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Supervisor: Prof. Robert de Mello Koch

DYNAMICS OF GIANT GRAVITONS FROM YOUNG DIAGRAMS

David Bekker

<David.Bekker@students.wits.ac.za>

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this dissertation we study the dynamics of excited giant gravitons. Giant gravitons are spherical membranes with a D3-brane dipole charge. Giant gravitons are excited by attaching open strings to them. We develop techniques to compute the one-loop anomalous dimensions of operators in the $\mathcal{N} = 4$ super Yang-Mills theory which are dual to open strings ending on boundstates of sphere giant gravitons. The results presented in this dissertation are applicable to excitations involving an arbitrary number of strings. We consider open strings which carry angular momentum on an S^3 embedded in the S^5 of the $AdS_5 \times S^5$ background. The problem of computing the one-loop anomalous dimensions is replaced by the problem of diagonalizing an interacting Cuntz oscillator Hamiltonian. We provide evidence that our Cuntz oscillator dynamics show how Chan-Paton factors emerge dynamically.

Acknowledgements

I would like to thank my parents, Dave and Jessica, for the financial and emotional support they have provided throughout my studies. I would also like to thank my supervisor, Robert, for the huge amount of time that he has invested in my education, and for the copious amounts of coffee that he provided, a *necessity* for the proper functioning of a physicist.

Contents

List of Figures

1 Introduction

The gauge theory/gravity correspondence[1],[2],[3] has given us intriguing insights into the quantum gravity problem. As we shall see in section 2.1, the claim is made that a quantum string field theory on Anti-deSitter (AdS) space with certain boundary conditions is equivalent to a Quantum Field Theory (QFT) defined on a brane. Thus, we should be able to use a $\mathcal{N}=4$ super Yang-Mills theory as a definition of quantum gravity on $\mathrm{AdS}_5\times\mathrm{S}^5$. There has been interesting progress in this direction recently, for more details see [4],[5]. However, the correspondence is not yet understood well enough for this to be possible. A detailed understanding of the gauge theory/gravity correspondence is frustrated by the fact that it is a weak/strong coupling duality in the 't Hooft coupling. At weak 't Hooft coupling the field theory may be treated perturbatively, but the spacetime of the dual quantum gravity is highly curved. In the opposite limit of strong 't Hooft coupling we have to face the difficult problem of strongly coupled quantum field theory. The dual quantum gravity however, simplifies, because in this limit the curvature of the spacetime is small. For this reason, most computations which can be carried out on both sides of the correspondence (and hence clearly shed light on the correspondence) compute quantities that are protected by symmetry - typically supersymmetry (see [6] and references therein). The number of these tests and the agreement found is impressive. However, computing and comparing protected quantities is not satisfying - to probe dynamical features of the correspondence it would be nice to be able to compare quantities that are not protected by any symmetries. This is in general, a formidable problem. In [7], the notion of an *almost BPS state* was introduced. These states are systematically small deformations of states that are protected. For this reason, for almost BPS states, it is possible to reliably extrapolate from weak to strong coupling. A good example of almost BPS states are the BMN loops[8]. By studying BMN loops it has been possible to probe truly stringy aspects of the gauge theory/gravity correspondence (see [9] and references therein).

In this dissertation, we will be examining giant gravitons. Giant gravitons are half-BPS states, and are the source of many quantities that are accessible on both sides of the correspondence. In addition, they are examples of protected non perturbative objects, which make them interesting from a string theory perspective. A giant graviton is a spherical D3 brane extended in the sphere[10] or in the AdS space [11],[12],[13] of the AdS×S background. They are (classically) stable due to the presence of the five form flux which produces a force that exactly balances their tension. The force of the five form flux is a velocity dependent (similar to the Lorentz force law) force. Thus, the size of the giant graviton is determined by it's angular momentum. The dual description of giant gravitons is in terms of Schur polynomials in the Higgs fields[14],[15], which we will review in section 2.2.

Our interest in giant gravitons is due to the fact that excited giant gravitons provide a rich source of nearly BPS states. We can excite giant gravitons by attaching open strings to them. The gauge theory operator dual to an excited sphere giant is known and the anomalous dimension of this operator reproduces the expected open string spectrum[16][1]. This has been extended and the operators dual to an arbitrary system of excited giant gravitons is now known[20]. The dual operators, restricted Schur polynomials, beautifully reproduce the restrictions imposed on excitations of the brane system by the Gauss law. Further, these excited giant gravitons have recently been

[1]See [17],[18],[19] for further studies of non-BPS excitations that have been interpreted as open strings attached to giant gravitons.

identified as the microstates of near-extremal black holes in $AdS_5 \times S^5$[21]. Although the evidence for identifying the restricted Schur polynomials as the operators dual to excited giant gravitons is convincing, much remains to be done. For example, we do not yet understand the detailed mechanism allowing Chan-Paton factors, expected for strings attached to a bound state of giant gravitons, to emerge from the super Yang-Mills theory. In this dissertation, our goal is to explore this issue, by providing techniques which allow the computation of the anomalous dimensions of excited giant gravitons, to one loop. We will argue that the Chan-Paton factors emerge from the symmetric group labels of the restricted Schur polynomials.

The computation of anomalous dimensions of operators in $\mathcal{N} = 4$ super Yang-Mills theory has progressed considerably. Much of the recent progress was sparked by a remarkable paper of Minahan and Zarembo[22] which shows that the spectrum of one loop anomalous dimensions of operators dual to closed string states, in a sub sector of the theory, gives rise to an integrable $SO(6)$ spin chain. This result can be generalized to include the full set of local operators of the theory[23]. The integrable spin chain model describing the full planar one loop spectrum of anomalous dimensions can be solved by Bethe-Ansatz techniques[23]. Clearly, it is desirable to find a similar approach for operators dual to open strings. A naive generalization is frustrated by the fact that, since the open string and giant can exchange momentum, the number of sites of the open string lattice becomes a dynamical variable[2]. This was circumvented in [25] by introducing a Cuntz oscillator chain. Restricting to the $SU(2)$ sector, the spin chain is obtained by mapping one of the matrices, say Z, into a spin up and the other, say Y, into a spin down. In contrast to this, the Cuntz chain uses the Ys to set

[2] An exception to this is an open string attached to a maximal giant graviton[24].

up a lattice which is populated by the Zs. Thus the number of sites in the Cuntz chain is fixed.

The power of the spin chain goes beyond the computation of anomalous dimensions. Indeed, the low energy description of the spin chain relevant for closed string states appearing on the field theory side matches perfectly with the low energy limit of the string action in $\mathrm{AdS}_5\times\mathrm{S}^5$[26]. This is an important result because it shows how a string action can emerge from large N gauge theory. For the open string, the coherent state expectation value of the Cuntz chain Hamiltonian reproduces the open string action for an open string attached to a sphere giant in $\mathrm{AdS}_5\times\mathrm{S}^5$[25],[17], for an open string attached to an AdS giant in $\mathrm{AdS}_5\times\mathrm{S}^5$[27] and for an open string attached to a sphere giant in a deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ background[28]. Recently[29], the worldsheet theory of an open string attached to a maximal giant has been studied. Evidence that the system is integrable at two loops has been obtained.

The fact that the open string can exchange momentum with the giant is reflected in the fact that there are sources and sinks (at the endpoints of the string) for the particles on the chain. The structure of these boundary interactions is complicated: since the brane can exchange momentum with the string, the brane will in general be deformed by these boundary interactions. The goal of this dissertation is to determine this Cuntz chain Hamiltonian for multiple strings attached to an arbitrary system of giant gravitons. In particular, this entails accounting for back reaction on the giant graviton. To compute the Cuntz chain Hamiltonian, we need the two point functions of restricted Schur polynomials. It is an involved combinatoric task to compute the two point functions of restricted Schur polynomials. The required technology to compute these correlators, in the free field limit, has recently been developed in [30][3]. This was then extended to one loop, for operators dual to giants with a single string attached[32]. In this dissertation, we extend the existing technology, allowing the one loop computation of correlators dual to giant graviton systems with an arbitrary number of strings attached.

The construction of the operators dual to excitations described by strings stretching between the branes requires the construction of an "intertwiner" [30]. One of the results of this dissertation, is to provide a general construction of the intertwiner. This construction is given in section 4.

The dissertation is organized as follows.

In section 2 we review the AdS/CFT correspondence and Schur polynomials. In section 2.1, we review the history behind the holographic principle, and examine the concrete realisation of this principle, the AdS/CFT correspondence. In section 2.2, we examine the operators dual to giant gravitons, the Schur polynomials. We also introduce the graphical notation used throughout this dissertation.

In section 3 we define the restricted character. In section 3.1 we examine how the restricted character is calculated. In section 3.2, we introduce a graphical notation known as "strand diagrams", which is designed to simplify the computation of restricted characters. In section 3.3, we give examples illustrating how strand diagrams are used. In section 3.4 we illustrate how we can use restricted characters to calculate characters of S_n, thus providing an important check of our restricted character formula. In section 3.5, we use strand diagrams to write down the irreducible matrix representations of S_n.

In section 4 we introduce the operators dual to states with open strings

[3]For some earlier related work, see[31].

stretching between giant gravitons. In section 4.1, we give the definition of an "intertwiner" for a system of two branes, with strings stretching between them. In section 4.2, we generalize the result given in section 4.1. In section 4.3, we consider a system of three branes, with three strings stretched between them.

In section 5 we derive identities that can be used to obtain the Cuntz chain Hamiltonian that accounts for the $O(g_{YM}^2)$ correction to the anomalous dimension of our operators. In section 5.1, we discuss the derivation of a hopping identity. In section 5.2, we examine the identities which are relevant to hopping off the first site of the string, as there is a difference if the first site or the last site of the string participates in the hopping. In section 5.3, we examine the identities which are relevant to hopping off the last site of the string. In section 5.4, we discuss the extensive numerical tests that were performed in order to test our identities. In section 5.5, we rewrite our operators in terms of Cuntz chain states by using the state-operator correspondence.

In section 6 we simply quote the six normalization factors that enter the relation between the restricted Schur polynomials and the normalized Cuntz chain states for the excited two giant graviton bound state.

In section 7 we present the main technical result of the dissertation.

In section 8 we describe a new "physical basis". The Hamiltonian we derive in section 7 treats string 1 and string 2 differently, which arises because when we build our operators, we treat the two strings differently. The new physical basis is singled out by requiring the two strings to be treated on an equal footing.

We conclude the main sections with a discussion of the main results.

The appendices provide background for the interested reader. Appendix

A reviews the properties of Young diagrams. Appendix B gives a simple example on how to calculate characters. This example gives useful background to the more general formula given in the main sections. In appendix C we review the subgroup swap rule and in appendix D we show how the subgroup swap rule is used in conjunction with the reduction rule by calculating a two point function. In addition, we show how to determine which terms in the two point function are subleading.

The new results obtained in this dissertation have been posted on the arXiv[48] and have been submitted for publication to the Journal of High Energy Physics.

2 Background

2.1 AdS/CFT Correspondence

The AdS/CFT correspondence is a concrete realisation of the holographic principle. The holographic principle was initially proposed by Gerard 't Hooft in [42] and extended by Leonard Susskind in [46]. Although the vast majority of authors use the term "holographic principle" there are some authors who feel that it has been elevated to the status of 'principle' without sufficient merit, and that it should rather be viewed as a conjecture[45]. One might take this criticism to mean that there isn't sufficient evidence for the holographic principle. As we shall see, there does exist evidence which suggests that we should take this principle very seriously. The elevation from conjecture to principle does require us to assume that the properties of black holes we shall investigate apply generally[4].

A hologram (in the usual sense) is a case where a 2-dimensional surface can give rise to a 3-dimensional image. The holographic principle takes this idea, and applies it to the universe. It says that the number of degrees of freedom available do not scale like the volume of the system (as one would expect) but rather they scale like the surface area[47]. The analogy to a hologram now becomes clear. This seems to be an outrageous claim, how can we be expected to believe that degrees of freedom scale like surface area? To understand *why* this claim is made, we will need to examine black holes.

There are four laws of black hole mechanics

- The Zeroth Law

  A stationary black hole's horizon has a constant surface gravity, κ.

[4]There are heuristic arguments as to why we should take this assumption seriously, but they are beyond the scope of this dissertation.

- The First Law

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ.$$

- The Second Law

  The horizon area does not decrease with time, $\frac{dA}{dt} \geq 0$.

- The Third Law

  It is not possible to form a black hole with vanishing surface gravity.

The zeroth and third laws given above are stated in terms of the surface gravity. The surface gravity of a black hole is defined to be the force required to keep an object of unit mass stationary at the event horizon by an observer at infinity. The zeroth law tells us that this surface gravity is a constant over the entire event horizon. To gain some insight into the first law, we will first examine the second law.

The second law appears to be very similar to the law of thermodynamics which states that entropy cannot decrease[5]. Because of this similarity it seems natural that we identify the area of the horizon with entropy. Once we have made that connection, we can then make the analogy between surface gravity and temperature. To do this, let us now examine the first law (neglecting angular momentum and electric charge)

$$dM = \kappa \frac{dA}{8\pi G} \tag{2.1}$$

and compare it with the corresponding thermodynamics equation

$$dE = TdS. \tag{2.2}$$

It is now clear that surface gravity is playing the role of temperature. With

[5] Also known as the *area theorem*, which was proved by Stephen Hawking.

this insight, we can now understand that the third law is analogous to not being able to reach absolute zero in thermodynamics, and the zeroth law is equivalent to saying that a system in equilibrium will have a uniform temperature.

This analogy is interesting, but is it just an analogy? Clearly not, else we would not be discussing it! Initially, there was skepticism that this could be true due to the "no hair" theorem. This theorem states that all you need to specify the state of a black hole is to specify it's charge, angular momentum, and it's mass[6]. This is where the reader may be skeptical. How can an object that requires so few parameters to know it's state have such large entropy? Progress was made when Stephen Hawking showed that black holes should radiate, thus giving a way to calculate the temperature of a black hole. This temperature read off the predicted black body radiation spectrum, was the same as the 'temperature' in the analogy above. As was mentioned above, we will assume that this property of gravity is general, and therefore entropy is proportional to area, not volume.

This identity of a temperature and an entropy is astounding, and the implications are equally astounding. Indeed, as Jacobson has argued[44], it suggests that the Einstein equations are an equation of state! How would we begin to quantize the wave equation for sound travelling through air? We don't. Similarly, this identity suggests that we shouldn't quantize the Einstein equation. This isn't to say that the phenomenon that the Einstein equation describes is not quantum mechanical, but rather that we cannot recover this underlying quantum mechanical system by canonically quantizing the Einstein equation. We obviously require a new approach.

In [1] the proposal is made that a quantum string field theory on Anti-

[6] That is, in 3+1 dimensions.

deSitter (AdS) space with certain boundary conditions is equivalent to a Quantum Field Theory (QFT) defined on a brane. Here we are seeing the implementation of the holographic principle. What Maldacena has claimed is that a theory of quantum gravity defined in one spacetime is equivalent to another non-gravitational theory defined on another spacetime of lower dimension! Let us examine this in more detail.

The correspondence claims that a maximally supersymmetric $SU(N)$ Yang-Mills theory in 4-dimensional Minkowski spacetime is dual to a type IIB closed superstring theory[43]. The type IIB string theory lives in 10-dimesions, five of those dimensions form a 5-sphere (S^5), while the other five form a noncompact AdS space. This $\mathrm{AdS}_5 \times \mathrm{S}^5$ space has negative curvature, and N units of five form flux. We call the $SU(N)$ Yang-Mills theory "maximally supersymmetric" as it has the most supersymmetry possible for a non-gravitational theory. This field theory has no dimensionful parameters, it is invariant under conformal transformations. We therefore call it a Conformal Field Theory (CFT). The duality claims that we have two different theories describing the *same* physics. We could thus use the $\mathcal{N} = 4$ super Yang-Mills theory as a definition of quantum gravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

$\mathcal{N}$ indicates the number of spinor supercharges that the gauge theory has. $\mathcal{N} = 4$ is the maximal number of supercharges possible for our gauge theory which exists in four dimensions[6]. There are fifteen operators which generate the conformal symmetries. Of these generators, six are the Lorentz generators, four are the spacetime translation generators, another four are the conformal transformations, and the last generator gives rise to the scaling symmetry. These symmetries must appear in our dual description. That is, on the string theory side. We do find that the isometries of the AdS_5 space are generated by fifteen operators, and these operators satisfy the same Lie algebra as our field theory operators. There are also isometries associated with the S^5 space, and these correspond to symmetries in the field theory description due to scalar fields and fermions. This provides further confidence that the correspondence is correct.

We leave the interesting problem of proving the AdS/CFT correspondence as a challenging exercise for the reader.

2.2 Schur Polynomials

In this subsection we will review Schur Polynomials and their labelling, Young diagrams[7]. For a more detailed discussion the interested reader can consult [20; 30; 32].

Schur polynomials in the Higgs field have been proposed to be the operators dual to giant gravitons[14; 15]. The Schur polynomial is defined as follows

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_n}_{i_{\sigma(n)}} \tag{2.3}$$

where R is a Young diagram with n boxes, $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in representation R, and Z is built out of two of the six Higgs in $\mathcal{N} = 4$ super Yang-Mills theory. We can look up the values for $\chi_R(\sigma)$ in character tables for the symmetric group. Alternatively, we will show in section 3 that it is possible to calculate $\chi_R(\sigma)$ using the formula we develop to calculate restricted characters.

There is a link between S_n and $U(N)$ that arises because the symmetric group and the unitary group commute. To get irreps of the unitary group we first guess that they are related to the symmetric group. A natural choice is then to try symmetric and antisymmetric tensors. This is known as the Frobenius-Schur duality, which is a duality between the representation

[7] For a brief summary of Young diagrams and their properties see Appendix A

theory of the symmetric group and the unitary group. The symmetric and antisymmetric tensors are eigenfunctions of the symmetric group.

The Schur polynomial, $\chi_R(U)$, is usually evaluated on the field of unitary matrices, $U \in U(N)$. This gives us the character of $U(N)$ element U in the irrep labelled by the Young diagram R. In this dissertation we will be studying $\chi_R(Z)$, where Z is not an element of the group, but rather $Z \in u(N)$, the algebra. A sphere giant's angular momentum is cut off, so if we can find operators that also cut off angular momentum, then that will be convincing evidence that they are dual to sphere giants. The antisymmetric representation for a Young diagram is indeed cut off. The number of boxes in a Young diagram corresponds to the degree of the Schur polynomial that it labels. Angular momentum is a conserved quantity arising from the symmetry of the five sphere. The symmetry which leaves the five-sphere in the geometry invariant is the SO(6) rotations.

On the field theory side, the SO(6) rotations rotate the six Higgs fields of the theory into each other. This is called the $\mathcal{R}$ symmetry which has corresponding $\mathcal{R}$ charge. Under this $\mathcal{R}$ symmetry the field Z is charged with one unit, therefore n fields correspond to a charge n state. $\mathcal{R}$ charge is mapped into angular momentum, and therefore the Schur polynomial with degree n is dual to a state with angular momentum n.

Thus, each column of the Young diagram is interpreted as a membrane. The number of boxes in a given column is the angular momentum of the corresponding membrane. Since the angular momentum of the giant determines the size of the giant, the longer the column, the bigger the brane.

We excite giant gravitons by attaching open strings. The proposal was made in [20] to insert words $(W^{(a)})_i^j$ describing open strings into the operator describing the system of giant gravitons in order to achieve this. Each word corresponds to one open string.

$$\chi^{(k)}_{R,R_1}(Z,W^{(1)},...,W^{(k)}) = \frac{1}{(n-k)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))\mathrm{Tr}(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}), \tag{2.4}$$

where

$$\mathrm{Tr}(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}) = Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n-k}}_{i_{\sigma(n-k)}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}\cdots(W^{(k)})^{i_n}_{i_{\sigma(n)}}.$$

The representation R of the giant graviton system is a Young diagram with n boxes, i.e. it is a representation of S_n. R_1 is a Young diagram with $n-k$ boxes, i.e. it is a representation of S_{n-k}. $\Gamma_R(\sigma)$ is the matrix representing σ in irreducible representation R of the symmetric group S_n. If we imagine that the k words are all distinct, then this corresponds to a case where all the open strings are distinguishable. We no longer have a nice group theoretical interpretation for this object.

Under a state-operator map, we map states of the string theory to operators of the quantum field theory

$$\langle\phi|\psi\rangle \longleftrightarrow \langle\mathcal{O}^\dagger_\phi\mathcal{O}_\psi\rangle.$$

At large N we have

$$\langle\chi_{R,R_\alpha}\chi^\dagger_{S,S_\beta}\rangle \propto \delta_{RS}\delta_{R_\alpha,S_\beta},$$

so these operators are dual to orthogonal states, as required.

Consider an $S_{n-k}\otimes(S_1)^k$ subgroup of S_n. The representation R of S_n will subduce a (generically) reducible representation of the $S_{n-k}\otimes(S_1)^k$ subgroup. One of the irreducible representations appearing in this subduced representation is R_1. Tr_{R_1} is an instruction to trace only over the indices belonging to this irreducible component. If the representation R_1 appears more than once, things are more interesting. The example discussed in [20] illustrates this point nicely. Suppose $R \to R_1 \oplus R_2 \oplus R_2$ under restricting S_n to $S_{n-2} \times S_1 \times S_1$. Choose a basis so that

$$\Gamma_R(\sigma) = \begin{bmatrix} \Gamma_{R_1}(\sigma)_{i_1 j_1} & 0 & 0 \\ 0 & \Gamma_{R_2}(\sigma)_{i_2 j_2} & 0 \\ 0 & 0 & \Gamma_{R_2}(\sigma)_{i_3 j_3} \end{bmatrix}, \qquad \forall \sigma \in S_{n-2} \times S_1 \times S_1,$$

$$\Gamma_R(\sigma) = \begin{bmatrix} A^{(1,1)}_{i_1 j_1} & A^{(1,2)}_{i_1 j_2} & A^{(1,3)}_{i_1 j_3} \\ A^{(2,1)}_{i_2 j_1} & A^{(2,2)}_{i_2 j_2} & A^{(2,3)}_{i_2 j_3} \\ A^{(3,1)}_{i_3 j_1} & A^{(3,2)}_{i_3 j_2} & A^{(3,3)}_{i_3 j_3} \end{bmatrix}, \qquad \sigma \notin S_{n-2} \times S_1 \times S_1.$$

There are four suitable definitions for $\mathrm{Tr}_{R_2}(\Gamma_R(\sigma))$: $\mathrm{Tr}(A^{(2,2)})$, $\mathrm{Tr}(A^{(2,3)})$, $\mathrm{Tr}(A^{(3,2)})$ or $\mathrm{Tr}(A^{(3,3)})$. Interpret the operator obtained using $\mathrm{Tr}(A^{(2,3)})$ or $\mathrm{Tr}(A^{(3,2)})$ as dual to the system with the open strings stretching between the giants and the operator obtained using $\mathrm{Tr}(A^{(2,2)})$ or $\mathrm{Tr}(A^{(3,3)})$ as dual to the system with one open string on each giant. In general, identify the "on the diagonal" blocks with states in which the two open strings are each on a specific giant and the "off the diagonal" blocks as states in which the open strings stretch between two giants. Since the representation R_2 appears with a multiplicity two, there is no unique way to extract two R_2 representations out of R. Therefore the specific representations obtained will depend on the details of the subgroups used in performing the restriction. These subgroups are the set of elements of the permutation group that leave an index invariant, $\sigma(i) = i$. Choosing the index to be the index of an open string, we can associate the subgroups participating with specific open strings. The subgroups are specified by dropping boxes from R, so that we can now associate boxes in R with specific open strings. This leads to a convenient graphical notation which has been developed in [30; 32].

Graphically, when we wish to attach an open string to a giant graviton we place a number in one of the boxes of a Young diagram. The number can only be placed in a box such that the removal of that box still leaves a valid Young diagram. Consider the example in Figure 1. Two open strings have been attached, one on the large threebrane, one on the smaller threebrane. The numbering of the boxes indicates the order in which the strings would be removed. We can see that by removing the boxes labelled "1" and "2" we are still left with a valid Young diagram. Each string is given a certain orientation. This illustrates an important point. The end points of an open string carry charge[8], and so the way they are attached to the membrane is important. It doesn't make a difference in this example, but we will see in a moment why orientation is important. This figure corresponds to an "on the diagonal" block. A simple example of an on-diagonal block calculation is given in Appendix B.

The world volume of a giant graviton is a compact space, and the Gauss law on a compact space implies that the total charge must sum to zero. This restricts the number of ways in which we can attach strings to membranes. Consider Figure 2. The two strings that we have attached now stretch between the two membranes, the arrow-heads indicate that they have opposite orientation, thus ensuring that the total charge sums to zero. This figure corresponds to an "off the diagonal" block.

Now consider Figure 3. We now have three membranes, with three strings attached. Notice that the number of strings leaving a brane equals

[8] That is, color charge

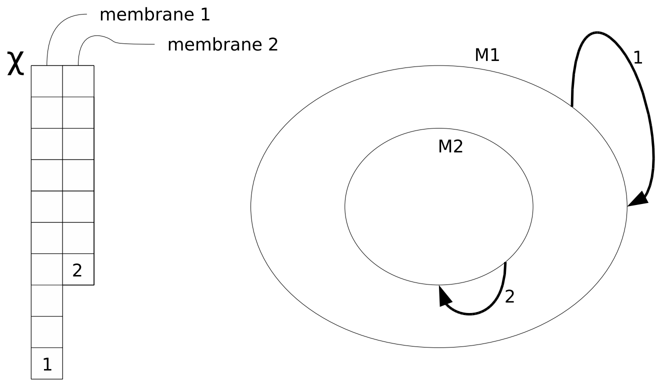


Figure 1: This figure illustrates the correspondence between our operators and giant gravitons. In this example, each membrane has one string attached.

the number of strings ending on a brane. This is the easiest way to check that the Gauss law is not being violated. Let's examine the string labelling in more detail. The labels placed in the Young diagram specify the sequence of irreps employed in subducing R_1. We place a pair of labels into each box, a lower label and an upper label[9]. The representations needed to subduce the row label of R_1 are obtained by starting with R. The second representation is obtained by dropping the box with upper label equal to 1; the third representation is obtained from the second by dropping the box with upper label equal to 2 and so on until the box with label k is dropped. The representations needed to subduce the column label are obtained in exactly the same way except that instead of using the upper label, we now use the lower label.

As an example of the restrictions imposed by the Gauss law, consider the state given by Figure 4. This state is forbidden. The number of strings leaving a brane does not equal the number of strings ending on a brane.

[9]For the example in Figure 1, the upper label and the lower label are the same, and we represent this by only inserting a single number.

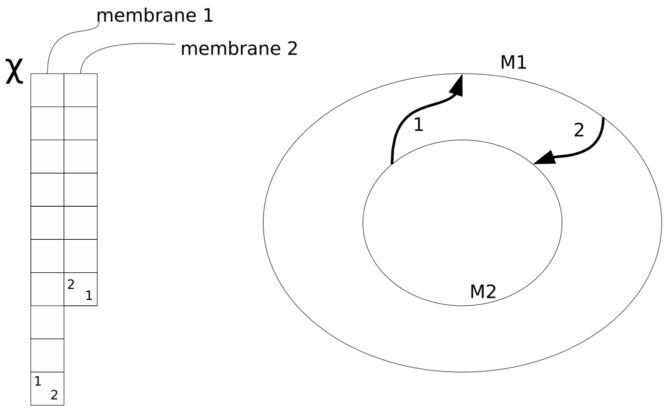


Figure 2: This figure illustrates the correspondence between our operators and giant gravitons. In this example, two strings stretch between the membranes with opposite orientation, in accordance with Gauss' law.

There is no restricted Schur polynomial that can reproduce this state. This shows that the restricted Schur polynomials correctly implement the Gauss law. For further details and explicit examples, we refer the reader to [30].

So far, we have considered the case where the all the strings are distinguishable. If, however, any of the strings are identical, then we need to decompose with respect to a larger subgroup, and pick a representation where the strings are indistinguishable. For example, consider a bound state of a giant system with three identical strings attached, we would consider an $S_{n-3} \otimes S_3$ subgroup of S_n. The restricted Schur polynomial would be given by $\chi^{(3)}_{R,R_1}$ with R an irrep of S_n and R_1 an irrep of $S_{n-3} \otimes S_3$. The S_3 subgroup would act by permuting the indices of the three identical strings; the S_{n-3} subgroup would act by permuting the indices of the Zs out of which the giant is composed. Write $R_1 = r_1 \times r_2$ with r_1 are irrep of S_{n-3} and r_2 an irrep of S_3.

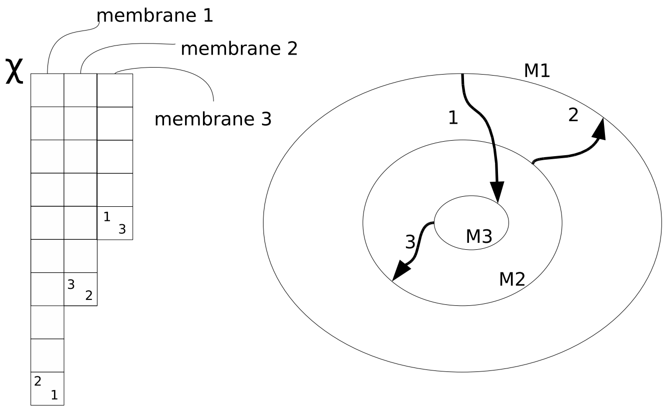


Figure 3: This figure illustrates the correspondence between our operators and giant gravitons. In this example, three strings stretch between three membranes. For each membrane, the number of strings leaving a brane equals the number of strings ending on a brane. This is a general requirement of the Gauss law.

As an example, if we take R to be an irrep of S_9

$$R = \yng(4,4,1) , \qquad \dim_R = 84$$

then we can have

$$R_1 = \yng(3,3) \otimes \yng(1,1,1), \quad \dim_{R_1} = 5, \qquad R_1 = \yng(3,3) \otimes \yng(2,1), \quad \dim_{R_1} = 10,$$

$$R_1 = \yng(4,2) \otimes \yng(3), \quad \dim_{R_1} = 9, \qquad R_1 = \yng(4,2) \otimes \yng(2,1), \quad \dim_{R_1} = 18,$$

$$R_1 = \yng(3,2,1) \otimes \yng(2,1), \qquad \dim_{R_1} = 32,$$

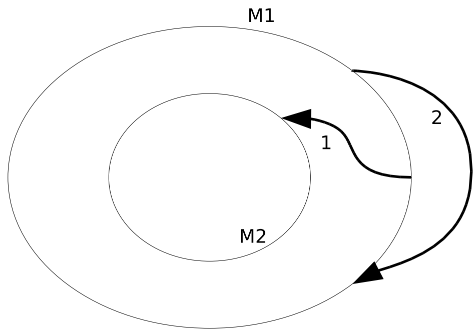


Figure 4: This figure illustrates the restrictions imposed by the Gauss law. This state is forbidden, the number of strings leaving a brane does not equal the number of strings ending on a brane. Consequently, there is no restricted Schur polynomial corresponding to this state.

or

$$R_1 = \yng(4,1,1) \otimes \yng(3), \qquad \dim_{R_1} = 10.$$

By summing the dimensions of these representations, it is easy to see that we have indeed listed all of the representations that are subduced by R.

We call the operator 2.4 a *restricted* Schur polynomial of representation R with representation R_1 for the restriction. This operator is dual to excited giant gravitons. The giant graviton system is dual to an operator containing a product of $O(N)$ fields; the open strings are dual to an operator containing a product of $O(\sqrt{N})$ fields. We will examine the case where:

- The number of strings is $O(1)$, that is, k is $O(1)$.
- The number of boxes in the Young diagram is $O(N)$, that is, n is $O(N)$.

3 Restricted Characters

Starting from S_n, define a chain of subgroups $\mathcal{G}_i$ $i = 1, ..., d$ as follows

$$\mathcal{G}_1 = \{\sigma \in S_n | \sigma(n) = n\} \tag{3.1}$$

$$\mathcal{G}_i = \{\sigma \in \mathcal{G}_{i-1} | \sigma(n-i+1) = n-i+1\}, \qquad i = 2, 3, ..., d. \tag{3.2}$$

In this section we will give a simple algorithm for the computation of

$$\chi_{R_1,R_2}\Big((p_1, p_2, ..., p_m)\Big) \equiv \mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R\big((p_1, p_2, ..., p_m)\big)\Big)$$

with R_1 and R_2 irreps of $\mathcal{G}_d$ subduced from R, $(p_1, p_2, ..., p_m)$ is an element of S_n specified using the cycle notation and $n - d < p_i \le n$ $\forall i$. We call χ_{R_1,R_2} a *restricted character*. If $R_1 = R_2$, we will simply write χ_{R_1}. We will see in section 4 that restricted characters determine the normalization of the intertwiners. Further, they are also needed in the derivation of the hopping identities that determine the interactions between strings and the branes to which they are attached.

In the next subsection we will derive the algorithm for the computation of the restricted character. The second subsection describes a graphical notation which considerably simplifies the computation. The remainder of this section then develops this diagrammatic notation further.

3.1 Computing Restricted Characters

Consider an irrep R of S_n labelled by a Young diagram which has at least two boxes, either of which can be dropped to leave a valid Young diagram[10]. Label these two boxes by 1 and 2. Denote the weights of these boxes by c_1

[10]See Appendix A for a review of Young diagrams.

and c_2. Denote the irrep of S_{n-2} obtained by dropping box 1 and then box 2 by R''_1. Denote the irrep of S_{n-2} obtained by dropping box 2 and then box 1 by R''_2. Our first task is to compute

$$\mathrm{Tr}_{R''_1,R''_2}\left(\Gamma_R\left((n,n-1)\right)\right).$$

Using the subgroup swap rule (see Appendix C), we can write

$$\begin{aligned}\chi_{R''_1}\left((n,n-1)\right) &= \left[1-\frac{1}{(c_1-c_2)^2}\right]\chi_{R''_2}\left((n,n-1)\right)+\frac{1}{(c_1-c_2)^2}\chi_{R''_1}\left((n,n-1)\right) \\ &+ \sqrt{1-\frac{1}{(c_1-c_2)^2}}\frac{1}{c_1-c_2}\left[\chi_{R''_1,R''_2}\left((n,n-1)\right)+\chi_{R''_2,R''_1}\left((n,n-1)\right)\right]\end{aligned} \quad (3.3)$$

A second application of the subgroup swap rule gives

$$\begin{aligned}\chi_{R''_2,R''_1}\left((n,n-1)\right) &= \left[1-\frac{1}{(c_1-c_2)^2}\right]\chi_{R''_1,R''_2}\left((n,n-1)\right)+\frac{1}{(c_1-c_2)^2}\chi_{R''_2,R''_1}\left((n,n-1)\right) \\ &+ \sqrt{1-\frac{1}{(c_1-c_2)^2}}\frac{1}{c_1-c_2}\left[\chi_{R''_2}\left((n,n-1)\right)-\chi_{R''_1}\left((n,n-1)\right)\right].\end{aligned} \quad (3.4)$$

Now, substituting the results[30]

$$\chi_{R''_1}\left((n,n-1)\right)=\frac{1}{c_1-c_2}\mathrm{dim}_{R''_1},\qquad \chi_{R''_2}\left((n,n-1)\right)=\frac{1}{c_2-c_1}\mathrm{dim}_{R''_2},$$

into (3.3) and (3.4) and solving, we obtain

$$\chi_{R''_1,R''_2}\left((n,n-1)\right)=\sqrt{1-\frac{1}{(c_1-c_2)^2}}\mathrm{dim}_{R''_1}=\chi_{R''_2,R''_1}\left((n,n-1)\right).$$

Next, consider an irrep of S_n labelled by Young diagram R . Choose three boxes in this Young diagram, and label them 1, 2 and 3 respectively. Choose the boxes so that dropping box 1 gives a legal Young diagram R' labelling an irrep of S_{n-1}, dropping box 1 and then box 2 gives a legal Young diagram

R'' labelling an irrep of S_{n-2}, and dropping box 1, then box 2 and then box 3 again gives a legal Young diagram R''' labelling an irrep of S_{n-3}. We will compute

$$\chi_{R'''}((n, n-2)) = \mathrm{Tr}_{R'''}(\Gamma_R((n, n-2)))$$

In what follows, we will frequently need to refer to vectors belonging to the carrier spaces of specific representations subduced by R when boxes are dropped from R. A convenient notation is to list the labels of the boxes that must be dropped from R in the order in which they must be dropped. Thus, the ket $|i, 123\rangle$ is the i^{th} ket belonging to the carrier space of the S_{n-3} irrep obtained by dropping box 1, then box 2 and then box 3 from R; the ket $|j, 231\rangle$ is the j^{th} ket belonging to the carrier space of the S_{n-3} irrep obtained by dropping box 2, then box 3 and then box 1 from R (assuming of course that the boxes can be dropped from R in this order, giving a legal Young diagram at each step). Start by writing

$$\begin{aligned}
\chi_{R'''}((n, n-2)) &= \sum_{i=1}^{\dim_{R'''}} \langle i, 123|\Gamma_R((n, n-2))|i, 123\rangle \\
&= \sum_{i=1}^{\dim_{R'''}} \langle i, 123|\Gamma_{R'}((n-1, n-2))\,\Gamma_R((n, n-1))\,\Gamma_{R'}((n-1, n-2))|i, 123\rangle.
\end{aligned}$$

Noting that $\Gamma_{R'}((n-1, n-2))|i, 123\rangle$ must belong to the carrier space of R', and using the completeness relation ($1_{R'}$ is the identity on the R' carrier space)

$$1_{R'} = \sum_{k=1}^{\dim_{R'}} |k, 1\rangle\langle k, 1|$$

we have

$$\chi_{R'''}((n,n-2)) = \sum_{i=1}^{\dim_{R'''}} \sum_{j,k=1}^{\dim_{R'}} \langle i,123|\Gamma_{R'}((n-1,n-2))|k,1\rangle\langle k,1|\Gamma_R((n,n-1))|j,1\rangle$$
$$\times\langle j,1|\Gamma_{R'}((n-1,n-2))|i,123\rangle.$$

Now, decompose R' into a direct sum of S_{n-2} irreps $R' = \oplus R''_\beta$. Use the label β to denote the box that must be dropped from R' to obtain R''_β. Thus, we can write

$$1_{R'} = \sum_{k=1}^{\dim_{R'}} |k,1\rangle\langle k,1| = \sum_\beta \sum_{k=1}^{\dim_{R''_\beta}} |k,1\beta\rangle\langle k,1\beta|$$

and hence

$$\chi_{R'''}((n,n-2)) = \sum_{i=1}^{\dim_{R'''}} \sum_{\beta_1,\beta_2} \sum_{k=1}^{\dim_{R''_{\beta_1}}} \sum_{j=1}^{\dim_{R''_{\beta_2}}} \langle i,123|\Gamma_{R'}((n-1,n-2))|k,1\beta_1\rangle.$$

$$\times\langle k,1\beta_1|\Gamma_R((n,n-1))|j,1\beta_2\rangle\langle j,1\beta_2|\Gamma_{R'}((n-1,n-2))|i,123\rangle$$

Now, introduce the operator $O(2)$ obtained by summing all two cycles of the S_{n-2} subgroup of which the R''_β are irreps. This operator is a Casimir of S_{n-2}. If the Young diagram R''_β has r_i boxes in the i^{th} row and c_i boxes in the i^{th} column, then when acting on the carrier space of R''_β we have[41]

$$O(2)|i,1\beta\rangle = \left[\sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}\right]|i,1\beta\rangle \equiv \lambda_\beta|i,1\beta\rangle.$$

Clearly, for the problem we study here, $\lambda_{\beta_1} = \lambda_{\beta_2}$ if and only if R_{β_1} and R_{β_2} have the same shape as Young diagrams. From the definition of the $\mathcal{G}_2$

subgroup given above, it is clear that

$$\big[O(2), \Gamma_R\left((n,n-1)\right)\big] = 0.$$

It is now a simple matter to see that

$$\begin{aligned}
\lambda_{\beta_1}\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle &= \langle k,1\beta_1|O(2)\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle \\
&= \langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)O(2)|j,1\beta_2\rangle \\
&= \lambda_{\beta_2}\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle.
\end{aligned}$$

so that $\langle k,1\beta_1|\Gamma_R\left((n,n-1)\right)|j,1\beta_2\rangle$ vanishes if R_{β_1} and R_{β_2} do not have the same shape. A completely parallel argument, using a Casimir of S_{n-3}, can be used to show that $\langle j,1\alpha_1\alpha_2|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle$ is only non-zero if $\alpha_1=2$, $\alpha_2=3$ or $\alpha_1=3$, $\alpha_2=2$. Thus,

$$\begin{aligned}
\chi_{R'''}\left((n,n-2)\right) &= \sum_{i=1,j,k}^{\dim_{R'''}}\Big[\langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,123\rangle\langle k,123|\Gamma_R\left((n,n-1)\right)|j,123\rangle \\
&\times\ \langle j,123|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle + \langle i,123|\Gamma_{R'}\left((n-1,n-2)\right)|k,132\rangle \\
&\times\ \langle k,132|\Gamma_R\left((n,n-1)\right)|j,132\rangle\langle j,132|\Gamma_{R'}\left((n-1,n-2)\right)|i,123\rangle\Big] \\
&= \left[\frac{1}{(c_2-c_3)^2}\frac{1}{c_1-c_2} + \left(1-\frac{1}{(c_2-c_3)^2}\right)\frac{1}{c_1-c_3}\right]\dim_{R'''}.
\end{aligned}$$

This example illustrates the general algorithm to be used to compute restricted characters:

- The group element whose trace is to be computed, can be decomposed into a product of two cycles of the form $\Gamma_R\left((i,i+1)\right)$. A complete set of states is inserted between each factor.
- Using appropriately chosen Casimirs, one can argue that the only non-zero matrix elements of each factor, are obtained when the order of

boxes dropped to obtain the carrier space of the bra matches the order of boxes dropped to obtain the carrier space of the ket, except for the $(n-i+1)^{\text{th}}$ and $(n-i+2)^{\text{th}}$ boxes, whose order can be swapped.

- We can plug in the known value of the restricted character, which we have computed for precisely the two cases arising in the previous point.

3.2 Strand Diagrams

Strand diagrams are a graphical notation designed to compute restricted characters. Strand diagrams keep track of two things:

- The order in which boxes are to be dropped and the identity (= position within the Young diagram) of the boxes.
- The group element whose trace we are computing.

If we are to drop n boxes, we draw a picture with n columns. The columns are populated by labelled strands - each strand represents one of the boxes that are to be dropped. We label the strands by the upper index in the box. This graphical notation should be familiar from the background given in section 2.2. Whatever appears in the first column is to be dropped first; whatever appears in the second column is to be dropped second and so on. The strands are ordered at the top of the diagram, according to the order in which they must be dropped to get the row index. The strands are ordered at the bottom of the diagram according to the column index. The strands move from the top of the diagram to the bottom of the diagram, without breaking, so that strand ends at the top connect to the corresponding strand ends at the bottom. To connect the strands (which in general are in a different order at the top and bottom of the diagram) we need to weave the strands, thereby allowing them to swap columns. The allowed swaps depend on the specific group element whose trace we are computing. To determine the allowed swaps, write the group element as a product of cycles of the form $(i, i+1)$. For example, we would write

$$(n, n-2) = (n, n-1)(n-1, n-2)(n, n-1)$$

Each time we drop a box, we are considering a new subgroup. The action of the permutation group can be visualized as a permutation of n indices. The subgroups are obtained by considering elements that hold certain indices fixed (see equations (3.1) and (3.2)). Choose the subgroups involved so that when box i is dropped, $n-i+1$ is held fixed. Clearly then, each column j is associated with the index $n-j+1$. Each cycle $(i, i+1)$ is drawn as a box which straddles the columns associated with indices i and $i+1$. When the strands pass through a box, they may do so without swapping or by swapping columns. Each box is associated with a factor. Imagine that the strands passing through the box, reading from left to right, are labelled n and m. The weights associated with these boxes are c_n and c_m respectively. If the strands do not swap inside the box the factor for the box is

$$f_{\text{no swap}} = \frac{1}{c_n - c_m}$$

If the strands do swap inside the box, the factor is

$$f_{\text{swap}} = \sqrt{1 - \frac{1}{(c_n - c_m)^2}}.$$

These factors should not be confused with the swap and no swap factors used in the subgroup swap rule (see Appendix C). In fact, the factors are completely reversed in this case. Denote the product of the factors, one from each box, by F. We have

$$\mathrm{Tr}_{R_1,R_2}\Big(\Gamma_R(\sigma)\Big) = \sum_i F_i \mathrm{dim}_{R_1}$$

where the index i runs over all possible paths consistent with the boundary conditions. With a little thought, the astute reader should be able to convince herself that this graphical rule is nothing but a convenient representation of the computation of the last subsection.

3.3 Strand Diagram Examples

In this section we will illustrate the use of strand diagrams in the computation of restricted characters. For our first example, we consider the computation of

$$\chi_1 = \mathrm{Tr}_{\young(\ \ {^1_3},\ {^2_1},{^3_2})}\Big(\Gamma_{\yng(3,2,1)}((6,4))\Big)$$

Writing $(6,4) = (6,5)(4,5)(6,5)$ we obtain the strand diagram shown in Figure 5. The factors for the upper most, middle and lower most boxes are

$$\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}, \qquad \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}, \qquad \frac{1}{c_2 - c_3}$$

respectively. Thus,

$$\begin{aligned}
\chi_1 &= \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\frac{1}{c_2 - c_3}\mathrm{dim}_{\yng(2,1)} \\
&= 2\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\sqrt{1 - \frac{1}{(c_1 - c_3)^2}}\frac{1}{c_2 - c_3}.
\end{aligned}$$

The alert reader may worry that our recipe is not unique. Indeed we could also have written $(6,4) = (4,5)(6,5)(4,5)$. In this case, we obtain the

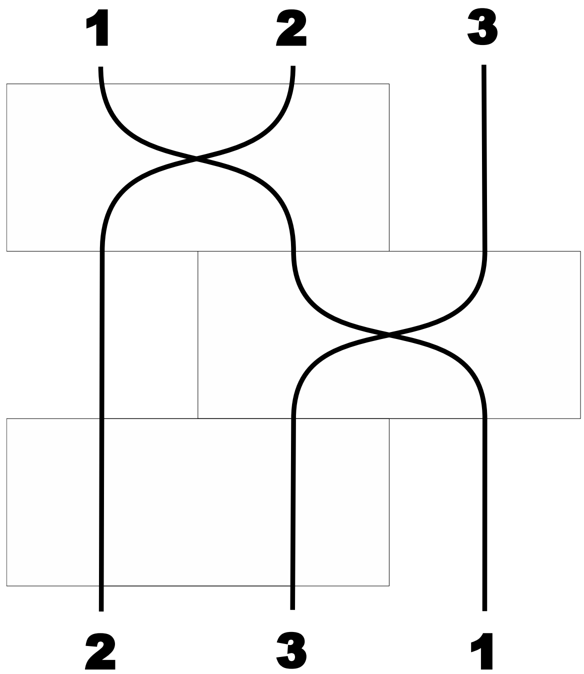


Figure 5: The strand diagram used in the computation of χ_1.

strand diagram given in Figure 6. In this case, the factors for the upper most, middle and lower most boxes are

$$\frac{1}{c_2 - c_3}, \qquad \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}, \qquad \sqrt{1 - \frac{1}{(c_1 - c_3)^2}}$$

respectively. This gives exactly the same value for χ_1.

Next, we consider the computation of

$$\chi_2 = \mathrm{Tr}_{\begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}}\Big(\Gamma_{\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}}\big((6,4)\big)\Big).$$

This example is interesting as more than one path contributes. Writing $(6,4) = (4,5)(6,5)(4,5)$ we obtain the strand diagrams shown in Figure 7.

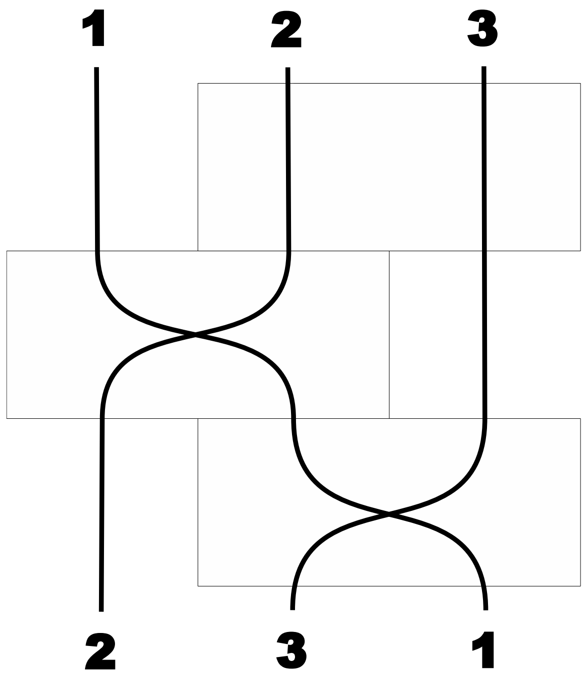


Figure 6: A second strand diagram that can be used in the computation of χ_1.

The product of factors for the diagram on the left is

$$\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right].$$

The product of factors for the diagram on the right is

$$\frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}.$$

Thus,

$$\begin{aligned}\chi_2 &= \left(\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right] + \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}\right)\dim \yng(2,1) \\ &= 2\left(\frac{1}{c_1 - c_3}\left[1 - \frac{1}{(c_2 - c_3)^2}\right] + \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_3)^2}\right).\end{aligned}$$

The reader can check that the same value for χ_2 is obtained by decomposing $(6,4) = (6,5)(4,5)(6,5)$.

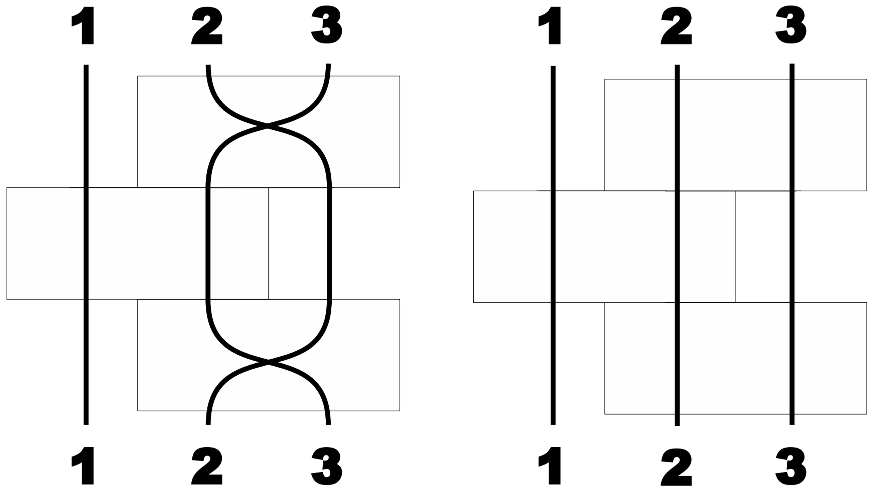


Figure 7: The strand diagrams used in the computation of χ_2.

Finally, consider

$$\chi_3 = \mathrm{Tr}_{\begin{smallmatrix} \square & \square & 1 \\ \square & 2 & \\ 3 & & \end{smallmatrix}}\Big(\Gamma_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \\ \square & & \end{smallmatrix}}(1)\Big).$$

Since we consider the identity element, the strand diagram has no boxes and hence $\chi_3 = \dim_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = 2$. Since $(4,5)(4,5) = 1$ we could also have written

$$\chi_3 = \mathrm{Tr}_{\begin{smallmatrix} \square & \square & 1 \\ \square & 2 & \\ 3 & & \end{smallmatrix}}\Big(\Gamma_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \\ \square & & \end{smallmatrix}}\big((4,5)(4,5)\big)\Big)$$

In this case there are two strand diagrams given in Figure 8. Summing the contributions from these two strand diagrams we obtain

$$\chi_3 = \frac{1}{(c_2-c_3)^2}\mathrm{dim}_{\tiny\yng(2,1)} + \left(1 - \frac{1}{(c_2-c_3)^2}\right)\mathrm{dim}_{\tiny\yng(2,1)} = \mathrm{dim}_{\tiny\yng(2,1)} = 2.$$

Once again, the two ways of writing the restricted character give the same result. Note that the trace

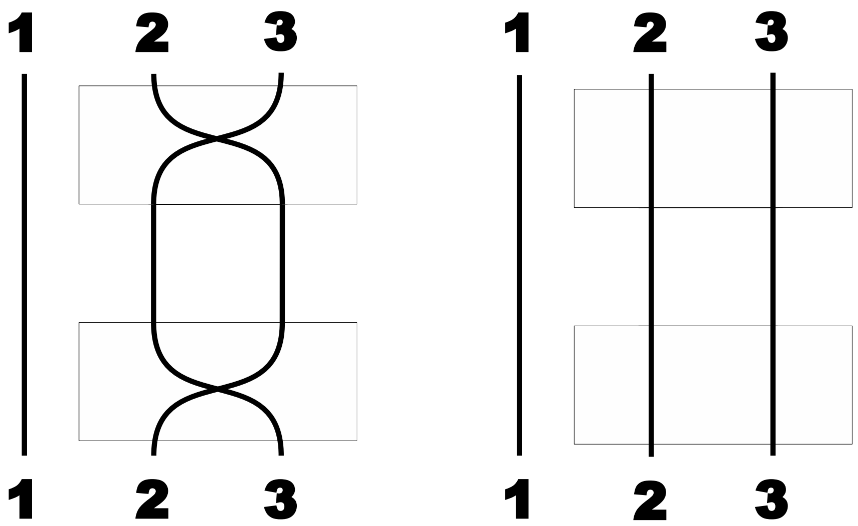


Figure 8: The strand diagrams used in the computation of χ_3.

$$\chi_3 = \mathrm{Tr}_{\tiny\young(\ \ {\substack{1\\2}},\ {\substack{2\\1}},3)}\left(\Gamma_{\tiny\yng(3,2,1)}(1)\right),$$

clearly vanishes because we are tracing the identity over an off the diagonal block. This is reflected graphically by the fact that there is no strand diagram that can be drawn - the order of strands at the top of the diagram does not match the order of strands at the bottom of the diagram and since we consider the identity element, the strand diagram has no boxes. The astute reader may be wondering whether it is possible to decompose the identity as $(6,5)(6,5)$. Indeed, this is possible, and gives us two strand diagrams of opposite sign, which cancel to zero as required. However, we will always use the *simplest* decomposition. For the identity, the simplest decomposition is to have no boxes at all.

3.4 Tests of the Restricted Character Results

By summing well chosen restricted characters, one can recover the characters of S_n which are known. This allows us to perform a number of tests, which our restricted character formulas pass. As an example, consider the computation of $\chi_R\left((6,7)\right)$ for

$$R = \begin{ytableau} \; & \; & \; & \; & \; & \; \\ \; \end{ytableau}\,.$$

From the character tables for S_7 we find $\chi_R\left((6,7)\right) = 4$. In terms of restricted characters

$$\chi_R\left((6,7)\right) = \chi_{\begin{ytableau} \; & \; & \; & \; & 2 & 1 \\ \; \end{ytableau}}\left((6,7)\right) + \chi_{\begin{ytableau} \; & \; & \; & \; & \; & 1 \\ 2 \end{ytableau}}\left((6,7)\right) + \chi_{\begin{ytableau} \; & \; & \; & \; & \; & 2 \\ 1 \end{ytableau}}\left((6,7)\right).$$

Using the algorithm given above, it is straight forward to verify that

$$\chi_{\begin{ytableau} \; & \; & \; & \; & 2 & 1 \\ \; \end{ytableau}}\left((6,7)\right) = \dim_{\begin{ytableau} \; & \; & \; & \; \\ \; \end{ytableau}} = 4,$$

$$\chi_{\begin{ytableau} \; & \; & \; & \; & \; & 1 \\ 2 \end{ytableau}}\left((6,7)\right) = \frac{1}{6}, \quad \chi_{\begin{ytableau} \; & \; & \; & \; & \; & 2 \\ 1 \end{ytableau}}\left((6,7)\right) = -\frac{1}{6},$$

which do indeed sum to give 4. The reader is invited to check some more examples herself.

As a further check of our methods, we have computed the restricted characters $\mathrm{Tr}_{R_1,R_2}\left(\Gamma_R\left[\sigma\right]\right)$ numerically. This was done by explicitly constructing the matrices $\Gamma_R\left[\sigma\right]$. Each representation used was obtained by induction. One induces a reducible representation; the irreducible representation that participates was isolated using projection operators built from the Casimir obtained by summing over all two cycles. See appendix B.2 of [30] for more details. The resulting irreducible representations were tested by verifying the multiplication table of S_n. The intertwiners were computed using the projection operators of [30] and the results of section 4; the normalization of the intertwiner was computed numerically.

3.5 Representations of S_n from Strand Diagrams

Using Strand diagrams, it is possible to write down the irreducible matrix representations of S_n. We will treat the simplest nontrivial example of S_3. First consider the $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ irrep. Start by numbering the boxes in the Young diagram labelling the irrep, with an ordering in which the boxes are to be removed, so that one is left with a legal Young diagram after each box is removed. These labelled Young diagrams are in one-to-one correspondence with the matrix indices of the matrices in the irrep. For our example,

$$i=1,\leftrightarrow \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad i=2,\leftrightarrow \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\;.$$

Each matrix element of $\Gamma_{\tiny\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}((12))$ is given by a single strand diagram

$$\left[\Gamma_{\tiny\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}((12))\right]_{11}=\mathrm{Tr}_{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}((12))=\frac{1}{c_1-c_2}=\frac{1}{2},$$

$$\left[\Gamma_{\ydiagram{2,1}}((12))\right]_{12} = \mathrm{Tr}_{\ytableaushort{3{\frac{1}{2}},{\frac{2}{1}}}}((12)) = \sqrt{1-\frac{1}{(c_1-c_2)^2}} = \frac{\sqrt{3}}{2},$$

$$\left[\Gamma_{\ydiagram{2,1}}((12))\right]_{21} = \mathrm{Tr}_{\ytableaushort{3{\frac{2}{1}},{\frac{1}{2}}}}((12)) = \sqrt{1-\frac{1}{(c_1-c_2)^2}} = \frac{\sqrt{3}}{2},$$

and

$$\left[\Gamma_{\ydiagram{2,1}}((12))\right]_{22} = \mathrm{Tr}_{\ytableaushort{32,1}}((12)) = \frac{1}{c_1-c_2} = -\frac{1}{2},$$

so that

$$\Gamma_{\ydiagram{2,1}}((12)) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

In exactly the same way we obtain

$$\Gamma_{\ydiagram{2,1}}((23)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These two elements can now be used to generate the complete irrep.

Next consider $\ydiagram{3}$. There is only one valid labelling $\ytableaushort{321}$, so that the representation is one dimensional. It is straight forward to obtain

$$\mathrm{Tr}_{\ytableaushort{321}}((12)) = \frac{1}{c_1-c_2} = 1, \qquad \mathrm{Tr}_{\ytableaushort{321}}((23)) = \frac{1}{c_2-c_3} = 1,$$

which are the correct results.

Finally, consider $\ydiagram{1,1,1}$. Again, there is only one valid labelling so that the representation is again one dimensional. We find

$$\mathrm{Tr}_{\ytableaushort{3,2,1}}((12)) = \frac{1}{c_1-c_2} = -1, \qquad \mathrm{Tr}_{\ytableaushort{3,2,1}}((23)) = \frac{1}{c_2-c_3} = -1,$$

which are again the correct results.

4 Intertwiners

Intertwiners are used to construct operators dual to states with open strings stretching between giant gravitons. In this section we provide a general discussion of intertwiners and their construction.

4.1 Strings stretching between two branes

The Gauss Law is a strict constraint on the allowed excited brane configurations[20]: since the branes we consider have a compact world volume, the total charge on any given brane must vanish. This implies that to construct a state with strings stretching between two branes, we need at least two strings in the brane plus string system. Thus, in constructing the restricted Schur polynomial, we will need to remove at least two boxes. For concreteness, consider the case of two sphere giants, so that our restricted Schur polynomial is built with the Young diagram R that has two columns and each column has $O(N)$ boxes. R has a total of $n = O(N)$ boxes. Denote the two boxes to be removed in constructing the restricted Schur polynomial[11] by box 1 and box 2. In order to attach strings stretching between these two giants, the two boxes must obviously belong to different columns. We will assume that box 1 belongs to column 1 and box 2 to column 2. After restricting S_n to an S_{n-1} subgroup, representation R subduces irrep R' (whose Young diagram is obtained by removing box 1 from R) and irrep S' (whose Young diagram is obtained by removing box 2 from R). If we now further restrict to an S_{n-2} subgroup, one of the irreps subduced by R' is R'' (whose Young diagram is obtained by removing box 2 from R') and one of the irreps subduced by S' is S'' (whose Young diagram is obtained by removing box 1 from S'). Note that R'' and S'' have the same Young diagram (and hence the same

[11] See [20; 30; 32] for a detailed discussion of restricted Schur polynomials.

dimension) but act on distinct states in the carrier space of R. The two possible intertwiners we can define map between the states belonging to R'' and the states belonging to S''.

The precise form of the intertwiners depends on the basis used for the S_{n-2} irreps $\Gamma_{R''}(\sigma)$ and $\Gamma_{S''}(\sigma)$. In writing down the intertwiner, we assume that $\Gamma_{R''}(\sigma)$ and $\Gamma_{S''}(\sigma)$ represent σ with the same matrix. With this assumption, it is possible to put the elements of the basis of the carrier space of R'' into one to one correspondence with the elements of the basis of the carrier space of S'': $|i, R''\rangle \leftrightarrow |i, S''\rangle$. We will use this correspondence below. In a suitable basis, we have

$$\Gamma_R(\sigma) = \begin{bmatrix} \Gamma_{R''}(\sigma) & 0 & \cdots \\ 0 & \Gamma_{S''}(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix},$$

for $\sigma \in S_{n-2}$. In constructing the restricted Schur polynomial, we also consider more general $\sigma \in S_n$. In this case, if $\sigma \notin S_{n-2}$, $\Gamma_R(\sigma)$ will not be block diagonal. Even in this more general case, we will use the labels of the S_{n-2} subduced subspaces to label the carrier space of irrep R. Denote the projection operator that projects from the carrier space of R to the R'' subspace by $P_{R\to R'\to R''}$, and the projection operator that projects from the carrier space of R to the S'' subspace by $P_{R\to S'\to S''}$. Clearly, the intertwiner which maps from S'' to R'' must take the form

$$I_{R'',S''} = P_{R\to R'\to R''} O P_{R\to S'\to S''} = \begin{bmatrix} 0 & M & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}. \tag{4.1}$$

The second possible intertwiner that we can construct is given by

$$I_{S'',R''} = P_{R\to S'\to S''} O P_{R\to R'\to R''} = \begin{bmatrix} 0 & 0 & \cdots \\ M & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

We want to find a unique specification for O so that M is simply the identity matrix. For $\sigma \in S_{n-2}$ we have

$$\Gamma_R(\sigma) I_{R'',S''} = \begin{bmatrix} 0 & \Gamma_{R''}(\sigma)M & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$I_{R'',S''}\Gamma_R(\sigma) = \begin{bmatrix} 0 & M\Gamma_{S''}(\sigma) & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

Now, by assumption, $\Gamma_{R''}(\sigma) = \Gamma_{S''}(\sigma)$ since we have $\sigma \in S_{n-2}$. Thus,

$$\big[\Gamma_R(\sigma), I_{R'',S''}\big] = \begin{bmatrix} 0 & \big[\Gamma_{R''}(\sigma), M\big] & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}. \tag{4.2}$$

Applying Schur's Lemma (for irrep R'') to the right hand side implies that M is the identity matrix if and only if $\big[\Gamma_R(\sigma), I_{R'',S''}\big] = 0$ for all $\sigma \in S_{n-2}$. Clearly, for $\sigma \in S_{n-2}$ we have $\big[\Gamma_R(\sigma), P_{R\to R'\to R''}\big] = \big[\Gamma_R(\sigma), P_{R\to S'\to S''}\big] = 0$ so that

$$0 = \big[\Gamma_R(\sigma), I_{R'',S''}\big] = P_{R\to R'\to R''}\big[\Gamma_R(\sigma), O\big]P_{R\to S'\to S''}.$$

Thus, we will require

$$\left[\Gamma_R(\sigma), O\right] = 0, \qquad \forall \sigma \in S_{n-2}. \tag{4.3}$$

If we specify a condition that determines the normalization of the intertwiner, then this normalization condition and (4.3) provide the specification for O that we were looking for. The normalization of the intertwiner is fixed by demanding that

$$\mathrm{Tr}(M) = \dim_{R''},$$

with $\dim_{R''}$ the dimension of irrep R''. This provides a unique definition of the intertwiner.

For the example we are considering here, imagine that the S_{n-1} subgroup is obtained as

$$\mathcal{G} = \{\sigma \in S_n | \sigma(n) = n\},$$

and further that the S_{n-2} subgroup is obtained as

$$\mathcal{H} = \{\sigma \in \mathcal{G} | \sigma(n-1) = n-1\}.$$

Then the intertwiner is given by

$$I_{R'',S''} = \mathcal{N} P_{R\to R'\to R''} \Gamma_R(n, n-1) P_{R\to S'\to S''},$$

with

$$\mathcal{N}^{-1} = \frac{\mathrm{Tr}_{R'',S''}(\Gamma_R(n,n-1))}{\dim_{R''}} \equiv \sum_{i=1}^{\dim_{R''}} \frac{\langle R'', i|\Gamma_R(n,n-1)|S'', i\rangle}{\dim_{R''}}.$$

This last equation makes use of the correspondence between the bases of the carrier spaces R'' and S''. Using the technology developed in section 3, we find

$$\frac{\mathrm{Tr}_{R'',S''}(\Gamma_R(n,n-1))}{\dim_{R''}} = \sqrt{1 - \frac{1}{(c_1-c_2)^2}},$$

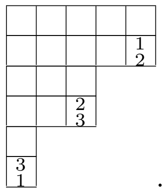
where c_1 and c_2 are the weights associated with box 1 and box 2 respectively. Note that the above trace is invariant under simultaneous similarity transformations of R'' and S''. It will however, change under general similarity transformations so that this last result is dependent on our choice of basis.

4.2 The General Construction

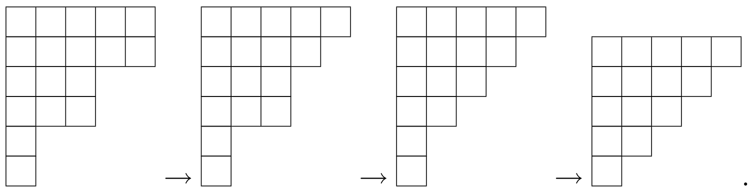
In the previous subsection we developed our discussion of the intertwiner using a system of two branes with strings stretching between them. Our conclusion however, is completely general. For any system of branes with strings stretching between the branes, the intertwiner is always given, up to normalization, by the product (projection operator)×(group element)×(projection operator). The Gauss Law forces the net charge on any given brane's worldvolume to vanish. This implies that for every string leaving a brane's worldvolume, there will be a string ending on the worldvolume. Thus, starting with any particular brane with a stretched string attached, we can follow the string to the next brane, switch to the stretched string leaving that brane, follow it and so on, until we again reach the first brane. If we move along k stretched strings before returning to the starting point, the group element is $\Gamma_R(n, n-k+1)$. The normalization factor easily follows using the results of section 3.

4.3 Example

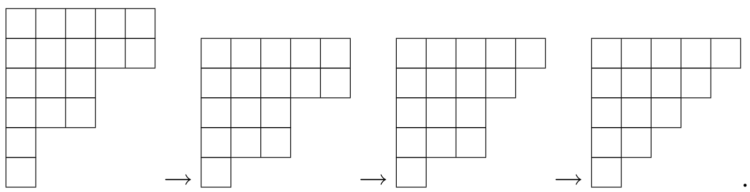
Consider the excited brane system described by the diagram (see 2.2 for a summary of our graphical notation)



The boxes are labelled by the upper index in each box and the weight of box i is denoted c_i. The projector $P_{R\to R_1'''}$ projects through the following sequence of irreps



The projector $P_{R\to R_2'''}$ projects through the following sequence of irreps



The intertwiner is now given by

$$I_{12} = \mathcal{N} P_{R\to R_2'''} \Gamma_R \left((n, n-2)\right) P_{R\to R_1'''},$$

where

$$\mathcal{N}^{-1}=\frac{\mathrm{Tr}_{R_2''',R_1'''}\left(\Gamma_R\left((n,n-2)\right)\right)}{\dim_{R_1'''}}=\frac{1}{c_2-c_3}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_1-c_3)^2}},$$

is easily computed using the methods of section 3. In fact, the strand diagram we use is the same one given by figure 5. To understand the order of the projection operators, note that

$$\begin{aligned}\mathrm{Tr}_{R_1''',R_2'''}\Big(\Gamma_R(\sigma)\Big) &= \sum_i \langle i,R_1'''|\Gamma_R(\sigma)|i,R_2'''\rangle \\ &= \mathrm{Tr}(\mathcal{N}^{-1}P_{R\to R_2'''}\Gamma_R(n,n-2)P_{R\to R_1'''}\Gamma_R(\sigma)),\end{aligned}$$

so that the row (column) index of the trace is column (row) index of the intertwiner respectively.

5 Hopping Identity

In this section, we derive identities that can be used to obtain the Cuntz chain Hamiltonian that accounts for the $O(g_{YM}^2)$ correction to the anomalous dimension of our operators. To construct the "hop off" process, we use the fact that whenever a Z field hops past the borders of the open string word W, the resulting restricted Schur polynomial decomposes into a sum of two types of systems, one is a giant with a closed string and another is a string-giant system where the giant is now bigger. In the large N limit only the second type needs to be considered. The identities we derive in this section express this decomposition. The irreps which play a role in the derivation of the identities are illustrated in Figure 9. The basic structure of the derivation of these identities is very similar. For this reason, we explicitly derive an identity in the next subsection and simply state the remaining identities. In contrast to the case of a single string attached[32], here it does make a difference if the first or last sites of the string participate in the hopping. The identities needed in these two cases are listed separately. We have performed extensive numerical checks of the identities, which we describe next. Finally, we explain how to express the leading large N form of the identities, in terms of states of the Cuntz chain.

5.1 Derivation of a Hopping Identity

Our starting point is the restricted Schur polynomial

$$\chi^{(2)}_{R,R''}\Big|_1\Big|_2 = \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{R''}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}(W^{(1)})^{i_n}_{i_{\sigma(n)}}.$$

There are two labelled boxes in R; dropping box 1 gives irrep R'; dropping box 2 gives irrep R''. Since R' is an irrep of the S_{n-1} subgroup $\mathcal{G}_1$ =

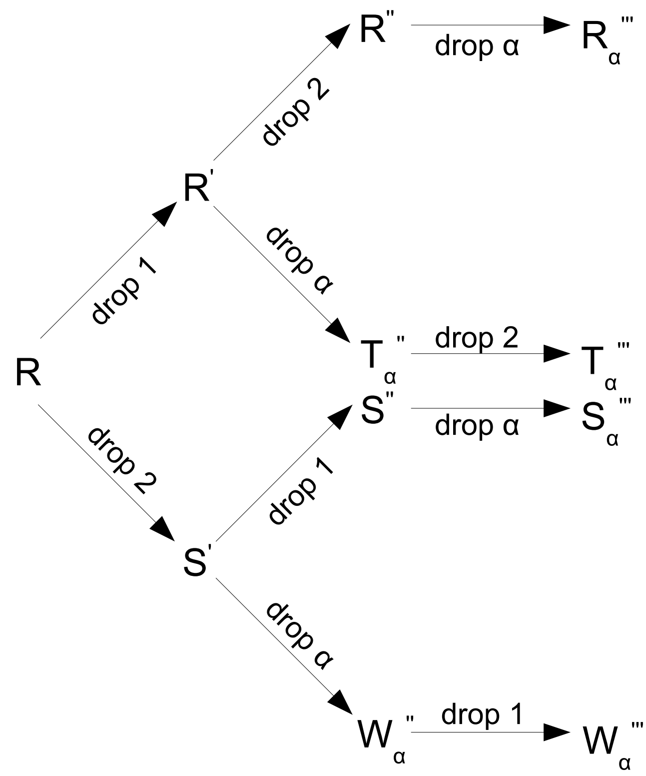


Figure 9: This figure shows the irreps that are used in the hopping identities. Starting from R, the figure shows which irrep is obtained when boxes in R are dropped.

$\{\sigma \in S_n | \sigma(n) = n\}$, we say that the open string described by the word $W^{(1)}$ is associated to box 1. Since R'' is an irrep of the S_{n-2} subgroup $\mathcal{G}_2 = \{\sigma \in \mathcal{G}_1 | \sigma(n-1) = n-1\}$, we say that the open string described by the word $W^{(2)}$ is associated with box 2. Notice that, in the chain of subductions used to define the restricted Schur polynomial, the box associated with $W^{(1)}$ is dropped before the box associated to $W^{(2)}$. We have indicated this with the notation $\Big|_1\Big|_2$. Rewrite the sum over S_n as a sum over $\mathcal{G}_1$ and its cosets

$$\begin{aligned}
\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\Big[\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}(W^{(1)}) \\
&+ \mathrm{Tr}_{R''}\left(\Gamma_R((1,n)\sigma)\right)(W^{(1)}Z)^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}+\cdots+ \\
&+ \mathrm{Tr}_{R''}\left(\Gamma_R((n-2,n)\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots (W^{(1)}Z)^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}+ \\
&+ \mathrm{Tr}_{R''}\left(\Gamma_R((n-1,n)\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}((W^{(1)}W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\Big].
\end{aligned}$$

The first term on the right hand side is

$$\frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}\mathrm{Tr}(W^{(1)}) = \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}(W^{(1)}).$$

Using the methods of section 3, we know that

$$\mathrm{Tr}_{R''}\left(\Gamma_R((n-1,n)\sigma)\right) = \frac{1}{c_1-c_2}\mathrm{Tr}_{R''}\left(\Gamma_{R'}(\sigma)\right),$$

so that the last term on the right hand side is

$$\begin{aligned}
&\frac{1}{(n-2)!}\sum_{\sigma\in\mathcal{G}_1}\mathrm{Tr}_{R''}\left(\Gamma_R((n,n-1)\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(1)}W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}} \\
&\qquad = \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)}).
\end{aligned}$$

Focus on the remaining terms on the right hand side. Each of these terms makes the same contribution. We need to evaluate

$$\mathrm{Tr}_{R''}\left(\Gamma_R((j,n)\sigma\right) = \sum_{i=1}^{\dim_{R''}}\langle i,12|\Gamma_R((j,n))\Gamma_{R'}(\sigma)|i,12\rangle.$$

Using the techniques of section 3, it is straight forward to show that (the sum on α in the next equation is a sum over all boxes that can be removed from R'' to leave a valid Young diagram; the relevant S_{n-3} subgroup is given by $\{\sigma\in\mathcal{G}_2|\sigma(j)=j\}$)

$$\begin{aligned}
\mathrm{Tr}_{R''}\left(\Gamma_R((j,n)\sigma\right) &= \sum_\alpha\sum_{i,k=1}^{\dim_{R'''_\alpha}}\langle i,12\alpha|\Gamma_R((j,n))|k,12\alpha\rangle\langle k,12\alpha|\Gamma_{R'}(\sigma)|i,12\alpha\rangle\\
&+ \sum_\alpha\sum_{i,k=1}^{\dim_{R'''_\alpha}}\langle i,12\alpha|\Gamma_R((j,n))|k,1\alpha 2\rangle\langle k,1\alpha 2|\Gamma_{R'}(\sigma)|i,12\alpha\rangle\\
&= \sum_\alpha\frac{1}{c_1-c_\alpha}\left[1+\frac{1}{(c_1-c_2)(c_2-c_\alpha)}\right]\mathrm{Tr}_{R'''_\alpha}(\Gamma_{R'}(\sigma))\\
&+ \sum_\alpha\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\mathrm{Tr}_{T'''_\alpha,R'''_\alpha}(\Gamma_{R'}(\sigma)).
\end{aligned}$$

Thus, summing the remaining $n-2$ terms we obtain

$$\begin{aligned}
&\textstyle\sum_\alpha \quad \frac{1}{c_1-c_\alpha}\left[1+\frac{1}{(c_1-c_2)(c_2-c_\alpha)}\right]\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1\\
+&\textstyle\sum_\alpha \quad \frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha,R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1.
\end{aligned}$$

A straight forward application of the subgroup swap rule gives

$$\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1=\Bigg[\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})$$

$$+\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})+\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\frac{1}{c_2-c_\alpha}\Big(\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})$$

$$+\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big)\Bigg]\Big|_1\Big|_2,$$

$$\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big|_2\Big|_1=\Bigg[\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})$$

$$-\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})+\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\frac{1}{c_2-c_\alpha}\Big(\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})$$

$$-\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Big)\Bigg]\Big|_1\Big|_2.$$

Thus, we finally obtain

$$\begin{aligned}
\chi^{(2)}_{R,R''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \chi^{(1)}_{R',R''}(Z, W^{(2)})\mathrm{Tr}(W^{(1)}) + \frac{1}{c_1 - c_2}\chi^{(1)}_{R',R''}(Z, W^{(1)}W^{(2)}) \\
&+ \sum_\alpha \Bigg[\frac{1}{c_1 - c_\alpha}\left(1 - \frac{1}{(c_2 - c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\
&+ \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\
&+ \frac{1}{c_1 - c_2}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R' \to R'''_\alpha T'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\
&+ \frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R' \to T'''_\alpha R'''_\alpha}(Z, W^{(1)}Z, W^{(2)})\Bigg]\Big|_1\Big|_2 .
\end{aligned}$$

The above identity is relevant for interactions in which the impurity hops out of the last site of the string. For the hopping interaction in which the impurity hops out of the first site of the string, the right hand side of our identity should be written in terms of $ZW^{(1)}$. This identity is easily derived by rewriting the sum over S_n in terms of right cosets of $\mathcal{G}_1$ instead of left cosets as we have done above.

The identity derived above is relevant for the description of interactions in which string 1 exchanges momentum with the branes in the boundstate. To derive identities that allow string 2 to exchange momentum with the branes in the boundstate, we first use the subgroup swap rule to swap strings 1 and 2. We then rewrite the sum over S_n in terms of a sum over S_{n-1} and its cosets and then employ character identities as above. We give a complete set of identities in the next two subsections.

5.2 Identities Relevant to Hopping off the first site of the string

$$
\begin{aligned}
\chi^{(2)}_{R,R''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \chi^{(1)}_{R',R''}(Z, W^{(2)})\mathrm{Tr}(W^{(1)}) + \frac{1}{c_1 - c_2}\chi^{(1)}_{R',R''}(Z, W^{(2)}W^{(1)}) \\
&+ \sum_\alpha \Bigg[\frac{1}{c_1 - c_\alpha}\left(1 - \frac{1}{(c_2 - c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \frac{1}{c_1 - c_2}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)})\Bigg]\Big|_1\Big|_2 .
\end{aligned} \tag{5.1}
$$

The form of this identity is rather intuitive. The first term on the right hand side contributes to the process in which the bound state emits string 1; the second term describes the process in which the two open strings join to form one long open string. In both of these processes, the box which string 1 occupied on the left hand side does not appear on the right hand side. These two processes will not contribute to our Cuntz chain Hamiltonian; they are relevant for the description of interactions which change the number of open strings attached to the boundstate and do not contribute at the leading order of the large N expansion.

It is instructive to consider the form of this identity for well separated branes. For well separated branes, we have $|c_1 - c_2| \gg 1$. For $|c_1 - c_\alpha| \sim 1$, $|c_2 - c_\alpha| \gg 1$ so that of the last four terms only the first one contributes, giving

$$
\approx \frac{1}{c_1 - c_\alpha}\chi^{(2)}_{R',T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}).
$$

Thus, string 2 stays in box 2 and string 1 is close to where it started. Note that dropping terms of order $(c_1 - c_2)^{-1}$ or $(c_\alpha - c_2)^{-1}$ we obtain

$$\chi^{(2)}_{R,R''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 \approx \chi^{(1)}_{R',R''}(Z, W^{(2)})\mathrm{Tr}(W^{(1)}) + \sum_\alpha \frac{1}{c_1 - c_\alpha}\chi^{(2)}_{R',T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}),$$

which is the identity of [32].

Next, consider the stretched string identities

$$\begin{aligned}
\chi^{(2)}_{R\to R''S''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(1)}_{R',R''}(Z, W^{(2)}W^{(1)}) \\
+ \quad & \sum_\alpha \Bigg[\frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \qquad (5.2) \\
+ \quad & \frac{1}{c_1 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)})\Bigg]\Big|_1\Big|_2,
\end{aligned}$$

$$\begin{aligned}
\chi^{(2)}_{R\to S''R''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(1)}_{S',S''}(Z, W^{(2)}W^{(1)}) \\
+ \quad & \sum_\alpha \Bigg[\frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \qquad (5.3) \\
+ \quad & \frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_1 - c_\alpha)^2}}\sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\chi^{(2)}_{S'\to W'''_\alpha S'''_\alpha}(Z, ZW^{(1)}, W^{(2)})\Bigg]\Big|_1\Big|_2.
\end{aligned}$$

Notice that in contrast to (5.1), (5.2) and (5.3) do not have a term on the right hand side corresponding to emission of string 1. This is what we would expect for an operator dual to a state with two strings stretching between branes, since if string 1 is emitted, it leaves a state with string 2 stretched between branes; this state is not allowed as it violates the Gauss Law. The process in which the two open strings join at their endpoints is allowed. In this process, it is the box with the upper 1 label that is removed. Thus, we can identify the Chan-Paton label for the side of the string defining the first lattice site of the Cuntz chain with the upper label for the string, in our diagrammatic notation. This corresponds to the first label of the restricted Schur polynomial. We will see further evidence for this interpretation when we interpret the final form of the Hamiltonian.

If we again consider the limit of two well separated branes, we find that (5.2) becomes

$$\begin{aligned}\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 &\approx \chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})\\ &+ \sum_\alpha \frac{1}{c_1-c_\alpha}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,ZW^{(1)},W^{(2)})\Big|_1\Big|_2.\end{aligned}$$

In this case, the box with upper 1 label and lower 2 label moves from box 1 to box α (which are close to each other in the Young diagram) and box with upper 2 label and lower 1 label stays where it is.

The first three identities that we have discussed corresponded to an interaction in which an impurity from the first site of string 1 interacts with the brane. The next three identities that we discuss correspond to an interaction in which an impurity from the first site of string 2 interacts with the brane. The first three terms of the identity

$$\begin{aligned}\chi^{(2)}_{R,R''}(\ Z\ ,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(1)}_{S',S''}(Z,W^{(1)})\mathrm{Tr}(W^{(2)})\\ &+ \frac{1}{(c_1-c_2)^2}\chi^{(1)}_{R',R''}(Z,W^{(1)})\mathrm{Tr}(W^{(2)})+\frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})\\ &+ \sum_\alpha\left[\frac{1}{c_2-c_\alpha}\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right.\\ &+ \frac{1}{c_2-c_\alpha}\frac{1}{(c_1-c_2)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},ZW^{(2)}) \qquad (5.4)\\ &+ \left.\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)},ZW^{(2)})\right]\Big|_1\Big|_2\end{aligned}$$

change the number of open strings attached to the boundstate. The first two terms correspond to gravitational radiation; for both of these terms, string 2 is emitted as a closed string. The third term corresponds to a process in which the two open strings join to give a single open string. The order of the open string words in this term is not the same as the order in the corresponding term of (5.1). The term above is natural because it is the first site of string 2 that is interacting; the order in (5.1) also looks natural because in that case it is the first site of string 1 that is interacting. Notice that the above identity is rather different to (5.1). Physically this is surprising - since in both cases it is the first site of the string interacting, these identities should presumably look identical. This mismatch between the two identities is a consequence of the fact that we have treated string 1 and string 2 differently when constructing the operator. See section 8 for further discussion of this point.

If we again consider the limit of two well separated branes, we find that (5.4) becomes (take $|c_1 - c_2| \gg 1$, $|c_1 - c_\alpha| \gg 1$ and $|c_2 - c_\alpha| \sim 1$)

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 \;\approx\;& \chi^{(1)}_{S',S''}(Z, W^{(1)}W^{(2)}) \\ +\;& \sum_\alpha \frac{1}{c_2 - c_\alpha}\chi^{(2)}_{S',S'''_\alpha}(Z, W^{(1)}, ZW^{(2)}).\end{aligned}$$

This again reproduces the identity of [32]. Thus, the content of the formula for well separated branes matches the corresponding limit of (5.1). This is satisfying, because in this limit the order in which the strings are attached does not matter. This follows because the swap factor of [32] behaves as $|c_1 - c_2|^{-1}$.

The remaining two identities are stretched string identities. In contrast to what we found above, there are terms corresponding to gravitational radiation in these identities. We interpret this as a signal that there is some mixing between the operators we have defined (which as explained above, made some arbitrary choices) to get to a "physical basis". See section 8 for more details. The first term in both identities

$$
\begin{aligned}
\chi^{(2)}_{R\to R''S''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)}) \\
&+ \frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}(W^{(2)}) \\
&+ \sum_\alpha\Bigg[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_1}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)}) \qquad (5.5) \\
&+ \frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_1)^2}}\chi^{(2)}_{S'\to S'''_\alpha W'''_\alpha}(Z,W^{(1)},ZW^{(2)}) \\
&+ \frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},ZW^{(2)})\Bigg]\Big|_1\Big|_2,
\end{aligned}
$$

$$
\begin{aligned}
\chi^{(2)}_{R\to S''R''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)}) \\
&+ \frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}(W^{(2)}) \\
&+ \sum_\alpha\Bigg[\frac{1}{c_2-c_\alpha}\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R''_\alpha}(Z,W^{(1)},ZW^{(2)}) \qquad (5.6) \\
&+ \frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_2)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)},ZW^{(2)}) \\
&+ \frac{1}{c_2-c_1}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},ZW^{(2)})\Bigg]\Big|_1\Big|_2,
\end{aligned}
$$

corresponds to two open strings joining to form one long open string. The order of the open string words in these terms again looks natural given that it is the first site of string 2 that is interacting. They will again not contribute in the leading order of the large N expansion. It is satisfying that the content of the large distance limit of (5.5)

$$\chi^{(2)}_{R\to R''S''}(Z,W^{(1)},W^{(2)})\Big|_1\Big|_2 \approx \chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)}) + \sum_\alpha \frac{1}{c_2-c_\alpha}\chi^{(2)}_{S'\to S'''_\alpha W'''_\alpha}(Z,W^{(1)},ZW^{(2)}),$$

is in complete agreement with the large distance limit of (5.2).

5.3 Identities Relevant to Hopping off the last site of the string

In this subsection, impurities hop between the last site of the strings and the threebrane. There are again six possible identities that we could consider. The first three identities describe an interaction between the last site of string 1 and the threebrane. The first identity

$$\begin{aligned}\chi^{(2)}_{R,R''}(\;Z\;,W^{(1)},W^{(2)})\Big|_1\Big|_2 &= \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}(W^{(1)}) + \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})\\ &+ \sum_\alpha \Bigg[\frac{1}{c_1-c_\alpha}\left(1-\frac{1}{(c_2-c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\\ &+ \frac{1}{c_1-c_2}\frac{1}{(c_2-c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\\ &+ \frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\\ &+ \frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\Bigg]\Big|_1\Big|_2\end{aligned} \tag{5.7}$$

can be obtained from (5.1) by (i) swapping the labels on the twisted string states on the right hand side and (ii) swapping the order of the open string words in the second term on the right hand side. This is exactly what we would expect - it is now the last site of the string that is interacting; to swap the first and last sites, we must swap Chan-Paton indices i.e. we must swap the labels on the twisted string states. The discussion of this identity now parallels the discussion of (5.1) and is not repeated.

Consider next the stretched string identities

$$\chi^{(2)}_{R\to S''R''}(\ Z\ ,W^{(1)},W^{(2)})\Big|_1\Big|_2 = \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,W^{(1)}W^{(2)})$$
$$+\ \sum_\alpha\left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right. \tag{5.8}$$
$$\left.+\ \frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right]\Big|_1\Big|_2,$$

$$\chi^{(2)}_{R\to R''S''}(\ Z\ ,W^{(1)},W^{(2)})\Big|_1\Big|_2 = \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z,W^{(1)}W^{(2)})$$
$$+\ \sum_\alpha\left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right. \tag{5.9}$$
$$\left.+\ \frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_\alpha)^2}}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S'\to S'''_\alpha W'''_\alpha}(Z,W^{(1)}Z,W^{(2)})\right]\Big|_1\Big|_2.$$

It is satisfying that identity (5.8) can be obtained from (5.2) and (5.9) from (5.3) by swapping the labels for stretched string states on both sides, and reversing the order of the open string words in the first term on the right hand side. The discussion of these identities now parallel the discussion of (5.2) and (5.3) and is not repeated.

The remaining three identities describe an interaction between the last site of string 2 and the threebrane. The identity

$$\begin{aligned}
\chi^{(2)}_{R,R''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(1)}_{S',S''}(Z,W^{(1)})\mathrm{Tr}(W^{(2)}) \\
&+ \frac{1}{(c_1-c_2)^2}\chi^{(1)}_{R',R''}(Z,W^{(1)})\mathrm{Tr}(W^{(2)}) + \frac{1}{c_1-c_2}\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)}) \\
&+ \sum_\alpha \Bigg[\frac{1}{c_2-c_\alpha}\left(1-\frac{1}{(c_1-c_2)^2}\right)\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},W^{(2)}Z) \\
&+ \frac{1}{c_2-c_\alpha}\frac{1}{(c_1-c_2)^2}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},W^{(2)}Z) \qquad (5.10) \\
&+ \frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)},W^{(2)}Z)\Bigg]\Big|_1\Big|_2
\end{aligned}$$

can be obtained from (5.4) by (i) swapping the labels on the twisted string states on the right hand side and (ii) swapping the order of the open string words in the second term on the right hand side. Finally, the stretched string identities

$$\begin{aligned}
\chi^{(2)}_{R\to R''S''}(\quad Z \quad , W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)}) \\
&+ \frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{R',R''}(Z,W^{(1)})-\chi^{(1)}_{S',S''}(Z,W^{(1)})\right)\mathrm{Tr}(W^{(2)}) \\
&+ \sum_\alpha \Bigg[\frac{1}{c_2-c_\alpha}\frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},W^{(2)}Z) \qquad (5.11) \\
&+ \frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_2)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,W^{(1)},W^{(2)}Z) \\
&- \frac{1}{c_1-c_2}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},W^{(2)}Z)\Bigg]\Big|_1\Big|_2
\end{aligned}$$

$$\chi^{(2)}_{R\to S''R''}(\quad Z \quad, W^{(1)}, W^{(2)})\Big|_1\Big|_2 = \sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z, W^{(2)}W^{(1)})$$
$$- \quad \frac{1}{c_1-c_2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\left(\chi^{(1)}_{S',S''}(Z,W^{(1)}) - \chi^{(1)}_{R',R''}(Z,W^{(1)})\right)\mathrm{Tr}(W^{(2)})$$
$$+ \quad \sum_\alpha \left[\frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_1}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{S',S'''_\alpha}(Z,W^{(1)},W^{(2)}Z)\right. \tag{5.12}$$
$$+ \quad \frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\sqrt{1-\frac{1}{(c_\alpha-c_1)^2}}\chi^{(2)}_{S'\to W'''_\alpha S'''_\alpha}(Z,W^{(1)},W^{(2)}Z)$$
$$+ \quad \left.\frac{1}{c_1-c_2}\frac{1}{c_1-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(2)}_{R',R'''_\alpha}(Z,W^{(1)},W^{(2)}Z)\right]\Big|_1\Big|_2$$

can be obtained from (5.4) and (5.5) by swapping the labels for stretched string states on both sides, and reversing the order of the open string words in the first term on the right hand side.

5.4 Numerical Test

An important result of this dissertation are the identities presented in the previous two subsections, since they determine the hop off interaction. The hop on interaction follows from the hop off interaction by Hermitian conjugation and the kissing interaction by composing the hop on and the hop off interactions. Thus, the complete boundary interaction and the corresponding back reaction on the brane are determined by these identities. For this reason, we have tested the identities numerically. In this subsection we will explain the check we have performed.

Our formulas are identities between restricted Schur polynomials. They must hold if we evaluate them for *any*[12] numerical value of the matrices Z and W. Our check entails evaluating our identities for randomly gen-

[12] In particular, not necessarily Hermitian.

erated matrices $W^{(1)}$, $W^{(2)}$ and Z, to check their validity. Evaluating a restricted Schur polynomial entails evaluating a restricted character as well as a product of traces of a product of the matrices $W^{(1)}$, $W^{(2)}$ and Z.

The restricted character $\mathrm{Tr}_{R'',S''}\left(\Gamma_R\left[\sigma\right]\right)$ or $\mathrm{Tr}_{R''}\left(\Gamma_R\left[\sigma\right]\right)$ was computed by explicitly constructing the matrices $\Gamma_R\left[\sigma\right]$. Each representation used was obtained by induction. One induces a reducible representation; the irreducible representation that participates was isolated using projection operators built from the Casimir obtained by summing over all two cycles. See appendix B.2 of [30] for more details. The resulting irreducible representations were tested by verifying the multiplication table of S_n. The restricted trace is then evaluated with the help of a projection operator or an intertwiner. The intertwiner was computed using the results of section 4.

The trace $\mathrm{Tr}(\sigma Z^{\otimes n-1}W^{(1)}W^{(2)}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}} (W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}} (W^{(1)})^{i_n}_{i_{\sigma(n)}}$ for any given $\sigma \in S_n$ is easily expressed as a product of traces of powers of Z, $W^{(1)}$ and $W^{(2)}$.

In total we verified over 50 specific instances of our identities, which provides a significant check of each identity.

5.5 Identities in terms of Cuntz Chain States

The state-operator correspondence is available for any conformal field theory. Using this correspondence, we can trade our (local) operators for a set of states. Concretely, this involves quantizing with respect to radial time. Considering a fixed "radial time" slice we obtain a round sphere. The states dual to the restricted Schur polynomial operators are the states of our Cuntz chain. Thus, we need to rewrite the identities obtained in this section as statements in terms of the states of the Cuntz oscillator chain. The states of the Cuntz oscillator chain are normalized. Normalized states correspond to operators whose two point function is normalized. Using the technology of [30] it is a simple task to compute the free equal time correlators of the restricted Schur polynomials. After making use of the free field correlators to write our identities in terms operators with unit two point functions, we find that not all terms are of the same order in N. We drop all terms which are subleading in N. These terms are naturally interpreted in terms of string splitting and joining processes, so that they will be important when interactions that change the number of open strings are considered.

The discussion for all of the identities above is rather similar, so we will be content to discuss a specific example which illustrates the general features. Consider the right hand side of (5.1). From the equal time correlator (there are a total of h_i fields in open string word $W^{(i)}$; f_R is the product of the weights of the Young diagram R; d_R is the dimension of R as an irrep of the symmetric group; n_R is the number of boxes in Young diagram R)

$$\langle \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}(W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger \mathrm{Tr}(W^{(1)})^\dagger\rangle$$

$$= \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} h_1 N^{h_1+h_2-1} n_{R''} f_{R'} \frac{d_{R''}}{d_{R'}} \qquad (5.13)$$

we know that the operator $\chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}(W^{(1)})$ corresponds to the state (all Cuntz chain states are normalized to 1)

$$\sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} h_1 N^{h_1+h_2-1} n_{R''} f_{R'} \frac{d_{R''}}{d_{R'}}}|R',R'',W^{(2)};W^{(1)}\rangle.$$

The result (5.13) is not exact. When computing

$$\langle \mathrm{Tr}(W^{(1)})\mathrm{Tr}(W^{(1)})^\dagger\rangle$$

we have only summed the leading planar contribution. When computing

$$\langle \chi^{(1)}_{R',R''}(Z,W^{(2)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\rangle$$

we have only kept the F_0 contribution in the language of [30]. We have also factorized

$$\langle \chi^{(1)}_{R',R''}(Z,W^{(2)})\mathrm{Tr}(W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\mathrm{Tr}(W^{(1)})^\dagger\rangle$$

as

$$\langle \chi^{(1)}_{R',R''}(Z,W^{(2)})\chi^{(1)}_{R',R''}(Z,W^{(2)})^\dagger\rangle\langle\mathrm{Tr}(W^{(1)})\mathrm{Tr}(W^{(1)})^\dagger\rangle$$

which is valid at large N. Similarly, (again we sum only the leading order at large N)

$$\langle \chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})^\dagger\rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} N^{h_1+h_2-1}n_{R''}f_{R'}\frac{d_{R''}}{d_{R'}}$$

implies that $\chi^{(1)}_{R',R''}(Z,W^{(2)}W^{(1)})$ corresponds to the state

$$\sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} N^{h_1+h_2-1}n_{R''}f_{R'}\frac{d_{R''}}{d_{R'}}}|R',R'',W^{(2)}W^{(1)})\rangle.$$

Finally, the correlators (again we sum only the leading order at large N)

$$\langle \chi^{(2)}_{R',T'''_\alpha}(Z,ZW^{(1)},W^{(2)})\chi^{(2)}_{R',T'''_\alpha}(Z,ZW^{(1)},W^{(2)})^\dagger\rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1}n^2_{R'}\frac{d_{T'''_\alpha}}{d_{R'}}f_{R'},$$

$$\langle \chi^{(2)}_{R',R'''_\alpha}(Z,ZW^{(1)},W^{(2)})\chi^{(2)}_{R',R'''_\alpha}(Z,ZW^{(1)},W^{(2)})^\dagger\rangle = \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1}n^2_{R'}\frac{d_{R'''_\alpha}}{d_{R'}}f_{R'},$$

$$\langle \chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,ZW^{(1)},W^{(2)})\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z,ZW^{(1)},W^{(2)})^\dagger\rangle$$

$$= \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{T'''_\alpha}}{d_{R'}} f_{R'},$$

$$\langle \chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)})^\dagger \rangle$$

$$= \left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{T'''_\alpha}}{d_{R'}} f_{R'}$$

imply the correspondences

$$\chi^{(2)}_{R',T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{T'''_\alpha}}{d_{R'}} f_{R'}} |R', T'''_\alpha, ZW^{(1)}, W^{(2)}\rangle,$$

$$\chi^{(2)}_{R',R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{R'''_\alpha}}{d_{R'}} f_{R'}} |R', R'''_\alpha, ZW^{(1)}, W^{(2)}\rangle,$$

$$\begin{aligned} \chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) &\longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{T'''_\alpha}}{d_{R'}} f_{R'}} \\ &\times \quad |R', T'''_\alpha R'''_\alpha, ZW^{(1)}, W^{(2)}\rangle, \end{aligned}$$

$$\begin{aligned} \chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) &\longleftrightarrow \sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+1+n_{T'''_\alpha}} N^{h_1+h_2-1} n_{R'}^2 \frac{d_{T'''_\alpha}}{d_{R'}} f_{R'}} \\ &\times \quad |R', R'''_\alpha T'''_\alpha, ZW^{(1)}, W^{(2)}\rangle. \end{aligned}$$

Consider the factor

$$n_{R'}^2 \frac{d_{R'''_\alpha}}{d_{R'}} = \frac{(\text{hooks})_{R'}}{(\text{hooks})_{R'''_\alpha}},$$

where $(\text{hooks})_R$ is the product of the hook lengths of Young diagram R. It is straight forward to compute this ratio of hook lengths, which is generically of order N^2 implying that $\frac{d_{R'''_\alpha}}{d_{R'}}$ is of order 1. Using this observation, it is equally easy to verify that $\frac{d_{T'''_\alpha}}{d_{R'}}$ and $\frac{d_{R''}}{d_{R'}}$ are also both $O(1)$. Given these results, it is simple to see that the sum of operators

$$
\begin{aligned}
\chi^{(1)}_{R',R''}(\quad Z \quad &, W^{(2)})\mathrm{Tr}(W^{(1)}) + \frac{1}{c_1 - c_2}\chi^{(1)}_{R',R''}(Z, W^{(2)}W^{(1)}) \\
&+ \sum_\alpha \left[\frac{1}{c_1 - c_\alpha}\left(1 - \frac{1}{(c_2 - c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z, ZW^{(1)}, W^{(2)})\right. \\
&+ \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \frac{1}{c_1 - c_2}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z, ZW^{(1)}, W^{(2)}) \\
&+ \left.\frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, ZW^{(1)}, W^{(2)})\right]\Big|_1\Big|_2
\end{aligned}
$$

corresponds to the following sum of normalized states

$$
\begin{aligned}
&\sqrt{\left(\frac{4\pi\lambda}{N}\right)^{h_1+h_2+n_{R''}} N^{h_1+h_2-1} n_{R'}^2 f_{R'}}\left[\sqrt{\frac{h_1 d_{R''}}{n_{R'} d_{R'}}}|R', R'', W^{(2)}; W^{(1)}\rangle\right. \\
&+ \frac{1}{c_1 - c_2}\sqrt{\frac{d_{R''}}{n_{R'} d_{R'}}}|R', R'', W^{(2)}W^{(1)}\rangle \\
&+ \sum_\alpha \left[\frac{1}{c_1 - c_\alpha}\left(1 - \frac{1}{(c_2 - c_\alpha)^2}\right)\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R', T'''_\alpha, ZW^{(1)}, W^{(2)}\rangle\right. \\
&+ \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_\alpha)^2}\sqrt{\frac{d_{R'''_\alpha}}{d_{R'}}}|R', R'''_\alpha, ZW^{(1)}, W^{(2)}\rangle \\
&+ \frac{1}{c_1 - c_2}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R', T'''_\alpha R'''_\alpha, ZW^{(1)}, W^{(2)}\rangle \\
&+ \left.\left.\frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\sqrt{\frac{d_{T'''_\alpha}}{d_{R'}}}|R', R'''_\alpha T'''_\alpha, ZW^{(1)}, W^{(2)}\rangle\right]\right].
\end{aligned}
$$

Recalling that $h_1 = O(\sqrt{N})$ and $n_{R'} = O(N)$, it is clear that the first two terms are subleading. These two terms correspond to gravitational radiation (first term) and string joining (second term); they are the only terms that correspond to an interaction that changes the number of open strings attached to the excited giant system. Although we have illustrated things with an example, this conclusion is general - for all of the identities obtained in this section, terms that do not correspond to two strings attached to the giant system can be dropped in the leading large N limit.

6 State/Operator Map

In this section we will simply quote the six normalization factors that enter the relation between the restricted Schur polynomials and the normalized Cuntz chain states relevant for the excited two giant graviton bound state[13]. The normalization factors are not exact - we simply quote the leading large N value of these normalizations. These factors are determined completely by the $F_0^{(1)}F_0^{(2)}$ contribution in the language of [30]. The factor f_R is the product of weights of the Young diagram R. The open string word $W^{(1)}$ contains a total number of h_1 Higgs fields; the open string word $W^{(2)}$ contains a total number of h_2 Higgs fields.

State	Normalization
$\lvert b_0-1,b_1,11,22\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\lvert b_0-1,b_1,22,11\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\lvert b_0-1,b_1,12,21\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\lvert b_0-1,b_1,21,12\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}$
$\lvert b_0-2,b_1+2,22,22\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{b_1+3}{b_1+1}}$
$\lvert b_0,b_1-2,11,11\rangle$	$\left(\frac{4\pi\lambda}{N}\right)^{\frac{2b_0+b_1+h_1+h_2-2}{2}} b_0\sqrt{f_R}\sqrt{N^{h_1+h_2-2}}\sqrt{\frac{b_1-1}{b_1+1}}$

[13]See section 7 for the restricted Schur polynomials corresponding to these states.

7 Cuntz Chain Hamiltonian

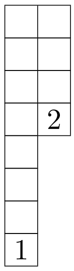
In this section we will derive the form of the terms in the Hamiltonian describing the string boundary interactions. This will allow us to compute the complete Cuntz chain Hamiltonian. The bulk Hamiltonian which is know from [17] is given in equation (7.1).

7.1 Background and Definitions

To make our discussion concrete, we mostly consider the specific example of two strings attached to a bound state of two sphere giants. Note however, that most of the formulas we derive (and certainly the techniques we develop) are applicable to the general problem. Both the strings and the branes that we consider are distinguishable. In this case there are a total of six possible states. For a bound state of two sphere giant gravitons, we need to consider restricted Schur polynomials labelled by Young diagrams with two columns each with $O(N)$ boxes[14]. Denote the number of boxes in the first column by b_0+b_1 and the number of boxes in the second column by b_0. It is natural to interpret the number of boxes in each column as the momentum of each giant. We can use the state operator correspondence (see sections 5.5 and 6 for further discussion) to associate a Cuntz chain state with each restricted Schur polynomial. The Cuntz chain states have six labels in total: the first two labels are b_0 and b_1 which determine the momenta of the two giants; the next two labels are the branes on which the endpoints of string one are attached and the final two labels are the branes on which the endpoints of string two are attached. We label the strings by '1' and '2'. The brane corresponding to column 1 of the Young diagram is labelled 'b' (for big brane) and the brane corresponding to column 2 of the Young diagram is

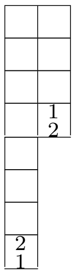
[14]See section 2.2 for a brief review of Schur Polynomials

labelled ‘l’ (for little brane). Since the second column of a Young diagram can never contain more boxes that the first column, and since the radius of the giant graviton is determined by the square root of its angular momentum, these are accurate labels. Consider a state with string 1 on big brane and string 2 on little brane. The restricted Schur polynomial (written using the graphical notation of [30],[32]) together with the corresponding Cuntz chain state are (in this case, $b_0 = 3$ and $b_1 = 4$)



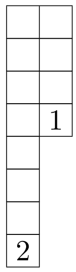
$$\longleftrightarrow |3, 4, bb, ll\rangle.$$

We will call states with strings stretching between branes “stretched string states”. When labelling the Cuntz chain state corresponding to a stretched string state, we will write the end point label corresponding to the upper index first. Thus,

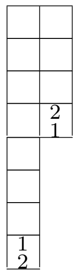


$$\longleftrightarrow |3, 4, lb, bl\rangle.$$

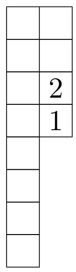
The remaining four states are



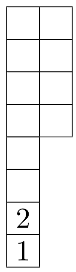
$$\longleftrightarrow |3,4,ll,bb\rangle$$



$$\longleftrightarrow |3,4,bl,lb\rangle,$$



$$\longleftrightarrow |2,6,ll,ll\rangle$$



$$\longleftrightarrow |4,2,bb,bb\rangle.$$

To obtain operators dual to giant gravitons, we take b_0 to be $O(N)$ and b_1 to be $O(1)$. We want to compute the matrix of anomalous dimensions to one loop and at large N. To compute this matrix, we need to compute the two point functions of restricted Schur polynomials. This is a hard problem: since the number of fields in the giant graviton is $O(N)$, huge combinatoric factors pile up as the coefficient of non-planar diagrams and the usual the planar approximation fails. We need to contract all of the fields in the giant gravitons exactly. The two open strings are described by the words $W^{(1)}$ and $W^{(2)}$. The six Higgs fields ϕ^i $i = 1, ..., 6$, of the $\mathcal{N} = 4$ super Yang-Mills theory can be grouped into the following complex combinations

$$Z = \phi^1 + i\phi^2, \qquad Y = \phi^3 + i\phi^4, \qquad X = \phi^5 + i\phi^6.$$

The giant gravitons are built out of the Z field; the open string words out of the Z and Y fields. Thus, the open strings carry a component of angular momentum on the S^3 that the giant wraps, as well a component parallel to the giant's angular momentum. We will normalize things so that the action of $\mathcal{N}=4$ super Yang-Mills theory on $R\times S^3$ is (we consider the Lorentzian theory and have set the radius of the S^3 to 1)

$$S=\frac{N}{4\pi\lambda}\int dt\int_{S^3}\frac{d\Omega_3}{2\pi^2}\left(\frac{1}{2}(D\phi^i)(D\phi^i)+\frac{1}{4}\big([\phi^i,\phi^j]\big)^2-\frac{1}{2}\phi^i\phi^i+\dots\right).$$

With these conventions,

$$\langle Z^\dagger_{ij}(t)Z_{kl}(t)\rangle=\frac{4\pi\lambda}{N}\delta_{il}\delta_{jk}=\langle Y^\dagger_{ij}(t)Y_{kl}(t)\rangle.$$

The open string words can be labelled as

$$(W(\{n_1,n_2,\cdots,n_{L-1}\}))^i_j=(YZ^{n_1}YZ^{n_2}Y\cdots YZ^{n_{L-1}}Y)^i_j,$$

where $\{n_1,n_2,\cdots,n_{L-1}\}$ are the Cuntz lattice occupation numbers. The giant is built out of Zs; the first and last letters of the open string word W are not Zs. We will always use L to denote the number of Y fields in the open string word and $J=n_1+n_2+\cdots+n_{L-1}$ to denote the number of Z fields in the open string word. The number of fields in each word is $J+L\approx L$ in the case that $J\ll L$ which we will assume in this discussion. For the words $W^{(1)},W^{(2)}$ to be dual to open strings, we need to take $L\sim O(\sqrt{N})$. We do not know how to contract the open strings words exactly; when contracting the open string words, only the planar diagrams are summed. To suppress the non-planar contributions we take $\frac{L^2}{N}\ll 1$. To do this we consider a double scaling limit in which the first limit takes $N\to\infty$ holding $\frac{L^2}{N}$ fixed and the second limit takes the effective genus counting parameter $\frac{L^2}{N}$ to zero. Taking the limits in this way corresponds, in the dual string theory, to taking the string coupling to zero, in the string theory constructed in a fixed giant graviton background. Since the two strings are distinguishable they are represented by distinct words and hence, in the large N limit, we have

$$\langle W^{(i)}(W^{(j)})^{\dagger}\rangle \propto \delta^{ij}.$$

When computing a correlator of two restricted Schur polynomials, the fields belonging to the giants in the two systems of excited giant gravitons are contracted amongst each other, the fields in the first open string of each are contracted amongst each other and the fields in the second open string are contracted amongst each other. We drop the contributions coming from contractions between Zs in the open strings and Zs associated to the brane system, as well as contractions between Zs in different open string words. When computing two point functions in free field theory, if the number of boxes in the representation R is less than[15] $O(N^2)$ and the numbers of Z's in the open string is $O(1)$, the contractions between any Zs in the open string and the rest of the operator are suppressed in the large N limit[34]. Contractions between Zs in different open string words are non planar and are hence subleading (clearly there are no large combinatoric factors that modify this).

An important parameter of our excited giant graviton system is $N - b_0$. This parameter can scale as $O(N)$, $O(\sqrt{N})$ or $O(1)$. We will see that when $N - b_0$ is $O(1)$ the sphere giant boundary interaction is $O(\frac{1}{N})$, when $N - b_0$ is $O(\sqrt{N})$ the boundary interaction is $O(\frac{1}{\sqrt{N}})$ and when $N - b_0$ is $O(N)$, the boundary interaction is $O(1)$. Since we want to explore the dynamics arising from the boundary interaction, we will assume that $N - b_0$ is $O(N)$. The

[15]When the number of operators in the Young diagram is $O(N^2)$, the operator is dual to an LLM geometry[33].

subspace of states reached by attaching two open strings to a giant graviton boundstate system is dynamically decoupled (from subspaces obtained by attaching a different number of open strings) at large N. It is possible to move out of this subspace by the process in which the word W "fragments" thereby allowing Ys to populate more than a single box in R. In the dual string theory this corresponds to a splitting of the original string into smaller strings, which are still attached to the giant. This process was considered in [30] and from that result we know that it does not contribute in the large N limit. One could also consider the process in which the open string detaches from the brane boundstate and is emitted as a closed string state, so that it no longer occupies any box in R. This process (decay of the excited giant boundstate by gravitational radiation) also does not contribute in the large N limit[16; 30].

Since the giant boundstate and the open string can exchange momentum, the value of J is not a parameter that we can choose, but rather, it is determined by the dynamics of the problem. Cases in which J becomes large correspond to the situation in which a lot of momentum is transferred from the giant to the open string, presumably signaling an instability. See [17] for a good physical discussion of this instability. In cases where J is large, back reaction is important and the approximations we are employing are no longer valid. This will happen when J becomes $O(\sqrt{N})$ since the assumption that we can drop non-planar contributions when contracting the open string words breaks down. Essentially this is because as more and more Zs hop onto the open string, it is starting to grow into a state which is eventually best described as a giant graviton itself. One can also no longer neglect the contractions between any Zs in the open string and the rest of the operator, presumably because the composite system no longer looks like a string plus giant (which can be separated nicely) but rather, it starts to look like one large deformed threebrane. Thus, the fact that our approximation breaks down has a straight forward interpretation: We have set up our description by assuming that the operator we study is dual to a threebrane with an open string attached. This implies that our operator can be decomposed into a "threebrane piece" and a "string piece". These two pieces are treated very differently: when contracting the threebrane piece, all contractions are summed; when contracting the string piece, only planar contractions are summed. Contractions between the two pieces are dropped. When a large number of Zs hop onto the open string our operator is simply not dual to a state that looks like a threebrane with an open string attached and our approximations are not valid. We are not claiming that this operator can not be studied using large N techniques - it may still be possible to set up a systematic $1/N$ expansion. We are claiming that the diagrams we have summed do not give this approximation.

It is useful to decompose the potential for the scalars into D terms and F terms. The advantage of this decomposition is that it is known that at one loop, the D term contributions cancel with the gauge boson exchange and the scalar self energies[35]. Consequently we will only consider the planar interactions arising from the F term. The F term interaction preserves the number of Y's (the lattice is not dynamical) and allows impurities (the Zs) to hop between neighboring sites. The bulk interactions are described by the Hamiltonian

$$H_{bulk} = 2\lambda \sum_{l=1}^{L} \hat{a}_l^\dagger \hat{a}_l - \lambda \sum_{l=1}^{L-1} (\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_l \hat{a}_{l+1}^\dagger), \tag{7.1}$$

where

$$\hat{a}_i\hat{a}_i^\dagger = I, \qquad \hat{a}_i^\dagger\hat{a}_i = I - |0\rangle\langle 0|.$$

The interested reader is referred to [17] for the derivation of this result. To obtain the full Hamiltonian, we need to include the boundary interactions arising from the string/brane system interaction. This interaction introduces sources and sinks for the impurities at the boundaries of the lattice. The boundary interaction allows Zs to hop from the string onto the giant, or from the giant onto the string. Since the number of Zs gives the angular momentum of the system in the plane that the giant is orbiting in, the boundary interaction allows the string and the brane to exchange angular momentum.

7.2 Hop Off Interaction

We start by deriving the hop off interaction. The F term vertex allows a Z and a Y to change position within a word. The hopping interaction corresponds to the situation in which a Z hops past the Y marking the end point of the string, i.e. a Z hops off the string and onto the giant. Concretely, when acting on either open string, this hop takes

$$W(\{n_1, n_2, \cdots, n_{L-1}\}) \to ZW(\{n_1 - 1, n_2, \cdots, n_{L-1}\}) \quad \text{or}$$

$$W(\{n_1, n_2, \cdots, n_{L-1}\}) \to W(\{n_1, n_2, \cdots, n_{L-1} - 1\})Z.$$

To determine the corresponding term in the interaction Hamiltonian, we need to be able to express objects like $\chi^{(2)}_{R,R''}(Z, ZW^{(1)}, W^{(2)})$ in terms of $\chi^{(2)}_{S,S''}(Z, W^{(1)}, W^{(2)})$ where S is a Young diagram with one more box than

R''[16]. This is easily achieved by inverting the identities derived in section 5. To get the hop off interaction in the Hamiltonian, we rewrite the identities in terms of normalized Cuntz chain states.

$+1 \to 1$ *Hop off Interaction:* This term in the Hamiltonian describes the hop off process in which a Z hops out of the first site of string 1. We write $+1 \to 1$ to indicate that the string before the hop has one extra Z in its first site.

$$H_{+1\to 1}\begin{bmatrix} |b_0-1,b_1,bb,ll\rangle \\ |b_0-1,b_1,ll,bb\rangle \\ |b_0-1,b_1,bl,lb\rangle \\ |b_0-1,b_1,lb,bl\rangle \\ |b_0-2,b_1+2,ll,ll\rangle \\ |b_0,b_1-2,bb,bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_1\begin{bmatrix} |b_0-1,b_1+1,bb,ll\rangle \\ |b_0,b_1-1,ll,bb\rangle \\ |b_0-1,b_1+1,bl,lb\rangle \\ |b_0,b_1-1,lb,bl\rangle \\ |b_0-1,b_1+1,ll,ll\rangle \\ |b_0,b_1-1,bb,bb\rangle \end{bmatrix},$$

where

$$M_1 = \begin{bmatrix} -(b_1)_1^2 & \frac{1}{b_1(b_1+1)^2} & 0 & \frac{(b_1)_0}{b_1+1} & \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{b_1(b_1+1)} \\ -\frac{1}{(b_1+2)(b_1+1)^2} & -(b_1)_1^2 & -\frac{(b_1)_2}{b_1+1} & 0 & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{(b_1)_1}{b_1+1} \\ -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & \frac{(b_1)_1}{b_1+1} & -(b_1)_1(b_1)_2 & 0 & -\frac{b_1}{(b_1+1)^2} & \frac{1}{(b_1+1)^2} \\ -\frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{b_1(b_1+1)} & 0 & -(b_1)_0(b_1)_1 & \frac{1}{(b_1+1)^2} & \frac{b_1+2}{(b_1+1)^2} \\ -\frac{(b_1)_2}{b_1+1} & 0 & \frac{1}{b_1+2} & 0 & -(b_1)_1(b_1)_2 & 0 \\ 0 & \frac{(b_1)_0}{b_1+1} & 0 & -\frac{1}{b_1} & 0 & -(b_1)_0(b_1)_1 \end{bmatrix},$$

and

$$(b_1)_n = \frac{\sqrt{b_1+n-1}\sqrt{b_1+n+1}}{b_1+n}.$$

[16]The number of primes on the label of the restricted Schur polynomial indicates how many boxes are dropped, i.e. R'' is obtained by dropping two boxes from R.

The term in the Hamiltonian describing the process in which the Z hops out of the last site of string 1 is described by swapping the labels of the endpoints of the open strings. Concretely, it is given by

$$H_{1+\rightarrow 1}\begin{bmatrix} |b_0-1,b_1,bb,ll\rangle \\ |b_0-1,b_1,ll,bb\rangle \\ |b_0-1,b_1,lb,bl\rangle \\ |b_0-1,b_1,bl,lb\rangle \\ |b_0-2,b_1+2,ll,ll\rangle \\ |b_0,b_1-2,bb,bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_1\begin{bmatrix} |b_0-1,b_1+1,bb,ll\rangle \\ |b_0,b_1-1,ll,bb\rangle \\ |b_0-1,b_1+1,lb,bl\rangle \\ |b_0,b_1-1,bl,lb\rangle \\ |b_0-1,b_1+1,ll,ll\rangle \\ |b_0,b_1-1,bb,bb\rangle \end{bmatrix},$$

where M_1 is the matrix given above. We write $1+\rightarrow 1$ to indicate that the string before the hop has one extra Z in its last site.

$+2\rightarrow 2$ *Hop off Interaction:* This term in the Hamiltonian describes the hop off process in which a Z hops out of the first site of string 2.

$$H_{+2\rightarrow 2}\begin{bmatrix} |b_0-2,b_1+1,bb,ll\rangle \\ |b_0-1,b_1-1,ll,bb\rangle \\ |b_0-2,b_1+1,bl,lb\rangle \\ |b_0-1,b_1-1,lb,bl\rangle \\ |b_0-2,b_1+1,ll,ll\rangle \\ |b_0-1,b_1-1,bb,bb\rangle \end{bmatrix} = -\lambda\sqrt{1-\frac{b_0}{N}}M_2\begin{bmatrix} |b_0-1,b_1,bb,ll\rangle \\ |b_0-1,b_1,ll,bb\rangle \\ |b_0-1,b_1,bl,lb\rangle \\ |b_0-1,b_1,lb,bl\rangle \\ |b_0-1,b_1,ll,ll\rangle \\ |b_0-1,b_1,bb,bb\rangle \end{bmatrix},$$

where

$$M_2 = \begin{bmatrix} -(b_1)_1^2 & -\frac{1}{(b_1+2)(b_1+1)^2} & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{(b_1)_1}{b_1+1} & 0 & -\frac{(b_1)_2}{b_1+1} \\ \frac{1}{b_1(b_1+1)^2} & -(b_1)_1^2 & \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{(b_1+1)b_1} & \frac{(b_1)_0}{b_1+1} & 0 \\ 0 & -\frac{(b_1)_2}{b_1+1} & -(b_1)_1(b_1)_2 & 0 & 0 & \frac{1}{b_1+2} \\ \frac{(b_1)_0}{b_1+1} & 0 & 0 & -(b_1)_0(b_1)_1 & -\frac{1}{b_1} & 0 \\ -\frac{(b_1)_1}{b_1(b_1+1)} & -\frac{(b_1)_1}{b_1+1} & \frac{1}{(b_1+1)^2} & \frac{b_1+2}{(b_1+1)^2} & -(b_1)_1(b_1)_0 & 0 \\ \frac{(b_1)_1}{b_1+1} & -\frac{(b_1)_1}{(b_1+1)(b_1+2)} & -\frac{b_1}{(b_1+1)^2} & \frac{1}{(b_1+1)^2} & 0 & -(b_1)_2(b_1)_1 \end{bmatrix},$$

Notice that these interactions (as is the case for all of the boundary interactions) are highly suppressed for a maximal giant[24]. The term in the Hamiltonian describing the process in which the Z hops out of the last site of string 2 is described by swapping the labels of the endpoints of the open strings.

The function $(b_1)_n$ also appears in the Hamiltonian relevant for a single string attached to a giant[32]. Notice that $(b_1)_n$ vanishes when $b_1 = 1 - n$, but tends to 1 very rapidly as b_1 is increased from this value. The diagonal terms in the Hamiltonian with a $(b_1)_1$ factor will thus vanish when $b_1 = 0$. The radius of each giant is determined by their momentum. Since b_1 is the difference in momentum of the two giants, $b_1 = 0$ corresponds to coincident giants. Thus, $(b_1)_n$ is switching off short distance interactions. The hop off Hamiltonian does not generate illegal Young diagrams from legal ones precisely because these interactions are switched off.

Finally, note that the structure of the hop on and hop off interactions, clearly reflect the fact that the open strings attached to the giants are orientable.

7.3 Hop On Interaction

Since $\mathcal{N}=4$ super Yang-Mills theory is a unitary conformal field theory, we know that the spectrum of anomalous dimensions of the theory is real. This implies that the energy spectrum of our Cuntz chain Hamiltonian must be real and hence the Hamiltonian must be Hermitian. Thus, the hop on term in the Hamiltonian is given by the Hermitian conjugate of the hop off term.

To give an example, we will now derive the term in the Hamiltonian describing the process in which a Z from the brane hops into the first site of string 1. Let $|\psi\rangle$ denote the state with a brane of momentum $P_{\text{brane}} = P$ and a string of momentum $P_{\text{string}} = p$ and $|\phi\rangle$ denote the state with $P_{\text{brane}} = P+1$ and $P_{\text{string}} = p-1$. Then,

$$H_{+1\to 1}|\psi\rangle = -\lambda\sqrt{1-\frac{b_0}{N}}M_1|\phi\rangle,$$

and

$$\langle\phi'|H_{+1\to 1}|\psi\rangle = -\lambda\sqrt{1-\frac{b_0}{N}}\langle\phi'|M_1|\phi\rangle = -\lambda\sqrt{1-\frac{b_0}{N}}(M_1)_{\phi'\phi}.$$

Daggering we find (keep in mind that M_1 is real)

$$\begin{aligned}\langle\psi|H_{1\to +1}|\phi'\rangle &= (\langle\phi'|H_{+1\to 1}|\psi\rangle)^\dagger \\ &= -\lambda\sqrt{1-\frac{b_0}{N}}\langle\phi|(M_1)^T|\phi'\rangle \\ &= -\lambda\sqrt{1-\frac{b_0}{N}}\left((M_1)^T\right)_{\phi\phi'}.\end{aligned}$$

Thus we obtain

$$H_{1\to +1}|\phi\rangle = -\lambda\sqrt{1-\frac{b_0}{N}}N_1|\psi\rangle,$$

with $N_1 = (M_1)^T$.

7.4 Kissing Interaction

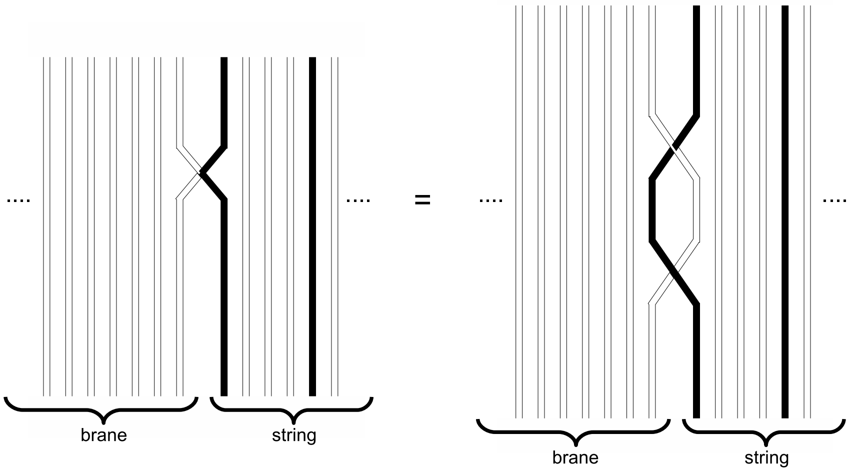


Figure 10: The Feynman diagram on the left shows the kissing interaction. The white ribbons are Z fields, the black ribbons are Y fields. The interacting black ribbon shown marks the beginning of the string; there are 3 Zs in the first site of the string. The Feynman diagram on the right shows a hop on interaction followed by a hop off interaction. If you shrink the composite hop on/hop off interaction to a point, you recover the kissing interaction.

The kissing interaction corresponds to the Feynman diagram shown on the left in Figure 10. Notice that the number of Z fields in the giant is unchanged by this process so that the string and brane do not exchange momentum by this process. As far as the combinatorics goes, we can model the kissing interaction as a hop on (the string) followed by a hop off. We know both the hop on and hop off terms so the kissing interaction follows. This is illustrated by the Feynman diagram shown on the right in Figure 10. The kissing interaction must be included for both endpoints of both strings.

A straight forward computation easily gives

$$H_{\rm kissing} = \lambda \left(1 - \frac{b_0}{N} \right) \mathbf{1},$$

for each endpoint of either string. In this formula **1** is the identity operator.

The fact that the kissing interaction comes out proportional to the identity operator is a non-trivial check of our hop on and hop off interactions. Indeed, the contraction of the F term vertex which leads to the kissing interaction removes an adjacent Z and Y and then replaces them in the same order. Thus, the kissing interaction had to come out proportional to the identity. The careful reader may worry that this is not in fact true - indeed, the restricted Schur polynomial includes terms in which the open string word is traced and terms in which the two open string words are multiplied. For these terms there is no Z next to the word to "do the kissing". Precisely these terms were considered in section 5.5. They do not contribute at large N.

8 Interpretation

The operators we are studying are dual to giant gravitons with open strings attached. Since the giant gravitons have finite volume, the Gauss Law implies that the total charge on each giant must vanish - there must be the same number of strings leaving each brane as there are arriving on each brane. These operators do indeed satisfy these non-trivial constraints[20], providing convincing evidence for the proposed duality. The low energy dynamics of the open strings attached to the giant gravitons is a Yang-Mills theory. This new *emergent* $3+1$ dimensional Yang-Mills theory is not described as a local field theory on the S^3 on which the original Yang-Mills theory is defined - it is local on a new space, the world volume of the giant gravitons[20],[37]. This new space emerges from the matrix degrees of freedom participating in the Yang-Mills theory. Reconstructing this emergent gauge theory may provide a simpler toy model that will give us important clues into reconstructing the full $\mathrm{AdS}_5\times\mathrm{S}^5$ quantum gravity. In this section, our goal is to make contact with this emergent Yang-Mills dynamics.

8.1 Dynamical Emergence of Chan-Paton Factors

Return to the $H_{+1\to 1}$ hop off interaction obtained in section 7.2. Recall that this corresponds to the interaction in which a Z hops out of the first site of string 1. If we expand the matrix M_1 for large b_1, we find

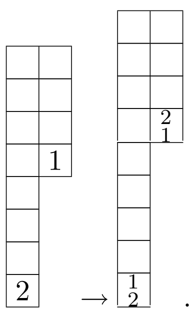
$$M_1 = \sum_{n=0}^{\infty} M_1(n) b_1^{-n}.$$

The leading order $M_1(0)$ is simply $-\mathbf{1}$ with $\mathbf{1}$ the 6×6 identity matrix. The Z simply hops off the string and onto the brane without much rearranging of the system. This is the dominant process. Next, consider the term of order b_1^{-1}. It is simple to compute

$$M_1(1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

The radius of the giant graviton R_g is related to its momentum P by $R_g = \sqrt{\frac{P}{N}}$. The giant orbits with a radius $R = \sqrt{1 - R_g^2}$ (see [10]). For the two giants in the bound state we are considering we have $P_1 = b_0$ and $P_2 = b_0 + b_1$. Using the fact that $b_0 = O(N)$ and $b_1 = O(1)$ it is simple to verify that both the difference in the radii of the two giants and the difference in the radii of their orbits is proportional to b_1. Thus, a b_1^{-1} dependence indicates a potential with an inverse distance dependence which is the correct dependence for massless particles moving in $3+1$ dimensions. In Figure 11 we have represented the transitions implied by $M_1(1)$ graphically. Transitions between any two adjacent Young diagrams are allowed.

As an example, consider the transition



The upper label of string 1 has moved. In all of the transitions shown, the

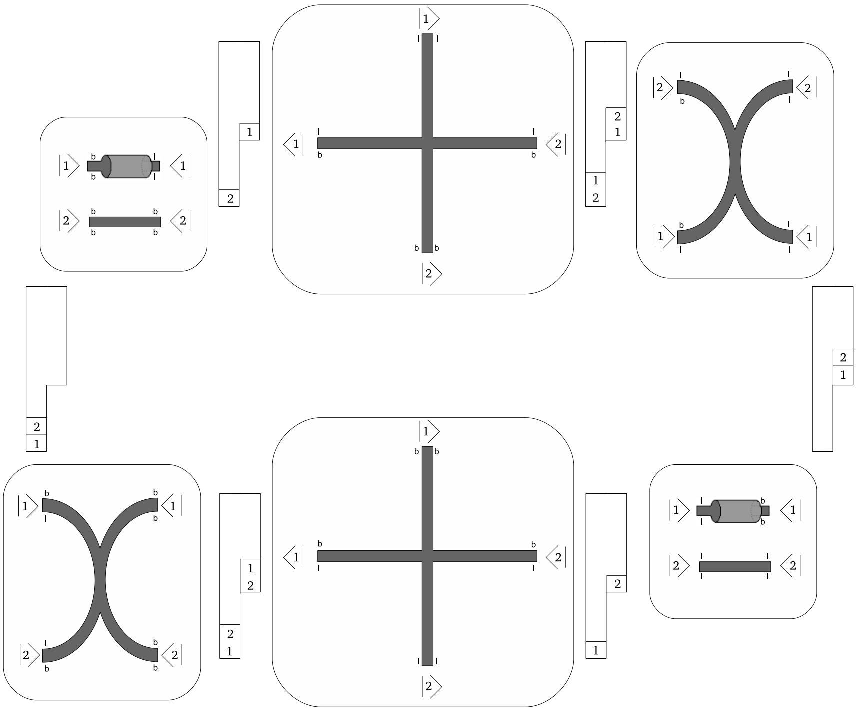


Figure 11: The order b_1^{-1} terms in the hop off interaction. This interaction allows a transition between the operators described by any two adjacent Young diagrams. The figures between the Young diagram show the open string diagram relevant for the clockwise transition. The kets are associated to the open string states before the transition; the bras to the states after the transition. The end point labels 'b' and 'l' are for big brane and little brane.

upper index of string 1 always moves, so that it is natural to associate the upper index of string 1 with the first site of string one, and to look for an interpretation of this interaction in terms of open string processes that involve the upper index of string 1. The figures between the Young diagram show that there is indeed a natural interpretation for these transitions. *It is clear that our Cuntz oscillator dynamics illustrates how the Chan-Paton factors for open strings propagating on multiple branes arise dynamically.* Drawing all possible ribbon diagrams correctly accounts for both $M_1(0)$ and $M_1(1)$.

8.2 Physical Basis

Although the interpretation of the b_1^{-1} terms is encouraging, there are extra higher order corrections ($M_1(2)b_1^{-2}$, $M_1(3)b_1^{-3}$ and higher orders) that do not seem to have a natural open string interpretation. In addition to this, the interaction we have obtained depends on the open string words describing each open string, the Young diagram describing the brane bound state system as well as the order in which the strings were attached. This dependence on the order in which the strings are attached is not physically sensible.

It is natural to expect that the resolution to these two puzzles is connected. Recall that when constructing the restricted Schur polynomial we have assumed that when computing reductions, string 1 is removed first and string 2 second. This arbitrary choice defines a basis for the Cuntz oscillator chain. We interpret the unphysical features of our interactions, described in the previous paragraph, as reflecting a property of the basis it is written in and not as an inherent problem with the interaction. In this section we will define a new physical basis, singled out by the requirement that the boundary interaction does not depend on the order in which the open strings are attached.

A few comments are in order. A basis for the $\frac{1}{2}$ BPS states (giants with no open strings attached) is provided by the taking traces of Z or by taking subdeterminants or by the Schur polynomials. These are three perfectly acceptable bases, since using any single one of these bases we can generate, by taking linear combinations of the elements of the basis considered, a member from every $\frac{1}{2}$ BPS multiplet[14]. From a physical point of view, these different bases are not on an equal footing: the Schur polynomial is the most useful. Indeed, the Schur polynomials diagonalize the matrix of two point correlators (Zamolodchikov metric) so that they can be put into correspondence with the (orthogonal) states of a Fock space. In the same way, the basis for excited giants gravitons we have been considering is a perfectly acceptable basis. However, it is the operators in the physical basis (defined below) that have a good physical interpretation.

Denote our two strings by string A and string B. The state obtained by attaching string A first will be denoted by $|b_0, b_1, x_A y_A, x_B y_B\rangle$, where $x_A y_A$ are the endpoints of string A and $x_B y_B$ are the endpoints of string B. The state obtained by attaching string B first will be denoted by $|b_0, b_1, x_B y_B, x_A y_A\rangle\rangle$. In each subspace of sharp giant graviton momentum (definite b_0 and b_1), we can write the following relation between these two sets of states

$$\begin{bmatrix} |b_0, b_1, bb, ll\rangle \\ |b_0, b_1, ll, bb\rangle \\ |b_0, b_1, bl, lb\rangle \\ |b_0, b_1, lb, bl\rangle \\ |b_0, b_1, ll, ll\rangle \\ |b_0, b_1, bb, bb\rangle \end{bmatrix} = PT \begin{bmatrix} |b_0, b_1, bb, ll\rangle\rangle \\ |b_0, b_1, ll, bb\rangle\rangle \\ |b_0, b_1, bl, lb\rangle\rangle \\ |b_0, b_1, lb, bl\rangle\rangle \\ |b_0, b_1, ll, ll\rangle\rangle \\ |b_0, b_1, bb, bb\rangle\rangle \end{bmatrix},$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and}$$

$$T = \begin{bmatrix} \left(1-\frac{1}{(b_1+1)^2}\right) & \frac{1}{(b_1+1)^2} & -\frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & 0 & 0 \\ \frac{1}{(b_1+1)^2} & \left(1-\frac{1}{(b_1+1)^2}\right) & \frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & \frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & 0 & 0 \\ \frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & \left(1-\frac{1}{(b_1+1)^2}\right) & -\frac{1}{(b_1+1)^2} & 0 & 0 \\ \frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)}\sqrt{1-\frac{1}{(b_1+1)^2}} & -\frac{1}{(b_1+1)^2} & \left(1-\frac{1}{(b_1+1)^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix T is determined by the subgroup swap rule of [30]. It is satisfying that $PT \times PT = 1$. It is straight forward to check that

$$H_{+1\to 1} = A_{2\to 1}\, PT\, H_{+2\to 2}\, A_{1\to 2}\, PT,$$

where

$$\begin{bmatrix} |b_0-2, b_1+2, bb, ll\rangle \\ |b_0-1, b_1, ll, bb\rangle \\ |b_0-2, b_1+2, bl, lb\rangle \\ |b_0-1, b_1, lb, bl\rangle \\ |b_0-2, b_1+2, ll, ll\rangle \\ |b_0-1, b_1, bb, bb\rangle \end{bmatrix} = A_{2\to 1} \begin{bmatrix} |b_0-1, b_1, bb, ll\rangle \\ |b_0-1, b_1, ll, bb\rangle \\ |b_0-1, b_1, bl, lb\rangle \\ |b_0-1, b_1, lb, bl\rangle \\ |b_0-1, b_1, ll, ll\rangle \\ |b_0-1, b_1, bb, bb\rangle \end{bmatrix}, \qquad \text{and}$$

$$\begin{bmatrix} |b_0-2, b_1+1, bb, ll\rangle \\ |b_0-1, b_1-1, ll, bb\rangle \\ |b_0-2, b_1+1, bl, lb\rangle \\ |b_0-1, b_1-1, lb, bl\rangle \\ |b_0-2, b_1+1, ll, ll\rangle \\ |b_0-1, b_1-1, bb, bb\rangle \end{bmatrix} = A_{1\to 2} \begin{bmatrix} |b_0-2, b_1+1, bb, ll\rangle \\ |b_0-2, b_1+1, ll, bb\rangle \\ |b_0-2, b_1+1, bl, lb\rangle \\ |b_0-2, b_1+1, lb, bl\rangle \\ |b_0-3, b_1+3, ll, ll\rangle \\ |b_0-1, b_1-1, bb, bb\rangle \end{bmatrix}.$$

Denote the similarity transformation which takes us to the physical basis by S. In this basis, we denote $H_{+1\to 1}$ by $\hat{H}_{+1\to 1}$ and $H_{+2\to 2}$ by $\hat{H}_{+2\to 2}$.

Clearly

$$\hat{H}_{+1\to 1} = SH_{+1\to 1}S^{-1}, \qquad \hat{H}_{+2\to 2} = SH_{+2\to 2}S^{-1}.$$

The transformation S is now determined by the requirement

$$\hat{H}_{+1\to 1} = P\hat{H}_{+2\to 2}P.$$

We have not yet been able to solve this equation for S. Due to the presence of $A_{1\to 2}$ and $A_{2\to 1}$ in the relation between $H_{+1\to 1}$ and $H_{+2\to 2}$, it seems that S must mix subspaces of different giant momenta (b_0, b_1). In this case the physical basis will not have sharp giant momentum and hence the resulting states will not have a definite radius. This is not too surprising: the open strings will pull "dimples" out of the giant graviton's world volume so that the giant with an open string attached does not have a definite radius. We leave the interesting question of determining the transformation S for the future.

9 Discussion

A bound state of giant gravitons can be excited by attaching open strings. The problem of computing the anomalous dimensions of these operators can be replaced with the problem of diagonalizing a Cuntz oscillator Hamiltonian. In this dissertation we have developed the technology needed to construct this Cuntz oscillator Hamiltonian to one loop. Firstly, we have given an algorithmic construction of the operators dual to excitations described by open strings which stretch between the branes. This involved giving an explicit construction of the intertwiner which is used to construct the relevant restricted Schur polynomial. Secondly, we have developed methods that allow an efficient evaluation of any restricted character. Our method expresses the restricted character graphically as a sum of strand diagrams. Finally, we have explained how to derive the boundary interaction terms from identities satisfied by the restricted Schur polynomials. Since the excited giant graviton operators are small excitations of BPS states, we expect that our results can be extrapolated to strong coupling and hence can be compared with results from the dual string theory. The form of our Cuntz oscillator Hamiltonian provides evidence that the excitations of the giant gravitons have the detailed interactions of an emergent gauge theory. In particular, we have demonstrated the dynamical emergence of the Chan-Paton factors of the open strings. We have also started to clarify the dictionary relating the states of the Cuntz oscillator chain to the states of string field theory on D-branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. Although we have mainly considered a bound state of two sphere giants with two open strings attached, our methods are applicable to an arbitrary bound state of giant gravitons with any number of open strings attached.

Our result is a generalization of the spin chains considered so far in the literature: usually the spin chain gives a description of closed strings. Our Cuntz oscillator Hamiltonian describes the dynamics of an open string interacting with a giant graviton. Both the state of the string (described by the Cuntz chain occupation numbers) and the state of the giant graviton (the shape of the Young diagram) are dynamical in our approach.

It is worth emphasizing that the new emergent gauge symmetry is distinct from the original gauge symmetry of the theory[20]. The excited giant graviton operators[20] are obtained by taking a trace over the indices of the symmetric group matrix $\Gamma_R(\sigma)$ appearing in the sum

$$\frac{1}{(n-k)!}\sum_{\sigma\in S_n}\Gamma_R(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}),\qquad \text{where}$$

$$\mathrm{Tr}(\sigma Z^{\otimes n-k}W^{(1)}\cdots W^{(k)}) = Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n-k}}_{i_{\sigma(n-k)}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}\cdots(W^{(k)})^{i_n}_{i_{\sigma(n)}}.$$

The color indices of the original super Yang-Mills theory are all traced: every term in the above sum is a color singlet with respect to the gauge symmetry of the original Yang-Mills theory. The color indices of the new gauge theory arise from the labelling of the partial trace over $\Gamma_R(\sigma)$. In some sense we are "substituting" symmetric group indices for the original gauge theory indices. We call this mechanism "*color substitution*".

There are a number of directions in which this work can be extended. For Young diagrams with m columns we expect an emergent Yang-Mills theory with gauge group $U(m)$. It would be nice to repeat the calculations we performed here in that setting. Another interesting calculation would involve studying the dynamics of two giant gravitons with strings stretched between them. In general, the boundary terms will certainly have different values at each boundary (as anticipated in [17]) in which case there will be a net flow of Zs from one brane to the other. This flow of Z's will produce a force between the two giants, conjectured to be an attractive force in[17].

A very concrete application of our methods is the construction of the gauge theory operator dual to the fat magnon[38][17]. The fat magnon is a bound state of a giant graviton and giant magnons (fundamental strings). Essentially, due to the background five form flux, the giant magnon becomes fat by the Myers effect[39]. The fat magnon has the same anomalous dimension as the giant magnon. It would be nice to explicitly recover this anomalous dimension using our technology.

Finally, there is now a proposal for gauge theory operators dual to brane-anti-brane states[40]. This proposal was made, at the level of the free field theory, by identifying the operators that diagonalize the two point functions of operators built from Z and $Z^\dagger$. Since these states are non-supersymmetric, corrections when the coupling is turned on are expected to be important for the physics. It would be interesting to extend the technology developed in this dissertation to this non-supersymmetric setting.

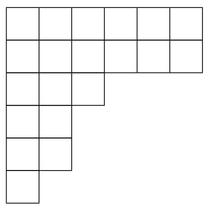
[17] The fat magnon in the plane wave background is the hedgehog of [19]

A Young diagrams

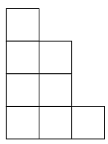
Young diagrams label the Schur polynomials which we study. The Young diagrams are in a one to one correspondence with the irreducible representations of the symmetric group. Hence, a Young diagram with n boxes is a representation of S_n. When constructing a Young diagram, two rules need to be followed:

- In every row, the number of boxes must equal, or decrease, from left to right
- In every column, the number of boxes must equal, or decrease, from top to bottom

The following is a valid Young diagram

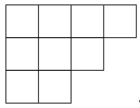
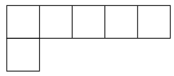
$R =$ 

whereas this Young diagram, is clearly not valid

$R =$ 

We now define the weight of a box in a Young diagram. The box in column i and row j will has weight $N + i - j$. As an example, consider the following

Young diagrams

$R_1 =$  , $R_2 =$ 

We define f_R to be the product of weights in the Young diagram. So for the above diagrams we have

$$f_{R_1} = N(N+1)(N+2)(N+3)(N-1)(N)(N+1)(N-2)(N-1),$$

$$f_{R_2} = N(N+1)(N+2)(N+3)(N+4)(N-1).$$

The hook length for a box is defined as the number of boxes immediately below a box plus the number of boxes immediately to the right of the box plus the box itself. This is best seen in Figure 12, where the arrow indicate the boxes which are to be counted in order to find the weight of the top, left-most box. This box has a weight of 7. Consider the example used in

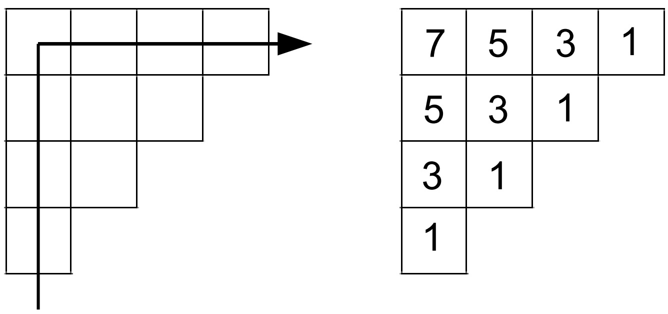


Figure 12: This figure shows how the hook length for a Young tableau is calculated.

Figure 12. The Young diagram, R, is an irrep of S_{10}. The dimension of R is given by

$$\dim_R = \frac{n!}{\Pi hooks} \tag{A.1}$$

So for the example we are examining

$$\begin{aligned} \dim_R &= \frac{10!}{7\cdot 5\cdot 5\cdot 3\cdot 3\cdot 3\cdot 1\cdot 1\cdot 1\cdot 1} \\ &= 768. \end{aligned}$$

B Calculating Characters

Tracing over a diagonal element of $\Gamma_R(\sigma)$ is not difficult, and is best seen with a simple example. It is hoped that this example will illustrate some important ideas which will help the reader when she comes to the more difficult case of intertwiners, which are discussed in section 4. Consider the following Schur polynomial with one string attached

$$\chi_{\young(\ 1,\ )}(\sigma) \equiv \mathrm{Tr}_{\yng(1,1)}(\Gamma_{\yng(2,1)}(\sigma)) \tag{B.1}$$

where $\Gamma_{\yng(2,1)}(\sigma)$ is a 2 by 2 matrix shown in Figure 13.

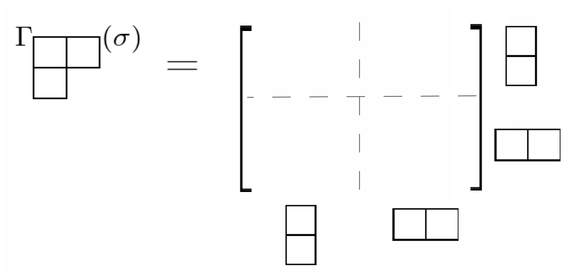


Figure 13: Matrix representing $\Gamma_R(\sigma)$.

We define an operator, $\hat{O} = (12)$, which is the sum of all 2 cycles in S_2.

The result of this operator acting on a state is

$$\hat{O}\left|\yng(1,1)\right\rangle = -\left|\yng(1,1)\right\rangle$$

$$\hat{O}\left|\yng(2)\right\rangle = \left|\yng(2)\right\rangle .$$

We can now build a projection operator that will only trace over the required block in $\Gamma_{\yng(2,1)}$. The operator must select out the block we need, setting everything else to zero. The obvious projection operators are

$$\frac{1}{2}(1+\hat{O})\left|\yng(1,1)\right\rangle = 0 \tag{B.2}$$

$$\frac{1}{2}(1+\hat{O})\left|\yng(2)\right\rangle = \left|\yng(2)\right\rangle \tag{B.3}$$

so in this case $P_{\yng(2,1),\yng(2)} = 1/2(1+(12))$.

Similarly, it is easy to verify that $P_{\yng(2,1),\yng(1,1)} = 1/2(1-(12))$.

Now,

$$\mathrm{Tr}_{\yng(1,1)}(\Gamma_{\yng(2,1)}(\sigma)) = \mathrm{Tr}(P_{\yng(2,1),\yng(1,1)} \cdot \Gamma_{\yng(2,1)}).$$

And looking at Figure 14 we can see that we have now selected out the required block in Γ.

C Subgroup Swap Rule

In this appendix, we review the subgroup swap rule. The reader requiring a more detailed explanation can consult Appendix D of [30]. Consider the definition of a restricted Schur polynomial given in section 2.2. We need to

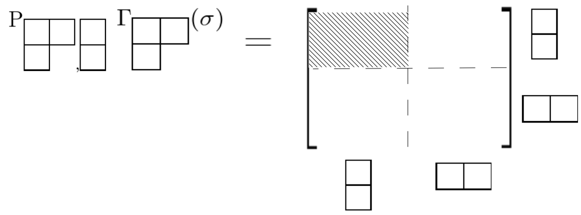


Figure 14: Projection operator selecting out a block on the diagonal.

specify the sequence of subgroups used to perform the restrictions. Consider the restricted Schur polynomial used in section 5.1

$$\chi^{(2)}_{R,R''}\Big|_1\Big|_2 = \frac{1}{(n-2)!}\sum_{\sigma\in S_n} \mathrm{Tr}_{R''}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-2}}_{i_{\sigma(n-2)}}(W^{(2)})^{i_{n-1}}_{i_{\sigma(n-1)}}(W^{(1)})^{i_n}_{i_{\sigma(n)}}.$$

The labelling on the left hand side tells us to first restrict with respect to the subgroup that leaves the index of $W^{(1)}$ inert, and then with respect to the subgroup that leaves the index of $W^{(2)}$ inert. In general, we will get a different polynomial if we were to restrict first with respect to the subgroup that leaves the index of $W^{(2)}$ inert, and then with respect to the subgroup that leaves the index of $W^{(1)}$ inert.

There is a relation between these two sets of polynomials, which is known as the "subgroup swap rule". This rule is especially important in the calculation of two point functions. The subgroup swap rule can be used to swap any two strings, but they must be next to each other. For example, we can swap strings n and $n-1$, but we cannot swap strings n and $n-2$. We would first have to swap strings n and $n-1$, then we would be able to swap n and $n-2$.

We will be using the weights of the boxes of the Young diagrams in the subgroup swap rule. All weights are defined by the Young diagram before the swap. The weight of the box labelled with upper index 1 is denoted by

c_1^U and the weight of the box labelled with lower index 1 is denoted by c_1^L. Similarly for index 2. We can now define “swap” and “no-swap” factors, the meaning of which will become clear with an example.

The upper and lower no-swap factors are given by

$$N_U = \sqrt{1 - \frac{1}{(c_1^U - c_2^U)^2}}, \qquad N_L = \sqrt{1 - \frac{1}{(c_1^L - c_2^L)^2}}.$$

The upper and lower swap factors are given by

$$S_U = \frac{1}{c_1^U - c_2^U}, \qquad S_L = \frac{1}{c_1^L - c_2^L}.$$

To understand how these factors are used, let us consider the following example. For two strings, the subgroup swap rule is

$$\chi_{\tiny\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}(n,n-1)\Big|_1\Big|_2 = \left[N_L N_U \chi_{\tiny\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}(n,n-1) + S_U N_L \chi_{\tiny\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & {}^{2}_{1} \\ \hline {}^{1}_{2} \\ \cline{1-1}\end{array}}(n,n-1) \right]\Big|_2\Big|_1$$

$$+ \left[S_L N_U \chi_{\tiny\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & {}^{1}_{2} \\ \hline {}^{2}_{1} \\ \cline{1-1}\end{array}}(n,n-1) + S_U S_L \chi_{\tiny\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & 2 \\ \hline 1 \\ \cline{1-1}\end{array}}(n,n-1) \right]\Big|_2\Big|_1$$

where the factors are given by the weights on the Young diagram as follows

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & c_1 \\ \hline c_2 \\ \cline{1-1}\end{array}$$

then

$$S_U = S_L = \frac{1}{c_1 - c_2},$$

$$N_U = N_L = \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}.$$

From the example it is now clear that whenever an index is swapped, we include a swap factor, and whenever there is no swap, we include a no-swap factor. When we review the reduction rule, and the two point function in the next appendix, we will see how the subgroup swap rule is used in computations.

We will conclude this appendix with an example which uses three strings. We will swap strings 2 and 3. For this case, the subgroup swap rule is

$$\begin{aligned}
\chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\3}},{\substack{3\\1}}}}(n,n-1)\Big|_1\Big|_2\Big|_3 &= \left[N_L N_U \chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\3}},{\substack{3\\1}}}}(n,n-1) + S_U N_L \chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{3},{\substack{2\\1}}}}(n,n-1) \right] \Big|_1\Big|_3\Big|_2 \\
&+ \left[S_L N_U \chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{2\\1}},{3}}}(n,n-1) + S_U S_L \chi_{\ytableaushort{{}{}{\substack{1\\2}},{}{\substack{3\\1}},{\substack{2\\3}}}}(n,n-1) \right] \Big|_1\Big|_3\Big|_2
\end{aligned}$$

where

$$N_U = \sqrt{1 - \frac{1}{(c_2^U - c_3^U)^2}}, \qquad N_L = \sqrt{1 - \frac{1}{(c_2^L - c_3^L)^2}}.$$

The upper and lower swap factors are given by

$$S_U = \frac{1}{c_2^U - c_3^U}, \qquad S_L = \frac{1}{c_2^L - c_3^L}.$$

D A Two Point Function Example

In this appendix we show how one calculates a two point function in order to illustrate the subgroup swap rule, and the reduction rule for Schur polynomials. For a complete description, consult [30].

Consider the following two point function

$$\langle\chi^{(2)}_{R,R''}(Z,W^{(1)},W^{(2)}\chi^{(2)\dagger}_{R,R''}(Z,W^{(1)},W^{(2)}\rangle = \alpha F^1_0F^2_0+\beta F^1_0F^2_1+\gamma F^1_1F^2_0+\delta F^1_1F^2_1, \tag{D.1}$$

where we use the language of [30].

D.1 Coefficient of $F^1_0F^2_0$

This coefficient is simply given by

$$\frac{n!}{(n-2)!}\frac{\dim_{R''}}{\dim_R}f_R.$$

D.2 Coefficient of $F^1_1F^2_0$

This coefficient is given by $\langle D_1\chi_{R,R''}(D_1\chi_{R,R''})^\dagger\rangle$ where D_1 indicates reduction with respect to the box labelled by string 1. At this point, we will simply state what the reduction rule is, the reader requiring more details can examine [30]. The reduction of a Schur polynomial sums all possible Schur polynomials that can be obtained by removing a single box that still leaves a valid Young diagram. The weight of the the removed box multiplies the reduced Schur polynomial.

$$D\chi_{(\lambda_1,\lambda_2,\dots,\lambda_r)} = (N+\lambda_1-1)\chi_{(\lambda_1-1,\lambda_2,\dots,\lambda_r)}+\cdots+(N+\lambda_r-r)\chi_{(\lambda_1,\lambda_2,\dots,\lambda_r-1)}. \tag{D.2}$$

So for the coefficient of $F^1_1F^2_0$, the operator D_1 is an instruction to remove the box with string 1 attached

$$\langle D_1\chi_{R,R''}(D_1\chi_{R,R''})^\dagger\rangle \quad = \quad \langle c_1\chi_{R',R''}c_1\chi^\dagger_{R',R''}\rangle$$

$$\begin{aligned} &= c_1^2 \frac{(n-1)!}{(n-2)!} \frac{\dim_{R''}}{\dim_{R'}} f_{R'} \\ &= c_1^2 (n-1) \frac{\dim_{R''}}{\dim_{R'}} f_{R'}. \end{aligned}$$

D.3 Coefficient of $F_0^1 F_1^2$

This coefficient is given by $\langle D_2 \chi_{R,R''} (D_2 \chi_{R,R''})^\dagger \rangle$ where D_2 indicates reduction with respect to the box labelled by string 2. In this case, we have to make use of the subgroup swap rule, as we cannot remove the box labelled by string 2 before removing the box labelled by string 1. Therefore, before calculating the coefficient, we need to calculate

$$\begin{aligned} D_2 \chi_{R,R''} \Big|_1 \Big|_2 &= D_2 (N_U N_L \chi_{R,S''} + S_U S_L \chi_{R,S''}) \Big|_2 \Big|_1 \\ &= D_2 \left(\left(1 - \frac{1}{(c_1 - c_2)^2} \right) \chi_{R,S''} + \frac{1}{(c_1 - c_2)^2} \chi_{R,R''} \right) \Big|_2 \Big|_1 \\ &= \left(1 - \frac{1}{(c_1 - c_2)^2} \right) c_2 \chi_{S',S''} + \frac{1}{(c_1 - c_2)^2} c_1 \chi_{R',R''}. \end{aligned}$$

We have neglected stretched string states since we will be removing a box, which will cause those states to disappear due to the Gauss law. Now, we can calculate the coefficient of $F_0^1 F_1^2$

$$\begin{aligned} \langle D_2 \chi_{R,R''} D_2 \chi_{R,R''}^\dagger \rangle &= \Big\langle \left[1 - \frac{1}{(c_1 - c_2)^2} c_2 \chi_{S',S''} + \frac{1}{(c_1 - c_2)^2} c_1 \chi_{R',R''} \right] \\ &\times \left[\left(1 - \frac{1}{(c_1 - c_2)^2} \right) c_2 \chi_{S',S''} + \frac{1}{(c_1 - c_2)^2} c_1 \chi_{R',R''} \right]^\dagger \Big\rangle \\ &= \left(1 - \frac{1}{(c_1 - c_2)^2} \right)^2 c_2^2 \langle \chi_{S',S''} \chi_{S',S''}^\dagger \rangle + \frac{1}{(c_1 - c_2)^4} c_1^2 \langle \chi_{R',R''} \chi_{R',R''}^\dagger \rangle \\ &= \left(1 - \frac{1}{(c_1 - c_2)^2} \right) c_2^2 \left[n \frac{\dim_{S''}}{\dim_{S'}} f_{S'} F_0 + c_1 f_{S'} F_1 \right] \\ &+ \frac{1}{(c_1 - c_2)^4} c_1^2 \left[n \frac{\dim_{R''}}{\dim_{R'}} f_{R'} F_0 + c_2 f_{R'} F_1 \right]. \end{aligned}$$

D.4 Coefficient of $F_1^1 F_1^2$

For this coefficient we can simply use the reduction rule given in (D.2)

$$\begin{aligned}\langle D_1 D_2 \chi_{R,R''} (D_1 D_2 \chi_{R,R''})^\dagger \rangle &= \langle (c_1 c_2)^2 \chi_{R''} c_1 \chi^\dagger_{R''} \rangle \\ &= (c_1 c_2)^2 f_{R''}.\end{aligned}$$

D.5 Determination of Subleading Terms

To determine which terms are subleading, we can make use of the identity given in section 5.1,

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z, W^{(1)}, W^{(2)})\Big|_1\Big|_2 &= \chi^{(1)}_{R',R''}(Z, W^{(2)})\mathrm{Tr}(W^{(1)}) + \frac{1}{c_1 - c_2}\chi^{(1)}_{R',R''}(Z, W^{(1)}W^{(2)}) \\ &+ \sum_\alpha \Bigg[\frac{1}{c_1 - c_\alpha}\left(1 - \frac{1}{(c_2 - c_\alpha)^2}\right)\chi^{(2)}_{R',T'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\ &+ \frac{1}{c_1 - c_2}\frac{1}{(c_2 - c_\alpha)^2}\chi^{(2)}_{R',R'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\ &+ \frac{1}{c_1 - c_2}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to R'''_\alpha T'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \\ &+ \frac{1}{c_1 - c_\alpha}\frac{1}{c_2 - c_\alpha}\sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}\chi^{(2)}_{R'\to T'''_\alpha R'''_\alpha}(Z, W^{(1)}Z, W^{(2)}) \Bigg] \Big|_1\Big|_2.\end{aligned}$$

By using the definition, $(W^{(1)})^i_j \equiv (Y^J)^i_j$, we find

$$\begin{aligned}\langle \mathrm{tr}(W^{(1)})\mathrm{tr}(W^{(1)})^\dagger \rangle &= \langle \mathrm{tr}(Y^J)\mathrm{tr}(Y^{\dagger J}) \rangle \\ &= \frac{N}{J+1}\left(\frac{(N+J)!}{N!} - \frac{(N-1)!}{(N-J-1)!}\right).\end{aligned}$$

The other two point functions will use the same methods given above. This is a simple, but tedious, calculation, so we do not give the full derivation here, we simply state the result. Comparing the terms on the left hand side with the terms on the right hand side, we find that the first two terms on the right hand side of the identity are subleading.

This analysis overestimates the size of the subleading terms when the open string includes impurities (by a factor of $O(J)$).

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