AFFINE KAC-MOODY SUPERALGEBRAS

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by
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Αφιερθμενο στους γονεις μου

χωρις την βοηθεια των οποιον δεν θα ειχε πραγματοποιηθει ποτε.

(Dedicated to my parents)

ABSTRACT

This thesis is a report of my research on the affine Kac-Moody superalgebras $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$ and $C^{(2)}(\ell+1)$. These are infinite dimensional complex Lie superalgebras and are canonically associated to irreducible non-reduced affine root systems. They were initially introduced by Kac(1978).

First the axiomatic foundation of irreducible affine root systems is summarised. Then, starting with a Cartan matrix corresponding to the class of irreducible non-reduced affine root systems, the above superalgebras are constructed at an abstract level in terms of "generators" and "relations".

The main interest lies in their explicit realisation which leads to the complete description of their root structure. This realisation is presented for all of the above superalgebras and is based on the finite dimensional basic simple classical complex Lie superalgebras $B(0/\ell)$, $A(2\ell-1/0)$, $A(2\ell/0)$ and $C(\ell+1)$. In particular, the determination of the root structure of $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$ and $C^{(2)}(\ell+1)$ involves certain automorphisms of the $A(2\ell-1/0)$, $A(2\ell/0)$ and $C(\ell+1)$ superalgebras. These automorphisms are derived and provide a neat way to determine the root structure.

Having achieved their realisation, a description of their highest weight representations is presented which facilitates the investigation of their relation with the Virasoro algebra.

This relation is demonstrated by performing the Sugawara construction. It is proved that these affine

superalgebras possess a semidirect sum structure with the Virasoro algebra. Of special interest in physical applications of these affine superalgebras might be the calculation of the values of the central charge of the Virasoro algebra that has been achieved.

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CHAPTER 1

INTRODUCTION

In recent years affine algebras and superalgebras have become a important field of research and a line of communication between mathematicians and particle physicists. The remarkable success and richness of the theory of affine Kac-Moody algebras and simple Lie superalgebras has initiated in the last few years the study of affine Kac-Moody superalgebras. Although their theory and their applications have not yet been investigated in the same depth as for the affine Lie algebras, one may say that they have an even richer mathematical structure which might allow for many interesting physical applications. There is, however, a class of affine Kac-Moody algebras whose structure and representations have been consistently developed and which constitute a natural generalization of affine Kac-Moody algebras. These are denoted by $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. A consistent exposition and investigation of their structure, representations and relation with the Virasoro algebras will be the central content of this thesis.

The development of the theory of affine Kac-Moody algebras was initiated by Kac and Moody independently in the late sixties (Kac(1968), Moody(1967,1968,1969)). The motivating idea was to generalise the definition of the Cartan matrix of finite dimensional semi-simple complex Lie algebras and then attempt to construct Lie algebras in terms of "generators" and "relations". Clearly this was a generalisation of the process that Serre(1966) followed to prove that all semi-simple complex Lie algebras can be obtained starting from a Cartan matrix, instead of reaching it as an end point (as was the case in Cartan's classification of semi-simple complex Lie algebras).

The result of these attempts was the discovery of two types of infinite dimensional Lie algebras which are known as affine and indefinite Lie algebras. Moreover all the semi-simple complex Lie algebras together with the affine and indefinite Lie algebras, can be obtained from the generalised Cartan matrices that they set up, and they constitute the unique set of algebras obtained in this way. They are known as Kac-Moody algebras and are special case of contragredient Lie algebras. In particular, the semi simple Lie algebras and the affine Kac-Moody algebras form the class of contragredient Lie algebras of finite growth.

The theory of affine Kac-Moody algebras has been considerably developed during the last twenty years. Their structure is very similar to that of semi-simple Lie algebras and many of the features of the latter algebras are encountered in the former. The most striking new features though are the infinite dimensionality and the concept of imaginary roots. A consistent description of affine Kac-Moody algebras can be found in Kac(1985) and Cornwell(1989).

A few years after the discovery of affine Lie algebras, the study of Lie superalgebras was initiated mainly for physical reasons. Their complete description first appeared in Kac's celebrated paper "Lie superalgebras" (1977). These are Z_2-graded vector spaces endowed with a generalised Lie product and generalised Jacobi identity. It is the class of (finite dimensional) basic classical simple Lie superalgebras that are of great importance both because of the remarkable resemblance of their structure with that of simple Lie algebras and their wide application in physics. Moreover they can be obtained from a particularly chosen Cartan matrix in terms of generators and relations. Kac, generalizing the concept of contragredient Lie algebras to the superalgebra case, proved that the basic simple Lie superalgebras constitute the class of finite dimensional simple contragredient Lie superalgebras of finite growth. For an extensive presentation of basic simple Lie superalgebras one may be referred to the above article or to Cornwell(1989), Scheunert(1979), Kac(1977b).

With this almost parallel development of the above two theories the obvious question to arise was whether one can obtain infinite dimensional Lie superalgebras of a similar type to the affine Kac-Moody algebras whose structures will be determined from basic classical simple Lie superalgebras.

In 1972 when the basic concepts of affine Kac-Moody algebras were still under development, Macdonald presented an axiomatic description and classification of irreducible affine root systems. He also generalised the Weyl denominator formula of finite reduced irreducible root systems (which are the root systems of semi-simple Lie algebras) to the case of affine irreducible reduced root systems. Although the concept of affine root systems was new, their Weyl groups, being the affine Weyl groups of finite root systems, had been known for a long time (see Bourbaki, "Groupes et algebre de Lie" ch. 4,5 et 6). It turned out that the new formula gave rise to multivariable identities associated to each one of the affine root systems, the simplest examples being that of Jacobi's triple product identity. Moreover they revealed the relation of affine irreducible reduced root systems with the famous Dedekind's η-function, $\eta(X)$. The above article not only initiated various applications in pure mathematics, in topics like modular forms, theta functions, etc., but also the study of three theories: the integrable highest weight representations of affine Kac-Moody algebras, the affine Kac-Moody superalgebras and their integrable highest weight representations.

With the real root systems of affine Kac-Moody algebras being the affine irreducible reduced root systems of Macdonald's classification, Kac(1974, 1978)) showed that Macdonald's formula was to be interpreted as the Weyl denominator formula of their trivial representations. In particular the concept of integrable highest weight representations was first introduced together with the construction of their character formula. It should be noted that the concept of imaginary roots was absent from Macdonald's description and in order to achieve the generalisation he aimed at, he had to introduce certain factors which in Kac's articles appeared naturally and which corresponded to the imaginary roots.

Together with the classification of irreducible reduced affine root system, Macdonald also classified the non-reduced irreducible affine root systems. In 1978 in an article by Kac entitled "Infinite Dimensional Algebras, Dedekind's η-Function, Classical Mobius Function and the Very Strange Formula", these root systems were canonically associated with four classes of infinite dimensional Lie superalgebras of finite growth which are denoted by $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. These names are not accidental. Their explicit realization is based on the basic classical simple complex Lie superalgebras $B(0/\ell)$, $A(2\ell-1/0)$, $A(2\ell/0)$, and $C(\ell+1)$ respectively. The set of $B^{(1)}(0/\ell)$ affine superalgebras are called <u>untwisted</u> because no non-trivial automorphism of $B(0/\ell)$ is needed in their realization. The rest of the sets are called <u>twisted</u> because there are certain non-trivial automorphisms of $A(2\ell-1/0)$, $A(2\ell/0)$, and $C(\ell+1)$ involved.

In the above article Kac established the abstract structure of these superalgebras and outlined a method for their explicit realization. One of the main objectives of this thesis is to apply this method and give an explicit realization of $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$.

The structure of all of the above affine superalgebras is exceedingly similar to that of the case of affine Kac-Moody algebras and most of the concepts and theorems of the latter transfer smoothly to the former. This of course does not imply that their superalgebraic nature plays a secondary role. The generalised Cartan matrices are again the starting point and are of affine type. These are the Cartan matrices obtained from the Dynkin diagrams of affine non-reduced irreducible root systems. Moreover Kac described their integrable highest weight representations and constructed the character formula for them. For the trivial representations this becomes the generalised Weyl denominator identity for the non-reduced irreducible affine root systems.

Although the development of affine Kac-Moody algebras began in a pure mathematical context it soon accelerated because of physical reasons and in particular because of the increasing interest in two dimensional conformal field theories. Affine Kac-Moody algebras (specially untwisted) and their integrable highest weight representations became one of the essential parts of string theories (for an extensive review see for example Green et al.(1988), Goddard et al. (1986), Lepowsky(1983)). They appear for example as the Lie algebra of currents of fields defined in two dimensions. Integrable highest weight representations of them are obtained for example via the vertex construction or the fermionic construction.

Although the affine Lie algebras have been used widely, this is not the case of the affine superalgebras. Only recently have they attracted the interest of mathematical physicists. Untwisted superalgebras are involved for example in the study of symplectic bosons (Goddard et al. (1987)) which themselves appear in constructing superconformal ghosts of fermionic string theories (see Friedan et al. (1986)). Nevertheless, because of the boson-fermion correspondence that they provide, the study of conformal field theories based on them looks quite promising. Obviously, in this context, one can go further in the more attractive idea of constructing conformal theories with twisted superalgebras following the example of twisted string models (see Nepomechie(1986)).

All of the above considerations are of little value if one has not established the relation of affine (super)algebras with the central ingredient of any conformal field theory, the Virasoro algebra. The Virasoro algebra arises naturally as the central extension of the infinite dimensional Lie algebra of the conformal group in two dimensions. The unitary irreducible highest weight representations of the Virasoro algebra have been studied extensively. These representations are labeled by the specific values of the central charge C_v and the eigenvalue h of the Virasoro operator L_0 (see chapter 6). The affine Kac-Moody algebras and the Virasoro algebras are related in a semi-direct sum algebraic structure, which is established via the Sugawara construction. The latter involves obtaining an appropriate expression of the Virasoro operators bilinear in operators of some representation of the affine algebra such that the Virasoro algebra will be satisfied. This process has an interesting consequence. It provides us with representations of the Virasoro algebra which are completely determined by those of the affine algebra. Thus whether or not a conformal field theory that incorporates an affine Lie algebra is physically meaningful depends on the representations of the affine algebras.

This brings us to an other main objective of this thesis, which is to establish the connection between the twisted Kac-Moody superalgebras $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and

$C^{(2)}(\ell+1)$ and the Virasoro algebra. The case of untwisted superalgebras has already been treated previously both from algebraic and field theoretical point of view but it will also be presented.

Following the historical development presented a while ago, in chapter two we refer first to the definition and some important properties of the generalised Cartan matrices whose complete theory can be found in Kac(1985). This presentation was necessitated by the fact that these matrices are the corner stone both of root systems and affine algebras. In particular, given an indecomposable symmetrisable affine Cartan matrix, via the uniqueness (up to isomorphism) of its realization we can obtain on one hand all irreducible affine root systems and on the other hand we can generate from it an affine Lie (super)algebra. In addition, a considerable part of this chapter is also devoted to the axiomatic foundation of affine irreducible root systems as was presented by Macdonald(1972), mainly because it was the classification of non-reduced irreducible root systems that led Kac to associate them with the affine superalgebras.

In chapter three we set up the abstract structure of the affine Kac-Moody superalgebras $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. We followed a more modern approach than the one presented by Kac in the original paper. Clearly it was the structure of the derived superalgebras of the above algebras that was given by Kac. Our apprcach is the same as that of affine Kac-Moody algebras that appears in Cornwell(1989) and Kac(1985). In fact the method presented is valid for any contragradient Lie (super)algebra.

Chapter four is devoted to the explicit realizations of $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. Certain special automorphisms have been used to obtain the root structure of $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. However, the process presented here is of general validity. It can be applied to any basic simple Lie superalgebra to obtain untwisted Kac-Moody superalgebras and, if the former possesses outer automorphisms, to obtain twisted Kac-Moody superalgebras. This is suggested from the works of Frappat et al. (1989), Serganova(1985) and Van der Leur(1985).

Chapter five is devoted to the presentation of integrable irreducible representations of $B^{(1)}(0/\ell)$ $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$. Again we slightly deviate from Kac's original exposition, where the subject was treated for the derived superalgebras of the above class.

Chapter six establishes a connection of the affine superalgebras with possible physical applications in that it demonstrate the relation of these superalgebras with the Virasoro algebra. This has been done with the use of Sugawara construction and the elements of the representation theory of chapter five. The Sugawara construction of untwisted superalgebras is not confined merely to the case of $B^{(1)}(0/\ell)$ but applies to any untwisted superalgebra based on a basic simple Lie superalgebra. This construction has appeared a number of times in the literature and it is treated here in less detail, but in a formulation that is more consistent with other developments. However much more detail will be given on the Sugawara construction based on the twisted superalgebras $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and

$C^{(2)}(\ell+1)$. Because of its originality and its complicated nature, this construction will be explicitly presented. Following the physical nomenclature we have distinguished two cases, namely the Ramond case and the Neveu-Schwarz case. Some interesting results concerning the values of the central charge of the Virasoro algebra seem to suggest that the expressions obtained are of a more general nature in that they incorporate the cases of the Sugawara construction for affine Kac-Moody algebras and untwisted Kac-Moody superalgebras.

Finally certain concluding remarks can be found in chapter seven. Certain tables with Dynkin diagrams and Cartan matrices of (super)algebras can be found at the end of this thesis.

As a result of this thesis the following articles have been published:

(a) "Supercharacters and superdimensions of the irreducible representations of $B(0/\ell)$ orthosymplectic simple Lie superalgebra"
I. Tsohantjis and J. F. Cornwell, International Journal of Theoretical Physics, **29**, 351(1989);

(b) "The complete root systems of the affine Kac-Moody superalgebras"
I. Tsohantjis and J. F. Cornwell, Journal of Mathematical Physics, **31**, 1817(1990);

(c) "Sugawara type constructions of the Virasoro algebra based on the twisted affine Kac-Moody superalgebras"
I. Tsohantjis and J. F. Cornwell, to appear in Journal of Mathematical Physics.

CHAPTER 2

GENERALIZED CARTAN MATRICES AND AFFINE ROOT SYSTEMS

2.1 Introduction

In this chapter the aim is to present the basic concepts of the axiomatic foundation of irreducible affine root system, as given by Macdonald(1972) and the generalised Cartan matrices as classified by Kac(1985). In the process we shall briefly summarize the irreducible reduced and non-reduced finite root systems which will help to develop the formulation of the affine ones.

2.2 Generalised Cartan Matrices A

Consider any square matrix **A** with entries in $\mathbb{C}$ whose rows and columns are labeled by an index set $I = \{0,\dots,n-1\}$.

Definition 2.1 Realization of **A**

A realization of a $n \times n$ matrix **A** of rank ℓ, is the set $\{\mathcal{H}, \Pi^{\vee}, \Pi\}$, where $\mathcal{H}$ is a complex vector space, $\Pi^{\vee}$ is a subset of $\mathcal{H}$ which consists of n elements H_{α_j} (for all $j \in I$) of $\mathcal{H}$, and Π is a set of n linear functional α_j (for all $j \in I$) of the dual space $\mathcal{H}^*$, defined on $\mathcal{H}$, such that

(a) $$\alpha_j(H_{\alpha_k}) = A_{kj} \qquad (2.1)$$

for all k, $j \in I$,

(b) the dimension of $\mathcal{H}$ is $\dim \mathcal{H} = 2n-\ell$,

(c) the elements H_{α_j} (for all $j \in I$) are linearly independent

(d) the elements α_j are linearly independent.

A is called symmetrizable if it can be written as a product of a symmetric matrix and a non-singular diagonal matrix and it is called indecomposable if it does not have the block form

$$\begin{pmatrix} \underset{\sim}{A}^{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}^{22} \end{pmatrix}, \qquad (2.2)$$

where $\underset{\sim}{A}^{11}$ and $\underset{\sim}{A}^{22}$ are non-trivial submatrices, nor can it be put in this form by any reordering of the index set I.

The requirement of A being symmetrizable is equivalent to the condition that for any sequence $i_1, i_2, ..., i_k$, such that $0 \le i_1,..., i_k \le n-1$ the following relation should hold:

$$A_{i_1 i_2}.......A_{i_k i_1}. = A_{i_2 i_1}........A_{i_1 i_k}. \qquad (2.3)$$

Note that when $\det\mathbf{A} = 0$ the elements H_{α_j} and α_j (for all $j \in I$) on their own, do not form base of $\mathcal{H}$ and $\mathcal{H}^*$ respectively. Clearly $n-\ell$ bases elements have to be added in each of the above spaces. When $\det\mathbf{A} \neq 0$ then $\dim \mathcal{H} = \dim \mathcal{H}^* = \ell$ and the elements H_{α_j} and α_j form bases of $\mathcal{H}$ and $\mathcal{H}^*$ respectively.

Two realizations $\{\mathcal{H}, \Pi^v, \Pi\}$ and $\{\mathcal{H}_1, \Pi^v{}_1, \Pi_1\}$ are isomorphic if there exists a vector space isomorphism ϕ such that $\phi(\mathcal{H}) = \mathcal{H}_1$, $\phi(\Pi^v) = \Pi^v{}_1$ and $\phi(\Pi) = \Pi_1$. If the matrix **A** is not indecomposable then its realization is a direct sum of realizations in the following sense. Let us assume that **A** can be put in the form (2.2) such that $\mathbf{A}^1$ and $\mathbf{A}^2$ are indecomposable. Then a realization of **A** is given by

$$\{ \mathcal{H}_1 \oplus \mathcal{H}_2, \Pi^{v}_1 \times \{0\} \cup \{0\} \times \Pi^{v}_2 , \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2 \},$$

and is called direct sum of the realizations $\{\mathcal{H}_1, \Pi^{v}_1, \Pi_1\}$ of $\mathbf{A}^1$ and $\{\mathcal{H}_2, \Pi^{v}_2, \Pi_2\}$ of $\mathbf{A}^2$.

Proposition 2.1

For every square matrix **A** there exists a unique (up to isomorphism) realization of **A**, and two such matrices **A** and **B** are said to be isomorphic if and only if one can be obtained from the other by a permutation of the index set I.

Proof (see Kac (1985))

Kac's classification on a particular set of matrices is given in the following theorem.

Proposition 2.2

Let **A** be a $n \times n$ indecomposable real matrix, with I its index set, such that its entries are subject to the following constraints

(i) $A_{jk} \leq 0$ for $j \neq k$ $j,k \in I$,

(ii) for $j \neq k$ ($j,k \in I$.) $A_{jk} = 0$ if and only if $A_{kj} = 0$

Let **u** and **v** be column vectors, and adopt the convention that $\mathbf{u} > 0$ means that all the $u_i > 0$ ($i \in I$) there being a similar convention for $\mathbf{u} < 0$. Then **A** satisfies one and only one of the following three possibilities at a time :

(a) there exists a vector **u** > 0 such that **Au** > 0.

If **v** is a vector such that **Av** ≥ 0 then **v** ≥ 0. Moreover det**A** ≠ 0 and all its principal minor are positive. These matrices are called generalised Cartan matrices of finite type.

(b) there exists a vector **u** > 0 such that **Au** = 0.

If **v** is a vector such that **Av** ≥ 0 then **Av** = 0. Moreover det**A** = 0 and all its proper principal minors are positive. These matrices are called generalised Cartan matrices of affine type.

(c) there exists a vector **u** > 0 such that **Au** < 0. If **v** is a vector such that **Av** > 0 and **v** ≥ 0 then **v** = 0. These matrices are called generalised Cartan matrices of indefinite type.

Proof (see Kac (1985))

The matrices of interest form a subset of those involved in the above proposition and are defined as follows.

Definition 2.2 Generalized Cartan matrix

Let **A** be a $n \times n$ matrix, with integer entries, rank ℓ, together with an index set $I = \{0,1,...,n-1\}$, which labels the rows and columns of **A** such that the following conditions are satisfied:

(i) $A_{jj} = 2$ for all $j \in I$

(ii) A_{jk} is zero or a negative integer for $j \neq k$ $j,k \in I$,

(iii) for $j \neq k$ $(j,k \in I)$ $A_{jk} = 0$ if and only if $A_{kj} = 0$ $(j,k \in I)$.

It is obvious that for the matrices **A** of the above definition, proposition 2.2 also applies. Then with the requirement that **A**, is symmetrizable and indecomposable, part (a) of proposition 2.2, provides us with the Cartan matrices of all the semi-simple complex Lie algebras together with the basic classical complex Lie superalgebra $B(0/\ell)$. Part (b) provides us with the Cartan matrices of all the affine Kac-Moody algebras. Finally part (c) provides us with the Cartan matrices of the indefinite Kac-Moody algebras.

Moreover as we shall see, there exists a class of affine Kac-Moody superalgebras which are associated with a generalized Cartan matrix of the type considered in definition 2.1 and fall under part (b) of proposition 2.2.

As is well known, a very useful way of visualizing all these cases is by associating a graph, called Dynkin diagram, to each one of the Cartan matrices corresponding to the (super)algebras just stated. It consists of a number of vertices equal to the dimension of Cartan matrix. Each vertex is associated with a simple root. Two vertices i,j are connected by lines if $A_{ij} \neq 0$. The construction of these diagrams is based in the following rules :

(i) To each $i \in I$ assign a vertex drawn as a circle

(ii) Draw L_{ij} lines from the vertex i to the vertex j where

$$L_{ij} = \max\{ |A_{ij}|, |A_{ji}| \}$$

(iii) Add an arrow from the vertex i to j if $|A_{ji}| > 1$

(iv) If $|A_{ij}A_{ji}| > 4$, draw a thick solid line. (2.4)

Given a Dynkin diagram we can construct, up to isomorphism, the Cartan matrix, making use of the above rules.

Before closing this subsection it is worth making some remarks on the Cartan matrices of the basic simple classical

Lie superalgebras other than $B(0/\ell)$ and their affine partners that have appeared in the literature (see for example Kac(1977), Serganova(1983), Frappat et al.(1987)). These are also associated with Dynkin diagrams which are constructed using the same rules as above.

Consider first the case of the basic simple classical Lie superalgebras other than $B(0/\ell)$. Their structure and classification can be found in Cornwell(1989), Kac(1977), Scheunert(1978). The first important remark to be made is that these superalgebras accept more than one, non-isomorphic Cartan matrices because they accept more than one non-equivalent (under the action of the Weyl group) system of simple roots. Although each of these matrices is indecomposable and symmetrisable and satisfies part (a) of proposition 2.2, they are not generalized Cartan matrices in the sence of the definition 2.2. They fail for example to satisfy condition (i) of this definition since they always possess at least one diagonal entry $A_{ii} = 0$.

Things are more complicated for the case of the affine partners of the above superalgebras, other than the ones that we examine in this thesis. Their Cartan matrices, nevertheless are indecomposable, symmetrisable, satisfy (b), and have det**A** = 0, however certain of their proper principal minors are not positive, and the requirement (a) of proposition 2.2 is not satisfied.

Let us now describe first the finite irreducible root systems. For a detailed account see Helgason (1978), Humphreys (1972) and N. Bourbaki Group et algebres de Lie ch.VI. (1968).

2.3 Finite irreducible root systems

Let E be a finite dimensional vector space over $\mathbb{R}$ equipped with a symmetric, positive definite non-degenerate bilinear form $< , >$. Let $\varepsilon \neq 0$ be any element of E. A reflection S_ε along ε, in E, is an invertible linear transformation such that $S_\varepsilon \varepsilon = -\varepsilon$ and its fixed point set (i.e. the set $\{ \varepsilon' \in E \mid S_\varepsilon \varepsilon' = \varepsilon' \}$) constitutes a hyperplane P_ε in E for which $P_\varepsilon = \{ \varepsilon' \in E \mid <\varepsilon', \varepsilon> = 0\}$. The action of S_ε on E is well defined by

$$(S_\varepsilon \varepsilon') = \varepsilon' - (2<\varepsilon, \varepsilon'>/<\varepsilon, \varepsilon>) \varepsilon \qquad (2.5)$$

and since any such transformation S_ε preserves the bilinear form in E, S_ε is said to be orthogonal (For a detailed exposition on this realization of S_ε see, for example, Bourbaki(1968) ch.V). A finite root system in E is defined as follows.

Definition 2.3 Finite Root system

A finite root system Δ in E is a finite set of non-zero vectors α of E which satisfy the following conditions

(i) Δ spans E

(ii) for each $\alpha \in \Delta$ there exists a reflection S_α along α defined as in (2.5) and leaving Δ invariant

(iii) the number $2<\alpha, \beta>/<\alpha, \alpha>$ $(\alpha, \beta \in \Delta)$ is an integer.

Two root systems Δ and Δ' defined in the vector spaces E and E' respectively, are said to be isomorphic if there exists a vector space isomorphism $E \rightarrow E'$ sending $\Delta \rightarrow \Delta'$.

Proposition 2.3

Let α and β be proportional roots, i.e. $\alpha = m\beta$ ($m \in \mathbb{R}$). Then m takes the values $\pm\frac{1}{2}$, ± 1, ± 2.

Proof (see Helgason (1978)).

A root $\alpha \in \Delta$ such that $\frac{1}{2}\alpha \notin \Delta$ is called indivisible.

Definition 2.4 Weyl group of Δ

The group generated by the reflections S_α for all $\alpha \in \Delta$ and leaves Δ invariant is called the Weyl group of Δ.

Definition 2.5 Reduced and non reduced finite root system

The subsets Δ_1 and Δ_2 of a root system Δ that are defined by

$$\Delta_1 = \{ \alpha \in \Delta \mid \alpha/2 \notin \Delta \} \qquad \Delta_2 = \{ \alpha \in \Delta \mid 2\alpha \notin \Delta \} \qquad (2.6)$$

are said to be reduced finite root systems in E. That is the only proportional roots in Δ_1 and Δ_2 are those for which $m = \pm 1$. If both of subsets Δ_1 and Δ_2 are proper, Δ is said to be non-reduced.

The following proposition embodies the most important properties of a finite root system.

Proposition 2.4

Let α, β be any roots of Δ.

(i) if α, β are linearly independent roots then

(a) $0 \leq 4\langle\beta, \alpha\rangle^2/\langle\alpha, \alpha\rangle\langle\beta, \beta\rangle \leq 3$

(b) if $2\langle\alpha, \beta\rangle/\langle\alpha, \alpha\rangle > 0$ then $\alpha - \beta \in \Delta$

(c) if $2\langle\alpha, \beta\rangle/\langle\alpha, \alpha\rangle < 0$ then $\alpha + \beta \in \Delta$.

(ii) if α, β are not proportional roots, then the set of roots of the form $\beta + k\alpha$ is in the α-string containing β for every integer k that satisfies the relation $-p \le k \le q$. That is $\beta+k\alpha$ is an arithmetic progression

$$\beta\text{-}p\alpha, \ldots, \beta-\alpha, \beta, \ldots, \beta+q\alpha.$$

Moreover, p and q are such that

$$p - q = 2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle$$

Proof (see Helgason (1978)).

Definition 2.6 Basis of Δ

A subset Π of Δ which is such that

(i) the elements of Δ form a basis of E, and

(ii) each $\alpha \in \Delta$ can be written as a linear combination of elements from Π with the coefficients all positive or all negative integers is said to form a basis of Δ.

Consider the subset Π of the above definition and denote its elements by α_i ,for all i =1,2,...,ℓ where ℓ is the dimension of E

Proposition 2.5

(a) Each root system has a basis Π, any two bases Π, Π' are conjugate under a unique element from the Weyl group of Δ and the integer $2\langle\alpha_i, \alpha_j\rangle/\langle\alpha_i, \alpha_i\rangle$ is non-positive for all α_i, α_j that belong to Π.

(b) The Weyl group is generated by reflections relative to the simple roots.

Proof (see Helgason (1978)).

Definition 2.7 Irreducible finite root system

A root system Δ is called irreducible if it can not be decomposed in two disjoint non empty orthogonal subsets with respect to the form $<\ ,\ >$ on V.

It can be proved that any root system decomposes uniquely as the union of irreducible root systems. Its basis elements also decompose in corresponding mutually orthogonal subsets. Then the vector space E, in which the root system is defined, accepts a direct sum decomposition of mutually orthogonal subspaces too.

Let us briefly comment on the classification of finite irreducible root systems.

Let $\Pi = \{\alpha_1, \alpha_2,..., \alpha_\ell\}$ be a basis of a root system Δ which might be irreducible or not. Then the integers $2<\alpha_i,\alpha_j>/<\alpha_i, \alpha_i>$ for all $\alpha_i, \alpha_j \in \Pi$ are the entries of a matrix **A** which is called the Cartan matrix of the root system Δ. From proposition 2.4 and 2.5 we deduce that the only possible values of its non-diagonal entries are 0, -1, -2, -3. It can be easily checked that **A** is a generalised Cartan matrix of finite type. If Δ is irreducible then **A** is indecomposable. If Δ is reducible then **A** accepts a decomposition as a direct sum of indecomposable submatrices of **A** which are the Cartan matrices of the irreducible root systems. Let Δ be irreducible.

Then it can be shown (c.f. Humphreys(1972)) that the Weyl group acts irreducibly on the vector space E (if this was not the case then E would be a direct sum of two mutually orthogonal non-empty subspaces which would be invariant under the action of W).

It should be noted that the Cartan matrix depends on the ordering of the basis Π but this does not create any complications since W acts transitively on the set of bases and thus the Cartan matrix is independent of Π. It can be shown that given two bases $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ and $\Pi' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_\ell\}$, if $<\alpha_i, \alpha_j> = <\alpha'_i, \alpha'_j>$, then a bijection $\alpha_i \to \alpha'_i$, extends uniquely to an isomorphism $E \to E'$ mapping $\Delta \to \Delta'$. This together with proposition 2.4 shows that the Cartan matrix determines Δ completely. Thus classifying all the irreducible indecomposable finite type Cartan matrices is equivalent to classifying the irreducible finite root systems and then using the rules of the previous section we can construct their Dynkin diagrams.

Another equivalent method is by using connected Coxeter graph (see Humphreys(1972), N. Bourbaki Group(1968)). This is defined to consists of n vertices (n being to equal the number of simple roots) such that the ith vertex is connected with the jth by $4<\alpha_i,\alpha_j><\alpha_j,\alpha_i>/<\alpha_i, \alpha_i> <\alpha_j, \alpha_j>$ lines, for all α_i, $\alpha_j \in \Pi$. Then from these graphs it is possible to obtain the Dynkin diagrams of all irreducible root systems and thus the systems themselves with their Cartan matrices. In fact the underlying theory of this method is related to the Weyl group. It can be proved that the Weyl group of irreducible reduced finite root systems is an irreducible Coxeter group of finite order. These are groups generated by reflections in an Euclidean space and might be of finite or infinite order. They are defined as follows .

Definition 2.8 Coxeter group

The group generated by a finite set of elements S_i ($i \in I = \{1, \ldots, n\}$) such that

$$(S_i)^2 = 1 \text{ and } (S_i S_j)^{m_{ij}} = 1 \tag{2.7}$$

where m_{ij} are positive integers or ∞ is called a Coxeter group.

These groups are associated with what we called above Coxeter graph. They have been classified by Coxeter(1934).

The Dynkin diagrams and the root systems of all the finite irreducible reduced root systems are listed in table I together with a system of simple roots for each one of them.

There is only one finite irreducible non-reduced root system which does not correspond to any complex simple Lie algebra. From the known structure of the reduced root systems it is now easy to determine this non-reduced one. Let Δ be non-reduced. Take the subsets Δ_1 and Δ_2 of Δ which are defined in (2.6). Since Δ_1 , Δ_2 and Δ have the same Weyl group, from table I we conclude that Δ_1, Δ_2 should be B_ℓ or C_ℓ and this is the only case. Thus we have only one non-reduced system, denoted as BC_ℓ which is given in table II together with a basis which is that of B_ℓ (since the simple roots should be indivisible). Clearly the Cartan matrix of BC_ℓ is also of finite type. From the known structure of the basic classical Lie superalgebra $B(0/\ell)$(see Cornwell(1989)) we can easily identify its root system with BC_ℓ .

2.4 Affine irreducible root systems

Let us now describe the affine irreducible reduced and non reduced root systems. For a detailed exposition see Macdonald(1972) and also Bourbaki(1968) ch.V, VI. Whenever necessary we will recall some notions on affine spaces (c.f. Mac Lane and Birkhoff ch. XII).

<u>Definition 2.9</u> Affine spaces.

Let V be a finite dimensional vector space over a field K. Then an affine space E over K is an non-empty set whose elements are called points, on which the vector space V acts, the action being described by a function $V \times E \to E$ defined as $(v, p) \to v+p \in E$ for any vector v of V and any point p of P such that the following conditions are satisfied:

(i) for any vectors v, v' of V and any point p of E

$0 + p = p, \qquad (v + v') + p = v + (v' + p)$

(ii) for any two points p and q of E there exists one and only one vector v of V such that

$v + q = p$

(iii) the dimension of E is the dimension of V.

From the definition of the action of of V on E it can be easily deduced that the elements of V translate the points of E. Thus V is called the space of translations of E. The symbol + denotes both the action of V on E and the usual sum of two vectors. The dimension of E is the dimension of V. Given any finite dimensional vector space over K we can construct an affine space E by regarding any vector of V as a point in E and V as the space of translations. From the definition we deduce that the difference of two points is a vector and thus there exists a map $E \times E \rightarrow V$. Given a point p_0 of E, the map $p \rightarrow p - p_0$ is a bijection of E on to the translation space V. We can identify E with V using this map by considering p_0 to be the origin in E.

An important characteristic of E is that given a list of points $p_0, p_1, \ldots, p_\ell$ of E any other point can be uniquely written as

$$p = \sum_{i=1}^{\ell} k_i p_i + p_0 \quad \text{where} \quad \sum_{i=1}^{\ell} k_i = 1 \qquad (2.8)$$

where k_i are scalars. In particular this set of points constitute a frame in E if we chose p_0 as an origin in E and if the vectors $p_i - p_0$ form a basis of the translation space V of E (see Mac Lane and Birkhoff(1978)). By definition every linear transformation from an affine space E to an affine space E' (over the same field as E) that preserves relations (2.8) is called an affine transformation. For example every translation is an affine transformation.

Proposition 2.6

Let E and E' be two affine spaces over K and V, V' their corresponding vector spaces of translations over K. Then to each affine transformation $f: E \rightarrow E'$ there exist a unique linear transformation $(Df): V \rightarrow V'$ such that

$$f(p + v) = (Df)(v) + f(p) \qquad (2.9)$$

for all $v \in V$ and all $p \in E$. Also to each linear transformation $(Df): V \rightarrow V'$ and two points $p_0 \in E$ and $p_0' \in E'$, there exists exactly one affine transformation f: $E \to E'$ with $f(p_0) = f(p'_0)$ defined by

$$f(p_0 + v) = (Df)(v) + f(p'_0) \tag{2.10}$$

for all $v \in V$.

Proof (see Mac Lane and Birkhoff).

Following Macdonald (1972), we call f affine linear and Df derivative of f. In the special case where f is the map f: $E \to K$ (here K is assumed to be an affine space too) we say that f is an affine linear function defined on E if and only if there exists a linear form Df: $V \to K$ such that the above theorem is satisfied. Taking an origin p_0 in E and identify E with V, the above theorem implies that every affine linear function f is such that any point p: $p \to \lambda + (Df)(p)$ where $f(p_0) = \lambda \in K$. Then the set F of affine linear function f is a vector space over K whose dimension is dim E +1.

By definition, the dual space V^* of a vector space V over K is $V^* = Hom_K(V,K)$ that is, it consists of all linear forms $\omega: V \to K$. Then D is a linear map from F to V^* and its kernel is the subset of F of all constant affine linear functions f (i.e. functions such that $f(p + v) - f(p) = 0$ for all p of E and all v of V). Note that Df does not mean $D \circ f$.

From now on we shall proceed assuming that $K = \mathbb{R}$ and that the vector space V is of dimension ℓ, and is equipted with a symmetric positive definite non-degenerate bilinear form $<\ ,\ >$. The lenght of a vector v is given as usual by $|v| = <v,v>^{\frac{1}{2}}$ Then E is an Euclidean space and for any two points p and q of E we denote by $|p-q|$ the distance function on E. By Riesz representation theorem (2.9) takes the form

$$f(p + v) = < Df, v > + f(p) \tag{2.11}$$

and Df is called the gradient of f. If f is a constant function then Df = 0 and f is called isotropic. We identify V and V* in terms of the the bilinear form of V.

We can define a bilinear form $F \times F \rightarrow \mathbb{R}$ on F by

$$< f, f' > = < Df, Df' > \tag{2.12}$$

This is a positive symmetric bilinear form because the left hand side of (2.12) is the bilinear form on V* induced by the one defined on V (note that $Df \in V^*$). If f is a constant function then <f, f> = 0, so the bilinear form is positive semi definite.

Definition 2.10 Affine hyperplane

The set P_f of points p of E which satisfy the condition $f(p) = k$ ($k \in \mathbb{R}$), where f is a non-isotropic affine linear function f, constitute an affine hyperplane P_f in E. That is,

$$P_f = \{ p \in E \mid f(p) = k \}.$$

Let us briefly recall some constructions that appear in E because of the existence of hyperplanes which we will encounter later.

Let P be a locally finite ensemble of hyperplanes of E. For any two points p and q of E the equivalence relation "For every hyperplane P_f of P either p and q belong to P_f or p and q are contained in the same open subspace of E limited by P_f", partitions E in classes of equivalence. We call these classes facettes relative to P. Obviously the set of facettes is locally finite too. Consider a facette and a point p of it. A necessary and sufficient condition for a hyperplane to contain this facette is that p has to belong in this hyperplane. It can be shown that the number of hyperplanes from the set P, in which this point belongs is finite. They have as intersection an affine subspace of E. We call this subspace the affine support of the facette. Then (see Bourbaki(1968), ch.V, p.58) the facette is an open convex set of its support. Any facette C which is not contained in any of the hyperplanes of P is called a chamber relative to the set P. We call face of a chamber C every facette which is contained in the closure $\bar{C}$ of the chamber C and whose support is a hyperplane in E. Then every hyperplane which is a the support of a face is called a wall of the chamber C. It is clear that every wall of a chamber C belongs in the set P and is the support of one an only one face of C. It can also be shown that every hyperplane from the set P is the wall of at least one chamber C. We can demonstrate some of the above notions with an example that will be useful in what will follow. We can define affine linear functions $f_0, f_1, \ldots, f_\ell$ on E which assign a real number k_i ($i = 0, 1, \ldots, \ell$) to each point p of E . For each of the affine functions define affine hyperplanes in E, $P_0, P_1, \ldots, P_\ell$, by $f_i(p) = 0$. The set of points of E for which $f_i(p) > 0$ for all $i = 0, 1, \ldots, \ell$ is called open simplex and constitutes a chamber C relative to the set P of hyperplanes $P_0, P_1, \ldots, P_\ell$. Its closure $\bar{C}$, is the set of points such that $f_i(p) \geq 0$ for all $i = 0, 1, \ldots, \ell$. Finally these hyperplanes are the walls of C.

Consider now linear invertible transformation $w: E \to E$. These transformations are often called affine isometries or rigid motions of the Euclidean space E. From proposition (2.9) we deduce that for all points p of E and all vectors v of the translation space V

$$w(p + v) = w(p) + w^*(v) \qquad (2.13)$$

where w^* is the unique linear transformation $V \to V$ associated with w. In addition w^* preserves the bilinear form defined on the translation space V of the Euclidean space E and w preserves the distances in E.

Consider the set P of hyperplanes defined by $P_f = \{ p \in E \mid f(p) = 0 \}$ for all non-constant affine linear functions f. We are interested in those of the affine linear transformations that leave invariant the hyperplanes P_f and are involutive. Such transformations will be denoted by S_f and are called orthogonal reflections with respect to the hyperplane P_f. Their action on any point p of E is defined by

$$S_f(p) = p - 2\{f(p)/\langle f, f\rangle\}Df \qquad (2.14)$$

where $\langle \, , \, \rangle$ is as defined in (2.12).
Obviously this is involutive and if $p \in P_f$ then $S_f(p) = p$. By transposition S_f acts on any f' of F as

$$S_f(f') = f' - 2\{\langle f', f\rangle/\langle f, f\rangle\}f \qquad (2.15)$$

With the definition (2.14) the set consisting of S_f (for all non-constant f) together with the identity reflection forms a group which will be denoted by W.

From the definition of an affine transformation (2.11) it is clear that to each such affine transformation S_f there should corresponds a unique linear transformation $V \rightarrow V$, the translation space of E (which in the general case of (2.13) was denoted by w^*) which preserves the inner product in V. We call this the derivative of S_f and denote it by DS_f. Then from (2.9), (2.11), (2.13) and (2.14) we obtain that

$$(DS_f)(v) = v - 2\{\langle v, Df\rangle / \langle Df, Df\rangle\}Df. \quad (2.16)$$

Using (2.16) we can easily obtain that $(DS_f)^2(v) = v$ and that $\langle (DS_f)(v), (DS_f)(v')\rangle = \langle v, v'\rangle$. Direct observation of (2.5) shows that

$$DS_f = S_{Df} \quad (2.17)$$

where $Df \in V$. Thus S_f induces an orthogonal reflection in V with respect to a hyperplane in V which consists of those vectors v of V such that $\langle Df, v\rangle = 0$.

Consider a set P of hyperplanes P_f (f being non-constant) defined as before in the Euclidean space E and the group W consisting of reflections S_f (for all f such that P_f belongs in P) such that the following conditions are satisfied:

(i) for every S of W and every P_f of P, $S(P_f)$ belongs in P;

(ii) W having the topology of a discrete group, acts properly on E.

It can be shown that P is locally finite (see Bourbaki ch.V, p.72 (1968)), and thus all constructions encountered a while ago (facettes, chambers, etc.) can be applied.

Consider a chamber C of the Euclidean space which is defined relative to those hyperplanes satisfying the conditions (i), (ii) above. Then the following proposition is very important in the foundation of affine root systems that will follow.

Proposition 2.7

(a) For every p of E, there exists an element S of W such that S(p) belongs in $\bar{C}$,

(b) for any chamber C' of E there exists an element S of W such that S(C') = C,

(c) W is generated by a set of orthogonal reflections relative to the walls of C.

Proof (see Bourbaki ch.V, p.73-74 (1968))

We are now in a position to give the definition of an affine root system on E.

Definition 2.11 Affine root system

Let Δ^{af} be the subsetof the set F of affine linear functions which satisfies the following conditions:

(i) Δ^{af} spans F and the elements of Δ^{af} are non-isotropic with respect to the form (2.12),

(ii) the reflections S_f (for all $f \in \Delta^{af}$) defined by (2.14) leave Δ^{af} invariant,

(iii) the quantity $2\langle f, f'\rangle / \langle f', f'\rangle$ is an integer for all f, f' of Δ^{af},

(iv) the group W^{af} generated by S_f (for all f of Δ^{af}) given as in (2.14-15), (as a discrete group) acts properly on E,

Then Δ^{af} is called an affine root system on E.

From now on we shall denote the members of Δ^{af} by greek letters α, β,..., etc. We call W the affine Weyl group of Δ^{af}. As in the case of finite root systems if $k\alpha$ is an affine root proportional to the affine root α then $k = \pm\frac{1}{2}, \pm 1, \pm 2$. The definitions of reduced and non-reduced affine root systems are exactly the same as for the finite ones. The rank of Δ^{af} is defined to be the dimension of E. If Δ^{af} and $\Delta^{af'}$ are two affine root systems defined on E and E' respectively, then an isomorphism of Δ^{af} onto $\Delta^{af'}$ is a bijection of Δ^{af} onto $\Delta^{af'}$ induced by an affine linear isometry of E onto E'. We call direct sum of affine root systems the affine root system which is the union of a finite number of mutually orthogonal(with respect to (2.12)) affine root systems i.e. $\Delta^{af} = \cup_i \Delta_i^{af}$. An affine root system D^{af} is said to be similar with an affine root system Δ^{af} if D^{af} is isomorphic to the direct sum $\cup_i k_i\Delta_i^{af}$, where $\Delta^{af} = \cup_i \Delta_i^{af}$ and k_i are non-zero real numbers. As in the case of finite root systems, every affine root system is expressible as the direct sum of a finite family of irreducible affine root system. This decomposition is unique to within isomorpism. We call dual affine root system Δ^{af*} the one obtained from Δ^{af} by substituting each root α of Δ^{af} by $2\alpha/<\alpha, \alpha>$.

For each affine root α let P be the set of hyperplanes P_α in E defined by

$$P_\alpha = \{ p \in E \mid \alpha(p) = 0 \}.$$

It is clear from the definition of the affine root system that all such hyperplanes satisfy conditions (i) and (ii) mentioned above and thus P is locally finite. Then all the constructions mentioned above (i.e. facettes, chambers,e.t.c.) can be demonstrated. In particular the chambers of Δ^{af} relative to P are defined as follows.

Definition 2.12 Chambers of Δ^{af}.

Consider the set E - $\cup_\alpha P_\alpha$. It is open in E and since E is locally connected the connected components of this set are also open. These connected components are the chambers of the root system Δ^{af} relative to the hyperplanes P_α.

It is not difficult to see that proposition 2.7 directly apply to the chambers of the root system. In particular, all the chamber of the affine root system are W-equivalent. Moreover It can be shown that the Weyl group of Δ^{af} acts faithfully and transitively on the set of chambers (see Macdonald(1972) and N. Bourbaki(1968), p.74 theorem1).

Assume from now on that Δ^{af} is irreducible, choose a chamber C once and for all and points in C, $p_0, p_1, ..., p_\ell$ (ℓ= dim E), called vertices, such that every other point in C is written as $p = \sum_{i=0}^{\ell} k_i p_i$ with $\sum_{i=0}^{\ell} k_i = 1$ and all $k_i > 0$ (obviously this is a property of any affine space). Consider now the set of indivisible affine roots α (i.e. such that $\frac{1}{2}\alpha \notin \Delta^{af}$) with the properties that

(i) $\alpha(p) > 0$ for all p of C,

(ii) P_α is a wall of C.

Then the following proposition provides a basis for Δ^{af}.

Proposition 2.8

(a) The set consisting of the indivisible affine roots with the properties (i), (ii) above is a basis for the irreducible affine root system Δ^{af} and consists of $\ell+1$ elements $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ which are called simple affine roots. It is also a basis of the vector space F.

(b) Each affine root α is written as a linear combination of the basis elements with integer coefficients which are all positive or all negative. In the first case α is called positive and in the second case negative.

Proof (see Macdonald (1972))

From proposition 2.7 (c) the following proposition is straightforward

Proposition 2.9

The Weyl group of Δ^{af} is generated by reflections S_{α_j} for all j=0, 1,..., ℓ, that is, reflections relative the walls of C.

Up till now nothing has been said about the relation of the the affine root systems with the finite ones that we saw in the previous section. The next proposition reveals their connection.

Let F_i (for each i=0, 1,..., ℓ) be the set of affine linear functionals from F that vanish at a vertex p_i of the chamber C of Δ^{af} and denote with Δ_i^{af} the subset of Δ^{af} which contains those roots that vanish at p_i. Also let W_i be the subgroup of W which fixes p_i.

Proposition 2.10

(a) the set Δ_i^{af} forms a finite root system in F_i which is reduced if Δ^{af} is reduced.

(b) Subtracting the simple root α_i of Δ^{af} from the set of the simple roots of Δ^{af} we get a basis of this finite root system.

(c) The Weyl group is a subgroup of the Weyl group of Δ^{af}

Proof (see Macdonald (1972))

With the use of above theorem Macdonald achieved one way of classifying all the irreducible reduced affine root systems in terms of the known finite irreducible root systems and their affine Weyl group. One can notice that if we associate a Dynkin diagram with the irreducible affine reduced root system, (using the same rules as in the finite root system case) the above proposition implies that removing any vertex from it the remaining diagram should be that of a finite reduced system. Now although, it has not been explicitly stated above, the Weyl group of the affine irreducible reduced root systems as constructed by Macdonald, is an infinite order irreducible group generated by reflection in the affine Euclidean space E. All such groups have been classified and found to correspond to the affine Weyl groups of irreducible finite root systems. Moreover they are irreducible infinite order Coxeter groups. Macdonald achieved the classification by obtaining the Dynkin diagrams of the irreducible affine reduced root systems from the Coxeter graphs that are associated with the affine Weyl groups.

A more explicit construction and classification of the affine irreducible reduced and non-reduced root systems that Macdonald also achieved was based on the notion of the gradient of the affine root system.

Definition 2.13 Gradient of an affine root system

The set $\Delta = D\Delta^{af} = \{ D\alpha$; for all α of $\Delta^{af} \}$ is called the gradient of the affine root system Δ^{af}.

Proposition 2.11

Δ is a finite root system in V, the translation space of E. If Δ^{af} is irreducible Δ is too. The map D: $S \rightarrow DS$, for all $S \in W$, is a homomorphism of the affine Weyl group to the Weyl group of the finite root system. The kernel of this map is the group of translation which is a subgroup of the affine Weyl group.

Proof (see Macdonald (1972))

Note that if Δ^{af} is reduced then Δ can be either reduced or non-reduced.

Definition 2.14 Special point for Δ^{af}

A point p of E is called special point for Δ^{af} if there exists affine roots vanishing at p, whose gradients form a basis of Δ.

Proposition 2.12

(a) There exists a special point for Δ^{af} which is also a special point for its Weyl group.

(b) If C is a chamber of Δ^{af} then there exists a vertex of C which is special point for Δ^{af}.

(c) Let Δ^{af} be irreducible. Fix a chamber C and a basis $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell \}$ corresponding to C. There exists a simple

affine root α_i, such that the gradients of the elements from Π - $\{\alpha_i\}$ form a basis of the finite root system $\Delta = D\Delta^{af}$.
Proof (see Macdonald (1972)).

Note that if we consider the finite root system Δ_i^{af} of proposition 2.14(c), obtained by taking the affine roots that vanish at a point p_i (not necessarily a special one) of the chamber C, and consider its gradient $\Delta_i = D\Delta_i^{af}$, then Δ_i is a subsystem of the finite root system $\Delta = D\Delta^{af}$ and the gradient map D: $\Delta_i^{af} \rightarrow \Delta_i$ is an isomorphism of finite root system. Then we can prove the following.

<u>Proposition 2.13</u>

Assume that Δ^{af} is reduced and irreducible and a vertex p_i of C is a special point of Δ^{af}. Let Δ_i be the set of gradients of the affine roots which vanish on p_i. Then Δ_i is the set of indivisible roots of the finite root system $\Delta = D\Delta^{af}$.

Let p_i be a special point for an irreducible affine root system Δ^{af} and $D\alpha_i$ (i=0, 1,...,ℓ) be the gradients of the simple roots of Δ^{af}. Then by proposition 2.12 , the elements $D\alpha_j$ (for $j \neq i$) form a basis of a finite irreducible root system. Since $< D\alpha_i , D\alpha_j > \leq 0$ with $j \neq i$, $<-D\alpha_i , D\alpha_j > \geq 0$. Thus $-D\alpha_i$ is a positive root of the finite root system Δ. Thus $-D\alpha_i$ can be written as

$$-D\alpha_i = \sum_{j \neq i} k_j \, D\alpha_j \qquad (2.18)$$

where the coefficients k_j are positive integers and $D\alpha_j$ are the basis of Δ. Consequently we can write (2.18) as

$$\sum_{j=0}^{\ell} k_j \, D\alpha_j = 0 \qquad (2.19)$$

where $k_i = 1$ and k_j (for $j \neq i$) are as above. Now we can state a very important proposition.

Proposition 2.14

Let p_i be a special point of an irreducible reduced affine root system Δ^{af} . Then there exists a constant, positive on the chamber C, affine linear function g defined on E which is given by

$$\gamma = \sum_{j=0}^{\ell} k_j \alpha_j \qquad (2.20)$$

where k_j are positive integers such that for j=i k_j =1. Every other constant function belonging to the lattice generated by the simple roots of Δ^{af} is an integral multiple of γ.

Proof (see Macdonald (1972)).

Note that from the definition of an affine root system it is obvious that γ does not belong in Δ^{af}.

Proposition 2.15

For each affine root α, let α_+ be the unique affine root such that $f_\alpha = \alpha_+ - \alpha$ is constant , positive and as small as possible.

(a) If $\alpha \in \Delta^{af}$ and $k \in \mathbb{R}$, then $\alpha + k \in \Delta^{af}$ if and only if k is an integral multiple of f_α .

(b) If $\alpha \in \Delta^{af}$ and $S \in W^{af}$, then $f_{S(\alpha)} = f_\alpha$.

(c) If Δ^{af} is reduced, $\alpha \in \Delta^{af}$ and $2\alpha + k \in \Delta^{af}$ for some $k \in \mathbb{R}$, then $k = m f_\alpha$ where m is an odd integer.

(d) f_α is a positive integral multiple of γ.

Proof (see Macdonald (1972)).

All of the above analysis makes obvious the direct connection of finite and affine root systems. We shall now describe explicitly the irreducible reduced affine root systems.

2.5 Classification of irreducible reduced affine root systems

Proposition 2.16

Let Δ be a reduced or non-reduced finite root system in a finite dimensional real vector space V equipped with a symmetric non-degenerate positive definite bilinear form $< , >$. Let E be the Euclidean space whose space of translations is V. For each $\alpha \in \Delta$ and each $j \in \mathbf{Z}$ the set of affine linear functions defined on E of the form

$$f(p) = j + <\alpha, p> \tag{2.21}$$

where

$$j \in \mathbf{Z} \text{ if } \tfrac{1}{2}\alpha \notin \Delta \quad \text{or} \quad j \in 2\mathbf{Z}+1 \text{ if } \tfrac{1}{2}\alpha \in \Delta. \tag{2.22}$$

is a reduced affine root system.

Proof (see Macdonald (1972)).

Clearly the above proposition together with the analysis on gradient root systems gives the classification of all irreducible reduced affine root systems since we know the irreducible reduced and nonreduced finite ones. We just have to choose a point p_i of the chamber C and consider it as an origin in E. Then the vector space of translations V is identified with E by means of the identification of a point a p of E with the vector $p-p_i$ in V. In this way an affine linear function f on E is identify with a linear functional defined on V and sending every vector to $v \to f(p_i) + \langle Df, v \rangle \in \mathbb{R}$. Then we can write the affine linear function as $f(p_i) + Df$. If $f(p_i) = 0$, the affine linear function is identified with the linear functional Df on V. Applying this method to affine root systems, it is easily seen that Df would belong to the finite root system which is the gradient of the affine one.

All the reduced irreducible affine root systems are listed in table III together with their Dynkin diagrams and a system of simple roots.

We can deduce a Cartan matrix **A** for the irreducible reduced affine root system by

$$A_{ij} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle = 2\langle D\alpha_i, D\alpha_j \rangle / \langle D\alpha_i, D\alpha_i \rangle \quad (2.23)$$

for all $i,j = 0,1,\ldots, \ell$ (the left hand side being a consequence of (2.12)), where α_i are the simple roots.

Clearly **A** is indecomposable since if it was not the Dynkin diagram would be disconnected i.e. the root system would not be irreducible. It is symmetrisable as it can be easily deduced from the left hand side of (2.23). Also, from the left hand side notice that $A_{ii} = 2$ since α_i are non-constant functions and the form <,> is positive definite. Making use of proposition 2.12(a) and (c) , A_{ij} is a negative integer for all i, j = 0,1,...,ℓ. and $A_{ij} = 0$ implies that $A_{ji} = 0$ since the matrix is symmetrizable. Consider the function given by (2.19) and take the expressions $2\langle\gamma, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$ for all i = 0,1,..., ℓ. Then from (2.12) and (2.23) we obtain

$$\mathbf{Ak} = 0 \quad \text{and} \quad \mathbf{k} > 0$$

where **k** is a column vector with entries $k_j > 0$. Thus the matrix **A** of the affine irreducible reduced root system is a Cartan matrix of affine type.

In view of the known classification of the irreducible reduced affine root systems it is a straightforward matter to enumerate all the non-reduced irreducible ones. We shall use the tool of the gradient of the affine root systems.

Let Δ^{af} be an irreducible non-reduced affine root system. Let Δ_1^{af} be the irreducible reduced root system which consists of roots α of Δ^{af} such that $\frac{1}{2}\alpha \notin \Delta^{af}$ and let Δ_2^{af} be the irreducible reduced root subsystem which consists of roots α of Δ^{af} such that $2\alpha \notin \Delta^{af}$. Note that Δ_1^{af}, Δ_2^{af} and Δ^{af} have the same Weyl group. Consider now the gradient root systems $\Delta_1 = D\Delta_1^{af}$, $\Delta_2 = D\Delta_2^{af}$ and $\Delta = D\Delta^{af}$. Since Δ^{af} is non-reduced according to proposition Δ is non reduced either (but it is still irreducible). From table II we can identify Δ with BC_ℓ since it is the only irreducible finite non-reduced root system. Similarly, for Δ_1 and Δ_2 defined by (2.6) respectively, direct observation of table I shows that $\Delta_1 = B_\ell$ and $\Delta_2 = C_\ell$. Then from table III we can easily deduce that Δ_1^{af} should be one of $A_{2\ell}^{(2)}$, $B_\ell^{(1)}$ or $D_{\ell+1}^{(2)}$ and Δ_2^{af} should be one of $A_{2\ell}^{(2)}$, $A_{2\ell-1}^{(2)}$ or

$C_\ell^{(1)}$.

The final step makes use of the fact that Δ_1^{af}, Δ_2^{af} must have the same Weyl group. This occurs only for the pairs Δ_1^{af} - Δ_2^{af} given by

(a) $A_{2\ell}^{(2)}$ - $C_\ell^{(1)}$ ($\ell \geq 1$), (b) $D_{\ell+1}^{(2)}$ - $A_{2\ell}^{(2)}$ ($\ell \geq 1$),

(c) $B_\ell^{(1)}$ - $A_{2\ell-1}^{(2)}$ ($\ell \geq 3$) and (d) $D_{\ell+1}^{(2)}$ - $C_\ell^{(1)}$ ($\ell \geq 1$).

We can assign a Cartan matrix to each class in the usual way and the corresponding Dynkin diagrams are obtained using the same rules as in the finite case. With the same method as in the reduced case we can see that the Cartan matrix of these root systems is of affine type too.

Up to similarity, these root systems are listed in table IV with their Dynkin diagrams. The black nodes in the diagrams of table IV denote that there is a non-simple root which is twice the simple one corresponding to that node. These black nodes will be identified later as the odd simple roots of the superalgebras. Note that proposition 2.12 (c) is valid for these non-reduced root systems too. Consequently if we remove one of the vertices of the Dynkin diagram then the remaining part is a Dynkin diagram corresponding to a non-reduced irreducible or a reduced irreducible or a direct sum of a reduced irreducible and a non-reduced irreducible finite root system.

Their respective names in terms of the superalgebras with which they will be identified, are $B^{(1)}(0/\ell)$, $A^{(4)}(0/2\ell)$, $A^{(2)}(0/2\ell-1)$ and $C^{(2)}(\ell+1)$ (the correspondence being from (a) to (d)). In the next two chapters we shall see how this can be done.

CHAPTER 3

ABSTRACT STRUCTURE OF AFFINE KAC-MOODY SUPERALGEBRAS

3.1 Introduction

In this chapter the aim is to set up at an abstract level a complex affine Kac-Moody superalgebra, whose structure will be determined solely from a particularly chosen generalized Cartan matrix **A** of affine type and its unique, up to isomorphism, realization. It should be noted that the method that will be demonstrated in section 3.2 of this chapter is closely related to that of Kac-Moody algebras (see Kac (1985), Cornwell (1989)). We shall concentrate only on these affine superalgebras that appear in Kac (1978) although many of their properties apply to other contragradient Lie superalgebras, affine or simple finite dimensional.

3.2 Abstract construction.

Let **A** be a $(\ell+1)\times(\ell+1)$ an indecomposable matrix, with entries in **Z**, rank ℓ, together with an index set $I = \{0,1,\ldots,\ell\}$, which labels the rows and columns of **A** and a non-empty subset τ of I (which may not necessarily be a proper one) such that the following conditions are satisfied:

(i) $A_{jj} = 2$ for all $j \in I$ (3.1)

(ii) $A_{jk} \leq 0$ for $j \neq k$ $j,k \in I$ (3.2)

(iii) if $j \in \tau$ then A_{jk} is a non-positive even integer (3.3)

(iv) for $j \neq k$ $(j,k \in I)$ $A_{jk} = 0$ if and only if $A_{kj} = 0$ (3.4)

In what follows it will always be assumed that **A** is symmetrizable, i.e. can be written as a product of a symmetric matrix and a non-singular diagonal matrix as follows

$$\mathbf{A} = \mathbf{DB} \quad \text{and} \quad A_{jk} = \varepsilon_j B_{jk} \quad \text{for all } j,k \in I. \tag{3.5}$$

where **D** is the diagonal matrix with entries ε_j and **B** is the symmetric matrix with entries B_{jk}. Finally we set $\det\mathbf{A} = 0$ and demand that every principal minor of **A** is positive. Thus we take **A** to be of affine type.

With all of the above assumptions it can be easily checked that **A** is the Cartan matrix for the irreducible non-reduced affine root systems of chapter 2. In table V, all these Cartan matrices and their corresponding Dynkin diagrams are presented. The entries of these Cartan matrices have been determined relative to the enumeration of the vertices of the Dynkin diagrams as indicated. The integers above the vertices are the entries of the unique, up to a constant factor, vector **v** such that $\mathbf{Av} = \mathbf{0}$.

According to the definition 2.1 of chapter 2, a realization of **A** is a complex vector space $\mathcal{H}$ of dimension $\ell+2$, together with a set of linearly independent elements

H_{α_j} (for all $j \in I$) and a set of linearly independent elements α_j (for all $j \in I$) of the dual space $\mathcal{H}^*$ such that

$$\alpha_k(H_{\alpha_j}) = A_{jk}. \tag{3.6}$$

We denote by Q the lattice in $\mathcal{H}^*$ defined by elements of the form $\alpha = \sum_{i \in I} k_i \alpha_i$ where k_i are integers and let Q_+ denote the subset of Q consisting of elements α such that k_i are positive integers. We call <u>height</u> of a linear functional $\alpha \in Q$, and denoted by $\underline{ht\alpha}$, the integer $\underline{ht\alpha} = \sum_{i \in I} k_i$.

We shall associate now with this Cartan matrix a Lie superalgebra whose Cartan subalgebra will be $\mathcal{H}$ and whose simple roots will be identified with the linear functionals α_j (for all $j \in I$).

Consider first an auxiliary complex Lie superalgebra $\tilde{\mathcal{L}}_s'$ whose set of generators are given by the basis elements of $\mathcal{H}$ and the $2(\ell+1)$ elements E_{α_j}, $E_{-\alpha_j}$ (for all $j \in I$). The defining relations of $\tilde{\mathcal{L}}_s'$ are as follows:

$$[E_{\alpha_j}, E_{-\alpha_k}] = \delta_{jk} H_{\alpha_j} \qquad \text{(for } j, k \in I) \tag{3.7}$$

$$[h, E_{\alpha_k}] = \alpha_k(h) E_{\alpha_k} \qquad \text{(for all } k \in I) \tag{3.8}$$

$$[h, E_{-\alpha_k}] = -\alpha_k(h) E_{-\alpha_k} \qquad \text{(for all } k \in I) \tag{3.9}$$

$$[h, h'] = 0 \qquad \text{(for all } h, h' \in \mathcal{H}). \tag{3.10}$$

The $\mathbf{Z}_2$ grading is defined by

$$\deg h = 0 \text{ (for all } h \in \mathcal{H}), \tag{3.11}$$

$$\deg E_{\alpha_k} = \deg E_{-\alpha_k} = 0 \text{ for all } k \in I \backslash \tau \tag{3.12}$$

$$\deg E_{\alpha_k} = \deg E_{-\alpha_k} = 1 \text{ for all } k \in \tau. \tag{3.13}$$

It should be noted that with $h = H_{\alpha_j}$, (3.8) and (3.9) reduce, to

$$[H_{\alpha_j}, E_{\alpha_k}] = A_{jk} E_{\alpha_k} \quad (3.14)$$

$$[H_{\alpha_j}, E_{-\alpha_k}] = -A_{jk} E_{-\alpha_k}. \quad (3.15)$$

Here and in what follows [,] denotes the commutator or the anticommutator as is appropriate. $\tilde{\mathcal{L}}'_s$ contains also all the generalized Lie products of the form:

$$[E_{\alpha_k}, E_{\alpha_{k'}}],\ [E_{\alpha_k}, [E_{\alpha_{k'}}, E_{\alpha_{k''}}]], \text{ and so on,} \quad (3.16)$$

together with those of the form:

$$[E_{-\alpha_k}, E_{-\alpha_{k'}}],\ [E_{-\alpha_k}, [E_{-\alpha_{k'}}, E_{-\alpha_{k''}}]], \text{ etc..} \quad (3.17)$$

all subject to the generalized Jacobi identity. We denote by $\mathcal{L}'_-$ and $\mathcal{L}'_+$ the subsuperalgebras of $\tilde{\mathcal{L}}'_s$ generated by E_{α_k} and E_{α_k} (for all $k \in I$) respectively.

The first thing that we have to establish is that $\tilde{\mathcal{L}}'_s$ has a decomposition of the form

$$\tilde{\mathcal{L}}'_s = \mathcal{L}'_- \oplus \mathcal{H} \oplus \mathcal{L}'_+ \quad (3.18)$$

and the spaces $\mathcal{L}'_-$, $\mathcal{L}'_+$ are freely generated by the elements $E_{-\alpha_k}$ and E_{α_k} (for all $k \in I$) respectively.

To achieve this we have to define a graded representation of $\tilde{\mathcal{L}}'_s$. Let T(V) be the tensor superalgebra over the $\mathbf{Z}_2$-graded complex vector space V (see Scheunert (1979)), whose basis elements are denoted by v_k (for all $k \in I$). Clearly T(V) is by construction a $\mathbf{Z}$-graded associative superalgebra of the form

$$T(V) = \Sigma_{j \geq 0} \oplus T^j(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots \quad (3.19)$$

where

$T^0(V) = \mathbb{C}$, $T^1(V) = V$, $T^2(V) = V \otimes V$, etc, (3.20)

with a consistent Z_2 grading inherited from V. Define an action of the generators of $\tilde{\mathcal{L}}_s'$ on T(V) as follows :

$$E_{-\alpha_k}(v) = v_k \otimes v \text{ for any } v \in T(V) \text{ and all } k \in I \quad (3.21)$$

$$h(I) = \lambda(h) \quad (3.22)$$

$$h(v_k \otimes v) = -a_k(h)E_{-\alpha_k} \otimes v + v_k \otimes h(v), \text{ for } v \in T^{j-1}(V) \quad (3.23)$$

for all $k \in I$, $h \in \mathcal{H}$ and where λ is a linear functional on $\mathcal{H}$,

$$E_{\alpha_k}(I) = 0 \quad (3.24)$$

$$E_{\alpha_k}(v_k \otimes v) = \delta_{jk} H_{\alpha_j}(v) + (-1)^{\deg E_{\alpha_k} \deg v_k} v_k \otimes E_{\alpha_k}(v) \quad (3.25)$$

for $v \in T^{j-1}(V)$, for all $j, k \in I$ and I is the unit in T(V).

Now it can be easily proved, that this action provides a representation of $\tilde{\mathcal{L}}_s'$ on T(V). This can be done by checking that relations (3.7) to (3.10) are satisfied. Moreover any product of elements E_{α_k}, $E_{-\alpha_k}$ and h lies in $\mathcal{L}_-' + \mathcal{H} + \mathcal{L}_+'$.

Consider an element a of $\tilde{\mathcal{L}}_s'$ of the form $a = \ell_- + h + \ell_+$ where ℓ_-, h, ℓ_+ are elements of the $\mathcal{L}_-'$, $\mathcal{H}$, $\mathcal{L}_+'$ subspaces respectively. Assume that a =0. Then in the representation defined above $a(I) = \ell_+(I) + \lambda(h) I + \ell_-(I) = \lambda(h)I + \Phi(\ell_-)I = 0$. The structure of T(V) implies that $\lambda(h)I = 0$ and $\Phi(\ell_-)I = 0$. Since $\lambda(h)I = 0$ should hold for every $\lambda \in \mathcal{H}^*$ we deduce that h = 0. Now notice that under the map $E_{-\alpha_k} \rightarrow v_k$ the associative superalgebra T(V) is isomorphic with the universal enveloping algebra $U(\mathcal{L}_-')$ of $\mathcal{L}_-'$ and the map $a_- \rightarrow a_-(I)$ for all elements a_- of $\mathcal{L}_-'$ is the canonical even linear mapping of $\mathcal{L}_-'$ into $U(\mathcal{L}_-')$. Consequential if $\ell_-(I) = 0$ then $\ell_- = 0$

and thus $\ell_+ = 0$ too. Relation (3.18) is satisfied.

Since $U(\mathcal{L}'_-)$ is the universal enveloping superalgebra of $\mathcal{L}'_-$ by Poincare-Birkhoff-Witt theorem $\mathcal{L}'_-$ is freely generated by $E_{-\alpha_k}$. Next let us define the map

$$\tilde{\Phi}(E_{-\alpha_k}) = - E_{\alpha_k} \text{ for all } k \in I$$

$$\tilde{\Phi}(E_{\alpha_k}) = -(-1)^{\deg E_{\alpha_k}} E_{-\alpha_k} \text{ for all } k \in I \qquad (3.26)$$

$$\tilde{\Phi}(h) = -h \text{ for all } h \in \mathcal{H}$$

It can be easily checked that the map $\tilde{\Phi}$ can be uniquely extended to become a graded automorphism of $\tilde{\mathcal{L}}'_s$ of order 4. Using this automorphism observe that $\mathcal{L}'_+$ is also freely generated by E_{α_k} for all $k \in I$.

Now by relations (3.8), (3.9) each of the generalized products in (3.16) and (3.17) are eigenspaces of adh with eigenvalues,

$$\alpha(h) = \Sigma_{k \in I} \; \kappa_k^{\alpha} \alpha_k(h) \qquad (3.27)$$

where κ_k^{α} are all negative or all positive integers and $|\kappa_k^{\alpha}|$ is the number of times that the element $E_{-\alpha_k}$ or E_{α_k} appears in the commutators. That is, α belongs in the root lattice Q. We shall denote the subspace corresponding to the linear functional α by $\mathcal{L}'_{\alpha}$. Moreover since $\mathcal{L}'_+$ and $\mathcal{L}'_-$ are spanned by elements of the form (3.16) and (3.17) respectively we obtain that

$$\mathcal{L}'_+ = \Sigma_{\alpha \in Q_+} \oplus \mathcal{L}'_{\alpha} \qquad (3.28)$$

$$\mathcal{L}'_- = \Sigma_{\alpha \in Q_+} \oplus \mathcal{L}'_{-\alpha} \qquad (3.29)$$

and $\tilde{\mathcal{L}}'_s = (\Sigma_{\alpha \in Q_+} \oplus \mathcal{L}'_{\alpha}) \oplus \mathcal{H} \oplus (\Sigma_{\alpha \in Q_+} \oplus \mathcal{L}'_{-\alpha})$ (3.30) where $\alpha \neq 0$. From relations (3.27) and (3.28) we deduce that for all $k \in I$ the generators E_{α_k} and $E_{-\alpha_k}$ are members of $\tilde{\mathcal{L}}'_{\alpha_k}$ and $\tilde{\mathcal{L}}'_{-\alpha_k}$ respectively, $\dim \tilde{\mathcal{L}}'_{\alpha_k} = \dim \tilde{\mathcal{L}}'_{-\alpha_k} = 1$. By making the obvious estimate that $\dim \mathcal{L}_\alpha \leq (\dim \mathbf{A})^{\mathrm{ht}\alpha}$, it follows $\dim \mathcal{L}_\alpha < \infty$.

Any ideal I of $\tilde{\mathcal{L}}'_s$ has the form

$$I = \sum_{\alpha \in Q} \oplus \left(\mathcal{L}'_\alpha \cap I \right) \qquad (3.31)$$

and it is obviously $\mathbf{Z}_2$ graded. The sum of all ideals that intersect $\mathcal{H}$ trivially is the unique maximal ideal R that intersects $\mathcal{H}$ trivially and it can be easily checked that it possesses the decomposition :

$$R = \left(\mathcal{L}'_- \cap R \right) \oplus \left(\mathcal{L}'_+ \cap R \right) \qquad (3.32)$$

We define the complex Kac-Moody superalgebra $\tilde{\mathcal{L}}_s$ based on the affine Cartan matrix $\mathbf{A}$, to be the factor algebra

$$\tilde{\mathcal{L}}_s = \tilde{\mathcal{L}}'_s / R \qquad (3.33)$$

Consequently $\tilde{\mathcal{L}}_s$ has no non-trivial ideals with trivial intersection with $\mathcal{H}$. We retain the same notation for the elements E_{α_k} , $E_{-\alpha_k}$ and h_{α_k} (for all $k \in I$), under the natural homomorphism of $\tilde{\mathcal{L}}_s$ onto $\tilde{\mathcal{L}}'_s/R$. The commutative subalgebra $\mathcal{H}$ of $\tilde{\mathcal{L}}_s$ is still referred to as its Cartan subalgebra. Also the set of elements a_α of $\tilde{\mathcal{L}}'_s$ that have the property

$$[\, h , a_\alpha \,] = \alpha(h) a_\alpha \qquad (3.34)$$

for all $h \in \mathcal{H}$ is again said to form the root subspace $\tilde{\mathcal{L}}'_{s\alpha}$ corresponding to the root α. The set of all non-zero roots of $\tilde{\mathcal{L}}_s$ will be denoted by Δ, the subset of positive roots by Δ_+, and the subset of negative roots by Δ_-. Clearly $\Delta_+ = \Delta \cap Q_+$, $\Delta_- = \Delta \cap -Q_+$. and $\Delta = \Delta_+ \cup \Delta_-$. The set of linear functionals α_j

for all $j \in I$, are said to be the simple roots of $\tilde{\mathcal{L}}_s$. If the root subspace $\tilde{\mathcal{L}}_{s\alpha}$ belongs to the odd (even) part of $\tilde{\mathcal{L}}_s$ then α is said to be odd (even) root. We shall denote by Δ^0 and Δ^1 the set of even and odd roots of $\tilde{\mathcal{L}}_s$. Also relation (3.30) takes the form

$$\tilde{\mathcal{L}}_s = (\Sigma_{\alpha \in \Delta_+} \oplus \mathcal{L}_\alpha) \oplus \mathcal{H} \oplus (\Sigma_{\alpha \in \Delta_+} \oplus \mathcal{L}_{-\alpha}). \qquad (3.35)$$

Finally the map $\tilde{\Phi}$ defined by (3.26) induces a Cartan automorphism ϕ of $\tilde{\mathcal{L}}_s$, of order 4, defined as in (3.26). This maps the root subspaces corresponding to positive roots to those corresponding to negative roots and thus, If $\alpha \in \Delta_+$ then $-\alpha \in \Delta_-$ and vice versa, and $\Delta_+ = - \Delta_-$ .We call the generators $E_{-\alpha_k}$, E_{α_k} and H_{α_k} (for all $k \in I$) Chevalley generators. In the same way as for the Kac-Moody algebras we can prove the following.

Proposition 3.1

Let a belong to $\mathcal{L}_+$ be such that $[a , E_{-\alpha_k}] = 0$ for all $k \in I$. Then $a = 0$. Also if a belongs to $\mathcal{L}_-$ and satisfies the relation $[a , E_{\alpha_k}] = 0$, for all $k \in I$, then $a = 0$.

Proposition 3.2

With the generalized Cartan matrix defined as above, the following relations hold in $\tilde{\mathcal{L}}_s$:

$$(\mathrm{ad}\ E_{\alpha_j})^{(1-A_{jk})} E_{\alpha_k} = 0 \qquad (3.36)$$

and

$$(\mathrm{ad}\ E_{\alpha_j})^{(1-A_{jk})} E_{-\alpha_k} = 0 \qquad (3.37)$$

Proof

To prove that (3.36) and (3.37) hold, it is sufficient to show that

$$(\mathrm{ad}E_{-\alpha_i})(\mathrm{ad}\, E_{\alpha_j})^{(1-A_{jk})}\, E_{\alpha_k} = 0 \qquad (3.38)$$

$$(\mathrm{ad}E_{\alpha_i})(\mathrm{ad}\, E_{-\alpha_j})^{(1-A_{jk})}\, E_{-\alpha_k} = 0 \qquad (3.39)$$

for all i, j, $k \in I$ and $j \neq k$, and then make use of proposition 3.1. Relations (3.38) and (3.39) can be shown to be true with the use of the generalized Jacobi identity and relations (3.7-15). Moreover if we prove (3.38) then (3.39) is obtained by using the Cartan automorphism ϕ.

From relations (3.36) and (3.37) and using the Leibniz formula $D^m[x,y] = \sum^{n}_{m=0} \binom{n}{m} [D^m x, D^{n-m} y]$ where D is $(\mathrm{ad}E_{\pm\alpha_k})$(for $k \in I \backslash \tau$) or $(\mathrm{ad}E_{\pm\alpha_k})^2$ (for $k \in \tau$) we can find that $(\mathrm{ad}E_{\pm\alpha_k})(k \in I)$ are <u>locally nilpotent</u> on $\tilde{\mathcal{L}}_s$.

As in the case of affine Kac- Moody algebras, the following proposition holds.

<u>Proposition 3.3</u>

The set of elements h of $\mathcal{H}$ such that

$$\alpha_k(h) = 0 \qquad \text{for all } k \in I \qquad (3.40)$$

form the centre $\mathbf{C}$ of the Lie superalgebra $\tilde{\mathcal{L}}_s$ and dim $\mathbf{C}$ =1. Any $h \in \mathbf{C}$ is given by

$$h = \sum_{i \in I} n_i H_{\alpha_i} \quad \text{with} \quad \sum_{i \in I} A_{ij} n_i = 0 \qquad (3.41)$$

where n_i being real numbers.

The existence of the centre is a direct consequence that the generalized Cartan matrix is of affine type (see chapter 2 section 2.2). This can be shown by the second of the relations (3.41), if we write it as $\mathbf{An} = 0$ where $\mathbf{n}$ is a $(\ell+1) \times 1$ vector with entries n_i. Then because $\mathbf{A}$ is of affine type, there exists $\mathbf{n} > 0$ such that $\mathbf{An} = \mathbf{0}$. With appropriate scaling n_i can be taken to be positive integers.

We must make now the following important remark. Since $\mathbf{A}$ is of affine type, $\mathcal{H}$ can have the decomposition

$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'' \tag{3.42}$$

where $\mathcal{H}'$ is the set of all linear combinations of H_{α_j} (for all $j \in I$) and $\mathcal{H}''$ is a complementary subspace with dimension one. It can be easily seen that the subspace of $\tilde{\mathfrak{L}}_s$ generated by the Chevalley generators, satisfying relations (3.7, 3.14-15) and (3.34-35), together with all their commutators is a subsuperalgebra of $\tilde{\mathfrak{L}}_s$ which differs with $\tilde{\mathfrak{L}}_s$ in that it does not contain the subspace $\mathcal{H}''$. This indicates that this subsuperalgebra is the <u>derived</u> superalgebra , $[\tilde{\mathfrak{L}}_s , \tilde{\mathfrak{L}}_s]$, of $\tilde{\mathfrak{L}}_s$. In Kac's original paper, both the abstract form and the explicit realization of the Lie superalgebras presented there concerned the derived superalgebra . Clearly the situation is the same as in the construction of Kac-Moody algebras that appeared in the original papers of Kac and Moody. Later Kac(1985) demonstrated a general method of abstract construction for the Kac-Moody algebras which includes the subspace $\mathcal{H}''$ straight from the beginning.

This method with slight modifications to fit the superalgebra case is adopted throughout this thesis. Clearly this additional subspace guarantees the non-degeneracy of the bilinear form on $\mathcal{H}$ and $\mathcal{H}^*$ as we shall see bellow.

3.3 The supersymmetric bilinear form

The next step is to define a supersymmetric invariant non-degenerate bilinear form on $\tilde{\mathcal{L}}_s$. That is, a bilinear form that satisfies the following conditions

(i) $B(a, b) = (-1)^{\deg a \deg b} B(b,a)$ (i.e supersymetric)

(ii) $B(a, [b, c]) = B([a, b], c)$ (i.e invariant)

(iii) there does not exists an $a \in \tilde{\mathcal{L}}_s$ with $a \neq 0$, such that $B(a, b) = 0$ for all $b \in \tilde{\mathcal{L}}_s$.

Due to the fact that $\tilde{\mathcal{L}}_s$ is allowed to be infinite dimensional the process of constructing the desired form should be carried out in stages. First let us state a theorem due to Kac (1978) that guarantees the existence of a supersymmetric invariant bilinear form on the derived superalgebra $[\tilde{\mathcal{L}}_s, \tilde{\mathcal{L}}_s]$, and is strictly related to the properties of the Cartan matrix **A**.

Proposition 3.4

Let **A** be an indecomposable generalized Cartan matrix. If **A** is symmetrisable then the derived contragradient Lie superalgebra with this Cartan matrix has a unique up to a constant factor bilinear supersymmetric invariant form such that

$$B(H_{\alpha_j} , H_{\alpha_j}) > 0 \quad \text{for all } j \in I \quad \text{and} \tag{3.43}$$

$$B(\mathcal{L}_\alpha , \mathcal{L}_\beta) = 0 \text{ if } \alpha \neq - \beta \text{ for any roots } \alpha , \beta \text{ of } \tilde{\mathcal{L}}_s. \tag{3.44}$$

Proof (see Kac (1968,1978))

Note that Kac's theorem does not imply non-degeneracy of the bilinear form.

Clearly this theorem does not state anything about how the bilinear form should be defined on the elements of the complementary subspace $\mathcal{H}''$. Thus first we have to define B(,) consistently on the whole of the Cartan subalgebra $\mathcal{H}$ and then extending it to the whole of $\tilde{\mathcal{L}}_s$ such that the above theorem would still be true. The process is identical with that of affine Kac-Moody algebras.

We define B(,) on $\mathcal{H}$ by

$$B(h , H_{\alpha_j}) = \alpha_j(h)\varepsilon_j \quad \text{for all } j\in I \quad \text{and all } h \in \mathcal{H} \qquad (3.45)$$

$$B(h , h') = 0 \quad \text{and all } h, h' \in \mathcal{H}'' \qquad (3.46)$$

$$B(h , H_{\alpha_j}) = B(H_{\alpha_j} , h) \quad \text{for all } j\in I \quad \text{and all } h \in \mathcal{H}'' \qquad (3.47)$$

where ε_j are the non zero diagonal elements of the matrix **D** (see 3.5) which can be taken to be real and positive. In particular from relations $\alpha_j(H_{\alpha_k}) = A_{kj}$, (3.5) and (3.47) for $h = H_{\alpha_k}$ it is found that

$$B(H_{\alpha_k} , H_{\alpha_j}) = B_{kj}\, \varepsilon_k\, \varepsilon_j \qquad (\text{for all } j, k\in I) \qquad (3.48)$$

where B_{kj} are the entries of the symmetric matrix **B** of (3.5).

<u>Proposition 3.5</u>

The bilinear form defined by (3.45) to (3.48) is non-degenerate on $\mathcal{H}$.

Proof (see Cornwell (1989))

This proposition allow us to define for each linear functional α on $\mathcal{H}$ a unique up to a constant factor element h_α of $\mathcal{H}$ by

$$B(h_\alpha , h) = \alpha(h) \tag{3.49}$$

and thus for any h_α, h_β we have

$$h_\alpha + h_\beta = h_{\alpha+\beta} \tag{3.50}$$

This bilinear form induces a symmetric bilinear form on $\mathcal{H}^*$ defined by

$$< \alpha , \beta > = B(h_\alpha , h_\beta)$$

and
$$\alpha(h_\beta) = \beta(h_\alpha) = < \beta , \alpha >. \tag{3.51}$$

Then by (3.45) and (3.49) we can define "Weyl" type generators h_{α_k} of $\mathcal{H}'$, as

$$h_{\alpha_k} = \varepsilon_k^{-1} H_{\alpha_k} \tag{3.52}$$

Also, it is not difficult to show that

$$< \alpha_k , \alpha_j > = B_{kj} = \varepsilon_j^{-1} A_{jk} , \quad < \alpha_j , \alpha_j > = 2/ \varepsilon_j \tag{3.53}$$

and thus

$$A_{jk} = (2<\alpha_j, \alpha_k>)/<\alpha_j, \alpha_j> \qquad \text{and} \tag{3.54}$$

$$H_{\alpha_k} = 2<\alpha_k, \alpha_k>^{-1} h_{\alpha_k} \tag{3.55}$$

Proposition 3.6

The $\mathbb{C}$-valued bilinear form defined by (3.45) to (3.48) is the unique, up to a constant multiplicative factor, consistent invariant supersymetric non-degenerate bilinear form on $\tilde{\mathcal{L}}_s$ and such that (3.43) and (3.44) are satisfied and so for any root vector a_α of $\tilde{\mathcal{L}}_s$

$$[\, a_\alpha \,,\, a_{-\alpha} \,] = B(\, a_\alpha \,,\, a_{-\alpha}\,)\, h_\alpha. \tag{3.56}$$

Proof This can be proved by checking each of the assumptions of the theorem, inductively, on $\tilde{\mathcal{L}}_s$, viewed as a Z-graded Lie superalgebra. (See last section of this chapter for the Z-grading of $\tilde{\mathcal{L}}_s$). Clearly the proof is the same as the one appearing for Kac-Moody algebras in Kac(1985) , the only difference here is that we demand that B(,) has to be supersymmetric.

3.4 The Weyl group

Consider now any linear functional β defined on $\mathcal{H}$ and define the linear transformations S_{α_j} (for all $j \in I$) on the elements of $\mathcal{H}^*$ as

$$(S_{\alpha_j}\, \beta)(h) = \beta(h) - \beta(H_{\alpha_j})\, \alpha_j(h) = \beta(h) - \frac{2<\alpha_j\, \beta>}{<\alpha_j\, \alpha_j>}\, \alpha_j(h) \tag{3.57}$$

for all $h \in \mathcal{H}$. Clearly the S_{α_j} are reflections on $\mathcal{H}^*$ relative to the simple roots $\alpha_j(h)$, the fixed point set of each S_{α_j} being the set $\{\, \lambda \in \mathcal{H}^* \mid <\lambda, \alpha_j> = 0 \,\}$ and $(S_{\alpha_j}\, \alpha_j)(h) = -\alpha_j(h)$. These reflections are called <u>fundamental</u>. If $\beta = \alpha_k(h)$ then

$$(S_{\alpha_j}\, \alpha_k)(h) = \alpha_k(h) - A_{kj}\, \alpha_j(h) \quad \text{for all } h \in H \text{ and } j \in I. \tag{3.58}$$

The set generated by the identity operator, the $\ell + 1$ fundamental reflections S_{α_j} and all products of the fundamental reflections S_{α_j} forms a group and is called the

Weyl group W of the affine Kac-Moody superalgebra $\tilde{\mathcal{L}}_s$.

Using the definition (3.57) it can be easily checked that the form < , >, induced on $\mathcal{H}^*$ from the bilinear form on $\mathcal{H}$ is invariant under the action of the Weyl group.

For any $\alpha \in \Delta$ such that $<\alpha, \alpha> > 0$, we define an operator S_α acting on any linear functional β of $\mathcal{H}^*$ as

$$(S_\alpha \beta)(h) = \beta(h) - 2<\alpha, \beta><\alpha, \alpha>^{-1} \alpha(h) \qquad (3.59)$$

and has the obvious properties

(i) $(S_\alpha \alpha)(h) = - \alpha(h)$,

(ii) $S_\alpha (S_\alpha \beta) = \beta$ for any linear functional β on $\mathcal{H}$,

(iii) $< S_\alpha \beta , S_\alpha \gamma > = < \beta, \gamma >$ for any β, γ defined on $\mathcal{H}$,

(iv) for any two linear functional β, γ on $\mathcal{H}$ and any two complex numbers λ and μ

$S_\alpha (\lambda\beta + \mu\gamma) = \lambda(S_\alpha \beta) + \mu(S_\alpha \gamma)$.

Clearly S_α defined above is a reflection relative to the root α.

As we shall see in the next section all the roots α that satisfy the condition $<\alpha, \alpha> > 0$ are the roots with the property that there exist an element S of the Weyl group and a simple root α_k such that $\alpha = S \alpha_j$. Then this last relation and (3.57), imply that $S_\alpha = S S_{\alpha_j} S^{-1}$. Thus S_α being a product of fundamental reflections lies in W.

The structure of the Weyl group will be better established, when we shall examine the explicit realization of the affine Kac-Moody superalgebras in the next chapter.

Finally, recall the definition of the Coxeter group in the previous chapter. With the fundamental reflections defined as in (3.57) and the Cartan matrices given in table V, we can verify (c.f. Kac (1985), proposition 3.13) that the Weyl group of $\tilde{\mathcal{L}}_s$ is an infinite order Coxeter group (see definition 2.8 chapter 2). Moreover for any pair of fundamental reflections $(S_{\alpha_j}, S_{\alpha_k})$ ($j \neq k, \in I$) the order m_{jk} of the product $S_{\alpha_j} S_{\alpha_k}$ is related to the entries of the Cartan matrix as follows:

if $A_{jk} A_{kj} = 0, 1, 2, 3$ or ≥ 4, then $m_{jk} = 2, 3, 4, 6$ or ∞

respectively, the convention being that $(S_{\alpha_j} S_{\alpha_k})^{\infty} = 1$.

It should be noted that whenever $m_{jk} = \infty$, this implies that $(S_{\alpha_j} S_{\alpha_k})$ is a translation (see of N. Bourbaki(1968) § 3, n° 4, chapter V). For example from the Cartan matrix of $B^{(1)}(0/1)$, in table V, $(S_{\alpha_0} S_{\alpha_1})$ is a translation as will become apparent in chapter 4.

3.5 The root system of $\tilde{\mathcal{L}}_s$.

Let us now investigate some fundamental properties of root system of the affine superalgebras of table V.

Proposition 3.7

If α_j is an odd simple root of $\tilde{\mathcal{L}}_s$ then $2\alpha_j$ is also a root of $\tilde{\mathcal{L}}_s$. Moreover $3\alpha_j$ is not a root of $\tilde{\mathcal{L}}_s$.

Proof One has to notice that

$[E_{-\alpha_j}, [E_{\alpha_j}, E_{\alpha_j}]] = 2\alpha_j(H_{\alpha_j})E_{\alpha_j}$ and that $[E_{-\alpha_j}, (\text{ad}E_{\alpha_j})^2 E_{\alpha_j}] = 0$ implies that there does not exist a simple odd root such that $3\alpha_j$ is a root.

This can be easily generalised to all the odd roots of $\tilde{\mathcal{L}}_s$.

Definition 3.1 Real roots of $\tilde{\mathcal{L}}_s$

A root α of $\tilde{\mathcal{L}}_s$ is called real if there exists an element S of the Weyl group and a simple root α_j ($j \in I$) or a root $2\alpha_j$ ($j \in \tau$) such that

$$\alpha = S\alpha_j \text{ or } \alpha = 2S\alpha_j . \qquad (3.60)$$

The set of real, positive real and negative real roots will be denoted by Δ^r, Δ^r_+ and Δ^r_- respectively.

Proposition 3.8

The real roots are characterized by the following properties:

(a) if $\alpha \in \Delta^r$ then $< \alpha, \alpha > >0$;

(b) if $\alpha \in \Delta^r$ and $\beta \in \Delta$ then there exist two non-negative integers p and q (which depend on α and β) such that $\beta+k\alpha$ is in the α-string containing β for every integer k that satisfies the relation $-p \leq k \leq q$. That is $\beta+k\alpha$ is an arithmetic progression

$$\beta-p\alpha,..., \beta-\alpha, \beta,..., \beta+q\alpha$$

Moreover, p and q are such that

$$p - q = 2<\beta, \alpha><\alpha, \alpha>^{-1}$$

and

$$S_\alpha\beta = \beta - 2<\beta, \alpha><\alpha, \alpha>^{-1} \alpha$$

is a non-zero root . Moreover dim $\tilde{\mathfrak{L}}_{S_\alpha\beta}$ = dim $\tilde{\mathfrak{L}}_\beta$;

(c) the set Δ^r is invariant with respect to the Weyl group and dim$\tilde{\mathfrak{L}}_{S\beta}$ = dim $\tilde{\mathfrak{L}}_\beta$ for any $S \in W$ and any $\beta \in \Delta^r$.

(d) the root subspaces $\tilde{\mathfrak{L}}_{s\alpha}$ of $\tilde{\mathfrak{L}}_s$ are all one dimensional for any $\alpha \in \Delta^r$;

(e) if α is real even root of $\tilde{\mathfrak{L}}_s$ then $k\alpha$ is a real root of $\tilde{\mathfrak{L}}_s$ if and only if $k = \pm 1$. Similarly if α is real odd root of $\tilde{\mathfrak{L}}_s$ then $k\alpha$ is a real root of $\tilde{\mathfrak{L}}_s$ if and only if and $k = \pm 1, \pm 2$.

Proof (see Cornwell (1989)).

Part (a) follows immediate from the definition of the real roots, the invariance of the form < , > under the Weyl group and the fact that $<\alpha_j , \alpha_j> > 0$ for all simple roots α_j .

To prove (b) we just have to notice that the adjoint representation is an integrable representation (see chapter 5). Then (b) follows from similar steps as in the affine algebra case by making use of proposition 5.1 and 5.2 (see also Kac(1985), (1978),(1968) or Cornwell(1989)).

Part (c) follows from (b) and (d) is obvious for the simple roots and for the rest of the roots it follows from (c).

Part (e) is evidently true for the simple even and odd roots and for the rest of the roots we can prove it by using the definition of the real roots and part (c).

One thing that has to be pointed out is that property (e) includes the cases $k = \pm 2$. In the finite and affine Kac-Moody algebra case these values do not appear since the finite or affine root systems, are reduced root systems. That is if α is a root of these root systems then $\frac{1}{2}\alpha$ is not. The appearance of these values here reveals the non-reduced character of the root system of the superalgebras under consideration (see chapter 2).

<u>Definition 3.2</u> Imaginary roots of $\tilde{\mathcal{L}}_s$

A root α that is not real is called imaginary. That is there <u>does not</u> exist any element of W such that when acting on a simple root gives α.

The set of imaginary, positive imaginary and negative imaginary roots will be denoted by Δ^i, Δ^i_+ and Δ^i_- respectively. The following proposition can be proved in exactly the same way as in Kac(1968) or Cornwell (1989), and provides us with a criterion for the existence of imaginary roots.

Proposition 3.9

Every one of the following three properties is equivalent to a root α being imaginary:

(a) if $\alpha \in \Delta^i_+$ then there exists an $S \in W$ such that $S\alpha = \beta \in \Delta^i_+$ and $< \beta, \alpha_j > \leq 0$ for all $j \in I$; (3.61)

(b) if $\alpha \in \Delta$ then $\alpha \in \Delta^i$ if and only if $< \alpha, \alpha > \leq 0$

(c) if $\alpha \in \Delta^i$ then $k\alpha \in \Delta$ for any integer k.

If $\alpha \in \Delta^i_+$ and $S \in W$ then $S\alpha \in \Delta^i_+$. The set of imaginary roots is Weyl-invariant.

Proof (See Kac(1978))

From part (a) and relation (3.27) we can write (3.61) as

$$< \beta, \alpha_j > = \Sigma_{k \in I} \; \kappa_k < \alpha_k, \alpha_j > \leq 0$$

where $\kappa_k \geq 0$ for all $k \in I$, or in a matrix form $\mathbf{k} \geq \mathbf{0}$ and $\mathbf{Ak} \leq \mathbf{0}$. Then since $\mathbf{A}$ is an affine matrix, by proposition 2.2(b) chapter 2, $\mathbf{Ak} \leq \mathbf{0}$ implies that $\mathbf{Ak} = \mathbf{0}$. Thus the affine Lie superalgebra possesses imaginary roots. Moreover, $<\beta, \alpha_j> = 0$ for all $j \in I$ and $< \beta, \beta > = 0$ for every $\beta \in \Delta^i$.

Let us concentrate for the moment on the subspace $\mathcal{H}''$ of the Cartan subalgebra. As we saw above the subspace $\mathcal{H}''$ is one dimensional and $\mathcal{H}$ is the direct sum of $\mathcal{H}'$ and $\mathcal{H}''$. Its basis will be denoted by d and will be called the scaling element or derivation of $\tilde{\mathcal{L}}_s$. It is defined to be such that

$$\alpha_j(d) = 1 \text{ if } j=0 \quad \text{and} \quad \alpha_j(d) = 0 \text{ if } j = 1, 2, \ldots, \ell. \qquad (3.62)$$

With these definitions it is clear that d does not belong in $\mathcal{H}'$. From relations (3.50) to (3.50) we have that

$$B(d\ ,\ d)\ =0 \tag{3.63}$$

$$B(d\ ,\ H_{\alpha_k}) = 2/<\alpha_0\ ,\ \alpha_0> \text{ if } k=0 \text{ and } B(d, H_{\alpha_k}) = 0 \text{ if } k=1,\ 2,....\ell. \tag{3.64}$$

The basis of $\mathcal{H}$ has thus been consistently established.

Consider a vector $\mathbf{k} > \mathbf{0}$. Since $\mathbf{A}$ is of affine type $\mathbf{Ak} = \mathbf{0}$ and we can assume that the entries k_i ($i \in I$) of $\mathbf{k}$ after appropriate scaling are positive integers. By the uniqueness up to a constant factor, of such a vector we can take k_i ($i \in I$ to be for example, the labels of the Dynkin diagrams of table V (this is usually the convention followed in the affine algebra case). Now define a linear functional δ on $\mathcal{H}$ by

$$\delta = \Sigma_{i\in I}\ k_i\alpha_i \tag{3.65}$$

It can be easily seen from (3.6) and the affine character of the Cartan matrix that

$$\delta(H_{\alpha_i}) = 0 \quad \text{for all } i\in I. \tag{3.66}$$

and consequently, $\delta(h) = 0$, for all $h \in \mathcal{H}'$. Moreover by (3.65), (3.66) and (3.62)

$$\delta(d) = k_0 \tag{3.67}$$

Moreover by (3.49), (3.50) the element h_δ of $\mathcal{H}$ corresponding to δ is given by

$$h_\delta = \Sigma_{i\in I}\ k_i\, h_{\alpha_i}\ =\ \Sigma_{i\in I}\ k_i\, \varepsilon_i^{-1}\ H_{\alpha_i} \tag{3.68}$$

where (3.56) has also been used. Then it is trivial to show that $\alpha_k(h_\delta) = 0$ and so by proposition 3.3 <u>h_δ belongs in the</u>

center $\mathfrak{C}$ of $\tilde{\mathfrak{L}}_s$.

Relation (3.66) states nothing but $< \delta, \alpha_j > = 0$ for all $j \in I$ and $< \delta, \delta > = 0$. Moreover all the conditions of proposition 3.6 are satisfied and thus δ is an imaginary root . In addition, by proposition 3.9(c), $j\delta$ ($j \in \mathbb{Z}$) is also an imaginary root and thus $< j\delta, j\delta > = 0$ too.

What remains now is to complete the basis of $\mathcal{H}^*$ by defining in a consistent way a basis for $\mathcal{H}''^*$ corresponding to the scaling element d. Let Λ_0 be the linear functional defined on $\mathcal{H}$ as

$$\Lambda_0(H_{\alpha_k}) = 1 \text{ if } k=0 \quad \text{and } \Lambda_0(H_{\alpha_k}) = 0 \text{ if } k=1, 2,..., \ell. \tag{3.69}$$

Using (3.49), (3.63-64) and the above relations, the element h_{Λ_0} of $\mathcal{H}''$ corresponding to Λ_0 is given by

$$h_{\Lambda_0} = \tfrac{1}{2}<\alpha_0 , \alpha_0> d. \tag{3.70}$$

Finally observe that because of (3.46) and (3.51)

$$<\Lambda_0 , \Lambda_0> = 0. \tag{3.71}$$

Also from (3.65) and (3.67)

$$\text{and} \quad \delta(h_{\Lambda_0}) = \tfrac{1}{2}<\alpha_0 , \alpha_0>k_0. \tag{3.72}$$

Both the role of d or h_{Λ_0} and h_δ will become more apparent in the explicit realization of the affine superalgebras. One thing that can be said concerns the functional Λ_0. Λ_0 is not a root of the superalgebra since it does not have the property (3.28). In the next chapter amongst other things, we shall determine explicitly the root structure of the affine Kac-Moody superalgebras.

3.6 $\tilde{\mathcal{L}}_s$ viewed as a Z-graded Lie superalgebra.

Definition 3.3 M-graded Lie superalgebras.

An M-grading of a Lie superalgebra $\mathcal{L}$ with respect to an abelian group M is a decomposition of $\mathcal{L}$ in to a direct sum of subspaces as:

$$\mathcal{L} = \Sigma_{m \in G} \oplus \mathcal{L}_m \tag{3.73}$$

such that

$$[\mathcal{L}_m, \mathcal{L}_n] \subset \mathcal{L}_{m+n} \tag{3.74}$$

$$\dim \mathcal{L}_m < \infty \tag{3.75}$$

Then the Lie superalgebra is called M-graded.

An element a of $\mathcal{L}$ which belongs to the subspace $\mathcal{L}_m$ is said to be homogeneous of degree m. A subspace $\mathcal{L}'$ of $\mathcal{L}$ is said to be M-graded if $\mathcal{L}' = \Sigma_{m \in M} \oplus (\mathcal{L}' \cap \mathcal{L}_m)$. For example the superalgebra itself is graded with respect to $\mathbf{Z}_2$. The root decomposition (3.30) is a Q-grading on $\mathcal{L}$.

Z-grading plays a very important role in the theory of Lie algebras or superalgebras. It actually initiated the study of what is now known as contragredient Lie algebras or superalgebras (see Kac 1968,1977). These are Z-graded Lie (super)algebras that are associated with an arbitrary matrix **A** and a set of relations (3.7-15). The affine superalgebras together with the basic classical superalgebras are special cases of contragredient Lie superalgebras.

Consider now the expression (3.38) appearing in the definition of the height of a root. Let a_α be an element of the root subspace $\tilde{\mathfrak{L}}_{s\alpha}$ corresponding to the root α of height $j \in Z$. Setting $\deg a_\alpha = \mathrm{ht}\,\alpha = j$ (for all roots α) and $\deg h = 0$ (for all $h \in \mathcal{H}$) we can introduce the structure of a $\mathbf{Z}$-grading in $\tilde{\mathfrak{L}}_s$ described by

$$\tilde{\mathfrak{L}}_s = \Sigma_{j \in Z} \oplus \tilde{\mathfrak{L}}_j, \qquad \tilde{\mathfrak{L}}_j = \Sigma_{\alpha | \mathrm{ht}\alpha = j} \oplus \tilde{\mathfrak{L}}_\alpha \tag{3.76}$$

where

$$\tilde{\mathfrak{L}}_0 = \mathcal{H}, \qquad \tilde{\mathfrak{L}}_{-1} = \Sigma_{j=0}^{\ell} \mathbb{C}(E_{-\alpha_j}), \qquad \tilde{\mathfrak{L}}_{+1} = \Sigma_{j=0}^{\ell} \mathbb{C}(E_{\alpha_j}) \tag{3.77}$$

$$\tilde{\mathfrak{L}}_+ = \Sigma_{j \geq 1} \oplus \mathfrak{L}_j, \qquad \tilde{\mathfrak{L}}_- = \Sigma_{j \geq 1} \oplus \mathfrak{L}_{-j} \tag{3.78}$$

This is called the <u>principal grading</u> of $\tilde{\mathfrak{L}}_s$.

CHAPTER 4

EXPLICIT REALIZASION OF AFFINE KAC-MOODY SUPERALGEBRAS

4.1 Introduction

The explicit realization of affine Kac-Moody superalgebras is a natural generalization of the realization of the affine Kac-Moody algebras (see Cornwell(1989), Kac (1985)).

It should be noted that for any basic classical simple complex Lie superalgebra and not only for $A(2\ell-1/0)$, $A(2\ell/0)$, $C(\ell+1)$ and $B(0/\ell)$, there exists an affine Kac-Moody superalgebra that has appeared in the literature. The difference with our superalgebras is that the resulting Cartan matrices are not those of the definition given in chapter 3. It has been demonstrated (see Serganova(1983), Van der Leur(1986)) that, with an appropriately chosen definition of a Cartan matrix, all the basic classical simple complex Lie superalgebras and their affine (untwisted and twisted) counterparts can be obtained, (including the ones that we are examining here), together with a complete classification of their Dynkin diagrams. This set constitutes all the contragradient Lie superalgebras of finite growth that exist. Since a lot of the characteristics of the twisted superalgebras that we shall investigate are essentially the same as those of the untwisted superalgebra $B^{(1)}(0/\ell)$ we shall demonstrate them explicitly only for the untwisted superalgebras.

All information needed on the structure of $A(2\ell-1/0)$, $A(2\ell/0)$, $C(\ell+1)$ and $B(0/\ell)$ needed in the explicit realization will be given on the process. In particular their Dynkin diagrams with the distinguished choice of simple roots can be found in table V at the end of this chapter. For more information see Cornwell(1989).

In presenting the method of constructing untwisted and twisted superalgebras in sections 4.2 A and 4.3 B we shall deal with the more general case of constructing such superalgebras based on any basic classical simple complex Lie superalgebra.

This set of basic classical simple complex Lie superalgebra consits of

$A(r/s)(= sl(r+1/s+1;\ \mathbb{C}))(r>s\geq0)$, $A(r/r)(=sl(r+1/r+1;\ \mathbb{C}))(r\geq1)$,

$B(r/s)(= osp(2r+1/2s\ ;\ \mathbb{C}))(r\geq0$ and $s\geq1)$,

$C(s)(=osp(2/2s-2;\ \mathbb{C}))(s\geq2\ )$,

$D(r/s)(= osp(2r/2s;\ \mathbb{C}))(r\geq2$ and $s\geq1)$, $D(2/1;\ \alpha)(\alpha\neq 0,1,\infty)$

F(4) and G(3). For more information see Cornwell(1989).

For a basic classical simple complex Lie superalgebra $\tilde{\mathfrak{L}}^0_s$ it is assumed that $B^0(\ ,\ )$ is the Killing form (or if the Killing form is identically zero then this is any other supersymmetric invariant non-degenerate bilinear form), that $\tilde{\mathfrak{L}}^0_s$ has rank ℓ^0, that $\mathcal{H}^0$ is its Cartan subalgebra, that α^0_k (for $k = 1,2,\ldots,\ell^0$) are its simple roots and that Δ^0, Δ^0_+, and Δ^0_- are its non-zero, positive, and negative root systems respectively. We fix a maximal solvable subalgebra for $\tilde{\mathfrak{L}}^0_s$ such that the set of simple roots will be the distinguished one.

A Weyl-type canonical basis will be chosen for $\tilde{\mathcal{L}}^0_s$ (see for example Cornwell(1989)). Its elements will be denoted by $h^0_{\alpha^0_k} \in \mathcal{H}^0$ (for $k = 1,2,\ldots,\ell^0$), together with $e^0_{\alpha^0}$ (for all $\alpha^0 \in \Delta^0$), and these are assumed to satisfy the usual commutation and anti-commutation relations. In a realization of $\tilde{\mathcal{L}}^0_s$ in which the elements of $\tilde{\mathcal{L}}^0_s$ are represented by supermatrices, with $e^0_{\alpha^0}$ being represented by $\underset{\sim}{e}^0_{\alpha^0}$, the convention will be adopted that

$$\underset{\sim}{e}^0_{-\alpha^0} = -(\underset{\sim}{e}^0_{\alpha^0})^{st} \tag{4.1}$$

for all α^0 of Δ^0_+, the superscripts st indicating that the supertranspose must be taken.

Since $B^0(\ ,\ )$ a is symmetric non-degenerate bilinear form on $\mathcal{H}^0$, for each linear functional α^0 on $\mathcal{H}^0$ there exists an element $h^0_{\alpha^0}$ of $\mathcal{H}^0$ that is defined by by

$$B^0(h^0_{\alpha^0}, h^0) = \alpha^0(h^0) \quad \text{for all } h^0 \in \mathcal{H}^0. \tag{4.2}$$

Then a symmetric non-degenerate bilinear form $<\ ,\ >^0$ may be defined in the dual space $\mathcal{H}^{0^*}$ of functionals defined on $\mathcal{H}^0$ by

$$<\alpha^0, \beta^0>^0 = B^0(h^0_{\alpha^0}, h^0_{\beta^0}) \tag{4.3}$$

for any pair of linear functionals α^0 and β^0 on $\mathcal{H}^0$. In addition $h^0_{\alpha^0} + h^0_{\beta^0} = h^0_{\alpha^0+\beta^0}$. Contrary to the case of simple Lie algebras $<\alpha^0, \alpha^0>^0$ is neither always real nor always positive.

4.2 Explicit realization of affine untwisted Kac-Moody superalgebras $\tilde{\mathcal{L}}_s^{(1)}$

A. Basic concepts and definitions

Let $\mathbb{C}[t,t^{-1}]$ be the associative algebra of Laurent polynomials in the indeterminate t. We define first the loop superalgebra corresponding to the basic classical complex simple Lie superalgebra $\tilde{\mathcal{L}}_s^0$ as

$$\tilde{\mathcal{L}}_{Loop}(\tilde{\mathcal{L}}_s^0) = \mathbb{C}[t,t^{-1}] \otimes \tilde{\mathcal{L}}_s^0. \tag{4.2.1}$$

This is an infinite dimensional complex Lie superalgebra and the generalized Lie product is given by

$$[\, t^j \otimes a^0 ,\; t^k \otimes b^0 \,] = t^{j+k} \otimes [\, a^0, b^0 \,] \tag{4.2.2}$$

for all integers j and k and all $a^0, b^0 \in \tilde{\mathcal{L}}_s^0$, where the generalized Lie product of the right-hand side of (4.2.2) is that of $\tilde{\mathcal{L}}_s^0$. The Z_2 graduation is defined such that
$\deg\,(t^j \otimes a^0) = \deg a^0$ for any homogeneous element a^0 of $\tilde{\mathcal{L}}_s^0$.

This superalgebra may be extended by introducing an additional even element c, with the generalized Lie product being modified to become

$$[\, t^j \otimes a^0 ,\, t^k \otimes b^0 \,] = t^{j+k} \otimes [\, a^0, b^0 \,] + j\delta^{j+k,0}\, B^0(a^0,b^0)c \tag{4.2.3}$$

for all integers j and k and all $a^0, b^0 \in \tilde{\mathcal{L}}_s^0$, and where it is assumed that

$$[\, t^j \otimes a^0 ,\, c \,] = 0 \tag{4.2.4}$$

for all integers j and all $a^0 \in \tilde{\mathfrak{L}}_s^0$. This latter Lie superalgebra may be enlarged by adding a further even element d, for which it is assumed that

$$[d, t^j \otimes a^0] = j\, t^j \otimes a^0, \qquad (4.2.5)$$

for all integers j and all $a^0 \in \tilde{\mathfrak{L}}_s^0$, and that

$$[d, c] = 0 . \qquad (4.2.6)$$

Clearly d acts as the operator $t \dfrac{d}{dt}$ on the loop superalgebra and it can be easily seen that it is actually a superderivation of it and is the extension of the derivation of the algebra of Laurent polynomials from $\mathbb{C}[t,t^{-1}]$ to $\tilde{\mathfrak{L}}_s^{(1)}$. Also it can be shown that the additional sum on the right hand side of (4.2.3) is the extension of the loop superalgebra by a two-cocycle which for our case has this particular form. The untwisted complex Lie superalgebra $\tilde{\mathfrak{L}}_s^{(1)}$ is defined to be

$$\tilde{\mathfrak{L}}_s^{(1)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus (\mathbb{C}[t,t^{-1}] \otimes \tilde{\mathfrak{L}}_s^0). \qquad (4.2.7)$$

Equation (4.2.3) shows that the set of elements $t^0 \otimes a^0$, where $a^0 \in \tilde{\mathfrak{L}}_s^0$, form a subalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$ that is isomorphic to $\tilde{\mathfrak{L}}_s^0$. The maximal abelian subalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$ is given by

$$\mathcal{H}^{(1)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus (t^0 \otimes \mathcal{H}^0) \qquad (4.2.8)$$

where $\mathcal{H}^0$ is the Cartan subalgebra of the basic classical complex Lie superalgebra $\tilde{\mathfrak{L}}_s^0$. Clearly dim $\mathcal{H}^{(1)} = \ell^0+2$ where ℓ^0 is the dimension of $\mathcal{H}^0$. By construction $\tilde{\mathfrak{L}}_s^{(1)}$ has a one dimensional center $\mathbf{C} = (\mathbb{C}c)$.

The derived superalgebra $[\tilde{\mathfrak{L}}_s^{(1)}, \tilde{\mathfrak{L}}_s^{(1)}]$ is easily seen to be given by

$$[\, \tilde{\mathfrak{L}}_s^{(1)}, \tilde{\mathfrak{L}}_s^{(1)} \,] = (\, \mathbb{C}[t,t^{-1}] \otimes \tilde{\mathfrak{L}}_s^{0} \,) \oplus \mathbb{C} \,. \qquad (4.2.9)$$

It can be checked that the following relations define a unique (up to a constant) <u>supersymmetric invariant non-degenerate bilinear form</u> $B^{(1)}(\ ,\ )$ on $\tilde{\mathfrak{L}}_s^{(1)}$:

$$B^{(1)}(t^j \otimes a^0, t^k \otimes b^0) = \delta_{j+k,0}\, B^0(a^0, b^0), \qquad (4.2.10)$$

$$B^{(1)}(\, t^j \otimes a^0, c) = B^{(1)}(\, t^j \otimes a^0, d) = 0 \,, \qquad (4.2.11)$$

$$B^{(1)}(c,c) = B^{(1)}(d,d) = 0 \,, \qquad (4.2.12)$$

$$B^{(1)}(c,d) = 1 \,, \qquad (4.2.13)$$

for all $a^0, b^0 \in \tilde{\mathfrak{L}}^0$ and all integers j and k. Clearly $B^{(1)}(\ ,\ )$ coincides with $B^0(\ ,\ )$ on the subalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$ that is isomorphic to $\tilde{\mathfrak{L}}_s^0$.

The bilinear form defined above, being symmetric non-degenerate on $\mathcal{H}^{(1)}$, induces a symmetric non-degenerate biliear form on the space of linear functionals $\mathcal{H}^{(1)^*}$(the dual of $\mathcal{H}^{(1)}$). Then for every linear functional α of $\mathcal{H}^{(1)^*}$, there exist an element h_α of $\mathcal{H}^{(1)}$ defined by

$$B^{(1)}(h, h_\alpha) = \alpha(h) \qquad (4.2.14)$$

for all $h \in \mathcal{H}^{(1)}$. Consequently for any two functionals α, β the induced form $<\ ,\ >^{(1)}$ on $\mathcal{H}^{(1)^*}$ is defined by

$$<\alpha, \beta>^{(1)} = B^{(1)}(h_\alpha, h_\beta) \qquad (4.2.15)$$

Using the above two relations we deduce that

$$h_\alpha + h_\beta = h_{\alpha+\beta} \qquad (4.2.16)$$

$$\alpha(h_\beta) = \beta(h_\alpha) = <\alpha, \beta>^{(1)} \tag{4.2.17}$$

for all h_α, h_β of $\mathcal{H}^{(1)}$ and all α and β of $\mathcal{H}^{(1)^*}$.

Every linear functional α^0 defined on $\mathcal{H}^0$ can be extended to become a linear functional on $\mathcal{H}^{(1)}$ by the following definitions

$$\alpha^0(t^0 \otimes h_{\alpha_k^0}) = \alpha^0(h_{\alpha_k^0}) \tag{4.2.18}$$

$$\alpha^0(c) = 0 \qquad \alpha^0(d) = 0 \tag{4.2.19}$$

for all $k = 1,2,\dots,\ell^0$. Consequently the same is true for all the roots of $\tilde{\mathcal{L}}_s^0$. Next let us define a linear functional δ on $\mathcal{H}^{(1)}$ by

$$\delta(t^0 \otimes h_{\alpha_k^0}) = 0 \text{ (for all } k = 1,2,\dots,\ell^0)$$

$$\delta(c) = 0 \qquad \delta(d) = 1. \tag{4.2.20}$$

A non-zero linear functional α defined on $\mathcal{H}^{(1)}$ is called a <u>root</u> of $\tilde{\mathcal{L}}_s^{(1)}$ if there exists at least one element a_α of $\tilde{\mathcal{L}}_s^{(1)}$ such that

$$[h, a_\alpha] = \alpha(h)\, a_\alpha \tag{4.2.21}$$

for all $h \in \mathcal{H}^{(1)}$. The set of elements a_α, for each such α, that satisfy the above relation form the <u>root subspace</u> $\tilde{\mathcal{L}}_{s\alpha}^{(1)}$ of $\tilde{\mathcal{L}}_s^{(1)}$.

Now the complete root system of $\tilde{\mathcal{L}}_s^{(1)}$ can be found by using the defining commutations relations (1.1.2-6) of $\tilde{\mathcal{L}}_s^{(1)}$.

Let $e^0_{\alpha^0}$ be basis vectors corresponding to the root α^0 of $\tilde{\mathcal{L}}_s^0$ and $h^0_{\alpha_k^0}$ (for all $k = 1,2,\dots,\ell^0$) be the basis elements of $\mathcal{H}^0$. Then from the commutation relations (4.2.2-6) we get:

$$[\, t^0 \otimes h^0_{\alpha^0_k} \,,\, t^j \otimes e^0_{\alpha^0}] \;=\; \alpha^0(t^0 \otimes h^0_{\alpha^0_k})\; (t^j \otimes e^0_{\alpha^0}) \;, \qquad (4.2.22)$$

$$[\, c \,,\, t^j \otimes e^0_{\alpha^0}] \;=\; 0 \,, \qquad (4.2.23)$$

$$[\, d \,,\, t^j \otimes e^0_{\alpha^0}] \;=\; j(t^j \otimes e^0_{\alpha^0}) \,, \qquad (4.2.24)$$

for any $\alpha^0 \in \Delta^0$ and for any integer j (and for $k = 1,\ldots,\ell^0$). Taking in to account the definition of δ and the extensions of α^0, the above relations become

$$[\, h \,,\, t^j \otimes e^0_{\alpha^0}] \;=\; \{j\delta(h) + \alpha^0(h)\}\; (t^j \otimes e^0_{\alpha^0}) \qquad (4.2.25)$$

for <u>all</u> $h \in \mathcal{H}^{(1)}$. Thus $t^j \otimes e^0_{\alpha^0}$ corresponds to a root $j\delta + \alpha^0$ of $\tilde{\mathcal{L}}^{(1)}_s$. Moreover it is obvious that the root subspace $\tilde{\mathcal{L}}^{(1)}_{s(j\delta+\alpha^0)}$ has dimension $\dim \tilde{\mathcal{L}}^{(1)}_{s(j\delta+\alpha^0)} = 1$ (except if $\tilde{\mathcal{L}}^0_s = A(1/1)$ in which case the odd root subspaces have dimension two) with the root basis vector being $t^j \otimes e^0_{\alpha^0}$. Similarly, for any $\beta^0 \in \Delta^0$ and any non-zero integer j

$$[\, t^0 \otimes h^0_{\alpha^0_k} \,,\, t^j \otimes h^0_{\beta^0}] \;=\; 0 \,, \qquad (4.2.26)$$

$$[\, c \,,\, t^j \otimes h^0_{\beta^0}] \;=\; 0 \,, \qquad (4.2.27)$$

$$[\, d \,,\, t^j \otimes h^0_{\beta^0}] \;=\; j(t^j \otimes h^0_{\beta^0}) \,, \qquad (4.2.28)$$

and so, by the definition of δ,

$$[\, h, t^j \otimes h^0_{\beta^0}] \;=\; j\delta(h)\; (t^j \otimes h^0_{\beta^0}) \qquad (4.2.29)$$

for all $h \in \mathcal{H}^{(1)}$. Thus $t^j \otimes h^0_{\beta^0}$ corresponds to a root $j\delta$ of $\tilde{\mathcal{L}}^{(1)}_s$. Moreover there are ℓ^0 linearly independent elements with this property, namely $h^0_{\alpha^0_k}$ (for $k = 1,2,\ldots,\ell^0$), and, as there are no further elements of $\tilde{\mathcal{L}}^{(1)}_s$ to consider, the root subspace of

$j\delta$ must have dimension ℓ^0 (for $j \neq 0$).

It follows from (4.2.14), and (4.2.10) to (4.2.11) that

$$h_{\alpha^0} = t^0 \otimes h^0_{\alpha^0} , \qquad (4.2.30)$$

for each $\alpha^0 \in \Delta^0$ (and its extension), and hence by (4.2.15),(4.3), (4.2.10), and (4.2.16) to (4.2.18) that

$$<\alpha^0,\beta^0>^{(1)} = <\alpha^0,\beta^0>^0 \qquad (4.2.31)$$

for every pair $\alpha^0,\beta^0 \in \Delta^0$ (and their extensions). Also (4.2.12), (4.2.13), and (4.2.19) to (4.2.21) imply that

$$h_\delta = c \qquad (4.2.32)$$

That is, δ is the root of $\tilde{\mathfrak{L}}^{(1)}_s$ corresponding to the basis element c. The above analysis reveals that the system of non-zero roots of $\tilde{\mathfrak{L}}^{(1)}_s$ is given by

$$\Delta^{(1)} = \{ j\delta + \alpha^0 \text{ for all } \alpha^0 \in \Delta^0 \text{ and } j \in \mathbf{Z},$$

$$j\delta \text{ for all } j \in \mathbf{Z}\text{-}\{0\} \} \qquad (4.2.33)$$

and that $\tilde{\mathfrak{L}}^{(1)}_s$ admits the decoposition

$$\tilde{\mathfrak{L}}^{(1)}_s = \mathcal{H}^{(1)} \oplus \{ \sum_{\alpha \in \Delta^{(1)}} \oplus \tilde{\mathfrak{L}}^{(1)}_{s\alpha} \} \qquad (4.2.34)$$

We shall describe as <u>even</u> roots the roots $j\delta$ and the roots $j\delta + \alpha^0$, where α^0 is an even root of $\tilde{\mathfrak{L}}^{(1)}_s$. The <u>odd</u> roots are of the form $j\delta + \alpha^0$, where α^0 is an odd root of $\tilde{\mathfrak{L}}^{(1)}_s$(Note that these definitions are direct consequences of the $\mathbf{Z}_2$-graduation of $\tilde{\mathfrak{L}}^{(1)}_s$ as defined above).

As in the case of affine untwisted Kac-Moody algebras the <u>simple roots</u> $\alpha_0, \alpha_1, \ldots , \alpha_\ell$ of the affine untwisted Kac-Moody superalgebra are taken to be

$$\alpha_0 = \delta - \alpha^0_H,$$

$$\alpha_k = \alpha^0_k \text{ for } k = 1,2, \ldots, \ell^0, \qquad (4.2.35)$$

where the α^0_k of (4.2.35) are the extensions of the simple roots of $\tilde{\mathfrak{L}}^0_s$ and α^0_H is the highest root of $\tilde{\mathfrak{L}}^0_s$. Then the set $\Delta^{(1)}_+$ of positive roots of $\tilde{\mathfrak{L}}^{(1)}_s$ is given by

$$\Delta^{(1)}_+ = \{ j\delta + \alpha^0 \text{ for all } \alpha^0 \in \Delta^0 \text{ and } j \in \mathbf{Z}_+ - \{0\},$$

$$j\delta \text{ and } j \in \mathbf{Z}_+ - \{0\},$$

$$\alpha^0 \text{ for all } \alpha^0 \in \Delta^0_+ \}$$

and we have a similar expression for the set $\Delta^{(1)}_-$ of negative roots of $\tilde{\mathfrak{L}}^{(1)}_s$. The only exception is $A^{(1)}(1/1)$ in the case each odd root is both negative and positive. Then for $\tilde{\mathfrak{L}}^{(1)}_s$ we have the following root decomposition of :

$$\tilde{\mathfrak{L}}^{(1)}_s = \{ \textstyle\sum_{j<0} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(j\delta)} \} \oplus \{ \sum_{j<0} \sum_{\alpha^0 \in \Delta^0} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(j\delta+\alpha^0)} \}$$

$$\oplus \{ \textstyle\sum_{\alpha^0 \in \Delta^0_+} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(-\alpha^0)} \} \oplus \mathcal{H}^{(1)} \oplus \{ \sum_{\alpha^0 \in \Delta^0_+} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(\alpha^0)} \}$$

$$\oplus \{ \textstyle\sum_{j>0} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(j\delta)} \} \oplus \{ \sum_{j>0} \sum_{\alpha^0 \in \Delta^0} \oplus \tilde{\mathfrak{L}}^{(1)}_{s(j\delta+\alpha^0)} \}.$$

Proposition 4.1

$\tilde{\mathfrak{L}}^{(1)}_s$ does not contain any non-trivial ideal $\mathcal{R}$ such that $\mathcal{R} \cap \mathcal{H}^{(1)} = 0$.

Proof . Assume that there exists such an ideal $\mathcal{R}$. Obviously it is graded with respect to the root decomposition. Consider an element $t^j \otimes a_{\alpha^0}$, of $\mathcal{R}$,corresponding to some root $j\delta + \alpha^0$ of

$\tilde{\mathcal{L}}_s^{(1)}$. Then for an element $t^{-j} \otimes a_{-\alpha^0}$ we should have that $[\, t^j \otimes a_{\alpha^0} , t^{-j} \otimes a_{-\alpha^0} \,] \in \mathcal{R}$. But clearly $[\, t^j \otimes a_{\alpha^0} , t^{-j} \otimes a_{-\alpha^0} \,] \in \mathcal{H}^{(1)}$ and thus $[\, t^j \otimes a_{\alpha^0} , t^{-j} \otimes a_{-\alpha^0}] \in \mathcal{R} \cap \mathcal{H}^{(1)}$. Since we have assumed that $\mathcal{R} \cap \mathcal{H}^{(1)} = 0$, $[\, t^j \otimes a_{\alpha^0} , t^{-j} \otimes a_{-\alpha^0} \,]$ should be zero which is a contradiction and thus $\mathcal{R} = 0$.

B. The $B^{(1)}(0/\ell)(\ell \geq 1)$ untwisted Kac-Moody superalgebra

The rest of the analysis we will be focused on $\tilde{\mathcal{L}}_s^0 = B(0/\ell)$ alone. Obviously all of the above considerations apply unaltered in this case too so we will demonstrate only those elements that will establish the isomorphism of $B^{(1)}(0/\ell)$ with the affine Kac-Moody superalgebra $\tilde{\mathcal{L}}_s$ of chapter 3 with Dynkin diagram and Cartan matrix given in figures 1,2 of table V. All information needed on $B(0/\ell)$ can be found in appendix A(3) and table II chapter 2.(See also Cornwell(1989)).

(a) The root system of $B^{(1)}(0/\ell)$

The simple roots α_j (for all j=0,1,..,ℓ) of $B^{(1)}(0/\ell)$ according to the previous analysis are the extensions of the even simple roots $\alpha_1^0, ..., \alpha_{\ell-1}^0$, of $B(0/\ell)$ and the odd simple root α_ℓ^0 of $B(0/\ell)$, together with α_0. Since the highest root of $B(0/\ell)$ is even and is given by

$$\alpha_H^0 = 2 \sum_{r=1}^{\ell} \alpha_r^0 , \tag{4.2.36}$$

it follows that

$$\alpha_0 = \delta - 2 \sum_{r=1}^{\ell} \alpha_r^0 \tag{4.2.37}$$

which is an <u>even</u> root of $B^{(1)}(0/\ell)$.

If α_k^0 is the extension of any simple root of $B(0/\ell)$, then from (4.2.10), (4.2.15), (4.2.30-32) folows that

$$<\delta,\alpha_k^0> = 0 \quad \text{for all } k = 1,...,\ell \tag{4.2.38}$$

$$<j\delta, j\delta> = 0 \qquad (4.2.39)$$

$$<j\delta+\alpha^0, j\delta+\alpha^0> = <\alpha^0, \alpha^0>^0 > 0 \qquad (4.2.40)$$

for every integer j and every non-zero root α^0 of $B(0/\ell)$. Every non-zero root of the form $j\delta$ will be called "imaginary". and every root of the form $j\delta+\alpha^0$ "real". The adaptation of these names is a direct consequence of propositions 3.5 and 3.6 of chapter 3. The latter are easily checked to be valid if we take in to account the properties of the root system of $B(0/\ell)$ and the structure of the Weyl group of $B^{(1)}(0/\ell)$ which will be demonstrated below.

We can express the roots of $B(0/\ell)$ in terms of the linearly independent functionals ε_j ($i \leq j \leq \ell$) defined on $\mathcal{H}^0$ (see table II and Cornwell(1989)). Then the set of real roots $\Delta^{(1)}_{rel}$ of $B^{(1)}(0/\ell)$ is given by

$$\Delta^{(1)}_{rel} = \{ m\delta \pm (\varepsilon_i \pm \varepsilon_j) \text{ with } 1 \leq i < j \leq \ell,$$

$$m\delta \pm \varepsilon_i \text{ with } 1 \leq i \leq \ell \text{ , } m\delta \pm 2\varepsilon_i \text{ with } 1 \leq i \leq \ell \text{ , } m \in \mathbf{Z}\} \qquad (4.2.41)$$

and the simple roots are given by

$$\alpha_0 = \delta - 2\varepsilon_1 \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad 1 \leq i \leq \ell -1 \quad \alpha_\ell = \varepsilon_\ell \qquad (4.2.42)$$

(b) The Cartan matrix

We define the Cartan matrix **A** of $B^{(1)}(0/\ell)$ to be given by

$$A_{ij} = 2<\alpha_i, \alpha_j>/<\alpha_i, \alpha_i> \text{ for all } i,j = 0,1,\ldots,\ell,$$

where $< , > \equiv < , >^{(1)}$ is evaluated taking in to account (4.2.31) and the information given in appendix A(3). The following verify that this is actually an affine Cartan matrix according to definition of chapter 3.

(i) The fact that it symmetrizable is straightforward. From the above relation **A** can be written as **A** = **DB** where **D** is non-singular and has diagonal entries $\varepsilon_i = 2/<\alpha_i, \alpha_i>$ (i = 0,1,...,ℓ). The matrix **B** has entries $B_{ij} = <\alpha_i, \alpha_j>$ and is obviously symmetric.

(ii) $A_{ii} = 2$ (i = 0,1,...,ℓ).

(iii) For i,j = 1,...,ℓ, A_{ij} are the entries of the Cartan matrix of B(0/ℓ), which is a Cartan matrix of finite type. In particular the root system of B(0/ℓ) is the unique finite non-reduced irreducible root system of type BC_ℓ and with α_ℓ being the odd simple root, $2\alpha_\ell$ is also a (even) root of B(0/ℓ). Moreover $A_{\ell j}$ (j = 1,...,ℓ) is an even non-negative integer.

(iv) We have to show that A_{j0} and A_{0j} are also non-negative integers. The highest root of B(0/ℓ) is nothing but the highest weight of the adjoint representation of B(0/ℓ) and thus it can be written in terms of the fundamental weight Λ_1. As $\overset{0}{\alpha}_H = 2\,\Lambda_1 = 2\sum_{r=1}^{\ell} \overset{0}{\alpha}_r$.

$$A_{j0} = 2<\alpha_j, \alpha_0> /<\alpha_j, \alpha_j> = -4 \sum_{r=1}^{\ell} <\overset{0}{\alpha}_j,\ \overset{0}{\alpha}_r\ >^0/<\overset{0}{\alpha}_j, \overset{0}{\alpha}_j>^0$$

Direct observation of appendix A(3) shows that

$A_{j0} = -4$ if $\ell = 1$ and for $\ell \geq 2$ $A_{j0} = -2$ if j=1 and 0 in any other case.

= 1, for j=1 and 0 in any other case. Thus $A_{10} = -4$ and $A_{j0} =$ 0 for j=2,...ℓ.

For A_{0j} (j = 1,...,ℓ), similar arguments show that $A_{0j} \neq 0$ if j=1 and in this case $A_{01} = -1$. $A_{0j} = 0$ in any other case.

(v) Finally we have to establish that **A** is of affine type.

From relation

$$\delta = \alpha_0 + \alpha_H^0 = \alpha_0 + 2 \sum_{r=1}^{\ell^0} \alpha_r^0 .$$

Let H_{α_k}, $k = 0,1,...,\ell$, be the Chevalley basis of $\mathcal{H}^{(1)}$ (see below). Then, since $\delta(H_{\alpha_k}) = 0$ for all $k = 0,1,...,\ell$, $\sum_{r=0}^{\ell} \delta(H_{\alpha_k}) = 0$ which by the use of the above relation and the definition of the Cartan matrix, imply that $\mathbf{Ak} = \mathbf{0}$, where $\mathbf{k}$ is a vector with entries $k_0 = 1$ and $k_i = 2$ for all $i=1,2,...,\ell$. Thus $\mathbf{A}$ is of affine type and det $\mathbf{A} = 0$.

It is easily checked that the the generalized Dynkin diagrams of $B^{(1)}(0/1)$ and $B^{(1)}(0/\ell)$ (for $\ell \geq 2$) evaluated using the above Cartan matrices are those of Figures 1 and 2 respectively.

(c) The Weyl group

<u>Definition 4.2</u> Scaled root lattice Q^v of $B^{(1)}(0/\ell)$

The scaled root lattice Q^v of $B^{(1)}(0/\ell)$ is defined to be the set of all linear functional α^v defined on $\mathcal{H}^{(1)}$ which have the form

$$\alpha^v = \sum_{j=0}^{\ell} k_j \alpha_j^v \tag{4.2.43}$$

where α_j^v is given by

$$\alpha_j^v = \{2/<\alpha_j , \alpha_j>\}\alpha_j \qquad \text{for all } j = 0,1,..., \ell \tag{4.2.44}$$

and k_j takes any integral value.

Since neither $B(0/\ell)$ nor $B^{(1)}(0/\ell)$ are simply laced, in general $\alpha^v + \beta^v \neq (\alpha + \beta)^v$.

We can investigate now the structure of the the Weyl group. Since we have identified the Cartan matrix, from section 3.4 we deduce that it is generated by reflections relative to the simple roots as found above which act on any functional β of $\mathcal{H}^{(1)^*}$ as in (3.56).

For every element α^v of the scaled root lattice Q^v of $B^{(1)}(0/\ell)$ and every linear functional β defined on $\mathcal{H}^{(1)}$, consider the following linear operator T_{α^v} acting on β that is defined by

$$T_{\alpha^v}(\beta) = \beta + <\beta, \delta> \alpha^v - \{ <\beta, \alpha^v> + \tfrac{1}{2} <\alpha^v, \alpha^v><\beta, \delta>\}\delta. \quad (4.2.45)$$

In particular if $\beta = \alpha$, where $\alpha \in \Delta^{(1)}$, or $\beta = \delta$ then

$$T_{\alpha^v}(\alpha) = \alpha - <\alpha, \alpha^v>\delta \quad \text{and } T_{\alpha^v}(\delta) = \delta . \quad (4.2.46)$$

As in the case of affine Kac-Moody algebras (see Cornwell(1989), Kac(1978)) the following properties can be easily established

<u>Proposition 4.2</u>

(a) For every two elements α^v and β^v of the scaled root lattice Q^v

$$T_{\alpha^v} T_{\beta^v} = T_{\alpha^v+\beta^v} \quad (4.2.47)$$

(b) For every element α^v of the scaled root lattice Q^v

$$T_{\alpha^v} = \prod_{j=1}^{\ell} (T_{\alpha_j^v})^{k_j} = \prod_{j=1}^{\ell} (S_{\delta-\alpha_j} S_{\alpha_j})^{k_j} , \quad (4.2.48)$$

where α_j^v are the scaled simple roots of $B^{(1)}(0/\ell)$, which are extensions of the simple roots of $B(0/\ell)$, and k_j are the integers of the expansion (4.2.43). In particular

$$T_{\alpha_j^v} = S_{\delta-\alpha_j} S_{\alpha_j} \qquad (j=1,2,\ldots,\ell). \qquad (4.2.49)$$

(c) The set T of elements of the form (4.2.42) is an invariant abelian subgoup of the Weyl group of $B^{(1)}(0/\ell)$. Moreover, the Weyl group of $B^{(1)}(0/\ell)$ has the semi-direct product structure

$$W = T \circledS W_s \qquad (4.2.50)$$

where W_s is the Weyl group of $B(0/\ell)$ (i.e. the Weyl group of the even part, C_ℓ , of $B(0/\ell)$).

Proof (see Cornwell(1989))

(d) The Chevalley generators

The basis vectors of the root subspaces corresponding to the simple roots and their negatives are given by

$$e_{\pm\alpha_k} = t^0 \otimes e^0_{\pm\alpha_k^0} \quad \text{for all } k = 1,2,\ldots,\ell, \quad e_{\pm\alpha_0} = t^{\pm 1} \otimes e^0_{\mp\alpha^0_H}$$

(4.2.51)

Now note that

$$[\, e^0_{\alpha^0_H} , e^0_{-\alpha^0_H} \,] = B^0(e^0_{\alpha^0_H} , e^0_{-\alpha^0_H}) h^0_{\alpha^0_H} \qquad (4.2.52)$$

and thus

$$[\, e_{\alpha_0} , e_{-\alpha_0} \,] = - t^0 \otimes [\, e^0_{\alpha^0_H} , e^0_{-\alpha^0_H} \,] + B^0(e^0_{\alpha^0_H} , e^0_{-\alpha^0_H})c$$

$$= B^0(e^0_{\alpha^0_H} , e^0_{-\alpha^0_H}) \{ - t^0 \otimes h^0_{\alpha^0_H} + c \} \qquad (4.2.53)$$

Consequently we may define an element h_{α_0} of $\mathcal{H}^{(1)}$ corresponding to the simple root α_0 by

$$h_{\alpha_0} \;=\; c - t^0 \otimes h^0_{\alpha^0_H} \;. \qquad (4.2.54)$$

We can normalize $e^0_{\alpha^0_H}$ and $e^0_{-\alpha^0_H}$ appropriately such that

$$B^0(E^0_{\alpha^0_H}, E^0_{-\alpha^0_H}) = 2/<\alpha^0_H, \alpha^0_H>^0 \qquad (4.2.55)$$

where $E^0_{\alpha^0_H}$, $E^0_{-\alpha^0_H}$ denote the normalized vectors $e^0_{\alpha^0_H}$ and $e^0_{-\alpha^0_H}$. A similar argument can be applied to $e_{\pm\alpha_k}$ (for all k = 1,2,...,ℓ) with $B^0(E^0_{\alpha^0_k}, E^0_{-\alpha^0_k}) = 2/<\alpha^0_k, \alpha^0_k>^0$. Then we get the "Chevalley" type basis vectors $E_{\pm\alpha_0}$ and $E_{\pm\alpha_k}$ are given by

$$E_{\pm\alpha_k} = t^0 \otimes E^0_{\pm\alpha^0_k} \quad \text{for all } k = 1,2,...,\ell \quad \text{and} \quad E_{\pm\alpha_0} = t^{\pm 1} \otimes E^0_{\mp\alpha^0_H}$$

$$H_{\alpha_0} = \{ 2/<\alpha_0, \alpha_0> \} h_{\alpha_0} = 2/<\alpha^0_H, \alpha^0_H>^0 \{- t^0 \otimes h^0_{\alpha^0_H} + c \}$$

$$H_{\alpha_k} = \{ 2/<\alpha_k, \alpha_k> \} h_{\alpha_k} = 2/<\alpha^0_k, \alpha^0_k>^0 \{t^0 \otimes h^0_{\alpha^0_k} \} \qquad (4.2.56)$$

for all k = 1,2,...,ℓ. Then it can be easily deduced that all relations (3.7) to (3.10) are satisfied and that the elements (4.2.56) generate $B^{(1)}(0/\ell)$. Moreover the sets $\{H_{\alpha_k}$ for all k = 0,1,2,...,$\ell\}$, $\{ \alpha_k$ for all k = 0,1,2,...,$\ell\}$ together with the $(\ell+2)$-dimensional complex vector space $\mathcal{H}^{(1)}$ provide a realization of the affine Cartan matrix of $B^{(1)}(0/\ell)$.

Finally in accordance with chapter 3, let Λ_0 be the linear functional defined on $\mathcal{H}^{(1)}$ by

$$\Lambda_0(H_{\alpha_k}) = 1 \text{ if } k=0, \quad \text{and } \Lambda_0(H_{\alpha_k}) = 0 \text{ if } k=1, 2,...,\ell. \qquad (4.2.57)$$

Let h_{Λ_0} be the corresponding element on $\mathcal{H}^{(1)}$. It is easily obtained that

$$h_{\Lambda_0} = \tfrac{1}{2}<\alpha^0_H, \alpha^0_H>^0 d,$$

$$\Lambda_0(h_\delta) = \tfrac{1}{2}\langle \overset{0}{\alpha}_H, \overset{0}{\alpha}_H \rangle^0 \text{ and } \Lambda_0(h_{\alpha^0}) = 0 ,$$

$$\langle \Lambda_0 , \Lambda_0 \rangle = 0. \tag{4.2.58}$$

for any $\overset{0}{\alpha} \in \Delta^0$ (and its extension). The set $\{ \Lambda_0 , \alpha_k$ for all k $= 0,1,2,\ldots,\ell\}$ provide a basis of $\mathcal{H}^{(1)}$.

Finally the <u>even</u> part of $B^{(1)}(0/\ell)$ is easily recognized to be $B^{(1)}(0/\ell)_0 = C_\ell^{(1)}$.

4.3 Explicit realization of affine twisted Kac-Moody superalgebras $\tilde{\mathcal{L}}_s^{(m)}$

A. Basic concepts and definitions

Let $\tilde{\mathcal{L}}_s^0$ be one of the basic classical simple complex Lie superalgebras. Let ϕ be an graded automorphism of $\tilde{\mathcal{L}}_s^0$ of finite order $q \neq 1$, such that under its action, $\tilde{\mathcal{L}}_s^0$ is decomposed as

$$\tilde{\mathcal{L}}_s^0 = \oplus \sum_{p=0}^{q-1} \tilde{\mathcal{L}}_{sp}^{0(q)} . \qquad (4.3.1)$$

where $\tilde{\mathcal{L}}_{sp}^{0(q)}$ are the subspaces of $\tilde{\mathcal{L}}_s^0$ that consists of all the elements a^0 of $\tilde{\mathcal{L}}_s^0$ such that

$$\phi(a^0) = e^{2\pi pi/q} a^0 , \qquad (4.3.2)$$

where p = 0,1, ... ,q-1. That is, $\tilde{\mathcal{L}}_{sp}^{0(q)}$ are eigenspaces of ϕ with corresponding eigenvalues $e^{2\pi pi/q}$ and (4.3.1)describes a Z_q-graduation of $\tilde{\mathcal{L}}_s^0$. It follows that $\tilde{\mathcal{L}}_{s0}^{0(q)}$ is a Lie (super)algebra, and that for each p taking the value 1,2, ..., q-1 the subspace $\tilde{\mathcal{L}}_{sp}^{0(q)}$ provides a carrier space for a representation Γ^p of $\tilde{\mathcal{L}}_{s0}^{0(q)}$ by the prescription

$$[a_{0r}^0 , a_{pr'}^0] = \sum_{r''=1}^{n_p} \Gamma^p(a_{0r}^0)_{r''r'} a_{pr''}^0 \qquad (4.3.3)$$

for all a_{0r}^0 of $\tilde{\mathcal{L}}_{s0}^{0(q)}$, where n_p is the dimension of $\tilde{\mathcal{L}}_{sp}^{0(q)}$ and a_{pr}^0 (for r = 1,2, ... ,n_p) are the basis elements of $\tilde{\mathcal{L}}_{sp}^{0(q)}$. We assume that ϕ leaves invariant at least one simple component of the even part of $\tilde{\mathcal{L}}_s^0$.

The main interest is the case in which ϕ is an <u>outer</u> automorphism. The structure of the group Out($\tilde{\mathcal{L}}_s^0$) of outer automorphisms of $\tilde{\mathfrak{L}}_s^0$ was demonstrated by Serganova(1983,1985). Moreover Serganova has shown that if ϕ belongs to the connected component of the identity of the group of automorphisms of $\tilde{\mathfrak{L}}_s^0$, the twisted loop superalgebra defined by

$$\sum_{p=0}^{q-1} \sum_{j=-\infty}^{\infty} (t^{qj+p} \otimes \tilde{\mathfrak{L}}_{sp}^{0(q)})$$

is not a subsuperalgebra of the loop superalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$, but is actually isomorphic to the loop superalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$. These untwisted and twisted loop superalgebras were termed infinite-dimensional contragredient Lie superalgebras, and their root systems, together with all their inequivalent systems of simple roots (and Dynkin diagrams) were presented by Serganova. Later in the work of Van der Leur(1986) a more consistent description of them was presented , in which it was shown that they are the only infinite dimensional contragredient Lie superalgebras of finite growth. Because of the their profound similarity with the affine Kac-Moody algebras, they were termed affine too. In this connection it may be noted that Frappat et al (1989) have shown that it is sometimes possible by using non-distinguished sets of simple roots to construct generalized Dynkin diagrams for the basic simple Lie superalgebras which possess rotational symmetries that do correspond to outer automorphisms of these superalgebras. However this is not possible in every case, the simplest example where it cannot be done being A(2/0).

In addition they showed that by folding symmetric generalized Dynkin diagrams of untwisted Kac-Moody superalgebras we can construct twisted Kac-Moody superalgebras. In what follows we shall outline the explicit realization of twisted Kac-Moody superalgebras and then we shall study the affine Kac-Moody superalgebras $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$.

We can associate with the outer automorphism ϕ a subsuperalgebra $\tilde{\mathcal{L}}_s^{(q)}$ of $\tilde{\mathcal{L}}_s^{(1)}$ whose set of basis elements consists of c, d, and, for p = 0,1,..., and q-1, of all $t^j \otimes a^0$ for every integer j that is such that j mod q = p and every basis element $a^0_{pr} \in \tilde{\mathcal{L}}^{0(q)}_{sp}$. This may be summarized by the statement that

$$\tilde{\mathcal{L}}_s^{(q)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus \sum_{p=0}^{q-1} \sum_{j=-\infty;\ j \bmod q = p}^{\infty} (t^j \otimes \tilde{\mathcal{L}}^{0(q)}_{sp}) \tag{4.3.4}$$

or equivalently

$$\tilde{\mathcal{L}}_s^{(q)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus \sum_{p=0}^{q-1} \sum_{j=-\infty}^{\infty} (t^{qj+p} \otimes \tilde{\mathcal{L}}^{0(q)}_{sp}) \tag{4.3.5}$$

(4.3.4) will be called <u>affine twisted Kac-Moody superalgebra</u>. Its <u>derived superalgebra</u> is simply

$$[\tilde{\mathcal{L}}_s^{(q)}, \tilde{\mathcal{L}}_s^{(q)}] = (\mathbb{C}c) \oplus \sum_{p=0}^{q-1} \sum_{j=-\infty}^{\infty} (t^{qj+p} \otimes \tilde{\mathcal{L}}^{0(q)}_{sp}).$$

The generalized Lie products of $\tilde{\mathcal{L}}_s$ are then those inherited from $\tilde{\mathcal{L}}_s^{(1)}$ and so are given by (4.2.2) to (4.2.6). Let $\mathcal{H}^0$ be the Cartan subalgebra of $\tilde{\mathcal{L}}_s^0$ and consider the subset $\mathcal{H}^{0(q)}$ of elements of $\mathcal{H}^0$ given by

$$\mathcal{H}^{0(q)} = \mathcal{H}^0 \cap \tilde{\mathcal{L}}^{0(q)}_{s0}. \tag{4.3.6}$$

It can be easlily seen that the maximal set of commuting elements in (4.3.4) is then given by

$$\mathcal{H}^{(q)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus (t^0 \otimes \mathcal{H}^{0(q)}) \qquad (4.3.7)$$

and thus consitutes the Cartan subalgebra of $\tilde{\mathfrak{L}}_s^{(q)}$.

The supersymmetric invariant non-degenerate bilinear form B(,) of $\tilde{\mathfrak{L}}_s^{(q)}$ may be taken to be such that

$$B^{(q)}(a,b) = \mu\, B^{(1)}(a,b) \qquad (4.3.8)$$

for all a,b $\in \tilde{\mathfrak{L}}_s^{(q)}$, μ being an arbitrary constant which may be chosen in any way and $B^{(1)}(a,b)$ is as in (4.2.10-13). As the subset of elements of $\tilde{\mathfrak{L}}_s^{(q)}$ of the form $t^0 \otimes a^0$ (for all the elements a^0 of $\tilde{\mathfrak{L}}_s^0$) form a subalgebra that is isomorphic to $\tilde{\mathfrak{L}}_s^0$, a particularly convenient choice is to let μ be such that $B^{(q)}(\; , \;)$ coincides with the supersymetric invariant non-degenarate bilinear form $B_{s0}^{0(q)}(\; , \;)$ of $\tilde{\mathfrak{L}}_{s0}^{0(q)}$, that is, so that

$$B^{(q)}(t^0 \otimes a^0, t^0 \otimes b^0) = B_{s0}^{0(q)}(a^0,b^0) \text{ for all } a^0,b^0 \text{ of } \tilde{\mathfrak{L}}_{s0}^{0(q)}. \qquad (4.3.9)$$

Since $B^0(\; , \;)$ is invariant under the automorphism ϕ it is not difficult to show that for any two basis elements a^0_{pr} ans $a^0_{p'r'}$ of the subspaces $\tilde{\mathfrak{L}}_{sp}^{0(q)}$ and $\tilde{\mathfrak{L}}_{sp'}^{0(q)}$ respectively,

$$B^0(a^0_{pr}, a^0_{p'r'}) \neq 0 \text{ if and only if } (p+p') \bmod q = 0 \qquad (4.3.10)$$

for all p,p' = 1,...q-1, r = 1,...n_p, and r' = 1,...,$n_{p'}$.

Every linear functional α^0 defined on $\mathcal{H}^{(1)}$ that is an extension of a linear functional defined on $\mathcal{H}^0$ can be restricted to become a linear functional on $\mathcal{H}^{(q)}$ by the following definitions

$$\alpha^0(t^0 \otimes h) = \alpha^0(h), \qquad \alpha^0(c) = 0, \qquad \alpha^0(d) = 0 \qquad (4.3.11)$$

for all h $\in \mathcal{H}^{0(q)}$. Consequently all the roots of $\tilde{\mathfrak{L}}_s^0$ are restricted to $\mathcal{H}^{0(q)}$. Next let us define a linear functional δ on $\mathcal{H}^{(q)}$ by

$$\delta(t^0 \otimes h) = 0 \text{ (for all } h \in \mathcal{H}^{0(q)}), \quad \delta(c) = 0, \quad \delta(d) = 1. \quad (4.3.12)$$

With these definitions the determination of the root structure of $\tilde{\mathcal{L}}_s^{(q)}$ follows the same steps as in the untwisted case.

A non-zero linear functional α defined on $\mathcal{H}^{(q)}$ is called a <u>root</u> of $\tilde{\mathcal{L}}_s^{(q)}$ if there exists at least one element a_α of $\tilde{\mathcal{L}}_s^{(q)}$ such that

$$[\, h, a_\alpha] = \alpha\,(h)\, a_\alpha \qquad (4.3.13)$$

for all $h \in \mathcal{H}^{(q)}$. The set of elements a_α, for each such α, that satisfy the above relation form the <u>root subspace</u> $\tilde{\mathcal{L}}_{s\alpha}^{(q)}$ of $\tilde{\mathcal{L}}_s^{(q)}$.

We denote by $\Delta_0^{0(q)}$ the set of roots of $\tilde{\mathcal{L}}_{s0}^{0(q)}$ and by $\Delta_p^{0(q)}$ (p = 1,...,q-1) the set of weights of the representations that the pth subspace provide for $\tilde{\mathcal{L}}_{s0}^{0(q)}$, and by $\Delta^{0(q)}$ the set of all roots and weights from $\Delta_p^{0(q)}$ (p = 1,...,q-1). All the elements of the above sets are defined on $\mathcal{H}^{0(q)}$. With the obvious modification of (4.2.22) to (4.2.25) becomes

$$[\, h\, ,\, t^j \otimes e_\alpha] \;=\; \{j\delta(h) + \alpha(h)\}\; (t^j \otimes e_\alpha\,) \qquad (4.3.14)$$

for <u>all</u> $h \in \mathcal{H}^{(q)}$ all integers j such that jmodq=p and where e_α is the element of the 0th subspace corresponding to the root of $\tilde{\mathcal{L}}_{s0}^{0(q)}$ or an element of the pth (p = 1,...,q-1) subspace corresponding to the weight $\alpha \in \Delta_p^{0(q)}$ (p = 1,...,q-1). Obviously $t^j \otimes e_\alpha$ corresponds to the root $j\delta + \alpha$ of $\tilde{\mathcal{L}}_s^{(q)}$. Any such root will be called 'real'.

Similarly, for any $\beta \in \Delta_0^{0(q)}$ and $h_\beta \in \mathcal{H}^{0(q)}$ and any non-zero integer j such that jmodq = 0, (4.2.29) imply that

$$[\ h,\ t^j \otimes h_\beta\ ] \ = \ j\delta(h)\ (t^j \otimes h_\beta) \qquad (4.3.15)$$

for all $h \in \mathcal{H}^{(q)}$. Thus $t^j \otimes h_\beta$ corresponds to the root $j\delta$ of $\tilde{\mathfrak{L}}_s^{(1)}$ such that jmodq = 0. Moreover there are ℓ linearly independent elements with this property, corresponding to the basis elements of $\mathcal{H}^{0(q)}$. Thus the root subspace of $j\delta$ such that jmodq = 0 must have dimension ℓ (for $j \neq 0$).

Finally for any zero weight of the q-1 representations similar reasoning implies that there exist roots $j\delta$ such that jmodq = p (p = 1,2,...,q-1) and their multiplicity is the dimension of the corresponding weight subspaces. The roots of the form $j\delta$
are called 'imaginary'.

Since $B^{(q)}(\ ,\ )$ is symmetric non-degenerate in $\mathcal{H}^{(q)}$ for every linear functional α defined on $\mathcal{H}^{(q)}$ an element h_α of $\mathcal{H}^{(q)}$ may be defined such that

$$B^{(q)}(h,\ h_\alpha) = \alpha(h) \qquad (4.3.16)$$

and thus a bilinear form on the dual space $\mathcal{H}^{(q)*}$ is defined by

$$\langle\alpha,\ \beta\rangle^{(q)} \ = B^{(q)}(h_\alpha,\ h_\beta) \qquad (4.3.17)$$

The definition of $B^{(q)}(\ ,\ )$ together with (1.1.5-6) and (4.3.17) imply that

$$h_\delta \ = \ (1/\mu)\ c, \qquad (4.3.18)$$

$$\langle j\delta,\ j\delta\rangle^{(q)} = 0 \quad \text{and} \quad \langle j\delta+\alpha,\ j\delta+\beta\ \rangle^{(q)} = \langle\alpha,\ \beta\rangle^{(q)} \qquad (4.3.19)$$

where α and β are extensions of non-zero linear functionals defined on $\mathcal{H}^{0(q)}$.

In what follows we shall establish using particular automorphisms of A(2ℓ-1/0), A(2ℓ/0), and C(ℓ+1), that the corresponding superalgebras $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$ (whose structures are given by (4.3.5)) are isomorphic to the affine superalgebras $\tilde{\mathfrak{L}}_s$ of chapter 3, and have the Dynkin diagrams and Cartan matrices as indicated in figures 3 to 8. The existence and the order of the automorphisms together with the explicit realization just described was demonstrated by Kac(1978) for $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$.

Consider the Dynkin diagrams of figures 3 to 8 corresponding to the Kac-Moody superalgebra $\tilde{\mathfrak{L}}_s^{(m)}$ and choose the node corresponding to <u>any</u> simple root α_K of $\tilde{\mathfrak{L}}_s^{(m)}$. Suppose that the corresponding numerical mark is N_K. Let q be the integer defined by

$$q = mN_K \, . \qquad (4.3.20)$$

Then (see Kac(1978)) there exists an automorphism ϕ of $\tilde{\mathfrak{L}}_s^{0}$ of order q such that (4.3.1) and (4.3.2) hold and such that $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ is a Lie (super)algebra whose Dynkin diagram is the one that remains from the Dynkin diagram of $\tilde{\mathfrak{L}}_s^{(m)}$ when the kth node is removed together with all the lines attached to it. Inspection of Figures 3 to 8 in table V shows that the only possible values of q are 1, 2, and 4 and that $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ is either simple or is the direct sum of simple Lie superalgebras. Clearly if $\tilde{\mathfrak{L}}_s^{(m)}$ has only one odd simple root and if the chosen node corresponds to this odd simple root, then $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ will contain no non-trivial odd part, and so in this case $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ will be a semi-simple Lie algebra. Also the Γ^1 representation which the $\tilde{\mathfrak{L}}_{s1}^{0(q)}$ subspace provides on $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ is irreducible.

We fix an enumeration of the nodes of the Dynkin diagram as shown in Figures 3 to 8. The choice of the node to be removed is the far right one for $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$, the middle one for $A^{(2)}(3/0)$ and the far left one for $C^{(2)}(\ell+1)$. Thus q = 4 or 2 for $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$ respectively. Moreover,direct observation of the Dynkin diagrams of $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$ shows that $\tilde{\mathfrak{L}}_{s0}^{0(q)} = D_\ell$, B_ℓ and $B(0/\ell)$ respectively. It should be noted that the above reasoning can also be applied for $B^{(1)}(0/\ell)$ by chosing for example the far right node with N_K = 2. This implies that q = 2. But $B(0/\ell)$ does not posseses any outer automorphisms and thus $B^{(2)}(0/\ell) = B^{(1)}(0/\ell)$.

The obvious choice of the automorphism ϕ of order 4 of the simple Lie superalgebra $\tilde{\mathfrak{L}}_s^0$ (= $A(2\ell-1/0)$, $A(2\ell/0)$) is the "canonical" 4-fold automorphism ψ that is defined by

$$\psi(h^0) = -h^0 \text{ (for all } h^0 \text{ of } \mathcal{H}^0), \tag{4.3.21}$$

$$\psi(e^0_{\alpha^0}) = e^0_{-\alpha^0}$$

$$\text{(if } \alpha^0 (\in \Delta^0) \text{ is even or is odd and negative)}, \tag{4.3.22}$$

and

$$\psi(e^0_{\alpha^0}) = -e^0_{-\alpha^0} \text{ (if } \alpha^0 (\in \Delta^0) \text{ is odd and positive)} \tag{4.3.23}$$

(see Scheunert (1978)). With this choice (4.3.2) implies that

(i) the basis elements of $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ may be taken to be:

$e^0_{\alpha^0} + e^0_{-\alpha^0}$, for all even positive roots $\alpha^0 \in \Delta^0$;

(ii) the basis elements of $\tilde{\mathcal{L}}^{0(q)}_{s1}$ may be taken to be:

$e^0_{\alpha^0} + ie^0_{-\alpha^0}$, for all odd positive roots $\alpha^0 \in \Delta^0$;

(iii) the basis elements of $\tilde{\mathcal{L}}^{0(q)}_{s2}$ may be taken to be:

$e^0_{\alpha^0} - e^0_{-\alpha^0}$, for all even positive roots $\alpha^0 \in \Delta^0$,

and

$h^0_{\alpha^0_k}$, for $k = 1,2,\ldots,\ell^0$;

(iv) the basis elements of $\tilde{\mathcal{L}}^{0(q)}_{s3}$ may be taken to be:

$e^0_{\alpha^0} - ie^0_{-\alpha^0}$, for all odd positive roots $\alpha^0 \in \Delta^0$.

This automorphism has been used by Golitzin(1988) to find the simple roots and generators of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$.

Although these basis elements are very straightforward, the difficulties start arising with this choice of automorphism when one tries to determine explicitly the complete root structure of the Kac-Moody superalgebra $\tilde{\mathcal{L}}^{(m)}_s$. The problem is that if the Cartan subalgebra of $\tilde{\mathcal{L}}^{(m)}_s$ is chosen to be in $\tilde{\mathcal{L}}^{0(q)}_{s0}$, (as in the case $q = 1$), it <u>cannot</u> consist of c, d and elements of the form $t^0 \otimes h^0_{\alpha^0_k}$ (for $k = 1,2,\ldots,\ell^0$), for the elements $h^0_{\alpha^0_k}$ are not members of $\tilde{\mathcal{L}}^{0(q)}_{s0}$. Instead the simplest choice is c, d, and certain linear combinations of $t^0 \otimes (e^0_{\alpha^0} + e^0_{-\alpha^0})$ (for the even positive roots $\alpha^0 \in \Delta^0$). To find the roots it is then necessary to evaluate the generalized Lie products of these with all the elements of the sets (i) to (iv) above, taking appropriate linear combinations of the latter in order satisfy the root equation (4.3.13). Not only is this messy, it also makes no direct use of the known root structure of the simple Lie superalgebra $\tilde{\mathcal{L}}^0_s$. Indeed the situation here is very similar to the one that occurs in the standard method of determination of the Iwasawa and Langlands decompositions of the simple Lie algebras, and the resolution of the problem is based on essentially the same idea as that of the "direct" determination of these decompositions that was given by Cornwell (1975, 1979).

The most general 4-fold automorphism ϕ of $\tilde{\mathcal{L}}_s^0$ (= $A(2\ell-1/0)$, $A(2\ell/0)$) has the form

$$\phi = \theta^{-1} \psi \theta \tag{4.3.24}$$

where ψ is the canonical 4-fold automorphism of $\tilde{\mathcal{L}}_s^0$ defined above and θ is any automorphism of $\tilde{\mathcal{L}}_s^0$. If θ can be chosen so that enough elements of the form $t^0 \otimes h^0_{\alpha_k^0}$ lie in $\tilde{\mathcal{L}}_{s0}^{0(q)}$ then the roots of the Kac-Moody superalgebra will be very easy to obtain. In investigating this condition it is useful to note that if the simple Lie superalgebra $\tilde{\mathcal{L}}_s^0$ is expressed in terms of supermatrices with the graded partitioning

$$\underset{\sim}{M} = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{pmatrix} \tag{4.3.25}$$

then

$$\psi(\underset{\sim}{M}) = -\underset{\sim}{M}^{st} = \begin{pmatrix} -\tilde{\underset{\sim}{A}} & \tilde{\underset{\sim}{C}} \\ -\tilde{\underset{\sim}{B}} & -\tilde{\underset{\sim}{D}} \end{pmatrix}, \tag{4.3.26}$$

where $\tilde{\underset{\sim}{A}}$ denotes the ordinary transpose of $\underset{\sim}{A}$.

As for the two-fold automorphism of $C(\ell+1)$, one could naturally use ψ^2 but as we shall see this would only lead to $\tilde{\mathcal{L}}_{s0}^{0(q)} = C_\ell$ and not to $B(0/\ell)$.

Incidentally, it is clear that the canonical 4-fold automorphism ψ of $\tilde{\mathcal{L}}_s^0$ is not associated with any rotation of the usual generalized Dynkin diagram of $\tilde{\mathcal{L}}_s^0$ based on the distinguished simple roots, because for

$A(2\ell-1/0)$, $A(2\ell/0)$, and $C(\ell+1)$ the generalized Dynkin diagrams exhibited in Figures 10, 11 and 12 possess no symmetries.

The choice of θ will first be investigated first for the Kac-Moody superalgebras of the form $A^{(2)}(2\ell-1/0)$ (for ℓ = 2,3,...).

B. Root structure of $A^{(2)}(2\ell-1/0)$ (for ℓ = 2,3,...)

(a) The 4-fold automorphism

An explicit realization of the simple Lie superalgebra $A(2\ell-1/0)$ is provided by $s\ell(2\ell/1)$, considered as a complex superalgebra, $s\ell(2\ell/1)$ being defined as the set of $(2\ell+1)\times(2\ell+1)$ complex supermatrices with the grading partitioning

$$\underset{\sim}{M} = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{pmatrix}, \qquad (4.3.27)$$

that are subject to the supertrace condition that

$$\text{str}\,\underset{\sim}{M} = \underset{\sim}{0} . \qquad (4.3.28)$$

(Here $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are of dimensions $2\ell\times2\ell$, $2\ell\times1$, $1\times2\ell$, and 1×1 respectively). The rank ℓ^0 of $A(2\ell-1/0)$ is given by $\ell^0 = 2\ell$. The generalized Dynkin diagram of $A(2\ell-1/0)$ is shown in Figure 10, which indicates that its distinguished simple roots α_k^0 are even for k = 1,2,...,$2\ell-1$, but that $\alpha_{2\ell}^0$ is odd.

With the bilinear form $B^0(\ ,\ )$ being defined by

$$B^0(\underset{\sim}{M},\underset{\sim}{N}) = 2(2\ell-1)\,\text{str}\,(\underset{\sim}{M}\underset{\sim}{N}) , \qquad (4.3.29)$$

the basis elements of its Cartan subalgebra $\mathcal{H}^0$ may be taken to be

$$\underset{\sim}{h}^0_{\alpha_k^0} = \{1/2(2\ell-1)\}\,\{\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+1,k+1}\}, \text{ (for k = 1,2,...,}2\ell-1) \qquad (4.3.30a)$$

and

$$\underset{\sim}{h}^0_{\alpha_{2\ell}^0} = \{1/2(2\ell-1)\}\,\{\underset{\sim}{e}_{2\ell,2\ell} + \underset{\sim}{e}_{2\ell+1,2\ell+1}\} . \qquad (4.3.30b)$$

Here $\underset{\sim}{e}_{r,s}$ is the matrix of dimension $(2\ell+1)\times(2\ell+1)$ that is defined by

$$(\underset{\sim}{e}_{r,s})_{jk} = \delta_{rj}\delta_{sk} \quad \text{(for } j,k = 1,2,\ldots,2\ell+1\text{)}, \tag{4.3.31}$$

so that with this choice all the matrices of $\mathcal{H}^0$ are diagonal. The positive even roots $\beta^0_{(j,k)}$ and positive odd roots $\delta^0_{(j)}$ of $A(2\ell-1/0)$ are given in terms of the distinguished set of simple roots $\overset{0}{\alpha}_1, \overset{0}{\alpha}_2, \ldots, \overset{0}{\alpha}_{2\ell}$ of $A(2\ell-1/0)$ by

$$\beta^0_{(j,k)} = \sum_{r=j}^{k-1} \overset{0}{\alpha}_r \quad \text{(for } j,k = 1,2, \ldots, 2\ell, \text{ with } j < k\text{)}, \tag{4.3.32a}$$

and

$$\delta^0_{(j)} = \sum_{r=j}^{2\ell} \overset{0}{\alpha}_r \quad \text{(for } j = 1,2, \ldots, 2\ell\text{)}, \tag{4.3.32b}$$

for which the corresponding basis elements of $A(2\ell-1/0)$ may be taken to be

$$\overset{0}{\underset{\sim}{e}}_{\beta^0_{(j,k)}} = \overset{0}{\underset{\sim}{e}}_{\overset{0}{\alpha}_j+\overset{0}{\alpha}_{j+1}+\ldots+\overset{0}{\alpha}_{k-1}} = \underset{\sim}{e}_{j,k} \quad \text{(for } j,k=1,2, \ldots, 2\ell;\ j < k\text{)} \tag{4.3.33a}$$

$$\overset{0}{\underset{\sim}{e}}_{\delta^0_{(j)}} = \overset{0}{\underset{\sim}{e}}_{\overset{0}{\alpha}_j+\overset{0}{\alpha}_{j+1}+\ldots+\overset{0}{\alpha}_{2\ell}} = \underset{\sim}{e}_{j,2\ell+1} \quad \text{(for } j = 1,2, \ldots, 2\ell\text{)}. \tag{4.3.33b}$$

The basis elements belonging to the corresponding negative roots may be chosen in accordance with (4.1).

Taking the node corresponding to the odd simple root α_ℓ of $A^{(2)}(2\ell-1/0)$ for $\ell \geq 3$, and to the odd simple root α_1 of $A^{(2)}(2\ell-1/0)$ (= $A^{(2)}(3/0)$) for $\ell = 2$, as the corresponding numerical mark has value 2, (c.f. Figures 3 and 4) $q = 4$. It follows from (4.3.2) that if the automorphism (4.3.26) is employed then the subalgebra $\tilde{\mathcal{L}}^{0(4)}_{s0}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\tilde{\underset{\sim}{A}} = \underset{\sim}{A}, \quad -\tilde{\underset{\sim}{B}} = \underset{\sim}{C}, \quad \tilde{\underset{\sim}{C}} = \underset{\sim}{B}, \quad \text{and} \quad -\tilde{\underset{\sim}{D}} = \underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1×1, imply that

$$-\tilde{\underset{\sim}{A}} = \underset{\sim}{A},\ \underset{\sim}{B} = \underset{\sim}{0},\ \underset{\sim}{C} = \underset{\sim}{0},\ \text{and}\ \underset{\sim}{D} = \underset{\sim}{0}. \qquad (4.3.34)$$

Thus subalgebra $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ is isomorphic to the set of $2\ell \times 2\ell$ complex antisymmetric matrices and hence is isomorphic to the simple complex Lie algebra D_ℓ, which is simple if $\ell > 2$ but is only semi-simple if $\ell = 2$, for then $D_2 = A_1 \oplus A_1$. As expected from the comments of the previous section, none of the basis elements of the Cartan subalgebra $\mathcal{H}^0$ of $A(2\ell-1/0)$ are members of this $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ (because all the members of this $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ are <u>non</u>-diagonal matrices).

A realization of D_ℓ in which the basis elements of the Cartan subalgebra of D_ℓ are given by diagonal matrices is given by the $2\ell \times 2\ell$ complex matrices $\underset{\sim}{A}'$ that satisfy the condition

$$\tilde{\underset{\sim}{A}}'\underset{\sim}{G} + \underset{\sim}{G}\underset{\sim}{A}' = \underset{\sim}{0}\ , \qquad (4.3.35)$$

where

$$\underset{\sim}{G} = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{1}_\ell \\ \underset{\sim}{1}_\ell & \underset{\sim}{0} \end{pmatrix}. \qquad (4.3.36)$$

This realization will be referred to as the "canonical" form of D_ℓ. These matrices $\underset{\sim}{A}'$ are related to the $2\ell \times 2\ell$ antisymmetric matrices $\underset{\sim}{A}$ by

$$\underset{\sim}{T}^{-1}\underset{\sim}{A}\underset{\sim}{T} = \underset{\sim}{A}', \qquad (4.3.37)$$

where $\underset{\sim}{T}$ is a certain $2\ell \times 2\ell$ complex matrix that maps the Lie algebra so(2N) in to its canonical form and satisfies the condition

$$\tilde{\underset{\sim}{T}}\underset{\sim}{T} = \underset{\sim}{G} \qquad (4.3.38)$$

(see Cornwell(1975) for the actual form of $\underset{\sim}{T}$). In what follows only (4.3.38) is needed).

This mapping can be extended to an automorphism of θ of $\tilde{\mathcal{L}}_s^0$ (= $s\ell(2\ell/1)$) by the definition

$$\theta(\underset{\sim}{M}) = \begin{pmatrix} \underset{\sim}{T} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_1 \end{pmatrix} \underset{\sim}{M} \begin{pmatrix} \underset{\sim}{T}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_1 \end{pmatrix} \qquad (4.3.39)$$

for all $\underset{\sim}{M}$ of $s\ell(2\ell/1)$. Then, by (4.3.24), (4.3.38) and (4.3.39),

$$\phi\left(\begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{pmatrix}\right) = \begin{pmatrix} -\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} & \underset{\sim}{G}\tilde{\underset{\sim}{C}} \\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} & -\tilde{\underset{\sim}{D}} \end{pmatrix}. \qquad (4.3.40)$$

The 4 subspaces $\tilde{\mathcal{L}}_{sp}^{0(4)}$ (for p = 0,1,2,3) corresponding to the automorphism ϕ of (4.3.40) will now be considered in turn:

(b) The subspace $\tilde{\mathcal{L}}_{s0}^{0(4)}$

By (4.3.2) the subalgebra $\tilde{\mathcal{L}}_{s0}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = \underset{\sim}{A},\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = \underset{\sim}{C},\ \underset{\sim}{G}\tilde{\underset{\sim}{C}} = \underset{\sim}{B},\ \text{and } -\tilde{\underset{\sim}{D}} = \underset{\sim}{D},$$

which when taken together, along with the fact that D is 1x1, imply that

$$\tilde{\underset{\sim}{A}}\underset{\sim}{G} + \underset{\sim}{G}\underset{\sim}{A} = \underset{\sim}{0},\ \underset{\sim}{B} = \underset{\sim}{0},\ \underset{\sim}{C} = \underset{\sim}{0},\ \text{and } \underset{\sim}{D} = \underset{\sim}{0}, \qquad (4.3.41)$$

and so is isomorphic to the canonical form of D_ℓ.

Before proceeding it will be useful to recall some properties of the canonical form of D_ℓ (see Konuma et al(1963) and Cornwell(1975)). Its Killing form $B^{D_\ell}(\ ,\ )$ is given by

$$B^{D_\ell}(\underset{\sim}{A},\underset{\sim}{A}') = 2(\ell-1)\ \mathrm{tr}(\underset{\sim}{A}\underset{\sim}{A}'), \qquad (4.3.42)$$

(for all $\underset{\sim}{A}$ and $\underset{\sim}{A}'$ of the canonical form). Thus, by (4.3.3),

$$B^0\left(\begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix},\begin{pmatrix} \underset{\sim}{A}' & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}\right)=\{(2\ell-1)/(\ell-1)\}B^{D_\ell}(\underset{\sim}{A},\underset{\sim}{A}') \qquad (4.3.43)$$

for all $\underset{\sim}{A}$ and $\underset{\sim}{A}'$ of the canonical form. This implies that (4.3.8) is satisfied if

$$\mu = (\ell-1)/(2\ell-1)\ . \qquad (4.3.44)$$

Denoting the simple roots of D_ℓ by $\alpha_k^{D_\ell}$ (for k = 1,2, ... ,ℓ), the corresponding basis elements of the Cartan subalgebra $\mathcal{H}^{D_\ell}$ of D_ℓ defined by

$$B^{D_\ell}(\underset{\sim}{h}^{D_\ell}_{\alpha_k^{D_\ell}},\underset{\sim}{h}) = \alpha_k^{D_\ell}(\underset{\sim}{h}) \text{ for all } \underset{\sim}{h} \in \mathcal{H}^{D_\ell}. \qquad (4.3.45)$$

are

$$\underset{\sim}{h}^{D_\ell}_{\alpha^{D_\ell}_k} = \{1/4(\ell-1)\}\ \{\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+\ell,k+\ell} - \underset{\sim}{e}_{k+1,k+1} + \underset{\sim}{e}_{k+\ell+1,k+\ell+1}\},$$

(for $k = 1,2, \ldots ,\ell-1$) (4.3.46)

and

$$\underset{\sim}{h}^{D_\ell}_{\alpha^{D_\ell}_\ell} = \{1/4(\ell-1)\}\ \{\underset{\sim}{e}_{\ell-1,\ell-1} - \underset{\sim}{e}_{2\ell-1,2\ell-1} + \underset{\sim}{e}_{\ell,\ell} - \underset{\sim}{e}_{2\ell,2\ell}\}\ . \quad (4.3.47)$$

The associated root subspace basis elements are

$$\underset{\sim}{e}^{D_\ell}_{\alpha^{D_\ell}_k} = \{1/2(\ell-1)\}\{\underset{\sim}{e}_{k,k+1} - \underset{\sim}{e}_{k+\ell+1,k+\ell}\}, \text{(for } k = 1,2,\ldots,\ell-1) \quad (4.3.48)$$

and

$$\underset{\sim}{e}^{D_\ell}_{\alpha^{D_\ell}_\ell} = \{1/2(\ell-1)\}\ \{\underset{\sim}{e}_{\ell-1,2\ell} - \underset{\sim}{e}_{\ell,2\ell-1}\}\ , \quad (4.3.49)$$

the normalization factors being chosen so that

$$B^{D_\ell}(\underset{\sim}{e}^{D_\ell}_{\alpha^{D_\ell}_k}, \underset{\sim}{e}^{D_\ell}_{-\alpha^{D_\ell}_k}) = -1, \quad (4.3.50)$$

where, as usual,

$$\underset{\sim}{e}^{D_\ell}_{-\alpha^{D_\ell}_k} = -\tilde{\underset{\sim}{e}}^{D_\ell}_{\alpha^{D_\ell}_k}\ . \quad (4.3.51)$$

The <u>diagonal</u> basis elements of $\tilde{\mathcal{L}}^{0(4)}_{s0}$ will be considered first. As they may be taken to consist of the set $\{\underset{\sim}{e}_{k,k}-\underset{\sim}{e}_{k+\ell,k+\ell}$ $|$ for $k = 1,2, \ldots ,\ell\}$, it follows that they are all members of the Cartan subalgebra $\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$) of D_ℓ (as expected). As

$$\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+\ell,k+\ell} = 2(2\ell-1)\sum\nolimits_{r=k}^{k+\ell-1} \underset{\sim}{h}^{0}_{\alpha^0_r} \quad (4.3.52)$$

(for $k = 1,2, \ldots ,\ell$, by (4.3.30a)), the most general element of $\mathcal{H}^{0(4)}$ has the form

$$\sum\nolimits_{k=1}^{\ell} \kappa_k(\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+\ell,k+\ell}) = 2(2\ell-1)\sum\nolimits_{k=1}^{2\ell-1} \rho_k \underset{\sim}{h}^{0}_{\alpha^0_k}\ , \quad (4.3.53)$$

where $\kappa_1,\kappa_2, \ldots , \kappa_\ell$ are any complex numbers, and where

$$\rho_k = \sum\nolimits_{r=1}^{k} \kappa_r \quad \text{(for } k = 1,2, \ldots ,\ell),$$

and

$$\rho_{k+\ell} = \sum_{r=k+1}^{\ell} \kappa_r \quad (\text{for } k = 1,2, \ldots ,\ell-1).$$

Thus on $\mathcal{H}^{0(4)}$ the simple roots of A(2ℓ-1/0) are given by

$$\alpha_k^0(h) = \kappa_k - \kappa_{k+1} \quad (\text{for } k = 1,2, \ldots ,\ell-1), \tag{4.3.54}$$

$$\alpha_\ell^0(h) = \kappa_1 + \kappa_\ell , \tag{4.3.55}$$

$$\alpha_{k+\ell}^0(h) = -(\kappa_k - \kappa_{k+1}) \quad (\text{for } k = 1,2, \ldots ,\ell-1), \tag{4.3.56}$$

$$\alpha_{2\ell}^0(h) = -\kappa_\ell . \tag{4.3.57}$$

However, from (4.3.46) and (4.3.47)

$$\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+\ell,k+\ell} = 4(\ell-1)\left\{\sum_{r=k}^{k+\ell-1} \underset{\sim}{h}^{D_\ell}_{\alpha_r^{D_\ell}} - \tfrac{1}{2}\underset{\sim}{h}^{D_\ell}_{\alpha_{\ell-1}^{D_\ell}} - \tfrac{1}{2}\underset{\sim}{h}^{D_\ell}_{\alpha_\ell^{D_\ell}}\right\}$$

(for k = 1,2, ... ,ℓ-2),

$$\underset{\sim}{e}_{\ell-1,\ell-1} - \underset{\sim}{e}_{2\ell-1,2\ell-1} = 2(\ell-1)\left\{\underset{\sim}{h}^{D_\ell}_{\alpha_{\ell-1}^{D_\ell}} + \underset{\sim}{h}^{D_\ell}_{\alpha_\ell^{D_\ell}}\right\},$$

and

$$\underset{\sim}{e}_{\ell,\ell} - \underset{\sim}{e}_{2\ell,2\ell} = 2(\ell-1)(-\underset{\sim}{h}^{D_\ell}_{\alpha_{\ell-1}^{D_\ell}} + \underset{\sim}{h}^{D_\ell}_{\alpha_\ell^{D_\ell}}) ,$$

so

$$\sum_{k=1}^{\ell} \kappa_k(\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+\ell,k+\ell}) = 4(\ell-1)\sum_{k=1}^{\ell} \mu_k \underset{\sim}{h}^{D_\ell}_{\alpha_k^{D_\ell}} , \tag{4.3.58}$$

where

$$\mu_k = \sum_{r=1}^{k} \kappa_r \quad (\text{for } k = 1,2, \ldots ,\ell-2),$$

$$\mu_{\ell-1} = \tfrac{1}{2}\sum_{r=1}^{\ell-1} \kappa_r - \tfrac{1}{2}\kappa_\ell,$$

and

$$\mu_\ell = \tfrac{1}{2}\sum_{r=1}^{\ell} \kappa_r .$$

Thus for $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{D_\ell})$

$$\alpha_k^{D_\ell}(h) \;=\; \kappa_k - \kappa_{k+1} \;\; (\text{for } k = 1,2, \ldots ,\ell-1), \qquad (4.3.59)$$

and

$$\alpha_\ell^{D_\ell}(h) \;=\; \kappa_{\ell-1} + \kappa_\ell \;. \qquad (4.3.60)$$

Comparison of (4.3.54) to (4.3.57) with (4.3.59) and (4.3.60) then shows that for $h \in \mathcal{H}^{0(4)}$ ($= \mathcal{H}^{D_\ell}$) of D_ℓ the simple roots $\alpha_k^{D_\ell}$ of D_ℓ and α_k^0 of A(2ℓ-1/0) are related by

$$\alpha_k^0(h) \;=\; -\alpha_{k+\ell}^0(h) \;=\; \alpha_k^{D_\ell}(h) \;\; \text{for } k = 1,2, \ldots ,\ell-1, \qquad (4.3.61)$$

$$\alpha_\ell^0(h) \;=\; \sum_{r=1}^{\ell-2}\alpha_r^{D_\ell}(h) + \alpha_\ell^{D_\ell}(h) \;, \qquad (4.3.62)$$

and

$$\alpha_{2\ell}^0(h) \;=\; \tfrac{1}{2}\alpha_{\ell-1}^{D_\ell}(h) - \tfrac{1}{2}\alpha_\ell^{D_\ell}(h) \;. \qquad (4.3.63)$$

(When $\ell = 2$ the first term of (4.3.62) do not appear).

Finally it follows from (4.2.10-11), (4.3.8), (4.3.16) (4.3.46-47), and (4.3.52) that corresponding elements of the Cartan subalgebra of the Kac-Moody superalgebra are

$$h_{\alpha_k^{D_\ell}} \;=\; t^0\otimes h^{D_\ell}_{\alpha_k^{D_\ell}} \;=\; \{(2\ell-1)/2(\ell-1)\}\, t^0\otimes\{\underset{\sim}{h}^0_{\alpha_k^0} - \underset{\sim}{h}^0_{\alpha_{k+\ell}^0}\} \quad (4.3.64)$$

(for $k = 1,2, \ldots , \ell-1$), and

$$h_{\alpha_\ell^{D_\ell}} = t^0\otimes h^{D_\ell}_{\alpha_\ell^{D_\ell}} = \{(2\ell-1)/2(\ell-1)\}\;\; t^0\otimes\left\{\underset{\sim}{h}^0_{\alpha_{\ell-1}^0} + 2\sum_{r=\ell}^{2\ell-2}\underset{\sim}{h}^0_{\alpha_r^0} + \underset{\sim}{h}^0_{\alpha_{2\ell-1}^0}\right\}.$$

(4.3.65)

The non-diagonal basis elements of $\tilde{\mathcal{L}}_{s0}^{0(4)}$ will now be examined. They fall into 4 sets:

(i) For $j,k = 1,2, \ldots , \ell$, with $j < k$:

$$\underset{\sim}{e}_{j,k} - \underset{\sim}{e}_{k+\ell,j+\ell} \;=\; \underset{\sim}{e}^0_{\beta^0_{(j,k)}} + \underset{\sim}{e}^0_{-\beta_{(j+\ell,k+\ell)}} \;, \qquad (4.3.66)$$

where $\underset{\sim}{e}^{0}_{\beta^{0}_{(j,k)}}$ and $\underset{\sim}{e}^{0}_{-\beta^{0}_{(j+\ell,k+\ell)}}$ are given by (4.1) and (4.3.33a).

As (4.3.61) implies that

$$\beta^{0}_{(j,k)}(h) = -\beta^{0}_{(j+\ell,k+\ell)}(h) = \sum_{r=j}^{k-1} \alpha_r^{D_\ell}(h) , \qquad (4.3.67)$$

(for j,k = 1,2, ... , ℓ, with j < k, and for all h $\in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$)), the basis element (4.3.66) corresponds to the root $\beta^{0}_{(j,k)}(h)$ of D_ℓ.

(ii) For j,k = 1,2, ... , ℓ, with j < k:

$$-\underset{\sim}{e}_{k,j} + \underset{\sim}{e}_{j+\ell,k+\ell} = \underset{\sim}{e}^{0}_{-\beta_{(j,k)}} + \underset{\sim}{e}^{0}_{\beta^{0}_{(j+\ell,k+\ell)}} , \qquad (4.3.68)$$

which corresponds to the root $-\beta^{0}_{(j,k)}(h)$ of D_ℓ, where $\beta^{0}_{(j,k)}(h)$ is given by (4.3.67).

(iii) For j,k = 1,2, ... , ℓ, with j < k:

$$\underset{\sim}{e}_{j,k+\ell} - \underset{\sim}{e}_{k,j+\ell} = \underset{\sim}{e}^{0}_{\beta^{0}_{(j,k+\ell)}} - \underset{\sim}{e}^{0}_{\beta^{0}_{(k,j+\ell)}} , \qquad (4.3.69)$$

where $\underset{\sim}{e}^{0}_{\beta^{0}_{(j,k+\ell)}}$ and $\underset{\sim}{e}^{0}_{\beta^{0}_{(k,j+\ell)}}$ are again given by (4.3.33a). As (4.3.61) and (4.3.62) imply that for all h $\in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$)

$$\beta^{0}_{(j,k+\ell)}(h) = \beta^{0}_{(k,j+\ell)}(h)$$

$$\sum_{r=j}^{k-1} \alpha_r^{D_\ell}(h) + 2\sum_{r=k}^{\ell-2} \alpha_r^{D_\ell}(h) + \alpha_{\ell-1}^{D_\ell}(h) + \alpha_\ell^{D_\ell}(h)$$

(for j,k = 1,2, ... , ℓ-2, with j < k),

$$\sum_{r=j}^{\ell-2} \alpha_r^{D_\ell}(h) + \alpha_{\ell-1}^{D_\ell}(h) + \alpha_\ell^{D_\ell}(h)$$

=

(for j = 1,2, ... , ℓ-2, and k = ℓ-1),

(4.3.70)

$$\sum_{r=j}^{\ell-2} \alpha_r^{D_\ell}(h) + \alpha_\ell^{D_\ell}(h)$$

(for $j = 1,2, \ldots, \ell-2$, and $k = \ell$), and

$\alpha_{\ell}^{D_\ell}(h)$, (for $j = \ell-1$ and $k = \ell$),

the basis element (4.3.69) corresponds to the root $\beta^0_{(j,k+\ell)}(h)$ of D_ℓ.

(iv) For $j,k = 1,2, \ldots, \ell$, with $j < k$:

$$\underset{\sim}{e}_{j+\ell,k} - \underset{\sim}{e}_{k+\ell,j} = -\underset{\sim}{e}^{0}_{-\beta_{(k,j+\ell)}} + \underset{\sim}{e}^{0}_{-\beta_{(j,k+\ell)}}, \qquad (4.3.71)$$

which corresponds to the root $-\beta^0_{(j,k+\ell)}(h)$ of D_ℓ, where $\beta^0_{(j,k+\ell)}(h)$ is given by (4.3.70).

As expected the elements of (4.3.40), (4.3.42), (4.3.43), and (4.3.71) are even members of $A(2\ell-1/0)$. It is easily checked that the set of $2\ell(\ell-1)$ non-zero roots of (i) to (iv) above, together with the ℓ zero roots, are all weights of the adjoint representation of D_ℓ. For $\ell \geq 4$ the highest weight is

$$\Lambda = \Lambda_2^{D_\ell} = \alpha_1^{D_\ell} + 2\sum_{k=2}^{\ell-2} \alpha_k^{D_\ell} + \alpha_{\ell-1}^{D_\ell} + \alpha_\ell^{D_\ell}, \qquad (4.3.72)$$

while for $\ell = 2$ and 3 the second term on the right-hand side of (4.3.72) does not appear and

$$\Lambda = \Lambda_{\ell-1}^{D_\ell} + \Lambda_\ell^{D_\ell} = \sum_{k=1}^{\ell} \alpha_k^{D_\ell} \qquad (4.3.73)$$

as expected (see Appendix A(1)).

(c) The subspace $\tilde{\mathcal{L}}_{s1}^{0(4)}$

By (4.3.2) the subspace $\tilde{\mathcal{L}}_{s1}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = i\underset{\sim}{A}, \quad -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = i\underset{\sim}{C}, \quad \underset{\sim}{G}\tilde{\underset{\sim}{C}} = i\underset{\sim}{B}, \text{ and } -\tilde{\underset{\sim}{D}} = i\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1×1, imply that

$$\underset{\sim}{A} = \underset{\sim}{0}, \quad \underset{\sim}{D} = \underset{\sim}{0}, \text{ and } \underset{\sim}{C} = i\tilde{\underset{\sim}{B}}\underset{\sim}{G}. \qquad (4.3.74)$$

The basis elements of $\tilde{\mathcal{L}}_{s1}^{0(4)}$ fall into 2 sets:

(i) For $j = 1,2, \ldots, \ell$:

$$\underset{\sim}{e}_{j,2\ell+1} + i\underset{\sim}{e}_{2\ell+1,j+\ell} = \underset{\sim}{e}^{0}_{\delta^0_{(j)}} - i\underset{\sim}{e}^{0}_{-\delta^0_{(j+\ell)}}, \qquad (4.3.75)$$

where $\underset{\sim}{e}^{0}_{\delta^0_{(j)}}$ and $i\underset{\sim}{e}^{0}_{-\delta^0_{(j+\ell)}}$ are given by (4.1) and (4.3.33b). For all

$h \in \mathcal{H}^{0(4)}$ ($= \mathcal{H}^{D_\ell}$) (4.3.61-63), imply that

$$\delta^0_{(j)}(h) = -\delta^0_{(j+\ell)}(h)$$

$$= \begin{cases} \sum_{r=j}^{\ell-2} \alpha_r^{D_\ell}(h) + \tfrac{1}{2}\alpha_{\ell-1}^{D_\ell}(h) + \tfrac{1}{2}\alpha_\ell^{D_\ell}(h), & \text{for } j \leq \ell-2, \\ \tfrac{1}{2}\alpha_{\ell-1}^{D_\ell}(h) + \tfrac{1}{2}\alpha_\ell^{D_\ell}(h), & \text{for } j = \ell-1, \\ -\tfrac{1}{2}\alpha_{\ell-1}^{D_\ell}(h) + \tfrac{1}{2}\alpha_\ell^{D_\ell}(h), & \text{for } j = \ell. \end{cases} \qquad (4.3.76)$$

In all cases the basis element (4.3.75) corresponds to the root $\delta^0_{(j)}(h)$ of D_ℓ.

(ii) For $j = 1,2, \ldots, \ell$:

$$-\underset{\sim}{e}_{2\ell+1,j} + i\underset{\sim}{e}_{j+\ell,2\ell+1} = \underset{\sim}{e}^{0}_{-\delta^0_{(j)}} + i\underset{\sim}{e}^{0}_{\delta^0_{(j+\ell)}}, \qquad (4.3.77)$$

which corresponds to the weight $-\delta^0_{(j)}(h)$ of D_ℓ ($\delta^0_{(j)}(h)$ being as in (4.3.76)).

These weights all belong to a 2ℓ-dimensional irreducible representation of D_ℓ with highest weight

$$\Lambda = \Lambda_1^{D_\ell} = \sum_{k=1}^{\ell-2} \alpha_k^{D_\ell} + \tfrac{1}{2}\alpha_{\ell-1}^{D_\ell} + \tfrac{1}{2}\alpha_\ell^{D_\ell}, \qquad (4.3.78)$$

(where for $\ell = 2$ the first term on the right-hand side of (4.3.78) does not appear). It should be noted that all the elements of (4.3.75) and (4.3.77) are odd members of

$A(2\ell-1/0)$, so all the elements of $\tilde{\mathcal{L}}_{s1}^{0(4)}$ are odd.

(d) The subspace $\tilde{\mathcal{L}}_{s2}^{0(4)}$

By (4.3.2) the subalgebra $\tilde{\mathcal{L}}_{s2}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = -\underset{\sim}{A},\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = -\underset{\sim}{C},\ \underset{\sim}{G}\tilde{\underset{\sim}{C}} = -\underset{\sim}{B},\ \text{and } -\tilde{\underset{\sim}{D}} = -\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1x1, imply that

$$\tilde{\underset{\sim}{A}}\underset{\sim}{G} - \underset{\sim}{G}\underset{\sim}{A} = \underset{\sim}{0},\ \underset{\sim}{B} = \underset{\sim}{0},\ \underset{\sim}{C} = \underset{\sim}{0}, \qquad (4.3.79)$$

with $\underset{\sim}{D}$ being determined only by the supertrace condition tr $\underset{\sim}{A}$ = tr $\underset{\sim}{D}$. On using (4.3.30), the diagonal basis elements of $\tilde{\mathcal{L}}_{s2}^{0(4)}$ may be taken to consist of the set

$$\{1/2(2\ell-1)\}\{\underset{\sim}{e}_{k,k} + \underset{\sim}{e}_{k+\ell,k+\ell} + 2\underset{\sim}{e}_{2\ell+1,2\ell+1}\}$$

$$= \sum_{r=k}^{k+\ell-1} \underset{\sim}{h}^{0}_{\alpha_r^0} + 2\sum_{r=k+\ell}^{2\ell} \underset{\sim}{h}^{0}_{\alpha_r^0} \qquad (4.3.80)$$

(for $k = 1,2, \ldots ,\ell$), which each corresponds to zero weight of D_ℓ.

The non-diagonal basis elements of $\tilde{\mathcal{L}}_{s2}^{0(4)}$ fall into 6 sets:

(i) For $j,k = 1,2, \ldots , \ell$, with $j < k$:

$$\underset{\sim}{e}_{j,k} + \underset{\sim}{e}_{k+\ell,j+\ell} = \underset{\sim}{e}^{0}_{\beta^0_{(j,k)}} - \underset{\sim}{e}^{0}_{-\beta_{(j+\ell,k+\ell)}}, \qquad (4.3.81)$$

where $\underset{\sim}{e}^{0}_{\beta^0_{(j,k)}}$ and $\underset{\sim}{e}^{0}_{-\beta_{(j+\ell,k+\ell)}}$ are given by (4.1) and (4.3.32a),

and

$\beta^0_{(j,k)}(h)$ (= $-\beta^0_{(j+\ell,k+\ell)}(h)$) is given for all $h \in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$) by (4.3.67), so this basis element (4.3.81) again corresponds to the root $\beta^0_{(j,k)}(h)$ of D_ℓ.

(ii) For j,k = 1,2, ... , ℓ, with j < k:

$$-\underset{\sim}{e}_{k,j} - \underset{\sim}{e}_{j+\ell,k+\ell} = \underset{\sim}{e}^{0}_{-\beta_{(j,k)}} - \underset{\sim}{e}^{0}_{\beta^{0}_{(j+\ell,k+\ell)}} , \qquad (4.3.82)$$

which corresponds to the root $-\beta^{0}_{(j,k)}(h)$ of D_{ℓ}, where $\beta^{0}_{(j,k)}(h)$ is given by (4.3.67).

(iii) For j = 1,2, ... , ℓ:

$$\underset{\sim}{e}_{j,j+\ell} = \underset{\sim}{e}^{0}_{\beta^{0}_{(j,j+\ell)}} , \qquad (4.3.82)$$

where $\underset{\sim}{e}^{0}_{\beta^{0}_{(j,j+\ell)}})$ is given by (4.3.32a), which corresponds to the weight $\beta^{0}_{(j,j+\ell)}$ of D_{ℓ}. By a further application of (4.3.61) and (4.3.62) $\beta^{0}_{(j,j+\ell)}(h)$ can be rewritten for $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{D_{\ell}})$ as

$$\beta^{0}_{(j,j+\ell)}(h) = \begin{cases} 2\sum_{r=j}^{\ell-2}\alpha_r^{D_\ell}(h) + \alpha_{\ell-1}^{D_\ell}(h) + \alpha_{\ell}^{D_\ell}(h) & (\text{for } j = 1,2, \dots , \ell-2), \\ \alpha_{\ell-1}^{D_\ell}(h) + \alpha_{\ell}^{D_\ell}(h) & (\text{for } j = \ell-1), \\ -\alpha_{\ell-1}^{D_\ell}(h) + \alpha_{\ell}^{D_\ell}(h) & (\text{for } j = \ell). \end{cases} \qquad (4.3.84)$$

(iv) For j = 1,2, ... , ℓ:

$$-\underset{\sim}{e}_{j+\ell,j} = \underset{\sim}{e}^{0}_{-\beta_{(j,j+\ell)}} , \qquad (4.3.59)$$

which corresponds to the weight $\beta^{0}_{(j,j+\ell)}(h)$ of D_{ℓ}, where $\beta^{0}_{(j,j+\ell)}(h)$ is given by (4.3.84).

(v) For j,k = 1,2, ... , ℓ, with j < k:

$$\underset{\sim}{e}_{j,k+\ell} + \underset{\sim}{e}_{k,j+\ell} = \underset{\sim}{e}^{0}_{\beta^{0}_{(j,k+\ell)}} + \underset{\sim}{e}^{0}_{\beta^{0}_{(k,j+\ell)}} , \qquad (4.3.86)$$

where $\underset{\sim}{e}^{0}_{\beta^{0}_{(j,k+\ell)}}$ and $\underset{\sim}{e}^{0}_{\beta^{0}_{(k,j+\ell)}}$ are again given by (4.3.32a). As

$\beta^0_{(j,k+\ell)}(h)$ (= $\beta^0_{(k,j+\ell)}(h)$) is given for all $h \in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$) by (4.3.70), so this basis element (4.3.86) again corresponds to the root $\beta^0_{(j,k+\ell)}(h)$ of D_ℓ.

(vi) For j,k = 1,2, ... , ℓ, with j < k:

$$-\underset{\sim}{e}_{k+\ell,j} - \underset{\sim}{e}_{j+\ell,k} = \underset{\sim}{e}_{-\beta^0_{(j,k+\ell)}} + \underset{\sim}{e}_{-\beta^0_{(k,j+\ell)}}, \qquad (4.3.87)$$

which corresponds to the root $-\beta^0_{(j,k+\ell)}(h)$ of D_ℓ, where $\beta^0_{(j,k+\ell)}(h)$ is given by (4.3.70).

These $2\ell^2+\ell$ weights belong to a representation of D_ℓ which is the direct sum of the trivial 1-dimensional irreducible representation with highest weight $\Lambda = 0$ and the $(2\ell^2+\ell-1)$-dimensional irreducible representation with highest weight

$$\Lambda = 2\Lambda_1^{D_\ell} = 2\sum_{k=1}^{\ell-2}\alpha_k^{D_\ell} + \alpha_{\ell-1}^{D_\ell} + \alpha_\ell^{D_\ell}, \qquad (4.3.88)$$

(where for ℓ = 2 the first term on the right-hand side of (4.3.88) does not appear). It should be noted that all the elements of $\tilde{\mathcal{L}}^{0(4)}_{s2}$ are even members of $A(2\ell-1/0)$.

(e) The subspace $\tilde{\mathcal{L}}^{0(4)}_{s3}$

By (4.3.2) the subspace $\tilde{\mathcal{L}}^{0(4)}_{s3}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\,\tilde{\underset{\sim}{A}}\,\underset{\sim}{G} = -i\underset{\sim}{A},\quad -\tilde{\underset{\sim}{B}}\,\underset{\sim}{G} = -i\underset{\sim}{C},\quad \underset{\sim}{G}\,\tilde{\underset{\sim}{C}} = -i\underset{\sim}{B},\ \text{and}\ -\tilde{\underset{\sim}{D}} = -i\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1x1, imply that

$$\underset{\sim}{A} = 0,\quad \underset{\sim}{D} = 0,\ \text{and}\ \underset{\sim}{C} = -i\tilde{\underset{\sim}{B}}\,\underset{\sim}{G}\,. \qquad (4.3.89)$$

The basis elements of $\tilde{\mathcal{L}}^{0(4)}_{s3}$ fall into 2 sets:

(i) For j = 1,2, ... , ℓ:

$$\underset{\sim}{e}_{j,2\ell+1} - i\underset{\sim}{e}_{2\ell+1,j+\ell} = \underset{\sim}{e}^{0}_{\delta^{0}_{(j)}} + i\underset{\sim}{e}^{0}_{-\delta^{0}_{(j+\ell)}} , \qquad (4.3.90)$$

where $\underset{\sim}{e}^{0}_{\delta^{0}_{(j)}}$ and $i\underset{\sim}{e}^{0}_{-\delta^{0}_{(j+\ell)}}$ are given by (4.1) and (4.3.32b). As

$\delta^{0}_{(j)}(h)$

$(= -\delta^{0}_{(j+\ell)}(h))$ is given for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{D_\ell})$ by (4.3.76) so the basis element (4.3.90) again corresponds to the root $\delta^{0}_{(j)}(h)$ of D_ℓ.

(ii) For $j = 1,2, \ldots , \ell$:

$$-\underset{\sim}{e}_{2\ell+1,j} - i\underset{\sim}{e}_{j+\ell,2\ell+1} = \underset{\sim}{e}^{0}_{-\delta^{0}_{(j)}} - i\underset{\sim}{e}^{0}_{\delta^{0}_{(j+\ell)}} , \qquad (4.3.91)$$

which corresponds to the weight $-\delta^{0}_{(j)}(h)$ of D_ℓ, $\delta^{0}_{(j)}(h)$ being as in (4.3.76).

These two sets of weights are exactly the same as for $\tilde{\mathcal{L}}^{0(4)}_{s1}$, so they all belong to a 2ℓ-dimensional irreducible representation of D_ℓ with highest weight Λ is given by (4.3.84) and (4.3.83). All the elements of $\tilde{\mathcal{L}}^{0(4)}_{s3}$ are odd.

(f) The roots of $A^{(2)}(2\ell-1/0)$

Defining $\delta(h)$ as in (4.2.20), it follows from the above analysis and relations that the roots $\alpha(h)$ and the corresponding basis elements e_α of $A^{(2)}(2\ell-1/0)$ are as follows:

(i) $\alpha(h) = 4J\delta(h)$, (for $J = 0,\pm1,\pm2, \ldots$). There are ℓ linearly independent basis elements $e^{(k)}_{\alpha}$ corresponding to this root which may be labeled by an additional superscript, so that

$$e_\alpha^{(k)} = \{(2\ell-1)/2(\ell-1)\}\ t^{4J}\otimes\{\underset{\sim}{h}^0_{\alpha_k^0} - \underset{\sim}{h}^0_{\alpha^0_{k+\ell}}\}\ (\text{for } k = 1,2, \ldots , \ell-1),$$

and

$$e_\alpha^{(\ell)} = \{(2\ell-1)/2(\ell-1)\}\ t^{4J}\otimes\left\{\underset{\sim}{h}^0_{\alpha^0_{\ell-1}} + 2\sum_{r=\ell}^{2\ell-2} \underset{\sim}{h}^0_{\alpha_r^0} + \underset{\sim}{h}^0_{\alpha^0_{2\ell-1}}\right\},$$

(which reduce to (4.3.64) and (4.3.65) in the special case J = 0).

(ii) $\alpha(h) = 4J\delta(h) \pm \beta^0_{(j,k)}(h)$, (for j,k = 1,2, ... , ℓ, with j < k, and for J = 0,±1,±2, ...), where $\beta^0_{(j,k)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.67) and

$$e_\alpha = t^{4J}\otimes\{\underset{\sim}{e}^0_{\pm\beta^0_{(j,k)}} + \underset{\sim}{e}^0_{\mp\beta^0_{(j+\ell,k+\ell)}}\}.$$

(iii) $\alpha(h) = 4J\delta(h) \pm \beta^0_{(j,k+\ell)}(h)$, (for j,k = 1,2, ... , ℓ, with j < k, and for J = 0,±1,±2, ...), where $\beta^0_{(j,k+\ell)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.70) and

$$e_\alpha = t^{4J}\otimes\{\underset{\sim}{e}^0_{\pm\beta^0_{(j,k+\ell)}} - \underset{\sim}{e}^0_{\mp\beta^0_{(k,j+\ell)}}\}.$$

(iv) $\alpha(h) = (4J+1)\delta(h) \pm \delta^0_{(j)}(h)$, (for j= 1,2, ... , ℓ, and for J = 0,±1,±2, ...), where $\delta^0_{(j)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.76) and

$$e_\alpha = t^{4J+1}\otimes\{\underset{\sim}{e}^0_{\pm\delta^0_{(j)}} \mp i\underset{\sim}{e}^0_{\mp\delta^0_{(j+\ell)}}\}.$$

(v) $\alpha(h) = (4J+2)\delta(h)$, (for J = 0,±1,±2, ...). There are ℓ linearly independent basis elements $e_\alpha^{(k)}$ corresponding to this root which may be labeled by an additional superscript, so that

$$e_\alpha^{(k)} = t^{4J+2}\otimes\left\{\sum_{r=k}^{k+\ell-1} \underset{\sim}{h}^0_{\alpha_r^0} + 2\sum_{r=k+\ell}^{2\ell} \underset{\sim}{h}^0_{\alpha_r^0}\right\},(\text{for } k= 1,2,\ldots, \ell);$$

(vi) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j,k)}(h)$, (for j= 1,2, ... , ℓ, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j,k)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.67) and

$$e_\alpha = t^{4J+2} \otimes \{\underset{\sim}{e}_{\pm\beta^0_{(j,k)}} - \underset{\sim}{e}_{\mp\beta^0_{(j+\ell,k+\ell)}}\}.$$

(vii) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j,j+\ell)}(h)$, (for j = 1,2, ... , ℓ, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j,j+\ell)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.84) and $e_\alpha = t^{4J+2} \otimes \underset{\sim}{e}_{\pm\beta^0_{(j,j+\ell)}}$.

(viii) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j,k+\ell)}(h)$, (for j,k = 1,2, ... , ℓ, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j,k+\ell)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.70) and

$$e_\alpha = t^{4J+2} \otimes \{\underset{\sim}{e}_{\pm\beta^0_{(j,k+\ell)}} + \underset{\sim}{e}_{\mp\beta^0_{(k,j+\ell)}}\}.$$

(ix) $\alpha = (4J+3)\delta(h) \pm \delta^{0}_{(j)}(h)$, (for j= 1,2, ... , ℓ, and for $J = 0,\pm1,\pm2,\ldots$), where $\delta^{0}_{(j)}(h)$ is the extension of the weight of D_ℓ that is given by (4.3.76) and

$$e_\alpha = t^{4J+3} \otimes \{\underset{\sim}{e}_{\pm\delta^0_{(j)}} \pm i\underset{\sim}{e}_{\mp\delta^0_{(j+\ell)}}\}.$$

(x) $\alpha(h) = 0$, with c and d as basis elements.

With μ chosen as in (4.3.44) , it follows that

$$\langle\alpha^0,\beta^0\rangle = \langle\alpha^0,\beta^0\rangle^{D_\ell} \qquad (4.3.92)$$

where on the right-hand side of (4.3.66) α^0 and β^0 are any pair of linear functionals defined on $\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{D_\ell}$), the evaluation being performed with respect to the Killing form of D_ℓ, and where on the left-hand side of (4.3.92) α^0 and β^0 denote the corresponding extensions to the Cartan subalgebra of the Kac-Moody superalgebra $A^{(2)}(2\ell-1/0)$, the evaluation being performed with respect to its supersymmetric bilinear invariant form B(,). As D_ℓ is a semi-simple Lie algebra, $<\alpha^0,\alpha^0>^{D_\ell} > 0$ for every non-zero linear functional α^0 defined on $\mathcal{H}^{D_\ell}$, so $<\alpha^0,\alpha^0> > 0$ for the corresponding extension.

Moreover (4.3.18) imply that

$$h_\delta = \{(2\ell-1)/(\ell-1)\}\ c\ . \tag{4.3.93}$$

Thus, if α_k^0 is the extension of any simple root of $\tilde{\mathcal{L}}_s^0$, then

$$<\delta,\alpha_k^0> = 0 \tag{4.3.94}$$

and

$$<j\delta,j\delta> = 0\ . \tag{4.3.95}$$

Thus $<j\delta,j\delta> = 0$ for integer j, so every non-zero root of $A^{(2)}(2\ell-1/0)$ belonging to the sets (i) and (v) is "imaginary". Moreover, because $<j\delta+\alpha^0,j\delta+\alpha^0> = <\alpha^0,\alpha^0>^{D_\ell}$ and because $<\alpha^0,\alpha^0>^{D_\ell} > 0$ for linear functional α^0 and its corresponding extension (as has just been noted), it follows that every root of $A^{(2)}(2\ell-1/0)$ belonging to the sets (ii), (iii), (iv), (vi), (vii), (viii), and (ix) is "real". All the elements mentioned in the above sets are even, except for those in the sets (iv) and (ix), which are odd.

In relating these roots to the simple roots of the Kac-Moody superalgebra $A^{(2)}(2\ell-1/0)$ it is necessary to consider the cases $\ell = 2$ and $\ell > 2$ separately because the labeling of the generalized Dynkin diagrams of $A^{(2)}(2\ell-1/0)$ is different in the two cases.

For $A^{(2)}(3/0)$ (i.e. for $\ell = 2$) the simple roots may be taken to be

$$\alpha_0 = \alpha_1^{D_2}, \quad \alpha_1 = \delta - \alpha_H^0, \quad \alpha_2 = \alpha_2^{D_2},$$

where

$$\alpha_H^0 = \Lambda_1^{D_2} = \tfrac{1}{2}\sum_{k=1}^{2} \alpha_k^{D_2} \tag{4.3.96}$$

is the highest weight of the representation of $\tilde{\mathcal{L}}_{s0}^{0(4)}$ for which $\tilde{\mathcal{L}}_{s1}^{0(4)}$ is the carrier space and $\alpha_1^{D_2}$ and $\alpha_2^{D_2}$ are the extensions of the simple roots α_1^0 and α_2^0 of D_2. As e_{α_1} appears in the set (iv) it follows that e_{α_1} is odd, so α_1 is an odd root of the Kac-Moody superalgebra $A^{(2)}(3/0)$. All the other simple roots of $A^{(2)}(3/0)$ are even.

For $A^{(2)}(2\ell-1/0)$ for $\ell > 2$ the simple roots may be taken to be

$$a_\ell = \delta - a_H^0, \tag{4.3.97}$$

and

$$\alpha_k = \alpha_{\ell-k}^{D_\ell} \quad (\text{for } k = 0,1, \ldots ,\ell-1) \tag{4.3.98}$$

where

$$\alpha_H^0 = \Lambda_1^{D_\ell} = \sum_{k=1}^{\ell-2} \alpha_k^{D_\ell} + \tfrac{1}{2}\alpha_{\ell-1}^{D_\ell} + \tfrac{1}{2}\alpha_\ell^{D_\ell} \tag{4.3.99}$$

is the highest weight of the representation of $\tilde{\mathcal{L}}_{s0}^{0(4)}$ for which $\tilde{\mathcal{L}}_{s1}^{0(4)}$ is the carrier space and the $\alpha_k^{D_\ell}$ are the extensions of the simple roots of D_ℓ. As e_{α_ℓ} appears in the set (iv) it follows that e_{α_ℓ} is odd, so α_ℓ is an odd root of the Kac-Moody superalgebra $A^{(2)}(2\ell-1/0)$ (for $\ell > 2$). All the other simple roots of $A^{(2)}(2\ell-1/0)$ are even (for $\ell > 2$).

In terms of these simple roots the positive and negative roots of $A^{(2)}(2\ell-1/0)$ are defined as in the case of $B^{(1)}(0/\ell)$. The quantities $<\overset{0}{\alpha},\overset{0}{\alpha}>^{D_\ell}$ can be computed from appendix A(1). It is then easily checked with arguments very similar to the ones presented for $B^{(1)}(0/\ell)$ that the Cartan matrices of $A^{(2)}(3/0)$ and $A^{(2)}(2\ell-1/0)$ (for $\ell \geq 3$) evaluated using definition (2.15) are of affine type. Their corresponding Dynkin diagrams are those given in Figures 3 and 4.

In terms of the linearly independent functionals ε_j ($i \leq j \leq \ell$) defined on $\mathcal{H}^{D_\ell}$(see Cornwell(1985), and table I chapter 2) the roots of $A^{(2)}(2\ell-1/0)$ are given by

$$\Delta = \{\ 2m\delta \pm (\varepsilon_i \pm \varepsilon_j) \text{ with } 1\leq i < j \leq \ell\ ,\ (2m+1)\delta \pm \varepsilon_i \text{ with } 1\leq i \leq \ell$$
$$(4m+2)\delta \pm 2\varepsilon_i \text{ with } 1\leq i \leq \ell \text{ all with } m\in \mathbb{Z}$$
$$\text{and } 2m\delta \text{ with } j\neq 0 \text{ and } m\in \mathbb{Z}\} \tag{4.3.100}$$

The basis is given by

$$\alpha_0 = \varepsilon_{\ell-1} + \varepsilon_\ell \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1\leq i \leq \ell -1 \quad \alpha_\ell = \delta - \varepsilon_1 \tag{4.3.101}$$

C. Root structure of $A^{(4)}(2\ell/0)$ (for ℓ = 2,3,...)

(a) The 4-fold automorphisms

The general line of argument for $A^{(4)}(2\ell/0)$ is very similar to that given for $A^{(2)}(2\ell-1/0)$ in the previous section, so its presentation can be given more briefly. An explicit realization of the simple Lie superalgebra $A(2\ell/0)$ is provided by $s\ell(2\ell+1/1)$, considered as a complex superalgebra, where $s\ell(2\ell+1/1)$ is defined as the set of $(2\ell+2)\times(2\ell+2)$ complex supermatrices that satisfy the condition (4.3.28). The grading partitioning may be taken to be as in (4.3.27), but now $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are of dimensions $(2\ell+1)\times(2\ell+1)$, $(2\ell+1)\times1$, $1\times(2\ell+1)$, and 1×1 respectively. The rank ℓ^0 of $A(2\ell/0)$ is given by

$$\ell^0=2\ell+1. \qquad (4.3.102)$$

The generalized Dynkin diagram of $A(2\ell/0)$ is shown in Figure 11, which indicates that its distinguished simple roots α_k^0 are even for k = 1,2,...,2ℓ, but that $\alpha_{2\ell+1}^0$ is odd. With the bilinear form

$B^0(\ ,\ )$ being defined by

$$B^0(\underset{\sim}{M},\underset{\sim}{N}) = 4\ell \ \mathrm{str}\,(\underset{\sim}{M}\underset{\sim}{N}), \qquad (4.3.103)$$

the basis elements of its Cartan subalgebra $\mathcal{H}^0$ may be taken to be

$$\underset{\sim}{h}^0_{\alpha_k^0} = \{1/4\ell\}\{\underset{\sim}{e}_{k,k} - \underset{\sim}{e}_{k+1,k+1}\}, \text{ (for } k = 1,2,\ldots,2\ell) \qquad (4.3.104)$$

and

$$\underset{\sim}{h}^{0}_{\alpha^{0}_{2\ell+1}} = \{1/4\ell\}\{\underset{\sim}{e}_{2\ell+1,2\ell+1} + \underset{\sim}{e}_{2\ell+2,2\ell+2}\} . \qquad (4.3.105)$$

Now $\underset{\sim}{e}_{r,s}$ is the matrix of dimension $(2\ell+2)\times(2\ell+2)$ that is defined by

$$(\underset{\sim}{e}_{r,s})_{jk} = \delta_{rj}\delta_{sk} \quad (\text{for } j,k = 1,2,\ldots,2\ell+2), \qquad (4.3.106)$$

so that with this choice all the matrices of $\mathcal{H}^0$ are again diagonal. The positive even roots $\beta^{0}_{(j,k)}$ and positive odd roots $\delta^{0}_{(j)}$ of A(2ℓ/0) are given in terms of the distinguished set of simple roots $\alpha^{0}_{1}, \alpha^{0}_{2}, \ldots, \alpha^{0}_{2\ell+1}$ of A(2ℓ/0) by

$$\beta^{0}_{(j,k)} = \sum_{r=j}^{k-1} \alpha^{0}_{r} \quad (\text{for } j,k = 1,2, \ldots, 2\ell+1;\ j < k), (4.3.107a)$$

and

$$\delta^{0}_{(j)} = \sum_{r=j}^{2\ell+1} \alpha^{0}_{r} \quad (\text{for } j = 1,2, \ldots, 2\ell+1), \qquad (4.3.107b)$$

for which the corresponding basis elements of A(2ℓ/0) may be taken to be

$$\underset{\sim}{e}^{0}_{\beta^{0}_{(j,k)}} = \underset{\sim}{e}^{0}_{\alpha^{0}_{j}+\alpha^{0}_{j+1}+\ldots+\alpha^{0}_{k-1}} = \underset{\sim}{e}_{j,k} \quad (\text{for } j,k = 1,2, \ldots, 2\ell+1;\ j < k),$$

(4.3.108a)

and

$$\underset{\sim}{e}^{0}_{\delta^{0}_{(j)}} = \underset{\sim}{e}^{0}_{\alpha^{0}_{j}+\alpha^{0}_{j+1}+\ldots+\alpha^{0}_{2\ell+1}} = \underset{\sim}{e}_{j,2\ell+2} \quad (\text{for } j = 1,2, \ldots, 2\ell+1).$$

(4.3.108b)

The basis elements corresponding to the corresponding negative roots may be chosen as stated in the introduction. (For further information on A(2ℓ/0) see Cornwell(1989)).

Taking the node corresponding to the odd simple root α_ℓ of $A^{(4)}(2\ell/0)$ for $\ell \geq 1$, as the corresponding numerical mark has value 1, (c.f. Figures 5 and 6) $q = 4$ again. It follows that if the automorphism ψ is employed then the subalgebra $\tilde{\mathfrak{L}}_{s0}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions (4.3.34), so that the subalgebra $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ is isomorphic to the set of $(2\ell+1)\times(2\ell+1)$ complex antisymmetric matrices and hence is isomorphic to the simple complex Lie algebra B_ℓ. As expected none of the basis elements of the Cartan subalgebra $\mathcal{H}^0$ of $A(2\ell/0)$ are members of this $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ (because all the members of this $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ are non-diagonal matrices).

A realization of B_ℓ in which the basis elements of the the Cartan subalgebra of B_ℓ are given by diagonal matrices is given by the $(2\ell+1)\times(2\ell+1)$ complex matrices $\underset{\approx}{A}'$ that satisfy the condition (4.3.9), but where now

$$\underset{\approx}{G} = \begin{pmatrix} \underset{\sim}{1}_1 & \underset{\approx}{0} & \underset{\approx}{0} \\ \underset{\approx}{0} & \underset{\approx}{0} & \underset{\sim}{1}_\ell \\ \underset{\approx}{0} & \underset{\sim}{1}_\ell & \underset{\approx}{0} \end{pmatrix}. \tag{4.3.109}$$

This realization will be referred to as the "canonical" form of B_ℓ. These matrices $\underset{\approx}{A}'$ are related to the $(2\ell+1)\times(2\ell+1)$ antisymmetric matrices $\underset{\approx}{A}$ by

$$\underset{\sim}{T}^{-1} \underset{\approx}{A} \underset{\sim}{T} = \underset{\approx}{A}', \tag{4.3.110}$$

where $\underset{\sim}{T}$ is a certain $(2\ell+1)\times(2\ell+1)$ complex matrix such that (4.3.110) maps the Lie algebra so(2N+1) into the canonical form of B_ℓ, and satisfies the condition

$$\tilde{\underset{\sim}{T}} \underset{\sim}{T} = \underset{\approx}{G}, \tag{4.3.111}$$

$\underset{\approx}{G}$ being as defined in (4.3.109) (see Cornwell(1975) for the actual form of $\underset{\sim}{T}$ in this case too). In what follows again, only (4.3.111) is needed). This mapping can be extended to an automorphism of θ of $\tilde{\mathcal{L}}_s^0$ (= $s\ell(2\ell+1/1)$) as in the previous case.

The 4 subspaces $\tilde{\mathcal{L}}_{sp}^{0(4)}$ (for p = 0,1,2,3) corresponding to the automorphism ϕ of (4.3.14) will now be considered in turn.

(b) The subspaces $\tilde{\mathcal{L}}_{s0}^{0(4)}$

By (53) the subalgebra $\tilde{\mathcal{L}}_{s0}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions (4.3.41) (with $\underset{\sim}{G}$ given by (4.3.109)),and so is isomorphic to the canonical form of B_ℓ.

Some properties of the canonical form of B_ℓ (c.f. Konuma et al(1963) and Cornwell(1975)) will first be summarized. Its Killing form $B^{B_\ell}(\, , \,)$ is given by

$$B^{B_\ell}(\underset{\sim}{A},\underset{\sim}{A}') = (2\ell-1)\ \mathrm{tr}(\underset{\sim}{A}\underset{\sim}{A}') , \qquad (4.3.112)$$

(for all $\underset{\sim}{A}$ and $\underset{\sim}{A}'$ of the canonical form). Thus, by (4.3.103),

$$B^0\left(\begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix},\begin{pmatrix} \underset{\sim}{A}' & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}\right) = \{4\ell/(2\ell-1)\}\ B^{B_\ell}(\underset{\sim}{A},\underset{\sim}{A}')$$

$$(4.3.113)$$

for all $\underset{\sim}{A}$ and $\underset{\sim}{A}'$ of the canonical form. This implies that (4.3.8) is satisfied if

$$\mu = (2\ell-1)/(4\ell) . \qquad (4.3.114)$$

Denoting the simple roots of B_ℓ by $\alpha_k^{B_\ell}$ (for k = 1,2, ... ,ℓ), the corresponding basis elements of the Cartan subalgebra $\mathcal{H}^{B_\ell}$ of B_ℓ defined by

$$B^{B_\ell}(\underset{\sim}{h}^{B_\ell}_{\alpha_k^{B_\ell}}, \underset{\sim}{h}) = \alpha_k^{B_\ell}(\underset{\sim}{h}) \text{ for all } \underset{\sim}{h} \in \mathcal{H}^{B_\ell}. \qquad (4.3.115)$$

are

$$\underset{\sim}{h}^{B_\ell}_{\alpha_k^{B_\ell}} =\{1/(2\ell-1)\}\ \{\underset{\sim}{e}_{k+1,k+1} - \underset{\sim}{e}_{k+\ell+1,k+\ell+1} - \underset{\sim}{e}_{k+2,k+2} + \underset{\sim}{e}_{k+\ell+2,k+\ell+2}\},$$

$$\text{(for } k = 1,2, \ldots ,\ell-1) \qquad (4.3.116)$$

and

$$\underset{\sim}{h}^{B_\ell}_{\alpha^{B_\ell}_\ell} = \{1/(2\ell-1)\}\,\{\underset{\sim}{e}_{\ell+1,\ell+1} - \underset{\sim}{e}_{2\ell+1,2\ell+1}\}\,. \qquad (4.3.117)$$

The associated root subspace basis elements are

$$\underset{\sim}{e}^{B_\ell}_{\alpha^{B_\ell}_k} = \{1/2(2\ell-1)\}\,\{\underset{\sim}{e}_{k+1,k+2} - \underset{\sim}{e}_{k+\ell+2,k+\ell+1}\},$$

(for k = 1,2, . . . ,ℓ-1) (4.3.118)

and

$$\underset{\sim}{e}^{B_\ell}_{\alpha^{B_\ell}_\ell} = \{1/2(2\ell-1)\}\,\{\underset{\sim}{e}_{1,2\ell+1} - \underset{\sim}{e}_{\ell+1,1}\}\,, \qquad (4.3.119)$$

the normalization factors being chosen so that

$$B^{B_\ell}(\underset{\sim}{e}^{B_\ell}_{\alpha^{B_\ell}_k}, \underset{\sim}{e}^{B_\ell}_{-\alpha^{B_\ell}_k}) = -1, \qquad (4.3.120)$$

where, as usual,

$$\underset{\sim}{e}^{B_\ell}_{-\alpha^{B_\ell}_k} = -\underset{\sim}{\tilde{e}}^{B_\ell}_{\alpha^{B_\ell}_k}\,. \qquad (4.3.120)$$

The <u>diagonal</u> basis elements of $\tilde{\mathfrak{L}}^{0(4)}_{s0}$ will be considered first. As they may be taken to consist of the set $\{\underset{\sim}{e}_{k+1,k+1}-\underset{\sim}{e}_{k+\ell+1,k+\ell+1} \mid \text{for } k = 1,2, \ldots ,\ell\}$, it follows that they are all members of the Cartan subalgebra $\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$) of B_ℓ (as expected). Thus the most general element of $\mathcal{H}^{0(4)}$ is of the form

$$\sum_{k=1}^{\ell} \kappa_k(\underset{\sim}{e}_{k+1,k+1}-\underset{\sim}{e}_{k+\ell+1,k+\ell+1})\,, \qquad (4.3.122)$$

where $\kappa_1, \kappa_2, \ldots , \kappa_\ell$ are any complex numbers, which can be rewritten, by (4.3.104), as

$$4\ell \sum_{k=2}^{2\ell} \rho_k \underset{\sim}{h}^{0}_{\alpha^0_k}\,, \qquad (4.3.123)$$

where

$$\rho_k = \sum_{r=1}^{k-1} \kappa_r \quad (\text{for } k = 2, \ldots ,\ell+1), \qquad (4.3.124)$$

and

$$\rho_{k+\ell} = \sum_{r=k}^{\ell} \kappa_r \quad (\text{for } k = 2, \ldots, \ell). \tag{4.3.125}$$

Thus on $\mathcal{H}^{0(4)}$ the simple roots of A(2ℓ/0) are given by

$$\alpha_1^0(h) = -\kappa_1 , \tag{4.3.126}$$

$$\alpha_k^0(h) = \kappa_{k-1} - \kappa_k \quad (\text{for } k = 2,3, \ldots, \ell), \tag{4.3.127}$$

$$\alpha_{\ell+1}^0(h) = \kappa_1 + \kappa_\ell , \tag{4.3.128}$$

$$\alpha_{k+\ell}^0(h) = -(\kappa_{k-1} - \kappa_k) \quad (\text{for } k = 2,3, \ldots, \ell), \tag{4.3.129}$$

$$\alpha_{2\ell+1}^0(h) = -\kappa_\ell ,$$

which implies that on $\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$)

$$\alpha_k^0(h) = -\alpha_{k+\ell}^0(h) \quad (\text{for } k = 2,3, \ldots, \ell), \tag{4.3.130}$$

$$\alpha_1^0(h) = -\tfrac{1}{2}\sum_{k=2}^{\ell+1} \alpha_k^0(h) , \tag{4.3.131}$$

and

$$\alpha_{2\ell+1}^0(h) = -\tfrac{1}{2}\alpha_{\ell+1}^0(h) + \tfrac{1}{2}\sum_{k=2}^{\ell} \alpha_k^0(h) . \tag{4.3.132}$$

Consideration of a similar argument for the simple roots $\alpha_k^{B_\ell}$ of B_ℓ then shows that on the Cartan subalgebra $\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$) of B_ℓ the simple roots $\alpha_k^{B_\ell}$ of B_ℓ and α_k^0 of A(2ℓ/0) are related by

$$\alpha_1^0(h) = -\sum_{r=1}^{\ell} \alpha_r^{B_\ell}(h) , \tag{4.3.133}$$

$$\alpha_k^0(h) = -\alpha_{k+\ell}^0(h) = \alpha_{k-1}^{B_\ell}(h) \quad (\text{for } k = 2,3, \ldots, \ell), \tag{4.3.134}$$

$$\alpha_{\ell+1}^0(h) = \sum_{r=1}^{\ell-1} \alpha_r^{B_\ell}(h) + 2\alpha_\ell^{B_\ell}(h) , \tag{4.3.135}$$

$$\alpha^{0}_{2\ell+1}(h) \quad = \quad -\alpha^{B_\ell}_{\ell}(h) \ , \qquad (4.3.136)$$

Finally it follows from (4.3.104), (4.3.105), (4.3.116), and (4.3.117) that corresponding elements of the Cartan subalgebra of the Kac-Moody superalgebra are

$$h_{\alpha^{B_\ell}_k} \quad = \quad t^0 \otimes h^{B_\ell}_{\alpha^{B_\ell}_k} \quad = \quad \{2\ell/(2\ell-1)\} \ t^0 \otimes \{\underset{\sim}{h}^{0}_{\alpha^0_{k+1}} - \underset{\sim}{h}^{0}_{\alpha^0_{k+\ell+1}}\}$$

$$(4.3.137)$$

(for $k = 1,2, \ldots , \ell-1$), and

$$h_{\alpha^{B_\ell}_\ell} \quad = \quad t^0 \otimes h^{B_\ell}_{\alpha^{B_\ell}_\ell} \quad = \quad \{2\ell/(2\ell-1)\} \ t^0 \otimes \left\{ \sum\nolimits_{r=\ell+1}^{2\ell} \underset{\sim}{h}^{0}_{\alpha^0_r} \right\}. (4.3.138)$$

The <u>non-diagonal</u> basis elements of $\tilde{\mathfrak{L}}^{0(4)}_{s0}$ will now be examined. They fall into 6 sets:

(i) For $j = 1,2, \ldots , \ell$:

$$\underset{\sim}{e}_{1,j+1} - \underset{\sim}{e}_{j+\ell+1,1} \quad = \quad \underset{\sim}{e}^{0}_{\beta^0_{(1,j+1)}} + \underset{\sim}{e}^{0}_{-\beta_{(1,j+\ell+1)}} \ , \qquad (4.3.139)$$

where $\underset{\sim}{e}^{0}_{\beta^0_{(1,j+1)}}$ and $\underset{\sim}{e}^{0}_{-\beta_{(1,j+\ell+1)}}$ are given by (4.1) and (4.3.108a). As (4.3.133) and (4.3.135) imply that

$$\beta^0_{(1,j+1)}(h) \quad = \quad -\beta^0_{(1,j+\ell+1)}(h) \quad = \quad -\sum\nolimits_{r=j}^{\ell} \alpha^{B_\ell}_r(h) \ , \quad (4.3.140)$$

(for $j = 1,2, \ldots , \ell$, and for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$), the basis element (4.3.138) corresponds to the root $\beta^0_{(1,j+1)}(h)$ of B_ℓ.

(ii) For $j = 1,2, \ldots , \ell$:

$$- \underset{\sim}{e}_{j+1,1} + \underset{\sim}{e}_{1,j+\ell+1} \quad = \quad \underset{\sim}{e}^{0}_{-\beta_{(1,j+1)}} + \underset{\sim}{e}^{0}_{\beta^0_{(1,j+\ell+1)}} \ , \qquad (4.3.141)$$

which corresponds to the root $-\beta^0_{(1,j+1)}(h)$ of B_ℓ, where $\beta^0_{(1,j+1)}(h)$ is given by (4.3.140)).

(iii) For $j,k = 1,2, \ldots , \ell$, with $j < k$:

$$\underset{\sim}{e}_{j+1,k+1} - \underset{\sim}{e}_{k+\ell+1,j+\ell+1} \quad = \quad \underset{\sim}{e}^{0}_{\beta^0_{(j+1,k+1)}} + \underset{\sim}{e}^{0}_{-\beta_{(j+\ell+1,k+\ell+1)}} \ . \ (4.3.142)$$

where $\underset{\sim}{e}_{\overset{0}{\beta}_{(j+1,k+1)}}^{0}$ and $\underset{\sim}{e}_{-\overset{0}{\beta}_{(j+\ell+1,k+\ell+1)}}^{0}$ are given by (4.1) and (4.3.108a).

As (4.3.134) implies that

$$\overset{0}{\beta}_{(j+1,k+1)}(h) \;=\; -\overset{0}{\beta}_{(j+\ell+1,k+\ell+1)}(h) \;=\; \sum_{r=j}^{k-1}\alpha_r^{B_\ell}(h) \;, \quad (4.3.143)$$

(for j,k = 1,2, ... , ℓ, with j < k, and for all h $\in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$), the basis element (4.3.142) corresponds to the root $\overset{0}{\beta}_{(j+1,k+1)}(h)$ of B_ℓ. (This set does not appear when ℓ = 1).

(iv) For j,k = 1,2, ... ,ℓ, with j < k:

$$-\underset{\sim}{e}_{k+1,j+1} + \underset{\sim}{e}_{j+\ell+1,k+\ell+1} \;=\; \underset{\sim}{e}_{-\overset{0}{\beta}_{(j+1,k+1)}}^{0} + \underset{\sim}{e}_{\overset{0}{\beta}_{(j+\ell+1,k+\ell+1)}}^{0} \,, \quad (4.3.144)$$

which corresponds to the root
$-\overset{0}{\beta}_{(j+1,k+1)}(h)$ of B_ℓ, where $\overset{0}{\beta}_{(j+1,k+1)}(h)$ is as in (4.3.143). (This set does not appear when ℓ = 1).

(v) For j,k = 1,2, ... , ℓ, with j < k:

$$\underset{\sim}{e}_{j+1,k+\ell+1} - \underset{\sim}{e}_{k+1,j+\ell+1} \;=\; \underset{\sim}{e}_{\overset{0}{\beta}_{(j+1,k+\ell+1)}}^{0} - \underset{\sim}{e}_{\overset{0}{\beta}_{(k+1,j+\ell+1)}}^{0} \,. \quad (4.3.145)$$

where $\underset{\sim}{e}_{\overset{0}{\beta}_{(j+1,k+\ell+1)}}^{0}$ and $\underset{\sim}{e}_{\overset{0}{\beta}_{(k+1,j+\ell+1)}}^{0}$ are given by (4.1) and (4.3.108a). As (4.3.134) and (4.3.135) imply that

$$\overset{0}{\beta}_{(j+1,k+\ell+1)}(h) = \overset{0}{\beta}_{(k+1,j+\ell+1)}(h) = \sum_{r=j}^{k-1}\alpha_r^{B_\ell}(h) + 2\sum_{r=k}^{\ell}\alpha_r^{B_\ell}(h), \quad (4.3.146)$$

(for j,k = 1,2, ... , ℓ, with j < k, and for all h $\in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$), the basis element (4.3.142) corresponds to the root $\overset{0}{\beta}_{(j+1,k+\ell+1)}(h)$ of B_ℓ. (This set does not appear when ℓ = 1).

(vi) For j,k = 1,2, ... , ℓ, with j < k:

$$-\underset{\sim}{e}_{k+\ell+1,j+1} + \underset{\sim}{e}_{j+\ell+1,k+1} = \underset{\sim}{e}^{0}_{-\beta(j+1,k+\ell+1)} - \underset{\sim}{e}^{0}_{-\beta(k+1,j+\ell+1)} .$$

(4.3.147)

This corresponds to $-\beta^{0}_{(j+1,k+\ell+1)}(h)$ of B_ℓ, where $\beta^{0}_{(j+1,k+1)}(h)$ is given by (4.3.146)). (This set does not appear when $\ell = 1$).

As expected the elements of (4.3.139), (4.3.141-144), and (4.3.147) are <u>even</u> members of $A(2\ell/0)$.

It is easily checked that the set of $2\ell^2$ non-zero roots of (i) to (vi) above, together with the ℓ zero roots, are all weights of the adjoint representation of B_ℓ. For $\ell \geq 2$ its highest weight is

$$\Lambda = \Lambda_2^{B_\ell} = \alpha_1^{B_\ell} + 2\sum_{k=2}^{\ell} \alpha_k^{B_\ell}, \qquad (4.3.148)$$

while for $\ell = 1$ it is

$$\Lambda = 2\Lambda_1^{B_\ell} = \alpha_1^{B_\ell} \qquad (4.3.149)$$

as expected (Appendix A(2)).

(c) The subspace $\tilde{\mathcal{L}}^{0(4)}_{s1}$

By (4.3.2) the subspace $\tilde{\mathcal{L}}^{0(4)}_{s1}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = i\underset{\sim}{A},\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = i\underset{\sim}{C},\ \underset{\sim}{G}\tilde{\underset{\sim}{C}} = i\underset{\sim}{B},\ \text{and}\ -\tilde{\underset{\sim}{D}} = i\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1x1, imply that

$$\underset{\sim}{A} = \underset{\sim}{0},\ \underset{\sim}{D} = \underset{\sim}{0},\ \text{and}\ \underset{\sim}{C} = i\tilde{\underset{\sim}{B}}\underset{\sim}{G} . \qquad (4.3.150)$$

The basis elements of $\tilde{\mathcal{L}}^{0(4)}_{s1}$ fall into 3 sets:

(i) For $j = 1,2, \dots, \ell$:

$$\underset{\sim}{e}_{j+1,2\ell+2} + i\underset{\sim}{e}_{2\ell+2,j+\ell+1} = \overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+1)}} - i\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+\ell+1)}}, \quad (4.3.151)$$

where $\overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+1)}}$ and $\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+\ell+1)}}$ are given by (4.1) and (4.3.108b). As (4.3.134), (4.3.3135), and (4.3.136) imply that for $j = 1,2, \dots, \ell$ and for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$

$$\delta^0_{(j+1)}(h) = -\delta^0_{(j+\ell+1)}(h) = \sum_{r=j}^{\ell} \alpha_r^{B_\ell}(h), \quad (4.3.152)$$

the basis element (4.3.50) corresponds to the weight $\delta^0_{(j+1)}(h)$ of B_ℓ.

(ii) For $j = 1,2, \dots, \ell$:

$$-\underset{\sim}{e}_{2\ell+2,j+1} + i\underset{\sim}{e}_{j+\ell+1,2\ell+2} = \overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+1)}} + i\overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+\ell+1)}}, \quad (4.3.153)$$

which corresponds to the weight $-\delta^0_{(j+1)}(h)$ of B_ℓ, where $\delta^0_{(j+1)}(h)$ is given by (4.3.152).

(iii) The single basis element:

$$\underset{\sim}{e}_{1,2\ell+2} + i\underset{\sim}{e}_{2\ell+2,1} = \overset{0}{\underset{\sim}{e}}_{\delta^0_{(1)}} - i\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(1)}}, \quad (4.3.154)$$

where $\overset{0}{\underset{\sim}{e}}_{\delta^0_{(1)}}$ and $i\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(1)}}$ are given by (4.1). However, by (4.3.130) to (4.3.132) $\delta^0_{(1)}(h) = 0$ for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$, so (4.3.154) corresponds to a zero weight of B_ℓ.

These weights all belong to a $(2\ell+1)$-dimensional irreducible representation of B_ℓ with highest weight

$$\Lambda = \Lambda_1^{B_\ell} = \sum_{k=1}^{\ell} \alpha_k^{B_\ell}. \quad (4.3.155)$$

It should be noted that all the elements of (4.3.151), (4.3.153), and (4.3.154) are odd members of $A(2\ell/0)$, so all the elements of $\tilde{\mathfrak{L}}_{s1}^{0(4)}$ are odd.

(d) The subspace $\tilde{\mathcal{L}}_{s2}^{0(4)}$

By (53) the subalgebra $\tilde{\mathcal{L}}_{s2}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = -\underset{\sim}{A},\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = -\underset{\sim}{C},\ \underset{\sim}{G}\tilde{\underset{\sim}{C}} = -\underset{\sim}{B},\ \text{and}\ -\tilde{\underset{\sim}{D}} = -\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1x1, imply that

$$\tilde{\underset{\sim}{A}}\underset{\sim}{G} - \underset{\sim}{G}\underset{\sim}{A} = \underset{\sim}{0},\ \underset{\sim}{B} = \underset{\sim}{0},\ \underset{\sim}{C} = \underset{\sim}{0}, \qquad (4.3.156)$$

with $\underset{\sim}{D}$ being determined only by the supertrace condition tr $\underset{\sim}{A}$ = tr $\underset{\sim}{D}$. On using (4.3.104) and (4.3.105), the diagonal basis elements of $\tilde{\mathcal{L}}_{s2}^{0(4)}$ may be taken to consist of the 2 sets:

(i) The single basis element

$$(1/4\ell)\{\underset{\sim}{e}_{1,1} + \underset{\sim}{e}_{2\ell+2,2\ell+2}\} = \sum_{r=1}^{2\ell+1} \underset{\sim}{h}^{0}_{\alpha_r^0}. \qquad (4.3.157)$$

(ii) For k = 1,2, ... ,ℓ:

$$(1/4\ell)\{\underset{\sim}{e}_{k+1,k+1} + \underset{\sim}{e}_{k+\ell+1,k+\ell+1} + 2\underset{\sim}{e}_{2\ell+2,2\ell+2}\}$$

$$= \sum_{r=k+1}^{k+\ell} \underset{\sim}{h}^{0}_{\alpha_r^0} + 2\sum_{r=k+\ell+1}^{2\ell+1} \underset{\sim}{h}^{0}_{\alpha_r^0}. \qquad (4.3.158)$$

Each of these corresponds to zero weight of B_ℓ, so that the zero weight has multiplicity $\ell+1$.

The non-diagonal basis elements of $\tilde{\mathcal{L}}_{s2}^{0(4)}$ fall into 8 sets:

(i) For j = 1,2, ... , ℓ:

$$\underset{\sim}{e}_{1,j+1} + \underset{\sim}{e}_{j+\ell+1,1} = \underset{\sim}{e}^{0}_{\beta^0_{(1,j+1)}} - \underset{\sim}{e}^{0}_{-\beta_{(1,j+\ell+1)}}, \qquad (4.3.159)$$

where $\underset{\sim}{e}^{0}_{\beta^0_{(1,j+1)}}$ and $\underset{\sim}{e}^{0}_{-\beta_{(1,j+\ell+1)}}$ are given by (4.1) and (4.3.108a), and$\beta^0_{(1,j+1)}(h)(= -\beta^0_{(1,j+\ell+1)}(h))$ is given for all h ∈

$\mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$) by (4.3.140), so this basis element (4.3.159) again corresponds to the weight $\beta^0_{(1,j+1)}(h)$ of B_ℓ.

(ii) For j = 1,2, ... , ℓ:

$$- \underset{\sim}{e}_{j+1,1} - \underset{\sim}{e}_{1,j+\ell+1} = \underset{\sim}{e}^0_{-\beta_{(1,j+1)}} - \underset{\sim}{e}^0_{\beta^0_{(1,j+\ell+1)}}, \qquad (4.3.160)$$

which corresponds to the weight $-\beta^0_{(1,j+1)}(h)$ of B_ℓ, where $\beta^0_{(1,j+1)}(h)$ is given by (4.3.140).

(iii) For j,k = 1,2, ... , ℓ, with j < k:

$$\underset{\sim}{e}_{j+1,k+1} + \underset{\sim}{e}_{k+\ell+1,j+\ell+1} = \underset{\sim}{e}^0_{\beta^0_{(j+1,k+1)}} - \underset{\sim}{e}^0_{-\beta_{(j+\ell+1,k+\ell+1)}}. \qquad (4.3.161)$$

where $\underset{\sim}{e}^0_{\beta^0_{(j+1,k+1)}}$ and $\underset{\sim}{e}^0_{-\beta_{(j+\ell+1,k+\ell+1)}}$ are given by (4.1) and (4.3.108a), and $\beta^0_{(j+1,k+1)}(h)$ (= $-\beta^0_{(j+\ell+1,k+\ell+1)}(h)$) is given for all $h \in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$) by (4.3.143), so this basis element (4.3.161) again corresponds to the weight $\beta^0_{(j+1,k+1)}(h)$ of B_ℓ. (This set does not appear when ℓ = 1).

(iv) For j,k = 1,2, ... ,ℓ, with j < k:

$$-\underset{\sim}{e}_{k+1,j+1} - \underset{\sim}{e}_{j+\ell+1,k+\ell+1} = \underset{\sim}{e}^0_{-\beta_{(j+1,k+1)}} - \underset{\sim}{e}^0_{\beta^0_{(j+\ell+1,k+\ell+1)}}.$$

(4.3.162)

This corresponds to the weight $-\beta^0_{(j+1,k+1)}(h)$ of B_ℓ, where $\beta^0_{(j+1,k+1)}(h)$ is given by (4.3.143). (This set does not appear when ℓ = 1).

(v) For j,k = 1,2, ... , ℓ, with j < k:

$$\underset{\sim}{e}_{j+1,k+\ell+1} + \underset{\sim}{e}_{k+1,j+\ell+1} = \underset{\sim}{e}^0_{\beta^0_{(j+1,k+\ell+1)}} + \underset{\sim}{e}^0_{\beta^0_{(k+1,j+\ell+1)}}. \qquad (4.3.163)$$

where $\underset{\sim}{e}^0_{\beta^0_{(j+1,k+\ell+1)}}$ and $\underset{\sim}{e}^0_{\beta^0_{(k+1,j+\ell+1)}}$ are given by (4.1) and (4.3.7a), and $\beta^0_{(j+1,k+\ell+1)}(h)$ (= $\beta^0_{(k+1,j+\ell+1)}(h)$) is given for all $h \in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$) by (4.3.146), so this basis element (4.3.163) again corresponds to the weight $\beta^0_{(j+1,k+\ell+1)}(h)$ of B_ℓ. (This set does not appear when $\ell = 1$).

(vi) For j,k = 1,2, ... , ℓ, with $j < k$:

$$- \underset{\sim}{e}_{k+\ell+1,j+1} - \underset{\sim}{e}_{j+\ell+1,k+1} = \underset{\sim}{e}^0_{-\beta_{(j+1,k+\ell+1)}} + \underset{\sim}{e}^0_{-\beta_{(k+1,j+\ell+1)}} .$$

(4.3.164)

This corresponds to the weight $-\beta^0_{(j+1,k+\ell+1)}(h)$ of B_ℓ, where $\beta^0_{(j+1,k+\ell+1)}$ is given by (4.3.146). (This set does not appear when $\ell = 1$).

(vii) For j = 1,2, ... ,ℓ:

$$\underset{\sim}{e}_{j+1,j+\ell+1} = \underset{\sim}{e}^0_{\beta^0_{(j+1,j+\ell+1)}} , \qquad (4.3.165)$$

where $\underset{\sim}{e}^0_{\beta^0_{(j+1,j+\ell+1)}}$ is given by (4.1) and (4.3.108a). Thus the

basis

element (4.3.165) corresponds to the weight $\beta^0_{(j+1,j+\ell+1)}(h)$ of B_ℓ, where (4.3.134) and (4.3.135) imply that

$$\beta^0_{(j+1,j+\ell+1)}(h) = 2\sum\nolimits_{r=j}^{\ell} \alpha_r^{B_\ell}(h) , \qquad (4.3.166)$$

(for j = 1,2, ... , ℓ, and for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$).

(viii) For j = 1,2, ... ,ℓ:

$$-\underset{\sim}{e}_{j+\ell+1,j+1} = \underset{\sim}{e}^0_{-\beta_{(j+1,j+\ell+1)}} , \qquad (4.3.167)$$

which corresponds to the weight $-\beta^0_{(j+1,j+\ell+1)}(h)$ of B_ℓ, where $\beta^0_{(j+1,j+\ell+1)}(h)$ is given by (4.3.166).

These $2\ell^2+3\ell+1$ weights belong to a representation of B_ℓ which is the direct sum of the trivial 1-dimensional irreducible representation with highest weight $\Lambda = 0$ and the

$(2\ell^2+3\ell)$-dimensional irreducible representation with highest weight

$$\Lambda = 2\Lambda_1^{B_\ell} = 2\sum_{k=1}^{\ell}\alpha_\ell^{B_\ell}. \qquad (4.3.168)$$

It should be noted that all the elements of $\tilde{\mathcal{L}}_{s2}^{0(4)}$ are even members of $A(2\ell/0)$.

(e) The subspace $\tilde{\mathcal{L}}_{s3}^{0(4)}$

By (53) the subspace $\tilde{\mathcal{L}}_{s3}^{0(4)}$ consists of the supermatrices whose submatrices satisfy the conditions

$$-\underset{\sim}{G}\tilde{\underset{\sim}{A}}\underset{\sim}{G} = -i\underset{\sim}{A},\ -\tilde{\underset{\sim}{B}}\underset{\sim}{G} = -i\underset{\sim}{C},\ \underset{\sim}{G}\tilde{\underset{\sim}{C}} = -i\underset{\sim}{B},\ \text{and } -\tilde{\underset{\sim}{D}} = -i\underset{\sim}{D},$$

which when taken together, along with the fact that $\underset{\sim}{D}$ is 1x1, imply that

$$\underset{\sim}{A} = 0,\ \underset{\sim}{D} = 0,\ \text{and } \underset{\sim}{C} = -i\tilde{\underset{\sim}{B}}\underset{\sim}{G}\,. \qquad (4.3.169)$$

The basis elements of $\tilde{\mathcal{L}}_{s3}^{0(4)}$ fall into 3 sets:

(i) For $j = 1,2, \dots, \ell$:

$$\underset{\sim}{e}_{j+1,2\ell+2} - i\underset{\sim}{e}_{2\ell+2,j+\ell+1} = \overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+1)}} + i\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+\ell+1)}}\ , \qquad (4.3.170)$$

where $\overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+1)}}$ and $\overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+\ell+1)}}$ are given by (4.1) and (4.3.108b),

and

$\delta^0_{(j+1)}(h)$ $(= -\delta^0_{(j+\ell+1)}(h))$ is given for all $h \in \mathcal{H}^{0(4)}$ $(= \mathcal{H}^{B_\ell})$ by (4.3.152), so this basis element (4.3.170) again corresponds to the weight $\delta^0_{(j+1)}(h)$ of B_ℓ.

(ii) For $j = 1,2, \dots, \ell$:

$$-\underset{\sim}{e}_{2\ell+2,j+1} - i\underset{\sim}{e}_{j+\ell+1,2\ell+2} = \overset{0}{\underset{\sim}{e}}_{-\delta^0_{(j+1)}} - i\overset{0}{\underset{\sim}{e}}_{\delta^0_{(j+\ell+1)}}\ , \qquad (4.3.171)$$

which corresponds to the root $-\delta^0_{(j+1)}(h)$ of B_ℓ, where $\delta^0_{(j+1)}(h)$ is given by (4.3.152).

(iii) The single basis element:

$$\underset{\approx}{e}_{1,2\ell+2} - i\underset{\approx}{e}_{2\ell+2,1} = \overset{0}{\underset{\approx}{e}}_{\delta^0_{(1)}} + i\overset{0}{\underset{\approx}{e}}_{-\delta^0_{(1)}} , \qquad (4.3.172)$$

where $\overset{0}{\underset{\approx}{e}}_{\delta^0_{(1)}}$ is given by (4.1) and (4.3.107b). As $\delta^0_{(1)}(h) = 0$ for all $h \in \mathcal{H}^{0(4)}$ (= $\mathcal{H}^{B_\ell}$), (4.3.71) corresponds to a zero weight of B_ℓ.

These weights all belong to a $(2\ell+1)$-dimensional irreducible representation of B_ℓ whose highest weight is given by (4.3.155). All the elements of $\tilde{\mathfrak{L}}^{0(4)}_{s3}$ are odd.

(f) The roots and root basis vectors of $A^{(4)}(2\ell/0)$

Defining $\delta(h)$ as before, it follows that the roots $\alpha(h)$ and the corresponding basis elements e_α of $A^{(4)}(2\ell/0)$ are as follows:

(i) $\alpha = 4J\delta(h)$, (for $J = 0,\pm1,\pm2, \ldots$).

There are ℓ linearly independent basis elements $e_\alpha^{(k)}$ corresponding to this root which may be labeled by an additional superscript, so that

$$e_\alpha^{(k)} = \{2\ell/(2\ell-1)\}\, t^{4J}\otimes\{\underset{\sim}{h}^0_{\alpha^0_{k+1}} - \underset{\sim}{h}^0_{\alpha^0_{k+\ell+1}}\} \text{ (for } k = 1,2, \ldots , \ell-1),$$

and

$$e_\alpha^{(\ell)} = \{2\ell/(2\ell-1)\}\; t^{4J}\otimes\left\{\sum_{r=\ell+1}^{2\ell} \underset{\sim}{h}^0_{\alpha^0_r}\right\},$$

(which reduce to (4.3.137) and (4.3.138) in the special case $J = 0$).

(ii) $\alpha(h) = 4J\delta(h) \pm \beta^0_{(1,j+1)}(h)$,

(for $j = 1,2, \ldots , \ell$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^0_{(1,j+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.140) and

$$e_\alpha = t^{4J}\otimes\{\underset{\sim}{e}^0_{\pm\beta_{(1,j+1)}} + \underset{\sim}{e}^0_{\mp\beta_{(1,j+\ell+1)}}\}.$$

(iii) $\alpha(h) = 4J\delta(h) \pm \beta^0_{(j+1,k+1)}(h)$,

(for $j,k = 1,2, \ldots , \ell$, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^0_{(j+1,k+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.143) and

$$e_\alpha = t^{4J} \otimes \{ \underset{\sim}{e}^{0}_{\pm\beta_{(j+1,k+1)}} + \underset{\sim}{e}^{0}_{\mp\beta_{(j+\ell+1,k+\ell+1)}} \}.$$

(iv) $\alpha(h) = 4J\delta(h) \pm \beta^{0}_{(j+1,k+\ell+1)}(h)$,

(for j,k = 1,2, ... , ℓ, with j < k, and for J = 0,±1,±2, ...), where $\beta^{0}_{(j+1,k+\ell+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.146) and

$$e_\alpha = t^{4J} \otimes \{ \underset{\sim}{e}^{0}_{\pm\beta_{(j+1,k+\ell+1)}} - \underset{\sim}{e}^{0}_{\mp\beta_{(k+1,j+\ell+1)}} \}.$$

(v) $\alpha(h) = (4J+1)\delta(h) \pm \delta^{0}_{(j+1)}(h)$, (for j = 1,2, ... , ℓ, and for J = 0,±1,±2, ...), where $\delta^{0}_{(j+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.152) and

$$e_\alpha = t^{4J+1} \otimes \{ \underset{\sim}{e}^{0}_{\pm\delta_{(j+1)}} \mp i\underset{\sim}{e}^{0}_{\mp\delta_{(j+\ell+1)}} \}.$$

(vi) $\alpha(h) = (4J+1)\delta(h)$, (for J = 0,±1,±2, ...), with

$$e_\alpha = t^{4J+1} \otimes \{ \underset{\sim}{e}^{0}_{\delta^{0}_{(1)}} - i\underset{\sim}{e}^{0}_{-\delta_{(1)}} \}.$$

(vii) $\alpha(h) = (4J+2)\delta(h)$, (for J = 0,±1,±2, ...). There are $\ell+1$ linearly independent basis elements $e^{(k)}_\alpha$ corresponding to this root which may be labeled by an additional superscript, so that

$$e^{(k)}_\alpha = t^{4J+2} \otimes \left\{ \sum_{r=k+1}^{k+\ell} \underset{\sim}{h}^{0}_{\alpha^0_r} + 2\sum_{r=k+\ell+1}^{2\ell+1} \underset{\sim}{h}^{0}_{\alpha^0_r} \right\} \text{,(for k= 1,2, .}\ell\text{)}$$

and

$$e^{(\ell+1)}_\alpha = t^{4J+2} \otimes \sum_{r=1}^{2\ell+1} \underset{\sim}{h}^{0}_{\alpha^0_r} ;$$

(viii) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(1,j+1)}(h)$, (for j = 1,2, ... , ℓ, and for J = 0,±1,±2, ...), where $\beta^{0}_{(1,j+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.140) and

$$e_\alpha = t^{4J+2} \otimes \{ \underset{\sim}{e}^{0}_{\pm\beta_{(1,j+1)}} - \underset{\sim}{e}^{0}_{\mp\beta_{(1,j+\ell+1)}} \}.$$

(ix) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j+1,k+1)}(h)$,

(for $j,k = 1,2, \ldots, \ell$, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j+1,k+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.143) and

$$e_\alpha = t^{4J+2} \otimes \{ \underset{\sim}{e}^{0}_{\pm\beta_{(j+1,k+1)}} - \underset{\sim}{e}^{0}_{\mp\beta_{(j+\ell+1,k+\ell+1)}} \}.$$

(x) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j+1,k+\ell+1)}(h)$,

(for $j,k = 1,2, \ldots, \ell$, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j+1,k+\ell+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.4146) and

$$e_\alpha = t^{4J+2} \otimes \{ \underset{\sim}{e}^{0}_{\pm\beta_{(j+1,k+\ell+1)}} + \underset{\sim}{e}^{0}_{\mp\beta_{(k+1,j+\ell+1)}} \} .$$

(xi) $\alpha(h) = (4J+2)\delta(h) \pm \beta^{0}_{(j+1,j+\ell+1)}(h)$, (for $j = 1,2, \ldots, \ell$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0}_{(j+1,j+\ell+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.166) and

$$e_\alpha = t^{4J+2} \otimes \underset{\sim}{e}^{0}_{\pm\beta_{(j+1,j+\ell+1)}} .$$

(xii) $\alpha(h) = (4J+3)\delta(h) \pm \delta^{0}_{(j+1)}(h)$, (for $j = 1,2, \ldots, \ell$, and for $J = 0,\pm1,\pm2, \ldots$), where $\delta^{0}_{(j+1)}(h)$ is the extension of the weight of B_ℓ that is given by (4.3.152) and

$$e_\alpha = t^{4J+3} \otimes \{ \underset{\sim}{e}^{0}_{\pm\delta_{(j+1)}} \pm i\underset{\sim}{e}^{0}_{\mp\delta_{(j+\ell+1)}} \} .$$

(xiii) $\alpha(h) = (4J+3)\delta(h)$, (for $J = 0,\pm1,\pm2, \ldots$), with

$$e_\alpha = t^{4J+3} \otimes \{ \underset{\sim}{e}_{\delta^{0}_{(1)}} + i\underset{\sim}{e}^{0}_{-\delta_{(1)}} \}.$$

(xiv) $\alpha(h) = 0$, with c and d as basis elements.

With μ chosen as in (4.3.114), it follows that

$$<\alpha^0,\beta^0> \;=\; <\alpha^0,\beta^0>^{B_\ell} \qquad (4.3.173)$$

where on the right-hand side of (4.3.173) α^0 and β^0 are any pair of linear functionals defined on $\mathcal{H}^{0(4)}$ ($= \mathcal{H}^{B_\ell}$), the evaluation being performed with respect to the Killing form of B_ℓ, and where on the left-hand side of (4.3.173) α^0 and β^0 denote the corresponding extensions to the Cartan subalgebra of the Kac-Moody superalgebra $A^{(4)}(2\ell/0)$, the evaluation being performed with respect to its supersymmetric bilinear invariant form B(,). As B_ℓ is a simple Lie algebra, $<\alpha^0,\alpha^0>^{B_\ell} > 0$ for every non-zero linear functional α^0 defined on $\mathcal{H}^{B_\ell}$, so $<\alpha^0,\alpha^0> > 0$ for the corresponding extension. Moreover (4.3.114) imply that

$$h_\delta \;=\; \{4\ell/(2\ell-1)\}\, c\ . \qquad (4.3.174)$$

Thus, if α_k^0 is the extension of any simple root of $\tilde{\mathfrak{L}}_s^0$, then

$$<\delta,\alpha_k^0> \;=\; 0 \qquad (4.3.175)$$

$$<j\delta,j\delta> \;=\; 0\ . \qquad (4.3.176)$$

Thus $<j\delta,j\delta> = 0$ for integer j, so every non-zero root of $A^{(4)}(2\ell/0)$ belonging to the sets (i), (vi), (vii), and (xiii) is "imaginary". Moreover, because $<j\delta+\alpha^0,j\delta+\alpha^0> = <\alpha^0,\alpha^0>^{B_\ell}$ and because $<\alpha^0,\alpha^0>^{B_\ell} > 0$ for linear functional α^0 and its corresponding extension (as has just been noted), it follows that every root of $A^{(4)}(2\ell/0)$ belonging to the sets (ii), (iii), (iv), (v), (viii), (ix), (x), (xi), and (xii) is "real". All the elements mentioned in the above sets are even, except for those in the sets (v), (vi), (xii) and (xiii), which are odd.

For $A^{(4)}(2\ell/0)$ for $\ell \geq 1$ the simple roots may be taken to be

$$\alpha_\ell = \delta - \alpha_H^0 , \quad (4.3.177)$$

$$\alpha_k = \alpha_{\ell-k}^{B_\ell} \text{ (for } k = 0,1, \dots ,\ell-1) \quad (4.3.178)$$

where

$$\alpha_H^0 = \Lambda_1^{B_\ell} = \sum_{k=1}^{\ell} \alpha_k^{B_\ell} \quad (4.3.179)$$

is the highest weight of the representation of $\tilde{\mathcal{L}}_{s0}^{0(4)}$ for which $\tilde{\mathcal{L}}_{s1}^{0(4)}$ is the carrier spaceand the $\alpha_k^{B_\ell}$ are the extensions of the simple roots of B_ℓ. As e_{α_ℓ} appears in the set (v) it follows that e_{α_ℓ} is odd, so α_ℓ is an odd root of the Kac-Moody superalgebra $A^{(4)}(2\ell/0)$. All the other simple roots of $A^{(4)}(2\ell/0)$ are even.

It is then easily checked that the Cartan matrices of $A^{(4)}(2/0)$ and $A^{(4)}(2\ell/0)$ (for $\ell \geq 2$) evaluated using the definition on section (4.1.5) correspond to the generalized Dynkin diagrams given in Figures 5 and 6 respectively. The quantities $<\alpha^0,\alpha^0>^{B_\ell}$ can be found in appendix A(2).

In terms of the linearly independent functionals ε_j ($i \leq j \leq \ell$) defined on $\mathcal{H}^{B_\ell}$(see Cornwell(1984) and table I, chapter 2) the roots of $A^{(4)}(2\ell/0)$ are given by

$$\Delta = \{ 2m\delta \pm (\varepsilon_i \pm \varepsilon_j) \text{ with } 1 \leq i < j \leq \ell ,\ m\delta \pm \varepsilon_i \text{ with } 1 \leq i \leq \ell$$
$$(4m+2)\delta \pm 2\varepsilon_i \text{ with } 1 \leq i \leq \ell \text{ all with } m \in \mathbb{Z}$$
$$\text{and } m\delta \text{ with } m \neq 0 \text{ and } m \in \mathbb{Z}\} \quad (4.3.180)$$

The basis is given by

$$\alpha_0 = \varepsilon_\ell \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad 1 \leq i \leq \ell -1 \quad \alpha_\ell = \delta - \varepsilon_1 \quad (4.3.181)$$

D. Root structure of $C^{(2)}(\ell+1)$ (for $\ell = 1,2,3,\ldots$)

(a) The 2-fold automorphisms

An explicit realization of the simple Lie superalgebra $C(\ell+1)$ is provided by the orthosymplectic algebra $osp(2/2\ell;\mathbb{C})$, considered as a complex superalgebra, where $osp(2/2\ell;\mathbb{C})$ is defined as the set of $(2\ell+2)\times(2\ell+2)$ complex supermatrices with the grading partitioning

$$\underset{\sim}{M} = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{C} & \underset{\sim}{D} \end{pmatrix}, \qquad (4.3.182)$$

that are subject to the condition that

$$\underset{\sim}{M}^{st}\underset{\sim}{K} + (-1)^{\deg \underset{\sim}{M}}\underset{\sim}{K}\,\underset{\sim}{M} = \underset{\sim}{0}, \qquad (4.3.183)$$

where

$$\underset{\sim}{K} = \begin{pmatrix} \underset{\sim}{G} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{J} \end{pmatrix}, \qquad (4.3.184)$$

with

$$\underset{\sim}{G} = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{1}_1 \\ \underset{\sim}{1}_1 & \underset{\sim}{0} \end{pmatrix}, \qquad (4.3.185)$$

and

$$\underset{\sim}{J} = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{1}_\ell \\ -\underset{\sim}{1}_\ell & \underset{\sim}{0} \end{pmatrix}, \qquad (4.3.186)$$

(Here $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, $\underset{\sim}{D}$, $\underset{\sim}{K}$, $\underset{\sim}{G}$ and $\underset{\sim}{J}$ are of dimensions 2×2, $2\times2\ell$, $2\ell\times2$, $2\ell\times2\ell$, $2\ell+2)\times(2\ell+2)$, 2×2, and $2\ell\times2\ell$ respectively). The condition (4.3.2) implies that

$$\tilde{\underset{\sim}{A}}G + GA = 0, \qquad (4.3.187)$$

$$\tilde{D}J + JD = 0, \qquad (4.3.188)$$

and

$$\tilde{B}G - JC = 0. \qquad (4.3.189)$$

The rank ℓ^0 of $C(\ell+1)$ is given by $\ell^0 = \ell+1$.

The generalized Dynkin diagram of $C(\ell+1)$ is shown in Figure 12, which indicates that its distinguished simple roots α_k^0 are even for $k = 2,3,\dots,\ell+1$, but that α_1^0 is odd. With the bilinear form $B^0(\ ,\ )$ being defined by

$$B^0(M,N) = -2\ell \operatorname{str}(MN), \qquad (4.3.190)$$

the basis elements of its Cartan subalgebra $\mathcal{H}^0$ may be taken to be

$$h_{\alpha_1^0}^0 = -\{1/4\ell\}\{e_{1,1} - e_{2,2} + e_{3,3} - e_{\ell+3,\ell+3}\}, \qquad (4.3.191)$$

$$h_{\alpha_k^0}^0 = \{1/4\ell\}\{e_{k+1,k+1} - e_{k+\ell+1,k+\ell+1} - e_{k+2,k+2} + e_{k+\ell+2,k+\ell+2}\}, \qquad (4.3.192)$$

(for $k = 2,3,\dots,\ell$) and

$$h_{\alpha_{\ell+1}^0}^0 = \{1/2\ell\}\{e_{\ell+2,\ell+2} - e_{2\ell+2,2\ell+2}\}. \qquad (4.3.193)$$

Again $e_{r,s}$ is the matrix of dimension $(2\ell+2)\times(2\ell+2)$ that is defined by (4.3.186), so that with this choice all the matrices of $\mathcal{H}^0$ are again <u>diagonal</u>.

The positive even roots $\beta_{(j,k)}^{0+}$ and $\beta_{(j,k)}^{0-}$ and positive odd roots $\delta_{(j)}^{0+}$ and $\delta_{(j)}^{0-}$ of $C(\ell+1)$ are given in terms of the distinguished set of simple roots $\alpha_1^0, \alpha_2^0, \dots, \alpha_{\ell+1}^0$ of $C(\ell+1)$ by

$$\beta^{0-}_{(j,k)} = \sum_{r=j+1}^{k} \overset{0}{\alpha}_r \text{ (for } j,k = 1,2, \ldots ,\ell, \ j < k), \qquad (4.3.194)$$

$$\beta^{0+}_{(j,k)} = \sum_{r=j+1}^{k} \overset{0}{\alpha}_r + 2\sum_{r=k+1}^{\ell} \overset{0}{\alpha}_r + \overset{0}{\alpha}_{\ell+1}$$

$$\text{(for } j,k = 1,2, \ldots ,\ell-1, \ j < k), \qquad (4.3.195)$$

$$\beta^{0+}_{(j,\ell)} = \sum_{r=j+1}^{\ell} \overset{0}{\alpha}_r + \overset{0}{\alpha}_{\ell+1} \text{ (for } j = 1,2, \ldots , \ell-1), \qquad (4.3.196)$$

$$\beta^{0+}_{(j,j)} = 2\sum_{r=j+1}^{\ell} \overset{0}{\alpha}_r + \overset{0}{\alpha}_{\ell+1} \text{ (for } j = 1,2, \ldots , \ell-1), \qquad (4.3.197)$$

$$\beta^{0+}_{(\ell,\ell)} = \overset{0}{\alpha}_{\ell+1} , \qquad (4.3.198)$$

$$\delta^{0-}_{(j)} = \sum_{r=1}^{j} \overset{0}{\alpha}_r \text{ (for } j = 1,2, \ldots ,\ell), \qquad (4.3.199)$$

$$\delta^{0+}_{(j)} = \sum_{r=1}^{j} \overset{0}{\alpha}_r + 2\sum_{r=j+1}^{\ell} \overset{0}{\alpha}_r + \overset{0}{\alpha}_{\ell+1} \text{ (for } j = 1,2, \ldots ,\ell-1), \qquad (4.3.200)$$

and

$$\delta^{0+}_{(\ell)} = \sum_{r=1}^{\ell+1} \overset{0}{\alpha}_r . \qquad (4.3.201)$$

The corresponding basis elements of $C(\ell+1)$ may be taken to be

$$\overset{0}{\underset{\sim}{e}}_{\beta^{0-}_{(j,k)}} = \underset{\sim}{e}_{j+2,k+2} - \underset{\sim}{e}_{k+\ell+2,j+\ell+2} , \text{ (for } j,k = 1,2, \ldots ,\ell; \ j < k), \qquad (4.3.202)$$

$$\overset{0}{\underset{\sim}{e}}_{\beta^{0+}_{(j,k)}} = \underset{\sim}{e}_{j+2,k+\ell+2} + \underset{\sim}{e}_{k+2,j+\ell+2} , \text{ (for } j,k = 1,2, \ldots ,\ell; \ j \leq k), \qquad (4.3.203)$$

$$\overset{0}{\underset{\sim}{e}}_{\delta^{0-}_{(j)}} = \underset{\sim}{e}_{1,j+2} + \underset{\sim}{e}_{j+\ell+2,2} , \text{ (for } j = 1,2, \ldots ,\ell), \qquad (4.3.204)$$

and

$$\underset{\sim}{e}^{0}_{\delta^{0+}_{(j)}} = \underset{\sim}{e}_{1,j+\ell+2} - \underset{\sim}{e}_{j+2,2} \text{ , (for } j = 1,2, \ldots ,\ell). \qquad (4.3.205)$$

The basis elements corresponding to the corresponding negative roots may be chosen in accordance with (4.1). (For further information on $C(\ell+1)$ see Cornwell(1989)).

Taking the node corresponding to the odd simple root α_0 of $C^{(2)}(\ell+1)$, as the corresponding numerical mark has value 1, (c.f. Figures 7 and 8) $q = 2$. Moreover inspection of Figures 7 and 8 shows that the generalized Dynkin diagram with the chosen node and attached lines removed corresponds to $B(0/\ell)$, the subalgebra $\tilde{\mathcal{L}}^{0(2)}_{s0}$ has to be isomorphic to $B(0/\ell)$.

The complex simple Lie superalgebra $B(0/\ell)$ may be realized as $osp(1/2\ell;\mathbb{C})$, which is the set of $(2\ell+1)\times(2\ell+1)$ supermatrices $\underset{\sim}{m}$ of the form

$$\underset{\sim}{m} = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{b} \\ \underset{\sim}{c} & \underset{\sim}{D} \end{pmatrix}, \qquad (4.3.206)$$

where $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are submatrices of dimensions $1\times 2\ell$ and $2\ell\times 1$ respectively that experience the constraint

$$\tilde{\underset{\sim}{b}} - Jc = 0 , \qquad (4.3.207)$$

and D is a $2\ell\times 2\ell$ submatrix such that

$$\tilde{\underset{\sim}{D}}J + JD = 0 , \qquad (4.3.208)$$

($\underset{\sim}{J}$ being defined in (4.3.186)). This will be called the canonical form of $B(0/\ell)$.

One possible two-fold automorphism of $C(\ell+1)$ is provided by ψ^2, where ψ is the automorphism of (4.3.26). However, as

$$\psi^2(\underset{\sim}{M}) = -(-\underset{\sim}{M}^{st})^{st} = \begin{pmatrix} \underset{\sim}{A} & -\underset{\sim}{B} \\ -\underset{\sim}{C} & \underset{\sim}{D} \end{pmatrix},$$

it follows from (53) that if this automorphism is employed then the subalgebra $\tilde{\mathcal{L}}_{s0}^{0(2)}$ would consist of the supermatrices with $\underset{\sim}{B} = \underset{\sim}{C} = \underset{\sim}{0}$, and with $\underset{\sim}{A}$ and $\underset{\sim}{D}$ satisfying (4.3.187) and (4.3.188) respectively, so that the subalgebra $\tilde{\mathcal{L}}_{s0}^{0(2)}$ would isomorphic to the even part of $C(\ell+1)$, and not to the superalgebra $B(0/\ell)$. Consequently ψ^2 is <u>not</u> an appropriate choice of automorphism.

As will be demonstrated explicitly in the next subsection the correct choice is actually given by

$$\phi(\underset{\sim}{M}) = \underset{\sim}{L}^{-1}(-\underset{\sim}{M}^{st})\underset{\sim}{L} \qquad (4.3.209)$$

where

$$\underset{\sim}{L} = \begin{pmatrix} \underset{\sim}{1}_2 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{J} \end{pmatrix},$$

so that

$$\phi(\underset{\sim}{M}) = \begin{pmatrix} -\tilde{\underset{\sim}{A}} & \tilde{\underset{\sim}{C}}\underset{\sim}{J} \\ -\underset{\sim}{J}^{-1}\tilde{\underset{\sim}{B}} & -\underset{\sim}{J}^{-1}\tilde{\underset{\sim}{D}}\underset{\sim}{J} \end{pmatrix}. \qquad (4.3.210)$$

It is easily checked that this provides a two-fold automorphism of $C(\ell+1)$.

The 2 subspaces $\tilde{\mathcal{L}}_{sp}^{0(2)}$ (for $p = 0,1$) corresponding to the automorphism ϕ of (4.3.28) will now be considered in turn:

(b) The subspaces $\tilde{\mathcal{L}}^{0(2)}_{s0}$

By (53) the subalgebra $\tilde{\mathcal{L}}^{0(2)}_{s0}$ consists of the supermatrices whose submatrices satisfy the conditions $-\tilde{\mathbf{A}} = \mathbf{A}$ and $\tilde{\mathbf{C}}\mathbf{J} = \mathbf{B}$ in addition to (4.3.187), (4.3.188), and (4.3.189). Together these imply that $\mathbf{A} = \mathbf{0}$ and that

$$\mathbf{B} = 2^{-\frac{1}{2}}\begin{pmatrix} -\mathbf{b} \\ \mathbf{b} \end{pmatrix} \text{ and } \mathbf{C} = 2^{-\square}\begin{pmatrix} -\mathbf{c} & \mathbf{c} \end{pmatrix}, \tag{4.3.211}$$

where $\mathbf{b}$ and $\mathbf{c}$ are submatrices of dimensions $1\times 2\ell$ and $2\ell\times 1$ respectively that experience the constraint (4.3.207). It is easily checked that subject to these conditions the mapping

$$\Psi\left(\begin{pmatrix} \mathbf{0} & \mathbf{b} \\ \mathbf{c} & \mathbf{D} \end{pmatrix}\right) = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{4.3.212}$$

is an isomorphic mapping of $B(0/\ell)$ onto $\tilde{\mathcal{L}}^{0(2)}_{s0}$, (the factors of $2^{-\frac{1}{2}}$ in (4.3.211) being inserted to help give this result).

Some properties of the canonical form of $B(0/\ell)$ and its image under the mapping (4.3.212) will first be summarized (the conventions being those of Cornwell(1989)). The Killing form $B^{B(0/\ell)}(\ ,\ )$ is given by

$$B^{B(0/\ell)}(\mathbf{m},\mathbf{m}') = -(2\ell+1)\ \mathrm{str}(\mathbf{m}\,\mathbf{m}')\ , \tag{4.3.213}$$

(for all $\mathbf{m}$ and $\mathbf{m}'$ of the canonical form of $B(0/\ell)$). Then, by (4.3.212),

$$B^{B(0/\ell)}(\mathbf{m},\mathbf{m}') = -(2\ell+1)\ \mathrm{str}(\Psi(\mathbf{m})\Psi(\mathbf{m}'))\ , \tag{4.3.214}$$

and so, by (4.3.190),

$$B^{0}(\Psi(\mathbf{m}),\Psi(\mathbf{m}')) = \{2\ell/(2\ell+1)\}\ B^{B(0/\ell)}(\mathbf{m},\mathbf{m}') \tag{4.3.215}$$

for all $\mathbf{m}$ and $\mathbf{m}'$ of the canonical form. This implies that (4.3.8) is satisfied if

$$\mu \;=\; (2\ell+1)/(2\ell)\,. \tag{4.3.216}$$

Denoting the simple even roots of $B(0/\ell)$ by $\alpha_k^{B(0/\ell)}$ (for $k = 1,2, \ldots,\ell-1$) and the simple odd root of $B(0/\ell)$ by $\alpha_\ell^{B(0/\ell)}$, the corresponding basis elements of the Cartan subalgebra $\mathcal{H}^{B(0/\ell)}$ of $B(0/\ell)$ defined by

$$B^{B(0/\ell)}(\underset{\sim}{h}^{B(0/\ell)}_{\alpha_k^{B(0/\ell)}}, \underset{\sim}{h}) \;=\; \alpha_k^{B(0/\ell)}(\underset{\sim}{h}) \quad \text{for all } \underset{\sim}{h} \in \mathcal{H}^{B(0/\ell)} \tag{4.3.217}$$

are

$$\underset{\sim}{h}^{B(0/\ell)}_{\alpha_k^{B(0/\ell)}} = \{1/(2\ell+1)\}\{\underset{\sim}{e}_{k+1,k+1} - \underset{\sim}{e}_{k+\ell+1,k+\ell+1} - \underset{\sim}{e}_{k+2,k+2} + \underset{\sim}{e}_{k+\ell+2,k+\ell+2}\},$$

$$(\text{for } k = 1,2, \ldots,\ell-1) \tag{4.3.218}$$

and

$$\underset{\sim}{h}^{B(0/\ell)}_{\alpha_\ell^{B(0/\ell)}} \;=\; \{1/(2\ell+1)\}\;\{\underset{\sim}{e}_{\ell+1,\ell+1} - \underset{\sim}{e}_{2\ell+1,2\ell+1}\}\,. \tag{4.3.219}$$

Thus

$$\Psi(\underset{\sim}{h}^{B(0/\ell)}_{\alpha_k^{B(0/\ell)}}) = \{1/(2\ell+1)\}\{\underset{\sim}{e}_{k+2,k+2} - \underset{\sim}{e}_{k+\ell+2,k+\ell+2} - \underset{\sim}{e}_{k+3,k+3} + \underset{\sim}{e}_{k+\ell+3,k+\ell+3}\},$$

$$(\text{for } k = 1,2, \ldots,\ell-1) \tag{4.3.220}$$

and

$$\Psi(\underset{\sim}{h}^{B(0/\ell)}_{\alpha_\ell^{B(0/\ell)}}) \;=\; \{1/(2\ell+1)\}\;\{\underset{\sim}{e}_{\ell+2,\ell+2} - \underset{\sim}{e}_{2\ell+2,2\ell+2}\}\,. \tag{4.3.221}$$

The <u>diagonal</u> basis elements of $\tilde{\mathfrak{L}}_{s0}^{0(2)}$ will be considered first. As they may be taken to consist of the set $\{\underset{\sim}{e}_{k+2,k+2} - \underset{\sim}{e}_{k+\ell+2,k+\ell+2} \mid \text{for } k = 1,2, \ldots, \ell\}$, it follows that they are all members of the Cartan subalgebra $\mathcal{H}^{0(2)}$ (= $\mathcal{H}^{B(0/\ell)}$) of $B(0/\ell)$ (as expected). By (4.3.192) the most general element of $\mathcal{H}^{0(2)}$ has the form

$$\sum_{k=1}^{\ell}\kappa_k(\underset{\sim}{e}_{k+2,k+2}-\underset{\sim}{e}_{k+\ell+2,k+\ell+2}) = 4\ell\sum_{k=2}^{\ell+1}\rho_k\underset{\sim}{h}^0_{\alpha_k^0},$$

where $\kappa_1, \kappa_2, \dots, \kappa_\ell$ are any complex numbers, and where

$$\rho_k = \sum_{r=1}^{k-1}\kappa_r \quad (\text{for } k = 2, \dots, \ell),$$

and

$$\rho_{\ell+1} = \tfrac{1}{2}\sum_{r=1}^{\ell}\kappa_r .$$

Thus on $\mathcal{H}^{0(2)}$ the simple roots of $C(\ell+1)$ are given by

$$\alpha_1^0(h) = -\kappa_1 , \qquad (4.3.222)$$

$$\alpha_k^0(h) = \kappa_{k-1} - \kappa_k \quad (\text{for } k = 2,3, \dots, \ell), \qquad (4.3.223)$$

and

$$\alpha_{\ell+1}^0(h) = 2\kappa_\ell , \qquad (4.3.224)$$

which implies that on $\mathcal{H}^{0(2)}$ $(= \mathcal{H}^{B(0/\ell)})$

$$\alpha_{\ell+1}^0(h) = -2\sum_{k=1}^{\ell}\alpha_k^0(h) . \qquad (4.3.225)$$

However, as (4.3.220) and (4.3.221) imply that

$$\sum_{k=1}^{\ell}\kappa_k(\underset{\sim}{e}_{k+2,k+2}-\underset{\sim}{e}_{k+\ell+2,k+\ell+2}) = 2(2\ell+1)\sum_{k=1}^{\ell}\mu_k\Psi(\underset{\sim}{h}^{B(0/\ell)}_{\alpha_k^{B(0/\ell)}})$$

,

with

$$\mu_k = \sum_{r=1}^{k}\kappa_r \quad (\text{for } k = 1,2, \dots, \ell),$$

it follows that on $\mathcal{H}^{0(2)}$ $(= \mathcal{H}^{B(0/\ell)})$

$$\alpha_k^{B(0/\ell)}(h) = \kappa_k + \kappa_{k-1} \quad (\text{for } k = 1,2, \dots, \ell-1), \qquad (4.3.226)$$

and

$$\alpha_\ell^{B(0/\ell)}(h) = \kappa_\ell . \qquad (4.3.227)$$

Comparison of (4.3.222) to (4.3.224) with (4.3.226) and (4.3.227) then shows that on the Cartan subalgebra on $\mathcal{H}^{0(2)}$

(= $\mathcal{H}^{B(0/\ell)}$) of B(0/ℓ) the simple roots $\alpha_k^{B(0/\ell)}$ of B(0/ℓ) and α_k^0 of C(ℓ+1) are related by

$$\alpha_1^0(h) = -\sum_{r=1}^{\ell}\alpha_r^{B(0/\ell)}(h) , \tag{4.3.228}$$

$$\alpha_k^0(h) = \alpha_{k-1}^{B(0/\ell)}(h) \text{ (for } k = 2,3, \dots ,\ell), \tag{4.3.229}$$

and

$$\alpha_{\ell+1}^0(h) = 2\alpha_\ell^{B(0/\ell)}(h) . \tag{4.3.230}$$

Finally it follows from (4.2.10) to (4.2.11), (4.3.8), (4.3.16), (4.3.192), (4.3.193), (4.3.220), and (4.3.221) that corresponding elements of the Cartan subalgebra of the Kac-Moody superalgebra are

$$h_{\alpha_k^{B(0/\ell)}} = t^0\otimes\Psi(\underset{\sim}{h}^{B(0/\ell)}_{\alpha_k^{B(0/\ell)}}) = \{2\ell/(2\ell+1)\}\ t^0\otimes\underset{\sim}{h}^0_{\alpha^0_{k+1}} \tag{4.3.231}$$

(for k = 1,2, ... , ℓ).

The <u>non-diagonal</u> basis elements of $\tilde{\mathcal{L}}_{s0}^{0(2)}$ will now be examined. They fall into 6 sets:

(i) For j,k = 1,2, ... , ℓ, with j < k: For the basis element $\underset{\sim}{e}^0_{\beta^{0-}_{(j,k)}}$ of (4.3.202), it is implied by (4.3.194) and (4.3.229) that this corresponds to the root

$$\beta^{0-}_{(j,k)}(h) = \sum_{r=j}^{k-1} \alpha_r^{B(0/\ell)}(h) , \tag{4.3.232}$$

of B(0/ℓ) (for j,k = 1,2, ... , ℓ, with j < k, and for all $h \in \mathcal{H}^{0(2)}$ (= $\mathcal{H}^{B(0/\ell)}$)).

(ii) For j,k = 1,2, ... ,ℓ, with j < k:

$$-\underset{\sim}{e}_{k+2,j+2} + \underset{\sim}{e}_{j+\ell+2,k+\ell+2} = \underset{\sim}{e}^0_{-\beta^{0-}_{(j,k)}} , \tag{4.3.233}$$

which corresponds to the root $-\beta^{0-}_{(j,k)}(h)$ of $B(0/\ell)$, where $\beta^{0-}_{(j,k)}(h)$ is as in (4.3.232).

(iii) For $j,k = 1,2, \ldots , \ell$, with $j \leq k$: For the basis element $\overset{0}{\underset{\sim}{e}}_{\beta^{0+}_{(j,k)}}$ of (4.3.203), it is implied by (4.3.195), (4.3.196), (4.3.229), and (4.3.230)

that this corresponds to the root

$$\beta^{0+}_{(j,k)}(h) = \sum_{r=j}^{k-1} \alpha_r^{B(0/\ell)}(h) + 2\sum_{r=k}^{\ell} \alpha_r^{B(0/\ell)}(h) , \qquad (4.3.234)$$

of $B(0/\ell)$ (for $j,k = 1,2, \ldots , \ell$, with $j \leq k$, and for all $h \in \mathcal{H}^{0(2)}$ ($= \mathcal{H}^{B(0/\ell)}$), the first term on the right-hand side of (4.3.234) not appearing if $j = k$).

(iv) For $j,k = 1,2, \ldots ,\ell$, with $j \leq k$:

$$-\underset{\sim}{e}_{k+\ell+2,j+2} - \underset{\sim}{e}_{j+\ell+2,k+2} = \overset{0}{\underset{\sim}{e}}_{-\beta^{0+}_{(j,k)}} , \qquad (4.3.235)$$

which corresponds to the root $-\beta^{0+}_{(j,k)}(h)$ of $B(0/\ell)$, where $\beta^{0+}_{(j,k)}(h)$ is as in (4.3.234).

(v) For $j = 1,2, \ldots , \ell$:

$$-\underset{\sim}{e}_{j+2,1} + \underset{\sim}{e}_{j+2,2} + \underset{\sim}{e}_{2,j+\ell+2} - \underset{\sim}{e}_{1,j+\ell+2} = -\overset{0}{\underset{\sim}{e}}_{\delta^{0+}_{(j)}} + \overset{0}{\underset{\sim}{e}}_{-\delta^{0-}_{(j)}} , \qquad (4.3.236)$$

where $\overset{0}{\underset{\sim}{e}}_{\delta^{0-}_{(j)}}$ and $\overset{0}{\underset{\sim}{e}}_{-\delta^{0+}_{(j)}}$ are given by (4.1), (4.3.204) and (4.3.205). As (4.3.199), (4.3.200), (4.3.201), (4.3.228), (4.3.229), and (4.3.230) imply that

$$\delta^{0+}_{(j)}(h) = -\delta^{0-}_{(j)}(h) = \sum_{r=j}^{\ell} \alpha_r^{B(0/\ell)}(h) , \qquad (4.3.237)$$

(for $j = 1,2, \ldots , \ell$, and for all $h \in \mathcal{H}^{0(2)}$ ($= \mathcal{H}^{B(0/\ell)}$)), the basis element (4.3.236) corresponds to the root $\delta^{0+}_{(j)}(h)$ of $B(0/\ell)$.

(vi) For $j = 1,2, \ldots , \ell$:

$$\underset{\sim}{e}_{1,j+2} - \underset{\sim}{e}_{2,j+2} + \underset{\sim}{e}_{j+\ell+2,2} - \underset{\sim}{e}_{j+\ell+2,1} = \underset{\sim}{e}^{0}_{\delta^{0-}_{(j)}} + \underset{\sim}{e}^{0}_{-\delta^{0+}_{(j)}}, \quad (4.3.238)$$

where $\underset{\sim}{e}^{0}_{\delta^{0-}_{(j)}}$ and $\underset{\sim}{e}^{0}_{-\delta^{0+}_{(j)}}$ are given by , (4.3.204) and (4.3.205), which corresponds to the root $-\delta^{0+}_{(j)}(h)$ of $B(0/\ell)$, where $\delta^{0+}_{(j)}(h)$ is as in (4.3.238).

All the elements of the above sets are even members of $C(\ell+1)$, except for those of (4.3.236) and (4.3.239), which are odd.

It is easily checked that the set of $2\ell(\ell+1)$ non-zero roots of (i) to (vi) above, together with the ℓ zero roots, are all weights of the adjoint representation of $B(0/\ell)$, whose highest weight is

$$\Lambda = 2\Lambda_1^{B(0/\ell)} = 2\sum_{k=1}^{\ell} \alpha_k^{B(0/\ell)}. \quad (4.3.240)$$

(c) The subspace $\tilde{\mathcal{L}}^{0(2)}_{s1}$:

By (4.3.2) the subspace $\tilde{\mathcal{L}}^{0(2)}_{s1}$ consists of the supermatrices whose submatrices satisfy the conditions $\tilde{\underset{\sim}{A}} = \underset{\sim}{A}$, $\tilde{\underset{\sim}{D}}\underset{\sim}{J} - \underset{\sim}{J}\underset{\sim}{D} = \underset{\sim}{0}$, and $\tilde{\underset{\sim}{C}}\underset{\sim}{J} = -\underset{\sim}{B}$ in addition to (4.3.187), (4.3.188), and (4.3.189). Together these imply that $\underset{\sim}{D} = \underset{\sim}{0}$, that

$$\underset{\sim}{B} = 2^{-\frac{1}{2}}\begin{pmatrix} \underset{\sim}{b} \\ \underset{\sim}{b} \end{pmatrix} \text{ and } \underset{\sim}{C} = 2^{-\frac{1}{2}}\begin{pmatrix} \underset{\sim}{c} & \underset{\sim}{c} \end{pmatrix}, \quad (4.3.241)$$

where $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are submatrices of dimensions $1\times 2\ell$ and $2\ell\times 1$ respectively that experience the constraint (4.3.207), and that

$$\underset{\sim}{A} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (4.3.242)$$

where a is any complex number. Thus $\tilde{\mathcal{L}}^{0(2)}_{s1}$ possesses a single diagonal basis element

$$(1/4\ell)\{\underset{\sim}{e}_{1,1} - \underset{\sim}{e}_{2,2}\} = -\tfrac{1}{2}\underset{\sim}{h}_{\alpha^0_{\ell+1}}^0 - \sum_{r=1}^{\ell} \underset{\sim}{h}_{\alpha^0_r}^0 , \qquad (4.3.243)$$

which corresponds to a zero weight of B(0/ℓ).

The non-diagonal basis elements of $\tilde{\mathcal{L}}_{s1}^{0(2)}$ fall into 2 sets:

(i) For j = 1,2, ... , ℓ:

$$-\underset{\sim}{e}_{j+2,1} - \underset{\sim}{e}_{j+2,2} + \underset{\sim}{e}_{2,j+\ell+2} + \underset{\sim}{e}_{1,j+\ell+2} = \underset{\sim}{e}^0_{\delta^{0+}_{(j)}} + \underset{\sim}{e}^0_{-\delta^{0-}_{(j)}} , \qquad (4.3.244)$$

where $\underset{\sim}{e}^0_{\delta^{0-}_{(j)}}$ and $\underset{\sim}{e}^0_{-\delta^{0+}_{(j)}}$ are given by (4.1), (4.3.204) and (4.3.205), and $\delta^{0+}_{(j)}(h)(=-\delta^{0-}_{(j)}(h))$ is given for j = 1,2, ... , ℓ, and for all $h \in \mathcal{H}^{0(2)}$ $(= \mathcal{H}^{B(0/\ell)})$ by (4.3.238), the basis element (4.3.244) corresponds to the weight $\delta^{0+}_{(j)}(h)$ of B(0/ℓ).

(ii) For j = 1,2, ... , ℓ:

$$\underset{\sim}{e}_{1,j+2} + \underset{\sim}{e}_{2,j+2} + \underset{\sim}{e}_{j+\ell+2,2} + \underset{\sim}{e}_{j+\ell+2,1} = \underset{\sim}{e}^0_{\delta^{0-}_{(j)}} - \underset{\sim}{e}^0_{-\delta^{0+}_{(j)}} , \qquad (4.3.245)$$

where $\underset{\sim}{e}^0_{\delta^{0-}_{(j)}}$ and $\underset{\sim}{e}^0_{-\delta^{0+}_{(j)}}$ are given by (4.1), (4.3.204) and (4.3.205), which corresponds to the root $-\delta^{0+}_{(j)}(h)$ of B(0/ℓ), where $\delta^{0+}_{(j)}(h)$ is as in (4.3.238).

The diagonal basis element (4.3.243) is an even element of C(ℓ+1), but all the non-diagonal elements of the sets (i) and (ii) are odd members of C(ℓ+1).

They form the carrier space of an irreducible representation of B(0/ℓ) of dimension 2ℓ+1 whose highest weight is

$$\Lambda = \Lambda_1^{B(0/\ell)} = \sum_{k=1}^{\ell} \alpha_k^{B(0/\ell)} . \qquad (4.3.246)$$

(See Tsohantjis and Cornwell(1990) for a discussion of the supercharacters and superdimensions of B(0/ℓ)).

(d) The roots of $C^{(2)}(\ell+1)$

Defining $\delta(h)$ as in subsection 4.2, it follows that the roots $\alpha(h)$ and the corresponding basis elements e_α of $C^{(2)}(\ell+1)$ are as follows:

(i) $\alpha = 2J\delta(h)$, (for $J = 0,\pm1,\pm2, \ldots$). There are ℓ linearly independent basis elements $e_\alpha^{(k)}$ corresponding to this root which may be labeled by an additional superscript, so that

$$e_\alpha^{(k)} = \{2\ell/(2\ell+1)\}\ t^{2J}\otimes \underset{\sim}{h}^0_{\alpha^0_{k+1}}\ \text{(for } k = 1,2, \ldots , \ell),$$

(which reduces to (4.3.231) in the special case $J = 0$).

(ii) $\alpha(h) = 2J\delta(h) \pm \beta^{0-}_{(j,k)}(h)$, (for $j,k = 1,2, \ldots , \ell$, with $j < k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0-}_{(j,k)}(h)$ is the extension of the weight of $B(0/\ell)$ that is given by (4.3.232) and

$$e_\alpha = t^{2J}\otimes \underset{\sim}{e}^0_{\pm\beta^-_{(j,k)}}\ .$$

(iii) $\alpha(h) = 2J\delta(h) \pm \beta^{0+}_{(j,k)}(h)$, (for $j,k = 1,2, \ldots , \ell$, with $j \leq k$, and for $J = 0,\pm1,\pm2, \ldots$), where $\beta^{0+}_{(j,k)}(h)$ is the extension of the weight of $B(0/\ell)$ that is given by (4.3.234) and

$$e_\alpha = t^{2J}\otimes \underset{\sim}{e}^0_{\pm\beta^+_{(j,k)}}\ .$$

(iv) $\alpha(h) = 2J\delta(h) \pm \delta^{0+}_{(j)}(h)$, (for $j = 1,2, \ldots , \ell$, and for $J = 0,\pm1,\pm2, \ldots$), where $\delta^{0+}_{(j)}(h)$ is the extension of the weight of $B(0/\ell)$ that is given by (4.3.238) and

$$e_\alpha = t^{2J}\otimes\{ \underset{\sim}{e}^0_{-\delta^-_{(j)}} \mp \underset{\sim}{e}^0_{\pm\delta^+_{(j)}}\}.$$

(v) $\alpha(h) = (2J+1)\delta(h)$, (for $J = 0,\pm1,\pm2, \ldots$), with

$$e_\alpha = t^{2J+1}\otimes\{\tfrac{1}{2}\underset{\sim}{h}^0_{\alpha^0_{\ell+1}} + \sum_{r=1}^{\ell} \underset{\sim}{h}^0_{\alpha^0_r}\}.$$

(vi) $\alpha(h) = (2J+1)\delta(h) \pm \delta^{0+}_{(j)}(h)$, (for $j = 1,2, \ldots, \ell$, and for $J = 0,\pm1,\pm2, \ldots$), where $\delta^{0+}_{(j)}(h)$ is the extension of the weight of $B(0/\ell)$ that is given by (4.3.238) and

$$e_\alpha = t^{2J}\otimes\{ \underset{\sim}{e}^{0}_{-\delta^{-}_{(j)}} \pm \underset{\sim}{e}^{0}_{\pm\delta^{+}_{(j)}}\}.$$

(vii) $\alpha(h) = 0$, with c and d as basis elements.

With μ chosen as in (4.3.216) , it follows that

$$<\alpha^0,\beta^0> \;=\; <\alpha^0,\beta^0>^{B(0/\ell)} \qquad (4.3.247)$$

where on the right-hand side of (4.3.247) α^0 and β^0 are any pair of linear functionals defined on $\mathcal{H}^{0(2)}$ (= $\mathcal{H}^{B(0/\ell)}$), the evaluation being performed with respect to the Killing form of $B(0/\ell)$, and where on the left-hand side of (4.3.247) α^0 and β^0 denote the corresponding extensions to the Cartan subalgebra of the Kac-Moody superalgebra $C^{(2)}(\ell+1)$, the evaluation being performed with respect to its supersymmetric bilinear invariant form B(,).

As $<\alpha^0,\alpha^0>^{B(0/\ell)} > 0$ for every non-zero linear functional α^0 defined on $\mathcal{H}^{B(0/\ell)}$, then $<\alpha^0,\alpha^0> > 0$ for the corresponding extension. Moreover (4.3.216) imply that

$$h_\delta \;=\; \{2\ell/(2\ell+1)\}\, c\ . \qquad (4.3.248)$$

Thus, if α^0_k is the extension of any simple root of $\tilde{\mathcal{L}}^0_s$, then

$$<\delta,\alpha^0_k> \;=\; 0 \qquad (4.3.249)$$

and

$$<j\delta,j\delta> \;=\; 0\ . \qquad (4.3.250)$$

Thus $<j\delta,j\delta> = 0$ for integer j, so every non-zero root of $C^{(2)}(\ell+1)$ belonging to the sets (i) and (v) is "imaginary". Moreover, because $<j\delta+\alpha^0,j\delta+\alpha^0> = <\alpha^0,\alpha^0>^{B(0/\ell)}$ and because $<\alpha^0,\alpha^0>^{B(0/\ell)} > 0$ for linear functional α^0 and its corresponding extension (as has just been noted), it follows that every non-zero root of $C^{(2)}(\ell+1)$ belonging to all the above sets except (i) and (v) is "real". All the elements mentioned in the above sets are even, except for those in the sets (iv) and (vi), which are odd.

For $C^{(2)}(\ell+1)$ the simple roots may be taken to be

$$\alpha_0 = \delta - \alpha_H^0, \qquad (4.3.251)$$

and

$$\alpha_k = \alpha_k^{B(0/\ell)} \text{ (for } k = 1, \ldots ,\ell) \qquad (4.3.252)$$

where

$$\alpha_H^0 = \Lambda_1^{B(0/\ell)} = \sum_{k=1}^{\ell} \alpha_k^{B(0/\ell)} \qquad (4.3.253)$$

is the highest weight of the representation of $\tilde{\mathcal{L}}_s^0$ for which $\tilde{\mathcal{L}}_{s1}^{0(2)}$ is the carrier space and the $\alpha_k^{B(0/\ell)}$ are the extensions of the simple roots of $B(0/\ell)$. As e_{α_0} and e_{α_ℓ} appear in the sets (vi) and (iv) respectively, it follows that e_{α_0} and e_{α_ℓ} are odd, so α_0 and α_ℓ are odd roots of the Kac-Moody superalgebra $C^{(2)}(\ell+1)$. All the other simple roots of $C^{(2)}(\ell+1)$ are even.

It is then easily checked that the Cartan matrices of $C^{(2)}(2)$ and $C^{(2)}(\ell+1)$ (for $\ell \geq 2$) when evaluated correspond to the generalized Dynkin diagrams given in Figures 7 and 8. The quantities $<\alpha^0,\alpha^0>^{B(0/\ell)}$ can be found in appendix A(3).

In terms of the linearly independent functionals ε_j $(i \leq j \leq \ell)$ defined on $\mathcal{H}^{B(0/\ell)}$ (see Cornwell(1989)) the roots of $C^{(2)}(\ell+1)$ are given by

$$\Delta = \{\ 2m\delta \pm (\varepsilon_i \pm \varepsilon_j) \text{ with } 1 \leq i < j \leq \ell\ ,\ m\delta \pm \varepsilon_i \text{ with } 1 \leq i \leq \ell$$
$$2m\delta \pm 2\varepsilon_i \text{ with } 1 \leq i \leq \ell \text{ all with } m \in \mathbb{Z}$$
$$\text{and } m\delta \text{ with } m \neq 0 \text{ and } m \in \mathbb{Z}\} \qquad (4.3.254)$$

The basis is given by

$$\alpha_0 = \delta - \varepsilon_1\ \varepsilon_\ell \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 1 \leq i \leq \ell-1 \quad \alpha_\ell = \varepsilon_\ell \qquad (4.3.255)$$

This brings us in to the end of the description of the structure of the affine Kac-Moody superalgebras. All the theory of the sections 4.2.B regarding the $B^{(1)}(0/\ell)$ can be applied with minor but straight forward modifications to the twisted superalgebras as well.

Before leaving this chapter it would be worth making some remarks. Had we chosen one of the far left nodes of the Dynkin diagrams of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ we would eventually have come up with a second order automorphism for $A^{(2)}(2\ell-1/0)$, although that of $A^{(4)}(2\ell/0)$ would still be fourth order. However both cases would be different from previously. $\tilde{\mathfrak{L}}_{s0}^{0(q)}$ would be $B(0/\ell)$ in both cases but the root systems of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ would not be the same.

If we denote with $\tilde{\mathfrak{L}}_{Loop}(\tilde{\mathfrak{L}}^{(q)})$ the loop algebra of the affine Kac-Moody superalgebra $\tilde{\mathfrak{L}}^{(q)}$ then the even parts of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ and $C^{(2)}(\ell+1)$ are given by

$$A^{(4)}(2\ell/0)_0 = \mathbb{C}[t^2,t^{-2}]\tilde{\mathfrak{L}}_{Loop}(A_{2\ell}^{(2)}) \oplus \sum_{j=-\infty}^{\infty} t^{4J+2} \otimes \sum_{r=1}^{2\ell+1} \underset{\sim}{h}_{\alpha_r^0}^{0} \oplus (\mathbb{C}c) \oplus (\mathbb{C}d)$$

$$C^{(2)}(\ell+1)_0 = \mathbb{C}[t^2,t^{-2}]\tilde{\mathcal{L}}_{Loop}(C_\ell^{(1)}) \oplus \sum_{j=-\infty}^{\infty} t^{2J+1} \otimes \underset{\sim}{c}\,;$$

$$A^{(2)}(2\ell-1/0)_0 = \mathbb{C}[t^2,t^{-2}]\tilde{\mathcal{L}}_{Loop}(A_{2\ell-1}^{(2)}) \oplus \sum_{j=-\infty}^{\infty} t^{4J+2} \otimes \{\sum_{r=k}^{k+\ell-1} \underset{\sim}{h}_{\alpha_r^0}^0$$

$$+ 2\sum_{r=k+\ell}^{2\ell} \underset{\sim}{h}_{\alpha_r^0}^0\} \oplus (\mathbb{C}c) \oplus (\mathbb{C}d),$$

for k being <u>one</u> of the $k = 1,2, \dots, \ell$ and $\underset{\sim}{c}$ being the basis of the one dimensional abelian subalgebra of $C(\ell+1)_0$.

CHAPTER 5

HIGHEST WEIGHT REPRESENTATIONS OF AFFINE KAC-MOODY SUPERALGEBRAS

5.1 Introduction

In this chapter we shall describe the structure of highest weight representations of the complex affine Kac-Moody superalgebras $\tilde{\mathcal{L}}_s$, where $\tilde{\mathcal{L}}_s$ denotes one of the $B^{(1)}(0/\ell)$ (for $\ell \geq 1$), $A^{(2)}(2\ell\text{-}1/0)$ (for $\ell \geq 2$), $A^{(4)}(2\ell/0)$ (for $\ell \geq 1$), and $C^{(2)}(\ell+1)$ (for $\ell \geq 1$). These representations are almost identical with those of the affine Kac-Moody algebras (see Kac(1985), Cornwell(1990)), although some of their features are the same as for the representations of the basic classical simple complex Lie superalgebras. In fact, although it will be not explicitly stated again, the carrier spaces will be Z_2-graded and all the operators acting on them will preserve this grading. The analysis will be confined mainly to the very interesting class of integrable irreducible highest weight representations. It should be noted that the description of such representations is not restricted only to these superalgebras but also to any affine Kac-Moody algebra as well. In particular the result on complete reducibility (see section 5.3) was first obtained in Kac(1978) within a wider content including the affine Kac-Moody algebra and superalgebras. The integrable irreducible highest weight representations of the affine Kac-Moody algebra and superalgebras that appeared in Kac(1978) are particular examples of a more general theory of representations using Verma modules. This theory refined and applied in the case of affine Kac-Moody algebras can be found in Kac(1985) (see also Dixmier(1974)). A byproduct of the representation theory of these superalgebras is certain multivariable identities for the non-reduced affine root systems which were not included in Macdonald analysis of reduced ones but which can be found in Kac(1978). Since the superalgebras under consideration are infinite dimensional, their representations are in general infinite dimensional, though the weight subspaces will be finite dimensional. The general notions (universal enveloping superalgebra, induced modules, etc.) of Lie superalgebra representations can be found in Kac(1977), Scheunert(1978) and Cornwell(1990).

5.2 Basic notions and definitions

The starting point in the representation theory is as usual the universal enveloping superalgebra of the affine Kac-Moody superalgebra $\tilde{\mathcal{L}}_s$. This is will be denoted by $U(\tilde{\mathcal{L}}_s)$ and can be regarded as the infinite dimensional Z_2-graded complex vector space of polynomials in the elements of the superalgebra $\tilde{\mathcal{L}}_s$. It follows from the Poincare-Birkhoff-Witt theorem that the basis elements of $U(\tilde{\mathcal{L}}_s)$(for some fixed ordering of the basis of $\tilde{\mathcal{L}}_s$) are given by the set of polynomials of the form

$$(e^{(1)}_{-\beta_1})^{r_{1,1}}(e^{(2)}_{-\beta_1})^{r_{1,2}}\ldots(e^{(\nu)}_{-\beta_1})^{r_{1,\nu}}(e^{(1)}_{-\beta_2})^{r_{2,1}}(e^{(2)}_{-\beta_2})^{r_{2,2}}\ldots(e^{(\mu)}_{-\beta_2})^{r_{2,\mu}}\ldots\ldots\times$$

$$(e^{(1)}_{-\gamma_1})^{s_{1,1}}(e^{(2)}_{-\gamma_1})^{s_{1,2}}\ldots(e^{(\nu)}_{-\gamma_1})^{s_{1,\nu}}(e^{(1)}_{-\gamma_2})^{s_{2,1}}(e^{(2)}_{-\gamma_2})^{s_{2,2}}\ldots(e^{(\mu)}_{-\gamma_2})^{s_{2,\mu}}\ldots\times$$

$$(h_1)^{p_1}(h_2)^{p_2}\ldots\ldots(h_{\ell+2})^{p_{\ell+2}}\times$$

$$(e^{(1)}_{\beta_1})^{r'_{1,1}}(e^{(2)}_{\beta_1})^{r'_{1,2}}\ldots(e^{(\nu)}_{\beta_1})^{r'_{1,\nu}}(e^{(1)}_{\beta_2})^{r'_{2,1}}(e^{(2)}_{\beta_2})^{r'_{2,2}}\ldots(e^{(\mu)}_{\beta_2})^{r'_{2,\mu}}\ldots\ldots\times$$

$$(e^{(1)}_{\gamma_1})^{s'_{1,1}}(e^{(2)}_{\gamma_1})^{s'_{1,2}}\ldots(e^{(\nu)}_{\gamma_1})^{s'_{1,\nu}}(e^{(1)}_{\gamma_2})^{s'_{2,1}}(e^{(2)}_{\gamma_2})^{s'_{2,2}}\ldots(e^{(\mu)}_{\gamma_2})^{s'_{2,\mu}}\ldots \quad (5.1)$$

where, $(e^{(k)}_{\pm\beta_i})$ ($k = 1,\ldots,$ mult $\tilde{\mathcal{L}}_{\pm\beta_i}$) are basis of the even root subspace $\tilde{\mathcal{L}}_{\pm\beta_i}$ corresponding to the root $\pm\beta_i$, $(e^{(kl)}_{\pm\gamma_i})$($k=1,\ldots,$mult $\tilde{\mathcal{L}}_{\pm\gamma_i}$) are basis of the odd root subspaces $\tilde{\mathcal{L}}_{\pm\gamma_i}$ corresponding to the odd roots $\pm\gamma_i$, h_i are basis elements of the Cartan subalgebra of $\tilde{\mathcal{L}}_s$, $r_{i,j}$, $r'_{i,j}$ and p_i are non-negative integers and $s'_{i,j}$ $s_{i,j} \in \{0,1\}$.

Because of the triangular decomposition of $\tilde{\mathcal{L}}_s$, $\mathcal{L}_- \oplus \mathcal{H} \oplus \mathcal{L}_+$ $U(\tilde{\mathcal{L}}_s)$ can also be put in the form

$$U(\tilde{\mathcal{L}}_s) = U(\mathcal{L}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{L}_+) \quad (5.2)$$

where $U(\mathcal{L}_-)$, $U(\mathcal{H})$ and $U(\mathcal{L}_+)$ are the universal enveloping superalgebras of the negative root subspace, the Cartan subalgebra and the positive root subspace of $\tilde{\mathcal{L}}_s$ respectively.

Now consider the sub-superalgebra $\mathcal{B}$ of $\tilde{\mathcal{L}}_s$ given by

$$\mathcal{B} = \mathcal{H} \oplus \mathcal{L}_+ \quad (5.3)$$

We call $\mathcal{B}$ the Borel sub-superalgebra of $\tilde{\mathcal{L}}_s$.

Using the method of the induced representation we can construct highest weight irreducible representations of $\tilde{\mathcal{L}}_s$ induced by a particular representations of $\mathcal{B}$.

Let V_Λ be a one dimensional complex vector space with basis denoted by $\psi(\Lambda)$. Assume that a trivial Z_2-gradiation is defined on V_Λ by $(V_\Lambda)_0 = V_\Lambda$ and $(V_\Lambda)_1 = \emptyset$ (so deg $\psi(\Lambda) = 0$). Let $\Phi(h)$ and $\Phi(a_\alpha)$ (for all h of $\mathcal{H}$ and all $\alpha \in \Delta_+$ be operators acting on V_Λ such that

(i) $\Phi(h)\psi(\Lambda) = \Lambda(h)\psi(\Lambda)$ for all $h \in \mathcal{H}$;

(ii) $\Phi(a_\alpha)\psi(\Lambda) = 0$ for all $\alpha \in \Delta_+$; (5.4)

where $\Lambda(h)$ is a linear functional defined on $\mathcal{H}$.
The operators are assumed to be Z_2-graded. That is deg $\Phi(h)$ = 0 for all $h \in \mathcal{H}$ and deg $\Phi(a_\alpha)$ = 0 or 1 depending on whether the α is even or odd root of $\tilde{\mathcal{L}}_s$. Clearly this action defines a graded representation of the subsuperalgebra $\mathcal{B}$. The pair (Φ, V_Λ) consisting of the operators Φ, as defined above, and the vector space V_Λ is often called an even $\mathcal{B}$-module. Since (Φ, V_Λ) is a $\mathcal{B}$-module it becomes naturally an $U(\mathcal{B})$-module, where $U(\mathcal{B})$ is the universal enveloping superalgebra of $\mathcal{B}$.

Consider now the Z_2-graded space

$$U(\tilde{\mathcal{L}}_s) \otimes V_\Lambda \tag{5.5}$$

and for any element a of $\tilde{\mathcal{L}}_s$ define operators $\Psi(a)$ acting on the above space as

$$\Psi(a)(u \otimes v) = (au) \otimes v \tag{5.6}$$

for all $u \in U(\tilde{\mathcal{L}}_s)$ and all $v \in V_\Lambda$. Clearly this action defines a graded representation of $\tilde{\mathcal{L}}_s$ and $U(\tilde{\mathcal{L}}_s) \otimes V_\Lambda$ is its carrier space. Now let $a \in U(\tilde{\mathcal{L}}_s)$ and $b \in U(\mathcal{B})$. The elements of $U(\tilde{\mathcal{L}}_s) \otimes V_\Lambda$ of the form

$$(ab)\otimes v - a \otimes \Phi(b)v, \tag{5.7}$$

where $\Phi(b)$ are operators defined in (5.4) and $v \in V_\Lambda$, generate a subspace I of $U(\tilde{\mathfrak{L}}_s) \otimes V_\Lambda$ and it can be easily checked by applying (5.6) on (5.7) that it is an invariant subspace. We can form now the quotient space $U(\tilde{\mathfrak{L}}_s) \otimes V_\Lambda / I$ and consider this as the carrier space of the representation. This is called the tensor product space of $U(\tilde{\mathfrak{L}}_s)$ and V_Λ over $U(\mathcal{B})$ and is denoted by $U(\tilde{\mathfrak{L}}_s) \otimes_{U(\mathcal{B})} V_\Lambda$. Under the canonical projection of $U(\tilde{\mathfrak{L}}_s) \otimes V_\Lambda$ on to $U(\tilde{\mathfrak{L}}_s) \otimes V_\Lambda / I$, I projects on 0, and thus in $U(\tilde{\mathfrak{L}}_s) \otimes_{U(B)} V_\Lambda$

$$(ab)\otimes v = a \otimes \Phi(b)v. \tag{5.8}$$

The pair $(\Psi(a), \tilde{V}(\Lambda))$ for all a of $\tilde{\mathfrak{L}}_s$, where

$$\tilde{V}(\Lambda) = U(\tilde{\mathfrak{L}}_s) \otimes_{U(B)} V_\Lambda, \tag{5.9}$$

and $\Psi(a)$ are operators acting on the carrier space $\tilde{V}(\Lambda)$ by left multiplication on the $U(\tilde{\mathfrak{L}}_s)$ component, form a graded representation of $\tilde{\mathfrak{L}}_s$ induced by the representation of $\mathcal{B}$ as defined in (5.4). It is called the induced $\tilde{\mathfrak{L}}_s$- module. From Poincare-Birkhoff-Witt theorem the basis element of $\tilde{V}(\Lambda)$ can be found to be

$$(e^{(1)}_{-\beta_1})^{r_{1,1}}(e^{(2)}_{-\beta_1})^{r_{1,2}}\dots(e^{(\nu)}_{-\beta_1})^{r_{1,\nu}}(e^{(1)}_{-\beta_2})^{r_{2,1}}(e^{(2)}_{-\beta_2})^{r_{2,2}}\dots(e^{(\mu)}_{-\beta_2})^{r_{2,\mu}}\dots\dots\times$$

$$(e^{(1)}_{-\gamma_1})^{s_{1,1}}(e^{(2)}_{-\gamma_1})^{s_{1,2}}\dots(e^{(\nu)}_{-\gamma_1})^{s_{1,\nu}}(e^{(1)}_{-\gamma_2})^{s_{2,1}}(e^{(2)}_{-\gamma_2})^{s_{2,2}}\dots(e^{(\mu)}_{-\gamma_2})^{s_{2,\mu}}\dots\otimes\psi(\Lambda) \tag{5.10}$$

where, $(e^{(k)}_{-\beta_i})$ (k = 1,..., mult $\tilde{\mathfrak{L}}_{-\beta_i}$) are basis of the even root subspace $\tilde{\mathfrak{L}}_{-\beta_i}$ corresponding to the negative root $-\beta_i$, $e^{(k)}_{-\gamma_i}$ (k=1,...,mult $\tilde{\mathfrak{L}}_{-\gamma_i}$)are basis of the odd root subspaces $\tilde{\mathfrak{L}}_{-\gamma_i}$ corresponding to the negative odd roots $-\gamma_i$, $r_{i,j}$ are non-negative integers and $s_{i,j} \in \{0,1\}$.

It is not difficult to show, by acting on the basis elements with $\Phi(h)$, that $\tilde{V}(\Lambda)$ accepts the following decomposition

$$\tilde{V}(\Lambda) = \oplus_\lambda \tilde{V}_\lambda \tag{5.11}$$

where $\tilde{V}_\lambda$ is the subspace of $\tilde{V}(\Lambda)$ spaned by the elements (5.10) such that the linear functional λ defined on $\mathcal{H}$ is given by

$$\lambda = \Lambda - \{(r_{1,1} + r_{1,2} \ldots\ldots r_{1,\nu})\beta_1 + (r_{2,1} + r_{2,2} \ldots\ldots r_{2,\mu})\beta_2 + +\ldots\ldots + (s_{1,1} + s_{1,2} \ldots\ldots s_{1,\nu})\gamma_1 + (s_{2,1} + s_{2,2} \ldots\ldots s_{2,\mu})\gamma_2 + \ldots\}. \tag{5.12}$$

Since all the quantities appearing in the brackets are non-negative the above expression can be simplified writing

$$\lambda = \Lambda - \Sigma_{i=0}^{\ell} k_i\alpha_i , \tag{5.13}$$

where k_i are non negative integers and α_i are the simple roots of $\tilde{\mathfrak{L}}_s$. We define by $D(\Lambda)$ the set consisting of all linear functional λ defined on $\mathcal{H}$ and having the form (5.13). Then (5.11) can be written as

$$\tilde{V}(\Lambda) = \oplus_{\lambda \in D(\Lambda)} \tilde{V}_\lambda . \tag{5.14}$$

The linear functional λ is called a <u>weight</u> of the representation if the corresponding supspace $\tilde{V}_\lambda \neq 0$, and $\tilde{V}_\lambda$ is called a <u>weight subspace</u>. The dimension of $\tilde{V}_\lambda$ is the multiplicity of the weight λ.

Generally $\tilde{V}(\Lambda)$ contains proper invariant subspaces graded with respect to (5.14). The quotient space of $\tilde{V}(\Lambda)$ with such a graded subspace is the carrier space of a <u>highest weight representation</u>. The union of all these subspaces constitute the unique maximal invariant subspace $R(\Lambda)$. Then the space $V(\Lambda) = \tilde{V}(\Lambda)/R(\Lambda)$ is the carrier space of an <u>irreducible representation</u> of $\tilde{\mathfrak{L}}_s$ with <u>highest weight Λ</u>. In this case if λ is any weight of the representation then $\lambda \leq \Lambda$.

Definition 5.1 The category $\mathcal{M}$

The category $\mathcal{M}$ is defined to be the set of representations (Φ, V) of $\tilde{\mathcal{L}}_s$ whose carrier space V satisfy the following conditions

(i) $\Phi(h)\psi(\lambda) = \lambda(h)\psi(\lambda)$ for all $h \in \mathcal{H}$ and $\psi(\lambda) \in V_\lambda$

(ii) $\dim V_\lambda < \infty$

(iii) $\Phi(a_\alpha)V_\lambda \subset V_{\lambda+\alpha}$ $(\alpha \in \Delta)$

(iv) $\mathrm{ch}V = \sum_{\lambda \in \mathcal{H}^*}(\dim V_\lambda)e^\lambda \in \mathcal{E}$

where ch denotes the character of the representation and $\mathcal{E}$ is the space of all functions on $\mathcal{H}^*$ which vanish outside the union of a finite number of sets of the form $D(\Lambda)$ (for more information of the space $\mathcal{E}$ see Kac(1978, 1985 §9.7) and Dixmier(1977 §7.5)).

It can be shown that both $\tilde{V}(\Lambda)$ and $V(\Lambda)$ belong in $\mathcal{M}$. In the Kac-Moody algebra case the module $(\Phi, \tilde{V}(\Lambda))$ is the equivalent of a Verma module (see Kac(1985) ch. 9)

Definition 5.2 The category $\mathcal{M}_0$

The category $\mathcal{M}_0$ is a subcategory of $\mathcal{M}$ which consists of those representations from $\mathcal{M}$ for which the operators $\Phi(E_{-\alpha_i})$ (for all $i \in I$) are locally nilpotent. That is $\Phi(E_{-\alpha_i})^n\psi(\lambda) = 0$ for some positive integer n, all $i \in I$ and every weight vector $\psi(\lambda)$ of the carrier space of the representation.

From the definition of the Cartan matrix given in chapter 3, we obtain the following proposition which plays a very important role in the study of the representations of $\tilde{\mathcal{L}}_s$.

Proposition 5.1

Let **A** be a Cartan matrix as defined in chapter 3 and let **A'** be the Cartan matrix obtained from **A** by dividing the ith row and multiplying the ith column by 2 for every $i \in \tau$. Then the affine Kac-Moody superalgebra $\tilde{\mathcal{L}}_s$ corresponding to **A** contains a subalgebra with generators $E'_{\alpha_j} = E_{\alpha_j}$, $E'_{-\alpha_j} = E_{-\alpha_j}$, $H'_{\alpha_j} = H_{\alpha_j}$, for all $j \in I \backslash \tau$ and

$$E'_{\alpha_j} = \tfrac{1}{4}[E_{\alpha_j}, E_{\alpha_j}], \; E'_{-\alpha_j} = \tfrac{1}{4}[E_{-\alpha_j}, E_{-\alpha_j}], \; H'_{\alpha_j} = \tfrac{1}{2}H_{\alpha_j}, \text{ for all } j \in \tau$$

which is isomorphic to a factor algebra of the Lie algebra whose Cartan matrix is **A'**.

A direct consequence of the above proposition on the structure of representations of $\tilde{\mathcal{L}}_s$ is revealed by the following proposition.

Proposition 5.2

Let (Φ, V) be a representation of $\tilde{\mathcal{L}}_s$ such that

(i) $V = \oplus_\lambda V_\lambda$;

(ii) $\Phi(H_{\alpha_j})\, \psi(\lambda) = \lambda(H_{\alpha_j})\, \psi(\lambda)$ for all $j \in I$ and $\psi(\lambda) \in V_\lambda$;

(iii) $\Phi(a_\alpha) V_\lambda \subset V_{\lambda+\alpha}$

(iv) $\Phi(E_{\alpha_j})$ and $\Phi(E_{-\alpha_j})$ $(i \in I)$ are locally nilpotent on V;

then, with respect to the three dimensional subalgebra generated by E'_{α_j}, $E'_{-\alpha_j}$ and H'_{α_j} (and being isomorphic to A_1), (Φ, V) is a direct sum of finite dimensional representations of this subalgebra.

Proof Let $\psi(\lambda) \in V_\lambda$. Then the subspace of V of the form $V' = \sum_{m\, n \geq 0} \mathbb{C}\{(\Phi(E'_{\alpha_j}))^m (\Phi(E'_{-\alpha_j}))^n\, \psi(\lambda),\ m,n \in \mathbb{Z}_+\}$ is finite dimensional since $\Phi(E'_{\alpha_j})$ and $\Phi(E'_{-\alpha_j})$ are locally nilpotent on

V. It can be easily proved that the action of $\Phi(E'_{\alpha_j})$, $\Phi(E'_{-\alpha_j})$ and $\Phi(H'_{\alpha_j})$ leaves V' invariant. This proves the proposition.

This direct sum structure of a representation of $\tilde{\mathcal{L}}_s$ defined as above is an extremely useful property in describing the weight system of such representations. In fact, as we will see shortly, the representation of $\tilde{\mathcal{L}}_s$ with the properties of this proposition are the most interesting ones.

5.3 Integrable highest weight representations.

<u>Definition 5.3</u> Integrable representations

A representation (Φ, V) of $\tilde{\mathcal{L}}_s$ is called integrable if the following two <u>nilpotency conditions</u> are satisfied

(a) $\Phi(E_{\alpha_j})^n\psi = 0$

(b) $\Phi(E_{-\alpha_j})^{n'}\psi = 0$

for some positive integers n and n', all $j\in I$ and every ψ of the carrier space V of the representation.

If the integrable representation is of highest weight Λ then condition (a) is redundant and we can say that the highest weight representation is integrable if and only if condition (b) is satisfied. This is a consequence of the fact that $ad(E_{\alpha_j})$ $(i\in I)$ is locally nilpotent on $\tilde{\mathcal{L}}_s$ and that $\Phi(E_{\alpha_j})\psi(\Lambda) = 0$. A highest weight representation together with condition (b) of the above theorem is often called <u>quasisimple</u> (see Kac(1978)).

It should be noted that a representation of $\tilde{\mathcal{L}}_s$ can be integrable without being a highest weight one. An example of this case is the adjoint representation of $\tilde{\mathcal{L}}_s$. Its weights are the roots of $\tilde{\mathcal{L}}_s$. It is not a highest weight one, since it contains roots of the form $j\delta$ for all integers j. Since we saw in chapter 3 that $ad(E_{\alpha_j})$ and $ad(E_{-\alpha_j})$(for all $j \in I$) are locally nillpotent on $\tilde{\mathcal{L}}_s$ and since there exists a finite number of times that $ad(E_{\alpha_j})$ and $ad(E_{-\alpha_j})$ have to act on the generators of $\tilde{\mathcal{L}}_s$ to give zero, the adjoint representation of $\tilde{\mathcal{L}}_s$ is integrable (see Kac(1985) ch. 3).

<u>Definition 5.4</u> Dominant highest weight

A highest weight Λ is called dominant if $\Lambda(H_{\alpha_j})$ is a non-negative integer for all $j \in I$. In particular if $j \in \tau$, then $\Lambda(H_{\alpha_j})$ must be even.

<u>Proposition 5.3</u>

(a) The irreducible highest weight representation $(\Phi, V(\Lambda))$ (where $V(\Lambda) = \tilde{V}(\Lambda)/R(\Lambda)$) is integrable if and only if Λ is dominant.

(b) If $(\Phi, V(\Lambda))$ (where $V(\Lambda) = \tilde{V}(\Lambda)/R(\Lambda)$) is an integrable irreducible highest weight representation with dominant highest weight Λ then,

$$V(\Lambda) = \tilde{V}(\Lambda) / \sum_{i=0}^{\ell} U(\tilde{\mathcal{L}}_s) \{\Phi(E_{-\alpha_j})^{\Lambda(H_j)+1}\} V_\Lambda \qquad (5.15)$$

where $U(\tilde{\mathcal{L}}_s)$ is the universal enveloping superalgebra of $\tilde{\mathcal{L}}_s$ and $H_i = H_{\alpha_j}$ for all $j \in I$.

Proof To obtain the first part merely involves showing that $\Phi(E_{-\alpha_j})^{\Lambda(H_i)+1}\,\psi(\Lambda) \in R(\Lambda)$ if $\Lambda(H_{\alpha_j})$ satisfies the requirements of definition 5.3 or else, $\Phi(E_{-\alpha_j})^m\,\psi(\Lambda) \notin R(\Lambda)$ for some j and all m. The second part is a consequence of the first part. It is the equivalent of Harish-Chandra's theorem (For details see Kac(1978,1985 ch.10 or Dixmier ch.7).

Since for integrable irreducible highest weight representation of $\tilde{\mathfrak{L}}_s$ the highest weight is dominant, all these representation can be described by the set of non-negative integers given by

$$2<\Lambda, \alpha_k>/<\alpha_k, \alpha_k> = n_k \quad \text{(for all } k = 0, 1,..., \ell). \qquad (5.16)$$

If $n_k = 0$ for all k= 0, 1,..., ℓ then the above equation accept the solution of the form $\Lambda = \mathbb{C}(\delta)$. This corresponds to the trivial representation of the derived superalgebra $[\tilde{\mathfrak{L}}_s, \tilde{\mathfrak{L}}_s]$ with highest weight $\Lambda(\mathcal{H}') = 0$ (where $\mathcal{H}' = \sum_{i=0}^{\ell} \mathbb{C}(H_{\alpha_i})$ is the Cartan subalgebra of derived superalgebra), and to a family of one dimensional irreducible representations of $\tilde{\mathfrak{L}}_s$ defined by

$$\Phi(d)\,\psi(\Lambda) = \Lambda(d)\psi(\Lambda) = \mu\,\delta(d)\,\psi(\Lambda) = \mu\,\psi(\Lambda) \quad \mu \in \mathbb{C}$$

$$\Phi(a)\ \psi(\Lambda) = 0 \text{ for all } a \in [\tilde{\mathfrak{L}}_s, \tilde{\mathfrak{L}}_s]. \qquad (5.17)$$

In fact an irreducible representation of $\tilde{\mathfrak{L}}_s$ will correspond to the direct product of the above representation with one of highest weight $\Lambda(\mathcal{H}') \neq 0$. Also note that the trivial irreducible representation is the one for which $\Lambda(\mathcal{H}) = 0$.

The above analysis suggests a less formal proof of the following proposition.

Proposition 5.4

The restriction of an integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ on to the derived superalgebra $[\tilde{\mathcal{L}}_s, \tilde{\mathcal{L}}_s]$ remains irreducible.

Proof It is essentially the same as that appeared in Kac(1985)).

Definition 5.5 Standard irreducible representation

We call standard irreducible representation of $\tilde{\mathcal{L}}_s$ an integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ whose highest weight is given by

$$\Lambda = \Sigma_{j=0}^{\ell} \; n_j \, \Lambda_j$$

where n_j are nonnegative integers for which at least one is non-zero and Λ_j are the fundamental weights (see (5.18).

From definition 5.3 it follows that, the representation considered in proposition 5.2 is an integrable representation and thus belongs in $\mathcal{M}_0$. Comparison of the definition 5.2 of the category $\mathcal{M}_0$ and the definition of the representation considered in proposition 5.2, shows that this latter one belongs in $\mathcal{M}_0$. Moreover every highest weight representation which belongs in $\mathcal{M}_0$ is integrable and satisfies proposition 5.2. Finally the irreducible highest weight representations of $\tilde{\mathcal{L}}_s$ with dominant highest weight are by proposition 5.3(a) integrable and thus belong in $\mathcal{M}_0$ too.

After these remarks, using proposition 5.2 and 5.3 and the second order Casimir operator (see section 5.5) it is possible to prove the equivalent of Weyl's complete reducibility theorem in the affine Kac-Moody algebra case.

Proposition 5.5

Any integrable representation of $\tilde{\mathcal{L}}_s$ belonging in $\mathcal{M}_0$ is completely reducible.

Proof (See Kac(1978))

In fact this proposition was derived by Kac for the Kac-Moody algebras as well, and it was the major step regarding their representation theory. A more elaborate exposition can be found in Kac (1985, ch. 10)

5.4 The weight systems of integrable irreducible highest weight representations.

We shall investigate now certain properties of the weight systems of the above representations and in particular those of definition 5.5 (i.e. the standard ones).
The following proposition embodies some of the most important properties of the weight systems of the integrable irreducible highest weight representations. It is heavily based on propositions 5.2 and 5.3(a).

Proposition 5.6

Let (Φ,V) be an integrable irreducible highest weight representations of $\tilde{\mathcal{L}}_s$. Then

(a) if λ is a weight of the representation then $\lambda+\alpha$ is also a weight for each non-zero root α such that $\Phi(e_\alpha)\psi(\lambda) \neq 0$;

(b) for any weight λ and any real root α, $2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$ is an integer;

(c) for every simple root α_i and a weight λ of the representation, there exist two non-negative integers p and q such that $\lambda + k\alpha_i$ is in the α_i-string of weights containing λ for every integer k that satisfies the relations $-p \le k \le q$ and $p-q = 2<\lambda,\alpha_i>/<\alpha_i, \alpha_i>$. The same is true for every <u>real</u> root α.

(d) The weight system of an integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ is invariant under the Weyl group of $\tilde{\mathcal{L}}_s$;

Proof (see Kac(1978), (1985), Cornwell(1990)).

For the imaginary roots, although one can notice that part (a) and (d) are still apply, the situation in general is different as the following theorem states.

<u>Proposition 5.7</u>

Let (Φ, V) be an integrable highest weight representations of $\tilde{\mathcal{L}}_s$. If λ is a weight of the representation and α is an imaginary root of $\tilde{\mathcal{L}}_s$ then $<\lambda, \alpha> \ge 0$. Moreover if $<\lambda, \alpha> > 0$ then $\lambda - k\alpha$ is a weight of the representation for any non-negative integer k.

Proof (See Kac(1978)).

<u>Definition 5.6</u> Fundamental weights

The linear functional of $\mathcal{H}^*$ defined by

$$2< \Lambda_j, \alpha_i >/< \alpha_j, \alpha_j > = \delta_{ij} \qquad \Lambda_j(d) = 0 \tag{5.18}$$

for all $i,j = 0, 1, \ldots, \ell$ are called fundamental weights.

In particular, comparison of the above definition with (3.67) shows that Λ_0 is the linear functional that corresponds to the scaling element d.

If we consider the submatrix $\mathbf{A^0}$ of the Cartan matrix of $\mathbf{A}$ of $\tilde{\mathcal{L}}_s$ (see table V) obtained by deleting the row and column of $\mathbf{A}$ corresponding to an odd node of the Dynkin diagram then $\mathbf{A^0}$ is the Cartan matrix of the semi-simple Lie algebra D_ℓ, B_ℓ, or the basic simple Lie superalgebra $B(0/\ell)$. Then for all $i,j \in I-\{k\}$ where k is the index of the removed node, these relations (5.18) are nothing but the defining relations of the fundamental weights of the above algebras. Then it is not difficult to show that are satisfied with

$$\Lambda_j = \Lambda_j^0 + m_j \Lambda_k, \tag{5.19}$$

where

$$m_j = - \sum_{i \in I-\{k\}} A_{ki}((\mathbf{A^0})^{-1})_{ij} \tag{5.20}$$

and $j \in I-\{k\}$ with $k=0$ or ℓ (see table V).

From relations (5.19) and (5.20) and the Cartan matrices of table V, we can obtain all the possible fundamental weights of $A^{(2)}(2\ell-1/0)$ (for $\ell \geq 2$), $A^{(4)}(2\ell/0)$ (for $\ell \geq 1$), and $C^{(2)}(\ell+1)$ (for $\ell \geq 1$). In fact it not difficult to see that all of the above analysis of integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ when restricted to their subalgebras D_ℓ, B_ℓ, or $B(0/\ell)$ is nothing but the theory of finite dimensional irreducible highest weight representations of these algebras. In particular no problem arises for $B(0/\ell)$ since all of its representation of this kind are typical. Certain construction of standard irreducible highest weight representation of $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$, and $C^{(2)}(\ell+1)$ can be found in Feingold and Frenkel(1985) and Golitzin(1986).

Definition 5.7 Maximal weight

Let (Φ, V) be an integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ with dominant highest weight Λ and δ be the imaginary root of $\tilde{\mathcal{L}}_s$. A weight λ of the representation is called maximal if $\lambda + \delta$ is not a weight of the representation. It will be denoted by λ_{max}. Moreover if $\lambda_{max}(H_{\alpha_i})$ (for all $i \in I$) is a non-negative integer then λ_{max} is called maximal dominant weight.

Proposition 5.8

For every integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ with dominant highest weight Λ:

(a) the highest weight Λ is a maximal dominant weight;

(b) if λ is a weight of the representation then λ - $k\delta$ is also a weight for all non-negative integer k.

(c) Any maximal dominant weight λ_{max} has the form

$$\lambda_{max} = \Lambda - \sum_{i=o}^{\ell} k_i\alpha_i = \Lambda \tag{5.21}$$

where k_i are non-negative integers and α_i are the simple roots.

(d) there exists only a finite number of maximal dominant weights.

(e) with respect to the finite set of maximal dominant weights λ_i (i=1,2,..., m) the set of the weights of the representation is given by

$$\{S(\lambda_i) - k\delta, \text{ for all } S \in W \text{ and any non-negative integer } k\}. \tag{5.22}$$

Proof (See Kac(1978)).

<u>Definition 5.8</u> Level of a standard irreducible highest weight representation

The level of a irreducible representation of highest weight Λ is defined to be the number given by

$$\text{level}(\Lambda) = 2\Lambda(h_\delta)/< \alpha_k, \alpha_k > \tag{5.23}$$

where h_δ is the central element (appropriately normalized) of the affine superalgebra and α_k corresponds to one of the simple roots of the Dynkin diagram of $\tilde{\mathcal{L}}_s$ selected for the explicit realization (see chapter 4). In particular since $< \Lambda_k, \delta > = \frac{1}{2}< \alpha_k, \alpha_k >$ for k=0 or ℓ (see chapter 4) then

$$\text{level}(\Lambda_k) = 1$$

Generally if Λ is dominant then the level is always a non-negative integer.

Consider the eigenvalues of the central element c in some irreducible highest weight representation of $\tilde{\mathcal{L}}_s$. Since c belongs in the Cartan subalgebra of $\tilde{\mathcal{L}}_s$ and commutes with all the elements of $\tilde{\mathcal{L}}_s$, it follows that

$$\Phi(c) = c_\Lambda \, I$$

$$\Phi(c)\, \psi(\lambda) = c_\Lambda \, \psi(\lambda)$$

for every weight λ of the representation and where c_Λ is a number that depends on the highest weight Λ, and I is the identity operator.

From the second of the above relations we can easily see that with $\lambda = \Lambda$ the eigenvalues c_Λ are given by

$$c_\Lambda = \Lambda(c) \tag{5.24}$$

Then, consideration of the expressions of the element h_δ obtained from the previous chapter (4.2.32, 4.3.93, 4.3.174, 4.3.248) and for a standard irreducible representation of $\tilde{\mathcal{L}}_s$ it follows that

$$c_\Lambda = \mu\langle\Lambda, \delta\rangle \qquad (5.25)$$

Expressing δ in terms of the simple roots found in the previous chapter, and with Λ given as in (5.18) it follows that c_Λ is always a positive number except if $\Lambda = 0$. Then the level can be expressed as level(Λ) $= 2c_\Lambda / \langle \alpha_k, \alpha_k \rangle$.

The final proposition is related with the tensor product of two irreducible representation.

Proposition 5.9

Let $\Gamma = \Gamma(\Lambda_1) \otimes \Gamma(\Lambda_2)$ be a tensor product of two irreducible representation of $\tilde{\mathcal{L}}_s$ with dominant highest weights Λ_1, Λ_2.

Then Γ is completely reducible and the highest weights of the irreducible components of Γ have the form $\Lambda_1 + \lambda_2^i - j\delta$ for $j \geq 0$ and $\{\lambda_2^i\}$ is a finite set of weights of $\Gamma(\Lambda_2)$ and t takes all non-negative integral values.

Proof (see Kac(1978)).

5.5 The generalized Casimir operator

We define a linear functional ρ on $\mathcal{H}$ by

$$2\langle\rho, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle = 1 \text{ for all } i \in I \text{ and } \rho(d) = 0. \qquad (5.26)$$

Definition 5.9 The generalized Casimir operator

Let e^k_α and $e^k_{-\alpha}$ be dual basis with respect to the invariant supersymmetric non-degenerate bilinear form of $\tilde{\mathcal{L}}_s$ corresponding to the positive roots α and k =1,..., m, m being the multiplicity of the root subspace $\tilde{\mathcal{L}}_\alpha$. That is $B(e^j_\alpha , e^k_{-\alpha}) = \delta_{jk}$ for all $\alpha \in \Delta^+$. We also choose h_i (i=1,2,..., $\ell+2$) to be dual basis of $\mathcal{H}$. Finally let H_ρ be the element of H corresponding to the functional ρ defined above. Then in analogy with the affine algebraic case the second order Casimir operator of an integrable irreducible highest weight representation of $\tilde{\mathcal{L}}_s$ is defined by

$$C_2 = 2\Phi(H_\rho)+ \sum_{i=1}^{\ell+2} \Phi(h_i)^2 + 2\sum_{\alpha\in\Delta^+} \sum_{k=1}^{m} \Phi(e^k_{-\alpha})\Phi(e^k_\alpha) \tag{5.27}$$

where Φ are the operators of the representation. Actually this is a more general definition of any representation that belongs in the category $\mathcal{M}$. In particular, it is the property of the representations that belong in the category $\mathcal{M}$ to be restricted,(that is, if for every weight vector $\psi(\lambda)$ of the representation $\Phi(e_\alpha)\psi(\lambda) = 0$ for all but a finite number of positive roots α $\tilde{\mathcal{L}}_s$) which allows the definition (5.27). It can be easily checked that C_2 commutes with all the operators of the representation belonging in the category $\mathcal{M}$. Since the representations that belong to this category are restricted, the third term when acting on any weight vector $\psi(\lambda)$ gives a finite result. Also since the first two terms when acting in any $\psi(\lambda)$ give $<\lambda+2\rho, \lambda>$, relation (5.27) impliesthat

$$C_2\,\psi(\lambda) = <\lambda+2\rho,\, \lambda>\psi(\lambda) + 2\sum_{\alpha\in\Delta^+} \sum_{k=1}^{m} \Phi(e^k_{-\alpha})\Phi(e^k_{\alpha})\psi(\lambda) \quad (5.28)$$

For a highest weight integrable irreducible representation

$$C_2\,\psi(\lambda) = <\Lambda+2\rho,\, \Lambda>\psi(\lambda) \quad (5.29)$$

for any $\psi(\lambda)$ of the representation with highest weight Λ.

5.6 The character formula

We shall now give the character formula for integrable irreducible representation of dominant highest weight Λ. The construction of the character formula follows the same steps as for the affine algebras with only minor modifications related with the existence of odd roots. The underlying general theory of characters of representations of infinite dimensional algebras or superalgebras is a consistent modification of that of finite dimensional representations of finite dimensional algebras or superalgebras.

Consider the space $\mathcal{E}$ introduced in section 5.2 above and define the function L on $\mathcal{H}^*$ by

$$L = \{e^{\rho} \prod_{\alpha\in\Delta_0^+} (1- e^{-\alpha})\} / \prod_{\alpha\in\Delta_1^+} (1+ e^{-\alpha}) \quad (5.30)$$

where each root is taken with its multiplicity.

Let β be any linear functional defined on $\mathcal{H}$ given by

$$\beta = - \sum_{\alpha\in\Delta_+} k_\alpha\, \alpha$$

where k_α is a non-negative integer if $\alpha \in \Delta_0^+$ and $k_\alpha = 0,1$ if $\alpha \in \Delta_1^+$. The Kostant function $K(\beta)$ defined on $\mathcal{H}^*$ is the number of finite sets $\{k_\alpha\}$ in the above expression. It can be shown that $KL = e^{\rho}$.

Let ε, be a homomorphism from the Weyl group to the multiplicative group {1,-1}, defined by $\varepsilon(S_{\alpha_j}) = -1$ for all $j \in I$, S_{α_j} being the generators of W. Note that for any element S of W, $\varepsilon(S) = (-1)^{l(S)}$ where l(S) is the number of factors in the shortest expression of S in terms of S_{α_j}. It can be proved that for any $S \in W$, $S(L) = \varepsilon(S)L$.

For the module $\tilde{V}(\Lambda)$ constructed in the beginning of the chapter it can be easily obtained using the weight structure of $\tilde{V}(\Lambda)$ in relation with the properties of the space $\mathcal{E}$ that

$$ch\tilde{V}(\Lambda) = \Sigma_{\lambda \in D(\Lambda)} K(\lambda - \Lambda)e^{\lambda} \tag{5.31}$$

and so since $KL = e^{\rho}$,

$$L\, ch\tilde{V}(\Lambda) = e^{\Lambda+\rho}. \tag{5.32}$$

Proposition 5.10

For an integrable irreducible representation of dominant highest weight Λ of an affine Kac -Moody superalgebra $\tilde{\mathcal{L}}_s$ the character formula is given by

$$chV = L^{-1} \Sigma_{S \in W} \varepsilon(S)e^{S(\Lambda+\rho)} \tag{5.33}$$

or equivalently in its Weyl form

$$chV = \Sigma_{S \in W} \varepsilon(S)e^{S(\Lambda+\rho)} / \Sigma_{S \in W} \varepsilon(S)e^{S(\rho)} \tag{5.34}$$

Proof (See Kac(1978))

To obtain (5.34) we have used relation (5.33) and the fact that chV = 1 in the case where $\Lambda = 0$. In particular for $\Lambda = 0$, (5.30) becomes the equivalent of Weyl's denominator formula and is given by

$$\Pi_{\alpha\in\Delta_0^+} (1- e^{-\alpha})/ \Pi_{\alpha\in\Delta_1^+} (1+ e^{-\alpha}) = \Sigma_{S\in W} \varepsilon(S)e^{S(\rho)-\rho} \quad (5.35)$$

Then the Kostant's formula is obtained by multipling both sides of (5.33) with $Ke^{-\rho}L$

$$chV_\lambda = \Sigma_{S\in W} \varepsilon(S) K((\lambda+\rho)-S(\Lambda+\rho)) \quad (5.36)$$

Also the character formula (5.34) can be rewritten using the weights λ_i - $j\delta$ ($j\in Z$) of multiplicity $m_i(j)$, where λ_i is the i-th dominant maximal weight (see proposition), as :

$$chV(\Lambda) = \Sigma_{i=1}^{s} \{ \Sigma_{j=0}^{\infty} m_i(j)e^{-j\delta)} \}\{ \Sigma_{S\in W} e^{S(\lambda_i)} \}/s \quad (5.37)$$

where s is the number of the dominant maximal weight. Finally it can be shown that (5.34) is equivalent to the so called 'star' formula

$$\Sigma_{S\in W} \varepsilon(S)dimV_{\lambda+\rho-S(\rho)} = 0 \text{ for } \lambda \in R(\Lambda) - \Lambda \quad (5.38)$$

where $R(\Lambda)$ is the set of all weights of the module $V(\Lambda)$.

Kac(1978) showed that from (5.30), under certain manipulations involving the root structure of the superalgebra, we can obtain the multivariable identities for non-reduced irreducible root system that did not appear in Macdonalds(1972) work.

As it is well known the carrier space of an integrable irreducible representations carries a contravariant hermitian form which allows for the definition of unitarity of representations.

Clearly a similar similar notion for affine superalgebras is still lacking. Actually, following the example of finite dimensional Lie superalgebras, it would be interesting to investigate the existence and reducibility of representations whose carrier spaces is endowed with a hermitian or superhermitian form. This would also demonstrate how we can define a suitable adjoint operation .

CHAPTER 6

SUGAWARA CONSTRUCTIONS OF THE AFFINE KAC-MOODY SUPERALGEBRAS

6.1. Introduction

Affine Kac-Moody (super)algebras together with the Virasoro algebras play a central role in two dimensional conformal field theories. It is well known that together they possess a semidirect sum algebraic structure. This is usually demonstrated by expressing the Virasoro generators, which are related to the energy momentum tensor in two dimensions, bilinearly in operators of some representation of the Kac-Moody (super)algebra, the latter being considered to correspond to currents. This idea dates back to Sugawara(1968). Physical reasons demand that the Sugawara construction has to be unitary. Thus the interest is in unitary irreducible highest weight representations of the Kac-Moody algebras which give unitary representations of the Virasoro algebra. The main objective of this chapter is to investigate the case of the Sugawara construction of the twisted Kac-Moody superalgebras $C^{(2)}(\ell+1)$ $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$, but we shall also demonstrate the Sugawara construction of untwisted Kac-Moody superalgebras (including $B^{(1)}(0/\ell)$). As the Sugawara construction requires great care even in the simplest case of an untwisted affine Kac-Moody algebra, its extension to the much more complicated situation of a twisted affine Kac-Moody superalgebra is inevitably appreciably more elaborate.

6.2. The Virasoro algebra

The Virasoro algebra (Virasoro(1970)) is the infinite-dimensional Lie algebra with basis elements L_J (for J = 0, ±1 ±2,....), and C_V, satisfing the following commutation relations relations:

$$[L_J , L_K] = (J-K) L_{J+K} + (1/12) C_V J(J^2-1)\delta_{J+K,0} , \quad (6.2.1)$$

$$[L_J , C_V] = 0 \qquad \text{(for all } J = 0,\pm1,\pm2,....) . \quad (6.2.2)$$

together with the Jacobi identity.

This algebra first appeared in the dual resonance models for hadrons, or what is thought now to be the early days of string theories (see Scherk(1970)). It arises naturally as an extension of the infinite-dimensional Lie algebra of the conformal group in two dimensions, the latter being given by (6.2.1) without the second term. In this context the Virasoro algebra is one of the basic ingredients of any two dimensional theory that possess conformal invariance. Together with its unitary highest weight representations, it has attracted the interest of mathematicians and physicists since its appearance, and it has beeen studied extensively both on its own or in relation with physics. (See for example Kac and Raina(1977), Friedan, Qiu and Shenker(1984b), Gorman et.al.(1989), Goddard, Kent and Olive(1986), Goddard and Olive(1988)).

Treating the basis of the Virasoro algebra as operators, let V be a carrier space on which they act, provided with an inner product $(\psi, \psi) \geq 0$, for all $\psi \in$ V allowing the possibility of $(\psi, \psi) = 0$ with $\psi \neq 0$. Assuming that V has no proper invariant subspace the operators form an irreducible representation of the Virasoro algebra, in which case

$$C_v = c_v I, \qquad (6.2.3)$$

where c_v is a constant known as the <u>central charge</u> of the Virasoro algebra. If in addition there exists a vector $\psi(h)$ of V such that

$$L_J \psi(h) = 0 \qquad \text{for all } J > 0, \qquad (6.2.4)$$

$$L_0 \psi(h) = h\psi(h) . \qquad (6.2.5)$$

then the set of Virasoro operators is said to form an irreducible highest weight representation, the highest weight vector being $\psi(h)$. Then all the other basis elements of V are obtained from $\psi(h)$ by succesive action of L_J according to the prescription

$$L_{J_1} L_{J_2} L_{J_3} \ldots L_{J_n} \psi(h) \qquad (6.2.6)$$

where $n \in \mathbb{Z}_+$ and $(J_1, J_2, J_3, \ldots, J_n)$ is any set of negative integers which satisfy the relation $J_1 \leq J_2 \leq J_3 \leq \ldots \leq J_n$.
Unitarity is achieved by demanding that

$$(L_J)^\dagger = L_{-J} \text{ and } (C_v)^\dagger = C_v \qquad (6.2.7)$$

in which case c_v is always real.
For unitary irreducible highest weight representations of the Virasoro algebra for which the inner product on V is non-negative it has been established (see Friedan, Qiu and

Shenker(1984, 1984b) that the only possible values of c_v and h are restricted to be either

$$c_v \geq 1 \text{ and } h \geq 0 \,, \tag{6.2.8}$$

or

$$c_v = 1 - \frac{6}{(m+2)(m+3)} \tag{6.2.9}$$

and

$$h = \frac{\{(m+3)p-(m+2)q\}2-1}{4(m+2)(m+3)} \,, \tag{6.2.10}$$

for $m \in Z_+$, $p = 1,2,....,m+1$, and $q = 1,2,....,p$. L_0 has always a spectrum which is bounded below.

Affine Kac-Moody algebras and superalgebras, as well as being important mathematically, play a vital role in the study of two-dimensional physical systems. In particular untwisted Kac-Moody algebras (see Goddard and Olive(1986)) arise naturally in the study of current algebras in two space-time dimensions and when the space itself is compact (e.g. S^1). The simplest case is that of the current algebra of free massless fermions fields defined on the circle, where their current algebra is recognized as an untwisted Kac-Moody algebra with the central extension term identified as the so called Schwinger term and representing second order quantum effects. A more complicated example is that of current algebras of boson fields.

Affine Kac-Moody algebras together with the Virasoro algebras are the basic ingredients of any two dimensional comformal field theory. They are related by means of a semi-direct sum structure given by the relations

$$[\, L_J \,,\, \Phi(t^k \otimes a_s) \,] \;=\; -k\Phi(t^{J+k} \otimes a_s)\;, \qquad (6.2.11)$$

$$[\, L_J \,,\, \Phi(c) \,] \;=\; 0, \qquad (6.2.12)$$

for all k, $J \in Z$ and where $\Phi(t^k \otimes a_s)$, $\Phi(c)$ are operators representing $t^k \otimes a_s$ and c, a_s being any basis elements of the simple Lie algebra on which the affine algebra is based. The above relations imply that it is the derived algebra of the affine algebra that possesses such structure with the Virasoro algebra.

This structure is usually demonstrated by expressing the Virasoro generator, bilinearly in operators of some representation of the Kac-Moody algebra. This idea originated from Sugawara(1968) who proposed that the energy-momentum tensor of four-dimensional theories can be expressed bilinearly in terms of currents taking into account the Schwinger term. In this context the Virasoro generators correspond to the energy-momentum tensor and the Kac-Moody generators to the currents.

That the Sugawara construction exists and satisfies the relations (6.2.11-12) for untwisted and twisted algebras and untwisted superalgebras has already been demonstrated. (see Goddard and Olive(1986), Goddard, Nahm and Olive(1985), Nepomechie(1986), Zheng and Kim(1990), Goddard and Olive and Waterson(1986), Hennigson(1990)).

Another objective of the Sugawara construction is to find the eigenvalues of C_V and L_0 which might be of physical interest, particularly in string theories. In this process unitary irreducible highest weight representations of the

Virasoro algebra are obtained from those of the affine Kac-Moody algebra or superalgebra, which themselves may be determined by the representations of their underlying algebras or superalgebras (see Goddard, Kent and Olive(1985,1986), Bernard and Thierry-Mieg(1987)). Consequently the eigenvalues of C_V and L_0 depend on the highest weights of these latter representations. It should be noted also that the existence of the Sugawara construction is guaranteed by the existence of a second order Casimir operator of the underlying algebras or superalgebras.

Finally it should be noted that there also exists supersymmetric extensions of the Virasoro algebra, namely the Virasoro superalgebras which contain the Virasoro algebra as their even part (for details on this subject see Cornwell(1989)).

6.3. Sugawara construction for affine untwisted Kac-Moody superalgebras

The Sugawara construction of the untwisted Kac-Moody superalgebras obtained from basic simple Lie superalgebras was first carried out in Goddard et al(1987) in a field theoretical content closely related to string theories. This was an intermediate stage for a more important result to which we shall briefly refer because it reveals the crucial role of affine (super)algebras in physics.

It is known that from a N-dimensional representation of a finite dimensional compact Lie algebra $\mathcal{L}_c$ described by real antisymmetric matrices we can obtain a representation of an untwisted Kac-Moody algebra associated with $\mathcal{L}_c$, by intoducing N fermi fields (periodic or antiperiodic, i.e.Ramond or Neveu-Schwarz) defined on the unit circle in the complex plane. The Kac-Moody generators will then be bilinear in these fields. The Sugawara construction of the untwisted algebra will then give Virasoro operators quatrilinear in the fields. There exists also a consrtuction of the Virasoro algebra based on the energy-momentum of the free fermion fields which is bilinear in the fields. The equality of these two constructions is achieved by means of the symmetric space theorem.(For details see Goddard and Olive(1986)).

Applying the same proccess, we can obtain now representations of an untwisted Kac-Moody algebra from real N'-dimensional symplectic representations of a possible non-compact real Lie algebra $\mathcal{L}$ using N' boson fields. Then the equality of the two constructions of the Virasoro algebra is provided by means of the superalgebra theorem which states that these two constructions are equal if and only if the above mentioned representation is the one provided by the odd part of a superalgebra whose even part contains $\mathcal{L}$ and possessing a second order Casimir operator. In addition it was also shown that the superalgebra theorem holds in the most general case where $\mathcal{L}$ is a reductive Lie algebra.

At this stage the Sugawara construction for a Lie superalgebra possessing a second order Casimir operator has to be carried out. Then in order to put these two cases described above, together, we can start with orthosymplectic representations of a Lie superalgebra and express the even and odd generators of an untwisted Kac-Moody algebra associated with it, bilinearly in the fermionic and bosonic fields. Then it has been demonstrated that the Sugawara construction of this untwisted superalgebra equals the sum of the bilinear constructions of the Virasoro algebras of the free fermions and the symplectic bosons by means of the supersymmetric space theorem. (For details see Goddard Olive and Waterson (1987)).

It should be noted that the demand of equality between the two Virasoro constructions in any of the cases referred to above is very crucial. In the case of fermions for example, by chosing an appropriate symmetric space as the symmetry group of our theory we can reduce the study of interacting fermions to free ones. This in turn has unexpected consequences related to a highly non-linear theory, namely the Wess-Zumino model, which now is quantum equivalent to a free fermion theory.

We shall now briefly demonstrate the Sugawara construction of untwisted Kac-Moody superalgebra using algebraic methods.

Let $\tilde{\mathfrak{L}}_s^{(1)}$ be a complex untwisted Kac-Moody superalgebra as described in chapter 4. Let m be the dimension of the even part and n the dimension of the odd part of $\tilde{\mathfrak{L}}_s^0$, where $\tilde{\mathfrak{L}}_s^0$ is one of the basic simple complex Lie superalgebras. We can choose an even basis of $\tilde{\mathfrak{L}}_s^0$ which, with respect to the Killing form (or any other supersymmetric non-degenerate bilinear form, if the Killing form is identically zero) of $\tilde{\mathfrak{L}}_s^0$ can be normalized as

$$B^0(a_p, a^{\#}_{p'}) = \delta_{pq} \quad \text{for all } p,q = 1,\ldots,m, \qquad (6.3.1)$$

where $a^{\#}_p$ is called the "dual" of a_p and is defined such that $a^{\#}_p = -a_p$. Then $B^0(a_p, a_{p'}) = -\delta_{pq}$. Such a choice is given for example by

(i) iH_j for all $j = 1,2,\ldots,\ell^0$, where $B^0(H_j, H_k) = \delta_{jk}$, and H_j are basis of the Cartan subalgebra of $\tilde{\mathcal{L}}^0_s$ and ℓ^0 is its rank,

(ii) $i\,\{2B^0(a_\alpha, a_{-\alpha})\}^{-\frac{1}{2}}(a_\alpha + a_{-\alpha})$, $-\{2B^0(a_\alpha, a_{-\alpha})\}^{-\frac{1}{2}}(a_\alpha - a_{-\alpha})$ for each positive even root α of $\tilde{\mathcal{L}}^0_s$.

There exists a particularly convenient choice (see Cornwell(1989)) for the odd basis elements $b_1, b_2, \ldots, b_n$ of $\tilde{\mathcal{L}}^0_s$, which is such that the matrix $\underset{\sim}{B}^0$ defined by

$$(\underset{\sim}{B}^0)_{pq} = B^0(b_p, b_q) \qquad \text{(for all } p,q = 1,\ldots,n) \qquad (6.3.2)$$

is given by

$$(\underset{\sim}{B}^0)_{pq} = -\underset{\sim}{J}_{pq}, \qquad (6.3.3)$$

where $\underset{\sim}{J}$ is a $n \times n$ antisymmetric matrix of the form

$$\underset{\sim}{J} = \mathrm{diag}\,(\underset{\sim}{b}, \underset{\sim}{b}, \underset{\sim}{b}, \ldots\ldots)$$

and

$$\underset{\sim}{b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (6.3.4)$$

It can be easily seen that with the choice of basis

$$(\underset{\sim}{B}^0)^{-1} = -\underset{\sim}{B}^0 = \underset{\sim}{J}$$

Such a choice is realized by

(iii) $b_{\alpha+} = \{2B^0(a_{-\alpha}, a_\alpha)\}^{-\frac{1}{2}}(a_\alpha + a_{-\alpha})$

(iv) $b_{\alpha-} = \{2B^0(a_{-\alpha}, a_\alpha)\}^{-\frac{1}{2}}(a_\alpha - a_{-\alpha})$

for every positive odd root α.

Then for each graded representation of $\tilde{\mathcal{L}}_s^0$, the second order Casimir operator has the form

$$C_2 = -\sum_{r=1}^{m} \Phi(a_r)\Phi(a_r) - \sum_{p=1}^{n} J_{pq}\, \Phi(b_p)\Phi(b_q) \ , \qquad (6.3.5)$$

and identifing a_r and b_p as above, it takes the form

$$C_2 = \sum_{j=1}^{\ell 0} \Phi(H_j)^2 + \sum_{\alpha\in\Delta_0^+} \Phi(h_\alpha) - \sum_{\alpha\in\Delta_1^+} \Phi(h_\alpha) + \sum_{\alpha\in\Delta^+} 2\Phi(a_{-\alpha})\Phi(a_\alpha) \,/\, B^0(\, a_\alpha \,,\, a_{-\alpha}\,). \qquad (6.3.6)$$

With this form of C_2 and for the adjoint representation of $\tilde{\mathcal{L}}_s^0$ it can be proved that the eigenvalue of the second order Casimir operator is 1 if $\tilde{\mathcal{L}}_s^0$ has non-zero Killing form and is 0 if it has identically zero Killing form.

Consider now the bosonic operators $\Phi(t^j \otimes a_p)$, $\Phi(c)$, $\Phi(d)$ and the fermionic operators $\Phi(t^k \otimes b_q)$, for all p =1,...,m, q = 1,...,n, and $j \in Z$ and $k \in Z$ or $k \in Z+\frac{1}{2}$. Following the terminology of Goddard et al(1987), and by analogy with the usual nomenclature for the Virasoro algebras, the situation for which $j,k \in Z$ will be called the "Ramond" case, and that with $j \in Z$ and $k \in Z+\frac{1}{2}$ will be referred to as the "Neveu-Schwarz" case. (In this description the untwisted affine Kac-Moody superalgebras $B^{(1)}(0/\ell)$ (for ℓ = 1,2, ...) are examples of the Ramond case). Let $V(\Lambda)$ be a carrier space of a highest weight representation of $\tilde{\mathcal{L}}_s^{(1)}$ on which the above operators act, the highest weight vector being $\psi(\Lambda)$. The action of the operators on $V(\Lambda)$ is described by

$$\Phi(t^j \otimes a_p)\, \psi(\Lambda) = 0 \qquad \text{for all } p = 1,\dots,m \text{ and all } j > 0,$$

$$\Phi(t^j \otimes b_q)\, \psi(\Lambda) = 0 \qquad \text{for all } q = 1,\dots,n \text{ and all } j > 0,$$

in the "Ramond" case and

$$\Phi(t^{j+\frac{1}{2}} \otimes b_q)\, \psi(\Lambda) = 0 \qquad \text{for all } q = 1,\dots,n \text{ and all } j \geq 0,$$

in the "Neveu-Schwarz" case, where $j \in \mathbf{Z}$, and

$$\Phi(c)\, \psi(\Lambda) = c_\Lambda \psi(\Lambda)\ ,$$

$$\Phi(d)\, \psi(\Lambda) = \Lambda(d)\ \psi(\Lambda)\ . \qquad (6.3.7)$$

Let $\psi(\lambda)$ be any weight vector of V other than $\psi(\Lambda)$. Then there exists a non-negative integer K depending on the weight λ such that

$$\Phi(t^j \otimes a_p)\, \psi(\lambda) = 0$$

$$\Phi(t^{j+\frac{1}{2}\varepsilon} \otimes b_q)\, \psi(\lambda) = 0 \qquad (6.3.8)$$

for all $j > K-\frac{1}{2}\varepsilon$, ($j\in\mathbf{Z}$) $p = 1,\dots,m$, $q = 1,\dots,n$ and where $\varepsilon = 0$ and 1 for the Ramond and Neveu-Schwarz cases respectively. Finally the generalized Lie products between the operators Φ are those of chapter 4 for $\tilde{\mathfrak{L}}_s^{(1)}$. Every $\psi(\lambda)$ will be obtained from $\psi(\Lambda)$ by the action of a linear combination of products of a finite number of operators on it. On the subsuperalgebra of $\tilde{\mathfrak{L}}_s^{(1)}$ which is isomorphic to $\tilde{\mathfrak{L}}_s^0$ this representation will provided a highest weight representation of $\tilde{\mathfrak{L}}_s^0$. Moreover on the even part of $\tilde{\mathfrak{L}}_s^0$ this will provide a highest weight representation too.

Then the Sugawara construction for L_J is given by

$$L_J \psi = \{ L_J^{even} + L_J^{odd} \} \psi = - \frac{1}{\kappa} \sum_{p=1}^{m} \sum_{j=-\infty}^{\infty} \{:\Phi(t^{J+j} \otimes a_p)\Phi(t^{-j} \otimes a_p):\}\psi$$

$$+ \frac{1}{\kappa} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{j=-\infty}^{\infty} J_{pq}\{:\Phi(t^{J+j} \otimes b_p)\, \Phi(t^{-j} \otimes b_q):\}\, \psi$$

$$+ \eta\varepsilon\, \delta_{J,0}\, \psi\,, \qquad (6.3.9)$$

for every $\psi \in V(\Lambda)$ and where $\varepsilon = 0$ and 1 for the Ramond and Neveu-Schwarz cases respectively, κ is an appropriate "normalization" constant to be found together with η. Clearly κ has to be inserted to give the desired relations (6.3.12a,b) below, thereby establishing the semi-direct sum of the Virasoro algebra with $\tilde{\mathcal{L}}_s^{(1)}$. The first sum is identified as L_J^{even} and the last two sums as L_J^{odd}. The normal ordering for the bosonic operators is given by

$$: \Phi(t^j \otimes a_p)\, \Phi(t^k \otimes a_q) : =$$

$$\Phi(t^j \otimes a_p)\, \Phi(t^k \otimes a_q)\,, \text{ if } j < k,$$

$$\tfrac{1}{2}\{\Phi(t^j \otimes a_p)\, \Phi(t^k \otimes a_q) + \Phi(t^k \otimes a_q)\, \Phi(t^j \otimes a_p)\,, \text{ if } j = k,$$

$$\Phi(t^k \otimes a_q)\, \Phi(t^j \otimes a_p)\,, \text{ if } j > k. \qquad (6.3.10)$$

where $j, k \in \mathbf{Z}$, and for the fermionic operators

$$: \Phi(t^j \otimes b_p)\, \Phi(t^k \otimes b_q) : =$$

$$\Phi(t^j \otimes b_p)\, \Phi(t^k \otimes b_q)\,, \text{ if } j < k,$$

$$\tfrac{1}{2}\{\Phi(t^j \otimes b_p)\, \Phi(t^k \otimes b_q) - \Phi(t^k \otimes b_q)\, \Phi(t^j \otimes b_p)\,, \text{ if } j = k,$$

$$- \Phi(t^k \otimes b_q)\, \Phi(t^j \otimes b_p)\,, \text{ if } j > k. \qquad (6.3.11)$$

where j, k$\in$ $\mathbf{Z}$, in the Ramond case and j, k$\in$ $\mathbf{Z}+\frac{1}{2}$ in the Neveu-Schwarz case.

It can be proved that with definition (6.3.9)

$$[\, L_J \,,\, \Phi(t^j \otimes a_r) \,] \;=\; -j\Phi(t^{J+j} \otimes a_r)\,, \qquad (6.3.12a)$$

$$[\, L_J \,,\, \Phi(t^k \otimes b_q) \,] \;=\; -k\Phi(t^{J+k} \otimes b_q)\,, \qquad (6.3.12b)$$

for all j $\in$ $\mathbf{Z}$ and all k $\in$ $\mathbf{Z}$ or $\mathbf{Z}+\frac{1}{2}$,

$$[\, L_J \,,\, \Phi(c) \,] \;=\; 0, \qquad (6.3.13)$$

provided that κ is given by

$$\kappa \;=\; 2c_\Lambda+1 = 2c_\Lambda + C_2(\mathrm{ad}) \qquad (6.3.14)$$

if $\tilde{\mathfrak{L}}^0_s$ has non-zero Killing form and

$$\kappa \;=\; 2c_\Lambda = 2c_\Lambda + C_2(\mathrm{ad}) \qquad (6.3.15)$$

if $\tilde{\mathfrak{L}}^0_s$ has identically zero Killing form, where $C_2(\mathrm{ad})$ is the eigenvalue of C_2 in the adjoint representation.

Having established the above results we can check that (for $J+K \neq 0$) $[L_J \,,\, L_K \,]\, \psi(\lambda) = (J-K)L_{J+K}\, \psi(\lambda)$ in both the Ramond and Neveu-Schwarz case. Finally the full commutation relations of the Virasoro algebra are obtained by evaluating the commutator $[L_J \,,\, L_K \,]\, \psi(\Lambda)$ with $J+K = 0$. Since this step is the most crucial one we should distinguish the two cases.

(a) The Ramond case

Without loss of generality we may assume that $J > 0$ and thus $K = -J < 0$. By evaluating first $[L_J \,,\, L^{\mathrm{even}}_{-J} \,]\, \psi(\Lambda)$ it is found that

$$[L_J \,,\, L^{\mathrm{even}}_{-J} \,]\, \psi(\Lambda) = \{-(2J/\kappa)\textstyle\sum_{p=1}^{m} \Phi(t^0 \otimes a_p)\, \Phi(t^0 \otimes a_p)$$

$$+ (1/12)J(J^2-1)(2c_\Lambda m/\kappa)\} \ \psi(\Lambda). \qquad (6.3.16)$$

However $L_0^{even}\,\psi(\Lambda)$ is easily shown, by considering the L_0^{even} of (6.3.9) and taking into account the normal ordering and the action on $\psi(\Lambda)$, to be given by

$$L_0^{even}\,\psi(\Lambda) = -\frac{1}{\kappa}\sum_{p=1}^{m} \Phi(t^0 \otimes a_p)\,\Phi(t^0 \otimes a_p)\,\psi(\Lambda) \qquad (6.3.17)$$

For $[L_J\,, L_{-J}^{odd}]\,\psi(\Lambda)$ we get

$$[L_J\,, L_{-J}^{odd}]\,\psi(\Lambda) = \{(2J/\kappa)\sum_{p=1}^{n}\sum_{q=1}^{n} J_{pq}\{\Phi(t^0 \otimes b_p)\,\Phi(t^0 \otimes b_q)\}$$

$$- (1/12)J(J^2-1)(2c_\Lambda n/\kappa)\} \ \psi(\Lambda). \qquad (6.3.18)$$

By considering the L_0^{odd} of (6.3.9) and taking into account the normal ordering and the action on $\psi(\Lambda)$ we can easily find that

$$L_0^{odd}\,\psi(\Lambda) = \frac{1}{\kappa}\sum_{p=1}^{n}\sum_{q=1}^{n} J_{pq}\{\Phi(t^0 \otimes b_p)\,\Phi(t^0 \otimes b_q\}\,\psi(\Lambda) (6.3.19)$$

Thus comparing (6.3.16) and (6.3.18) shows that in the Ramond case

the value of the central charge c_V is given by

$$c_V = 2c_\Lambda(m-n)/\kappa\,, \qquad (6.3.20)$$

and the eigenvalue ofL_0 is given by

$$L_0\,\psi(\Lambda) = \{C_{2S}(\Lambda_s^0)/\kappa\}\,\psi(\Lambda) \qquad (6.3.21)$$

where $C_{2S}(\Lambda^0)$ is the eigenvalue of the second order Casimir operator in the representation of the Lie superalgebra $\tilde{\mathcal{L}}_s^0$ with highest weight Λ^0.

(b) The Neveu-Schwarz case

Again without loss of generality, we may assume that $J > 0$ and thus $K = -J < 0$. By evaluating first $[L_J , L^{even}_{-J}]\,\psi(\Lambda)$ we get exactly the same result as in the Ramond case.
For $[L_J , L^{odd}_{-J}]\,\psi(\Lambda)$ we get

$$[L_J , L^{odd}_{-J}]\psi(\Lambda) = \{(2nc_\Lambda/\kappa)\{-(1/12)J(J^2-1) - (1/8)J\}\psi(\Lambda).$$

(6.3.22)

$L_0^{odd}\,\psi(\Lambda)$ is easily shown, by considering the L_0^{odd} of (6.3.9) and taking into account the normal ordering and the action on $\psi(\Lambda)$, to be given by

$$L_0^{odd}\,\psi(\Lambda) = -(nc_\Lambda/8\kappa)\,\psi(\Lambda) \qquad (6.3.23)$$

Thus in the Neveu-Schwarz case the central charge is as before but the eigenvalue of L_0 is given by

$$L_0\psi(\Lambda) = (1/8\kappa)\{\,8C_{20}(\Lambda_0^0) - nc_\Lambda\}\,\psi(\Lambda) \qquad (6.3.24)$$

where $C_{20}(\Lambda_0^0)$ is the eigenvalue of the second order Casimir operator of the representation of the even part of $\tilde{\mathfrak{L}}_s^0$ with highest weight Λ_0^0.

The final stage is the determination of the relation of $\Phi(d)$ with the Virasoro algebra. It can be easily observed that $[\,L_0, \Phi(d)\,]$ commutes with all the operators of the representation of $\tilde{\mathfrak{L}}_s^{(1)}$ and thus by Schur's lemma and the eigenvalues of L_0 found above

$$L_0\psi(\Lambda) = \{-\Phi(d) + \Lambda(d) + h\}\psi(\Lambda)\,, \qquad (6.3.25)$$

where h is given by (6.3.21) or (6.3.24).

Some important remarks are in order. We drescribe as "critical representations" of $\tilde{\mathfrak{L}}_s^{(1)}$ those representations for which the Sugawara construction fails. This is the case where the constant $\kappa = 0$ i.e. when $c_\Lambda = -\frac{1}{2}$ or $c_\Lambda = 0$. Consider $B^{(1)}(0/\ell)$. According to chapter 5, in a standard irreducible highest weight representation of $B^{(1)}(0/\ell)$ the eigenvalue c_Λ of such representation is given by $c_\Lambda = < \Lambda , \delta >$ where Λ is the highest weight of the representation. From relations (5.19), (5.20) of chapter 5 we deduce that c_Λ is always positive and thus no such representation is critical. Examples of critical representations are provided by certain representations of the real untwisted superalgebra $osp^{(1)}(2\ell+1/\ell; \mathbb{R})$(see Goddard, Olive and Waterson(1987))

The existence of unitary irreducible highest weight representations of $\tilde{\mathcal{L}}_s^{(1)}$, which will lead to unitary representations of the Virasoro algebra, have recently been investigated in Jarvis and Zhang((1988),(1989)). They considered irreducible representations of the untwisted superaglebras $\tilde{\mathcal{L}}_s^{(1)}$, built from unitary irreducible representations of $\tilde{\mathcal{L}}_s^0$. It was demonstrated that with appropriately chosen adjoint operation on the elements of a real untwited Kac-Moody algebra based on a real compact or non-compact form of $\tilde{\mathcal{L}}_s^0$, unitarity will restrict the values c_Λ and c_V. Constraints on c_Λ were found and come both from the even and odd roots of $\tilde{\mathcal{L}}_s^0$. It was shown that the only candidates for unitary representations are untwisted superalgebras obtained (i) from the compact real forms of A(r/0) and C(ℓ+1), su(r+1/1) and $osp(2/2\ell; \mathbb{R})$ respectively, (ii) the non-compact real forms of A(1/s), D(r/2) and B(r/1), su(1,1/s+1), $osp(2r/2; \mathbb{R})$ and $osp(2r+1/2; \mathbb{R})$ respectively and (iii) non-compact real forms of D(2/1; α), F(4) and G(3) whose even parts are given by su(2) ⊕ su(2) ⊕ su(1,1), su(1,1) ⊕ so(7) and G_2 ⊕ su(1,1) respectively. The representations of the untwisted superalgebras are built from those of the superalgebras mentioned above, the latter being such that those of case (i) are unitary finite dimensional irreducible and of highest weight but in the rest of the cases are unitary irreducible but infinite dimensional. From case (ii) it is obvious that we cannot obtain unitary representations of the Virasoro algebra constructed as in (6.3.9) with $\tilde{\mathfrak{L}}_s^0$ being osp(1/2ℓ; $\mathbb{R}$)($\ell \neq 1$). In the case of osp(1/2; $\mathbb{R}$) we do obtain such representations but we have to consider infinite dimensional representations of osp(1/2; $\mathbb{R}$). Finally it should be noted that unitarity of representations of $\tilde{\mathfrak{L}}_s^0$ have recently been investigated by Gould and Zhang(1990).

6.4. Sugawara construction for $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$

6.4.1. Choice of basis for the $\tilde{\mathcal{L}}_p^{0(4)}$ (p=0,1,2,3) subspaces.

The first step in the construction is to define appropriate normalisation amongst the basis of $\tilde{\mathcal{L}}_p^{0(4)}$ subspaces with respect to the Killing form of $A(2\ell-1/0)$ and of $A(2\ell/0)$. A convenient choice is to work is with the dual basis, $a^{\#}_{pr}$, of the $\tilde{\mathcal{L}}_p^{0(4)}$ subspaces defined by

$$B^0(a_{pr}\,,\, a^{\#}_{p'r'}) \;=\; \delta_{pp'}\,\delta_{rr'} \qquad (6.4.1)$$

for all p,p' = 0,1,2,3, and all r =1,2,...,n_p, r'=1,2,...,$n_{p'}$, $B^0(\ ,\ )$ being the Killing forms of $A(2\ell-1/0)$ or $A(2\ell/0)$. Recall that these as given by:

$$B^0(\underset{\sim}{M},\underset{\sim}{N}) \;=\; 2(2\ell-1)\ \mathrm{str}(\underset{\sim}{M},\underset{\sim}{N}) \qquad (6.4.2)$$

for all $\underset{\sim}{M},\underset{\sim}{N}$ supermatrices of $s\ell(2\ell/1)$ (= $A(2\ell-1/0)$) or

$$B^0(\underset{\sim}{M},\underset{\sim}{N}) \;=\; 4\ell\ \mathrm{str}(\underset{\sim}{M},\underset{\sim}{N}) \qquad (6.4.3)$$

for all $\underset{\sim}{M},\underset{\sim}{N}$ supermatrices of $s\ell(2\ell+1/1)$ (= $A(2\ell/0)$) .

(i) For p = 0, $\tilde{\mathcal{L}}_0^{0(4)} = D_\ell$ or B_ℓ, which has dimension $n_0 = \ell(2\ell-1)$ or $\ell(2\ell+1)$ respectively. With basis elements denoted by a_{0r} (r = 1,...,n_0), relation (4.1) gives:

$$B^0(a_{0r}\,,\, a^{\#}_{0r'}) \;=\; \delta_{rr'}\,, \quad \text{for all } r,r'=1,\ldots,n_0, \qquad (6.4.4)$$

where $a_{0r'}^{\#}$ is the dual basis of a_{0r}. Since $\tilde{\mathcal{L}}_0^{0(4)}$ consists of even elements, we can always choose the basis a_{0r} such that

$$B^0(a_{0r}, a_{0r'}) = -\delta_{rr'}, \quad \text{for all } r,r' = 1,\dots,n_0, \qquad (6.4.5)$$

and thus their duals can be defined by

$$a_{0r}^{\#} = -a_{0r}, \quad \text{for all } r=1,\dots,n_0. \qquad (6.4.6)$$

It is also assumed that these basis satisfy the following relations between the Killing forms of $A(2\ell-1/0)$ or $A(2\ell/0)$ and D_ℓ or B_ℓ respectively(see chapter 4):

$$B^0(a_{0r}, a_{0r'}) = (2\ell-1)/(\ell-1)\ B^{D_\ell}(a_{0r}, a_{0r'}) = -\delta_{rr'}, \quad (6.4.7)$$

for all $r,r' = 1,\dots,n_0 = \ell(2\ell-1)$.

$$B^0(a_{0r}, a_{0r'}) = (4\ell)/(2\ell-1)\ B^{B_\ell}(a_{0r}, a_{0r'}) = -\delta_{rr'} \quad (6.4.8)$$

for all $r,r' = 1,\dots,n_0 = \ell(2\ell+1)$.

(ii) For the $\tilde{\mathcal{L}}_1^{0(4)}$ and $\tilde{\mathcal{L}}_3^{0(4)}$ subspaces of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$, which have dimensions $n_1 = n_3 = 2\ell$ and $2\ell+1$ respectively, all the elements are odd. Consequently the basis elements will be denoted by b_{1p} and b_{3p}, and relation (6.4.1) becomes

$$B^0(b_{1p}, b_{1p'}^{\#}) = \delta_{pp'}, \quad \text{for all } p,p' = 1,\dots,n_1, \qquad (6.4.9)$$

$$B^0(b_{3p}, b_{3p'}^{\#}) = \delta_{pp'}, \quad \text{for all } p,p' = 1,\dots,n_3, \qquad (6.4.10)$$

where, as before, $b_{1p}^{\#}$ and $b_{3p}^{\#}$ are the duals of b_{1p} and b_{3p} respectively. It can be easily checked that $b_{1p}^{\#}$ should belong in $\tilde{\mathcal{L}}_3^{0(4)}$ and $b_{3p}^{\#}$ in $\tilde{\mathcal{L}}_1^{0(4)}$. Thus we can define

$$b_{1p}^{\#} = -b_{3p}\ , \qquad (6.4.11)$$

$$b_{3p}^{\#} = b_{1p}\ . \qquad (6.4.12)$$

(Note that such choice is consistent with (2.33a)).

(iii) For the $\tilde{\mathcal{L}}_2^{0(4)}$ subspace of $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$, which has dimension $n_2 = 2\ell^2+\ell$ and $2\ell^2+3\ell+1$ respectively, all the elements are even. Denoting its basis elements by a_{2s} for all $s = 1,\ldots,n_2$, relation (6.4.1) becomes

$$B^0(a_{2s},a_{2s'}^{\#}) = \delta_{ss'}\ , \qquad (6.4.13)$$

where $a_{2s}^{\#}$ is the dual of a_{2s}. Since a_{2s} are even basis we can always orthonormalize them with respect to the Killing form $B^0(\ ,\ )$ by requiring that

$$B^0(a_{2s},a_{2s'}) = -\delta_{ss'} \qquad (6.4.14)$$

and thus define $a_{2s}^{\#} = -a_{2s}$. In the case of $\tilde{\mathcal{L}}_2^{0(4)}$ attention should be concentrated on the fact that it provides a representation of $\tilde{\mathcal{L}}_0^{0(4)}$ which is the direct sum of the trivial representation with the $(2\ell^2+\ell-1)$ or $(2\ell^2+3\ell)$-dimensional irreducible representation of D_ℓ or B_ℓ respectively. Thus a particular basis element has to be picked out from the commutative subspace of $\tilde{\mathcal{L}}_2^{0(4)}$, which together with the rest of the basis will satisfy (4.1) and be the basis of the trivial representation. The choice of this element in terms of the basis found in chapter 4 is unique in a sense that will become apparent from the analysis below.

Let $C_2(ad)$ be the quadratic Casimir operator of

$A(2\ell-1/0)$ or $A(2\ell/0)$ in the adjoint representation. If a_{2s} is a basis element of $\tilde{\mathfrak{L}}_2^{0(4)}$ then

$$C_2(\text{ad})\, a_{2s} = a_{2s}\,, \qquad (6.4.15)$$

with basis elements a_{0r}, a_{2s}, b_{1p}, and b_{3p} satisfying (6.4.1). $C_2(\text{ad})$ is given by

$$C_2(\text{ad}) = -\Sigma_{r=1}^{n_0}\Phi(a_{0r})\Phi(a_{0r}) - \Sigma_{s=1}^{n_2}\Phi(a_{2s})\Phi(a_{2s})$$

$$-\Sigma_{p=1}^{n_1}\Phi(b_{1p})\Phi(b_{1p}^{\#}) - \Sigma_{p=1}^{n_3}\Phi(b_{3p})\Phi(b_{3p}^{\#})\,, \qquad (6.4.16)$$

where Φ are operators of $A(2\ell-1/0)$ or $A(2\ell/0)$ belonging in the adjoint representation. From (6.4.16) it follows that

$$C_2(\text{ad})\, a_{2s'} = -\Sigma_{r=1}^{n_0}\Phi(a_{0r})\Phi(a_{0r})\, a_{2s'} - \Sigma_{s=1}^{n_2}\Phi(a_{2s})\Phi(a_{2s})\, a_{2s'}$$

$$-\Sigma_{p=1}^{n_1}\Phi(b_{1p})\Phi(b_{1p}^{\#})\, a_{2s'} - \Sigma_{3p}^{n_3}\Phi(b_{3p})\Phi(b_{3p}^{\#})\, a_{2s'}. (6.4.17)$$

The first sum can be evaluated to give

$$-\Sigma_{r=1}^{n_0}\Phi(a_{0r})\Phi(a_{0r})\, a_{2s'} = -\Sigma_{r=1}^{n_0}\Sigma_{s''=1}^{n_2}(\underset{\sim}{\Gamma}^2(a_{0r})\underset{\sim}{\Gamma}^2(a_{0r}))_{s''s'}\, a_{2s''}\,,$$

$$(6.4.18)$$

where Γ^2 is the representation of D_ℓ or B_ℓ provided by $\tilde{\mathfrak{L}}_2^{0(4)}$. The second sum can be evaluated to give

$$-\Sigma_{s=1}^{n_2}\Phi(a_{2s})\Phi(a_{2s})\, a_{2s'} = -\Sigma_{r=1}^{n_0}\Sigma_{s''=1}^{n_2}(\underset{\sim}{\Gamma}^2(a_{0r})\underset{\sim}{\Gamma}^2(a_{0r}))_{s''s'}\, a_{2s''}\,.$$

$$(6.4.19)$$

Similarly the third sum yields

$$-\Sigma_{p=1}^{n_1}\Phi(b_{1p})\Phi(b_{1p}^{\#})\, a_{2s'} = -\Sigma_{p=1}^{n_1}[\, b_{1p}\,, [\, a_{2s'}\,, b_{3p}\,]], \qquad (6.4.20)$$

and the fourth sum can be evaluated to produce

$$- \sum_{p=1}^{n_3} \Phi(b_{3p})\, \Phi(b_{3p}^{\#})\, a_{2s'} \;=\; \sum_{p=1}^{n_3} [\; b_{3p}\,, [\, a_{2s'}\,, b_{1p}\,]\,]$$

$$= \; -\sum_{p=1}^{n_3} [\, b_{1p}\,, [\, a_{2s'}\,, b_{3p}\,]\,]. \quad (6.4.21)$$

Thus it follows from (6.4.16) to (6.4.21) that

$$-2 \sum_{p=1}^{n_1} [\, b_{1p}\,, [\, a_{2s'}\,, b_{3p}\,]\,]$$

$$= \; a_{2s'} + 2 \sum_{r=1}^{n_0} \sum_{s''=1}^{n_2} (\, \underset{\sim}{\Gamma}^2(a_{0r}) \underset{\sim}{\Gamma}^2(a_{0r})\,)_{s''s'}\, a_{2s''}\,. \quad (6.4.22)$$

If the basis element $a_{2s'}$ corresponds to the trivial representation then the second term on the r.h.s. gives zero and

$$\sum_{p=1}^{n_1} [\, b_{1p}\,,[\, a_{2s'}\,, b_{3p}\,]\,] \;=\; -\tfrac{1}{2}\, a_{2s'}\,. \qquad (6.4.23)$$

On the other hand, if $a_{2s'}$ belongs to the non-trivial irreducible representation of D_ℓ or B_ℓ, then

$$\sum_{r=1}^{n_0} \sum_{s''=1}^{n_2'} (\, \Gamma^2(a_{0r})\Gamma^2(a_{0r}))_{s''s'}\, a_{2s''} \;=\; -\, \mu^{-1}(n_0/n_2')\, \gamma_2\, a_{2s'}\,, \qquad (6.4.24)$$

(see Appendix B (1)), where γ_2 is the Dynkin index of the non-trivial representation of D_ℓ or B_ℓ , which is given by(see Cornwell(1989):

$$\gamma_2 \;=\; (2\ell^2+\ell-1)/\{(2\ell-1)(\ell-1)\} \quad \text{or} \quad (2\ell^2+3\ell)/\{(2\ell-1)\ell\} \quad (6.4.25)$$

respectively, and μ is given by (see (6.4.7) and (6.4.8)):

$$\mu \;=\; (2\ell-1)/(\ell-1) \quad \text{or} \quad 4\ell/(2\ell-1)) \qquad (6.4.26)$$

respectively. Then, from (6.4.22) and (6.4.24)-(6.4.26), it follows that

$$\Sigma_{p=1}^{n_1} [\, b_{1p}, [\, a_{2s'}, b_{3p} \,]\,] = \{1/2(2\ell-1)\}\, a_{2s'} \quad , \text{ if } \tilde{\mathcal{L}}_0^{0(4)} = D_\ell \quad (6.4.27)$$

$$\Sigma_{p=1}^{n_1} [\, b_{1p}, [\, a_{2s'}, b_{3p} \,]\,] \;=\; (1/4\ell)\, a_{2s'} \quad , \text{ if } \tilde{\mathcal{L}}_0^{0(4)} = B_\ell \quad .(6.4.28)$$

A basis of the trivial representation of D_ℓ , that would satisfy (6.4.23), (6.4.1), and belong in $\tilde{\mathcal{L}}_2^{0(4)}$, can be chosen to be

$$\frac{1}{2\sqrt{\ell}}\, \underset{\sim}{c}^0 \;, \qquad (6.4.29)$$

where $\underset{\sim}{c}^0$ is the basis element of the even Abelian part of $A(2\ell\text{-}1/0)$ given by $\underset{\sim}{c}^0 = -\,(1/2\ell\text{-}1)\mathrm{diag}\,(\underset{\sim}{1}_{2\ell},1)$. Similarly for the trivial representation of B_ℓ the basis element can be chosen to be

$$\frac{1}{\sqrt{2(2\ell+1)}}\, \underset{\sim}{c}^0 \;, \qquad (6.4.30)$$

where $\underset{\sim}{c}^0$ is the basis element of the even Abelian part of $A(2\ell/0)$ given by $\underset{\sim}{c}^0 = -\,(1/2\ell)\mathrm{diag}\,(\underset{\sim}{1}_{2\ell+1},1)$. Note that $\underset{\sim}{c}^0$ commutes with any even element of $A(2\ell\text{-}1/0)$ or $A(2\ell/0)$.

6.4.2. Sugawara Construction for $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ in the "Ramond" case

Let us now introduce the following operators which will assumed to belong to a highest weight irreducible representation of $A^{(2)}(2\ell-1/0)$ or $A^{(4)}(2\ell/0)$ (as appropriate):

$$\Phi(t^{4j} \otimes a_{0r}) \quad (\text{for all } r = 1,\dots,n_0 \text{ and all } j\in \mathbf{Z}),$$

$$\Phi(t^{4j+2} \otimes a_{2s}) \quad (\text{for all } s = 1,\dots,n_2 \text{ and all } j\in \mathbf{Z}), \qquad (6.4.31)$$

$$\Phi(c)\ ,\ \Phi(d)\ ,$$

which are all considered to be even, and

$$\Phi(t^{4j+1} \otimes b_{1p}) \quad (\text{for all } p = 1,\dots,n_1 \text{ and all } j\in \mathbf{Z}),$$

$$\Phi(t^{4j+3} \otimes b_{3p}) \quad (\text{for all } p = 1,\dots,n_3 \text{ and all } j\in \mathbf{Z}), \qquad (6.4.32)$$

which are all considered to be odd. Because all the exponents j are assumed here to be integers, this will be referred to as the "Ramond" case. These operators will act in a carrier space $V(\Lambda)$ with a highest weight vector $\psi(\Lambda)$ such that:

$$\Phi(t^{4j} \otimes a_{0r})\, \psi(\Lambda) = 0 \qquad \text{for all } r = 1,\dots,n_0 \text{ and all } j > 0,$$

$$\Phi(t^{4j+p} \otimes a_{pr})\, \psi(\Lambda) = 0$$

(for all $j \geq 0$, and all $r = 1,\dots,n_p$, with $p = 1,2,3$),

$$\Phi(c)\, \psi(\Lambda) = c_\Lambda \psi(\Lambda)\ ,$$

$$\Phi(d)\, \psi(\Lambda) = \Lambda(d)\ . \qquad (6.4.33)$$

Let $\psi(\lambda)$ be any weight vector of V other than $\psi(\Lambda)$. Then there exists a positive integer K depending on the weight λ such that

$$\Phi(t^{4j+p} \otimes a_{pr})\,\psi(\lambda) = 0 \qquad (6.4.34)$$

for all $j > \frac{1}{4}(K-p)$, ($j \in \mathbf{Z}$), $p = 0,1,2,3$, and $r = 1,\ldots,n_p$. Finally the generalized Lie products are given by

$$\begin{aligned}
[\,\Phi(t^j \otimes a_{pr})\,,\,\Phi(t^k \otimes a_{p'r'})\,] &= \Phi(t^{j+k} \otimes [\,a_{pr}\,,\,a_{p'r'}\,]) \\
&\quad + j\delta^{j+k,o}B^0(\,a_{pr},a_{p'r'})\,\Phi(c)\,, \\
[\,\Phi(d)\,,\,\Phi(t^j \otimes a_{pr})\,] &= j\Phi(t^j \otimes a_{pr})\,, \\
[\,\Phi(c)\,,\,\Phi(t^j \otimes a_{pr})\,] &= 0\,, \\
[\,\Phi(c)\,,\,\Phi(d)\,] &= 0, \qquad (6.4.35)
\end{aligned}$$

for all j mod 4 = p, k mod 4 = p', for p,p' = 0,...,3, for r = 1,...,n_p, and r'=1,...,$n_{p'}$

We can now define the following highest weight representation of the Virasoro algebra using the operators of $A^{(2)}(2\ell-1/0)$ or $A^{(4)}(2\ell/0)$ discussed above:

$$\begin{aligned}
L_J = \frac{1}{\kappa}\Big\{ &\textstyle\sum_{j=-\infty}^{\infty} \sum_{r=1}^{n_0} :\Phi(t^{4J+4j} \otimes a_{0r})\,\Phi(t^{-4j} \otimes a^{\#}_{0r}): \\
&+ \textstyle\sum_{j=-\infty}^{\infty} \sum_{s=1}^{n_2} :\Phi(t^{4J+4j+2} \otimes a_{2s})\,\Phi(t^{-4j-2} \otimes a^{\#}_{2s}): \\
&- \textstyle\sum_{j=-\infty}^{\infty} \sum_{p=1}^{n_1} :\Phi(t^{4J+4j+1} \otimes b_{1p})\,\Phi(t^{-4j-1} \otimes b^{\#}_{1p}): \\
&- \textstyle\sum_{j=-\infty}^{\infty} \sum_{p=1}^{n_3} :\Phi(t^{4J+4j+3} \otimes b_{3p})\,\Phi(t^{-4j-3} \otimes b^{\#}_{3p}): \Big\} \\
&+ \nu\,\delta_{J,O}I \qquad (6.4.36)
\end{aligned}$$

The normal ordering , : : ,is defined by

$$:\Phi(t^{4J+4j+p} \otimes a_{pr})\Phi(t^{-4j-p} \otimes a^{\#}_{pr}): =$$

$$\Phi(t^{4J+4j+p} \otimes a_{pr})\Phi(t^{-4j-p} \otimes a^{\#}_{pr}) \ , \quad \text{if } 4J+4j+p < -4j-p,$$

$$\tfrac{1}{2}\{\Phi(t^{4J+4j+p} \otimes a_{pr})\Phi(t^{-4j-p} \otimes a^{\#}_{pr}) + \Phi(t^{-4j-p} \otimes a^{\#}_{pr})\Phi(t^{4J+4j+p} \otimes a_{pr})\}$$

$$\text{if } 4J+4j+p = -4j-p \ ,$$

$$\Phi(t^{-4j-p} \otimes a^{\#}_{pr})\Phi(t^{4J+4j+p} \otimes a_{pr}) \ , \quad \text{if } 4J+4j+p > -4j-p, \quad (6.4.37)$$

for all bosonic operators with $p = 0$ or 2 and with $r = 1,\ldots,n_0$ or n_2, and by

$$:\Phi(t^{4J+4j+p} \otimes b_{pr})\Phi(t^{-4j-p} \otimes b^{\#}_{pr}): =$$

$$\Phi(t^{4J+4j+p} \otimes b_{pr})\Phi(t^{-4j-p} \otimes b^{\#}_{pr}) \ , \quad \text{if } 4J+4j+p < -4j-p,$$

$$\tfrac{1}{2}\{\Phi(t^{4J+4j+p} \otimes b_{pr})\Phi(t^{-4j-p} \otimes b^{\#}_{pr}) - \Phi(t^{-4j-p} \otimes b^{\#}_{pr})\Phi(t^{4J+4j+p} \otimes b_{pr})\}$$

$$\text{if } 4J+4j+p = -4j-p \ ,$$

$$- \Phi(t^{-4j-p} \otimes b^{\#}_{pr})\Phi(t^{4J+4j+p} \otimes b_{pr}) \ , \quad \text{if } 4J+4j+p > -4j-p \quad (6.4.38)$$

for all fermionic operators with $p = 1$ and 3 with and $r = 1,\ldots,n_1$ or n_3.

In what follows we shall prove that the Virasoro generators defined as above and the $A^{(2)}(2\ell\text{-}1/0)$ (or $A^{(4)}(2\ell/0)$ as appropriate) superalgebra together form a semi-direct sum, with the L_J satisfying the relations (2.1). We shall also find the values of c_V and the eigenvalues of L_0 .

(a) Evaluation of the product $[\ L_J\ ,\ \Phi(t^{4j'} \otimes a_{0r'})\ ]\ \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

After some algebra it can be shown that

$$[\, L_J\, ,\, \Phi(t^{4j'} \otimes a_{0r'})\,]\ \psi(\lambda) =$$

$$-\frac{1}{\kappa}\Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{r=1}^{n_0}\Big\{\Phi(t^{4J+4j} \otimes a_{0r})\Phi(t^{-4j+4j'} \otimes [\, a_{0r},\, a_{0r'}\,])$$

$$+\ \Phi(t^{4J+4j+4j'} \otimes [\, a_{0r},\, a_{0r'}\,])\Phi(t^{-4j} \otimes a_{0r})\Big\}$$

$$-\Sigma_{j=-\infty}^{\infty} \Sigma_{s=1}^{n_2}\Big\{\Phi(t^{4J+4j+2} \otimes a_{2s})\Phi(t^{-4j-2+4j'} \otimes [\, a_{2s}\, ,\, a_{0r'}\,])$$

$$+\ \Phi(t^{4J+4j+2+4j'} \otimes [\, a_{2s},\ a_{0r'}\,])\Phi(t^{-4j-2} \otimes a_{2s})\Big\}$$

$$+\Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n_1}\Big\{\Phi(t^{4J+4j+1} \otimes b_{1p})\Phi(t^{-4j-1+4j'} \otimes [\, b_{3p}\, ,\, a_{0r'}\,])$$

$$+\ \Phi(t^{4J+4j+1+4j'} \otimes [\, b_{1p},\, a_{0r'}\,]\,)\ \Phi(t^{-4j-1}\otimes b_{3p})\Big\}$$

$$-\ \Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n_3}\Big\{\Phi(t^{4J+4j+3} \otimes b_{3p})\Phi(t^{-4j-3+4j'} \otimes [\, b_{1p},\, a_{0r'}])$$

$$+\ \Phi(t^{4J+4j+3+4j'} \otimes [\ b_{3p}\, ,\, a_{0r'}\,])\Phi(t^{-4j-3} \otimes b_{1p})\Big\}\Big\}\psi(\lambda)$$

$$-\ (4j'/\kappa)\ 2c_\Lambda\, \Phi(t^{4J+4j'} \otimes a_{0r'})\ \psi(\lambda)\ , \qquad (6.4.39)$$

where (6.4.35) and (6.4.36) have been used. In order to evaluate the infinite sums in (6.4.39), we have to introduce partial sums and make use of (6.4.34)(For more information on this method see Cornwell(1989), Knizhnik Zamolodchikov(1984), Goddard and Olive(1985), Todorov(1985), Goodman et al(1984,1984b)).

Then the infinite sums in (6.4.39) can be written as

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} \quad = \qquad (6.4.40)$$

$$\lim_{m\to\infty}\Big\{\frac{1}{\kappa}\Big\{-\Sigma_{j=j'-K/4}^{m}\Sigma_{r=1}^{n_0}\ \Phi(t^{4J+4j} \otimes a_{0r})\ \Phi(t^{-4j+4j'} \otimes [\ a_{0r},\, a_{0r'}])$$

$$- \sum_{j=-K/4}^{m} \sum_{r=1}^{n_0} \Phi(t^{4J+4j+4j'} \otimes [\, a_{0r}, a_{0r'}]) \, \Phi(t^{-4j} \otimes a_{0r})$$

$$- \sum_{j=j'-(K+2)/4}^{m} \sum_{s=1}^{n_2} \Phi(t^{4J+4j+2} \otimes a_{2s}) \, \Phi(t^{-4j-2+4j'} \otimes [\, a_{2s}, a_{0r'}])$$

$$+ \sum_{j=-(K+2)/4}^{m} \sum_{s=1}^{n_2} \Phi(t^{4J+4j+2+4j'} \otimes [\, a_{2s}, a_{0r'}]) \, \Phi(t^{-4j-2} \otimes a_{2s})$$

$$+ \sum_{j=j'-(K+1)/4}^{m} \sum_{p=1}^{n_1} \Phi(t^{4J+4j+1} \otimes b_{1p}) \, \Phi(t^{-4j-1+4j'} \otimes [\, b_{3p}, a_{0r'}])$$

$$+ \sum_{j=-(K+1)/4}^{m} \sum_{p=1}^{n_1} \Phi(t^{4J+4j+1+4j'} \otimes [\, b_{1p}, a_{0r'}]) \, \Phi(t^{-4j-1} \otimes b_{3p})$$

$$- \sum_{j=j'-(K+3)/4}^{m} \sum_{p=1}^{n_3} \Phi(t^{4J+4j+3} \otimes b_{3p}) \Phi(t^{-4j-3+4j'} \otimes [\, b_{1p}, a_{0r'}])$$

$$+ \sum_{j=-(K+3)/4}^{m} \sum_{p=1}^{n_3} \Phi(t^{4J+4j+3+4j'} \otimes [\, b_{3p}, a_{0r'}]) \Phi(t^{-4j-3} \otimes b_{1p}) \} \} \, \psi(\lambda).$$

Now observe that

$$[\, a_{0r} , a_{0r'}] = \sum_{r''=1}^{n_0} f_{rr'}^{r''} \, a_{0r''} \qquad (6.4.41)$$

where $f_{rr'}^{r''}$ are antisymmetric structure constants of D_ℓ or B_ℓ,

$$[\, a_{0r'} , a_{2s}] = \sum_{s'=1}^{n_2'} \Gamma^2(a_{0r'})_{s's} \, a_{2s'} , \qquad (6.4.42)$$

where $\underset{\sim}{\Gamma}^2(a_{0r'})$ denotes the non-trivial representation of D_ℓ or B_ℓ whose carrier space is the $\tilde{\mathcal{L}}_2^{0(4)}$ subspace, and where $\Gamma^2(a_{0r'})_{s's} = -\Gamma^2(a_{0r'})_{ss'}$. (It should be noted that $n_2' = 2\ell^2+\ell-1$ or $2\ell^2+3\ell$ in the two cases). Also

$$[\, a_{0r} , b_{1p}] = \sum_{p'=1}^{n_1} \Gamma^1(a_{0r'})_{p'p} \, b_{1p'} ,$$

$$[\, a_{0r} , b_{3p}] = \sum_{p'=1}^{n_3} \Gamma^3(a_{0r'})_{p'p} \, b_{3p'} , \qquad (6.4.43)$$

where $\underset{\sim}{\Gamma}^1(a_{0r'})$ and $\underset{\sim}{\Gamma}^3(a_{0r'})$ denote the representations of D_ℓ or B_ℓ whose carrier spaces are provided by the $\tilde{\mathcal{L}}_1^{0(4)}$ and $\tilde{\mathcal{L}}_3^{0(4)}$

subspaces respectively. It can be easily checked using the invariance property of the Killing form that $\Gamma^1(a_{0r'})_{pp'} = -\Gamma^3(a_{0r'})_{p'p}$. Defining $\underset{\sim}{D}(a_{0r'}) = \underset{\sim}{\Gamma}^1(a_{0r'})$, relations (6.4.41) can be written as

$$[\, a_{0r} \,,\, b_{1p} \,] \;=\; \Sigma^{n_1}_{p'=1}\, D(a_{0r'})_{p'p} b_{1p'} \,,$$

$$[\, a_{0r} \,,\, b_{3p} \,] \;=\; -\,\Sigma^{n_3}_{p'=1}\, D(a_{0r'})_{pp'}\, b_{3p'} \,. \qquad (6.4.44)$$

Using the above definitions, and after some algebra, (6.4.40) becomes

$$\lim_{m\to\infty}\{A_m \psi(\lambda)\}$$

$$= \lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'+1}\,\Sigma^{n_0}_{r=1}\Sigma^{n_0}_{r''=1}\, f^{r''}_{rr'}\Big\{\Phi(t^{4J+4j'}\otimes[\, a_{0r''} \,,\, a_{0r} \,])$$

$$+\,\Phi(t^{-4j}\otimes a_{0r})\Phi(t^{4J+4j+4j'}\otimes a_{0r''}) + (4J+4j+4j')\,\delta^{4J+4j',0}\, c_\Lambda\, B^0(a_{0r''},a_{0r})\Big\}\Big\}$$

$$-\,\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'+1}\,\Sigma^{n'_2}_{s=1}\,\Sigma^{n'_2}_{s''=1}\,\Gamma^2(a_{0r'})_{s's}\,\Big\{\Phi(t^{4J+4j'}\otimes[\, a_{2s} \,,\, a_{2s'} \,])$$

$$+\Phi(t^{-4j+2}\otimes a_{2s'})\Phi(t^{4J+4j+4j'+2}\otimes a_{2s})+(4J+4j+4j'+2)\;\delta^{4J+4j',0}c_\Lambda B^0(a_{2s},a_{2s'})\Big\}\Big\}$$

$$-\,\frac{1}{\kappa}\Big\{\,\Sigma^{m}_{j=m-j'+1}\,\Sigma^{n_1}_{p=1}\Sigma^{n_1}_{p'=1}\, D(a_{0r'})_{p'p}\,\Big\{\;\Phi(t^{4J+4j'}\otimes[\, b_{1p'} \,,\, b_{3p} \,])$$

$$-\Phi(t^{-4j-1}\otimes b_{3p})\Phi(t^{4J+4j+4j'+1}\otimes b_{1p'})+(4J+4j+4j'+1)\;\delta^{4J+4j',0}c_\Lambda B^0(b_{1p'},b_{3p})\Big\}\Big\}$$

$$-\,\frac{1}{\kappa}\Big\{\;\Sigma^{m}_{j=m-j'+1}\,\Sigma^{n_3}_{p=1}\,\Sigma^{n_3}_{p'=1}\;D(a_{0r'})_{pp'}\,\Big\{\Phi(t^{4J+4j'}\otimes[\, b_{3p'} \,,\, b_{1p} \,])$$

$$-\,\Phi(t^{-4j-3}\otimes b_{1p})\,\Phi(t^{4J+4j+4j'+3}\otimes b_{3p'})$$

$$+\;(4J+4j+4j'+3)\;\delta^{4J+4j',0}\, c_\Lambda\, B^0(\, b_{3p'} \,,\, b_{1p})\Big\}\Big\}\Big\}\;\psi(\lambda).(6.4.45)$$

It may now be observed that

$$\Sigma_{r=1}^{n_0} \Sigma_{r''=1}^{n_0} f_{rr'}^{r''} B^0(a_{0r''}, a_{0r}) = 0,$$

$$\Sigma_{s=1}^{n_2'} \Sigma_{s'=1}^{n_2'} \Gamma^2(a_{0r'})_{s's} B^0(a_{2s}, a_{2s'}) = 0, \qquad (6.4.46)$$

$$\Sigma_{p=1}^{n_1} \Sigma_{p'=1}^{n_1} D(a_{0r'})_{p'p} B^0(b_{1p'}, b_{3p}) = 0,$$

$$\Sigma_{p=1}^{n_3} \Sigma_{p'=1}^{n_3} D(a_{0r'})_{pp'} B^0(b_{3p'}, b_{1p}) = 0,$$

(because of the antisymmetry of $f_{rr'}^{r''}$ and the fact that the trace of any representation of a semi simple Lie algebra is zero). Consequently, for sufficiently large m, (4.45) becomes

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} \qquad (6.4.47)$$

$$= \lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{r=1}^{n_0} \Sigma_{r''=1}^{n_0} f_{rr'}^{r''} \Phi(t^{4J+4j'} \otimes [a_{0r''}, a_{0r}])\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{s=1}^{n_2'} \Sigma_{s'=1}^{n_2'} \Gamma^2(a_{0r'})_{s's} \Phi(t^{4J+4j'} \otimes [a_{2s}, a_{2s'}])\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{p=1}^{n_1} \Sigma_{p'=1}^{n_1} D(a_{0r'})_{p'p} \Phi(t^{4J+4j'} \otimes [b_{1p'}, b_{3p}])\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{p=1}^{n_3} \Sigma_{p'=1}^{n_3} D(a_{0r'})_{pp'} \{\Phi(t^{4J+4j'} \otimes [b_{3p'}, b_{1p}])\Big\}\Big\}\psi(\lambda).$$

It can be easily checked that (see Appendix B(1))

$$\Sigma_{r=1}^{n_0} \Sigma_{r''=1}^{n_0} f_{rr'}^{r''}[a_{0r''}, a_{0r}] = \mu^{-1} a_{0r'}, \qquad (6.4.48)$$

$$\Sigma_{s=1}^{n_2'} \Sigma_{s'=1}^{n_2'} \Gamma^2(a_{0r'})_{s's}[a_{2s}, a_{2s'}] = \gamma_2 \mu^{-1} a_{0r'} \qquad (6.4.49)$$

$$\Sigma_{p=1}^{n_1} \Sigma_{p'=1}^{n_1} D(a_{0r'})_{p'p}[b_{1p'}, b_{3p}])$$

$$= \Sigma_{p=1}^{n_3} \Sigma_{p'=1}^{n_3} D(a_{0r'})_{pp'}[b_{3p'}, b_{1p}] = -\gamma_D \mu^{-1} a_{0r'}, \qquad (6.4.50)$$

where μ and γ_2 are as in (6.4.25) and (6.4.26) and γ_D is the Dynkin index of the representation of D_ℓ or B_ℓ , provided by the $\tilde{\mathfrak{L}}_1^{0(4)}$ and $\tilde{\mathfrak{L}}_3^{0(4)}$ subspaces, given by

$$\gamma_D = \{2(\ell-1)\}^{-1} \quad \text{for } D_\ell , \qquad \gamma_D = (2\ell-1)^{-1} \quad \text{for } B_\ell . \qquad (6.4.51)$$

Thus applying (6.4.48)-(6.4.51) to (6.4.47) and taking the limit $m\to\infty$, we obtain (on noting that $\sum_{j=m-j'+1}^{m} 1 = j'$)

$$[L_J , \Phi(t^{4j'}\otimes a_{0r'})] \psi(\lambda) =$$

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} - (8j'/\kappa)\ c_\Lambda\, \Phi(t^{4J+4j'} \otimes a_{0r'})$$

$$= - j'\, \Phi(t^{4J+4j'} \otimes a_{0r'}) , \qquad (6.4.52)$$

provided that

$$\kappa = 8c_\Lambda+1 . \qquad (6.4.53)$$

(b) Evaluation of $[\, L_J\, ,\, \Phi(t^{4j'+2} \otimes a_{2s'})\,]\, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

(1) Consider first the case where $a_{2s'}$ is not a basis element of the trivial representation of D_ℓ or B_ℓ , providedby the subspace $\tilde{\mathcal{L}}_2^{0(4)}$. Relation (6.4.39) will have the form

$$
\begin{aligned}
&[\, L_J\, ,\ \Phi(t^{4j'+2} \otimes a_{2s'})\ ]\ \psi(\lambda) \\
&= \Big\{ -\frac{1}{\kappa}\Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{r=1}^{n_0} \Big\{ \Phi(t^{4J+4j} \otimes a_{0r}) \Phi(t^{-4j+4j'+2} \otimes [\, a_{0r},\, a_{2s'}\,]\,) \\
&\qquad + \Phi(t^{4J+4j+4j'+2} \otimes [\, a_{0r},\, a_{2s'}\,])\ \Phi(t^{-4j} \otimes a_{0r}) \Big\} \Big\} \\
&- \frac{1}{\kappa}\ \Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{s=1}^{n_2'} \Big\{ \Phi(t^{4J+4j+2} \otimes a_{2s})\ \Phi(t^{-4j-+4j'} \otimes\ [a_{2s}, a_{2s'}]) \\
&\qquad + \Phi(t^{4J+4j+4+4j'} \otimes [\, a_{2s}, a_{2s'}])\ \Phi(t^{-4j-2} \otimes a_{2s}) \Big\} \Big\} \\
&+ \frac{1}{\kappa}\Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n_1} \Big\{ \ \Phi(t^{4J+4j+1} \otimes b_{1p})\ \Phi(t^{-4j+1+4j'} \otimes [\, b_{3p}\, ,\, a_{2s'}]\,) \\
&\qquad + \Phi(t^{4J+4j+1+4j'} \otimes [\, b_{1p},\, a_{2s'}])\ \Phi(t^{-4j-1} \otimes b_{3p}) \Big\} \Big\} \\
&- \frac{1}{\kappa}\ \Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n_3} \Big\{ \Phi(t^{4J+4j+3} \otimes b_{3p})\ \Phi(t^{-4j-1+4j'} \otimes [\, b_{1p},\, a_{2s'}]\,) \\
&\qquad + \Phi(t^{4J+4j+5+4j'} \otimes [\, b_{3p},\, a_{2s'}])\ \Phi(t^{-4j-3} \otimes b_{1p}) \Big\} \Big\} \Big\}\ \ \psi(\lambda) \\
&\qquad - \{4(2j'+1)/\kappa\}\ c_\Lambda\, \Phi(t^{4J+4j'+2} \otimes a_{2s'})\ \psi(\lambda). \qquad (6.4.54)
\end{aligned}
$$

As in the previous case we shall use partial sums to evaluate the infinite sums in (6.4.54)

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} \qquad (6.4.55)$$

$$= \lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=j'+\frac{1}{4}(2-K)}\Sigma^{n_0}_{r=1}\Phi(t^{4J+4j}\otimes a_{0r})\,\Phi(t^{-4j+4j'+2}\otimes[a_{0r},a_{2s'}])$$

$$+\Sigma^{m}_{j=-\frac{1}{4}K}\Sigma^{n_0}_{r=1}\Phi(t^{4J+4j+4j'+2}\otimes[a_{0r},a_{2s'}])\,\Phi(t^{-4j}\otimes a_{0r})\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=j'-\frac{1}{4}K}\Sigma^{n_2}_{s=1}\Phi(t^{4J+4j+2}\otimes a_{2s})\,\Phi(t^{-4j+4j'}\otimes[a_{2s},a_{2s'}])$$

$$+\Sigma^{m}_{j=-\frac{1}{4}(K+2)}\Sigma^{n_2}_{s=1}\Phi(t^{4J+4j+4+4j'}\otimes[a_{2s},a_{2s'}])\,\Phi(t^{-4j-2}\otimes a_{2s})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=j'+\frac{1}{4}(1-K)}\Sigma^{n_1}_{p=1}\Phi(t^{4J+4j+1}\otimes b_{1p})\,\Phi(t^{-4j+1+4j'}\otimes[b_{3p},a_{2s'}])$$

$$+\Sigma^{m}_{j=-\frac{1}{4}(K+1)}\Sigma^{n_1}_{p=1}\Phi(t^{4J+4j+3+4j'}\otimes[b_{1p},a_{2s'}])\,\Phi(t^{-4j-1}\otimes b_{3p})\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=j'-\frac{1}{4}(K+1)}\Sigma^{n_3}_{p=1}\Phi(t^{4J+4j+3}\otimes b_{3p})\,\Phi(t^{-4j-1+4j'}\otimes[b_{1p},a_{2s'}])$$

$$+\Sigma^{m}_{j=-\frac{1}{4}(K+3)}\Sigma^{n_3}_{p=1}\Phi(t^{4J+4j+5+4j'}\otimes[b_{3p},a_{2s'}])\,\Phi(t^{-4j-3}\otimes b_{1p})\Big\}\Big\}\psi(\lambda).$$

Now since $[a_{2s}, a_{2s'}] \in \tilde{\mathcal{L}}_0^{0(4)}$, define the quantities $W^r_{ss'}$ by

$$[a_{2s}, a_{2s'}] = \Sigma^{n_0}_{r=1} W^r_{ss'}\, a_{0r}. \qquad (6.4.56)$$

Then, using (6.4.42) and the invariance property of the Killing form, it is easily shown that:

$$W^r_{ss'} = \Gamma^2(a_{0r})_{s's} . \qquad (6.4.57)$$

Similarly, defining the quantities $B^{p'}_{s'p}$ and $T^{p'}_{s'p}$ by

$$[a_{2s'}, b_{3p}] = \Sigma^{n_1}_{p'=1} B^{p'}_{s'p}\, b_{1p'}, \qquad (6.4.58)$$

$$[a_{2s'}, b_{1p}] = \Sigma^{n_3}_{p'=1} T^{p'}_{s'p}\, b_{3p'}, \qquad (6.4.59)$$

it can be easily shown using the invariance property of the Killing form that

$$B^{p'}_{s'p} = B^{p}_{s'p'} \quad \text{and} \quad T^{p'}_{s'p} = T^{p}_{s'p'} . \qquad (6.4.60)$$

With the help of (4.56)-(4.60), and after some algebra, (4.55) becomes

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} = \qquad (6.4.61)$$

$$\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'}\ \Sigma^{n_0}_{r=1}\ \Sigma^{n_2'}_{s=1}\ \Gamma^2(a_{0r})_{s's}\Big\{\Phi(t^{4J+4j'+2}\otimes[a_{0r}, a_{2s}])$$

$$+\ \Phi(t^{-4j-2}\otimes a_{2s})\ \Phi(t^{4J+4j+4j'+4}\otimes a_{0r})\Big\}\Big\}$$

$$+\frac{1}{\kappa}\Big\{\ \Sigma^{m}_{j=m-j'+1}\ \Sigma^{n_2'}_{s=1}\ \Sigma^{n_0}_{r=1}\ \Gamma^2(a_{0r})_{s's}\Big\{\Phi(t^{4J+4j'+2}\otimes[\ a_{2s}\ ,\ a_{0r}\ ])$$

$$+\ \Phi(t^{-4j}\otimes a_{0r})\ \Phi(t^{4J+4j+4j'+2}\otimes a_{2s})\Big\}\Big\}$$

$$+\frac{1}{\kappa}\Big\{\ \Sigma^{m}_{j=m-j'}\ \Sigma^{n_1}_{p=1}\ \Sigma^{n_1}_{p'=1}\ B^{p'}_{s'p}\ \Big\{\Phi(t^{4J+4j'+2}\otimes[\ b_{1p}\ ,\ b_{1p'}\ ])$$

$$-\ \Phi(t^{-4j-3}\otimes b_{1p'})\ \Phi(t^{4J+4j+4j'+5}\otimes b_{1p})\Big\}\Big\}$$

$$-\ \frac{1}{\kappa}\Big\{\ \Sigma^{m}_{j=m-j'+1}\ \Sigma^{n_3}_{p=1}\ \Sigma^{n_3}_{p'=1}\ T^{p'}_{s'p}\ \Big\{\Phi(t^{4J+4j'+2}\otimes[\ b_{3p}\ ,\ b_{3p'}\ ])$$

$$-\ \Phi(t^{-4j-1}\otimes b_{3p'})\ \Phi(t^{4J+4j+4j'+3}\otimes b_{3p})\ \Big\}\Big\}\Big\}\psi(\lambda)\ .$$

For sufficiently large m the second , fourth, sixth and eighth terms of (6.4.61) give zero and, of course, $\Sigma^{m}_{j=m-j'}\ 1 = j' + 1$. Now observe that

$$\Sigma^{n_0}_{r=1}\ \Sigma^{n_2'}_{s=1}\ \Gamma^2(a_{0r})_{s's}\,[\ a_{0r}\ ,\ a_{2s}\ ]\ =\ (n_0/n_2')\ \mu^{-1}\ \gamma_2\ a_{2s'}\ , \qquad (6.4.62)$$

and that

$$\Sigma^{n_1}_{p=1}\ \Sigma^{n_1}_{p'=1}\ B^{p'}_{s'p}\,[\ b_{1p}\ ,\ b_{1p'}\ ]\ =\ \Sigma^{n_1}_{p=1}\,[\ b_{1p}\ ,\ [\ a_{2s'}\ ,\ b_{3p}\ ],\qquad (6.4.63)$$

which is given by (6.4.27) or (6.4.28). Also,

$$\sum_{p=1}^{n_3}\sum_{p'=1}^{n_3} T_{s'p}^{p'} [\, b_{3p} , b_{3p'} \,] = \sum_{p=1}^{n_3}[\, b_{3p} , [\, a_{2s'} , b_{1p} \,]$$

$$= -\sum_{p=1}^{n_1} [\, b_{1p} , [\, a_{2s'} , b_{3p} \,] \,. \qquad (6.4.64)$$

(because of the Jacobi identities). Then, substituting (6.4.62)-(6.4.64) into (6.4.61) and after some algebra, we finally deduce that

$$[\, L_J , \Phi(t^{4j'+2} \otimes a_{2s'}) \,] \, \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} \; - \{(4j'+2)/\kappa\} \, 2c_\Lambda \, \Phi(t^{4J+4j'+2} \otimes a_{2s'}) \, \psi(\lambda)$$

$$= - \tfrac{1}{4}(4j'+2) \, \Phi(t^{4J+4j'+2} \otimes a_{2s'}) \, \psi(\lambda) \,, \qquad (6.4.65)$$

provided κ is given by (6.4.53).

(2) Consider the case where $a_{2s'}$ is a basis element of the trivial representation. If $a_{2s'}$ is the basis of the trivial representation of D_ℓ or B_ℓ mentioned above then the only non-zero contributions in the sums are obtained from the last four sums of (6.4.54). These can be dealt in exactly the same way as above, but now (6.4.63) is given by (6.4.23) instead of (6.4.27) or (6.4.28). Again we get the same results as in (6.4.65).

(c) Evaluation of $[\, L_J , \Phi(t^{4j'+1} \otimes b_{1p'}) \,] \, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

Using definition (6.4.36), properties (6.4.35), and applying the method of partial sums we get

$$[\, L_J\, ,\, \Phi(t^{4j'+1} \otimes b_{1p'})\,]\, \psi(\lambda) = \lim_{m\to\infty}\left\{A_m \psi(\lambda)\right\}$$

$$- \{(4j'+1)/\kappa\}\, 2c_\Lambda\, \Phi(t^{4J+4j'+1} \otimes b_{1p'})\, \psi(\lambda) \quad (6.4.66)$$

where, by properties (6.4.34), we find that

$$\lim_{m\to\infty}\left\{A_m \psi(\lambda)\right\} = \quad (6.4.67)$$

$$= \lim_{m\to\infty}\Big\{ -\frac{1}{\kappa}\Big\{ \Sigma^{m}_{j=j'+\frac{1}{4}(1-K)}\, \Sigma^{n_0}_{r=1} \Phi(t^{4J+4j} \otimes a_{0r})\, \Phi(t^{-4j+4j'+1} \otimes [\, a_{0r}, b_{1p'}])$$

$$+ \Sigma^{m}_{j=-\frac{1}{4}K}\, \Sigma^{n_0}_{r=1}\, \Phi(t^{4J+4j+4j'+1} \otimes [\, a_{0r}, b_{1p'}\,])\, \Phi(t^{-4j} \otimes a_{0r})\Big\}$$

$$- \frac{1}{\kappa}\Big\{ \Sigma^{m}_{j=j'-\frac{1}{4}(K+1)}\, \Sigma^{n_2}_{s=1} \Phi(t^{4J+4j+2} \otimes a_{2s})\, \Phi(t^{-4j+4j'-1} \otimes [a_{2s}, b_{1p'}])$$

$$+ \Sigma^{m}_{j=-\frac{1}{4}(K+2)}\, \Sigma^{n_2}_{s=1}\, \Phi(t^{4J+4j+3+4j'} \otimes [a_{2s}, b_{1p'}])\, \Phi(t^{-4j-2} \otimes a_{2s})\Big\}$$

$$+ \frac{1}{\kappa}\Big\{ \Sigma^{m}_{j=j'-\frac{1}{4}K}\, \Sigma^{n_1}_{p=1} \Phi(t^{4J+4j+1} \otimes b_{1p})\, \Phi(t^{-4j+4j'} \otimes [b_{3p}, b_{1p'}])$$

$$- \Sigma^{m}_{j=-\frac{1}{4}(K+1)}\, \Sigma^{n_1}_{p=1}\, \Phi(t^{4J+4j+2+4j'} \otimes [\, b_{1p}, b_{1p'}]\, \Phi(t^{-4j-1} \otimes b_{3p})\Big\}$$

$$- \frac{1}{\kappa}\Big\{ \Sigma^{m}_{j=j'-\frac{1}{4}(K+2)}\, \Sigma^{n_3}_{p=1} \Phi(t^{4J+4j+3} \otimes b_{3p})\, \Phi(t^{-4j-2+4j'} \otimes [\, b_{1p}, b_{1p'}])$$

$$- \Sigma^{m}_{j=-\frac{1}{4}(K+3)}\, \Sigma^{n_3}_{p=1}\, \Phi(t^{4J+4j+4+4j'} \otimes [\, b_{3p}, b_{1p'}])\, \Phi(t^{-4j-3} \otimes b_{1p})\Big\}\Big\}\psi(\lambda)\Big\}.$$

Defining the quantities $A^{r}_{p'p}$ by

$$[\, b_{3p}\, ,\, b_{1p'}\,] = \Sigma^{n_0}_{r=1}\, A^{r}_{p'p}\, a_{0r}\ , \quad (6.4.68)$$

and using (6.4.44) and the invariance property of the Killing form, we deduce that

$$A^{r}_{p'p} = D(a_{0r})_{pp'}\ . \quad (6.4.69)$$

Similarly defining $S^{s}_{pp'}$ and $R^{p}_{sp'}$ by

$$[\, b_{1p} \, , b_{1p'} \,] \;=\; \Sigma_{s=1}^{n_2} S^{s}_{pp'} a_{2s} \,, \qquad (6.4.70)$$

$$[\, a_{2s} \, , b_{1p'} \,] \;=\; \Sigma_{p=1}^{n_3} R^{p}_{sp'} b_{3p} \,, \qquad (6.4.71)$$

it is easily checked that

$$R^{p}_{sp'} \;=\; R^{p'}_{sp} \text{ and } S^{s}_{pp'} \;=\; -R^{p'}_{sp} \,. \qquad (6.4.72)$$

Using (6.4.68)-(6.4.72) in (6.4.67) and after some algebra we find that

$$\lim_{m\to\infty}\left\{A_m \psi(\lambda)\right\} \qquad (6.4.73)$$

$$= \lim_{m\to\infty}\Big\{\frac{1}{\kappa}\Big\{\Sigma_{j=m-j'}^{m} \Sigma_{r=1}^{n_0}\Sigma_{p=1}^{n_1} D(a_{0r})_{pp'}\Big\{\Phi(t^{4J+4j'+1}\otimes [a_{0r}, b_{1p}\,])$$

$$+ \Phi(t^{-4j-3} \otimes b_{1p})\, \Phi(t^{4J+4j+4j'+4} \otimes a_{0r})$$

$$- \frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{r=1}^{n_0} \Sigma_{p=1}^{n_1} D(a_{0r})_{pp'} \Big\{\Phi(t^{4J+4j'+1} \otimes [\, b_{1p} \, , a_{0r} \,])$$

$$+ \Phi(t^{-4j} \otimes a_{0r})\, \Phi(t^{4J+4j+4j'+1} \otimes b_{1p})\Big\}\Big\}$$

$$+ \frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m} \Sigma_{p=1}^{n_1(=n_3)} \Sigma_{s=1}^{n_2} R^{p'}_{sp}\Big\{\Phi(t^{4J+4j'+1} \otimes [\, a_{2s} \, , b_{3p} \,])$$

$$+ \Phi(t^{-4j-1} \otimes b_{3p})\, \Phi(t^{4J+4j+4j'+2} \otimes a_{2s})\Big\}\Big\}$$

$$- \frac{1}{\kappa}\Big\{\Sigma_{j=m-j'+1}^{m}\Sigma_{p=1}^{n_3}\Sigma_{s=1}^{n_2} R^{p}_{sp'} \Big\{\Phi(t^{4J+4j'+1} \otimes [\, b_{3p} \, , a_{2s} \,])$$

$$+ \Phi(t^{-4j-2} \otimes a_{2s})\, \Phi(t^{4J+4j+4j'+3} \otimes b_{3p})\Big\}\Big\}\psi(\lambda)\Big\}.$$

For sufficiently large m the second, fourth, sixth, and eighth terms give zero. In order to proceed the following relations are needed:

$$\Sigma_{r=1}^{n_0}\Sigma_{p=1}^{n_1} D(a_{0r})_{pp'} [\, a_{0r} , b_{1p} \,] \;=\; -\,\mu^{-1}\,(n_0/n_1)\,\gamma_D\, b_{1p'} \quad (6.4.74)$$

(see Appendix B(1)), where μ, γ_D are given by (6.4.26) and (6.4.51) for the two cases being considered,

$$\Sigma_{p=1}^{n_1(=n_3)} \Sigma_{s=1}^{n_2} R_{sp}^{p'} [\, a_{2s} , b_{3p} \,] = \Sigma_{s=1}^{n_2} [\, a_{2s} , [\, a_{2s} , b_{1p'} \,]\,]$$

$$= \Sigma_{p=1}^{n_1(=n_3)} [\, b_{3p} , [\, b_{1p} , b_{1p'} \,]\,] , \qquad (6.4.75)$$

where (6.4.70) and (6.4.71) have been used. Using the Jacobi identity, relation (6.4.68), and the fact that

$$\Sigma_{p=1}^{n_1(=n_3)} [\, b_{3p} , b_{1p} \,] \;=\; 0 , \qquad (6.4.76)$$

(6.4.75) becomes

$$\Sigma_{p=1}^{n_1(=n_3)} [\, b_{3p} , [\, b_{1p} , b_{1p'} \,]\,] \;=\; -\,\Sigma_{p=1}^{n_1(=n_3)} [\, b_{1p} , [\, b_{1p'} , b_{3p} \,]\,]$$

$$= -\,\Sigma_{r=1}^{n_0}\, \Sigma_{p=1}^{n_1(=n_3)}\, \Sigma_{r=1}^{n_0} A_{pp'}^{r} [\, a_{0r} , b_{1p} \,]$$

$$= \;\; \Sigma_{p=1}^{n_1(=n_3)} \Sigma_{r=1}^{n_0}\; D(a_{0r})_{pp'} [\, a_{0r} , b_{1p} \,]$$

$$= \; -\mu^{-1}\,(n_0/n_1)\,\gamma_D\, b_{1p'} \; . \quad (6.4.77)$$

Finally, substituting (6.4.74)-(6.4.77) in (6.4.73), we find (after some algebra) that

$$[\, L_J , \Phi(t^{4j'+1} \otimes b_{1p'}) \,]\, \psi(\lambda)$$

$$= \quad \lim_{m\to\infty}\{A_m \psi(\lambda)\} \;-\; \{(4j'+1)/\kappa\}\; 2c_\Lambda\, \Phi(t^{4J+4j'+1} \otimes b_{1p'})\, \psi(\lambda)$$

$$= -\,\tfrac{1}{4}(4j'+1)\;\; \Phi(t^{4J+4j'+1} \otimes b_{1p'})\, \psi(\lambda) \; , \qquad (6.4.78)$$

provided that $\kappa \;=\; 8c_\Lambda+1$.

(d) Evaluation of $[\, L_J\, ,\, \Phi(t^{4j'+3} \otimes b_{3p'})\,]\, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

Using definition (6.4.36), properties (6.4.35), and applying the method of partial sums we get

$$[\, L_J\, ,\, \Phi(t^{4j'+3} \otimes b_{3p'})\,]\, \psi(\lambda) = \lim_{m\to\infty}\{A_m \psi(\lambda)\}$$

$$- \{(4j'+3)/\kappa\}\, 2c_\Lambda\, \Phi(t^{4J+4j'+1} \otimes b_{1p'})\, \psi(\lambda) \quad (6.4.79)$$

where, by properties (6.4.34),

$$\lim_{m\to\infty}\{A_m \psi(\lambda)\} = \quad (6.4.80)$$

$$\lim_{m\to\infty}\Big\{- \frac{1}{\kappa}\Big\{\sum_{j=j'+\frac{1}{4}(3-K)}^{m} \sum_{r=1}^{n_0}\Phi(t^{4J+4j} \otimes a_{0r})\, \Phi(t^{-4j+4j'+3} \otimes [\, a_{0r},\, b_{3p'}\,])$$

$$+ \sum_{j=-\frac{1}{4}K}^{m} \sum_{r=1}^{n_0} \Phi(t^{4J+4j+4j'+3} \otimes [\, a_{0r},\, b_{3p'}\,])\, \Phi(t^{-4j} \otimes a_{0r})\Big\}$$

$$- \frac{1}{\kappa}\Big\{\sum_{j=j'+\frac{1}{4}(1-K)}^{m} \sum_{s=1}^{n_2}\Phi(t^{4J+4j+2} \otimes a_{2s})\, \Phi(t^{-4j+4j'+1} \otimes [a_{2s}, b_{3p'}])$$

$$+\sum_{j=-\frac{1}{4}(K+2)}^{m} \sum_{s=1}^{n_2} \Phi(t^{4J+4j+5+4j'} \otimes [a_{2s}, b_{3p'}])\, \Phi(t^{-4j-2} \otimes a_{2s})\Big\}$$

$$+ \frac{1}{\kappa}\Big\{\sum_{j=j'+\frac{1}{4}(2-K)}^{m} \sum_{p=1}^{n_1}\Phi(t^{4J+4j+1} \otimes b_{1p})\, \Phi(t^{-4j+4j'+2} \otimes [b_{3p}, b_{3p'}])$$

$$- \sum_{j=-\frac{1}{4}(K+1)}^{m} \sum_{p=1}^{n_1} \Phi(t^{4J+4j+4+4j'} \otimes [\, b_{1p},\, b_{3p'}]\, \Phi(t^{-4j-1} \otimes b_{3p})\Big\}$$

$$- \frac{1}{\kappa}\Big\{ \sum_{j=j'-\frac{1}{4}K}^{j=m} \sum_{p=1}^{n_3}\Phi(t^{4J+4j+3} \otimes b_{3p})\, \Phi(t^{-4j+4j'} \otimes [\, b_{1p},\, b_{3p'}])$$

$$-\sum_{j=-\frac{1}{4}(K+3)}^{m} \sum_{p=1}^{n_3}\Phi(t^{4J+4j+6+4j'} \otimes [\, b_{3p},\, b_{3p'}\,])\, \Phi(t^{-4j-3} \otimes b_{1p})\Big\}\Big\}\Big\}\psi(\lambda).$$

Defining the quantities $N^r_{pp'}$ by

$$[\, b_{1p}\, ,\, b_{3p'}\,] = \sum_{r=1}^{n_0} N^r_{pp'}\, a_{0r}\, , \quad (6.4.81)$$

it follows from relation (6.4.49) and the invariance property of the Killing form that

$$N^{r}_{pp'} = D(a_{0r})_{p'p} . \qquad (6.4.82)$$

Similarly defining $M^{p''}_{sp'}$ and $P^{S}_{pp'}$ by

$$[\, a_{2s} \,,\, b_{3p'} \,] = \Sigma^{n_1}_{p''=1} M^{p''}_{sp'}\, b_{1p''}. \qquad (6.4.83)$$

$$[\, b_{3p} \,,\, b_{3p'} \,] = \Sigma^{n_2}_{s=1} P^{s}_{pp'}\, a_{2s} . \qquad (6.4.84)$$

it is easily shown using the invariance property of the Killing form that

$$P^{S}_{pp'} = M^{p'}_{sp} . \qquad (6.4.85)$$

On applying (6.4.81)-(6.4.85) to (6.4.80), and after some algebra, we get

$$\lim_{m\to\infty}\left\{A_m \psi(\lambda)\right\} = \qquad (6.4.86)$$

$$\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'}\ \Sigma^{n_0}_{r=1}\Sigma^{n_1}_{p=1} D(a_{0r})_{p'p}\Big\{\Phi(t^{4J+4j'+3} \otimes [\, a_{0r} \,,\, b_{3p}\,])$$

$$+ \Phi(t^{-4j-1} \otimes b_{3p})\Phi(t^{4J+4j+4j'+4} \otimes a_{0r})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'+1}\ \Sigma^{n_0}_{r=1}\ \Sigma^{n_1}_{p=1} D(a_{0r})_{p'p}\Big\{\Phi(t^{4J+4j'+3} \otimes [\, b_{3p} \,,\, a_{0r}\,])$$

$$+ \Phi(t^{-4j} \otimes a_{0r})\Phi(t^{4J+4j+4j'+3} \otimes b_{3p})\Big\}\Big\}$$

$$+\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'}\Sigma^{n_1=n_3}_{p=1}\Sigma^{n_2}_{s=1} M^{p}_{sp'}\Big\{\ \Phi(t^{4J+4j'+3} \otimes [\, a_{2s},\, b_{1p}\,])$$

$$+ \Phi(t^{-4j-3} \otimes b_{1p})\Phi(t^{4J+4j+4j'+6} \otimes a_{2s})\Big\}$$

$$-\frac{1}{\kappa}\Big\{\Sigma^{m}_{j=m-j'}\Sigma^{n_3}_{p=1}\Sigma^{n_2}_{s=1} M^{p}_{sp'}\Big\{\ \Phi(t^{4J+4j'+3} \otimes [\, b_{1p} \,,\, a_{2s}\,])$$

$$+ \Phi(t^{-4j-2} \otimes a_{2s})\Phi(t^{4J+4j+4j'+5} \otimes b_{1p})\}\}\}\psi(\lambda)\}.$$

For sufficiently large m the second, fourth, sixth, and eighth terms give zero. Now observe that

$$\Sigma_{r=1}^{n_0} \Sigma_{p=1}^{n_3} D(a_{0r})_{p'p} [a_{0r} , b_{3p}]) = \mu^{-1} (n_0/n_3) \gamma_D b_{3p'} \quad (6.4.87)$$

(see Appendix B(1)). Also,

$$\Sigma_{p=1}^{n_1(=n_3)} \Sigma_{s=1}^{n_2} M_{sp'}^{p} [a_{2s} , b_{1p}]) = \Sigma_{s=1}^{n_2} [a_{2s} ,[a_{2s} , b_{3p'}]] \quad (6.4.88)$$

$$\Sigma_{p=1}^{n_1(=n_3)} [b_{1p}, [b_{3p} ,b_{3p'}]] = - \Sigma_{p=1}^{n_1=n_3} \Sigma_{s=1}^{n_2} P_{pp'}^{S} [a_{2s} , b_{1p}] \quad (6.4.89)$$

Then, using (6.4.85), from (6.4.89) and (6.4.88) we obtain

$$\Sigma_{s=1}^{n_2} [a_{2s} ,[a_{2s} , b_{3p'}]] = - \Sigma_{p=1}^{n_1(=n_3)} [b_{1p} , [b_{3p} ,b_{3p'}]]. \quad (6.4.90)$$

Using now the Jacobi identity on the left of (6.4.90) in relation with (6.4.76), it is easily seen that

$$\Sigma_{s=1}^{n_2} [a_{2s} ,[a_{2s} , b_{3p'}]] = \Sigma_{p=1}^{n_1(=n_3)} [b_{3p} , [b_{1p} ,b_{3p'}]] . \quad (6.4.91)$$

Finally, because of (6.4.81), (6.4.82) and (6.4.44), we obtain

$$\Sigma_{s=1}^{n_2} [a_{2s} ,[a_{2s} , b_{3p'}]] = \Sigma_{p=1}^{n_1(=n_3)} [b_{3p} , [b_{1p} ,b_{3p'}]] \quad (6.4.92)$$

$$= \Sigma_{p=1}^{n_1(=n_3)} \Sigma_{r=1}^{n_0} D(a_{0r})_{p'p} [b_{3p} , a_{0r}] = - \mu^{-1} (n_0/n_3) \gamma_D b_{3p'} .$$

Thus, on applying (6.4.87)-(6.4.92) on (6.4.86), we find after some algebra that

$$[\, L_J \,,\, \Phi(t^{4j'+3} \otimes b_{3p'}) \,]\, \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} \;-\; \{(4j'+3)/\kappa\}\; 2c_\Lambda\, \Phi(t^{4J+4j'+1} \otimes b_{3p'})\, \psi(\lambda)$$

$$= -\tfrac{1}{4}((4j'+3)\;\; \Phi(t^{4J+4j'+3} \otimes b_{3p'})\, \psi(\lambda)\;, \qquad (6.4.93)$$

provided that $\kappa = 8c_\Lambda + 1$.

The results of (a)-(d) can be summarized in the single formula:

$$[\, L_J \,,\, \Phi(t^j \otimes a_{pr}) \,] = -\tfrac{1}{4} j\, \Phi(t^{4J+j} \otimes a_{pr})\;, \qquad (6.4.94)$$

which is valid for all j such that j mod p = 4, all p = 0,1,2,3, all r = 1,2,...,n_p, and J = 0,±1,....., provided that

$$\kappa \;=\; 8c_\Lambda + 1\;. \qquad (6.4.95)$$

Having established the (6.4.94), it is a matter of lengthy but essentially trivial algebraic manipulations to check that

$$[\, L_J \,,\, L_K \,]\, \psi \;=\; (J-K)\, L_{J+K}\, \psi \qquad (6.4.96)$$

for any $\psi \in V$ and all J,K $\in$ Z, such that J+K $\neq$ 0. As usual the interest is in the case where J+K = 0, which we will now examine closely. Let $\psi(\Lambda)$ be the highest weight vector. From the definition (6.4.36) and the properties (6.4.35) we get

$$[\, L_J \,,\, L_{-J} \,]\, \psi(\Lambda)$$

$$= \frac{1}{4\kappa}\Big\{\Sigma_{j=-\infty}^{\infty}\; \Sigma_{r=1}^{n_0}\; \Big\{\; \Phi(t^{-4J+4j} \otimes a_{0r})\, \Phi(t^{4J-4j} \otimes a_{0r}^{\#})\;\; (4j)$$

$$- \Phi(t^{4j} \otimes a_{0r})\, \Phi(t^{-4j} \otimes a^{\#}_{0r}) \quad (-4J+4j)\Big\}$$

$$+ \Sigma^{\infty}_{j=-\infty} \Sigma^{n_2}_{s=1} \Big\{ \Phi(t^{-4J+4j+2} \otimes a_{2s})\, \Phi(t^{4J-4j-2} \otimes a^{\#}_{2s}) \quad (4j+2)$$

$$- \Phi(t^{4j+2} \otimes a_{2s})\, \Phi(t^{-4j-2} \otimes a^{\#}_{2s}) \quad (-4J+4j+2)\Big\}$$

$$- \Sigma^{\infty}_{j=-\infty} \Sigma^{n_1}_{p=1} \Big\{ \Phi(t^{-4J+4j+1} \otimes b_{1p})\, \Phi(t^{4J-4j-1} \otimes b^{\#}_{1p}) \quad (4j+1)$$

$$- \Phi(t^{4j+1} \otimes b_{1p})\, \Phi(t^{-4j-1} \otimes b^{\#}_{1p}) \quad (-4J+4j+1)\Big\}$$

$$- \Sigma^{\infty}_{j=-\infty} \Sigma^{n_3}_{p=1} \Big\{ \Phi(t^{-4J+4j+3} \otimes b_{3p})\, \Phi(t^{4J-4j-3} \otimes b^{\#}_{3p}) \quad (4j+3)$$

$$- \Phi(t^{4j+3} \otimes b_{3p})\, \Phi(t^{-4j-3} \otimes b^{\#}_{3p}) \quad (-4J+4j+3)\Big\} \ \psi(\Lambda). \quad (6.4.97)$$

Because of the properties (6.4.33), the sums that appear to be infinite are actually finite, and each one of them can be spit up in to three subsums as

$$\Sigma^{\infty}_{j=-\infty} \; = \; \Sigma^{J-1}_{j=0} + \Sigma^{J}_{j=J} + \Sigma^{\infty}_{j=J+1} \qquad (6.4.98)$$

Then the first and the second terms of (6.4.97) will give a contribution of the form

$$2J\Big\{\frac{1}{\kappa} \Sigma^{n_0}_{r=1} \Big\{ \Phi(t^0 \otimes a_{0r})\, \Phi(t^0 \otimes a^{\#}_{0r})\Big\}\psi(\Lambda) + \frac{4c_\Lambda n_0}{6\kappa} J(J^2-1)\psi(\Lambda), \qquad (6.4.99)$$

the second and the third terms of (6.4.97) will give a contribution of the form

$$\Big\{\frac{4c_\Lambda n_2}{6\kappa} J(J^2-1) + \frac{c_\Lambda n_2}{\kappa} J \Big\} \psi(\Lambda) , \qquad (6.4.100)$$

the fifth and the sixth terms of (6.4.97) will give a contribution of the form

$$\left\{- \frac{4c_\Lambda n_1}{6\kappa} J(J^2-1) - \frac{3c_\Lambda n_1}{4\kappa} J\right\} \psi(\Lambda) , \qquad (6.4.101)$$

and finally the last two terms of (6.4.97) will give a contribution of the form

$$\left\{ - \frac{4c_\Lambda n_3}{6\kappa} J(J^2-1) - \frac{3c_\Lambda n_3}{4\kappa} J\right\} \psi(\Lambda) . \qquad (6.4.102)$$

Consequently

$$[L_J , L_{-J}] \psi(\Lambda) = \{2J L_0 + \tfrac{1}{12} J(J^2-1)c_v\} \psi(\Lambda) , \quad (6.4.103)$$

where the values of the central charge of the Virasoro algebra are given by

$$c_V = \frac{8(m-n)c_\Lambda}{8c_\Lambda+1} = \frac{8(m-n)c_\Lambda}{\kappa} , \qquad (6.4.104)$$

with m and n being the even and odd dimension respectively of $A(2\ell-1/0)$ or $A(2\ell/0)$. Clearly m-n is the superdimension of $A(2\ell-1/0)$ or $A(2\ell/0)$, which is positive. $L_0 \psi(\Lambda)$ is found to be

$$L_0 \psi(\Lambda) = \{\frac{1}{\kappa}\Sigma_{r=1}^{n_0} \Phi(t^0 \otimes a_{0r}) \Phi(t^0 \otimes a_{0r}^{\#}) + \nu I \} \psi(\Lambda) \quad (6.4.105)$$

where ν is given by

$$\nu = \frac{(m-n)c_\Lambda}{4\kappa} \qquad (6.4.106)$$

and a_{0r} are the basis elements of D_ℓ or B_ℓ. From (6.4.105) we deduce that the eigenvalues of L_0 are given by

$$\mu \frac{C_2(\Lambda^0)}{\kappa} + \frac{(m-n)c_\Lambda}{4\kappa} , \qquad (6.4.107)$$

where $C_2(\Lambda^0)$ is the value of the second-order Casimir operator of the representation of D_ℓ or B_ℓ with highest weight Λ^0. The factor μ in front of the first term in (6.4.107) is the same as that of (6.4.26) and has to be inserted to allow for the appearance of $C_2(\Lambda^0)$. Clearly, any highest weight Λ of $A^{(2)}(2\ell-1/0)$ or $A^{(4)}(2\ell/0)$ is reduced on the Cartan subalgebra of $\tilde{\mathcal{L}}_0^{0(2)}$ (= D_ℓ or B_ℓ) to Λ^0.

6.4.3. Sugawara Construction for $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ in the "Neveu-Schwarz" case

The first question to be discussed is the appropriate definition of the "Neveu-Schwarz" superalgebras corresponding to twisted Kac-Moody superalgebras. For an untwisted Kac-Moody superalgebra the corresponding Neveu-Schwarz superalgebra is obtained by allowing the exponent j of the odd elements of the superalgebra to take values in $\mathbf{Z}+\frac{1}{2}$ (as was mentioned in Section 6.3), and thus merely corresponds to replacing j by $j+\frac{1}{2}$ in the expressions for the basis elements of the Kac-Moody superalgebra. However we cannot follow exactly the same procedure for the twisted superalgebras. The reason for this is associated with the closure of the twisted superalgebra when the exponent j of the odd elements takes half-integer values.

To see this consider the values of j for the odd elements of the $\tilde{\mathcal{L}}_1^{0(4)}$ and $\tilde{\mathcal{L}}_3^{0(4)}$ subspaces. These are given by j mod 4 = 1 and j mod 4 = 3, or, equivalently, by $j = 4k+1$ and $j = 4k+3$ (for $k \in \mathbf{Z}$) respectively. In the Neveu-Schwarz case one might think that they should be replaced by $j = 4k+1+\frac{1}{2}$ and $j = 4k+3+\frac{1}{2}$ respectively (where again $k \in \mathbf{Z}$). However with the values of j of the $\tilde{\mathcal{L}}_0^{0(4)}$ and $\tilde{\mathcal{L}}_2^{0(4)}$ subspaces given by $j = 4k$ and $j = 4k+2$ ($k \in \mathbf{Z}$) respectively, it can be easily checked that

$$[\, t^{4k+2} \otimes \tilde{\mathfrak{L}}_2^{0(4)} \,,\, t^{4k'+1+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_1^{0(4)} \,] \;\notin\; t^{4m+3+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_3^{0(4)} \,,$$

$$[\, t^{4k+2} \otimes \tilde{\mathfrak{L}}_2^{0(4)} \,,\, t^{4k'+3+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_3^{0(4)} \,] \;\notin\; t^{4m+1+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_1^{0(4)} \,,$$

(6.4.108)

$$[\, t^{4k+1+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_1^{0(4)} \,,\, t^{4k'+3+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_3^{0(4)} \,] \;\notin\; t^{4m} \otimes \tilde{\mathfrak{L}}_0^{0(4)} \,,$$

$$[\, t^{4k+1+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_1^{0(4)} \,,\, t^{4k'+1+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_1^{0(4)} \,] \;\notin\; t^{4m+2} \otimes \tilde{\mathfrak{L}}_2^{0(4)} \,,$$

$$[\, t^{4k+3+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_3^{0(4)} \,,\, t^{4k'+3+\frac{1}{2}} \otimes \tilde{\mathfrak{L}}_3^{0(4)} \,] \;\notin\; t^{4m+2} \otimes \tilde{\mathfrak{L}}_2^{0(4)} \,,$$

(6.4.109)

(for all k, k', and m taking values in $\mathbf{Z}$).

Nevertheless there is a unique solution to the problem of obtaining a closed superalgebra with half-integer exponents, as we shall now demonstrate. Consider the following values of j:

for the $\tilde{\mathfrak{L}}_0^{0(4)}$ subspace let $j = 2k$ ($k \in \mathbf{Z}$),

for the $\tilde{\mathfrak{L}}_2^{0(4)}$ subspace let $j = 2k+1$ ($k \in \mathbf{Z}$),

for the $\tilde{\mathfrak{L}}_1^{0(4)}$ subspace let $j = 2k+\frac{1}{2}$ ($k \in \mathbf{Z}$), (6.4.110)

for the $\tilde{\mathfrak{L}}_3^{0(4)}$ subspace let $j = 2k+1+\frac{1}{2}$ ($k \in \mathbf{Z}$).

The corresponding generalized loop algebra may then be taken to be

$$\sum_{p=0}^{q-1} \sum_{j=-\infty \text{ with } j \bmod \frac{1}{2}q = \frac{1}{2}p}^{\infty} \{\, t^j \otimes \tilde{\mathfrak{L}}_p^{0(4)} \} \,. \qquad (6.4.111)$$

Thus the "Neveu-Schwarz" version of the twisted superalgebra is the unique subalgebra of the untwisted one whose loop algebra given by (6.4.111)

We define operators

$\Phi(t^{2j} \otimes a_{0r})$ (for all $r = 1,\dots,n_0$ and all $j \in \mathbf{Z}$),

$\Phi(t^{2j+1} \otimes a_{2r})$ (for all $r = 1,\dots,n_2$ and all $j \in \mathbf{Z}$),

$\Phi(c)$, and $\Phi(d)$ (6.4.112)

to be <u>even</u>, and

$\Phi(t^{2j+\frac{1}{2}} \otimes a_{1r})$ (for all $r = 1,\dots,n_1$ and all $j \in \mathbf{Z}$),

$\Phi(t^{2j+1+\frac{1}{2}} \otimes a_{3r})$ (for all $r = 1,\dots,n_3$ and all $j \in \mathbf{Z}$) (6.4.113)

to be <u>odd</u>. These operators will be assumed to act in a carrier space $V(\Lambda)$ with a highest weight vector $\psi(\Lambda)$ such that:

$$\begin{aligned}
&\Phi(t^{2j} \otimes a_{0r})\, \psi(\Lambda) = 0 \quad (\text{for all } j > 0),\\
&\Phi(t^{2j+1} \otimes a_{2r})\, \psi(\Lambda) = 0 \quad (\text{for all } j \geq 0),\\
&\Phi(t^{2j+\frac{1}{2}} \otimes a_{1r})\, \psi(\Lambda) = 0 \quad (\text{for all } j \geq 0),\\
&\Phi(t^{2j+1+\frac{1}{2}} \otimes a_{3r})\, \psi(\Lambda) = 0 \quad (\text{for all } j \geq 0),\\
&\Phi(c)\, \psi(\Lambda) = c_\Lambda \psi(\Lambda),\\
&\Phi(d)\, \psi(\Lambda) = \Lambda(d)
\end{aligned} \qquad (6.4.114)$$

(for all $p = 0,1,2,3$ and $r = 1,\dots,n_p$).

Let $\psi(\lambda)$ be any weight vector of V other than $\psi(\Lambda)$,then there exists a positive integer K depending on the weight λ such that

$$\Phi(t^{2j+\frac{1}{2}p} \otimes a_{pr})\, \psi(\lambda) = 0 \qquad (6.4.115)$$

for all $j > \frac{1}{2}(K-\frac{1}{2}p)$ $(j \in \mathbf{Z})$, with $p = 0,1,2,3$, and $r = 1,\dots,n_p$. Finally the generalized Lie products are as in (6.4.35).

From these operators a highest weight representation of the Virasoro algebra can be obtained by the following definition:

$$L_J = \frac{1}{\kappa}\Big\{ \sum_{j=-\infty}^{\infty} \sum_{r=1}^{n_0} :\Phi(t^{2J+2j} \otimes a_{0r})\, \Phi(t^{-2j} \otimes a^{\#}_{0r}):$$

$$+ \sum_{j=-\infty}^{\infty} \sum_{s=1}^{n_2} :\Phi(t^{2J+2j+1} \otimes a_{2s})\, \Phi(t^{-2j-1} \otimes a^{\#}_{2s}):$$

$$- \sum_{j=-\infty}^{\infty} \sum_{p=1}^{n_1} :\Phi(t^{2J+2j+\frac{1}{2}} \otimes b_{1p})\, \Phi(t^{-2j-\frac{1}{2}} \otimes b^{\#}_{1p}):$$

$$- \sum_{j=-\infty}^{\infty} \sum_{p=1}^{n_3} :\Phi(t^{2J+2j+1+\frac{1}{2}} \otimes b_{3p})\Phi(t^{-4j-1-\frac{1}{2}} \otimes b^{\#}_{3p}: \Big\}$$

$$+ \eta\, \delta_{J,O} I \tag{6.4.116}$$

the normal ordering being as in (6.4.37) and (6.4.38). Repeating the same procedure as in subsections (a)-(d) above, we find that

$$[\, L_J\, ,\, \Phi(t^{2j+\frac{1}{2}p} \otimes a_{pr})\,]\, \psi(\lambda) = -\tfrac{1}{2}(2j+\tfrac{1}{2}p)\, \Phi(t^{2J+2j+\frac{1}{2}p} \otimes a_{pr}) \tag{6.4.117}$$

provided κ is given by

$$\kappa = 4c_\Lambda+1 \,. \tag{6.4.118}$$

Similarly it is found that

$$[\, L_J\, ,\, L_K\,]\, \psi(\lambda) = (J-K)\, L_{J+K}\, \psi(\lambda) \tag{6.4.119}$$

for $J+K \neq 0$ and any $\psi(\lambda)$ of $V(\Lambda)$.

As in the Ramond case, the main interest lies in the L_0 term. In this case relation (6.4.97) becomes

$$[\, L_J\, ,\, L_{-J}\,]\, \psi(\Lambda) =$$

$$\frac{1}{2\kappa}\Big\{ \Sigma_{j=-\infty}^{\infty} \Sigma_{r=1}^{n_0} \{ \Phi(t^{-2J+2j} \otimes a_{0r}) \Phi(t^{2J-2j} \otimes a_{0r}^{\#}) \ (2j)$$

$$- \Phi(t^{2j} \otimes a_{0r}) \Phi(t^{-2j} \otimes a_{0r}^{\#}) \ (-2J+2j)\Big\}$$

$$+ \Sigma_{j=-\infty}^{\infty} \Sigma_{s=1}^{n_2}\Big\{\Phi(t^{-2J+2j+1} \otimes a_{2s}) \Phi(t^{2J-2j-1} \otimes a_{2s}^{\#}) \ (2j+1)$$

$$- \Phi(t^{2j+2} \otimes a_{2s}) \Phi(t^{-2j-2} \otimes a_{2s}^{\#}) \ (-2J+2j+1)\Big\}$$

$$- \Sigma_{j=-\infty}^{\infty} \Sigma_{p=1}^{n_1}\Big\{\Phi(t^{-2J+2j+\frac{1}{2}} \otimes b_{1p}) \Phi(t^{2J-2j-\frac{1}{2}} \otimes b_{1p}^{\#})(2j+ \tfrac{1}{2})$$

$$+ \Phi(t^{2j+\frac{1}{2}} \otimes b_{1p}) \Phi(t^{-2j-\frac{1}{2}} \otimes b_{1p}^{\#})(-2J+2j+\tfrac{1}{2})\Big\}$$

$$- \Sigma_{j=-\infty}^{\infty}\Sigma_{p=1}^{n_3}\Big\{\Phi(t^{-2J+2j+1+\frac{1}{2}} \otimes b_{3p}) \Phi(t^{2J-2j-1-\frac{1}{2}} \otimes b_{3p}^{\#}(2j+1+\tfrac{1}{2})$$

$$+ \Phi(t^{2j+1+\frac{1}{2}} \otimes b_{3p})\Phi(t^{-2j-1-\frac{1}{2}} \otimes b_{3p}^{\#})(-2J+2j+1+\tfrac{1}{2})\Big\}\psi(\Lambda).$$

(6.4.120)

Because of the properties (6.4.34), the sums that appear to be infinite are actually finite, and each one of them can be spit up into three subsums as in (6.4.98). The first and the second terms of (6.4.120) will then give a contribution of the form

$$2J\Big\{\frac{1}{\kappa}\Sigma_{r=1}^{n_0} \{ \Phi(t^0 \otimes a_{0r}) \Phi(t^0 \otimes a_{0r}^{\#}) \}\psi(\Lambda) + \frac{4c_\Lambda n_0}{12\kappa} J(J^2-1)\psi(\Lambda),$$

(6.4.121)

the second and the third terms of (6.4.120) will give a contribution of the form

$$\Big\{ \frac{4c_\Lambda n_2}{12\kappa} J(J^2-1) + \frac{c_\Lambda n_2}{2\kappa} J \Big\} \ \psi(\Lambda) , \qquad (6.4.122)$$

the fifth and the sixth terms of (6.4.120) will give a contribution of the form

$$\left\{ -\frac{4c_\Lambda n_1}{12\kappa} J(J^2-1) - \frac{3c_\Lambda n_1}{8\kappa} J \right\} \psi(\Lambda), \tag{6.4.123}$$

and finally the last two terms of (6.4.120) will give a contribution of the form

$$\left\{ -\frac{4c_\Lambda n_3}{12\kappa} J(J^2-1) - \frac{3c_\Lambda n_3}{8\kappa} J \right\} \psi(\Lambda). \tag{6.4.124}$$

Consequently

$$[\, L_J , L_{-J} \,] \, \psi(\Lambda) = \{2J \, L_0 + \tfrac{1}{12} J(J^2-1)c_v\} \, \psi(\Lambda) \,, \tag{6.4.125}$$

where the values of the central charge of the Virasoro algebra are given by

$$c_V = \frac{4(m-n)c_\Lambda}{4c_{\Lambda+1}} = \frac{4(m-n)c_\Lambda}{\kappa} , \tag{6.4.126}$$

with m and n being the even and odd dimension respectively of $A(2\ell-1/0)$ or $A(2\ell/0)$. It follows that

$$L_0 \, \psi(\Lambda) = \{ \frac{1}{\kappa} \Sigma_{r=1}^{n_0} \, \Phi(t^0 \otimes a_{0r}) \, \Phi(t^0 \otimes a_{0r}^{\#}) + \eta \, I \} \, \psi(\Lambda), \tag{6.4.127}$$

where η is given by

$$\eta = \frac{(m-n)c_\Lambda}{8\kappa} \tag{6.4.128}$$

and the a_{0r} are the basis elements of D_ℓ or B_ℓ. From (6.4.127) we deduce that the eigenvalues of L_0 are given by

$$\mu \frac{C_2(\Lambda^0)}{\kappa} + \frac{(m-n)c_\Lambda}{8\kappa} , \tag{6.4.129}$$

where $C_2(\Lambda^0)$ is the value of the second-order Casimir operator of the representation of D_ℓ or B_ℓ with highest weight Λ^0 and κ is given by (6.4.118). The factor μ is given by (6.4.26).

6.5 Sugawara Construction for $C^{(2)}(\ell+1)$

In this Section we shall denote by a_{0r} ($r = 1,2,\ldots,n_0$) and b_{0p} ($p = 1,2,\ldots,n_1$) the even and odd basis elements of $\tilde{\mathcal{L}}_0^{0(2)} = B(0/\ell)$ respectively, where $n_0 = \ell(2\ell+1)$ is the dimension of C_ℓ and $n_1 = 2\ell$ is the dimension of the irreducible representation of C_ℓ provided by the odd subspace of $B(0/\ell)$. In addition the single even basis element (see chapter 4) of $\tilde{\mathcal{L}}_1^{0(2)}$ will be denoted by c' and the odd basis elements of $\tilde{\mathcal{L}}_1^{0(2)}$ will be denoted by b_{1p} ($p = 1,2,\ldots,n_1$). We shall work again with dual basis elements defined with respect to the Killing form of $C(\ell+1)$ that is given by It is always possible to choose the even basis elements of $\tilde{\mathcal{L}}_0^{0(2)}$ in such a way that $B^0(a_{0r},a_{0r'}) = -\delta_{rr'}$, which implies that we shall take

$$a_{0r}^{\#} = -a_{0r}. \tag{6.5.1}$$

For the odd basis elements of $\tilde{\mathcal{L}}_0^{0(2)}$ the situation is more complicated. The odd basis elements can be chosen such that the $n_1 \times n_1$ matrix $\underset{\sim}{B}$ with entries given by

$$(\underset{\sim}{B})_{pq} = B^0(b_{0p},b_{0q}) \quad , \tag{6.5.2}$$

(for all p , $q = 1,\ldots,n_1$ and where $B^0(\ ,\ )$ is the Killing form of $C(\ell+1)$), is antisymmetric and its nonzero entries take values +1 or -1. (For more information on this choice of odd basis see Cornwell(1989)). Defining $b_{0p}^{\#}$ by

$$b_{0p}^{\#} = \Sigma_{q=1}^{n_1}(\underset{\sim}{B})_{pq}\, b_{0q} \ , \tag{6.5.3}$$

it can be easily checked that $B^0(b_{0p},b_{0p}^{\#}) = 1$, for all $p = 1,2,\ldots,n_1$.

Turning now to the $\tilde{\mathcal{L}}_1^{0(2)}$ subspace, the even basis element may be chosen to be

$$c' = \frac{1}{2\sqrt{\ell}} \underset{\sim}{c}^0 \qquad (6.5.4)$$

where $\underset{\sim}{c}^0$ is the basis element of the Abelian part of $C(\ell+1)$ that is given by

$$\underset{\sim}{c}^0 = \underset{\sim}{e}_{11} - \underset{\sim}{e}_{22} \, . \qquad (6.5.6)$$

so that

$$(c')^{\#} = -c' \, , \qquad (6.5.5)$$

The 2ℓ odd basis elements of $\tilde{\mathcal{L}}_1^{0(2)}$ can be chosen in the same way as those of the $\tilde{\mathcal{L}}_0^{0(2)}$ subspace. To this end let $\underset{\sim}{B}'$ be the $n_1 \times n_1$ antisymmetric matrix with entries given by

$$(\underset{\sim}{B}')_{pq} = B^0(b_{0p}, b_{0q}) \quad (\text{for all } p \, , \, q = 1,\ldots,n_1 \,), \qquad (6.5.7)$$

where $B^0(\, , \,)$ is the Killing form of $C(\ell+1)$, and its nonzero entries take values +1 or -1. Consequently we can take $\underset{\sim}{B}' = \underset{\sim}{B}$, so $\underset{\sim}{B}$ will be used henceforth in this role. Thus

$$b_{1p}^{\#} = \sum_{q=1}^{n_1} (\underset{\sim}{B})_{pq} \, b_{1q} \, . \qquad (6.5.8)$$

6.5.1 Sugawara Construction for $C^{(2)}(\ell+1)$ in the "Ramond" case

As in the previous section let us introduce the following operators belonging to an irreducible highest weight representation of $C^{(2)}(\ell+1)$:

$$\Phi(t^{2j} \otimes a_{0r}) , \quad \Phi(t^{2j+1} \otimes c') , \quad \Phi(c) , \quad \Phi(d) \qquad (6.5.9)$$

(for all $r = 1,\ldots,n_0(=\ell(2\ell+1))$, and all $j \in \mathbf{Z}$), which are all even, and

$$\Phi(t^{2j} \otimes b_{0r}) , \quad \Phi(t^{2j+1} \otimes b_{1r}) \qquad (6.5.10)$$

(for all $r = 1,\ldots,n_1(=2\ell)$, and all $j \in \mathbf{Z}$), which are odd. As before, because all the exponents j are assumed here to be integers, this will be referred to as the "Ramond" case. These operators satisfy relations (6.4.35) with the appropriate values of j and k and act on a vector space $V(\Lambda)$, with highest weight vector $\psi(\Lambda)$, according to the prescription:

$$\Phi(t^{2j} \otimes a_{0r}) \psi(\Lambda) = 0 \quad , \text{ for all } r = 1,\ldots,n_0, \text{ and all } j > 0,$$

$$\Phi(t^{2j+1} \otimes c') \psi(\Lambda) = 0 \quad , \text{ for all } j \geq 0,$$

$$\Phi(t\backslash s(2j) \otimes b_{0p}) \psi(\Lambda) = 0 , \text{ for all } p = 1,\ldots,n_1 \text{ and all } j>0,$$

$$\Phi(t^{2j+1} \otimes b_{1p}) \psi(\Lambda) = 0 \quad , \text{ for all } p = 1,\ldots,n_1', \text{ and all } j\geq 0,$$

$$\Phi(c) \psi(\Lambda) = c(\Lambda) \psi(\Lambda) = c_\Lambda \psi(\Lambda) ,$$

$$\Phi(d)\ \psi(\Lambda) = \Lambda(d)\ \psi(\Lambda) \qquad \text{for all } j \in \mathbf{Z}. \qquad (6.5.11)$$

Let $\psi(\lambda)$ be any weight vector of V other than $\psi(\Lambda)$. Then there exists a positive integer K depending on the weight λ such that

$$\Phi(t^j \otimes a_{pr})\ \psi(\lambda) = 0 \qquad (6.5.12)$$

for all integers j such that j > K and j mod 2 = p, where p = 0 and 1, and where r = 1,...,n_p.

We define the generators L_J in the Ramond case by

$$L_J = \frac{1}{\kappa}\Big\{\Sigma_{j=-\infty}^{\infty} \Big\{\Sigma_{r=1}^{n_0} :\Phi(t^{2J+2j} \otimes a_{0r})\ \Phi(t^{-2j} \otimes a^{\#}_{0r}):$$

$$- \Sigma_{p=1}^{n_1} :\Phi(t^{2J+2j} \otimes b_{0p})\ \Phi(t^{-2j} \otimes b^{\#}_{0p}): \Big\}$$

$$+ \Sigma_{j=-\infty}^{\infty} \Big\{ :\Phi(t^{2J+2j+1} \otimes c'\,)\ \Phi(t^{-2j-1} \otimes c'^{\,\#}):$$

$$- \Sigma_{p=1}^{n_1'} :\Phi(t^{2J+2j+1} \otimes b_{1p})\ \Phi(t^{-2j-1} \otimes b^{\#}_{1p}): \Big\}\Big\}$$

$$+ \nu\, \delta_{J,O}\, I, \qquad (6.5.13)$$

where the normal ordering : : is defined as follows:

$$:\Phi(t^{2J+2j+p} \otimes a)\ \Phi(t^{-2j-p} \otimes a^{\#}): =$$

$$\Phi(t^{2J+2j+p} \otimes a)\ \Phi(t^{-2j-p} \otimes a^{\#}), \quad \text{if } 2J+2j+p < -2j-p,$$

$$\tfrac{1}{2}\{\Phi(t^{2J+2j+p} \otimes a)\ \Phi(t^{-2j-p} \otimes a^{\#}) + \Phi(t^{-2j-p} \otimes a^{\#})\ \Phi(t^{2J+2j+p} \otimes a)\}$$

$$\text{if } 2J+2j+p = -2j-p$$

$$\Phi(t^{-2j-p} \otimes a^{\#})\Phi(t^{2J+2j+p} \otimes a), \quad \text{if } 2J+2j+p > -2j-p, \qquad (6.5.14)$$

for all bosonic operators with $p = 0$ and 1 and $a = a_{0r}$ or c', and

$$: \Phi(t^{2J+2j+p} \otimes b_{pr})\, \Phi(t^{-2j-p} \otimes b^{\#}_{pr}) : =$$

$$\Phi(t^{2J+2j+p} \otimes b_{pr})\, \Phi(t^{-2j-p} \otimes b^{\#}_{pr}), \quad \text{if } 2J+2j+p < -2j-p,$$

$$\tfrac{1}{2}\{\Phi(t^{2J+2j+p} \otimes b_{pr})\Phi(t^{-2j-p} \otimes b^{\#}_{pr}) - \Phi(t^{-2j-p} \otimes b^{\#}_{pr})\Phi(t^{2J+2j+p} \otimes b_{pr})\}$$

$$\text{if } 2J+2j+p = -2j-p \,,$$

$$- \Phi(t^{-2j-p} \otimes b^{\#}_{pr})\, \Phi(t^{2J+2j+p} \otimes b_{pr}), \quad \text{if } 2J+2j+p > -2j-p, \tag{6.5.15}$$

for all fermionic operators with $p = 0$ and 1 and with $r = 1,\dots,n_1$.

In order to prove that (6.5.13) satisfy the Virasoro algebra and to find the values of ν , c_V and the eigenvalues of L_0 we proceed as before evaluating separately the products $[\, L_J \,, \Phi \,]$ from the various subspaces of the superalgebra.

(a) Evaluation of $[\, L_J \,, \Phi(t^{2j'} \otimes a_{0r'})\,]\, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

Using (6.5.13), (6.4.35), and the fact that c' commutes with all the even basis elements, we get

$$[\, L_J \,, \Phi(t^{2j'} \otimes a_{0r'})\,]\, \psi(\lambda)$$

$$= \frac{1}{\kappa}\Big\{\Sigma_{j=-\infty}^{\infty} \Big\{-\Big\{ \Sigma_{r=1}^{n_0} \;\Phi(t^{2J+2j} \otimes a_{0r})\, \Phi(t^{-2j+2j'} \otimes [\, a_{0r} \,, a_{0r'} \,])$$

$$\Phi(t^{2J+2j+2j'} \otimes [\, a_{0r} \,, a_{0r'} \,])\, \Phi(t^{-2j} \otimes a_{0r}) \Big\}\Big\}$$

$$+ \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} \Big\{ (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j} \otimes b_{0p})\, \Phi(t^{-2j+2j'} \otimes [\, b_{0q} \,, a_{0r'} \,])$$

$$+ (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+2j'} \otimes [\, b_{0p} , a_{0r'}])\, \Phi(t^{-2j} \otimes b_{0q})\}$$

$$+ \sum_{j=-\infty}^{\infty} \sum_{p=1}^{n_1'} \sum_{q=1}^{n_1'} \{(\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+1} \otimes b_{1p})\, \Phi(t^{-(2j+1)+2j'} \otimes [\, b_{1q}, a_{0r'}])$$

$$+ (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+1+2j'} \otimes [\, b_{1p} , a_{0r'}])\, \Phi(t^{-2j-1} \otimes b_{1q})\}\}\ \psi(\lambda)$$

$$- (2j'/\kappa)\ 2c_{\Lambda}\, \Phi(t^{2J+2j'} \otimes a_{0r'})\, \psi(\lambda) \ . \qquad (6.5.16)$$

Using the method of partial sums and the properties (6.5.12) the infinite sums above may be evaluated as

$$\lim_{m\to\infty}\{A_m \psi(\lambda)\} =$$

$$\lim_{m\to\infty}\{-\frac{1}{\kappa}\{\sum_{j=j'-\frac{1}{2}K}^{m} \sum_{r=1}^{n_0} \Phi(t^{2J+2j} \otimes a_{0r})\, \Phi(t^{-2j+2j'} \otimes [a_{0r} , a_{0r'}])$$

$$+ \sum_{j=-\frac{1}{2}K}^{m} \sum_{r=1}^{n_0} \Phi(t^{2J+2j+2j'} \otimes [\, a_{0r} , a_{0r'}])\, \Phi(t^{-2j} \otimes a_{0r})\}$$

$$+ \frac{1}{\kappa}\{\sum_{j=j'-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j} \otimes b_{0p})\, \Phi(t^{-2j+2j'} \otimes [\, b_{0q} , a_{0r'}])$$

$$+ \sum_{j=-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+2j'} \otimes [b_{0p}, a_{0r'}])\, \Phi(t^{-2j} \otimes b_{0q})\}$$

$$+ \frac{1}{\kappa}\{\sum_{j=j'-\frac{1}{2}(K+1)}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq} \Phi(t^{2J+2j+1} \otimes b_{1p}) \Phi(t^{-2j-1+2j'} \otimes [b_{1q}, a_{0r'}])$$

$$+ \sum_{j=-\frac{1}{2}(K+1)}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq} \Phi(t^{2J+2j+1+2j'} \otimes [b_{1p} , a_{0r'}]) \Phi(t^{-2j-1} \otimes b_{1q})\}\}\, \psi(\lambda)$$

(6.5.17).

Now we have to make use of the following relations:

$$\sum_{r=1}^{n_0} [\ a_{0r} , a_{0r'}] = \sum_{r=1}^{n_0} \sum_{r''=1}^{n_0} f_{rr'}^{r''}\, a_{0r''} , \qquad (6.5.18)$$

where $f_{rr'}^{r''}$ are antisymmetric structure constants of C_ℓ ,

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,[b_{0p},\,a_{0r'}\,] = -\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,\underset{\sim}{D}(a_{0r'})_{sp}b_{0s}\,, \tag{6.5.19}$$

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,[b_{1p},\,a_{0r'}\,] = -\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,\underset{\sim}{D}(a_{0r'})_{sp}b_{1s}\,, \tag{6.5.20}$$

where $\underset{\sim}{D}(a_{0r'})$ is the irreducible representation of C_ℓ, provided by the odd subspace of $B(0/\ell)$ and the odd subspace $\tilde{\mathcal{L}}_1^{0(2)}$. Then relation (6.5.17) becomes

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} =$$

$$\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}K}^{m}\sum_{r=1}^{n_0}\sum_{r''=1}^{n_0} f_{rr'}^{r''}\,\Phi(t^{2J+2j}\otimes a_{0r})\,\Phi(t^{-2j+2j'}\otimes a_{0r''})$$

$$+\sum_{j=-\frac{1}{2}K}^{m}\sum_{r=1}^{n_0}\sum_{r''=1}^{n_0} f_{rr'}^{r''}\,\Phi(t^{2J+2j+2j'}\otimes a_{0r''})\,\Phi(t^{-2j}\otimes a_{0r})\Big\}$$

$$-\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}K}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\underset{\sim}{D}(a_{0r'})_{sp}\Phi(t^{2J+2j}\otimes b_{0p})\Phi(t^{-2j+2j'})\otimes b_{0s})$$

$$+\sum_{j=-\frac{1}{2}K}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,\underset{\sim}{D}(a_{0r'})_{sp}\,\Phi(t^{2J+2j+2j'}\otimes b_{0s})\,\Phi(t^{-2j}\otimes b_{0q})\Big\}$$

$$-\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}(K+1)}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,\underset{\sim}{D}(a_{0r'})_{sp}\;\times$$

$$\Phi(t^{2J+2j+1}\otimes b_{1p})\Phi(t^{-2j-1+2j'}\otimes b_{1s})$$

$$+\sum_{j=-\frac{1}{2}(K+1)}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\,\underset{\sim}{D}(a_{0r'})_{sp}\;\times$$

$$\Phi(t^{2J+2j+1}\otimes b_{1s})\,\Phi(t^{-2j-1+2j'}\otimes b_{1q})\Big\}\Big\}\Big\}\psi(\lambda). \tag{6.5.21}$$

The first two terms of the above relation become

$$+\frac{1}{\kappa}\Big\{\sum_{j=m-j'+1}^{m}\sum_{r=1}^{n_0}\sum_{r''=1}^{n_0} f_{rr'}^{r''}\Big\{\Phi(t^{2J+2j'}\otimes[\,a_{0r}\,,\,a_{0r''}\,])$$

$$+\ \Phi(t^{-2j}\otimes\ a_{0r''})\,\Phi(t^{2J+2j+2j'}\otimes a_{0r})\Big\}\psi(\lambda)\ , \qquad (6.5.22)$$

where we have used the fact that

$$\sum_{r=1}^{n_0}\sum_{r''=1}^{n_0} f_{rr'}^{r''}\,B(a_{0r},a_{0r''}) \;=\; 0\,. \qquad (6.5.23)$$

The next two terms become

$$+\frac{1}{\kappa}\Big\{\sum_{j=m-j'+1}^{m}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{D}(a_{0r'})\underset{\sim}{B})_{sq}\ \times \qquad (6.5.24)$$

$$\Big\{-\Phi(t^{-2j}\otimes b_{0q})\,\Phi(t^{2J+2j+2j'}\otimes b_{0s}) + \Phi(t^{2J+2j'}\otimes[\,b_{0s}\,,\,b_{0q}\,])\Big\}\psi(\lambda)$$

where we have used the relation

$$\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}(\underset{\sim}{D}(a_{0r'})\underset{\sim}{B})_{sq}\,\underset{\sim}{B}_{sq} \;=\; \mathrm{tr}\,\underset{\sim}{D}(a_{0r'}) \;=\; 0\ , \qquad (6.5.25)$$

and the fact that $(\underset{\sim}{B}^{-1})_{pq} = (\underset{\sim}{B})_{qp}$. Finally the last two terms may be treated in the same way as (6.5.24) giving

$$\frac{1}{\kappa}\Big\{\sum_{j=m-j'+1}^{m}\sum_{q=1}^{n_1}\sum_{s=1}^{n_1}\ (\underset{\sim}{D}(a_{0r'})\underset{\sim}{B})_{sq}\ \ \times \qquad (6.5.26)$$

$$\Big\{-\Phi(t^{-2j}\otimes b_{1q})\Phi(t^{2J+2j+2j'}\otimes b_{1s}) + \Phi(t^{2J+2j'}\otimes[b_{1s}\,,\,b_{1q}\,])\Big\}\psi(\lambda).$$

Now observe that

$$\sum_{r=1}^{n_0}\sum_{r'''=1}^{n_0} f_{rr'}^{r''}\,[\,a_{0r}\,,\,a_{0r''}\,]$$

$$=\sum_{r'''=1}^{n_0}\mathrm{tr}(\underset{\sim}{a}\underset{\sim}{d}(a_{0r'})\underset{\sim}{a}\underset{\sim}{d}(a_{0r'''}))a_{0r'''} = \sum_{r'''=1}^{n_0} B^{C_\ell}(a_{0r'},a_{0r'''})\ a_{0r'''}$$

$$(6.5.27)$$

where $f_{rr'}^{r''} = ad(a_{0r})_{r''r'}$, $\underset{\sim}{a}\underset{\sim}{d}(\ \ )$ denotes the adjoint representation, tr the trace, and $B^{C_\ell}(\ \ ,\ )$ is the Killing form of C_ℓ. The relation between the Killing form of $C(\ell+1)$ and that of $B(0/\ell)$ is given by

$$B^{C(\ell+1)}(\ ,\ ) = \frac{2\ell}{2\ell+1} B^{B(0/\ell)}(\ ,\ ), \tag{6.5.28}$$

and since

$$B^{B(0/\ell)}(\ ,\ ) = (1-\gamma)B^{C_\ell}(\ ,\ ) = \frac{2\ell+1}{2(\ell+1)} B^{C_\ell}(\ ,\ ), \tag{6.5.29}$$

then

$$B^{C(\ell+1)}(\ ,\ ) = \{\ell/(\ell+1)\}\ B^{C_\ell}(\ ,\ ). \tag{6.5.30}$$

Thus

$$\sum_{r'''=1}^{n_0} B^{C_\ell}(a_{0r'}, a_{0r'''})\ a_{0r'''} = -\{(\ell+1)/\ell\}\ a_{0r'}. \tag{6.5.31}$$

Defining the quantities A^r_{sq} by

$$[\,b_{0s}, b_{0q}\,] = \sum_{r=1}^{n_0} A^r_{sq}\ a_{0r}, \tag{6.5.32}$$

it can be easily proved that

$$A^r_{sq} = (\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{sq}. \tag{6.5.33}$$

Thus from (6.5.31), (6.5.32) and (6.5.33)

$$\sum_{q=1}^{n_1}\sum_{s=1}^{n_1} (\underset{\sim}{D}(a_{0r'})\underset{\sim}{B})_{sq}\,[\,b_{0s}, b_{0q}\,] = \gamma_D\,\{(\ell+1)/\ell\}\ a_{0r'},$$

(6.5.34)

where γ_D is the Dynkin index of the 2ℓ-dimensional representation of C_ℓ provided by the odd part of $\tilde{\mathcal{L}}_0^{0(2)}$, which is given by

$$\gamma_D = (\{1/2(\ell+1)\}\ . \tag{6.5.35}$$

Similarly

$$\Sigma_{q=1}^{n_1} \Sigma_{s=1}^{n_1} (\underset{\sim}{D}(a_{0r'})\, \underset{\sim}{B})_{sq} [\, b_{1s}\, ,\, b_{1q}\,] \;=\; \gamma_D \{(\ell+1)/\ell\}\, a_{0r'}\, ,$$

(6.5.36)

where we have defined

$$[\, b_{1s}\, ,\, b_{1q}\,] \;=\; \Sigma_{r=1}^{n_0}\; B_{sq}^{r}\, a_{0r}\, , \tag{6.5.37}$$

and proved that

$$B_{sq}^{r} \;=\; (\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{sq}\; . \tag{6.5.38}$$

Applying all of the above results to (6.5.21) and taking the limit $m \to \infty$ we get

$$[\; L_J\, ,\, \Phi(t^{2j'} \otimes a_{0r'})\;]\; \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} - (2j'/\kappa)\;\; 2c_\Lambda\;\, \Phi(t^{2J+2j'} \otimes a_{0r'})\;\, \psi(\lambda)$$

$$= \;\; -j'\Phi(t^{2J+2j'} \otimes a_{0r'})\;\, \psi(\lambda)\; , \tag{6.5.39}$$

provided that

$$\kappa \;=\; 4c_\Lambda + 1\; . \tag{6.5.40}$$

(b) Evaluation of $[\, L_J\, , \Phi(t^{2j'} \otimes b_{0t})\,]\, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

$$[\, L_J\, , \Phi(t^{2j'} \otimes b_{0t})\,]\, \psi(\lambda)$$

$$= \frac{1}{\kappa}\Big\{ \Sigma_{j=-\infty}^{\infty} \Big\{ -\Big\{ \Sigma_{r=1}^{n_0} \Phi(t^{2J+2j} \otimes a_{0r})\, \Phi(t^{-2j+2j'} \otimes [\, a_{0r}\, , b_{0t}\,])$$

$$+ \Phi(t^{2J+2j+2j'} \otimes [\, a_{0r}\, , b_{0t}\,])\;\; \Phi(t^{-2j} \otimes a_{0r}) \Big\}\Big\}$$

$$+ \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} \Big\{ (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j} \otimes b_{0p})\, \Phi(t^{-2j+2j'} \otimes [\, b_{0q}\, , b_{0t}\,])$$

$$- (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+2j'} \otimes [\, b_{0p}\, , a_{0t}\,]\,)\, \Phi(t^{-2j} \otimes b_{0q}) \Big\}\Big\}$$

$$+ \Sigma_{j=-\infty}^{\infty} \Big\{ -\Phi(t^{2J+2j+1} \otimes c')\, \Phi(t^{-(2j+1)+2j'} \otimes [\, c'\, , b_{0t}\,])$$

$$- \Phi(t^{2J+2j+1+2j'} \otimes [\, c', b_{0t}\,])\, \Phi(t^{-(2j+1)} \otimes c') \Big\}$$

$$+ \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} \Big\{ (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+1} \otimes b_{1p})\, \Phi(t^{-(2j+1)+2j'} \otimes [\, b_{1q}\, , b_{0t}\,])$$

$$- (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j+1+2j'} \otimes [\, b_{1p}\, , b_{0t}\,])\, \Phi(t^{-2j-1} \otimes b_{1q}) \Big\}\Big\}\Big\}\; \psi(\lambda)$$

$$- (2j'/\kappa)\;\, 2c_{\Lambda}\, \Phi(t^{2J+2j'} \otimes b_{0t})\; \psi(\lambda)\;, \qquad (6.5.41)$$

where we have made use of the relation

$$\Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq} (\underset{\sim}{B})_{qt}\, b_{0p} = b_{0t} = -\Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq} (\underset{\sim}{B})_{pt}\, b_{0q}$$

Using the method of partial sums and the properties (6.5.14) we get

$$\lim_{m\to\infty}\{A_m \psi(\lambda)\} =$$

$$\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma_{j=j'-\frac{1}{2}K}^{m}\ \Sigma_{r=1}^{n_0}\ \Phi(t^{2J+2j}\otimes a_{0r})\ \Phi(t^{-2j+2j'}\otimes[a_{0r}\,,b_{0t}\,])$$

$$+\Sigma_{j=-\frac{1}{2}K}^{m}\ \Sigma_{r=1}^{n_0}\ \Phi(t^{2J+2j+2j'}\otimes[\,a_{0r}\,,b_{0t}\,])\ \Phi(t^{-2j}\otimes a_{0r})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\Sigma_{j=j'-\frac{1}{2}K}^{m}\Sigma_{p=1}^{n_1}\Sigma_{q=1}^{n_1}(\underset{\sim}{B}^{-1})_{pq}\ \Phi(t^{2J+2j}\otimes b_{0p})\Phi(t^{-2j+2j'}\otimes[b_{0q},\,b_{0t}\,])$$

$$-\Sigma_{j=-\frac{1}{2}K}^{m}\Sigma_{p=1}^{n_1}\Sigma_{q=1}^{n_1}\ (\underset{\sim}{B}^{-1})_{pq}\ \Phi(t^{2J+2j+2j'}\otimes[\,b_{0p},\,b_{0t}\,])\ \Phi(t^{-2j}\otimes b_{0q})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\Sigma_{j=j'-\frac{1}{2}(K+1)}^{m}\Sigma_{p=1}^{n_1}\Sigma_{q=1}^{n_1}\ (B^{-1})_{pq}\ \Phi(t^{2J+2j+1}\otimes b_{1p})\Phi(t^{-2j-1+2j'}\otimes[\,b_{1q}\,,b_{0t}\,])$$

$$-\ \Sigma_{j=-\frac{1}{2}(K+1)}^{m}\Sigma_{p=1}^{n_1}\Sigma_{q=1}^{n_1}\ (B^{-1})_{pq}\ \Phi(t^{2J+2j+1+2j'}\otimes[\,b_{1p}\,,\,b_{0t}\,]\,)\ \Phi(t^{-2j-1}\otimes b_{1q})$$

$$-\ \Sigma_{j=j'-\frac{1}{2}(K+1)}^{m}\ \Phi(t^{2J+2j+1}\otimes c')\ \Phi(t^{-2j-1+2j'}\otimes[\,c'\,,\,b_{0t}\,])$$

$$-\Sigma_{j=-\frac{1}{2}(K+1)}^{m}\Phi(t^{2J+2j+1+2j'}\otimes[c',\,b_{0t}\,])\Phi(t^{-2j-1}\otimes c'\,)\Big\}\Big\}\Big\}\psi(\lambda).\quad(6.5.42)$$

Now defining $A_s^{(t)}$, T_{tp} and T_{tq} by

$$[\,c'\,,\,b_{0t}\,]\ =\ \Sigma_{s=1}^{n_1}\ A_s^{(t)}\ b_{1s}\,,\qquad(6.5.43)$$

$$[\,b_{1p}\,,\,b_{0t}\,]\ =\ T_{tp}\,c'\,,\qquad(6.5.44)$$

it can be easily checked using the invariance property of the Killing form that

$$T_{tp} = -\,\Sigma_{s=1}^{n_1}\ A_s^{(t)}\ (\underset{\sim}{B})_{sp}\,,\qquad(6.5.45)$$

In addition, with the definitions

$$[\,a_{0r}\,,\,b_{0t}\,]\ =\ \Sigma_{s=1}^{n_1}\ \underset{\sim}{D}(a_{0r})_{st}\,b_{0s}\,,\qquad(6.5.46)$$

and

$$[\,b_{0p}\,,\,b_{0t}\,]\ =\ \Sigma_{r=1}^{n_0}\ A_{pt}^{r}\,a_{0r}\,,\qquad(6.5.47)$$

it can be checked that

$$A^{r}_{pt} = (\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{tp} . \qquad (6.5.48)$$

On applying (6.5.43)-(6.5.48) to relation (6.5.42) it is found that

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} =$$

$$\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}K}^{m}\ \sum_{r=1}^{n_0}\sum_{s=1}^{n_1} \underset{\sim}{D}(a_{0r})_{st}\ \Phi(t^{2J+2j}\otimes a_{0r})\Phi(t^{-2j+2j'}\otimes b_{0s}])$$

$$+ \sum_{j=-\frac{1}{2}K}^{m}\ \sum_{r=1}^{n_0}\sum_{s=1}^{n_1} \underset{\sim}{D}(a_{0r})_{st}\ \Phi(t^{2J+2j+2j'}\otimes b_{0s}\ ])\ \Phi(t^{-2j}\otimes a_{0r})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}K}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{r=1}^{n_0}(\underset{\sim}{B}^{-1})_{pq}(\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{qt}\Phi(t^{2J+2j}\otimes b_{0p})\ \Phi(t^{-2j+2j'}\otimes a_{0r})$$

$$-\sum_{j=-\frac{1}{2}K}^{m}\sum_{p=1}^{n_1}\sum_{q=1}^{n_1}\sum_{r=1}^{n_0}(\underset{\sim}{B}^{-1})_{pq}\ (\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{pt}\ \Phi(t^{2J+2j+2j'}\otimes a_{0r})\ \Phi(t^{-2j}\otimes b_{0q})\Big\}$$

$$+\frac{1}{\kappa}\Big\{-\sum_{j=j'-\frac{1}{2}(K+1)}^{m}\ \Big\{\sum_{s=1}^{n_1}A_s^{(t)}\ \Phi(t^{2J+2j+1}\otimes c')\ \Phi(t^{-2j-1+2j'}\otimes b_{1s})\Big\}$$

$$-\sum_{j=-\frac{1}{2}(K+1)}^{m}\Big\{\sum_{s=1}^{n_1}A_s^{(t)}\ \Phi(t^{2J+2j+1+2j'}\otimes b_{1s}\ ])\ \Phi(t^{-2j-1}\otimes c')\Big\}$$

$$+\sum_{j=j'-\frac{1}{2}(K+1)}^{m}\ \Big\{\sum_{q=1}^{n_1}\ A_s^{(t)}\ \Phi(t^{2J+2j+1}\otimes b_{1s})\ \Phi(t^{-2j-1+2j'}\otimes c')\Big\}$$

$$+\sum_{j=-\frac{1}{2}(K+1)}^{m}\Big\{\sum_{p=1}^{n_1}\ A_s^{(t)}\ \Phi(t^{2J+2j+1+2j'}\otimes c')\Phi(t^{-2j-1}\otimes b_{1s})\Big\}\Big\}\psi(\lambda)\Big\}.$$

(6.5.49)

In order to proceed we need the following relations, together with those defined above:

$$[\,c'\,,[\,c'\,,b_{0t}\,]\,] = \frac{1}{4\ell}\ b_{0t}\,, \qquad (6.5.50)$$

(which can be obtained from (6.5.5) and the fact that b_{0t} is an odd supermatrix),

$$\sum_{r=1}^{n_0}\sum_{q=1}^{n_1}\underset{\sim}{D}(a_{0r})_{qt}\,[\,a_{0r}\,,\,b_{0q}\,] \;=\; \sum_{r=1}^{n_0}\sum_{s=1}^{n_1}(\underset{\sim}{D}(a_{0r})\underset{\sim}{D}(a_{0r}))_{st}\,b_{0s}\,,$$

(6.5.51)

and

$$\sum_{r=1}^{n_0}\sum_{s=1}^{n_1}(\underset{\sim}{D}(a_{0r})\underset{\sim}{D}(a_{0r}))_{st}\,b_{0s} \;=\; -\{(2\ell+1)/4\ell\}\,b_{0t} \quad (6.5.52)$$

(see Appendix B(2)). Then, substituting (6.5.50) and (6.5.52) into (6.5.49), performing some algebraic manipulations, and taking the limit, it is found that

$$[\,L_J\,,\,\Phi(t^{2j'}\otimes b_{0t})\,]\,\psi(\lambda)$$

$$=\;\lim_{m\to\infty}\{A_m\psi(\lambda)\}\;-\;(2j'/\kappa)\;2c_\Lambda\,\Phi(t^{2J+2j'}\otimes b_{0t})\,\psi(\lambda)$$

$$=\;-j'\,\Phi(t^{2J+2j'}\otimes b_{0t})\,\psi(\lambda)\,, \quad (6.5.53)$$

provided that $\kappa = 4c_\Lambda + 1$.

(c) Evaluation of $[\,L_J\,,\,\Phi(t^{2j'+1}\otimes b_{1t})\,]\,\psi(\lambda)$, for $\psi(\lambda)\in V(\Lambda)$:

As in the previous case

$$[\,L_J\,,\,\Phi(t^{2j'+1}\otimes b_{1t})\,]\,\psi(\lambda)$$

$$=\;\lim_{m\to\infty}\{A_m\psi(\lambda)\}\;-\;\{(2j'+1)/\kappa\}\;2c_\Lambda\,\Phi(t^{2J+2j'+1}\otimes b_{1t})\,\psi(\lambda)\,,$$

(6.5.54)

where using the method of partial sums and the properties (6.5.12) we get

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\}\;=$$

$$\lim_{m \to \infty}\Big\{-\frac{1}{\kappa}\Big\{\sum_{j=j'+\frac{1}{2}(1-K)}^{m} \sum_{r=1}^{n_0}\Phi(t^{2J+2j}\otimes a_{0r})\,\Phi(t^{-2j+2j'+1}\otimes [\,a_{0r}\,,b_{1t}\,])$$

$$+\sum_{j=-\frac{1}{2}K}^{m} \sum_{r=1}^{n_0} \Phi(t^{2J+2j+2j'+1}\otimes [\,a_{0r}\,,b_{1t}\,])\,\Phi(t^{-2j}\otimes a_{0r})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\sum_{j=j'+\frac{1}{2}(1-K)}^{m} \sum_{p=1}^{n_1}\sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\,\Phi(t^{2J+2j}\otimes b_{0p})\,\Phi(t^{-2j+2j'+1}\otimes [\,b_{0q},\,b_{1t}\,])$$

$$-\sum_{j=-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1}\sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\,\Phi(t^{2J+2j+2j'+1}\otimes [\,b_{0p}\,,b_{1t}\,])\,\Phi(t^{-2j}\otimes b_{0q})\Big\}$$

$$+\frac{1}{\kappa}\Big\{\sum_{j=j'-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1}\sum_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\,\Phi(t^{2J+2j+1}\otimes b_{1p})\Phi(t^{-2j-1+2j'+1}\otimes [b_{1q}\,,b_{1t}])$$

$$-\sum_{j=-\frac{1}{2}(K+1)}^{m} \sum_{p=1}^{n_1}\sum_{q=1}^{n_1} (B^{-1})_{pq}\,\Phi(t^{2J+2j+2+2j'}\otimes [\,b_{1p}\,,b_{1t}\,]\,)\,\Phi(t^{-2j-1}\otimes b_{1q})$$

$$-\sum_{j=j'-\frac{1}{2}K}^{m} \Phi(t^{2J+2j+1}\otimes c')\,\Phi(t^{-2j+2j'}\otimes [\,c'\,,b_{1t}\,])$$

$$-\sum_{j=-\frac{1}{2}(K+1)}^{m} \Phi(t^{2J+2j+1+2j'}\otimes [\,c'\,,b_{1t}\,])\,\Phi(t^{-2j-1}\otimes c')\Big\}\Big\}\Big\}\psi(\lambda)\;. \tag{6.5.55}$$

Now defining $R_s^{(t)}$, T_{tp} and T_{tq} by

$$[\,c'\,,b_{1t}\,] \;=\; \sum_{s=1}^{n_1} R_s^{(t)}\, b_{0s}\,, \tag{6.5.56}$$

$$[\,b_{0q}\,,b_{1t}\,] \;=\; T_{tq}\, c'\,, \tag{6.5.57}$$

it can be easily checked using the invariance property of the Killing form that

$$T_{tp} = -\sum_{s=1}^{n_1} R_s^{(t)}\, (\underset{\sim}{B})_{sp}\,, \tag{6.5.58}$$

In addition, with the definitions

$$[\,a_{0r}\,,b_{1t}\,] \;=\; \sum_{s=1}^{n_1} \underset{\sim}{D}(a_{0r})_{st}\, b_{1s}\,, \tag{6.5.59}$$

and

$$[\,b_{1p}\,,b_{1t}\,] \;=\; \Sigma_{r=1}^{n_o} B_{pt}^{r}\, a_{0r}\,, \qquad (6.5.60)$$

it can be checked that

$$B_{pt}^{r} \;=\; (\underset{\sim}{B}\underset{\sim}{D}(a_{0r}\,))_{tp}\,. \qquad (6.5.61)$$

Thus relation (6.5.55) becomes

$$\lim_{m\to\infty}\{A_m\psi(\lambda)\} \qquad (6.5.62)$$

$$=\lim_{m\to\infty}\Big\{-\frac{1}{\kappa}\Big\{\Sigma_{j=j'+\frac{1}{2}(1-K)}^{m}\Sigma_{r=1}^{n_0}\,\Sigma_{s=1}^{n_1}\underset{\sim}{D}(a_{0r})_{st}\Phi(t^{2J+2j}\otimes a_{0r})\Phi(t^{-2j+2j'+1}\otimes b_{1s})$$

$$+\,\Sigma_{j=-\frac{1}{2}K}^{m}\,\Sigma_{r=1}^{n_0}\,\Sigma_{s=1}^{n_1}\,\underset{\sim}{D}(a_{0r})_{st}\,\Phi(t^{2J+2j+2j'+1}\otimes\, b_{1s}\,)\,\Phi(t^{-2j}\otimes\, a_{0r})\Big\}$$

$$+\,\frac{1}{\kappa}\Big\{\Sigma_{j=j'-\frac{1}{2}K}^{m}\,\Sigma_{p=1}^{n_1}\,\Sigma_{q=1}^{n_1}\,\Sigma_{r=1}^{n_0}\,(\underset{\sim}{B}^{-1})_{pq}\,(\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{qt}$$

$$\times\,\Phi(t^{2J+2j+1}\otimes\, b_{1p})\,\Phi(t^{-2j+2j'}\otimes\, a_{0r})$$

$$-\,\Sigma_{j=-\frac{1}{2}(K+1)}^{m}\,\Sigma_{p=1}^{n_1}\,\Sigma_{q=1}^{n_1}\,\Sigma_{r=1}^{n_0}\,(\underset{\sim}{B}^{-1})_{pq}\,(\underset{\sim}{B}\underset{\sim}{D}(a_{0r}))_{pt}$$

$$\times\,\Phi(t^{2J+2j+2j'+2}\otimes\, a_{0r})\,\Phi(t^{-2j-1}\otimes\, b_{1q})\Big\}$$

$$+\,\frac{1}{\kappa}\Big\{-\Sigma_{j=j'-\frac{1}{2}K}^{m}\Big\{\,\Sigma_{s=1}^{n_1}R_s^{(t)}\,\Phi(t^{2J+2j+1}\otimes\, c')\;\Phi(t^{-2j+2j'}\otimes\, b_{0s})\Big\}$$

$$-\,\Sigma_{j=-\frac{1}{2}(K+1)}^{m}\Big\{\,\Sigma_{s=1}^{n_1}\,R_s^{(t)}\,\Phi(t^{2J+2j+2+2j'}\otimes\, b_{0s})\Phi(t^{-2j-1}\otimes\, c')\Big\}$$

$$+\,\Sigma_{j=j'+\frac{1}{2}(1-K)}^{m}\Big\{\Sigma_{s=1}^{n_1}\,R_s^{(t)}\,\Phi(t^{2J+2j}\otimes\, b_{0s})\;\Phi(t^{-2j+2j'+1}\otimes\, c')\Big\}$$

$$+\,\Sigma_{j=-\frac{1}{2}K}^{m}\,)\Big\{\Sigma_{s=1}^{n_1}\,R_s^{(t)}\,\Phi(t^{2J+2j+1+2j'}\otimes\, c')\;\Phi(t^{-2j}\otimes\, b_{0s})\Big\}\Big\}\psi(\lambda)\Big\}\,.$$

In order to proceed we need the following relations together with those defined above:

$$[\, c' , [\, c' , b_{1t} \,]\,] \;=\; \frac{1}{4\ell}\, b_{1t} \,, \qquad (6.5.63)$$

(which can be obtained from (6.5.5) and the fact that b_{1t} is an odd supermatrix),

$$\sum_{r=1}^{n_0}\sum_{q=1}^{n_1} \underset{\sim}{D}(a_{0r})_{qt}\, [\, a_{0r} , b_{1q} \,] \;=\; \sum_{r=1}^{n_0}\sum_{s=1}^{n_1} (\underset{\sim}{D}(a_{0r})\underset{\sim}{D}(a_{0r}))_{st}\, b_{1s} \,,$$

$$(6.5.64)$$

$$\sum_{r=1}^{n_0} \sum_{s=1}^{n_1} (\underset{\sim}{D}(a_{0r})\underset{\sim}{D}(a_{0r}))_{st}\, b_{1s} \;=\; -\{(2\ell+1)/4\ell\}\, b_{1t} \qquad (6.5.65)$$

(see Appendix B(2)). Then substituting (6.5.63) and (6.5.65) into (6.5.62), after some further algebraic manipulations and taking the limit, we find that

$$[\, L_J \,, \Phi(t^{2j'+1} \otimes b_{1t}) \,]\, \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} \;-\; \frac{2j'+1}{\kappa}\, 2c_\Lambda\, \Phi(t^{2J+2j'+1} \otimes b_{1t})\, \psi(\lambda)$$

$$= -\tfrac{1}{2}(2j'+1)\;\; \Phi(t^{2J+2j'+1} \otimes b_{1t})\, \psi(\lambda) \,, \qquad (6.5.66)$$

provided that $\kappa = 4c_\Lambda + 1$.

(d) Evaluation of $[\, L_J\,, \Phi(t^{2j'+1} \otimes c'\,)\,]\, \psi(\lambda)$, for $\psi(\lambda) \in V(\Lambda)$:

Because

$$[\, L_J\,, \Phi(t^{2j'+1} \otimes c')\,]\, \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} - \{(2j'+1)/\kappa\}\; 2c_\Lambda\; \Phi(t^{2J+2j'+1} \otimes c')\; \psi(\lambda)\;, \tag{6.5.67}$$

where

$$A_m \psi(\lambda) = \tag{6.5.68}$$

$$= \frac{1}{\kappa}\Big\{\sum_{j=j'+\square(1-K)}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (B^{-1})_{pq}\, \Phi(t^{2J+2j} \otimes b_{0p})\, \Phi(t^{-2j+2j'+1} \otimes [\, b_{0q}\;, c'])$$

$$+ \sum_{j=-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (B^{-1})_{pq} \Phi(t^{2J+2j+2j'+1} \otimes [\, b_{0p},\, c']\,) \Phi(t^{-2j} \otimes b_{0q})$$

$$+ \sum_{j=j'-\frac{1}{2}K}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (B^{-1})_{pq}\, \Phi(t^{2J+2j+1} \otimes b_{1p})\, \Phi(t^{-2j+1+2j'} \otimes [\, b_{1q}\;, c'])$$

$$+ \sum_{j=-\frac{1}{2}(K+1)}^{m} \sum_{p=1}^{n_1} \sum_{q=1}^{n_1} (B^{-1})_{pq} \Phi(t^{2J+2j+2+2j'} \otimes [b_{1p}, c']) \Phi(t^{-2j-1} \otimes b_{1q})\Big\}\, \psi(\lambda).$$

The above sum can be evaluated observing that

$$[\, b_{0p}\,, c'\,] \;=\; -\sum_{s=1}^{n_1} A_s^{(p)}\, b_{1s}\;, \tag{6.5.69}$$

$$[\, b_{1p}\,, c'\,] \;=\; -\sum_{s=1}^{n_1} R_s^{(p)}\, b_{0s}\;, \tag{6.5.70}$$

Defining the quantities T_{pq} by

$$[\, b_{1p}\,, b_{0q}\,] \;=\; T_{pq}\, c'\;, \tag{6.5.71}$$

it can be easily deduced from the invariance property of the Killing form and (6.5.69) to(6.5.71) that generally

$$\Sigma_{s=1}^{n_1} A_s^{(q)} B_{ps} = -\Sigma_{s=1}^{n_1} R_s^{(p)} B_{sq} \ . \qquad (6.5.72)$$

With the help of (6.5.69) to (6.5.72), and after some standard algebraic manipulations, after taking the limit $m \to \infty$ (6.5.68) becomes

$$\Big\{ -\frac{(j'+1)}{\kappa} \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j'+1} \otimes [\,[\,c'\,,\, b_{1p}\,]\,,\, b_{1q}]\,)$$

$$-\frac{j'}{\kappa} \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, \Phi(t^{2J+2j'+1} \otimes [\,[\,c',\, b_{1p}\,]\,,\, b_{0q}])\Big\}\, \psi(\lambda).$$

(6.5.73)

But it can be shown (see Appendix B(3)) that

$$\Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, [\,[\,c'\,,\, b_{1p}\,]\,,\, b_{1q}]$$

$$= \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, [\,[\,c',\, b_{1p}\,]\,,\, b_{0q}] = \tfrac{1}{2}\,, \qquad (6.5.74)$$

Thus, finally,

$$[\, L_J\,, \Phi(t^{2j'+1} \otimes c')\,]\, \psi(\lambda)$$

$$= \lim_{m\to\infty}\{A_m \psi(\lambda)\} - \{(2j'+1)/\kappa\}\, 2c_\Lambda\, \Phi(t^{2J+2j'+1} \otimes c')\, \psi(\lambda)$$

$$= -\tfrac{1}{2}(2j'+1)\, \Phi(t^{2J+2j'+1} \otimes c')\, \psi(\lambda)\,, \qquad (6.5.75)$$

provided that $\kappa = 4c_\Lambda + 1$.

The results of sections (a) to (d) can be summarized in the formula

$$[\, L_J\,, \Phi(t^j \otimes a_{pr})\,] = -\tfrac{1}{2} j\, \Phi(t^{2J+j} \otimes a_{pr})\,, \qquad (6.5.76)$$

for all j such that j mod 2 = p, for p = 0 and 1, for all r = $1,2,\ldots,n_p$, and J = $0,\pm1,\ldots\ldots$, provided that

$$\kappa = 4c_\Lambda+1 . \qquad (6.5.77)$$

Having established the validity of (6.5.76) and (6.5.87), it is a matter of straightforward algebraic manipulations to check that

$$[L_J , L_K]\psi = (J\text{-}K) L_{J+K}\psi \qquad (6.5.78)$$

for any $\psi\in V$ and all $J,K \in Z$ such that $J+K \neq 0$.

As usual the interest is in the case where $J+K = 0$. It can be checked using the results (6.5.76) the generalized Lie products of the operators Φ, properties (6.5.12), relations (6.5.11) and (6.4.98) that

$$[L_J , L_{-J}]\psi(\Lambda)$$

$$= 2J\Big\{\frac{1}{\kappa}\Sigma_{r=1}^{n_0}\Phi(t^0\otimes a_{0r})\Phi(t^0\otimes a_{0r}^{\#}) - \frac{1}{\kappa}\Sigma_{p=1}^{n_1}\Phi(t^0\otimes b_{0p})\Phi(t^0\otimes b_{0p}^{\#})\Big\}\psi(\Lambda)$$

$$+\Big\{\frac{4(n_0\text{-}n_1)c_\Lambda}{12\kappa}J(J^2\text{-}1) + \frac{4(1\text{-}n_1)c_\Lambda}{12\kappa}J(J^2\text{-}1) + \frac{(1\text{-}n_1)c_\Lambda}{2\kappa}J\Big\}\ \psi(\Lambda)$$

$$= 2J\{L_0 + \frac{1}{12}J(J^2\text{-}1)c_v\}\ \psi(\Lambda) , \qquad (6.5.79)$$

where n_0 is the dimension of C_ℓ, $n_1 = 2\ell$,

$$c_v = \frac{4(m\text{-}n)c_\Lambda}{\kappa} , \qquad (6.5.80)$$

with κ being as in (6.5.78), and m and n being the even and odd dimensions of $C(\ell+1)$ respectively. Clearly m-n is the superdimension of $C(\ell+1)$.

$L_0\ \psi(\Lambda)$ is then found to be

$$L_0\psi(\Lambda) = \frac{1}{\kappa}\Big\{\Big\{\ \Sigma_{r=1}^{n_0}\Phi(t^0\otimes a_{0r})\Phi(t^0\otimes a_{0r}^{\#})$$

$$-\Sigma_{p=1}^{n_1}\Phi(t^0\otimes b_{0p})\Phi(t^0\otimes b_{0p}^{\#})\Big\} + \nu I\Big\}\ \psi(\Lambda) , \qquad (6.5.81)$$

where ν is given by

$$\nu = -\frac{\{\text{supdim}\, \underset{\sim}{\Gamma}(B(0/\ell))\}\, c_\Lambda}{4\kappa} = \frac{(1-n_1)c_\Lambda}{4\kappa}, \qquad (6.5.82)$$

$\underset{\sim}{\Gamma}(B(0/\ell))$ being the representation of $B(0/\ell)$ whose carrier space is the $\tilde{\mathcal{L}}_1^{0(2)}$ subspace and supdim $\underset{\sim}{\Gamma}(B(0/\ell))$ being its superdimension. From (6.5.81) we deduce that the eigenvalues of L_0 are given by

$$\{ (2\ell+1)/2\ell \} \frac{C_2(\Lambda^0)}{\kappa} + \frac{(1-n_1)c_\Lambda}{4\kappa}, \qquad (6.5.83)$$

where $C_2(\Lambda^0)$ is the value of the second order Casimir operator of the representation of $B(0/\ell)$ with highest weight Λ^0 and κ is given by (6.5.77). The factor in front of the first term in (6.5.83) is the same as that of (6.5.28) and has to be inserted to allow for the appearance of $C_2(\Lambda^0)$. Clearly, any highest weight Λ of $C^{(2)}(\ell+1)$ is reduced on the Cartan subalgebra of $\tilde{\mathcal{L}}_0^{0(2)}(=B(0/\ell))$ to Λ^0.

6.5.2 Sugawara Construction for $C^{(2)}(\ell+1)$ in the "Neveu-Schwarz" case

The possibility of constructing "Neveu-Schwarz" type superalgebras based on $C^{(2)}(\ell+1)$ (for ℓ = 1,2,3,...) will now be investigated. Consider first the values of j for the odd elements of the $\tilde{\mathcal{L}}_0^{0(2)}$ and $\tilde{\mathcal{L}}_1^{0(2)}$ subspaces. In the Ramond case these are given by j mod 2 = 0 and j mod 2 = 1 respectively, or, equivalently, by j = 2k and j = 2k+1 respectively (for $k \in \mathbf{Z}$). For the Neveu-Schwarz case the simplest modification would be to replace these by $j = 2k+\frac{1}{2}$ and $j = 2k+1+\frac{1}{2}$ respectively (for $k \in \mathbf{Z}$). However, it can be easily checked that

$$[\ t^{2k+\frac{1}{2}} \otimes \tilde{\mathcal{L}}_{0\ \mathrm{odd}}^{0(2)}\ ,\ t^{2k'+\frac{1}{2}} \otimes \tilde{\mathcal{L}}_{0\ \mathrm{odd}}^{0(2)}\ ] \notin t^{2m} \otimes \tilde{\mathcal{L}}_{0\ \mathrm{even}}^{0(2)}\ ,$$

$$[\ t^{2k+\frac{1}{2}} \otimes \tilde{\mathcal{L}}_{0\ \mathrm{odd}}^{0(2)}\ ,\ t^{2k'+1+\frac{1}{2}} \otimes \tilde{\mathcal{L}}_{1\ \mathrm{odd}}^{0(2)}] \notin t^{4m+1} \otimes \tilde{\mathcal{L}}_{1\ \mathrm{even}}^{0(2)}\ , \quad (6.5.84)$$

and so on, for k, k', and m all taking values in $\mathbf{Z}$, so that closure is not achieved with this choice. Closer examination shows that closure can only be obtained by taking the exponents to be j for the $\tilde{\mathcal{L}}_{0\ \mathrm{even}}^{0(2)}$ and $\tilde{\mathcal{L}}_{1\ \mathrm{even}}^{0(2)}$ subspaces and $j+\frac{1}{2}$ for the $\tilde{\mathcal{L}}_{0\ \mathrm{odd}}^{0(2)}$ and $\tilde{\mathcal{L}}_{1\ \mathrm{odd}}^{0(2)}$ subspaces (with $j \in \mathbf{Z}$ in all cases). The resulting loop algebra has the form

$$\Sigma_{p=0}^{1}\ \Sigma_{j=-\infty}^{\infty}\{\ t^{j+\frac{1}{2}\varepsilon} \otimes \tilde{\mathcal{L}}_{p}^{0(2)}\}\ , \quad (6.5.85)$$

where ε = 0 or 1 depending on whether the basis element of $\tilde{\mathcal{L}}_p^{0(2)}$ is even or odd respectively. However, it can be easily seen that this is precisely the loop algebra of the Neveu-Schwarz version of the untwisted superalgebra $C^{(1)}(\ell+1)$. That is, there are essentially <u>no</u> new "Neveu-Schwarz" type superalgebras based on the twisted superalgebras $C^{(2)}(\ell+1)$ (for ℓ =1,2,3,...). Of course, for the superalgebra (6.5.85) the results of Section III apply. That is, with L_J given by

$$L_J = \frac{1}{\kappa}\Big\{\Sigma_{j=-\infty}^{\infty}\Big\{ \Sigma_{r=1}^{n_0} : \Phi(t^{J+j}\otimes a_{0r})\, \Phi(t^{-j}\otimes a^{\#}_{0r}) :$$

$$- \Sigma_{p=1}^{n_1} : \Phi(t^{J+j+\frac{1}{2}} \otimes b_{0p})\Phi(t^{-(j+\frac{1}{2})}\otimes b^{\#}_{0p}) :$$

$$+ \; : \Phi(t^{J+j}\otimes c')\, \Phi(t^{-j}\otimes c'^{\#}) :$$

$$- \Sigma_{p=1}^{n_1'} : \Phi(t^{J+j+\frac{1}{2}}\otimes b_{1p})\, \Phi(t^{-(j+\frac{1}{2})}\otimes b^{\#}_{1p}) : \Big\}\Big\}$$

$$+ \eta\, \delta_{J,O}\, I ,$$

(6.5.86)

the Virasoro algebra is satisfied provided

$$c_V = \frac{2(m-n)c_\Lambda}{\kappa} , \qquad (6.5.87)$$

$$\kappa = 2c_\Lambda +1 , \qquad (6.5.88)$$

and

$$\eta = - \frac{nc_\Lambda}{8\kappa} , \qquad (6.5.89)$$

where m is the even dimension and n the odd dimension of $C(\ell+1)$.

CHAPTER 7

CONCLUSION

The main objective of this thesis was on one hand to give a complete description of the root system of the affine Kac-Moody superalgebras $B^{(1)}(0/\ell)$, $A^{(2)}(2\ell-1/0)$, $A^{(4)}(2\ell/0)$ and $C^{(2)}(\ell+1)$ and on the other hand to demonstrate the relation of these superalgebras with the Virasoro algebra.

There are still fields of research related with these affine superalgebras that one might look at. The explicit knowledge of their root subspaces as was found in chapter 4 will facilitate for example, research in the classification of involutive automorpisms of these superalgebras and determination of their possible real forms.

Another field of research is related to the construction of Vertex operators of affine Kac-Moody algebras. Such construction have already appeared in the literature (see for example Frappat et all (1988)) but their use in conformal field theory is still at a speculative level.

An interesting problem that remains open is that of unitarity of the representations of these affine superalgebras, mainly in view of possible unitary representations of the Virasoro algebra that can be obtained by them. This is itself an extensive topic and is directly related to the definition of a consistent adjoint operation.

Let us finally comment on the results obtained from the Sugawara construction

The analysis of chapter 6 shows that all the Sugawara type constructions in the Ramond case can be put in a general form as follows:

$$[L_J , \Phi(t^j \otimes a_{pr})] = -(j/q) \; \Phi(t^{qJ+j} \otimes a_{pr}) \qquad (7.1)$$

for all j such that j mod q = p, (for p = 0,...,q-1), and

$$c_V = \{2qc_\Lambda(m-n)/\kappa\} \; , \qquad (7.2)$$

provided that the value of the normalization constant of the Virasoro generators is given by

$$\kappa = 2qc_\Lambda + 1 \; . \qquad (7.3)$$

The corresponding results for the Ramond type untwisted affine Kac-Moody superalgebras, the twisted affine Kac-Moody algebras, and the untwisted affine Kac-Moody algebras may then all be regarded as being given by special cases of these formulae. Indeed, with q = 1 these relations reduce to the corresponding relations obtained for the untwisted affine Kac-Moody superalgebras. Similarly, when the superalgebra involves no odd part but $q \neq 1$, then we obtain the corresponding relations for the twisted affine Kac-Moody algebras (see Tsohantjis and Cornwell(1990)). Finally, when the superalgebra involves no odd part and q = 1, these relations reduce to the corresponding relations of the untwisted affine Kac-Moody algebras (see Tsohntjis and Cornwell(1990)). All of the above considerations would imply that the "universal" formula for the Virasoro central charge that has recently been obtained (see Gorman et.al.(1989)) can be extended to include the above values too.

By contrast, in the Neveu-Schwarz case there are no such general relations. Indeed, for the superalgebras $C^{(2)}(\ell+1)$, there are essentially no Neveu-Schwarz versions, while for $A^{(2)}(2\ell-1/0)$ and $A^{(4)}(2\ell/0)$ the results (4.117), (4.118), and (4.126) indicate that there are no natural general formulae which reduce to those of the untwisted case when $q = 1$.

From the relations obtained in chapter 6 for the eigenvalues of L_0 and c_v, it is possible to calculate numerically all these eigenvalues for the standard irreducible representations. By making use of (5.18), (5.19), (5.20), (5.25), and the expressions of δ found in chapter 4, we can calculate c_Λ which will give us c_v immediately be means of (7.1-3). Then (6.3.21), (6.3.24), (6.4.107), (6.4.129) and (6.5.83) can be found by calculating the eigenvalue of the Casimir operator involved in these relations, the latter being a trivial procedure for irreducible highest weight representations.

Another interesting observation from (6.5.82) is that the Ramond construction for $C^{(2)}(2)$ gives zero value for c_V and thus the Virasoro algebra is reduced to a "rigid" conformal algebra. This is exactly the same result that has been obtained previously (see Jarvis and Zhang(1988),(1989)), for the case of $C^{(1)}(2)$. Unitarity of a highest weight irreducible representation of the Virasoro algebra would constrained the eigenvalues (5.83) of L_0 to be such that

$$\frac{3C_2(\Lambda^0)}{2\kappa} - \frac{c_\Lambda}{4\kappa} = 0 \qquad (7.4)$$

thus constraining the value of c_Λ to be $c_\Lambda = 6C_2(\Lambda^0)$.

We should mention that the relationship of L_0 to the operator $\Phi(d)$ can be easily obtained, since for $q = 2$ and 4 the quantity $\{L_0 + (1/q)\Phi(d)\}$ commutes with all the elements of the twisted superalgebras considered here both in the Ramond and in the Neveu-Schwarz cases. Thus

$$L_0\psi(\Lambda) = \{-(1/q)\Phi(d) + (1/q)\Lambda(d) + h\}\psi(\Lambda)\,, \qquad (7.5)$$

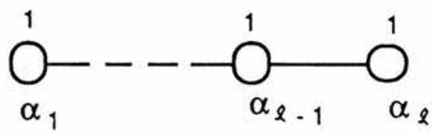
where $\Lambda(d)$ is the eigenvalue of $\Phi(d)$ and h is the eigenvalue of L_0.

TABLE I

Finite irreducible reduced root system and their Dynkin diagrams

In this table ε_i (i = 1,..., n) denote orthonormal unit vectors of $E = \mathbb{R}^n$. The numbers above the vertices of the Dynkin diagrams are the coefficients of the expansion of the heightest root in terms of the simple roots.

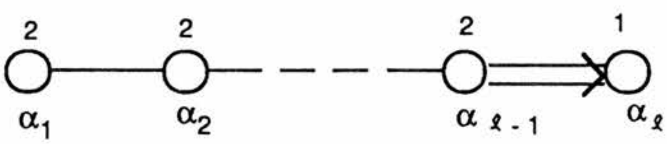
Type A_ℓ ($\ell \geq 1$)



(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell$

(ii) $\Delta = \{ \pm(\varepsilon_i - \varepsilon_j)$, for $1 \leq i < j \leq \ell + 1 \}$

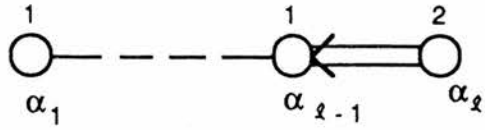
Type B_ℓ ($\ell \geq 1$)



(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$ $\alpha_\ell = \varepsilon_\ell$

(ii) $\Delta = \{ \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $\pm \varepsilon_i$, for $1 \leq i \leq \ell \}$

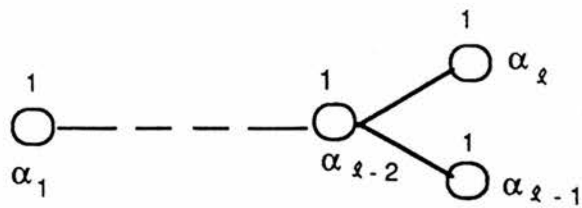
Type C_ℓ $(\ell \geq 1)$



(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$ $\alpha_\ell = 2\varepsilon_\ell$

(ii) $\Delta = \{ \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $\pm 2\varepsilon_i$, for $1 \leq i \leq \ell \}$

Type D_ℓ $(\ell \geq 3)$

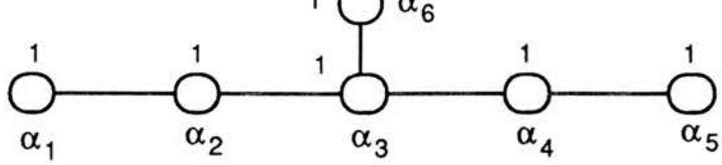


(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$ $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$

(ii) $\Delta = \{ \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell \}$

For the next three types let $E = \mathbb{R}^8$ and we define $e_i = \varepsilon_i - (1/9) \sum_{i=0}^{8} \varepsilon_i$, for $\sum_{i=0}^{8} e_i = 0$

Type E_6



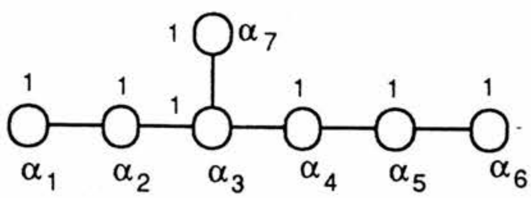
(i) Basis: $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le 5)$ $\alpha_6 = e_4 + e_5 + e_6$

(ii) $\Delta = \{ \pm(e_i - e_j)$, for $1 \le i < j \le 6$,

$\pm(e_i + e_j + e_k)$, for $1 \le i < j < k \le 6$,

$\pm(\sum_{i=0}^{6} e_i) \}$

Type E_7



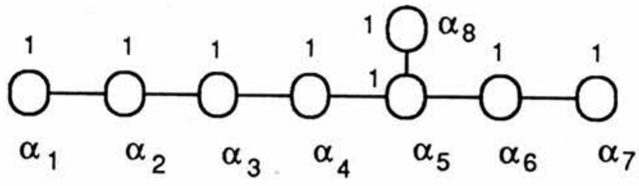
(i) Basis: $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le 6)$ $\alpha_7 = e_5 + e_6 + e_7$

(ii) $\Delta = \{ \pm(e_i - e_j)$, for $1 \le i < j \le 7$,

$\pm(e_i + e_j + e_k)$, for $1 \le i < j < k \le 7$,

$\pm(e_1 + \ldots + e_i + \ldots + e_7)$, for $1 \le i \le 7 \}$

Type E_8

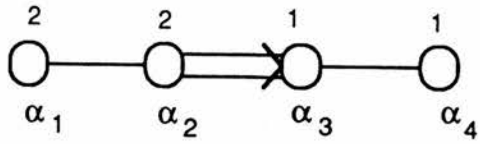


(i) Basis: $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le 7)$ $\alpha_8 = e_6 + e_7 + e_8$

(ii) $\Delta = \{ \pm(e_i - e_j)$, for $1 \le i < j \le 8$,

$\pm(e_i + e_j + e_k)$, for $1 \le i < j < k \le 8 \}$

Type F_4 (Here $E = \mathbb{R}^3$)



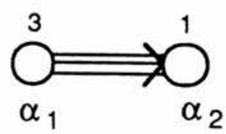
(i) Basis: $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$,

$\alpha_3 = \varepsilon_4 - \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ $\alpha_4 = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$

(ii) $\Delta = \{ \pm\varepsilon_i$, for $1 \leq i < 4$, $\pm(e_i + e_j)$, for $1 \leq i < j \leq 4$,

$\pm \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \}$

Type G_2 (Here $E = \mathbb{R}^3$)



(i) Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2$ $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

(ii) $\Delta = \{ \pm (\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3,\ 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3,\ 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3$

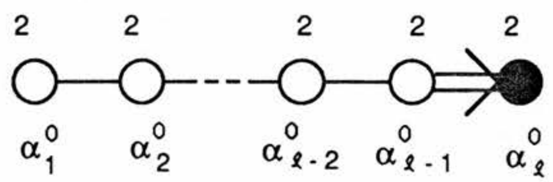
$2\varepsilon_3 - \varepsilon_1 - \varepsilon_2) \}$

TABLE II

Finite irreducible non-reduced root system and its Dynkin diagrams.

In this table ε_i ($i = 1,\ldots, \ell$) denote orthonormal unit vectors of $E = \mathbb{R}^\ell$. The numbers above the vertices of the Dynkin diagrams are the coefficients of the expansion of the highest root in terms of the simple roots.

<u>Type BC_ℓ ($\ell \geq 1$) or $B(0/\ell)$</u>



(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell$ $\alpha_\ell = \varepsilon_\ell$

(ii) $\Delta = \{ \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $\pm\varepsilon_i$, for $1 \leq i \leq \ell$,
$\pm 2\varepsilon_i$, for $1 \leq i \leq \ell$, $\}$

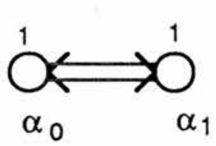
TABLE III

Affine irreducible reduced root system and their Dynkin diagrams

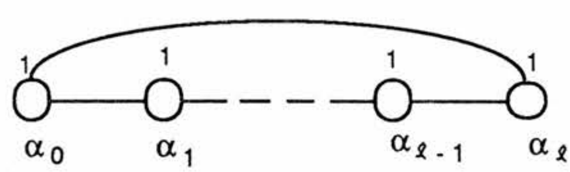
In this table ε_i (i = 1,..., n) denote orthonormal unit vectors of $E = \mathbb{R}^n$. The numbers above the vertices of the Dynkin diagrams are the coefficients of the expansion (2.19) in terms of the simple roots. They are often called numerical marks.

Type $A_\ell^{(1)} (\ell \geq 1)$

$A_1^{(1)}$



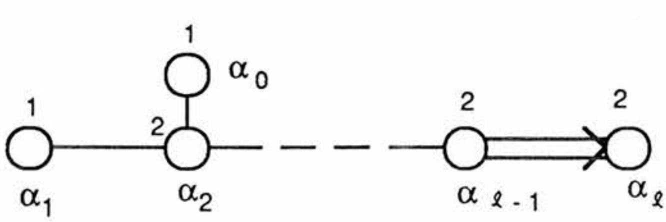
$A_\ell^{(1)} (\ell \geq 2)$



(i) Basis: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for..$1 \leq i \leq \ell$ $\alpha_0 = \delta - \varepsilon_1 + \varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta \pm (\varepsilon_i - \varepsilon_j)$, for $1 \leq i < j \leq \ell + 1$, $m \in \mathbf{Z} \}$

Type $B_\ell^{(1)} (\ell \geq 3)$

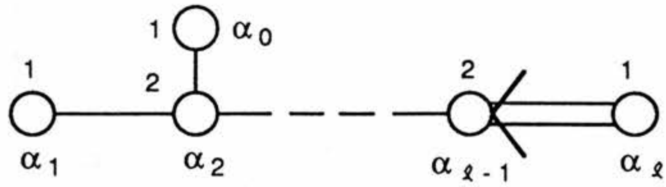


(i) Basis: $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$,

$\alpha_\ell = \varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta \pm (\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $m\delta \pm \varepsilon_i$, for $1 \leq i \leq \ell$, $m \in \mathbf{Z} \}$

Type $A^{(2)}_{2\ell-1}$ $(\ell \geq 3)$

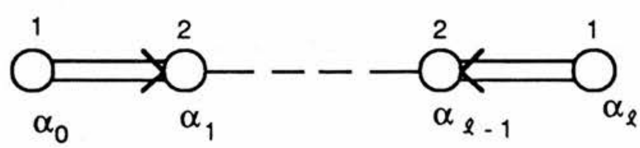


(i) Basis: $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$,

$\alpha_\ell = 2\varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta \pm (\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $2m\delta \pm 2\varepsilon_i$, for $1 \leq i \leq \ell$, $m \in \mathbf{Z} \}$

Type $C^{(1)}_\ell$ $(\ell \geq 2)$

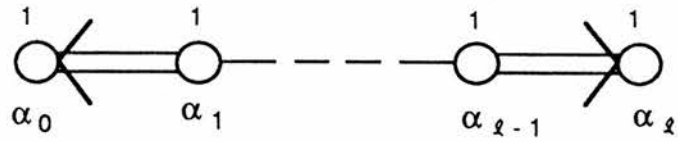


(i) Basis: $\alpha_0 = \delta - 2\varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$,

$\alpha_\ell = 2\varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta \pm (\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $m\delta \pm 2\varepsilon_i$, for $1 \leq i \leq \ell$, $m \in \mathbf{Z} \}$

Type $D_\ell^{(2)} (\ell \geq 2)$



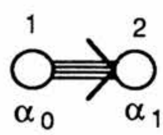
(i) Basis: $\alpha_0 = \frac{1}{2}\delta - \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell-1$, $\alpha_\ell = \varepsilon_\ell$

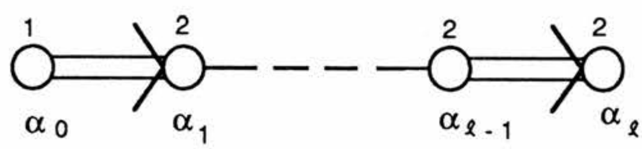
(ii) $\Delta^r = \{ m\delta \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $\frac{1}{2}m\delta \pm \varepsilon_i$, for $1 \leq i \leq \ell$, $m \in \mathbf{Z}\}$

Type $A_{2\ell}^{(2)} (\ell \geq 1)$

$A_2^{(2)}$

$A_{2\ell}^{(2)} (\ell \geq 2)$

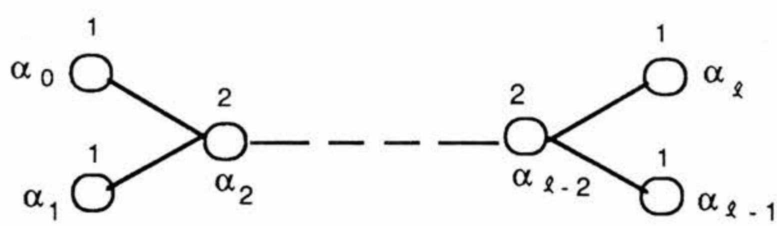




(i) Basis: $\alpha_0 = \delta - 2\varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell-1$, $\alpha_\ell = \varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta\pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \leq i < j \leq \ell$, $(2m+1)\delta \pm 2\varepsilon_i$, for $1 \leq i \leq \ell$,
$m\delta \pm \varepsilon_i$, for $1 \leq i \leq \ell$, $m \in \mathbf{Z}$ }

Type $D_\ell^{(1)} (\ell \geq 4)$



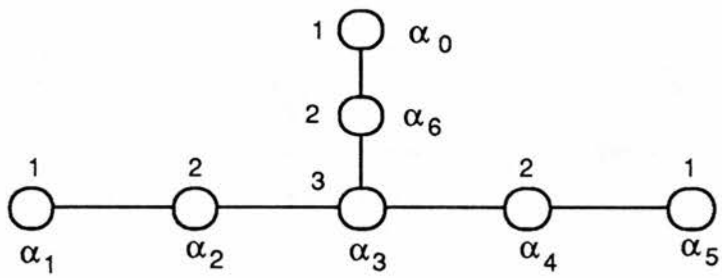
(i) Basis: $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \le i \le \ell - 1$,

$\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$

(ii) $\Delta^r = \{ m\delta \pm(\varepsilon_i \pm \varepsilon_j)$, for $1 \le i < j \le \ell$, $m \in \mathbf{Z} \}$

For the next three types let $E = \mathbb{R}^8$ and define $e_i = \varepsilon_i - (1/9) \sum_{i=0}^{8} \varepsilon_i$, for $\sum_{i=0}^{8} e_i = 0$

Type $E_6^{(1)}$

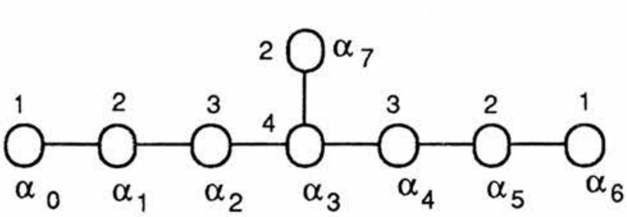


(i) Basis: $\alpha_0 = \delta - (\sum_{i=0}^{6} e_i)$, $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le 5)$,

$\alpha_6 = e_4 + e_5 + e_6$

(ii) $\Delta^r = \{ m\delta \pm(e_i - e_j)$, for $1 \le i < j \le 6$,
$m\delta \pm(e_i + e_j + e_k)$, for $1 \le i < j < k \le 6$, $m\delta \pm(\sum_{i=0}^{6} e_i)$, $m \in \mathbf{Z}\}$

Type $E_7^{(1)}$



(i) Basis: $\alpha_0 = \delta - (\sum_{i=0}^{6} e_i)$, $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le 6)$

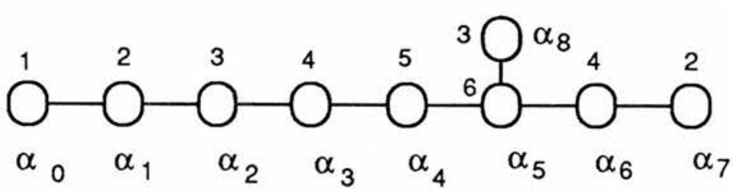
$\alpha_7 = e_5 + e_6 + e_7$

(ii) $\Delta^r = \{ m\delta \pm(e_i - e_j)$, for $1 \le i < j \le 7$,

$m\delta \pm(e_i + e_j + e_k)$ for $1 \le i < j < k \le 7$,

$m\delta \pm(e_1 + \quad + e_i + \quad + e_7)$ for $1 \le i \le 7$, $m \in \mathbb{Z}\}$

Type $E_8^{(1)}$



(i) Basis: $\alpha_0 = \delta + e_0 - e_1$, $\alpha_i = e_i - e_{i+1}$, for $1 \le i \le 7$

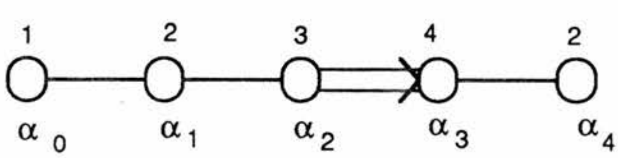
$\alpha_8 = e_6 + e_7 + e_8$

(ii) $\Delta^r = \{ m\delta \pm(e_i - e_j)$, for $1 \le i < j \le 8$,

$m\delta \pm(e_i + e_j + e_k)$ for $1 \le i < j < k \le 8$, $m \in \mathbb{Z}\}$

Type $F_4^{(1)}$

$E = \mathbb{R}^4$



(i) Basis: $\alpha_0 = \delta + \varepsilon_1 - \varepsilon_2$, $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$,

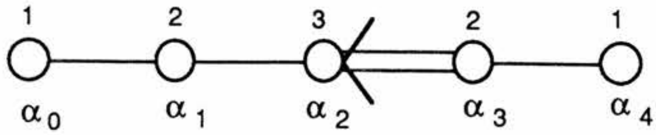
$\alpha_3 = \varepsilon_4 - \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, $\alpha_4 = - \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$

(ii) $\Delta^r = \{\ m\delta \pm\varepsilon_i,$ for $1 \leq i < 4,\quad m\delta \pm(e_i + e_j),$ for $1 \leq i < j \leq 4,$

$m\delta \pm \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4),\ m \in \mathbf{Z}\}$

<u>Type</u> $E_6^{(2)}$

$E = \mathbb{R}^4$



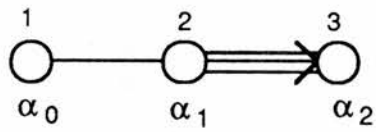
(i) Basis: $\alpha_0 = \delta + \varepsilon_1 - \varepsilon_2,\quad \alpha_1 = \varepsilon_2 - \varepsilon_3,\quad \alpha_2 = \varepsilon_3 - \varepsilon_4,$

$\alpha_3 = 2\varepsilon_4,\quad \alpha_4 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$

(ii) $\Delta^r = \{\ 2m\delta \pm 2\varepsilon_i,$ for $1 \leq i < 4,\quad m\delta \pm(\varepsilon_i + \varepsilon_j)$ for $1 \leq i < j \leq 4,$

$2m\delta \pm (\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4),\ m \in \mathbf{Z}\ \}$

<u>Type</u> $G_2^{(1)}$ (Here $E = \mathbb{R}^3$)



(i) Basis: $\alpha_0 = \delta + \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3,\quad \alpha_1 = \varepsilon_1 - \varepsilon_2,$

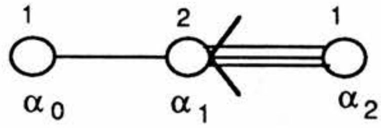
$\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

(ii) $\Delta^r = \{\ m\delta \pm (\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3,\ 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3,\ 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3$

$2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\ ),\ m \in \mathbf{Z}\ \}$

Type $D_4^{(3)}$

$E = \mathbb{R}^3$



(i) Basis: $\alpha_0 = \delta - \varepsilon_1 - \varepsilon_3$, $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = 3\,\varepsilon_2$

(ii) $\Delta^r = \{\ m\delta \pm (\varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_3,\ \varepsilon_1-\varepsilon_3),$

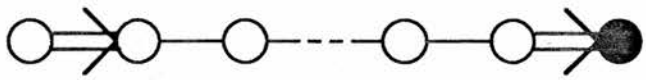
$3m\delta \pm (\ 2\varepsilon_1-\varepsilon_2 - \varepsilon_3,\ 2\varepsilon_2 - \varepsilon_1-\varepsilon_3,\ 2\varepsilon_3 - \varepsilon_1-\varepsilon_2\ ),\ m\in \mathbb{Z}\ \}$

TABLE IV

Affine irreducible non-reduced root systems

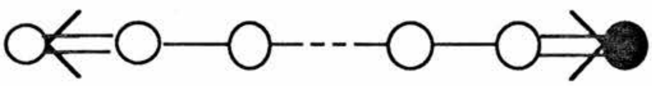
In this table ε_i ($i = 1,\ldots, \ell$) denote orthonormal unit vectors of $E = \mathbb{R}^\ell$. The black nodes denote those simple roots such that twice of them is a root of the system.

Type $B^{(1)}(0/\ell)$ ($\ell \geq 1$)



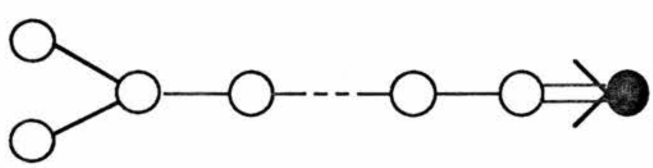
(i) $\Delta^r = \{\ m\delta \pm (\varepsilon_i \pm \varepsilon_j),$ for $1 \leq i < j \leq \ell$, $m\delta \pm \varepsilon_i,$ for $1 \leq i \leq \ell$, $m\delta \pm 2\varepsilon_i,$ for $1 \leq i \leq \ell$, $m \in \mathbf{Z}\ \}$

Type $A^{(4)}(2\ell/0)$ ($\ell \geq 1$)



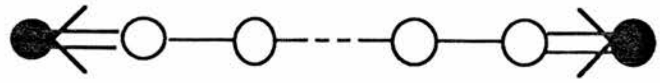
(i) $\Delta^r = \{\ m\delta \pm (\varepsilon_i \pm \varepsilon_j),$ for $1 \leq i < j \leq \ell$ $\frac{1}{2}m\delta \pm \varepsilon_i,$ for $1 \leq i \leq \ell$, $(2m+1)\delta \pm 2\varepsilon_i,$ for $1 \leq i \leq \ell$, $m \in \mathbf{Z}\ \}$

Type $A^{(2)}(2\ell\text{-}1/0)$ ($\ell \geq 3$)



(ii) $\Delta^r = \{\ m\delta \pm (\varepsilon_i \pm \varepsilon_j),$ for $1 \leq i < j \leq \ell$, $m\delta \pm \varepsilon_i$ for $1 \leq i \leq \ell$, $2m\delta \pm 2\varepsilon_i,$ for $1 \leq i \leq \ell$, $m \in \mathbf{Z}\ \}$

Type $C^{(2)}(\ell+1)$ $(\ell \geq 1)$

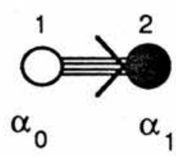


(i) $\Delta^r = \{\ m\delta \pm (\varepsilon_i \pm \varepsilon_j),$ for $1 \leq i < j \leq \ell,$ $\tfrac{1}{2}m\delta \pm \varepsilon_i,$ for $1 \leq i \leq \ell$,

$m\delta \pm 2\varepsilon_i,$ for $1 \leq i \leq \ell$, $m \in \mathbf{Z}\ \}$

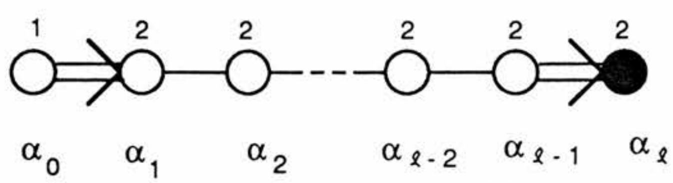
TABLE V

Figure 1. Generalised Dynkin diagram and Cartan matrix of $B^{(1)}(0/1)$



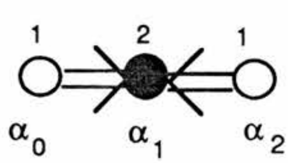
$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

Figure 2. Generalised Dynkin diagram and Cartan matrix of $B^{(1)}(0/\ell)(\ell \geq 2)$



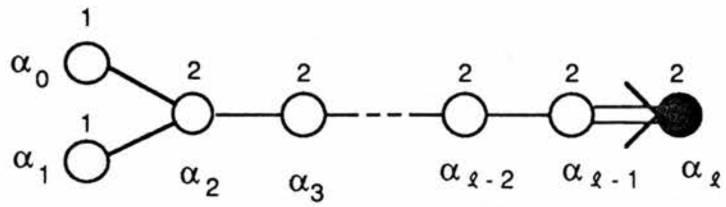
$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -2 & 2 \end{pmatrix}$$

Figure 3. Generalised Dynkin diagram and Cartan matrix of $A^{(2)}(3/0)$



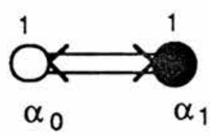
$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Figure 4. Generalised Dynkin diagram and Cartan matrix of $A^{(2)}(2\ell-1/0)(\ell \geq 2)$



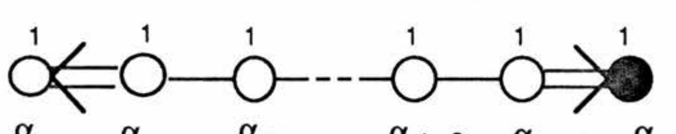
$$A=\begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

Figure 5. Generalised Dynkin diagram and Cartan matrix of $A^{(2)}(2/0)$



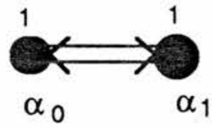
$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Figure 6. Generalised Dynkin diagram and Cartan matrix of $A^{(4)}(2\ell/0)(\ell \geq 2)$



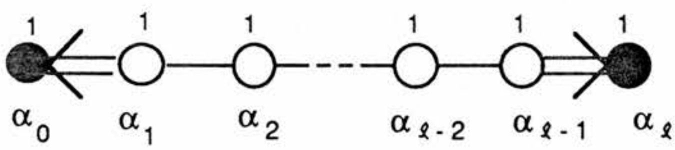
$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

Figure 7. Generalised Dynkin diagram and Cartan matrix of $C^{(2)}(2)$



$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Figure 8. Generalised Dynkin diagram and Cartan matrix of $C^{(2)}(\ell+1)(\ell \geq 2)$



$$A = \begin{pmatrix} 2 & -2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

Figure 9. Generalised Dynkin diagram of $B(0/\ell)(\ell \geq 1)$

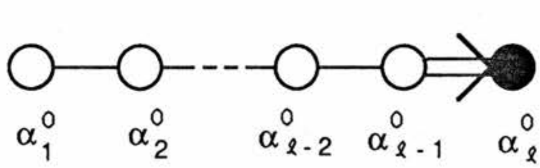


Figure 10. Generalised Dynkin diagram of $A(2\ell-1/0)(\ell \geq 2)$

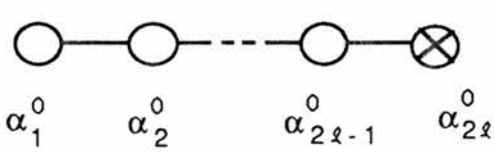


Figure 11. Generalised Dynkin diagram of A(2ℓ/0)(ℓ≥1)

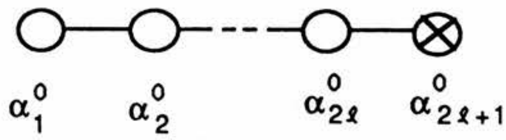
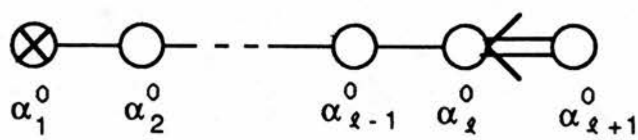


Figure 12. Generalised Dynkin diagram of C(ℓ+1)(ℓ≥1)



APPENDIX A

(1) The $\ell(\ell-1)$ positive roots of D_ℓ ($\ell \geq 2$) are given by:

$$\sum_{r=j}^{k-1} \alpha_r + 2\sum_{r=k}^{\ell-2} \alpha_r + \alpha_{\ell-1} + \alpha_\ell,$$

$$\sum_{r=j}^{k-1} \alpha_r \quad ,$$

with $j,k = 1,2, \ldots, \ell-2$, and $j < k$,

$$\sum_{r=j}^{\ell-2} \alpha_r + \alpha_{\ell-1} + \alpha_\ell,$$

$$\sum_{r=j}^{\ell-2} \alpha_r + \alpha_{\ell-1}$$

$$\sum_{r=j}^{\ell-2} \alpha_r + \alpha_\ell,$$

$$\sum_{r=j}^{\ell-2} \alpha_r$$

with $j = 1,2, \ldots, \ell-2$, together α_ℓ, $\alpha_{\ell-1}$.

The quantities $\langle\alpha_j, \alpha_k\rangle$ of D_ℓ are given by:

$$\langle\alpha_j, \alpha_k\rangle = \begin{cases} 1/2(\ell-1), & \text{with } j = k \ (j = 1,2,\ldots,\ell) \\ -1/4(2\ell-1), & \text{with } j = k \pm 1 \ (j = 1,2,\ldots,\ell-3); \\ & j=\ell-2 \text{ with } k = \ell-1, \ell;\ k = \ell-2 \text{ with } j=\ell-1, \ell, \\ 0 & \text{for all other values.} \end{cases}$$

The fundamental weights of D_ℓ are given by:

$$\Lambda_j = \begin{cases} \sum_{p=1}^{\ell-2} \alpha_p + \tfrac{1}{2}\alpha_{\ell-1} + \tfrac{1}{2}\alpha_\ell, & \text{with } j = 1, \\ \sum_{p=1}^{j-1} p\alpha_p + \sum_{p=j}^{\ell-2} j\alpha_p + \tfrac{1}{2}j\alpha_{\ell-1} + \tfrac{1}{2}j\alpha_\ell, & \text{with } j = 2, \ldots, \ell-2, \end{cases}$$

$$\tfrac{1}{2}\{\ \textstyle\sum_{p=1}^{\ell-2} p\alpha_p + \tfrac{1}{2}\ell\alpha_{\ell-1} + (\tfrac{1}{2}\ell-1)\alpha_\ell\ \} \quad \text{with } j = \ell-1,$$

$$\tfrac{1}{2}\{\ \textstyle\sum_{p=1}^{\ell-2} p\alpha_p + (\tfrac{1}{2}\ell-1)\alpha_{\ell-1} + \tfrac{1}{2}\ell\alpha_\ell\ \} \quad \text{with } j = \ell.$$

(2) The ℓ^2 positive roots of B_ℓ ($\ell\geq1$)are given by

$\sum_{p=j}^{\ell} \alpha_p$ with $j = 1,2, \ldots ,\ell$,

$\sum_{p=j}^{k-1} \alpha_p + 2\sum_{p=k}^{\ell} \alpha_p$ with $j, k = 1,2, \ldots ,\ell$ and $j<k$,

$\sum_{p=j}^{k-1} \alpha_p$ with $j, k = 1,2, \ldots ,\ell$ and $j<k$.

The quantities $<\alpha_j, \alpha_k>$ of B_ℓ are given by:

$$<\alpha_j, \alpha_k> = \begin{cases} 1/(2\ell-1), & \text{with } j = k,\ (j = 1,2,\ldots,\ell-1) \\ 1/2(2\ell-1), & \text{with } j = k = \ell \\ -1/(2\ell-1), & \text{with } j = k \pm 1, \text{ with } (j, k= 1,2,\ldots,\ell) \\ 0 & \text{for all other values.} \end{cases}$$

The foundamental weights of B_ℓ are given by:

$$\Lambda_j = \begin{cases} \sum_{p=1}^{\ell} \alpha_p & \text{with } j = 1, \\ \sum_{p=1}^{j-1} p\alpha_p + \sum_{p=j}^{\ell} j\alpha_p & \text{with } j = 1,2, \ldots ,\ell-1, \\ (1/2)\sum_{p=1}^{\ell} p\alpha_p & \text{with } j = \ell. \end{cases}$$

(3) The positive roots of $B(0/\ell)$($\ell\geq1$) are given by

(a) even positive roots

$\sum_{p=j}^{k-1} \alpha_p$ with $j, k = 1,2,\ldots,\ell$ and $j<k$,

$\sum_{p=j}^{k-1} \alpha_p + 2\sum_{p=j}^{\ell} \alpha_p$ with $j, k = 1,2,\ldots,\ell$ and $j<k$,

$2\sum_{p=j}^{\ell} \alpha_p$ with $j = 1,2,\ldots,\ell$,

(b) odd positive roots

$\sum_{p=j}^{\ell} \alpha_p$ with $j = 1,2,\ldots,\ell$.

The quantities $<\alpha_j, \alpha_k>$ of $B(0/\ell)$are given by:

$$<\alpha_j, \alpha_k> = \begin{cases} 1/(2\ell+1), & \text{with } j = k, \ (j = 1,2,\ldots,\ell-1) \\ 1/2(2\ell+1), & \text{with } j = k = \ell \\ -1/(2\ell+1), & \text{with } j = k \pm 1, \text{ with } (j, k= 1,2,\ldots,\ell) \\ 0 & \text{for all other values.} \end{cases}$$

Appendix B

(1) Proof of (6.4.24), (6.4.48), (6.4.49) , (6.4.50) , (6.4.74) and (6.4.87):

Let $\Psi(a_{0r})$ ($r = 1,\ldots,n_0$) denote operators belonging to a non-trivial representation (of dimension is n_p) that the $\tilde{\mathcal{L}}_p^{0(4)}$ subspace provides for $\tilde{\mathcal{L}}_0^{0(4)}$ (for p = 1,2,3). Then the Casimir operator of this representation, C_2, will have the form[10] :

$$C_2 = -\Sigma_{r=1}^{n_0} \Psi(a_{0r})\Psi(a_{0r}) \tag{B.1}$$

and its eigenvalues are given by

$$C_2(\Lambda) = -\{1/n_p\} \operatorname{tr}\left\{ \Sigma_{r=1}^{n_0} \underset{\sim}{\Gamma}(a_{0r})\underset{\sim}{\Gamma}(a_{0r})\right\} . \tag{B.2}$$

Thus, denoting the basis of the pth subspace by a_{ps},

$$C_2\, a_{ps'} = -\Sigma_{r=1}^{n_0} \Sigma_{s''=1}^{n_p} (\underset{\sim}{\Gamma}(a_{0r})\underset{\sim}{\Gamma}(a_{0r}))_{s''s'}\, a_{ps''}$$

$$= -\{1/n_p\} \operatorname{tr}\left\{ \Sigma_{r=1}^{n_0} \underset{\sim}{\Gamma}(a_{0r})\underset{\sim}{\Gamma}(a_{0r})\right\} a_{ps'} . \tag{B.3}$$

Relations (6.4.24) ,(6.4.74) and (6.4.87) then follow immediately if we also take in to account the relations (6.4.7) or (6.4.8) and the fact that

$$\operatorname{tr}\left\{ G(a_{0r})G(a_{0r'})\right\} = \gamma\, B^{\tilde{\mathcal{L}}_0^{0(4)}}(a_{0r}, a_{0r'}) , \tag{B.4}$$

where γ is the Dynkin index of the representation and $B^{\tilde{\mathcal{L}}_0^{0(4)}}(\ ,\ )$ is the Killing form of $\tilde{\mathcal{L}}_0^{0(4)}$. The proof of (6.4.49) goes through as above if we observe that, on defining $S_{ss'}^r$ by

$$[\, a_{2s} , a_{2s'}] = \Sigma_{r=1}^{n_0} S_{ss'}^r\, a_{0r} , \tag{B.5}$$

the invariance property of the Killing form implies that

$$S^{r}_{ss'} \;=\; -\,\Gamma^2(a_{0r}))_{ss'}\,. \tag{B.6}$$

Similarly for (6.4.50) we observe that, on defining $A^{r}_{pp'}$ by

$$[\, b_{1p'}\;,\, b_{3p}\,] = \Sigma^{n_0}_{r=1}\; A^{r}_{p'p}\; a_{0r}\,, \tag{B.7}$$

the invariance property of the Killing form implies that

$$A^{r}_{p'p} \;=\; \underset{\sim}{D}(a_{0r})_{pp'}\;, \tag{B.8}$$

with the rest of the proof following the same steps as above. For (6.4.48) the only new feature is that $f^{r''}_{rr'} = ad(a_{0r})_{r''r'}$. The result (6.4.48) then follows using the same arguments as above and the fact that the Dynkin index of the adjoint representation is 1.

(2) Proof of (6.5.52) and (6.5.65):

Consider the Casimir operator of $C(\ell+1)$ in the adjoint representation, which is given by

$$C_2(ad) \;=\; \Sigma^{n_0}_{r=1}\,\Psi(a_{0r})\,\Psi(a^{\#}_{0r}) \;-\; \Sigma^{n_1}_{p=1}\,\Psi(b_{0p})\,\Psi(b^{\#}_{0p})$$

$$+\;\Psi(c')\,\Psi(c'^{\#}) \;-\; \Sigma^{n_1'}_{p=1}\,\Psi(b_{1p})\,\Psi(b^{\#}_{1p})\;, \tag{B.9}$$

(where the duals are as defined in section V.A.). Then from the relation $C_2(ad)\; b_{0t} = b_{0t}$, we find using (5.43) to (5.48) and (5.50) that

$$\Sigma^{n_0}_{r=1}\,\Psi(a_{0r})\,\Psi(a^{\#}_{0r})\; b_{0t} \;=\; -\Sigma^{n_0}_{r=1}\,\Sigma^{n_1}_{s=1}(\underset{\sim}{D}(a_{0r})\underset{\sim}{D}(a_{0r}))_{st}\; b_{0s}\,,$$

(B.10)

$$- \Sigma_{p=1}^{n_1} \Psi(b_{0p})\, \Psi(b_{0p}^{\#})\, b_{0t} \;=\; -\Sigma_{r=1}^{n_0} \Sigma_{s=1}^{n_1} (\underset{\sim}{D}(a_{0r}) \underset{\sim}{D}(a_{0r}))_{st}\, b_{0s}\,,$$

(A.11)

$$- \Sigma_{p=1}^{n_1'} \Psi(b_{1p})\, \Psi(b_{1p}^{\#})\, b_{0t} \;=\; -(1/4\ell)\, b_{0t}\,, \qquad \text{(B.12)}$$

$$\Psi(c')\, \Psi(c'^{\#})\, b_{0t} \;=\; -(1/4\ell)\, b_{0t}\,. \qquad \text{(B.13)}$$

Then (6.5.52) is obvious. Also (6.5.65) follows by exactly the same steps on using (6.5.56) to (6.5.61) and (6.5.63).

(3) Proof of (6.5.74):

On using the identity $C_2(ad)c' = c'$, relations (6.5.69)-(6.5.72) and the equality

$$B(\,[\,c', b_{0q}\,]\,, [\,c', b_{0p}\,]\,) = B(\,[\,c', b_{1q}\,]\,, [\,c', b_{1p}\,]\,) \qquad \text{(B.14)}$$

we obtain

$$- \Sigma_{p=1}^{n_1} \Psi(b_{0p})\, \Psi(b_{0p}^{\#}) c' - \Sigma_{p=1}^{n_1'} \Psi(b_{1p})\, \Psi(b_{1p}^{\#}) c'$$

$$= 2 \left\{ \Sigma_{t=1}^{n_1} \Sigma_{s=1}^{n_1} \Sigma_{p=1}^{n_1} \Sigma_{q=1}^{n_1} (\underset{\sim}{B}^{-1})_{pq}\, (\underset{\sim}{B})_{ts}\, A_s^{(q)}\, A_t^{(p)} c' \right\} = c', \quad \text{(B.15)}$$

thus proving (6.5.74).

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