Hyperbolic groups of Fibonacci type and T(5) cyclically presented groups[⋆]

Ihechukwu Chinyere[a], Gerald Williams[a,*]

[a]*Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, UK.*

Abstract

Building on previous results concerning hyperbolicity of groups of Fibonacci type, we give an almost complete classification of the (non-elementary) hyperbolic groups within this class. We are unable to determine the hyperbolicity status of precisely two groups, namely the Gilbert-Howie groups $H(9,4), H(9,7)$. We show that if $H(9,4)$ is torsion-free then it is not hyperbolic. We consider the class of T(5) cyclically presented groups and classify the (non-elementary) hyperbolic groups and show that the Tits alternative holds.

Keywords: hyperbolic group, Tits alternative, cyclically presented group, Fibonacci group, small cancellation theory
2000 MSC: 20F05, 20F06, 20F67.

1. Introduction

The *groups of Fibonacci type* $G_n(m,k)$, defined by the presentations

$$P_n(m,k) = \langle x_0, \dots, x_{n-1} \mid x_i x_{i+m} = x_{i+k} \ (0 \le i < n) \rangle$$

($0 \le m, k < n$, subscripts mod n, $n \ge 2$), were introduced in [31],[6]. This family of groups contains many well-studied groups, such as the *Fibonacci groups* $F(2,n) = G_n(1,2)$ ([13]), the *Sieradski groups* $S(2,n) = G_n(2,1)$ ([44]), and the *Gilbert-Howie groups* $H(n,m) = G_n(m,1)$ ([22]). Much is known about their algebraic, geometric and topological properties (see, for example, [31],[6],[1],[30], the survey [50], and this article's references). For instance the aspherical presentations $P_n(m,k)$ were classified in [22],[48],[38] and (with the exceptions of the groups $H(9,4), H(9,7)$), the finite groups $G_n(m,k)$ were classified in [22],[48],[7],[41],[49]. For $n \ge 10$ there is a uniform pattern to these classifications that breaks down when $n \le 9$ (namely, for $n \ge 10$, the finite

groups $G_n(m, k)$ are cyclic and the non-aspherical presentations are those that correspond to these groups, the Fibonacci groups, and the Sieradski groups – see [50, Theorems 13,25,26]).

With the exceptions of the groups $H(9, 4)$, $H(9, 7)$ we classify the hyperbolic and non-elementary hyperbolic groups $G_n(m, k)$ and show that the Tits alternative holds. We also show that if $H(9, 4)$ is torsion-free then it is not hyperbolic. (Recall that a class of groups satisfies the Tits alternative if each group in the class either contains a non-abelian free subgroup or is virtually solvable.) In all but one of the cases where the group contains a non-abelian free subgroup, we show that it is SQ-universal. (A group G is *SQ-universal* if every countable group can be embedded in a quotient of G.) We set out the appropriate background precisely in Section 2, but essentially the key prior results are a classification of the hyperbolic Fibonacci groups in [23],[8], a result concerning hyperbolicity of the groups $H(n, 3)$ from [29], and the classification of the hyperbolic T(6) cyclically presented groups given in [11], together with computations using KBMAG [27] and MAF [46]. These results reduce the problem to the case $k = sn/p$ where $p \in \{3, 4, 5\}$ and $(s, p) = 1$. We prove the following:

Theorem A. *Suppose $n \geq 13$, $0 \leq m, k < n$, where $(n, m, k) = 1$. If $k = sn/p$ for some $p \in \{3, 4, 5\}$, $(s, p) = 1$ then $G_n(m, k)$ is non-elementary hyperbolic.*

As we show in Section 2, this implies the following classification of non-elementary hyperbolic and SQ-universal groups $G_n(m, k)$ for $n \geq 13$:

Corollary B. *Suppose $n \geq 13$, $0 \leq m, k < n$, and suppose $(n, m, k) = 1$, $k \neq 0, m \neq k$. Define $A = k, B = k - m$ and let $G = G_n(m, k)$. Then one of the following holds:*

(a) $A \equiv B \mod n$, *in which case* $G \cong \mathbb{Z}_{2^n - 1}$;

(b) $A \equiv n/2 \mod n$ *or* $B \equiv n/2 \mod n$, *in which case* $G \cong \mathbb{Z}_{2^{n/2} - (-1)^{m+n/2}}$;

(c) $A + B \equiv 0 \mod n$, *in which case* $G \cong S(2, n)$ *which is not hyperbolic and is SQ-universal;*

(d) $A + B \equiv n/2 \mod n$, *in which case* $G \cong H(n, n/2 + 2)$ *which is not hyperbolic and is SQ-universal;*

(e) $A \not\equiv n/2$, $B \not\equiv n/2$, $A + B \not\equiv 0$, $A - B \not\equiv 0$, $A + B \not\equiv n/2 \mod n$, *in which case G is non-elementary hyperbolic and hence SQ-universal.*

The hypothesis $(n, m, k) = 1$ prevents the group $G_n(m, k)$ decomposing as a free product in a natural way (see [1, Lemma 1.2]) and the hypothesis $k \neq 0, m \neq k$ (which is equivalent to $A \not\equiv 0, B \not\equiv 0 \mod n$) avoids the trivial groups $G_n(m, 0), G_n(k, k)$.

For $3 \leq n \leq 12$ we summarize the hyperbolicity and the Tits alternative status for the isomorphism classes of the groups $G_n(m, k)$ in Table 1; the only unresolved cases for hyperbolicity and for the Tits alternative are the groups $H(9, 4), H(9, 7)$. When combined with Corollary B we have the following:

Corollary C (The Tits alternative.)**.** *Suppose $n \geq 2$, $0 \leq m, k < n$, where $(n, m, k) = 1$ and let $G = G_n(m, k)$, and if $n = 9$ suppose $G \not\cong H(9, 4)$ or $H(9, 7)$.*

(a) *If $n \geq 11$ then G is either SQ-universal or finite cyclic;*

(b) *if $n \leq 10$ then G either contains a non-abelian free subgroup or is virtually solvable.*

We now discuss consequences of our results for T(5) cyclically presented groups. The *cyclically presented group $G_n(w)$* is the group defined by the *cyclic presentation*

$$P_n(w) = \langle x_0, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w) \rangle$$

where $w(x_0, \ldots, x_{n-1})$ is a cyclically reduced word in the free group F_n (of length $l(w)$) with generators $x_0, \ldots, x_{n-1}$ and $\theta : F_n \to F_n$ is the *shift automorphism* of F_n given by $\theta(x_i) = x_{i+1}$ for each $0 \leq i < n$ (subscripts mod n, $n \geq 2$). Therefore $G_n(m, k) = G_n(x_0 x_m x_k^{-1})$.

If a presentation satisfies T(5) then, as observed by Pride ([43, Section 5],[20, Lemma 3.1]), every piece has length 1. Thus if $P_n(w)$ satisfies T(5) and $l(w) > 3$ then it satisfies C(4)-T(5) and hence $G_n(w)$ is hyperbolic by [20, Corollary 4.1], and therefore non-elementary hyperbolic by [12], and hence SQ-universal by [15], [42]. If the length $l(w) = 1$ then $G_n(w)$ is trivial, and if $l(w) = 2$ then $G_n(w)$ is the free product of copies of $\mathbb{Z}$ or $\mathbb{Z}_2$. Therefore we must consider the case $l(w) = 3$. When w is a positive word of length 3 the star graph of $P_n(w)$ is bipartite so, by [24], $P_n(w)$ satisfies T(5) if and only if it satisfies T(6), in which case the non-elementary hyperbolic groups $G_n(w)$ were classified in [11, Theorem A], and $G_n(w)$ is SQ-universal if and only if $n \neq 7$ by [28] (see [39, Corollary 3.6], [18, Corollary 5.2]) and contains a non-abelian free subgroup when $n = 7$ by [17].

Thus we may assume that w is a non-positive word of length 3, and so $G_n(w) \cong G_n(m, k)$ for some $0 \leq m, k < n$. As in [29, Theorem 10], [11, Table 2], consideration of the girth of the star graph of $P_n(m, k)$ shows that the presentation satisfies T(5) if and only if $A \pm B \not\equiv 0 \mod n$ and $tA \not\equiv 0, tB \not\equiv 0 \mod n$ for any $1 \leq t \leq 4$. Corollary B(d),(e) and Table 1 then classify the T(5) hyperbolic groups $G_n(m, k)$ and show that the Tits alternative holds (note that there are no unresolved cases here, since the cyclic presentations of $H(9, 4)$ and $H(9, 7)$ do not satisfy $T(5)$). We record this in the case $n \geq 11$ in the following corollary:

Corollary D. *Suppose $n \geq 11$, $0 \leq m, k < n$, $(n, m, k) = 1$ and that $P_n(m, k)$ satisfies $T(5)$. Then (a) $G_n(m, k)$ is SQ-universal and (b) $G_n(m, k)$ is non-elementary hyperbolic if and only if $2k - m \not\equiv n/2 \mod n$.*

For $n \leq 10$ the T(5) groups $G_n(m, k)$ are as follows: $F(2, 5) \cong \mathbb{Z}_{11}$, $F(2, 7) \cong \mathbb{Z}_{29}$, $H(7, 3)$ (which is virtually $\mathbb{Z}^8$), $H(10, 7)$ (which is not hyperbolic and contains a non-abelian free subgroup), and $F(2, 9)$, $H(9, 3)$, $F(2, 10)$, $H(10, 3)$, $H(10, 5)$ (which are non-elementary hyperbolic and hence SQ-universal).

In Section 2 we survey known results which, either directly or indirectly, concern hyperbolicity of groups $G_n(m, k)$. We extend them to show that groups known to be hyperbolic are in fact non-elementary hyperbolic, that certain previously unaddressed groups are non-elementary hyperbolic and that others are not hyperbolic. With the exceptions of the groups $H(9, 4)$ and $H(9, 7)$ this reduces the question of hyperbolicity to the situation of Theorem A and leads to the classification given in Corollary B. In Section 3 we prove Theorem A in the case $n > 7p$, and report results of computations performed using KBMAG that deal with the cases $13 \leq n \leq 7p$.

2. Review and recontextualization of prior results

We express some of our results in terms of congruences mod n in parameters $A = k, B = k - m$. Note that $(n, m, k) = 1$ if and only if $(n, A, B) = 1$. We summarise the results of this section for $n \geq 13$ in Table 2 and for $3 \leq n \leq 12$ in Table 1. A hyperbolic group is elementary hyperbolic if it is virtually cyclic and is non-elementary hyperbolic otherwise. Recall that finite groups are elementary hyperbolic, groups that contain $\mathbb{Z} \oplus \mathbb{Z}$ or any Baumslag-Solitar group are not hyperbolic, and that non-elementary hyperbolic groups are SQ-universal by [15],[42]. Since a torsion-free group is virtually $\mathbb{Z}$ if and only if it is isomorphic to $\mathbb{Z}$ (see, for example, [35, Lemma 3.2]) any non-trivial, torsion-free, hyperbolic group with finite abelianisation is non-elementary hyperbolic. We use these results freely. The isomorphisms stated below can be obtained from the isomorphisms given in [1, Section 1].

2.1. The Fibonacci groups $F(2, n)$

If $(n, A, B) = 1$ and either $A - 2B \equiv 0 \bmod n$ or $B - 2A \equiv 0 \bmod n$ then $G_n(m, k) \cong F(2, n) = G_n(1, 2)$ and the corresponding cyclic presentation satisfies the small cancellation condition T(5) when $n = 7$ or $n \geq 9$.

Theorem 2.1 ([23, Theorem C]). *Suppose $n \geq 8$ is even. Then $F(2, n)$ is the fundamental group of an orientable hyperbolic manifold.*

Theorem 2.2 ([23, Statement (P3)]). *The group $F(2, 6)$ is virtually $\mathbb{Z}^3$.*

(In fact, a computation in GAP [19] shows that the derived subgroup of $F(2, 6)$ is isomorphic to $\mathbb{Z}^3$.) We now turn to the case n odd. The group $F(2, 9)$ was proved infinite in [40],[25]; a computation in KBMAG shows that $F(2, 9)$ is hyperbolic (see [26, Section 13.4]). For $n \geq 9$ and odd, $F(2, n)$ was proved infinite in [9, Theorem 4.0]. For $n \geq 11$ and odd the group $F(2, n)$ has recently been proved hyperbolic in [8].

Theorem 2.3 ([8]). *Suppose $n \geq 11$ is odd. Then $F(2, n)$ is hyperbolic.*

We also have $F(2, 2) = 1$, $F(2, 3) \cong Q_8$, $F(2, 4) \cong \mathbb{Z}_5$, $F(2, 5) \cong \mathbb{Z}_{11}$, $F(2, 7) \cong \mathbb{Z}_{29}$. Using these results, we summarize the hyperbolicity and SQ-universality status of the groups $F(2, n)$ in the following corollary, where we show that the hyperbolic groups are non-elementary hyperbolic.

4

Corollary 2.4. *(a) If $n \in \{2, 3, 4, 5, 7\}$ then $F(2, n)$ is finite;*

(b) if $n = 6$ then $F(2, n)$ is virtually $\mathbb{Z}^3$ and hence is neither hyperbolic nor SQ-universal;

(c) if $n \geq 8$ then $F(2, n)$ is non-elementary hyperbolic and hence SQ-universal.

Proof. The cases $n \leq 7$ follow from Theorem 2.2 and remarks immediately before this corollary.

Suppose $n \geq 8$ and even. Then $G = F(2, n)$ is hyperbolic (by Theorem 2.1), torsion-free (Statement (P4) of [23]), and has non-trivial finite abelianisation ([14]), so it is non-elementary hyperbolic.

Suppose then that $n \geq 9$ and odd. Since $G = F(2, n)$ is infinite and hyperbolic (by Theorem 2.3, and KBMAG in the case $n = 9$) it is either non-elementary hyperbolic or virtually $\mathbb{Z}$. Suppose for contradiction that G is virtually $\mathbb{Z}$. Then G maps onto $\mathbb{Z}$ or $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ (see, for example, [33, Theorem 17]). Since G^{ab} is finite G does not map onto $\mathbb{Z}$. Suppose, for contradiction, that $\phi : G \to D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is an epimorphism. By [1, Proposition 3.1] the element $g = x_0 x_1 \ldots x_{n-1}$ has order 2 in G and the derived subgroup $G' = \langle g \rangle^G$. Now $\phi(G') = D'_\infty$ so $\phi(g) = 1$ or is an element of order 2 in D'_∞. But the derived subgroup D'_∞ is free so has no order 2 elements, so $\phi(g) = 1$. But since $G' = \langle g \rangle^G$, if $\phi(g) = 1$ then $\phi(G)$ is abelian, a contradiction. $\square$

2.2. The Sieradski groups $S(2, n)$

If $(n, A, B) = 1$ and $A + B \equiv 0 \bmod n$ then $G_n(m, k) \cong S(2, n) = G_n(2, 1)$.

Theorem 2.5 ([44],[5])**.** *Suppose $n \geq 2$. Then $S(2, n)$ is isomorphic to a properly discontinuous cocompact group of isometries that acts without fixed points on a space X_n where $X_n = S^3$ for $n < 6$, $X_6 = Nil$, $X_n = \widetilde{SL(2, \mathbb{R})}$ for $n \geq 7$.*

Theorem 2.6 ([37, Theorem 3.2.2.1])**.** *Suppose $n \geq 7$. Then $S(2, n)$ is SQ-universal.*

(It had been previously shown in [45, Theorem C] that if $n \geq 7$ then $S(2, n)$ contains a non-abelian free subgroup.)

Theorem 2.7 ([45, Theorem D])**.** *Suppose $n = 6$. Then $S(2, n)$ is an infinite metabelian group that contains $\mathbb{Z}^2$.*

Cocompact groups of isometries of $\widetilde{SL(2, \mathbb{R})}$ are not hyperbolic by [4] so, together with the observation that $S(2, 2) \cong \mathbb{Z}_3$, $S(2, 3) \cong Q_8$, $S(2, 4) \cong SL(2, 3)$, $S(2, 5) \cong SL(2, 5)$ (the special linear groups), we have the following classification.

Corollary 2.8. *(a) If $n < 6$ then $S(2, n)$ is finite;*

(b) if $n = 6$ then $S(2, n)$ is not hyperbolic and is not SQ-universal;

(c) if $n \geq 7$ then $S(2, n)$ is not hyperbolic and is SQ-universal.

2.3. The Gilbert-Howie groups $H(n, 3)$

If $(n, A, B) = 1$ and either $A + 2B \equiv 0 \mod n$ or $B + 2A \equiv 0 \mod n$ then $G_n(m, k) \cong H(n, 3) = G_n(3, 1)$ and the corresponding cyclic presentation satisfies the small cancellation condition T(5) when $n = 7$ or $n \geq 9$.

Theorem 2.9 ([29, Theorem 13, Corollary 14]). *Suppose $n \geq 11$, $n \neq 12, 14$. Then $H(n, 3)$ is non-elementary hyperbolic and hence SQ-universal.*

The groups $H(10, 3)$, $H(12, 3)$, $H(14, 3)$ can be shown to be hyperbolic using KBMAG. The group $H(9, 3)$ was shown to be infinite in [7, Lemma 15] and Alun Williams [47] has shown it is hyperbolic using MAF, and Derek Holt and Christopher Chalk [10] have confirmed this using KBMAG.

Theorem 2.10 ([47]). *The group $H(9, 3)$ is hyperbolic.*

We also have $H(2, 3) = 1$, $H(3, 3) \cong \mathbb{Z}_7$, $H(4, 3) \cong \mathbb{Z}_5$, $H(5, 3) \cong \mathbb{Z}_{11}$, $H(6, 3) \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$ and $H(8, 3)$ is a finite solvable group of order $3^{10} \cdot 5$ and of derived length 3 [1, Table 4]. The group $H(7, 3)$ is virtually $\mathbb{Z}^8$, as observed by R.M. Thomas [22, page 129] (the second derived subgroup of $H(7, 3)$ is isomorphic to $\mathbb{Z}^8$). We summarize the hyperbolicity and SQ-universality status of the groups $H(n, 3)$ in the following corollary, where we show that the hyperbolic groups are non-elementary hyperbolic.

Corollary 2.11. (a) *If $n \in \{2, 3, 4, 5, 6, 8\}$ then $H(n, 3)$ is finite;*

(b) *if $n = 7$ then $H(n, 3)$ is not hyperbolic and is not SQ-universal;*

(c) *if $n \geq 9$ then $H(n, 3)$ is non-elementary hyperbolic and hence SQ-universal.*

Proof. (a),(b) follow from the comments immediately before this corollary, so consider (c). By Theorem 2.9 we may assume $n \in \{9, 10, 12, 14\}$, in which case the group $H(n, 3)$ is hyperbolic, as reported above. If $n \geq 10$ then, since its defining cyclic presentation is aspherical [22, Theorem 3.2], $H(n, 3)$ is torsion-free, and given that it has non-trivial finite abelianisation, it is non-elementary hyperbolic. Suppose then that $n = 9$. Since $G = H(9, 3)$ is infinite and hyperbolic (by [7, Lemma 15] and Theorem 2.10) it is either non-elementary hyperbolic or virtually $\mathbb{Z}$. Suppose for contradiction that G is virtually $\mathbb{Z}$. Then G maps onto $\mathbb{Z}$ or $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$. But since $G^{\mathrm{ab}} \cong \mathbb{Z}_7$ is finite of odd order, G does not map onto $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$, a contradiction. $\square$

2.4. The Gilbert-Howie groups $H(n, n/2 + 2)$

If $(n, A, B) = 1$ and $A + B \equiv n/2 \mod n$ (where n is even) then the group $G_n(m, k) \cong H(n, n/2 + 2)$ (see [11, Lemma 3.3]) and the corresponding cyclic presentation satisfies the small cancellation condition T(6) when $n = 8$ or $n \geq 12$.

Theorem 2.12 ([11, Lemma 3.6]). *Suppose $n \geq 8$ is even, $n \neq 10$. Then $H(n, n/2 + 2)$ is not hyperbolic.*

6

Note that for $n \geq 8$, $n \neq 10$ the groups $H(n, n/2 + 2)$ are SQ-universal by [29, Corollary 11]. We now extend this to the case $n = 10$. We need the following known result concerning solvable Baumslag-Solitar groups.

Proposition 2.13. *Let* $B = BS(1, q) = \langle a, b \mid bab^{-1} = a^q \rangle$ *where* $q \neq 1$. *In any proper quotient of* B, *the image of either* a *or* b *has finite order.*

Proof. Let $\phi : B \to Q$ be an epimorphism where Q is a proper quotient of B. As shown in [3, page 154] any element of B can be written in the form $b^{-r}a^sb^t$ for some integers r, s, t where $r, t \geq 0$. Suppose $g = b^{-r}a^sb^t$ is a non-trivial element of $\ker(\phi)$.

Then $\phi(b^{-r}a^sb^t) = 1$ so $\phi(a^s) = \phi(b^{r-t})$. Now the defining relation of B implies $a^s = b^{-1}a^{qs}b$ and so $\phi(a^s) = \phi(b^{-1}a^{qs}b)$ and hence $\phi(b^{r-t}) = \phi(b^{-1}a^{qs}b)$; that is, $\phi(a^{qs}) = \phi(b^{r-t})$. Therefore $\phi(a^s) = \phi(a^{qs})$ so $\phi(a^{(q-1)s}) = 1$. If $s \neq 0$ then $\phi(a)$ has finite order, so assume $s = 0$. Therefore $g = b^{t-r}$ and $t \neq r$ since $g \neq 1$ and b has infinite order in B. Then $\phi(b^{t-r}) = 1$ so $\phi(b)$ has finite order, as required. $\qquad\square$

Lemma 2.14. *Let* $G = H(10, 7)$. *Then*

(a) G *is not hyperbolic;*

(b) G *contains a non-abelian free subgroup.*

Proof. (a) Let $a = x_0x_5$, $b = x_1x_0x_2x_1x_3$. Figure 1 depicts a van Kampen diagram with boundary $abab^{-1}$, and hence the relation $abab^{-1} = 1$ holds in G. Let L be the subgroup of G generated by a and b. Therefore L is a quotient of the Baumslag-Solitar group $BS(1, -1)$ (or Klein bottle group) $K = \langle a, b \mid abab^{-1} \rangle$.

If $\phi : G \to G_5(x_0x_2x_1^{-1}) \cong SL(2, 5)$ is the natural epimorphism given by $x_i \mapsto x_{i \bmod 5}$ then using GAP we can show that $\phi(a) \neq 1$ and $\phi(b) \neq 1$ and hence a, b are non-trivial elements of G. By [22, Theorem 3.2] G is torsion-free, so L is non-trivial and torsion-free. But by Proposition 2.13 in any proper quotient of L the image of either a or b has finite order so $K \cong L$. Hence G contains the Baumslag-Solitar group $BS(1, -1)$, so is not hyperbolic.

(b) By eliminating the odd numbered generators of $H(10, 7)$ and defining $y_j = x_{2j}$ we obtain the 5-generator cyclic presentation $G_5(y_0^{-1}y_1y_4^{-1}y_2y_4^{-1})$. Using this presentation, a computation using KBMAG shows that the subgroup of G generated by y_0^2, y_4^2 is (of infinite index and) free of rank 2. $\qquad\square$

Observing $H(2, 3) = 1$, $H(4, 4) \cong \mathbb{Z}_{15}$ and $H(6, 5) \cong F(2, 6)$, which is virtually $\mathbb{Z}^3$, we have the following:

Corollary 2.15. (a) *If* $n \in \{2, 4\}$ *then* $H(n, n/2 + 2)$ *is finite;*

(b) *if* $n = 6$ *then* $H(n, n/2 + 2)$ *is virtually* $\mathbb{Z}^3$, *and hence is neither hyperbolic nor SQ-universal;*

(c) *if* $n = 10$ *then* $H(n, n/2 + 2)$ *is not hyperbolic and contains a non-abelian free subgroup;*

(d) *if* $n \geq 8$, $n \neq 10$ *then* $H(n, n/2 + 2)$ *is not hyperbolic and is SQ-universal.*
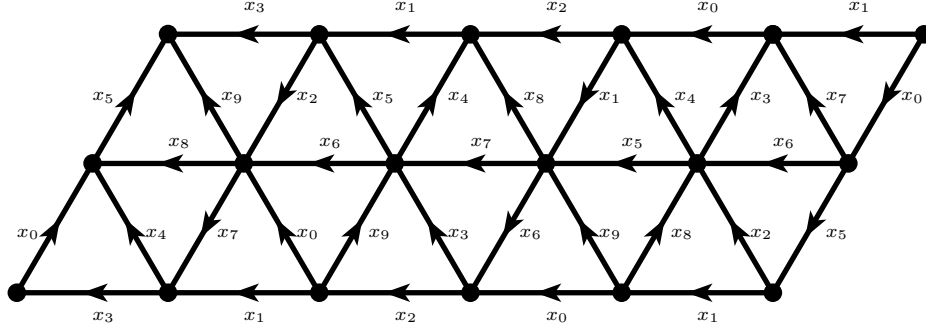
Figure 1: A van Kampen diagram over the presentation $P_{10}(x_0 x_7 x_1^{-1})$ of the group $H(10, 7)$, with boundary label $abab^{-1}$ where $a = x_0 x_5$, $b = x_1 x_0 x_2 x_1 x_3$.

2.5. Finite cyclic groups $G_n(m, k)$

If $(n, A, B) = 1$ and either $A \equiv n/2 \bmod n$ or $B \equiv n/2 \bmod n$ then $G_n(m, k)$ is a finite cyclic group:

Theorem 2.16 ([48, Lemma 3]). *Suppose $n \geq 2$ is even and either $k \equiv n/2$ or $(k - m) \equiv n/2 \bmod n$. Then $G_n(m, k) \cong \mathbb{Z}_{2^{n/2} - (-1)^{m+n/2}}$.*

If $(n, A, B) = 1$ and $A \equiv B \bmod n$ then $G_n(m, k)$ is a finite cyclic group:

Theorem 2.17 ([1, Lemma 1.1(1)]). *Suppose $n \geq 2$ and $m \equiv 0 \bmod n$. Then $G_n(m, k) \cong \mathbb{Z}_{2^n - 1}$.*

If $A \equiv 0 \bmod n$ or $B \equiv 0 \bmod n$ then $G_n(m, k)$ is trivial.

2.6. The T(6) hyperbolic groups $G_n(m, k)$

The non-elementary hyperbolic T(6) groups $G_n(m, k)$ were classified in [11]. The non-hyperbolic groups are those in Theorem 2.12, with the remainder being non-elementary hyperbolic, and hence SQ-universal. We record this in the following theorem (note that T(6) groups do not arise if $n < 8$).

Theorem 2.18 ([11, Theorem B]). *Suppose $n \geq 8$, $(n, A, B) = 1$ and that none of the following hold (congruences mod n): $B \equiv n/2$, $B \equiv \pm n/3$, $B \equiv \pm n/4$, $B \equiv sn/5$ $(s, 5) = 1$, $A \equiv n/2$, $A \equiv \pm n/3$, $A \equiv \pm n/4$, $A \equiv sn/5$ $(s, 5) = 1$, $A \pm B \equiv 0$, $A \pm 2B \equiv 0$, $B \pm 2A \equiv 0$, $A + B \equiv n/2$. Then $G_n(m, k)$ is non-elementary hyperbolic, and hence SQ-universal.*

2.7. The groups $G_n(m, k)$ with $n \leq 12$

Corollary B will show that a uniform pattern of behaviour holds for the groups $G_n(m, k)$ when $n \geq 13$ so in this section we describe all groups $G_n(m, k)$ with $(n, m, k) = 1$ and $n \leq 12$, up to isomorphism. In particular, in Lemma 2.22 we show that the uniform pattern breaks down for $n \leq 12$.

For $n \leq 6$ all groups $G_n(m,k)$ were identified in [31] and (with a few unresolved cases) the isomorphism classes of the groups $G_n(m,k)$ with $n \leq 27$ were obtained in [7, Section 4]. We record the (non-elementary) hyperbolicity and the Tits alternative status of the groups $G_n(m,k)$ with $0 < m, k < n$, $m \neq k$, $(n,m,k) = 1$ for $n \leq 12$ in Table 1. This updates much of [50, Section 4] and extends it to include the cases $n = 10, 11, 12$. (For conciseness we omit the case $m = 0$ from Table 1, since this is finite cyclic by Theorem 2.17.) We need the following corollary to results from [11],[29] concerning the $T(6)$ group $H(8,4)$.

Theorem 2.19 ([11, Corollary 2.8],[29, Corollary 11]). *The group $H(8,4)$ is not hyperbolic and it is SQ-universal.*

The groups $H(9,4), H(9,7)$ and their defining cyclic presentations are known to be challenging objects of study (see, for example, [50, Section 4]). Their cyclic presentations correspond to unresolved asphericity cases in [16] and have recently been proved non-aspherical in [38], where it is also shown that $H(9,4)$ is not a 3-manifold group and that $H(9,7)$ is a 3-manifold group if and only if it is cyclic of order 37.

It is an open question as to whether or not $H(9,4)$ is torsion-free ([38, Question 3, page 89]). (Whereas $H(9,7)$ has non-trivial torsion [38, Corollary B.1].) If $H(9,4)$ is torsion-free, then the Baumslag-Solitar relation found in [38, Section 5.4] allows us to prove that it is not hyperbolic.

Lemma 2.20. *Suppose every finite subgroup of $H(9,4)$ has order at most 2. Then $H(9,4)$ contains the Baumslag-Solitar group $BS(1,-8)$ and hence is not hyperbolic.*

Proof. As in the proof of [38, Proposition 5.5.2], since every finite subgroup of $H(9,4)$ has order at most 2, every finite subgroup of $E = H(9,4) \rtimes_\theta \langle t \mid t^9 \rangle = \langle y, t \mid t^9, y^2 t y^{-1} t^3 \rangle$ has order dividing 18 (where θ is the shift automorphism). Then as in [38, Section 5.4] there is a Baumslag-Solitar relation $bab^{-1} = a^{-8}$ in E in which $a = t^4 y$ and b have infinite order.

Let L be the subgroup of E generated by a and b. Then L is a quotient of the Baumslag-Solitar group $BS(1,-8) = \langle a, b \mid bab^{-1} = a^{-8} \rangle$. But by Proposition 2.13 in any proper quotient of $BS(1,-8)$ the image of at least one of the generators has finite order, so $L \cong BS(1,-8)$. Hence E contains the Baumslag-Solitar group $BS(1,-8)$, so is not hyperbolic. Since $H(9,4)$ is a finite index subgroup of E, it is not hyperbolic. $\square$

We also have the following:

Example 2.21. Computations in KBMAG show that the groups $H(10,5)$, $H(12,4)$, $H(12,9)$, $H(12,10)$, $G_{12}(1,3)$, $G_{12}(1,9)$ are hyperbolic. Further, the corresponding cyclic presentations are aspherical by [22, Theorem 3.2],[48, Theorem 2] so the groups are torsion-free; since they also have finite and non-trivial abelianisation, they are non-elementary hyperbolic.

In the following lemma we consider the group $H(12,5)$. This shows, in particular, that Corollary B (specifically part (e)) cannot be extended to $n = 12$.

Lemma 2.22. *Let* $G = H(12, 5)$. *Then*

   (a) G *is not hyperbolic;*

   (b) G *is SQ-universal.*

Proof. (a) A computation in GAP shows that the subgroup K of G generated by the set of elements $\{x_0^{-2}, x_2^{-2}, x_3 x_0^{-1}, x_3^{-1} x_0^{-1}, x_4 x_1, x_4^{-1} x_1^{-1}, x_1^{-4}\}$ is normal, with $G/K \cong D_8$, and contains a pair of commuting elements that project onto generators of the free abelian group of rank 2. Hence G contains a free abelian group of rank 2, so is not hyperbolic.

   (b) Using Magma [2] the group G can be shown to be large, and hence SQ-universal. $\qquad\blacksquare$

2.8. The outstanding cases

   Sections 2.1–2.7 (summarized in Tables 1,2) show that the cases that remain to be considered are $A \equiv sn/p \bmod n$ or $B \equiv sn/p \bmod n$, where $p \in \{3, 4, 5\}$, $(s, p) = 1$, and $n \geq 13$. By applying the isomorphism $G_n(m, k) \cong G_n(n - m, n - m + k)$ ([1, Lemma 1.1(3)]) we may assume $A \equiv sn/p \bmod n$; that is $k \equiv sn/p \bmod n$. This case is dealt with in Theorem A; Corollary B then follows from the discussion above.

n	Name	Group	Order	Non-elem. hyp.	Tits alt.
3	$F(2,3)$	Q_8	8	No	finite
4	$F(2,4)$	cyclic	5	No	finite
	$S(2,4)$	$SL(2,3)$	24	No	finite
5	$F(2,5)$	cyclic, T(5)	11	No	finite
	$S(2,5)$	$SL(2,5)$	120	No	finite
6	$F(2,6)$	3-mfd. gp.	∞	No	virtually $\mathbb{Z}^3$
	$S(2,6)$	3-mfd. gp.	∞	No	metabelian
	$H(6,3)$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$	56	No	finite
	$H(6,4)$	cyclic	9	No	finite
	$G_6(1,3)$	cyclic	7	No	finite
7	$F(2,7)$	cyclic, T(5)	29	No	finite
	$S(2,7)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(7,3)$	T(5)	∞	No	virtually $\mathbb{Z}^8$
8	$F(2,8)$	3-mfd. gp.	∞	Yes	SQ-univ.
	$S(2,8)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(8,3)$	solvable	$3^{10} \cdot 5$	No	finite
	$H(8,4)$	T(6)	∞	No	SQ-univ.
	$H(8,5)$	cyclic	17	No	finite
	$H(8,6)$	T(6)	∞	No	SQ-univ.

n	Name	Group	Order	Non-elem. hyp.	Tits alt.
9	$F(2,9)$	T(5)	∞	Yes	SQ-univ.
	$S(2,9)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(9,3)$	T(5)	∞	Yes	SQ-univ.
	$H(9,4)$		?	Lem. 2.20	?
	$H(9,7)$		?	?	?
10	$F(2,10)$	3-mfd. gp., T(5)	∞	Yes	SQ-univ.
	$S(2,10)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(10,3)$	T(5)	∞	Yes	SQ-univ.
	$H(10,7)$	T(5)	∞	No	Free subgroup
	$H(10,4)$	T(6)	∞	Yes	SQ-univ.
	$H(10,5)$	T(5)	∞	Yes	SQ-univ.
	$H(10,6)$	cyclic	33	No	finite
	$G_{10}(1,6)$	cyclic	31	No	finite
11	$F(2,11)$	T(5)	∞	Yes	SQ-univ.
	$S(2,11)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(11,3)$	T(5)	∞	Yes	SQ-univ.
	$H(11,4)$	T(6)	∞	Yes	SQ-univ.
	$H(11,8)$	T(6)	∞	Yes	SQ-univ.
12	$F(2,12)$	3-mfd. gp., T(5)	∞	Yes	SQ-univ.
	$S(2,12)$	3-mfd. gp.	∞	No	SQ-univ.
	$H(12,3)$	T(5)	∞	Yes	SQ-univ.
	$H(12,8)$	T(6)	∞	No	SQ-univ.
	$H(12,7)$	cyclic	65	No	finite
	$H(12,6)$	T(6)	∞	Yes	SQ-univ.
	$H(12,5)$		∞	No	SQ-univ.
	$H(12,4)$		∞	Yes	SQ-univ.
	$H(12,9)$		∞	Yes	SQ-univ.
	$H(12,10)$		∞	Yes	SQ-univ.
	$G_{12}(1,3)$		∞	Yes	SQ-univ.
	$G_{12}(1,9)$		∞	Yes	SQ-univ.

Table 1: All groups $G_n(m,k)$ with $0 < m, k < n$, $m \neq k$, $(n, m, k) = 1$ and $3 \leq n \leq 12$, up to isomorphism.

3. The proof of Theorem A

In Section 3.1 we prove Theorem A in the case $n > 7p$, and in Section 3.2 we report results of computations that prove it in the cases $13 \leq n \leq 7p$.

3.1. Proof of Theorem A in the case $n > 7p$

Following the method of proof of [29, Theorem 13] we show that G has a linear isoperimetric function [21, Theorem 3.1]. That is, we show that there

Congruence mod n	Group	Non-elem. hyp.	Tits alt.
$B \equiv n/2$ or $A \equiv n/2$	$\mathbb{Z}_{2^{\frac{n}{2}}-(-1)^{m+\frac{n}{2}}}$	No	finite
$B \equiv sn/p$ or $A \equiv sn/p$, $(s,p)=1$, $p \in \{3,4,5\}$		Yes	SQ-univ.
$A + B \equiv 0$	$S(2,n)$	No	SQ-univ.
$A - B \equiv 0$	$\mathbb{Z}_{2^n-1}$	No	finite
$A - 2B \equiv 0$ or $B - 2A \equiv 0$	$F(2,n)$, T(5)	Yes	SQ-univ.
$A + 2B \equiv 0$ or $B + 2A \equiv 0$	$H(n,3)$, T(5)	Yes	SQ-univ.
$B + A \equiv n/2$	$H(n,\frac{n}{2}+2)$, T(6)	No	SQ-univ.
None of the above	T(6)	Yes	SQ-univ.

Table 2: Groups $G_n(m,k)$ with $0 \le m,k < n$, $m \neq k$, $(n,m,k)=1$ and $n \ge 13$, in terms of congruences (mod n) in parameters $A = k, B = k - m$.

is a linear function $f : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ and all freely reduced words $W \in F_n$ with length at most N that represent the identity of G we have Area$(W) \le f(N)$, where Area(W) denotes the minimum number of faces in a reduced van Kampen diagram over P with boundary label W. Without loss of generality we may assume that the boundary of such a van Kampen diagram is a simple closed curve. Throughout this section D will denote a reduced van Kampen diagram over the presentation $P_n(m,k)$ whose boundary is a simple closed curve.

Let Γ be the star graph ([34, page 61]) of the cyclic presentation $P_n(x_0 x_m x_k^{-1})$. Then Γ has vertices x_i, x_i^{-1} and edges $x_i - x_{i+m}^{-1}$, $x_i - x_{i+B}$, $x_i^{-1} - x_{i+A}^{-1}$ $(0 \le i < n)$, which we will refer to as edges of type X, Y, Z, respectively. Each face in D is a triangle, as shown in Figure 2, where the corner labels X, Y, Z correspond to the edge types of Γ. Note that labels of cycles in Γ cannot contain a cyclic subword XX, and hence labels of vertices cannot contain the cyclic subword XX. We say that a vertex of D is a *boundary vertex* if it lies on the boundary ∂D, and is an *interior vertex* otherwise. We write I, B, F to denote the sets of interior vertices, boundary vertices, and faces of D.

Writing π to denote 180, we assign angles between 0 and π to the corners of faces of D and define the curvature of a face f by $\kappa(f) = -\pi +$ (sum of angles in f), the curvature of an interior vertex v by $\kappa(v) = 2\pi -$ (sum of angles at v), and the curvature of a boundary vertex $\hat{v}$ by $\kappa(\hat{v}) = \pi -$(sum of angles at $\hat{v}$). We use the following consequence of the Gauss-Bonnet Theorem:

Lemma 3.1. *If for every reduced van Kampen diagram over a presentation defining a group G there exists an angle assignment on each corner at every*
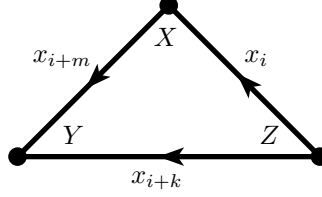
Figure 2: A typical face in a van Kampen diagram over the presentation $P_n(x_0 x_m x_k^{-1})$.

vertex such that

$$\sum_{v \in I} \kappa(v) \le 0 \quad \text{and} \quad \sum_{f \in F} \kappa(f) \le -|F|$$

then G has a linear isoperimetric function.

Proof. It follows from the Gauss-Bonnet Theorem that

$$\sum_{v \in I} \kappa(v) + \sum_{\hat{v} \in B} \kappa(\hat{v}) + \sum_{f \in F} \kappa(f) = 2\pi$$

(see [36, Section 4] and the references therein). Therefore the hypotheses imply

$$2\pi \le \sum_{v \in I} 0 + \sum_{\hat{v} \in B} (\pi - \text{sum of angles at } \hat{v}) + \sum_{f \in F} (-1)$$
$$= |B|\pi - \sum_{\hat{v} \in B} (\text{sum of angles at } \hat{v}) - |F|$$

so

$$\sum_{\hat{v} \in B} \text{sum of angles at } \hat{v} \le (|B| - 2)\pi - |F|.$$

Now let $\gamma\pi$ $(0 < \gamma < 1)$ be the minimum angle assigned to a corner of a boundary vertex. Then the sum of angles over the boundary vertices is bounded below by $\gamma\pi|B|$ and hence $\gamma\pi|B| \le (|B| - 2)\pi - |F|$ so $|F| \le 180(1-\gamma)|B| - 360$. If W is the boundary label of D then $\text{Area}(W) \le |F|$ so $\text{Area}(W) \le 180(1-\gamma)|B| - 360$ and hence $f(N) = 180(1-\gamma)N - 360$ is a linear isoperimetric function satisfied by G. $\square$

The following lemma and corollary identify the labels of interior vertices of degree less than 8:

Lemma 3.2. *Let $p \in \{3, 4, 5\}$, $n > 7p$, $1 \le m, k < n$, $m \ne k$, where $(n, m, k) = 1$ and set $A = k$, $B = k - m$. If $A = sn/p$ where $(s, p) = 1$ then any interior vertex of D of degree less than 8 has label Z^3 or Z^6, when $p = 3$, label Z^4 when $p = 4$, or label Z^5 when $p = 5$.*

Proof. As explained in the proof of [29, Theorem 10], if the star graph Γ contains a cycle C of length less than 8 then C involves α Z edges and β Y edges where $\alpha A \pm \beta B \equiv 0 \bmod n$ and $\{\alpha, \beta\} \in \{\{t, 0\} \ (1 \leq t \leq 7), \{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 2\}, \{2, 3\}\}$. In particular, $0 \leq \alpha, \beta \leq 7$ and at most one of α, β is zero.

Since $pA \equiv 0 \bmod n$ and $\alpha A \pm \beta B \equiv 0 \bmod n$, we have $p\beta B \equiv 0 \bmod n$. Therefore $n | p\beta B$ so there exists r such that $rn = p\beta B$ so $B = rn/p\beta = (r/d)n/(p\beta/d)$ where $d = (r, p\beta)$. Setting $u = r/d$, $q = p\beta/d$, gives $(u, q) = 1$ and $B \equiv un/q \bmod n$ and so $q | n$. Now write $n = [p, q]N$ where $[p, q]$ denotes the lowest common multiple of p and q. Then $1 = (n, m, k) = (n, A, B)$ implies

$$\left([p, q]N, sN\frac{[p, q]}{p}, uN\frac{[p, q]}{q} \right) = 1$$

so $N = 1$ and hence $n = [p, q]$. Since $p\beta$ is a common multiple of p, q we then have $n = [p, q]$ divides $p\beta$.

If $\beta \neq 0$ then, by the above, n divides $p\beta$ so $n \leq p\beta \leq 7p$, a contradiction. Thus we may assume $\beta = 0$ and hence $\alpha \neq 0$. Then C is of the form Z^α where $1 \leq \alpha \leq 7$ and $\alpha(sn/p) \equiv 0 \bmod n$, so $p | \alpha$. Thus $\alpha = 3$ or 6, if $p = 3$, $\alpha = 4$ if $p = 4$, and $\alpha = 5$ if $p = 5$. $\square$

Lemma 3.2 indicates that interior vertices of degree at most 6 will be important to us. This motivates our choice of angles: we assign angle $2\pi/q = 360/q$ to each corner at an interior vertex of D of degree q and angle $\pi/6 - 1 = 29$ to each corner of a boundary vertex. In particular, if v is an interior vertex of D then $\kappa(v) = 0$ so by Lemma 3.1 it suffices to show that the sum of curvatures of faces of D is at most $-|F|$. With this angle assignment, only certain faces of D can have positive curvature, and so we colour the faces of D to keep track of such faces. Let $\mathcal{V}$ denote the set of all interior vertices of D of degree $d \leq 6$. If a face f of D is incident to a vertex in $\mathcal{V}$ we call f a *black* face; otherwise f is a *white* face. If f is a white face then $\kappa(f) < -180 + 3(52) = -24$, so only black faces have positive curvature. The strategy of the proof is to identify a collection of 'lanes' of faces (that include all black faces of D) where in each such lane there are sufficiently many faces of negative curvature to compensate for the faces of positive curvature so that, overall, the average curvature of faces in a lane is at most -1.

If a face in D is incident to vertices u, v, w we denote that face by uvw. The following lemma allows us to conclude that certain configurations of black and white faces are impossible in D.

Lemma 3.3. *Let vv_1v_2, uv_1v_2 be distinct faces of D. If the corner of vv_1v_2 at v is labelled Z then the corners of vv_1v_2 at v_1, v_2 are not labelled Z and the corner of uv_1v_2 at u is not labelled Z.*

Proof. Let vv_1v_2, uv_1v_2 be as in the statement and suppose the corner of vv_1v_2 at v is labelled Z. Then the edges vv_1, vv_2 are oriented away from v (see Figure 2). Without loss of generality, suppose the edge v_1v_2 is oriented towards v_2. Then the corner of vv_1v_2 at v_1 is labelled X and the corner at v_2 is labelled
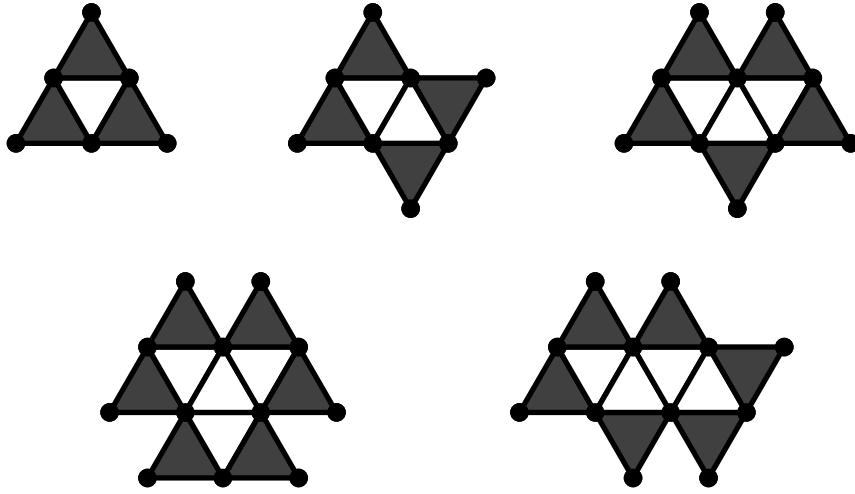
Figure 3: Some impossible configurations of black and white faces in D.

Y. If the corner of uv_1v_2 at u is labelled Z then the edges uv_1, uv_2 are oriented away from u, so the label of the corner of uv_1v_2 at v_1 is X, and so the label of the vertex v_1 contains the cyclic subword XX, which is impossible since D is reduced. □

Example 3.4. In Figure 3 we give some examples of impossible configurations of a sub-diagram of D. We now explain how Lemma 3.3 can be used to show how the first configuration (with four black faces and one white face) is impossible. Similar arguments show that the remaining configurations are also impossible. Figure 4 depicts this first configuration, with three black faces B_0, B_1, B_2 and one white face, where the vertices are labelled $v_0, v_1, v_2, w_0, w_1, w_2$ and certain corners are labelled $c_0, c_1, c_2, d_0, d_1, d_2$, as shown. Now vertices w_0, w_1, w_2, being incident to a white face, are not elements of $\mathcal{V}$. Hence v_0, v_1, v_2, being the remaining vertices of the black faces, are elements of $\mathcal{V}$, and so the corner labels c_0, c_1, c_2 are each Z. Then by Lemma 3.3 none of the corner labels d_0, d_1, d_2 are Z, a contradiction, since the corner labels of faces are X, Y, and Z.

Inspired by Klyachko's Car Crash Lemma [32], we consider an ant taking a walk on D. The ant must obey the following rules:

- It begins its journey at the interior of a black face called *home* and never walks into a black face and never crosses a boundary edge of D; if home is adjacent to a white face f, then the ant must walk into f.

- It must move out of a white face f into an adjacent white face if f is adjacent to a black face different from home.
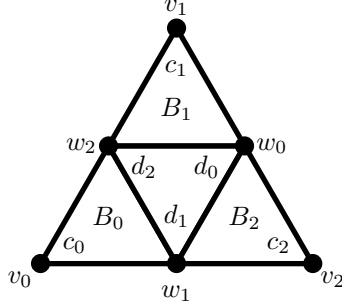
- It is not allowed to backtrack.

Figure 4: Application of Lemma 3.3.

We now observe that an ant following these rules does not visit a face twice. Suppose for contradiction that W is the first white face that is visited more than once by an ant. Write $W' = W$ for the second occurrence of this face. Then W has neighbours V_-, W_+, B where W_+ is white, B is black, and V_- could be black or white, and V_- is visited before W. Similarly W' has neighbours W'_-, W'_+, B' where W'_-, W'_+ are white and B' is black, and W'_- is visited before W'. Therefore $V_- = W'_-$ or $V_- = W'_+$, so V_- is white and V_- is visited more than once, a contradiction (since W is the first white face that is visited more than once). Therefore an ant's journey must end; we call the face where it ends *destination*. (Note that it is possible for an ant never to leave home, in which case destination is home, a black face; otherwise destination is a white face.)

Definition 3.5. (a) An *ant walk* is the set of white faces in the path of an ant following the rules above.

(b) An *ant lane* is the set consisting of home and all white faces in an ant walk, together with all the black faces adjacent to these white faces. If an ant lane L has $b > 0$ black faces and $w \geq 0$ white faces then we say that L is of *type* (b, w).

(c) A white face of an ant lane is called a *junction* if it is not adjacent to a black face.

(d) An ant lane is *maximal* if it is not properly contained in another ant lane.

Note that more than one ant walk can give rise to the same ant lane, for instance in an ant lane with two black faces and one white boundary face each black face is home for an ant walk corresponding to this ant lane. Any black face of D is home for some ant walk, so every black face is in some ant lane. Note that a junction must be destination, but destination need not be a junction. Since an ant does not visit a white face more than once, all the white faces in an ant lane are distinct. By definition a black face can have at most one white neighbour so all the black faces in an ant lane are also distinct; in particular, an ant lane has finitely many faces. Note that if an ant lane L has no white faces,
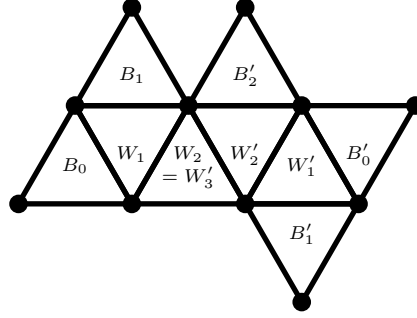
16

Figure 5: Maximal ant lanes $\{B_0, B_1; W_1\} \cup \{W_2\}$, $\{B'_0, B'_1; W'_1\} \cup \{B'_2; W'_2\} \cup \{W'_3\}$ meeting at the junction $W_2 = W'_3$.

then by Lemma 3.3 it has exactly one black face, which is a boundary face, so it is of type $(1, 0)$. An ant lane can consist of one black face and one white face, in which case it is of type $(1, 1)$.

Now we consider an ant lane L with $b \geq 2$ black faces $B_0, B_1, \ldots, B_{b-1}$ and $w \geq 1$ white faces $W_1, W_2, \ldots, W_w$ where the subscripts of the W_j's are labelled such that W_i is adjacent to W_{i-1} for each $2 \leq i \leq w$. Then W_1 is adjacent to home, and at least one other black face. However, by Example 3.4, W_1 cannot be adjacent to two black faces other than home, so W_1 is adjacent to home and exactly one other black face. The face W_w is destination and so one of the following holds: (a) W_w has no black neighbours (b) W_w has exactly one black neighbour and is a boundary face (c) W_w has 2 black neighbours.

Therefore (relabelling the subscripts of the B_j's if necessary) L is of one of the following forms:

(a) $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{w-1}; W_{w-1}\} \cup \{W_w\}$;

(b) $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{w-1}; W_{w-1}\} \cup \{B_w; W_w\}$;

(c) $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{w-1}; W_{w-1}\} \cup \{B_w, B_{w+1}; W_w\}$;

where the set notation $\{B_t, B_{t+1}, \ldots, B_u; W_s\}$ means that W_s is adjacent to $B_t, B_{t+1}, \ldots, B_u$. In particular $w \leq b \leq w+2$ and B_i is adjacent to W_i for each $1 \leq i \leq w-1$. Note that in (a) W_w is a junction; in (b) W_w is a boundary face and is not a junction; in (c) W_w is not a junction. To illustrate these concepts, in Figure 5 we give an example of two maximal ant lanes that meet at a common junction.

We summarize the above discussion as follows:

Corollary 3.6. *Let L be an ant lane of type (b, w). Then $(b, w) = (1, 0)$ or $(1, 1)$ or one of the following holds:*

(a) *$b = w \geq 2$, $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{b-1}; W_{b-1}\} \cup \{W_b\}$, and W_b is a junction;*

(b) $b - 1 = w \geq 1$, $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{b-2}; W_{b-2}\} \cup \{B_{b-1}; W_{b-1}\}$ and W_{b-1} is a boundary face and is not a junction;

(c) $b - 2 = w \geq 1$, $L = \{B_0, B_1; W_1\} \cup \{B_2; W_2\} \cup \cdots \cup \{B_{b-3}; W_{b-3}\} \cup \{B_{b-2}, B_{b-1}; W_{b-2}\}$ and W_{b-2} is not a junction.

Now we prove the following lemma about junctions:

Lemma 3.7. *(a) An ant lane contains at most one junction.*

(b) If $f \in L \cap L'$ where L, L' are distinct maximal ant lanes of type (b, w), (b', w'), respectively, then f is a junction and $w, w' > 1$.

(c) A junction is contained in at most three maximal ant lanes.

Proof. (a) A junction is destination, and an ant lane with a white face has at most one destination, so the ant lane has at most one junction. (b) Suppose that $f \in L \cap L'$ is white. If f has a neighbour that is black and is not home for an ant walk corresponding to either L or L' then, in a corresponding ant walk, an ant arriving at f along the ant lane L (resp. L') can continue its journey into a white face of the ant lane L' (resp. L), and hence L (resp. L') is not maximal, a contradiction. If the only black neighbour of f is home for an ant walk corresponding to L or L' then one of these ant lanes is contained in the other, so is not maximal, a contradiction. Therefore f has no black neighbours so f is a junction. If $w = 1$ or $w' = 1$ then f is adjacent to a black face, a contradiction, so $w, w' > 1$. Suppose then that f is black; then f is adjacent to exactly one white face W. Then $W \in L \cap L'$ which, as we have just shown, implies that W is a junction. But then W is a junction that is adjacent to a black face, a contradiction.

(c) If more than three maximal ant lanes share a junction then at least two of them, L, L' (say), must share at least one more white face W, which is a junction by part (b). Therefore L, L' have two junctions, which is impossible by (a). $\square$

We now introduce a 'join' operation # on maximal ant lanes: if L, L' are two distinct maximal ant lanes (of types (b, w), (b', w'), respectively) with a common junction that is not a junction of any other ant lane then we define $L \# L'$ to be the set of black and white faces of L, L' with the junction identified, and if L, L', L'' are three distinct maximal ant lanes (of types (b, w), (b', w'), (b'', w''), respectively) with a common junction then we define $L \# L' \# L''$ to be the set of black and white faces of L, L', L'' with the junction identified. Since $L \# L'$ has $b + b'$ black faces and $w + w' - 1$ white faces, we say that it is of *type $(b + b', w + w' - 1)$* and since $L \# L' \# L''$ has $b + b' + b''$ black faces and $w + w' + w'' - 2$ white faces, we say that it is of *type $(b + b' + b'', w + w' + w'' - 2)$*. We define $\mathcal{L}$ to be the set consisting of all maximal ant lanes with no junction, together with the join of any two ant lanes with a common junction that is not the junction of a third ant lane, together with the join of any three maximal ant lanes with a common junction.

Corollary 3.8. *Let $L \in \mathcal{L}$. Then exactly one of the following holds:*

(a) *L is of type (b, b) $(b \geq 1)$, $(b, b-1)$ $(b \geq 1)$, or $(b, b-2)$ $(b \geq 3)$;*

(b) *$L = (b, b) \# (b', b')$ where $b, b' \geq 2$, and so is of type $(b + b', b + b' - 1)$;*

(c) *$L = (b, b) \# (b', b') \# (b'', b'')$ where $b, b', b'' \geq 2$, and so is of type $(b + b' + b'', b + b' + b'' - 2)$.*

We define $\kappa(L)$ to be the sum of curvatures $\kappa(f)$ of faces $f \in L$. By Lemma 3.7 no face is common to two distinct elements of $\mathcal{L}$ so

$$\sum_{f \in \cup_{L \in \mathcal{L}} L} \kappa(f) = \sum_{L \in \mathcal{L}} \kappa(L). \tag{1}$$

Let $\kappa_{\mathcal{L}}$ denote the maximum average curvature of faces of an element of $\mathcal{L}$. Then, since every black face is in some maximal ant lane, faces that are not in ant lanes are white, and therefore have curvature at most -24, we have

$$\begin{aligned}
\sum_{f \in F} \kappa(f) &= \sum_{f \notin \cup_{L \in \mathcal{L}} L} \kappa(f) + \sum_{f \in \cup_{L \in \mathcal{L}} L} \kappa(f) \\
&\leq \sum_{f \notin \cup_{L \in \mathcal{L}} L} (-24) + \sum_{f \in \cup_{L \in \mathcal{L}} L} \kappa(f) \\
&= \sum_{f \notin \cup_{L \in \mathcal{L}} L} (-24) + \sum_{L \in \mathcal{L}} \kappa(L) \qquad \text{by (1)} \\
&\leq \sum_{f \notin \cup_{L \in \mathcal{L}} L} (-24) + \sum_{f \in \cup_{L \in \mathcal{L}} L} \kappa_{\mathcal{L}}. \tag{2}
\end{aligned}$$

Lemma 3.9. *Suppose each interior vertex of D either has degree at least 8 or label Z^d where $3 \leq d \leq 6$. Then $\kappa_{\mathcal{L}} \leq -1$.*

Proof. Let $L \in \mathcal{L} \subseteq D$. Then L is of type $(1, 0)$ or type (b, w) for some $1 \leq w \leq b \leq w + 2$. If L is of type $(1, 0)$ then $L = \{B_0\}$, where B_0 is a boundary face by Lemma 3.3, in which case $\kappa_{\mathcal{L}} = \kappa(L) = \kappa(B_0) \leq -180 + 120 + 2(29) = -2$, so assume $w \geq 1$. If f is a black face then $\kappa(f) \leq -180 + 2(60) + 2(45) = 30$ and if f is white then $\kappa(f) \leq -180 + 3(45) = -45$. Therefore $\kappa(L) \leq 30b - 45w$, so the average curvature of faces in L is at most $(30b - 45w)/(b + w)$. If this exceeds -1 then $44w < 31b$, but since $w \leq b \leq w + 2$ we have $0 \leq w < b \leq 6$.

If $w = b - 1$ then $44w < 31b$ implies $2 \leq b \leq 3$. By Corollary 3.8 L is a maximal ant lane (rather than a join of ant lanes). In particular W_{b-1} is a boundary face. In the case $b = 3, w = 2$ the average face curvature $\kappa(L)/5 \leq (-5(180) + 3(120) + 2(45) + 2(29))/5 \leq -22$; in the case $b = 2, w = 1$ the average face curvature $\kappa(L)/3 \leq (-3(180) + 2(120) + 3(45) + 4(29))/3 \leq -16$; in the case $b = 1, w = 0$ the average face curvature is at most $-180 + 120 + 2(29) = -2$. It remains to consider the case $w = b - 2$ with $3 \leq b \leq 6$. By Corollary 3.8 either L is a maximal ant lane of type $(3, 1)$, $(4, 2)$, $(5, 3)$, $(6, 4)$; or L is the join of three maximal ant lanes of type $(2, 2)$ and so it has type $(6, 4)$. Recalling that

the vertices of white faces all have degree greater than 6, we see that ant lanes of these types are the configurations given in Figure 3. But by Example 3.4 these configurations cannot arise. □

Proof of Theorem A for $n > 7p$. By Lemma 3.2 each interior vertex of D either has degree at least 8 or label Z^d where $3 \leq d \leq 6$, and hence $\kappa_{\mathcal{L}} \leq -1$ by Lemma 3.9. By (2) we then have

$$\sum_{f \in F} \kappa(f) \leq \sum_{f \notin \cup_{L \in \mathcal{L}} L} (-24) + \sum_{f \in \cup_{L \in \mathcal{L}}} (-1) \leq \sum_{f \notin \cup_{L \in \mathcal{L}} L} (-1) + \sum_{f \in \cup_{L \in \mathcal{L}}} (-1) = -|F|$$

so G has a linear isoperimetric function by Lemma 3.1, and hence G is hyperbolic. Since the presentation $P_n(m, k)$ is aspherical (by [22, Theorem 3.2], [48, Theorem 2]), the group G is torsion-free, and since it also has finite non-trivial abelianisation, it is non-elementary hyperbolic. □

3.2. Proof of Theorem A in the cases $13 \leq n \leq 7p$

Let $G = G_n(m, k)$ where $k = sn/p$, $(s, p) = 1$. Writing $n = Np$ we have $N \leq 7$ and $N \geq 5$ if $p = 3$, $N \geq 4$ if $p = 4$, $N \geq 3$ if $p = 5$. Since $1 = (n, m, k) = (pN, m, sN)$ we have $(m, N) = 1$ and hence $(m, n) = 1$ or p if $p \in \{3, 5\}$ and $(m, n) = 1, 2,$ or 4 if $p = 4$. If $(m, n) = 1$ then $G = G_{pN}(m, sN) \cong G_{pN}(1, tN)$ for some $(t, p) = 1$. If $(m, n) = p$ then $G = G_{pN}(m, sN) \cong G_{pN}(p, tN)$ for some $(t, p) = 1$, $(p, N) = 1$. If $p = 4$ and $(m, n) = 2$ then $G = G_{4N}(m, sN) \cong G_{4N}(2, tN)$ for some $(t, 4) = 1$ and odd N. Therefore G is isomorphic to one of the following: $G_{5N}(1, tN)$ $(1 \leq t \leq 4, 3 \leq N \leq 7)$; $G_{5N}(5, tN)$ $(1 \leq t \leq 4, N \in \{3, 4, 6, 7\})$; $G_{4N}(1, tN)$ $(t \in \{1, 3\}, N \in \{4, 5, 6, 7\})$; $G_{4N}(2, tN)$ $(t \in \{1, 3\}, N \in \{5, 7\})$; $G_{4N}(4, tN)$ $(t \in \{1, 3\}, N \in \{5, 7\})$; $G_{3N}(1, tN)$ $(t \in \{1, 2\}, N \in \{5, 6, 7\})$; $G_{3N}(3, tN)$ $(t \in \{1, 2\}, N \in \{5, 7\})$.

Using KBMAG we can show that each of these groups is hyperbolic. In each case the defining cyclic presentation is aspherical, by [22, Theorem 3.2], [48, Theorem 2], so the groups are torsion-free; since they also have finite, non-trivial abelianisation, they are non-elementary hyperbolic.

Acknowledgements

References

[1] V. G. Bardakov and A. Yu. Vesnin. On a generalization of Fibonacci groups. *Algebra Logika*, 42(2):131–160, 255, 2003.

[2] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[3] Marcus Brazil. Growth functions for some nonautomatic Baumslag-Solitar groups. *Trans. Amer. Math. Soc.*, 342(1):137–154, 1994.

[4] J. W. Cannon and Daryl Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. *Trans. Amer. Math. Soc.*, 330(1):419–431, 1992.

[5] A. Cavicchioli, F. Hegenbarth, and A. C. Kim. A geometric study of Sieradski groups. *Algebra Colloq.*, 5(2):203–217, 1998.

[6] Alberto Cavicchioli, Friedrich Hegenbarth, and Dušan Repovš. On manifold spines and cyclic presentations of groups. In *Knot theory (Warsaw, 1995)*, volume 42 of *Banach Center Publ.*, pages 49–56. Polish Acad. Sci. Inst. Math., Warsaw, 1998.

[7] Alberto Cavicchioli, E. A. O'Brien, and Fulvia Spaggiari. On some questions about a family of cyclically presented groups. *J. Algebra*, 320(11):4063–4072, 2008.

[8] Christopher P. Chalk. Fibonacci groups, $F(2, n)$, are hyperbolic for n odd and $n \geq 11$. *J. Group Theory*, 24(2):359–384, 2021.

[9] Christopher P. Chalk. Fibonacci groups with aspherical presentations. *Comm. Algebra*, 26(5):1511–1546, 1998.

[10] Christopher P. Chalk and Derek F. Holt. Personal communication, 8th September 2020.

[11] Ihechukwu Chinyere and Gerald Williams. Hyperbolicity of $T(6)$ cyclically presented groups. *arXiv preprint arXiv:2006.09018*, 2020.

[12] Donald J. Collins. Free subgroups of small cancellation groups. *Proc. London Math. Soc. (3)*, 26:193–206, 1973.

[13] J. Conway. Advanced problem 5327. *Am. Math. Mon.*, 72(8):915, 1965.

[14] J. Conway. Solution to advanced problem 5327. *Am. Math. Mon.*, 74(91):91–93, 1967.

[15] Thomas Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.*, 83(3):661–682, 1996.

[16] Martin Edjvet. On the asphericity of one-relator relative presentations. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(4):713–728, 1994.

[17] Martin Edjvet and James Howie. Star graphs, projective planes and free subgroups in small cancellation groups. *Proc. London Math. Soc. (3)*, 57(2):301–328, 1988.

[18] Martin Edjvet and Gerald Williams. The cyclically presented groups with relators $x_i x_{i+k} x_{i+l}$. *Groups Geom. Dyn.*, 4(4):759–775, 2010.

[19] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.2*, 2019.

[20] S. M. Gersten and H. B. Short. Small cancellation theory and automatic groups. *Invent. Math.*, 102(2):305–334, 1990.

[21] Steve M. Gersten. Isoperimetric and isodiametric functions of finite presentations. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 79–96. Cambridge Univ. Press, Cambridge, 1993.

[22] N. D. Gilbert and James Howie. LOG groups and cyclically presented groups. *J. Algebra*, 174(1):118–131, 1995.

[23] H. Helling, A. C. Kim, and J. L. Mennicke. A geometric study of Fibonacci groups. *J. Lie Theory*, 8(1):1–23, 1998.

[24] Patricia Hill, Stephen J. Pride, and Alfred D. Vella. On the $T(q)$-conditions of small cancellation theory. *Israel J. Math.*, 52(4):293–304, 1985.

[25] Derek F. Holt. An alternative proof that the Fibonacci group $F(2,9)$ is infinite. *Experiment. Math.*, 4(2):97–100, 1995.

[26] Derek F. Holt, Bettina Eick, and Eamonn A. O'Brien. *Handbook of computational group theory*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005.

[27] D.F. Holt. KBMAG (Knuth-Bendix in Monoids and Automatic groups), 2000. http://www.warwick.ac.uk/staff/D.F.Holt/download/kbmag2/.

[28] James Howie. On the SQ-universality of $T(6)$-groups. *Forum Math.*, 1(3):251–272, 1989.

[29] James Howie and Gerald Williams. Tadpole Labelled Oriented Graph groups and cyclically presented groups. *J. Algebra*, 371:521–535, 2012.

[30] James Howie and Gerald Williams. Fibonacci type presentations and 3-manifolds. *Topology Appl.*, 215:24–34, 2017.

[31] D. L. Johnson and H. Mawdesley. Some groups of Fibonacci type. *J. Austral. Math. Soc.*, 20(2):199–204, 1975.

[32] Anton A. Klyachko. A funny property of sphere and equations over groups. *Comm. Algebra*, 21(7):2555–2575, 1993.

[33] Daciberg Lima Gonçalves and John Guaschi. *The classification of the virtually cyclic subgroups of the sphere braid groups*. Springer Briefs in Mathematics. Springer, Cham, 2013.

[34] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin-New York, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[35] Dugald Macpherson. Permutation groups whose subgroups have just finitely many orbits. In *Ordered groups and infinite permutation groups*, volume 354 of *Math. Appl.*, pages 221–229. Kluwer Acad. Publ., Dordrecht, 1996.

[36] Jonathan P. McCammond and Daniel T. Wise. Fans and ladders in small cancellation theory. *Proc. London Math. Soc. (3)*, 84(3):599–644, 2002.

[37] Kirk McDermott. *Topological and Dynamical Properties of Cyclically Presented Groups.* PhD thesis, Oregon State University, 2017.

[38] Matthias C. Merzenich. *A Computational Method for Building Spherical Pictures and Theoretic Results from Explicit Constructions.* PhD thesis, Oregon State University, 2020.

[39] Esamaldeen Mohamed and Gerald Williams. An investigation into the cyclically presented groups with length three positive relators. *Experimental Mathematics*, 0(0):1–15, 2019.

[40] M. F. Newman. Proving a group infinite. *Arch. Math. (Basel)*, 54(3):209–211, 1990.

[41] R. W. K. Odoni. Some Diophantine problems arising from the theory of cyclically-presented groups. *Glasg. Math. J.*, 41(2):157–165, 1999.

[42] A. Yu. Olshanskiĭ. SQ-universality of hyperbolic groups. *Mat. Sb.*, 186(8):119–132, 1995.

[43] Stephen J. Pride. Star-complexes, and the dependence problems for hyperbolic complexes. *Glasgow Math. J.*, 30(2):155–170, 1988.

[44] Allan J. Sieradski. Combinatorial squashings, 3-manifolds, and the third homology of groups. *Invent. Math.*, 84(1):121–139, 1986.

[45] Richard M. Thomas. On a question of Kim concerning certain group presentations. *Bull. Korean Math. Soc.*, 28(2):219–224, 1991.

[46] Alun Williams. MAF (Monoid Automata Factory). Software package https://sourceforge.net/projects/maffsa/, 2016.

[47] Alun Williams. Personal communication, 3rd July 2016.

[48] Gerald Williams. The aspherical Cavicchioli-Hegenbarth-Repovš generalized Fibonacci groups. *J. Group Theory*, 12(1):139–149, 2009.

[49] Gerald Williams. Unimodular integer circulants associated with trinomials. *Int. J. Number Theory*, 6(4):869–876, 2010.

[50] Gerald Williams. Groups of Fibonacci type revisited. *Internat. J. Algebra Comput.*, 22(8):1240002, 19 pages, 2012.