a)

THE ORIGIN AND METHODS OF SOLUTION OF DIFFERENCE EQUATIONS IMPORTANT IN THEORETICAL PHYSICS

A Thesis presented by
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to the
University of St.Andrews
In application for the Degree of
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DECLARATION

I hereby declare that the accompanying Thesis is my own composition and that it is based upon research carried out by myself, and that no part of it has previously been presented in application for a higher degree.

<u>CERTIFICATE</u>

I certify that Gwynne James Morgan, B.Sc., has spent four terms as a research student in the Department of Theoretical Physics of the United College of St. Salvador and St. Leonard in the University of St. Andrews; that he has fulfilled the conditions of the University Court Ordinance 338 (St.Andrews No.51) and the supplementary Senate Regulations and that he is qualified to submit the following Thesis in application for the Degree of Master of Science.

Research Supervisor.

CAREER

I obtained the degree of B.Sc, in the Honours School of Physics, at the University of Manchester in 1962. From October 1962 to October 1963 I was a student in the Department of Theoretical Physics of St. Andrews University. During this time I attended a course of lectures on Theoretical Physics and began work on this M.Sc. Thesis under the supervision of Professor R.B. Dingle. Since 1963 I have been working as a Theoretical Physicist in the Laboratories of Zenith Radio Research Corporation (U.K.) Limited. Since 1964 I have been preparing a Thesis for the degree of Doctor of Philosophy to be submitted to the Department of Mathematics, Westfield College of the University of London.

ACKNOWLEDGMENTS

I would like to thank my supervisor, Professor R.B. Dingle, for introducing me to this interesting subject. I also wish to express my thanks for his encouragement, patience and tuition. Sections II (2c), III (5), IV (1), VI (1), VI (2c), VI (4), VI (7), and VII (1), are based or developed from unpublished work by Professor Dingle.

I am grateful to my employers, Zenith Radio Research Corporation (U.K.) Ltd., for providing facilities for the typing and reproduction of this thesis. I am also grateful to Miss W.S. Embers, Miss I. Fowle, and especially Miss G.E. Minting for carrying out this work.

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NOTATION

$\Gamma(n) = \underline{|n+1}$: Gamma function.

$\left[n\right]_s = \dfrac{\Gamma(n+1)}{\Gamma(n+s+1)}$: Factorial function.

1. INTRODUCTION

(1) PURPOSE OF RESEARCH

It has been remarked[1] that problems in Theoretical Physics have been largely formulated in terms of differential equations and that difference equations have been avoided. An alternative formulation, often permitting different approximations to be made, may also give increased insight into a problem. Some problems are expressed naturally in terms of difference equations, there being no differential alternative. In these cases it is clearly necessary for new methods to be developed since difference equations have been rather neglected. The object of this study is to examine and develop the use of difference equations in physical problems.

(2) THE NATURE OF DIFFERENCE EQUATIONS[2,3,4,5,6].

In mathematical analysis two types of function may be distinguished. Functions of a continuous variable, belonging to the realm of the Infinitesimal Calculus, may be studied with the aid of the differential equations they satisfy. Difference equations perform an analogous role in Finite Difference Calculus, where one is concerned with functions whose argument can take on only discrete equally spaced values. A fundamental operator in Finite Calculus is the shift operator defined by,

$$\underset{w}{E}^{s} f(n) = f(n+sw) \tag{1}$$

where w is a constant termed the interval, s is a positive or negative integer and f(n) is an arbitrary function. An s'th order, linear difference equation may be written in the form,

$$\left[A_s(n)\underset{w}{E}^{s} + A_{s-1}(n)\underset{w}{E}^{s-1} + \cdots\cdots + A_0(n)\underset{w}{E}^{0} \right] f(n) = V(n) \tag{2}$$

The functions $A_r(n)$ and $V(n)$ are given, and the equation is to be solved for $f(n)$ If $V(n)$ is zero the equation is said to be homogeneous, otherwise the equation is inhomogeneous. A difference equation may also be expressed in terms of the difference operator which is defined by,

$$\underset{\omega}{\Delta} f(n) = \frac{1}{\omega}\left(E_{\omega}^{1} - 1\right) f(n) \tag{3}$$

This operator is analagous to the differential operator in ordinary calculus. There are many analogies which may be drawn between operators and methods in the two forms of calculus, some of which have been exploited in this thesis. Here we are mainly concerned with second order equations.

(3) CONTENTS

In Section II we give some examples of how difference equations arise in Theoretical Physics. We give the simplest derivations wherever possible without introducing extra complicating factors. For example in II (1a) on the transport of electrons in a polar semiconductor, we give no discussion of scattering by accoustic lattice modes.

We go on (III) to give some standard methods of difference calculus, most of which are employed later in the thesis.

Most of the functions of Mathematical Physics which satisfy a second order differential equation are functions of z (the differential variable) and a parameter n (eigen values of the differential equation), i.e. $f_n(z)$ The eigen value n takes on discrete equally spaced values, and $f_n(z)$ satisfies a second order difference equation (or recurrence relation) with n as the difference variable. In Section IV we show how to calculate series solutions in rising powers of z. Such solutions are termed convergent expansions. We also show how expansions may be found in terms of inverse powers of z. These expansions are of the Stokes' type. They are frequently asymptotic in nature.

However when a function is a polynomial in z the series may terminate. The method employed to obtain these results is an iterative one. Other methods could be used to generate the same results. For example the factorial series method of Boole, the Laplace method of transforms or the technique of cont~~a~~inued fractions. The iterative procedure seems rather simpler than the first two methods and has more general applications.

Section V deals with finding Stokes' expansions in the variable n by expressing the difference operators in terms of differential operators.

The most important part (VI) is that dealing with a finite difference analogue to the well known W.K.B. method for differential equations. This method enables Green-type expansions to be calculated, which are asymptotic in nature. It should be noted that the W.K.B. method for difference equations is rather more difficult for practical purposes than the corresponding differential equation case. We develop a generalised W.K.B. analogue which could be used to generate what are termed uniform expansions. We find that the well known Euler-Maclaurin summation formula is equivalent to a W.K.B. approximation to a first order difference equation and derive a generalised summation formula.

Section VII is concerned with perturbation expansions. Firstly we give a perturbation theory which is applicable to the Schroedinger equation when it is expressed in terms of the difference equation which the Frobenius coefficients satisfy. A second type of perturbation theory is briefly dealt with. Almost the only well investigated difference equations are the recurrence relations for the standard functions. It is therefore desirable to construct a method for solving an equation which deviates slightly from these standard types. We give a way of approaching this problem.

Section VIII contains tables of the difference equations for some standard functions and section IX is a brief discussion of some other methods of solving difference equations.

II. THE DERIVATION OF SOME DIFFERENCE EQUATIONS IMPORTANT IN THEORETICAL PHYSICS

This section consists of two main parts. Firstly we consider difference equations arising in specific physical problems and secondly equations which occur in general mathematical methods of importance to physicists.

(1) DIFFERENCE EQUATIONS ARISING IN SPECIFIC PHYSICAL PROBLEMS

(a) ELECTRONIC CONDUCTION IN POLAR SEMICONDUCTORS

The problem of electron transport through a crystal in which the lattice waves may be polarised has been studied theoretically and experimentally since the 1930's. The optical modes in a polar material gives rise to an electric field which scatters electrons drifting through the crystal under the action of an external field. The salient features of the problem are contained in the simplified model of Howarth and Sondheimer[7] though more accurate treatments have been given since[8]. Howarth and Sondheimer's derivations of the Boltzmann equation for electrons interacting only with optical modes will be outlined. The Boltzmann equation reduces to an inhomogeneous, second order difference equation because of the special assumptions made.

The fundamental assumption of most theories of this phenomenon is that all the optical modes have the same frequency ν_0, irrespective of wave length. It is this assumption which eventually leads to a difference equation. Howarth and Sondheimer[7] consider the case of electrons distributed in a parabolic energy band, the energy of an electron with wave vector $\underline{k}$ being given by,

$$E = \frac{\hbar^2}{2m} |\underline{k}|^2 \tag{4}$$

where m is the effective mass of an electron. The standard form for the Boltzmann equation when a constant electric field $\mathcal{E}$ and a temperature gradient $\frac{\partial T}{\partial x}$ is applied along the x-axis is,

$$-c_1 \frac{\partial f_0}{\partial E}\left(e\mathcal{E} + T\frac{\partial}{\partial x}\frac{\eta}{T} + \frac{E}{T}\frac{\partial T}{\partial x}\right) = \left[\frac{\partial f}{\partial t}\right]_{\text{COLL.}} \tag{5}$$

where c_1 is the component of an electron's velocity in the x direction, $-e$ is the electron charge, η is the Fermi level, $f_0 = \left\{\exp\left(\frac{E-\eta}{kT}\right) + 1\right\}^{-1}$ is the equilibrium Fermi distribution function, and $\left[\frac{\partial f}{\partial t}\right]_{\text{COLL}}$ is the rate of change of the electron distribution function $f(\underline{k})$ due to "collisions" with optical modes.

It is convenient and standard practice to make the substitution for $f(\underline{k})$,

$$f(\underline{k}) = f_0 - \phi(\underline{k})\frac{\partial f_0}{\partial E} \tag{6}$$

where $-\phi(\underline{k})\frac{\partial f_0}{\partial E}$ represents the departure of the electron distribution function from equilibrium. The collision term may be approximated by

$$\left[\frac{\partial f}{\partial t}\right]_{\text{COLL.}} = -\frac{1}{kT}\int V(\underline{k},\underline{k}')\left\{\phi(\underline{k}) - \phi(\underline{k}')\right\} d\underline{k}' \tag{7}$$

where the expression has been linearised by dropping terms of the order $\phi^2(\underline{k})$ The function $V(\underline{k},\underline{k}')$ is given by,

$$V(\underline{k},\underline{k}') = W(\underline{k},\underline{k}')\, f_0(\underline{k}')\left\{1 - f_0(\underline{k})\right\} = V(\underline{k}',\underline{k}) \tag{8}$$

where $W(\underline{k},\underline{k}')$ is the transition probability calculated by first order perturbation theory. H - S used values for $W(\underline{k},\underline{k}')$ calculated by Frölich.[9] Because total energy must be conserved in the collision process an electron with wave vector $\underline{k}$ can only be scattered to a state $\underline{k} \pm \underline{q}$ either absorbing or emitting a quantum of energy $h\nu$. The matrix elements which were used by H - S are

$$W(\underline{k}, \underline{k}+\underline{q}) = \frac{e^4 N}{2\pi a^3 |\underline{q}|^2 M \nu_0} \delta\left(E(\underline{k}+\underline{q}) - E(\underline{k}) - h\nu_0\right) \tag{9a}$$

$$W(\underline{k}, \underline{k}-\underline{q}) = \frac{e^4 (N+1)}{2\pi a^3 |\underline{q}|^2 M \nu_0} \delta\left(E(\underline{k}-\underline{q}) - E(\underline{k}) + h\nu_0\right) \tag{9b}$$

where a is the interiomic distance, M is the reduced mass of the ions, and N is the equilibrium Bose-Einstein function given by

$$N = \left\{ \exp\left(\frac{h\nu_0}{kT}\right) - 1 \right\}^{-1} \tag{10}$$

If the substitution

$$\phi(\underline{k}) = |\underline{k}| \cos\theta \, C(E) \tag{11}$$

is made, where θ is the angle $\underline{k}$ makes with the x-axis, the collision term is greatly simplified. This final reduction in fact yields a difference equation for the function C(E). Since this process is a little involved we quote H-S's result. The equation is,

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}} = \frac{-e^4 |\underline{k}| \cos\theta}{a^3 M \nu_0 E \frac{dE}{d|\underline{k}|}} \frac{\partial f_0}{\partial E} \, L \, C(E) \tag{12}$$

where L is an operator acting on C(E) and is defined by

$$\begin{aligned} L\,C(E) = {} & (N+1) \frac{f_0(E+h\nu_0)}{f_0(E)} \Bigg[C(E+h\nu_0) \left\{ (2E+h\nu_0) \sinh^{-1}\sqrt{\frac{E}{h\nu_0}} - \sqrt{[E(E+h\nu_0)]} \right\} \\ & \qquad - 2E\, C(E) \sinh^{-1}\sqrt{\frac{E}{h\nu_0}} \Bigg] \\ & + H(E-h\nu_0)\, N \frac{f_0(E-h\nu_0)}{f_0(E)} \Bigg[C(E-h\nu_0) \left\{ (2E-h\nu_0) \cosh^{-1}\sqrt{\frac{E}{h\nu_0}} - \sqrt{[E(E-h\nu_0)]} \right\} \\ & \qquad - 2E\, C(E) \cosh^{-1}\sqrt{\frac{E}{h\nu_0}} \Bigg] \end{aligned} \tag{13}$$

The positive value of $\cosh^{-1}\sqrt{\frac{E}{h\nu_0}}$ is understood and $H(E)$ is Heaviside's function. Equation (12) has the form

$$\alpha(E)C(E+h\nu_0) + \beta(E)C(E) + \gamma(E)C(E-h\nu_0) = \delta(E) \tag{14}$$

where α, β, γ and δ are rather complicated functions of E. The more detailed treatments[8] of the problem result in an equation very similar to (12) provided the assumption of a single frequency ν_0 is made. It is felt that a discussion of all the approximations that lead to equation (12) are outside the scope of this thesis. For example the form of the matrix elements, the applicability of first order perturbation theory and the assumption of a single frequency ν_0 have all been considered in the literature[8,10]. This special case of the Boltzmann equation represents rather a more general type of problem where 'particles' interact with a single energy level. Since it is rare for the Boltzmann equation to reduce to a comparatively simple form it is worthwhile attempting to solve it with a fair degree of rigour.

Attempts[7,8,11] at solving equation (12) in the past have involved expressing the difference equation as an integral equation instead of tackling the difference equation directly. Solutions have been obtained for temperatures much greater and much less than the Debye temperature. At intermediate temperatures solutions obtained by the standard variational method of transport theory[7,8] are extremely complicated. It is hoped that methods developed in this thesis will overcome these difficulties.

(b) <u>WAVE FILTERS</u>[12]

Consider a sequence of condensers linked by inductances as in Figure 1.

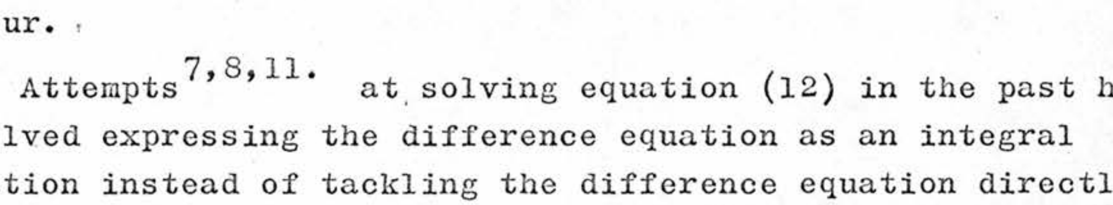

Fig. I

Let C_n be the capacity of the n'th condenser and let $L_{n,n+1}$ be the values of the inductance connecting C_n and C_{n+1} Let V_n, Q_n be the voltage and charge on the n'th condenser. The current from C_p to C_q is to be denoted by $i_{p,q}$. We derive a second order difference equation for Q_n following Brillouin[12]. The rate of change with time of Q_n is given by

$$\frac{d}{dt} Q_n = \left[i_{n-1,n} - i_{n,n+1} \right] \tag{15}$$

If this equation is differentiated with respect to time we have

$$\frac{d^2}{dt^2} Q_n = \left[\frac{d}{dt} i_{n-1,n} - \frac{d}{dt} i_{n,n+1} \right] \tag{16}$$

The time differential of the current can be expressed in terms of the charges as follows:

$$\frac{d}{dt} i_{n-1,n} = \frac{1}{L_{n-1,n}} \left[V_{n-1} - V_n \right] = \frac{1}{L_{n-1,n}} \left[\frac{Q_{n-1}}{C_{n-1}} - \frac{Q_n}{C_n} \right]. \tag{17}$$

$$\frac{d}{dt} i_{n,n+1} = \frac{1}{L_{n,n+1}} \left[V_n - V_{n+1} \right] = \frac{1}{L_{n,n+1}} \left[\frac{Q_n}{C_n} - \frac{Q_{n+1}}{C_{n+1}} \right] \tag{18}$$

The combination of equations (16), (17), and (18) gives

$$\frac{d^2 Q_n}{dt^2} = \frac{Q_{n+1}}{L_{n,n+1} C_{n+1}} + \frac{Q_{n-1}}{L_{n-1,n} C_{n-1}} - \frac{Q_n}{C_n} \left[\frac{1}{L_{n,n+1}} + \frac{1}{L_{n-1,n}} \right] \tag{19}$$

If the explicit time dependence for Q_n is assumed to be

$$Q_n(t) = Q_n e^{-i\omega t} \tag{20}$$

we have

$$\frac{Q_{n+1}}{L_{n,n+1} C_{n+1}} + \frac{Q_{n-1}}{L_{n-1,n} C_{n-1}} - \frac{1}{C_n} \left[\frac{1}{L_{n,n+1}} + \frac{1}{L_{n-1,n}} \right] Q_n + \omega^2 Q_n = 0 \tag{21}$$

This is a very simple example of how difference equations arise to wave filter theory. The equation is of second order because each condenser is linked to only the first nearest neighbour. Higher order equations are obtained by having linkages to more distant neighbours. Difference equations also can be derived for series of elements more complex than a single condenser [12]

(c) <u>A ONE-DIMENSIONAL LATTICE OF MASSES LINKED BY SPRINGS</u>[13]

The mechanical system depicted in Figure 2 is very similar to the electrical one discussed in part (b). Let the mass of the n'th mass be M_n and let the spring connecting M_n and M_{n+1} have a force constant $\beta_{n,n+1}$

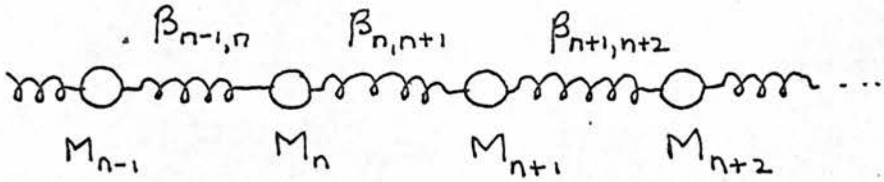

Fig. 2.

Let the displacement of the n'th mass, from its static equilibrium position, be U_n. The equation of motion for the mass M_n is

$$M_n \frac{d^2}{dt^2} U_n = \beta_{n,n+1} U_{n+1} + \beta_{n-1,n} U_{n-1} - \left[\beta_{n,n+1} + \beta_{n-1,n}\right] U_n \tag{22}$$

If a time dependence of the form

$$U_n(t) = U_n e^{-i\omega t} \tag{23}$$

is assumed, we have

$$\beta_{n,n+1} U_{n+1} + \beta_{n-1,n} U_{n-1} - \left[\beta_{n,n+1} + \beta_{n-1,n}\right] U_n + M_n \omega^2 U_n = 0 \tag{24}$$

This equation is very similar in form to (21). The analogies between mechanical and electrical systems are discussed in reference 12 Higher order equations can easily be derived for the case when the masses are linked by springs connecting more than first nearest neighbours.

One dimensional lattices have been studied in great detail[13] since they furnish a guide to the behaviour of more complex three dimensional lattices.

(d) THE INTERACTION OF WAVES [14]

The interference between two interacting wave forms is a subject which encompasses a large section of physical phenomena. Slater[14] has given a review of the many situations where this occurs. Examples of this phenomenon are: an electron moving in a periodic lattice potential, X-rays being scattered by the periodic distribution of charge surrounding atoms in a crystal, and light waves being scattered by ultrasonic sound.

We will concentrate on the last case and give Slater's analysis of the problem, which is applicable to all of the examples cited. This analysis is for two waves within an infinite container within which Born-Von Karmen[15] boundary conditions are imposed. A difference equation formulation is obtained. In practice experiments which are made involve sound being passed through a transparent substance which has a finite volume and the boundary conditions are more complicated. The difference equation formulation of this more practical problem will be given following the original derivation by Raman and Nath[16,17]. The author is indebted to M.V. Berry[18] for supplying him with details of his research on this subject.

(a) THE CASE OF BORN-VON KARMEN BOUNDARY CONDITIONS.[14]

The imposition of Born-Von Karmen boundary conditions is merely a convenient method of normalising the problem. To put matters in their simplest form we consider the sound wave to be travelling in the x direction with a wave vector k_1. The light wave is to be considered as initially travelling in the y direction with a propagation vector K_0 with electric field polarised in the z direction. For simplicity we assume the sound to be unaffected by the light wave. The light has a much higher frequency than the sound wave and so we consider the sound as giving rise to a stationary variation of the refractive index of the transparent substance. We have

$$\mu = \mu_0 + \mu_1 \cos K_1 x \quad , \tag{25}$$

where μ_0 is the unperturbed refractive index and $\mu_1 \cos K_1 x$ is the periodic variation produced by the sound wave. The wave equation for the component of the electric field, E_z is

$$\frac{\partial^2}{\partial x^2} E_z + \frac{\partial^2}{\partial y^2} E_z = \frac{\mu^2}{c^2} \frac{\partial^2}{\partial t^2} E_z \tag{26}$$

where c is the velocity of light. Let ω_0 be the frequency of the light, then if it is assumed that E_z can be expanded in the form,

$$E_z = \sum_{n=-\infty}^{\infty} A_n \exp i \left[K_0 y + n K_1 x - \omega_0 t \right] \tag{27}$$

where n is an integer, use of the othogonality relation for plane waves gives a difference equation for A_n. The equation is,

$$\left[\mu_0^2 - c^2 \left(\frac{K_0 + n K_1}{\omega_0} \right)^2 \right] A_n + \mu_0 \mu_1 \left[A_{n+1} + A_{n-1} \right] = 0 \tag{28}$$

where terms $O\left(\frac{\mu_1^2}{\mu_0^2}\right)$ have been ignored.

The physical interpretation of the A_n is that a plane light wave with wave vector $(0, K_0, 0)$ is scattered into states with vectors $(nK_1, K_0, 0)$ and with amplitudes A_n.

Equation (26) is basically Mathieu's differential equation if the y dependence of E_z is separated, and the A_n are the Fourier components of the solution to Mathieu's equation. Higher order difference equations are generated if the sound wave contains more than a single plane wave. The solution of (28) for a fixed value of K_o is only possible for certain allowed values of ω_o In other words the sound wave causes dispersion of the light and gives a band structure for $\omega_o^2(K_o)$.

(β) THE EXPERIMENTAL SITUATION[16,17,18].

Experiments on this subject involve measuring the intensity of the diffracted light beams produced when light is shone on a transparent substance. It has been pointed out[18] that theoretical treatments of the problem which have been given, fail to produce formulae valid over the whole range of the parameters involved. Again the problem will be considered for a simple geometry since all the essential features are contained therein. The notation is the same as in (α), with the sound wave considered as a stationary disturbance. The geometrical arrangement is depicted in Figure 3.

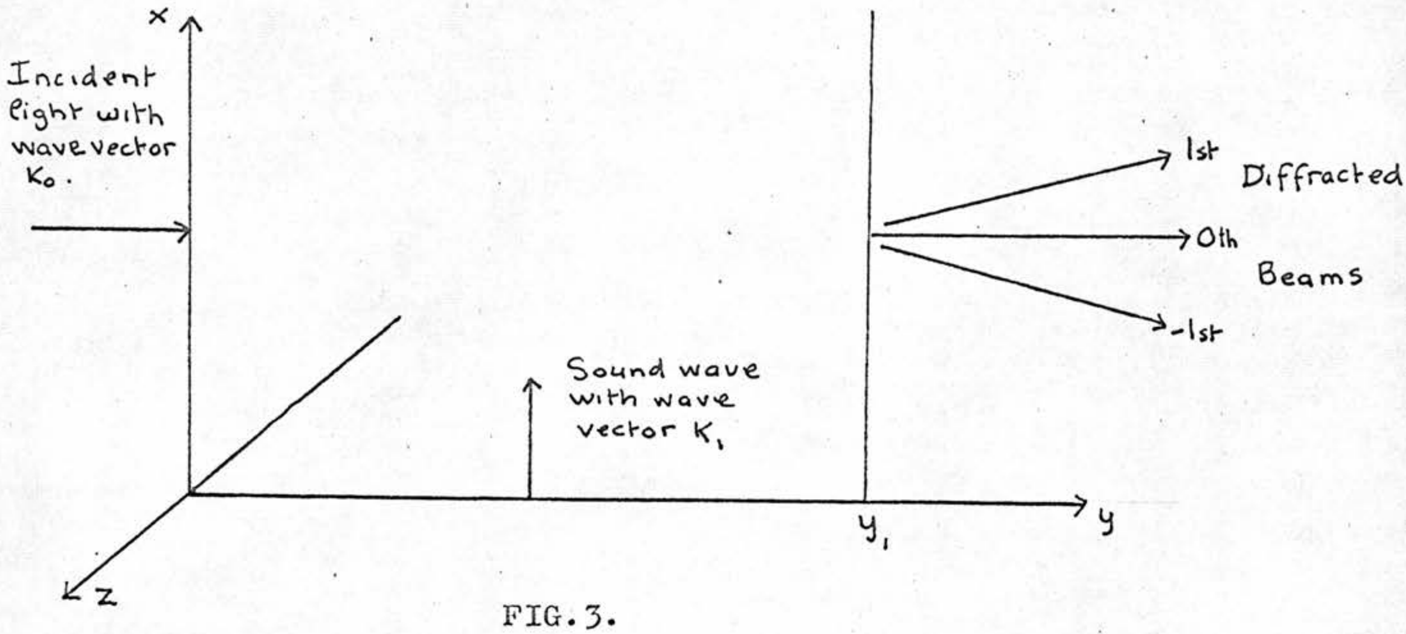

FIG.3.

The transparent substance is to be considered as an infinite slab of thickness y_1. The time dependence can be separated by writing,

$$E_z(x,y,t) = E_z(x,y)e^{-i\omega t}. \tag{29}$$

If terms $O\left(\frac{\mu_1^2}{\mu_0^2}\right)$ are omitted we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -\frac{\omega^2}{c^2}\left[\mu_0^2 + 2\mu_0\mu_1 \cos k_1 x\right] E_z \tag{30}$$

$E_z(x,y)$ is no longer separable because of the geometry. Instead of (27) the substitution

$$E_z(x,y) = \sum_{n=-\infty}^{\infty} X_n(y)\, e^{i n k_1 x} , \tag{31}$$

can be made. By substituting for E_z and making the transformations

$$\begin{aligned} \xi &= k_0 \mu_1 y , \\ \Psi_n(\xi) &= i^n e^{i\frac{\mu_0}{\mu_1}\xi} X_n(y) \end{aligned} \tag{32}$$

we obtain

$$2\frac{d}{d\xi}\Psi_n(\xi) - \Psi_{n-1}(\xi) + \Psi_{n+1}(\xi) = i\rho n^2 \Psi_n(\xi) \tag{33}$$

where

$$\rho = \frac{k_1^2}{\mu_0 \mu_1 k_0^2} \tag{34}$$

In obtaining (33) a term proportional to $\frac{\mu_1}{\mu_0}\frac{d^2}{d\xi^2}\Psi_n(\xi)$ has been neglected since $\frac{\mu_1}{\mu_0}$ is assumed small. For $\xi < 0$ no scattering can occur and this is represented by the boundary condition

$$\Psi_n(0) = \delta_{n0} \tag{35}$$

If the scattering is observed at $\xi_1 = k_0\mu_1 y_1$ then $\Psi_n(\xi_1)$ is the amplitude of the n'th diffracted beam outside the slab.

Equation (33) is a differential-difference equation but the derivitive may be removed by taking the Laplace transform of (33). if

$$L\,\psi_n(\xi) = \int_0^\infty e^{-p\xi}\,\psi_n(\xi)\,d\xi = S_n(p)\,, \tag{36}$$

then

$$L\,\frac{d}{d\xi}\psi_n(\xi) = p\,S_n(p) - \psi_n(0)\,, \tag{37}$$

and a pure difference equation is obtained for $S_n(p)$ which is

$$-S_{n-1}(p) + (2p - ipn^2)S_n(p) + S_{n+1}(p) = 2\,\delta_{n0}\,. \tag{38}$$

It is easily shown from (38) that we must have

$$S_{-n}(p) = (-)^n S_n(p) \tag{39}$$

so that

$$\begin{aligned} &p S_0 + S_1 = 1 \\ &-S_{n-1} + (2p - ipn^2)S_n + S_{n+1} = 0 \quad (n > 0)\,. \end{aligned} \tag{40}$$

The experimental ranges of ξ, and ρ are[18]

$$0 < \xi < 100 \tag{41}$$

and

$$10^{-2} < \rho < \infty \tag{42}$$

Methods developed in this thesis may prove of value for solving (38) when ρ is small.

(2) DIFFERENCE EQUATIONS ARISING IN GENERAL MATHEMATICAL METHODS

We consider three examples here. Firstly, we consider the Frobenius power series method of solving differential equations where difference equations result for the unknown coefficients in the power series. This is typical of how series expansion methods often yield a difference formulation of an equation. Secondly, we consider how the standard functions, defined by a second order differential equation, are functions of a parameter which takes on equally spaced values. Second order difference equations with the parameter as difference variable exist for most of the standard functions and the difference equation may be taken as a starting point for the study of their properties. Finally, at a rather more advanced level, we give a precis of R.B. Dingle's work[1] showing how difference equations may be used to investigate the convergence of perturbation series methods for solving Schroedinger's equation.

(a) THE METHOD OF FROBENIUS[19,20].

One of the most fundamental and important methods of solving differential equations is that due to Frobenius. As an example we first consider the application to Legendre's differential equation which is

$$(1-x^2)\frac{d^2}{dx^2}f(x) - 2x\frac{d}{dx}f(x) + \ell(\ell+1)f(x) = 0 \qquad (43)$$

We may attempt to find a solution of (43) as a series of rising powers of x. Hence we write

$$f(x) = \sum_{n=0}^{\infty} a_n x^{k+n} \qquad (44)$$

where n takes on integral values, a_n and k are to be determined. Substitution of (44) into (43) gives

$$\sum_n a_n (n+k)(n+k+1)x^{n+k-2} - \sum_n a_n\left[(n+k)(n+k-1) - 2(n+k) - \ell(\ell+1)\right] = 0 \qquad (45)$$

Every separate coefficient of each power of x must vanish because (45) is to be satisfied for all values of x. Doing this for (45) we find

$$K(K-1)a_0 = 0 \tag{46}$$

which is termed the indicial equation with roots $K=0,1$ if $a\neq 0$. In general we have

$$a_{n+2}(n+K+2)(n+K+1) = a_n\left[(n+K)(n+K+1) - \ell(\ell+1)\right] \tag{47}$$

which is a first order difference equation. The majority of the well investigated functions also have first order equations for the Frobenius coefficients and it is simple to solve a first order equation. The functions which present more of a problem are those with a second order equation for the coefficients. An example are the functions satisfying Poiseuille's differential equation, which is

$$\frac{d^2}{dr^2}P_\omega(r) + \frac{1}{r}\frac{d}{dr}P_\omega(r) + 4\omega^2(1-r^2)P_\omega(r) = 0 \tag{48}$$

If the substitution

$$P_\omega(r) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n r^{2n}}{n!\,n!} \tag{49}$$

is made the following second order equation results

$$a_{n+1} - \omega^2 n^2 a_{n-1} = \omega^2 a_n \tag{50}$$

In section VII a method of carrying out perturbation theory for difference equations satisfied by Frobenius coefficients will be given .

(b) RECURRENCE RELATIONS FOR STANDARD FUNCTIONS WHICH SATISFY SECOND ORDER DIFFERENTIAL EQUATIONS

The standard functions of mathematical physics, defined by second order differential equations, in addition satisfy second order difference equations or recurrence relations as they are usually known in this context. For example the Legendre polynomials denoted by $P_n(x)$ satisfy the equation

$$(n+1)\,P_{n+1}(x) + n\,P_{n-1}(x) = 2x\,(n+\tfrac{1}{2})\,P_n(x) \tag{51}$$

where the order of the polynomial n acts as a difference variable and the differential equation variable x acts as a parameter in the difference equation. The associated Legendre polynomials $P_n^m(x)$ satisfy two difference equations, one with n as difference variable and another with the minor order m as variable. The general form of the equations for the standard function is

$$A(n)\,f_{n+1}(x) + C(n)\,f_{n-1}(x) = B(n,x)\,f_n(x) \tag{52}$$

It is very useful for classification and calculational purposes to define "normal" forms for a general second order equation. By making the correct substitution of the form,

$$f_n(x) = Q(n)\,F_n(x) \tag{53}$$

it is possible to transform (52) to

$$F_{n+1}(x) + F_{n-1}(x) = 2\,G(n,x)\,F_n(x). \tag{54}$$

The general technique of finding the function $Q(n)$ will be given in section III. Two other normal forms have been found convenient. They are

$$Y_{n+1}(x) - Y_{n-1}(x) = 2\tau(n,x)\,Y_n(x) \tag{55}$$

and

$$U_{n+1}(x) - 2U_n(x) + U_{n-1}(x) = \gamma(n,x)\,U_{n-1}(x) \tag{56}$$

In section VIII a list has been compiled of the difference equations satisfied by some of the well known functions and the functions which give the normal form (54). The point of view of this thesis is that difference equations constitute an alternative to differential equations or integral representations, as a starting point for carrying out analysis.

(c) THE ROLE OF DIFFERENCE EQUATIONS IN PERTURBATION THEORY[1].

We give here an outline of the way in which difference equations may be used to investigate the convergence properties of a perturbation expansion. We first consider the iterative form of perturbation theory for the one particle Schroedinger equation for the case of no degeneracy. This particular theory is convenient for the purpose in hand since the r'th term in the perturbation expansion may be written down by inspection.

Let the Schroedinger equation for an 'unperturbed' system with known energy eigen values E_q be symbolised by

$$(E_q - H)\,\psi_q = 0 \tag{57}$$

where H is the Hamiltonion operator and q is a label which takes integer values. It is desired to solve a perturbed equation with eigen values $E_q + \epsilon$ but with unperturbed boundary conditions, i.e.

$$(E_q + \epsilon - H - h)\,\phi = 0 \tag{58}$$

where h is the perturbation to H and is assumed small. Equation (57) may be refashioned to a form convenient for successive approximation by putting the terms $(\epsilon - h)\phi$ on the right hand side and then inverting the operator $(E_q - H)$. Using the orthogonality of the functions Ψ_q it is easily shown that we must have

$$\phi = \Psi_q + \sum_{p \neq q} \frac{(\Psi_p, (h-\epsilon)\phi)}{(E_q - E_p)} \Psi_p \tag{58}$$

and

$$(\Psi_q, (h-\epsilon)\phi) = 0 \tag{59}$$

where (A,B) denotes a matrix element. If (58) is iterated beginning with the zeroth approximation $\phi = \Psi_q$ we find

$$\phi = \Psi_q + \sum_{\alpha \neq 0} \frac{(0,\alpha)}{(E_q - E_{q+\alpha})} \Psi_{q+\alpha} + \sum_{\alpha \neq 0} \sum_{\beta \neq 0} \frac{(0,\alpha)(\alpha,\beta)}{(E_q - E_{q+\alpha})(E_q - E_{q+\beta})} \Psi_{q+\beta}$$

$$+ \sum_{\alpha \neq 0} \sum_{\beta \neq 0} \sum_{\gamma \neq 0} \frac{(0,\alpha)(\alpha,\beta)(\beta,\gamma)}{(E_q - E_{q+\alpha})(E_q - E_{q+\beta})(E_q - E_{q+\gamma})} \Psi_{q+\gamma} + \cdots \tag{60}$$

$$= \sum_{r=0}^{\infty} \phi^{(r)},$$

where

$$(\alpha, \beta) = (\Psi_{q+\beta}, (h-\epsilon) \Psi_{q+\alpha}). \tag{61}$$

The general form form $\phi^{(r)}$ is obvious by inspection. The new eigen values are determined by the equation which results on substituting for ϕ in (59), which gives

$$0 = (0,0) + \sum_{\alpha \neq 0} \frac{(0,\alpha)(\alpha,0)}{(E_q - E_{q+\alpha})} + \sum_{\alpha \neq 0} \sum_{\beta \neq 0} \frac{(0,\alpha)(\alpha,\beta)(\beta,0)}{(E_q - E_{q+\alpha})(E_q - E_{q+\beta})} + \cdots \tag{62}$$

The r'th term in the series (62) can be interpreted diagramatically as corresponding to all the possible ways of moving from o back to o in r moves with the position o not allowed as an intermediate move. In general it is difficult to investigate the properties of the series (62) in the above form. The problem can be simplified by expressing each term in the series in terms of its neighbours. If the function $P_r(\alpha)$ is defined by

$$P_0(\alpha) = \delta_{\alpha,0}\ , \quad P_1(\alpha) = \frac{(0,\alpha)}{(E_q - E_{q+\alpha})}\ , \quad P_2(\beta) = \sum_{\alpha\neq 0} \frac{(0,\alpha)(\alpha,\beta)}{(E_q - E_{q+\alpha})(E_q - E_{q+\beta})} \tag{63}$$

$$P_3(\gamma) = \sum_{\alpha\neq 0}\sum_{\beta\neq 0} \frac{(0,\alpha)(\alpha,\beta)(\beta,\gamma)}{(E_q - E_{q+\alpha})(E_q - E_{q+\beta})(E_q - E_{q+\gamma})}\ ,$$

and so on, Then in general

$$(E_q - E_{q+\gamma})\, P_{r+1}(\gamma) = \sum_{\beta\neq 0 \text{ unless } r=0} (\beta,\gamma)\, P_r(\beta) \tag{64}$$

Equation (63) is a partial difference equation in the variables β, γ. Both (60) and (62) can be expressed in terms of the function $P_r(\gamma)$ i.e.

$$\phi = \sum_{r=0}^{\infty} \sum_{\gamma\neq 0 \text{ unless } r=0} P_r(\gamma)\, \Psi_{q+\gamma} \tag{65}$$

and

$$\sum_{r=0}^{\infty} \lim_{\gamma\to 0} (E_q - E_{q+\gamma})\, P_{r+1}(\gamma) = 0 \tag{66}$$

The reduction of (64) to an ordinary difference equation is obtained by writing

$$P(\gamma) = \sum_{r=0}^{\infty} \bar{\rho}^{\,r}\, P_r(\gamma). \tag{67}$$

$P(\gamma)$ is called the generating function of $P_r(\gamma)$[2,3].
Multiplying (64) by $\bar{\rho}^r$ and summing over r it is found that

$$\rho\,(E_q - E_{q+\gamma})\,P(\gamma) = \sum_{\beta \neq 0 \text{ except for terms in } \rho^0} (\beta,\gamma)\,P(\beta) \qquad (68)$$

In general the number of solutions to (68) will be equal to the order of the difference equation and any solution can be multiplied by an arbitrary function of ρ. The required solution must be the one permitting an expansion of the form (67). The artificial selector variable may be chosen to suit the nature of a particular problem. If (68) can be solved either approximately or exactly it is an elegant way of investigating the convergence properties of a perturbation series. R.B. Dingle[1]. has illustrated the power of this method by applying it to the problem of determining the eigen values of the periodic Mathieu functions.

III SOME BASIC METHODS OF FINITE DIFFERENCE CALCULUS[2,3].

In this section a short account is given of standard methods which are used in the main part of the thesis.

(1) OPERATORS

The shift operator $\underset{\omega}{E}^{s}$ is defined by

$$\underset{\omega}{E}^{s} f(n) = f(n+s\omega) \tag{69}$$

where s can be a positive or negative integer and $f(n)$ is an arbitrary function of n. The difference operator $\underset{\omega}{\Delta}$ is defined by

$$\underset{\omega}{\Delta} f(n) = \frac{1}{\omega}\left[f(n+\omega) - f(n)\right] \tag{70}$$

is analagous to the differential operator $\frac{d}{dn}$ and in the limit as ω tends to zero

$$\lim_{\omega \to 0} \underset{\omega}{\Delta} \equiv \frac{d}{dn}. \tag{71}$$

$\underset{\omega}{E}$ and $\underset{\omega}{\Delta}$ can be expressed in terms of differential operators by using Taylor's theorem, i.e.

$$\underset{\omega}{E}^{1} f(n) = f(n) + \omega D f(n) + \frac{\omega^2}{2!} D^2 f_n + \cdots \tag{72}$$

where D^r denotes $\frac{d^r}{dn^r}$ Hence

$$\underset{\omega}{E}^{1} \equiv e^{\omega D} \tag{73}$$

and

$$\underset{\omega}{\Delta} \equiv \frac{1}{\omega}\left[e^{\omega D} - 1\right]. \tag{74}$$

The second order difference operator is defined by

$$\underset{\omega}{\Delta}^2 f(n) = \underset{\omega}{\Delta}\left(\underset{\omega}{\Delta} f(n)\right) = \frac{1}{\omega^2}\left[f(n+2\omega) - 2f(n+\omega) + f(n)\right] \tag{75}$$

and $\underset{\omega}{\Delta}^s$ means multiplying by $\underset{\omega}{\Delta}$ times. The interval ω in these operations can be made equal to unity by changing the variable If we write

$$p = \frac{1}{\omega} n + c \tag{76}$$

where C is any constant then

$$\underset{\omega}{\Delta} f(n) = \frac{1}{\omega} \underset{1}{\Delta} F(p). \tag{77}$$

(2) <u>SUMMATION</u>

Consider the inhomogeneous equation

$$\underset{\omega}{\Delta} f(n) = \phi(n) \tag{78}$$

The solution to (78) can be written symbolically as

$$f(n) = \underset{\omega}{\Delta}^{-1} \phi(n) \tag{79}$$

The process of inverting $\underset{\omega}{\Delta}$ is analogous to inverting $\frac{d}{dn}$ and is called indefinite summation. The nature of $\underset{\omega}{\Delta}^{-1}$ can be found by Andre's method [2]. We consider (78) to be equivalent to the system of equations

$$\begin{aligned} f(n) - f(n-\omega) &= \omega\, \phi(n-\omega) \\ f(n-\omega) - f(n-2\omega) &= \omega\, \phi(n-2\omega) \\ &\cdots\cdots \\ f(n_0+\omega) - f(n_0) &= \omega\, \phi(n_0) \end{aligned} \tag{80}$$

Adding these equations it is found that

$$f(n) = f(n_0) + \omega\left\{\phi(n_0) + \phi(n_0+\omega) \cdot \cdot \cdot \cdot + \phi(n-\omega)\right. \tag{81}$$

In finite difference calculus it is convenient to use the notation

$$\sum_{n=n_0}^{n} \phi(n) = \phi(n_0) + \phi(n_0+\omega) \cdot \cdot \cdot \cdot + \phi(n-\omega) \tag{82}$$

Hence

$$f(n) = f(n_0) + \omega \sum_{n_0}^{n} \phi(n) \tag{83}$$

Equation (83) may be written as

$$f(n) = \omega \sum_{c}^{n} \phi(n) \tag{84}$$

where c is an arbitrary constant of summation. We will use Norlund's[3,5] notation for the indefinite sum, and this is defined by

$$\omega \sum_{n=c}^{n} \phi(n) \equiv \mathop{\mathsf{S}}_{c}^{n} \phi(n) \underset{\omega}{\Delta} n \tag{85}$$

Any arbitrary function of period ω is also a solution to (78) but these solutions are to be considered as understood in this thesis.

(3) FIRST ORDER DIFFERENCE EQUATIONS

The general solution to a first order difference equation will be required in later sections. A homogeneous equation has the form

$$f(n+\omega) - p(n)\, f(n) = 0 \tag{86}$$

Taking logarithms

$$\ln f(n+\omega) - \ln f(n) = \ln p(n) \tag{87}$$

Hence by summing both sides

$$\ln f(n) = \frac{1}{\omega} \mathop{S}_{c}^{n} \ln p(n) \underset{\omega}{\Delta} n \tag{88}$$

or

$$f(n) = \exp \frac{1}{\omega} \mathop{S}_{c}^{n} \ln p(n) \underset{\omega}{\Delta} n \tag{89}$$

Consider now the complete equation

$$f(n) - p(n) f(n) = q(n) \tag{90}$$

When $q(n)$ is zero the solution of (90) is given by equation (89) and will be called $f_1(n)$ Hence by writing

$$f(n) = f_1(n) g(n) \tag{91}$$

we have

$$f_1(n+\omega) g(n+\omega) - p(n) f_1(n) g(n) = q(n) \tag{92}$$

Now $f_1(n+\omega) = p(n) \; f(n)$ so that

$$g(n+\omega) - g(n) = \frac{q(n)}{p(n) f_1(n)} \tag{93}$$

Therefore the complete solution to (90) is

$$f_c(n) = f_1(n)\left[1 + \frac{1}{\omega} \overset{n}{\underset{c}{S}} \frac{q(n)}{p(n)\, f_1(n)} \underset{\omega}{\Delta} n\right] \tag{94}$$

(4) THE NORMAL FORM FOR A SECOND ORDER EQUATION

We wish to express a general equation

$$A(n) f(n+\omega) - B(n) f(n) + C(n) f(n-\omega) = 0 \tag{95}$$

as

$$F(n+\omega) + F(n-\omega) = 2\, б(n) F(n) \tag{96}$$

If we write

$$f(n) = Q(n) F(n) \tag{97}$$

then

$$Q(n+\omega) A(n) F(n+\omega) + Q(n-\omega) C(n) F(n-\omega) = B(n) Q(n) F(n) \tag{98}$$

To obtain form (96) we must have

$$Q(n+\omega) = \frac{C(n)}{A(n)} Q(n-\omega) \tag{99}$$

This is a first order equation with interval 2ω. Hence from (89)

$$Q(n) = \exp \frac{1}{2\omega} \overset{n}{\underset{n_0}{S}} \ln \frac{C(n+\omega)}{A(n+\omega)} \underset{2\omega}{\Delta} n \tag{100}$$

and

$$G(n) = \frac{B(n)\,Q(n)}{2\,A(n)\,Q(n+\omega)} \tag{101}$$

The arbitrary constant n_0 can be given any convenient value. The same type of argument may be used to obtain the other normal forms (55) and (56).

(5) INHOMOGENEOUS SECOND ORDER EQUATIONS

The complete second order equation in normal form is

$$F(n+\omega) + F(n-\omega) - 2\,G(n)\,F(n) = V(n) \tag{102}$$

There are s independent solutions to an s'th order equation[2,3,4]. Let the two independent solutions to (102), when $V(n)$ is zero, be denoted by, $F_1(n)$ and $F_2(n)$ i.e.

$$F_1(n+\omega) + F_1(n-\omega) = 2\,G(n)\,F_1(n) \tag{103}$$

and

$$F_2(n+\omega) + F_2(n-\omega) = 2\,G(n)\,F_2(n) \tag{104}$$

Multiply (103) by $F_2(n)$ and (104) by $F_1(n)$ and subtract, then

$$F_1(n)\,F_2(n+\omega) - F_1(n+\omega)\,F_2(n) = F_1(n-\omega)\,F_2(n) - F_1(n)\,F_2(n-\omega) \tag{105}$$

Equation (105) must be true for all n, therefore

$$C = F_1(n)\,F_2(n+\omega) - F_1(n+\omega)\,F_2(n) \tag{106}$$

must be independent of n The quantity C is Casorati's determinant[3] for the special normal form considered and is analogous to

the Wronskian in differential equation theory.
By using Lagrange's[3] method of variation of parameters we show that the normal form (102) leads to simple expressions for $F(n)$ in terms of $F_1(n)$ and $F_2(n)$. Assume that a solution exists to (102) of the form

$$F(n) = P(n)\,F_1(n) + Q(n)\,F_2(n). \tag{107}$$

Substitution of (107) into (102) gives

$$\begin{aligned}&[P(n+w)-P(n)]\,F_1(n+w) - [P(n)-P(n-w)]\,F_1(n-w)\\ &+ [Q(n+w)-Q(n)]\,F_2(n+w) - [Q(n)-Q(n-w)]\,F_2(n-w) = V(n)\end{aligned} \tag{108}$$

Because there are two disposable functions $P(n)$ and $Q(n)$ we can make them satisfy one other condition. We can assume that

$$[P(n+w)-P(n)]\,F_1(n) + [Q(n+w)-Q(n)]\,F_2(n) = 0 \tag{109}$$

hence

$$[P(n+w)-P(n)]\,F_1(n+w) + [Q(n+w)-Q(n)]\,F_2(n+w) = V(n) \tag{110}$$

Eliminating $[P(n+w)-P(n)]$ we have

$$\begin{aligned}Q(n+w)-Q(n) &= V(n)\,F_1(n)\Big/\big(F_1(n)F_2(n+w)-F_1(n+w)F_2(n)\big) \\ &= V(n)\,F_1(n)/C\end{aligned} \tag{111}$$

Hence

$$Q(n) = \frac{1}{C\omega} \mathop{S}^{n} V(n)\,F_1(n) \mathop{\Delta}_{\omega} n \tag{112}$$

and using (109) we have

$$F(n) = \frac{1}{\omega C}\left\{ F_2(n) \mathop{S}^{n} V(n) F_1(n) \mathop{\Delta}_{\omega} n - F_1(n) \mathop{S}^{n} V(n) F_2(n) \mathop{\Delta}_{\omega} n \right\} \tag{113}$$

IV. THE CALCULATION OF SERIES SOLUTIONS IN RISING AND INVERSE POWERS OF z FOR A FUNCTION $f_n(z)$ DEFINED BY A SECOND ORDER DIFFERENCE EQUATION WHERE n IS THE DIFFERENCE VARIABLE AND z A PARAMETER OF THE EQUATION.

The functions we will investigate are those studied initially from the differential equations they satisfy. For example the Bessel function $J_n(z)$ satisfies the differential equation,

$$\frac{d^2}{dz^2}J_n(z) + \frac{1}{z}\frac{d}{dz}J_n(z) + \left(1-\frac{n^2}{z^2}\right)J_n(z) = 0 \tag{114}$$

in which z is the independent variable and n a parameter, but it also satisfies the difference equation

$$J_{n+1}(z) + J_{n-1}(z) = 2\frac{n}{z}J_n(z) \tag{115}$$

where n is the variable and z a parameter. We show how solutions in rising powers of z (convergent expansions) and inverse powers of z (Stokes'[21] expansions) can be calculated from the difference equation by iterative methods. Two classical methods of solving difference equations are the methods of Boole[2,3] and Laplace[2,3] which are applicable to equations with rational coefficients. The equations we consider fall into this category but the iterative method can be applied to equations with more general coefficients. A brief comparison of the iterative method with the above two approaches and with the technique of continued fractions[22,23] is given in section IX. The method we use involves writing a difference equation as a summation equation (analogous to an integral equation).

Consider a general order equation represented symbolically by

$$L\,u_n(z) = 0 \tag{116}$$

where L operates on n. We seek to rewrite this equation in the form

$$L_1 U_n(z) = L_2 U_n(z) \tag{117}$$

Quite formally we can iterate this equation by ignoring the right hand side to begin with. We may do this without consideration of the magnitude of the right hand side with the justification that after iteration we can examine the convergence properties of the resulting series. The procedure then is to first solve the equation

$$L_1 U_n(z) = 0 \tag{118}$$

the solution to which we call $U_n^0(z)$. The first requirement of L_1 is that it must not be the identity operator. The next term called $U_n^1(z)$ is the solution to

$$L_1 U_n^1(z) = L_2 U_n^0(z) \tag{119}$$

or

$$U_n^1(z) = L_1^{-1} L_2 U_n^0(z) \tag{120}$$

Hence

$$U_n(z) = U_n^0(z) + U_n^1(z) + \cdots \tag{121}$$

We must therefore be able to find the inverse L_1^{-1} to L_1. In the examples to be considered L_1^{-1} takes the form of an indefinite summation and hence the iteration series depends on the constants of summation involved. The solution also depends on the range and exact values of the variable n. In the following examples we consider n to be known. In general the constants of summation and values would have to be determined from boundary conditions or some subsidiary equation. The splitting of the operator L into L_1 and L_2 such that we obtain a series of the required form is largely a matter of experiment but the various possibilities can be quickly investigated. If L_1 is a second order operator we generate two solutions of the equation. If L_1 be of first order we obtain one solution and then have to find another L_1 to find a second solution.

(1) SERIES IN RISING POWERS OF z

To obtain a series in rising powers of z it is necessary to write a difference equation in the form (119) with a positive power of z on the right hand side. These series are termed convergent since by taking enough terms the series converges to the exact answer. The series can terminate for a polynomial solution.

(a) BESSEL FUNCTIONS

The difference equation for Bessel functions of real argument is

$$f_{n+1}(z) + f_{n-1}(z) = 2\frac{n}{z} f_n(z), \qquad (122)$$

where n is not necessarily integral. Let us consider the case when n is not an integer and begin by arranging the above equation in the form

$$f_n(z) - \frac{z}{2n} f_{n-1}(z) = \frac{z}{2n} f_{n+1}(z) \qquad (123)$$

The prescription is to first solve the zeroth equation

$$f^0_n(z) - \frac{z}{2n} f_{n-1}(z) = 0 \qquad (124)$$

The solution is the standard result

$$f_n^{\circ}(z) = \exp \mathop{S}_{c}^{n} \ln \frac{z}{2(n+1)} \Delta n = A(z) \left(\frac{z}{2}\right)^n / \Gamma(n+1) . \tag{125}$$

It is convenient to remove this zeroth order term by writing

$$f_n(z) = f_n^{\circ}(z) \; F_n(z) \tag{126}$$

so that after substitution we have

$$F_{n+1}(z) - F_n(z) = \left(\frac{z}{2}\right)^2 / \left[(n+2)(n+1)\right] \; F_{n+2}(z) \tag{127}$$

This exact equation for $F_n(z)$ is nicely expressed by a summation equation

$$F_n(z) = \left(\frac{z}{2}\right)^2 \mathop{S}_{n_0'}^{n} \frac{F_{n+2}(z)}{(n+2)(n+1)} \Delta n \equiv 1 + \left(\frac{z}{2}\right)^2 \mathop{S}_{n_0}^{n} \frac{F_{n+2}(z)}{(n+2)(n+1)} \Delta n \tag{128}$$

We are at liberty to write the equation in either of the above forms because of the arbitrary constant of summation. $A(z)$ and n_0 correspond to one of the two arbitrary constants of a second order equation.

The iterative process is begun by putting $F_n(z) = 1$ for all n on the right hand side

$$F_n(z) = 1 + \left(\frac{z}{2}\right)^2 \mathop{S}_{n_0}^{n} [n]_{-2} \Delta n + \cdots = 1 - \left(\frac{z}{2}\right)^2 \Big[[n]_{-1} \Big]_{n_0}^{n} \tag{129}$$

At this stage we identify the series with $J_n(z)$ by taking $n_0 = \infty$

$$F_n(z) = 1 - \left(\frac{z}{2}\right)^2 [n]_{-1} + \cdots \tag{130}$$

The next term is given by substitution of the above into the summation equation.

$$F_n(z) = 1 - \left(\tfrac{z}{2}\right)^2 [n]_{-1} - \left(\tfrac{z}{2}\right)^4 \overset{n}{\underset{\infty}{S}} [n]_{-3}\,\Delta n + \cdots$$

$$= 1 - \left(\tfrac{z}{2}\right)^2 [n]_{-1} + \left(\tfrac{z}{2}\right)^4 [n]_{-2}/2 + \cdots \tag{131}$$

In this case the general term is easily found by inspection so that

$$F_n(z) = \frac{A(z)\left(\frac{z}{2}\right)^n}{\Gamma(n+1)} \left\{ 1 - \left(\tfrac{z}{2}\right)^2 [n]_{-1} \cdots + \frac{(-)^p \left(\frac{z}{2}\right)^{2p} [n]_{-p}}{\Gamma(p+1)} \cdots \right\} \tag{132}$$

The standard definition of $J_n(z)$ has $A(z)=1$. In general $A(z)$, n_0 and the exact values of n would have to be determined from boundary conditions. A second solution is generated by applying the same argument to the form

$$f_{n+1}(z) - \frac{2n}{z} f_n(z) = - f_{n-1}(z) \tag{133}$$

This second solution $f_n'(z)$ is

$$f_n'(z) = A'(z)\left(\tfrac{z}{2}\right)^{-n} \Gamma(n) \left\{ 1 + \left(\tfrac{z}{2}\right)^2 \frac{1}{[n-1]_{-1}} + \cdots \left(\tfrac{z}{2}\right)^{2p} \frac{1}{\Gamma(p+1)\,[n-1]_p} + \cdot \right\} \tag{134}$$

If we take $A'(z)=1$

$$f_n'(z) = \frac{\pi}{\sin n\pi} J_{-n}(z) \tag{135}$$

The point is that the second solution of Bessel's differential equation for non-integral n satisfies the difference equation

$$f_{n+1}(z) + f_{n-1}(z) = -\frac{2n}{z} f_n(z) \tag{136}$$

The same procedure may be followed through for Bessel Functions of integral order, indeed the argument is identical except that the second solution has to be synthesised from $J_n(z)$ and $J_{-n}(z)$.

$$Y_n(z) = \lim_{n \to \text{integer}} (\sin n\pi)^{-1} \left\{ J_n(z) \cos n\pi - J_{-n}(z) \right\} \tag{137}$$

(b) THE HERMITE FUNCTIONS

We shall seek the polynomial solutions to the equation

$$f_{n+1}(z) + 2n\, f_{n-1}(z) = 2z\, f_n(z). \tag{138}$$

We remove the zeroth term by writing

$$f_n(z) = A(z) \left(\pm 2i \right)^n \Gamma\left(\tfrac{n+1}{2}\right) F_n(z) \tag{139}$$

Substituting

$$\begin{aligned} q &= n/2 \\ F_n &= F'_q \end{aligned} \tag{140}$$

we find

$$F'_q = 1 \mp \frac{iz}{2} \mathop{\mathrm{S}}_{q_0}^{q} \frac{\Gamma(q+1)}{\Gamma(q+\frac{3}{2})} F'_{q+1/2}\, \Delta q. \tag{141}$$

This is best expressed in terms of the generalised factorial defined by

$$[q]_s = \Gamma(q+1) / \Gamma(q+1-s). \tag{142}$$

The next approximation obtained by putting $F_q' = 1$ in the right hand side of (141) is

$$F_q' = 1 \mp \frac{iz}{2}\left[2[q]_{\frac{1}{2}}\right]_{q_0}^{q} + \cdots \tag{143}$$

Let us choose q_0 such that $[q_0]_{\frac{1}{2}} = 0$. The next terms are

$$F_q' = 1 \pm \left(\frac{iz}{2}\right)2[q]_{\frac{1}{2}} + \left(\frac{iz}{2}\right)^2 2[q]_1 \mp \left(\frac{iz}{2}\right)^3 \frac{2\cdot 2}{3}[q]_{3/2} + \cdots \tag{144}$$

If we take a linear combination of the solutions associated with the $\pm$ signs and take $A = \frac{1}{2\sqrt{\pi}}$ we obtain

$$\begin{aligned} H_n(z) = \frac{\Gamma\left(\frac{n+1}{2}\right)(2i)^n}{2\sqrt{\pi}}\Big\{ & 1 - \left(\frac{iz}{2}\right)2[q]_{\frac{1}{2}} + \left(\frac{iz}{2}\right)^2 2[q]_1 + \cdots \\ & + (-)^n\left(1 + \left(\frac{iz}{2}\right)2[q]_{\frac{1}{2}} + \left(\frac{iz}{2}\right)^2 2[q]_1 + \cdots\right)\Big\} \end{aligned} \tag{145}$$

(c) LEGENDRE FUNCTIONS

These functions satisfy

$$(n+1)f_{n+1}(z) + n f_{n-1}(z) = 2z\left(n+\tfrac{1}{2}\right)f_n(z) \tag{146}$$

We could investigate this equation for general values of n but we shall restrict ourselves to $n = 0, 1, \cdots \infty$ and look for the polynomial solutions termed Legendre Polynomials. It is clear from inspection that the form

$$f_{n+1}(z) + \frac{n}{n+1}f_{n-1}(z) = 2z\frac{(n+\frac{1}{2})}{(n+1)}f_n(z) \tag{147}$$

will give a series in rising powers of z since z does not occur explicitly on the left hand side. The solution to

$$f^0_{n+1}(z) + \frac{n}{(n+1)}f^0_{n-1}(z) = 0 \tag{148}$$

is

$$f^{o}_{n}(z) = A(z)(\pm i)^{n}\,\Gamma\left(\tfrac{n+1}{2}\right)/\Gamma\left(\tfrac{n+2}{2}\right) \tag{149}$$

We now write

$$f_{n}(z) = A(z)(\pm i)^{n}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\,F_{n}(z) \tag{150}$$

to obtain

$$F_{n+2}(z) - F_{n}(z) = \pm\,\frac{z}{i}\left(n+\tfrac{3}{2}\right)\frac{\Gamma^{2}\left(\frac{n+2}{2}\right)}{\Gamma^{2}\left(\frac{n+3}{2}\right)}\,F_{n+1}(z) \tag{151}$$

It is convenient to make the transformations $n=2q$, $x=\frac{z}{i}$, $F_n=F'_q$ whence

$$F'_{q} = 1 \pm 2x \int_{q_0}^{q}\left(q+\tfrac{3}{4}\right)\frac{\Gamma^{2}(q+1)}{\Gamma^{2}\left(q+\frac{3}{2}\right)}\,F'_{q+\frac{1}{2}}\,\Delta q. \tag{152}$$

Substituting $F'_{q+\frac{1}{2}}=1$ in the right hand side gives

$$F'_{q} = 1 \pm 2x \int_{q_0}^{q}\frac{\left(q+\frac{3}{4}\right)\Gamma^{2}(q+1)}{\Gamma^{2}\left(q+\frac{3}{2}\right)}\,\Delta q + \cdots \tag{153}$$

This gives

$$F'_{q} = 1 \pm 2x\left[\frac{\Gamma^{2}(q+1)}{\Gamma^{2}\left(q+\frac{1}{2}\right)}\right]_{q_0}^{q} + \cdots \tag{154}$$

We take q_0 such that $\Gamma(q_0+1)/\Gamma(q_0+\frac{1}{2}) = 0$. If we take $A(z)=\frac{1}{2\sqrt{\pi}}$ and add the solutions associated with $\pm$ signs we have

$$P_{n}(z) = \frac{1}{2\sqrt{\pi}}\,i^{n}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\Big\{1 + 2\left(\tfrac{z}{i}\right)\Gamma^{2}(q+1)/\Gamma^{2}\left(q+\tfrac{1}{2}\right) + \cdots \tag{155}$$

$$+ (-)^{n}\left(1 - 2\left(\tfrac{z}{i}\right)\Gamma^{2}(q+1)/\Gamma^{2}\left(q+\tfrac{1}{2}\right) + \cdots\right)\Big\}.$$

(d) FOURIER COEFFICIENTS OF MATHIEU FUNCTIONS

One form for Mathieu's differential equation is

$$\frac{d^2y}{dx^2} + 4(\alpha - 4q\cos 2x)\,y = 0 \tag{156}$$

The substitution

$$y = e^{isx} \sum_{r=-\infty}^{\infty} b_r e^{i2rx} \tag{157}$$

gives a three term recurrence relation for the coefficients b_r

$$b_{r+1} + b_{r-1} = \frac{[\alpha - (r+\frac{s}{2})^2]}{2q}\, b_r \tag{158}$$

The substitutions

$$r + s/2 = n \qquad b_r = f_n \tag{159}$$

finally gives us

$$f_{n+1} + f_{n-1} = \frac{[\alpha - n^2]}{2q}\, f_n \tag{160}$$

We shall seek a series in rising powers of q, q occurring in the equation much as z does in, for example, the Bessel equation. The summations do not succumb easily. We give the first two terms. The zeroth equation is,

$$f^0_{n+1} + \frac{(n+\sqrt{\alpha})(n-\sqrt{\alpha})}{2q}\, f^0_n = 0 \tag{161}$$

whence

$$f^0_n = A(\alpha, q)\,\Gamma(n+\sqrt{\alpha})\,\Gamma(n-\sqrt{\alpha})\,/\,(-2q)^n \tag{162}$$

If the zeroth term is removed by writing

$$f_n = f_n^o \, F_n \tag{163}$$

we find

$$F_n = 1 - 4q^2 \sum_{n_0}^{n} \frac{F_{n-1}}{(n^2-\alpha)([n-1]^2-\alpha)} \Delta n \tag{164}$$

The next term is given by writing $F_n = 1$ in the right hand side.

$$F_n = 1 - 4q^2 \sum_{n_0}^{n} \frac{1}{(n^2-\alpha)([n-1]^2-\alpha)} \Delta n + \cdots \tag{165}$$

This sum can be obtained by decomposing the summand into linear partial fractions as follows :-

$$\frac{1}{(n+\sqrt{\alpha})(n-\sqrt{\alpha})} = \frac{1}{2\sqrt{\alpha}} \left(\frac{1}{(n-\sqrt{\alpha})} - \frac{1}{(n+\sqrt{\alpha})} \right) \tag{166}$$

and

$$\frac{1}{(n^2-\alpha)([n-1]^2-\alpha)} = \frac{1}{4\alpha} \left[\frac{1}{(n+\sqrt{\alpha})(n-1+\sqrt{\alpha})} + \frac{1}{(n-\sqrt{\alpha})(n-1-\sqrt{\alpha})} \right.$$

$$\left. + \frac{1}{(1+2\sqrt{\alpha})} \left\{ \frac{1}{(n-1-\sqrt{\alpha})} - \frac{1}{(n+\sqrt{\alpha})} \right\} + \frac{1}{(1-2\sqrt{\alpha})} \left\{ \frac{1}{(n-1+\sqrt{\alpha})} - \frac{1}{(n-\sqrt{\alpha})} \right\} \right] \tag{167}$$

The first two terms are straight forward factorial sums. The indefinite sum of $(n)^{-1}$ is called the Psi function[3], i.e.

$$\sum_{n_0}^{n} \frac{1}{(n+c)} \Delta n = \Big[\psi(n+c-1, \omega=1) \Big]_{n_0}^{n} \tag{168}$$

Finally we have

$$F_n = 1 + \frac{q^2}{\alpha}\left[\frac{2(n-1)}{([n-1]^2-\alpha)} - \frac{1}{(1+2\sqrt{\alpha})}\left\{\psi(n-2-\sqrt{\alpha}) - \psi(n-1+\sqrt{\alpha})\right\}\right.$$

$$\left. - \frac{1}{(1-2\sqrt{\alpha})}\left\{\psi(n-2+\sqrt{\alpha}) - \psi(n-1-\sqrt{\alpha})\right\}\right]_{n_0}^{n} + \cdot \tag{169}$$

The simplest result appears to come from taking $n_0 = \infty$. For this value of n_0 the term in brackets is zero.

We have found one solution in rising powers of q clearly another such solution results from the form

$$f_n + \frac{2q}{(n+\sqrt{\alpha})(n-\sqrt{\alpha})} f_{n-1} = -\frac{2q}{(n+\sqrt{\alpha})(n-\sqrt{\alpha})} f_{n+1}. \tag{170}$$

The summation equation follows directly from substituting

$$f_n = \left\{A(\alpha, q)\,(-2q)^n / \Gamma(n+1+\sqrt{\alpha})\Gamma(n+1-\sqrt{\alpha})\right\} F_n \tag{171}$$

and we find

$$F_n = 1 + 4q^2 \mathop{S}_{n_0}^{n} \frac{F_{n+2}\quad \Delta n}{([n+2]^2-\alpha)([n+1]^2-\alpha)} \tag{172}$$

On substituting $F_n = 1$ in the right hand side we find the sums are very similar to those for the first form.

$$F_n = 1 - \frac{q^2}{\alpha}\left[\frac{2(n+1)}{([n+1]^2-\alpha)} - \frac{1}{(1+2\sqrt{\alpha})}\left\{\psi(n-\sqrt{\alpha}) - \psi(n+1+\sqrt{\alpha})\right\}\right.$$

$$\left. - \frac{1}{(1-2\sqrt{\alpha})}\left\{\psi(n+\sqrt{\alpha}) - \psi(n+1-\sqrt{\alpha})\right\}\right]_{n_0}^{n} + \cdots \tag{173}$$

Again $n_0 = \infty$ appears to be most appropriate since the function inside the brackets vanishes at this limit.

(2) SERIES IN INVERSE POWERS OF z

In the foregoing examples the series in rising powers of z resulted from obtaining a summation equation with a positive power of z on the right hand side. Similarly the inverse power series are obtained from a summation equation with a negative power of z on the right hand side. These solutions are called Stokes' type expansions[21] and are frequently asymptotic in character. For a polynomial however the series can terminate.

(a) BESSEL FUNCTIONS

The desired form is

$$f_{n+1}(z) + f_{n-1}(z) = 2\frac{n}{z} f_n(z) \tag{174}$$

and the solution to the zeroth order equation is

$$f_n^o(z) = A(z)\, e^{\pm i\frac{n\pi}{2}} \tag{175}$$

Hence if we make the substitutions

$$\begin{aligned} f_n(z) &= f_n^o(z) F_n(z), \\ n &= 2q, \\ x &= iz, \\ F_n(z) &= F'_q(z), \end{aligned} \tag{176}$$

we obtain the summation equation

$$F'_q(z) = 1 \pm \frac{4}{x} \mathop{S}_{q_0}^{q} (q+\tfrac{1}{2}) F'_{q+\frac{1}{2}} \Delta q \tag{177}$$

We begin by substituting

$$F'_q(x) = 1 \tag{178}$$

in the right hand side.

$$F'_q = 1 \pm \frac{4}{x} \mathop{S}_{q_0}^{q} (q+\tfrac{1}{2})\Delta q + \cdot \cdot \tag{179}$$

At this stage we find agreement with the Stokes' expansion for Bessel functions by choosing $q_0 = \frac{1}{4}$

$$F'_q = 1 \pm \frac{4}{x}\left[(q+\tfrac{1}{2})(q-\tfrac{1}{2})\right]_{\frac{1}{4}}^{q} + \cdot \cdot = 1 \pm \frac{2}{x}(q+\tfrac{1}{4})(q-\tfrac{1}{4}) + \cdot \cdot \tag{180}$$

The next term is

$$F'_q = 1 \pm \frac{2}{x}(q+\tfrac{1}{4})(q-\tfrac{1}{4}) + \frac{4}{x^2}\mathop{S}_{\frac{1}{4}}^{q}(q+\tfrac{3}{4})(q+\tfrac{1}{2})(q+\tfrac{1}{4})\Delta q + \cdot \cdot$$
$$= 1 \pm \frac{2}{x}(q+\tfrac{1}{4})(q-\tfrac{1}{4}) + \frac{1}{x^2}(q+\tfrac{1}{4})(q-\tfrac{1}{4})(q+\tfrac{3}{4})(q-\tfrac{3}{4}) + \cdot \cdot \tag{181}$$

or

$$f_n(z) = A(z)e^{\pm\frac{n\pi}{2}}\left\{1 \pm \frac{(4n^2-1)}{8iz} + \frac{(4n^2-1)(4n^2-9)}{128\,(iz)^2} + \cdots\right\} \tag{182}$$

The Bessel function $J_n(z)$ is in fact a linear combination of the above solutions

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{\pi}{4})} e^{i\frac{n\pi}{2}}\left\{1 + \frac{(4n^2-1)}{8iz} + \cdots\right\}$$
$$+ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{\pi}{4})} e^{-i\frac{n\pi}{2}}\left\{1 - \frac{(4n^2-1)}{8iz} + \cdots\right\} \tag{183}$$

The same basic method can be used, of course, for modified Bessel functions.

(b) <u>HERMITE FUNCTIONS</u>

Consider the form

$$f_{n+1}(z) - 2z f_n(z) = -2n f_{n-1}(z) \tag{184}$$

Iteration of the above equation, of course, gives a quite general solution until we specify the summation constants. This we shall do to obtain identification with Hermite polynomials. Omitting the preliminary steps we substitute

$$f_n(z) = A(z)\,(2z)^n\,F_n(z) \tag{185}$$

which yields the summation expression

$$F_n(z) = 1 - \frac{1}{2z^2} \mathop{S}_{n_0}^{n} n\,F_{n-1}(z)\,\Delta n \tag{186}$$

The first term is therefore

$$F_n(z) = 1 - \frac{1}{2z^2} \mathop{S}_{n_0}^{n} [n]_1\,\Delta n + \cdots = 1 - \frac{1}{2z^2}\left[\frac{[n]_2}{2}\right]_{n_0}^{n} + \cdots \tag{187}$$

The polynomial solution corresponds to $n_0 = 0$ and n a positive integer. The summations are particularly simple and we find

$$F_n(z) = 1 - \frac{1}{2 \cdot 2z^2}[n]_2 + \frac{1}{4 \cdot 2 \cdot 4z^4}[n]_4 + \cdots \tag{188}$$

or

$$f_n(z) = A(z)\,(2z)^n \left\{ 1 - \frac{n(n-1)}{\underline{|1}\;(4z^2)} + \frac{n(n-1)(n-2)(n-3)}{\underline{|2}\;(4z^2)^2} + \cdots \right\} \tag{189}$$

The choice $A(z)=1$ completes the identification with the standard definition of Hermite polynomials. The form

$$f_n(z) - \frac{n}{z} f_{n-1}(z) = \frac{1}{2z} f_{n+1}(z) \qquad (190)$$

if processed in the same way also gives a series in inverse powers of z corresponding to the second solution of Hermite's differential equation.

(c) FOURIER COEFFICIENTS OF MATHIEU FUNCTIONS

The form which gives a summation equation with an inverse power of q on the right hand side is

$$f_{n+1} + f_{n-1} = \frac{(\alpha - n^2)}{2q} f_n \qquad (191)$$

We may remove the zeroth approximation by writing

$$f_n = A(\alpha, q)\, e^{\pm \frac{in\pi}{2}} F_n \qquad (192)$$

and by substituting

$$\begin{aligned} n &= 2r \\ F_n &= F'_r \end{aligned} \qquad (193)$$

we find

$$F'_r = 1 \pm \frac{2i}{q} \mathop{S}_{r_0}^{r} \left[(r+\tfrac{1}{2})^2 - \frac{\alpha}{4} \right] F'_{r+\frac{1}{2}} \Delta r \qquad (194)$$

We now find the next two terms by taking $r_0 = 0$ on the basis that it gives a simple looking series, i.e.

$$F'_r = 1 \pm \frac{2i}{q} \mathop{S}_{r_0}^{r} \left[(r+\tfrac{1}{2})^2 - \frac{\alpha}{4} \right] \Delta r + \cdots \qquad (195)$$

$$= 1 \pm \frac{i}{6q} \left[4r^3 - (3\alpha+1) r \right] + \cdots$$

The next term is

$$F'_r = 1 \pm \frac{ir}{6q}\left[4r^2 - (3\alpha+1)\right] - \frac{1}{3q^2}\sum_0^r \left[4\left(r+\tfrac{1}{2}\right)^5 - (4\alpha+1)\left(r+\tfrac{1}{2}\right)^3 + \frac{\alpha(3\alpha+1)}{4}\left(r+\tfrac{1}{2}\right)\right]\Delta r + \cdots$$

$$= 1 \pm \frac{ir}{6q}\left[4r^2 - (3\alpha+1)\right] - \frac{r^2}{72q^2}\left[16r^4 - 6(12\alpha+13)r^2 + (9\alpha^2+15\alpha+19)\right] + \cdots \tag{196}$$

or

$$f_n = A(\alpha q)\, e^{\pm\frac{n\pi}{2}}\Big\{1 \pm \frac{in}{12q}\left[n^2 - (3\alpha+1)\right] - \frac{n^2}{288q^2}\left[n^4 - \frac{3}{2}(12\alpha+13)n^2 + (9\alpha^2+15\alpha+19)\right] + \cdots \tag{197}$$

(d) FROBENIUS SERIES FOR POISEUILLE FUNCTIONS

We shall first consider the form

$$f_{n+1} - \omega^2 f_n = \omega^2 n^2 f_{n-1} \tag{198}$$

and seek a series in inverse powers of ω. As usual we remove the zeroth iberate and write

$$f_n = A(\omega)\, \omega^{2n} F_n \tag{199}$$

to give

$$F_n = 1 + \frac{1}{\omega^2}\sum_{n_0}^{n} n^2 F_{n-1}\,\Delta n \tag{200}$$

The next term is therefore

$$F_n = 1 + \frac{1}{\omega^2}\sum_{n_0}^{n}\left\{[n]_2 + [n]_1\right\}\Delta_n + \cdots = 1 + \frac{1}{\omega^2}\left[\frac{[n]_3}{3} + \frac{[n]_2}{2}\right]_{n_0}^{n} + \qquad (201)$$

The situation is the same encountered with the Mathieu equation as regards choice of n_0. We shall consider the series resulting from the choice $n_0 = 0$.

$$F_n = 1 + \frac{1}{\omega^2}\left[\frac{[n]_3}{3} + \frac{[n]_2}{2}\right] + \cdots \qquad (202)$$

The next term is

$$F_n = 1 + \frac{1}{\omega^2}\left[\frac{[n]_3}{3} + \frac{[n]_2}{2}\right] + \frac{1}{\omega^4}\left[\frac{[n]_6}{18} + 11\frac{[n]_5}{30} + 3\frac{[n]_4}{8}\right] + \cdots \qquad (203)$$

Hence

$$f_n = A(\omega)\,\omega^{2n}\left\{1 + \frac{1}{\omega^2}\left[\frac{[n]_3}{3} + \frac{[n]_2}{2}\right] + \frac{1}{\omega^4}\left[\frac{[n]_6}{18} + 11\frac{[n]_5}{30} + \frac{3[n]_4}{8}\right] + \cdots\right\} \qquad (204)$$

The other form which gives an inverse power series in ω is

$$f_n + n^2 f_{n-1} = \frac{1}{\omega^2} f_{n+1} \qquad (205)$$

The summation equation results from substituting

$$f_n = A(\omega)(-)^n \Gamma^2(n+1) F_n \tag{206}$$

which gives

$$F_n = 1 - \frac{1}{\omega^2} \overset{n}{\underset{n_0}{S}} (n+2)^2 F_{n+2} \Delta n \tag{207}$$

The first term is

$$F_n = 1 - \frac{1}{\omega^2} \overset{n}{\underset{n_0}{S}} \left\{ [n+2]_2 + [n+1]_1 + [n]_0 \right\} \Delta n + \cdots \tag{208}$$

We again consider the case $n_0 = 0$ whence

$$F_n = 1 - \frac{1}{\omega^2} \left\{ \frac{[n+2]_3}{3} + \frac{[n+1]_2}{2} + [n]_1 \right\} + \cdots \tag{209}$$

and the next term is

$$F_n = 1 - \frac{1}{\omega^2} \left\{ \frac{[n+2]_3}{3} + \frac{[n+1]_2}{2} + [n]_1 \right\}$$

$$+ \frac{1}{\omega^4} \left\{ \frac{[n+5]_6}{18} - \frac{[n+4]_5}{30} + \frac{5}{24}[n+3]_4 + \frac{[n+2]_3}{2} + 2[n+1]_2 + 4[n]_1 \right\} + \cdots \tag{210}$$

(e) LEGENDRE FUNCTIONS

We shall again restrict the discussion to the polynomial solutions. Consider the form

$$f_{n+1}(z) - 2z\frac{(n+\frac{1}{2})}{(n+1)} f_n(z) = -\frac{n}{(n+1)} f_{n-1}(z). \tag{211}$$

We proceed straight to the summation equation by substituting

$$f_n(z) = A(z)\,(2z)^n\,\Gamma(n+\tfrac{1}{2})\,/\,\Gamma(n+1)\quad F_n(z) \tag{212}$$

to find

$$F_n(z) = 1 - \frac{1}{(2z)^2} \sum_{n_0}^{n} \frac{n^2}{(n^2-\frac{1}{4})}\, F_{n-1}\,\Delta n \tag{213}$$

The first term is

$$F_n(z) = 1 - \frac{1}{(2z)^2} \sum_{n_0}^{n} \frac{n^2}{(n^2-\frac{1}{4})}\,\Delta n \quad + \cdots \tag{214}$$

and taking $n_0=0$ we find

$$F_n(z) = 1 - \frac{1}{(2z)^2}\,\frac{n(n-1)}{(n-\frac{1}{2})} \quad + \cdots \tag{215}$$

The next term is

$$F_n(z) = 1 - \frac{1}{(2z)^2}\,\frac{n(n-1)}{(n-\frac{1}{2})} + \frac{1}{(2z)^4} \sum_{0}^{n} \frac{n^2(n-1)(n-2)}{(n^2-\frac{1}{4})(n-\frac{3}{2})}\,\Delta n \ + \cdot \tag{216}$$

The summation is a little awkward but may be performed by breaking down the summand into factorials and inverse factorials .

$$F_n(z) = 1 - \frac{1}{(2z)^2}\,\frac{n(n-1)}{(n-\frac{1}{2})} + \frac{1}{(2z)^4}\,\frac{n(n-1)(n-2)(n-3)}{(n-\frac{1}{2})(n-\frac{3}{2})} + \cdots \tag{217}$$

By choosing

$$A(z) = \pi^{1/2} \tag{218}$$

we achieve identification with the polynomials $P_n(z)$. The alternative form

$$f_n(z) + \frac{n}{2z(n+\frac{1}{2})} f_{n-1}(z) = \frac{(n+1)}{2z(n+\frac{1}{2})} f_{n+1}(z). \tag{219}$$

yields the infinite series $Q_n(z)$ for $n = 0, 1 \cdots \infty$

V. INVERSE POWERS SERIES SOLUTIONS IN THE DIFFERENCE VARIABLE n BY THE FROBENIUS METHOD

Any difference equation can be considered as a differential equation of infinite order by replacing difference operations by Taylor expansions, i.e.

$$f(n+sw) \equiv \exp\left(sw\frac{d}{dn}\right) f(n) \tag{220}$$

If $f(n)$ is regular and defined for complex n, we may adopt this point of view[24]. Let us consider standard form (54), i.e.

$$2\cosh(\omega D)\, f(n) = 2\,\sigma(n)\, f(n) \tag{221}$$

where

$$D = \frac{d}{dn} \tag{222}$$

We seek to write the above standard form as

$$L_1 F(n) = \left[\alpha_0 + \alpha_1 n^{-1} + \alpha_2 n^{-2} + \cdots\right] L_2 F(n) \tag{223}$$

where L_1 and L_2 are difference operators which do not contain the variable n explicitly and

$$f(n) = T(n)\, F(n) \tag{224}$$

The Frobenius[19,20] method is to try the solution

$$u(n) = n^{s}\left[a_0 + a_1 n^{-1} + a_2 n^{-2} + \cdots\right] \tag{225}$$

and to determine the coefficients a_s by equating the coefficients of the separate powers of n to zero. This by definition is an inverse power series called a Stokes' expension. The corresponding rising series cannot be contemplated since the coefficient of each power of n will contain an infinite number of terms. The method is best illustrated by example and we consider the equations for the periodic Mathieu functions.

(a) FOURIER COEFFICIENTS OF MATHIEU FUNCTIONS

The relevant equation is

$$f_{n+1} + f_{n-1} = \frac{(\alpha - n^2)}{2q} f_n \tag{226}$$

In section IV we found that the substitution

$$f_n = A(\alpha q)\,\Gamma(n+\sqrt{\alpha})\,\Gamma(n-\sqrt{\alpha})\,(-2q)^{-n}\,F_n \tag{227}$$

gives

$$F_{n+1} - F_n = -4q^2 (n^2-\alpha)^{-1}([n-1]^2-\alpha)^{-1} F_{n-1} \tag{228}$$

Expanding $(n^2-\alpha)^{-1}$ and $([n-1]^2-\alpha)^{-1}$ we have

$$\begin{aligned} \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 &= 0 \\ \alpha_4 &= -4q^2 \\ \alpha_5 &= -8q^2 \\ \alpha_6 &= -4(2\alpha+3)q^2 \\ \alpha_7 &= -4(6\alpha+4)q^2 . \end{aligned} \tag{229}$$

On substituting

$$F_n = n^\sigma [a_0 + a_1 n^{-1} + \cdots] \tag{230}$$

we first have

$$a_1 n^{\sigma-1} = 0 . \tag{231}$$

which is usually called the indicial equation. We must clearly have $\sigma = 0$. The coefficient a_0 is arbitrary. We find

$$\begin{aligned} a_1 &= a_2 = 0 \\ a_3 &= \tfrac{4}{3} q^2 a_0 \\ a_4 &= 4q^2 a_0 \\ a_5 &= 4q^2\left(\tfrac{29}{15} + \tfrac{\alpha}{5}\right) a_0 \end{aligned} \tag{232}$$

Higher powers of q enter in a_6 etc. The Frobenius method is equivalent to an iterative procedure. We may write the equation for F_n as

$$\frac{d}{dn} F_n = -\frac{4q^2}{n^4}\left\{1 + 2n^{-1} + \cdots\right\} e^{-\frac{d}{dn}} F_n - \left\{e^{\frac{d}{dn}} - 1 - \frac{d}{dn}\right\} F_n \tag{233}$$

and iterate by first putting the right hand side equal to zero. Constants of integration now appear at every stage just as constants of summation were involved in the previous section. In fact what we have done is equivalent to replacing the summation in Section IV (1d) by Euler-Maclaurin[2,3] sum formulae. The zeroth approximation to (233) is

$$F_n \simeq \text{constant} = a_0 \tag{234}$$

Substituting this in the right hand side we have

$$F_n \simeq a_0 - 4q^2 a_0 \int_{n_0}^{n} n^{-4}\, dn \tag{235}$$

We may choose $n_0 = \infty$ to obtain agreement with the Frobenius series. This somewhat justifies the choice $n_0 = \infty$ in IV (1d). Hence one solution of the Mathieu difference equation is

$$f_n = A(\alpha, q)\, \frac{\Gamma(n+\sqrt{\alpha})\,\Gamma(n-\sqrt{\alpha})}{(-2q)^n} \left\{1 + \tfrac{4}{3} q^2 n^{-3} + 4q^2 n^{-4} + \cdots\cdots \right\} a_0 \tag{236}$$

A second solution may be found by substituting

$$f_n = \frac{A(\alpha, q)\,(-2q)^n\, F_n}{\Gamma(n+1+\sqrt{\alpha})\,\Gamma(n+1-\sqrt{\alpha})} \tag{237}$$

This type of expansion is valuable when n is large and q, α are small. It could of course be obtained directly from equation (169) by expanding each term as a power series in n.

VI. W.K.B. METHODS FOR DIFFERENCE EQUATIONS

The well known W.K.B. method for differential[25,26]. equations involves exploiting the fact that when the coefficients of a differential equation are roughly constant, the solution is expected to behave in an exponential manner. With the generalised W.K.B.[27] method the solution of a differential equation can be expressed in terms of the known solution to an arbitrary differential equation. The ordinary W.K.B. method leads to Green-type expansions which are asymptotic while the generalised form gives what are called uniform expansions which have a more general validity.

The problem we consider here is how to develop a similar formalism for difference equations. The principles of the W.K.B. method could be expressed in terms of general operator language but this would not be very helpful when we come to consider a specific type of equation. It seems likely that these principles could be used to investigate other forms of equation such as integral equations or mixed differential-difference equations, e.g.

$$\frac{d}{dz} f_n(z) = 2z\, f_n(z) - f_{n+1}(z) \tag{238}$$

which is the equation the Hermite polynomials satisfy.

For second order difference equations we give a method directly analogous to the ordinary W.K.B. method for differential equations. This method is not practical for the equations for the standard functions because summations are involved which are very difficult. However, one could easily construct equations to which this method would be applicable. The second approach is to treat the difference equation as a differential equation of infinite order. By this method we are able to develop more practical W.K.B. approximations. The application of these results to the standard functions gives exactly the same Green type expansions that can be obtained from the differential equation these functions satisfy. We also give a generalised theory for difference equations.

(1) FINITE DIFFERENCE ANALOGUE TO THE W.K.B. METHOD

Consider the differential equation

$$\frac{d^2}{dx^2} f(x) = \gamma(x) f(x) \tag{239}$$

When $\gamma(x)$ is a constant the two solutions of (239) are

$$f(x) = \exp \pm \sqrt{\gamma}\, x \tag{240}$$

The basis of the W.K.B. method for differential equations is the assumption that when $\gamma(x)$ varies slowly $f(x)$ has the form

$$f(x) = \exp Z(x) \tag{241}$$

where derivdtives of $Z(x)$ higher than the first are expected to be small.

For second order difference equations the most convenient standard forms have been found to be

$$\left[\underset{\omega}{E}^{1} + \underset{\omega}{E}^{-1} \right] f(n) = 2\,\sigma(n) f(n) \qquad \text{a)}$$

$$\left[\underset{\omega}{E}^{1} - \underset{\omega}{E}^{-1} \right] y(n) = 2\,\tau(n)\, y(n) \qquad \text{b)} \tag{242}$$

$$\left[\underset{\omega}{E}^{2} - 2\underset{\omega}{E}^{1} + 1 \right] u(n) = \gamma(n)\, u(n) \qquad \text{c)}$$

When σ, τ, γ are independent of n the solutions of the above equations are also exponentials, i.e.

$$f(n) = \exp \tfrac{1}{\omega} n \ln\left(\sigma \pm \sqrt{\sigma^2 - 1}\right)$$

$$y(n) = \exp \tfrac{1}{\omega} n \ln\left(\tau \pm \sqrt{\tau^2 + 1}\right) \qquad (243$$

$$u(n) = \exp \tfrac{1}{\omega} n \ln\left(1 \pm \sqrt{\gamma}\right)$$

Hence when $\sigma(n)$ etc. are slowly varying we expect the solutions to equations (243) to be of an exponential nature in the same way as for the differential equation.

Consider equation (242 a). If we substitute

$$f(n) = \exp \frac{z(n)}{\omega} \tag{244}$$

the non-linear equation for $z(n)$ is

$$e^{\underset{\omega}{\Delta} z} + e^{-\underset{\omega}{\Delta} z + \omega \underset{\omega}{E}^{-1} \underset{\omega}{\Delta}^{2} z} = 2\sigma \tag{245}$$

The first difference of $z(n)$ is expected to be roughly independent of n when $\sigma(n)$ is slowly varying. Hence neglecting $\underset{\omega}{\Delta}^{2} z$ we find

$$\cosh \underset{\omega}{\Delta} z \simeq \sigma$$

or (246)

$$z \simeq \mathop{S}_{n_0}^{n} \ln.(\sigma \pm \sqrt{\sigma^2 - 1}) \underset{\omega}{\Delta} n$$

There are a number of different ways of obtaining higher approximations. One method is to write the exact solution as

$$f(n) = u(n) \exp. \frac{z(n)}{\omega} \tag{247}$$

when $z(n)$ now denotes the approximation (245). Substitution yields

$$\underset{\omega}{E}^{1} u\, e^{\underset{\omega}{E}^{1} \frac{z}{\omega}} + \underset{\omega}{E}^{-1} u\, e^{\underset{\omega}{E}^{-1} \frac{z}{\omega}} = 2\sigma u\, e^{\frac{z}{\omega}} \tag{248}$$

The above exact equation can be rewritten as

$$\omega^2 \underset{\omega}{E}^{-1} \underset{\omega}{\Delta}^2 u\, e^{-\underset{\omega}{\Delta} z + \omega \underset{\omega}{E}^{1} \underset{\omega}{\Delta}^2 z} + \underset{\omega}{E}^{1} u \left\{ e^{\underset{\omega}{\Delta} z} - e^{-\underset{\omega}{\Delta} z + \omega \underset{\omega}{E}^{1} \underset{\omega}{\Delta}^2 z} \right\}$$

$$= 2u \left\{ \sigma - e^{-\Delta z + \omega \underset{\omega}{E}^{1} \Delta^2 z} \right\} \qquad (249)$$

The function $u(n)$ is expected to vary slowly and hence $\underset{\omega}{\Delta}^2 u$ will be neglected. The expression for the second approximation will then depend on how we treat the term $\exp \omega \underset{\omega}{E}^{1} \underset{\omega}{\Delta}^2 z$. We first note that

$$\underset{\omega}{\Delta} z = \ln (\sigma \pm \sqrt{\sigma^2 - 1})$$

and that (250)

$$\omega \underset{\omega}{E}^{-1} \underset{\omega}{\Delta}^2 z = \ln \left(\frac{\sigma \pm \sqrt{\sigma^2 - 1}}{\underset{\omega}{E}^{1} (\sigma \pm \sqrt{\sigma^2 - 1})} \right)$$

In finding the best form for the second approximation we shall be guided by the principle of simplicity. The approximation equation for $u(n)$ may now be written as

$$\omega \underset{\omega}{\Delta} u \left\{ e^{\underset{\omega}{\Delta} z} - e^{-\underset{\omega}{\Delta} z + \omega \underset{\omega}{E}^{1} \underset{\omega}{\Delta}^2 z} \right\} \simeq u \left\{ 2\sigma - e^{-\underset{\omega}{\Delta} z + \omega \underset{\omega}{E}^{1} \underset{\omega}{\Delta}^2 z} - e^{\underset{\omega}{\Delta} z} \right\} \qquad (251)$$

It is reasonable to ignore $(\underset{\omega}{\Delta}^2 z)^2$ and higher powers, also $(\underset{\omega}{\Delta} u \underset{\omega}{\Delta}^2 z)$ etc, so that we are left with

$$\omega \Delta u \left[e^{\underset{\omega}{\Delta} z} - e^{-\underset{\omega}{\Delta} z} \right] \simeq -\omega u \left[e^{-\underset{\omega}{\Delta} z} \underset{\omega}{E}^{-1} \underset{\omega}{\Delta}^2 z \right] \qquad (252)$$

An important point to notice is that the above equation corresponds to keeping terms containing only a single power of ω. This illustrates how ω may be used as a parameter which marshals together terms of comparable magnitude. Thus we have

$$u \simeq \exp \frac{1}{\omega} \mathop{S}_{n_0}^{n} \ln \left\{ 1 - \omega \frac{e^{-\frac{\Delta}{\omega} z} \, \underset{\omega}{E}^{-1} \Delta^2 z}{\left[e^{\frac{\Delta}{\omega} z} - e^{-\frac{\Delta}{\omega} z} \right]} \right\} \underset{\omega}{\Delta} n$$

$$= \exp \frac{1}{\omega} \mathop{S}_{n_0}^{n} \ln \left\{ 1 \mp \frac{\ln \left(\dfrac{\sigma \pm \sqrt{\sigma^2 - 1}}{\underset{\omega}{E}^{-1} (\sigma \pm \sqrt{\sigma^2 - 1})} \right)}{2 \sqrt{\sigma^2 - 1} \, (\sigma \pm \sqrt{\sigma^2 - 1})} \right\} \underset{\omega}{\Delta} n \tag{253}$$

This may not be the most suitable form depending on whether the summation can be performed in a particular case. We may simplify the above by writing

$$\ln \left(\frac{\sigma \pm \sqrt{\sigma^2 - 1}}{\underset{\omega}{E}^{-1} (\sigma \pm \sqrt{\sigma^2 - 1})} \right) \simeq \frac{(\underset{\omega}{E}^{-1} - 1)(\sigma \mp \sqrt{\sigma^2 - 1})}{\sigma \mp \sqrt{\sigma^2 - 1}} \tag{254}$$

so that the expression for $u(n)$ becomes

$$u(n) \simeq \exp \frac{1}{\omega} \mathop{S}_{n_0}^{n} \ln \left\{ 1 \pm \frac{(1 - \underset{\omega}{E}^{-1})(\sigma \mp \sqrt{\sigma^2 - 1})}{2 \sqrt{\sigma^2 - 1}} \right\} \underset{\omega}{\Delta} n. \tag{255}$$

The summations are frequently very difficult in practice. This makes it hard to ascertain which form for $u(n)$ will be most convenient. If required an iteration procedure could be set up to obtain higher approximations. The other two standard forms may be treated in an identical fashion. The first two approximations to a), b) and c) are

$$f(n) \simeq \exp \pm \frac{1}{\omega} \overset{n}{\underset{n_0}{S}} \left\{ \ln.(\sigma \pm \sqrt{\sigma^2 - 1}) + \ln.\left[1 \mp \frac{\ln.\left(\frac{\sigma \pm \sqrt{\sigma^2 - 1}}{E_\omega^{-1}(\sigma \pm \sqrt{\sigma^2-1})} \right)}{2\sqrt{\sigma^2 - 1}\,(\sigma \pm \sqrt{\sigma^2 - 1})} \right] \right\} \underset{\omega}{\Delta} n \qquad \text{a)}$$

$$y(n) \simeq \exp \pm \frac{1}{\omega} \overset{n}{\underset{n_0}{S}} \left\{ \ln.(\tau \pm \sqrt{\tau^2 + 1}) + \ln.\left[1 \pm \frac{\ln.\left(\frac{\tau \pm \sqrt{\tau^2 + 1}}{E_\omega^{-1}(\tau \pm \sqrt{\tau^2+1})} \right)}{2\sqrt{\tau^2 + 1}\,(\tau \pm \sqrt{\tau^2 + 1})} \right] \right\} \underset{\omega}{\Delta} n \qquad \text{b)} \quad (256)$$

$$U(n) \simeq \exp \pm \frac{1}{\omega} \overset{n}{\underset{n_0}{S}} \left\{ \ln.(1 \pm \sqrt{\gamma}) + \ln\left[1 \mp \frac{(1 \pm \sqrt{\gamma}) \ln\left(\frac{E_\omega'(1 \pm \sqrt{\gamma})}{1 \pm \sqrt{\gamma}} \right)}{2\sqrt{\gamma}} \right] \right\} \underset{\omega}{\Delta} n \qquad \text{c)}$$

We may check that no glaring errors have been made in the foregoing formulae by the following device. If we write

$$\sigma(n) = 1 + \tfrac{1}{2}\omega^2 R(n) \qquad (257)$$

and let ω tend to zero, equation a) tends to the differential equation

$$\frac{d^2 f(n)}{dn^2} = R(n) f(n). \qquad (258)$$

The finite difference approximations give

$$f(n) \simeq R(n)^{-1/4} \exp \pm \int_{n_0}^{n} \sqrt{R(n)}\, dn \qquad (259)$$

in this limit, which is the standard W.K.B. result[25] for a second order differential equation.

If one attempts to use the preceding approximations to find expansions for the well known functions the summations encountered are very difficult. For example the Bessel function $J_n(z)$ has $\sigma(n) = \frac{n}{z}$ but the simple looking summation

$$\overset{n}{S} \ln.\left(\frac{n}{z} \pm \sqrt{\left(\frac{n}{z}\right)^2 - 1} \right) \Delta n \qquad (260)$$

cannot be performed in any simple way. We shall return briefly to these summations at a later stage.

It is possible to obtain slightly different approximation by substituting

$$f(n) = \exp \frac{1}{\omega}\left[z_0 + \omega z_1 + \omega^2 z_2 \cdots\right] \tag{261}$$

in equation (242). By equating the coefficients of each power of ω separately to zero a series of equations is obtained which can be solved in succession.

(2) INFINITE ORDER DIFFERENTIAL EQUATION APPROACH.

In Section V we have used the fact that a difference equation is equivalent to a differential equation of infinite order under certain restrictions. This enables us to treat difference equations in a more practical manner. This is because a differential equation of any order may be considered within the framework of the usual W.K.B. method. We shall consider three methods of approach to this problem and then consider the connection with the direct finite difference analogue.

In the following description attention will be confined to the normal form a). The corresponding results for b) and c) will be quoted since they may be derived by identical arguments.

a) METHOD I.

In quantum mechanics the W.K.B. method is frequently applied to Schroedinger's equation

$$\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0 \tag{262}$$

One method of solution[28] is to substitute

$$\psi(x) = \exp \frac{1}{\hbar^2}\left[S_0 + \hbar S_1 + \hbar^2 S_2 \cdots\right] \tag{263}$$

If the coefficients of each power of $\hbar$ are equated to zero, a series of equations results which may be solved for S_0, S_1 etc. in turn. This suggests that for equation (242 a)

$$2\cosh\left(\omega \frac{d}{dn}\right) f(n) = 2\,\sigma(n)\, f(n) \tag{264}$$

The substitution

$$f(n) = \exp \frac{1}{\omega}\left[Z_0 + \omega Z_1 + \omega^2 Z_2 \cdots\right] \tag{265}$$

might be suitable.

We will anticipate a little and state that it is frequently convenient and necessary from a practical point of view to be able to expand $\sigma(n)$ as

$$\sigma(n) = a(n) + \omega^2 b(n) + \omega^4 c(n) + \cdots \tag{266}$$

with odd powers of ω missing from the expansion. This is a purely formal procedure at this stage. In practice we find that $\sigma(n)$ is often a complicated function and that results are simplified by expanding $\sigma(n)$ as a sum of simpler terms which decrease in magnitude. This will be clarified by examples later. The interval ω is a very useful parameter for deriving results. Let $(Z_r^s)^v$ denote the v'th power of the s derivitive of Z_r. We now have

$$2\cosh\left(\omega \frac{d}{dn}\right) \exp\frac{1}{\omega}\left[Z_0 + \omega Z_1 \cdots\right] = 2\left[a(n) + \omega^2 b(n) \cdots\right] \exp\frac{1}{\omega}\left[Z_0 + \cdots\right] \tag{267}$$

to solve by equating the coefficients of the powers of ω to zero. The equations which determine Z_0, Z_1, Z_2 Z_3, are,

I COEFFICIENTS OF w^0

$$e^{Z_0^1} + e^{-Z_0^1} = 2a$$

II COEFFICIENTS OF w^1

$$e^{Z_0^1}\left[Z_1^1 + \frac{Z_0^2}{2}\right] + e^{-Z_0^1}\left[-Z_1^1 + \frac{Z_0^2}{2}\right] = 0$$

III COEFFICIENTS OF w^2

$$e^{Z_0^1}\left[Z_2^1 + \frac{(Z_1^1)^2}{2} + \frac{(Z_0^2)^2}{8} + \frac{Z_1^1 Z_0^2}{2} + \frac{Z_1^2}{2} + \frac{Z_0^3}{6}\right]$$

$$+ e^{-Z_0^1}\left[-Z_2^1 + \frac{(Z_1^1)^2}{2} + \frac{(Z_0^2)^2}{8} - \frac{Z_1^1 Z_0^2}{2} + \frac{Z_1^2}{2} - \frac{Z_0^3}{6}\right] = 2b \qquad (268)$$

IV COEFFICIENTS OF w^3

$$e^{Z_0^1}\left[Z_3^1 + \frac{(Z_1^1)^3}{6} + Z_1^1 Z_2^1 + \frac{(Z_0^2)^3}{48} + \frac{Z_2^2}{2} + \frac{Z_0^2 Z_1^2}{4} + \frac{Z_1^3}{6} + \frac{Z_0^4}{24}\right.$$

$$\left. + \frac{Z_1^1 (Z_0^2)^2}{8} + \frac{Z_1^1 Z_1^2}{2} + \frac{Z_2^1 Z_0^2}{2} + \frac{(Z_1^1)^2 Z_0^2}{4} + \frac{Z_1^1 Z_0^3}{6} + \frac{Z_0^2 Z_0^3}{12}\right]$$

$$+ e^{-Z_0^1}\left[-Z_3^1 - \frac{(Z_1^1)^3}{6} + Z_1^1 Z_2^1 + \frac{(Z_0^2)^3}{48} + \frac{Z_2^2}{2} + \frac{Z_0^2 Z_1^2}{4} - \frac{Z_1^3}{6} + \frac{Z_0^4}{24}\right.$$

$$\left. - \frac{Z_1^1 (Z_0^2)^2}{8} - \frac{Z_1^1 Z_1^2}{2} - \frac{Z_2^1 Z_0^2}{2} + \frac{(Z_1^1)^2 Z_0^2}{4} + \frac{Z_1^1 Z_0^3}{6} - \frac{Z_0^2 Z_0^3}{12}\right] = 0$$

The algebra involved to find the next term is very complicated.

The solutions to the preceding equations are

$$Z_0 = \int_{n_0}^{n} \ln.(a \pm \sqrt{a^2-1})\, dn$$

$$Z_1 = \ln.(a^2-1)^{-\frac{1}{4}}$$

$$Z_2 = \pm \int_{n_0}^{n} \frac{2a_2(a^2-1)(a^2+2) - a a_1^2(2a^2+13) + 24b(a^2-1)^2}{24\ (a^2-1)^{5/2}}\, dn \tag{269}$$

$$Z_3 = -\int_{n_0}^{n} \frac{a a_3(a^2-1)^2 - a_1 a_2(a^2-1)(7a^2+2) + a a_1^3(8a^2+7) + 4ab_1(a^2-1)^3 - 4a_1 b(a^2-1)^2(a^2+1)}{8\ (a^2-1)^4}\, dn$$

For the case $w=1$

$$f(n) \simeq \frac{e^{\int_{n_0}^{n} \ln.(a \pm \sqrt{a^2-1})dn}}{(a^2-1)^{\frac{1}{4}}} \exp \pm \int_{n_0}^{n} \frac{2a_2(a^2-1)(a^2+2) - a a_1^2(2a^2+13) + 24b(a^2-1)^2}{24\ (a^2-1)^{\frac{5}{2}}}\, dn$$

$$\times \exp - \int_{n_0}^{n} \frac{a a_3(a^2-1)^2 - a_1 a_2(a^2-1)(7a^2+2) + a a_1^3(8a^2+7) + 4ab_1(a^2-1)^3 - 4a_1 b(a^2-1)^2(a^2+1)}{8\ (a^2-1)^4}\, dn \tag{270}$$

The corresponding result for equation b) when $\tau = g + w^2 h + w^4 j \cdots$ is

$$y(n) \simeq \frac{e^{\int_{n_0}^{n} \ln(g \pm \sqrt{g^2+1})dn}}{(g^2+1)^{\frac{1}{4}}} \exp \mp \int_{n_0}^{n} \frac{2g_2(g^2+1)(g^2-2) - g g_1^2(2g^2-13) + 24h(g^2+1)^2}{24\ (g^2+1)^{5/2}}\, dn \tag{271}$$

$$\times \exp \int_{n_0}^{n} \frac{g g_3(g^2+1)^2 - g_1 g_2(g^2+1)(7g^2-2) + g g_1^3(8g^2-7) - 4gh_1(g^2+1)^3 + 4g_1 h(g^2+1)^2(g^2-1)}{8\ (g^2+1)^4}\, dn$$

For the third form c) when

$$\gamma = m + w^2 p + \cdots \tag{272}$$

we find the first three terms are given by

$$U(n) \simeq \frac{e^{\int_{n_0}^{n} \ln(1 \pm \sqrt{m})\, dn}}{m^{\frac{1}{4}}\,(1 \pm \sqrt{m})^{\frac{1}{2}}} \tag{273}$$

$$\times \exp \pm \int_{n_0}^{n} \frac{8\, m_2 m (1 \pm \sqrt{m})(4m \pm 6m^{\frac{1}{2}} + 3) - m_1^2 (\pm 44 m^{3/2} + 82m \pm 63 m^{\frac{1}{2}} + 21) + 96 p m^2 (1 \pm \sqrt{m})}{192\, m^{\frac{5}{2}}\,(1 \pm \sqrt{m})^2}\, dn$$

C.J. Carlton[29] has investigated two alternative standard forms as far as the third approximation. For rapid convergence of the series $Z = Z_0 + \omega Z_1 + \cdot$ we require $a_1\,(a^2-1)^{-\frac{1}{2}}$ to be small and all subsequent derivdtives of a, b, etc. to be small compared with $(a^2-1)^{-\frac{1}{2}}$.

b) <u>METHOD II</u>

We now consider a second approach which is equivalent to expanding the exponentials of the third and higher approximations in the preceding method. Consider form a) again and write as before

$$f(n) = \exp \frac{Z(n)}{\omega} \tag{274}$$

whereupon we find

$$\exp\left[\exp\left(\omega \frac{d}{dn}\right) \frac{Z}{\omega}\right] + \exp\left[\exp\left(-\omega \frac{d}{dn}\right) \frac{Z}{\omega}\right] = 2\left[a + \omega^2 b \cdots\right] \exp \frac{Z}{\omega} \tag{275}$$

The first approximation is to ignore all derivatives of $Z(n)$ higher than the first, also $b(n)$ etc. This is because $Z(n)$ is expected to be roughly proportional to n and $b(n)$ is assumed to be small compared with $a(n)$. Hence

$$2 \cosh \omega Z_1 \simeq 2a \tag{276}$$

or

$$Z \simeq \int_{n_0}^{n} \ln.\,(a \pm \sqrt{a^2-1})\, dn. \tag{277}$$

This is identical with the previous first approximation. We now write

$$f(n) = U(n)\, e^{\frac{1}{\omega}\int^{n} \ln\left(a \pm \sqrt{a^2-1}\right)dn} = U(n)\, e^{\frac{1}{\omega} Z(n)} \tag{278}$$

After substitution into form a) we neglect $U_1 Z_2$ etc, u_2 etc, Z_3 etc, Z_2^3 etc. b is also omitted on the grounds that in practice it is often possible to arrange that b is two orders smaller than a. We obtain

$$\omega u_1\left[e^{Z_1} - e^{-Z_1}\right] + u\left[e^{Z_1} + e^{-Z_1}\right] \simeq 2a\left[1 - \frac{\omega Z_2}{2}\right]$$

or (279)

$$\omega\, 2\sqrt{a^2-1}\, u_1 \simeq \mp\, \omega\, \frac{a Z_2}{2}\, u$$

Hence

$$u \simeq (a^2-1)^{-\frac{1}{4}} \tag{280}$$

This again agrees with the previous result. Further approximations can now be obtained by comparing the exact equation for U with the approximate one. The exact equation for U is

$$e^{-\frac{\omega Z_2}{2}}\left[E_\omega^{1} u\, e^{E_\omega^{1}\frac{Z}{\omega}} + E_\omega^{-1} u\, e^{E_\omega^{-1}\frac{Z}{\omega}} - 2\left(a + \omega^2 b + \cdots\right) u\, e^{\frac{Z}{\omega}}\right] = 0 \tag{281}$$

while the approximate equation is

$$\pm\, \omega\, 2\left(a^2-1\right)^{\frac{1}{2}} e^{\frac{Z}{\omega}} u_1 + \omega\, \frac{a Z_2}{2}\, e^{\frac{Z}{\omega}} u = 0 \tag{282}$$

The exact equation may then be rewritten in a form suitable for iteration as

$$u_1 + \frac{aa_1}{2(a^2-1)}\,u = u_1 + \frac{aa_1}{2(a^2-1)}\,u$$

$$\mp \frac{1}{2\omega(a^2-1)^{\frac{1}{2}}}\left[E_{\omega}' u\, e^{\frac{E'}{\omega}\frac{Z}{\omega} - \frac{Z}{\omega} - \frac{\omega Z_2}{2}} + E_{\omega}' u\, e^{\frac{E'}{\omega}\frac{Z}{\omega} - \frac{Z}{\omega} - \frac{\omega Z_2}{2}} - 2(a + \omega^2 b + \cdot\cdot)\, u\, e^{-\frac{\omega Z_2}{2}} \right] \tag{283}$$

In the preceding method the interval ω was found to be a very convenient expansion parameter. Again here the powers of ω arrange the terms on the right hand side for us in orders of magnitude

$$u_1 + \frac{aa_1}{2(a^2-1)}\,u = \mp\omega\left[\frac{u_2 a}{2(a^2-1)^{\frac{1}{2}}} \pm \frac{uZ_3}{6} - \frac{u\,a\,Z_2^2}{8(a^2-1)^{\frac{1}{2}}} - \frac{ub}{(a^2-1)^{\frac{1}{2}}}\right] \tag{284}$$

$$\mp\omega^2\left[\pm\frac{u_3}{6} + \frac{u_1 a Z_3}{6(a^2-1)^{\frac{1}{2}}} + \frac{u\,a\,Z_4}{24(a^2-1)^{\frac{1}{2}}} + \frac{u\,a\,Z_2^3}{48(a^2-1)^{\frac{1}{2}}} + \frac{ubZ_2}{2(a^2-1)^{\frac{1}{2}}}\right] \mp \cdots$$

Now

$$Z = \int^{n} \ln.\left(a \pm \sqrt{a^2-1}\right) dn \tag{285}$$

and we take out the leading term of $U(n)$ by writing

$$U(n) = (a^2-1)^{-\frac{1}{4}}\, V(n) \tag{286}$$

to obtain an equation for $V(n)$.

$$\frac{V}{n} = \pm\,\omega\left[\frac{-aV_2}{2(a^2-1)^{\frac{1}{2}}} + \frac{a_1 a^2 V_1}{2(a^2-1)^{3/2}} + \frac{V}{24}\left\{\frac{2a_2(a^2-1)(a^2+2) - aa_1^3(2a^2+13) + 24b(a^2-1)^2}{(a^2-1)^{5/2}}\right\}\right]$$

$$\omega^2\left[-\frac{V_3}{6} + \frac{a_1 a V_2}{4(a^2-1)} - \frac{V_1}{24}\left\{\frac{a_1^2(5a^2+6) - 2a_2 a(a^2-1)}{(a^2-1)^2}\right\}\right.$$

$$\left. + \frac{V_2}{48}\left\{\frac{2a_3 a(a^2-1)^2 - 4a_2 a_1(a^2-1)(2a^2+3) + aa_1^3(6a^2+29) - 24ba_1(a^2-1)^2}{(a^2-1)^3}\right\}\right] + \cdots \tag{287}$$

The iteration procedure is then to substitute $V=1$ in the right hand side of the above equation. This form of the solution is rather different to the one obtained before as then the 3rd and 4th approximations were exponentials. Expansion of the exponentials gives an answer equivalent to the above. Putting $V=1$ in the right hand side of (287) we have

$$V(n) \simeq 1 \pm \omega\int_{n_0}^{n}\frac{2a_2(a^2-1)(a^2+2) - aa_1^2(2a^2+13) + 24b(a^2-1)^2}{24(a^2-1)^{\frac{5}{2}}} \tag{288}$$

$$+\,O(\omega^2)\;\ldots\ldots$$

To obtain the next term explicitly is rather awkward because of the double integrals involved.

c) <u>METHOD III</u>

The following presentation is rather more concise. It is equivalent to Method I. We use a slightly different notation. We again consider the standard form

$$2\cosh\left(\omega\frac{d}{dn}\right)f(n) = 2(a+\omega^2 b+\cdots)f(n). \tag{289}$$

If we substitute $f = \exp\frac{1}{\omega}\int^n \rho(n)\,dn$ and ignore all derivatives of ρ and the terms involving b etc. we have, as before

$$e^{\rho} + e^{-\rho} \simeq 2a \tag{290}$$

or

$$\rho = \ln\left(a \pm \sqrt{a^2-1}\right).$$

The problem of obtaining further approximations for ρ may be expressed in terms of a transcendental equation for ρ. This is straight forward when it is realised what terms are comparable in magnitude. In Method I we allowed the interval ω to arrange terms for us. What in fact we are saying is that $\sigma = a + \omega^2 b + \cdot$ is roughly constant and we are assuming that successive derivatives of a, b etc. become smaller and smaller. We can write the exact equation for ρ as

$$\exp \sum_{m=0}^{\infty} \frac{(\omega D)^m}{\lfloor m+1} \rho(n) + \exp - \sum_{m=0}^{\infty} \frac{(-\omega D)^m}{\lfloor m+1} \rho(n) = 2(a + \omega^2 b \cdot\cdot) \qquad (291)$$

where $D = \frac{d}{dn}$. We use the notation

$$\frac{d^m}{dn^m}\rho(n) = \rho_m(n) . \qquad (292)$$

This expression can be rearranged as

$$\cosh\left(\rho + \frac{\omega^2 \rho_2}{\lfloor 3} + \frac{\omega^4 \rho_4}{\lfloor 5}\right) \exp\left(\frac{\omega \rho_1}{\lfloor 2} + \frac{\omega^3 \rho_3}{\lfloor 4} + \cdot\cdot\right) = a + \omega^2 b + \cdot\cdot\cdot \qquad (293)$$

or

$$\rho = \cosh^{-1}\left\{(a + \omega^2 b \cdots) \exp - \left[\left(\frac{\omega \rho_1}{\lfloor 2} + \frac{\omega^3 \rho_3}{\lfloor 4} \cdots\right)\right]\right\} - \frac{\omega^2 \rho_2}{\lfloor 3} - \frac{\omega^4 \rho_4}{\lfloor 5} \qquad (294)$$

We now expand the $\cosh^{-1}$ term about a to obtain

$$\rho = \cosh^{-1} a - \omega\left(\frac{a\rho_1}{2(a^2-1)^{\frac{1}{2}}}\right) - \omega^2\left(\frac{a\rho_1^2}{8(a^2-1)^{3/2}} - \frac{b}{(a^2-1)^{\frac{1}{2}}} + \frac{\rho_2}{6}\right)$$
$$- \omega^3\left(\frac{a\rho_1^3(2a^2+1)}{48(a^2-1)^{5/2}} - \frac{b\rho_1}{2(a^2-1)^{3/2}} + \frac{a\rho_3}{24(a^2-1)^{\frac{1}{2}}}\right) + \omega^4\left(\cdots\right. \qquad (295)$$

When the above transcendental equation is iterated it is seen that the above scheme of ordering corresponds to ordering of the derivatives of a, b and c e.t.c. Including the two possible signs for $\cosh^{-1}$ we find

$$\rho = \pm \cosh^{-1} a - \frac{\omega a a_1}{2(a^2-1)} \pm \omega^2 \left[\frac{2a_2(a^2-1)(a^2+2) - a a_1^2(2a^2+13) + 24b(a^2-1)^2}{24(a^2-1)^{5/2}} \right]$$

$$- \omega^3 \left[\frac{a a_3 (a^2-1)^2 - a_1 a_2 (a^2-1)(7a^2+2) + a a_1^3 (8a^2+7) + 4ab_1(a^2-1)^3 - 4a_1 b (a^2-1)^2 (a^2+1)}{8(a^2-1)^4} \right] + \cdots \tag{296}$$

Equation (295) is exactly equivalent to equation (269).

(3) THE CONNECTION BETWEEN THE INFINITE ORDER DIFFERENTIAL EQUATION AND THE FINITE DIFFERENCE RESULTS

In Section VI(1) approximations were derived in terms of indefinite summations. An approximate method of performing sums in the Euler-Maclaurin formula which is[2,3].

$$\mathop{S}_{a}^{n} f(n) \mathop{\Delta}_{\omega} n = \int_a^n f(n)\,dn + \sum_{i=1}^{\infty} \frac{B_i}{\underline{|i}} \omega^i \left[D^{i-1} f(n) - D^{i-1} f(a) \right] \tag{297}$$

where

$$D^{i-1} = \frac{d^{i-1}}{dn^{i-1}} \tag{298}$$

nd B_i is the i'th Bernoulli number. This method frequently gives n asymptotic expansion for the sum required. For example the amma function is the solution to the first order difference quation

$$\Gamma(n+1) = n\,\Gamma(n) \tag{299}$$

o that

$$\Gamma(n) = \exp \mathop{S}_{a}^{n} \ln(n) . \Delta n \tag{300}$$

The E-M formula applied to this sum gives the Stirling asymptotic expansion for the Gamma function. If we now consider expression (256 a) and expand all differences as differential operators and then collect terms of the same power in ω we have

$$f(n) \simeq \exp \frac{1}{\omega} \mathop{S}_{n_0}^{n} \ln\left(\sigma \pm \sqrt{\sigma^2-1}\right) \underset{\omega}{\Delta} n \pm \mathop{S}_{n_0}^{n} \frac{\frac{d}{dn}\left(\sigma \pm \sqrt{\sigma^2-1}\right)}{2\sqrt{\sigma^2-1}} \underset{\omega}{\Delta} n + \cdots \tag{301}$$

The E-M formula now gives us $\left(B_1 = -\frac{1}{2}\right)$,

$$f(n) \simeq \exp \frac{1}{\omega} \int_{n_0}^{n} \ln\left(\sigma \pm \sqrt{\sigma^2-1}\right) dn - \frac{1}{2}\left[\ln\left(\sigma \pm \sqrt{\sigma^2-1}\right)\right]_{n_0}^{n} \tag{302}$$

$$\pm \int_{n_0}^{n} \frac{\frac{d}{dn}\left(\sigma \pm \sqrt{\sigma^2-1}\right)}{2\sqrt{\sigma^2-1}} dn + O(\omega) + \cdots$$

This is easily reduced to

$$f(n) \simeq \exp \frac{1}{\omega} \int_{n_0}^{n} \ln\left(\sigma \pm \sqrt{\sigma^2-1}\right) dn - \frac{1}{2} \int_{n_0}^{n} \frac{\sigma \sigma_1}{(\sigma^2-1)} dn + \cdots \tag{303}$$

$$\propto (\sigma^2-1)^{-\frac{1}{4}} \exp \frac{1}{\omega} \int_{n_0}^{n} \ln\left(\sigma \pm \sqrt{\sigma^2-1}\right) dn + \cdots$$

which is identical with the result obtained from the infinite order differential equation.

(4) CONNECTION FORMULAE FOR LINKING EXPONENTIAL AND TRIGONOMETRICAL REGIONS AND THE BEHAVIOUR NEAR TURNING POINTS.

We shall consider the specific standard form

$$f_{n+1}(z) + f_{n-1}(z) = 2\,\sigma(n,z)\, f_n(z) \tag{304}$$

for which the first two approximations are

$$f_n(z) = (\sigma^2-1)^{-\frac{1}{4}} \exp \pm \left| \int^{n} \cosh^{-1}\sigma \, dn \right| \tag{305}$$

This approximation is accurate when $\sigma_1(\sigma^2-1)^{-\frac{1}{2}}$ is small but fails when

$$\sigma = \pm 1 \tag{306}$$

since the expression becomes infinite at these 'turning' points. We assume that the function $\sigma(n)$ can be considered roughly linear in n across the turning points. We show how solutions may be linked using this hypothesis. When

$$|\sigma| < 1 \tag{307}$$

it is advantageous to write the two independent solutions to the difference equations as

$$\tilde{f}_n = \begin{matrix}\sin\\ \cos\end{matrix} \left| \int^n \cos^{-1}\sigma \, dn \right| \Big/ (1-\sigma^2)^{\frac{1}{4}} \tag{308}$$

This is because $\cosh^{-1}\sigma$ becomes complex (or pure imaginary for real σ) and the solutions would not be analytically continuous near $|\sigma| = 1$. The turning point at

$$\sigma = -1 \tag{309}$$

need not be considered separately, since by writing

$$f_n = (-)^n f_n' \tag{310}$$

the difference equation for f_n' is

$$f_{n+1}'(z) + f_{n-1}'(z) = -2\,\sigma(n,z)\, f_n'(z) \tag{311}$$

and the turning point is now at $-\sigma(n,z)=-1$. Let $n_o(z)$ be one of the roots of

$$\sigma(n,z)=1 \tag{312}$$

Close to the region where $\sigma(n,z)=1$ it is reasonable to assume that

$$\sigma \simeq 1 + \frac{n-n_o(z)}{F(z)} \tag{313}$$

This should be fairly general but would not be correct for example if $\sigma=1$ corresponded to a maximum or a minimum of the function as well. Clearly

$$\frac{1}{F(z)} = \left[\frac{d\,\sigma(n,z)}{dn}\right]_{n_o(z)} \tag{314}$$

The equation which the Bessel functions $J_p(x)$, $Y_p(x)$ satisfy is

$$D_{p+1}(x) + D_{p-1}(x) = 2\left[1-\left(\frac{p-x}{x}\right)\right]D_p(x) \tag{315}$$

while the approximate equation for $f_n(z)$ is

$$f_{n+1}(z) + f_{n-1}(z) = 2\left[1-\left(\frac{n-n_o(z)}{F(z)}\right)\right]f_n(z) \tag{316}$$

Hence the identification

$$x = F(z) \tag{317}$$

$$p = n - F(z) - n_o(z)$$

gives the behaviour of $f_n(z)$ near $\sigma = 1$ as

$$f^1_n(z) = J^{(F(z))}_{n+F(z)-n_0(z)}$$

and (318)

$$f^2_n(z) = Y^{(F(z))}_{n+F(z)-n_0(z)} .$$

It is known from differential equation analysis how to connect Bessel functions across a turning point. This is summarised by the following tables[27].

$\frac{p}{x} > 1$	$\frac{p}{x} = 1$	$\frac{p}{x} < 1$
$\dfrac{\exp -\left[p\cosh^{-1}\frac{p}{x} - \sqrt{p^2 - x^2}\,\right]}{2\left(\frac{p^2}{x^2} - 1\right)^{\frac{1}{4}}}$	$\sqrt{\frac{\pi x}{2}}\, J_p(x)$	$\dfrac{\cos\left[p\cos^{-1}\frac{p}{x} - \sqrt{x^2+p^2} + \frac{\pi}{4}\right]}{\left(1 - \frac{p^2}{x^2}\right)^{\frac{1}{4}}}$

TABLE I

$\frac{p}{x} > 1$	$\frac{p}{x} = 1$	$\frac{p}{x} < 1$
$\dfrac{\exp\left[p\cosh^{-1}\frac{p}{x} - \sqrt{p^2 - x^2}\,\right]}{\left(\frac{p^2}{x^2} - 1\right)^{\frac{1}{4}}}$	$-\sqrt{\frac{\pi x}{2}}\, Y_p(x)$	$\dfrac{\sin\left[p\cos^{-1}\frac{p}{x} - \sqrt{x^2+p^2} + \frac{\pi}{4}\right]}{\left(1 - \frac{p^2}{x^2}\right)^{\frac{1}{4}}}$

TABLE II

Away from the turning point the W.K.B. approximations for $J^{(F)}_{n+F-n_0}$ and $Y^{(F)}_{n+F-n_0}$ must tend to the W.K.B. approximations for $f_n(z)$ so that the general linkages are given by the following tables

$\sigma > 1 \quad n > n_0$	$\sigma = 1 \quad n = n_0$	$\sigma < 1 \quad n < n_0$
$\dfrac{\exp -\left[\int_{n_0}^{n}\cosh^{-1}\sigma\, dn\right]}{2\left(\sigma^2 - 1\right)^{\frac{1}{4}}}$	$\sqrt{\frac{\pi F(z)}{2}}\, J_{n+F(z)-n_0(z)}(F(z))$	$\dfrac{\cos\left[\int_{n_0}^{n}\cos^{-1}\sigma\, dn + \frac{\pi}{4}\right]}{\left(1 - \sigma^2\right)^{\frac{1}{4}}}$

TABLE III

$\sigma > 1 \quad n > n_0$	$\sigma = 1 \quad n = n_0$	$\sigma < 1 \quad n < n_0$
$\dfrac{\exp\left[\int_{n_0}^{n} \cosh^{-1}\sigma\, dn\right]}{(\sigma^2 - 1)^{\frac{1}{4}}}$	$-\sqrt{\dfrac{\pi F(z)}{2}}\; Y_{n+F(z)-n_0(z)}(F(z))$	$\dfrac{\sin\left[\int_{n_0}^{n} \cos^{-1}\sigma\, dn + \frac{\pi}{4}\right]}{(1-\sigma^2)^{\frac{1}{4}}}$

TABLE IV

The above formulae apply to the case when $\sigma(nz)$ is greater than one for n greater than $n_0(z)$. The alternative situation is when $\sigma(nz)$ is greater than one for n less than $n_0(z)$. Both types of behaviour are included in the following formulae

$$\frac{1}{2(\sigma^2-1)^{\frac{1}{4}}} \exp\left[-\left|\int_{n_0}^{n} \cosh^{-1}\sigma\, dn\right|\right] \rightleftharpoons \frac{1}{(1-\sigma^2)^{\frac{1}{4}}} \sin\left[\left|\int_{n_0}^{n} \cos^{-1}\sigma\, dn\right| + \frac{\pi}{4}\right]$$

$$\frac{1}{(\sigma^2-1)^{\frac{1}{4}}} \exp\left[+\left|\int_{n_0}^{n} \cosh^{-1}\sigma\, dn\right|\right] \rightleftharpoons \frac{1}{(1-\sigma^2)^{\frac{1}{4}}} \cos\left[\left|\int_{n_0}^{n} \cos^{-1}\sigma\, dn\right| + \frac{\pi}{4}\right] \tag{319}$$

The corresponding results for the differential equation

$$\frac{d^2\psi}{dq^2} = f(q)\,\psi(q) \tag{320}$$

are [27]

$$\frac{1}{2f^{\frac{1}{4}}} \exp\left[-\left|\int_{q_0}^{q} \sqrt{f}\, dq\right|\right] \rightleftharpoons \frac{1}{(-f)^{\frac{1}{4}}} \sin\left[\left|\int_{q_0}^{q} \sqrt{-f}\, dq\right| + \frac{\pi}{4}\right]$$

$$\frac{1}{f^{\frac{1}{4}}} \exp\left[+\left|\int_{q_0}^{q} \sqrt{f}\, dq\right|\right] \rightleftharpoons \frac{1}{(-f)^{\frac{1}{4}}} \cos\left[\left|\int_{q_0}^{q} \sqrt{-f}\, dq\right| + \frac{\pi}{4}\right] \tag{321}$$

where q_0 is a root of the equation

$$f(q_0) = 0 \tag{322}$$

(5) THE GENERALISED W.K.B. METHOD FOR DIFFERENCE EQUATIONS

It was shown some years ago[27] how to express the solution of a second order differential equation

$$\frac{d^2 F(y)}{dy^2} = \Gamma(y) F(y) \tag{323}$$

in terms of the known solution, to a second equation

$$\frac{d^2 f(x)}{d^2 x^2} = \gamma(x) f(x) \tag{324}$$

when $\Gamma(y)$ has a similar analytical behaviour to $\gamma(x)$. It has been found possible to develop a similar method for difference equations. The two equations which are to be compared we write in the form

$$\frac{1}{2}\left\{f(x+\omega)+f(x-\omega)\right\} = \cosh\left(\omega \frac{d}{dx}\right) f(x) = \gamma(x) f(x)$$

$$\frac{1}{2}\left\{F(y+\omega)+F(y-\omega)\right\} = \cosh\left(\omega \frac{d}{dy}\right) F(y) = \Gamma(y) F(y) \tag{325}$$

The solution $f(x)$ we assume is known. The argument now proceeds in exactly the same way as for a differential equation of finite order. We assume that for some range of y, $\Gamma(y)$ is functionally similar to $\gamma(x)$. We substitute

$$F(y) = f(x(y)) \tag{326}$$

expecting $x(y)$ to be roughly proportional to y. We then obtain

$$\cosh\left(\omega \frac{d}{dy}\right) F(y) = \Gamma(y) f(x) = \cosh\left(\omega \frac{d}{dy}\right) f(x). \tag{327}$$

Let

$$\alpha \equiv \cosh\left(w\frac{d}{dx}\right) \quad , \quad \beta \equiv \cosh\left(w\frac{d}{dy}\right) \tag{328}$$

whence

$$w\frac{d}{dx} \equiv \pm \cosh^{-1}\alpha \quad , \quad w\frac{d}{dy} \equiv \pm \cosh^{-1}\beta . \tag{329}$$

The problem is to express β in terms of $\frac{d}{dx}$ then $\frac{d}{dx}$ in terms of α and finally to obtain the result of a function of α operating on $f(x)$. We first note that

$$\cosh\left(w\frac{d}{dy}\right) \equiv \cosh\left(\frac{dx}{dy} w\frac{d}{dx}\right) + \frac{w}{2}\frac{d^2x}{dy^2}\sum_{p=1}^{\infty}\frac{w^{2p-1}}{\lfloor 2p-2}\left(\frac{dx}{dy}\right)^{2p-2}\frac{d^{2p-1}}{dx^{2p-1}} + w^2\left[\ldots\right. \tag{330}$$

where

$$\cosh\left(\frac{dx}{dy} w\frac{d}{dx}\right) \equiv 1 + \frac{w^2}{\lfloor 2}\left(\frac{dx}{dy}\right)^2\frac{d^2}{dx^2} + \frac{w^4}{\lfloor 4}\left(\frac{dx}{dy}\right)^4\frac{d^4}{dx^4} + \cdots \tag{331}$$

To a first approximation we may neglect second and higher derivatives of x. We are left with the expression

$$\cosh\left(w\frac{dx}{dy}\frac{d}{dx}\right)f(x) \equiv \cosh\left(\pm\frac{dx}{dy}\cosh^{-1}\alpha\right) f(x) \tag{332}$$

We may expand the above operator as a power series in α so that we first require α^s acting on $f(x)$. We find

$$\alpha^s f(x) = \gamma^s(x) f(x) + w\, O\left(\frac{d\gamma}{dx}\right) f(x) + \cdots \tag{333}$$

To make further progress we assume that $\frac{d\gamma}{dx}$ and higher derivatives are small compared with γ. We then have

$$\Gamma(y) f(x) \simeq \cosh\left(\pm\frac{dx}{dy}\cosh^{-1}\gamma(x)\right) f(x) \tag{334}$$

or

$$\int_{y_0}^{y} \cosh^{-1} \Gamma(y)\, dy \simeq \int_{x_0}^{x} \cosh^{-1} \gamma(x)\, dx \cdot \tag{335}$$

where y_0 and x_0 are equivalent points. To obtain the next approximation we write

$$F(y) = Z(y) f(x) \tag{336}$$

where x now denotes $x(y)$ determined from the first order approximation. The exact equation for Z is

$$\Gamma(y) Z(y) f(x) \doteq \cosh\left(\omega \frac{d}{dy}\right) Z(y) f(x) \tag{337}$$

We now assume that Z is slowly varying with y. Now

$$\begin{aligned} &\cosh\left(\omega \frac{d}{dy}\right) Z(y) f(x) \\ &= Z(y) \cosh\left(\omega \frac{d}{dy}\right) f(x) + \omega \frac{dZ}{dy}(y) \sinh\left(\omega \frac{d}{dy}\right) f(x) + \omega^2(\cdots \end{aligned} \tag{338}$$

The next approximation is to ignore derivatives of Z higher than the second, derivatives of Z higher than the first and derivatives of γ higher than the first. If we use the results

$$\alpha^s f(x) \doteq \gamma^s(x) f(x) + \frac{\omega}{2} \frac{d\gamma}{dx} \sinh\left(\pm \cosh^{-1}\gamma\right) s(s-1) \gamma^{s-2}(x) f(x) + \omega^2(\cdots$$

$$Q(\alpha) f(x) = Q(\gamma) f(x) + \frac{\omega}{2} \frac{d\gamma}{dx} \sinh\left(\pm \cosh^{-1}\gamma\right) \frac{d^2}{d\gamma^2} Q(\gamma)\; f(x) + \omega^2(\cdots \tag{339}$$

where $Q(\alpha)$ is an arbitrary function of α and

$$\frac{\omega}{2} \frac{d^2x}{dy^2} \sum_{p=1}^{\infty} \frac{\omega^{2p-1}}{\lfloor 2p-2} \left(\frac{dx}{dy}\right)^{2p-2} \frac{d^{2p-1}}{dx^{2p-1}} f(x) = \pm \frac{\omega}{2} \frac{d^2x}{dy^2} \cosh^{-1}\gamma \cosh\left(\pm \frac{dx}{dy} \cosh^{-1}\gamma\right) f(x) + \omega^2(\cdots \tag{340}$$

we find

$$\Gamma(y)\, z(y) \simeq \cosh\left(\pm \frac{dx}{dy} \cosh^{-1} \gamma(x)\right) z(y)\, f(x)$$

$$+ \omega z(y) \left\{ \frac{1}{2} \frac{d}{dy}\left[\sinh\left(\pm \frac{dx}{dy} \cosh^{-1} \gamma(x)\right)\right] - \frac{1}{2} \frac{d\gamma}{dx} \frac{dx}{dy} \frac{\gamma(x)}{(\gamma^2(x)-1)} \sinh\left(\pm \frac{dx}{dy} \cosh^{-1} \gamma(x)\right)\right\} f(x \tag{341}$$

$$+ \omega \frac{dz(y)}{dy} \sinh\left(\pm \frac{dx}{dy} \cosh^{-1} \gamma(x)\right) f(x)$$

The approximation we have made corresponds to retaining only terms containing a single power of ω. Since the first term on the right hand side cancels out the left hand side we find

$$\frac{dz}{z} \simeq -\frac{1}{2} \frac{\frac{d}{dy}\left[\sinh\left(\frac{dx}{dy} \cosh^{-1} \gamma\right)\right]}{\sinh\left(\frac{dx}{dy} \cosh^{-1} \gamma\right)} + \frac{1}{2} \frac{d\gamma}{dy} \frac{\gamma}{(\gamma^2-1)} dy \tag{342}$$

or

$$z \simeq \left[\sinh\left(\frac{dx}{dy} \cosh^{-1} \gamma\right)\right]^{-\frac{1}{2}} |\gamma^2-1|^{\frac{1}{4}} \tag{343}$$

The algebra to obtain higher terms is rather formidable. To summarise we have found that

$$F(y) \simeq \left[\sinh\left(\frac{dx}{dy} \cosh^{-1} \gamma\right)\right]^{-\frac{1}{2}} |\gamma^2-1|^{\frac{1}{4}} f(x) \tag{344}$$

where $x(y)$ is determined from

$$\int_{y_0}^{y} \cosh^{-1} \Gamma(y)\, dy = \int_{x_0}^{x} \cosh^{-1} \gamma(x)\, dx \tag{345}$$

(6) A GENERALISED EULER-MACLAURIN SUMMATION FORMULA

In V (3) we found that the E-M formula enabled us to express the finite difference W.K.B. formula in terms of the infinite differential equation results. The reason for this is that the E-M formula is itself a W.K.B. approximation to a first order difference equation. This may easily be shown using the methods of this section. It is interesting, therefore, to derive a generalised E-M formula using the theory of the generalised W.K.B. method for difference equations. Consider the first order equation

$$f(x+\omega) = \exp\left(\omega \frac{d}{dx}\right) f(x) = \gamma(x)\, f(x) \tag{346}$$

The solution is

$$f(x) = \exp \frac{1}{\omega} \mathop{\mathrm{S}}_{x_0}^{x} \ln \gamma(x) \mathop{\Delta}_{\omega} x \tag{347}$$

Let us assume that $f(x)$ is a thoroughly investigated function. Consider a second equation

$$F(y+\omega) = \exp\left(\omega \frac{d}{dy}\right) F(y) = \Gamma(y)\, F(y) \tag{348}$$

with solution

$$F(y) = \exp \frac{1}{\omega} \mathop{\mathrm{S}}_{y_0}^{y} \ln \Gamma(y) \mathop{\Delta}_{\omega} y \tag{349}$$

Let the summation be one which is neither simple nor tabulated and let us therefore try to express $F(y)$ in terms of $f(x)$ in the case when $\gamma(x)$ and $\Gamma(y)$ have a similar behaviour. As before let

$$\alpha \equiv \exp\left(\omega \frac{d}{dx}\right) \quad , \quad \beta \equiv \exp\left(\omega \frac{d}{dy}\right) \tag{350}$$

whence

$$w\frac{d}{dx} \equiv \ln\alpha \qquad , \quad \omega\frac{d}{dy} \equiv \ln\beta \tag{351}$$

The procedure is the same as before so that using the following two expressions

$$\exp\left(\omega\frac{d}{dy}\right) = \exp\left(\frac{dx}{dy}\omega\frac{d}{dx}\right) + \frac{\omega}{2}\sum_{s=2}^{\infty}\omega^{s-1}\frac{s(s-1)}{\underline{|s}}\frac{d^2x}{dy^2}\left(\frac{dx}{dy}\right)^{s-2}\frac{d^{s-1}}{dx^{s-1}} + \omega^3(\cdots$$

$$Q(\alpha)F(x) = Q(\gamma)f(x) + \frac{\omega}{2}\frac{d\gamma}{dx}\gamma\frac{d^2}{d\gamma^2}Q(\gamma)f(x) + \omega^2(\cdots \tag{352}$$

we find

$$F(y) \simeq \exp\left[\frac{1}{2}\ln\gamma\left(1-\frac{dx}{dy}\right]\right. f(x) \tag{353}$$

where $x(y)$ is determined by

$$\int_{y_0}^{y}\ln\Gamma(y)\,dy = \int_{x_0}^{x}\ln\gamma(x)\,dx \tag{354}$$

Putting in constants of integration we may then write

$$\frac{1}{\omega}\mathop{S}_{y_0}^{y}\ln\Gamma(y)\mathop{\Delta}_{\omega}y \simeq \frac{1}{\omega}\mathop{S}_{x_0}^{x}\ln\gamma(x)\mathop{\Delta}_{\omega}x + \left[\frac{1}{2}\ln\gamma(x)\left(1-\frac{dx}{dy}\right)\right]_{y_0}^{y} \tag{355}$$

For the special case when

$$\ln\gamma = 1 \tag{356}$$

we recover the first two terms of the E-M formula since

$$x - x_0 = \int_{y_0}^{y} \ln \Gamma(y)\, dy$$

$$\frac{dx}{dy} = \ln \Gamma(y) \tag{357}$$

then

$$\frac{1}{\omega} \mathop{S}_{y_0}^{y} \ln \Gamma(y) \underset{\omega}{\Delta} y \simeq \frac{1}{\omega} \int_{y_0}^{y} \ln \Gamma(y) dy - \frac{1}{2} \left[\ln \Gamma(y) \right]_{y_0}^{y} \tag{358}$$

(7) W.K.B. EXPANSIONS FOR SOME STANDARD FUNCTIONS [30,31]

The W.K.B. method for difference equations can be used to find expansion formulae for the standard functions of mathematical physics. These expansions are found to be identical with the Green type expansions that can be obtained from the differential equations the functions satisfy. The functions $\sigma(n,x)$ given in Tables V and VI are rather complicated involving products and quotients of Gamma functions. We will require the asymptotic expansion

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta} \left\{ 1 + \frac{A_1}{n} + \frac{\left(\frac{A_1^2}{2} + A_2\right)}{n^2} + \cdots \right. \tag{359}$$

where

$$A_1 = \frac{1}{2} \left[\alpha(\alpha-1) - \beta(\beta-1) \right]$$

$$A_2 = -\frac{1}{12} \left[\alpha(\alpha-1)(2\alpha-1) - \beta(\beta-1)(2\beta-1) \right] \tag{360}$$

This form is easily obtained from Stirlings [2,3] expansion for the Gamma function. In deriving the W.K.B. formulae it was assumed that we could write

$$\sigma(n) = a(n) + \omega^2 b(n) + \cdots \tag{361}$$

For the functions $\sigma(n)$ given in Tables V and VI the interval is equal to unity and there is no natural expansion of the form (361). In actual fact it is fairly easy to expand the $\sigma(n)$ of Tables V and VI such that a simple W.K.B. expansion is obtained which agrees with differential equation results. We obtain W.K.B. approximations for the modified Bessel functions, for the Hermite polynomials, the Whittaker functions and the Frobenius coefficients of the Mathieu functions.

a) THE MODIFIED BESSEL FUNCTIONS $K_n(x), I_n(x)$

The functions $K_n(x)$ and $(-1)^{-n} I_n(x)$ both satisfy the equation

$$y_{n+1}(x) - y_{n-1}(x) = \frac{2n}{x} y_n(x) \tag{362}$$

Using the notation of method I we take $\tau = g = \frac{n}{x}$ so that $g_1 = \frac{1}{x}$ and $g_2 = g_3 = 0$ etc. Ignoring constants of integration we have

$$\frac{\exp \int^n \ln(g \pm \sqrt{g^2+1})\,dn}{(g^2+1)^{\frac{1}{4}}} = \frac{e^{n \ln.\left(\frac{n}{z} \pm \left(\frac{n^2}{z^2}+1\right)^{\frac{1}{2}}\right) - (\pm)(n^2+z^2)^{\frac{1}{2}}}}{\left(\frac{n^2}{z^2}+1\right)^{\frac{1}{4}}} \tag{363}$$

R.B. Dingle has shown that the third and fourth approximations are converted to polynomial form by changing to new integration variable $\alpha = \frac{x}{(n^2+x^2)^{\frac{1}{2}}}$. It is then found that

$$\exp\left[\text{3rd approximation} + \text{4th approximation}\right]$$

$$= \exp\left[\pm \frac{\alpha}{24x}(2-5\alpha^2) - \frac{\alpha^4}{16x^2}(4-5\alpha^2)\right] \tag{364}$$

The variable change which is introduced in infinitesimal calculus for analysis of Bessel functions is not α but

$$q = \frac{n}{(n^2+x^2)^{\frac{1}{2}}}$$

or (365)

$$\alpha = \sqrt{1-q^2}$$

Thus we can write

$$\exp\left[\pm \frac{\alpha}{24x}(2-5\alpha^2) - \frac{\alpha^4}{16x^2}(4-5\alpha^2)\right]$$

$$= \exp\left[\pm \frac{q}{24n}(5q^2-3) + \frac{q^2}{16n^2}(1-q^2)(5q^2-1)\right] \tag{366}$$

$$\simeq 1 \pm \frac{q}{24n}(5q^2-3) + \frac{q^2}{1152n^2}(385q^4-462q^2+81)$$

The $+$'ve sign is associated with $K_n(x)$ and the $-$'ve sign with $(-1)^{-n} I_n(x)$. Hence

$$K_n(x) = C_1\left(\frac{n^2}{x^2}+1\right)^{-\frac{1}{4}} \exp\left(n \sinh \frac{n}{x} - (n^2+x^2)^{\frac{1}{2}}\right) \tag{367}$$

$$\times\left[1 + \frac{q}{24n}(5q^3-3) + \dots\right]$$

$$I_n(x) = C_2\left(\frac{n^2}{x^2}+1\right)^{-\frac{1}{4}} \exp\left(-n \sinh \frac{n}{x} + (n^2+x^2)^{\frac{1}{2}}\right)$$

$$\times\left[1 - \frac{q}{24n}(5q^3-3) + \dots\right] \tag{368}$$

where C_1 and C_2 are integration constants which must be taken as

$$C_1 = \left(\frac{x}{2}\right)^{\frac{1}{2}} \pi^{\frac{1}{2}}$$

$$C_2 = \left(\frac{x}{2}\right)^{\frac{1}{2}} \pi^{-\frac{1}{2}} \tag{369}$$

to obtain agreement with the standard definitions.

b) THE HERMITE FUNCTIONS

For these functions we have

$$\sigma(n,x) = \frac{x}{2}\ \Gamma\left(\tfrac{n+1}{2}\right) \Big/ \Gamma\left(\tfrac{n+2}{2}\right) \tag{370}$$

Using (359) we expand this in inverse powers of n i.e. for large values of n

$$\sigma(n x) = \frac{x}{2}\left(\frac{n}{2}\right)^{-\frac{1}{2}}\left[1 - \left(\frac{n}{2}\right)^{-1}\frac{1}{8} + \left(\frac{n}{2}\right)^{-2}\frac{1}{128} + \cdots\right]$$

$$= \frac{x}{2}\left(\frac{n}{2}\right)^{-\frac{1}{2}}\left[1 - \frac{1}{4n} + \frac{1}{32n^2} + \cdots\right] \tag{371}$$

It is natural to use the power of n to order the terms in equation (370) We could put

$$a(n) = \frac{x}{2}\left(\frac{n}{2}\right)^{-\frac{1}{2}}$$

$$b(n) = -\frac{x}{2}\left(\frac{n}{2}\right)^{-\frac{1}{2}}\frac{1}{4n} \tag{372}$$

but it turns out that to obtain the simplest form for the expansion it is necessary to eliminate the term in n^{-1}. This can be achieved by changing the difference variable to p i.e.

$$n = p + k \tag{373}$$

where k is a constant. The expansion of $\sigma(n,x)$ in terms of p is

$$\sigma(n,x) \Rightarrow \sigma(p x) = \frac{x}{2}\left(\frac{p}{2}\right)^{-\frac{1}{2}}\left[1 + \left(\frac{1}{4} + \frac{k}{2}\right)\frac{1}{p} + \left(\frac{1}{32} + \frac{3}{8}k^2 + \frac{3}{8}k\right)\frac{1}{p^2} \cdots\right] \tag{374}$$

The coefficient of p^{-1} inside the brackets may be made zero by choosing

$$K = -\tfrac{1}{2} \tag{375}$$

We then have

$$\sigma(px) = \frac{x}{2}\left(\frac{p}{2}\right)^{-\frac{1}{2}}\left[1 - \frac{1}{16p^2} + \;\cdots\right] \tag{376}$$

We now take

$$a(p) = \frac{\sqrt{2}\,x}{2}\,p^{-\frac{1}{2}} \;, \quad b(p) = -\frac{\sqrt{2}\,x}{32}\,p^{-\frac{5}{2}}\,. \tag{377}$$

The first two W.K.B. approximations are given by

$$\begin{aligned} &\frac{\exp \int^{p} \ln\left(a \pm \sqrt{a^2-1}\right)dp}{(a^2-1)^{\frac{1}{4}}} \\ &= \left(\frac{x^2}{2p}-1\right)^{-\frac{1}{4}} \exp\left[p \ln\left(\frac{x \pm (x^2-2p)^{\frac{1}{2}}}{(2p)^{\frac{1}{2}}}\right) \mp \frac{x}{2}\left(x^2-2p\right)^{\frac{1}{2}}\right] \end{aligned} \tag{378}$$

The third approximation has been calculated. Calculation is handicapped by the fact that it does not seem possible to define a new variable which converts the integration to polynomial form. However, the variable change

$$\alpha = \left(\frac{x^2}{2p}-1\right)^{-\frac{1}{2}} \tag{379}$$

makes the integration reasonably systematic. We find

$$\begin{aligned} \exp\,[\text{3rd approximation}] &= \exp\left[\pm\frac{(\alpha^2+1)^{\frac{1}{2}}}{24x^2}\left(\frac{1}{\alpha^2}-4-5\alpha^2\right)\right] \\ &\simeq 1 \pm \frac{(\alpha^2+1)^{\frac{1}{2}}}{24x^2}\left(\frac{1}{\alpha^2}-4-5\alpha^2\right) + \cdots \end{aligned} \tag{380}$$

In terms of the quantity

$$q = x\,(x^2 - 2p)^{-\frac{1}{2}} \tag{381}$$

we have

$$\exp\,[\text{3rd approximation}] \simeq 1 \mp \frac{1}{48p}\left[5q^3 - 6q\right] + \cdots \tag{382}$$

which is in agreement with the known Green type solutions to Hermite's differential equation.

c) THE WHITTAKER FUNCTION $W_n^m(x)$

This function is similar to the Hermite functions and the same difficulties arise. It has not been found possible to define a subsidiary variable which converts the integration of the third and fourth approximations to polynomial form. From Table V we have

$$\sigma(n,x) = \left(\frac{x}{4} - \frac{n}{2}\right)\frac{\Gamma\left(\frac{n-m+\frac{1}{2}}{2} - 1\right)\Gamma\left(\frac{n+m+\frac{1}{2}}{2} - 1\right)}{\Gamma\left(\frac{n-m+\frac{1}{2}}{2} - \frac{1}{2}\right)\Gamma\left(\frac{n+m+\frac{1}{2}}{2} - \frac{1}{2}\right)} \tag{383}$$

We will consider the two main cases. Firstly when n is large compared with m and secondly when $|n^2 - m^2|$ is large but m cannot be considered as small.

a) $n \gg m$

Making use of equation (359) we find

$$\sigma(n,x) = \frac{1}{2}\left(\frac{x}{2} - n\right)\left(\frac{n}{2}\right)^{-1}\left[1 + \left(\frac{n}{2}\right)^{-1} + \cdots\right] \tag{384}$$

By changing the variable to

$$p = n - 2$$

and writing (385)

$$z = x - 4$$

we find

$$\sigma(n,x) \to \sigma(p,z) = \frac{1}{p}\left(\frac{z}{2} - p\right)\left[1 + \left(\frac{m^2}{2} - \frac{1}{8}\right)\frac{1}{p^2} + \cdots\right] \tag{386}$$

We take

$$a(p,z) = \left(\frac{z}{2} - p\right)\frac{1}{p}\ , \quad b(p,z) = \left(\frac{z}{2} - p\right)\left(\frac{m^2}{2} - \frac{1}{8}\right)\frac{1}{p^2} \tag{387}$$

We find the first two approximations are given by

$$\begin{aligned} W_p^m(z) \simeq \left[\frac{z - 4pz}{4p^2}\right]^{-\frac{1}{4}} \exp\Big[\, & p \ln\left(\left(\frac{z}{2p} - 1\right) \pm \left(\frac{z^2 - 4pz}{4p^2}\right)^{\frac{1}{2}}\right) \\ & \mp \frac{1}{2}\left(z^2 - 4pz\right)^{\frac{1}{2}} \Big] \end{aligned} \tag{388}$$

where

$$w_p^m(z) = 2^n\, \Gamma\left(\frac{n - m + \frac{1}{2}}{2} - 1\right) \Gamma\left(\frac{n + m + \frac{1}{2}}{2} - 1\right) W_n^m(x) \tag{389}$$

Taking an integration variable

$$\alpha = \left(\left(\frac{z}{2p} - 1\right)^2 - 1\right)^{-\frac{1}{2}} \tag{390}$$

we find

exp [3rd approximation]

$$= \exp\left[\mp \frac{1}{12z}\left(10\alpha^3 + 7\alpha + (12m^2-1)\frac{1}{\alpha} + 2(\alpha^2+1)^{\frac{3}{2}} + 8\alpha^2(\alpha^2+1)^{\frac{1}{2}}\right)\right] \tag{391}$$

In differential equation analysis of Whittaker functions a convenient subsidiary variable is the quantity

$$q = \left(\frac{z}{z-4p}\right)^{\frac{1}{2}} \tag{392}$$

We have

$$\alpha = \frac{q^2-1}{2q} \tag{393}$$

On substitution into (391) we find

exp [3rd approximation]

$$\simeq 1 \mp \frac{1}{96p}\left[5q^3 - 6q + q^{-1}(48m^2-3)\right] + \cdots \tag{394}$$

which agrees completely with the differential equation results.

β) $|n^2-m^2|$ large

When m cannot be considered as small the function $\sigma(p,z)$ can be expanded as

$$\sigma(p,z) = \left(\frac{z}{2}-p\right)(p^2-m^2)^{-\frac{1}{2}}\left[1 - \frac{1}{8}\frac{(p^2+m^2)}{(p^2-m^2)^2} + \cdot\cdot\right] \tag{395}$$

We take

$$a(p,z) = \left(\frac{z}{2}-p\right)(p^2-m^2)^{-\frac{1}{2}}\ ,\ b(p,z) = -\frac{1}{8}\left(\frac{z}{2}-p\right)\frac{(p^2+m^2)}{(p^2-m^2)^{5/2}} \qquad (396)$$

The first two W.K.B. approximations are found to be

$$\begin{aligned} &(a^2-1)^{-\frac{1}{4}} \exp \int^p \ln(a \pm \sqrt{a^2-1})\,dp \\ &= \left[\left(\frac{z}{2}-p\right)^2(p^2-m^2)^{-1} - 1\right]^{-\frac{1}{4}} \exp\left[p \ln\left(\left(\frac{z}{2}-p\right) \pm \left(\frac{z^2}{4}+m^2-pz\right)^{\frac{1}{2}}\right)\right. \\ &\quad \left. - p\ln(p^2-m^2) \pm \left(\frac{z^2}{4}+m^2-zp\right)^{\frac{1}{2}} \pm \int^p \frac{m^2\left(\frac{z}{2}-p\right)}{(p^2-m^2)\left(\frac{z^2}{4}+m^2-zp\right)^{\frac{1}{2}}}\,dp\right] \end{aligned} \qquad (397)$$

The integration in equation (397) is a little difficult but can be performed by changing the variable to

$$y = (2m^2-pz)/z(p^2-m^2)^{\frac{1}{2}} \qquad (398)$$

We then obtain

$$\begin{aligned} &(a^2-1)^{-\frac{1}{4}} \exp \int^p \ln(a \pm \sqrt{a^2-1})\,dp = \left[\left(\frac{z}{2}-p\right)^2(p^2-m^2)^{-1} - 1\right]^{-\frac{1}{4}} \\ &\times \exp\left[\pm p \cosh^{-1}\left(\frac{\frac{z}{2}-p}{(p^2-m^2)^{\frac{1}{2}}}\right) \mp \left(\frac{z^2}{4}+m^2-zp\right)^{\frac{1}{2}}\right. \\ &\qquad \left. \pm\, m \cosh^{-1}\left|\frac{2m^2-pz}{2(p^2-m^2)}\right|\right] \end{aligned} \qquad (399)$$

It is rather difficult to obtain the third and fourth terms due to the lack of a suitable subsidiary variable.

d) THE FOURIER COEFFICIENTS OF THE PERIODIC MATHIEU FUNCTIONS

The difference equation is

$$f_{n+1} + f_{n-1} = \left(\frac{\alpha - n^2}{2q}\right) f_n \tag{400}$$

The integrations for this function are non-trivial. We give three appropriate forms for the integral involved in the first W.K.B. term. It is advantageous to write

$$f_n' = (-)^n f_n \tag{401}$$

so that

$$f'_{n+1} + f'_{n-1} = \left(\frac{n^2 - \alpha}{2q}\right) f'_n \tag{402}$$

If we take $\sigma(n) = \frac{n^2 - \alpha}{2q}$ we have

$$f_n = (-)^n f_n' = \left[\left(\frac{n^2-\alpha}{4q}\right)^2 - 1\right]^{-\frac{1}{4}} \exp \pm \int^n \cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) dn \tag{403}$$

The integral is an indefinite elliptic integral. The three different forms will now be given. The most suitable form in a particular situation depends on the value of the parameters involved.

(i) Integration by parts gives

$$\pm \int^n \cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) dn = \int^n \ln\left(\frac{n^2-\alpha}{4q} \pm \left[\left(\frac{n^2-\alpha}{4q}\right)^2 - 1\right]^{\frac{1}{2}}\right) dn$$

$$= \pm n \cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) \mp 2 \int^n \frac{n^2\, dn}{\left((n^2-\alpha)^2 - 16q^2\right)^{\frac{1}{2}}} \tag{404}$$

gives

$$ {}^{\epsilon}\!\int (\epsilon + 1 + 2\sinh^2\epsilon)^{\frac{1}{2}} d\epsilon = \sqrt{2}\,(1-k^2)\ {}^{\phi}\!\int \frac{\sec^2\phi \; d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}} \qquad (411)$$

We obtain after integrating by parts

$$\pm {}^{n}\!\int \cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) dn = \pm\, n\cosh\left(\frac{n^2-\alpha}{4q}\right) \mp \frac{2}{n}\left((n^2-\alpha^2)-16q^2\right)^{\frac{1}{2}}$$

$$\pm\ 4\sqrt{2q}\ E\left(\cos^{-1}\left(\frac{\alpha+4q}{n^2}\right)^{\frac{1}{2}}, \sqrt{\tfrac{1}{2}\left(1-\frac{\alpha}{4q}\right)}\right) \qquad (412)$$

$$\mp\ \frac{1}{2}\left(1+\frac{\alpha}{4q}\right) F\left(\cos^{-1}\left(\frac{\alpha+4q}{n^2}\right)^{\frac{1}{2}}, \sqrt{\tfrac{1}{2}\left(1-\frac{\alpha}{4q}\right)}\right)$$

where F and E are elliptic integrals of the first and second kind.[32]

(iii) Consider again the term ${}^{u}\!\int \left(\frac{u+\xi}{1-u^2}\right)^{\frac{1}{2}} du$. The substitutions

$$\begin{aligned} u &= \cosh 2\epsilon \\ \cosh\epsilon &= \cosh\epsilon_0 \operatorname{cosec}\theta \\ \xi &= -\cosh 2\epsilon_0 \\ k &= \operatorname{sech}\epsilon_0 \end{aligned} \qquad (413)$$

give

$$ {}^{u}\!\int \left(\frac{u+\xi}{u^2-1}\right)^{\frac{1}{2}} du = -2^{3/2}\cosh\epsilon_0 \ {}^{\theta}\!\int \frac{\cot^2\theta \; d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} \qquad (414)$$

By integrating by parts we obtain

$$\pm {}^{n}\!\int \cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) dn = \pm\, n\cosh^{-1}\left(\frac{n^2-\alpha}{4q}\right) \mp 2n\left(\frac{n^2-\alpha-4q}{n^2-\alpha+4q}\right)^{\frac{1}{2}}$$

$$\mp\ 2(4q-\alpha)^{\frac{1}{2}} E\left(\sin^{-1}\sqrt{\frac{4q-\alpha}{n^2-\alpha+4q}}\ ,\ \sqrt{\frac{8q}{4q-\alpha}}\right) \qquad (415)$$

where E is an elliptical integral of the second kind.

VII PERTURBATION THEORY
(1) A PERTURBATION METHOD FOR SOLVING THE DIFFERENCE EQUATION FOR FROBENIUS COEFFICIENTS

Most of the standard functions of mathematical physics, satisfying a second order differential equation, have Frobenius coefficients which satisfy a first order difference equation. Perturbation theory seeks to express the solution of a perturbed equation in terms of the solutions of standard unperturbed equation. We show how the Frobenius coefficients of a perturbed equation can be expressed in terms of the unperturbed coefficients. We begin with the specific physical problem of an electron in a simple harmonic oscillator potential plus a constant electric field.

Schroedinger's equation for an electron in a harmonic oscillator potential takes the form

$$\frac{d^2}{dq^2}\psi + \frac{2m}{\hbar^2}\left(E - 2\pi^2\nu^2q^2\right)\psi = 0 \tag{416}$$

where the symbols have their usual meaning. In dimensionless form this equation becomes

$$\frac{d^2\psi_n}{dx^2} + \left(n+\tfrac{1}{2} - \tfrac{1}{4}x^2\right)\psi_n = 0 \tag{417}$$

where $E = h\nu(n+\frac{1}{2})$ and $x = q\left(4\pi M\nu/\hbar\right)^{\frac{1}{2}}$. For large values of x

$$\psi_n \simeq e^{-\frac{1}{4}x^2} \tag{418}$$

Hence putting

$$\psi_n = H_n(x)\, e^{-\frac{1}{4}x^2} \tag{419}$$

the equation for $H_n(x)$ is

$$\frac{d^2 H_n}{dx^2} - x \frac{d}{dx} H_n + n H_n = 0 \tag{420}$$

The Frobenius method is to write

$$H_n(x) = \sum_{\mu} a_n(\mu) x^{\mu} \tag{421}$$

which leads to the difference equation

$$a_n(\mu+2) = -\frac{(n-\mu)}{(\mu+1)(\mu+2)} a_n(\mu) \tag{422}$$

The $H_n(x)$ are Hermite polynomials (the definition here is slightly different to that used in Table V) with

$$a_n(\mu) = (-2)^{\frac{1}{2}(\mu-n)} \frac{\lfloor n}{\lfloor \mu \; \lfloor \frac{1}{2}(n-\mu)} \tag{423}$$

Equation (420) can be expressed in terms of the shift operator as

$$\left\{ (\mu+1)(\mu+2) E^2 + (n-\mu) \right\} a_n(\mu) = 0 \tag{424}$$

This is to be regarded as the unperturbed equation of the theory and is a simple case of the general unperturbed form

$$\left\{ N_n - U \right\} a_n(\mu) = 0 \tag{425}$$

when N_n is the n'th eigenvalue and U is a difference operator. For the simple harmonic oscillator

$$N_n = n \tag{426}$$

and

$$U = \mu - (\mu+1)(\mu+2)E^2 \tag{427}$$

When a constant electric field F is applied the perturbing potential is eFq and equation (417) becomes

$$\frac{d^2\phi}{dx^2} + \left(n' + \tfrac{1}{2} - \tfrac{1}{4}x^2 - cx\right)\phi = 0 \tag{428}$$

where the perturbed energy E' is

$$E' = h\nu\left(n' + \tfrac{1}{2}\right)$$

and (429)

$$c = \frac{eF}{h\nu}\left(\frac{\hbar}{4\pi M\nu}\right)^{\frac{1}{2}}$$

If the exponential behaviour is removed by writing

$$\phi = y(x)\, e^{-\frac{1}{4}x^2} \tag{430}$$

then

$$y(x) = \sum_{\mu} b(\mu)\, x^{\mu} \tag{431}$$

where the coefficients $b(\mu)$ satisfy

$$\left\{(\mu+1)(\mu+2)E^2 + (n'-\mu) - cE^{1}\right\} b(\mu) = 0 \tag{432}$$

Equation (432) is to be regarded as the perturbed equation which in general takes the form

$$\left\{(N_n+3) - (U+u)\right\} b(\mu) = 0 \tag{433}$$

where η is the eigenvalue shift due to the perturbing operator For the simple harmonic oscillator in an electric field

and
$$\eta = n' - n$$
$$u = c E' \tag{434}$$

The derivation of the general theory for equations (425) and (433) will be given. The application of the theory to the simple harmonic oscillator will then be considered. The general perturbed and unperturbed equations are

$$\{ N_n - U \} a_n(\mu) = 0 \tag{435}$$

and

$$\{ N_n - U \} b(\mu) = \{ u - \eta \} b(\mu) \tag{436}$$

The perturbation theory can be developed in much the same way as the theory for differential equations. When the perturbing terms are small the zeroth approximation $b^0(\mu)$ is obviously

$$b^0(\mu) = a_n(\mu) \tag{437}$$

This leaves an uncompensated term $(u-\eta)a_n(\mu)$ on the right hand side of equation (436). We must add a correction term $b_1(\mu)$ to $b^0(\mu)$ The fundamental assumption of the theory is that $(u-\eta)a_n(\mu)$ may be expanded as

$$(u-\eta) a_n(\mu) = \sum_\alpha (0,\alpha)\, a_{n+\alpha}(\mu) \tag{438}$$

where the coefficients $(0,\alpha)$ are independent of μ and operators acting on μ. By taking a correcting term

$$b'(\mu) = \sum_{\alpha \neq 0} \frac{(0,\alpha)}{(N_n - N_{n+\alpha})} a_{n+\alpha}(\mu) \tag{439}$$

the only term which is uncompensated is

$$(0,0)\,a_n(\mu) \tag{440}$$

The introduction of $b'(\mu)$ itself leaves uncompensated terms equal to

$$(u-\zeta)\,b'(\mu) = \sum_{\alpha\neq 0} \frac{(0,\alpha)}{(N_n - N_{n+\alpha})}(u-\zeta)\,a_{n+\alpha}(\mu)$$

$$= \sum_{\alpha\neq 0} \sum_{\beta} \frac{(0,\alpha)(\alpha,\beta)}{(N_n - N_{n+\alpha})}\,a_{n+\beta}(\mu) \tag{441}$$

Except for the term

$$\sum_{\alpha\neq 0} \frac{(0,\alpha)(\alpha,0)}{(N_n - N_{n+\alpha})}\,a_n(\mu) \tag{442}$$

$(u-\zeta)b_1(\mu)$ can be compensated by a second correction $b^2(\mu)$ given by

$$b_2(\mu) = \sum_{\alpha\neq 0}\sum_{\beta\neq 0} \frac{(0,\alpha)\,(\alpha,\,\beta)}{(N_n - N_{n+\alpha})(N_n - N_{n+\beta})}\,a_{n+\beta}(\mu) \tag{443}$$

Continuation of this line of reasoning leaves a series of uncompensated terms which must be zero if $b(\mu) = b^0(\mu) + b'(\mu) + \cdots$ is to be a solution to equation (436). Placing the sum of uncompensated terms equal to zero determines the eigenvalue shifts, i.e.

$$0 = (0,0) + \sum_{\alpha\neq 0} \frac{(0,\alpha)}{(N_n - N_{n+\alpha})}(\alpha,0) + \sum_{\alpha\neq 0}\sum_{\beta\neq 0} \frac{(0,\alpha)(\alpha,\beta)}{(N_n - N_{n+\alpha})(N_n - N_{n+\beta})}(\beta,0)$$

$$+ \sum_{\alpha\neq 0}\sum_{\beta\neq 0}\sum_{\gamma\neq 0} \frac{(0,\alpha)(\alpha,\beta)(\beta,\gamma)}{(N_n - N_{n+\alpha})(N_n - N_{n+\beta})(N_n - N_{n+\gamma})}(\gamma,0) + \cdots \tag{444}$$

The structure of this equation is completely analogous to the differential equation perturbation theory outlined in Section II 2c). The perturbed coefficients are given by

$$b(\mu) = a_n(\mu) + \sum_{\alpha\neq 0} \frac{(0,\alpha)}{(N_n - N_{n+\alpha})} a_{n+\alpha}(\mu) + \sum_{\alpha\neq 0}\sum_{\beta\neq 0} \frac{(0,\alpha)(\alpha,\beta)}{(N_n - N_{n+\alpha})(N_n - N_{n+\beta})} a_{n+\beta}(\mu)$$

$$+ \sum_{\alpha\neq 0}\sum_{\beta\neq 0}\sum_{\gamma\neq 0} \frac{(0,\alpha)(\alpha,\beta)(\beta,\gamma)}{(N_n - N_{n+\alpha})(N_n - N_{n+\beta})(N_n - N_{n+\gamma})} a_{n+\gamma}(\mu) + \cdots \qquad (445)$$

The fact that $b(\mu)$ is expressed in terms of the unperturbed $a_n(\mu)$ ensures that the function $y(x)$ will satisfy the same boundary conditions as $H_n(x)$.

For the simple harmonic oscillator problem the unperturbed coefficients are given by (423) i.e.

$$a_n(\mu) = (-2)^{\frac{1}{2}(\mu-n)} \frac{\lfloor n}{\lfloor \mu \, \lfloor \frac{1}{2}(\mu-n)} \qquad (446)$$

Now

$$(u-\eta)\, a_n(\mu) = c\, a_n(\mu-1) - \eta\, a_n(\mu) = \sum_{\alpha} (0,\alpha)\, a_{n+\alpha}(\mu)$$

and (447)

$$a_n(\mu-1) = a_{n+1}(\mu) + n\, a_{n-1}(\mu).$$

Hence the only non-zero coefficients $(0,\alpha)$ are

$$(0,-1) = cn,$$
$$(0,0) = -\eta, \qquad (448)$$
$$(0,+1) = c.$$

Using the eigenvalue equation (444) we have

$$\eta = \frac{(0,-1)(-1,0)}{N_n - N_{n-1}} + \frac{(0,1)(1,0)}{N_n - N_{n+1}} + \cdots \tag{449}$$
$$= c^2 n - c^2(n+1) = -c^2$$

Thus

$$E' - E = h\nu(n' - n) = h\nu\eta = -h\nu c^2 = \frac{-e^2F^2}{8\pi^2 M\nu^2} \tag{450}$$

The complete analogy between the preceding theory and the theory for differential equations means that the method outlined in Section II 2c) can be used to analyse the perturbation series. Although the difference equation method has the same basic calculational problems of ordinary perturbation theory it is more fundamental. Firstly the method is no less general than the usual methods since the only unperturbed functions which can be usefully employed in the usual theory are precisely those which have coefficients satisfying a first order difference equation. Secondly, it is not necessary to invoke extra conditions such as the orthonormality of the functions used.

The greatest merit of the method is it is not essential to take an unperturbed difference equation associated with a differential equation. For example the function

$$a_n(\mu) = \frac{\lfloor\mu}{\lfloor\mu - n} \tag{451}$$

satisfies a difference equation

$$\{\mu(E^{-1} - 1) + n\}a_n(\mu) = 0 \tag{452}$$

and one can say that n is an eigenvalue and that for $a_n(\mu)$ to be a polynomial in μ we must have $n=0,1,2\cdots$. The perturbation method would enable an equation deviating from equation (452) to be investigated.

(2) PERTURBATION THEORY FOR AN EQUATION DEVIATING FROM THE EQUATION FOR A STANDARD FUNCTION

The difference equations for the standard functions given in Tables V and VI have the general form

$$L_n f_n(x) = x\, f_n(x) \tag{453}$$

where L_n is an operator independent of x. The range and values of n (the eigenvalues of the associated differential equation) we can consider to be determined from boundary conditions. Consider a perturbed equation

$$L_N F_N(z) - z\, F_N(z) = V_N\, F_N(z) \tag{454}$$

where V_N is an operator which we can assume is independent of z. The boundary conditions are assumed unchanged. If we write

$$N = n + \epsilon \tag{455}$$

where ϵ is the eigenvalue shift and

$$F_N(z) = D_n(z) \tag{456}$$

we have a perturbed equation of the form

$$L_n D_n(z) - z\, D_n(z) = V_n'(\epsilon)\, D_n(z) \,. \tag{457}$$

We now exploit the fact that x appears in the equation like a continuous eigenvalue. We assume that the unperturbed solutions satisfy a completeness relation, i.e.

$$\sum_m W(m)\, f_m^*(x')\, f_m(x) = \delta(x-x') \tag{458}$$

which looks like an orthogonality condition in n space. $W(m)$ is some weighting function. The first approximation is

$$D_n^0(z) = f_n(z). \tag{459}$$

Substituting this approximation into the right hand side of (457) we must solve

$$L_n\, D_n^1(z) - z D_n^1(z) = V_n'(\epsilon)\, f_n(z) \tag{460}$$

We make the assumption that we can write

$$\begin{aligned} D_n^1(z) &= \int P(z,z')\, f_n(z')\, dz' \\ V_n'(\epsilon)\, f_n(z) &= \int Q(z,z',\epsilon)\, f_n(z')\, dz'. \end{aligned} \tag{461}$$

This must certainly be true if $D_n(z)$ satisfies the same boundary conditions as $f_n(z)$. Substituting into (460) we find

$$P(zz') = Q(z,z',\epsilon)/(z-z') \tag{462}$$

or

$$D_n^1(z) = \int \frac{Q(z,z',\epsilon)}{(z-z')}\, f_n(z')\, dz' \tag{463}$$

The integrand contains a singularity at $z=z'$. To remove the singularity we take

$$Q(z,z,\epsilon)=0 \tag{464}$$

as a condition which determines ϵ and take the principal value of the integral, i.e.

$$D'_n(z) = \int \frac{Q(z,z',\epsilon)}{(z-z')} f_n(z')\,dz' \tag{465}$$

This is very similar to the form of the perturbation series in II 3c) if summations are replaced by integrals and vice versa. We will not consider the perturbation series further since it proceeds in the same way as previously. We consider the example of the simple harmonic oscillator in a perturbing electric field.

The unperturbed and perturbed differential equations in the previous section were

$$\frac{d^2}{dx^2}\psi_n + \left(n+\tfrac{1}{2}-\tfrac{x^2}{4}\right)\psi_n = 0 \tag{466}$$

and

$$\frac{d^2}{dx^2}\phi_n + \left(N+\tfrac{1}{2}-\tfrac{x^2}{4} - cx\right)\phi_n = 0 \tag{467}$$

These two functions have difference equation analogues.

$$\psi_{n+1}(x) + n\,\psi_{n-1}(x) = x\,\psi_n(x) \tag{468}$$

and

$$\phi'_{N+1}(z) + (N+c^2)\,\phi'_{N-1}(z) = z\,\phi'_N(z) \tag{469}$$

where

$$z = x + 2c$$
$$\phi'_N(z) = \phi_N(x) \tag{470}$$

Writing

$$N = n + \epsilon$$

and (471)

$$\phi_N(z) = \chi_n(z)$$

we have

$$\psi_{n+1}(x) + n\,\psi_{n-1}(x) - x\,\psi_n(x) = 0 \tag{472}$$

and

$$\chi_{n+1}(z) + n\,\chi_{n-1}(z) - z\,\chi_n(z) = -(\epsilon + c^2)\,\chi_{n-1}(z).$$

We can write

$$-(\epsilon + c^2)\,\psi_{n-1}(z) = \int Q(z, z', \epsilon)\,\psi_n(z')\,dz' \tag{473}$$

where

$$Q(z, z', \epsilon) = -\sum_m \frac{1}{\lfloor m} \left[\psi_m(z')(\epsilon + c^2)\,\psi_{m-1}(z)\right] \tag{474}$$

This follows from

$$\delta(z - z') = \sum_m \frac{1}{\lfloor m}\,\psi_m(z')\,\psi_m(z) \tag{475}$$

For this very simple example

$$Q(z, z, \epsilon) = 0 = \epsilon + c^2$$

or (476)

$$\epsilon = -c^2.$$

Also

$$\chi_n(z) = \psi_n(z) \tag{477}$$

This is not a very satisfying or rigourous test of the perturbation theory. The difficulty is that in general one cannot write a perturbed differential equation in a difference form because in general the eigenvalue shifts are not independent of n. One can still use the type of method outlined above for a perturbed difference equation not associated with a differential equation.

VIII DIFFERENCE EQUATIONS FOR SOME WELL KNOWN FUNCTIONS AND THE NORMAL FORM FOR THESE EQUATIONS 30,31

The equations are classified by the functions $A(n)$, $B(n,x)$ and $C(n)$ of equation (52). Some functions of two parameters satisfy two difference equations one with n as the difference variable and another with m. Table V gives the equations in n and Table VI those in the minor order m. The notation for the functions is as follows :-

$T_n(x)$: Tschebyscheff polynomials.

$J_n(x)$: Bessel function of the first kind.

$Y_n(x)$: Bessel function of the second kind.

$H_n^1(x), H_n^2(x)$: Hankel functions of the first and second kind.

$I_n(x)$: Modified Bessel function of the first kind.

$K_n(x)$: Modified Bessel function of the third kind.

$P_n^m(x)$: Associated Legendre polynomials.

$L_n^m(x)$: Associated Laguerre polynomials.

$W_n^m(x)$: Whittaker's functions.

$C_n^m(x)$: Gegenbauer functions.

$F_n^m(x)$: Confluent hypergeometric functions.

$H_n(x)$: Hermite polynomials.

$D_n(x)$: Parabolic cylinder functions.

TABLE V

$A(n)$	$B(n,x)$	$C(n)$	$Q(n)$	$G(n,x)$	
1	$2x$	1	1	x	$T_n(x)$.
1	$2\frac{n}{x}$	1	1	$\frac{n}{x}$	$J_n(x), Y_n(x), H'_n(x), H^2_n(x)$.
1	$-2\frac{n}{x}$	-1	i^n	$-\frac{n}{ix}$	$I_n(x)$.
1	$2\frac{n}{x}$	-1	i^n	$\frac{n}{ix}$	$K_n(x)$.
1	$2x$	$2n$	$2^n\ \Gamma\left(\frac{n+1}{2}\right)$	$\frac{x}{2}\ \Gamma\left(\frac{n+1}{2}\right) / \Gamma\left(\frac{n+2}{2}\right)$	$H_n(x)$, $D_n(x)$.
$(n+1)$	$2x\left(n+\frac{1}{2}\right)$	n	$\Gamma\left(\frac{n+1}{2}\right) / \Gamma\left(\frac{n+2}{2}\right)$	$\frac{x}{2}\left(n+\frac{1}{2}\right)\Gamma^2\left(\frac{n+1}{2}\right) / \Gamma^2\left(\frac{n+2}{2}\right)$	$P_n(x)$
$(n+1-m)$	$2x\left(n+\frac{1}{2}\right)$	$(n-m)$	$\Gamma\left(\frac{n-m+1}{2}\right) / \Gamma\left(\frac{n-m+2}{2}\right)$	$\frac{x}{2}\left(n+\frac{1}{2}\right)\Gamma^2\left(\frac{n-m+1}{2}\right) / \Gamma^2\left(\frac{n-m+2}{2}\right)$	$P^m_n(x)$
$(n+1)$	$2\left(n+m+\frac{1}{2}-\frac{x}{2}\right)$	$(n+m)$	$\Gamma\left(\frac{n+m+1}{2}\right) / \Gamma\left(\frac{n+2}{2}\right)$	$\left(\frac{n}{2}+\frac{m+1-x}{4}\right) \times \dfrac{\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+m+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}$	$L^m_n(x)$
1	$2\left(\frac{x}{2}-n\right)$	$\left(n-m-\frac{1}{2}\right) \times \left(n+m-\frac{1}{2}\right)$	$2^n\ \Gamma\left(\frac{n-m}{2}\right)\Gamma\left(\frac{n+m}{2}\right)$	$\left(\frac{x}{4}-\frac{n}{2}\right) \times \dfrac{\Gamma\left(\frac{n-m}{2}\right)\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n+m+1}{2}\right)}$	$W^m_n(x)$
$(n+1)$	$2x(n+m)$	$(n+2m-1)$	$\Gamma\left(\frac{n+2m}{2}\right) / \Gamma\left(\frac{n+2}{2}\right)$	$\frac{x}{2}(n+m) \times \dfrac{\Gamma\left(\frac{n+2m}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+2m+1}{2}\right)}$	$C^m_n(x)$

$A(n)$	$B(n,x)$	$C(n)$	$Q(n)$	$\text{Б}(n,x)$	
n	$2\left(n+\frac{x}{2}-\frac{m}{2}\right)$	$(n-m)$	$\Gamma\left(\frac{n-m+1}{2}\right) / \Gamma\left(\frac{n+1}{2}\right)$	$\left(\frac{n}{2}+\frac{x}{4}-\frac{m}{4}\right) \times \dfrac{\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)}$	$F_n^m(x)$

TABLE VI

$A(m)$	$B(m,x)$	$C(m)$	$Q(m)$	$\text{Б}(m,x)$	
1	$2mx\left(1-x^2\right)^{-\frac{1}{2}}$	$(n+m)(n+1-m)$	$2^m\,\Gamma\left(\frac{m+n+1}{2}\right)\Gamma\left(\frac{m-n}{2}\right)$	$\frac{mx}{2}\left(1-x^2\right)^{-\frac{1}{2}} \times \dfrac{\Gamma\left(\frac{m+n+1}{2}\right)\Gamma\left(\frac{m-n}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)\Gamma\left(\frac{m-n+1}{2}\right)}$	$P_n^m(x)$
$(m-n)$	$m(m-1+x)$	$m(m-1)$	$2^{-\frac{m}{2}}\,\Gamma(m) / \Gamma\left(\frac{m-n+1}{2}\right)$	$\dfrac{2^{-3/2}(m-1+x)\,\Gamma\left(\frac{m-n}{2}\right)}{\Gamma\left(\frac{m-n+1}{2}\right)}$	$F_n^m(x)$
1	$2\left(\frac{m+x}{2\sqrt{x}}\right)$	$(m+n)$	$2^{-\frac{m}{2}}\,\Gamma\left(\frac{m+n+1}{2}\right)$	$\dfrac{\left(\frac{m+x}{4\sqrt{2x}}\right)\Gamma\left(\frac{m+n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$	$x^{\frac{m}{2}}\,L_n^m(x)$

IX SOME OTHER METHODS OF SOLVING DIFFERENCE EQUATIONS

We give a brief discussion of three other methods of solving difference equations and the type of solution which can be found using them. The three methods are, operational method of Boole[2,3,6] the Laplace method of transforms[2,3] and the technique of continued fractions[22,23]

1) Boole's operational method.

In difference calculus the natural analogue to a power series is a series of factorials. We have used the notation

$$[n]_s = n(n-1)(n-2)\cdots\cdots(n-s+1) \qquad (s>0)$$

$$[n]_{-s} = \left([n+s]_s\right)^{-1} = \left((n+s)(n+s-1)\cdots(n+1)\right)^{-1} \quad (s<0) \qquad (478)$$

$$[n]_0 = 1$$

For general n and s we can use the extended definition

$$[n]_s = \frac{\Gamma(n+1)}{\Gamma(n-s+1)} \qquad (479)$$

The factorial has the useful property

$$\Delta [n]_s = s\,[n]_{s-1} \qquad (480)$$

but does not satisfy the law of indices, i.e.

$$[n]_s\,[n]_r \neq [n]_{s+r}. \qquad (481)$$

The theorem of unique development[3] states that a function which can be developed in a series of factorials can be so developed in only one manner. This naturally leads us to seek solutions to difference equations in the form of series of rising factorials ($s \geq 0$) or inverse factorials ($s \leq 0$). These solutions are analogous to the Frobenius solutions of differential equation theory. They can be constructed by methods due to Boole[6] and Milne-Thomson[3] for difference equations with rational coefficients of the variable n. The methods are quite complicated to execute stemming from the fact that factorials do not satisfy the indices law. Some of the solutions found in Section IV are factorial series, for example the solutions in rising powers of z for $J_n(z)$. These solutions could have been discovered using Boole's method. The iterative approach seems more useful since we are able to find a Stokes' type solution for $J_n(z)$ which would not have been possible using Boole's method. The iterative method is not restricted to equations with rational coefficients.

2) Laplace transform method.

The Laplace transform method involves transforming from a difference equation to a differential equation and obtaining an integral representation for the solution to the difference equation. We can illustrate this by the well known example of the Gamma function. $\Gamma(n)$ satisfies the first order difference equation

$$\Gamma(n+1) = n\,\Gamma(n) \qquad (482)$$

We write

$$\Gamma(n) = \int_a^b t^{n-1}\,\gamma(t)\,dt \qquad (483)$$

where $\gamma(t)$ and the limits are to be determined. Now

$$\Gamma(n+1) = \int_a^b t^n\,\gamma(t)\,dt. \qquad (484)$$

The term $n\Gamma(n)$ can be written as

$$n\Gamma(n) = \left[t^n \gamma(t) \right]_a^b - \int_a^b t^n \frac{d}{dt}\gamma(t)\, dt \tag{485}$$

by integrating by parts. Equation (482) is therefore equivalent to

$$\int_a^b t^n \left\{ \frac{d}{dt}\gamma(t) + \gamma(t) \right\} dt - \left[t^n \gamma(t) \right]_a^b = 0 \tag{486}$$

The integral can be disposed of by putting the integrand equal to zero for all t,

$$\frac{d\,\gamma(t)}{dt} + \gamma(t) = 0 \tag{487}$$

Therefore

$$\gamma(t) = C.e^{-t} \tag{488}$$

where C is an arbitrary constant. We can obtain the limits a, b and satisfy equation (482) by making $t^n\gamma(t)$ zero at the limits. We require a and b to be the roots of

$$t^n \gamma(t) = 0 \tag{489}$$

We may therefore take $a=0$ and $b=\infty$. This gives us

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \tag{490}$$

taking $C=1$. It is well known that the asymptotic expansion for $\Gamma(n)$ can be found from the integral representation by the Method of Steepest Descents[19]. Stokes' and Green-type asymptotic expansions can be found from integral representations for the well known functions[30,33]. It seems likely (though this has not been proved) that by using the Laplace method to find an integral representation for say $J_n(z)$ we could then use standard methods to find the Stokes' and Green type expansions. The main restriction on the method is that a difference equation with a polynomial coefficient of order S gives rise to an S'th order differential equation. We must be able to solve the differential equation for the method to be useful. Therefore, we can also only deal with difference equations with polynomial coefficients.

3) Continued Fractions[22,23]

The theory of continued fractions is a very powerful tool by no means restricted to finding solutions to difference equations. The development

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots \cfrac{a_n}{b_n + \cdots}}} \tag{491}$$

is known as a continued fraction. Rodgers[22] notation is

$$b_0 + \frac{a_1}{b_1 +}\;\frac{a_2}{b_2 +}\;\cdots\;\frac{a_n}{b_n +}\;\cdots \tag{492}$$

The solution to a difference equation can be developed as a continued fraction. Consider the equation for $J_n(z)$.

$$J_{n+1}(z) + J_{n-1}(z) = 2\frac{n}{z}J_n(z) \tag{493}$$

If we define

$$G_n(z) = J_n(z)/J_{n-1}(z) \tag{494}$$

we have

$$G_n(z) = \frac{1}{2\frac{n}{z} - G_{n+1}(z)} \tag{495}$$

and

$$J_n(z) = \exp \overset{n}{S} \ln . G_{n+1}(z)\, \Delta n \tag{496}$$

Equation (495) can be written as

$$G_n(z) = \frac{z/2}{n -} \; \frac{z/2}{(n+1) -} \; \frac{z/2}{(n+2) -} \cdots \frac{z/2}{(n+s) - z/2\, G_{n+1+s}(z)} \tag{497}$$

For small z and large n this development will converge very rapidly. By expanding out the continued fraction as a power series in and then calculating $J_n(z)$ the convergent expansion for $J_n(z)$ can be obtained. The second solution to (494) can be found by considering the continued fraction development of $G_n'(z) = \frac{J_n(z)}{J_{n+1}(z)}$.

An approximate value for the continued fraction can be obtained by carrying out the development to say the s'th term and then substituting an approximate value for $G_{n+1+s}(z)$ from some approximate formula for $J_n(z)$ obtained from say a W.K.B. formula or Stokes' expansion.

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