac{1}{2}\dot{\tau}(t)\ (x\,\ln x\, -\, x)\, +\, a_2\, x\, +\, a_3, \tag{3.43}$$

where a_2 and a_3 are arbitrary constants. By substitution of (3.41) with a_1 in (3.43) into (3.36), we have that the temporal infinitesimal reduces to

$$\tau(t) = a_4 + a_5 t, \tag{3.44}$$

and the spatial infinitesimal becomes

$$\xi(t,x) = \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\,(a_4 + a_5 t)\, +\, a_2\, x\, +\, \frac{1}{2}a_5(x\ln x - x). \tag{3.45}$$

We demonstrate that the finite transformations are recoverable by considering the non-trivial case, i.e. $a_4 = 1$ and $a_i = 0$ for $i = 2$ and 5

$$\frac{d\,\overline{x}}{d\epsilon} = \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\,\overline{x} \tag{3.46}$$

$$\frac{d\,\overline{t}}{d\epsilon} = 1. \tag{3.47}$$

Solving (3.47) gives

$$\bar{t} = t. \tag{3.48}$$

Solving for $\overline{x}$ leaves us with the following

$$\overline{x} = e^{\frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\,\epsilon}\, x. \tag{3.49}$$

The Itô formula trivially gives the form invariant SDE

$$\begin{aligned} d\overline{X}(t,\omega) &= \mu\, X(t,\omega)\, e^{\frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\,\epsilon}\, dt + \sigma\, X(t,\omega)\, e^{\frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\,\epsilon} dW(t,\omega) \\ &= \mu\, \overline{X}(\bar{t},\omega)\, d\bar{t} \,+\, \sigma\, \overline{X}(\bar{t},\omega)\, d\overline{W}(\bar{t},\omega), \end{aligned} \tag{3.50}$$

which makes sense, since there is no random time change for this example. However, in [4], the symmetries found do not maintain the properties of the transformed Wiener processes. In trying to show that the finite transformations are retrievable from the infinitesimal ones, the work of [4] uses the spatial and temporal infinitesimal values directly up to order $\mathcal{O}(\epsilon)$ and falsely claim that the higher order infinitesimals will agree with the Itô formula applied to the finite group transformations, which are found using the Lie equations.

Focusing on the non-trivial finite group of transformations which was found in [4], i.e.

$$\overline{x} = (\epsilon + x^{-\gamma})^{-\frac{1}{\gamma}} \tag{3.51}$$

and

$$\bar{t} = t\,\left(1 \,+\, \epsilon\, x^{\gamma}\right)^{-2}, \tag{3.52}$$

where $\gamma = \frac{2\mu}{\sigma^2} - 1$, which we are able to derive using the spatial and temporal infinitesimals,

$$\tau = 2\,t\,x^{\gamma} \tag{3.53}$$

and

$$\xi = \frac{x^{\gamma\,-\,1}}{\gamma} \tag{3.54}$$

respectively, we see that the invariant condition is not satisfied. We have that the Itô formula gives the following for the transformed spatial process

$$d\overline{X}(\bar{t},\omega) = \Gamma(\overline{x})(t,\omega)\, dt \,+\, Y(\overline{x})(t,\omega)\, dW(t,\omega) \tag{3.55}$$

$$\begin{aligned} &= (\epsilon \,+\, X^{-\gamma})^{-\frac{1}{\gamma}-2}\, X^{-\gamma}(\mu\,(\epsilon \,+\, X^{-\gamma}) \,-\, \frac{\epsilon\sigma^2}{2}\,(1 \,+\, \gamma))dt \\ &\quad +\, \sigma\,(\epsilon \,+\, X^{-\gamma})^{-\frac{1}{\gamma}-1}\, X^{-\gamma}\, dW(t,\omega), \end{aligned} \tag{3.56}$$

while for the transformed time index, we have

$$d\bar{t}(\bar{t},\omega) = \Gamma(\bar{t})(t,\omega)\, dt \,+\, Y(\bar{t})(t,\omega)\, dW(t,\omega) \tag{3.57}$$

$$\begin{aligned} &= (1 \,+\, \epsilon\, X^{\gamma})^{-2}\Big(1 \,-\, 2\,t\,\gamma\mu\,\epsilon\, X^{\gamma}\,(1 \,+\, \epsilon\, X^{\gamma})^{-1} \\ &\quad -\, \frac{t\,\sigma^2}{2}\,\left(2\,\gamma\,(\gamma \,-\, 1)\,\epsilon\, X^{\gamma}\,(1 \,+\, \epsilon\, X^{\gamma})^{-1} \,-6\,\gamma^2\,\epsilon^2\, X^{2\gamma}\,(1 \,+\, \epsilon\, X^{\gamma})^{-2}\right)\Big)dt \\ &\quad -\, 2\,\sigma\,t\,\gamma\,\epsilon\, X^{\gamma}\,(1 \,+\, \epsilon\, X^{\gamma})^{-3}\, dW(t,\omega). \end{aligned} \tag{3.58}$$

The Itô formula for the transformed time index can never be recovered in that of the transformed spatial processes because the variable t appears nowhere in (3.56). We continue with a brief study of the relations between the three generators of symmetry which we have found, viz.

$$\begin{aligned} H_1 &= \frac{\partial}{\partial t} + \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\, \frac{\partial}{\partial x} \\ H_2 &= x\, \frac{\partial}{\partial x} \\ H_3 &= t\frac{\partial}{\partial t} + \frac{1}{2}\left((\mu - \frac{1}{2}\sigma^2)tx + x\ln x - x\right)\frac{\partial}{\partial x}. \end{aligned} \tag{3.59}$$

The Lie bracket relations are

$$[H_1, H_2] = 0,\ [H_1, H_3] = H_1 + \frac{1}{4}(\mu - \frac{1}{2}\sigma^2)\, H_2,\ [H_2, H_3] = \frac{1}{2}H_2. \tag{3.60}$$

Therefore we have a three-dimensional algebra of symmetry generators.

3.4.2 Example 2

We next look at

$$dX(t,\omega) = \mu\, dt + dW(t,\omega) \tag{3.61}$$

with the initital condition $X(0,\omega) = x_0$. The determining equations are

$$\Gamma(\xi(t,x)) = H(\mu) + \mu\,\Gamma(\tau(t)),$$

or

$$\frac{\partial \xi(t,x)}{\partial t} + \mu\,\frac{\partial \xi(t,x)}{\partial x} + \frac{\partial^2 \xi(t,x)}{2\partial x^2} = \mu\,\frac{d\tau(t)}{d\,t} \tag{3.62}$$

and

$$Y(\xi(t,x)) = \frac{1}{2}\Gamma(\tau(t)),$$

which in conjunction with Ünal's [3] extra condition forces the following relation between the spatial and temporal infinitesimals

$$\frac{\partial \xi(t,x)}{\partial\, x} = \frac{1}{2}\dot{\tau}(t). \tag{3.63}$$

Substituting $\xi(t,x)$ from (3.63) into (3.62), causes the equation (3.62) to become

$$\frac{\partial \xi(t,x)}{\partial t} = \frac{1}{2}\mu\dot{\tau}(t), \tag{3.64}$$

which due to the compatibility of (3.63) and (3.64) results in

$$\tau(t,x) = a_1 + a_2 t, \tag{3.65}$$

where a_1 and a_2 are arbitrary constants. Then (3.63) and (3.64) imply

$$\xi = \frac{1}{2}a_2 x + \frac{1}{2}\mu a_2 t + a_3, \tag{3.66}$$

where a_3 is an arbitrary constant. The finite group of transformations for $a_2 = 2$ and rest zero are solved from the following equations

$$\frac{d\,\overline{x}}{d\epsilon} = \mu\,\overline{t} + \overline{x}, \quad \overline{x}\Big|_{\epsilon=0} = x \tag{3.67}$$

and

$$\frac{d\,\overline{t}}{d\epsilon} = 2\overline{t}, \quad \overline{t}\Big|_{\epsilon=0} = t. \tag{3.68}$$

The temporal group of transformations is easily solved to give

$$\overline{t} = t\,e^{2\epsilon}, \tag{3.69}$$

which induces the following ODE from (3.67)

$$\frac{d\,\overline{x}}{d\epsilon} - \overline{x} = \mu\,t\,e^{2\,\epsilon}. \tag{3.70}$$

Solving gives

$$\overline{x} = (x\,-\,\mu\,t)\,e^{\epsilon} + \mu\,t\,e^{2\epsilon}. \tag{3.71}$$

Applying Itô's formula to (3.71) yields

$$\begin{aligned} d\overline{X}(\overline{t},\omega) &= (\mu\,dt + dW(t,\omega) - \mu\,dt)\;e^{\epsilon} + \mu\,dt\,e^{2\,\epsilon} \\ &= \mu\,e^{2\,\epsilon}\,dt\;+\;e^{\epsilon}\,dW(t,\omega) \\ &= \mu\,d\overline{t}\;+\;d\overline{W}(\overline{t},\omega). \end{aligned} \tag{3.72}$$

Thus form invariance is maintained. This is not true for the following transformations which were found in [4]:

$$\overline{x} = x - \frac{1}{2\,\mu}\ln\left(1 - 2\,\mu\,\epsilon\,e^{2\,\mu\,x}\right) \tag{3.73}$$

and

$$\overline{t} = t\left(1 - 2\,\mu\,\epsilon\,e^{2\,\mu\,x}\right)^{-2}. \tag{3.74}$$

The Itô formula for the spatial transformations is

$$\begin{aligned} d\overline{X}(\overline{t},\omega) = &\Bigg(\mu\big(1\,+\,\frac{4\,\mu^2\,\epsilon\,e^{2\,\mu\,x}}{2\,\mu\,(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})}\big) \\ &+\frac{8\,\mu^3\,\epsilon\,e^{2\,\mu\,x}}{4\,\mu\,(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^2}\Bigg)\,dt \\ &+\left(1\,+\,\frac{4\,\mu^2\,\epsilon\,e^{2\,\mu\,x}}{2\,\mu\,(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})}\right)dW(t,\omega). \end{aligned} \tag{3.75}$$

As with the previous example there is no presence of a time index in the drift or diffusion coefficients. Calculating the Itô formula for the transformed time index gives

$$\begin{aligned} d\overline{t}(\overline{t},\omega) = &\Bigg(\frac{1}{(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^2} + t\,\mu\big(\frac{8\,\mu^2\,\epsilon\,e^{2\,\mu\,x}}{(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^3}\big) \\ &+\frac{8\,t\,\mu^3\,\epsilon\,e^{2\,\mu\,x}}{(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^3} + \frac{48\,t\,\mu^4\epsilon^2\,e^{4\,\mu\,x}}{(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^4}\Bigg)\,dt \\ &+\frac{8\,\mu^2\,\epsilon\,t\,e^{2\,\mu\,x}}{(1-2\,\mu\,\epsilon\,e^{2\,\mu\,x})^3}dW(t,\omega). \end{aligned} \tag{3.76}$$

The consequence of failing to comply with condition (3.10), leads to symmetries which do not leave the transformed spatial process form invariant. Though [4] demonstrates that the transformed Wiener process obeys the random time change formula, form invariance for the transformed spatial process is deficient.

The symmetry generators we obtain are

$$H_1 = \frac{\partial}{\partial t}$$
$$H_2 = \frac{\partial}{\partial x}$$
$$H_3 = 2\,t\,\frac{\partial}{\partial t} + (\mu\,t\,+\,x)\frac{\partial}{\partial x} \tag{3.77}$$

which have the following Lie bracket relations

$$[H_1, H_2] = 0,\ [H_1, H_3] = 2H_1 + \mu H_2,\ [H_2, H_3] = H_2. \tag{3.78}$$

3.4.3 Example 3

We now consider

$$dX(t,\omega) = f(t)\,dt + g(t)\,dW(t,\omega) \tag{3.79}$$

with the initital condition $X(0,\omega) = x_0$. Here the determining equations are

$$\Gamma(\xi(t,x)) = H(f(t)) + f(t)\,\Gamma(\tau(t)),$$

which gives

$$\frac{\partial\xi(t,x)}{\partial t} + f(t)\,\frac{\partial\xi(t,x)}{\partial x} + \frac{g^2(t)}{2}\,\frac{\partial^2\xi(t,x)}{\partial x^2} = \tau(t)\,\frac{\partial f(t)}{\partial t} + f(t)\,\frac{d\tau(t)}{d\,t} \tag{3.80}$$

and

$$Y(\xi(t,x)) = H(g(t)) + \frac{g(t)}{2}\,\Gamma(\tau(t,x)).$$

Applying Ünal's [3] extra condition forces the following relation between the spatial and temporal infinitesimals

$$\frac{\partial\xi(t,x)}{\partial\,x} = \tau(t)\,\frac{\partial\ln g(t)}{\partial t} + \frac{1}{2}\dot{\tau}(t). \tag{3.81}$$

Substituting $\xi(t,x)$ from (3.81) into (3.80) results in the equation (3.80) becoming

$$\frac{\partial\xi(t,x)}{\partial t} + f(t)\,\tau(t)\,\frac{\partial\ln g(t)}{\partial t} = \tau(t)\frac{\partial f(t)}{\partial t} + f(t)\frac{1}{2}\dot{\tau}(t), \tag{3.82}$$

Integrating (3.81) w.r.t. the spatial coordinate x and then using the time differentiated result of this calculation to perform a comparison within (3.82) via separation of coefficients w.r.t. the spatial variable x, leads to the following

$$\tau(t)\,\frac{\partial\ln g(t)}{\partial t} + \frac{1}{2}\dot{\tau}(t) = C_0 \tag{3.83}$$

and

$$\dot{m}(t) = -f(t)\,\tau(t)\,\frac{\partial\ln g(t)}{\partial t} + \tau(t)\frac{\partial f(t)}{\partial t} + f(t)\frac{1}{2}\dot{\tau}(t), \tag{3.84}$$

where C_0 and $m(t)$ are an arbitrary constant and arbitrary function of time respectively, which constitute the spatial infinitesimal as

$$\xi(t,x) = C_0\,x + m(t). \tag{3.85}$$

Solving for the temporal infinitesimal from (3.83) gives

$$\tau(t) = g^{-2}(t)\,\left(C_1\,+\,2C_0\int^t g^2(s)\,ds\right), \tag{3.86}$$

where C_1 is a further constant, which finally results in (from (3.84))

$$m(t) = C_1\frac{f(t)}{g^2(t)} + C_0\int^t f(s)ds + 2C_0\int^t \frac{d}{dt'}\left(\frac{f(t')}{g^2(t')}\right)\left(\int^{t'} g^2(s)ds\right)\,dt' + C_2, \tag{3.87}$$

in which C_2 is a constant. There arises three symmetry generators which have in them functions $f(t)$ and $g(t)$. They are

$$\begin{aligned} H_1 &= \frac{\partial}{\partial x} \\ H_2 &= 2g^{-2}(t)\int^t g^2(s)ds\frac{\partial}{\partial t} + \Big(x + \int^t f(s)ds \\ &\quad + 2\int^t \frac{d}{dt'}\left(\frac{f(t')}{g^2(t')}\right)\left(\int^{t'} g^2(s)ds\right)\,dt'\Big)\frac{\partial}{\partial x} \\ H_3 &= g^{-2}(t)\frac{\partial}{\partial t} + \frac{f(t)}{g^2(t)}\frac{\partial}{\partial x}. \end{aligned} \tag{3.88}$$

The commutators are

$$[H_1, H_2] = H_1,\ [H_1, H_3] = 0,\ [H_2, H_3] = -2H_3. \tag{3.89}$$

Here the generators span a three-dimensional Lie algebra.

3.4.4 Example 4

This example seeks to investigate a SODE which has a diffusion coefficient strictly as a function of time, *viz.*

$$dX(t,\omega) = \mu\,dt + t\,dW(t,\omega) \tag{3.90}$$

with the initital condition $X(0,\omega) = x_0$. Thus the determining equations are

$$\Gamma(\xi(t,x)) = \mu\,\Gamma(\tau(t,x)),$$

which by the condition (3.10) gives

$$\frac{\partial\xi(t,x)}{\partial t} + \mu\,\frac{\partial\xi(t,x)}{\partial x} + \frac{t^2}{2}\,\frac{\partial^2\xi(t,x)}{\partial x^2} = \mu\,\frac{d\tau(t)}{d\,t} \tag{3.91}$$

and

$$Y(\xi(t,x)) = \tau(t)\,+\,\frac{t}{2}\,\Gamma(\tau(t)).$$

Applying Ünal's extra condition forces the following relation between the spatial and temporal infinitesimals

$$\frac{\partial \xi(t,x)}{\partial x} = \frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t). \tag{3.92}$$

Integrating (3.92) with respect to the spatial variable gives

$$\xi(t,\, x) = \left(\frac{\tau(t)}{t} + \frac{\dot{\tau}(t)}{2} \right) x \,+\, C_0(t), \tag{3.93}$$

where C_0 is an arbitrary function. This follows from the fact that the right-hand side of (3.92) is a purely a function of time; thus the second derivative of the spatial infinitesimal will be zero. Substituting $\xi(t,x)$ from (3.93) into (3.91), causes the equation (3.91) to become

$$\dot{C}_0(t) + \left(\frac{\dot{\tau}(t)}{t} - \frac{\tau(t)}{t^2} + \frac{\ddot{\tau}(t)}{2} \right) x + \mu \left(\frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t) \right) = \mu\dot{\tau}(t), \tag{3.94}$$

Separation of coefficients gives

$$\frac{d}{dt}\left(\frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t) \right) = 0, \tag{3.95}$$

which means that we have the following equation

$$\frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t) = C_1, \tag{3.96}$$

where C_1 is an arbitrary constant and thus the following results

$$\tau(t) = \frac{C_2}{t^2} + \frac{2\,C_1}{3}\,t. \tag{3.97}$$

Enforcing the following condition

$$\Gamma(\tau) = \text{Constant}, \tag{3.98}$$

whence we have that C_2 must be zero, i.e.

$$\tau(t) = \frac{2\,C_1}{3}\,t, \tag{3.99}$$

$$C_0(t) = -\mu\left(\frac{C_1}{3}\,t \right) + C_3. \tag{3.100}$$

The only solution which satisfies both equations is when the temporal infinitesimal is zero.

The spatial infinitesimal reduces to

$$\xi(t,\, x) = C_1\, x - \mu\left(\frac{C_1}{3}\,t \right) + C_3. \tag{3.101}$$

However, the equation (3.98) is only met if C_1 is zero. Hence the symmetries are

$$H_1 = \frac{\partial}{\partial x} \tag{3.102}$$

and

$$H_2 = \frac{2\,t}{3}\,\frac{\partial}{\partial t} + \left(x - \mu\,\frac{t}{3} \right)\frac{\partial}{\partial x}. \tag{3.103}$$

The ODEs for finding the finite transformations are (subject to initial conditions)

$$\frac{d\,\bar{t}}{d\,\epsilon} = \frac{2\,\bar{t}}{3} \tag{3.104}$$

which solves as

$$\bar{t} = t\,e^{\frac{2}{3}\epsilon} \tag{3.105}$$

with remaining the ODE being

$$\frac{d\,\overline{x}}{d\,\epsilon} = \overline{x} - \frac{\mu\,t}{3}\,e^{\frac{2}{3}\epsilon} \tag{3.106}$$

which easily gives

$$\overline{x} = x\,e^{\epsilon} - \mu\,t\left(e^{\epsilon} - e^{\frac{2}{3}\epsilon}\right). \tag{3.107}$$

The Itô SODEs associated with these finite transformations are

$$d\bar{t}(t,\omega) = e^{\frac{2}{3}\epsilon}\,dt \tag{3.108}$$

$$d\,\overline{X}(t,\omega) = \mu\,e^{\frac{2}{3}\epsilon}\,dt + e^{\epsilon}\,t\,dW(t,\omega) \tag{3.109}$$

which maintains form invariance, since by the random time change formula we have

$$d\overline{W}(t,\omega) = e^{\frac{1}{3}\epsilon}dW(t,\omega). \tag{3.110}$$

Thus in our transformed probability space, we have

$$d\,\overline{X}(\bar{t},\omega) = \mu\,d\bar{t} \,+\, \bar{t}\,d\overline{W}(\bar{t},\omega). \tag{3.111}$$

3.4.5 Example 5

We investigate and adjusted version of the previous SODE which has zero drift, i.e. a Martingale

$$dX(t,\omega) = t\,dW(t,\omega) \tag{3.112}$$

with the initital condition $X(0,\omega) = x_0$. Thus the determining equations are

$$\Gamma(\xi(t,x)) = 0,$$

which by the condition (3.10) gives

$$\frac{\partial\xi(t,x)}{\partial t} + \frac{t^2}{2}\,\frac{\partial^2\xi(t,x)}{\partial x^2} = 0 \tag{3.113}$$

and

$$Y(\xi(t,x)) = \tau(t) \,+\, \frac{t}{2}\,\Gamma(\tau(t)).$$

Applying Ünal's extra condition forces the following relation between the spatial and temporal infinesimals

$$\frac{\partial\xi(t,x)}{\partial\,x} = \frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t) \tag{3.114}$$

Integrating (3.114) with respect to the spatial variable yields

$$\xi(t,\,x) = \left(\frac{\tau(t)}{t} + \frac{\dot{\tau}(t)}{2}\right)\,x\,+\,C_0, \tag{3.115}$$

where C_0 is an arbitrary constant. As with the previous example, it follows from the fact that the right-hand side of (3.114) is a purely a function of time. Substituting $\xi(t,x)$ from (3.115) into (3.113), causes the equation (3.113) to become the following linear ODE

$$\left(\frac{\dot{\tau}(t)}{t} - \frac{\tau(t)}{t^2} + \frac{\ddot{\tau}(t)}{2}\right)\,x = 0. \tag{3.116}$$

Separation of coefficients gives

$$\frac{d}{dt}\left(\frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t)\right) = 0, \tag{3.117}$$

which means that we have to solve the following equations

$$\frac{\tau(t)}{t} + \frac{1}{2}\dot{\tau}(t) = C_1, \tag{3.118}$$

where C_1 is an arbitrary constant. The integrating factor t^2 furnishes the temporal infinitesimal as

$$\tau(t) = C_1\,t\,+\,\frac{C_2}{t^2}, \tag{3.119}$$

where C_1 is an arbitrary constant. Thus the spatial infinitesimal is

$$\xi(t,\,x) = \left(\frac{3\,C_1}{2}\right)\,x\,+\,C_0. \tag{3.120}$$

The only interesting case is $C_2 = 1$; with the remaining arbitrary constants being zero, i.e.

$$\overline{x} = x \tag{3.121}$$

and

$$\frac{d\bar{t}}{d\epsilon} = \frac{1}{\bar{t}^2}, \tag{3.122}$$

which solves as (using initial conditions)

$$\bar{t} = \left(2\,\epsilon\,+\,t^3\right)^{\frac{1}{3}}. \tag{3.123}$$

Moving to the probability space we have

$$d\bar{t}(t,\omega) = \frac{t^2\,dt}{(2\,\epsilon\,+\,t^3)^{\frac{2}{3}}}. \tag{3.124}$$

The random time change formula now gives the following transformed Wiener process

$$d\overline{W}(\bar{t},\omega) = \frac{t\,dW(t)}{(2\,\epsilon\,+\,t^3)^{\frac{1}{3}}}. \tag{3.125}$$

The transformed spatial process is

$$\begin{aligned} d\overline{X}(\bar{t},\omega) &= dX(t,\omega) && (3.126)\\ &= t\,dW(t,\omega) && (3.127)\\ &= (2\,\epsilon\,+\,t^3)^{\frac{1}{3}}\,d\overline{W}(\bar{t},\omega), && (3.128) \end{aligned}$$

as a consequence of (3.125); thus we finally have

$$d\overline{X}(\bar{t},\omega) = \bar{t}\,d\overline{W}(\bar{t},\omega), \tag{3.129}$$

as a result of (3.123). Form invariance is preserved.

(3.130)

3.5 Conclusion

The method which Meleshko et al. [4] followed overlooked the fact that the temporal group transformation was also subject to the Itô formula. The random time change definition in conjunction with the Itô formula applied to the temporal group transformation, is enough to derive a condition which is consistent with what Ünal [3] had derived, where the Itô multiplication table was used in combination with the random time change formula instead.

The methodology of [4] was incomplete: the Itô formula was precluded in the temporal Lie point transformation analysis which we have remedied. Overlooking this formula gave rise to transformations that did not provide form invariance. We re-considered two examples from [4] to demonstrate this.

We also note that as a consequence of the equations

$$Y(\xi) = H(G) + \frac{1}{2}G\,\Gamma(\tau) \tag{3.131}$$

and

$$Y(\tau) = 0 \tag{3.132}$$

for first order SODEs with one underlying Wiener process, the temporal infinitesimal must at most be a linear function of time. This ensures that the condition

$$\Gamma(e^{\epsilon H}(t)) = e^{\epsilon \Gamma(\tau)}, \tag{3.133}$$

which we derived in Fredericks and Mahomed [5] will always be satisfied. It is this condition that enables us to reconstruct the finite transformations from the infinitesimal ones, which maintain form invariance in each of the two examples from [4]. The Itô formula is an important component of the temporal Lie point symmetry transformation analysis. Without it, reconstructing form invariant finite transformations from the infinitesimal ones is impossible.

Our third example demonstrates that the symmetry generators found still form a Lie algebra even though the condition (3.133) is not satisfied. However, it is not guaranteed that the finite transformations that are recoverable from the infinitesimal ones will guarantee form invariance.

The differences in the way in which the determining equations appear in Meleshko et al. [4] and Fredericks and Mahomed [5] is superficial. The examples so far have been for SDEs driven by a single Wiener process. Trying to find a random time change formula for multidimensional Wiener processes is tackled in the next chapter.

From the examples thus far, it seems that the temporal infinitesimals were all projective, but this is only because the SDEs were driven by one Wiener process. The example in the second chapter has so far been the only exception, because there were two Wiener processes driving the Brownian motion on a circle in that example.

Chapter 4

An Alternative 'W-symmetries' Approach to Lie Point Symmetries of Scalar First-Order Itô Stochastic Ordinary Differential Equations

Extending the symmetry generator to include the infinitesimal transformations of the Wiener process for Itô SDEs has successfully been done in this chapter. The impact of this work leads to an intuitive understanding of the random time change formulae in the context of Lie point symmetries without having to consult much of the intense Itô calculus theory needed to derive it formerly (see Øksendal [8, 9]).

4.1 Introduction

A seminal work by Gaeta [7] incorporates the Wiener process into the symmetry operator. However, [7] conditions the transformed Wiener process to be consistent with the original process in terms of its momenta, i.e. the instantaneous mean and variance of the transformed process are forced to be exactly the instantaneous mean and variance of the original Wiener process. This chapter enforces the philosophy of a property invariance instead.

In [7], the Wiener process assimilation into the symmetry operator, forces an additional term to appear in the determining equations. In particular, the determining equation which handles the form invariance of the diffusion component of the transformed spatial process. This additional term can actually be coalesced into the symmetry analysis, by using Itô's formula on a system of SODEs which is built upon both the Wiener and spatial processes.

Instead of viewing the spatial processes individually and applying the random time change formula to the Wiener processes driving the spatial ones, as has been done in Gaeta and Quintero [1], Gaeta [7], Wafo Soh and Mahomed [2], Ünal [3] and Fredericks and Mahomed [5], we include the system of Wiener processes as part of the system of spatial processes. This change of thought adjusts the drift and diffusion operators that have been used so far.

This naturally introduces a Wiener infinitesimal in our symmetry operator. Thus the symmetry analysis takes place in the Banach space; the Wiener process is viewed as a variable in the Banach space. The Itô formula transports the group transformations of these variables to processes which exist in the probability space. The derivation of the random time change formula will then be easily achieved by using the Itô formula in conjunction with a Wiener property invariance criterion.

This work seeks to reconcile the work of Gaeta [7] with that of Fredericks and Mahomed [5]. We firstly introduce the SODEs as a system made up of both the spatial and Wiener processes. The operators which are inherently built upon the Itô formula for these SODEs will be re-defined. With these operators we introduce the transformation methodology used by the Lie point symmetry approach. An invariance argument on the properties of the transformed Wiener process should still give rise to the condition (3.10), which we derived in the previous chapter. The Random Time Change formula used in [5], Wafo Soh and Mahomed [2], Ünal [3] and Meleshko et al. [4], will be derived from a Lie point symmetry approach. Examples that were considered in [7] will then be done again and compared.

4.2 Review of Gaeta [7]

SODEs are non-deterministic, see Øksendal [9]. The driver of this randomness is the Wiener process $W(t,\omega)$. The Wiener process is a family of random variables indexed by time; its sample paths or possible realisations are denoted by ω. So for a particular realisation ω there is a time index, t, following it. (Books by Brzeźniak and Zastawniak [15], Freidlin [14] and Revuz and Yor [13] explain these concepts well). The SODEs which a random N-dimensional spatial process, $\mathbf{X}(t,\omega)$ satisifies, will intuitively be viewed as

$$d\mathbf{X}(t,\omega) = \mathbf{f}\,(\mathbf{X}(\mathbf{t},\omega),\,\mathbf{t})dt + \mathbf{G}\,(\mathbf{X}(\mathbf{t},\omega),\,\mathbf{t})\;d\mathbf{W}(t,\omega), \tag{4.1}$$

where $\mathbf{f}(\mathbf{X}(t,\omega),\,t)$ and $\mathbf{G}(\mathbf{X}(t,\omega),\,t)$ are the instantaneous N-dimensional mean and $N\times M$-dimensional standard diffusion of our random spatial process, respectively. These momenta are associated with a spatial measure $\mathbb{P}$. In the work by Gaeta [7], the following infinitesimal transformations were made in the Banach space,

$$\overline{x}_j \;=\; x_j \;+\; \epsilon\,\xi_j(\mathbf{x},\,t) + \mathcal{O}(\epsilon^2) \tag{4.2}$$

$$\bar{t} \;=\; t \;+\; \epsilon\,\tau(t) + \mathcal{O}(\epsilon^2) \tag{4.3}$$

and

$$\overline{w}_l \;=\; w_l \;+\; \epsilon\,\gamma_l(\mathbf{w},\,t) + \mathcal{O}(\epsilon^2),\;\text{where } j = 1,N \text{ and } l = 1,M. \tag{4.4}$$

Thus our symmetry operator is

$$H = \tau(t)\,\frac{\partial}{\partial t} \;+\; \sum_{l=1}^{M}\gamma_l(\mathbf{w},\,t)\,\frac{\partial}{\partial w_l} \;+\; \sum_{j=1}^{N}\xi_j(\mathbf{x},\,t)\frac{\partial}{\partial x_j}. \tag{4.5}$$

The spatial infinitesimal is chosen to be independent of the Wiener variables, for physical reasons; the temporal infinitesimal was forced to be projective based upon the Fokker-Plank *ansatz* used in ealier work by Gaeta and Quintero [1]; and the Wiener infinitesimal was forced to be independent of the spatial variables. Gaeta [7] referred to this Wiener transformation as an 'internal' transformation because of this spatial independence. To ensure that the Wiener transformation remains identical to the original Wiener process, only constant orthogonal transformations are considered by [7]

$$\overline{w}_l = \sum_{m=1}^{M} K_l^m\,w_m,\; l = 1,M \tag{4.6}$$

where

$$\mathbf{K}\,\mathbf{K}^T = I \tag{4.7}$$

and I is the $M\,\times\,M$-identity matrix. He further introduces an $M\,\times\,M$-dimensional antisymmetric matrix, $\mathbf{B}$ such that

$$\mathbf{B}^T \;=\; -\mathbf{B} \tag{4.8}$$

and

$$\overline{w}_l \;=\; w_l \;+\; \epsilon \sum_{m=1}^{M} B_l^m \, w_m + \mathcal{O}(\epsilon^2), \; l = 1, M. \tag{4.9}$$

In Gaeta's previous work, by Gaeta and Quintero [1], the random time change formula of Øksendal [9, 8] was not used but a portion of the formula was re-derived, *viz.* ,

$$\overline{w}_l(t) = (1 \,+\, \epsilon \frac{\dot{\tau}(t)}{2}) \, w_l + \mathcal{O}(\epsilon^2). \tag{4.10}$$

Still not apparently knowing of the random time change formula of [9, 8], Gaeta [7] uses the transformed diffusion component to absorb the scalar term which was derived in [1], i.e. $\epsilon \frac{\dot{\tau}(t)}{2}$ under the introduction of the following form invariance argument

$$d\overline{\mathbf{X}}(\bar{t}, \omega) = \mathbf{f}\left(\overline{\mathbf{X}}(\bar{t}, \omega), \, \bar{t}\right) d\bar{t} + \mathbf{G}\left(\overline{\mathbf{X}}(\bar{t}, \omega), \, \bar{t}\right) \, d\overline{\mathbf{W}}(\bar{t}, \omega), \tag{4.11}$$

where, the transformed drift component is

$$\mathbf{f}\left(\overline{\mathbf{X}}(\bar{t}, \omega), \, \bar{t}\right) \,=\mathbf{f} \,+\, \epsilon \, H(\mathbf{f}\,(\mathbf{X}(t, \omega), \, t)) + \mathcal{O}(\epsilon^2) \tag{4.12}$$

and the transformed diffusion component is given by

$$\mathbf{G}\left(\overline{\mathbf{X}}(\bar{t}, \omega), \, \bar{t}\right) \,=\mathbf{G} \,+\, \epsilon \, H(\mathbf{G}\,(\mathbf{X}(t, \omega), \, t)) \,+\, \epsilon \frac{\dot{\tau}(t)}{2} \, \mathbf{I} + \mathcal{O}(\epsilon^2). \tag{4.13}$$

The problem of having this absorbed term, is that the Lie point transformation is not being strictly followed.

The Itô SODEs associated with the transformations made in the Banach space are

$$\begin{aligned} d\overline{\mathbf{X}}(\bar{t}, \omega) &= d\mathbf{X}(t, \omega) + \epsilon \left(\frac{\partial \xi_j}{\partial t} + \sum_{r=1}^{N} f_r \, \frac{\partial \xi_j}{\partial x_r} \,+\, \frac{1}{2} \sum_{l=1}^{M} \sum_{i,\, j=1}^{N} G_i^l G_j^l \, \frac{\partial^2 \xi_j}{\partial x_i \partial x_j} \right) dt \\ &\quad + \epsilon \sum_{r=1}^{N} G_r^k \, \frac{\partial \xi_j}{\partial x_r} \, dW_k(t, \omega) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.14}$$

$$d\bar{t}(t, \omega) = dt + \epsilon \dot{\tau}(t) \, dt + \mathcal{O}(\epsilon^2) \tag{4.15}$$

and

$$d\overline{W}_l(\bar{t}, \omega) \;=\; dW_l(t, \omega) \;+\; \epsilon \sum_{k=1}^{M} B_l^k \, dW_k(t, \omega) + \mathcal{O}(\epsilon^2), \;\; l = 1, M. \tag{4.16}$$

Expanding these terms to $\mathcal{O}(\epsilon)$ in (4.11) gives the following determining equations

$$\Gamma_{(x)}(\boldsymbol{\xi}) \;= H(\mathbf{f}) + \mathbf{f}\Gamma_{(x)}(\tau) \tag{4.17}$$

and

$$Y_{(x)}^k(\xi_j) = H(G_j^k) + G_j^k \frac{\Gamma_{(x)}(\tau)}{2} + \sum_{p=1}^{M} G_p^k \, B_j^p, \quad j = 1, \, N \tag{4.18}$$

where

$$\Gamma_{(x)} = \frac{\partial}{\partial t} + \sum_{r=1}^{N} f_r \, \frac{\partial}{\partial x_r} + \frac{1}{2} \sum_{l=1}^{M} \sum_{i,\, j=1}^{N} G_i^l G_j^l \, \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.19}$$

$$Y_{(x)}^k = \sum_{r=1}^{N} G_r^k \, \frac{\partial}{\partial x_r}, \quad k = 1, M. \tag{4.20}$$

The projective nature of the temporal infinitesimal simplifies the determining equations further

$$\Gamma_{(x)}(\xi_j) \;=\; H(\mathbf{f}) + \mathbf{f}\,\dot{\tau}(t) \tag{4.21}$$

and

$$Y^k_{(x)}(\xi_j) \;=\; H(G^k_j) + \frac{G^k_j}{2}\,\dot{\tau} + \sum_{p\,=\,1}^{M} G^k_p\,B^p_j, \tag{4.22}$$

where $k = 1, M$ and $j = 1, N$. The difference between these determining equations and those of Fredericks and Mahomed [5] is that the temporal infinitesimal does not necessarily have to be projective. The additional terms $\sum_{p\,=\,1}^{M} G^k_p\,B^p_j$ also do not appear in the determining equations of [5], which are the result of the Itô SODEs for the transformed Wiener process, i.e. (4.16), where the Itô formula used in [7] is based on the operators

$$\Gamma_{(w)} = \frac{\partial}{\partial t} + \sum_{l\,=\,1}^{M} \frac{\partial}{\partial w_l} + \frac{1}{2}\sum_{l,\,m\,=\,1}^{M} \frac{\partial^2}{\partial w_l \partial w_m} \tag{4.23}$$

and

$$Y^k_{(w)} = \frac{\partial}{\partial w_k}, \qquad k = 1, M. \tag{4.24}$$

The use of two different systems of operators is apparent, where one pair is applied to functions associated to the spatial process and the remaining pair used for Wiener processes. Our objective is to reconcile these two pairs of operators into one pair of operators; and allow a form invariant argument on the transformed Wiener properties to guide us to what the characteristics of the transformed temporal variable should be.

4.3 Coupled System of SODEs

The methodology of Gaeta [7] highlights the fact that there are two different probability measures involved, i.e. a Wiener process measure $\mathbb{Q}$ and a measure associated with the spatial process $\mathbb{P}$.

The application of Lie point symmetry transformations on Itô SODEs transforms the momenta of these processes; thus transforming the measures as well. The Wiener infinitesimal $\boldsymbol{\gamma}$ is used to transform the Wiener process measure $\mathbb{Q}$ to a new measure $\overline{\mathbb{Q}}$; and the spatial infinitesimal $\boldsymbol{\xi}$ is used to transform the spatial process measure $\mathbb{P}$ to a new measure $\overline{\mathbb{P}}$.

These measures are independent of one another because of the inherent difference in the characteristics of the momenta which they represent. For example, the instantaneous drift of the Wiener process is zero, while that of the spatial process (4.1) is not. This explains, from a measure theoretic context, why the Wiener transformation is chosen to be independent of the spatial process, i.e. the Wiener infinitesimal is not a function of the spatial coordinates; and why the spatial transformation is independent of the realizations of the Wiener process, i.e. the spatial infinitesimal is independent of the Wiener variables. However, as these transformed processes are following their different paths of realisation, a transformed time index will be needed to follow both the transformed Wiener and spatial processes. As a result, we will assume that the scalar temporal infinitesimal, τ be a function of time, the Wiener variables and the spatial variables.

4.3.1 Random Time Change Formula

With the above in mind we first derive the random time change formula using the Wiener infinitesimal. Thus the system which we analyze first is the M-dimensional Wiener process. The Lie point theorem on the time index and Wiener variable in the Banach space, respectively gives us

$$\bar{t} = e^{\epsilon\, H}(t) \tag{4.25}$$

and

$$\overline{\mathbf{w}} = e^{\epsilon\, H}(\mathbf{w}), \tag{4.26}$$

which are associated with the following respective Itô SODEs

$$d\bar{t}(t,\omega) = \Gamma_{(w)}(e^{\epsilon\, H}(t))\, dt \,+\, Y^{l}_{(w)}(e^{\epsilon\, H}(t))\, dW_l(t,\omega) \tag{4.27}$$

and

$$d\overline{\mathbf{W}}(\bar{t},\omega) = \Gamma_{(w)}(e^{\epsilon\, H}(\mathbf{w}))\, dt \,+\, Y^{l}_{(w)}(e^{\epsilon\, H}(\mathbf{w}))\, dW_l(t,\omega). \tag{4.28}$$

The probabilistic nature of the transformed time index should remain form invariant in the following sense

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\bar{t}(t,\omega)\right] = d\bar{t}, \tag{4.29}$$

since this is trivially satisfied by the original differential time index, dt. This gives rise to the following condition

$$Y^{l}_{(w)}(e^{\epsilon\, H}(t)) \;= 0 \tag{4.30}$$

for $l = 1, M$, which is a more generalized version of the condition than that by Ünal [3]. In actual fact, we have derived a random time change formula, where

$$\bar{t} = \int^{t} \Gamma_{(w)}\left(e^{\epsilon\, H}(s)\right)\, ds. \tag{4.31}$$

The condition (4.29) also forces

$$\Gamma_{(w)}(e^{\epsilon\, H}(t)) = Constant. \tag{4.32}$$

An invariance argument on the instantaneous drift component of the Wiener process gives

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{\mathbf{W}}(\bar{t},\omega)\middle| W = w\right] = 0. \tag{4.33}$$

As a consequence we have the following new condition

$$\Gamma_{(w)}(e^{\epsilon\, H}(\mathbf{w})) = 0. \tag{4.34}$$

A similar invariance argument for the Itô isometry condition or instantaneous variance furnishes

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{W}_l(\bar{t},\omega)d\overline{W}_m(\bar{t},\omega)\middle| X = x, W = w\right] = \delta^{l}_{m} d\bar{t} \tag{4.35}$$

$$= \Gamma_{(w)}(e^{\epsilon\, H}(t))\, dt \,+\, Y^{l}_{(w)}(e^{\epsilon\, H}(t))\, dW_l \tag{4.36}$$

thus providing

$$\sum_{r=1}^{M} Y^{r}_{(w)}(e^{\epsilon\, H}(w_l))dW_r \sum_{s=1}^{M} Y^{s}_{(w)}(e^{\epsilon\, H}(w_m))dW_s = \delta^{l}_{m} d\bar{t} \tag{4.37}$$

$$\sum_{r=1}^{M} Y^{r}_{(w)}(e^{\epsilon\, H}(w_l))Y^{r}_{(w)}(e^{\epsilon\, H}(w_m))dt = \delta^{l}_{m}\Gamma_{(w)}(e^{\epsilon\, H}(t))\, dt \,+\, Y^{r}_{(w)}(e^{\epsilon\, H}(t))\, dW_m \tag{4.38}$$

which by comparing Riemann and Itô integrals, also forces the condition (4.30)

$$Y^l_{(w)}(e^{\epsilon\, H}(t)) \;= 0 \tag{4.39}$$

for $l = 1, M$; with the transformed Wiener process being given as

$$d\overline{\mathbf{W}}(\bar{t}, \omega) = \int^t Y^l_{(w)}(e^{\epsilon\, H}(\mathbf{w}))\, dW_l(s, \omega), \tag{4.40}$$

where

$$\sum_{r=1}^{M} Y^r_{(w)}(e^{\epsilon\, H}(w_l))Y^r_{(w)}(e^{\epsilon\, H}(w_m)) \;= \delta^l_m \Gamma_{(w)}(e^{\epsilon\, H}(t)). \tag{4.41}$$

Differentiating (4.41) with respect to ϵ gives

$$\sum_{r=1}^{M} Y^r_{(w)}(w_l)Y^r_{(w)}(\gamma_m)) + \sum_{r=1}^{M} Y^r_{(w)}(w_m)Y^r_{(w)}(\gamma_l)) + \mathcal{O}(\epsilon) \;= \delta^l_m \Gamma_{(w)}(\tau) + \mathcal{O}(\epsilon). \tag{4.42}$$

which simplifies further as

$$Y^l_{(w)}(w_l)Y^l_{(w)}(\gamma_m)) + Y^m_{(w)}(w_m)Y^m_{(w)}(\gamma_l)) + \mathcal{O}(\epsilon) \;= \delta^l_m \Gamma_{(w)}(\tau) + \mathcal{O}(\epsilon), \tag{4.43}$$

since $Y^r_{(w)}(w_l) = \delta^r_l$.

Equation (4.43) evaluated at $\epsilon = 0$ simply is

$$Y^l_{(w)}(\gamma_m) + Y^m_{(w)}(\gamma_l) \;= \delta^l_m \Gamma_{(w)}(\tau). \tag{4.44}$$

The case where $l \neq m$ naturally leads to the antisymmetric matrix $\mathbf{B}$ which was defined in [7]. We extend the M-dimensional system by coupling it with an N-dimensional spatial process

$$d\mathbf{X}(t, \omega) = \mathbf{f}\left(\mathbf{X}(\mathbf{t}, \omega),\, \mathbf{t}\right)dt + \mathbf{G}\left(\mathbf{X}(\mathbf{t}, \omega),\, \mathbf{t}\right)\, d\mathbf{W}(t, \omega). \tag{4.45}$$

As a result of the covariance property of a scalar Wiener process, any arbitrary function of the random spatial process is subject to Itô calculus. The traditional Itô calculus based on the SODEs (4.45) only, purports that an arbitrary function, which is at least once and twice differentialable w.r.t. time and space, respectivley, will satisfy the SODEs

$$dF\left(\mathbf{X}(t, \omega),\, t\right) = \Gamma_{(x)}(F)\left(\mathbf{X}(t, \omega),\, t\right) dt + Y^k_{(x)}(F)\left(\mathbf{X}(t, \omega),\, t\right)\, dW_k(t, \omega), \tag{4.46}$$

where

$$\Gamma_{(x)}(F) \;=\; \frac{\partial}{\partial t} + \sum_{r=1}^{N} f_r \frac{\partial}{\partial x_r} + \frac{1}{2}\sum_{l=1}^{M}\sum_{i,\,j=1}^{N} G^l_i G^l_j \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.47}$$

$$Y^k_{(x)}(F) \;=\; \sum_{r=1}^{N} G^k_r \frac{\partial}{\partial x_r}\;, \text{where } k \;=\; 1, M. \tag{4.48}$$

With the same arbitrary function $F(\mathbf{x},\, t)$ we define the following operators which are based upon our coupled system of M-dimensional Wiener processes and (4.45)

$$\Gamma \;=\; \frac{\partial}{\partial t} + \sum_{r=1}^{N} f_r \frac{\partial}{\partial x_r} + \frac{1}{2}\sum_{l,\,m=1}^{M} \frac{\partial^2}{\partial w_l \partial w_m} + \frac{1}{2}\sum_{l=1}^{M}\sum_{i,\,j=1}^{N} G^l_i G^l_j \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.49}$$

$$Y^k \;=\; \frac{\partial}{\partial w_k} + \sum_{r=1}^{N} G^k_r \frac{\partial}{\partial x_r}. \tag{4.50}$$

The application of these operators on our arbitrary function $F(\mathbf{x},\, t)$ still satisfies equation (4.51) since it is not a function of the Wiener variables, $\mathbf{w}$, i.e.

$$dF\left(\mathbf{X}(t, \omega),\, t\right) = \Gamma(F)\left(\mathbf{X}(t, \omega),\, t\right) dt + Y^k(F)\left(\mathbf{X}(t, \omega),\, t\right)\, dW_k(t, \omega). \tag{4.51}$$

This calculus is needed for the construction of the Lie point algorithm for SODEs as can be seen in Wafo Soh and Mahomed [2] and Fredericks and Mahomed [5]. By the above extension we will make use of group transformations to derive the random time change formula used in the previous chapter for spatial processes with our newly defined operators.

4.3.2 Group Transformations

The construction begins with a one-parameter group of transformations of the time index t, the spatial variable $\mathbf{x}$ and Wiener variable $\mathbf{w}$, respectively,

$$\bar{t} = \theta(\mathbf{x},\,\mathbf{w},\,t,\,\epsilon), \quad \overline{x} = \varphi(\mathbf{x},\,t,\,\epsilon) \quad \text{and } \overline{w} = \vartheta(\mathbf{w},\,t,\,\epsilon), \tag{4.52}$$

with the following relation to the infinitesimals

$$\frac{\partial\theta}{\partial\epsilon} = \tau(\theta,\varphi,\vartheta), \quad \frac{\partial\varphi}{\partial\epsilon} = \xi(\theta,\varphi,\vartheta) \quad \text{and } \frac{\partial\vartheta}{\partial\epsilon} = \mu(\theta,\varphi,\vartheta). \tag{4.53}$$

The initial boundary conditions at $\epsilon = 0$ are

$$\bar{t}\big|_{\epsilon=0} = t, \quad \overline{\mathbf{X}}(\bar{t},\omega)\big|_{\epsilon=0} = \mathbf{X}(t,\omega) \quad \text{and } \overline{\mathbf{W}}(\bar{t},\omega)\big|_{\epsilon=0} = \mathbf{W}(t,\omega). \tag{4.54}$$

Hence the symmetry operator is given by

$$H = \tau(\mathbf{x},\,\mathbf{w},\,t)\,\frac{\partial}{\partial t} + \sum_{l=1}^{M}\gamma_l(\mathbf{w},\,t)\,\frac{\partial}{\partial w_l} + \sum_{j=1}^{N}\xi_j(\mathbf{x},\,t)\frac{\partial}{\partial x_j}, \tag{4.55}$$

which is different to the one used by Gaeta [7] because the temporal infinitesimal is non-projective. Thus group transformations can be expressed in terms of the symmetry operator as

$$\bar{t} = e^{\epsilon H}(t), \tag{4.56}$$

$$\overline{\mathbf{x}} = e^{\epsilon H}(\mathbf{x}) \tag{4.57}$$

and

$$\overline{\mathbf{w}} = e^{\epsilon H}(\mathbf{w}). \tag{4.58}$$

The Itô SODEs are associated to these group of transformations (4.52) by the following

$$d\bar{t}(t,\omega) = \Gamma(e^{\epsilon H}(t))\,dt \,+\, Y^l(e^{\epsilon H}(t))\,dW_l(t,\omega), \tag{4.59}$$

$$d\overline{\mathbf{W}}(\bar{t},\omega) = \Gamma(e^{\epsilon H}(\mathbf{w}))\,dt \,+\, Y^l(e^{\epsilon H}(\mathbf{w}))\,dW_l(t,\omega) \tag{4.60}$$

and

$$d\overline{\mathbf{X}}(\bar{t},\omega) = \Gamma(e^{\epsilon H}(\mathbf{x}))\,dt \,+\, Y^l(e^{\epsilon H}(\mathbf{x}))\,dW_l(t,\omega). \tag{4.61}$$

The infinitesimal SODEs are given as

$$d\tau(t,\omega) = \Gamma(\tau(t,\omega))\,dt + Y^l(\tau(t,\omega))\,dW_l(t,\omega), \tag{4.62}$$

$$d\boldsymbol{\xi}(t,\omega) = \Gamma(\boldsymbol{\xi}(t,\omega))\,dt + Y^l(\boldsymbol{\xi}(t,\omega))\,dW_l(t,\omega) \tag{4.63}$$

and

$$d\boldsymbol{\mu}(t,\omega) = \Gamma(\boldsymbol{\mu}(t,\omega))\,dt + Y^l(\boldsymbol{\mu}(t,\omega))\,dW_l(t,\omega), \tag{4.64}$$

where Γ and Y are operators which have been defined in (4.49) and (4.50), respectively. These operators are in fact the instantaneous mean and standard deviation of the temporal, spatial and Wiener infintesimals τ, $\boldsymbol{\xi}$ and $\boldsymbol{\mu}$ respectively.

4.3.3 Wiener Invariance Properties

Before deriving the determining equations we apply an invariance argument to the rudimentary properties of the Wiener process, *viz.* the instanteous mean and variance of the Wiener process which are

$$\mathbb{E}_{\mathbb{Q}}\left[d\mathbf{W}(t,\omega)\middle|\mathbf{W}=\mathbf{w}\right]=0 \tag{4.65}$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[dW_l(t,\omega)dW_m(t,\omega)\middle|\mathbf{W}=\mathbf{w}\right]=\delta^l_m dt, \tag{4.66}$$

respectively, for a scalar Wiener process. The form invariance of the instantaneous mean of the transformed Wiener process under the new measure $\overline{\mathbb{Q}}$ is expressed as

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{\mathbf{W}}(\bar{t},\omega)\middle|\mathbf{X}=\mathbf{x},\mathbf{W}=\mathbf{w}\right]=0, \tag{4.67}$$

which in conjunction with (4.60) gives

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[\Gamma(e^{\epsilon H}(\mathbf{w}))\,dt\,+\,Y^l(e^{\epsilon H}(\mathbf{w}))\,dW_l(t,\omega)\middle|\mathbf{X}=\mathbf{x},\mathbf{W}=\mathbf{w}\right]=0. \tag{4.68}$$

Now on (4.65) use (4.68) to derive the following condition

$$\Gamma(e^{\epsilon H}(\mathbf{w}))=0, \tag{4.69}$$

which is a consequence of the linearity property of the expectation operator and the fact that the expected value of an Itô integrand is always zero. The condition (4.69), is similar to the condition (4.34) that we found earlier for the $\mathbf{M}$-dimensional system of Wiener processes. We next apply the form invariance argument to the instantaneous variance of the transformed Wiener process under the transformed Wiener measure, i.e.

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{W}_l(\bar{t},\omega)d\overline{W}_m(\bar{t},\omega)\middle|X=x,W=w\right]=\delta^l_m d\bar{t}, \tag{4.70}$$

which after a simple substitution from (4.60) produces

$$\sum_{r=1}^{M}Y^r(e^{\epsilon H}(w_l))(t,\omega)Y^r(e^{\epsilon H}(w_m))(t,\omega)\,dt=\delta^l_m\,d\bar{t}, \tag{4.71}$$

which forces the following differential relation

(4.72)

$$\sum_{r=1}^{M}Y^r(e^{\epsilon H}(w_l))(t,\omega)Y^r(e^{\epsilon H}(w_m))(t,\omega)\,dt=\Gamma(e^{\epsilon H}(t))\,dt\,+\,Y^k(e^{\epsilon H}(t))\,dW_k(t,\omega),\ l=1,M. \tag{4.73}$$

Comparing the Wiener and Riemann integrals we have a generalization of the condition developed by Ünal [3] for the instantaneous standard deviation of the temporal infinitesimal, i.e.

$$Y^l(e^{\epsilon H}(t))=0,\ l=1,M, \tag{4.74}$$

as well as the following relation

$$\sum_{r=1}^{M}Y^r(e^{\epsilon H}(w_l))(t,\omega)Y^r(e^{\epsilon H}(w_m))(t,\omega)\ =\Gamma(e^{\epsilon H}(t)). \tag{4.75}$$

Thus we have derived the following generalized random time change formula

$$d\bar{t}(t,\omega) = \int^{t} \Gamma(e^{\epsilon H}(s))\, ds \tag{4.76}$$

with

$$\Gamma(e^{\epsilon H}(t)) = Constant \tag{4.77}$$

when we apply the probabilistic invariance principle to the transformed time index differential, i.e.

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\bar{t}(t,\omega)\right] = d\bar{t}. \tag{4.78}$$

The generalized Wiener transformation is

$$\overline{W}_r(\bar{t},\omega) = \int^{t} \sum_{m=1}^{M} Y^m(e^{\epsilon H}(w_r))dW_m(s,\omega), \tag{4.79}$$

where

$$\sum_{r=1}^{M} Y^r(e^{\epsilon H}(w_l))(t,\omega)Y^r(e^{\epsilon H}(w_m))(t,\omega) \; = \delta^l_m \Gamma(e^{\epsilon H}(t)). \tag{4.80}$$

This is a generalized form of the random time change formula derived in Øksendal [8], Øksendal [9] and Meleshko et al. [4] and used in [5], [2], [3] and [4].

(4.81)

4.3.4 Form Invariance of the Spatial Process

In order to find the condition (3.17) that ensures the recovery of the finite transformations from the infinitesimal transformations, we need the following form invariant argument

$$d\overline{\mathbf{X}}(\bar{t},\omega) = \mathbf{f}\left(\overline{\mathbf{X}}(\bar{t},\omega),\, \bar{t}\right) d\bar{t} + \mathbf{G}\left(\overline{\mathbf{X}}(\bar{t},\omega),\, \bar{t}\right)\; d\overline{\mathbf{W}}(\bar{t},\omega), \tag{4.82}$$

where, the transformed drift component is

$$\mathbf{f}\left(\overline{\mathbf{X}}(\bar{t},\omega),\, \bar{t}\right) \; = e^{\epsilon H}(\mathbf{f}) \tag{4.83}$$

and the transformed diffusion component is given by

$$\mathbf{G}\left(\overline{\mathbf{X}}(\bar{t},\omega),\, \bar{t}\right) \; = e^{\epsilon H}(\mathbf{G}). \tag{4.84}$$

There is no absorption for the transformed diffusion component as was done in Gaeta [7].

If we now expand the drift component of (4.82), we deduce

$$\begin{aligned}\mathbf{f}\left(\overline{\mathbf{X}}(\bar{t},\omega),\, \bar{t}\right)\, d\bar{t} = \Bigg\{ & \mathbf{f}(t,\, \mathbf{X}(t)) \,+\, \epsilon\, \left(\Gamma(H(t)) \,+\, H\right)\, \mathbf{f}(t,\, \mathbf{X}(t)) \\ & + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \Bigg(\left(\Gamma(H(t)) \,+\, H\right)^k\, \mathbf{f}(t,\, \mathbf{X}(t)) \\ & + \sum_{j=0}^{k-2} \binom{k}{k\,-\,j} H^j(\,\mathbf{f}(t,\, \mathbf{X}(t)))\, \Big(\Gamma(H^{k-j}(t)) \,-\, [\Gamma(H(t))]^{k-j}\Big)\Bigg)\Bigg\}\, dt.\end{aligned} \tag{4.85}$$

The equations (2.23) and (4.85) will be exactly identical even though the operators used here and in Chapter 1 are different.

The following condition is needed to ensure the recovery of the finite transformations from the infinitesimal transformations

$$e^{\epsilon\,\Gamma(H(t))}(t,\ \mathbf{X}(t)) \;=\; \Gamma\left(e^{\epsilon\,H}(t)(t,\ \mathbf{X}(t))\right). \tag{4.86}$$

This condition ensures that the higher order terms depend solely on the first order term associated with $\mathcal{O}(\epsilon)$. All the ordered terms contribute in the construction of the finite transformations; the zeroth and first order terms, contribute towards the construction of the infinitesimal transformations. This also forces the instantaneous drift coefficient of the temporal infinitesimal to be a constant, which was demonstrated in the first chapter, i.e.

$$\Gamma(\tau) = C, \text{ where } C \text{ is an arbitrary constant.} \tag{4.87}$$

Condition (4.86) also simplifies (4.80) to

$$\overline{\mathbf{W}}(\bar{t},\omega) = \int^{t} \sum_{m=1}^{M} Y^{m} e^{\epsilon H}(\mathbf{w}) dW_l(s,\omega), \tag{4.88}$$

where

$$\sum_{r=1}^{M} Y^{r}(e^{\epsilon H}(w_l))(t,\omega) Y^{r}(e^{\epsilon H}(w_m))(t,\omega) = \delta^{l}_{m} e^{\epsilon\Gamma(H(t))}. \tag{4.89}$$

Expanding the diffusion component of (4.82) gives

$$\begin{aligned}\sum_{l=1}^{M} G^{l}_{j}\left(\overline{\mathbf{X}}(\bar{t},\omega),\,\bar{t}\right)\, d\overline{W}_l(\bar{t},\omega) &= \sum_{l=1}^{M} G^{l}_{j}\, dW_l + \epsilon\left(\sum_{l=1}^{M} H(G^{l}_{j})\, dW_l + \sum_{l,m=1}^{M} G^{l}_{j}\, Y^{m}(H(w_l)) dW_m\right)\\ &\quad + \sum_{k=2} \frac{\epsilon^{k}}{k!}\left(\sum_{r=0}^{k}\binom{k}{r} H^{k-r}(G^{l}_{j})\, Y^{m}(H^{r}(w_l)) dW_m\right),\end{aligned} \tag{4.90}$$

where

$$H^{k}(G^{l}_{j})\, Y^{m}(H^{0}(w_l)) dW_m = H^{k}(G^{l}_{j})\, Y^{m}(w_l) dW_m = H^{k}(G^{l}_{j})\, dW_l. \tag{4.91}$$

4.3.5 Determining equations

We derive the determining equations for furnishing the spatial infinitesimals by differentiating the equations associated with the drift and diffusion components of the transformed spatial process, i.e. the drift components and diffusion components of (4.61) with respect to ϵ; and compare the results with the ϵ-differentiated equations of (4.85) and (4.90), at $\epsilon = 0$. This methodology is the method used in previous chapters, but with different operators. As a result we have the following determining equations

$$\Gamma(\boldsymbol{\xi}) \;=\; (\Gamma(\tau) \,+\, H)\,\mathbf{f} \tag{4.92}$$

$$Y^{l}(\xi_j) \;=\; \left(H(G^{l}_{j}) + \sum_{m=1}^{M} G^{m}_{j}\, Y^{l}(H(w_m))\right)\; l = 1, M \text{ and } j = 1, N. \tag{4.93}$$

We derive the determining equations for furnishing the Wiener infinitesimal by differentiating the equations associated with the drift and diffusion components of the transformed Wiener process, i.e. the drift components and diffusion components of (4.69) and (4.75), with respect to ϵ, which gives

$$\Gamma(\gamma_l) + \mathcal{O}(\epsilon) = 0,\ l = 1, M \tag{4.94}$$

and

$$Y^l(\gamma_j) + Y^j(\gamma_l) + \mathcal{O}(\epsilon) = \delta_l^j \Gamma(\tau) + \mathcal{O}(\epsilon). \tag{4.95}$$

Evaluating at $\epsilon = 0$, establishes the following determining equations

$$\Gamma(\boldsymbol{\gamma}) = 0, \tag{4.96}$$

as well as the following relation

$$Y^l(\gamma_l) = \frac{\Gamma(\tau)}{2} \text{ summation not implied, } l = 1, M \tag{4.97}$$

and

$$Y^l(\gamma_j) = -Y^j(\gamma_l), \; j \neq l. \tag{4.98}$$

Equation (4.98) develops the antisymmetric matrix, $\mathbf{B}$ as was done earlier for the M-dimensional system alone.

Remarks. Prior knowledge of the random time change formula is not required when using an adjusted 'W-symmetries' approach of Gaeta [7]. The rate of change is the drift component of the Itô SODEs of transformed temporal infinitesimal. By comparing our new method with that of [7] we have that his M matrix used in (4.6) does not have to be orthogonal. By using the invariant property philosophy, we can define his matrix as

$$\mathbf{M} = \sum_{m=1}^{M} \Big(Y^m(e^{\epsilon H}(\mathbf{w})) \Big) \mathbf{I}, \tag{4.99}$$

where $\mathbf{I}$ is the identity matrix. The components of antisymmetric matrix $\mathbf{B}$ are given as

$$B_i^i = 0, \tag{4.100}$$

$$B_m^l = Y^l(\gamma_m), \tag{4.101}$$

where

$$Y^l(\gamma_m) = -Y^m(\gamma_l). \tag{4.102}$$

Thus the Wiener infinitesimal transformation can be given as

$$\overline{w}_l = \left(1 + \epsilon \frac{\Gamma(\tau)}{2}\right) w_l + \epsilon \sum_{m=1}^{M} B_l^m dW_m. \tag{4.103}$$

The absorption methodology that was used in [7] is unnecessary. By using a property invariance principal, the absorbed term, $\epsilon \frac{\Gamma(\tau)}{2} w_l$, comes into existence in a natural way. We continue with examples. The first two are based on examples done by Gaeta [7].

4.4 Examples

Example 4.1.

This example was done in the first chapter. Here we look at it again. We have

$$dX_1(t) = \frac{a_1}{X_1}\,dt + dW_1(t), \tag{4.104}$$

$$dX_2(t) = a_2\,dt + dW_2(t). \tag{4.105}$$

The determining equations are simply given as

(4.106)

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_1} + a_2\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_2^2} =$$

$$\frac{a_1}{x_1}\left(\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1} + a_2\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2^2}\right)$$

$$+\xi_1\left(-\frac{a_1}{x_1^2}\right), \tag{4.107}$$

and

$$\frac{\partial \xi_2(\mathbf{x},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_1} + a_2\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_2^2} =$$

$$a_2\left(\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1} + a_2\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2^2}\right), \tag{4.108}$$

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_1} + G_1^1\frac{\partial \xi_1}{\partial x_1} + G_2^1\frac{\partial \xi_1}{\partial x_2} = H(G_1^1) + G_1^1 Y^1(\gamma_1) + G_1^2 Y^1(\gamma_2)$$

$$\frac{\partial \xi_1}{\partial x_1} = H(G_1^1) + G_1^1 Y^1(\gamma_1) \tag{4.109}$$

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_2} + G_1^2\frac{\partial \xi_1}{\partial x_1} + G_2^2\frac{\partial \xi_1}{\partial x_2} = H(G_1^2) + G_1^1 Y^2(\gamma_1) + G_1^2 Y^2(\gamma_2)$$

$$\frac{\partial \xi_1}{\partial x_2} = G_1^1 Y^2(\gamma_1) \tag{4.110}$$

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_1} + G_1^1\frac{\partial \xi_2}{\partial x_1} + G_2^1\frac{\partial \xi_2}{\partial x_2} = H(G_2^1) + G_2^1 Y^1(\gamma_1) + G_2^2\, Y^1(\gamma_2)$$

$$\frac{\partial \xi_2}{\partial x_1} = Y^1(\gamma_2) \tag{4.111}$$

$$\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_2} + G_1^2\frac{\partial \xi_2}{\partial x_1} + G_2^2\frac{\partial \xi_2}{\partial x_2} = H(G_2^2) + G_2^1 Y^2(\gamma_1) + G_2^2\, Y^2(\gamma_2)$$

$$\frac{\partial \xi_2}{\partial x_2} = H(G_2^2) + G_2^2\, Y^2(\gamma_2) \tag{4.112}$$

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1} + G_1^1\frac{\partial \tau}{\partial x_1} + G_2^1\frac{\partial \tau}{\partial x_2} = 0,$$

which implies that

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1} + \frac{\partial \tau}{\partial x_1} = 0 \tag{4.113}$$

and

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2} + G_1^2\frac{\partial \tau}{\partial x_1} + G_2^2\frac{\partial \tau}{\partial x_2} = 0,$$

which reduces to

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2} + \frac{\partial \tau}{\partial x_2} = 0. \tag{4.114}$$

From (4.113) and (4.114) we have

$$\tau(\mathbf{x},\mathbf{w},t) = F_1(x_1 - w_1) + F_2(x_2 - w_2) + a(t), \tag{4.115}$$

where F_1, F_2 and a are arbitrary functions. The condition (2.39) forces

$$\Gamma(\tau) = c_0. \tag{4.116}$$

Thus we have

$$\dot{a}(t) + \frac{a_1}{x_1} F_1'(x_1 - w_1) + a_2 F_2'(x_2 - w_2) + F_1'' + F_2'' = c_0. \tag{4.117}$$

Differentiating (4.117) with respect to time gives that $a(t)$ is the linear function of time

$$a(t) = c_1\, t \,+\, c_2. \tag{4.118}$$

Thus (4.117) becomes

$$\frac{a_1}{x_1} F_1'(x_1 - w_1) + a_2 F_2'(x_2 - w_2) + F_1'' + F_2'' = c_0 \,-\, c_1. \tag{4.119}$$

A simple comparison by coefficients of $\frac{1}{x_1}$, imposes that F_1 be a constant.

The equation (4.119) evolves into an ODE with dependent variable F_2 and independent $(x_2 - w_2)$ which solves as

$$F_2(x_2 - w_2) = \frac{(c_0 \,-\, c_1)}{(a_2)^2}\,(a_2\,(x_2 - w_2) - 1) + e^{-a_2\,(x_2 - w_2)} c_3. \tag{4.120}$$

This gives rise to the temporal infinitesimal as

$$\tau(\mathbf{x}, \mathbf{w}, t) = c_1\, t \,+\, c_2 \,+\, \frac{(c_0 \,-\, c_1)}{(a_2)^2}\,(a_2\,(x_2 - w_2) - 1) + e^{-a_2\,(x_2 - w_2)} c_3. \tag{4.121}$$

The Wiener infinitesimals are associated with the following determining equations

(4.122)

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial x_1} + a_2\frac{\partial \gamma_1(\mathbf{w},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_2^2} = 0, \tag{4.123}$$

and

$$\frac{\partial \gamma_2(\mathbf{w},t)}{\partial t} + \frac{a_1}{x_1}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial x_1} + a_2\frac{\partial \gamma_2(\mathbf{w},t)}{\partial x_2} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial x_1^2} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_2^2} = 0, \tag{4.124}$$

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_1} + G_1^1\frac{\partial \gamma_1}{\partial x_1} + G_2^1\frac{\partial \gamma_1}{\partial x_2} = \frac{\Gamma(\tau)}{2}$$

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_1} + \frac{\partial \gamma_1}{\partial x_1} = \frac{c_0}{2} \tag{4.125}$$

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_2} + G_1^2\frac{\partial \gamma_1}{\partial x_1} + G_2^2\frac{\partial \gamma_1}{\partial x_2} = -\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_1} + G_1^1\frac{\partial \gamma_2}{\partial x_1} + G_2^1\frac{\partial \gamma_2}{\partial x_2} \tag{4.126}$$

$$\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_2} + G_2^2\frac{\partial \gamma_2}{\partial x_2} = \frac{c_0}{2}. \tag{4.127}$$

Thus we have

$$\gamma_1(\mathbf{w}, t) = \frac{c_0}{2} w_1 + L_1(w_2,\, t) \tag{4.128}$$

and

$$\gamma_2(\mathbf{w}, t) = \frac{c_0}{2} w_2 + L_2(\, w_1,\, t). \tag{4.129}$$

The use of equation (4.126) forces the equality

$$\left(\frac{\partial}{\partial w_2} + \frac{\partial}{\partial x_2}\right) L_1(w_2,\, t) = -\left(\frac{\partial}{\partial w_1} + \frac{\partial}{\partial x_1}\right) L_2(w_1,\, t) = \text{Constant}, \tag{4.130}$$

as neither side is the function of the other. Further deduction concludes that both functions be linear with respect to the arguments, i.e.

$$L_1(w_2,\, t) = c_5\, w_2 \,+\, c_8\, t \,+\, c_6 \tag{4.131}$$

and

$$L_2(w_1,\, t) = c_{10}\, w_1 \,+\, c_9\, t \,+\, c_7. \tag{4.132}$$

From relation (4.126) we have the following

$$L_1(x_2,\, w_2,\, t) = c_5\, w_2 \,+\, c_8\, t \,+\, c_6 \tag{4.133}$$

and

$$L_2(x_1,\, w_1,\, t) = -c_5\, w_1 \,+\, c_9\, t \,+\, c_7. \tag{4.134}$$

Applying conditions (4.123) and (4.124) yields

$$c_8 = 0 \tag{4.135}$$

and

$$c_9 = 0. \tag{4.136}$$

Thus we finally have the following summary of the Wiener infinitesimals

$$\gamma_1(\mathbf{w}, t) = \frac{c_0}{2} w_1 + c_5\, w_2 \,+\, c_6 \tag{4.137}$$

and

$$\gamma_2(\mathbf{w}, t) = \frac{c_0}{2} w_2 - c_5\, w_1 \,+\, c_7. \tag{4.138}$$

As a result, the determining equations (4.107), (4.108), (4.109), (4.110), (4.111) and (4.112) become

$$\frac{\partial \xi_1}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_1}{\partial x_1} + a_2 \frac{\partial \xi_1}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_1}{\partial x_2^2} = \frac{a_1}{x_1}\, c_0 + \xi_1 \left(-\frac{a_1}{x_1^2}\right) \tag{4.139}$$

$$\frac{\partial \xi_2}{\partial t} + \frac{a_1}{x_1}\frac{\partial \xi_2}{\partial x_1} + a_2 \frac{\partial \xi_2}{\partial x_2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_1^2} + \frac{1}{2}\frac{\partial \xi_2}{\partial x_2^2} = a_2\, c_0 \tag{4.140}$$

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1}{2}\, c_0 \tag{4.141}$$

$$\frac{\partial \xi_1}{\partial x_2} = c_5 \tag{4.142}$$

$$\frac{\partial \xi_2}{\partial x_1} = -c_5 \tag{4.143}$$

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\, c_0 + . \tag{4.144}$$

From (4.141), (4.142), (4.143) and (4.143) we observe the following

$$\xi_1 = \frac{1}{2}\, c_0\, x_1 \,+\, c_5\, x_2 \,+\, F_1(t) \tag{4.145}$$

and

$$\xi_2 = \frac{1}{2}\, c_0\, x_2 \,-\, c_5\, x_1 + \, F_2(t). \tag{4.146}$$

The consequences of (4.145) reduces (4.139) to

$$\dot{F}_1(t) + \frac{a_1\, c_0}{2\, x_1} + a_2\, c_5 \;=\; \frac{a_1\, c_0}{x_1} \;-\; \left(\frac{1}{2}\, c_0\, x_1 \,+\, c_5\, x_2 \,+\, F_1(t)\right) \frac{a_1}{x_1^2}. \tag{4.147}$$

By comparison of coefficients we have that $F_1(t)$ and c_5 must be zero, i.e.

$$\xi_1 = \frac{1}{2}\, c_0\, x_1. \tag{4.148}$$

For the remaining spatial infinitesimal we have that (4.146) simplifies (4.140) to

$$\dot{F}_2(t) \,+\, a_2 \frac{1}{2}\, c_0 = a_2\, c_0. \tag{4.149}$$

The function $F_2(t)$ is forced to be linear with respect to time

$$F_2(t) = \frac{a_2\, c_0}{2}\, t \,+\, c_4. \tag{4.150}$$

In summarizing the results, we have

$$\xi_1 = \frac{1}{2}\, c_0\, x_1, \tag{4.151}$$

$$\xi_2 = \frac{1}{2}\, c_0\, x_2 \,+\, \frac{a_2\, c_0}{2}\, t \,+\, c_4, \tag{4.152}$$

$$\tau(t) = c_1\, t \,+\, c_2 \,+\, \frac{(c_0 \,-\, c_1)}{(a_2)^2}\, (a_2\,(x_2 - w_2) - 1) + e^{-a_2\,(x_2 - w_2)} c_3, \tag{4.153}$$

with the Wiener infinitesimal being

$$\gamma_1(\mathbf{w}, t) = \frac{c_0}{2} w_1 \,+\, c_5 \tag{4.154}$$

and

$$\gamma_2(\mathbf{w}, t) = \frac{c_0}{2} w_2 \,+\, c_6. \tag{4.155}$$

The seven symmetries are

$$H_0 = \frac{(a_2\,(x_2 - w_2) - 1)}{a_2^2} \frac{\partial}{\partial t} + \frac{x_1}{2} \frac{\partial}{\partial x_1} + \left(\frac{x_2}{2} + \frac{a_2\, t}{2}\right) \frac{\partial}{\partial x_2} + \frac{w_1}{2} \frac{\partial}{\partial w_1} + \frac{w_2}{2} \frac{\partial}{\partial w_2} \tag{4.156}$$

$$H_1 = \left(t - \frac{(a_2\,(x_2 - w_2) - 1)}{a_2^2}\right) \frac{\partial}{\partial t} \tag{4.157}$$

$$H_2 = \frac{\partial}{\partial t} \tag{4.158}$$

$$H_3 = e^{-a_2\,(x_2 - w_2)} \frac{\partial}{\partial t} \tag{4.159}$$

$$H_4 = \frac{\partial}{\partial x_2} \tag{4.160}$$

$$H_5 = \frac{\partial}{\partial w_1} \tag{4.161}$$

and

$$H_6 = \frac{\partial}{\partial w_2}. \tag{4.162}$$

These symmetries were not found by [7] for $a_1 \neq 0$.

In order to find new symmetries, [7] considered the case $a_1 = 0$, where the determining equation for the spatial infinitesimals are

$$\dot{F}_1(t) + a_2\, c_5 \; = 0. \tag{4.163}$$

By comparison of coefficients we have that $F_1(t)$ is a linear function of time, i.e.

$$F_1(t) = -a_2\, c_5\, t + c_7. \tag{4.164}$$

Whence the first spatial infinitesimal is given by

$$\xi_1 = \frac{1}{2}\, c_0\, x_1 + c_5\, \left(x_2 - a_2\, t\right) + c_7. \tag{4.165}$$

For the remaining spatial infinitesimal the determining equations remain unchanged, i.e.

$$\dot{F}_2(t) \; + \; a_2 \frac{1}{2}\, c_0 = a_2\, c_0. \tag{4.166}$$

The function $F_2(t)$ remains linear with respect to time

$$F_2(t) = \frac{a_2\, c_0}{2}\, t \; + \; c_4. \tag{4.167}$$

However the remaining spatial infinitesimal is different

$$\xi_2 = \frac{1}{2}\, c_0\, x_2 \; - \; c_5\, x_1 + \; \frac{a_2\, c_0}{2}\, t \; + \; c_4. \tag{4.168}$$

Thus giving rise to an additional symmetry which Gaeta [7] had found. The symmetries are

$$\begin{aligned} H_0 &= \frac{(a_2\,(x_2 - w_2) - 1)}{a_2^2}\frac{\partial}{\partial t} + \frac{x_1}{2}\frac{\partial}{\partial x_1} \\ &+ \left(\frac{x_2}{2} + \frac{a_2\, t}{2}\right)\frac{\partial}{\partial x_2} + \frac{w_1}{2}\frac{\partial}{\partial w_1} + \frac{w_2}{2}\frac{\partial}{\partial w_2} \end{aligned} \tag{4.169}$$

$$H_1 = \left(t - \frac{(a_2\,(x_2 - w_2) - 1)}{a_2^2}\right)\frac{\partial}{\partial t} \tag{4.170}$$

$$H_2 = \frac{\partial}{\partial t} \tag{4.171}$$

$$H_3 = e^{-a_2\,(x_2 - w_2)}\frac{\partial}{\partial t} \tag{4.172}$$

$$H_4 = \frac{\partial}{\partial x_2} \tag{4.173}$$

$$H_5 = \frac{\partial}{\partial w_1} \tag{4.174}$$

$$H_6 = \frac{\partial}{\partial w_2} \tag{4.175}$$

$$H_7 = \frac{\partial}{\partial x_1} \tag{4.176}$$

and

$$H_8 = (a_2\, t \; - \; x_2)\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + w_2\frac{\partial}{\partial w_1} - w_1\frac{\partial}{\partial w_2}. \tag{4.177}$$

The four out of the five symmetries which Gaeta [7] lists for $a_1 = 0$ are

$$H'_0 = H_1 + H_2 \tag{4.178}$$

$$= t\frac{\partial}{\partial t} + \frac{x_1}{2}\frac{\partial}{\partial x_1} + \left(\frac{x_2}{2} + \frac{a_2\,t}{2}\right)\frac{\partial}{\partial x_2} + \frac{w_1}{2}\frac{\partial}{\partial w_1} + \frac{w_2}{2}\frac{\partial}{\partial w_2} \tag{4.179}$$

$$H'_1 = H_2 \tag{4.180}$$

$$H'_2 = H_8 \tag{4.181}$$

and

$$H'_3 = H_4. \tag{4.182}$$

The fifth symmetry found by Gaeta [7] is

$$H'_5 = (\,x_2\;-\;a_2\,t\,)\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + w_2\frac{\partial}{\partial w_1} - w_1\frac{\partial}{\partial w_2} \tag{4.183}$$

which conserves form invariance. Thus Gaeta [7] obtained five of the nine symmetries given in (4.169) - (4.177).

We find the finite transformations (subject to initial conditions) for the fifth symmetry (4.183)

$$\frac{d\overline{x}_1}{d\epsilon} = (\,\overline{x}_2\;-\;a_2\,t\,), \tag{4.184}$$

$$\frac{d\overline{x}_2}{d\epsilon} = -\overline{x}_1, \tag{4.185}$$

$$\frac{d\overline{t}}{d\epsilon} = 0 \tag{4.186}$$

$$\frac{d\overline{w}_1}{d\epsilon} = \overline{w}_2 \tag{4.187}$$

and

$$\frac{d\overline{w}_2}{d\epsilon} = -\overline{w}_1. \tag{4.188}$$

From (4.184) and (4.185) we have

$$\frac{d^2\overline{x}_2}{d\epsilon^2} = a_2\,t\;-\;\overline{x}_2, \tag{4.189}$$

which solves as

$$\overline{x}_2 = (x_2 - a_2\,t)\,\cos(\epsilon) + b_0\,\sin(\epsilon) + a_2\,t, \tag{4.190}$$

where b_0 is an arbitrary constant. Thus (4.184) becomes

$$\frac{d\overline{x}_1}{d\epsilon} = (x_2 - a_2\,t)\,\cos(\epsilon) + b_0\,\sin(\epsilon) + a_2\,t\;-\;a_2\,t \tag{4.191}$$

which implies that the spatial transformation are

$$\overline{x}_1 = x_1\,\cos(\epsilon) + (x_2 - a_2\,t)\,\sin(\epsilon) \tag{4.192}$$

and

$$\overline{x}_2 = (x_2 - a_2\,t)\,\cos(\epsilon) - x_1\,\sin(\epsilon) + a_2\,t. \tag{4.193}$$

The first Wiener infinitesimal is solved in the same fashion as the first spatial transformation

$$\overline{w}_1 = w_1 \cos(\epsilon) + b_1 \sin(\epsilon). \tag{4.194}$$

This implies that

$$\frac{d\overline{w}_2}{d\epsilon} = -w_1 \cos(\epsilon) - b_1 \sin(\epsilon), \tag{4.195}$$

where b_1 is an arbitrary constant. Thus we can solve the Wiener transformation as

$$\overline{w}_1 = w_1 \cos(\epsilon) + w_2 \sin(\epsilon) \tag{4.196}$$

and

$$\overline{w}_2 = w_2 \cos(\epsilon) - w_1 \sin(\epsilon). \tag{4.197}$$

We now furnish the Itô SODEs associated with these finite transformations

$$\begin{aligned} d\,\overline{X}_1(\bar{t},\omega) &= \Gamma(\overline{x}_1)\,dt + Y^l(\overline{x}_1)dW_l(t,\omega) \\ &= (a_2\ -\ a_2\ )\sin(\epsilon)\,dt + (\cos(\epsilon))\ dW_1(t,\omega) + (\sin(\epsilon))\ dW_2(t,\omega) \\ &= d\overline{W}_1 \qquad (4.198) \\ d\,\overline{X}_2(\bar{t},\omega) &= \Gamma(\overline{x}_2)\,dt + Y^l(\overline{x}_2)dW_l(t,\omega) \\ &= ((a_2\ -\ a_2\ )\sin(\epsilon) + a_2\,t)\ dt + (\cos(\epsilon))\ dW_2(t,\omega) - (\sin(\epsilon))\ dW_1(t,\omega) \qquad (4.199) \\ &= a_2\,dt\ +\ d\overline{W}_2 \qquad (4.200) \end{aligned}$$

where the transformed Wiener processes are derived by Itô's formula as well

$$\begin{aligned} d\,\overline{W}_1(\bar{t},\omega) &= \Gamma(\overline{w}_1)\,dt + Y^l(\overline{w}_1)dW_l(t,\omega) \\ &= (\cos(\epsilon))\ dW_1(t,\omega) + (\sin(\epsilon))\ dW_2(t,\omega) \end{aligned} \tag{4.201}$$

and

$$\begin{aligned} d\,\overline{W}_2(\bar{t},\omega) &= \Gamma(\overline{w}_2)\,dt + Y^l(\overline{w}_2)dW_l(t,\omega) \\ &= (-\sin(\epsilon))\ dW_1(t,\omega) + (\cos(\epsilon))\ dW_2(t,\omega). \end{aligned} \tag{4.202}$$

Since $\overline{W}_1 = \overline{X}_1$, invariance is maintained.

Example 4.2.

The remaining comparison will be done with the following SODEs again from Gaeta [7]

$$dX_1(t) = X_2\,dt, \tag{4.203}$$

$$dX_2(t) = -k^2\,X_2\,dt + \sqrt{2\,k^2}dW(t), \tag{4.204}$$

where the instantaneous diffusion matrix $\mathbf{G}$, is

$$\mathbf{G} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.205}$$

The determining equations are simply

(4.206)

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial t} + x_2\,\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_1} - k^2\,x_2\,\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_2} + k^2\frac{\partial \xi_1(\mathbf{x},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_2^2} =$$
$$x_2\,\left(\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial t} + x_2\,\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1} - k^2\,x_2\,\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2} + k^2\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2^2}\right)$$
$$+\xi_2, \tag{4.207}$$

and

$$\frac{\partial \xi_2(\mathbf{x},t)}{\partial t} + x_2\,\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_1} - k^2\,x_2\,\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_2} + k^2\frac{\partial \xi_2(\mathbf{x},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_2^2} =$$
$$-k^2\,x_2\,\left(\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial t} + x_2\,\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_1} - k^2\,x_2\,\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2} + k^2\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial x_2^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2^2}\right)$$
$$-k^2\,\xi_2 \tag{4.208}$$

$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_1} + G_1^1\frac{\partial \xi_1}{\partial x_1} + G_2^1\frac{\partial \xi_1}{\partial x_2} = H(G_1^1) + G_1^1 Y^1(\gamma_1) + G_1^2 Y^1(\gamma_2)$$
$$0 = 0 \tag{4.209}$$
$$\frac{\partial \xi_1(\mathbf{x},t)}{\partial w_2} + G_1^2\frac{\partial \xi_1}{\partial x_1} + G_2^2\frac{\partial \xi_1}{\partial x_2} = H(G_1^2) + G_1^1 Y^2(\gamma_1) + G_1^2 Y^2(\gamma_2)$$
$$\frac{\partial \xi_1}{\partial x_2} = 0 \tag{4.210}$$
$$\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_1} + G_1^1\frac{\partial \xi_2}{\partial x_1} + G_2^1\frac{\partial \xi_2}{\partial x_2} = H(G_2^1) + G_2^1 Y^1(\gamma_1) + G_2^2 Y^1(\gamma_2)$$
$$0 = Y^1(\gamma_2) \tag{4.211}$$
$$\frac{\partial \xi_2(\mathbf{x},t)}{\partial w_2} + G_1^2\frac{\partial \xi_2}{\partial x_1} + G_2^2\frac{\partial \xi_2}{\partial x_2} = H(G_2^2) + G_2^1 Y^2(\gamma_1) + G_2^2 Y^2(\gamma_2)$$
$$G_2^2\,\frac{\partial \xi_2}{\partial x_2} = H(G_2^2) + G_2^2 Y^2(\gamma_2) \tag{4.212}$$
$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1} + G_1^1\frac{\partial \tau}{\partial x_1} + G_2^1\frac{\partial \tau}{\partial x_2} = 0,$$

which implies that

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_1} = 0 \tag{4.213}$$

and

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2} + G_1^2\frac{\partial \tau}{\partial x_1} + G_2^2\frac{\partial \tau}{\partial x_2} = 0,$$

which reduces to

$$\frac{\partial \tau(\mathbf{x},\mathbf{w},t)}{\partial w_2} + \sqrt{2\,k^2}\,\frac{\partial \tau}{\partial x_2} = 0. \tag{4.214}$$

Equations (4.213) and (4.214) imply

$$\tau(\mathbf{x},\mathbf{w},t) = F_1(x_2 - \sqrt{2\,k^2}\,w_2) + F_2(x_1) + a(t), \tag{4.215}$$

where F_1, F_2 and a are arbitrary functions. The condition (2.39) forces

$$\Gamma(\tau) = c_0. \tag{4.216}$$

Thus we have

$$\dot{a}(t) + x_2 F_2'(x_1) - k^2\,x_2\,F_1'(x_2 - \sqrt{2\,k^2}\,w_2) + k^2\,F_1'' + 2\,k^2 F_1'' + \frac{1}{2}F_2'' = c_0. \tag{4.217}$$

Differentiating (4.217) with respect to time gives that $a(t)$ is the following linear function of time

$$a(t) = c_1\, t \,+\, c_2. \tag{4.218}$$

Hence (4.217) becomes

$$x_2 F_2'(x_1) - k^2\, x_2\, F_1'(x_2 - \sqrt{2\,k^2}\, w_2) + k^2\, F_1'' + 2\, k^2 F_1'' + \frac{1}{2} F_2'' = c_0 \,-\, c_1. \tag{4.219}$$

A simple comparison by coefficients of x_2, imposes that both F_1 and F_2 be linear in terms of their arguments

$$F_1 = c_3\,(x_2 - \sqrt{2\,k^2}\, w_2) \,+\, c_4 \tag{4.220}$$
$$F_2 = k^2\, c_3\, x_1 + c_5 \tag{4.221}$$

and that the constants c_0 and c_1 be the identical, which implies

$$a(t) = c_0\, t \,+\, c_2. \tag{4.222}$$

It eventuates that the temporal infinitesimal is

$$\tau(\mathbf{x}, \mathbf{w}, t) = c_0\, t \,+\, c_2 \,+\, c_3\,(x_2 - \sqrt{2\,k^2}\, w_2) + k^2\, c_3\, x_1. \tag{4.223}$$

All that is left to obtain are the Wiener infinitesimals, before we can proceed with the spatial infinitesimals. The associated Wiener infinitesimal determining equations are

(4.224)

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial t} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_2^2} = 0, \tag{4.225}$$

and

$$\frac{\partial \gamma_2(\mathbf{w},t)}{\partial t} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_1^2} + \frac{1}{2}\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_2^2} = 0, \tag{4.226}$$

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_1} = \frac{c_0}{2}, \tag{4.227}$$
$$\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_2} = \frac{c_0}{2}, \tag{4.228}$$
$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_2} = -\frac{\partial \gamma_2(\mathbf{w},t)}{\partial w_1} \tag{4.229}$$

which in conjunction with (4.211) implies

$$\frac{\partial \gamma_1(\mathbf{w},t)}{\partial w_2} = 0. \tag{4.230}$$

Thus we have the following

$$\gamma_1(\mathbf{w},t) = \frac{c_0}{2}\, w_1 + L_1(\,t) \tag{4.231}$$

and

$$\gamma_2(\mathbf{w},t) = \frac{c_0}{2}\, w_2 + L_2(\,t). \tag{4.232}$$

The use of equation (4.225) forces the arbitrary L_1 function to be a constant

$$L_1(t) = c_7, \tag{4.233}$$

and the (4.226) forces the arbitrary function L_2 to follow suite

$$L_2 = c_8. \tag{4.234}$$

Thus we finally have the following summary of the Wiener infinitesimals

$$\gamma_1(\mathbf{w}, t) = \frac{c_0}{2}\, w_1 + c_7 \tag{4.235}$$

and

$$\gamma_2(\mathbf{w}, t) = \frac{c_0}{2}\, w_2 + c_8. \tag{4.236}$$

As a result, the determining equations (4.107), (4.208), (4.209), (4.210), (4.211) and (4.212) become

$$\frac{\partial \xi_1}{\partial t} + x_2\, \frac{\partial \xi_1}{\partial x_1} - k^2\, x_2\, \frac{\partial \xi_1}{\partial x_2} + k^2 \frac{\partial \xi_1}{\partial x_2^2} = x_2\, c_0 + \xi_2 \tag{4.237}$$

$$\frac{\partial \xi_2}{\partial t} + x_2\, \frac{\partial \xi_2}{\partial x_1} - k^2\, x_2\, \frac{\partial \xi_2}{\partial x_2} + k^2 \frac{\partial \xi_2}{\partial x_2^2} = -k^2\, x_2\; c_0 - k^2\, \xi_2 \tag{4.238}$$

$$0 = 0 \tag{4.239}$$

$$\frac{\partial \xi_1}{\partial x_2} = 0 \tag{4.240}$$

$$0 = 0 \tag{4.241}$$

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\, c_0. \tag{4.242}$$

From (4.239), (4.240), (4.241) and (4.241) we observe the following

$$\xi_1 = F_1(t) + F_2(x_1) \tag{4.243}$$

and

$$\xi_2 = \frac{1}{2}\, c_0\, x_2 \; + \; F_3(t) \; + \; F_4(x_1). \tag{4.244}$$

The consequences of (4.243) reduces (4.237) to

$$\dot{F}_1 + x_2\, F_2' = x_2\, c_0 + \frac{1}{2}\, c_0\, x_2 \; + \; F_3(t) \; + \; F_4(x_1). \tag{4.245}$$

By comparison of coefficients we have that

$$F_2' = \frac{3}{2}\, c_0 \tag{4.246}$$

and

$$\dot{F}_1 \; = \; F_3(t) \; + \; F_4(x_1). \tag{4.247}$$

Therefore we have

$$\xi_1 = \frac{3}{2}\, c_0\, x_1 + c_4 + F_1(t) \tag{4.248}$$

and

$$\xi_2 = \frac{1}{2}\, c_0\, x_2 \; + \; \dot{F}_1(t)\,. \tag{4.249}$$

Equation (4.238) becomes the following as a result

$$\ddot{F}_1 - \frac{k^2\, x_2\, c_0}{2} = -k^2\, x_2\; c_0 - k^2\, \left(\frac{1}{2}\, c_0\, x_2 \; + \; \dot{F}_1(t)\right). \tag{4.250}$$

The ensuing conclusions are

$$c_0 = 0 \tag{4.251}$$

and

$$F_1(t) = c_6 + c_5\, e^{-k^2\, t}. \tag{4.252}$$

Our infinitesimals are therefore

$$\xi_1 = c_4 + c_5\, e^{-k^2\, t}, \tag{4.253}$$

$$\xi_2 = -\, c_5\, k^2\, e^{-k^2\, t}, \tag{4.254}$$

$$\gamma_1 = c_7 \tag{4.255}$$

$$\gamma_2 = c_8 \tag{4.256}$$

and the temporal infinitesimal is

$$\tau(t) = \; c_2 \; + \; c_3\, (x_2 - \sqrt{2\, k^2}\, w_2) + k^2\, c_3\, x_1. \tag{4.257}$$

Thus the symmetries are

$$H_0 = \frac{\partial}{\partial t} \tag{4.258}$$

$$H_1 = \left(x_2 - \sqrt{2\, k^2}\, w_2 + k^2\, x_1\right)\, \frac{\partial}{\partial t} \tag{4.259}$$

$$H_2 = \frac{\partial}{\partial x_1} \tag{4.260}$$

$$H_3 = e^{-k^2\, t}\, \frac{\partial}{\partial x_1} - k^2\, e^{-k^2\, t}\, \frac{\partial}{\partial x_2} \tag{4.261}$$

$$H_4 = \frac{\partial}{\partial w_1} \tag{4.262}$$

and

$$H_5 = \frac{\partial}{\partial w_2}. \tag{4.263}$$

The symmetries which were found by Gaeta [7] were based on those of Gaeta and Quintero [1] and are

$$H_1' = \frac{\partial}{\partial x_1} \tag{4.264}$$

$$H_2' = -\frac{1}{k^2}\, e^{-k^2\, t}\, \frac{\partial}{\partial x_1} + e^{-k^2\, t}\, \frac{\partial}{\partial x_2} \tag{4.265}$$

and

$$H_3' = \frac{\partial}{\partial t}. \tag{4.266}$$

These again are sub-algebras of the ones we have found spanned by (4.258) - (4.263).

4.4.1 Stock Price Model

Consider the price of a single stock price $X(t,\omega)$ with instantaneous drift $\nu\, X(t,\omega)$ and standard deviation $\sigma\, X(t,\omega)$ as was done in Fredericks and Mahomed [6], i.e.

$$dX(t,\omega) = \nu\, X(t,\omega)\, dt + \sigma\, X(t,\omega)\, dW(t,\omega) \tag{4.267}$$

with the initial condition $X(0,\omega) = x_0$. The determining equations by Fredericks and Mahomed [5] were

$$\begin{aligned} \Gamma(\xi(t,x)) &= H(\nu\, x) - \nu\, x\, \Gamma(\tau(t)), \\ \frac{\partial \xi(t,x)}{\partial t} + \nu\, x \frac{\partial \xi(t,x)}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \xi(t,x)}{\partial x^2} &= \xi(t,x)\, \nu - \nu\, x \frac{d\tau(t)}{d\,t} \end{aligned} \tag{4.268}$$

and

$$\begin{aligned} Y(\xi(t,x)) &= \xi(t,x)\, \sigma - \frac{\sigma\, x}{2} \frac{d\tau(t)}{dt} \\ x \frac{\partial \xi(t,x)}{\partial\, x} &= \xi(t,x) - \frac{x}{2} \frac{d\tau(t)}{dt}. \end{aligned} \tag{4.269}$$

The infinitesimals were given as

$$\tau(t) = a_4 + a_5 t, \tag{4.270}$$

and

$$\xi(t,x) = \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\, (a_4 + a_5 t) \; + \; a_2\, x \; + \; \frac{1}{2} a_5 (x \ln x - x). \tag{4.271}$$

The three generators of symmetry were

$$H_1 = \frac{\partial}{\partial t} \; + \; \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\, \frac{\partial}{\partial x}, \tag{4.272}$$

$$H_2 = \; x\, \frac{\partial}{\partial x} \tag{4.273}$$

and

$$H_3 = \; t \frac{\partial}{\partial t} + \frac{1}{2} \left((\mu - \frac{1}{2}\sigma^2) t x + x \ln x - x \right) \frac{\partial}{\partial x}. \tag{4.274}$$

By applying the relation (4.97) we find the following relation for the Wiener infinitesimal

$$\frac{\partial \gamma}{\partial w} = a_5 \tag{4.275}$$

which implies that

$$\gamma = a_5\, w + A(t). \tag{4.276}$$

However, it also has to satisfy the condition (4.96) which after simplification is

$$\dot{A}(t) = 0. \tag{4.277}$$

Thus, the Wiener infinitesimal is given by

$$\gamma = a_5\, w + c_1 \tag{4.278}$$

which means that the 'W-symmetry' generators are given by

$$H_1 = \frac{\partial}{\partial t} + \frac{1}{2}(\mu - \frac{1}{2}\sigma^2)\, x\, \frac{\partial}{\partial x}, \tag{4.279}$$

$$H_2 = x\, \frac{\partial}{\partial x}, \tag{4.280}$$

$$H_3 = t\frac{\partial}{\partial t} + \frac{1}{2}\left((\mu - \frac{1}{2}\sigma^2)tx + x\ln x - x\right)\frac{\partial}{\partial x} + w\frac{\partial}{\partial w} \tag{4.281}$$

and the additional symmetry generator

$$H_4 = \frac{\partial}{\partial w}. \tag{4.282}$$

4.4.2 Blood Clotting Dynamics

Consider the one-dimensional SODEs model which models the position of platelets at time t

$$dX(t,\omega) = u(t)\, dt + \sigma\, dW(t,\omega). \tag{4.283}$$

The instantaneous drift is the velocity which satisfies the Stokes' equations

$$\frac{\partial u(t,x)}{\partial x} = 0 \tag{4.284}$$

and

$$\rho\, \dot{u}(t) + \frac{\partial p}{\partial x} = 0, \tag{4.285}$$

where the first and second components of (4.285) are the inertia (density, ρ, multiplied by acceleration, $\dot{u}(t)$) and pressure gradient of the platelets along an arteriole, respectively. The dot notation represents d/dt. The conservation of mass for these thrombocytes are handled by (4.284). By observation the pressure, p, due to the interactions between the platelets is linear in the spatial variable, x. Thus the instantaneous drift can be solved from (4.285) and can be expressed as $(A_1\rho)\, t\, +\, A_2$, where A_1 and A_2 are arbitrary constants which can be determined by experimentation. The determining equations associated with (4.283) in conjunction with (4.74) are thus

$$\Gamma(\xi(t,x)) = H(A_1\rho\, t\, +\, A_2) + (A_1\rho\, t\, +\, A_2)\; \Gamma(\tau(t)),$$
$$\frac{\partial\xi(t,x)}{\partial t} + (A_1\rho\, t\, +\, A_2)\,\frac{\partial\xi(t,x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2\xi(t,x)}{\partial x^2} = \tau(t)\, A_1\rho + (A_1\rho\, t\, +\, A_2)\,\frac{d\tau(t)}{d\,t} \tag{4.286}$$

and

$$Y(\xi(t,x)) = \frac{\sigma}{2}\frac{d\tau(t)}{dt}$$
$$\frac{\partial\xi(t,x)}{\partial\, x} = \frac{1}{2}\dot{\tau}(t). \tag{4.287}$$

The infinitesimals are easily solved for by substituting (4.287) into (4.286) to give

$$\tau(t) = c_1\, t + c_2 \tag{4.288}$$

and

$$\xi(t,x) = \frac{1}{2}\, c_1\, x + A_1\rho\left(\frac{3}{4}c_1\, t^2 + c_2\, t\right) + A_2\frac{c_1}{2}\, t + c_3. \tag{4.289}$$

The three generators of symmetry as per [5] are

$$H_1 = t\,\frac{\partial}{\partial t} + \left(\frac{x}{2} + \frac{3A_1\,\rho}{4}\,t^2 + \frac{A_2}{2}t\right)\,\frac{\partial}{\partial x}, \tag{4.290}$$

$$H_2 = \frac{\partial}{\partial t} + (A_1\,t\rho)\,\frac{\partial}{\partial x} \tag{4.291}$$

and

$$H_3 = \frac{\partial}{\partial x}. \tag{4.292}$$

If we now wish to extend to 'W-symmetries' by applying the relation (4.97) we find the following relation for the Wiener infinitesimal

$$\frac{\partial \gamma}{\partial w} = \frac{c_1}{2} \tag{4.293}$$

which implies that

$$\gamma(w,t) = \frac{c_1}{2}w + A(t). \tag{4.294}$$

However, it also has to satisfy the condition (4.96) which easily solves to give

$$A(t) = c_2. \tag{4.295}$$

Thus, the Wiener infinitesimal is given by

$$\gamma = \frac{c_1}{2} \;+ c_2 \tag{4.296}$$

which means that the 'W-symmetry' generators are

$$H_1 = t\,\frac{\partial}{\partial t} + \left(\frac{x}{2} + \frac{3A_1\,\rho}{4}\,t^2 + \frac{A_2}{2}t\right)\,\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial w}, \tag{4.297}$$

$$H_2 = \frac{\partial}{\partial t} + (A_1\,t\rho)\,\frac{\partial}{\partial x}, \tag{4.298}$$

$$H_3 = \frac{\partial}{\partial x} \tag{4.299}$$

and the additional symmetry generator

$$H_4 = \frac{\partial}{\partial w}. \tag{4.300}$$

4.4.3 Experimental Psychology

Consider the following simple linearised SODE which models small repetitive motions in humans with non-deterministic flactuations arising from highly populated weakly coupled neuronal cells

$$dX(t,\omega) = -\Big(a\,(X(t,\omega)) + 4\,b\,X(t,\omega))\Big)\,dt + \sigma\,dW(t,\omega). \tag{4.301}$$

The determining equations are

$$\Gamma(\xi(t,x)) = -H\Big(a\,x + 4\,b\,x\Big) \;+\; \Big(a\,x + 4,b\,x\Big)\,\Gamma(\tau(t))$$

$$\frac{\partial \xi(t,x)}{\partial t} - \Big(a\,x + 4\,b\,x\Big)\frac{\partial \xi(t,x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \xi(t,x)}{\partial x^2} = -\xi(t,x)\Big(a\; + 4\,b\Big) \;+\; \Big(a\,x + 4\,b\,x\Big)\,\dot{\tau}(t) \tag{4.302}$$

and

$$
\begin{aligned}
Y(\xi(t,x)) &= \frac{\sigma}{2}\frac{d\tau(t)}{dt} \\
\frac{\partial \xi(t,x)}{\partial\, x} &= \frac{1}{2}\dot{\tau}.
\end{aligned} \tag{4.303}
$$

The temporal infinitesimal has to be linear with respect to time due to the condition that the instantaneous drift of the temporal has to be constant to ensure the recoverability of the finite transformations from the infinitesimal ones, i.e

$$\tau(t) = c_1\, t\, + c_2. \tag{4.304}$$

Thus the spatial infinitesimal solves as

$$\xi(t,x) = \frac{c_1\, x}{2}\, + C(t) \tag{4.305}$$

and now we are able to find the arbitrary function $C(t)$, by looking at the resulting substitution of (4.305) into (4.302)

$$\dot{C}(t) = -C\,(a + 4\,b)\, + \left(a\,x + 4\,b\,x\right) c_1. \tag{4.306}$$

By comparison of coefficients we have that the arbitrary constant c_1 is zero. Hence we solve for the arbitrary function $C(t)$

$$C(t) = c_3\, e^{-a\,-\,4\,b}. \tag{4.307}$$

Our temporal and spatial infinitesimal respectively are

$$\tau = c_2 \tag{4.308}$$

and

$$\xi = c_3\, e^{-a\,-\,4\,b} \tag{4.309}$$

The two generators of symmetry are given as

$$H_1 = e^{-a\,-\,4\,b}\,\frac{\partial}{\partial x}, \tag{4.310}$$

and

$$H_2 = \frac{\partial}{\partial t}. \tag{4.311}$$

Including 'W-symmetries' via relation (4.97) leads to the following relation for the Wiener infinitesimal

$$\frac{\partial \gamma}{\partial w} = 0 \tag{4.312}$$

which implies that

$$\gamma(t,x) = A(t). \tag{4.313}$$

However, as in the two previous examples it has to satisfy the condition (4.96) which after simplification leads to

$$A(t) = c_4. \tag{4.314}$$

Thus, only the following symmetry is added

$$H_3 = \frac{\partial}{\partial w}. \tag{4.315}$$

Remark. The number of symmetry generators will always increase by at least one, when introducing the Wiener transformation into the Lie group transformation methodology.

$Algebra$	$Basis\,Operators$	$Representative\,Equations$
L_1	$H_1 = p$	$dX = f(x)\,dt + g(x)\,dW$
L_1	$H_1 = q$	$dX = f(t)\,dt + g(t)\,dW$
L_2	$H_1 = p, H_2 = q$	$dX = \alpha\,dt + \beta\,dW$
L_1	$H_1 = t\,q$	$dX = \frac{x}{t}dt + g(t)\,dW$
L_1	$H_1 = p + x\,q$	$dX = x\,e^{-t}\,dt + x\,e^{-t}\,dW$
L_1	$H_1 = t\,p + x\,q$	$dX = F\left(\ln\left(\frac{x}{t}\right)\right)\,dt + \sqrt{\frac{x}{t}}\,dW$
L_1	$H_1 = (1-a)\,t\,p + x\,q$	$dX = F\left(\ln\left(\frac{x}{(t(1-a))^{\frac{1}{1-a}}}\right)\right)\,dt + \sqrt{\frac{x}{(t(1-a))^{\frac{1}{1-a}}}}\,dW$

$Algebra$	$W - symmetry\,Basis\,Operators$	$Representative\,Equations$
L_2	$H_1 = p, H_2 = r$	$dX = f(x)\,dt + g(x)\,dW$
L_2	$H_1 = q, H_2 = r$	$dX = f(t)\,dt + g(t)\,dW$
L_3	$H_1 = p, H_2 = q, H_3 = r$	$dX = \alpha\,dt + \beta\,dW$
L_2	$H_1 = t\,q, H_2 = r$	$dX = \frac{x}{t}dt + g(t)\,dW$
L_2	$H_1 = p + x\,q, H_2 = r$	$dX = x\,e^{-t}\,dt + x\,e^{-t}\,dW$
L_2	$H_1 = t\,p + x\,q + \frac{w}{2}r, H_2 = r$	$dX = F\left(\ln\left(\frac{x}{t}\right)\right)\,dt + \sqrt{\frac{x}{t}}\,dW$
L_2	$H_1 = (1-a)\,t\,p + x\,q + \frac{(1-a)\,w}{2}r, H_2 = r$	$dX = F\left(\ln\left(\frac{x}{(t(1-a))^{\frac{1}{1-a}}}\right)\right)\,dt + \sqrt{\frac{x}{(t(1-a))^{\frac{1}{1-a}}}}\,dW$

4.5 Applications to 1-Dimensional Wiener SODEs

Classification of SODEs have only been done by Wafo Soh and Mahomed [2] for second-order SODEs. Here we show the classification as per [2] for first-order SODEs in Table 1. We also show how these basis operators change when introducing 'W-symmetries' in Table 2. The classification was slightly adjusted using the standard basis operator table from Wafo Soh and Mahomed [2]. The following representations are used

$$p = \partial/\partial t,\ q = \partial/\partial x \text{ and } r = \partial/\partial w$$

Table 1

The 'W-symmetries' basis operator table below extends the dimension in most cases.

Table 2

Remark. Here L_r means the r-dimensional algebra. Algebras in the first table are one-dimensional in most cases. The introduction of the Wiener infinitesimal into the Lie group analysis increases the dimension of all the algebras by one.

4.6 Conclusions

The use of the Lie point transformation methodology in conjunction with Itô's formula enables us to include the transformation of the Wiener process into the symmetry operator via a Wiener infinitesimal which performs the transformation in the Banach space. The Itô formula allows us to relate these transformations to their Itô SODEs counterparts.

The seminal work of Gaeta [7] is extended here to take into consideration temporal infinitesimals that are not necessarily projective. This is mainly due to a property invariance philosophy which apply to the transformed Wiener process. A generalized form of a condition which Ünal [3] had derived for a one-dimensional Wiener process, is easily extended as a result. By following an invariance argument on the Wiener process' properties, we were able to confirm the antisymmetric condition placed on the Wiener transformation for multidimensional Wiener processes by Gaeta [7].

However, the recovery of the finite transformations from the infinitesimal ones to preserve form invariance has not been guaranteed by any of the works in the past for multi-dimensional Wiener processes. This chapter derives the conditions necessary for this to be valid. We show that a particular example by Gaeta [7] does not ensure this.

Many of these insights and new conditions should be replicable for higher order SODEs. Wiener symmetries for multi-dimensional Wiener processes of higher order SODEs are an exciting unchartered area of investigation. The recovery of finite transformations that keep form invariancefor these higher order SODEs are important. This aspect is done in Chapter 5.

Chapter 5

Symmetries of nth Order Stochastic Ordinary Differential Equations

Symmetries of nth-order SODEs are studied. The determining equations of these SODEs are derived in an Itô calculus context. These determining equations are not stochastic in nature. SODEs are normally used to model nature (e.g. earthquakes) or for testing the safety and reliability of models in construction engineering when looking at the impact of random perturbations.

5.1 Introduction

Wafo Soh and Mahomed [2] gave an algorithm to obtain Lie point symmetries for both first- and nth-order SODEs. We briefly review their work and follow with an extension from point symmetries to generalised symmetries.

The first section begins with the transformations of the spatial, temporal and Wiener variables for an nth-order Itô process. These transformations have the same properties as stated in our previous chapter on first-order SODEs (see also Fredericks and Mahomed [5]).

Using the Itô formula in conjunction with the infinitesimal transformations which preserve form invariance, we derive conditions for nth-order SODEs that ensures the recovery of invariance preserving finite transformations from the infinitesimal ones. This has not been done in the past. The inclusion of a Wiener infinitesimal in the symmetry operator has also been precluded in the past. Faithfully following the 'Wiener- symmetry' methodology of the previous chapter, we construct a generalized random time change formula for multi-dimensional nth-order SODEs.

This is followed up with the development of recursive relations needed for finding the prolonged spatial infinitesimals in a SODEs context by using the concept of form invariance. This differs from the methodology used by [2], where the recursive relation defined was predefined from an ODE context. As a result we also derive a conditioning on these prolonged spatial infinitesimal variables. We further derive a conditioning on the diffusion coefficient of the temporal generalised symmetry τ, which is similar to that of Ünal [3]. We conclude the chapter with an introduction of operators which generalize the determining equations for SODEs of any order that is adaptable to both point and generalized symmetries.

5.2 Review of Wafo Soh and Mahomed [2] for nth-order SODEs

An nth-order Itô process has the following vector form

$$d\mathbf{X}^{(n-1)}(t) = \mathbf{f}(t,\mathbf{X}(t),\dot{\mathbf{X}}(t),\dots,\mathbf{X}^{(n-1)}(t))dt + \mathbf{G}(t,\mathbf{X}(t),\dot{\mathbf{X}}(t),\dots,\mathbf{X}^{(n-1)}(t))\,d\mathbf{W}(t) \tag{5.1}$$

$$dX_j^{(k)}(t) = X_j^{(k+1)}(t)dt, \tag{5.2}$$

$$X_j^{(0)}(t) = X_j(t) \tag{5.3}$$

for $k = 0, 1, \dots, n-2$. Since the instantaneous mean, $\mathbf{f}$, is an N-vector valued function, the index j runs from one to N, i.e. $j = 1, \dots, N$. The diffusion coefficent $\mathbf{G}$ is an $N \times M$-matrix valued function and $\mathbf{W}(t)$ is an M-dimensional standard Wiener process. From here onwards we denote $\{\mathbf{X}(t), \dot{\mathbf{X}}(t), \dots, \mathbf{X}^{(n-1)}(t)\}$ by $\mathcal{X}^{(n-1)}(t)$. The context of this processes is that both the instanteous drift and diffusion coefficients are Lipschitz continuous with respect to the right norm. A good example of the type of norm used for this is given by [15] in their seventh chapter.

The Lie point transformation methodology used by Wafo Soh and Mahomed [2] does all calculations to $\mathcal{O}(\epsilon)$. As a result the recoverability of the finite transformations, which keep invariance, from the infinitesimal ones is not verified. The symmetry operator H is the same as that used in chapter 2, with point symmetries

$$H = \tau(x,t)\,\frac{\partial}{\partial t} \; + \; \xi_j(x,t)\,\frac{\partial}{\partial x_j}, \tag{5.4}$$

where there is summation $j = 1, N$. However since we are dealing with nth-order SODEs, prolongation formulation is necessary. In the Banach space the transformation for the $(n-1)$th-order spatial derivative is

$$\overline{\mathbf{x}}^{(n-1)} = e^{\epsilon H^{[n-1]}}\, x^{(n-1)}, \tag{5.5}$$

$$\begin{aligned}\overline{\mathbf{x}}^{(k)} &= e^{\epsilon H^{[n-1]}}\, x^{(k)} \\ &= e^{\epsilon H^{[k]}}\, x^{(k)}, \quad k \, < \, n-1,\end{aligned} \tag{5.6}$$

where

$$H^{[n-1]} = H^{(n-2)} + \xi_j^{[n-1]} \frac{\partial}{\partial x_j^{(n-1)}},\; n \geq 1 \tag{5.7}$$

and

$$H^{[0]} = H. \tag{5.8}$$

Applying Itô's formula to a prolongation of a spatial infintesimal of arbitrary order , ${\xi_j}^{[r]}(t, \mathcal{X}^{(r-1)}(t))$, gives

$$d{\xi_j}^{[r]}(t,\mathcal{X}^{(r-1)}(t)) \;=\; f_{(\xi^{[r]})j}(t,\mathcal{X}^{(r-1)}(t))\,dt + G^l_{(\xi^{[r]})j}(t,\mathcal{X}^{(r-1)}(t))\,dW_l(t), \tag{5.9}$$

where

$$f_{(\xi^{[r]})j}(t,\mathcal{X}^{(r-1)}(t)) \;=\; \frac{\partial {\xi_j}^{[\gamma]}}{\partial t} + f_i \frac{\partial {\xi_j}^{[\gamma]}}{\partial x_i^{(n-1)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_p^k \frac{\partial^2 {\xi_j}^{[\gamma]}}{\partial x_i^{(n-1)} \partial x_p^{(n-1)}} + \sum_{\alpha=0}^{n-2} x_p^{(\alpha+1)} \frac{\partial {\xi_j}^{[r]}}{\partial x_p^{(\alpha)}},\ \text{where } n \,\geq\, 2 \tag{5.10}$$

$$G^l_{(\xi^{[r]})j}(t,\mathcal{X}^{(r-1)}(t)) \;=\; \frac{\partial {\xi_j}^{[\gamma]}}{\partial x_i^{(n-1)}} G_i^l r \text{ for each } j \text{ ranging from 1 to } N. \tag{5.11}$$

If the summation operator runs from a non-negative value, e.g. 0, to a negative one, i.e. -1, the outcome of the entire summation is set to zero. With this convention we are able to recover the It$\hat{0}$ formula for first order SODEs. Due to the repeated index summation convention, the spatial indices i and p both run from 1 to N in the summation; the Wiener indices l and k runs from 1 to M. Similarly, the Itô's formula for the temporal variable, $\tau(x,t)$, gives

$$d\tau = f_{(\tau)}(\mathbf{X}(t,\omega),t)\,dt + G^l_{(\tau)}(\mathbf{X}(t,\omega),t)\,dW_l(\mathbf{X}(t,\omega),t), \tag{5.12}$$

where

$$f_{(\tau)}(\mathbf{X}(t,\omega),t) = \frac{\partial\tau}{\partial t} + f_i\frac{\partial\tau}{\partial x_i^{(n-1)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_p^k \frac{\partial^2\tau}{\partial x_i^{(n-1)}\partial x_p^{(n-1)}} + \sum_{\alpha=0}^{n-2} x_p^{(\alpha+1)}\frac{\partial\tau}{\partial x_p^{(\alpha)}} \tag{5.13}$$

which reduces to the total derivative, since the temporal infinitesimal is a point transformation

$$f_{(\tau)}(\mathbf{X}(t,\omega),t) = D(\tau). \tag{5.14}$$

where the total derivative is defined as

$$D = \frac{\partial}{\partial t} + \sum_{\alpha=0}^{n-2}\frac{\partial}{\partial x_p^{(\alpha)}}x_p^{(\alpha+1)}. \tag{5.15}$$

The diffusion coefficient of the temporal infinitesimal, is given by

$$G^l_{(\tau)}(\mathbf{X}(t,\omega),t) = G_i^l\,\frac{\partial\tau}{\partial x_i^{(n-1)}} \tag{5.16}$$

reduces to zero as well because of the fact that we are dealing with point transformations, i.e.

$$G^l_{(\tau)}(\mathbf{X}(t,\omega),t) = 0. \tag{5.17}$$

The drift and diffusion coefficients of the $(n-1)$th-order spatial derivative are respectively transformed as

$$f_j(\overline{\mathcal{X}}^{(n-1)}(\bar{t}),\bar{t}) \quad = \quad f_j(\mathcal{X}^{(n-1)}(t),t) + \epsilon H^{[n-1]}\left(f_j(\mathcal{X}^{(n-1)}(t),t)\right) + \mathcal{O}(\epsilon^2) \tag{5.18}$$

and

$$G_j^l(\overline{\mathcal{X}}^{(n-1)}(\bar{t}),\bar{t}) \quad = \quad G_j^l(\mathcal{X}^{(n-1)}(t),t) + \epsilon H^{[n-1]}\left(G_j^l(\mathcal{X}^{(n-1)}(t),t)\right) + \mathcal{O}(\epsilon^2). \tag{5.19}$$

The Itô formula of the finite time index transformation is

$$d\bar{t} = \Gamma(e^{\epsilon H^{[n-1]}}(t))\,dt + Y^l(e^{\epsilon H^{[n-1]}}(t))dW_l. \tag{5.20}$$

which Wafo Soh and Mahomed [2] simply write as

$$d\bar{t} = dt\,(1 + \epsilon\,D(\tau)) + \mathcal{O}(\epsilon^2), \tag{5.21}$$

since the temporal infinitesimal is a point transformation. We also have that the transformed time index should keep invariance in the following probabilistic way

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\bar{t}(t,\omega)\right] = d\bar{t}(t,\omega). \tag{5.22}$$

This requires

$$Y^l(e^{\epsilon H^{[n-1]}}(t)) = 0 \quad \text{l = 1,M}, \tag{5.23}$$

which is automatically satisfied since τ is point transformation. Condition (5.22) also forces

$$D(e^{\epsilon H^{[n-1]}}(t)) = Constant, \tag{5.24}$$

which is overlooked in [2]. Thus the finite transformation of the Wiener process is

$$d\overline{W}_l(\bar{t},\omega) = \sqrt{D(e^{\epsilon H^{[n-1]}}(t))}dW_l(t,\omega) \tag{5.25}$$

which Wafo Soh and Mahomed [2] simplified as

$$d\overline{W}_l(\bar{t},\omega) = dW_l(t,\omega)\,(1 + \frac{\epsilon}{2}\,D(\tau)) + \mathcal{O}(\epsilon^2), \tag{5.26}$$

where [2] used a generalized binomial expansion of the square-root of the derivative of the transformed time index with respect to time. The Itô SODEs associated with Lie point nth-order spatial transformation is

$$d\overline{X}_j^{(n-1)}(\bar{t}) = dX_j^{(n-1)}(t) + \epsilon\left(f_{(\xi^{[n-1]})j}\,dt + G^l_{(\xi^{[n-1]})j}\,dW_l(t)\right) + \mathcal{O}(\epsilon^2)\,. \tag{5.27}$$

Wafo Soh and Mahomed [2] make the assumption that only the system of nth order SODEs, (5.1), remain invariant under the symmetry operator (5.4), which implies that

$$d\overline{X}_j^{(n-1)}(\bar{t}) = f_j(\bar{t},\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega))d\bar{t} + G^l_j(\bar{t},\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega))d\overline{W}_l(\bar{t}), \tag{5.28}$$

where we denote $\{\bar{\mathbf{X}}(\bar{t}), \dot{\bar{\mathbf{X}}}(\bar{t}), \ldots, \bar{\mathbf{X}}^{(r-1)}(\bar{t})\}$ by $\overline{\mathcal{X}}^{(r-1)}(\bar{t})$ for an arbitrary $r \in \mathbb{N}$.

Expanding the drift component $f_j(\bar{t},\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega))d\bar{t}$ of (5.28) using (5.18) and (5.21) gives

$$\begin{aligned}\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t))d\bar{t} = \Bigg\{&\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t)) \,+\, \epsilon\,\left(D(\tau) \,+\, H^{[n-1]}\right)\,\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t))\\ &+\,\sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\Bigg(\left(D(\tau) \,+\, H^{[n-1]}\right)^k\,\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t))\\ &+\,\sum_{j=0}^{k-2}\binom{k}{k\,-\,j}\,H^j(\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t)))\,\left(D(H^{k-j}(t)) \,-\, [D(\tau)]^{k-j}\right)\Bigg)\Bigg\}\,dt.\end{aligned} \tag{5.29}$$

In order for the finite transformations to keep invariance we require ϵ-terms of higher order to be solely dependent on the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ terms, this forces the condition

$$e^{\epsilon\,D(\tau)} \;=\; D\left(e^{\epsilon\,H^{[n-1]}}(t)\right), \tag{5.30}$$

which is satisfied as a result of condition (5.24). Whence the finite transformation becomes

$$d\overline{W}_l(\bar{t},\omega) = e^{\frac{\epsilon D(\tau)}{2}}\,dW_l(t,\omega). \tag{5.31}$$

The diffusion component of (5.28) can easily be expanded with the utility of (5.19) and (5.26)

$$\begin{aligned}G^l_j(t,\ \mathcal{X}^{(n-1)}(t))d\overline{W}_l = \Bigg\{&G^l_j(t,\ \mathcal{X}^{(n-1)}(t)) \,+\, \epsilon\,\left(\frac{D(\tau)}{2} \,+\, H^{[n-1]}\right)\,G^l_j(t,\ \mathcal{X}^{(n-1)}(t))\\ &+\,\sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\left(\frac{D(\tau)}{2} \,+\, H^{[n-1]}\right)^k\,G^l_j(t,\ \mathcal{X}^{(n-1)}(t))\Bigg\}\,dW_l.\end{aligned} \tag{5.32}$$

This allows us to make a comparison with the Itô SODEs associated with the nth-order spatial transformation (5.27), which furnishes the determining equations used by Wafo Soh and Mahomed [2], i.e.

(5.33)

$$f_{(\xi^{[n-1]})j} = \left(D(\tau) + H^{[n-1]}\right)\,f_j(\mathcal{X}^{(n-1)}(t), t) \tag{5.34}$$

and

$$G^l_{(\xi^{[n-1]})j} = \left(\frac{D(\tau)}{2} + H^{[n-1]}\right)\,G^l_j(\overline{\mathcal{X}}^{(n-1)}(\bar{t}), \bar{t}). \tag{5.35}$$

Constructing the prolonged variables was done by using pre-existing recursive relations based on the Lie point theory for ODEs, i.e.

$$\xi_j{}^{[k]} = D\xi_j{}^{[k-1]} - x_j^{(k)} D_\tau,\ \xi_j{}^{[0]} = \xi_j, \tag{5.36}$$

for $k \leq n$. The sketch of the methodology used for Lie point symmetries for nth-order SODEs by Wafo Soh and Mahomed [2] ends here. However, it is possible to construct the recursive relations using form invariance arguments on the SODEs described in equation (5.2),i.e.

$$d\overline{X}_j^{(k)}(\bar{t}) = \overline{X}^{(k+1)}(\bar{t})d\bar{t},$$

which expands as

$$d\overline{X}_j^{(k)}(\bar{t}) = X^{(k+1)}\, dt + \epsilon\, \left(\xi_j{}^{[k+1]} + x_j^{(k+1)}\, D(\tau)\right)\, dt + \mathcal{O}(\epsilon^2); \tag{5.37}$$

in conjunction with the Itô SODEs associated with the transformation of the kth-order spatial transformation, i.e.

$$d\overline{X}_j^{(k)}(\bar{t}) = dX_j^{(k)}(t) + \epsilon\, \left(f_{(\xi^{[k]})j}\, dt + G^l_{(\xi^{[k]})j}\, dW_l(t)\right) + \mathcal{O}(\epsilon^2), \tag{5.38}$$

which reduces to

$$d\overline{X}_j^{(k)}(\bar{t}) = dX_j^{(k)}(t) + \epsilon\, D(\xi_j^{[k]})\, dt + \mathcal{O}(\epsilon^2) \tag{5.39}$$

as a result of the fact that the prolongation infinitesimals of lower order $\xi_j^{[k]}$, are not a function of $\mathbf{x}^{(n-1)}$ for $k < (n-1)$. Thus the recursive relations defined by Wafo Soh and Mahomed [2] from an ODE context, are easily derived using a form invariance philosophy, *viz.*

$$D(\xi_j^{[k]}) = \xi_j{}^{[k+1]} + x_j^{(k+1)}\, D(\tau). \tag{5.40}$$

5.3 Generalized Symmetries

Instead of concerning ourselves with only point transformations with respect to the spatial infinitesimals, we consider making our spatial infinitesimals and thus our temporal infinitesimal generalized transformations as well. In conjunction with this we include a Wiener infinitesimal with the symmetry operator. The reasoning behind having the Wiener and spatial infinitesimals independent of one another is still valid, i.e. their respective probability spaces are independent of one another in terms of the measures associated with them.

We first re-derive the random time change formula using coupled operators as in chapter 3 but for a multi-dimensional nth-order SODEs. This entails the invariance of the transformed Wiener process with respect to its properties. A form invariance philosophy for the drift component of the transformed spatial infinitesimals in combination with the Itô formula is studied to establish the recoverability of the finite transformations from the infinitesimal ones.

The crux of this methodology is the coupling of two systems of SODEs - the system associated with the M-dimensional Wiener processes and the N-dimensional nth-order system of spatial SODEs. The probability space associated with the former is $(\Omega, \mathcal{F}, \mathbb{Q})$ and the probability space associated with the latter is $(\Omega, \mathcal{F}, \mathbb{P})$. The probability spaces associated with the transformed Itô SODEs of these two systems become $(\Omega, \mathcal{F}, \overline{\mathbb{Q}})$ and $(\Omega, \mathcal{F}, \overline{\mathbb{P}})$, respectively. The independence of these two spaces with respect to the measures is maintained by insisting on the independence of the spatial and Wiener infinitesimals.

5.3.1 Generalized Transformations

The construction of the symmetry and Itô formula operators form the basis of the Lie group transformations. They are used in combination to form the determining equations needed to unearth the infinitesimal transformations that allow the original probability spaces to evolve to new transformed probability spaces; whilst maintaining the probabilistic properties of both the Wiener and the spatial processes.

The generalized symmetry operator used here is,

$$H = \tau(\hat{\mathbf{x}}^{(n-1)}, t)\,\frac{\partial}{\partial t} + \sum_{j=1}^{N} \xi_j(\hat{\mathbf{x}}^{(n-1)}, t)\,\frac{\partial}{\partial x_j} + \sum_{l=1}^{M} w_l(\mathbf{w}, t)\,\frac{\partial}{\partial w_l}, \tag{5.41}$$

where $\hat{\mathbf{x}}^{(n-1)}$ signifies the collection $\{\mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(n-1)}\}$, and with the generalized prolonged symmetry operator having generalized spatial infinitesimals, i.e.

$$H^{[n-1]} = \tau(\hat{\mathbf{x}}^{(n-1)}, t)\,\frac{\partial}{\partial t} + \sum_{j=1}^{N} \xi_j(\hat{\mathbf{x}}^{(n-1)}, t)\,\frac{\partial}{\partial x_j} + \sum_{j=1}^{N}\sum_{\alpha=1}^{n-1} \xi_j^{[\alpha]}(\hat{\mathbf{x}}^{(n-1)}, t)\,\frac{\partial}{\partial x_j^{(\alpha)}} + \sum_{l=1}^{M} w_l(\mathbf{w}, t)\,\frac{\partial}{\partial w_l}. \tag{5.42}$$

We allow form and property invariance arguments to guide the behavior of our infinitesimals. In order for this to take place we use coupled operators as had been done in the previous chapter

$$\Gamma = \frac{\partial}{\partial t} + \sum_{i=1}^{N} f_i \frac{\partial}{\partial x_i^{(n-1)}} + \frac{1}{2}\sum_{i,p=1}^{N}\sum_{k=1}^{M} G_i^k G_p^k \frac{\partial^2}{\partial x_i^{(n-1)} \partial x_p^{(n-1)}} + \sum_{p=1}^{N}\sum_{\alpha=0}^{n-2} \frac{\partial}{\partial x_p^{(\alpha)}} x_p^{(\alpha+1)} + \frac{1}{2}\sum_{l=1}^{M} \frac{\partial^2}{\partial w_l^2} \tag{5.43}$$

which in essence gives the instantaneous drift of an arbitrary function of the spatial and Wiener variables. The following coupled operator gives the instantaneous diffusion

$$Y^l = \frac{\partial}{\partial w_l} + \sum_{i=1}^{N} G_i^l\,\frac{\partial^{[\gamma]}}{\partial x_i^{(n-1)}}. \tag{5.44}$$

Equipped with these operators we perform generalized transformations.

The construction begins with a one-parameter group transformation of the time index t, the spatial variables $\mathbf{x}$, the time derivatives of the spatial variables $\mathbf{x}^{(k)}$, where $k = 0, (n-1)$; and the Wiener variable $\mathbf{w}$, respectively,

$$\bar{t} = \theta(\hat{\mathbf{x}}^{(n-1)},\, \mathbf{w},\, t,\, \epsilon), \quad \overline{x} = \varphi(\hat{\mathbf{x}}^{(n-1)},\, t,\, \epsilon) \quad \overline{x}^{(k)} = \varphi^{[k]}(\hat{\mathbf{x}}^{(n-1)},\, t,\, \epsilon) \quad \text{and } \overline{w} = \vartheta(\mathbf{w},\, t,\, \epsilon), \tag{5.45}$$

with the following relation to the infinitesimals

$$\frac{\partial\theta}{\partial\epsilon} = \tau(\theta, \varphi, \vartheta), \quad \frac{\partial\varphi}{\partial\epsilon} = \xi(\theta, \varphi, \vartheta), \quad \frac{\partial\varphi^{[k]}}{\partial\epsilon} = \xi^{[k]}(\theta, \varphi, \vartheta) \quad \text{and } \frac{\partial\vartheta}{\partial\epsilon} = \gamma(\theta, \varphi, \vartheta). \tag{5.46}$$

The initial boundary conditions at $\epsilon = 0$ are

$$\bar{t}\big|_{\epsilon=0} = t, \quad \overline{\mathbf{X}}(\bar{t}, \omega)\big|_{\epsilon=0} = \mathbf{X}(t, \omega), \quad \overline{\mathbf{X}}^{(k)}(\bar{t}, \omega)\big|_{\epsilon=0} = \mathbf{X}^{(k)}(t, \omega) \quad \text{and } \overline{\mathbf{W}}(\bar{t}, \omega)\big|_{\epsilon=0} = \mathbf{W}(t, \omega), \tag{5.47}$$

Thus group transformations can be expressed in terms of the symmetry operator

$$\begin{aligned} \bar{t} &= e^{\epsilon H^{[n-1]}}(t) \\ &= e^{\epsilon H}(t) \end{aligned} \tag{5.48}$$

$$\overline{\mathbf{x}} = e^{\epsilon H^{[n-1]}}(\mathbf{x})$$

$$e^{\epsilon H}(\mathbf{x}) \tag{5.49}$$

$$\begin{aligned} \overline{\mathbf{x}}^{(k)} &= e^{\epsilon H^{[n-1]}}(\mathbf{x}^{(k)}) \\ &= e^{\epsilon H^{[k]}}(\mathbf{x}^{(k)}) \end{aligned} \tag{5.50}$$

and

$$\overline{\mathbf{w}} = e^{\epsilon H}(\mathbf{w}). \tag{5.51}$$

The Itô SODEs are associated to these group transformations (5.45) by the following

$$d\bar{t}(t,\omega) = \Gamma(e^{\epsilon H}(t))\,dt \;+\; Y^l(e^{\epsilon H}(t))\,dW_l(t,\omega) \tag{5.52}$$

$$d\overline{\mathbf{W}}(\bar{t},\omega) = \Gamma(e^{\epsilon H}(\mathbf{w}))\,dt \;+\; Y^l(e^{\epsilon H}(\mathbf{w}))\,dW_l(t,\omega) \tag{5.53}$$

$$d\overline{\mathbf{X}}^{(k)}(\bar{t},\omega) = \Gamma(e^{\epsilon H^{[k]}}(\mathbf{x}^{(k)}))\,dt \;+\; Y^l(e^{\epsilon H^{[k]}}(\mathbf{x}^{(k)}))\,dW_l(t,\omega) \tag{5.54}$$

and

$$d\overline{\mathbf{X}}(\bar{t},\omega) = \Gamma(e^{\epsilon H}(\mathbf{x}))\,dt \;+\; Y^l(e^{\epsilon H}(\mathbf{x}))\,dW_l(t,\omega). \tag{5.55}$$

The infinitesimal SODEs are given as

$$d\tau(t,\omega) = \Gamma(\tau(t,\omega))\,dt + Y^l(\tau(t,\omega))\,dW_l(t,\omega), \tag{5.56}$$

$$d\boldsymbol{\xi}(t,\omega) = \Gamma(\boldsymbol{\xi}(t,\omega))\,dt + Y^l(\boldsymbol{\xi}(t,\omega))\,dW_l(t,\omega) \tag{5.57}$$

$$d\boldsymbol{\xi}^{[k]}(t,\omega) = \Gamma(\boldsymbol{\xi}^{[k]}(t,\omega))\,dt + Y^l(\boldsymbol{\xi}^{[k]}(t,\omega))\,dW_l(t,\omega) \tag{5.58}$$

and

$$d\boldsymbol{\gamma}(t,\omega) = \Gamma(\boldsymbol{\gamma}(t,\omega))\,dt + Y^l(\boldsymbol{\gamma}(t,\omega))\,dW_l(t,\omega). \tag{5.59}$$

5.4 Property Invariance of Transformed Wiener Process

Invariance of the characteristics of our transformed standard Weiner process, $d\overline{W}_j(t)$, should still satisfy

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{\mathbf{W}}(\bar{t},\omega)\middle|W = w\right] = 0. \tag{5.60}$$

As a result we have the following new condition

$$\Gamma(e^{\epsilon\,H}(\mathbf{w})) = 0. \tag{5.61}$$

Differentiating (5.61) with respect to ϵ at $\epsilon = 0$ gives the following Wiener infinitesimal condition

$$\Gamma(\boldsymbol{\gamma}) = 0. \tag{5.62}$$

A similar invariance argument for the Itô isometry condition, or instantaneous variance yields

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\overline{W}_l(\bar{t},\omega)d\overline{W}_m(\bar{t},\omega)\middle|X = x, W = w\right] = \delta^l_m d\bar{t} \tag{5.63}$$

$$= \Gamma(e^{\epsilon\,H}(t))\,dt \;+\; Y^l(e^{\epsilon\,H}(t))\,dW_l \tag{5.64}$$

thus furnishing

$$\sum_{r=1}^{M} Y^r(e^{\epsilon\,H}(w_l))dW_r \sum_{s=1}^{M} Y^s(e^{\epsilon\,H}(w_m))dW_s = \delta^l_m d\bar{t} \tag{5.65}$$

$$\sum_{r=1}^{M} Y^r(e^{\epsilon\,H}(w_l))Y^r(e^{\epsilon\,H}(w_m))dt = \delta^l_m \Gamma_{(w)}(e^{\epsilon\,H}(t))\,dt \;+\; Y^r(e^{\epsilon\,H}(t))\,dW_m \tag{5.66}$$

which by comparing Riemann and Itô integrals, forces the condition

$$Y^l(e^{\epsilon\,H}(t)) \;= 0 \tag{5.67}$$

for $l = 1, M$. Differentiating (5.67) with respect to ϵ at $\epsilon = 0$, gives rise to the following temporal infinitesimal condition

$$Y^l(\tau) \ = 0, \tag{5.68}$$

which is a more generalized version of the condition by Ünal [3] for one-dimensional first-order Itô SODEs. In actual fact, we have derived a random time change formula, where

$$\bar{t} = \int^t \Gamma\left(e^{\epsilon\, H}(s)\right)\, ds \tag{5.69}$$

and by applying the probabilistic invariance condition

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\bar{t}(t,\omega)\right] = d\bar{t}(t,\omega) \tag{5.70}$$

on the transformed time index, which is automatically satisfied by the original differential time index, we get

$$Y^l(e^{\epsilon\, H}(t)) \ = 0 \tag{5.71}$$

being re-enforced and we also have the following deduction

$$\Gamma\left(e^{\epsilon\, H}(t)\right) = Constant. \tag{5.72}$$

The finite transformation of the Wiener process is given as

$$d\overline{\mathbf{W}}(\bar{t},\omega) = \int^t Y^l(e^{\epsilon\, H}(\mathbf{w}))\, dW_l(s,\omega), \tag{5.73}$$

where

$$\sum_{r=1}^{M} Y^r(e^{\epsilon\, H}(w_l))Y^r(e^{\epsilon\, H}(w_m)) \ = \delta^l_m \Gamma(e^{\epsilon\, H}(t)). \tag{5.74}$$

Differentiating (5.74) with respect to ϵ gives

$$\sum_{r=1}^{M} Y^r(w_l)Y^r(\gamma_m)) + \sum_{r=1}^{M} Y^r(w_m)Y^r(\gamma_l)) + \mathcal{O}(\epsilon) \ = \delta^l_m \Gamma(\tau) + \mathcal{O}(\epsilon). \tag{5.75}$$

which simplifies further as

$$Y^l(w_l)Y^l(\gamma_m)) + Y^m(w_m)Y^m(\gamma_l)) + \mathcal{O}(\epsilon) \ = \delta^l_m \Gamma(\tau) + \mathcal{O}(\epsilon), \tag{5.76}$$

since $Y^r(w_l) = \delta^r_l$.

Equation (5.76) evaluated at $\epsilon = 0$ simply is

$$Y^l(\gamma_m) + Y^m(\gamma_l) \ = \delta^l_m \Gamma(\tau). \tag{5.77}$$

The case where $l \neq m$ naturally leads to an antisymmetric matrix which is a generalized version of the point transformation antisymmetric marix, $\mathbf{B}$ which was defined by Gaeta [7] for first-order multi-dimensional Itô SODEs.

5.5 Form Invariance of the nth-order Spatial Process

In order to find a similar condition to that of (3.17) for nth-order Itô SODEs, that ensures the recovery of the invariance preserving finite transformations from the infinitesimal transformations, we need the following form invariant argument,

$$d\overline{\mathbf{X}}^{(n-1)}(\bar{t},\omega) = \mathbf{f}\left(\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega),\, \bar{t}\right) d\bar{t} + \mathbf{G}\left(\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega),\, \bar{t}\right)\, d\overline{\mathbf{W}}(\bar{t},\omega), \tag{5.78}$$

where, the transformed drift component is

$$\mathbf{f}\left(\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega),\, \bar{t}\right) = e^{\epsilon H}(\mathbf{f}), \tag{5.79}$$

and the transformed diffusion component is given by

$$\mathbf{G}\left(\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega),\, \bar{t}\right) = e^{\epsilon H}(\mathbf{G}). \tag{5.80}$$

If we now expand the drift component of (5.78), we get

$$\begin{aligned}\mathbf{f}\left(\overline{\mathcal{X}}^{(n-1)}(\bar{t},\omega),\, \bar{t}\right) d\bar{t} = \Bigg\{ & \mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t)) \,+\, \epsilon\ (\Gamma(H(t)) \,+\, H)\ \mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t)) \\ & + \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} \Bigg((\Gamma(H(t)) \,+\, H)^k\ \mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t)) \\ & + \sum_{j=0}^{k-2} \binom{k}{k\,-\,j} H^j(\mathbf{f}(t,\ \mathcal{X}^{(n-1)}(t))) \left(\Gamma(H^{k-j}(t)) \,-\, [\Gamma(H(t))]^{k-j}\right)\Bigg)\Bigg\}\ dt.\end{aligned} \tag{5.81}$$

The equations (2.23) and (5.81) are superficially identical, because the operators used here and in chapter one are different.

The following condition is needed to ensure the recovery of the finite transformations from the infinitesimal transformations

$$e^{\epsilon\,\Gamma(H(t))}(t,\ \mathcal{X}^{(n-1)}(t)) \,=\, \Gamma\left(e^{\epsilon\, H}(t)(t,\ \mathcal{X}^{(n-1)}(t))\right). \tag{5.82}$$

This condition ensures that the higher order terms depend solely on the first order term associated with $\mathcal{O}(\epsilon)$. All the order terms contribute the construction of the finite transformations; the zeroth and first order term, contribute towards the construction of the infinitesimal transformations. This also forces the instantaneous drift coefficient of the temporal infinitesimal to be a constant, which was demonstrated in the first chapter, i.e.

$$\Gamma(\tau) = C, \text{ where } C \text{ is an arbitrary constant.} \tag{5.83}$$

Condition (5.82), also simplifies (5.69), to

$$\overline{\mathbf{W}}(\bar{t},\omega) = \int^{t} \sum_{m=1}^{M} Y^m e^{\epsilon H}(\mathbf{w}) dW_m(s,\omega). \tag{5.84}$$

where

$$\sum_{r=1}^{M} Y^r(e^{\epsilon H}(w_l))(t,\omega) Y^r(e^{\epsilon H}(w_m))(t,\omega) = \delta^l_m e^{\epsilon\Gamma(H(t))}. \tag{5.85}$$

Thus expanding the diffusion component of (5.78) gives

$$\sum_{l=1}^{M} G_j^l\left(\overline{\mathbf{X}}(\bar{t},\omega),\,\bar{t}\right)\,d\overline{W}_l(\bar{t},\omega)=\sum_{l=1}^{M} G_j^l\,dW_l+\epsilon\left(\sum_{l=1}^{M} H(G_j^l)\,dW_l+\sum_{l,m=1}^{M} G_j^l\,Y^m(H(w_l))dW_m\right)$$
$$+\sum_{k=2}\frac{\epsilon^k}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}H^{k-r}(G_j^l)\,Y^m(H^r(w_l))dW_m\right), \tag{5.86}$$

where

$$H^k(G_j^l)\,Y^m(H^0(w_l))dW_m=H^k(G_j^l)\,Y^m(w_l)dW_m=H^k(G_j^l)\,dW_l. \tag{5.87}$$

5.5.1 Generalized Prolongation Formulae

All that remains to be derived is the prolongation formulae. We use form invariance for the lower order spatial derivative processes, i.e.

$$d\overline{X}_j^{(r)}(\bar{t})=\overline{X}_j^{(r+1)}(\bar{t})d\bar{t},\;\; j=1,N \text{ and } r=1,(n-2)$$

which simplifies to

$$d\overline{X}_j^{(r)}(\bar{t})=X_j^{(r+1)}\,dt\,+\,\left(\sum_{k=1}^{\infty}\frac{\epsilon^k}{k!}\left((\Gamma(H(t))\,+\,H)^k\;x_j^{(r+1)}\right)\right)(t,\omega)\,dt \tag{5.88}$$

with the use of the relation (5.82). The Itô SODEs associated with the transformation of the rth-order spatial transformation is

$$d\overline{X}_j^{(r)}(\bar{t})=dX_j^{(r)}(t)+\epsilon\left(\Gamma(\xi_j^{[r]})(t,\omega)\;dt+Y^l(\xi_j^{[r]})(t,\omega)\;dW_l(t)\right)$$
$$+\sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\left(\Gamma\left(H^k\left(x_j^{(r+1)}\right)\right)(t,\omega)dt+Y^l\left(H^k\left(x_j^{(r+1)}\right)\right)(t,\omega)dW_l\right), \tag{5.89}$$

which on comparison with (5.88) with respect to the Riemann and Itô integrals gives the following relations

$$\Gamma\left(H^k\left(x_j^{(r+1)}\right)\right)=(\Gamma(H(t))\,+\,H)^k\;x_j^{(r+1)},\;\text{ for each } r=1,(n-2) \text{ and } j=1,N \tag{5.90}$$

where $k=1,\infty$, and

$$Y^l\left(H^k\left(x_j^{(r+1)}\right)\right)=0,\;\text{ for each } r=1,(n-2);\, l=1,M \text{ and } j=1,N. \tag{5.91}$$

where $l=1,M$. The differentiation of both equations (5.88) and (5.89) with respect to ϵ at $\epsilon=0$, gives rise to the following determining equations for the prolonged spatial infinitesimals

$$\Gamma(\xi_j^{[r]})=(\Gamma(H(t))\,+\,H)\;x_j^{(r+1)}, \tag{5.92}$$

which is a generalized prolongation formula and the remaining determining equation is

$$Y^l\left(\xi_j^{[r]}\right)=0,\;\text{ for each } r=1,(n-2);\, l=1,M \text{ and } j=1,N. \tag{5.93}$$

In summary we have the following determining equations for multi-dimensional generalized nth-order SODEs

$$\Gamma(\xi_j^{[n-1]}) = \Big(\Gamma(H(t)) \; + \; H^{[n-1]}\Big) \; f_j \tag{5.94}$$

$$Y^l(\xi_j^{[n-1]}) = H(G_j^l) \; + \; \sum_{m=1}^{M} G_j^m \, Y^l(H(w_m)) \tag{5.95}$$

$$\Gamma(\tau) = \text{Constant} \tag{5.96}$$

$$Y^l(\tau) \; = 0 \tag{5.97}$$

$$\Gamma(\gamma) = 0 \tag{5.98}$$

$$Y^l(\gamma_m) + Y^m(\gamma_l) \; = \delta_m^l \Gamma(\tau) \tag{5.99}$$

$$\Gamma(\xi_j^{[r]}) = \Big(\Gamma(H(t)) \; + \; H^{[r]}\Big) \; x_j^{(r+1)} \tag{5.100}$$

and

$$Y^l \left(\xi_j^{[r]} \right) = 0, \tag{5.101}$$

where $r \leq n-2$, $r \, \in \, \mathbb{N}$.

$$\tag{5.102}$$

In our first example, we consider point symmetries only.

Example 5.1.

Consider the mass-spring linear oscillator response to random excitation (see [2])

$$d\dot{X} = -\omega^2 X dt + \sigma dW \tag{5.103}$$
$$dX = \dot{X} dt \tag{5.104}$$

which is associated with the following set of determining equations

$$-\omega^2 \Gamma_1(\tau) + H(-\omega^2 x) - \Gamma_1(\xi^{[1]}) = 0, \tag{5.105}$$
$$H(\sigma) + \frac{1}{2}\sigma\Gamma_1(\tau) - Y_1(\xi^{[1]}) = 0 \tag{5.106}$$

with the extra conditions

$$\sigma\frac{\partial \xi}{\partial \dot{x}} = 0 \tag{5.107}$$
$$\sigma\frac{\partial \tau}{\partial \dot{x}} = 0 \tag{5.108}$$

which means that $\tau(t, x, \dot{x}) = a(t, x)$ and $\xi(t, x) = c(t, x)$, where $a(t, x)$ and $c(t, x)$ are arbitrary functions. We use this and the fact that ω and σ are constants to get

$$-\omega^2 x\Gamma_1(a(t, x)) - \omega^2 c(t, x) - \Gamma_1(\xi^{[1]}) = 0, \tag{5.109}$$
$$\frac{1}{2}\sigma\Gamma_1(a(t, x)) - Y_1(\xi^{[1]}) = 0 \tag{5.110}$$

which expands to

$$-\omega^2 x\left(\dot{a}(t, x) + \dot{x}\frac{\partial a(t, x)}{\partial x}\right) - \omega^2 c(t, x) - \left(\frac{\partial \xi^{[1]}}{\partial t} - \omega^2 x\frac{\partial \xi^{[1]}}{\partial \dot{x}} + \frac{\sigma^2}{2}\frac{\partial^2 \xi^{[1]}}{\partial \dot{x}^2} + \dot{x}\frac{\partial \xi^{[1]}}{\partial x}\right) = 0, \tag{5.111}$$
$$\frac{\sigma}{2}\left(\dot{a}(t, x) + \dot{x}\frac{\partial a(t, x)}{\partial x}\right) - \left(\sigma\frac{\partial \xi^{[1]}}{\partial \dot{x}}\right) = 0 \tag{5.112}$$

we now find $\xi^{[1]}$ in terms of $a(t, x)$ and $c(t, x)$ by using (5.92) to get

$$\xi^{[1]} = \dot{c}(t, x) + \dot{x}\frac{\partial c(t, x)}{\partial x} - \left(\dot{a}(t, x) + \dot{x}\frac{\partial a(t, x)}{\partial x}\right)\dot{x} \tag{5.113}$$

substituting (5.113) into (5.111) gives

$$\begin{aligned} &-\omega^2 x\left(\dot{a}(t, x) + \dot{x}a'(t, x)\right) - \omega^2 c(t, x) - \Big(\ddot{c}(t, x) + \ddot{x}c'(t, x) + \dot{x}\dot{c}'(t, x) - \left(\ddot{a}(t, x) + \ddot{x}a'(t, x)\right. \\ &\left.+\dot{x}\dot{a}'(t, x)\right)\dot{x} - \ddot{x}\left(\dot{a}(t, x) + \dot{x}a'(t, x)\right) - \omega^2 x\left(c'(t, x) - \dot{a}(t, x) - 2\dot{x}a'(t, x)\right) \\ &+\frac{\sigma^2}{2}\left(-2a'(t, x)\right) + \dot{x}\left(\dot{c}'(t, x) + \dot{x}c''(t, x)\right) - \left(\dot{a}'(t, x) + \dot{x}a''(t, x)\right)\dot{x}\Big) = 0 \end{aligned} \tag{5.114}$$
$$\frac{\sigma}{2}\left(\dot{a}(t, x) + \dot{x}a'(t, x)\right) - \sigma\left(c'(t, x) - \dot{a}(t, x) - 2\dot{x}a'(t, x)\right) = 0 \tag{5.115}$$

where $'$ denotes the partial derivative with respect to the spatial variable x. From (5.115) we find that τ is a constant and that $c(t, x)$ is a linear function of the spatial coordinate. This simplifes (5.114) to

$$-\omega^2 c(t, x) - \left(\ddot{c}(t, x) + \ddot{x}c'(t, x) + \dot{x}\dot{c}'(t, x) - \omega^2 x\left(c'(t, x)\right) + \dot{x}\left(\dot{c}'(t, x) + \dot{x}c''(t, x)\right)\right) = 0, \tag{5.116}$$

which implies that

$$c(t, x) = c(t). \tag{5.117}$$

The consequences of which lead to the following

$$\ddot{c}(t) + \omega^2 c(t) = 0. \tag{5.118}$$

This solves as

$$c(t) = c_2 \cos \omega t + c_3 \sin \omega t. \tag{5.119}$$

Thus our symmetries are

$$\tau = c_1 \tag{5.120}$$

and

$$\xi = c_2 \cos \omega t + c_3 \sin \omega t. \tag{5.121}$$

In the work by Wafo Soh and Mahomed [2], cases were analyzed concerning ω. When $\omega = 0$ the symmetry infinitesimals they found were

$$\tau = c_1 t + c_2, \tag{5.122}$$

and

$$\xi = c_1 x + c_3 t + c_4, \tag{5.123}$$

thus arriving at the symmetries

$$H_1 = \frac{\partial}{\partial t}, \tag{5.124}$$

$$H_2 = \frac{\partial}{\partial x}, \tag{5.125}$$

$$H_3 = t\frac{\partial}{\partial x}, \tag{5.126}$$

$$H_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{5.127}$$

Next if $\omega \neq 0$ the symmetries are consistent with the symmetries we found, i.e.

$$\tau(t) = c_2, \tag{5.128}$$

$$\xi(t) = c_3 \cos \omega t + c_4 \sin \omega t, \tag{5.129}$$

which gives

$$H_1'' = \frac{\partial}{\partial t}, \tag{5.130}$$

$$H_2'' = \cos \omega t \frac{\partial}{\partial x}, \tag{5.131}$$

$$H_3'' = \sin \omega t \frac{\partial}{\partial x}. \tag{5.132}$$

We investigate what form invariance implies for this example. For the case $\omega = 0$ by Wafo Soh and Mahomed [2] a trivial symmetry H_3 gives the following prolonged spatial infinitesimal as

$$\xi^{[1]}(\bar{t}, \bar{x}) = \Gamma(t) - \dot{x}\Gamma(0) = 1, \tag{5.133}$$

thus as a result

$$\frac{d\bar{\dot{x}}}{d\epsilon} = 1 \tag{5.134}$$

which implies that

$$\bar{\dot{x}} = \dot{x} + \epsilon. \tag{5.135}$$

The spatial infinitesimal satisifies

$$\frac{d\overline{x}}{d\epsilon} = t, \tag{5.136}$$

since $\bar{t} = t$, which furnishes

$$\overline{x} = x + \epsilon\, t \tag{5.137}$$

which means that the Itô SODE associated with these group transformations is respectively

$$d\overline{\dot{X}} = d\dot{X} = \sigma\, dW \tag{5.138}$$

$$\begin{aligned} d\overline{X} &= \Gamma(x + \epsilon\, t)dt + Y(x + \epsilon\, t)dW \\ &= (\dot{x} + \epsilon)\, dt \\ &= \overline{\dot{X}}\, dt, \end{aligned} \tag{5.139}$$

which is not consistently maintaining form invariance. The most non-trivial symmetry for the case $\omega = 0$ by Wafo Soh and Mahomed [2], is H_4. The prolonged spatial infinitesimal is thus

$$\xi^{[1]}(\bar{t}, \overline{x}) = \Gamma(x) - \dot{x}\Gamma(t) = 0. \tag{5.140}$$

As a result, we have

$$\frac{d\bar{\dot{x}}}{d\epsilon} = 0 \tag{5.141}$$

which implies that

$$\bar{\dot{x}} = \dot{x}. \tag{5.142}$$

The spatial infinitesimal is gives rise the following relation

$$\frac{d\overline{x}}{d\epsilon} = \overline{x} \tag{5.143}$$

which furnishes

$$\overline{x} = x\, e^{\epsilon}. \tag{5.144}$$

A similar procedure gives

$$\bar{t} = t\, e^{\epsilon} \tag{5.145}$$

and the random time change formula gives

$$d\overline{W} = e^{\frac{1}{2}\epsilon} dW. \tag{5.146}$$

The Itô SODEs associated with the transforms above are respectively given as

$$d\overline{\dot{X}} = d\dot{X} = -\omega^2\, x\, dt + \sigma\, dW \tag{5.147}$$

$$= \sigma\, dW \text{ since } \omega = 0. \tag{5.148}$$

$$\begin{aligned} d\overline{X} &= \Gamma(x\, e^{\epsilon})dt + Y(x\, e^{\epsilon})dW \\ &= \dot{x}e^{\epsilon}(t, \omega)dt = \dot{X}\, d\bar{t} \end{aligned} \tag{5.149}$$

since

$$d\bar{t} = \Gamma(t\,e^{\epsilon})\,dt + Y(t\,e^{\epsilon})dW = e^{\epsilon}\,dt, \tag{5.150}$$

hence form invariance is absent or inconsistent. A non-trivial symmetry for the case $\omega \neq 0$, is H_2''. The prolonged spatial infinitesimal is therefore

$$\xi^{[1]}(\bar{t},\bar{x}) = \Gamma(\cos(\omega\,t)) - \dot{x}\Gamma(0) = -\omega\,\sin(\omega\,t) \tag{5.151}$$

As a consequence, we have

$$\frac{d\bar{\dot{x}}}{d\epsilon} = -\omega\,\sin(\omega\,t) \tag{5.152}$$

since $\bar{t} = t$, which implies that

$$\bar{\dot{x}} = \dot{x} - \epsilon\,\omega\,\sin(\omega\,t). \tag{5.153}$$

The spatial infinitesimal gives rise to the following relation

$$\frac{d\bar{x}}{d\epsilon} = \cos(\omega\,t) \tag{5.154}$$

which produces

$$\bar{x} = x + \epsilon\cos(\omega\,t). \tag{5.155}$$

The Itô SODEs associated with the transforms above are respectively given as

$$\begin{aligned}
d\bar{\dot{x}} &= \Gamma(\dot{x} - \epsilon\,\omega\,\sin(\omega\,t))dt + Y(\dot{x} - \epsilon\,\omega\,\sin(\omega\,t))dW \\
&= -\omega^2\,\left(X + \epsilon\,\cos(\omega\,t)\right)\,dt + (\sigma)\,dW && (5.156)\\
&= -\omega^2\,\overline{X}\,d\bar{t} + (\sigma)\,d\overline{W} && (5.157)\\
d\overline{X} &= \Gamma(X + \epsilon\cos(\omega\,t))dt + Y(x + \epsilon\cos(\omega\,t))dW \\
&= (\dot{x} - \epsilon\omega\sin(\omega\,t))\,(t,\omega)dt && (5.158)\\
&= \overline{\dot{X}}(t,\omega)\,d\bar{t}, && (5.159)
\end{aligned}$$

since

$$d\bar{t} = \Gamma(t)\,dt + Y(t)dW = dt \tag{5.160}$$

and

$$d\overline{W} = dW. \tag{5.161}$$

Thus form invariance is maintained by the finite transformations in this instance.

5.5.2 Revisiting the Canonical Forms for second-order Itô SODEs

The following representations are used in conjunction with the original table given by Wafo Soh and Mahomed [2]. We let

$$p = \partial/\partial t,\; q = \partial/\partial x \text{ and } r = \partial/\partial w$$

5.5.3 Table 3

Algebra	*Basis Operators*	*Representative Equations*
L_2	$H_0 = q, H_1 = t\,q$	$d\dot{X} = f(t)\,dt + g(t)\,dW$
L_2	$H_0 = p, H_1 = q$	$d\dot{X} = f(\dot{x})\,dt + g(\dot{x})\,dW$
L_2	$H_0 = q, H_1 = x\,q$	$d\dot{X} = f(t)\dot{x}\,dt + g(t)\dot{x}\,dW$
L_2	$H_0 = q, H_1 = t\,p + x\,q$	$d\dot{X} = t^{-1}f(\dot{x})\,dt + t^{-\frac{1}{2}}g(\dot{x})\,dW$

The introduction of 'W-symmetries' gives rise to at least one new symmetry, as seen below.

5.5.4 Table 4

Algebra	*Basis Operators*	*Representative Equations*
L_3	$H_0 = q, H_1 = t\,q, H_2 = r$	$d\dot{X} = f(t)\,dt + g(t)\,dW$
L_3	$H_0 = p, H_1 = q, H_2 = r$	$d\dot{X} = f(\dot{x})\,dt + g(\dot{x})\,dW$
L_3	$H_0 = q, H_1 = x\,q, H_2 = r$	$d\dot{X} = f(t)\dot{x}\,dt + g(t)\dot{x}\,dW$
L_3	$H_0 = q, H_1 = t\,p + x\,q + \frac{1}{2}\,r, H_2 = r$	$d\dot{X} = t^{-1}f(\dot{x})\,dt + t^{-\frac{1}{2}}g(\dot{x})\,dW$

Example 5.2.

Consider the following 2-dimensional version of a mass-spring linear oscillator response to random excitation (see [2]), without 'W-symmetries'

$$d\dot{\mathbf{X}}(t) = -\mathbf{\Omega}\mathbf{X}(t)dt + \mathbf{\Xi}d\mathbf{W}(t) \tag{5.162}$$

$$d\mathbf{X} = \dot{\mathbf{X}}dt \tag{5.163}$$

$$\text{where } \mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix},\ \mathbf{\Omega} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \text{ and } \mathbf{\Xi} = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.164}$$

which is associated with the following set of determining equations

$$\Gamma({\xi_1}^{[1]}) = -\omega_1^2 x_1 \Gamma(\tau) + H(-\omega_1^2 x_1), \tag{5.165}$$
$$Y^1({\xi_1}^{[1]}) = H(\sigma_{11}) + \frac{1}{2}\sigma_{11}\Gamma(\tau),$$

which becomes

$$Y^1({\xi_1}^{[1]}) = \frac{1}{2}\sigma_{11}\Gamma(\tau), \tag{5.166}$$

$$\Gamma({\xi_2}^{[1]}) = -\omega_2^2 x_2 \Gamma(\tau) + H(-\omega_2^2 x_2), \tag{5.167}$$

$$Y^1({\xi_2}^{[1]}) = 0, \tag{5.168}$$

with the extra conditions

$$\sigma_{11}\frac{\partial \xi_1}{\partial \dot{x}_1} = 0, \tag{5.169}$$

$$\sigma_{11}\frac{\partial \xi_2}{\partial \dot{x}_1} = 0 \tag{5.170}$$

and

$$\sigma_{11}\frac{\partial \tau}{\partial \dot{x}_1} = 0, \tag{5.171}$$

which means that $\tau(t, \mathbf{x}, \dot{\mathbf{x}}) = a(t, \mathbf{x}, \dot{x_2})$, $\xi_1(t, \mathbf{x}, \dot{\mathbf{x}}) = c_1(t, \mathbf{x}, \dot{x_2})$, $\xi_2(t, \mathbf{x}, \dot{\mathbf{x}}) = c_2(t, \mathbf{x}, \dot{x_2})$ and ${\xi_2}^{[1]}(t, \mathbf{x}, \dot{x}_2)$, where a, c_1 and c_2 are arbitrary functions. We use the fact that the instantaneous drift of the temporal infinitesimal has to be constant and the fact that Ω and Ξ are matrices comprised of only constants to get

$$\Gamma(\tau) = c_0 \tag{5.172}$$

which expands to

$$\frac{\partial a}{\partial t} - \omega^2\, x_2\, \frac{\partial a}{\partial \dot{x}_2} + \dot{x}_1 \frac{\partial a}{\partial x_1} + \dot{x}_2 \frac{\partial a}{\partial x_2} = c_0. \tag{5.173}$$

The use of comparison by coefficients of $\dot{x}_1$ gives us the following relations

$$\frac{\partial a}{\partial x_1} = 0 \tag{5.174}$$

and

$$\frac{\partial a}{\partial t} - \omega^2\, x_2\, \frac{\partial a}{\partial \dot{x}_2} + \dot{x}_2 \frac{\partial a}{\partial x_2} = c_0. \tag{5.175}$$

Hence we have that the temporal infinitesimal evolves to the following function

$$\tau = F_0\left(\frac{\dot{x}_2^2\, +\, \omega_2^2\, x_2^2}{2}\right) \,+\, c_0\, t \,+\, a_0. \tag{5.176}$$

The determining equations (5.165), (5.166) and (5.167) thus become

$$\frac{\partial {\xi_1}^{[1]}}{\partial t} - \omega_1^2\, x_1\, \frac{\partial {\xi_1}^{[1]}}{\partial \dot{x}_1} - \omega_1^2\, x_2\, \frac{\partial {\xi_1}^{[1]}}{\partial \dot{x}_2} + \frac{\sigma_{11}^2}{2}\, \frac{\partial {\xi_1}^{[1]}}{\partial \dot{x}_1^2} + \dot{x}_1 \frac{\partial {\xi_1}^{[1]}}{\partial x_1} + \dot{x}_2 \frac{\partial {\xi_1}^{[1]}}{\partial x_2} = -\omega_1^2 x_1\, c_0 - \omega_1^2 \xi_1, \tag{5.177}$$

$$\frac{\partial {\xi_1}^{[1]}}{\partial \dot{x}_1} = \frac{1}{2}\, c_0, \tag{5.178}$$

$$\frac{\partial {\xi_2}^{[1]}}{\partial t} - \omega_1^2\, x_1\, \frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_1} - \omega_1^2\, x_2\, \frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_2} + \frac{\sigma_{11}^2}{2}\, \frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_1^2} + \dot{x}_1 \frac{\partial {\xi_2}^{[1]}}{\partial x_1} + \dot{x}_2 \frac{\partial {\xi_2}^{[1]}}{\partial x_2} = -\omega_2^2 x_2\, c_0 - \omega_2^2 \xi_2. \tag{5.179}$$

The determining equations (5.165) and (5.167) simplify to

$$\frac{\partial {\xi_1}^{[1]}}{\partial t} - \omega_2^2\, x_2\, \frac{\partial {\xi_1}^{[1]}}{\partial \dot{x}_2} + \dot{x}_1 \frac{\partial {\xi_1}^{[1]}}{\partial x_1} + \dot{x}_2 \frac{\partial {\xi_1}^{[1]}}{\partial x_2} = -\frac{1}{2}\omega_1^2 x_1\, c_0 - \omega_1^2 \xi_1, \tag{5.180}$$

$$\frac{\partial {\xi_2}^{[1]}}{\partial t} - \omega_2^2\, x_2\, \frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_2} + \dot{x}_1 \frac{\partial {\xi_2}^{[1]}}{\partial x_1} + \dot{x}_2 \frac{\partial {\xi_2}^{[1]}}{\partial x_2} = -\omega_2^2 x_2\, c_0 - \omega_2^2 \xi_2. \tag{5.181}$$

From (5.181) we have that

$$\frac{\partial {\xi_2}^{[1]}}{\partial x_1} = 0 \tag{5.182}$$

by using comparison of coefficients with respect to $\dot{x}_1$ and the fact that ${\xi_2}^{[1]}(t, \mathbf{x}, \dot{x}_2)$, which implies that

$${\xi_2}^{[1]}(t, \mathbf{x}, \dot{x}_2) = {\xi_2}^{[1]}(t, x_2, \dot{x}_2). \tag{5.183}$$

This simplifies (5.181) further

$$\frac{\partial {\xi_2}^{[1]}}{\partial t} - \omega^2\, x_2\, \frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_2} + \dot{x}_2 \frac{\partial {\xi_2}^{[1]}}{\partial x_2} = -\omega_2^2 (x_2\, c_0 + \xi_2). \tag{5.184}$$

By using the prolongation formula (5.92) we have

$${\xi_1}^{[1]} = \frac{\partial c_1}{\partial t} - \omega_2^2\, x_2 \frac{\partial c_1}{\partial \dot{x}_2} + \dot{x}_1 \frac{\partial c_1}{\partial x_1} + \dot{x}_2 \frac{\partial c_1}{\partial x_2} - c_0\, \dot{x}_1. \tag{5.185}$$

We also have the relation (5.178) which forces

$$c_1 = \frac{3}{2}c_0\, x_1 + c_1(t, x_2, \dot{\mathbf{x}}). \tag{5.186}$$

Thus the first prolonged spatial infinitesimal becomes

$$\xi_1{}^{[1]} = \frac{\partial c_1}{\partial t} - \omega_2^2\, x_2 \frac{\partial c_1}{\partial \dot{x}_2} + \dot{x}_1 \frac{1}{2} c_0 + \dot{x}_2 \frac{\partial c_1}{\partial x_2}. \tag{5.187}$$

Therefore we have the following

$$\frac{\partial \xi_1{}^{[1]}}{\partial t} = \frac{\partial^2 c_1}{\partial t^2} - \omega_2^2\, \dot{x}_2 \frac{\partial c_1}{\partial \dot{x}_2} - \omega^2\, x_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial t} + \ddot{x}_1 \frac{1}{2} c_0 + \dot{x}_1 \frac{\partial^2 c_1}{\partial x_1 \partial t} + \ddot{x}_2 \frac{\partial c_1}{\partial x_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial t}, \tag{5.188}$$

$$\frac{\partial \xi_1{}^{[1]}}{\partial \dot{x}_2} = \frac{\partial^2 c_1}{\partial t \partial \dot{x}_2} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial \dot{x}_2^2} + \frac{\partial c_1}{\partial x_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial \dot{x}_2}, \tag{5.189}$$

$$\frac{\partial \xi_1{}^{[1]}}{\partial x_1} = 0, \tag{5.190}$$

and

$$\frac{\partial \xi_1{}^{[1]}}{\partial x_2} = \frac{\partial^2 c_1}{\partial t \partial x_2} - \omega_2^2 \frac{\partial c_1}{\partial \dot{x}_2} - \omega^2\, x_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial x_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2^2}. \tag{5.191}$$

Using (5.188), (5.189), (5.190) and (5.191) the equation (5.180) becomes

(5.192)

$$\begin{aligned} &\frac{\partial^2 c_1}{\partial t^2} - \omega_2^2\, \dot{x}_2 \frac{\partial c_1}{\partial \dot{x}_2} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial t} + \ddot{x}_2 \frac{\partial c_1}{\partial x_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial t} - \frac{1}{2} c_0\, \ddot{x}_1 - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial t \partial \dot{x}_2} \\ &\quad + \omega^4\, x_2^2 \frac{\partial^2 c_1}{\partial \dot{x}_2^2} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial x_2 \partial t} - \omega_2^2\, x_2\, \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial \dot{x}_2} + \\ &+ \dot{x}_2 \frac{\partial^2 c_1}{\partial t \partial x_2} - \omega_2^2\, \dot{x}_2 \frac{\partial c_1}{\partial \dot{x}_2} \\ &- \omega^2\, x_2\, \dot{x}_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial x_2} + \dot{x}_2^2 \frac{\partial^2 c_1}{\partial x_2^2} = -\omega_1^2 \left(\frac{1}{2} x_1\, c_0 + \frac{3}{2} c_0\, x_1 + c_1(t, x_2, \dot{\mathbf{x}}) \right). \end{aligned} \tag{5.193}$$

Comparison of coefficients with respect $\ddot{x}_1$ leads to c_0 being zero, i.e.

$$c_1(t,\, \mathbf{x},\, \dot{x}_2) = c_1(t,\, x_2,\, \dot{x}_2). \tag{5.194}$$

This leads to the following simplification of (5.193)

$$\begin{aligned} &\frac{\partial^2 c_1}{\partial t^2} - \omega_2^2\, \dot{x}_2 \frac{\partial c_1}{\partial \dot{x}_2} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial t} + \ddot{x}_2 \frac{\partial c_1}{\partial x_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial t} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial t \partial \dot{x}_2} \\ &\quad + \omega_2^4\, x_2^2 \frac{\partial^2 c_1}{\partial \dot{x}_2^2} - \omega_2^2\, x_2 \frac{\partial^2 c_1}{\partial x_2 \partial t} - \omega_2^2\, x_2\, \dot{x}_2 \frac{\partial^2 c_1}{\partial x_2 \partial \dot{x}_2} + \dot{x}_2 \frac{\partial^2 c_1}{\partial t \partial x_2} - \omega_2^2\, \dot{x}_2 \frac{\partial c_1}{\partial \dot{x}_2} \\ &- \omega^2\, x_2\, \dot{x}_2 \frac{\partial^2 c_1}{\partial \dot{x}_2 \partial x_2} + \dot{x}_2^2 \frac{\partial^2 c_1}{\partial x_2^2} = -\omega_1^2\, c_1(t,\, x_2,\, \dot{x}_2). \end{aligned} \tag{5.195}$$

Comparison of coefficients with respect to $\ddot{x}_2$, gives

$$\frac{\partial c_1}{\partial x_2} = 0, \tag{5.196}$$

which implies that $c_1(t,\, x_2,\, \dot{x}_2) = c_1(t,\, \dot{x}_2)$. This simplifies equation (5.195), to

$$\frac{\partial^2 c_1}{\partial t^2} - 2\,\omega_2^2\,\dot{x}_2\frac{\partial c_1}{\partial \dot{x}_2} - 2\,\omega_2^2\,x_2\frac{\partial^2 c_1}{\partial \dot{x}_2 \partial t} + \omega_2^4\,x_2^2\frac{\partial^2 c_1}{\partial \dot{x}_2^2} = -\omega_1^2\,c_1(t,\,\dot{x}_2). \tag{5.197}$$

Comparison with respect to coefficients of x_2^2 demands

$$c_1 = c_5(t)\,\dot{x}_2 \,+\, c_6(t). \tag{5.198}$$

As a result, we have that equation (5.197) becomes

$$\ddot{c}_5(t)\,\dot{x}_2 \,+\, \ddot{c}_6(t) - 2\,\omega_2^2\,\dot{x}_2\,c_5(t) - 2\,\omega_2^2\,x_2\dot{c}_5(t) = -\omega_1^2\left(c_5(t)\,\dot{x}_2 \,+\, c_6(t)\right). \tag{5.199}$$

Comparison of coefficients of x_2 gives that $c_5(t) = a_5$ where a_5 is constant. This changes equation (5.199) to

$$\ddot{c}_6(t) - 2\,\omega_2^2\,\dot{x}_2\,a_5 = -\omega_1^2\left(a_5\,\dot{x}_2 \,+\, c_6(t)\right). \tag{5.200}$$

In the same vein as before we find that a_5 is forced to be zero, which eventuates the following equation

$$\ddot{c}_6(t) = -\omega_1^2\,c_6(t). \tag{5.201}$$

Solving for c_6 we find that

$$c_6(t) = a_6\sin(\omega\,t) + a_7\cos(\omega\,t). \tag{5.202}$$

Thus the first spatial infinitesimal is

$$\xi_1 = a_6\sin(\omega\,t) + a_7\cos(\omega\,t) \tag{5.203}$$

The prolongation formula (5.92) similarly gives

$${\xi_2}^{[1]}(t, x_2, \dot{x}_2) = \frac{\partial c_2}{\partial t} - \omega_2^2\,x_2\frac{\partial c_2}{\partial \dot{x}_2} + \dot{x}_2\frac{\partial c_2}{\partial x_2}. \tag{5.204}$$

Thus we have the following

$$\frac{\partial {\xi_2}^{[1]}}{\partial t} = \frac{\partial^2 c_2}{\partial t^2} - \omega_2^2\,\dot{x}_2\frac{\partial c_2}{\partial \dot{x}_2} - \omega^2\,x_2\frac{\partial^2 c_2}{\partial \dot{x}_2 \partial t} + \ddot{x}_2\frac{\partial c_2}{\partial x_2} + \dot{x}_2\frac{\partial^2 c_2}{\partial x_2 \partial t}, \tag{5.205}$$

$$\frac{\partial {\xi_2}^{[1]}}{\partial \dot{x}_2} = \frac{\partial^2 c_2}{\partial t \partial \dot{x}_2} - \omega_2^2\,x_2\frac{\partial^2 c_2}{\partial \dot{x}_2^2} + \frac{\partial^2 c_2}{\partial x_2 \partial t} + \dot{x}_2\frac{\partial^2 c_2}{\partial x_2 \partial \dot{x}_2}, \tag{5.206}$$

and

$$\frac{\partial {\xi_2}^{[1]}}{\partial x_2} = \frac{\partial^2 c_2}{\partial t \partial x_2} - \omega_2^2\,\frac{\partial c_2}{\partial \dot{x}_2} - \omega^2\,x_2\frac{\partial^2 c_2}{\partial \dot{x}_2 \partial x_2} + \dot{x}_2\frac{\partial^2 c_2}{\partial x_2^2}. \tag{5.207}$$

Using (5.205), (5.206), and (5.207) the equation (5.184) becomes

$$\begin{aligned}
&\frac{\partial^2 c_2}{\partial t^2} - \omega_2^2\,\dot{x}_2\frac{\partial c_2}{\partial \dot{x}_2} - \omega_2^2\,x_2\frac{\partial^2 c_2}{\partial \dot{x}_2 \partial t} + \ddot{x}_2\frac{\partial c_2}{\partial x_2} + \dot{x}_2\frac{\partial^2 c_2}{\partial x_2 \partial t} \\
&- \omega_2^2\,x_2\,\frac{\partial^2 c_2}{\partial t \partial \dot{x}_2} + \omega_2^4\,x_2^2\frac{\partial^2 c_2}{\partial \dot{x}_2^2} - \omega_2^2\,x_2\,\frac{\partial^2 c_2}{\partial x_2 \partial t} - \omega_2^2\,x_2\,\dot{x}_2\frac{\partial^2 c_2}{\partial x_2 \partial \dot{x}_2} \\
&\dot{x}_2\frac{\partial^2 c_2}{\partial t \partial x_2} - \omega_2^2\,\dot{x}_2\,\frac{\partial c_2}{\partial \dot{x}_2} - \omega_2^2\,x_2\,\dot{x}_2\frac{\partial^2 c_2}{\partial \dot{x}_2 \partial x_2} + \dot{x}_2^2\frac{\partial^2 c_2}{\partial x_2^2} = -\omega_2^2(x_2\,c_0 + c_2).
\end{aligned} \tag{5.208}$$

Comparison of coefficients with respect $\ddot{x}_2$ leads to the fact that

$$c_2(t,\,\mathbf{x},\,\dot{x}_2) = c_2(t,\,\dot{x}_2). \tag{5.209}$$

This leads to the following simplification of (5.208)

$$\frac{\partial^2 c_2}{\partial t^2} - 2\omega_2^2\,\dot{x}_2\frac{\partial c_2}{\partial \dot{x}_2} \;-\; 2\omega_2^2\,x_2\frac{\partial^2 c_2}{\partial \dot{x}_2 \partial t} + \omega_2^4\,x_2^2\frac{\partial^2 c_2}{\partial \dot{x}_2^2} = -\omega_2^2\,c_2(t,\,\dot{x}_2). \tag{5.210}$$

Comparing coefficients with respect to x_2^2 we have that

$$\frac{\partial^2 c_2(t,\,\dot{x}_2)}{\partial \dot{x}_2^2} = 0 \tag{5.211}$$

which implies that

$$c_2 = c_3(t)\,\dot{x}_2 + c_4(t). \tag{5.212}$$

Thus equation (5.210) becomes

$$\ddot{c}_3(t)\,\dot{x}_2 + \ddot{c}_4(t) - 2\omega_2^2\,\dot{x}_2\,c_3(t) \;-\; 2\omega_2^2\,x_2\dot{c}_3 = -\omega_2^2\,(c_3(t)\,\dot{x}_2 + c_4(t)). \tag{5.213}$$

Comparison by coefficients with respect to x_2 furnishes

$$c_3 = a_3,\ \text{where } a_3 \text{ is an arbitrary constant.} \tag{5.214}$$

As a result we have

$$\ddot{c}_4(t) - 2\omega_2^2\,\dot{x}_2\,a_3 \;= -\omega_2^2(a_3\,\dot{x}_2 + c_4(t)). \tag{5.215}$$

Comparison by coefficients with respect to $\dot{x}_2$ implies

$$\tag{5.216}$$

$$a_3 \;=\; \frac{1}{2}(a_3\,) \tag{5.217}$$

and

$$\ddot{c}_4(t) = -\omega_2^2 c_4(t). \tag{5.218}$$

Equation (5.217) forces a_3 to be zero and solving for c_4 gives

$$c_4(t) = a_4\cos(\omega\,t) + a_5\sin(\omega\,t). \tag{5.219}$$

Thus the second spatial infinitesimal is

$$\xi_2 = a_4\cos(\omega\,t) + a_5\sin(\omega\,t). \tag{5.220}$$

In summary we have the infinitesimals being

$$\tau = F_0\left(\frac{\dot{x}_2^2\,+\,\omega_2^2\,x_2^2}{2}\right)\;+\;a_0 \tag{5.221}$$

$$\xi_1 = a_1\sin(\omega\,t) + a_2\cos(\omega\,t) \tag{5.222}$$

and

$$\xi_2 = a_3\cos(\omega\,t) + a_4\sin(\omega\,t). \tag{5.223}$$

Remarks. The generalized symmetry generators still form an algebra. Finding the generalized symmetry transformations is more involved than the point symmetry transformations.

5.6 Concluding Comments

Lie group analysis for nth-order Itô SODEs were first pursued in Wafo Soh and Mahomed [2]. Though it had only been done for point symmetries, it has lead to many interesting findings in this chapter. We have shown that it is possible to derive the prolongation formulas by using the philosophy of form invariance and we have been able to extend the algebras using the idea of 'W-symmetries', which were first introduced by Gaeta [7].

It has been shown that with the introduction of 'W-symmetries', we are able to derive a random time change formula for a multi-dimensional nth-order Itô SODEs. With the use of the philosophy that the properties of the Wiener processes should remain invariant under the Lie group transformations, we derive conditions on the temporal and lower level derivative spatial infinitesimals that are a generalization of the condition derived by Ünal [3] for one-dimensional SODEs.

The key to the success of this chapter is the idea of coupling the M-dimensional Wiener process with the N-dimensional nth-order spatial Itô process via one pair of operators as in the previous chapter. As a result the determining equations of both point and generalised symmetries can be handled by operators in one generalised set of determining equations. There is a wider scope for these operators. We intend to derive these results in an alternative methodology; eventually applying it to approximate SODEs as in Ibragimov et al. [10].

Unlike ODEs, the contact and point transformations for SODEs are not equivalent. This is highlighted in the example by Wafo Soh and Mahomed [2] above. By allowing the temporal infintesimal to be a function of the highest order spatial derivative, Itô's formula changes the characteristics of the system of determining equations - giving a larger group of transformations. This is not the case if the temporal infinitesimal is assumed to be projective.

Chapter 6

Symmetries of nth Order Multi-dimensional Approximate Stochastic Ordinary Differential Equations

The symmetries of high-order multi-dimensional SODEs are found using form invariance arguments on both the instantaneous drift and diffusion properties of the SODEs. We then apply this work to a generalised approximation analysis algorithm. The determining equations of SODEs are derived in an Itö calculus context.

6.1 Introduction

The modelling power of SODE has been applied to many diverse fields of research, from the modelling of turbulent diffusion to neuronal activity in the brain. Models such as these are often influenced by more than one Wiener process. In models such as these we assume these Wiener processes are independent of one another. As a result of this increase in the number of Wiener processes affecting the model, the form of the Itô formula is slightly different to the one used in Fredericks and Mahomed [6] and Fredericks and Mahomed [17]. The Itô formula is able to relate an arbitrary sufficiently smooth function $F(t,x)$ of time and space to a particular SODE, of which it is a solution. This formula, however, needs the SODE of the spatial random process $X(t,\omega)$ which drives the arbitrary function $F(X(t,\omega),\omega)$. The application of SODE to an approximate analysis algorithm has been done by Ibragimov et al. [10] for scalar SODEs of first-order. We extend this work for higher dimensions and order. We derive a similar conditioning on the temporal infinitesimal τ as had been done by Ünal [3] and Fredericks and Mahomed [17]. We introduce operators to write the determining equations in a neater form.

6.2 Derivation of the Determining Equations

Consider

$$dX^{(\beta)}(\bar{t}) = \mathbf{f}(t, \mathbf{X}(\bar{t}), \dot{\mathbf{X}}(\bar{t}), \ldots, \mathbf{X}^{(\beta)}(\bar{t}), R_\mu)dt + \mathbf{G}(t, \mathbf{X}(\bar{t}), \dot{\mathbf{X}}(\bar{t}), \ldots, \mathbf{X}^{(\beta)}(\bar{t}), R_\nu)d\mathbf{W}(t) \tag{6.1}$$

$$dX_i^{(k)}(t) = X_i^{(k+1)}dt, \tag{6.2}$$

$$X_i^{(0)}(t) = X_i(t) \tag{6.3}$$

for $k = 0, 1, \ldots, \beta - 1$. The function $\mathbf{f}$ is an approximate drift, which is an N vector-valued function, $i = 1, \ldots, N$. $\mathbf{G}$ is an $N \times M$ matrix-valued function approximating diffusion and $\mathbf{W}(t)$ is an M-dimensional Wiener process. Here $\mathbf{f}$ and $\mathbf{G}$ are defined as follows

$$\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, R_\mu) = \epsilon^{r\mu} \overset{r}{\mathbf{f}}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}), \tag{6.4}$$

where the repeated index r runs from 0 to R_μ, where R_μ is the largest positive integer such that $\mu R_\mu < 2\rho$ and

$$\mathbf{G}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, R_\nu) = \epsilon^{r\nu} \overset{r}{\mathbf{G}}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}) \tag{6.5}$$

where the repeated index r runs from 0 to R_ν; R_ν is the largest positive integer such that $\nu R_\nu < 2\rho$. The order of accuracy to which we choose to work is ρ.

The spatial and temporal variables of our infinitesimal generator

$$H = \tau(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, \rho)\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, \rho)\frac{\partial}{\partial x_j},$$

are defined as

$$\tau(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, \rho) = \epsilon^r \overset{r}{\tau}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}), \tag{6.6}$$

$$\xi(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}, \rho) = \epsilon^r \overset{r}{\xi}(t, \mathbf{x}, \dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\beta)}). \tag{6.7}$$

The repeated index runs from 0 to ρ, since throughout this article we will be working to $O(\epsilon^\rho)$. Using Itô's formula on the βth-prolongation of the spatial we get

$$\begin{aligned} d\xi_j^{[\beta]} &= \left(\frac{\partial \xi_j{}^{[\beta]}}{\partial t} + f_i \frac{\partial \xi_j{}^{[\beta]}}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{s=1}^{M} G_i^s G_k^s \frac{\partial^2 \xi_j{}^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \xi_j{}^{[\beta]}}{\partial x_k^{(\alpha)}}\right) dt + \frac{\partial \xi_j{}^{[\beta]}}{\partial x_i^{(\beta)}} G_i^k \, dW_k(t) \\ &= \left(\left(\frac{1}{2}\sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial \xi_j^{l[\beta]}}{\partial x_i} + \frac{\partial \overset{l}{\xi}{}_j^{[\beta]}}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_k^{(\alpha)}} \right) \epsilon^l + \ldots \right. \\ &\quad \left. + \frac{1}{2}\sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + 2p\nu} + \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + \nu(r+p)} + \overset{q}{f}_i \frac{\partial \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)}} \epsilon^{l + \mu q} \right) dt + \ldots \\ &\quad + \overset{p}{G}{}_i^s \frac{\partial \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_s^{(\beta)}} \epsilon^{l + \nu p} \, dW^{(i)} \end{aligned} \tag{6.8}$$

and on the temporal infinitesimal

$$\begin{aligned}
d\tau &= \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}}\right) dt + G_j^i \frac{\partial \tau}{\partial x_i^{(\beta)}}\, dW_t^{(j)} \\
&= \left(\left(\frac{1}{2}\sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} + \frac{\partial \overset{l}{\tau}}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \overset{l}{\tau}}{\partial x_k^{(\beta)}}\right) \epsilon^l + \frac{1}{2}\sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i \partial x_k} \epsilon^{l\,+\,2\,p\,\nu} + \ldots\right. \\
&\quad \left. + \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l\,+\,\nu(r\,+\,p)} + \overset{q}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} \epsilon^{l\,+\,\mu\,q}\right) dt + \ldots \\
&\quad + \overset{p}{G}{}_i^s \frac{\partial \overset{l}{\tau}}{\partial x_s^{(\beta)}} \epsilon^{l\,+\,\nu\,p}\, dW^{(i)}.
\end{aligned} \tag{6.9}$$

The repeated indices r, p, q, and l run from 0 to $R_\nu - 1$, R_ν, R_μ, and ρ respectivley in our repeated index summation convention; $r < p$. Thus, by substitution we get

$$\begin{aligned}
d\overline{X}^{(\beta)} &= dX^{(\beta)} + \theta\left(\frac{\partial \xi^{[\beta]}}{\partial t} + f_i \frac{\partial \xi^{[\beta]}}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2 \xi^{[\beta]}}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \xi^{[\beta]}}{\partial x_j^{(\alpha)}}\right) dt \ldots \\
&\quad + \theta G_j^i \frac{\partial \xi^{[\beta]}}{\partial x_i^{(\beta)}}\, dW_t^{(j)} + \mathcal{O}(\theta^2)
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
d\bar{t} &= dt + \theta\left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}}\right) dt + \ldots \\
&\quad + \theta G_j^i \frac{\partial \tau}{\partial x_i^{(\beta)}}\, dW_t^{(j)} + \mathcal{O}(\theta^2)
\end{aligned} \tag{6.11}$$

$$\begin{aligned}
d\overline{W}_{\bar{t}} &= dW_t^{(l)}\left(1 + \frac{\theta}{2}\left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}}\right) + \ldots\right. \\
&\quad \left. + \frac{\theta}{2} G_j^i \frac{\partial \tau}{\partial x_i^{(\beta)}} \frac{dW_t^{(j)}}{dt}\right) + \mathcal{O}(\theta^2).
\end{aligned} \tag{6.12}$$

The transformed time index should satisfy the following probabilistic condition which the original time differential index satisfies, i.e.

$$\mathbb{E}_{\overline{\mathbb{Q}}}\left[d\bar{t}(t,\omega)\right] = d\bar{t}(t,\omega). \tag{6.13}$$

As a result of this condition we have

$$G_j^i \frac{\partial \tau}{\partial x_i^{(\beta)}} = 0, \tag{6.14}$$

which in turn gives

$$G_j^i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} \quad = \quad 0, \tag{6.15}$$

which is true for all l from 0 to ρ and for all j from 1 to M. Thus (6.11) can now be written as

$$d\bar{t} \quad = \quad dt + \theta\left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2}\sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}}\right) dt + \mathcal{O}(\theta^2). \tag{6.16}$$

The condition (6.13) also forces

$$\Gamma(\tau) = Constant, \tag{6.17}$$

where

$$\Gamma = \frac{\partial}{\partial t} + f_i \frac{\partial}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k\,=\,1}^{M} G_i^k G_j^k \frac{\partial^2}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial}{\partial x_j^{(\alpha)}}. \tag{6.18}$$

The transformation of $\mathbf{f}$ and $\mathbf{G}$ under our prolongated infinitesimal generator $H^{[\beta]}$ is

$$\begin{aligned}
f_i(\bar{t}, \hat{\mathcal{X}}^{(\beta)}) &= f_i(t, \mathcal{X}^{(\beta)}) + \theta H^{[\beta]} f_i(t, \mathcal{X}^{(\beta)}) + \mathcal{O}(\theta^2) && (6.19)\\
&= f_i + \theta \left(\tau(t, \mathcal{X}^{(\beta)}) \frac{\partial f_i}{\partial t} + \xi_j^{[n]}(t, \mathcal{X}^{(\beta)}) \frac{\partial f_i}{\partial x_j^{(n)}} \right) + \mathcal{O}(\theta^2) \\
&= \epsilon^q \overset{q}{f}_i + \theta \, \overset{l}{\xi}{}_j^{[n]} \frac{\partial \overset{q}{f}_i}{\partial x_j^{(n)}} \epsilon^{l\,+\,\mu\,q} + \theta \, \overset{l}{\tau} \frac{\partial \overset{q}{f}_i}{\partial t} \epsilon^{l\,+\,\mu\,q} + \mathcal{O}(\theta^2) \\
&= \epsilon^q \overset{q}{f}_i + \theta \, \epsilon^{l\,+\,\mu\,q} \left(\overset{l}{\xi}{}_j^{[n]} \frac{\partial \overset{q}{f}_i}{\partial x_j^{(n)}} + \overset{l}{\tau} \frac{\partial \overset{q}{f}_i}{\partial t} \right) + \mathcal{O}(\theta^2) && (6.20)\\
G_k^i(\bar{t}, \hat{\mathcal{X}}^{(\beta)}) &= G_k^i(t, \mathcal{X}^{(\beta)}) + \theta H^{[\beta]} G_k^i(t, \mathcal{X}^{(\beta)}) + \mathcal{O}(\theta^2) && (6.21)\\
&= G_k^i + \theta \left(\tau(t, \mathcal{X}^{(\beta)}) \frac{\partial G_k^i}{\partial t} + \xi_j^{[n]}(t, \mathcal{X}^{(\beta)}) \frac{\partial G_k^i}{\partial x_j^{(n)}} \right) + \mathcal{O}(\theta^2) \\
&= \epsilon^p \overset{p}{G}{}_k^i + \theta \, \epsilon^{\nu\,p\,+\,l} \left(\overset{l}{\xi}{}_j^{[n]} \frac{\partial \overset{p}{G}{}_k^i}{\partial x_j^{(n)}} + \overset{l}{\tau} \frac{\partial \overset{p}{G}{}_k^i}{\partial t} \right) + \mathcal{O}(\theta^2), && (6.22)
\end{aligned}$$

where $\{X,\, \dot{X},\, \ldots,\, X^{(\beta)}\}$ is represented by $\mathcal{X}^{(\beta)}$ and the transformed set $\{\overline{X},\, \overline{\dot{X}},\, \ldots,\, \overline{X^{(\beta)}}\}$ is represented by $\hat{\mathcal{X}}^{(\beta)}$. The repeated indices q, p, l and n run from 0 to R_μ, R_ν, ρ and β respectively. Form invariance of $d\overline{\mathbf{X}}^{(\beta)}$ means the following for each of its components

$$d\overline{X}_m^{(\beta)} = f_m(\bar{t}, \hat{\mathcal{X}}^{(\beta)}) d\bar{t} + G_k^m(\bar{t}, \hat{\mathcal{X}}^{(\beta)}) d\overline{W}_{\bar{t}}^k. \tag{6.23}$$

Multiplying out the drift component gives

$$\begin{aligned}
f_m(\bar{t}, \hat{\mathcal{X}}^{(\beta)}) d\bar{t} = &\Bigg\{ \mathbf{f}(t,\ \mathcal{X}(t)) \ + \ \theta \ \left(\Gamma(\tau) \ + \ H^{[\beta]} \right) \ \mathbf{f}(t,\ \mathcal{X}(t)) \\
&+ \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \Bigg(\left(\Gamma(\tau) \ + \ H^{[\beta]} \right)^k \ \mathbf{f}(t,\ \mathcal{X}(t)) \\
&+ \sum_{j=0}^{k-2} \begin{pmatrix} k \\ k\ -\ j \end{pmatrix} H^j (\mathbf{f}(t,\ \mathcal{X}(t))) \ \left(\Gamma(H^{k-j}(t)) \ - \ [\Gamma(\tau)]^{k-j} \right) \Bigg) \Bigg\} \ dt.
\end{aligned} \tag{6.24}$$

In order for the finite transformations to keep invariance we need the following condition

$$e^{\epsilon\, \Gamma(\tau)} \ = \ \Gamma\left(e^{\epsilon\, H^{[\beta]}}(t) \right), \tag{6.25}$$

which is automatically satisfied as a result of (6.17). The relation (6.25) ensures that the higher θ-terms depend only on the $\mathcal{O}(1)$ and $\mathcal{O}(\theta)$ terms. As a result may ignore the higher order terms and construct them later once we have solved for the infinitesimals. Carrying on with the expansion of (6.23) we have

$$\begin{aligned}
d\overline{X}_m^{(\beta)} &= f_m dt + G_m^k dW_k + \dots \\
&+ \theta \left(f_m \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}} \right) + \tau \frac{\partial f_m}{\partial t} + \dots \right. \\
&+ \left. \xi_j^{[n]} \frac{\partial f_m}{\partial x_j^{(n)}} \right) dt + \dots \\
&+ \theta \left(\tau \frac{\partial G_m^l}{\partial t} + \xi_j^{[n]} \frac{\partial G_m^l}{\partial x_j^{(n)}} + \dots \right. \\
&+ \left. \frac{1}{2} G_m^l \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}} \right) \right) dW_l + \mathcal{O}(\theta^2).
\end{aligned} \tag{6.26}$$

Thus by comparing the terms that follow the θ in (6.10) and (6.26) we have that

$$f_m \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}} \right) + \tau \frac{\partial f_m}{\partial t} + \xi_j^{[n]} \frac{\partial f_m}{\partial x_j^{(n)}} + \dots$$

$$- \left(\frac{\partial {\xi_m}^{[\beta]}}{\partial t} + f_i \frac{\partial {\xi_m}^{[\beta]}}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2 {\xi_m}^{[\beta]}}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial {\xi_m}^{[\beta]}}{\partial x_j^{(\alpha)}} \right) = 0 \tag{6.27}$$

(6.28)

which can be written as

$$\begin{aligned}
&\epsilon^q \overset{q}{f}_m \left(\left(\frac{1}{2} \sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2 \overset{1}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} + \frac{\partial \overset{l}{\tau}}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \overset{l}{\tau}}{\partial x_k^{(\beta)}} \right) \epsilon^l + \dots \right. \\
&\left. + \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + \nu(r+p)} + \frac{1}{2} \sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + 2\nu p} + \overset{q}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} \epsilon^{l + \mu q} + \right) + \dots \\
&\overset{l}{\xi}{}_j^{[n]} \frac{\partial \overset{q}{f}_m}{\partial x_j^{(n)}} \epsilon^{\mu q + l} + \overset{l}{\tau} \frac{\partial \overset{q}{f}_m}{\partial t} \epsilon^{l + \mu q} + \dots \\
&- \left(\frac{1}{2} \sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_m^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial \overset{l}{\xi}{}_m^{[\beta]}}{\partial x_i^{(\beta)}} + \frac{\partial \overset{l}{\xi}{}_m^{[\beta]}}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \overset{l}{\xi}_m}{\partial x_k^{(\beta)}} \right) \epsilon^l - \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + \nu(r+p)} - \dots \\
&- \frac{1}{2} \sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \epsilon^{l + 2\nu p} - \overset{q}{f}_i \frac{\partial \overset{l}{\xi}{}_j^{[\beta]}}{\partial x_i^{(\beta)}} \epsilon^{l + \mu q} = 0,
\end{aligned} \tag{6.29}$$

and

$$\begin{aligned}
&\tau \frac{\partial G_k^m}{\partial t} + \xi_j^{[n]} \frac{\partial G_k^m}{\partial x_j^{(n)}} + \dots \\
&+ \frac{1}{2} G_k^m \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i^{(\beta)}} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2 \tau}{\partial x_i^{(\beta)} \partial x_j^{(\beta)}} + \sum_{\alpha=0}^{\beta-1} x_j^{(\alpha+1)} \frac{\partial \tau}{\partial x_j^{(\alpha)}} \right) + \dots \\
-\quad &G_k^i \frac{\partial {\xi_m}^{[\beta]}}{\partial x_i^{(\beta)}} = 0.
\end{aligned} \tag{6.30}$$

which we write as

$$\overset{l}{\xi}_j^{[n]} \frac{\partial \overset{p}{G}{}_k^m}{\partial x_j^{(\beta)}} \epsilon^{l\,+\,\nu\,p} + \overset{l}{\tau} \frac{\partial \overset{p}{G}{}_k^m}{\partial t} \epsilon^{l\,+\,\nu\,p} + \dots$$

$$+ \frac{1}{2} \overset{p}{G}{}_k^m \left(\left(\frac{1}{2} \sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i^{(\beta)}} + \frac{\partial \overset{l}{\tau}}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial \overset{l}{\tau}}{\partial x_k^{(\beta)}} \right) \epsilon^{\nu\,p\,+\,l} + \dots \right.$$

$$\left. \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i \partial x_k} \epsilon^{l\,+\,\nu(r\,+\,p)} + \frac{1}{2} \sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2 \overset{l}{\tau}}{\partial x_i \partial x_k} \epsilon^{l\,+\,2\,\nu\,p} + \overset{q}{f}_i \frac{\partial \overset{l}{\tau}}{\partial x_i} \epsilon^{l\,+\,\mu\,q} \right) - \overset{p}{G}{}_k^i \frac{\partial \overset{l}{\xi}{}_m^{[\beta]}}{\partial x_i^{(\beta)}} \epsilon^{l\,+\,\nu\,p} = 0. \tag{6.31}$$

Following the same methodology as above we implement a form invariance argument on (6.2)

$$d\overline{X}_i^{(k)}(t) = \overline{X}_i^{(k+1)} d\bar{t}. \tag{6.32}$$

Expanding (6.32) yields the following θ-order relations

$$\Gamma(\xi^{[k]}) = \left(\Gamma(\tau) + H^{[k+1]} \right) X_i^{(k+1)} \tag{6.33}$$

and

$$G_j^i \frac{\partial \xi^{[k]}}{\partial x_i^{(\beta)}} = 0, \tag{6.34}$$

which in turn means

$$G_j^i \frac{\partial \overset{l}{\xi}{}^{[k]}}{\partial x_i^{(\beta)}} = 0, \quad l = 1,\, R_\mu. \tag{6.35}$$

6.3 Operators

We can now rewrite (6.29) and (6.31) as

$$+\epsilon^{\mu\,q\,+\,l} \left(\overset{q}{f}_m \left(\overset{0}{\Gamma}(\overset{l}{\tau}) + \epsilon^{\nu(r\,+\,p)} \mho_{r\,p}(\overset{l}{\tau}) + \epsilon^{2\,\nu\,p} \Upsilon_p(\overset{l}{\tau}) + \epsilon^{\mu\,j} \Psi_j(\overset{l}{\tau}) \right) + \overset{l}{H}_\beta(\overset{q}{f}_m) \right) + \dots$$

$$-\epsilon^l \left(\overset{0}{\Gamma}(\overset{l}{\xi}{}_m^{[\beta]}) + \epsilon^{\nu(r\,+\,p)} \mho_{r\,p}(\overset{l}{\xi}{}_m^{[\beta]}) + \epsilon^{2\,\nu\,p} \Upsilon_p(\overset{l}{\xi}{}_m^{[\beta]}) + \epsilon^{\mu\,q} \Psi_q(\overset{l}{\xi}{}_m^{[\beta]}) \right) = 0 \tag{6.36}$$

and

$$\epsilon^{l\,+\,\nu\,p} \left(\overset{l}{H}_\beta(\overset{p}{G}{}_k^m) + \frac{1}{2} \overset{p}{G}{}_k^m \left(\overset{0}{\Gamma}(\overset{l}{\tau}) + \epsilon^{\nu(r\,+\,p)} \mho_{r\,p}(\overset{l}{\tau}) + \epsilon^{2\,\nu\,p} \Upsilon_p(\overset{l}{\tau}) + \epsilon^{\mu\,q} \Psi_q(\overset{l}{\tau}) \right) - \overset{p\,k}{Y}_\beta(\overset{l}{\xi}{}_m^{[\beta]}) \right) = 0, \tag{6.37}$$

respectively, where

$$\overset{0}{\Gamma} = \frac{1}{2}\sum_{s=1}^{M} \overset{0}{G}{}_i^s \overset{0}{G}{}_k^s \frac{\partial^2}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} + \overset{0}{f}_i \frac{\partial}{\partial x_i^{(\beta)}} + \frac{\partial}{\partial t} + \sum_{\alpha=0}^{\beta-1} x_k^{(\alpha+1)} \frac{\partial}{\partial x_k^{(\beta)}} \tag{6.38}$$

$$\mho_{r\,p} = \sum_{s=1}^{M} \overset{r}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \ (0 \le r < p \le R_\nu) \tag{6.39}$$

$$\Upsilon_p = \frac{1}{2}\sum_{s=1}^{M} \overset{p}{G}{}_i^s \overset{p}{G}{}_k^s \frac{\partial^2}{\partial x_i^{(\beta)} \partial x_k^{(\beta)}} \ (0 < p \le R_\nu) \tag{6.40}$$

$$\Psi_q = \overset{q}{f}_i \frac{\partial}{\partial x_i^{(\beta)}} \ (0 < q, j \le R_\mu) \tag{6.41}$$

$$\overset{p\ i}{Y}_\beta = \overset{p}{G}{}_i^k \frac{\partial}{\partial x_k^{(\beta)}} \ (0 \le p \le R_\nu) \tag{6.42}$$

$$\overset{l}{H}_\beta = \overset{l}{\tau} \frac{\partial}{\partial t} + \overset{l}{\xi}{}_j^{[n]} \frac{\partial}{\partial x_j^{(n)}} \ (0 \le n \le \beta). \tag{6.43}$$

Note that we cannot cancel out the terms ϵ^l and $\epsilon^{l + \nu p}$ in (6.36) and (6.37) respectively, in order to simplify them. These terms are a part of the summation convention implied by the repeated indices. These terms contribute to the order of error as a result of this implication.

We now apply our generalised methodology for find approximate symmetries to the Itô system considered in [10]. Our application should be consistent with the determining equations found in Ibragimov, Ünal, and Jogréus [10].

Example 1

For their approximate SODEs, $\beta = 0$, $\mu = 1$, $\nu = \frac{1}{2}$, $R_\mu = 1$, $R_\nu = 1$ and $\rho = 1$. Thus the diffusion coefficient $\mathbf{G}$, which was taken to be constant, and the drift $\mathbf{f}$ appeared as follows in the Itô system

$$d\mathbf{x} = \left(\overset{0}{\mathbf{f}} + \epsilon\overset{1}{\mathbf{f}}\right) dt + \sqrt{\epsilon}\mathbf{G}\, d\mathbf{W}_t \tag{6.44}$$

where the drift is a $N \times 1$ vector and the constant diffusion coefficient is a matrix with dimension $N \times M$. The determining equations are

$$\begin{aligned} & -\frac{1}{2}\sum_{s=1}^{M} \overset{1}{G}{}_k^s \overset{1}{G}{}_i^s \epsilon \left(\frac{\partial^2 \overset{0}{\xi}_j}{\partial x_i \partial x_k} + \epsilon \frac{\partial^2 \overset{1}{\xi}_j}{\partial x_i \partial x_k} \right) + \\ & (\overset{0}{\xi}_i + \epsilon \overset{1}{\xi}_i) \left(\frac{\partial \overset{0}{f}_j}{\partial x_i} + \epsilon \frac{\partial \overset{1}{f}_j}{\partial x_i} \right) - \overset{0}{f}_i \left(\frac{\partial \overset{0}{\xi}_j}{\partial x_i} + \epsilon \frac{\partial \overset{1}{\xi}_j}{\partial x_i} \right) - \epsilon \overset{1}{f}_i \left(\frac{\partial \overset{0}{\xi}_j}{\partial x_i} + \epsilon \frac{\partial \overset{1}{\xi}_j}{\partial x_i} \right) + \\ & (\overset{0}{\tau} + \epsilon \overset{1}{\tau}) \left(\frac{\partial \overset{0}{f}_j}{\partial t} + \epsilon \frac{\partial \overset{1}{f}_j}{\partial t} \right) - \frac{\partial \overset{0}{\xi}_j}{\partial t} - \epsilon \frac{\partial \overset{1}{\xi}_j}{\partial t} + (\overset{0}{f}_j + \epsilon \overset{1}{f}_j) \left(\frac{1}{2} \epsilon \sum_{s=1}^{M} \overset{1}{G}{}_k^s \overset{1}{G}{}_i^s \left(\frac{\partial^2 \overset{0}{\tau}}{\partial x_i \partial x_k} + \epsilon \frac{\partial^2 \overset{1}{\tau}}{\partial x_i \partial x_k} \right) + \right. \\ & \left. + \overset{0}{f}_i \left(\frac{\partial \overset{0}{\tau}}{\partial x_i} + \epsilon \frac{\partial \overset{1}{\tau}}{\partial x_i} \right) + \epsilon \overset{1}{f}_i \left(\frac{\partial \overset{0}{\tau}}{\partial x_i} + \epsilon \frac{\partial \overset{1}{\tau}}{\partial x_i} \right) + \frac{\partial \overset{0}{\tau}}{\partial t} + \epsilon \frac{\partial \overset{1}{\tau}}{\partial t} \right) = 0 \end{aligned} \tag{6.45}$$

and

$$\sqrt{\epsilon}(\overset{0}{\xi}_i + \epsilon\overset{1}{\xi}_i)\frac{\partial \overset{1}{G}{}^j_k}{\partial x_i} - \sqrt{\epsilon}\overset{1}{G}{}^i_k\left(\frac{\partial\overset{0}{\xi}_j}{\partial x_i} + \epsilon\frac{\partial\overset{1}{\xi}_j}{\partial x_i})\right) + \sqrt{\epsilon}(\overset{0}{\tau}+\epsilon\overset{1}{\tau})\frac{\partial \overset{1}{G}{}^j_k}{\partial t} + \tag{6.46}$$

$$\frac{1}{2}\sqrt{\epsilon}\overset{1}{G}{}^j_k\left(\frac{1}{2}\epsilon\left(\frac{\partial^2\overset{0}{\tau}}{\partial x_i\partial x_l}+\epsilon\frac{\partial^2\overset{1}{\tau}}{\partial x_i \partial x_l}\right)\sum_{s=1}^{M} \overset{1}{G}{}^s_i\overset{1}{G}{}^s_l + \overset{0}{f}_i\left(\frac{\partial\overset{0}{\tau}}{\partial x_i}+\epsilon\frac{\partial\overset{1}{\tau}}{\partial x_i}\right) + \epsilon\overset{1}{f}_i\left(\frac{\partial\overset{0}{\tau}}{\partial x_i}+\epsilon\frac{\partial\overset{1}{\tau}}{\partial x_i}\right) + \right.$$

$$\left.\frac{\partial\overset{0}{\tau}}{\partial t}+\epsilon\frac{\partial\overset{1}{\tau}}{\partial t}\right) = 0.$$

Now since we are working to order ρ we get the following groups of determining equations which are exactly what Ibragimov, Ünal, and Jogréus [10] get

$$-\frac{\partial\overset{0}{\xi}_j}{\partial t} - \overset{0}{f}_i\frac{\partial\overset{0}{\xi}_j}{\partial x_i} + \overset{0}{\xi}_i\frac{\partial\overset{0}{f}_j}{\partial x_i} + \overset{0}{\tau}\frac{\partial\overset{0}{f}_j}{\partial t} + \overset{0}{f}_j\overset{0}{f}_i\frac{\partial\overset{0}{\tau}}{\partial x_i} + \overset{0}{f}_j\frac{\partial\overset{0}{\tau}}{\partial t} = 0, \tag{6.47}$$

which we get by comparing coefficients with no ϵ's

$$\overset{0}{f}_j\overset{0}{f}_i\frac{\partial\overset{1}{\tau}}{\partial x_i} - \frac{\partial\overset{1}{\xi}_j}{\partial x_i}\overset{0}{f}_i + \overset{0}{f}_j\overset{1}{f}_i\frac{\partial\overset{0}{\tau}}{\partial x_i} + \overset{1}{f}_j\overset{0}{f}_i\frac{\partial\overset{0}{\tau}}{\partial x_i} + \frac{1}{2}\frac{\partial^2\overset{0}{\tau}}{\partial x_i\partial x_l}\sum_{s=1}^{M}\overset{1}{G}{}^s_i\overset{1}{G}{}^s_l\overset{0}{f}_j + \frac{\partial\overset{1}{\tau}}{\partial t}\overset{0}{f}_j + \overset{1}{\xi}_i\frac{\partial\overset{0}{f}_j}{\partial x_i} + \overset{0}{\xi}_i\frac{\partial\overset{1}{f}_j}{\partial x_i} - \overset{1}{f}_i\frac{\partial\overset{0}{\xi}_j}{\partial x_i}$$

$$-\frac{1}{2}\frac{\partial^2\overset{0}{\xi}_j}{\partial x_i\partial x_l}\sum_{s=1}^{M}\overset{1}{G}{}^s_i\overset{1}{G}{}^s_l + \frac{\partial\overset{0}{f}_j}{\partial t}\overset{1}{\tau} + \frac{\partial\overset{1}{f}_j}{\partial t}\overset{0}{\tau} - \frac{\partial\overset{1}{\xi}_j}{\partial t} + \frac{\partial\overset{0}{\tau}}{\partial t}\overset{1}{f}_j = 0, \tag{6.48}$$

which all share the same coefficient ϵ. In a similar fashion we get the following for $\sqrt{\epsilon}$ and ϵ respectively

$$\frac{1}{2}\overset{1}{G}{}^j_k\left(\frac{\partial\overset{0}{\tau}}{\partial t} + \overset{0}{f}_i\frac{\partial\overset{0}{\tau}}{\partial x_i}\right) - \overset{1}{G}{}^i_k\frac{\partial\overset{0}{\xi}_j}{\partial x_i} = 0, \tag{6.49}$$

and

$$-\overset{1}{G}{}^i_k\frac{\partial\overset{1}{\xi}_j}{\partial x_i} + \frac{1}{2}\overset{1}{G}{}^j_k\left(\frac{\partial\overset{1}{\tau}}{\partial t} + \overset{0}{f}_i\frac{\partial\overset{1}{\tau}}{\partial x_i}\right) + \frac{1}{2}\overset{1}{G}{}^j_k\overset{1}{f}_i\frac{\partial\overset{0}{\tau}}{\partial x_i} + \frac{\overset{1}{G}{}^j_k}{4}\sum_{s=1}^{M}\overset{1}{G}{}^s_i\overset{1}{G}{}^s_l\frac{\partial^2\overset{0}{\tau}}{\partial x_i\partial x_l} = 0. \tag{6.50}$$

Notice that we used (6.14), and the fact that G was constant to simplify the above.

Remark. Our application is consistent with that of [10] in this example.

Example 2

We consider

$$d\dot{X} = -\omega^2\, X\, dt + \sigma\, dW + \sqrt{\epsilon} X\, dW. \tag{6.51}$$

By applying the condition (6.34) we have that

$$\xi = \xi(t, x) \tag{6.52}$$

and that the prolongation formula (6.33) becomes

$$\xi^{[1]} = D(\xi) - \dot{x}D(\tau), \tag{6.53}$$

where D is the total time derivative operator. Our determining equations at ϵ^0 are

$$-\omega^2\, x\,\overset{0}{\Gamma}(\overset{0}{\tau}) + \overset{0}{H}(-\omega^2\, x) = \overset{0}{\Gamma}(\overset{0}{\xi}{}^{[1]}) \tag{6.54}$$

and

$$\overset{0}{H}(\overset{0}{G}) + \frac{1}{2}\overset{0}{G}\overset{0}{\Gamma}(\overset{0}{\tau}) = \overset{0}{Y}(\overset{0}{\xi}^{[1]}). \tag{6.55}$$

Our determining equations at ϵ are

$$-\omega^2\, x\, \overset{0}{\Gamma}(\overset{1}{\tau}) + \overset{1}{H}(-\omega^2\, x) - \omega^2\, x\, \Upsilon_1(\overset{0}{\tau}) = \overset{0}{\Gamma}(\overset{1}{\xi}^{[1]}) + \Upsilon_1(\overset{0}{\xi}) \tag{6.56}$$

and

$$\overset{1}{H}(\overset{0}{G}) + \frac{1}{2}\overset{0}{G}\overset{0}{\Gamma}(\overset{1}{\tau}) = \overset{0}{Y}(\overset{1}{\xi}^{[1]}). \tag{6.57}$$

Our determining equation at $\epsilon^{\frac{1}{2}}$ is

$$\overset{0}{H}(\overset{1}{G}) + \frac{1}{2}\overset{1}{G}\overset{0}{\Gamma}(\overset{0}{\tau}) + \frac{1}{2}\overset{0}{G}\mho_{10}(\overset{0}{\tau}) = \overset{1}{Y}(\overset{0}{\xi}^{[1]}) \tag{6.58}$$

and the final determining equation at $\epsilon^{\frac{3}{2}}$

$$\frac{1}{2}\overset{0}{G}\mho_{12}(\overset{0}{\tau}) + \frac{1}{2}\overset{1}{G}\overset{0}{\Gamma}(\overset{1}{\tau}) = \overset{1}{Y}(\overset{1}{\xi}^{[1]}). \tag{6.59}$$

Solving equations (6.54) and (6.55) for the infinitesimals give

$$\overset{0}{\tau} = C_0 \tag{6.60}$$

$$\overset{0}{\xi} = C_1\, \cos(\omega\, t) + C_2\, \sin(\omega\, t). \tag{6.61}$$

Therefore, equations (6.58) and (6.59) force

$$\overset{0}{\xi} = 0 \tag{6.62}$$

and

$$\overset{1}{\tau} = C_3. \tag{6.63}$$

From equation (6.56) we get

$$-\omega^2\overset{1}{\xi} = D^2(\overset{0}{\xi}) \tag{6.64}$$

which solves as

$$\overset{1}{\xi} = C_4\, \cos(\omega\, t) + C_5\, \sin(\omega\, t). \tag{6.65}$$

Therefore we have

$$\xi = \epsilon\,(C_4\, \cos(\omega\, t) + C_5\, \sin(\omega\, t)) \tag{6.66}$$

and

$$\tau = C_0 + \epsilon C_3. \tag{6.67}$$

6.4 Concluding Comments

In this more general approximate approach to higher order SODEs we derive the same conditioning as Ünal [3] did without recourse to the Itô's multiplication table for the transformed variables. Our results are consistent with that of [10] in the first order case. However, we have a generalization to nth order SODEs. We also applied our method to an example taken from [10] as well as another example.

Chapter 7

Conservation Laws for SDEs

A methodology for constructing conserved quantities with Lie symmetry infinitesimals in an Itô integral context is pursued. The basis of this construction relies on Lie bracket relations on both the instantaneous drift and diffusion operators.

7.1 Introduction

Conserved quantities in this context implies an entity which is constant on all sample paths for all time indices; their instantaneous drift and diffusion are zero. Trivially this implies that these conserved quantities are all Martingales, i.e. their expected value in the future or present is their eventuated values in the past. Methods for constructing conserved quantities of SODEs by using Lie transformations was analyzed for Stratonovich integral based SODEs by Misawa [18] and Albeverio and Fei [19]. The conserved quantity construction of Misawa [18] and Albeverio and Fei [19], preclude the necessity for Lagrangian or Hamiltonian theory. The philosophy followed, highlighted the interplay between the infinitesimals of the symmetry operator, H, and the conserved quantity itself.

The Itô integral construction of the conserved quantities was later attempted by Ünal [3]. In this attempt Ünal [3] uses both the FP equations and its associated SODEs to construct the conserved quantity.

Having reconciled the determining equations between Wafo Soh and Mahomed [2] and Ünal [3] via Fredericks and Mahomed [5], we can focus on the conserved quantity analysis of [3]. In the first chapter, we showed that the symmetries of the FP equations are projectable using the methodology of Mahomed and Momoniat [11]. This projectable nature of the temporal infinitesimal was an *anzats* that Gaeta and Quintero [1] enforced on both the FP equations and its associated SODEs.

The work of [3] shows that in the SODEs context, the temporal infinitesimal need not be a function of time only. This implies that the Lie algebra generated by the SODEs can have non-projectable symmetries which will not belong to the Lie algebra generated by the FP equation.

However, in constructing the conserved quantity for Itô integral based SODEs, [3] tries to combine the determining equations associated with SODEs, which allows the said infinitesimal to be non-projectable, with the determining equations based on the associated FP equation. However, in the first chapter we proved that the symmetries of the FP equation have to be projectable. Thus we have that only projectable symmetries will satisfy both FP equation and its associated SODEs, which is what was shown by Gaeta and Quintero [1].

In this chapter, we first revisit the conserved quantity results of Ünal [3] and juxtapose it with the new findings from our earlier chapters. This scrutiny will be followed by an attempt to construct a conserved quantity based upon the methodology of Albeverio and Fei [19] for Stratonovich integral SODEs.

7.2 Conserved Quantities for Itô Integrals Revisited

The system determining equations belonging to the FP equations can be rewritten in terms of the instantaneous drift and diffusion operators. The original equations are:

$$\frac{\partial(\tau A_{ik})}{\partial t} + \left(\xi_r \frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0 \tag{7.1}$$

$$\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r \frac{\partial \xi_i}{\partial x_r} - \xi_r \frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} + \dots$$
$$-2\left(A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right) = 0. \tag{7.2}$$

$$\left(\frac{\partial}{\partial t} + f_i \frac{\partial}{\partial x_i} - A_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right)\left(\alpha_2(t,\mathbf{x}) + \frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{7.3}$$

where

$$A_{ij} = -\frac{1}{2}\sum_{k=1}^{M} G_i^k G_j^k \tag{7.4}$$

and the dependent variables infinitesimal Φ of the FP equation has the following relation

$$\Phi = \alpha_1(t,\, \mathbf{x}) + u\,\alpha_2(t,\, \mathbf{x}) \tag{7.5}$$

which is associated to the FP symmetry operator as

$$H_{FP} = \tau(t)\frac{\partial}{\partial t} + \xi_j(t,\, \mathbf{x})\frac{\partial}{\partial x_j} + \Phi(t,\, \mathbf{x},\, u)\frac{\partial}{\partial u}. \tag{7.6}$$

Equation (7.1) can be written as

$$\sum_{l=1}^{M} G_i^l Y^l(\xi_k) + \sum_{l=1}^{M} G_k^l Y^l(\xi_i) = H\left(\sum_{l=1}^{M} G_i^l G_k^l\right) + \sum_{l=1}^{M} G_i^l G_k^l \Gamma(\tau). \tag{7.7}$$

Since τ is a projectable in this context, i.e. function of time only, we have that $\Gamma(\tau) = \frac{\partial \tau}{\partial t}$. Further simplification gives

$$Y^l(\xi_k) = H\left(G_k^l\right) + \frac{1}{2}\, G_k^l\, \Gamma(\tau), \text{ for } l = 1, M \text{ and } k = 1, N. \tag{7.8}$$

Equations (7.2) and (7.3) can likewise be written as

$$\Gamma(\xi_k) = \left(\Gamma(\tau) + H\right) f_k + \sum_{l=1}^{M} G_k^l Y^l \left(\alpha_2(t,\, \mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) \tag{7.9}$$

and

$$\Gamma\left(\alpha_2(t,\, \mathbf{x}) + \sum_{r=1}^{N}\frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{7.10}$$

respectively.

(7.11)

The projectable symmetries of the Itô SODEs satisfy the following determining equations

$$Y^l(\xi_k) = \left(\frac{1}{2}\,\Gamma(\tau) + H\right) G_k^l, \tag{7.12}$$

$$\Gamma(\xi_k) = \left(\Gamma(\tau) + H\right) f_k, \tag{7.13}$$

$$Y^l(\tau) = 0 \tag{7.14}$$

and

$$\Gamma(\tau) = Constant, \tag{7.15}$$

for $l = 1, M$ and $k = 1, N$. Since these projectable symmetries are a sub-algebra of that belonging to the FP equation, we have that the determining equations associated with the FP equation become

$$Y^l\left(\alpha_2(t,\, \mathbf{x}) + \sum_{r=1}^{N} \frac{\partial \xi_r}{\partial x_r}\right) = 0 \tag{7.16}$$

and

$$\Gamma\left(\alpha_2(t,\, \mathbf{x}) + \sum_{r=1}^{N} \frac{\partial \xi_r}{\partial x_r}\right) = 0, \tag{7.17}$$

for all $l = 1, M$. Thus for projectable symmetries of the Itô integral based SODEs we have that $\alpha_2(t, \mathbf{x}) + \sum_{r=1}^{N} \partial\xi_r/\partial r$ is a conserved quantity because both its instantaneous drift and diffusion is zero. This is different from what was derived in [3], where extra terms involving the spatial derivative of the temporal infinitesimal survive, because of the preclusion of the fact that the temporal infinitesimal has to be projectable in the FP equation context.

7.3 An Alternative Formulation

An alternative formulation for deriving conserved quantities from the Lie symmetries is adapted from Albeverio and Fei [19], which derived conserved quantities from the Lie infinitesimals for Stratonovich based SODEs. This allows us to use both the projectable and non-projectable Lie symmetries of the Itô SODEs.

We first need a relation between the instantaneous drift and diffusion operators and the the symmetry operator. The use of Lie brackets achieves this. The determining equations (7.12) and (7.13) based on the SODEs can be written in terms of Lie brackets, i.e.

$$[\Gamma, H]\,(f_k) = \Gamma(\tau)\,\Gamma(f_k) \tag{7.18}$$

and

$$\left[Y^l, H\right](G_k^l) = \frac{1}{2}\,\Gamma(\tau)Y^l(G_k^l)\; l = 1, M, \tag{7.19}$$

where $[\Gamma, H] = \Gamma(H) - H(\Gamma)$, and where condition (7.14) dictates that $Y^l(H) = \sum_{k=1}^{N} Y^l(\xi_k)\,\partial/\partial x_k$ for all $l = 1, M$. However the drift and diffusion coefficients of the SODEs are arbitrary, so we have the following

$$[\Gamma, H] \;=\; \Gamma(\tau)\,\Gamma \tag{7.20}$$

$$\left[Y^l, H\right] \;=\; \frac{1}{2}\,\Gamma(\tau)Y^l\; l = 1, M. \tag{7.21}$$

We next define $\mathcal{I} \equiv \{\, I\,(t, \mathbf{x})|\, dI = 0,\ \text{wherever (7.12) and (7.13) are satisfied}\}$.

Theorem 7.1. If $I \in \mathcal{I}$, i.e. satisfies $\Gamma(I) = 0$ and $Y^l(I) = 0$, then $H(I) \in \mathcal{I}$, where H satisfies (7.20) and (7.21).
Proof: from (7.20)

$$\Gamma(H(I)) \;=\; (\Gamma(H))\,(I) + H\,(\Gamma(I)) \tag{7.22}$$

$$=\; [\Gamma, H]\,(I) + H(\Gamma(I)) \tag{7.23}$$

$$=\; \Gamma(\tau)\Gamma(I) \tag{7.24}$$

$$=\; 0. \tag{7.25}$$

By looking at (7.21) we also get

$$\begin{aligned}
Y^l(H(I)) &= \left(Y^l(H)\right)(I) + H\left(Y^l(I)\right) && (7.26)\\
&= \left[Y^l, H\right](I) + H(Y^l(I)) && (7.27)\\
&= \frac{1}{2}\Gamma(\tau)Y^l(I) && (7.28)\\
&= 0. && (7.29)
\end{aligned}$$

Let $\mathcal{L}$ denote the set of all H satisfying (7.20) and (7.21). Having established $\mathcal{L}$ we now proceed with a proof that demonstrates that it is a complex Lie algebra.

Theorem 7.2. This set $\mathcal{L}$ forms a complex Lie algebra, i.e. for
$H_1,\ H_2,$ and $H_3 \in \mathcal{L}$:

$$\begin{aligned}
&(a) a_1 H_1 + a_2 H_2 \in \mathcal{L}\, \forall a_1, a_2 \in \mathcal{C} \setminus \{0\} && (7.30)\\
&(b)\, [H_1, H_2] \in \mathcal{L} && (7.31)\\
&(c)\, [H_1\, [H_2, H_3]] + [H_2\, [H_3, H_1]] + [H_3\, [H_1, H_2]] = 0 && (7.32)
\end{aligned}$$

Proof: let $H_1, H_2 \in \mathcal{L}$, i.e.,

$$\begin{aligned}
[\Gamma, H_i] &= \Gamma(\tau_i)\Gamma && (7.33)\\
\left[Y^l, H_i\right] &= \frac{1}{2}\Gamma(\tau_i)Y^l. && (7.34)
\end{aligned}$$

for $i = 1, 2$.
(a)

$$\begin{aligned}
[\Gamma, [a_1 H_1 + a_2 H_2]] &= [\Gamma, a_1 H_1] + [\Gamma, a_2 H_2] && (7.35)\\
&= \Gamma(\tau_1)\Gamma + \Gamma(\tau_2)\Gamma && (7.36)\\
&= \Gamma(\tau_1 + \tau_2)\,\Gamma && (7.37)
\end{aligned}$$

we also have

$$\begin{aligned}
\left[Y^l, [a_1 H_1 + a_2 H_2]\right] &= \left[Y^l, a_1 H_1\right] + \left[Y^l, a_2 H_2\right] && (7.38)\\
&= \frac{1}{2}\Gamma(\tau_1)Y^l + \frac{1}{2}\Gamma(\tau_2)Y^l && (7.39)\\
&= \frac{1}{2}\Gamma(\tau_1 + \tau_2)\,Y^l && (7.40)
\end{aligned}$$

(b)
By direct calculation

$$
\begin{aligned}
[\Gamma,[H_1,H_2]] &= [\Gamma, H_1H_2 - H_2H_1] && (7.41)\\
&= [\Gamma, H_1H_2] - [\Gamma, H_2H_1] && (7.42)\\
&= \Gamma H_1H_2 - H_1H_2\Gamma - \Gamma H_2H_1 + H_2H_1\Gamma && (7.43)\\
&= \Gamma H_1H_2 - H_2\Gamma H_1 - H_1\Gamma H_2 + H_2H_1\Gamma + \ldots && (7.44)\\
&- \Gamma H_2H_1 + H_1\Gamma H_2 + H_2\Gamma H_1 - H_1H_2\Gamma && (7.45)\\
&= [\Gamma H_1, H_2] - [H_1\Gamma, H_2] + \ldots && (7.46)\\
&- [\Gamma H_2, H_1] + [H_2\Gamma, H_1] && (7.47)\\
&= [[\Gamma, H_1], H_2] - [[\Gamma, H_2], H_1] && (7.48)\\
&= [\Gamma(\tau_1)\Gamma, H_2] - [\Gamma(\tau_2)\Gamma, H_1] && (7.49)\\
&= \Gamma(\tau_1)\Gamma(H_2) - H_2(\Gamma(\tau_1)\Gamma) + \ldots && (7.50)\\
&- \Gamma(\tau_2)\Gamma(H_1) + H_1(\Gamma(\tau_2)\Gamma) && (7.51)\\
&= \Gamma(\tau_1)[\Gamma, H_2] - H_2(\Gamma(\tau_1))\Gamma + \ldots && (7.52)\\
&- \Gamma(\tau_2)[\Gamma, H_1] + H_1(\Gamma(\tau_2))\Gamma && (7.53)\\
&= \Gamma(\tau_1)\Gamma(\tau_2)\Gamma - \Gamma(\tau_2)\Gamma(\tau_1)\Gamma + \ldots && (7.54)\\
&- H_2(\Gamma(\tau_1))\Gamma + H_1(\Gamma(\tau_2))\Gamma && (7.55)\\
&= (H_1(\Gamma(\tau_2)) - H_2(\Gamma(\tau_1)))\Gamma. && (7.56)
\end{aligned}
$$

A similar manipulation is used to get the following

$$
\begin{aligned}
\left[Y^l,[H_1,H_2]\right] &= \left[\left[Y^l, H_1\right], H_2\right] - \left[\left[Y^l, H_2\right], H_1\right] && (7.57)\\
&= \left[\frac{1}{2}\Gamma(\tau_1)Y^l, H_2\right] - \left[\frac{1}{2}\Gamma(\tau_2)Y^l, H_1\right] && (7.58)\\
&= \frac{1}{2}\Gamma(\tau_1)Y^l(H_2) - H_2\left(\frac{1}{2}\Gamma(\tau_1)Y^l\right) + \ldots && (7.59)\\
&- \frac{1}{2}\Gamma(\tau_2)Y^l(H_1) + H_1\left(\frac{1}{2}\Gamma(\tau_2)Y^l\right) && (7.60)\\
&= \frac{1}{2}\Gamma(\tau_1)\left[Y^l, H_2\right] - H_2\left(\frac{1}{2}\Gamma(\tau_1)\right)Y^l + \ldots && (7.61)\\
&- \frac{1}{2}\Gamma(\tau_2)\left[Y^l, H_1\right] + H_1\left(\frac{1}{2}\Gamma(\tau_2)\right)Y^l && (7.62)\\
&= \frac{1}{4}\Gamma(\tau_1)\Gamma(\tau_2)Y^l - \frac{1}{4}\Gamma(\tau_2)\Gamma(\tau_1)Y^l + \ldots && (7.63)\\
&- H_2\left(\frac{1}{2}\Gamma(\tau_1)\right)Y^l + H_1\left(\frac{1}{2}\Gamma(\tau_2)\right)Y^l && (7.64)\\
&= \frac{1}{2}(H_1(\Gamma(\tau_2)) - H_2(\Gamma(\tau_1)))Y^l. && (7.65)
\end{aligned}
$$

The proof for (c) is as follows

$$
\begin{aligned}
& [H_1[H_2,H_3]] + [H_2[H_3,H_1]] + [H_3[H_1,H_2]] = && (7.66)\\
= \; & [H_1, H_2H_3] - [H_1, H_3H_2] + \ldots && (7.67)\\
+ \; & [H_2, H_3H_1] - [H_2, H_1H_3] + \ldots && (7.68)\\
+ \; & [H_3, H_1H_2] - [H_3, H_2H_1] && (7.69)\\
= \; & H_1H_2H_3 - H_2H_3H_1 - H_1H_3H_2 + H_2H_3H_1 + \ldots && (7.70)\\
+ \; & H_2H_3H_1 - H_3H_1H_2 - H_2H_1H_3 + H_1H_3H_2 + \ldots && (7.71)\\
+ \; & H_3H_1H_2 - H_1H_2H_3 - H_3H_2H_1 + H_2H_1H_3 && (7.72)\\
= \; & 0 && (7.73)
\end{aligned}
$$

Thus $\mathcal{L}$ is a complex Lie algebra.

7.3.1 Conserved Quantities for First Order SODEs

From the previous work we have

$$[\Gamma, H] = \Gamma(\tau)\Gamma, \tag{7.74}$$

$$H(f_j) = \Gamma(\xi_j) - \Gamma(\tau) f_j, \tag{7.75}$$

and

$$\Gamma(\tau) = Constant \tag{7.76}$$

and we also have

$$\left[Y^l, H\right] = \frac{1}{2}\Gamma(\tau)Y^l, \tag{7.77}$$

$$Y^l(\tau) = 0, \tag{7.78}$$

$$Y^l(\Gamma(\tau)) = 0, \tag{7.79}$$

$$\left[\Gamma, Y^l\right](\tau) = 0 \tag{7.80}$$

for first order SODEs. We propose that for first order SODEs,

$$I = \sum_{j=1}^{N} \xi_j + \Gamma(\tau) + H(\phi) \tag{7.81}$$

is a conserved quantity, where ϕ (not yet specified) is at least twice continuous with respect to spacial and temporal variables. This implies the following

$$\begin{aligned} \Gamma(I) &= \Gamma(\xi_j) + \Gamma(H(\phi)) + \Gamma(\Gamma(\tau)) \\ &= \sum_{j=1}^{N} (\Gamma(\tau) + H) f_j + \Gamma(H\phi) \end{aligned}$$

which we get by using relation (7.75) and (7.76) for first order SODEs. Using (7.74) gives

$$\Gamma(I) = \sum_{j=1}^{N} (\Gamma(\tau) + H) f_j + \Gamma(\tau)\Gamma(\phi) + H\Gamma(\phi),$$

which simply means that

$$(H + \Gamma(\tau)) \left(\sum_{j=1}^{N} f_j + \Gamma(\phi) \right) = 0. \tag{7.82}$$

The function ϕ is chosen such that

$$\sum_{j=1}^{N} f_j + \Gamma(\phi) = 0. \tag{7.83}$$

Next we have to show that $Y^l I$ is zero. We have

$$\begin{aligned} Y^l I &= Y^l \left(\sum_{j=1}^{N} \xi_j \right) + Y^l (H\phi) + Y^l (\Gamma(\tau)) \\ &= \sum_{j=1}^{N} \left(\frac{1}{2}\Gamma(\tau) + H \right) G_j^l + \frac{1}{2}\Gamma(\tau)Y^l(\phi) + HY^l(\phi) \\ &= \left(H + \frac{1}{2}\Gamma(\tau) \right) \left(\sum_{j=1}^{N} G_j^l + Y^l(\phi) \right). \end{aligned}$$

This is because we proved that $Y^l(\Gamma(\tau)) = 0$. The main calculations above were arrived at in a similar manner to what we did before only now using (7.77). In summary, this forces ϕ to be chosen such that

$$\sum_{j=1}^{N} f_j + \Gamma(\phi) = 0, \tag{7.84}$$

and

$$\sum_{j=1}^{N} G_j^l + Y^l(\phi) = 0. \tag{7.85}$$

7.3.2 Conserved Quantities for nth-order SODEs

By defining an $\mathcal{I} \equiv \left\{ I\left(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}_{(\beta)}\right) \middle| \, dI = 0, \text{ wherever (5.94) and (5.95) are satisfied} \right\}$ we extend the methodology to nth-order SODEs. For nth-order SODEs, the temporal and spatial infinitesimals can be either point or generalized Lie symmetries. The drift and diffusion coefficients, $\mathbf{f}$ and $\mathbf{G}$, are respectively functions of at most the highest order spatial process $\mathbf{X^{(n)}}(t,\omega)$. We have

$$\begin{aligned}
\left[\Gamma, H^{[n]}\right] &= \Gamma(\tau)\Gamma, && (7.86)\\
\left[Y^l, H^{[n]}\right] &= \frac{1}{2}\Gamma(\tau)Y^l, && (7.87)\\
\Gamma(\xi_j^{[r]}) &= \xi_j^{[r+1]} + \Gamma(\tau)x_j^{(r+1)}, \; r = 0, \ldots, n-1, && (7.88)\\
\Gamma(\xi_j^{[n]}) &= H^{[n]}(f_j) + \Gamma(\tau)f_j, && (7.89)\\
Y^l(\xi_i^{[n]}) &= \left(H^{[n]} + \frac{1}{2}\Gamma(\tau)\right) G_j^i, && (7.90)\\
2\, G_l^i Y^l(\xi_j^{[n]}) &= H(G_k^i G_j^k) + \Gamma(\tau) G_k^i G_j^k, && (7.91)\\
\Gamma(\tau) &= \mathit{Constant} && (7.92)\\
Y^l(\tau) &= 0, && (7.93)\\
Y^l(\xi_j^{[k]}) &= 0, \text{ where } k < n && (7.94)\\
Y^l(\Gamma(\tau)) &= 0, && (7.95)\\
\left[\Gamma, Y^l\right](\tau) &= 0. && (7.96)
\end{aligned}$$

We construct the conserved quantity as

$$I \;=\; \sum_{j=1}^{N}\sum_{r=0}^{n} \xi^{[r]}{}_j + \Gamma(\tau) + H(\phi). \tag{7.97}$$

Again, ϕ will be defined later. Applying Γ on I we obtain

$$\begin{aligned}
\Gamma(I) &= \Gamma\left(\sum_{j=1}^{N}\sum_{r=0}^{n} \xi^{[r]}{}_j\right) + \Gamma\left(H(\phi)\right) + \Gamma(\Gamma(\tau))\\
&= \sum_{j=1}^{N}\left(\Gamma(\tau) + H^{[n]}\right) f_j + \sum_{j=1}^{N}\sum_{r=0}^{n-1}\left(\Gamma(\tau) + H^{[r+1]}\right) x_j^{(r+1)} + \Gamma(\tau)\Gamma(\phi) + H(\Gamma(\phi)),
\end{aligned}$$

which we get by using (7.89), (7.88), (7.86) and (7.92). This simplifies as

$$\Gamma(I) \;=\; \left(\Gamma(\tau) + H^{[n]}\right)\left(\sum_{j=1}^{N} f_j + \sum_{j=1}^{N}\sum_{r=0}^{n-1} x_j^{(r+1)} + \Gamma(\phi)\right)$$

Thus we have to choose ϕ such that

$$\sum_{j=1}^{N} f_j + \sum_{j=1}^{N}\sum_{r=0}^{n-1} x_j^{(r+1)} + \Gamma(\phi) = 0.$$

Next we have to show that $Y^l I$ is zero:

$$\begin{aligned}
Y^l I &= Y^l\left(\sum_{r=0}^{n} \xi^{[r]}{}_j + \Gamma(\tau) + (H\phi)\right) && (7.98)\\
&= \sum_{j=1}^{N}\left(\frac{1}{2}\Gamma(\tau) + H^{[n]}\right) G_j^l + \frac{1}{2}\Gamma(\tau)Y^l(\phi) + H^{[n]}(Y^l(\phi)), && (7.99)
\end{aligned}$$

which we get by implementing (7.90), (7.94), (7.87) and (7.92). In summary, this forces ϕ to be chosen such that

$$\sum_{j=1}^{N} f_j + \sum_{j=1}^{N}\sum_{r=0}^{n-1} x_j^{(r+1)} + \Gamma(\phi) = 0, \tag{7.100}$$

and

$$\sum_{j=1}^{N} G_j^l + Y^l(\phi) = 0. \tag{7.101}$$

7.3.3 Conserved Quantities based on the FP equation

Although we are limited to only projectable symmetries under the FP equation context, we can still derive some interesting results. By considering only the projectable symmetries of the associated SODEs, we showed that the FP determining equations simplify to

$$\Gamma\bigg(\alpha_2(t,\,\mathbf{x}) + \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}\bigg) = 0 \tag{7.102}$$

and

$$Y^l\bigg(\alpha_2(t,\,\mathbf{x}) + \sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}\bigg) = 0. \tag{7.103}$$

Focusing only on (7.102), we expand in the following way

$$\begin{aligned}
\Gamma(\alpha_2) &= -\Gamma(\sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}) &(7.104)\\
&= \sum_{j=1}^{N}\bigg[-\frac{\partial}{\partial x_j}\bigg(\Gamma(\xi_j)\bigg) + \frac{\partial f_k}{\partial x_j}\bigg(\frac{\partial \xi_j}{\partial x_k}\bigg) - \frac{\partial A_{rs}}{\partial x_j}\bigg(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\bigg)\bigg] &(7.105)\\
&= \sum_{j=1}^{N}\bigg[-\xi_i \frac{\partial^2 f_j}{\partial x_i\, x_j} - \frac{\partial \xi_i}{\partial x_j}\frac{\partial f_j}{\partial x_i} + \frac{\partial f_k}{\partial x_j}\bigg(\frac{\partial \xi_j}{\partial x_k}\bigg) - \frac{\partial A_{rs}}{\partial x_j}\bigg(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\bigg)\bigg] &(7.106)
\end{aligned}$$

which we get by using (7.12) and the fact that the temporal infinitesimal is projectable, i.e. a function of time only. Thus we have the following relation

$$\Gamma(\alpha_2) = \sum_{j=1}^{N}\bigg[-\xi_i \frac{\partial^2 f_j}{\partial x_i\, x_j} - \frac{\partial A_{rs}}{\partial x_j}\bigg(\frac{\partial^2 \xi_j}{\partial x_r\, x_s}\bigg)\bigg]. \tag{7.107}$$

In a similar fashion we focus on (7.103) which we modify as follows

$$\begin{aligned}
Y^l\bigg(\alpha_2(t,\,\mathbf{x})\bigg) &= Y^l\bigg(\sum_{j=1}^{N} \frac{\partial \xi_j}{\partial x_j}\bigg) &(7.108)\\
&= \sum_{j=1}^{N}\bigg[-\frac{\partial}{\partial x_j}\bigg(Y^l(\xi_j)\bigg) + \frac{\partial G_k^l}{\partial x_j}\bigg(\frac{\partial \xi_j}{\partial x_k}\bigg) &(7.109)\\
&= \sum_{j=1}^{N}\bigg[-\xi_i \frac{\partial^2 G_j^l}{\partial x_i\, x_j} - \frac{\partial \xi_i}{\partial x_j}\frac{\partial G_j^l}{\partial x_i} + \frac{\partial G_k^l}{\partial x_j}\bigg(\frac{\partial \xi_j}{\partial x_k}\bigg)\bigg] &(7.110)
\end{aligned}$$

which we get by using (7.13) and the fact that the temporal infinitesimal is projectable. Thus we have the following

$$Y^l\Big(\alpha_2(t,\,\mathbf{x})\Big)=\sum_{j=1}^{N}\left[-\xi_i\frac{\partial^2 G_j^l}{\partial x_i\, x_j}\right]. \tag{7.111}$$

If we can find an $\alpha_2(t,\,\mathbf{x})$ such that (7.107) and (7.111) are satisfied, then we can use the projectable symmetries of the SODEs to generate conserved quantities. Thus it is also possible to generate the conserved quantities from the determining equations of the associated Fokker-Plank equations, but only for the case where $\tau(t)$, $\xi(t,\mathbf{x})$ and $\Phi(t,\mathbf{x},u)$, which is what [1] used as an *ansatz* for both the SODEs and the FP equations.

7.4 Example

In Ünal [3] it was stated that the temporal infinitesimal of the form

$$H=\tau\frac{\partial}{\partial t} \tag{7.112}$$

was a conserved quantity. This is not true. By analyzing the resulting determining equations we have that

$$\Gamma(\tau)f_j+\tau\frac{\partial f_j}{\partial t}=0 \tag{7.113}$$

and

$$\frac{1}{2}\Gamma(\tau)G_j^l+\tau\frac{\partial G_j^l}{\partial t}=0. \tag{7.114}$$

In the concluding example in [3] we have the following SODEs

$$d\mathbf{X}(t)=\mathbf{f}\;dt+\mathbf{G}\;dW(t), \tag{7.115}$$

where $\mathbf{f}$ is the vector

$$\begin{pmatrix} -\frac{1}{2}X_1(t) \\ -\frac{1}{2}X_2(t) \end{pmatrix} \tag{7.116}$$

and $\mathbf{G}$ the vector

$$\begin{pmatrix} -X_2(t) \\ X_1(t) \end{pmatrix}. \tag{7.117}$$

In chapter 2 we found the following symmetries infinitesimals

$$\tau(t,\mathbf{X}(t))=C_0F_0\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right), \tag{7.118}$$

$$\xi_1(t,\mathbf{X}(t))=C_1F_1\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)X(t)_1+C_2F_2\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)X(t)_2 \tag{7.119}$$

and

$$\xi_2(t,\mathbf{X}(t))=C_1F_1\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)X(t)_2-C_2F_2\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)X(t)_1 \tag{7.120}$$

Since the temporal infinitesimal is not projectable, it does not belong to sub-algebra of the FP equation. Thus only the spatial infinitesimals are symmetries of the FP equation itself. The temporal infinitesimal in this case, is a conserved quantity because the drift and diffusion coefficients are not functions of time and the instantaneous drift of the temporal infinitesimal is zero, i.e. $\Gamma(\tau)=0$; thus the equations (7.113) and (7.114) are satisfied. We now construct conserved quantities using both our alternate method and the FP associated method above.

7.4.1 Alternative Method

Considering equation (7.85), we have

$$Y(\phi) = x_2 - x_1 \tag{7.121}$$

which implies that

$$-x_2\frac{\partial\phi}{\partial x_1} + x_1\frac{\partial\phi}{\partial x_2} = -x_1 + x_2, \tag{7.122}$$

which easily solves as

$$\phi = F_3\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)F_4(t) - (X_1(t) + X_2(t)). \tag{7.123}$$

Invoking relation (7.84) gives

$$F_3\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right)\dot{F}_4(t) = 0 \tag{7.124}$$

since $\Gamma(F_3\left(\frac{X(t)_2^2+X(t)_1^2}{2}\right)) = 0$. This forces the following simplification

$$\phi = F_3\left(\frac{X(t)_2^2 + X(t)_1^2}{2}\right) - (X_1(t) + X_2(t)). \tag{7.125}$$

The conserved quantity is constructed by utilizing the non-projectable temporal infinitesimal in this instance. By invoking the projectable symmetries only, we now implement the FP associated conserved quantity construction.

7.4.2 FP associated conserved quantity construction

The conserved quantity is of the form

$$I = \alpha_2 + \sum_{j=1}^{2}\frac{\partial\xi_j}{\partial x_j}. \tag{7.126}$$

Equation (7.111) becomes

$$Y(\alpha_2) = 0, \tag{7.127}$$

since $\mathbf{G}$ has linear components. Thus we have

$$\alpha_2 = F_5\left(u\right)F_6(t), \tag{7.128}$$

where

$$u = \frac{X(t)_2^2 + X(t)_1^2}{2}. \tag{7.129}$$

By bringing equation (7.107) into consideration we have

$$\Gamma(\alpha_2) = -\frac{\partial A_{22}}{\partial x_1}\frac{\partial^2 \xi_1}{\partial^2 x_2} - \frac{\partial A_{11}}{\partial x_2}\frac{\partial^2 \xi_2}{\partial^2 x_1} \tag{7.130}$$

where

$$-\frac{\partial A_{11}}{\partial x_2}\frac{\partial^2 \xi_2}{\partial^2 x_1} = -2\,x_1\,\left(C_1\,F_1''(u)\,x_1\,x_2^2 + C_1\,F_1'(u)\,x_1 + C_2\,F_2''(u)\,x_2^3 + 2C_2\,F_2'(u)\,x_2 + C_2\,F_2'(u)\,x_2\right), \tag{7.131}$$

and

$$-\frac{\partial A_{22}}{\partial x_1}\frac{\partial^2 \xi_1}{\partial^2 x_2} = -2\,x_2\,\left(C_1\,F_1''(u)\,x_2\,x_1^2 + C_1\,F_1'(u)\,x_2 - C_2\,F_2''(u)\,x_1^3 - 2C_2\,F_2'(u)\,x_1 - C_2\,F_2'(u)\,x_1\right). \tag{7.132}$$

Comparing coefficients of various combinations of the spatial variables which are independent of u, we get

$$F_1''(u) = 0 \tag{7.133}$$

which implies

$$F_1(u) = \frac{C_1\,u^2}{2} + C_2\,u + C_3 \tag{7.134}$$

and

$$F_2''(u) = 0 \tag{7.135}$$

which results in a similar looking quadratic

$$F_2(u) = \frac{C_4\,u^2}{2} + C_5\,u + C_6. \tag{7.136}$$

Thus we ultimately have

$$F_5(u)\dot{F}_6(t) = -2C_1\left(C_2\,u^2 + C_3\,u\right), \tag{7.137}$$

which we can solve as

$$F_5(u) = C_2\,u^2 + C_3\,u \tag{7.138}$$

and

$$F_6(t) = -2\,C_1\,t + C_8. \tag{7.139}$$

Eventually we can write our unknown variable α_2 as

$$\alpha_2 = (C_8 - 2\,C_1\,t)\left(C_2\,u^2 + C_3\,u\right), \tag{7.140}$$

which implies that our conserved quantity is

$$I = (C_8 - 2\,C_1\,t)\left(C_2\,u^2 + C_3\,u\right) + 2\,C_1\left(C_2\,u^2 + C_3\,u\right) + 2\,C_2\,F_2(u), \tag{7.141}$$

since

$$\frac{\partial \xi_1}{\partial x_1} = C_1\,F_1'(u)\,x_1^2 + C_2\,F_2(u) + C_2\,F_2'(u)\,x_1\,x_2 \tag{7.142}$$

and

$$\frac{\partial \xi_1}{\partial x_1} = C_1\,F_1'(u)\,x_2^2 + C_2\,F_2(u) - C_2\,F_2'(u)\,x_1\,x_2. \tag{7.143}$$

Remark. The two methods yield two unrelated conserved quantities. Neither of the two have been found in the past. It is also interesting to note that the last method further dictates the form of the arbitrary functions F_1 and F_2, which generate the two spatial infinitesimals.

7.5 Conclusion

We have reconciled the conserved quantity analysis of Ünal [3] with the latest findings concerning the symmetries of both the FP equations and its associated SODEs. We have shown two ways of constructing conserved quantities: one based on the projectable symmetries of the SODEs and thus a sub-algebra of the FP equation and the other method takes advantage of both the projectable and non-projectable symmetries of the SODEs alone. Both methods preclude the necessity for a Hamiltonian or Lagrangian framework.

Chapter 8

Conclusions

We have proved that the symmetry infinitesimals for the FP equation has to be projectable by using the work of Mahomed and Momoniat [11]. We have also shown that the symmetry transformations for the SODE need not be projective as those for the FP equation.

This body of work has successfully reconciled the work of Wafo Soh and Mahomed [2] and Ünal [3]. This means that both works agree on the determining equations needed to furnish the spatial and temporal infinitesimals, i.e.

$$\Gamma(\xi_j) = \Big(H + \Gamma(\tau)\Big) f_j \; j \,=\, 1,\, N, \tag{8.1}$$

$$Y^l(\xi_j) = \Big(H + \frac{1}{2}\Gamma(\tau)\Big) G^l_j \; l \,=\, 1,\, M \tag{8.2}$$

and

$$Y^l(\tau) = 0, \tag{8.3}$$

where

$$\Gamma = \frac{\partial}{\partial t} + f_j \frac{\partial}{\partial x_j} + \sum_{k=1}^{M} G^k_i\, G^k_j \frac{\partial}{\partial x_i \partial x_j} \tag{8.4}$$

and

$$Y^l = G^l_j \frac{\partial}{\partial x_j}, \tag{8.5}$$

where N is the dimension of the spatial process and M is the dimension of the Wiener process.

A condition that allows the Lie transformation theory to be used in an Itô SODE context was also found

$$\Gamma(\tau) = Constant. \tag{8.6}$$

It ensured that the finite transformations are recoverable from the infinitesimal ones to preserve invariance.

With the works of Wafo Soh and Mahomed [2], Ünal [3] and Meleshko et al. [4] being reconciled, we extended the Lie transformation theory to the Wiener process as well. This extension was based on the work of Gaeta [7]. However, unlike the previous work of Gaeta and Quintero [1] and Gaeta [7], we preclude the condition that the symmetry infinitesimals need to be projective. This gave rise to larger dimension algebras. The determining equations needed to produce our infinitesimals were

$$\Gamma(\xi_j) = \left(H + \Gamma(\tau)\right) f_j \; j \; = \; 1, \, N, \tag{8.7}$$

$$Y^l(\xi_j) = \left(H + \frac{1}{2}\Gamma(\tau)\right) G_j^l \; l \; = \; 1, \, M, \tag{8.8}$$

$$\Gamma(\gamma_l) = 0, \tag{8.9}$$

$$Y^l(\gamma_m) + Y^m(\gamma_l) = \delta_m^l \Gamma(\tau) \tag{8.10}$$

and

$$Y^l(\tau) = 0, \tag{8.11}$$

where

$$\Gamma = \frac{\partial}{\partial t} + f_j \frac{\partial}{\partial x_j} + \sum_{k=1}^{M} G_i^k \, G_j^k \frac{\partial}{\partial x_i \partial x_j} + \sum_{k=1}^{M} \delta_i^k \, \delta_j^k \frac{\partial}{\partial w_i \partial w_j} \tag{8.12}$$

and

$$Y^l = G_j^l \frac{\partial}{\partial x_j} + \delta_j^l \frac{\partial}{\partial w_j}. \tag{8.13}$$

This was easily extended to higher order SDEs

$$\Gamma(\xi_j^{[n-1]}) = \left(H + \Gamma(\tau)\right) f_j \; j \; = \; 1, \, N, \tag{8.14}$$

$$Y^l(\xi_j^{[n-1]}) = H\left(G_j^l\right) + \sum_{m=1}^{M} G_j^m \, Y^l\left(\gamma_m\right) \;\; l \; = \; 1, \, M, \tag{8.15}$$

$$\Gamma(\gamma_l) = 0, \tag{8.16}$$

$$Y^l(\gamma_m) + Y^m(\gamma_l) = \delta_m^l \Gamma(\tau), \tag{8.17}$$

$$Y^l(\tau) = 0, \tag{8.18}$$

$$\Gamma(\xi_j^{[r]}) = \left(H + \Gamma(\tau)\right) x_j^{(r+1)} \; j \; = \; 1, \, N, \tag{8.19}$$

and

$$Y^l(\xi_j^{[r]}) = 0, \; r \leq n-2 \tag{8.20}$$

where

$$\Gamma = \frac{\partial}{\partial t} + \sum_{i=1}^{N} f_i \frac{\partial}{\partial x_i^{(n-1)}} + \frac{1}{2} \sum_{i,p=1}^{N} \sum_{k\,=\,1}^{M} G_i^k G_p^k \frac{\partial^2}{\partial x_i^{(n-1)} \partial x_p^{(n-1)}} + \sum_{p=1}^{N} \sum_{\alpha=0}^{n-2} \frac{\partial}{\partial x_p^{(\alpha)}} x_p^{(\alpha+1)} + \frac{1}{2} \sum_{l=1}^{M} \frac{\partial^2}{\partial w_l^2} \tag{8.21}$$

and

$$Y^l = \frac{\partial}{\partial w_l} + \sum_{i=1}^{N} G_i^l \frac{\partial^{[\gamma]}}{\partial x_i^{(n-1)}}. \tag{8.22}$$

We then applied these results to approximate SODEs and to the construction of conserved quantities. The result of these applications gave rise to new methods of constructing conserved quantities and we used an example from Ünal [3] to demonstrate this.

Extending these results to numerical Monte Carlo integration techniques remains an open problem. This could provide us with a method of implementing variance reduction schemes (see Kloeden and Platen [20]).

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