INTEGRATION OF MODULES – II: EXPONENTIALS

DMITRIY RUMYNIN AND MATTHEW WESTAWAY

ABSTRACT. We continue our exploration of various approaches to integration of representations from a Lie algebra Lie(G) to an algebraic group G in positive characteristic. In the present paper we concentrate on an approach exploiting exponentials. This approach works well for over-restricted representations, introduced in this paper, and takes no note of G-stability.

If G is a connected simply-connected Lie group with Lie algebra $\mathfrak{g}$, the categories of finite-dimensional G-modules and $\mathfrak{g}$-modules are equivalent. In one direction the equivalence is a differentiation functor $\mathcal{D} : G-\text{Mod} \to \mathfrak{g}-\text{Mod}$. Its quasi-inverse is an integration (or exponentiation) functor $\mathcal{E} : \mathfrak{g}-\text{Mod} \to G-\text{Mod}$. Since every $x \in G$ can be written as a product of exponentials $x = \text{Exp}_G(\mathbf{a}_1)\text{Exp}_G(\mathbf{a}_2)\ldots\text{Exp}_G(\mathbf{a}_n)$, $\mathbf{a}_i \in \mathfrak{g}$, we can exponentiate a representation $\mathcal{E}(V,\theta) = (V,\Theta)$ by an explicit formula

$$\Theta\Big(\text{Exp}_G(\mathbf{a}_1)\text{Exp}_G(\mathbf{a}_2)\cdots\text{Exp}_G(\mathbf{a}_n)\Big) = \text{Exp}_{\text{GL}(V)}(\theta(\mathbf{a}_1))\cdots\text{Exp}_{\text{GL}(V)}(\theta(\mathbf{a}_n)).$$

This method works for a semisimple simply-connected algebraic group G over $\mathbb{C}$ and its category $G-\text{Mod}$ of rational representations. The key observation is that $\mathcal{E}(V,\theta)$ is rational. In this case G is generated by unipotent root subgroups U_α, thus, we can choose $\mathbf{a}_i \in \mathfrak{g}_{\alpha_i}$ in the exponentiation formula. Then $\theta(\mathbf{a}_i)$ is nilpotent, so $\text{Exp}_{\text{GL}(V)}(\theta(\mathbf{a}_i))$ is polynomial.

Curiously, we can use the same formula for exponentiation of representations for more general algebraic groups. However, we can no longer rely on the Lie group structure on G for proving that the exponentiation formula produces a well-defined group homomorphism $\Theta : G \to \text{GL}(V)$. It is a minor inconvenience in zero characteristic that turns into a major technical issue in positive characteristic.

The idea of using exponentials in positive characteristic goes back to Chevalley and his construction of finite groups of Lie type. Kac and Weisfeiler use exponentials in positive characteristic to study contragradient Lie algebras [VKa]. If G is an algebraic group over a field of positive characteristic, its Lie algebra $\mathfrak{g}$ is a restricted subalgebra of the commutator Lie algebra $U_0(\mathfrak{g})^{(-)}$ of the restricted enveloping algebra $U_0(\mathfrak{g})$. N.B., $U_0(\mathfrak{g})^{(-)}$ is the Lie algebra of the algebraic group $\text{GL}_1(U_0(\mathfrak{g}))$ but $\mathfrak{g}$ is not an algebraic subalgebra, i.e., not

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the Lie algebra of an algebraic subgroup. Let $\widehat{G} \leqslant \mathrm{GL}_1(U_0(\mathfrak{g}))$ be a minimal (possibly non-unique) algebraic subgroup[1] of $\mathrm{GL}_1(U_0(\mathfrak{g}))$ whose Lie algebra contains $\mathfrak{g}$.

An interesting fact about the group $\widehat{G}$ is that it acts on restricted $\mathfrak{g}$-modules. If we can relate the groups $\widehat{G}$ and G, we may be able to integrate representations of $\mathfrak{g}$. Notice that not all $\mathfrak{g}$-modules are integrable: the baby Verma modules (except Steinberg modules) have non-G-stable support varieties, hence, cannot be integrated. Yet $\widehat{G}$ acts on the baby Verma modules. This suggests that the relation between G and $\widehat{G}$ is delicate. We uncover this relation for some class of modules which we call *over-restricted*.

Now we reveal the detailed content of the present paper, emphasising the main results. We study exponentials on a restricted representation of a restricted Lie algebra in Section 1. These are particularly well-behaved when the representation is not only restricted but also *over-restricted*, a concept introduced in this section. Our first major result of the paper is Theorem 4 in this section. This yields Corollary 6, a notable general result which says that for an algebraic group G, with some mild restrictions, an over-restricted representation of its Lie algebra can be integrated to the group G.

In Section 2 we extend the concept of an over-restricted representation to Kac-Moody Lie algebras. The main result of this section is Theorem 8: an over-restricted representation of a Kac-Moody Lie algebra can be integrated to the Kac-Moody group. The set-up of this section is similar, yet slightly different from the over-restricted representations of Kac-Moody algebras discussed by the first author in another paper [Ru].

In Section 3 we study over-restricted representations of the higher Frobenius kernels of a semisimple algebraic group G. We switch to semisimple groups as there are some subtleties to overcome compared to the first Frobenius kernels. We stop short of proving an analogue of Theorem 4 for the higher Frobenius kernels. We formulate it as a conjecture instead.

In Section 4 we elaborate how our Higher Frobenius Conjecture applies to the Humphreys-Verma Conjecture, a well-known hypothesis that projective $U_0(\mathfrak{g})$-modules are G-modules.

We discuss examples of over-restricted representations in Section 5. We give several non-trivial examples of over-restricted representations of classical simple Lie algebras and propose the notion of an over-restricted enveloping algebra.

We draw conclusions for this and the first paper in the series [RuW] in Section 6. The final section 7 is a technical appendix with essential results on generic smoothness of morphisms of algebraic varieties.

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1. Over-restricted Representations

Let $(\mathfrak{g}, {}^{[p]})$ be a restricted Lie algebra over a field $\mathbb{K}$ of characteristic p, $U_0(\mathfrak{g})$ its restricted enveloping algebra, (V, θ) a restricted representation. Let $N_p(\mathfrak{g})$ be the p-nilpotent cone of $\mathfrak{g}$, i.e, the set of all $\mathbf{x} \in \mathfrak{g}$ such that $\mathbf{x}^{[p]} = 0$. Notice that for $\mathbf{x} \in N_p(\mathfrak{g})$ we have

[1] We have been informed that there exists an unpublished preprint by Haboush where a group similar to $\widehat{G}$ has been studied but we could not find it.

$\theta(\mathbf{x})^p = \theta(\mathbf{x}^{[p]}) = 0$. This allows us to define exponentials for each $\mathbf{x} \in N_p(\mathfrak{g})$:

$$e^{\theta(\mathbf{x})} = \sum_{k=0}^{p-1} \frac{1}{k!}\theta(\mathbf{x})^k \in \mathfrak{gl}(V) \ .$$

The element $e^{\theta(\mathbf{x})}$ is invertible because $(e^{\theta(\mathbf{x})})^{-1} = e^{\theta(-\mathbf{x})}$. We define *the pseudo-Chevalley group* G_V as the subgroup of $\mathrm{GL}(V)$ generated by all exponentials $e^{\theta(\mathbf{x})}$ for all $\mathbf{x} \in N_p(\mathfrak{g})$.

Proposition 1. *The following statements hold for any restricted finite-dimensional representation* (V, θ) *of* $\mathfrak{g}$*:*

(1) G_V *is a (Zariski) closed subgroup of* $\mathrm{GL}(V)$*.*
(2) *One can choose finitely many* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in N_p(\mathfrak{g})$ *such that the following map* f *is surjective:*

$$f : \mathbb{K}^n \to G_V, \quad f(a_1, a_2, \ldots, a_n) = e^{\theta(a_1\mathbf{x}_1)} \cdots e^{\theta(a_n\mathbf{x}_n)} \ .$$

Proof. It follows from the standard fact [Bo, Prop I.2.2] by choosing $I = N_p(\mathfrak{g})$, $V_{\mathbf{x}} = \mathbb{K}$, $f_{\mathbf{x}}(a) = e^{\theta(a\mathbf{x})}$ in Borel's notation. □

Two particular pseudo-Chevalley groups are worth separate discussion. Let $(U_0(\mathfrak{g}), \theta)$ be the left regular representation of $\mathfrak{g}$ on its restricted enveloping algebra. The exponential $e^{\theta(\mathbf{x})}$ is uniquely determined by its application to the identity

$$e^{\theta(\mathbf{x})}(1) = \sum_{k=0}^{p-1} \frac{1}{k!}\mathbf{x}^k \in U_0(\mathfrak{g}) \ .$$

This element should be called $e^{\mathbf{x}} \in U_0(\mathfrak{g})$. We can identify $e^{\theta(\mathbf{x})}$ with $e^{\mathbf{x}}$ because $G_{U_0(\mathfrak{g})}$ is a subgroup of $\mathrm{GL}_1(U_0(\mathfrak{g}))$ that, in turn, acts on $U_0(\mathfrak{g})$ by left multiplication:

$$G_{U_0(\mathfrak{g})} \leqslant \mathrm{GL}_1(U_0(\mathfrak{g})) \leqslant \mathrm{GL}(U_0(\mathfrak{g})_{\mathbb{K}}).$$

We define the group $\widehat{G}$ discussed in the introduction as $\widehat{G} := G_{U_0(\mathfrak{g})}$. It acts on restricted $\mathfrak{g}$-modules, hence, its structure is worth further investigation.

The element $e^{\mathbf{x}}$ is not group-like in $U_0(\mathfrak{g})$, yet it is close to it in the sense that

$$\Delta(e^{\mathbf{x}}) = e^{\mathbf{x}} \otimes e^{\mathbf{x}} + \mathcal{O}(\mathbf{x}^{\lfloor (p+1)/2 \rfloor})$$

where $\mathcal{O}(\mathbf{x}^m)$ denotes a sum of terms containing $\mathbf{x}^k$ with $k \geqslant m$. To make this precise, we say that a $U_0(\mathfrak{g})$-module V is *over-restricted* if $\theta(\mathbf{x})^{\lfloor (p+1)/2 \rfloor} = 0$ for all $\mathbf{x} \in N_p(\mathfrak{g})$. See Section 5 for some examples. Notice that if $p = 2$, then $\lfloor (p+1)/2 \rfloor = 1$ and this requirement is severe: $\theta(\mathbf{x}) = 0$.

Proposition 2. *Let* $(\mathfrak{g}, ad)$ *be the adjoint representation. If* (V, θ) *is an over-restricted representation, then*

$$\theta(e^{ad(\mathbf{x})}(\mathbf{y})) = e^{\theta(\mathbf{x})}\theta(\mathbf{y})e^{-\theta(\mathbf{x})}$$

for all $\mathbf{x} \in N_p(\mathfrak{g})$*,* $\mathbf{y} \in \mathfrak{g}$*.*

Proof. First, observe by induction that for each $k = 1, 2, \ldots, p-1$

$$\theta(\frac{1}{k!}ad(\mathbf{x})^k(\mathbf{y})) = \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!j!}\theta(\mathbf{x})^{k-j}\theta(\mathbf{y})\theta(\mathbf{x})^j \ .$$

For $k = 1$ this is just the definition of a representation:

$$\theta(ad(\mathbf{x})(\mathbf{y})) = \theta([\mathbf{x},\mathbf{y}]) = \theta(\mathbf{x})\theta(\mathbf{y}) - \theta(\mathbf{y})\theta(\mathbf{x}).$$

Going from k to $k+1$,

$$\theta(\frac{1}{(k+1)!}ad(\mathbf{x})^{k+1}(\mathbf{y})) = \frac{1}{k+1}(\theta(\mathbf{x})\theta(\frac{1}{k!}ad(\mathbf{x})^{k}(\mathbf{y})) - \theta(\frac{1}{k!}ad(\mathbf{x})^{k}(\mathbf{y}))\theta(\mathbf{x}))$$

$$= \sum_{j=0}^{k} \frac{(-1)^j}{k+1}\Bigg(\frac{1}{(k-j)!j!}\theta(\mathbf{x})^{k-j+1}\theta(\mathbf{y})\theta(\mathbf{x})^{j} - \frac{1}{(k-j)!j!}\theta(\mathbf{x})^{k-j}\theta(\mathbf{y})\theta(\mathbf{x})^{j+1}\Bigg)$$

$$= \frac{1}{(k+1)!}\theta(\mathbf{x})^{k+1}\theta(\mathbf{y}) + \sum_{i=1}^{k}\frac{(-1)^i}{(k+1)(k-i)!(i-1)!}\Bigg(\frac{1}{i}+\frac{1}{k+1-i}\Bigg)\cdot$$

$$\cdot\,\theta(\mathbf{x})^{k+1-i}\theta(\mathbf{y})\theta(\mathbf{x})^{i} + \frac{(-1)^{k+1}}{(k+1)!}\theta(\mathbf{y})\theta(\mathbf{x})^{k+1} = \sum_{i=0}^{k+1}\frac{(-1)^i}{(k+1-i)!i!}\theta(\mathbf{x})^{k+1-i}\theta(\mathbf{y})\theta(\mathbf{x})^{i}\ .$$

Finally,

$$\theta(e^{ad(\mathbf{x})}(\mathbf{y})) = \sum_{k=0}^{p-1}\theta(\frac{1}{k!}ad(\mathbf{x})^{k}(\mathbf{y})) = \sum_{i+j=0}^{p-1}\frac{(-1)^j}{i!j!}\theta(\mathbf{x})^{i}\theta(\mathbf{y})\theta(\mathbf{x})^{j} =$$

$$\sum_{i,j=0}^{p-1}\frac{(-1)^j}{i!j!}\theta(\mathbf{x})^{i}\theta(\mathbf{y})\theta(\mathbf{x})^{j} = \Big(\sum_{i=0}^{p-1}\frac{1}{i!}\theta(\mathbf{x})^{i}\Big)\theta(\mathbf{y})\sum_{j=0}^{p-1}\frac{(-1)^j}{j!}\theta(\mathbf{x})^{j} = e^{\theta(\mathbf{x})}\theta(\mathbf{y})e^{-\theta(\mathbf{x})},$$

where the third equality holds because (V,θ) is over-restricted: all missing terms are actually zero. □

The second vital example of a pseudo-Chevalley group is $G_{\mathfrak{g}}$, procured from the adjoint representation $(\mathfrak{g}, ad)$. This group is intricately connected with the pseudo-Chevalley groups of over-restricted representations:

Proposition 3. *If (V,θ) is a faithful over-restricted representation of $\mathfrak{g}$, then the assignment*

$$\phi : e^{\theta(N_p(\mathfrak{g}))} \to G_{\mathfrak{g}}, \quad \phi(e^{\theta(\mathbf{x})}) = e^{ad(\mathbf{x})}, \quad \mathbf{x} \in N_p(\mathfrak{g})$$

extends to a surjective homomorphism of groups $\phi : G_V \to G_{\mathfrak{g}}$ whose kernel is central and consists of $\mathfrak{g}$-automorphisms of V.

Proof. Proposition 1 yields the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n \in N_p(\mathfrak{g})$ for G_V and the elements $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_m \in N_p(\mathfrak{g})$ for $G_{\mathfrak{g}}$. Combining these elements together, we get surjective algebraic maps with common domain:

$$f : \mathbb{K}^m \to G_V,\ \widehat{f} : \mathbb{K}^m \to G_{\mathfrak{g}}, \quad f\Big((a_k)\Big) = \prod_k e^{\theta(a_k\mathbf{x}_k)},\ \widehat{f}\Big((a_k)\Big) = \prod_k e^{ad(a_k\mathbf{x}_k)}\ .$$

Let $H = (\mathbb{K},+)^{*m}$ be the free product of m additive groups. The maps f and $\widehat{f}$ extend to surjective group homomorphisms

$$f^{\sharp} : H \to G_V,\ \widehat{f}^{\sharp} : H \to G_{\mathfrak{g}}$$

so that both G_V and $G_{\mathfrak{g}}$ are quotients of H as abstract groups. Consider an element of the kernel $a_1 * \ldots * a_k \in \ker(f^\sharp)$ where a_i belongs to the $t(i)$-th component of the free product. Clearly,

$$I_V = f^\sharp(a_1 * \ldots * a_k) = e^{\theta(a_1 \mathbf{x}_{t(1)})} e^{\theta(a_2 \mathbf{x}_{t(2)})} \ldots e^{\theta(a_k \mathbf{x}_{t(k)})} .$$

Proposition 2 tells us that

$$\theta(e^{ad(a_1 \mathbf{x}_{t(1)})} e^{ad(a_2 \mathbf{x}_{t(2)})} \ldots e^{ad(a_k \mathbf{x}_{t(k)})}(\mathbf{y})) = \theta(\mathbf{y}) \text{ for all } \mathbf{y} \in \mathfrak{g}.$$

Since θ is injective it follows that $e^{ad(a_1 \mathbf{x}_{t(1)})} \ldots e^{ad(a_k \mathbf{x}_{t(k)})} = I_{\mathfrak{g}}$, so $a_1 * \ldots * a_k \in \ker(\widehat{f}^\sharp)$. It follows that the homomorphism ϕ is well-defined.

Consider $A = e^{\theta(a_1 \mathbf{x}_{t(1)})} \ldots e^{\theta(a_k \mathbf{x}_{t(k)})} \in \ker(\phi)$. By Proposition 2, $\theta(\mathbf{y}) = \theta(\phi(A)(\mathbf{y})) = A\theta(\mathbf{y})A^{-1}$ for all $\mathbf{y} \in \mathfrak{g}$. Hence, $A \in \mathrm{Aut}_{\mathfrak{g}}(V)$, so that A commutes with all $\theta(\mathbf{y})$. Consequently, A commutes with all $e^{\theta(\mathbf{x})}$, which are generators of G_V. Hence, A is central. □

It is natural to inquire whether the homomorphism ϕ is a homomorphism of algebraic groups. To prove this, we need a technical result, Theorem 17 about generic smoothness of polynomial maps in positive characteristic, established in the appendix (Section 7). We include the answer to this natural question in the main result of this section:

Theorem 4. *Suppose that the field $\mathbb{K}$ is algebraically closed. The following statements hold for a faithful over-restricted finite-dimensional representation (V, θ) of a finite-dimensional restricted Lie algebra $\mathfrak{g}$:*

(1) *The map $\phi : G_V \to G_{\mathfrak{g}}$ constructed in Proposition 3 is a homomorphism of algebraic groups.*
(2) *The Lie algebra $\mathrm{Lie}(G_V)$ is isomorphic to $\theta(\mathfrak{g}_0)$ where $\mathfrak{g}_0$ is the Lie subalgebra of $\mathfrak{g}$, generated by all $\mathbf{x} \in N_p(\mathfrak{g})$. Moreover, $\mathfrak{g}_0$ is a restricted Lie subalgebra of $\mathfrak{g}$.*
(3) *The differential $\mathbf{d}_1\eta$ of the natural representation $\eta : G_V \hookrightarrow \mathrm{GL}(V)$ is equal to $\theta|_{\mathfrak{g}_0}$.*
(4) *The differential $\mathbf{d}_1\phi$ is surjective. Its kernel is $\mathfrak{g}_0 \cap Z(\mathfrak{g})$ where $Z(\mathfrak{g})$ is the centre.*
(5) *The scheme-theoretic kernel $\ker \phi$ is a subgroup scheme of $\mathrm{Aut}_{\mathfrak{g}}(V)$, central in G_V.*
(6) *If $Z(\mathfrak{g}) = 0$, then $\ker \phi$ is discrete.*

Proof. (1) On top of the surjective maps $f : \mathbb{K}^m \to G_V$ and $\widehat{f} : \mathbb{K}^m \to G_{\mathfrak{g}}$, utilised in Proposition 3, by [Bo, Prop I.2.2] we can find $\mathbf{x}_{m+1}, \mathbf{x}_{m+2} \ldots, \mathbf{x}_k \in N_p(\mathfrak{g})$ such that the image G of the map

$$\widetilde{f} : \mathbb{K}^k \to G_V \times G_{\mathfrak{g}}, \quad f(a_1, a_2, \ldots, a_k) = (e^{\theta(a_1 \mathbf{x}_1)} \cdots e^{\theta(a_k \mathbf{x}_k)}, e^{ad(a_1 \mathbf{x}_1)} \cdots e^{ad(a_k \mathbf{x}_k)})$$

is a closed algebraic subgroup of $G_V \times G_{\mathfrak{g}}$. Extending f and $\widehat{f}$ in the obvious way to maps f' and $\widehat{f}'$ defined on $\mathbb{K}^k$, we see that $\widetilde{f} = (f', \widehat{f}')$. Hence, G is the graph of the group homomorphism $\phi : G_V \to G_{\mathfrak{g}}$.

Moreover, the first projection $\pi_1 : G \to G_V$ is bijective. Since f' is given by polynomials of degree less than p by construction, Theorem 17 ensures that f' is generically smooth. Since $\mathbf{d}\pi_1 \circ \mathbf{d}\widetilde{f} = \mathbf{d}f'$, the differential $\mathbf{d}\pi_1$ is surjective at some point. Since π_1 is a morphism of algebraic groups, the differential $\mathbf{d}\pi_1$ is surjective at all points. Hence, π_1 is an isomorphism of algebraic groups. Consequently, ϕ is a morphism of algebraic varieties (or groups) since $\phi = \pi_2\pi_1^{-1}$.

(2) Let $\mathfrak{g}_1$ be the linear span of all $\mathbf{x} \in N_p(\mathfrak{g})$. Let $(z_1, \ldots, z_k)$ be the standard coordinates on $\mathbb{K}^k$. For all $i = 1, \ldots k$ the calculation

$$\mathbf{d}_0 f'(\frac{\partial}{\partial z_i}) = \frac{d}{dt} e^{\theta(t\mathbf{x}_i)}|_{t=0} = \theta(\mathbf{x}_i)$$

implies that $\mathrm{Lie}(G_V) \supseteq \mathrm{Im}(\mathbf{d}_0 f') = \theta(\mathfrak{g}_1)$. It follows that $\mathrm{Lie}(G_V) \supseteq \theta(\mathfrak{g}_0)$.

By Theorem 17, the differential $\mathbf{d}_a f'$ is surjective at some point $a \in \mathbb{K}^k$. If $L_a : G_V \to G_V$ is the left multiplication by $f'(a)^{-1}$, then the Lie algebra $\mathrm{Lie}(G_V)$ is spanned by the elements

$$\mathbf{d}_{f'(a)} L_a\big(\mathbf{d}_a f'(\frac{\partial}{\partial z_i})\big) = \mathbf{d}_{f'(a)} L_a\big(\frac{d}{dt} e^{\theta(a_1\mathbf{x}_1)} \ldots e^{\theta(a_{i-1}\mathbf{x}_{i-1})} e^{\theta((a_i+t)\mathbf{x}_i)} e^{\theta(a_{i+1}\mathbf{x}_{i+1})} \ldots |_{t=0}\big) =$$
$$\mathbf{d}_{f'(a)} L_a\big(e^{\theta(a_1\mathbf{x}_1)} \ldots e^{\theta(a_{i-1}\mathbf{x}_{i-1})} e^{\theta(a_i\mathbf{x}_i)} \theta(\mathbf{x}_i) e^{\theta(a_{i+1}\mathbf{x}_{i+1})} \ldots\big) = e^{-\theta(a_n\mathbf{x}_n)} \ldots e^{-\theta(a_{i+1}\mathbf{x}_{i+1})} \theta(\mathbf{x}_i) \cdot$$
$$e^{\theta(a_{i+1}\mathbf{x}_{i+1})} \ldots e^{\theta(a_n\mathbf{x}_n)} = \theta\big(e^{-ad(a_n\mathbf{x}_n)} \ldots e^{-ad(a_{i+1}\mathbf{x}_{i+1})}(\mathbf{x}_i)\big).$$

The last equality holds because of Proposition 2. The element $e^{-ad(a_n\mathbf{x}_n)} \ldots e^{-ad(a_{i+1}\mathbf{x}_{i+1})}(\mathbf{x}_i)$ belongs to $\mathfrak{g}_0$ since all $\mathbf{x}_j$ belong there. Hence, this calculation shows $\mathrm{Lie}(G_V) \subseteq \theta(\mathfrak{g}_0)$.

It remains to argue that $\mathfrak{g}_0$ is a restricted Lie subalgebra of $\mathfrak{g}$. This is true because θ is an injective homomorphism of restricted Lie algebras, and both $\theta(\mathfrak{g}_0) = \mathrm{Lie}(G_V)$ and $\theta(\mathfrak{g})$ are restricted subalgebras of $\mathfrak{gl}(V)$.

(3) It follows from the same calculation as just above for $\mathbf{x} \in N_p(\mathfrak{g})$:

$$\mathbf{d}_1\eta(\mathbf{x}) = \frac{d}{dt} e^{\theta(t\mathbf{x})}|_{t=0} = \theta(\mathbf{x}).$$

(4) The same argument as in (1) shows that $\mathbf{d}_1\pi_2$ is surjective. Hence, $\mathbf{d}_1\phi = \mathbf{d}_1\pi_2 \circ \mathbf{d}_1\pi_1^{-1}$ is surjective as well.

The second statement follows from the observation that $\mathbf{d}_1\phi = ad|_{\mathfrak{g}_0}$. This can be checked on elements $\mathbf{x} \in N_p(\mathfrak{g})$ since they span $\mathfrak{g}_0$:

$$\mathbf{d}_1\phi(\mathbf{x}) = \frac{d}{dt} e^{ad(t\mathbf{x})}|_{t=0} = ad(\mathbf{x})\,.$$

(5) This follows from Proposition 3.

(6) It follows from (4) that the differential $\mathbf{d}_1\phi : \mathrm{Lie}(G_V) \to \mathrm{Lie}(G_{\mathfrak{g}})$ is an isomorphism of Lie algebras. Observe that G_V is connected because it is generated as a group by a connected set $e^{\theta(N_p(\mathfrak{g}))}$ containing the identity element. Hence, the kernel of ϕ is discrete. □

Let us state an immediate, rather curious corollary of the proof of part (2):

Corollary 5. *Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over an algebraically closed field that admits a faithful over-restricted representation. Let $\mathfrak{g}_1$ be the span of $N_p(\mathfrak{g})$. The following statements in the notation of the proof of Theorem 4.(2) are equivalent:*

(1) *$\mathfrak{g}_1$ is a restricted Lie subalgebra,*
(2) *for some choice of θ and f', the differential $\mathbf{d}_0 f'$ is surjective,*
(3) *for all choices of θ and f', the differential $\mathbf{d}_0 f'$ is surjective.*

Our terminology of pseudo-Chevalley groups is justified by the following example: consider the adjoint representation $\mathfrak{g}$ of a semisimple algebraic group G. Then, barring accidents in small characteristic, (for instance, if $p \geqslant 5$), $G_{\mathfrak{g}}$ is precisely the adjoint Chevalley group G_{ad}. Notice that the Chevalley group G_{ad} is generated by the exponentials of root vectors $\mathbf{e}_\alpha$. In characteristic zero $ad_{\mathbb{Z}}(\mathbf{e}_\alpha)^4 = 0$, while in positive characteristic $ad(\mathbf{e}_\alpha)^p = 0$ so the exponentials could be different. For instance, if G is of type G_2 in characteristic 3, then the Chevalley exponential $e_{\mathbb{Z}}^{\mathbf{e}_\alpha}$ of the short root vector $\mathbf{e}_\alpha$ contains the divided-power term $ad_{\mathbb{Z}}(\mathbf{e}_\alpha^{(3)})$ but our exponential stops at $ad(\mathbf{e}_\alpha)^2/2$. Similar difficulty appears for all groups in characteristic 2. It would be interesting to investigate this question further: what is the precise relation between $G_{\mathfrak{g}}$ and G_{ad} for simple algebraic groups in characteristic 2 (and the type G_2 group in characteristic 3).

Let us contemplate applications of Theorem 4 to integration of representations. Suppose $\mathfrak{g} = \text{Lie}(G)$ where G is a connected algebraic group (over an algebraically closed field $\mathbb{K}$). The adjoint group G_{ad} is defined as the image of the adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. Notice that G_{ad} is closed because the image of a morphism of algebraic groups is closed [Bo, I.1.4]. We can compare G_{ad} and $G_{\mathfrak{g}}$ as sets because both are algebraic subgroups of $\text{GL}(\mathfrak{g})$.

Corollary 6. *Suppose that $G_{ad} = G_{\mathfrak{g}}$. The following statements hold for a faithful over-restricted finite-dimensional representation (V, θ) of $\mathfrak{g} = \text{Lie}(G)$:*

1. *The representation (V, θ) yields a rational representation (V, Θ) of a central extension (that happens to be G_V) of G_{ad} such that $\mathbf{d}_1\Theta(\mathbf{x}) = \theta(\mathbf{x})$ for all $\mathbf{x} \in \mathfrak{g}_0$.*
2. *If (V, θ) is a brick (i.e., $\text{End}_{\mathfrak{g}} V = \mathbb{K}$), then (V, θ) yields a rational projective representation of G_{ad} such that $\mathbf{d}_1\Theta(\mathbf{x}) = \theta(\mathbf{x})$ for all $\mathbf{x} \in \mathfrak{g}_0$.*

We finish the section with an application to semisimple groups. Notice that it is true in characteristic 2 because over-restricted representations are direct sums of the trivial representation.

Corollary 7. *Suppose that G is a connected simply-connected semisimple algebraic group such that $Z(\mathfrak{g}) = 0$. Assume further that if $p = 3$, then G has no components of type G_2. Then a faithful over-restricted finite-dimensional representation (V, θ) of $\mathfrak{g}$ integrates to a rational representation of G.*

2. Kac-Moody Groups

Let $\mathcal{A} = (A_{i,j})_{n \times n}$ be a generalised Cartan matrix, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}(\mathcal{A})$ its corresponding complex Kac-Moody algebra. The divided powers integral form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ forges the Kac-Moody algebra over any commutative ring $\mathbb{A}$:

$$\mathfrak{g}_{\mathbb{Z}} := \mathfrak{g}_{\mathbb{C}} \cap \mathcal{U}_{\mathbb{Z}}\ , \quad \mathfrak{g}_{\mathbb{A}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{A}\ .$$

It inherits a triangular decomposition $\mathfrak{g}_{\mathbb{A}} = (\mathfrak{n}_- \otimes \mathbb{A}) \oplus (\mathfrak{h} \otimes \mathbb{A}) \oplus (\mathfrak{n}_+ \otimes \mathbb{A})$ from $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. If $\mathbb{K}$ is a field of characteristic p, the Lie algebra $\mathfrak{g}_{\mathbb{K}}$ is restricted with the p-operation

$$(\mathbf{h} \otimes 1)^{[p]} = \mathbf{h} \otimes 1, \quad (\mathbf{x} \otimes 1)^{[p]} = \mathbf{x}^p \otimes 1 \quad \text{where} \quad \mathbf{h} \in \mathfrak{h},\ \mathbf{x} \in \mathfrak{n}_\pm$$

where $\mathbf{x}^p$ is calculated inside the associative $\mathbb{Z}$-algebra $\mathcal{U}_{\mathbb{Z}}$ [M, Th. 4.39]. In particular, $(\mathbf{e}_\alpha \otimes 1)^{[p]} = 0$ for any real root vector $\mathbf{e}_\alpha$.

The Kac-Moody group is a functor $G_{\mathcal{A}}$ from commutative rings to groups. Its value on a field $\mathbb{F}$ can be described using the set of real roots Φ^{re}:

$$G_{\mathcal{A}}(\mathbb{F}) \ = \ \circledast_{\alpha \in \Phi^{re}} U_\alpha / \langle \text{ Tits' relations } \rangle, \quad U_\alpha = \{X_\alpha(t) \,|\, t \in \mathbb{F}\} \cong \mathbb{F}^+.$$

There are different ways to write Tits' relations: the reader should consult the classical papers [CCh, T] for succinct presentations.

While the precise relations are peripheral to our deliberations, the following fact is vital:

the group $G_{\mathcal{A}}(\mathbb{F})$ acts on the Lie algebra $\mathfrak{g}_{\mathbb{F}}$ via the adjoint action [M, R].

The adjoint action of each root subgroup U_α is exponential over $\mathbb{Z}$, reduced to the field $\mathbb{F}$:

$$\mathrm{Ad}(X_\alpha(t))(\mathbf{a}\otimes 1) = \text{“ } \mathrm{Exp}(ad(\mathbf{e}_\alpha \otimes t))(\mathbf{a}\otimes 1) \text{ ” } := \sum_{n=0}^{\infty} \big(\frac{1}{n!}\, ad(\mathbf{e}_\alpha)^n(\mathbf{a}) \otimes t^n \big).$$

Observe that the latter sum is well-defined: if $\mathbf{a} \in \mathfrak{g}_{\mathbb{Z}}$ then $\frac{1}{n!}\, ad(\mathbf{e}_\alpha)^n(\mathbf{a}) \in \mathfrak{g}_{\mathbb{Z}}$. The sum is actually finite: by writing $\mathbf{a} = \sum_\beta \mathbf{a}_\beta$ as a sum of elements from root subspaces we can see that there exists N such that $n\alpha + \beta$ is not a root for all $n > N$ and all β so that, consequently, $ad(\mathbf{e}_\alpha)^n(\mathbf{a}) = 0$ as soon as $n > N$. We denote the image of Ad by $G^{ad}_{\mathcal{A}}(\mathbb{F})$ and call it *the adjoint Kac-Moody group*.

Let $\mathbb{K}$ be a field of positive characteristic p. Each real root α yields an additive family of linear operators (in the sense that $Y_\alpha(t+s) = Y_\alpha(t)Y_\alpha(s)$) on a restricted representation (V, θ) of the Lie algebra $\mathfrak{g}_{\mathbb{K}}$:

$$Y_\alpha(t) := e^{\theta(\mathbf{e}_\alpha \otimes t)} = \sum_{k=0}^{p-1} \frac{1}{k!}\, \theta(\mathbf{e}_\alpha \otimes t)^k.$$

By G_V^{KM} we denote the group generated by $Y_\alpha(t)$ for all real roots α and $t \in \mathbb{K}$. Notice that G_V^{KM} is a subgroup of G_V, defined in Section 1. If $p > \max_{i\neq j}(-A_{ij})$, then $\mathfrak{g}_{\mathbb{K}}$ is generated by root vectors $\mathbf{e}_\alpha$ [Ro] and, consequently, we expect that $G_V^{KM} = G_V$ for all over-restricted faithful representations. It is an interesting problem to compare G_V^{KM} and G_V for an arbitrary representation. If (V, θ) is over-restricted, then Proposition 2 applies:

$$\theta\big(\mathrm{Ad}(X_\alpha(t))(\mathbf{y})\big) = Y_\alpha(t)\theta(\mathbf{y})Y_\alpha(-t) \tag{1}$$

for all $\mathbf{y} \in \mathfrak{g}_{\mathbb{K}}$. Here is the main result of this section, which is an adaptation of Proposition 3 (cf. [Ru, Theorem 1.2] for a graded version of this result):

Theorem 8. *If (V, θ) is a faithful over-restricted representation of $\mathfrak{g}_{\mathbb{K}}$, then the assignment $\phi(Y_\alpha(t)) = \mathrm{Ad}(X_\alpha(t))$ extends to a surjective homomorphism of groups $\phi : G_V^{KM} \to G^{ad}_{\mathcal{A}}(\mathbb{K})$, whose kernel is central and consists of $\mathfrak{g}_{\mathbb{K}}$-automorphisms of V.*

Proof. Let H be the free product of all additive groups U_α for all real roots α. Both G_V^{KM} and $G^{ad}_{\mathcal{A}}(\mathbb{K})$ are naturally quotients of H. From this point the rest of the proof repeats the proof of Proposition 3 word for word. □

As soon as there are few endomorphisms, the map ϕ in Theorem 4 can be "reversed" to define a projective representation of the Kac-Moody group.

Corollary 9. *If in the conditions of Theorem 4 the representation (V, θ) is a brick (see Cor. 6), then the assignment*

$$\Theta : G^{ad}_{\mathcal{A}}(\mathbb{K}) \to \mathrm{GL}(V),\ \ \Theta(\mathrm{Ad}(X_\alpha(t)) = Y_\alpha(t),$$

extends to a group homomorphism $G^{ad}_{\mathcal{A}}(\mathbb{K}) \to \mathrm{PGL}(V)$ and, thus, defines a projective representation of $G^{ad}_{\mathcal{A}}(\mathbb{K})$.

3. Higher Frobenius Kernels

In this section we take G to be a semisimple simply-connected split algebraic group over a field $\mathbb{K}$ of characteristic $p > 0$. Let Φ be the root system of G, $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Phi$ a basis of simple roots. The standard Chevalley basis of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is $\mathbf{e}_\alpha$, $\alpha \in \Phi$, $\mathbf{h}_i = [\mathbf{e}_{\alpha_i}, \mathbf{e}_{-\alpha_i}]$. In particular, $\mathfrak{g}$ is generated by $\mathbf{e}_\alpha$, $\alpha \in \Phi$. It is useful to keep in mind that $ad(\mathbf{e}_\alpha)^p = 0$ for all $\alpha \in \Phi$.

Let $G_{(n)}$ be the n-th Frobenius kernel of G, $\mathrm{Dist}(G_{(n)})$ the distribution algebra on it. $\mathrm{Dist}(G_{(n)})$ has a divided powers basis

$$\prod_{\alpha\in\Phi^+} \mathbf{e}_\alpha^{(m_\alpha)} \prod_{\beta\in\Pi} \binom{\mathbf{h}_\beta}{n_\beta} \prod_{\alpha\in\Phi^+} \mathbf{e}_{-\alpha}^{(m_{-\alpha})} \quad 0 \leqslant m_\alpha, n_\beta, m_{-\alpha} < p^n .$$

If $k < p$ then

$$\mathbf{e}^{(k)} = \frac{1}{k!}\mathbf{e}^k \in \mathrm{Dist}(G_{(1)}) \ni \binom{\mathbf{h}}{k} = \frac{1}{k!}\mathbf{h}(\mathbf{h}-1)\ldots(\mathbf{h}-k+1)$$

so that $\mathrm{Dist}(G_{(1)})$ is a subalgebra of $\mathrm{Dist}(G_{(n)})$, naturally isomorphic to $U_0(\mathfrak{g})$.

Let us now consider a representation (V, θ) of $G_{(n)}$. It is naturally a representation of $\mathrm{Dist}(G_{(n)})$ which we also denote by (V, θ). We define exponentials in an analogous way to the previous section:

$$Y_\alpha(t) = Y_\alpha^V(t) := e^{\theta(t\mathbf{e}_\alpha)} = \sum_{k=0}^{p^n-1} \theta(t^k \mathbf{e}_\alpha^{(k)}) \in \mathrm{End}(V), \quad Z_\alpha(t) = e^{t\mathbf{e}_\alpha} = \sum_{k=0}^{p^n-1} t^k \mathbf{e}_\alpha^{(k)} \in \mathrm{Dist}(G_{(n)})$$

where $t \in \mathbb{K}$ and $\alpha \in \Phi$. Both $Y_\alpha(t)$ and $Z_\alpha(t)$ are invertible. In fact, these are one-parameter subgroups: $Y_\alpha(t)Y_\alpha(s) = Y_\alpha(t+s)$ and $Z_\alpha(t)Z_\alpha(s) = Z_\alpha(t+s)$. Let us generate subgroups by them:

$$G_{(n),V} := \langle Y_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{K}\rangle \leqslant \mathrm{GL}(V), \ \widetilde{G} := \langle Z_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{K}\rangle \leqslant \mathrm{GL}_1(\mathrm{Dist}(G_{(n)})).$$

Conjugation by G equips $\mathrm{Dist}(G_{(n)})$ with a G-module structure, which we can then restrict to $G_{(n)}$-module and $\mathrm{Dist}(G_{(n)})$-module structures. We denote the corresponding representation of $\mathrm{Dist}(G_{(n)})$ by ad because it is a version of the adjoint representation; for instance, the "usual" adjoint representation on $\mathfrak{g}$ is a subrepresentation under $\mathfrak{g} \hookrightarrow U_0(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(G_{(n)})$ (cf. [J2, I.7.18, I.7.11(4)]). We also use ad to denote the representation of $\mathrm{Dist}(G)$ on $\mathrm{Dist}(G_{(n)})$; this restricts to the above ad on $\mathrm{Dist}(G_{(n)})$. We say that (V, θ) is *n-over-restricted* if $\theta(\mathbf{e}_\alpha^{(k)}) = 0$ for all $k \geqslant \lfloor (p^n+1)/2 \rfloor$ and all $\alpha \in \Phi$. Notice that if $p^n = 2$ then this condition forces (V, θ) to be a direct sum of copies of the trivial module.

Proposition 10. *(cf. Proposition 2) If (V, θ) is an n-over-restricted representation of $\mathrm{Dist}(G_{(n)})$, then*

$$\theta\Big(ad(Z_\alpha(t))(\mathbf{d})\Big) = Y_\alpha(t)\theta(\mathbf{d})Y_\alpha(-t)$$

for all $t \in \mathbb{K}$, $\alpha \in \Phi$ and $\mathbf{d} \in \mathrm{Dist}(G_{(n)})$.

Proof. We write ad using Sweedler's Σ-notation [J2, I.7.18]:

$$ad(\mathbf{x})(\mathbf{d}) = \sum_{(\mathbf{x})} \mathbf{x}_{(1)}\mathbf{d}S(\mathbf{x}_{(2)}) \ \text{ for all } \mathbf{x}, \mathbf{d} \in \mathrm{Dist}(G_{(n)}).$$

Since $\Delta(\mathbf{e}_\alpha^{(k)}) = \sum_{i+j=k} \mathbf{e}_\alpha^{(i)} \otimes \mathbf{e}_\alpha^{(j)}$ and $S(\mathbf{e}_\alpha^{(k)}) = (-1)^k \mathbf{e}_\alpha^{(k)}$, we get

$$\theta(ad(t^k \mathbf{e}_\alpha^{(k)})(\mathbf{d})) = \theta(\sum_{i+j=k} (-1)^j t^k \mathbf{e}_\alpha^{(i)} \mathbf{d} \mathbf{e}_\alpha^{(j)}) = \sum_{i+j=k} \theta(t^i \mathbf{e}_\alpha^{(i)}) \theta(\mathbf{d}) \theta((-t)^j \mathbf{e}_\alpha^{(j)}).$$

Hence,

$$\theta\Big(ad(Z_\alpha(t))(\mathbf{d})\Big) = \sum_{k=0}^{p^n-1} \sum_{i+j=k} \theta(t^i \mathbf{e}_\alpha^{(i)}) \theta(\mathbf{d}) \theta((-t)^j \mathbf{e}_\alpha^{(j)}).$$

On the other hand, we have

$$Y_\alpha(t) \theta(\mathbf{d}) Y_\alpha(-t) = \sum_{i,j=0}^{p^n-1} \theta(t^i \mathbf{e}_\alpha^{(i)}) \theta(\mathbf{d}) \theta((-t)^j \mathbf{e}_\alpha^{(j)}).$$

The result follows from the fact that V is n-over-restricted. □

It is useful to remind the reader that $\mathfrak{g}$ can be recovered inside $\text{Dist}(G_{(n)})$ as the set of primitive elements:

$$\mathfrak{g} = \text{Prim}(\text{Dist}(G_{(n)})) := \{\mathbf{d} \in \text{Dist}(G_{(n)}) \mid \Delta(\mathbf{d}) = \mathbf{d} \otimes 1 + 1 \otimes \mathbf{d}\}.$$

This explains why $\mathfrak{g}$ is a submodule of $\text{Dist}(G_{(n)})$ under the adjoint action: we leave it to the reader to check that $ad(\mathbf{x})(\mathbf{d}) \in \text{Prim}(\text{Dist}(G_{(n)}))$ for all $\mathbf{x} \in \text{Dist}(G_{(n)})$ and $\mathbf{d} \in \text{Prim}(\text{Dist}(G_{(n)}))$.

Proposition 11. *Let (V, θ) be an n-over-restricted representation of $\text{Dist}(G_{(n)})$, faithful on $\mathfrak{g}$. Then the assignment*

$$\phi(Y_\alpha^V(t)) = Y_\alpha^{\mathfrak{g}}(t) \ (= e^{ad(t\mathbf{e}_\alpha)})$$

extends to a surjective homomorphism of groups $\phi : G_{(n),V} \to G_{(n),\mathfrak{g}}$, whose kernel consists of $\mathfrak{g}$-automorphisms of V.

Proof. The fact that ϕ is a well-defined homomorphism is proved in a similar way as in Proposition 3. Let $H = *_\alpha U_\alpha$ be the free product of (additive) root subgroups. Both $G_{(n),V}$ and $G_{(n),\mathfrak{g}}$ are naturally quotients of H. If $W_{\beta_1}(t_1) * \ldots * W_{\beta_m}(t_m) \in \ker(H \to G_{(n),V})$ then

$$Y_{\beta_1}^V(t_1) \ldots Y_{\beta_m}^V(t_m) = I_V \ .$$

Proposition 10 tells us that for all $\mathbf{d} \in \mathfrak{g}$

$$\theta(ad(Z_{\beta_1}(t_1)) ad(Z_{\beta_2}(t_2)) \ldots ad(Z_{\beta_m}(t_m))(\mathbf{d})) = \theta(Y_{\beta_1}^{\mathfrak{g}}(t_1) \ldots Y_{\beta_m}^{\mathfrak{g}}(t_m)(\mathbf{d})) = \theta(\mathbf{d}).$$

Since θ is faithful on $\mathfrak{g}$, $Y_{\beta_1}^{\mathfrak{g}}(t_1) Y_{\beta_2}^{\mathfrak{g}}(t_2) \ldots Y_{\beta_m}^{\mathfrak{g}}(t_m) = I_{\mathfrak{g}}$, hence $W_{\beta_1}(t_1) * \ldots * W_{\beta_m}(t_m) \in \ker(H \to G_{(n),\mathfrak{g}})$. Thus, the homomorphism ϕ is well-defined.

Suppose $A = Y_{\beta_1}^V(t_1) \ldots Y_{\beta_m}^V(t_m) \in \ker(\phi)$. By above, $\theta(\mathbf{d}) = \theta(\phi(A)(\mathbf{d})) = A\theta(\mathbf{d})A^{-1}$ for all $\mathbf{d} \in \mathfrak{g}$. Hence, $A \in \text{Aut}_{\mathfrak{g}}(V)$. □

If the adjoint representation is n-over-restricted, we can identify the adjoint group G_{ad} with $G_{(n),\mathfrak{g}}$. Proposition 11 yields an exact sequence of abstract groups

$$1 \to Z_{(n),V} \to G_{(n),V} \xrightarrow{\phi} G_{ad} \to 1$$

where $Z_{(n),V}$ is the kernel of ϕ. To tie up loose ends we need to address the algebraic group properties of this sequence:

Higher Frobenius Conjecture. *Suppose that G is a semisimple connected algebraic group over an algebraically closed field $\mathbb{K}$. The following statements should hold for an n-over-restricted finite-dimensional representation (V, θ) of $G_{(n)}$, faithful on $\mathfrak{g}$:*

(1) *The map $\phi : G_{(n),V} \to G_{(n),\mathfrak{g}}$ constructed in Proposition 11 is a homomorphism of algebraic groups.*
(2) *If $(\mathfrak{g}, ad)$ is n-over-restricted then $\phi : G_{(n),V} \to G_{(n),\mathfrak{g}}$ is a central extension of algebraic groups.*
(3) *If $(\mathfrak{g}, ad)$ is n-over-restricted then (V, θ) extends to a rational representation of the simply-connected group G_{sc}.*

4. Applications of Higher Frobenius Conjecture

We consider G as in the previous section, and assume $\mathbb{K}$ to be algebraically closed. Let (P, θ) be a projective indecomposable $U_0(\mathfrak{g})$-module. The well-known Humphreys-Verma Conjecture [B, D, HV, So] (currently proved for $p \geqslant 2h-2$, where h is the Coxeter number [J1], cf. [J2, II.11.11]) states (P, θ) extends to a G-module. A similar statement for the higher Frobenius kernels follows from Humphreys-Verma Conjecture [J2, Remark II.11.18]. Let us examine what the Higher Frobenius Conjecture can contribute towards this long-standing conjecture.

Let T be a maximal torus of G. $TG_{(n)}$-modules are the same as $X(T)$-graded $G_{(n)}$-modules. We can control the condition of being n-over-restricted for them by monitoring their weights $X(V) = \{\lambda \in X(T) \mid V_\lambda \neq 0\}$. We define *the height of* V by the following formula:

$$\xi(V) := \inf\{n \in \mathbb{N} \mid \forall \alpha \in \Phi \quad X(V) \cap (X(V) + n\alpha) = \varnothing\}.$$

Clearly $\theta(\mathbf{e}_\alpha^{(\xi(V))}) = 0$ is guaranteed for a $TG_{(n)}$-module (V, θ). Hence, the next proposition immediately follows from the Higher Frobenius Conjecture:

Proposition 12. *Suppose that Higher Frobenius Conjecture holds for a connected simply-connected semisimple algebraic group G such that $Z(\mathfrak{g}) = 0$. Assume further that if $p^n = 3$, then G has no components of type G_2. Let (V, θ) be a $TG_{(n)}$-module, faithful as a $\mathfrak{g}$-module, such that $p^n \geqslant 2\xi(V) - 1$ if p is odd, or $p^n \geqslant 2\xi(V)$ if $p = 2$. Then (V, θ) can be extended to a G-module.*

It follows that if a $TG_{(1)}$-module can be extended to a $TG_{(n)}$-module for sufficiently large n, then it can be extended to a G-module. Due to particular significance of projective $U_0(\mathfrak{g})$-modules we state this observation for them as a proposition. Recall that $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ is the half-sum of positive roots. Let $a = \max_{1\leqslant i\leqslant r}(a_i)$ where $2\rho = \sum_{\alpha_i\in\Pi} a_i\alpha_i$ for $a_i \in \mathbb{Z}$.

Proposition 13. *Suppose that the Higher Frobenius Conjecture holds for a connected simply-connected semisimple algebraic group G such that $Z(\mathfrak{g}) = 0$. Let P be a projective indecomposable $U_0(\mathfrak{g})$-module. Suppose P extends to a rational $G_{(n)}$-module where*

$$n \geqslant \log_p(4a(p-1)+1).$$

if p is odd, or

$$n \geqslant \log_2(a+1) + 2$$

if $p = 2$. Then P extends to a G-module.

Table 1. Coxeter numbers and coefficients a

	A_{2l+1}	A_{2l}	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$2h-2$	$4l+2$	$4l$	$4n-2$	$4n-2$	$4n-6$	22	34	58	22	10
a	$(l+1)^2$	$l(l+1)$	n^2	$(n-1)(n+2)$	$(n+1)(n-2)$	42	96	270	42	10

Proof. It is known that P is a $TG_{(1)}$-module [J2, II.11.3]. Clearly, $\xi(P) \leqslant \xi(U_0(\mathfrak{g}))$. From the PBW-basis, it follows that the "top" grade of the grading on $U_0(\mathfrak{g})$ is attained by the element $\prod_{\alpha\in\Phi^+} \mathbf{e}_\alpha^{p-1}$. This has grade $2(p-1)\rho$. Similarly, the "bottom" grade is $-2(p-1)\rho$. Thus, $\xi(U_0(\mathfrak{g})) \leqslant 2(p-1)a+1$ and the condition in Proposition 12, when p is odd, becomes $p^n \geqslant 2\xi(U_0(\mathfrak{g}))-1$; for this to be true, it is enough that $p^n \geqslant 4a(p-1)+1$. When $p=2$, the condition becomes $2^{n-1} \geqslant \xi(U_0(\mathfrak{g}))$, for which it is enough that $2^{n-1} \geqslant 2a+1$ or equivalently $2^{n-2} \geqslant a+1$. □

For the reader's benefit we add two tables. The first contains the values of $2h-2$ and a. The second lists the smallest prime p_0 for all groups up to rank 8 so that extension of P to a rational $G_{(n)}$-module guarantees an extension to a rational G-module as soon as $p \geqslant p_0$ (the column is the type of G, the row is $G_{(n)}$). It also lists the smallest n such that extension to $G_{(n)}$ ensures extension to G for $p = 2, 3, 5$. Some of the entries are marked with the dagger †. This signifies the presence of a nontrivial centre $Z(\mathfrak{g}) \neq 0$.

5. Examples

The heights can be computed for Weyl modules. Let $V(\lambda)$ be the Weyl module with the highest weight $\lambda = \sum_i k_i \varpi_i$ written in the basis of fundamental weights. It follows from the description of $V(\lambda)$ by generators and relations [H, Theorem 21.4] that

$$\xi(V(\lambda)) \leqslant 1 + 2\max_i \frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} = 1 + \max_i k_i \ .$$

This means that the Weyl modules with $k_i \leqslant (p-1)/2$ for all $i = 1, \ldots, r$ are over-restricted. For instance, if $\mathfrak{g}$ is of type A_2 then (for $p>3$) the Weyl module $V(\frac{p-1}{2}\omega_1 + \frac{p-1}{2}\omega_2)$ is the only over-restricted Weyl module outside the first closed p-alcove (under the $\bullet$-action): indeed, $k_1 + k_2 = p-1 > p-2$. Thus, most (but not all) over-restricted modules are semisimple in this case.

On the other hand, if $\mathfrak{g}$ is of type G_2 and α_1 is short, then the over-restricted Weyl module $V(\frac{p-1}{2}\omega_1 + \frac{p-1}{2}\omega_2)$ lies inside the ninth p-alcove (if $p>3$):

$$k_1 + 2k_2 = \frac{3}{2}(p-1) < 2p-3, \ \ k_1 + 3k_2 = 2(p-1) > 2p-4, \ \ k_1 = \frac{p-1}{2} < p-1.$$

Ninth in this context means that there are eight dominant p-alcoves below it. Thus, in type G_2 there are many over-restricted non-semisimple modules.

Table 2. $G_{(n)}$-extension requirements in characteristic p

	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	2
A_1	3	$^\dagger 2$	$^\dagger 2$	$^\dagger 2$	$^\dagger G_{(3)}$
A_2	7	$^\dagger 3$	2	2	$G_{(4)}$
B_2	17	5	3	$^\dagger 2$	$^\dagger G_{(5)}$
G_2	41	7	3	3	$G_{(6)}$
A_3	17	5	3	$^\dagger 2$	$^\dagger G_{(5)}$
B_3	37	7	3	3	$^\dagger G_{(6)}$
C_3	41	7	3	3	$^\dagger G_{(6)}$
A_4	23	$^\dagger 5$	3	2	$G_{(5)}$
B_4	67	11	5	3	$^\dagger G_{(7)}$
C_4	71	11	5	3	$^\dagger G_{(7)}$
D_4	41	7	3	3	$G_{(6)}$
A_5	37	7	$^\dagger 3$	$^\dagger 3$	$G_{(6)}$
B_5	101	11	5	3	$^\dagger G_{(7)}$
C_5	113	11	5	3	$^\dagger G_{(7)}$
D_5	71	11	5	3	$G_{(7)}$

	$G_{(2)}$	$G_{(3)}$	$G_{(4)}$	$G_{(5)}$	2	3	5
F_4	167	13	7	5	$G_{(8)}$	$G_{(6)}$	$G_{(5)}$
A_6	47	$^\dagger 7$	5	3	$G_{(6)}$	$G_{(5)}$	$G_{(4)}$
B_6	149	13	5	5	$^\dagger G_{(8)}$	$G_{(6)}$	$G_{(4)}$
C_6	161	13	7	5	$^\dagger G_{(8)}$	$G_{(6)}$	$G_{(5)}$
D_6	113	11	5	3	$G_{(7)}$	$G_{(5)}$	$G_{(4)}$
E_6	167	13	7	5	$G_{(8)}$	$^\dagger G_{(6)}$	$G_{(5)}$
A_7	67	11	5	3	$^\dagger G_{(7)}$	$G_{(5)}$	$G_{(4)}$
B_7	193	17	7	5	$^\dagger G_{(8)}$	$G_{(6)}$	$G_{(5)}$
C_7	221	17	7	5	$^\dagger G_{(8)}$	$G_{(6)}$	$G_{(5)}$
D_7	161	13	7	5	$G_{(8)}$	$G_{(6)}$	$G_{(5)}$
E_7	383	23	7	5	$^\dagger G_{(9)}$	$G_{(7)}$	$G_{(5)}$
A_8	79	11	5	3	$G_{(7)}$	$^\dagger G_{(5)}$	$G_{(4)}$
B_8	257	17	7	5	$^\dagger G_{(9)}$	$G_{(6)}$	$G_{(5)}$
C_8	281	17	7	5	$^\dagger G_{(9)}$	$G_{(6)}$	$G_{(5)}$
D_8	221	17	7	5	$G_{(8)}$	$G_{(6)}$	$G_{(5)}$
E_8	1087	37	11	7	$G_{(11)}$	$G_{(7)}$	$G_{(6)}$

It is an interesting problem to achieve a detailed description of over-restricted modules. We can formulate some precise questions if we consider *the over-restricted enveloping algebra*

$$U_{\text{over}}(\mathfrak{g}) := U_0(\mathfrak{g})/\langle \mathbf{e}_\alpha^{\lfloor (p+1)/2 \rfloor},\ \alpha \in \Phi \rangle.$$

What is the centre of $U_{\text{over}}(\mathfrak{g})$? Can we describe the blocks of $U_{\text{over}}(\mathfrak{g})$ by quivers with relations? Which of its blocks are tame and which are finite?

6. Conclusion

What have we achieved in this and the preceding paper [RuW]? Suppose G is a semisimple algebraic group with Lie algebra $\mathfrak{g}$. Which concrete $\mathfrak{g}$-modules can we now extend to G-modules? One evident case is when (V, θ) is an indecomposable G-stable $\mathfrak{g}$-module such that G acts trivially on $\mathrm{Aut}_{\mathfrak{g}}(V, \theta)$. By combination of [RuW, Corollary 23], [RuW, Lemma 25] and the cohomology vanishing of the trivial module [J2, II.4.11], $H^2_{Rat}(G, G_{(1)}; A) = 0 = H^1_{Rat}(G, G_{(1)}; A)$ for all A, constituents of $\mathrm{Aut}_{\mathfrak{g}}(V, \theta)$. Thus, the $\mathfrak{g}$-module structure of such (V, θ) extends uniquely to a G-module structure.

It is possible to ensure triviality of the action if one can control the weights. The weights of simple constituents of $\mathrm{Aut}_{\mathfrak{g}}(V, \theta)$ must be divisible by p because $G_{(1)}$ acts trivially. On the other hand, the weights of $V \otimes V^*$ are differences of weights of V. Thus, we have a version of Proposition 12:

Proposition 14. *Let (V, θ) be a G-stable $TG_{(1)}$-module such that $p \geqslant 2\xi(V) - 1$. Then (V, θ) can be uniquely extended to a G-module.*

It would be interesting to extend this result to the higher Frobenius kernels.

7. Appendix: Generic Smoothness in Positive Characteristic

A morphism $\Psi : X \to Y$ of irreducible algebraic varieties over an algebraically closed field is called *smooth* if $d_{\mathbf{x}}\Psi : T_{\mathbf{x}}X \to T_{\Psi(\mathbf{x})}Y$ is surjective for all $\mathbf{x} \in X$. The morphism $\Psi : X \to Y$ is called *generically smooth* if there exists a dense open subset $U \subseteq X$ such that $d_{\mathbf{x}}\Psi$ is surjective for all $\mathbf{x} \in U$.

A generically smooth morphism is necessarily dominant. In the opposite direction, it is a standard fact that dominant morphisms are generically smooth in zero characteristic [S, II.6.2 Lemma 2], but it is manifestly untrue in positive characteristic. For instance, the Frobenius morphism, e.g., $\Psi(\mathbf{x}) = \mathbf{x}^p$ from the affine line to itself, has zero differential at every point.

The issue is best understood on the rational level. Let $\mathbb{K}(X)$ be the field of rational functions on the variety X.

Lemma 15. [Bo, Prop. AG.17.3] *Let $\Psi : X \to Y$ be a dominant morphism of irreducible algebraic varieties over an algebraically closed field $\mathbb{K}$. Then Ψ is generically smooth if and only if the pullback field extension $\mathbb{K}(Y) \xhookrightarrow{\Psi^\sharp} \mathbb{K}(X)$ is separable.*

Our aim is to contemplate a polynomial map

$$F = (F_j(x_1, \dots x_n))_{j=1}^m : \mathbb{K}^n \to \mathbb{K}^m.$$

Lemma 16. *Let Y be the Zariski closure of the image of the polynomial map F. Then there exist a dense Zariski-open set $U \subset \mathbb{K}^n$, a sequence of varieties $U_0 = U, U_1, \dots, U_k$, a sequence of algebraic morphisms $H_t : U_t \to U_{t+1}$ for $t = 0, \dots, k-1$ and an algebraic morphism $\widetilde{F} : U_k \to Y$ such that*

1. *on U the map F factors as $F|_U = \widetilde{F} \circ H_{k-1} \circ \dots H_0$,*
2. *for each t the map H_t is finite of degree p and purely inseparable,*
3. *the morphism $\widetilde{F} : U_k \to Y$ is smooth.*

Proof. Let $x_1, \ldots, x_n$ be the coordinate functions on $\mathbb{K}^n$, $z_1, \ldots, z_m$ the pull-backs to $\mathbb{K}^n$ of the coordinate functions on $\mathbb{K}^m$. Consider a maximal (in $\mathbb{K}(x_1, \ldots, x_n)$) separable extension $\widetilde{\mathbb{K}} \supset F^*\mathbb{K}(Y) = \mathbb{K}(z_1, \ldots, z_m)$. Hence, the gap extension $\mathbb{K}(x_1, \ldots, x_n) \supset \widetilde{\mathbb{K}}$ is purely inseparable. It can be decomposed as a tower of degree p purely inseparable extensions

$$\mathbb{K}_0 = \mathbb{K}(x_1, \ldots, x_n) \supset \mathbb{K}_1 \supset \cdots \supset \mathbb{K}_{k-1} \supset \mathbb{K}_k = \widetilde{\mathbb{K}}.$$

For each intermediate extension we can pick an element $y_t \in \mathbb{K}_0$ such that $y_t^p \in \mathbb{K}_t$ and $\mathbb{K}_{t-1} = \mathbb{K}_t(y_t)$.

Now the field $\widetilde{\mathbb{K}}$ is finitely-generated, so suppose $\widetilde{\mathbb{K}} = \mathbb{K}(w_1, \ldots, w_l)$ where the elements w_j are not necessarily algebraically independent. Let A_0 be the subalgebra of $\mathbb{K}_0$ generated by all w_j, x_j and y_j. Its spectrum is an open subset of $\mathbb{K}^n$. Let us define $A_t := A_0 \cap \mathbb{K}_t$. Let us examine the towers of algebras and their quotient fields

$$A_0 \supset A_1 \supset \cdots \supset A_k \ \text{ and } \ Q(A_0) = \mathbb{K}_0 \supset Q(A_1) \supset \cdots \supset Q(A_{k-1}) \supset Q(A_k) = \mathbb{K}_k.$$

While the equalities $Q(A_0) = \mathbb{K}_0$ and $Q(A_k) = \mathbb{K}_k$ follow from our construction, in general, only $Q(A_t) \subseteq \mathbb{K}_t$ can be immediately discerned. Notice, however, that $y_t \in A_{t-1}$ but $y_t \notin \mathbb{K}_t \supseteq Q(A_t)$. Thus, all extensions in the tower of the quotient fields are proper. Inevitably, by degree considerations, $Q(A_t) = \mathbb{K}_t$ for all t.

The spectra of the rings A_t and the algebraic maps defined by their inclusions, which we denote H_t, nearly satisfy the requirements of the lemma. The only issue is that the map $\operatorname{Spec}(A_k) \to Y$ is only generically smooth. Let U_k be a dense open subset of $\operatorname{Spec}(A_k)$ where this map is smooth. It remains to define all the varieties recursively: $U_t := H_t^{-1}(U_{t+1})$. $\square$

Now we have a tool to establish the key property: "small degree" polynomial maps are generically smooth.

Theorem 17. *Suppose that each degree* $\operatorname{Deg}_{x_t}(F_j(x_1, \ldots x_n))$ *of every component of a polynomial map* $F = (F_j(x_1, \ldots x_n))_{j=1}^m : \mathbb{K}^n \to \mathbb{K}^m$ *is less than* p*. Let* Y *be the Zariski closure of the image of the polynomial map* F*. Then the corestricted morphism* $\widehat{F} := F|^Y : \mathbb{K}^n \to Y$ *is generically smooth.*

Proof. Since the function $\mathbf{x} \mapsto \operatorname{Rank} d_{\mathbf{x}}\widehat{F}$ is lower semicontinuous, it suffices to find a single point $\mathbf{x} \in \mathbb{K}^n$ where the differential $d_{\mathbf{x}}\widehat{F}$ is surjective.

Lemma 16 yields the varieties U_t, the maps H_t, as well as various rational functions w_t, x_t, y_t and z_t. Near any point $\mathbf{x} \in U_0$ we can choose local parameters $X_t := x_t - x_t(\mathbf{x})$ so that the formal neighbourhood of $\mathbf{x}$ in U_0 is the formal spectrum of $B_0 = \mathbb{K}[[X_1, \ldots, X_n]]$. If $\widehat{F}(\mathbf{x})$ is smooth, we can choose local parameters near $\widehat{F}(\mathbf{x})$ from the coordinate functions $z_1, \ldots, z_m$ on $\mathbb{K}^m$. Without loss of generality, the local parameters are $Z_t := z_t - z_t(F(\mathbf{x}))$ for $t = 1, \ldots, s$ where $s = \dim Y \leqslant m$. In particular,

$$Z_t = F_t(x_1, \ldots, x_n) - z_t(F(\mathbf{x})) = F_t(X_1 + x_1(\mathbf{x}), \ldots, X_n + x_n(\mathbf{x})) - z_t(F(\mathbf{x})) = \widetilde{F}_t(X_1, \ldots, X_n),$$

where $\widetilde{F}_t$ is a polynomial without a free term of degree less than p in each variable so that near a generic $\mathbf{x}$ the map $\widehat{F}$ is described by the embedding

$$B_0 = \mathbb{K}[[X_1, \ldots, X_n]] \supseteq B_\infty := \mathbb{K}[[Z_1, \ldots, Z_s]]$$

on the level of formal neighbourhoods.

For a generic point $\mathbf{x} \in U_0$ all of its images $\mathbf{x}_t = H_{t-1}(H_{t-2} \cdots H_0(\mathbf{x}) \cdots) \in U_t$ are smooth. Let B_t be the ring of functions on the formal neighbourhood of $\mathbf{x}_t$, i.e., the formal neighbourhood is the formal spectrum of B_t. Since $\mathbf{x}_t$ is smooth, the ring B_t is the ring of formal power series: $B_t \cong \mathbb{K}[[X_1, \ldots, X_n]]$. Let us examine the tower of formal neighbourhoods

$$B_0 = \mathbb{K}[[X_1, \ldots, X_n]] \supset B_1 \supset \cdots \supset B_k \supset B_\infty = \mathbb{K}[[Z_1, \ldots, Z_s]].$$

In the notation of Lemma 16 we can observe that $\mathbb{K}_t^p \subseteq \mathbb{K}_{t+1}$. It follows that for a generic $\mathbf{x}$ we have the same inclusion on the formal level: $B_t^p \subseteq B_{t+1}$ for all $t < k$. As a corollary of the Kimura-Niitsima Theorem [KN, Cor. 2] (cf. [Ku, Section 15 and Exercise 15.4]), we can describe each map H_t on the formal level as

$$B_t = \mathbb{K}[[Y_1, \ldots, Y_n]] \supset B_{t+1} = \mathbb{K}[[Y_1^p, Y_2, Y_3, \ldots, Y_n]] \tag{2}$$

after a suitable choice of regular sequence of local parameters for B_t.

Let I_t be the maximal ideal of B_t. Now we are ready to prove that the differential that can be described as the natural map

$$d_{\mathbf{x}}\widehat{F} : (I_0/I_0^2)^* \longrightarrow (I_\infty/I_\infty^2)^*$$

is surjective. This is equivalent to injectivity of the natural map $I_\infty/I_\infty^2 \longrightarrow I_0/I_0^2$. Suppose that $d_{\mathbf{x}}\widehat{F}$ is not surjective. Then there exists a nonzero $(\alpha_1, \ldots, \alpha_s) \in \mathbb{K}^s$ such that $Z := \sum_j \alpha_j Z_j \in I_0^2$. However, $\widetilde{F}$ is smooth, hence $d_{\mathbf{x}_k}\widetilde{F}$ is surjective and $Z \notin I_k^2$. Going up the tower, we can find t such that $Z \notin I_{t+1}^2$ and $Z \in I_t^2$. Looking at the description of the floor of the tower in Equation (2), we can conclude that $Z \in B_t Y_1^p$. This is a contradiction because Z is a non-zero polynomial in X_j of degree less than p in each variable. $\square$

It would be quite useful to establish generic smoothness for a larger class of maps than we currently do in Theorem 17. To do that, more detailed information about the local behaviour of inseparable maps is essential. By a $p^\bullet$-basis of a ring R over a subring S we understand a sequence of elements $a_1, \ldots, a_n \in R$ together with a sequence of natural numbers $k_1, \ldots, k_n$ such that the elements $a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}$ (where $0 \leqslant m_i < p^{k_i}$ for all i) form an S-basis of R.

Higher Kunz' Conjecture. *Let $\mathbb{K}$ be a perfect field of characteristic p. Consider a higher Frobenius sandwich of commutative local regular $\mathbb{K}$-algebras*

$$R \geqslant S \geqslant R^q$$

where $q = p^s$ for some natural s. Then there should exist a $p^\bullet$-basis of R over S.

Certainly one can inquire whether this statement holds for a larger class of rings R and S but this is the generality we need. For $s = 1$ and regular local rings this is proved by Kimura and Niitsuma [KN].

We believe that the Higher Kunz Conjecture is key to the Higher Frobenius Conjecture.

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Email address: `D.Rumynin@warwick.ac.uk`

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK
Associated member of Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Russia

Email address: `M.P.Westaway@bham.ac.uk`

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK