Gerontogiannis, Dimitrios Michail D. (2021) *Ahlfors regularity, extensions by Schatten ideals and a geometric fundamental class of Smale space C*-algebras using dynamical partitions of unity.* PhD thesis.

Ahlfors regularity, extensions by Schatten ideals and a geometric fundamental class of Smale space C*-algebras using dynamical partitions of unity

by

Dimitrios Michail D. Gerontogiannis
(supervised by Dr Michael F. Whittaker and Prof. Joachim Zacharias)

Submitted in fulfilment of the requirements for the
Degree of Doctor of Philosophy

School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

March 2021

Abstract

In this thesis we study the Ahlfors regularity of Bowen's measure on Smale spaces and analyse Smale space C^*-algebras in the framework of Connes' noncommutative geometry using smooth extensions by Schatten ideals and summable Fredholm modules.

Bowen's construction of Markov partitions implies that Smale spaces are factors of topological Markov chains. The latter are equipped with Parry's measure which is Ahlfors regular. By extending Bowen's construction we create a tool for transferring, up to topological conjugacy, the Ahlfors regularity of the Parry measure down to the Bowen measure of the Smale space. An essential part of our method uses a refined notion of approximation graphs over compact metric spaces. Moreover, we obtain new estimates for the Hausdorff, box-counting and Assouad dimensions of a large class of Smale spaces.

In the noncommutative setting, given a Smale space, our generalised Markov partitions yield dynamical partitions of unity which produce explicit θ-summable Fredholm modules that represent a fundamental K-homology class for the Spanier-Whitehead duality of the stable and unstable Ruelle algebras of the Smale space. Therefore, we obtain an exhaustive description of the K-homology classes of Ruelle algebras in terms of Fredholm modules constructed by Markov partitions. Our method involves the construction of dynamical metrics on Smale space groupoids that give rise to smooth (holomorphically stable and dense) $*$-subalgebras of Smale space C^*-algebras. In particular, for every such smooth subalgebra of a Ruelle algebra, we show that every extension class in the BDF-theory group of the Ruelle algebra can be represented by an extension that on the smooth subalgebra reduces to an algebraic extension by a Schatten p-ideal. The value p is related to the dimension of the underlying Smale space. This provides a new approach to the noncommutative dimension theory of Smale spaces.

Contents

1 Introduction **1**

1.1 Smale spaces and their C*-algebras . . . 2

1.2 Ahlfors regularity of measure of maximal entropy and fractal dimension . . 3

1.3 Smooth extensions of Ruelle algebras and noncommutative dimension . . . 5

1.4 Spanier-Whitehead K-duality for Ruelle algebras and Fredholm modules . 7

1.5 Thesis structure . . . 8

2 Approximations of compact metric spaces **11**

2.1 Preliminaries . . . 11

2.1.1 Topological dynamical systems . . . 11

2.1.2 Dimension theory . . . 13

2.2 Approximations via rooted graphs . . . 15

2.2.1 Refining sequences . . . 15

2.2.2 Approximation graphs . . . 16

2.2.3 Essentiality of approximation graphs . . . 18

2.2.4 Dynamic approximation graphs . . . 20

2.2.5 Geometric approximation graphs and Assouad dimension . . . 22

3 Smale spaces, Ahlfors regularity and dimension **25**

3.1 Smale spaces . . . 25

3.1.1 Preliminaries . . . 25

3.1.2 Topological Markov chains . . . 27

3.1.3 Wieler solenoids . . . 30

3.1.4 Dyadic solenoid . . . 31

3.1.5 Metrics and smoothing of Smale spaces . . . 31

3.2 Markov partitions . . . 34

3.2.1 Basics on Markov partitions . . . 34

3.2.2 Approximation graphs from Markov partitions . . . 37

3.3 Geometric approximations of Smale spaces . . . 44

3.3.1 Proof of conditions (1)–(5) of Theorem 3.3.2 . . . 47

3.3.2 Proof of conditions (6), (7) of Theorem 3.3.2 . . . 53

3.4 Semi-conformal Smale spaces and Ahlfors regularity . . . 57

4 KK-theory and smoothness in C*-algebras **64**
4.1 K-theoretic preliminaries . . . 64
4.1.1 A primer on KK-theory . . . 64
4.1.2 Stinespring dilation of extensions of C^*-algebras . . . 69
4.1.3 Spanier-Whitehead K-duality . . . 71
4.2 K-duality and uniformly smooth C*-algebras . . . 73
4.2.1 Smooth extensions and summable Fredholm modules . . . 73
4.2.2 Slant products for simple, purely infinite C^*-algebras . . . 78
4.2.3 Holomorphic functional calculus and K-duality . . . 83

5 Smale space C*-algebras **93**
5.1 An introduction to Smale space C*-algebras . . . 93
5.1.1 Equivalence groupoids . . . 94
5.1.2 C^*-algebras . . . 97
5.2 KPW-extension class and the K-duality of Ruelle algebras . . . 102
5.2.1 K-theory duality class . . . 103
5.2.2 KPW-extension class . . . 105

6 Noncommutative dimension of Ruelle algebras **108**
6.1 Metrisation of Smale space groupoids . . . 108
6.1.1 Dependence of groupoids on Smale space metric . . . 109
6.1.2 Metrics and Lipschitz dynamics on Smale space groupoids . . . 112
6.1.3 Optimisation of uniform convergence rates . . . 130
6.1.4 Groupoid ultrametrics for topological Markov chains . . . 133
6.2 Smoothness of Ruelle algebras . . . 139
6.2.1 Lipschitz algebras on general étale groupoids . . . 140
6.2.2 Lipschitz subalgebras of Ruelle algebras . . . 145
6.2.3 Smooth extensions of Ruelle algebras . . . 155

7 Geometric K-duality for Ruelle algebras **159**
7.1 Building a KK_1-lift of the KPW-extension class . . . 159
7.1.1 The symmetric representation . . . 161
7.1.2 Intuition for the isometry . . . 163
7.1.3 Partitions of unity, aperiodic samples of Markov partitions . . . 167
7.1.4 The averaging isometries . . . 170
7.2 Fredholm modules for the KPW-extension class . . . 176

Conclusion **192**

Bibliography **193**

Acknowledgements

It has been an amazing adventurous journey! It had everything; ups and downs, slippery slopes, heat, cold and definitely fog. At times it was exhausting. Luckily, I was not alone! I had amazing people supporting me all along.

First and foremost, I would like to sincerely thank my advisors Joachim and Mike for offering me this wonderful opportunity to be their student. I am truly grateful for their guidance, endless support and encouragement. They are amazing spirits and my role models for being excellent mathematicians and teachers. I will be forever grateful.

Secondly, I would like to thank EPSRC for their financial support, without which it would have been impossible for me to undertake my doctoral studies in Glasgow. I would also like to thank the School of Mathematics and Statistics at the University of Glasgow for their assistance in attending several conferences and workshops where I had the chance to meet amazing people and places. Also, I am grateful to Tara Brendle and Andrew Wilson for giving me the chance to lecture in several courses in Mathematics and Statistics. Further, I would like to thank the College of Science and Engineering for providing support for my research visit at the University of Victoria, Canada.

A special thank you goes to Ian Putnam for hosting me at the University of Victoria during the Autumn of 2019. It has been amazing working with him. I would also like to thank Heath Emerson for many interesting conversations. Moreover, I am grateful to Magnus Goffeng for guiding me to the literature regarding smooth extensions by Schatten ideals. I would also like to thank Siegfried Echterhoff and Christian Voigt for taking time out of their busy schedule to read the thesis.

I would like to thank Jamie Gabe for all his guidance in C^*-algebras and KK-theory during the first years of my doctoral studies and for his friendship. Also, I would like to thank Christian Bönicke for many helpful discussions regarding C^*-algebras and for also being my friend. Moreover, I would like to thank the Analysis Group in Glasgow, and especially Stuart White, for the lively research environment they cultivate to young students and for all the encouragement and advices. I feel very lucky that I was part of it and have made really good friends. I am also grateful to Iakovos Androulidakis and Aristides Katavolos for introducing me to C^*-algebras during my studies at the University of Athens as an undergraduate and master student.

A special thanks goes to all my old and new friends in Glasgow that have made my stay here most enjoyable and interesting. Thank you again. Also, I would like to thank Maeve and her family for making my stay in Victoria most welcome. It was a very productive period. Further, I would like to thank my Mexican friends and our rock band “Los Guacamoles”...funny name, right?

I sincerely thank all my good friends in Greece for supporting me all these years to pursue my dreams. They are very important to me and too many to list here. They know who I am referring to and I'm indebted to all of them. Especially, I would like to thank my dear friends Petros Galanopoulos, Eva Kontidi, Kostas Spyrou and Kostas Tromaras for frequently just asking how my day was...Once I got caught by Dimitra staring at some tiles on a wall. According to her I was totally immersed - not paying attention to anything else - maybe for an hour, and she was really worried I'd gone mad. Well, she told all of our friends about it and since then they tease me for my wandering mind. To be fair, the tiles had a rectangular rather simple shape, and a totally indifferent grey color. Little did they know I was trying to solve a problem on Markov partitions that eventually led to the first part of my thesis. Long story short, according to them it never gets old to tease me about that...I love it.

I would like to thank, from the depths of my heart, my whole family which has been always there for me. Especially, my parents Dionisios and Roula, my sister Lina, my grandparents Eleni and Thomas, Dimitrios and Stavroula. I hope I have made you all proud. You have been always caring, supporting and advising me from as long as I remember. You are an invaluable part of my life and this endeavour would not have been possible without you. Thank you. Finally, my sincere appreciation goes to my dear Dimitra for her endless support, for believing in me, for patiently listening to my incomprehensible ideas and for always being by my side, fuelling my days with her cheerful spirit and smile.

To my family

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

In the 1970's Bowen [16–18], using Markov partitions, showed that topological Markov chains provide a combinatorial model of arbitrary precision for Smale spaces. Since then, Markov partitions have been an indispensable tool for studying the dimension theory [11,101] and homology [107] of Smale spaces. The pervading theme of this thesis is a certain generalisation of Bowen's Markov partitions that allows us to study the geometry of Smale spaces. In particular, our primary innovation is to inflate Bowen's Markov partitions to open covers realising the same combinatorial data, but which can also be used to approximate the metric structure of Smale spaces, by inductively refining them to open covers of smaller diameter admitting Lipschitz partitions of unity with controlled Lipschitz constants. In this thesis we use Markov partitions and their generalisation to

(i) study the Ahlfors regularity of Bowen's measure (measure of maximal entropy) on Smale spaces. As an application we calculate Hausdorff, box-counting and Assouad dimensions for a large class of Smale spaces;

(ii) construct and study smooth structures related to the Brown-Douglas-Fillmore Ext-groups of the stable and unstable Ruelle algebras associated to Smale spaces. This involves the construction of dynamical metrics on Smale space groupoids giving rise to smooth subalgebras of Ruelle algebras. For each smooth subalgebra, every class in the Ext-group of the corresponding Ruelle algebra can be represented by an extension (by the compacts) that can be reduced to an algebraic extension of the smooth subalgebra by a Schatten p-ideal. The value $p > 0$ is related to dimensional data of the underlying Smale space;

(iii) find an explicit θ-summable Fredholm module picture of the K-theoretic Spanier-Whitehead duality of Kaminker, Putnam and Whittaker between the stable and unstable Ruelle algebras [74]. In this way, we obtain a geometric description of the K-homology of both Ruelle algebras in terms of Fredholm modules constructed by Markov partitions.

Part (i) is a classical approach to dimension theory of Smale spaces, while part (ii) can be regarded as a noncommutative analogue. The original PhD problem was part (iii) only. However, we discovered that most of the machinery that we developed to solve it could, in fact, be used for parts (i) and (ii). We now give a brief introduction to Smale spaces and their C^*-algebras. These topics are described in detail in Sections 3.1 and 5.1. Then, we briefly discuss each of the aforementioned three parts and conclude the introduction with the structure of the thesis.

1.1 Smale spaces and their C*-algebras

In the 1970's, Ruelle [116] defined Smale spaces as the topological counterpart of Smale's famous non-wandering Axiom A systems [128]. Roughly speaking, a Smale space (X, d, φ) is a dynamical system consisting of a homeomorphism φ acting on a compact metric space (X, d) in a hyperbolic way; every $x \in X$ has a small neighbourhood homeomorphic to the product of two sets on which φ expands and contracts distances at least by a factor $\lambda_d > 1$ and $\lambda_d^{-1} < 1$, respectively. Other basic examples include the hyperbolic toral automorphisms, Wieler solenoids [134], aperiodic substitution tilings [6] and topological Markov chains. Bowen's Theorems on Markov partitions proved that the latter provide a combinatorial model of arbitrary precision for Smale spaces. In this thesis we focus on non-wandering Smale spaces, which due to Smale's Decomposition Theorem [116, Section 7.4] can be studied through their irreducible or mixing parts.

In the late 1980's, Ruelle in [117] constructed groupoid C^*-algebras from homoclinic equivalence relations on Smale spaces. Ruelle's work was later extended by Putnam [108] who constructed additional groupoid C^*-algebras from Smale spaces. Among these, the stable and unstable Ruelle algebras are crossed products by $\mathbb{Z}$ and higher dimensional analogues of stabilised Cuntz-Krieger algebras. Putnam and Spielberg [109] showed that Ruelle algebras are separable, simple, nuclear, C^*-stable, purely infinite and in the UCT class, hence are classified up to isomorphism by their K-theory. Kaminker, Putnam and Whittaker [74] also proved that for a given Smale space, the stable and unstable Ruelle algebras $\mathcal{R}^s$ and $\mathcal{R}^u$ are (Spanier-Whitehead) K-dual. This K-theoretic duality is given by a K-homology class $\Delta \in \mathrm{KK}_1(\mathcal{R}^s \otimes \mathcal{R}^u, \mathbb{C})$ and a K-theory class $\widehat{\Delta} \in \mathrm{KK}_1(\mathbb{C}, \mathcal{R}^s \otimes \mathcal{R}^u)$ such that

$$\widehat{\Delta} \otimes_{\mathcal{R}^u} \Delta = 1_{\mathcal{R}^s} \text{ and } \widehat{\Delta} \otimes_{\mathcal{R}^s} \Delta = -1_{\mathcal{R}^u}.$$

One of the various isomorphisms that we obtain via Kasparov product is

$$- \otimes_{\mathcal{R}^s} \Delta : \mathrm{KK}_j(\mathbb{C}, \mathcal{R}^s) \to \mathrm{KK}_{j+1}(\mathcal{R}^u, \mathbb{C}). \tag{1.1.1}$$

The class Δ is represented by an extension τ_Δ of $\mathcal{R}^s \otimes \mathcal{R}^u$ (that we call the KPW-extension), and a large portion of this thesis is dedicated to finding a geometric Fredholm module representative of Δ.

1.2 Ahlfors regularity of measure of maximal entropy and fractal dimension

The first main part of the thesis is about proving that, up to topological conjugacy, every Smale space admits natural Ahlfors regular measures. A Borel measure μ on a compact metric space (X, d) is Ahlfors s-regular if it is of the order r^s on every closed ball of radius r. In this case, the measure μ is comparable to the s-dimensional Hausdorff measure and the typically distinct Hausdorff, box-counting and Assouad dimensions of (X, d) are equal to s, see Subsection 2.1.2.

Ahlfors regular measures have been fundamental to the study of fractal structures. For instance, if an iterated function system has the open set condition, then it admits an Ahlfors regular measure [90]. Metric spaces that admit Ahlfors regular measures provide an abstract framework for the tools of harmonic analysis to be applied since, in particular, they are uniformly perfect and doubling, two very useful properties in analysis (see [64, Chapter 11] and the numerous references therein). In addition, Lebesgue's Differentiation Theorem holds [4, Theorem 5.2.6]. Moreover, Ahlfors regularity lies deep in the heart of fractional calculus [3,121] and opens a window to apply Connes' noncommutative machinery [28] in the study of metric spaces and dynamical systems. A recent example is the work of Goffeng and Mesland [59] who used the theory of Riesz potentials [139] to build interesting spectral triples on Cuntz algebras.

Mixing Smale spaces are equipped with Bowen's measure [18]; the unique invariant probability measure that is ergodic and maximises the topological entropy. The Bowen measure defined on topological Markov chains coincides with the quite tractable Parry measure [97]. A deeper connection between these two measures comes from Bowen's construction of Markov partitions of arbitrarily small diameter [16]. With this, given a mixing Smale space (X, d, φ) one can build a topological Markov chain (Σ, ρ, σ) and a factor map $\pi : (\Sigma, \rho, \sigma) \to (X, d, \varphi)$. Among many nice properties (see Theorem 3.2.5), the map π becomes a measure-theoretic isomorphism when both Smale spaces are equipped with Bowen's measure. Therefore, Parry's measure provides a combinatorial approximation of Bowen's measure.

Using the Perron-Frobenius Theorem one can show that Parry's measure on (Σ, ρ, σ) is Ahlfors regular, see Subsection 3.1.2. This fact is straightforward, mainly because Σ is equipped with an ultrametric ρ. On the other hand, Bowen's measure on (X, d, φ) is not necessarily Ahlfors regular, see Remark 3.4.9 about horseshoes in $\mathbb{R}^3$ and $\mathbb{R}^4$. However, if the metric d is homogeneous enough, for instance exhibits self-similarity, in Theorem 3.4.6 we show that it is possible to transfer the Ahlfors regularity of the Parry measure down to the Bowen measure using the factor map π. Then, by applying the work of Artigue [7] on the construction of self-similar metrics for expansive dynamical systems, we obtain the following.

Theorem 1. *Any mixing Smale space* (X, d, φ) *with topological entropy* $\mathrm{h}(\varphi)$ *admits a compatible self-similar metric* d' *with respect to which Bowen's measure of maximal entropy on* (X, d', φ) *is Ahlfors* s_0*-regular, where* $s_0 = 2\,\mathrm{h}(\varphi)/\log \lambda_{d'}$ *and* $\lambda_{d'} > 1$ *is the self-similar contraction constant of* (X, d', φ)*. Consequently, the Hausdorff, box-counting and Assouad dimensions satisfy*

$$\dim_H(X, d') = \dim_B(X, d') = \dim_A(X, d') = s_0.$$

As an application, we also obtain new estimates for the Hausdorff and box-counting dimensions (see Corollary 3.4.10) of Smale spaces, where φ and φ^{-1} are Lipschitz maps (with some restrictions).

The main tool for transferring Ahlfors regularity using factor maps is Theorem 3.3.2 which essentially provides a way to approximate the metric structure of Smale spaces. This theorem extends the work of Bowen on Markov partitions. More precisely, given a Smale space (X, d, φ) equipped with a Markov partition, using the dynamics, we build a refining sequence of open covers $(\mathcal{V}_n)_{n\geq 0}$ of X with diameters converging to zero. The sequence $(\mathcal{V}_n)_{n\geq 0}$ encodes various topological properties of the dynamical system from which the most important is derived from the Neighbouring Rectangles Lemma 3.2.16. Without having any assumption on the metric d, this lemma implies that

$$\sup_n \max_{V\in\mathcal{V}_n} \#\{W \in \mathcal{V}_n : W \cap V \neq \varnothing\} < \infty, \tag{1.2.1}$$

which manifests that Smale spaces are homogeneous on a topological level.

Moreover, the sequence $(\mathcal{V}_n)_{n\geq 0}$ encodes the metric d. Depending on the behaviour of φ, it is possible to estimate the rate of decay of the Lebesgue covering numbers and the diameters of $\mathcal{V}_n$, as n goes to infinity. The best estimates can be obtained in the case where (X, d, φ) has self-similar dynamics, which occurs when both Lipschitz constants of φ and φ^{-1} are equal to the contraction/expansion constant λ_d of (X, d, φ). In this case, the uniform upper bound in (1.2.1) yields that for every $r \in (0, \operatorname{diam}(X))$ with $n_r = \min\{n \in \mathbb{N} : \operatorname{diam}(V) \leq r, \text{ for every } V \in \mathcal{V}_n\}$ it holds that

$$\sup_{x\in X} \sup_r \#\{W \in \mathcal{V}_{n_r} : W \cap \overline{B}(x, r) \neq \varnothing\} < \infty. \tag{1.2.2}$$

This uniformity should be interpreted as a homogeneity condition of the metric space at every scale.

The other known examples of Smale spaces with an Ahlfors regular measure (these differ from Bowen's measure) are the (Euclidean) mixing parts of $C^{1+\varepsilon}$-conformal Axiom A systems, following Pesin in [101]. The tools that we develop here are new and can also be applied to the study of non-Euclidean Smale spaces, like Wieler solenoids [134], since the existing techniques are restricted to the Euclidean setting. Further, the coincidence of the Hausdorff and box-counting dimensions for self-similar Smale spaces can be also obtained using Barreira's work [11] on the dimension theory of Smale spaces with bi-Lipschitz local product structure, and with asymptotically conformal dynamics on stable and unstable sets. This is proved in Remark 3.4.7. However, Barreira's arguments are particularly designed to compute Hausdorff and box-counting dimensions, and cannot be used to calculate Assouad dimensions or give strong homogeneity results like Ahlfors regularity.

For a thorough discussion on Ahlfors regularity in hyperbolic dynamical systems, and how our ideas and techniques differ from all previous work, we refer the reader to the introduction of our paper [55] that consists of Chapters 2 and 3 of this thesis.

1.3 Smooth extensions of Ruelle algebras and non-commutative dimension

The second main part of the thesis concerns the study of p-smooth extensions of the stable and unstable Ruelle algebras. A p-smooth extension of a separable C^*-algebra A is an extension by the ideal of compact operators that can restrict to an algebraic extension of a dense $*$-subalgebra $\mathscr{A} \subset A$ by the Schatten p-ideal $\mathcal{L}^p$, where $p > 0$. For an explicit description of smooth extensions in terms of Busby invariants we refer to Definition 4.2.2. Note here that for $p \in (0,1)$ the ideal $\mathcal{L}^p$ is a quasi-Banach space.

This concept was introduced by Douglas [40] who studied 1-smooth extensions of finite complexes. Shortly after, it was studied by Douglas and Voiculescu [41] in the case of sphere extensions. They obtained that, for every $n \geq 2$, every $(n-1)$-smooth extension of $C(S^{2n-1})$, where $S^{2n-1} \subset \mathbb{C}^n$, should be trivial, and that p-smooth non-trivial extensions exist if $p > n$. Therefore, smoothness is strongly related with dimension. It seems natural then to consider the notion of uniformly $\mathcal{L}^p$-smooth C^*-algebras; these are separable, nuclear C^*-algebras A that have a dense $*$-subalgebra $\mathscr{A} \subset A$ such that every class in the extensions group $\operatorname{Ext}(A)$ has a representative that is p-smooth on $\mathscr{A}$.

Important examples of uniformly smooth C^*-algebras can be obtained from the work of Emerson and Nica [48] on the crossed product C^*-algebras $C(\partial\Gamma) \rtimes \Gamma$ of certain hyperbolic groups Γ, and the work of Goffeng and Mesland [58] on Cuntz-Krieger algebras. The first C^*-algebras are uniformly $\mathcal{L}^p$-smooth, for p depending on a family of Hausdorff dimensions of the Gromov boundary $\partial\Gamma$. The latter are uniformly $\mathcal{L}^p$-smooth for every $p > 0$, thus recovering the zero-dimensionality of the underlying shift space. In both cases, the authors worked on the level of K-homology and used K-duality to obtain uniform results. For details on the relation of summable Fredholm modules and smooth extensions we refer to Subsection 4.2.1.

Given a separable, nuclear C^*-algebra A the infimum of all $p > 0$ for which A is uniformly $\mathcal{L}^p$-smooth is related to the topological dimension of the (possible) underlying dynamical system of A. We refer to this infimum as the "degree of irregularity" of A, see Remark 4.2.6. Although it is not investigated here, it might be possible that the degree of irregularity satisfies dimension type conditions for general separable, nuclear C^*-algebras. If this is the case, it would be interesting to know if it relates to other notions of noncommutative dimension. In this light, we note that the degree of irregularity is not directly related with Connes' spectral-triple dimension theory, as it is finite for the aforementioned purely infinite C^*-algebras. The same can be said for the nuclear dimension which is another extremely successful notion of noncommutative dimension [137]. From [58] we know that the degree of irregularity of Cuntz-Krieger algebras is zero, while from [118] it follows that their nuclear dimension is one.

In this thesis, by also using the K-duality for Ruelle algebras, we prove directly (without passing to the K-homology picture) that the stable and unstable Ruelle algebras of a Smale space (X,d,φ) are uniformly $\mathcal{L}^p$-smooth, for p depending on the topological entropy and the λ-number $\lambda(X,\varphi)\in(1,\infty]$ of the Smale space; that is, the supremum of all contraction constants associated to compatible self-similar metrics on (X,d,φ). In Proposition 6.1.18 we prove that the λ-numbers are topological invariants and in Proposition 6.1.19 we show that they are related to the topological dimension of the corresponding Smale space, in particular, $\lambda(X,\varphi)=\infty$ if and only if X is zero-dimensional. The precise statement of the following theorem can be found in Corollaries 6.2.19 and 6.2.21.

Theorem 2. *Let (X,d,φ) be an irreducible Smale space. There exists a (topological) constant $d(\lambda(X,\varphi),\mathrm{h}(\varphi))\geq 0$ such that, the Ruelle algebras $\mathcal{R}^s$ and $\mathcal{R}^u$ are uniformly $\mathcal{L}^p$-smooth on holomorphically stable dense $*$-subalgebras, for every $p>d(\lambda(X,\varphi),\mathrm{h}(\varphi))$. In particular, if X is zero-dimensional, both Ruelle algebras are uniformly $\mathcal{L}^p$-smooth, for every $p>0$.*

We note that in the zero-dimensional case, the Ruelle algebras are stabilised Cuntz-Krieger algebras and hence their uniform smoothness condition does not follow explicitly from the work in [58]. Also, our approach is purely dynamical in nature and we work on the level of Smale space groupoids.

The main strategy for proving Theorem 2 is to prove that, for a given Smale space, the KPW-extension τ_Δ (see (1.1.1)) is p-smooth, for some $p>0$, and then try to lower the value of p as much as possible. This requires a novel approach. First, we use the Alexandroff-Urysohn-Frink Metrisation Theorem [54] to construct metrics on the (étale) Smale space groupoids (Theorem 6.1.4). These metrics generate the étale topologies and make the groupoid maps bi-Lipschitz or locally bi-Lipschitz, depending on whether these maps are homeomorphisms or local homeomorphisms. Consequently, these metrics produce dense (Lipschitz) $*$-subalgebras $\Lambda_s\subset\mathcal{R}^s$ and $\Lambda_u\subset\mathcal{R}^u$ for which we show (Theorems 6.2.16 and

6.2.20) that the KPW-extension τ_Δ is p-smooth on $\Lambda_s \otimes_{\mathrm{alg}} \Lambda_u$, for some $p > 0$. Then, we use holomorphic functional calculus to enlarge Λ_s and Λ_u to holomorphically stable $\star$-subalgebras $\mathrm{H}_s \subset \mathcal{R}^s$ and $\mathrm{H}_u \subset \mathcal{R}^u$ for which τ_Δ is still p-smooth on $\mathrm{H}_s \otimes_{\mathrm{alg}} \mathrm{H}_u$. We note that this last step forces us to work in the setting of quasi-Banach spaces since we also consider the Schatten p-ideals for $p \in (0,1)$. Moreover, the whole process so far depends on the Smale space metric, and it can be optimised (lowering the value of p) by considering more appropriate compatible metrics on the Smale space in the first place. In particular, by considering all self-similar metrics on (X, d, φ), the optimisation yields a topological constant $d(\lambda(X,\varphi), \mathrm{h}(\varphi)) \geq 0$ such that, for every $p > d(\lambda(X,\varphi), \mathrm{h}(\varphi))$, there is a self-similar metric d' and holomorphically stable dense $\star$-subalgebras $\mathrm{H}_{s,d'}$, $\mathrm{H}_{u,d'}$ constructed as above, so that τ_Δ is p-smooth on $\mathrm{H}_{s,d'} \otimes_{\mathrm{alg}} \mathrm{H}_{u,d'}$. For a detailed discussion on this matter we refer to Subsection 6.1.3. Finally, the proof of Theorem 2 follows from computing slant products of the form (1.1.1) on the level of Fredholm modules and extensions (Proposition 4.2.11 and Corollary 4.2.12). In the literature we were able to find these computations only in the setting of unital C^*-algebras [58], which do not apply in our case since both $\mathcal{R}^s$ and $\mathcal{R}^u$ are never unital. Here we circumvent the difficulty of lacking a unit by using approximate identities $(u_n)_{n\in\mathbb{N}}$ satisfying $u_{n+1}u_n = u_n$ and the fact that the (full) corners of simple, purely infinite C^*-algebras have nice K-theory [30].

1.4 Spanier-Whitehead K-duality for Ruelle algebras and Fredholm modules

The final main part of the thesis is about constructing θ-summable Fredholm modules that represent the K-homology duality classes Δ (see (1.1.1)). This is interesting for several reasons. For instance, it allows to find explicit descriptions of K-homology classes of the Ruelle algebras in terms of Fredholm modules. It also opens the window to study Lefschetz fixed point formulas for endomorphisms of Ruelle algebras in KK-theory [47]. Finally, it is a natural step in constructing interesting spectral triples [28, Chapter VI].

Abstract Fredholm module representatives exist due to the nuclearity of the Ruelle algebras, as it follows from the abstract machinery of the Choi-Effros Lifting Theorem and Stinespring's Dilation Theorem, see Subsection 4.1.2. However, in general, these representatives are not geometric nor θ-summable. Here, we focus on constructing concrete θ-summable representatives by using the geometry of Smale spaces. Finding Fredholm module representatives of K-homology duality classes for noncommutative C^*-algebras has been achieved by Goffeng and Mesland [58] in the case of Cuntz-Krieger algebras, using Fock space representations. Another instance can be found in the work of Echterhoff, Emerson and Kim [43]. It concerns crossed products of complete Riemannian manifolds by countable groups acting isometrically, co-compactly and properly. However, Ruelle algebras are not associated to Fock spaces, in general, and Smale spaces are typically fractal rather than smooth manifolds. Further, we should note that the work of Rennie, Robertson and Sims [118] on constructing K-homology duality classes (in terms of spectral triples) over crossed products by $\mathbb{Z}$, with coefficient algebras being K-dual to their opposites, does not apply in our case. One simple reason is that the coefficient algebras of $\mathcal{R}^s$ and $\mathcal{R}^u$ do not have K-duals in general, even in simple cases like the full 2-shift, see Subsection 5.1.2.

Finding a Fredholm module representative for such a class Δ is a technically difficult problem and requires most of the machinery that we used and developed for studying the Ahlfors regularity of Bowen's measure and the smoothness of extensions of the Ruelle algebras. In Section 7.1 we present the intuitive walkthrough that led us to construct θ-summable Fredholm module representatives for Δ. This is where we also compare our method with the one used in [58] for Cuntz-Krieger algebras. The general philosophy is to first represent $\mathcal{R}^s \otimes \mathcal{R}^u$ on a large Hilbert space in order to untwist τ_Δ and then compress back using the range projection of a certain isometry. The difficulty lies in constructing this isometry. We do so by using our generalisation of Markov partitions that we described in Section 1.2 of the introduction. Our isometry is very geometric in nature and can be related to Whitney's Embedding Theorem [132] of compact topological manifolds in Euclidean spaces of sufficiently large dimension, see Subsection 7.1.2. The precise statement of our result (Theorem 7.2.1) is too technical to be fully presented here. Nevertheless, we state it without belabouring the technicalities as follows.

Theorem 3. *The abstract* K*-homology duality class Δ of the KPW-extension τ_Δ has a θ-summable Fredholm module representative.*

In this way, using our calculation for slant products of the form (1.1.1) we obtain a complete description of the K-homology classes of both Ruelle algebras in terms of Fredholm modules constructed by Markov partitions.

1.5 Thesis structure

In Chapter 2, we build on the idea that compact metric spaces are quotients of Cantor spaces by associating the latter to infinite path spaces of rooted graphs. This allows us to study the fine structure of these quotients maps. In Section 2.1 we start with some preliminaries in topological dynamical systems and dimension theory. In Section 2.2 we introduce and study in detail the notion of approximation graphs which provides a convenient way to study refining sequences of covers. We investigate structural properties of approximation graphs and their behaviour under dynamics. Finally, we introduce the concept of geometric approximation graphs and a sufficient condition for a compact metric space to have finite Assouad dimension.

Chapter 3 is about the Ahlfors regularity of Bowen's measure and focuses on the first main part of the thesis, see Section 1.2 of the introduction. In Section 3.1 we provide a basic introduction to Smale spaces and present a detailed proof that the Parry measure is Ahlfors regular. We discuss metrics of expansive dynamical systems and make an observation that leads to new dimension estimates for Smale spaces with bi-Lipschitz homeomorphisms. In Section 3.2 we introduce the notion of Markov partitions and we show how to construct an approximation graph from a given Markov partition. The structural properties of such an approximation graph are presented in Proposition 3.2.6. One of the key tools regarding the Ahlfors regularity of Bowen's measure is the Neighbouring Rectangles Lemma 3.2.16. In Section 3.3, using Markov partitions, we build refining sequences of open covers that allow us to transfer the Ahlfors regularity of the shift space down to the Smale space. For such refining sequences we study the multiplicities, cardinalities and rates of decay of Lebesgue covering numbers and diameters of the covers. All these results establish Theorem 3.3.2. Finally, in Section 3.4 we study Smale spaces with some degree of homogeneity, namely, we introduce the concept of semi-conformal Smale spaces which include self-similar Smale spaces and Wieler solenoids. In this context, we prove Theorem 3.4.6 and obtain Corollary 3.4.8 which are about the Ahlfors regularity of Bowen's measure. We conclude with the new dimension estimates of Corollary 3.4.10.

Chapter 4 is the K-theoretic backbone of this thesis. Section 4.1 begins with a brief introduction to Kasparov's KK-theory and its relation with the Brown-Douglas-Fillmore Ext-groups. Then, we present the notion of Spanier-Whitehead K-duality. In Section 4.2 we introduce the notion of smooth extensions and summable Fredholm modules. We compute slants products in KK-theory for simple, purely infinite C^*-algebras which are not necessarily unital (Proposition 4.2.11). We conclude the section by developing several tools from holomorphic functional calculus that can be used to study the uniform smoothness of C^*-algebras with Spanier-Whitehead K-duals (Proposition 4.2.21).

In Chapter 5, we present the basic theory on Smale space C^*-algebras and their non-commutative topology. In Section 5.1 we introduce Smale space C^*-algebras and in Section 5.2 we present the work of Kaminker, Putnam and Whittaker on the Spanier-Whitehead K-duality between the stable and unstable Ruelle algebras.

Chapter 6 is about the uniform smoothness of the stable and unstable Ruelle algebras and focuses on the second main part of the thesis (Section 1.3). In Section 6.1, using the Alexandroff-Urysohn-Frink Metrisation Theorem, we construct dynamical metrics for equivalence groupoids of general Smale spaces (Theorem 6.1.4). Using an alternative method, in the case of topological Markov chains we build refined dynamical groupoid ultrametrics (Proposition 6.1.23). In Section 6.2, we show how to obtain dense (Lipschitz) $*$-subalgebras of C^*-algebras associated to general étale groupoids equipped with metrics, and use this result (together with the aforementioned Smale space groupoid metrics) to construct dense $*$-subalgebras of the stable and unstable Ruelle algebras (Propositions 6.2.6 and 6.2.7). Then, in Theorems 6.2.16 and 6.2.20 we obtain the smoothness of the KPW-extension, and in Corollaries 6.2.19 and 6.2.21 we deduce the uniform smoothness of the stable and unstable Ruelle algebras.

Chapter 7 constitutes the third main part of the thesis (Section 1.4) which is about constructing explicit θ-summable Fredholm module representatives of the K-homology duality class Δ that is given in terms of the KPW-extension (see (1.1.1)). In Section 7.1 we build the necessary tools for constructing the Fredholm module representatives. The key tool in this endeavour is Theorem 3.3.2 where we construct refining sequences of generalised Markov partitions, each corresponding to a sequence of Lipschitz partitions of unity with controlled Lipschitz constants and a particular choice function from the vertices of the associated approximation graph with values in the corresponding Smale space. Finally, in Section 7.2 we derive the main result (Theorem 7.2.1).

Chapter 2

Approximations of compact metric spaces

It is well-known that every compact metrisable space Z is the quotient of a Cantor space [122]. If Z has no isolated points, the Cantor space can be realised as the infinite path space of a rooted graph that corresponds to a refining sequence of open covers of Z with diameters converging to zero, satisfying certain conditions. The quotient map takes an infinite path to a point in Z using Cantor's Intersection Lemma. In this chapter, we extend this idea to general refining sequences of open or closed covers and study the fine structure of the associated quotient maps. Further, we study factor maps for expansive dynamical systems. We conclude with a criterion for a compact metric space to have finite Assouad dimension.

Throughout, let Z be an infinite topological space, $\psi : Z \to Z$ be a continuous map and denote the corresponding dynamical system by (Z, ψ). If Z is equipped with a metric d, the dynamical system will be denoted by (Z, d, ψ). However, if there is no risk of confusion the notation will be reduced to (Z, ψ).

2.1 Preliminaries

2.1.1 Topological dynamical systems

If Z is compact, the topological entropy of (Z, ψ) is defined using open covers in the following way. Let $\mathcal{U}$ be a finite open cover of Z and $\mathrm{N}(\mathcal{U})$ denote the minimal cardinality of a subcover. By continuity of ψ we have that $\psi^{-1}(\mathcal{U}) = \{\psi^{-1}(U) : U \in \mathcal{U}\}$ is also an open cover. Then using the *joint cover* notation

$$\mathcal{W} \vee \mathcal{W}' = \{W \cap W' : W \in \mathcal{W}, W' \in \mathcal{W}'\}, \tag{2.1.1}$$

for covers $\mathcal{W}$ and $\mathcal{W}'$ of Z, one can prove (see [2]) that the following limit exists and is finite

$$\mathrm{h}(\psi,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{N}(\bigvee_{i=0}^{n-1} \psi^{-i}(\mathcal{U})). \tag{2.1.2}$$

Definition 2.1.1. ([2]) The *topological entropy* of (Z,ψ) is defined by

$$\mathrm{h}(\psi) = \sup_{\mathcal{U}} \mathrm{h}(\psi,\mathcal{U}),$$

where the supremum is taken over all open covers of Z.

We will be interested in dynamical systems with topological recurrence conditions.

Definition 2.1.2. Let (Z,ψ) be a dynamical system.

(1) A point $z \in Z$ is called *non-wandering* if for every open neighbourhood U of z there is some $n \in \mathbb{N}$ such that $\psi^n(U) \cap U \neq \varnothing$. Moreover, we say that (Z,ψ) is *non-wandering* if every $z \in Z$ is non-wandering.

(2) (Z,ψ) is called *irreducible* if for every ordered pair of non-empty open sets $U, V \subset Z$, there is some $n \in \mathbb{N}$ such that $\psi^n(U) \cap V \neq \varnothing$.

(3) (Z,ψ) is called *mixing* if for every ordered pair of non-empty open sets $U, V \subset Z$, there is some $N \in \mathbb{N}$ such that $\psi^n(U) \cap V \neq \varnothing$, for every $n \geq N$.

The following is a simple consequence of irreducibility. Recall that Z is infinite.

Proposition 2.1.3. *If (Z,ψ) is irreducible and Z is Hausdorff then Z has no isolated points.*

We introduce some notation that will be used later. First let $\#S$ denote the cardinality of any finite set S. Suppose that (Z,d) is a compact metric space. For a finite cover $\mathcal{U}$ of Z, for which $\varnothing, Z \notin \mathcal{U}$, and $Y \subset Z$ we will be interested in the quantities

$$\overline{\mathrm{diam}}(\mathcal{U}) = \max_{U\in\mathcal{U}} \mathrm{diam}(U) \tag{Q1}$$

$$\underline{\mathrm{diam}}(\mathcal{U}) = \min_{U\in\mathcal{U}} \mathrm{diam}(U) \tag{Q2}$$

$$\mathrm{N}_{\mathcal{U}}(Y) = \{U \in \mathcal{U} : U \cap Y \neq \varnothing\} \tag{Q3}$$

$$\mathrm{m}(\mathcal{U}) = \max\{n : U_{i_1} \cap \ldots \cap U_{i_n} \neq \varnothing\} \tag{Q4}$$

where $U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$ are different, and in the case where $\mathcal{U}$ is open we will also consider

$$\mathrm{Leb}(\mathcal{U}) = \min_{z\in Z} \max_{U\in\mathcal{U}} d(z, Z \smallsetminus U) > 0. \tag{Q5}$$

The last quantity is the Lebesgue covering number of $\mathcal{U}$ and it holds that for every $z \in Z$ there is some $U \in \mathcal{U}$ so that the ball $B(z,\ell) \subset U$, where $\ell = \mathrm{Leb}(\mathcal{U})$.

2.1.2 Dimension theory

We introduce several types of dimension and comment on their relationship with one another. For further details see [50,90].

Definition 2.1.4. ([93, Def. I.4]) We say that Z has *covering dimension* $\dim Z \leq n$ if every finite open cover $\mathcal{U}$ has an open refinement $\mathcal{W}$ with $\mathrm{m}(\mathcal{W}) \leq n+1$. We say that $\dim Z = n$ if it is true that $\dim Z \leq n$ and it is false that $\dim Z \leq n-1$.

Suppose now that Z has a metric d. If $\{U_i\}$ is a countable (or finite) cover of Z with diameter at most δ, we say that $\{U_i\}$ is a δ-*cover* of Z. The Hausdorff measure and Hausdorff dimension are defined as follows. Let $s \geq 0$ and for every $\delta > 0$ define

$$\mathcal{H}^s_\delta(Z) = \inf\{\sum_i (\mathrm{diam}(U_i))^s : \{U_i\} \text{ is a } \delta\text{-cover of } Z\}. \tag{2.1.3}$$

One can show that the limit

$$\mathcal{H}^s(Z) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z) \tag{2.1.4}$$

exists and that $\mathcal{H}^s$ defines a measure, see [50, Section 2.1].

Definition 2.1.5. ([50, Section 2.1]) We call $\mathcal{H}^s$ the s-*dimensional Hausdorff measure* on (Z,d).

Definition 2.1.6. ([50, Section 2.2]) The *Hausdorff dimension* of (Z,d) is defined as

$$\dim_H Z = \inf\{s \geq 0 : \mathcal{H}^s(Z) = 0\}.$$

Let $\mathrm{N}_\delta(Z)$ be the smallest number of sets of diameter at most $\delta > 0$ which can cover Z.

Definition 2.1.7. ([50, Section 3.1]) The *lower* and *upper box-counting dimensions* of (Z,d) are defined as

$$\underline{\dim}_B Z = \liminf_{\delta\to 0} \frac{\log \mathrm{N}_\delta(Z)}{-\log\delta}$$
$$\overline{\dim}_B Z = \limsup_{\delta\to 0} \frac{\log \mathrm{N}_\delta(Z)}{-\log\delta}.$$

If these are equal, their common value is called the *box-counting dimension* and denoted by $\dim_B Z$. As we will shortly see, in many interesting cases the box-counting dimension coincides with the Hausdorff dimension.

Another important metric dimension was introduced by Assouad in [8–10] in the framework of bi-Lipschitz embeddability of metric spaces into Euclidean spaces. For a detailed exposition see [87]. Moreover, there is a vast literature on Assouad dimension, see [52,89,95]. We say that (Z,d) is *bi-Lipschitz embeddable* into some $\mathbb{R}^N$ if there exists a bi-Lipschitz map $f:(Z,d)\to\mathbb{R}^N$. Any such metric space should have the following doubling property [112, Lemma 9.4].

Definition 2.1.8. ([112, p.84]) A metric space (Z,d) is called K-*doubling*, where $K\geq 1$, if every ball of radius $2r$ can be covered by K balls of radius r, where K is independent of r.

Assouad, in an attempt to study the converse question; whether every K-doubling metric space admits a bi-Lipschitz embedding into some $\mathbb{R}^N$, obtained the following.

Theorem 2.1.9 ([10, Proposition 2.6])**.** *Let (Z,d) be K-doubling. For every $\varepsilon\in(0,1)$ there is a bi-Lipschitz embedding $f:(Z,d^{\varepsilon})\to\mathbb{R}^N$, for some $N\in\mathbb{N}$ that depends on K and ε.*

We note that d^{ε} is the metric defined by $d^{\varepsilon}(z,y)=d(z,y)^{\varepsilon}$. Hence Assouad's Theorem does not offer a bi-Lipschitz embedding of the actual space but of a *snowflaked* version of it. However, it turns out that there are doubling spaces which do not admit bi-Lipschitz embeddings [86,125]. The dependence of N on ε has been studied in [94].

Definition 2.1.10 ([10])**.** Let (Z,d) be a metric space. Suppose $s\geq 0$ and $C\geq 0$ are numbers such that

$$\#Y\leq C(b/a)^s$$

wherever $0<a\leq b$ and $Y\subset Z$ is a finite subset with $a\leq d(y,y')\leq b$ if $y,y'\in Y$ and $y\neq y'$. Then Z is called (C,s)-*homogeneous*. We say that Z is s-*homogeneous* if it is (C,s)-*homogeneous* for some C. The *Assouad dimension* is defined to be

$$\dim_{\mathrm{A}}Z=\inf\{s\in[0,\infty):Z\ \text{ is s-homogeneous}\}.$$

It is straightforward to show that $\dim_{\mathrm{A}}Z$ is finite if and only if Z is K-doubling [112, Lemma 9.4]. Specifically, if Z is (C,s)-homogeneous then it is $C2^s$-doubling. Before passing to the interplay of measure theory and dimension theory let us summarise the known relations between the dimensions discussed so far.

Proposition 2.1.11 ([50,90])**.** *For a totally bounded metric space (Z,d) it holds*

$$\dim Z\leq\dim_H Z\leq\underline{\dim}_B Z\leq\overline{\dim}_B Z\leq\dim_A Z.$$

We now introduce a class of measures with an important role in the study of metric spaces.

Definition 2.1.12 ([90, Def. 4.1.2]). A Borel measure μ on (Z,d) is called *D-doubling*, where $D \geq 1$, if

$$0 < \mu(\overline{B}(z,2r)) \leq D\mu(\overline{B}(z,r)) < \infty$$

for every $z \in Z$ and $r \in [0, \operatorname{diam} Z)$.

It turns out that for a complete metric space the doubling property is equivalent to the existence of a doubling measure, see [64, Section 13]. The doubling property is not uncommon, for instance if Z is a separable metrizable space with $\dim Z < \infty$ then there is a totally bounded metric so that $\dim_A Z = \dim Z$ [87, Theorem 4.3]. This is also true to some extent for doubling measures [87, Theorem 4.5]. Significantly more regular measures can be constructed on spaces that exhibit self-similarity, like the middle-third Cantor set or, more generally, sets generated by iterated function systems satisfying the open-set condition, see [90]. A prominent case of measures with the doubling property are the Ahlfors regular measures.

Definition 2.1.13. ([90, Def. 1.4.13]) A metric space (Z,d) is *Ahlfors s-regular* for some $s > 0$, if there exists a Borel measure μ on Z and $C > 0$ so that

$$C^{-1} r^s \leq \mu(\overline{B}(z,r)) \leq C r^s,$$

for all $z \in Z$, $r \in [0, \operatorname{diam} Z)$. Such μ is called *Ahlfors s-regular* (or Ahlfors regular).

Remark 2.1.14. If μ is an Ahlfors s-regular measure on (Z,d) then it is comparable to the s-dimensional Hausdorff measure $\mathcal{H}^s$, in the sense that μ is within constant multiples of $\mathcal{H}^s$. Therefore, $\mathcal{H}^s$ is strictly positive. A typical example of an Ahlfors regular space is the classical Sierpinski gasket on which the $\log 3/\log 2$-dimensional Hausdorff measure is Ahlfors $\log 3/\log 2$-regular, see [90, Example 8.3.4].

Proposition 2.1.15 ([90, Section 1.4]). *If the metric space (Z,d) is Ahlfors s-regular then $\dim_H Z = \dim_A Z = s$.*

2.2 Approximations via rooted graphs

2.2.1 Refining sequences

The next concept provides a way to study topological or dynamical properties on a compact metric space by means of finite approximations. It was first introduced in [2, Cor. p.314] as a way to study topological entropy. Here we adjust it to our needs.

Definition 2.2.1. Let (Z, d) be a compact metric space. A sequence $(\mathcal{V}_n)_{n\geq 0}$ of finite covers of Z, which are either all open or all closed with non-empty interiors, is called *refining* if, $\mathcal{V}_0 = \{Z\}$ and for every $n \geq 0$ any element of $\mathcal{V}_{n+1}$ lies inside some element of $\mathcal{V}_n$, so that

$$\lim_{n\to\infty} \overline{\operatorname{diam}}(\mathcal{V}_n) = 0.$$

It is straightforward to check that, in the case of open covers, the collection $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ forms a countable basis for the topology on Z and that $\underline{\operatorname{diam}}(\mathcal{V}_n) > 0$, for every $n \in \mathbb{N}$, if and only if Z does not have isolated-points. Interesting refining sequences exist over spaces Z that admit an expansive homeomorphism ψ; that is, there is some $\varepsilon_Z > 0$ so that if $d(\psi^n(x), \psi^n(y)) \leq \varepsilon_Z$ for every $n \in \mathbb{Z}$ then $x = y$.

Definition 2.2.2 ([130, Def. 5.10])**.** Let (Z, d) be a compact metric space and ψ be a homeomorphism. A *generator* for (Z, ψ) is a finite open cover $\mathcal{V}_1$ of Z such that for each bi-infinite sequence $\{V_i\}_{i\in\mathbb{Z}}$ of elements of $\mathcal{V}_1$, it holds that $\bigcap_{i\in\mathbb{Z}} \psi^{-i}(\operatorname{cl}(V_i))$ is at most one point.

It turns out that the existence of a generator is equivalent to the expansiveness of the system [130, Theorem 5.22]. Given a generator we obtain a refining sequence $(\mathcal{V}_n)_{n\geq 0}$ of open covers by setting $\mathcal{V}_0 = \{Z\}$ and for $n \in \mathbb{N}$

$$\mathcal{V}_n = \bigvee_{i=1-n}^{n-1} \psi^{-i}(\mathcal{V}_1), \tag{2.2.1}$$

since $\lim_{n\to\infty} \overline{\operatorname{diam}}(\mathcal{V}_n) = 0$. Also, $\mathrm{h}(\psi, \mathcal{V}_1) = \mathrm{h}(\psi)$. For these we refer to [130].

The notion of refining sequences was also used to prove that any compact metrizable space Z is the quotient of a Cantor space, basically, built from the non-isolated points of Z [122]. The dynamical analogue is that any expansive dynamical system (Z, ψ) is a factor of some (Σ, σ) where Σ is a compact zero-dimensional space and σ is a homeomorphism. If Z has no isolated points then Σ will be a Cantor space, see Corollary 2.2.9. The basic idea is that Σ corresponds to the path space of an infinite rooted graph induced by a refining sequence, as in (2.2.1).

2.2.2 Approximation graphs

Given a refining sequence of open or closed covers for a compact metric space (Z, d) we construct a rooted graph, with vertices the sets in the covers and edges defined by inclusion of the sets in preceding refinements. Such a graph will be called an approximation graph since its infinite path space will provide an approximation of (Z, d). We study how precise this approximation can be and how it behaves in the dynamical context.

This notion was previously used by Palmer in his thesis [96], who proved the existence of an abstract refining sequence of open covers whose approximation graph can be used to obtain the Hausdorff measure and Hausdorff dimension of (Z,d). However, Palmer did not study the structure of approximation graphs nor did he study dynamics on them. Here we extend Palmer's definition by including refining sequences of closed covers and make a few adjustments that suit our needs. A related but different concept known as approximating graphs has been used in [99] for ultrametric Cantor spaces and in [70] for transversals of substitution tilings.

Definition 2.2.3. Let (Z,d) be a compact metric space and $(\mathcal{V}_n)_{n\geq 0}$ be a refining sequence of Z. The corresponding *approximation graph* is the rooted graph $\Gamma = (\mathcal{V},\mathcal{E})$ where

(1) the set of vertices $\mathcal{V}$ is given by the disjoint union $\coprod_{n\geq 0}\mathcal{V}_n$;

(2) the set of edges $\mathcal{E}$ is given by the disjoint union $\coprod_{n\geq 0}\mathcal{E}_n$, where

$$\mathcal{E}_n = \{(v_{n+1},v_n) : v_{n+1}\in\mathcal{V}_{n+1}, v_n\in\mathcal{V}_n, v_{n+1}\subset v_n\}$$

is the set of edges that have sources in $\mathcal{V}_n$ and ranges in $\mathcal{V}_{n+1}$. The source map s is given by $s(v_{n+1},v_n) = v_n$, the range map r is given by $r(v_{n+1},v_n) = v_{n+1}$;

(3) the root is Z.

We consider approximation graphs that have *no sinks*; for every $v\in\mathcal{V}$ it holds that $s^{-1}(v)\neq\varnothing$. Also the *only source is the root*; for every $v\neq Z$ we have $r^{-1}(v)\neq\varnothing$. This is because $(\mathcal{V}_n)_{n\geq 0}$ is a refining sequence. An approximation graph with these two conditions is an example of a Bratteli diagram [65, Definition 2.1]. A symbolic description of Z comes by considering the *space of infinite paths*

$$\mathcal{P}_\Gamma = \{\widetilde{p} = (p_n)\in\prod_n\mathcal{E}_n : s(p_{n+1}) = r(p_n)\}. \tag{2.2.2}$$

For a finite path $\mu = \mu_0\mu_1\ldots\mu_\ell$ in Γ, where each $\mu_i\in\mathcal{E}_i$, denote by C_μ the *cylinder set*

$$C_\mu = \{\widetilde{p}\in\mathcal{P}_\Gamma : p_i = \mu_i, \text{ for } 0\leq i\leq\ell\} \tag{2.2.3}$$

which is non-empty since there are no sinks. The collection of all these sets forms a clopen basis for a compact Hausdorff topology on $\mathcal{P}_\Gamma$ [5, p.18]. Therefore, $\mathcal{P}_\Gamma$ is a compact zero-dimensional space. Moreover, for $v\in\mathcal{V}_n$ let $[v]$ denote the set of paths from the root Z to v. These sets are non-empty since there are no sources. Let

$$C_{[v]} = \bigcup_{\mu\in[v]} C_\mu \tag{2.2.4}$$

and because $[v]$ is finite, $C_{[v]}$ is clopen. The collection of these sets forms a sub-basis for the cylinder set topology on $\mathcal{P}_\Gamma$. Finally, let $\mathrm{C}(v) = \{v_{n+1}\in\mathcal{V}_{n+1} : v_{n+1}\subset v\}$ denote the *descendants* of $v\in\mathcal{V}_n$ in $\mathcal{V}_{n+1}$.

Remark 2.2.4. $\mathcal{P}_\Gamma$ is a Cantor space if and only if for every $n \geq 0$ and every $v \in \mathcal{V}_n$, there is a path from v to some $w \in \mathcal{V}_m$ where $m \geq n, \#(s^{-1}(w)) \geq 2$ [5, Lemma 6.4]. It follows that $\mathcal{P}_\Gamma$ is a Cantor space if and only if Z has no isolated points.

Due to Cantor's Intersection Theorem we can define the map $\pi_\Gamma : \mathcal{P}_\Gamma \to Z$ given by

$$\widetilde{p} \mapsto \bigcap_{n \geq 0} \mathrm{cl}(r(p_n)). \tag{2.2.5}$$

Proposition 2.2.5. *The map π_Γ is continuous and surjective. Consequently, it is a quotient map.*

Proof. Continuity of π_Γ follows because if U is an open neighbourhood of $\pi_\Gamma(\widetilde{p})$, since $\mathrm{diam}(r(p_n))$ tends to zero, there is some $n_0 \in \mathbb{N}$ such that $\mathrm{cl}(r(p_{n_0})) \subset U$ and hence $\pi_\Gamma(C_{p_0 \dots p_{n_0}}) \subset \mathrm{cl}(r(p_{n_0})) \subset U$.

The surjectivity is more interesting and is basically a consequence of König's Lemma [38, Lemma 8.1.2]. Let $z \in Z$ and for each $n \geq 0$ consider the sets in $\mathcal{V}_n$ that contain z, that is $\mathrm{N}_{\mathcal{V}_n}(\{z\})$. Let Γ_z be the sub-graph of Γ restricted on the set of vertices $\coprod_{n \geq 0} \mathrm{N}_{\mathcal{V}_n}(\{z\})$. It is an infinite rooted graph with vertices of finite degree and contains some infinite rooted tree Γ'_z. From König's Lemma, the tree Γ'_z has an infinite path $\widetilde{p} \in \mathcal{P}_\Gamma$ and $\pi_\Gamma(\widetilde{p}) = z$. □

2.2.3 Essentiality of approximation graphs

We introduce some structural properties of approximation graphs that can be used to construct spectral triples over compact metric spaces in the sense of Christensen and Ivan [25]. However, we do not deal with this here. First we define the *overlapping set* of $\mathcal{V}_n$ to be

$$\mathcal{Y}_n = \{\mathrm{cl}(v_n) \cap \mathrm{cl}(w_n) : v_n \neq w_n \in \mathcal{V}_n\}, \tag{2.2.6}$$

and its *essential part* to be

$$\mathcal{V}_n^{\mathrm{ess}} = \{v_n^{\mathrm{ess}} : v_n \in \mathcal{V}_n\} \text{ where } v_n^{\mathrm{ess}} = \mathrm{int}(v_n) \setminus \bigcup \mathcal{Y}_n. \tag{2.2.7}$$

Definition 2.2.6. An approximation graph Γ is called *regular* if for every $n \in \mathbb{N}$ and $v_n \in \mathcal{V}_n$ we have

$$v_n = \bigcup \{v_{n+1} \in \mathcal{V}_{n+1} : v_{n+1} \subset v_n\} \text{ and} \tag{E1}$$

$$v_n^{\mathrm{ess}} \neq \varnothing. \tag{E2}$$

If Γ consists of closed covers and $\bigcup \mathcal{V}_n^{\mathrm{ess}}$ is dense in Z, for every $n \in \mathbb{N}$, we will say that Γ is *essential*.

Any Γ induced by a generator (see equation (2.2.1)) satisfies condition (E1). Also, an arbitrary Γ can always be modified to satisfy it, but at the cost of increasing the cardinality of the covers. Condition (E2) is a type of regularity assumption on the covers. More precisely, if $\mathrm{cl}(v_n) = \mathrm{cl}(w_n)$ then $v_n = w_n$, because otherwise $\mathrm{cl}(v_n) \in \mathcal{Y}_n$ and $v_n^{\mathrm{ess}} = \varnothing$. Moreover, for every $v_n \in \mathcal{V}_n$ we have

$$\mathrm{int}(\pi_\Gamma(C_{[v_n]})) \neq \varnothing \tag{2.2.8}$$

since $\varnothing \neq \pi_\Gamma^{-1}(v_n^{\mathrm{ess}}) \subset C_{[v_n]}$. Indeed, if $z \in v_n^{\mathrm{ess}}$ and $\widetilde{p} \in \mathcal{P}_\Gamma$ is such that $\pi_\Gamma(\widetilde{p}) = z$, then $z \in \mathrm{cl}(r(p_n)) \cap v_n^{\mathrm{ess}}$ and hence $r(p_n) = v_n$. Finally, essentiality really means that for every $m \geq n$ and $v_m \in \mathcal{V}_m$ there is a unique $v_n \in \mathcal{V}_n$ such that $v_m \subset v_n$. Consequently, for every cylinder set it holds that

$$C_{\mu_0 \ldots \mu_\ell} = C_{[r(\mu_\ell)]}. \tag{2.2.9}$$

To see how well $\mathcal{P}_\Gamma$ approximates Z consider the set

$$\mathcal{I}_\Gamma = \{\widetilde{p} \in \mathcal{P}_\Gamma : \#\pi_\Gamma^{-1}(\widetilde{p}) = 1\} \tag{2.2.10}$$

on which π_Γ is injective.

Proposition 2.2.7. *For a regular approximation graph* Γ *we have*

$$\mathcal{I}_\Gamma = \pi_\Gamma^{-1}\Big(\bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n^{ess}\Big).$$

Proof. Let $z \in \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n^{\mathrm{ess}}$ and assume to the contrary that $\pi_\Gamma^{-1}(z) \notin \mathcal{I}_\Gamma$. Then there are two different $\widetilde{p}, \widetilde{q} \in \mathcal{P}_\Gamma$ such that $\pi_\Gamma(\widetilde{p}) = z = \pi_\Gamma(\widetilde{q})$. Let n_0 be the first time when $(r(p_{n_0}), s(p_{n_0})) \neq (r(q_{n_0}), s(q_{n_0}))$, meaning $s(p_{n_0}) = s(q_{n_0})$ and $r(p_{n_0}) \neq r(q_{n_0})$. Then $z \in \mathrm{cl}(r(p_{n_0})) \cap \mathrm{cl}(r(q_{n_0})) \in \mathcal{Y}_{n_0}$ which is a contradiction.

To prove that $\mathcal{I}_\Gamma \subset \pi_\Gamma^{-1}(\bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n^{\mathrm{ess}})$, let $\widetilde{p} \in \mathcal{I}_\Gamma$ and assume to the contrary that $\pi_\Gamma(\widetilde{p}) \notin \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n^{\mathrm{ess}}$. This means there is some $n_0 \in \mathbb{N}$ such that $\pi_\Gamma(\widetilde{p}) \in \bigcup \mathcal{Y}_{n_0}$. Consequently, for some $v_{n_0} \in \mathcal{V}_{n_0}$ that is different from $r(p_{n_0})$ we have $\pi_\Gamma(\widetilde{p}) \in \mathrm{cl}(v_{n_0}) \cap \mathrm{cl}(r(p_{n_0}))$. Since π_Γ is surjective, there is a path $\widetilde{q} \in \mathcal{P}_\Gamma$ such that $\pi_\Gamma(\widetilde{q}) = \pi_\Gamma(\widetilde{p})$ and specifically due to condition (E1) we can arrange that $r(q_{n_0}) = v_{n_0}$. This contradicts the fact that $\widetilde{p} \in \mathcal{I}_\Gamma$. □

Proposition 2.2.8. *Suppose* Γ *is an essential approximation graph. Then* $\mathcal{I}_\Gamma$ *is a dense* G_δ*-set.*

Proof. The fact that $\mathcal{I}_\Gamma$ is a G_δ-subset of $\mathcal{P}_\Gamma$ follows from the continuity of π_Γ. To show that $\mathcal{I}_\Gamma$ is dense we note that every cylinder set satisfies $C_{\mu_0 \ldots \mu_\ell} = C_{[r(\mu_\ell)]}$, see equation (2.2.9). Then from (2.2.8) we get that $\mathrm{int}(\pi_\Gamma(C_{\mu_0 \ldots \mu_\ell})) \neq \varnothing$ and hence for every open set $U \subset \mathcal{P}_\Gamma$ we have $\mathrm{int}(\pi_\Gamma(U)) \neq \varnothing$. The Baire Category Theorem guarantees that $D = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n^{\mathrm{ess}}$ is dense in Z, hence if $U \subset \mathcal{P}_\Gamma$ is open then $\mathrm{int}(\pi_\Gamma(U)) \cap D \neq \varnothing$. Therefore, $U \cap \mathcal{I}_\Gamma \neq \varnothing$. □

2.2.4 Dynamic approximation graphs

Suppose that (Z,d) admits an expansive homeomorphism ψ with expansivity constant $\varepsilon_Z > 0$. We will show how to build an essential approximation graph for Z whose infinite path space provides a symbolic representation for the ψ-orbits.

The first step is to construct a closed cover $\mathcal{V}_1$ of diameter at most ε_Z, where $\bigcup \mathcal{V}_1^{\text{ess}}$ is dense in Z. One way to do this is by considering a cover by open sets $\{U_1,\dots,U_\ell\}$ of diameter at most ε_Z, where each U_k is not covered by the closure of the other sets. Then let $u_1 = U_1$ and inductively define

$$u_{i+1} = U_{i+1} \setminus \bigcup_{j=1}^{i} \operatorname{cl}(u_j), \tag{2.2.11}$$

for $i = 1,\dots \ell-1$. It holds that each u_k is open in Z and if $k \neq l$ then $u_k \cap u_l = \varnothing$. Also, it is immediate that $\bigcup_{k=1}^{\ell} u_k$ is dense in Z. Let $\mathcal{V}_1 = \{\operatorname{cl}(u_1),\dots,\operatorname{cl}(u_\ell)\}$ and we claim that $\bigcup \mathcal{V}_1^{ess}$ is dense in Z. Indeed, for the overlapping set $\mathcal{Y}_1$, see equation (2.2.6), it holds that $\bigcup \mathcal{Y}_1$ is a closed nowhere-dense subset of Z. Therefore, since $\bigcup \mathcal{V}_1^{\text{ess}} \supset \bigcup_{k=1}^{\ell} u_k \setminus \bigcup \mathcal{Y}_1$ the conclusion follows.

The second step is to define a refining sequence $(\mathcal{V}_n)_{n\geq 0}$ of Z using the dynamics on $\mathcal{V}_1$. Let $\mathcal{V}_0 = \{Z\}$ and for $n \in \mathbb{N}$ consider

$$\mathcal{V}_n = \{v \in \bigvee_{i=1-n}^{n-1} \psi^{-i}(\mathcal{V}_1) : \operatorname{int}(v) \neq \varnothing\}. \tag{2.2.12}$$

Each $\mathcal{V}_n$ is a cover since $\bigcup \mathcal{V}_n$ is closed in Z and contains $\bigcup \mathcal{V}_n^{\text{ess}}$ which in a similar fashion is proved to be dense in Z. Moreover, from [130, Theorem 5.21] we have $\lim_{n\to\infty} \overline{\operatorname{diam}}(\mathcal{V}_n) = 0$. Consequently, the sequence $(\mathcal{V}_n)_{n\geq 0}$ is refining and induces an essential approximation graph Γ.

The final step is to define a (left) *shift map* $\sigma_\Gamma : \mathcal{P}_\Gamma \to \mathcal{P}_\Gamma$ that commutes with the quotient map $\pi_\Gamma : \mathcal{P}_\Gamma \to Z$. Let $\widetilde{p} \in \mathcal{P}_\Gamma$ and each coordinate p_n can be written uniquely in the form $(w_{-n}\dots w_0\dots w_n, w_{1-n}\dots w_0\dots w_{n-1})$, where each $w_i \in \psi^{-i}(\mathcal{V}_1)$ and every word $w_{-k}\dots w_0\dots w_k$ corresponds to $\bigcap_{i=-k}^{k} w_i$. Recursively define the path $\widetilde{q} \in \mathcal{P}_\Gamma$ with first coordinate $q_0 = (\psi(w_1), Z)$, and for $n \geq 0$ define

$$q_{n+1} = (\psi(w_{-n})\dots\psi(w_1)\dots\psi(w_{n+2}), r(q_n)), \tag{2.2.13}$$

and let $\sigma_\Gamma(\widetilde{p}) = \widetilde{q}$. The map σ_Γ is bijective with an inverse constructed by right shifting and continuous because for every $\widetilde{p}$ it holds that $\sigma_\Gamma(C_{p_0\dots p_n}) \subset C_{q_0\dots q_n}$. Since $\mathcal{P}_\Gamma$ is compact and Hausdorff the map σ_Γ is a homeomorphism. In this setting, the quotient map π_Γ

becomes a *factor map* $\pi_\Gamma : (\mathcal{P}_\Gamma, \sigma_\Gamma) \to (Z, \psi)$ since for any $\widetilde{p} \in \mathcal{P}_\Gamma$

$$\begin{aligned}
\pi_\Gamma(\sigma_\Gamma(\widetilde{p})) &= \bigcap_{n\geq 0} \mathrm{cl}(r(q_n)) \\
&= \bigcap_{n\geq 1} \psi(\mathrm{cl}(r(p_{n-1}))) \\
&= \psi(\bigcap_{n\geq 1} \mathrm{cl}(r(p_{n-1}))) \\
&= \psi(\pi_\Gamma(\widetilde{p})).
\end{aligned}$$

Corollary 2.2.9. *Every expansive dynamical system (Z, ψ) is the quotient of some $(\mathcal{P}_\Gamma, \sigma_\Gamma)$, where $\mathcal{P}_\Gamma$ is a compact zero-dimensional space constructed as above. If Z does not have isolated points then $\mathcal{P}_\Gamma$ is a Cantor space. Moreover, the factor map $\pi_\Gamma : (\mathcal{P}_\Gamma, \sigma_\Gamma) \to (Z, \psi)$ is injective on a dense G_δ-set.*

Factor maps have been studied extensively by Adler [1], particularly factor maps which are uniformly bounded to one. We will come back to this fact later. For now we can argue that our construction captures some of the dynamical behaviour of (Z, ψ).

Proposition 2.2.10. *Let Γ be an essential approximation graph. Then*

(1) if (Z, ψ) is irreducible so is $(\mathcal{P}_\Gamma, \sigma_\Gamma)$;

(2) if (Z, ψ) is mixing so is $(\mathcal{P}_\Gamma, \sigma_\Gamma)$.

Proof. We will only prove (1) since (2) is similar. Let $U, W \subset \mathcal{P}_\Gamma$ be non-empty and open. We need to find $N \in \mathbb{N}$ such that $\sigma_\Gamma^N(U) \cap W \neq \varnothing$. We can find small enough cylinder sets $C_\mu \subset U, C_{\mu'} \subset W$ and since Γ is essential, (2.2.8) implies that $\varnothing \neq \pi_\Gamma^{-1}(r(\mu_\ell)^{\mathrm{ess}}) \subset C_\mu$ and $\varnothing \neq \pi_\Gamma^{-1}(r(\mu_{\ell'})^{\mathrm{ess}}) \subset C_{\mu'}$. Since (Z, ψ) is irreducible, there is some $N \in \mathbb{N}$ such that $\psi^N(r(\mu_\ell)^{\mathrm{ess}}) \cap r(\mu_{\ell'})^{\mathrm{ess}} \neq \varnothing$. Then

$$\begin{aligned}
\varnothing &\neq \pi_\Gamma^{-1}(\psi^N(r(\mu_\ell)^{\mathrm{ess}}) \cap r(\mu_{\ell'})^{\mathrm{ess}}) \\
&= \pi_\Gamma^{-1}(\psi^N(r(\mu_\ell)^{\mathrm{ess}})) \cap \pi_\Gamma^{-1}(r(\mu_{\ell'})^{\mathrm{ess}}) \\
&= \sigma_\Gamma^N(\pi_\Gamma^{-1}(r(\mu_\ell)^{\mathrm{ess}})) \cap \pi_\Gamma^{-1}(r(\mu_{\ell'})^{\mathrm{ess}}) \\
&\subset \sigma_\Gamma^N(C_\mu) \cap C_{\mu'} \\
&\subset \sigma_\Gamma^N(U) \cap W.
\end{aligned}$$

This completes the proof. □

An essential approximation graph provides a combinatorial model for (Z, ψ) that allows us to study its topological structure. However, it is not the right tool to study the geometric or metric properties of (Z, ψ). The reason is that it consists of closed covers with nowhere-dense overlaps and hence all the Lebesgue numbers of the covers are zero. So the idea is that, given such a graph Γ', we try to enlarge it to a graph Γ consisting of open covers so that Γ' is isomorphic to a spanning subgraph of Γ; that is, a subgraph which contains every vertex of Γ.

Definition 2.2.11. An approximation graph $\Gamma = (\mathcal{V}, \mathcal{E})$ of open covers of Z will be called *metrically-essential* if there is an essential approximation graph $\Gamma' = (\mathcal{V}', \mathcal{E}')$ of Z with bijections

$$F_n : \mathcal{V}'_n \to \mathcal{V}_n,$$

given by $v'_n \mapsto v_n$ if $v'_n \subset v_n$, such that $\coprod_{n\geq 0} F_n : \Gamma' \to \Gamma$ is a graph homomorphism and the induced map $F : \mathcal{P}_{\Gamma'} \to \mathcal{P}_{\Gamma}$ satisfies $\pi_{\Gamma} \circ F = \pi_{\Gamma'}$.

Remark 2.2.12. A metrically-essential graph is not necessarily regular. Also, an essential graph is metrically essential only if Z is zero-dimensional.

2.2.5 Geometric approximation graphs and Assouad dimension

We introduce another class of approximation graphs which now encode geometric properties of their base spaces. Recall the notation (Q1-Q5) introduced in Subsection 2.1.1 and that $\mathrm{C}(v_n)$ denotes the set of descendants of $v_n \in \mathcal{V}_n$. We first define geometric approximation graphs and then discuss their properties.

Definition 2.2.13. An approximation graph $\Gamma = (\mathcal{V}, \mathcal{E})$ of open covers of Z will be called *geometric* if there exist constants $\lambda, \Lambda > 1$ with $\lambda \leq \Lambda$, constants $\eta, \theta > 0$ with $\eta \leq \theta$ and $C_\Gamma, N_\Gamma \in \mathbb{N}$ so that, for all $n \in \mathbb{N}$,

(1) $\overline{\mathrm{diam}}(\mathcal{V}_n) \leq \lambda^{-n+1}\theta$;

(2) $\mathrm{Leb}(\mathcal{V}_n) \geq \Lambda^{-n+1}\eta$;

(3) $\#\mathrm{C}(v_n) \leq C_\Gamma$, for every $v_n \in \mathcal{V}_n$;

(4) $\#\mathrm{N}_{\mathcal{V}_n}(v_n) \leq N_\Gamma$, for every $v_n \in \mathcal{V}_n$.

If $\lambda = \Lambda$, the approximation graph Γ will be called *homogeneous*. Moreover, a geometric approximation graph which is also metrically-essential (see Definition 2.2.11) will be called *geometrically-essential*.

Geometric approximation graphs are related to dimension theory and, in particular, with the following concepts.

From Theorem V.1 of [93] we see that Z has finite covering dimension at most m if and only if there is a sequence of arbitrarily small open covers $(\mathcal{U}_n)_{n\in\mathbb{N}}$ (not necessarily refining) with multiplicities $\mathrm{m}(\mathcal{U}_n) \leq m+1$, for all $n \in \mathbb{N}$. Due to condition (4), geometric approximation graphs are related to finite covering dimension. Note though that condition

(4) is stronger than having uniformly bounded multiplicities. From the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of Theorem V.1 it is possible to obtain a refining sequence by considering a subsequence. However, this increases the cardinality of the covers and the rate of decay of the Lebesgue numbers. Condition (1) can be satisfied though.

An important example of the above situation is the case where Z admits an expansive homeomorphism. In [91] Mañé proves that Z has covering dimension at most $(\#\mathcal{V}_1)^2-1$ by constructing arbitrarily small open covers with multiplicity at most $(\#\mathcal{V}_1)^2$. It turns out that the missing ingredient, which leads to a geometric approximation graph, is a Markov partition which we introduce in Section 3.2. An expansive dynamical system has a Markov partition if and only if it has a local product structure [53]. Assume now that Z admits an expansive homeomorphism $\psi : Z \to Z$ which is also Λ-bi-Lipschitz. The approximation graph Γ induced by a generator in (2.2.1) satisfies conditions (2) and (3). More precisely, $\mathrm{Leb}(\mathcal{V}_n) \geq \Lambda^{-n+1}\,\mathrm{Leb}(\mathcal{V}_1)$ and $\#\mathrm{C}(v_n) \leq (\#\mathcal{V}_1)^2$. Nonetheless, the other conditions are not necessarily satisfied and the upper bounds on $\#\mathrm{C}(v_n)$ may not be sharp.

The concept that is closest to our approximation graphs comes from the Nagata-Assouad dimension [86]. The main characterisation is that Z will have Nagata-Assouad dimension $\dim_{\mathrm{NA}} Z \leq n$ if and only if there is some $c > 0$ such that for every $r > 0$ there is a cover $\mathcal{U}_r$ of Z with $\mathrm{m}(\mathcal{U}_r) \leq n+1$, $\overline{\mathrm{diam}}(\mathcal{U}_r) \leq cr$ and $\mathrm{Leb}(\mathcal{U}_r) \geq r$, see [21, Proposition 2.2]. Therefore, a homogeneous approximation graph provides a discrete version of the above characterisation but in slightly stronger form that allows us to prove the following proposition. First, we should mention that Assouad dimension is an upper bound for the Nagata-Assouad dimension [39].

Proposition 2.2.14. *A compact metric space (Z, d) with a homogeneous approximation graph has finite Assouad dimension.*

Proof. Let $0 < a \leq b$ and consider a finite subset $Y \subset Z$ with $a \leq d(y, y') \leq b$ if $y, y' \in Y$ and $y \neq y'$. We claim that there are $s, C \geq 0$ independent of a, b and Y so that $\#Y \leq C(b/a)^s$. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a homogeneous approximation graph with constants as in Definition (2.2.13). First we prove the claim in the case where $b \leq \eta$. Define

$$n_0 = \min\{n \in \mathbb{N} : \lambda^{-n+1}\theta < a\}$$
$$m_0 = \max\{n \in \mathbb{N} : b \leq \lambda^{-n+1}\eta\}$$

and an easy computation shows that $n_0 = 1+\max\{1, \lceil \log_\lambda(\theta/a)\rceil\}$ and $m_0 = 1+\lfloor \log_\lambda(\eta/b)\rfloor$. Note that since $b \leq \eta$ we cannot have $\theta < a$. We have $m_0 < n_0$ since $\lambda^{-n_0+1}\theta < a \leq b \leq \lambda^{-m_0+1}\eta$, meaning that $n_0 - m_0 > \log_\lambda(\theta/\eta) \geq 0$. From our definition of n_0 there cannot be two elements of Y in one element of $\mathcal{V}_{n_0}$ since $\overline{\mathrm{diam}}(\mathcal{V}_{n_0}) \leq \lambda^{-n_0+1}\theta < a$. Also since $\mathrm{diam}(Y) \leq b \leq \lambda^{-m_0+1}\eta \leq \mathrm{Leb}(\mathcal{V}_{m_0})$, there is some element $v_{m_0} \in \mathcal{V}_{m_0}$ that contains Y. Actually, any element of $\mathcal{V}_{m_0}$ that intersects Y is a neighbour of v_{m_0} and from the definition of the geometric approximation graph there cannot be more than N_Γ neighbours. The descendants in $\mathcal{V}_{n_0}$ of the neighbours of v_{m_0} are the only ones that cover the whole Y, because if $v_{n_0} \in \mathcal{V}_{n_0}$ contains some $y \in Y$, then its ancestor in $\mathcal{V}_{m_0}$ should also contain y and hence be a neighbour of v_{m_0}. From condition (3) in the Definition 2.2.13 we conclude that $\#Y \leq N_\Gamma C_\Gamma^{n_0 - m_0}$. If $\theta = a$ then $n_0 = 2$ and $m_0 = 1$. If $\theta > a$ then we have

$$\begin{aligned} n_0 - m_0 &= \lceil \log_\lambda(\theta/a) \rceil - \lfloor \log_\lambda(\eta/b) \rfloor \\ &\leq 2 + \log_\lambda(\theta/a) - \log_\lambda(\eta/b) \\ &= 2 + \log_\lambda(\theta/\eta) + \log_\lambda(b/a), \end{aligned}$$

and for simplicity let $c = 2 + \log_\lambda(\theta/\eta) \geq 0$.

We now have $\#Y \leq N_\Gamma C_\Gamma^c C_\Gamma^{\log_\lambda(b/a)} = N_\Gamma C_\Gamma^c (b/a)^s$, for $s = 1/\log_{C_\Gamma}(\lambda)$. In the case where $b > \eta$, let K_η be the cardinality of a minimal cover $\{B(x_i, \eta/2)\}_{i \in I}$ of Z. Then apply the above construction for each $Y \cap B(x_i, \eta/2)$ and in general we get that

$$\#Y \leq C(b/a)^s, \tag{2.2.14}$$

for $C = K_\eta N_\Gamma C_\Gamma^c$ and $s = 1/\log_{C_\Gamma}(\lambda)$. □

Chapter 3

Smale spaces, Ahlfors regularity and dimension

This chapter constitutes one of our main contributions to the theory of Smale spaces. The main theme is Markov partitions; Bowen's special closed covers of a Smale space that can be used to encode, symbolically, its points. By extending Bowen's construction to open covers that behave like Markov partitions (Theorem 3.3.2), we prove that every mixing Smale space is topologically conjugate to a mixing Smale space on which Bowen's measure (the measure of maximal entropy) is Ahlfors regular (Corollary 3.4.8). Interesting on its own, the latter result also provides an abundance of Smale spaces with an Alhfors regular Bowen measure. Previous examples of Smale spaces with Ahlfors regular measures (these are different from Bowen's measure) are restricted to the Euclidean setting, in particular that of $C^{1+\varepsilon}$-conformal Axiom A systems [101]. Finally, our Ahlfors regularity result leads to new estimates for the Hausdorff, box-counting and Assouad dimensions for a large class of Smale spaces (Theorem 3.4.6 and Corollary 3.4.10).

3.1 Smale spaces

3.1.1 Preliminaries

Roughly speaking, a Smale space (X, φ) is a dynamical system consisting of a homeomorphism φ acting on a compact metric space X that is locally hyperbolic under φ, in the sense that every $x \in X$ has a small neighbourhood that can be decomposed into the product of contracting and expanding sets.

Definition 3.1.1 ([116, Section 7.1]). Let (X, d) be a compact metric space and $\varphi : X \to X$ be a homeomorphism. The dynamical system (X, φ) is a *Smale space* if there are constants $\varepsilon_X > 0$, $\lambda_X > 1$ (which depend on d) and a locally defined bi-continuous map, called the *bracket map*,

$$\{(x,y) \in X \times X : d(x,y) \leq \varepsilon_X\} \mapsto [x,y] \in X$$

that satisfies the axioms:

$$[x,x] = x, \tag{B1}$$
$$[x,[y,z]] = [x,z], \tag{B2}$$
$$[[x,y],z] = [x,z], \tag{B3}$$
$$\varphi([x,y]) = [\varphi(x),\varphi(y)]; \tag{B4}$$

for any $x,y,z \in X$, whenever both sides are defined. For $x \in X$ and $0 < \varepsilon \leq \varepsilon_X$ let

$$X^s(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon, [x,y] = y\} \tag{3.1.1}$$
$$X^u(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon, [y,x] = y\} \tag{3.1.2}$$

be the *local stable* and *unstable sets*. On these sets we have the *contraction axioms*:

$$d(\varphi(y),\varphi(z)) \leq \lambda_X^{-1} d(y,z), \text{ for any } y,z \in X^s(x,\varepsilon), \tag{C1}$$
$$d(\varphi^{-1}(y),\varphi^{-1}(z)) \leq \lambda_X^{-1} d(y,z), \text{ for any } y,z \in X^u(x,\varepsilon). \tag{C2}$$

Quite often, we will consider the Lipschitz constants $\mathrm{Lip}(\varphi)$ and $\mathrm{Lip}(\varphi^{-1})$ which are both greater than $\lambda_X > 1$ and, in general, are allowed to be infinite. In particular, we will focus on

$$\ell_X = \min\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\} \text{ and } \Lambda_X = \max\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}. \tag{3.1.3}$$

We say that the bracket map defines a *local product structure* on X because, for any $x \in X$ and $0 < \varepsilon \leq \varepsilon_X/2$, the bracket map

$$[\cdot,\cdot] : X^u(x,\varepsilon) \times X^s(x,\varepsilon) \to X \tag{3.1.4}$$

is a homeomorphism onto its image [107, Proposition 2.1.8]. Also, due to the uniform continuity, there is a constant $0 < \varepsilon'_X \leq \varepsilon_X/2$ such that, if $d(x,y) \leq \varepsilon'_X$, then both $d(x,[x,y]), d(y,[x,y]) < \varepsilon_X/2$ and hence

$$X^s(x,\varepsilon_X/2) \cap X^u(y,\varepsilon_X/2) = [x,y]. \tag{3.1.5}$$

Equation (3.1.5) together with a bracket independent description of the local stable and unstable sets (see [106, Subsection 4.1]) imply that the bracket map is unique on X (but of course depends on ε_X and λ_X). Moreover, an important property of Smale spaces is that they are *expansive* [107, Proposition 2.1.9]. Expansiveness immediately implies finiteness of covering dimension [91], topological entropy [130, Theorem 3.2] and upper box-counting dimension (for certain metrics) [51]. Moreover, Smale spaces (without any assumption on the metric) have finite Hausdorff dimension, see Ruelle's Exercise 1 in [116, Chapter 7]. According to Smale's program [128], the interesting dynamics of a Smale space lie in the non-wandering set which is the closure of its periodic points [15, Corollary 3.7] and which can be studied through its irreducible and mixing components [116, Section 7.4].

Theorem 3.1.2 (Smale's Decomposition Theorem)**.** *Assume that the Smale space* (X,φ) *is non-wandering. Then* X *can be decomposed into a finite disjoint union of clopen,* φ*-invariant, irreducible sets* $X_0,\dots,X_{N-1}$*. Each of these sets can be decomposed into a finite disjoint union of clopen sets* $X_{i0},\dots,X_{iN_i}$ *that are cyclically permuted by* φ*, and where* $\varphi^{N_i+1}|_{X_{ij}}$ *is mixing, for every* $0\leq j\leq N_i$*.*

As a corollary of Theorem 3.1.2 and Proposition 2.1.3 we note the following.

Corollary 3.1.3. *Assume that the Smale space* (X,φ) *is non-wandering and that all its irreducible parts are infinite. Then* X *has no isolated points.*

Every irreducible Smale space admits a distinguished measure, referred to as the Bowen measure (see [18], [81, Section 20]), which we denote by μ_{B}. Roughly speaking, it is exhibited as a limit distribution of periodic orbits and is the unique φ-invariant probability measure that maximises the topological entropy $\mathrm{h}(\varphi)$. Moreover, it is compatible with the bracket map [115].

Smale spaces are ubiquitous in the theory of expansive dynamical systems, see [53, Lemma 2]. Smale spaces were defined by Ruelle [116] to give a topological description of the non-wandering sets of differentiable dynamical systems satisfying *Axiom A* [128]. Zero dimensional Smale spaces are exactly the *subshifts of finite type* (SFT), see [19, Section 3], [107, Theorem 2.2.8]. More recently, Wieler characterised the Smale spaces with totally disconnected stable or unstable sets, now called *Wieler solenoids* [134]. Among these, the SFT play an important role in coding the orbits of a Smale space, see Section 3.2. Since any SFT is topologically conjugate to a *topological Markov chain* [19, Prop. 3.2.1], let us introduce the latter.

3.1.2 Topological Markov chains

We equip $\{1,\dots,N\}$ with the discrete topology and $\{1,\dots,N\}^{\mathbb{Z}}$ with the product topology that makes it a compact Hausdorff space. Let M be a square matrix indexed by N, with 0 and 1 entries, and consider the closed subspace of *allowable sequences*

$$\Sigma_M=\{x=(x_i)_{i\in\mathbb{Z}}\in\{1,\dots,N\}^{\mathbb{Z}}:M_{x_i,x_{i+1}}=1\}. \tag{3.1.6}$$

The *cylinder sets*

$$C_{\mu_{-n},\dots,\mu_m} = \{x \in \Sigma_M : x_i = \mu_i, \text{ for } -n \le i \le m\} \tag{3.1.7}$$

form a basis of clopen sets for the product topology on Σ_M. The number of fixed digits will be called the *rank of the cylinder*, which for the cylinder set in (3.1.7) is equal to $m+n+1$. If $m=n$ the cylinders will be called *symmetric*. The metric

$$d(x,y) = \inf\{2^{-n} : n \ge 0, \; x_i = y_i \text{ for } |i| < n\} \tag{3.1.8}$$

induces the product topology on Σ_M and it is straightforward to prove that it is actually an ultrametric. Also, with this metric, every symmetric cylinder of rank $2n-1$ is a ball of radius 2^{-n} around each of its points.

The shift map $\sigma_M : \Sigma_M \to \Sigma_M$ given by $\sigma_M(x)_i = x_{i+1}$ for any $i \in \mathbb{Z}$, is a homeomorphism and (Σ_M, σ_M) is called a *topological Markov chain*. It admits a Smale space structure with the bracket map defined by

$$([x,y])_n = \begin{cases} y_n, & \text{for } n \le 0 \\ x_n, & \text{for } n \ge 1 \end{cases} \tag{3.1.9}$$

for any $x, y \in \Sigma_M$ such that $d(x,y) \le 2^{-1}$. The expansivity constant is $\varepsilon_{\Sigma_M} = 2^{-1}$ and the contraction constant is $\lambda_{\Sigma_M} = 2$.

Remark 3.1.4. A word $x_1 \dots x_{n-1}$ can only be concatenated on the right by a letter x_n in $\{1,\dots,N\}$, if the value of M_{x_{n-1},x_n} is 1. This is called the *Markov property* and it appears in the general setting of Smale spaces.

We would like to estimate the number of symmetric cylinder sets of rank $2n-1$ for the shift space Σ_M. Denote this number by $N_M(2n-1)$. The estimation is obtained by replicating the computation for the topological entropy $\mathrm{h}(\sigma_M)$, see [81, p. 121]. However, we sketch the proof for completeness.

Lemma 3.1.5. *Let (Σ_M, σ_M) be a topological Markov chain with matrix M where no row contains only 0's. There exist constants $C, c > 0$ so that for every $\varepsilon \in (0,1)$ there is some $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$, we have*

$$ce^{2(\mathrm{h}(\sigma_M)-\varepsilon)n} < N_M(2n-1) < Ce^{2(\mathrm{h}(\sigma_M)+\varepsilon)n}.$$

Proof. Let $M = (m_{ij})_{i,j=1}^N$. Then

$$N_M(2n-1) = \sum_{i,j=1}^N m_{ij}^{2n-2}$$

and there is some $C' > 0$ such that $N_M(2n-1) < C'\|M^{2n-2}\|$. Also, since the numbers m_{ij}^{2n-2} are non-negative, there is some $c' > 0$ such that $N_M(2n-1) > c'\|M^{2n-2}\|$. Following the equalities (3.2.3) in [81, p. 121] one obtains that

$$\lim_{n\to\infty} \frac{1}{2n-2} \log \|M^{2n-2}\| = \mathrm{h}(\sigma_M)$$

and the result follows. $\square$

The Bowen measure on a mixing topological Markov chain (Σ_M, σ_M) has a very nice description, and is known as the Parry measure μ_{P} [81, Chapter 4]. Since (Σ_M, σ_M) is mixing, M is a primitive matrix, meaning that there is some power of M with only positive entries. For primitive matrices the Perron-Frobenius Theorem [81, Theorem 1.9.11] yields a unique (up to a scalar) eigenvector of strictly positive coordinates whose eigenvalue $\lambda_{\max} > 0$ is greater than the absolute value of all the other eigenvalues. Let u, v be the Perron-Frobenius eigenvectors for M and M^T, respectively, which are normalised so that

$$\sum_{i=1}^{N} v_i u_i = 1. \tag{3.1.10}$$

The distribution $p = (p_1, \ldots, p_N)$, with $p_i = v_i u_i$, induces the Parry measure.

Lemma 3.1.6. *Let (Σ_M, σ_M) be a mixing topological Markov chain. Then there is a constant $D > 0$ so that every non-empty symmetric cylinder set has Parry measure*

$$D^{-1}\lambda_{\max}^{-2n} \leq \mu_{\mathrm{P}}(C_{\mu_{-n},\ldots,\mu_n}^{-n,\ldots,n}) \leq D\lambda_{\max}^{-2n}.$$

Proof. Following the computations on page 176 of [81], if we look at (Σ_M, σ_M) as an edge shift then

$$\mu_{\mathrm{P}}(C_{\mu_{-n},\ldots,\mu_n}^{-n,\ldots,n}) = v_i u_j \lambda_{\max}^{-2n+2}$$

where $i = (\mu_{-n}, \mu_{-n+1})$ and $j = (\mu_{n-1}, \mu_n)$ are edges. Since the coordinates of u, v are all positive the result follows. $\square$

Corollary 3.1.7. *Let (Σ_M, σ_M) be a mixing topological Markov chain equipped with the metric defined in (3.1.8). Its Parry measure μ_{P} is Ahlfors s_0-regular and therefore,*

$$\dim_H \Sigma_M = \dim_B \Sigma_M = \dim_A \Sigma_M = s_0$$

where $s_0 = 2\,\mathrm{h}(\sigma_M)/\log(2)$.

3.1.3 Wieler solenoids

In [134] Wieler characterised Smale spaces with totally disconnected local stable sets as stationary inverse limits of (eventually) Ruelle expanding dynamical systems [116, p.138]. Wieler's results generalise the earlier work of Williams for expanding attractors [136] to the topological setting.

Let (Y,d) be a compact metric space and $g : Y \to Y$ be a continuous and surjective map. We say that (Y,g) satisfies *Wieler's axioms* if there are constants $\beta > 0, K \in \mathbb{N}$ and $\gamma \in (0,1)$ such that for every $x,y,z \in Y$ with $d(x,y) \leq \beta$ and $0 < \varepsilon \leq \beta$ we have

$$d(g^K(x), g^K(y)) \leq \gamma^K d(g^{2K}(x), g^{2K}(y)) \tag{W1}$$

$$g^K(B(g^K(z),\varepsilon)) \subset g^{2K}(B(z,\gamma\varepsilon)). \tag{W2}$$

The *Wieler solenoid* associated with such (Y,g) is the stationary inverse limit $(\widehat{Y},\widehat{g})$ where

$$\widehat{Y} = \{(y_0,y_1,y_2,\dots) : y_n \in Y, y_n = g(y_{n+1}),\ n \geq 0\} \tag{3.1.11}$$

and

$$\widehat{g}(y_0,y_1,y_2,\dots) = (g(y_0),y_0,y_1,\dots), \tag{3.1.12}$$

with metric $\widehat{d}$ given by

$$\widehat{d}(x,y) = \sum_{k=0}^{K-1} \gamma^{-k} d'(\widehat{g}^{-k}(x), \widehat{g}^{-k}(y)), \tag{3.1.13}$$

where $d'(x,y) = \sup\{\gamma^n d(x_n,y_n) : n \geq 0\}$. Wieler's main result is the following.

Theorem 3.1.8 ([134, Theorem A and B])**.**

(A) Any Wieler solenoid $(\widehat{Y},\widehat{g})$ is a Smale space with totally disconnected stable sets. Moreover, $(\widehat{Y},\widehat{g})$ is irreducible if and only if (Y,g) is non-wandering and has a dense forward orbit.

(B) Any irreducible Smale space with totally disconnected stable sets is topologically conjugate to a Wieler solenoid.

Remark 3.1.9. Following [134], the contraction constant $\lambda_{\widehat{Y}}$ of $(\widehat{Y},\widehat{g})$ is equal to γ^{-1}. Also, from the definition of the metric, we see that the map $\widehat{g}^{-1}$ is $\lambda_{\widehat{Y}}$-Lipschitz, while $\widehat{g}$ need not be Lipschitz. However, if g is Lipschitz then $\widehat{g}$ will be too. Moreover, one has

$$\widehat{d}(\widehat{g}(x),\widehat{g}(y)) = \lambda_{\widehat{Y}}^{-1}\widehat{d}(x,y),$$

whenever $y \in \widehat{Y}^s(x,\varepsilon_{\widehat{Y}})$.

3.1.4 Dyadic solenoid

We equip the unit circle $\mathbb{T} = \{e^{2\pi i\theta} : \theta \in [0,1)\}$ with the (normalised) arc length distance

$$d(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) = \min\{|\theta_1 - \theta_2|, 1 - |\theta_1 - \theta_2|\}, \tag{3.1.14}$$

and consider the *doubling map* $g : \mathbb{T} \to \mathbb{T}$ given by $g(z) = z^2$. The map g is expanding because, if $d(z,w) \leq 1/4$ then $d(g(z), g(w)) = 2d(z,w)$.

As in the case of Wieler solenoids, we construct the stationary inverse limit $(\widehat{\mathbb{T}}, \widehat{g})$ and equip it with the metric

$$\widehat{d}(x,y) = \sum_{n=0}^{\infty} 2^{-n} d(x_n, y_n), \tag{3.1.15}$$

which induces the product topology on $\widehat{\mathbb{T}}$. Note that the metric (3.1.15) is different from the metric (3.1.13) defined for Wieler solenoids. This inverse limit is called the *dyadic solenoid* and is topologically conjugate to the Smale-Williams solenoid [19]. In fact, $(\widehat{\mathbb{T}}, \widehat{g})$ is a Smale space with expansivity and contraction constants $\varepsilon_{\widehat{\mathbb{T}}} = 1/4$ and $\lambda_{\widehat{\mathbb{T}}} = 2$. If $\widehat{d}(x,y) \leq 1/4$ then, in particular, $d(x_0, y_0) \leq 1/4$ and hence there is a unique $|t| \leq 1/4$ such that $x_0 = y_0 e^{2\pi i t}$. The bracket map for such x, y is defined as the sequence

$$[x,y] = (y_0 e^{2\pi i t}, y_1 e^{\pi i t}, y_2 e^{\pi i t/2}, \dots) \tag{3.1.16}$$

which is in $\widehat{\mathbb{T}}$. The (largest) local stable set around $x \in \widehat{\mathbb{T}}$ consists of sequences whose 0-th coordinate is x_0. Therefore, local stable sets are Cantor sets and global stable sets are totally disconnected. On the other hand, the global unstable sets are one-dimensional. Moreover, one can check that the doubling map g is 2-Lipschitz, hence $\widehat{g}$ is 5/2-Lipschitz. Moreover, the map $\widehat{g}$ is the 2^{-1}-multiple of an isometry on local stable sets. Finally, the inverse $\widehat{g}^{-1}$ is 2-Lipschitz and the 2^{-1}-multiple of an isometry on local unstable sets.

3.1.5 Metrics and smoothing of Smale spaces

There is an abundance of Smale spaces (X, φ) where φ is a *bi-Lipschitz* homeomorphism, meaning $\Lambda_X < \infty$, where $\Lambda_X = \max\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}$. The obvious examples are the SFT or the non-wandering sets of Axiom A diffeomorphisms. This is often true for Wieler solenoids, see Remark 3.1.9. For such Smale spaces the following holds.

Lemma 3.1.10 ([119, p.234-235])**.** *Let (X, φ) be a Smale space with $\Lambda_X < \infty$. Then there exists $A_X > 0$ with*

$$A_X \leq \frac{\Lambda_X \lambda_X}{\lambda_X^2 - 1}$$

such that, for every $x, y \in X$ with $d(x,y) \leq \varepsilon'_X$, it holds that $d(x, [x,y]) \leq A_X d(x,y)$ and $d(y, [x,y]) \leq A_X d(x,y)$.

In [53], Fried showed that any expansive dynamical system (Z, ψ) admits a compatible hyperbolic metric d_{F} for which ψ becomes bi-Lipschitz. This means that any Smale space (X, d, φ) admits a compatible metric d_{F} for which φ is bi-Lipschitz. Now, the new dynamical system $(X, d_{\mathrm{F}}, \varphi)$ is still a Smale space. Indeed, the existence of a bracket map satisfying axioms (B1)-(B4) is not affected by changing to another compatible metric. Moreover, d_{F} is hyperbolic, meaning that the contraction axioms (C1) and (C2) will still be satisfied (possibly with a different constant). Therefore we obtain the following.

Theorem 3.1.11 ([53])**.** *Any Smale space is topologically conjugate to a Smale space with bi-Lipschitz dynamics.*

We note that Lemma 3.1.10 and Theorem 3.1.11 solved a question posed by Ruelle in [116, Appendix B.7]. Later, Fathi [51, Theorem 5.1] showed that Fried's metric d_{F}, defined on an expansive (Z, ψ), satisfies an additional property which can be used to obtain upper bounds for $\overline{\dim}_B(Z, d_{\mathrm{F}})$. In our case, *Fathi's property* is the following.

Theorem 3.1.12 ([51])**.** *For any Smale space $(X, d_{\mathrm{F}}, \varphi)$ equipped with Fried's metric there exist constants $k > 1, \xi > 0$ such that*

$$\max\{d_{\mathrm{F}}(\varphi(x), \varphi(y)), d_{\mathrm{F}}(\varphi^{-1}(x), \varphi^{-1}(y))\} \geq \min\{k d_{\mathrm{F}}(x,y), \xi\}$$

for every $x, y \in X$.

Fathi's property implies that the contraction axioms (C1) and (C2) of the Smale space $(X, d_{\mathrm{F}}, \varphi)$, hold more globally than just on local stable or unstable sets. This complicated statement will be extremely useful in the sequel. At this point we should say that Fried's metric d_{F} is not concretely related to the original metric d of (X, d, φ), even if (X, d, φ) has bi-Lipschitz dynamics. However, in the latter case it may be still possible to obtain Fathi's property for (X, d, φ) without changing the metric d.

More precisely, suppose that $\Lambda_X < \infty$ for the Smale space (X, d, φ). Also, assume that the contraction constant satisfies $\lambda_X > 2A_X$, where $A_X > 0$ is obtained from Lemma 3.1.10. Let $0 < \widetilde{\varepsilon_X} \leq \varepsilon'_X$ be small enough so that if $d(x,y) \leq \widetilde{\varepsilon_X}$ then $d(\varphi^i(x), \varphi^i(y)) \leq \varepsilon'_X$, for $i \in \{-1, 1\}$. For $x, y \in X$ with $d(x,y) \leq \widetilde{\varepsilon_X}$ one has

$$\begin{aligned}
d(x,y) &\leq d(x,[x,y]) + d(y,[x,y]) \\
&\leq \frac{1}{\lambda_X}(d(\varphi^{-1}(x), \varphi^{-1}([x,y])) + d(\varphi(y), \varphi([x,y]))) \\
&= \frac{1}{\lambda_X}(d(\varphi^{-1}(x), [\varphi^{-1}(x), \varphi^{-1}(y)]) + d(\varphi(y), [\varphi(x), \varphi(y)])) \\
&\leq \frac{A_X}{\lambda_X}(d(\varphi^{-1}(x), \varphi^{-1}(y)) + d(\varphi(x), \varphi(y))) \\
&\leq \frac{2A_X}{\lambda_X} \max\{d(\varphi(x), \varphi(y)), d(\varphi^{-1}(x), \varphi^{-1}(y))\}.
\end{aligned}$$

Consequently, we can choose $k = \lambda_X/(2A_X)$. Moreover, for every $x, y \in X$ we have $\max\{d(\varphi(x), \varphi(y)), d(\varphi^{-1}(x), \varphi^{-1}(y))\} \geq \Lambda_X^{-1} d(x,y)$ and hence we can choose $\xi = \Lambda_X^{-1}\widetilde{\varepsilon_X}$.

Remark 3.1.13. The computations above give a lower bound for A_X. In particular, given a Smale space (X, d, φ) with $\Lambda_X < \infty$ (no other restriction) one has

$$A_X \geq \frac{\lambda_X}{2\Lambda_X}.$$

Finding the best such k is crucial for obtaining good estimates for the Hausdorff and box-counting dimensions, see Section 3.4. The computations above indicate that for estimating k one should first try to estimate the constant A_X of Lemma 3.1.10. In general though, for $\lambda_X > 2A_X$ to be true it suffices to restrict to Smale spaces with $\lambda_X \in (1+\sqrt{2}, \infty)$ and $\Lambda_X \in [\lambda_X, (\lambda_X^2 - 1)/2)$.

Returning to the discussion of Fried's metric, Artigue [7] recently constructed compatible metrics on expansive dynamical systems for which the systems exhibit self-similarity. We describe his construction in the context of Smale spaces.

Let (X, d, φ) be a Smale space. One can construct the Smale space $(X, d_{\mathrm{F}}, \varphi)$ for which Fathi's property in Theorem 3.1.12 is satisfied for some $k > 1, \xi > 0$. Then Artigue defines the metric d_{A} given by

$$d_{\mathrm{A}}(x,y) = \max_{n\in\mathbb{Z}} \frac{d_{\mathrm{F}}(\varphi^n(x), \varphi^n(y))}{k^{|n|}} \tag{3.1.17}$$

and proves that

$$d_{\mathrm{F}} \leq d_{\mathrm{A}} \leq c(d_{\mathrm{F}})^{\gamma}, \tag{3.1.18}$$

where $\gamma = \log_{\Lambda_{X,\mathrm{F}}}(k) \in (0,1)$ ($\Lambda_{X,\mathrm{F}}$ is the maximum of the two Lipschitz constants for the metric d_{F}) and $c > 0$. Most importantly, the new contraction constant and the new Lipschitz constants of φ, φ^{-1} for the Smale space $(X, d_{\mathrm{A}}, \varphi)$ are equal to $k > 1$.

Definition 3.1.14. A Smale space (X, φ) is called *self-similar* if $\lambda_X = \Lambda_X$, where $\lambda_X > 1$ is its contraction constant and $\Lambda_X = \max\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}$.

Remark 3.1.15. One can observe that Artigue's construction works for any metric d that satisfies Fathi's property and hence the metric inequalities (3.1.18) hold with d in the place of d_{F}. For instance, we proved that this is true for Smale spaces with $\Lambda_X < \infty$ and $\lambda_X > 2A_X$. This fact will be used in Corollary 3.4.10.

As we can see from the next lemma, there is a plethora of self-similar Smale spaces.

Lemma 3.1.16. *Any Smale space is topologically conjugate to a self-similar Smale space.*

Obvious examples are the SFT. Self-similarity means that the dynamics is very tight (see [7, Remark 2.22] and compare with the case of SFT, Wieler solenoids and Lemma 3.1.10). Note that in a self-similar Smale space, φ and φ^{-1} act on local stable and unstable sets, respectively, as the λ_X^{-1}-multiple of an isometry.

3.2 Markov partitions

Roughly speaking, for an irreducible Smale space (X,φ) a Markov partition is a partition of the space X into closed subsets that have a local product structure and which overlap only on their boundaries. Such partitions yield a dynamically defined refining sequence, as in Subsection 2.2.4, which induces an essential approximation graph Π of X. From Proposition 2.2.8, the factor map $\pi_\Pi : (\mathcal{P}_\Pi, \sigma_\Pi) \to (X,\varphi)$ is injective on a dense G_δ-set and since (X,φ) is irreducible, $(\mathcal{P}_\Pi, \sigma_\Pi)$ will be too. This would be the end of the story if the so-called *shadowing property* did not hold for Smale spaces [15, Prop. 3.6]. This property sets up such a partition of X which in addition satisfies the crucial *Markov property*, see Remark 3.1.4. In this case, $(\mathcal{P}_\Pi, \sigma_\Pi)$ is a topological Markov chain and this is actually where the story begins!

Except for being essential and providing a combinatorial model for (X,φ), the graph Π satisfies conditions (1), (3) and (4) in the Definition 2.2.13 of geometric approximation graphs. However, it does not satisfy condition (2) since the overlaps in each cover of the refining sequence are nowhere-dense and hence the Lebesgue numbers are zero. Therefore, our plan is to recursively δ-fatten the closed covers in Π for some carefully chosen $\delta > 0$. In this way we will keep all the nice properties of Π and obtain a geometrically-essential approximation graph Π^δ. The Markov property makes this possible. The dynamical systems that admit Markov partitions have been characterised in [53] and are known as *finitely presented dynamical systems*. These contain sofic systems and pseudo-Anosov homeomorphisms. It is quite possible that the method we described can be generalised to this larger class of dynamical systems.

3.2.1 Basics on Markov partitions

We deal with the classical Markov partitions and not with topological partitions, as defined in [1,106]. This means that we also consider the boundaries of the partitions. However, the ideas are of the same nature. Also, we focus on irreducible (infinite) Smale spaces, but most of the statements still hold for non-wandering Smale spaces due to Smale's Decomposition Theorem 3.1.2.

First, let (X,φ) be a Smale space and recall the definition of $0 < \varepsilon'_X \leq \varepsilon_X/2$ from (3.1.5), and similarly let $0 < \varepsilon''_X < \varepsilon'_X/12$ be so small that whenever $d(x,y) \leq \varepsilon''_X$, we have

$$d(\varphi^i(x), \varphi^i(y)) \leq \varepsilon'_X/2, \;\; d([x,y],y) \leq \varepsilon'_X/4, \;\; d([x,y],x) \leq \varepsilon'_X/4, \tag{3.2.1}$$

for every $|i| \leq 2$.

Definition 3.2.1. A non-empty subset $R \subset X$ is called a *rectangle* if $\operatorname{diam}(R) \leq \varepsilon'_X$ and $[x,y] \in R$, for any $x,y \in R$.

If R is a rectangle and $x \in R$, let

$$X^s(x,R) = X^s(x, 2\varepsilon'_X) \cap R \text{ and } X^u(x,R) = X^u(x, 2\varepsilon'_X) \cap R. \tag{3.2.2}$$

From the bracket axioms and the definition of rectangles it holds that

$$R = [X^u(x,R), X^s(x,R)]. \tag{3.2.3}$$

In fact, the local product structure on X (see (3.1.4)) implies that

$$\operatorname{int}(R) = [\operatorname{int}(X^u(x,R)), \operatorname{int}(X^s(x,R))] \text{ and } \operatorname{cl}(R) = [\operatorname{cl}(X^u(x,R)), \operatorname{cl}(X^s(x,R))], \tag{3.2.4}$$

where the interiors and the closures are taken in $X^u(x, 2\varepsilon'_X)$ and $X^s(x, 2\varepsilon'_X)$. Define the *stable boundary* to be

$$\partial^s R = \{x \in R : X^s(x,R) \cap \operatorname{int}(R) = \varnothing\} \tag{3.2.5}$$

and the *unstable boundary* to be

$$\partial^u R = \{x \in R : X^u(x,R) \cap \operatorname{int}(R) = \varnothing\}. \tag{3.2.6}$$

It follows that

$$\partial^s R = [\partial X^u(x,R), X^s(x,R)] \text{ and } \partial^u R = [X^u(x,R), \partial X^s(x,R)], \tag{3.2.7}$$

where $\partial X^u(x,R)$ and $\partial X^s(x,R)$ are the boundaries of $X^u(x,R)$ and $X^s(x,R)$ as subsets of $X^u(x,2\varepsilon'_X)$ and $X^s(x, 2\varepsilon'_X)$, respectively. Also [15, Lemma 3.11] states that

$$\partial R = \partial^s R \cup \partial^u R. \tag{3.2.8}$$

Lemma 3.2.2. *For any rectangles $R, R' \subset X$ with $R \cap R' \neq \varnothing$ and diameter small enough, the set*

$$[R,R'] = \{[r,r'] : r \in R, \ \ r' \in R'\}$$

(1) is a rectangle;

(2) $\operatorname{int}([R,R']) = [\operatorname{int}(R), \operatorname{int}(R')]$ *and* $\operatorname{cl}([R,R']) = [\operatorname{cl}(R), \operatorname{cl}(R')]$.

Proof. Part (1) follows from the bracket axioms (B2) and (B3). For part (2) let $x \in R \cap R'$ and then $[R,R'] = [X^u(x,R), X^s(x,R')]$ using bracket axioms (B2), (B3) and equation (3.2.3). The local product structure implies that

$$\mathrm{int}([R,R']) = [\mathrm{int}(X^u(x,R)),\mathrm{int}(X^s(x,R'))] = [\mathrm{int}(R),\mathrm{int}(R')]$$

using axioms (B2), (B3) and equation (3.2.4). The proof for the closures is similar. □

A rectangle R is called *proper* if it is closed and $R = \mathrm{cl}(\mathrm{int}(R))$.

Definition 3.2.3 ([16, Section 3])**.** A *Markov partition* is a finite covering $\mathcal{R}_1 = \{R_1,\dots,R_\ell\}$ of X by non-empty proper rectangles such that

(1) $\mathrm{int}(R_i)\cap\mathrm{int}(R_j) = \varnothing$ for $i \neq j$;

(2) $\varphi(X^u(x,R_i)) \supset X^u(\varphi(x),R_j)$ and

(3) $\varphi(X^s(x,R_i)) \subset X^s(\varphi(x),R_j)$ when $x \in \mathrm{int}(R_i)\cap\varphi^{-1}(\mathrm{int}(R_j))$.

The following is Bowen's seminal theorem [16, Theorem 12].

Theorem 3.2.4 (Bowen's Theorem)**.** *If the Smale space (X,φ) is irreducible, then it has Markov partitions of arbitrarily small diameter.*

Consequently, if the Smale space (X,φ) is irreducible, then it is a factor of a topological Markov chain. More precisely, let $\mathcal{R}_1$ be a Markov partition for (X,φ) and M be the transition $\ell\times\ell$-matrix given by

$$M_{i,j} = \begin{cases} 1, & \text{if } \mathrm{int}(R_i)\cap\varphi^{-1}(\mathrm{int}(R_j)) \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem is due to Bowen. We refer the reader to [15, Theorem 3.18], [16, Prop. 30] and [17, Prop. 10]. Recall the Bowen measure μ_{B} from Subsection 3.1.1.

Theorem 3.2.5. *Define the map $\pi_M : (\Sigma_M,\sigma_M) \to (X,\varphi)$ by*

$$(x_i)_{i\in\mathbb{Z}} \mapsto \bigcap_{i\in\mathbb{Z}} \varphi^{-i}(R_{x_i}).$$

Then π_M is

(1) a factor map;

(2) injective on the residual set $X \setminus \bigcup_{i\in\mathbb{Z}}\varphi^i(\partial\mathcal{R}_1)$, where $\partial\mathcal{R}_1 = \bigcup_{R\in\mathcal{R}_1}\partial R$;

(3) a metric isomorphism between $(\Sigma_M,\sigma_M,\mu_{\mathrm{B}})$ and $(X,\varphi,\mu_{\mathrm{B}})$.

(4) For every $x\in X$, the pre-image $\pi_M^{-1}(x)$ has at most $(\#\mathcal{R}_1)^2$ elements.

(5) (Σ_M,σ_M) is irreducible (or mixing if (X,φ) is mixing).

3.2.2 Approximation graphs from Markov partitions

Let (X, φ) be an irreducible Smale space, $\mathcal{R}_1$ be a Markov partition with $\overline{\operatorname{diam}}(\mathcal{R}_1) \leq \varepsilon''_X$. We consider the sequence $(\mathcal{R}_n)_{n\geq 0}$ given by $\mathcal{R}_0 = \{X\}$ and

$$\mathcal{R}_n = \{R \in \bigvee_{i=1-n}^{n-1} \varphi^{-i}(\mathcal{R}_1) : \operatorname{int}(R) \neq \varnothing\}, \tag{3.2.9}$$

for $n \in \mathbb{N}$. In order to show that each $\mathcal{R}_n$ covers X, let $\mathcal{R}_n^o = \{\operatorname{int}(R) : R \in \mathcal{R}_n\}$ and observe that $\bigcup \mathcal{R}_1^o$ is dense in X. Moreover, since

$$\mathcal{R}_n^o = \bigvee_{i=1-n}^{n-1} \varphi^{-i}(\mathcal{R}_1^o) \tag{3.2.10}$$

we obtain that $\bigcup \mathcal{R}_n^o$ is dense in X. Therefore, $\mathcal{R}_n$ covers X. From [130, Theorem 5.21] we obtain that $(\mathcal{R}_n)_{n\geq 0}$ is a refining sequence and the corresponding graph $\Pi = (\mathcal{R}, \mathcal{A})$ has no sources and no sinks. Therefore, Π is an essential approximation graph (see Lemma 3.2.11) and (X, φ) is a factor of $(\mathcal{P}_\Pi, \sigma_\Pi)$. Also, $(\mathcal{P}_\Pi, \sigma_\Pi)$ is irreducible since (X, φ) is irreducible. This means that $\mathcal{P}_\Pi$ is a Cantor space, see Subsection 2.2.3.

Now we present the main result of this section. We note that part (1) will be proved in a slightly more general setting in Lemma 3.3.7, hence we defer the proof until then. Recall the notation (Q1-Q5) introduced in Subsection 2.1.1 and we have the following.

Proposition 3.2.6. *For every $n \in \mathbb{N}$, the cover $\mathcal{R}_n$ in (3.2.9) is a Markov partition. Moreover, there exists constants $\theta \in (0, \varepsilon_X]$ and $C, c > 0$ so that*

(1) $\overline{\operatorname{diam}}(\mathcal{R}_n) \leq \lambda_X^{-n+1}\theta$;

(2) $\mathrm{m}(\mathcal{R}_n) \leq \#(\mathcal{R}_1)^2$;

(3) the number of neighbouring rectangles is uniformly bounded, meaning

$$\sup_n \max_{R \in \mathcal{R}_n} \# \mathrm{N}_{\mathcal{R}_n}(R) < \infty;$$

(4) for every $\varepsilon \in (0,1)$, there is some $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, we have

$$ce^{2(\mathrm{h}(\varphi)-\varepsilon)n} < \#\mathcal{R}_n < Ce^{2(\mathrm{h}(\varphi)+\varepsilon)n};$$

(5) the approximation graph $\Pi = (\mathcal{R}, \mathcal{A})$, associated to the refining sequence of Markov partitions $(\mathcal{R}_n)_{n\geq 0}$, is essential, and

$$(\mathcal{P}_\Pi, \sigma_\Pi) = (\Sigma_M, \sigma_M),$$

where M is the transition matrix of $\mathcal{R}_1$.

The proof of Proposition 3.2.6 will be achieved by establishing the following lemmas.

Lemma 3.2.7. *For the Markov partition $\mathcal{R}_1$ with $\overline{diam}(\mathcal{R}_1) \leq \varepsilon''_X$, the following hold.*

(1) $\varphi(R), \varphi^{-1}(R)$ are rectangles, for any $R \in \mathcal{R}_1$.

(2) For any $R_i, R_j \in \mathcal{R}_1$ such that $\operatorname{int}(R_i) \cap \varphi^{-1}(\operatorname{int}(R_j)) \neq \varnothing$ we have

$$[\varphi^{-1}(R_j), R_i] = R_i \cap \varphi^{-1}(R_j), \tag{3.2.11}$$

and hence $R_i \cap \varphi^{-1}(R_j)$ is a proper rectangle. We will refer to this condition as the Markov property.

(3) If $R_i, R_j, R_k \in \mathcal{R}_1$ with $\varphi(\operatorname{int}(R_i)) \cap \operatorname{int}(R_j) \cap \varphi^{-1}(\operatorname{int}(R_k)) \neq \varnothing$ then we have

$$[\varphi^{-1}(R_k), \varphi(R_i)] = \varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k). \tag{3.2.12}$$

Consequently, $\varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k)$ is a proper rectangle equal to $\varphi(R_i) \cap \varphi^{-1}(R_k)$.

Proof. Part (1) follows from the fact that ε''_X is so small that $\operatorname{diam}(\varphi^{\pm 1}(\mathcal{R}_1)) \leq \varepsilon'_X/2$ and the φ-invariance of the bracket map. For part (2) it is clear that $R_i \cap \varphi^{-1}(R_j) \subset [\varphi^{-1}(R_j), R_i]$. For the reverse inclusion let $x \in \operatorname{int}(R_i) \cap \varphi^{-1}(\operatorname{int}(R_j))$ and then it should be clear that $[\varphi^{-1}(R_j), R_i] = [X^u(x, \varphi^{-1}(R_j)), X^s(x, R_i)]$. First we claim that

$$X^u(x, \varphi^{-1}(R_j)) = \varphi^{-1}(X^u(\varphi(x), R_j)). \tag{3.2.13}$$

Indeed, since $\varphi(x) \in R_j$ write $R_j = [X^u(\varphi(x), R_j), X^s(\varphi(x), R_j)]$ and hence

$$\varphi^{-1}(R_j) = [\varphi^{-1}(X^u(\varphi(x), R_j), \varphi^{-1}(X^s(\varphi(x), R_j))],$$

where $\varphi^{-1}(X^u(\varphi(x), R_j) \subset X^u(x, \varepsilon_X)$ and $\varphi^{-1}(X^s(\varphi(x), R_j)) \subset X^s(x, \varepsilon_X)$. Also,

$$\varphi^{-1}(R_j) = [X^u(x, \varphi^{-1}(R_j)), X^s(x, \varphi^{-1}(R_j))].$$

The claim follows since the bracket map is bijective around x.

Using conditions (2) and (3) in the definition of the Markov partition, it is easy to show that

$$[\varphi^{-1}(X^u(\varphi(x), R_j)), X^s(x, R_i)] \subset R_i \cap \varphi^{-1}(R_j).$$

We have $X^s(x, R_i) \subset R_i$ and

$$\varphi^{-1}(X^u(\varphi(x), R_j)) \subset X^u(x, R_i) \subset R_i.$$

Therefore,

$$[\varphi^{-1}(X^u(\varphi(x), R_j)), X^s(x, R_i)] \subset [R_i, R_i] = R_i,$$

since R_i is a rectangle. In the same way,

$$\varphi(X^s(x,R_i)) \subset X^s(\varphi(x),R_j) \subset R_j,$$

which gives $X^s(x,R_i) \subset \varphi^{-1}(R_j)$. Also $\varphi^{-1}(X^u(\varphi(x),R_j)) \subset \varphi^{-1}(R_j)$, and one gets

$$[\varphi^{-1}(X^u(\varphi(x),R_j)), X^s(x,R_i)] \subset [\varphi^{-1}(R_j),\varphi^{-1}(R_j)] = \varphi^{-1}(R_j).$$

This shows that $[\varphi^{-1}(R_j),R_i] \subset R_i \cap \varphi^{-1}(R_j)$, and hence $[\varphi^{-1}(R_j),R_i] = R_i \cap \varphi^{-1}(R_j)$. Consequently, $R_i \cap \varphi^{-1}(R_j)$ is proper since using Lemma 3.2.2 we obtain

$$\operatorname{int}([\varphi^{-1}(R_j),R_i]) = [\varphi^{-1}(\operatorname{int}(R_j)),\operatorname{int}(R_i)]$$

and $\operatorname{cl}(\operatorname{int}([\varphi^{-1}(R_j),R_i])) = [\operatorname{cl}(\varphi^{-1}(\operatorname{int}(R_j))),\operatorname{cl}(\operatorname{int}(R_i))] = [\varphi^{-1}(R_j),R_i]$.

Part (3) is proved in a similar fashion. First we observe that

$$R_i \cap \varphi^{-1}(R_j) \cap \varphi^{-2}(R_k) = [\varphi^{-1}(R_j) \cap \varphi^{-2}(R_k), R_i].$$

The inclusion

$$R_i \cap \varphi^{-1}(R_j) \cap \varphi^{-2}(R_k) \subset [\varphi^{-1}(R_j) \cap \varphi^{-2}(R_k), R_i]$$

should be clear. For the reverse inclusion let $x \in \varphi^{-1}(R_j) \cap \varphi^{-2}(R_k)$ and $y \in R_i$. In particular, $x \in \varphi^{-1}(R_j)$ and using part (2) we get $[x,y] \in [\varphi^{-1}(R_j),R_i] = R_i \cap \varphi^{-1}(R_j)$. Also, $\varphi(x) \in R_j \cap \varphi^{-1}(R_k)$ and $\varphi([x,y]) \in R_j$. Therefore,

$$\begin{aligned}
[x,y] = \varphi^{-1}[\varphi(x),\varphi[x,y]] &\in \varphi^{-1}[R_j \cap \varphi^{-1}(R_k), R_j] \\
&= \varphi^{-1}[[\varphi^{-1}(R_k),R_j],R_j] \\
&= \varphi^{-1}(R_j \cap \varphi^{-1}(R_k)) \\
&\subset \varphi^{-2}(R_k)
\end{aligned}$$

and consequently $[x,y] \in R_i \cap \varphi^{-1}(R_j) \cap \varphi^{-2}(R_k)$.

We now have that

$$\begin{aligned}
\varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k) &= \varphi(R_i \cap \varphi^{-1}(R_j) \cap \varphi^{-2}(R_k)) \\
&= \varphi[\varphi^{-1}(R_j) \cap \varphi^{-2}(R_k), R_i] \\
&= \varphi[\varphi^{-1}[\varphi^{-1}(R_k),R_j],R_i] \\
&= \varphi[[\varphi^{-2}(R_k),\varphi^{-1}(R_j)],R_i] \\
&= \varphi[\varphi^{-2}(R_k),R_i] \\
&= [\varphi^{-1}(R_k),\varphi(R_i)],
\end{aligned}$$

where the fourth equality makes sense because $\overline{\operatorname{diam}}(\mathcal{R}_1) \leq \varepsilon''_X$. The properness of the rectangle follows as in part (2) and $\varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k) = \varphi(R_i) \cap \varphi^{-1}(R_k)$ because $\varphi(R_i) \cap \varphi^{-1}(R_k) \subset [\varphi^{-1}(R_k), \varphi(R_i)]$. □

Remark 3.2.8. Let $R = \varphi(R_i) \cap \varphi^{-1}(R_k) = [\varphi^{-1}(R_k), \varphi(R_i)]$ be a rectangle as in part (2) of Proposition 3.2.7. Then it is not hard to see that

$$\partial^s R \subset \partial^s(\varphi^{-1}(R_k)) \text{ and } \partial^u R \subset \partial^u(\varphi(R_i)). \tag{3.2.14}$$

Similarly as in equation (3.2.13), the local product structure of the space implies that $X^u(x,R) = X^u(x,\varphi^{-1}(R_k))$, for $x \in R$. Consequently, it holds that

$$\begin{aligned}\partial^s R &= \partial^s[X^u(x,R), X^s(x,R)] \\ &= [\partial X^u(x,R), X^s(x,R)] \\ &\subset [\partial X^u(x,\varphi^{-1}(R_k)), X^s(x,\varphi^{-1}(R_k))] \\ &= \partial^s(\varphi^{-1}(R_k)).\end{aligned}$$

The unstable case is proved similarly.

The next lemma will allow us to apply Lemma 3.2.7 inductively.

Lemma 3.2.9. *For every $n \in \mathbb{N}$ one has*

$$\mathcal{R}_{n+1} = \{R \in \varphi(\mathcal{R}_n) \vee \mathcal{R}_n \vee \varphi^{-1}(\mathcal{R}_n) : \operatorname{int}(R) \neq \varnothing\}.$$

Proof. To prove that $\mathcal{R}_{n+1} \subset \{R \in \varphi(\mathcal{R}_n) \vee \mathcal{R}_n \vee \varphi^{-1}(\mathcal{R}_n) : \operatorname{int}(R) \neq \varnothing\}$ we just need the fact that the interior of a finite intersection is the intersection of the interiors. For the reverse inclusion it suffices to observe that if $R, S \in \mathcal{R}_1$ with $\operatorname{int}(R) \cap \operatorname{int}(S) \neq \varnothing$ then $R = S$. □

Lemma 3.2.10. *For every $n \in \mathbb{N}$ the cover $\mathcal{R}_n$ is a Markov partition.*

Proof. The statement holds for $n = 1$ since $\mathcal{R}_1$ is a Markov partition by definition. Assume that $\mathcal{R}_n$ is a Markov partition and the claim is that $\mathcal{R}_{n+1}$ is one too. Let $R \in \mathcal{R}_{n+1}$ and we certainly have $\operatorname{diam}(R) \leq \varepsilon''_X$. Using Lemma 3.2.9 and parts (1) and (3) of Proposition 3.2.7 we deduce that R is a proper rectangle. The fact that the interiors of any two rectangles in $\mathcal{R}_{n+1}$ are mutually disjoint follows from φ being a homeomorphism.

For conditions (2) and (3) of Definition 3.2.3, again by Lemma 3.2.9, it is enough to consider an arbitrary

$$x \in \operatorname{int}(\varphi(R_\ell) \cap R_i \cap \varphi^{-1}(R_j) \cap \varphi^{-1}(\varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k)))$$

for some $R_\ell, R_i, R_j, R_k \in \mathcal{R}_n$.

Then one has

$$\begin{aligned} X^u(\varphi(x), \varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k)) &\subset X^u(\varphi(x), R_j) \cap \varphi(R_i) \\ &\subset \varphi(X^u(x, R_i)) \cap \varphi(R_i) \cap R_j \\ &\subset \varphi(\varphi(X^u(\varphi^{-1}(x), R_\ell))) \cap \varphi(R_i) \cap R_j \\ &= \varphi(X^u(x, \varphi(R_\ell) \cap R_i \cap \varphi^{-1}(R_j))), \end{aligned}$$

where the second inclusion holds because $x \in \operatorname{int}(R_i) \cap \varphi^{-1}(\operatorname{int}(R_j))$ and the third because $x \in \varphi(\operatorname{int}(R_\ell)) \cap \operatorname{int}(R_i)$. For the last equality use the same argument as in (3.2.13).

For the stable part, we have

$$\begin{aligned} \varphi(X^s(x, \varphi(R_\ell) \cap R_i \cap \varphi^{-1}(R_j))) &\subset \varphi(X^s(x, R_i)) \cap R_j \\ &\subset X^s(\varphi(x), R_j) \cap \varphi(R_i) \cap R_j \\ &\subset \varphi^{-1}(X^s(\varphi^2(x), R_k)) \cap \varphi(R_i) \cap R_j \\ &= X^s(\varphi(x), \varphi(R_i) \cap R_j \cap \varphi^{-1}(R_k)), \end{aligned}$$

where the second inclusion holds because $x \in \operatorname{int}(R_i) \cap \varphi^{-1}(\operatorname{int}(R_j))$, the third because $\varphi(x) \in \operatorname{int}(R_j) \cap \varphi^{-1}(\operatorname{int}(R_k))$, and finally the last equality is true for the same reason as in the unstable case. □

Lemma 3.2.11. *The approximation graph* $\Pi = (\mathcal{R}, \mathcal{A})$ *is essential. Also* $(\mathcal{P}_\Pi, \sigma_\Pi) = (\Sigma_M, \sigma_M)$*, where* M *is the transition matrix of* $\mathcal{R}_1$*.*

Proof. The graph Π is essential because the rectangles in $\mathcal{R}_n$ are proper with mutually disjoint interiors. Finally, $(\mathcal{P}_\Pi, \sigma_\Pi) = (\Sigma_M, \sigma_M)$ following the definition of $\mathcal{R}_n$ and the cylinder sets of Σ_M. □

We now estimate the growth rate of $\#\mathcal{R}_n$ with respect to n. Recall that an irreducible Smale space has the Bowen measure which maximises the topological entropy.

Lemma 3.2.12 ([16, Theorem 33])**.** *For a topological Markov chain* (Σ_M, σ_M) *that is induced by a Markov partition of an irreducible Smale space* (X, φ) *one has* $\mathrm{h}(\varphi) = \mathrm{h}(\sigma_M)$*.*

Lemma 3.2.13. *There exist constants* $C, c > 0$ *such that for every* $\varepsilon \in (0,1)$ *there is some* $n_0 \in \mathbb{N}$ *so that, for* $n \geq n_0$*, we have*

$$c e^{2(\mathrm{h}(\varphi) - \varepsilon)n} < \#\mathcal{R}_n < C e^{2(\mathrm{h}(\varphi) + \varepsilon)n}.$$

Proof. For any rectangle $R \in \mathcal{R}_n$ its essential part is given by $R^{\mathrm{ess}} = \operatorname{int}(R)$. Now (2.2.8) together with the essentiality of Π imply that $R^{\mathrm{ess}} \subset \pi_\Pi(C_R) \subset R$, where C_R is the cylinder set of R in $\mathcal{P}_\Pi$. Since R is proper and π_Π is a closed map, we get $\pi_\Pi(C_R) = R$. Now, let

(Σ_M, σ_M) be the topological Markov chain that is induced by $\mathcal{R}_1$ and recall the definition of the factor map $\pi_M : (\Sigma_M, \sigma_M) \to (X, \varphi)$ where $\pi_M = \pi_\Pi$ from Lemma 3.2.11. Then every $R \in \mathcal{R}_n$ is the π_M-image of a cylinder set of rank $2n-1$. Actually, $\#\mathcal{R}_n = N_M(2n-1)$; the number of non-empty cylinder sets of Σ_M of rank $2n-1$. From Lemma 3.1.5, there exist constants $C, c > 0$ such that for every $\varepsilon \in (0,1)$ there is some $n_0 \in \mathbb{N}$ so that for $n \geq n_0$ we have

$$ce^{2(\mathrm{h}(\sigma_M)-\varepsilon)n} < \#\mathcal{R}_n < Ce^{2(\mathrm{h}(\sigma_M)+\varepsilon)n}.$$

Using Lemma 3.2.12 we obtain the result. □

Lemma 3.2.14. *For every $n \geq 0$ we have* $\mathrm{m}(\mathcal{R}_n) \leq (\#\mathcal{R}_1)^2$.

Proof. It follows from the same argument used in the proof of Lemma 3.2.13 and part (4) of Theorem 3.2.5. □

For proving condition (3) of Proposition 3.2.6 we will use the *diamond trick* found in [1]. More precisely, the map π_M would have a *diamond* if there were two sequences $x = (x_i)_{i\in\mathbb{Z}}$ and $y = (y_i)_{i\in\mathbb{Z}}$ with $\pi_M(x) = \pi_M(y)$ for which there exist indices $k < l < m$ such that $x_k = y_k, x_l \neq y_l$ and $x_m = y_m$. However, π_M does not have a diamond since $\overline{\mathrm{diam}}(\mathcal{R}_1) < \varepsilon_X/2$, see [1, Lemma 6.9]. So our strategy for proving condition (3) is by contradiction. We will assume that it does not hold and then we will be able to create a diamond for π_M.

We say that $R, S \in \mathcal{R}_n$ have a *common neighbour* if they intersect some $T \in \mathcal{R}_n$. Also, recall that for any $m \geq n$ and $R \in \mathcal{R}_m$ there is a unique $R' \in \mathcal{R}_n$ such that $R \subset R'$. We call R' an *ancestor* of R. The next lemma will follow from the Markov property.

Lemma 3.2.15. *There is some $N \in \mathbb{N}$ such that for every $R, S \in \mathcal{R}_{n+N}$ with a common neighbour, their ancestors in $\mathcal{R}_n$ intersect.*

Before proceeding to the proof let us make an observation. The above statement is equivalent to the one saying that there is some $N \in \mathbb{N}$ such that for every disjoint $R, S \in \mathcal{R}_n$, all their descendants in $\mathcal{R}_{n+N}$ do not have a common neighbour. Another way rephrasing this is to say that, for every disjoint $R, S \in \mathcal{R}_n$ we have $\mathrm{N}_{\mathcal{R}_{n+N}}(R) \cap \mathrm{N}_{\mathcal{R}_{n+N}}(S) = \varnothing$. Of course, any two disjoint $R, S \in \mathcal{R}_n$ can be separated by neighbourhoods since they are closed sets and X is normal. What Lemma 3.2.15 says is that *Smale spaces are normal in a highly controlled way.*

Proof of Lemma 3.2.15. We will use induction on n. First, let $N \in \mathbb{N}$ be the smallest number that

$$\mathrm{N}_{\mathcal{R}_{1+N}}(R) \cap \mathrm{N}_{\mathcal{R}_{1+N}}(S) = \varnothing,$$

for every disjoint $R, S \in \mathcal{R}_1$. This can be done since $(\mathcal{R}_n)_{n\geq 0}$ is refining and X is normal.

Assume now that

$$\mathrm{N}_{\mathcal{R}_{n+N}}(R) \cap \mathrm{N}_{\mathcal{R}_{n+N}}(S) = \varnothing,$$

for every disjoint $R, S \in \mathcal{R}_n$ and we claim that the same holds for $n+1$.

Let $R, S \in \mathcal{R}_{n+1}$ be disjoint. From Lemma 3.2.9 one can write $R = \varphi(R_i) \cap \varphi^{-1}(R_j)$ and $S = \varphi(S_i) \cap \varphi^{-1}(S_j)$, for $R_i, R_j, S_i, S_j \in \mathcal{R}_n$. Actually $R = [\varphi^{-1}(R_j), \varphi(R_i)]$ and $S = [\varphi^{-1}(S_j), \varphi(S_i)]$, and since they are disjoint we have that either $\varphi^{-1}(R_j) \cap \varphi^{-1}(S_j) = \varnothing$ or $\varphi(R_i) \cap \varphi(S_i) = \varnothing$. Then, from the inductive step,

$$\mathrm{N}_{\varphi(\mathcal{R}_{n+N})}(\varphi(R_i)) \cap \mathrm{N}_{\varphi(\mathcal{R}_{n+N})}(\varphi(S_i)) = \varnothing$$

or

$$\mathrm{N}_{\varphi^{-1}(\mathcal{R}_{n+N})}(\varphi^{-1}(R_j)) \cap \mathrm{N}_{\varphi^{-1}(\mathcal{R}_{n+N})}(\varphi^{-1}(S_j)) = \varnothing.$$

Hence, we obtain that $\mathrm{N}_{\mathcal{R}_{n+N+1}}(R) \cap \mathrm{N}_{\mathcal{R}_{n+N+1}}(S) = \varnothing$. Indeed, assume there is some $T = \varphi^{-1}(T_j) \cap \varphi(T_i) \in \mathrm{N}_{\mathcal{R}_{n+N+1}}(R) \cap \mathrm{N}_{\mathcal{R}_{n+N+1}}(S)$ with $T_i, T_j \in \mathcal{R}_{n+N}$ then,

$$\varphi(T_i) \in \mathrm{N}_{\varphi(\mathcal{R}_{n+N})}(\varphi(R_i)) \cap \mathrm{N}_{\varphi(\mathcal{R}_{n+N})}(\varphi(S_i))$$

and

$$\varphi^{-1}(T_j) \in \mathrm{N}_{\varphi^{-1}(\mathcal{R}_{n+N})}(\varphi^{-1}(R_j)) \cap \mathrm{N}_{\varphi^{-1}(\mathcal{R}_{n+N})}(\varphi^{-1}(S_j)).$$

But at least one of the two intersections is empty leading to a contradiction. □

Let $M_N = \max\{\#\mathrm{N}_{\mathcal{R}_n}(R) : 0 \leq n \leq N+1, R \in \mathcal{R}_n\}$.

Lemma 3.2.16 (Neighbouring Rectangles)**.** *For $n \geq N+2$ and every $R \in \mathcal{R}_n$ we have*

$$\# \mathrm{N}_{\mathcal{R}_n}(R) \leq (\#\mathcal{R}_1)^{2(N+1)}.$$

Consequently,

$$\# \mathrm{N}_{\mathcal{R}_n}(R) \leq \max\{(\#\mathcal{R}_1)^{2(N+1)}, M_N\}$$

for every $n \in \mathbb{N}$ and $R \in \mathcal{R}_n$.

Proof. Assume to the contrary that there is some $n \geq N+2$ and $R \in \mathcal{R}_n$ such that

$$\#\mathrm{N}_{\mathcal{R}_n}(R) \geq (\#\mathcal{R}_1)^{2(N+1)} + 1.$$

Let

$$(R^{(k)}_{-n+1}, \dots, R^{(k)}_{0}, \dots, R^{(k)}_{n-1}),$$

where $1 \leq k \leq \#\mathrm{N}_{\mathcal{R}_n}(R)$, be the sequences that denote the different elements of $\mathrm{N}_{\mathcal{R}_n}(R)$, where each term is a rectangle in $\mathcal{R}_1$. Since there can only be up to $(\#\mathcal{R}_1)^2$ different pairs $(R^{(k)}_i, R^{(k)}_j)$, from the pigeonhole principle there are two sequences

$$S = (R^{(k)}_{-n+1}, \dots, R^{(k)}_{0}, \dots, R^{(k)}_{n-1}) \text{ and } T = (R^{(j)}_{-n+1}, \dots, R^{(j)}_{0}, \dots, R^{(j)}_{n-1})$$

such that, for every $0 \leq i \leq N$,

$$R^{(k)}_{-n+1+i} = R^{(j)}_{-n+1+i} \text{ and } R^{(k)}_{n-1-i} = R^{(j)}_{n-1-i}. \tag{3.2.15}$$

Since S and T have a common neighbour, that is R, by Lemma 3.2.15 their ancestors

$$S' = (R^{(k)}_{-n+1+N}, \dots, R^{(k)}_{0}, \dots, R^{(k)}_{n-1-N})$$

and

$$T' = (R^{(k)}_{-n+1+N}, \dots, R^{(j)}_{0}, \dots, R^{(k)}_{n-1-N})$$

intersect. Observe that due to (3.2.15) both end-terms of S' and T' agree. At this point we should note that since $n \geq N+2$ the sequences S' and T' have at least three terms. Therefore, since $S \neq T$ again by (3.2.15) we conclude $S' \neq T'$ and hence there is some index i in between such that $R^{(k)}_i \neq R^{(j)}_i$.

Choose a point $x \in S' \cap T'$ and we can find bi-infinite sequences

$$(\dots, R^{(k)}_{-n+1+N}, \dots, R^{(k)}_{i}, \dots, R^{(k)}_{n-1-N}, \dots)$$

and

$$(\dots, R^{(k)}_{-n+1+N}, \dots, R^{(j)}_{i}, \dots, R^{(k)}_{n-1-N}, \dots)$$

which both map to x under Bowen's factor map, see Theorem 3.2.5. This means the factor map has a diamond, which is a contradiction. □

3.3 Geometric approximations of Smale spaces

Let (X, φ) be an irreducible Smale space, $\mathcal{R}_1$ be a Markov partition with $\overline{\mathrm{diam}}(\mathcal{R}_1) \leq \varepsilon''_X/2$ (see (3.2.1) for the definition of ε''_X). Proposition 3.2.6 yields the refining sequence of Markov partitions $(\mathcal{R}_n)_{n\geq 0}$ and the induced approximation graph $\Pi = (\mathcal{R}, \mathcal{A})$. Our goal is to modify Π so that we obtain a geometrically-essential approximation graph for (X, φ).

For now, consider an arbitrary

$$0 < \delta \leq \varepsilon''_X/4, \tag{3.3.1}$$

and for every $R \in \mathcal{R}_1$ define its *δ-fattening* to be

$$R^\delta = [R \cup (\partial^s R)^\delta, R \cup (\partial^u R)^\delta], \tag{3.3.2}$$

where

$$(\partial^s R)^\delta = \bigcup\{X^u(z,\delta) : z \in \partial^s R\} \text{ and } (\partial^u R)^\delta = \bigcup\{X^s(w,\delta) : w \in \partial^u R\}. \tag{3.3.3}$$

This is a well-defined rectangle with $\operatorname{diam}(R^\delta) \leq \varepsilon'_X$, since

$$\operatorname{diam}(R \cup (\partial^s R)^\delta \cup (\partial^u R)^\delta) \leq \operatorname{diam}(R) + 2\delta \leq \varepsilon''_X, \tag{3.3.4}$$

and a triangle inequality yields $\operatorname{diam}(R^\delta) \leq \varepsilon'_X/2 + 2\delta + \operatorname{diam}(R) < 7\varepsilon'_X/12$.

For $n \geq 2$ and $R \in \mathcal{R}_n$, written uniquely as $\bigcap_{i=1-n}^{n-1} \varphi^{-i}(R_{x_i})$ with $R_{x_i} \in \mathcal{R}_1$, define its δ-fattening by

$$R^\delta = \bigcap_{i=1-n}^{n-1} \varphi^{-i}(R^\delta_{x_i}). \tag{3.3.5}$$

As a result, for every $n \geq 0$ we obtain the covers

$$\mathcal{R}^\delta_n = \{R^\delta : R \in \mathcal{R}_n\}. \tag{3.3.6}$$

For $n \in \mathbb{N}$, each cover $\mathcal{R}^\delta_{n+1}$ refines $\mathcal{R}^\delta_n$ since for $R \in \mathcal{R}_{n+1}$ and $S \in \mathcal{R}_n$ such that $R \subset S$ it holds $R^\delta \subset S^\delta$. Indeed, writing $R = \bigcap_{i=-n}^{n} \varphi^{-i}(R_{x_i})$ and $S = \bigcap_{i=1-n}^{n-1} \varphi^{-i}(S_{y_i})$ for unique $R_{x_i}, S_{y_i} \in \mathcal{R}_1$, since $R \subset S$, we have $R_{x_i} = S_{y_i}$ for $|i| \leq n-1$ and hence $R^\delta \subset S^\delta$. Finally, with the same arguments that we used for $(\mathcal{R}_n)_{n\geq 0}$, we can show that the sequence $(\mathcal{R}^\delta_n)_{n\geq 0}$ is refining and induces the approximation graph Π^δ.

Remark 3.3.1. By choosing $\delta > 0$ small enough we can make $\mathcal{R}^\delta_1$ to behave like a Markov partition. Using the Markov property we will prove that this behaviour passes on each $\mathcal{R}^\delta_n$, since the latter are inductively defined by $\mathcal{R}^\delta_1$. Moreover, the inductive definition of the refining sequence $(\mathcal{R}^\delta_n)_{n\geq 0}$ will allow us to estimate the rate of decay of the Lebesgue covering numbers and the diameters of the covers $\mathcal{R}^\delta_n$.

Recall that $\ell_X = \min\{\operatorname{Lip}(\varphi), \operatorname{Lip}(\varphi^{-1})\}$ and $\Lambda_X = \max\{\operatorname{Lip}(\varphi), \operatorname{Lip}(\varphi^{-1})\}$. One of the key tools in this thesis is the following.

Theorem 3.3.2. *For every $n \in \mathbb{N}$, the open cover $\mathcal{R}^\delta_n$ in (3.3.6) behaves like the Markov partition $\mathcal{R}_n$, given that δ is sufficiently small. More precisely, there exist constants $\theta \in (0, \varepsilon_X]$ and $C, c > 0$ and $\delta_1 \in (0, \varepsilon''_X/4]$ that depend on $\mathcal{R}_1$ so that, for every $\delta \in (0, \delta_1]$, we have that*

(1) $\overline{\operatorname{diam}}(\mathcal{R}_n^\delta) \leq \lambda_X^{-n+1}\theta$;

(2) $\operatorname{m}(\mathcal{R}_n^\delta) \leq (\#\mathcal{R}_1)^2$;

(3) the number of neighbouring rectangles is uniformly bounded, meaning

$$\sup_n \max_{R^\delta \in \mathcal{R}_n^\delta} \# \operatorname{N}_{\mathcal{R}_n^\delta}(R^\delta) < \infty;$$

(4) for every $\varepsilon \in (0,1)$, there is some $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, we have

$$ce^{2(\operatorname{h}(\varphi)-\varepsilon)n} < \#\mathcal{R}_n^\delta < Ce^{2(\operatorname{h}(\varphi)+\varepsilon)n};$$

(5) the approximation graph Π^δ, associated to the refining sequence of δ-fat Markov partitions $(\mathcal{R}_n^\delta)_{n\geq 0}$, is metrically-essential.

If either φ or φ^{-1} are Lipschitz, there is some $\zeta \in (0,\theta]$ (independent of δ) so that

(6) $\underline{diam}(\mathcal{R}_n^\delta) \geq \underline{diam}(\mathcal{R}_n) \geq \ell_X^{-n+1}\zeta$.

If in addition φ is bi-Lipschitz, for every $\delta \in (0,\delta_1]$, there is some $\eta \in (0,\theta]$ so that

(7) $\operatorname{Leb}(\mathcal{R}_n^\delta) \geq \Lambda_X^{-n+1}\eta$.

From Theorem 3.3.2 and Lemma 3.1.16 we obtain the following.

Corollary 3.3.3. *The irreducible Smale space (X,φ) is topologically conjugate to a Smale space (Y,ψ) which admits a refining sequence that satisfies all conditions of Theorem 3.3.2 with $\lambda_Y = \ell_Y = \Lambda_Y$.*

Example 3.3.4. *Suppose that (X,φ) is an irreducible topological Markov chain equipped with the ultrametric (3.1.8). Moreover, assume that the Markov partition $\mathcal{R}_1$ consists of sufficiently small symmetric cylinder sets, and hence each $\mathcal{R}_n$ consists of smaller symmetric cylinder sets. Then, for every $0 < \delta \leq \varepsilon_X''/4$, the δ-fattening process of $(\mathcal{R}_n)_{n\geq 0}$ is trivial since $\mathcal{R}_n^\delta = \mathcal{R}_n$, for every $n \in \mathbb{N}$. More precisely, for every $R \in \mathcal{R}_1$ we have that $\partial^s R = \partial^u R = \varnothing$, and hence*

$$(\partial^s R)^\delta = (\partial^u R)^\delta = \varnothing.$$

Therefore, from (3.3.2) we have $R^\delta = [R,R] = R$. In order to see how Theorem 3.3.2 applies, recall that $\lambda_X = \ell_X = \Lambda_X = 2$. Finally, the multiplicities of the covers are clearly equal to one, and condition (4) follows from Lemma 3.1.5.

Example 3.3.5. *Suppose now that (X, φ) is the dyadic solenoid of Subsection 3.1.4. From [19, Prop. 2.3.4] we see that it is mixing. In order to build a Markov partition $\mathcal{R}_1$, consider the decomposition of the unit circle $\mathbb{T}$ into the closed upper half e_0 and the closed lower half e_1. Then, let*

$$E_0 = \{x \in X : x_1 \in e_0\}$$
$$E_1 = \{x \in X : x_1 \in e_1\},$$

and one can observe that $\operatorname{int}(\varphi^{-1}(E_i) \cap E_j) \neq \varnothing$, for all $0 \leq i, j \leq 1$. Each of the four sets corresponds to one of the quadrants of the circle in a clear way. With a bit more effort one can show that the cover $E = \{E_0, E_1\}$ yields a 2-to-1 factor map from the full-two shift to (X, φ). Roughly speaking, this map encodes $x \in X$ by a bi-infinite sequence containing all encodings for the dyadic expansions of the coordinates of x. However, the sets E_0, E_1 are not quite rectangles since their diameter is large. But for sufficiently large $m \in \mathbb{N}$ we get the Markov partition

$$\mathcal{R}_1 = \bigvee_{j=1-m}^{m-1} \varphi^{-j}(E),$$

on which we can apply the δ-fattening process. Since the local stable sets are Cantor sets, for every $R \in \mathcal{R}_1$ it holds that $\partial^u R = \varnothing$. As a result, the δ-fattening happens only on the local unstable sets and $R^\delta = [R \cup (\partial^s R)^\delta, R]$. The proof of Lemma 3.3.6 helps to visualise the rectangles.

Finally, let us understand some important parts of Theorem 3.3.2 in this case. From Subsection 3.1.4 we have that $\lambda_X = 2, \ell_X = 2$ and $\Lambda_X \leq 5/2$. Therefore, the diameters $\overline{\operatorname{diam}}(\mathcal{R}_n^\delta) \sim 2^{-n}$, the Lebesgue numbers $\operatorname{Leb}(\mathcal{R}_n^\delta) \gtrsim (5/2)^{-n}$ and the cardinalities $\#\mathcal{R}_n^\delta \sim 2^{2n}$. Moreover, from the construction of the factor map we have $\operatorname{m}(\mathcal{R}_n^\delta) \leq 2$. In fact, $\operatorname{m}(\mathcal{R}_n^\delta) = 2$ for all $n \in \mathbb{N}$, since the two fixed points of the shift space are mapped down to the fixed point $(1, 1, 1, \dots) \in E_0 \cap E_1$. Finally, since $(\mathcal{R}_n^\delta)_{n \geq 0}$ is a refining sequence of open covers, it holds that $\dim X \leq 1$ and because X is connected we get the already known fact that $\dim X = 1$.

3.3.1 Proof of conditions (1)-(5) of Theorem 3.3.2

Their proof consists of the following lemmas and corollaries. For the rest of this subsection we consider the refining sequences $(\mathcal{R}_n^\delta)_{n \geq 0}$ constructed in (3.3.6), for $\delta \in (0, \varepsilon_X''/4]$. But first, recall that if S is a rectangle in X and $x \in S$, the set $X^s(x, 2\varepsilon_X') \cap S$ is denoted by $X^s(x, S)$ and similarly $X^u(x, 2\varepsilon_X') \cap S$ is denoted by $X^u(x, S)$.

Lemma 3.3.6. *For $\delta \in (0, \varepsilon_X''/4]$ and $n \in \mathbb{N}$, the cover $\mathcal{R}_n^\delta$ consists of open rectangles.*

Proof. We just need to prove that every $R^\delta \in \mathcal{R}_1^\delta$ is open in X, and in the process we will demonstrate how it differs from $R \in \mathcal{R}_1$. For any $x \in R$ define

$$(\partial X^u(x,R))^\delta = \bigcup_{z \in \partial X^u(x,R)} X^u(z,\delta) \text{ and } (\partial X^s(x,R))^\delta = \bigcup_{w \in \partial X^s(x,R)} X^s(w,\delta), \tag{3.3.7}$$

along with the open rectangle

$$R_x^\delta := [X^u(x,R) \cup (\partial X^u(x,R))^\delta, X^s(x,R) \cup (\partial X^s(x,R))^\delta]. \tag{3.3.8}$$

The claim is that the rectangle $R^\delta = \bigcup_{x \in R} R_x^\delta$ and hence R^δ is open. Note here that if we let $R_x = [X^u(x,R), X^s(x,R)]$ for $x \in R$, then $R_x = R = R_y$ for any $x, y \in R$. However, in the δ-fattened version we may get $R_x^\delta \neq R_y^\delta$, for some $x \neq y \in R$, because there is no more control of the local stable and the unstable sets as soon as they get outside of R.

First we prove that

$$R \cup (\partial^s R)^\delta = \bigcup_{x \in R} X^u(x,R) \cup (\partial X^u(x,R))^\delta,$$

where it is straightforward to see that $R = \bigcup_{x \in R} X^u(x,R)$. The claim is that

$$(\partial^s R)^\delta = \bigcup_{x \in R} (\partial X^u(x,R))^\delta. \tag{3.3.9}$$

To see this, let $x \in R$ and if $z \in \partial X^u(x,R)$ then $z \in \partial^s R$ since, following the equations (3.2.7), it holds that $z \in X^s(z,R) = [z, X^s(x,R)] \subset [\partial X^u(x,R), X^s(x,R)] = \partial^s R$. As a result, $\bigcup_{x \in R} (\partial X^u(x,R))^\delta \subset (\partial^s R)^\delta$. For the other inclusion, we note that if $z \in \partial^s R$, then $z \in \partial X^u(z,R)$, for if $z \in \operatorname{int}(X^u(z,R))$ then $X^s(x,R) \cap \operatorname{int}(R) \neq \varnothing$. Similarly, it holds that

$$R \cup (\partial^u R)^\delta = \bigcup_{y \in R} X^s(y,R) \cup (\partial X^s(y,R))^\delta.$$

As a result,

$$\begin{aligned} R^\delta &= \bigcup_{x,y \in R} [X^u(x,R) \cup (\partial X^u(x,R))^\delta, X^s(y,R) \cup (\partial X^s(y,R))^\delta] \\ &= \bigcup_{[y,x] \in R} [X^u([y,x],R) \cup (\partial X^u([y,x],R))^\delta, X^s([y,x],R) \cup (\partial X^s([y,x],R))^\delta] \\ &= \bigcup_{[y,x] \in R} R_{[y,x]}^\delta \\ &= \bigcup_{x \in R} R_x^\delta. \square \end{aligned}$$

Lemma 3.3.7. *For $\delta \in (0, \varepsilon_X''/4]$ and $n \in \mathbb{N}$, it holds $\overline{\operatorname{diam}}(\mathcal{R}_n^\delta) \leq \min\{\lambda_X^{-n+1}\varepsilon_X, \varepsilon_X'\}$.*

Proof. Let $n \in \mathbb{N}$ and $R^\delta \in \mathcal{R}_n^\delta$ with some $x, y \in R^\delta$. From the definition of R^δ we have $d(\varphi^i(x), \varphi^i(y)) \leq \varepsilon'_X$, for all $|i| \leq n-1$. In particular, using the bracket axiom (B4), we have

$$[\varphi^{-n+1}(x), \varphi^{-n+1}(y)] = \varphi^{-n+1}([x,y])$$

and

$$d(\varphi^{-n+1}(x), [\varphi^{-n+1}(x), \varphi^{-n+1}(y)]) \leq \varepsilon_X/2.$$

Therefore, we have $\varphi^{-n+1}([x,y]) \in X^s(\varphi^{-n+1}(x), \varepsilon_X/2)$ and hence $[x,y] \in X^s(x, \lambda_X^{-n+1}\varepsilon_X/2)$. Similarly, $[x,y] \in X^u(y, \lambda_X^{-n+1}\varepsilon_X/2)$ and from the triangle inequality $d(x,y) \leq \lambda_X^{-n+1}\varepsilon_X$. $\square$

We want to show that our method produces covers which are closely related to the Markov partitions. First we study all the possible overlaps that can occur between elements of $\mathcal{R}_n^\delta$, for a given $\delta \in (0, \varepsilon''_X/4]$. For a finite cover $\mathcal{U} = \{U_1, \dots, U_\ell\}$ of X let

$$M(\mathcal{U}) = \{\{U_i\}_{i\in I} : I \subset \{1, \dots, \ell\} \text{ and } \bigcap_{i \in I} U_i \neq \varnothing\} \tag{3.3.10}$$

be the nerve of $\mathcal{U}$ and $M^\#(\mathcal{U}) = \{\#I : \{U_i\}_{i\in I} \in M(\mathcal{U})\}$. It should be clear that $\max(M^\#(\mathcal{U}))$ is the multiplicity of $\mathcal{U}$. For $\delta \in (0, \varepsilon''_X/4]$ and $n \in \mathbb{N}$, consider the map

$$M_n^\delta : M(\mathcal{R}_n) \to M(\mathcal{R}_n^\delta), \;\; \{R_i\}_{i\in I} \mapsto \{R_i^\delta\}_{i \in I} \tag{3.3.11}$$

and we aim to show that for small enough δ, if $\bigcap_{i\in I} R_i^\delta \neq \varnothing$ then $\bigcap_{i \in I} R_i \neq \varnothing$, meaning that the map M_n^δ is surjective.

Let $\mathcal{R}_1 = \{R_1, \dots, R_\ell\}$ and $E(\mathcal{R}_1) = \{I \subset \{1, \dots, \ell\} : \bigcap_{i\in I} R_i = \varnothing\}$. Then, there is $\delta_0 \in (0, \varepsilon''_X/4]$ such that for every $I \in E(\mathcal{R}_1)$ it holds $\bigcap_{i\in I} R_i^{\delta_0} = \varnothing$. Indeed, let $I \in E(\mathcal{R}_1)$ and then for every $i \in I$ there is some $\delta'_{I,i} > 0$ such that the $\delta'_{I,i}$-neighbourhood of R_i, denoted by $B(R_i, \delta'_{I,i})$, satisfies $B(R_i, \delta'_{I,i}) \cap B(\bigcap_{j \in I, j\neq i} R_j, \delta'_{I,i}) = \varnothing$. Also, there is $0 < \delta_{I,i} < \delta'_{I,i}$ such that $R_i^{\delta_{I,i}} \subset B(R_i, \delta'_{I,i})$ and $\bigcap_{j\in I, j \neq i} R_j^{\delta_{I,i}} \subset B(\bigcap_{j \in I, j \neq i} R_j, \delta'_{I,i})$. So choosing

$$\delta_0 = \min\{\varepsilon''_X/4, \min\{\delta_{I,i} : I \in E(\mathcal{R}_1), i \in I\}\} \tag{3.3.12}$$

has the required property.

Lemma 3.3.8. *For $\delta \in (0, \delta_0]$ and $n \in \mathbb{N}$, the map M_n^δ defined in (3.3.11) is surjective.*

Proof. For $n = 0$ it is trivially true and from the choice of δ the map M_1^δ is surjective. Let us assume M_n^δ is surjective and we claim that M_{n+1}^δ is too. Let $\{R_i^\delta\}_{i\in I} \in M(\mathcal{R}_{n+1}^\delta)$ and from Lemma 3.2.9 every

$$R_i^\delta = \varphi(R_{i1}^\delta) \cap R_{i2}^\delta \cap \varphi^{-1}(R_{i3}^\delta)$$

with $R_{i1}^\delta, R_{i2}^\delta, R_{i3}^\delta \in \mathcal{R}_n^\delta$ and $R_i = \varphi(R_{i1}) \cap R_{i2} \cap \varphi^{-1}(R_{i3}) \in \mathcal{R}_{n+1}$.

Then it holds

$$\bigcap_{i\in I} R_i^\delta = \bigcap_{i\in I} \varphi(R_{i1}^\delta) \cap R_{i2}^\delta \cap \varphi^{-1}(R_{i3}^\delta) \neq \varnothing$$

and we want to prove that

$$\bigcap_{i\in I} R_i = \bigcap_{i\in I} \varphi(R_{i1}) \cap R_{i2} \cap \varphi^{-1}(R_{i3}) \neq \varnothing.$$

Equivalently, using part (3) of Proposition 3.2.7 we want to prove that

$$\bigcap_{i\in I} [\varphi^{-1}(R_{i3}), \varphi(R_{i1})] \neq \varnothing.$$

From the inductive step we obtain that

$$\bigcap_{i\in I} \varphi(R_{i1}^\delta) \neq \varnothing \Rightarrow \bigcap_{i\in I} \varphi(R_{i1}) \neq \varnothing$$

and also

$$\bigcap_{i\in I} \varphi^{-1}(R_{i3}^\delta) \neq \varnothing \Rightarrow \bigcap_{i\in I} \varphi^{-1}(R_{i3}) \neq \varnothing.$$

Hence

$$\begin{aligned}\varnothing \neq [\bigcap_{i\in I} \varphi^{-1}(R_{i3}), \bigcap_{i\in I} \varphi(R_{i1})] \\ \subset \bigcap_{i\in I} [\varphi^{-1}(R_{i3}), \varphi(R_{i1})] \\ = \bigcap_{i\in I} R_i.\end{aligned}$$

Thus, by induction the maps M_n^δ are surjective. □

We now aim to find $\delta_1 \in (0, \delta_0]$ so that, for every $\delta \in (0, \delta_1]$, if $R, S \in \mathcal{R}_n$ with $R \neq S$ then $R \notin S^\delta$. In particular, for such δ, the δ-fattening maps

$$F_n^\delta : \mathcal{R}_n \to \mathcal{R}_n^\delta, \quad R \mapsto R^\delta \tag{3.3.13}$$

and the maps M_n^δ defined in (3.3.11) will be shown to be bijective, for every $n \in \mathbb{N}$.

This is a subtle procedure that requires the following concepts. First, for every $n \in \mathbb{N}$ and $R \in \mathcal{R}_n$ define

$$\partial^{s,o} R = \{x \in \partial^s R : X^u(x, R) \cap \operatorname{int}(R) \neq \varnothing\} \tag{3.3.14}$$

and

$$\partial^{u,o} R = \{x \in \partial^u R : X^s(x, R) \cap \operatorname{int}(R) \neq \varnothing\}. \tag{3.3.15}$$

Then, consider the sets

$$\mathrm{N}^s_{\mathcal{R}_n}(R) = \{S \in \mathrm{N}_{\mathcal{R}_n}(R) \smallsetminus \{R\} : S \cap \partial^{u,o} R = \varnothing\} \tag{3.3.16}$$

and

$$\mathrm{N}^u_{\mathcal{R}_n}(R) = \{S \in \mathrm{N}_{\mathcal{R}_n}(R) \smallsetminus \{R\} : S \cap \partial^{s,o} R = \varnothing\} \tag{3.3.17}$$

which are the *stable* and *unstable neighbours* of R, respectively. These sets provide a decomposition in the sense that

$$\mathrm{N}_{\mathcal{R}_n}(R) \smallsetminus \{R\} = \mathrm{N}^s_{\mathcal{R}_n}(R) \cup \mathrm{N}^u_{\mathcal{R}_n}(R). \tag{3.3.18}$$

Indeed, if $S \in \mathrm{N}_{\mathcal{R}_n}(R) \smallsetminus \{R\}$ with $S \notin \mathrm{N}^s_{\mathcal{R}_n}(R)$ and $S \notin \mathrm{N}^u_{\mathcal{R}_n}(R)$ then there are some $x \in S \cap \partial^{u,o} R$ and $y \in S \cap \partial^{s,o} R$, and hence $[x,y] \in S \cap \mathrm{int}(R)$. However, $S \cap \mathrm{int}(R) = \varnothing$ which results in a contradiction.

For every $R \in \mathcal{R}_1$ choose some $x_R \in \mathrm{int}(R)$ and then for all $T \in \mathrm{N}^s_{\mathcal{R}_1}(R)$ and $S \in \mathrm{N}^u_{\mathcal{R}_1}(R)$ we have $T \cap X^s(x_R, R) = \varnothing$ and $S \cap X^u(x_R, R) = \varnothing$. Consequently, there is some small enough $\delta' > 0$ so that

$$T^{\delta'} \cap X^s(x_R, R) = \varnothing \tag{3.3.19}$$

and

$$S^{\delta'} \cap X^u(x_R, R) = \varnothing \tag{3.3.20}$$

for every $R \in \mathcal{R}_1$ and $T \in \mathrm{N}^s_{\mathcal{R}_1}(R)$, $S \in \mathrm{N}^u_{\mathcal{R}_1}(R)$. Our last choice for how small δ should be is,

$$\delta_1 = \min\{\delta_0, \delta'\}. \tag{3.3.21}$$

Lemma 3.3.9. *Let $\delta \in (0, \delta_1]$. For every $n \in \mathbb{N}$ and $S, R \in \mathcal{R}_n$ with $S \in \mathrm{N}_{\mathcal{R}_n}(R) \smallsetminus \{R\}$ we have that*

(1) if $S \in \mathrm{N}^u_{\mathcal{R}_n}(R)$ and $x \in S \cap R$ then $X^s(x, R) \not\subset X^s(x, S^\delta)$;

(2) if $S \in \mathrm{N}^s_{\mathcal{R}_n}(R)$ and $x \in S \cap R$ then $X^u(x, R) \not\subset X^u(x, S^\delta)$.

Proof. We will prove part (1) by induction on n and the proof for part (2) is similar. Assume to the contrary that there are $S, R \in \mathcal{R}_1$ with $S \in \mathrm{N}^u_{\mathcal{R}_1}(R)$ and some $x \in S \cap R$ such that $X^s(x, R) \subset X^s(x, S^\delta)$. In particular, $X^s(x, R) \subset S^\delta$ and since R is a rectangle we have

$$[x, x_R] \in X^s(x, R) \cap X^u(x_R, R) \subset S^\delta.$$

However, from the choice of δ_1 in (3.3.21) we have $S^\delta \cap X^u(x_R, R) = \varnothing$ leading to a contradiction.

Suppose now that part (1) is true for some $n \in \mathbb{N}$. We claim that it is also true for $n+1$. Again assume to the contrary that there are $S, R \in \mathcal{R}_{n+1}$ with $S \in \mathrm{N}^u_{\mathcal{R}_{n+1}}(R)$ and some $x \in S \cap R$ such that $X^s(x, R) \subset X^s(x, S^\delta)$.

As usual write $R = [\varphi^{-1}(R_j), \varphi(R_i)]$ and $S = [\varphi^{-1}(S_j), \varphi(S_i)]$ for $R_i, R_j, S_i, S_j \in \mathcal{R}_n$. It holds that

$$X^s(x, R) = X^s(x, \varphi(R_i)),\ X^s(x, S) = X^s(x, \varphi(S_i))$$

and $X^s(x, S^\delta) \subset X^s(x, \varphi(S_i^\delta))$ since $S^\delta \subset \varphi(S_i^\delta)$. From the assumption we obtain that $X^s(x, \varphi(R_i)) \subset X^s(x, \varphi(S_i^\delta))$. Then applying the map φ^{-1} results to

$$X^s(\varphi^{-1}(x), R_i) \subset X^s(\varphi^{-1}(x), S_i^\delta)$$

since R_i, S_i^δ are rectangles and the bracket map is locally bijective. Clearly $\varphi^{-1}(x) \in S_i \cap R_i$ and we have to show that $S_i \in \mathrm{N}^u_{\mathcal{R}_n}(R_i)$ in order to get a contradiction from the inductive step.

First note that

$$X^s(x, S) \cap \operatorname{int}(X^s(x, R)) = \varnothing, \tag{3.3.22}$$

for if not, we would have

$$[X^u(x, S), X^s(x, S)] \cap [X^u(x, R), \operatorname{int}(X^s(x, R))] \neq \varnothing.$$

Then since the intersection of S and R can happen only on their boundaries, this is equivalent to $S \cap \partial^{s,o} R \neq \varnothing$, meaning that $S \notin \mathrm{N}^u_{\mathcal{R}_{n+1}}(R)$. Equation (3.3.22) implies that

$$X^s(\varphi^{-1}(x), S_i) \cap \operatorname{int}(X^s(\varphi^{-1}(x), R_i)) = \varnothing \tag{3.3.23}$$

and is easy to observe that $S_i \in \mathrm{N}_{\mathcal{R}_n}(R_i) \setminus \{R_i\}$. Now assume to the contrary that $S_i \notin \mathrm{N}^u_{\mathcal{R}_n}(R_i)$ and hence there is some $y \in S_i \cap \partial^{s,o} R_i$. Then

$$X^s(\varphi^{-1}(x), S_i) \cap X^u(y, \varepsilon_X/2) = [\varphi^{-1}(x), y]$$

since both $\varphi^{-1}(x), y \in S_i$ but also

$$\operatorname{int}(X^s(\varphi^{-1}(x), R_i)) \cap X^u(y, \varepsilon_X/2) = [\varphi^{-1}(x), y]$$

since $y \in \partial^{s,o} R_i$. However, this contradicts (3.3.23). $\square$

Lemma 3.3.10. *Let $\delta \in (0, \delta_1]$. For every $n \in \mathbb{N}$ and $S, R \in \mathcal{R}_n$ such that $S \neq R$ we have that $R \not\subset S^\delta$. Consequently, the maps F_n^δ and M_n^δ are bijective.*

Proof. Let $n \in \mathbb{N}$ and assume to the contrary that there are some $S, R \in \mathcal{R}_n$ with $S \neq R$ such that $R \subset S^\delta$. From Lemma 3.3.8 we obtain that $S \cap R \neq \varnothing$ and so for any $x \in S \cap R$

it holds $X^s(x,R) \subset X^s(x,S^\delta)$ and $X^u(x,R) \subset X^u(x,S^\delta)$. However, from Lemma 3.3.9 we have that $X^s(x,R) \notin X^s(x,S^\delta)$ or $X^u(x,R) \notin X^u(x,S^\delta)$. This gives the desired contradiction. □

Lemma 3.3.10 yields the following corollaries.

Corollary 3.3.11. *For every $\delta \in (0,\delta_1]$, the approximation graph Π^δ is metrically-essential.*

Proof. From Lemma 3.2.11 the underlying approximation graph Π is essential and from Lemma 3.3.10 the δ-fattening maps F_n^δ are bijective. It is easy to see that $\coprod_{n\geq 0} F_n^\delta : \Pi \to \Pi^\delta$ is a graph homomorphism which induces the map $F^\delta : \mathcal{P}_\Pi \to \mathcal{P}_{\Pi^\delta}$ that satisfies $\pi_{\Pi^\delta} \circ F^\delta = \pi_\Pi$ because, if $\widetilde{p} \in \mathcal{P}_\Pi$ then

$$\varnothing \neq \bigcap_{n\geq 0} r(p_n) \subset \bigcap_{n\geq 0} r(p_n^\delta).$$

This completes the proof. □

Corollary 3.3.12. *Let $\delta \in (0,\delta_1]$. For every $n \geq 0$ and $R \in \mathcal{R}_n$ the following hold.*

(1) $\#\mathcal{R}_n^\delta = \#\mathcal{R}_n$;

(2) $\#\,\mathrm{N}_{\mathcal{R}_n^\delta}(R^\delta) = \#\,\mathrm{N}_{\mathcal{R}_n}(R)$;

(3) $\mathrm{m}(\mathcal{R}_n^\delta) = \mathrm{m}(\mathcal{R}_n)$.

Proof. Observe that Lemma 3.3.10 implies that any $R^\delta \in \mathcal{R}_n^\delta$ is written uniquely in the form $\varphi(R_i^\delta) \cap R_j^\delta \cap \varphi^{-1}(R_k^\delta)$, for $R_i^\delta, R_j^\delta, R_k^\delta \in \mathcal{R}_n^\delta$. □

3.3.2 Proof of conditions (6), (7) of Theorem 3.3.2

Consider the refining sequence $(\mathcal{R}_n^\delta)_{n\geq 0}$ defined in (3.3.6), for some $\delta \in (0,\delta_1]$. The constant δ_1 is defined in (3.3.21). Our aim is to investigate the properties of $(\mathcal{R}_n^\delta)_{n\geq 0}$ in the case $\ell_X = \min\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}$ or $\Lambda_X = \max\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}$ is finite.

Recall again that, if S is a rectangle in X and $x \in S$, the set $X^s(x, 2\varepsilon'_X) \cap S$ is denoted by $X^s(x,S)$ and similarly $X^u(x,2\varepsilon'_X) \cap S$ is denoted by $X^u(x,S)$. Also, note that both sets are compact subsets of X.

Lemma 3.3.13. *There exists some $\eta > 0$ such that for every $n \geq 0$ it holds that*

$$\mathrm{Leb}(\mathcal{R}_n^\delta) \geq \Lambda_X^{-n+1}\eta.$$

Proof. The case $n=0$ is trivial. We claim that there exists some $\eta > 0$ such that for every $n \in \mathbb{N}$, we have $d(x, X \smallsetminus R^\delta) \geq \Lambda_X^{-n+1}\eta$, for every $x \in X$ and $R \in \mathrm{N}_{\mathcal{R}_n}(\{x\})$. Then we will have

$$\begin{aligned}\mathrm{Leb}(\mathcal{R}_n^\delta) &= \inf_{x\in X}\sup_{R^\delta\in\mathcal{R}_n^\delta} d(x, X \smallsetminus R^\delta)\\ &\geq \inf_{x\in X}\sup_{R\in \mathrm{N}_{\mathcal{R}_n}(\{x\})} d(x, X\smallsetminus R^\delta)\\ &\geq \Lambda_X^{-n+1}\eta.\end{aligned}$$

To prove the claim, let $R_{1,k}\in\mathcal{R}_1$, for $1\leq k\leq \#\mathcal{R}_1$. We have that $(X\smallsetminus R_{1,k}^\delta)\cap R_{1,k}=\varnothing$ and define $\eta_k>0$ so that $d(X\smallsetminus R_{1,k}^\delta, R_{1,k}) = 2\eta_k$. Let

$$\eta = \min\{\eta_k : 1\leq k\leq \#\mathcal{R}_1\}$$

and then, for every $x\in X$ and $R\in \mathrm{N}_{\mathcal{R}_1}(\{x\})$, it holds that $d(x, X\smallsetminus R^\delta)\geq\eta$. This proves the case $n=1$.

Assume that for some $n\in\mathbb{N}$ we have $d(x,X\smallsetminus R^\delta)\geq \Lambda_X^{-n+1}\eta$, for every $x\in X$ and $R\in \mathrm{N}_{\mathcal{R}_n}(\{x\})$. We claim that

$$d(x, X\smallsetminus(\varphi(R_i^\delta)\cap R_j^\delta\cap\varphi^{-1}(R_k^\delta)))\geq \Lambda_X^{-n}\eta,$$

for every $x\in X$ and $\varphi(R_i)\cap R_j\cap\varphi^{-1}(R_k)\in N_{\mathcal{R}_{n+1}}(\{x\})$ with $R_i,R_j,R_k\in\mathcal{R}_n$. For $x\in X$ and $\varphi(R_i)\cap R_j\cap\varphi^{-1}(R_k)\in \mathrm{N}_{\mathcal{R}_{n+1}}(\{x\})$ one has that $d(x, X\smallsetminus(\varphi(R_i^\delta)\cap R_j^\delta\cap\varphi^{-1}(R_k^\delta)))$ is equal to

$$\min\{d(x,X\smallsetminus\varphi(R_i^\delta)), d(x, X\smallsetminus R_j^\delta), d(x, X\smallsetminus\varphi^{-1}(R_k^\delta))\}$$

which is greater or equal to

$$\min\{\Lambda_X^{-1}\Lambda_X^{-n+1}\eta, \Lambda_X^{-n+1}\eta, \Lambda_X^{-1}\Lambda_X^{-n+1}\eta\}$$

which is $\Lambda_X^{-n}\eta$, concluding the induction argument. □

The next results show that Smale spaces with Lipschitz dynamics can be controlled in a refined way. Our approach makes use of the next lemma that holds for an arbitrary Smale space.

For any closed rectangle $R\subset X$ that has a local stable and unstable set of cardinality at least two, let

$$\underline{\mathrm{diam}}_s(R) = \inf\{\mathrm{diam}(X^s(x,R)) : x\in R\} \tag{3.3.24}$$

and

$$\underline{\mathrm{diam}}_u(R) = \inf\{\mathrm{diam}(X^u(x,R)) : x\in R\}. \tag{3.3.25}$$

It is clear that for every $x\in R$, the diameters $\mathrm{diam}(X^s(x,R))$ are non-zero, since all the local stable sets are mutually homeomorphic and hence of cardinality at least two. For the same reason the diameters $\mathrm{diam}(X^u(x,R))$ are non-zero. The fact that $\underline{\mathrm{diam}}_s(R), \underline{\mathrm{diam}}_u(R)>0$ follows from the compactness of R and the next lemma. But first, for a closed rectangle $R \subset X$, denote by $\mathcal{K}(R)$ the set of its compact subsets and by d_H the usual Hausdorff metric on $\mathcal{K}(R)$ which is described for the reader's convenience in the proof.

Lemma 3.3.14. *For every closed rectangle $R \subset X$, the maps from (R, d) to $(\mathcal{K}(R), d_H)$ given by $x \mapsto X^s(x, R)$ and $x \mapsto X^u(x, R)$ are continuous. In particular, the diameter maps $R \ni x \mapsto diam(X^s(x, R))$ and $R \ni x \mapsto diam(X^u(x, R))$ are continuous.*

Proof. We prove the unstable case and the stable case is similar. Let $y \in R$ and $(x_n)_{n\geq 0}$ be a sequence in R which converges to y. We will prove that

$$\lim_{n\to\infty} d_H(X^u(x_n, R), X^u(y, R)) = 0$$

where d_H is the Hausdorff distance that, since $X^u(x_n, R), X^u(y, R)$ are compact, is given by

$$d_H(X^u(x_n, R), X^u(y, R)) = \max\{C(x_n, y, R), C'(x_n, y, R)\}, \tag{3.3.26}$$

where

$$C(x_n, y, R) = \max\nolimits_{z\in X^u(x_n,R)} d(z, X^u(y, R))$$

and

$$C'(x_n, y, R) = \max\nolimits_{w\in X^u(y,R)} d(w, X^u(x_n, R)).$$

A straightforward computation shows that

$$|\text{diam}X^u(x_n, R) - \text{diam}X^u(y, R)| \leq 2d_H(X^u(x_n, R), X^u(y, R)) \tag{3.3.27}$$

and hence the map $R \ni x \mapsto \text{diam}X^u(x, R)$ is continuous.

We now claim that

$$\lim_{n\to\infty} C(x_n, y, R) = 0.$$

Assume to the contrary that it does not converge to zero. Then there is some $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\geq 0}$ in R such that

$$\max\nolimits_{z\in X^u(x_{n_k},R)} d(z, X^u(y, R)) \geq \varepsilon,$$

for every $k \geq 0$. Hence, there are some $z_{n_k} \in X^u(x_{n_k}, R)$ so that $d(z_{n_k}, X^u(y, R)) \geq \varepsilon$, for every $k \geq 0$. However, by compactness of R, there is a convergent subsequence $(z_{n_{k_\ell}})_{\ell\geq 0}$ that converges to some $z' \in R$. But then, $z_{n_{k_\ell}} = [z_{n_{k_\ell}}, x_{n_{k_\ell}}]$ converges to $[z', y] \in X^u(y, R)$ which leads to a contradiction.

Now we show that

$$\lim_{n\to\infty} C'(x_n, y, R) = 0.$$

Again assume to the contrary that it does not converge to zero. Then there is some $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\geq 0}$ in R such that

$$\max_{w\in X^u(y,R)} d(w, X^u(x_{n_k}, R)) \geq \varepsilon,$$

for every $k \geq 0$. Similarly, this means that there are $w_{n_k} \in X^u(y, R)$ such that

$$d(w_{n_k}, X^u(x_{n_k}, R)) \geq \varepsilon,$$

for every $k \geq 0$. Since $X^u(y, R)$ is compact, there is a converging subsequence $(w_{n_{k_\ell}})_{\ell\geq 0}$ that converges to some $w' \in X^u(y, R)$. In particular, there is some $\ell_0 \in \mathbb{N}$ such that $d(w_{n_{k_\ell}}, w') < \varepsilon/2$, for every $\ell \geq \ell_0$. Then $d(w', X^u(x_{n_{k_\ell}}, R)) \geq \varepsilon/2$, for every $\ell \geq \ell_0$, by using the inequality

$$|d(w_{n_{k_\ell}}, X^u(x_{n_{k_\ell}}, R)) - d(w', X^u(x_{n_{k_\ell}}, R))| \leq d(w_{n_{k_\ell}}, w').$$

However, for big enough ℓ, since the diameter of R is small, w' and $x_{n_{k_\ell}}$ will be close enough to be bracketed and hence

$$d(w', X^u(x_{n_{k_\ell}}, R)) \leq d(w', [w', x_{n_{k_\ell}}]) = d([w', y], [w', x_{n_{k_\ell}}]),$$

where the last expression converges to zero. This leads to a contradiction. □

We consider the positive quantities

$$\underline{\mathrm{diam}}_s \mathcal{R}_1 = \min\{\underline{\mathrm{diam}}_s(R) : R \in \mathcal{R}_1\} \tag{Q6}$$

$$\underline{\mathrm{diam}}_u \mathcal{R}_1 = \min\{\underline{\mathrm{diam}}_u(R) : R \in \mathcal{R}_1\} \tag{Q7}$$

Lemma 3.3.15. *Suppose that the homeomorphism φ^{-1} is Lipschitz. Then it holds that*

$$\underline{diam}(\mathcal{R}_n) \geq \mathrm{Lip}(\varphi^{-1})^{-n+1} \underline{diam}_s \mathcal{R}_1.$$

If φ is Lipschitz then

$$\underline{diam}(\mathcal{R}_n) \geq \mathrm{Lip}(\varphi)^{-n+1} \underline{diam}_u \mathcal{R}_1.$$

Proof. We just prove the first inequality and the second is similar. The case $n = 0$ is trivial. Instead of proving the statement directly we will prove, by induction on n, that for every $R \in \mathcal{R}_n$ and every $x \in R$ we have

$$\operatorname{diam}(X^s(x,R)) \geq \operatorname{Lip}(\varphi^{-1})^{-n+1}\underline{\operatorname{diam}}_s\mathcal{R}_1.$$

For $n = 1$ the above inequality is true from the definition of $\underline{\operatorname{diam}}_s\mathcal{R}_1$. Assume that it holds for some $n \in \mathbb{N}$ and we claim that for every $R \in \mathcal{R}_{n+1}$ and every $x \in R$ it holds that

$$\operatorname{diam}(X^s(x,R)) \geq \operatorname{Lip}(\varphi^{-1})^{-n}\underline{\operatorname{diam}}_s\mathcal{R}_1.$$

Write $R = [\varphi^{-1}(R_j), \varphi(R_i)]$, for $R_i, R_j \in \mathcal{R}_n$, and with a similar argument as in equation 3.2.13 one has $X^s(x,R) = \varphi(X^s(\varphi^{-1}(x), R_i))$. Then

$$\begin{aligned}
\operatorname{diam}(X^s(x,R)) &= \operatorname{diam}(\varphi(X^s(\varphi^{-1}(x), R_i))) \\
&\geq \operatorname{Lip}(\varphi^{-1})^{-1}\operatorname{diam}(X^s(\varphi^{-1}(x), R_i)) \\
&\geq \operatorname{Lip}(\varphi^{-1})^{-1}\operatorname{Lip}(\varphi^{-1})^{-n+1}\underline{\operatorname{diam}}_s\mathcal{R}_1 \\
&= \operatorname{Lip}(\varphi^{-1})^{-n}\underline{\operatorname{diam}}_s\mathcal{R}_1
\end{aligned}$$

where the first inequality is true because φ^{-1} is Lipschitz. □

From Remark 3.1.9 we obtain the following.

Corollary 3.3.16. *For any irreducible Wieler solenoid (X,φ) it holds that*

$$\underline{\mathit{diam}}(\mathcal{R}_n) \geq \lambda_X^{-n+1}\underline{\mathit{diam}}_s\mathcal{R}_1,$$

where λ_X is the contraction constant.

3.4 Semi-conformal Smale spaces and Ahlfors regularity

In this section we study regularity properties of the Bowen measure and derive dimension estimates for Smale spaces. In particular we focus on the following class of Smale spaces.

Definition 3.4.1. A Smale space (X,φ) is called *semi-conformal* if $\lambda_X = \ell_X$, where $\lambda_X > 1$ is its contraction constant and $\ell_X = \min\{\operatorname{Lip}(\varphi), \operatorname{Lip}(\varphi^{-1})\}$.

By definition a self-similar Smale space is semi-conformal. Also, any Wieler solenoid is semi-conformal, see Remark 3.1.9. Note that if φ^{-1} is λ_X-Lipschitz then φ acts as the λ_X^{-1}-multiple of an isometry on local stable sets. The dual happens if φ is λ_X-Lipschitz. In what follows $(\mathcal{R}_n)_{n\geq 0}$ will be a refining sequence of Markov partitions (see (3.2.9)) for an irreducible or mixing Smale space, with $\operatorname{diam}(\mathcal{R}_1) \leq \varepsilon''_X/2$.

Proposition 3.4.2. *Let (X,φ) be a mixing semi-conformal Smale space. There is a constant $K>0$ such that, for every $n\in\mathbb{N}$ and $R\in\mathcal{R}_n$, the Bowen measure satisfies*

$$K^{-1}\operatorname{diam}(R)^{s_0}\leq\mu_{\mathrm{B}}(R)\leq K\operatorname{diam}(R)^{s_0}$$

where $s_0=2\,\mathrm{h}(\varphi)/\log(\lambda_X)$.

Proof. Since (X,φ) is mixing, the corresponding topological Markov chain (Σ_M,σ_M) will be mixing, too. Recall that the Bowen measure on (Σ_M,σ_M) is the Parry measure μ_{P}.

Moreover, since (X,φ) is semi-conformal, from Theorem 3.3.2 and Lemma 3.3.15, we obtain constants $\theta\geq\zeta>0$ such that

$$\lambda_X^{-n+1}\zeta\leq\operatorname{diam}(R)\leq\lambda_X^{-n+1}\theta, \tag{3.4.1}$$

for every $n\in\mathbb{N}$ and $R\in\mathcal{R}_n$.

Let $R\in\mathcal{R}_n$ and $C\in\Sigma_M$ be the symmetric cylinder set of rank $2n-1$ such that $\pi_M(C)=R$. Theorem 3.2.5 says that π_M is a metric isomorphism between $(\Sigma_M,\sigma_M,\mu_{\mathrm{P}})$ and $(X,\varphi,\mu_{\mathrm{B}})$, hence $\mu_{\mathrm{B}}(R)=\mu_{\mathrm{P}}(C)$. From Lemma 3.1.6 there is $D>0$ so that

$$D^{-1}\lambda_{\max}^{-2n}\leq\mu_{\mathrm{P}}(C)\leq D\lambda_{\max}^{-2n}$$

where $\lambda_{\max}$ is the Perron-Frobenius eigenvalue. Then using the inequality (3.4.1) together with

$$K=\max\{D(\theta\lambda_X)^{s_0},D(\zeta\lambda_X)^{-s_0}\}$$

and

$$s_0=2\,\mathrm{h}(\varphi)/\log(\lambda_X)=\log_{\lambda_X}(\lambda_{\max}^2)$$

we obtain the result. □

Remark 3.4.3. Results similar to Proposition 3.4.2 have been obtained in the setting of Moran constructions [71, Def. 2.2] built from iterated function systems on complete metric spaces [71,72]. Roughly speaking, a Moran construction $\mathcal{M}$ on a complete metric space Z, is a countable poset (by inclusion) of closed, bounded subsets of Z with positive diameter, that has a unique maximum, the infimum of every chain is a point in Z, and where the elements of $\mathcal{M}$ correspond to finite words occurring in an one-sided subshift on finitely many symbols. Each Moran construction on Z describes a limit set in Z, and one aims to control the diameter of the sets in $\mathcal{M}$. Measures that satisfy the inequality of Proposition 3.4.2 are called semi-conformal [72]. We note that the aforementioned notion of Moran constructions, although it has similarities, is different than the one used in [11,101] that works in the Euclidean setting.

We believe that refining sequences of Markov partitions on Smale spaces correspond to some kind of inverse limits of Moran constructions, as these refining sequences produce two-sided subshifts. We intend to investigate this connection in a future project.

We now introduce a homogeneity property for Smale spaces. It is related to the *uniform finite clustering property* (UFCP) for Moran constructions used in [71,72], but is adjusted in the setting of refining sequences of Markov partitions.

Definition 3.4.4. A refining sequence of Markov partitions $(\mathcal{R}_n)_{n\geq 0}$ for a Smale space (X,φ) satisfies the *uniform finite clustering property* (UFCP) if

$$\sup_{x\in X}\sup_r \#\,\mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r)) < \infty,$$

where r takes values in $(0,\operatorname{diam}(X))$ and $n_r = \min\{n\in\mathbb{N} : \overline{\operatorname{diam}}(\mathcal{R}_n)\leq r\}$.

We are interested in semi-conformal Smale spaces which admit refining sequences of Markov partitions that satisfy UFCP. In particular the following holds.

Proposition 3.4.5. *Any refining sequence of Markov partitions $(\mathcal{R}_n)_{n\geq 0}$ for an irreducible self-similar Smale space (X,φ) satisfies UFCP.*

Proof. Let $x\in X$ and $0<r<\operatorname{diam}(X)$. We claim that $\#\,\mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r))$ is bounded above by a constant which does not depend on x and r. From Theorem 3.3.2, for sufficiently small $\delta>0$, we obtain the δ-fattening $(\mathcal{R}_n^\delta)_{n\geq 0}$ and constants $0<\eta\leq\theta$ such that

$$\overline{\operatorname{diam}}(\mathcal{R}_n^\delta)\leq \lambda_X^{-n+1}\theta \ \text{ and } \ \operatorname{Leb}(\mathcal{R}_n^\delta)\geq \lambda_X^{-n+1}\eta$$

for every $n\in\mathbb{N}$. Note that it suffices to prove the statement for $0<r<\eta/2$. Fix such r and let

$$m_r = \min\{n\in\mathbb{N} : \lambda_X^{-n+1}\theta\leq r\}.$$

It is easy to check that $m_r = 1+\lceil\log_{\lambda_X}(\theta/r)\rceil$ and that $m_r\geq n_r$. Therefore, one has that $\#\,\mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r))\leq \#\,\mathrm{N}_{\mathcal{R}_{m_r}}(\overline{B}(x,r))$. Moreover, from Lemma 3.3.10 the map

$$\mathrm{N}_{\mathcal{R}_{m_r}}(\overline{B}(x,r))\to \mathrm{N}_{\mathcal{R}_{m_r}^\delta}(\overline{B}(x,r))$$

given by $R\to R^\delta$ is injective. In particular,

$$\mathrm{N}_{\mathcal{R}_{m_r}}(\overline{B}(x,r))\leq \mathrm{N}_{\mathcal{R}_{m_r}^\delta}(\overline{B}(x,r)).$$

Define $\ell_r = 1+\lfloor\log_{\lambda_X}(\eta/(2r))\rfloor$ and one has $2r\leq\lambda_X^{-\ell_r+1}\eta\leq\operatorname{Leb}(\mathcal{R}_{\ell_r}^\delta)$. Then there is some $T^\delta\in\mathcal{R}_{\ell_r}^\delta$ which contains $\overline{B}(x,r)$ and clearly

$$\mathrm{N}_{\mathcal{R}^\delta_{\ell_r}}(\overline{B}(x,r)) \subset \mathrm{N}_{\mathcal{R}^\delta_{\ell_r}}(T^\delta),$$

where $\#\,\mathrm{N}_{\mathcal{R}^\delta_{\ell_r}}(T^\delta) \leq N_\Pi$, the uniform upper bound in the number of neighbouring rectangles, see Theorem 3.3.2.

We will work in a similar fashion as in Proposition 2.2.14. We begin by noting that $m_r \geq \ell_r$ and that every $R^\delta \in \mathrm{N}_{\mathcal{R}^\delta_{m_r}}(\overline{B}(x,r))$ is a descendant of depth $m_r - \ell_r$ of some element in $\mathrm{N}_{\mathcal{R}^\delta_{\ell_r}}(\overline{B}(x,r))$. We have $m_r - \ell_r \leq 2 + \log_{\lambda_X}(2\theta/\eta)$ and since there is also a uniform upper bound C_Π (see Theorem 3.3.2) on the number of descendants, due to the finite entropy, it holds

$$\#\,\mathrm{N}_{\mathcal{R}^\delta_{m_r}}(\overline{B}(x,r)) \leq N_\Pi C_\Pi^{2+\log_{\lambda_X}(2\theta/\eta)}.$$

As a result,

$$\#\,\mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r)) \leq N_\Pi C_\Pi^{2+\log_{\lambda_X}(2\theta/\eta)}. \qquad \square$$

We are now in a position to prove one of the main results of this thesis. The power of this can be seen in Corollary 3.4.8, where we obtain a sweeping result for all mixing Smale spaces.

Theorem 3.4.6. *Let (X,φ) be a mixing semi-conformal Smale space. Assume there is a refining sequence of Markov partitions that satisfies the UFCP. Then the Bowen measure is Ahlfors s_0-regular and therefore,*

$$\dim_H X = \dim_B X = \dim_A X = s_0$$

where $s_0 = 2\,\mathrm{h}(\varphi)/\log(\lambda_X)$. Moreover, the s_0-dimensional Hausdorff measure is strictly positive.

Proof. Suppose $(\mathcal{R}_n)_{n\geq 0}$ is a refining sequence of Markov partitions that satisfies the UFCP and let $M > 0$ such that $\#\,\mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r)) \leq M$ for every $x \in X$ and $0 < r < \mathrm{diam}(X)$. Now fix some $x \in X$ and we want to estimate the measure $\mu_{\mathrm{B}}(\overline{B}(x,r))$ for $0 < r < \mathrm{diam}(X)$, but since $\mathrm{diam}(X) < \infty$ it suffices to consider $0 < r < \underline{\mathrm{diam}}(\mathcal{R}_1)/2$. Using the constant $K > 0$ from Proposition 3.4.2 we obtain

$$\begin{aligned}
\mu_{\mathrm{B}}(\overline{B}(x,r)) &\leq \mu_{\mathrm{B}}(\textstyle\bigcup \mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r))) \\
&\leq \sum_R \mu_{\mathrm{B}}(R) \\
&\leq \sum_R K\,\mathrm{diam}(R)^{s_0} \\
&\leq MKr^{s_0}
\end{aligned}$$

where the sum is taken over all $R \in \mathrm{N}_{\mathcal{R}_{n_r}}(\overline{B}(x,r))$.

For the lower bound, take an infinite path in the approximation graph $\mathcal{P}_\Pi$ of $(\mathcal{R}_n)_{n\geq 0}$ that converges to x. From this path let R be the first rectangle that is contained in $\overline{B}(x,r)$. Then its first ancestor $\widehat{R}$ will not be contained and hence $\operatorname{diam}(\widehat{R}) > r$. Now we observe that $R \in \mathcal{R}_n$ for some $n \geq 2$, for if $R \in \mathcal{R}_1$ then $\underline{\operatorname{diam}}(\mathcal{R}_1) \leq \operatorname{diam}(R) \leq 2r < \underline{\operatorname{diam}}(\mathcal{R}_1)$. This means $\widehat{R} \in \mathcal{R}_{n-1}$ and $n-1 \geq 1$. Therefore, we can apply inequality (3.4.1) to obtain that $\operatorname{diam}(R) \geq c \operatorname{diam}(\widehat{R})$ for some $c \leq \zeta/(\lambda_X \theta)$.

As a result,

$$\mu_{\mathrm{B}}(\overline{B}(x,r)) \geq \mu_{\mathrm{B}}(R) \geq K^{-1} \operatorname{diam}(R)^{s_0} \geq K^{-1} c^{s_0} r^{s_0},$$

and hence the Bowen measure is Ahlfors s_0-regular. The rest follows from Proposition 2.1.15 and Remark 2.1.14. □

Remark 3.4.7. We now explain Theorem 3.4.6 by focusing on a mixing self-similar Smale space (X,d,φ). First, we should note that the coincidence of the Hausdorff and box-counting dimensions can be equivalently obtained using Barreira's techniques. Specifically, one can use Theorem 3.15 in [11], which concerns the dimension theory of Smale spaces with bi-Lipschitz local product structure and with asymptotically conformal dynamics on stable and unstable sets. However, Barreira's result cannot be related to Ahlfor regularity or Assouad dimension.

The stable and unstable sets of (X,d,φ) have asymptotically conformal dynamics in a strong sense. Moreover, using part (4) of Proposition 3.2.6 and the proof of Lemma 3.3.15, we obtain the dimensions of local stable and unstable sets as the roots $r_s = r_u = \mathrm{h}(\varphi)/\log(\lambda_X)$ of Bowen's equation for the topological pressure. What needs to be observed is that the local product structure is bi-Lipschitz. Then, we can add the dimensions and obtain $\dim_H X = \dim_B X = 2\,\mathrm{h}(\varphi)/\log(\lambda_X)$.

From [7, Lemma 4.3], every stable and unstable holonomy map in a sufficiently small rectangle is Lipschitz. In particular, for every $\varepsilon > 0$ there is $c > 0$ so that, for all $y \in X$ and $x, x' \in X^u(y,c)$, $z, z' \in X^s(y,c)$, it holds that

$$\begin{aligned} d([x,z],[x',z]) &\leq (1+\varepsilon) d(x,x') \\ d([x,z],[x,z']) &\leq (1+\varepsilon) d(z,z'). \end{aligned} \tag{3.4.2}$$

Moreover, the self-similar version of Lemma 3.1.10 (see [7, Remark 2.22]) is that for a (possibly) smaller $c > 0$, if $z, w \in X$ with $d(z,w) \leq c$, then

$$\begin{aligned} d(z,[z,w]) &\leq (1+\varepsilon) d(z,w) \\ d(w,[z,w]) &\leq (1+\varepsilon) d(z,w). \end{aligned} \tag{3.4.3}$$

Fix $\varepsilon > 0$ and let $c > 0$ be small enough so that both (3.4.2) and (3.4.3) hold. Then, it is straightforward to see that, for every $y \in X$, if $X^u(y,c) \times X^s(y,c)$ is equipped with the product metric, the bracket map $[\cdot,\cdot] : X^u(y,c) \times X^s(y,c) \to X$ is bi-Lipschitz onto its image, with a small constant depending on ε.

Further, from Theorem 3.4.6 we obtain that the Assouad dimension of a self-similar Smale space (X,d,φ) is finite. Note that this weaker result can also be obtained by Theorem 3.3.2 and Proposition 2.2.14. Now, Assouad's Theorem 2.1.9 asserts that for every $\varepsilon \in (0,1)$, the snowflaked metric space (X,d^{ε}) is bi-Lipschitz embeddable in a Euclidean space. Note that $(X,d^{\varepsilon},\varphi)$ is still a self-similar Smale space. However, it is not clear whether the embedding is a Smale space, because the contraction axioms (C1) and (C2) depend on the Lipschitz constant of the embedding. But even if it were a Smale space, the embedding may no longer be self-similar or even conformal in a broader sense, so that Pesin's techniques [101] on Ahlfors regularity could be applied. One would require some sort of isometric embedding in the Euclidean space, and this seems extremely difficult, if not unlikely. But even if such a fine embedding would exist, the methods in [101] would apply only on the snowflaked version $(X,d^{\varepsilon},\varphi)$.

Theorem 3.4.6, together with Proposition 3.4.5 and Lemma 3.1.16, yields the following result.

Corollary 3.4.8. *Any mixing Smale space is topologically conjugate to a mixing Smale space on which the Bowen measure is Ahlfors regular.*

Remark 3.4.9. Not all mixing Smale spaces have Ahlfors regular measures. Such Smale spaces exist in the context of non-conformal hyperbolic dynamical systems, where the Hausdorff and box-counting dimensions may not agree. An example of a Smale space whose dimensions do not coincide can be found in the work of Pollicott and Weiss [102] who studied the dimension theory of certain linear horseshoes in $\mathbb{R}^3$. We note that a horseshoe in $\mathbb{R}^3$ is constructed in a way similar to the classical Smale's horseshoe in $\mathbb{R}^2$. For a specific definition of the diffeomorphism in $\mathbb{R}^3$ we refer to [126]. Pollicott and Weiss considered a linear horseshoe (Λ, f) so that $\Lambda = F \times E$, where F is a certain self-affine limit set in the plane and E is a uniform Cantor set. The limit set F is constructed by two affine contractions A_0, A_1 on the unit square I, where $A_0(I), A_1(I)$ are disjoint rectangles in I placed in the lower left corner and the upper right corner of I, respectively, each having height $\lambda_1 < 1/2$ and width λ_2 equal to the reciprocal of the golden mean. The horseshoe (Λ, f) is a Smale space (after considering a Lipschitz equivalent adapted metric [19, Prop. 5.2.2]) and is topologically conjugate to the full-two shift (Σ_2, σ_2).

Now, the specific construction of F gives that $\dim_H F < \dim_B F$, and since $\dim_H \Lambda = \dim_H F + \dim_H E$, $\dim_B \Lambda = \dim_B F + \dim_B E$ and $\dim_H E = \dim_B E$, we obtain that $\dim_H \Lambda < \dim_B \Lambda$. This argument is independent of $\lambda_1 < 1/2$, and hence there is a family of horseshoes indexed by an open interval, whose Hausdorff and box-counting dimensions do not coincide. Therefore, every such Λ does not have Ahlfors regular measures. In [102] one can find several interesting linear horseshoes whose dimension depends on fine number theoretic properties of the contraction coefficients. Also, from [101, Section 16] one can build linear horseshoes in $\mathbb{R}^4$ whose Hausdorff dimension is strictly smaller than the box-counting dimension.

According to Theorem 3.4.6, in order to make the Bowen measure on the example horseshoe (Λ, f) Ahlfors regular, it suffices to change the metric of Λ to a self-similar one. Since (Λ, f) is topologically conjugate to (Σ_2, σ_2), one possibility is to equip Λ with a self-similar ultrametric of Σ_2. Another approach is to see whether (Λ, f), equipped with its original metric, satisfies Fathi's property (Theorem 3.1.12) and then follow the method discussed in Subsection 3.1.5. This depends on the Lipschitz and contraction constants of f, f^{-1}. Finally, one can follow the philosophy of Fried [53] and reconstruct a metric on (Λ, f) that will satisfy Fathi's property with parameters obtained from the original geometry of Λ. This approach is more abstract since it requires to view Λ as a uniform space. However, we believe it can produce natural metrics on Λ. In a future project we aim to construct natural self-similar metrics for specific classes of Smale spaces.

From Theorem 3.4.6, Proposition 3.4.5, Remark 3.1.15 and the metric inequalities (3.1.18) in Subsection 3.1.5 we obtain the following dimension estimates. Recall that $\Lambda_X = \max\{\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1})\}$.

Corollary 3.4.10. *Let (X, φ) be a mixing Smale space with $\Lambda_X < \infty$. Suppose that $\lambda_X > 2A_X$, where $A_X > 0$ is the constant obtained in Lemma 3.1.10. Then it holds*

$$\frac{2\,\mathrm{h}(\varphi)}{\log \Lambda_X} \leq \dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X \leq \frac{2\,\mathrm{h}(\varphi)}{\log \lambda_X - \log(2A_X)}.$$

We should point out again that since $A_X \leq (\Lambda_X \lambda_X)/(\lambda_X^2 - 1)$, for $\lambda_X > 2A_X$ to be true in general, it suffices to restrict to $\lambda_X \in (1 + \sqrt{2}, \infty)$ and $\Lambda_X \in [\lambda_X, (\lambda_X^2 - 1)/2)$. Then, considering the behaviour of the Hausdorff and box-counting dimensions with Hölder equivalent metrics, it is possible to obtain upper and lower bounds for Smale spaces with contraction constants in the interval $(1, 1 + \sqrt{2}]$. However, the goal should be to estimate the constant A_X. The Assouad dimension is not included in the inequality since it does not behave well with arbitrary Hölder transformations [64]. Moreover, the upper bound can also be obtained from [51, Theorem 5.3] and the discussion in Subsection 3.1.5. Finally, the lower bound $2\,\mathrm{h}(\varphi)/\log \Lambda_X \leq \dim_H X$ enhances the previous bound $\mathrm{h}(\varphi)/\log \Lambda_X \leq \underline{\dim}_B X$ obtained in [51, Theorem 5.6].

Chapter 4

KK-theory and smoothness in C*-algebras

This chapter contains all the necessary K-theoretic background and tools needed for this thesis. We begin with a basic introduction in Kasparov's KK-theory and Spanier-Whitehead K-duality. Then, we present the notion of smooth extensions and summable Fredholm modules over C^*-algebras. The main contribution of this chapter is that we calculate slant products in KK-theory for simple, purely infinite C^*-algebras which are not necessarily unital (Proposition 4.2.11). In the literature we were able to find these calculations only in the unital case. Further, we develop tools from holomorphic functional calculus that allow us to deduce uniform results about the smoothness of extensions of C^*-algebras that have Spanier-Whitehead K-duals (Proposition 4.2.21). This also requires to work in the context of quasi-Banach spaces.

4.1 K-theoretic preliminaries

In this section we introduce the K-theoretic framework on which the thesis is based. We begin with an explicit description of the Fredholm module picture of Kasparov's KK-theory since it will be important for the sequel. Then we present its relation with extensions of C^*-algebras. After this, we introduce some aspects of the Spanier-Whitehead duality in KK-theory. In this section we assume that the reader is familiar with Hilbert C^*-modules. For their theory we refer to [85].

4.1.1 A primer on KK-theory

The classical references are [14,69,80]. Let A and B be C^*-algebras. We say that A is $\mathbb{Z}_2$-*graded* (or simply *graded*) if it is endowed with an automorphism $\gamma_A \in \mathrm{Aut}(A)$ such that $\gamma_A^2 = 1$. This yields a direct sum decomposition $A = A^+ \oplus A^-$, where $A^+ = \{a \in A : \gamma_A(a) = a\}$

and $A^- = \{a \in A : \gamma_A(a) = -a\}$. We say that the grading is *trivial* if $A^- = \{0\}$, meaning that $\gamma_A = \mathrm{id}$. The *degree* of $a \in A$, denoted by $\deg(a)$, is equal to 0 if $a \in A^+$ and equal to 1 if $a \in A^-$. In this setting, we consider the *graded commutator* that for $x, y \in A$ is defined as

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx. \tag{4.1.1}$$

Moreover, a *graded* $*$*-homomorphism* between graded C^*-algebras is one that respects the gradings. Also, it is possible to define graded tensor products of graded C^*-algebras, see [14, Section 14]. In the context of trivially graded C^*-algebras, the adjective "graded" will be dropped.

Suppose B is $\mathbb{Z}_2$-graded by $\gamma_B \in \mathrm{Aut}(B)$ and let E be a (right) Hilbert B-module with B-valued inner product $\langle \cdot, \cdot \rangle_B$. We say that E is $\mathbb{Z}_2$*-graded* if it is equipped with a linear bijection $\gamma_E : E \to E$ such that $\gamma_E^2 = 1$, which additionally satisfies

(i) $\gamma_E(\eta b) = \gamma_E(\eta)\gamma_B(b)$;

(ii) $\langle \gamma_E(\xi), \gamma_E(\eta) \rangle_B = \langle \xi, \eta \rangle_B$,

for all $\eta, \xi \in E, b \in B$. In this case, we have a direct sum decomposition $E = E^+ \oplus E^-$ into the ± 1-eigenspaces of γ_E, respectively. The $\mathbb{Z}_2$-grading of E passes naturally to the adjointable operators $\mathcal{B}_B(E)$ and the two-sided ideal of compact operators $\mathcal{K}_B(E)$. Finally, it is possible to define graded tensor products of Hilbert C^*-modules, again see [14, Section 14]. For our purposes we can always assume that A and B are separable graded C^*-algebras.

Definition 4.1.1 ([14, Def. 17.1.1])**.** A *Kasparov* (A, B)*-bimodule* is a triple (E, ρ, F) such that

(1) E is a countably generated Hilbert B-module with grading γ_E;

(2) $\rho : A \to \mathcal{B}_B(E)$ is a graded $*$-homomorphism;

(3) the operator $F \in \mathcal{B}_B(E)$ is *odd*, meaning that $F\gamma_E = -\gamma_E F$, and satisfies

$$\rho(a)(F^* - F) \in \mathcal{K}_B(E),\ \rho(a)(F^2 - 1) \in \mathcal{K}_B(E),\ [\rho(a), F] \in \mathcal{K}_B(E),$$

for all $a \in A$.

The triple (E, ρ, F) is *degenerate* if the above three operators are 0, for all $a \in A$. The triple is said to be *normalised* if $F = F^*$ and $F^2 = 1$. The set of Kasparov (A, B)-bimodules is denoted by $\mathcal{E}(A, B)$ and the set of degenerate modules by $\mathcal{D}(A, B)$.

At this point, it is important to note that every countably generated Hilbert B-module E gets absorbed by the Hilbert B-module $\mathbb{H}_B = \{(b_n) \in \prod_{n\in\mathbb{N}} B : \sum_n b_n^* b_n \text{ converges}\}$, meaning that $E \oplus \mathbb{H}_B \cong \mathbb{H}_B$. This is Kasparov's Stabilisation Theorem [78].

We will be particularly interested in trivially graded C^*-algebras. However, this does not make the previous discussion unnecessary. To the contrary, gradings are an intrinsic part of KK-theory and particularly of the Kasparov product, which will be discussed shortly.

Assume for a moment that A, B are trivially graded. Then for any $(E, \rho, F) \in \mathcal{E}(A, B)$, following the notation of Definition 4.1.1, we additionally have that

(i) the eigenspaces E^+, E^- of γ_E are Hilbert B-submodules of E;

(ii) for every $a \in A$ the operator $\rho(a)$ is *even*, meaning $\rho(a) = \rho^+(a) \oplus \rho^-(a)$, where $\rho^\pm$ are representations of A on $E^\pm$;

(iii) the operator $F \in \mathcal{B}_B(E)$ is written as

$$F = \begin{pmatrix} 0 & \widetilde{T} \\ T & 0 \end{pmatrix}$$

where $T \in \mathcal{B}_B(E^+, E^-)$ and $\widetilde{T} \in \mathcal{B}_B(E^-, E^+)$.

There are various operations on Kasparov bimodules and for the convenience of the reader we present a few without proof. For a complete treatment of the subject we refer to [69, Section 2.1]. Suppose D is a separable graded C^*-algebra and $\mathcal{E} = (E, \rho, F) \in \mathcal{E}(A, B)$. If $\psi : D \to A$ is a graded $*$-homomorphism, then the *pull-back* operation yields the triple

$$\psi^*(\mathcal{E}) = (E, \rho \circ \psi, F) \in \mathcal{E}(D, B). \tag{4.1.2}$$

If $\psi : B \to D$ is a graded $*$-homomorphism, then the *push-forward* operation produces the triple

$$\psi_*(\mathcal{E}) = (E \otimes_\psi D, \rho \otimes_\psi \mathrm{id}, F \otimes_\psi \mathrm{id}) \in \mathcal{E}(A, D), \tag{4.1.3}$$

where $E \otimes_\psi D$ is the *internal tensor product*. Finally, the *external tensor product* operation produces the triple

$$\tau_D(\mathcal{E}) = (E \otimes D, \rho \otimes \mathrm{id}, F \otimes \mathrm{id}) \in \mathcal{E}(A \otimes D, B \otimes D), \tag{4.1.4}$$

where the graded tensor products of C^*-algebras have the spatial norm. Tensoring with D from the left gives the triple $\tau^D(\mathcal{E})$.

There is a natural notion of isomorphism ($\simeq$) and unitary equivalence between two Kasparov (A,B)-bimodules [69, Section 2.1]. We say that $\mathcal{E}_0, \mathcal{E}_1 \in \mathcal{E}(A,B)$ are *operator homotopic* if there is a path $(\mathcal{F}_t)_{t\in[0,1]}$, where $\mathcal{F}_t = (E,\rho,F_t) \in \mathcal{E}(A,B)$, and $t \mapsto F_t$ is norm continuous, so that $\mathcal{F}_0 \simeq \mathcal{E}_0$ and $\mathcal{F}_1 \simeq \mathcal{E}_1$. This can be turned into an equivalence relation ($\approx$) where, for any two $\mathcal{E}_0, \mathcal{E}_1 \in \mathcal{E}(A,B)$, we write $\mathcal{E}_0 \approx \mathcal{E}_1$ if there are $\mathcal{D}_0, \mathcal{D}_1 \in \mathcal{D}(A,B)$ such that $\mathcal{E}_0 \oplus \mathcal{D}_0$ and $\mathcal{E}_1 \oplus \mathcal{D}_1$ are unitarily equivalent to a pair of operator homotopic Kasparov (A,B)-bimodules. The $\approx$ equivalence class of $\mathcal{E} \in \mathcal{E}(A,B)$ is denoted by $[\mathcal{E}]$.

Definition 4.1.2 ([14, Def. 17.3.1])**.** The *Kasparov group* $\mathrm{KK}_0(A,B)$ is the abelian group $\mathcal{E}(A,B)/\approx$, with direct sum as operation and unit the class of the zero triple. Moreover, let $\mathrm{KK}_1(A,B) = \mathrm{KK}_0(A \otimes \mathbb{C}_1, B)$, where $\mathbb{C}_1$ is the Clifford algebra with one generator.

From *formal Bott periodicity* we have that $\mathrm{KK}_1(A \otimes \mathbb{C}_1, B) \cong \mathrm{KK}_0(A,B)$, see for example [14, Corollary 17.8.9]. All the operations on Kasparov-bimodules carry over to the level of KK-classes. Also, every KK-class has a normalised representative and Kasparov's Stabilisation Theorem implies that it suffices to consider only triples $(E,\rho,F) \in \mathcal{E}(A,B)$ with $E = \mathbb{H}_B$, see [14, Section 17].

Any graded $*$-homomorphism $\psi : A \to B$ determines an element $[\psi] \in \mathrm{KK}_0(A,B)$. More generally, if E is a countably generated Hilbert B-module, then every graded $*$-homomorphism $\psi : A \to \mathcal{K}_B(E)$ gives the class $[E,\psi,0] \in \mathrm{KK}_0(A,B)$. In particular, this applies when E is a strong Morita equivalence (A,B)-bimodule. If $A = \mathbb{C}$ and B is trivially graded, the groups $\mathrm{KK}_i(\mathbb{C},B)$ are isomorphic to the K-theory groups $\mathrm{K}_i(B)$. If A is trivially graded and $B = \mathbb{C}$, the groups $\mathrm{KK}_i(A,\mathbb{C})$ are exactly the K-homology groups $\mathrm{K}^i(A)$ that can be found in Higson and Roe [67, Chapter 8]. However, there is a slight difference on the level of triples, since KK_1-triples are defined over the graded C^*-algebra $A \otimes \mathbb{C}_1$ while K^1-triples are defined over the trivially graded A. For the next definition assume that A is trivially graded (or ungraded).

Definition 4.1.3 ([67, Def. 8.1.1])**.** An *odd Fredholm module* over A is a triple (H,ρ,F) such that

(1) H is a separable Hilbert space;

(2) $\rho : A \to \mathcal{B}(H)$ is a $*$-homomorphism;

(3) the operator $F \in \mathcal{B}(H)$ satisfies

$$\rho(a)(F^* - F) \in \mathcal{K}(H),\ \rho(a)(F^2 - 1) \in \mathcal{K}(H),\ [\rho(a), F] \in \mathcal{K}(H),$$

for all $a \in A$.

The triple (H,ρ,F) is *degenerate* if the above three operators are 0, for all $a \in A$. It is *normalised* if $F = F^*$ and $F^2 = 1$.

An *even Fredholm module* over A is an odd Fredholm module (H, ρ, F) equipped with a grading on H. This is exactly a Kasparov $(A, \mathbb{C})$-bimodule, which meets the conditions (i), (ii) and (iii) since A is trivially graded. In the same way as before, operator homotopy between even Fredholm modules gives rise to the $K^0(A)$ group. Similarly, operator homotopy between odd Fredholm modules produces $K^1(A)$. Also, every K-homology class can be normalised.

Before presenting the explicit relation between Kasparov $(A, \mathbb{C})$-bimodules and Fredholm modules over A, we should recall the definition of the Clifford algebra $\mathbb{C}_1$. Define $\mathbb{C}_1 = \{\lambda + \mu\alpha : \lambda, \mu \in \mathbb{C}\}$ to be the algebra over $\mathbb{C}$ such that $\alpha^2 = 1$ and where the grading is induced by $\gamma_{\mathbb{C}_1}(\lambda + \mu\alpha) = \lambda - \mu\alpha$. Equip $\mathbb{C}^2$ with the grading induced by $\mathrm{diag}(1, -1)$. Then $\mathbb{C}_1$ can be represented on $\mathbb{C}^2$ by the graded $*$-homomorphism $\rho_{\mathbb{C}_1}$ given by

$$\rho_{\mathbb{C}_1}(\lambda + \mu\alpha) = \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}. \tag{4.1.5}$$

Denote by σ_2 the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and we have the following.

Proposition 4.1.4 ([28, IV-A-Prop. 13]). *A normalised Kasparov $(A, \mathbb{C})$-bimodule is exactly an even normalised Fredholm module over A. Moreover, every normalised Kasparov $(A \otimes \mathbb{C}_1, \mathbb{C})$-bimodule is of the form $(H \otimes \mathbb{C}^2, \rho \otimes \rho_{\mathbb{C}_1}, F \otimes \sigma_2)$ for a unique normalised odd Fredholm module (H, ρ, F) over A.*

At the heart of KK-theory lies the *Kasparov product* which generalises the cup-cap product from topological K-theory. For separable graded C^*-algebras A, B, D there is a bilinear operation

$$\otimes_D : \mathrm{KK}_0(A, D) \times \mathrm{KK}_0(D, B) \to \mathrm{KK}_0(A, B) \tag{4.1.6}$$

which is associative and functorial in all possible ways (see [14, Section 18]), thus creating the KK-category, see [14, Section 22]. This turns $\mathrm{KK}_0(A, A)$ into a ring with identity denoted by 1_A. The KK-functor is homotopy invariant in both variables, C^*-stable and split exact [66]. The most general form of the product is given as

$$\otimes_D : \mathrm{KK}_0(A_1, B_1 \otimes D) \times \mathrm{KK}_0(D \otimes A_2, B_2) \to \mathrm{KK}_0(A_1 \otimes A_2, B_1 \otimes B_2) \tag{4.1.7}$$

with a slight abuse of notation; for $x \in \mathrm{KK}_0(A_1, B_1 \otimes D)$ and $y \in \mathrm{KK}_0(D \otimes A_2, B_2)$ we define $x \otimes_D y = \tau_{A_2}(x) \otimes_{B_1 \otimes D \otimes A_2} \tau^{B_1}(y)$, see (4.1.4) for the definition of τ_{A_2} and τ^{B_1}.

The Kasparov product generalises composition of graded $*$-homomorphisms. Also, if $\phi : D \to A$ and $\psi : B \to D$ are graded $*$−homomorphisms, by pull-back and push-forward we obtain maps $\phi^* : \mathrm{KK}_0(A, B) \to \mathrm{KK}_0(D, B)$ and $\psi_* : \mathrm{KK}_0(A, B) \to \mathrm{KK}_0(A, D)$ which are alternatively given by

$$\phi^*(x) = [\phi] \otimes_A x \text{ and } \psi_*(y) = y \otimes_B [\psi]. \tag{4.1.8}$$

In addition, the Kasparov product is *graded commutative*, meaning that for the flip map $\sigma_{12} : A_1 \otimes A_2 \to A_2 \otimes A_1$ and for $x \in \mathrm{KK}_i(A_1, B_1)$ and $y \in \mathrm{KK}_j(A_2, B_2)$ it holds that

$$\tau_{A_2}(x) \otimes_{B_1 \otimes A_2} \tau^{B_1}(y) = (-1)^{ij} (\sigma_{12})_* (\sigma_{12})^* (\tau_{A_1}(y) \otimes_{B_2 \otimes A_1} \tau^{B_2}(x)) \tag{4.1.9}$$

as an element of $\mathrm{KK}_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2)$, see [45, Lemma 2.1].

We say that A, B are KK_0-*equivalent* if there are $x \in \mathrm{KK}_0(A, B)$ and $y \in \mathrm{KK}_0(B, A)$ such that $x \otimes_B y = 1_A$ and $y \otimes_A x = 1_B$. Similarly, there is a KK_1-equivalence. A key tool in (complex) KK-theory is *Bott periodicity*, that is $\mathrm{KK}_0(A, C_0(\mathbb{R}) \otimes B) \cong \mathrm{KK}_1(A, B)$. There are numerous proofs of this fact and we point out a new one found in [49], where the author shows that the Bott-class $[\beta] \in \mathrm{KK}_1(\mathbb{C}, C_0(\mathbb{R}))$ and the Dirac-class $[d] \in \mathrm{KK}_1(C_0(\mathbb{R}), \mathbb{C})$ yield a KK_1-equivalence between $C_0(\mathbb{R})$ and $\mathbb{C}$.

4.1.2 Stinespring dilation of extensions of C^*-algebras

For simplicity assume that A and B are separable trivially graded C^*-algebras, with B being C^*-stable. An *extension* of A by B is a short exact sequence of C^*-algebras

$$0 \to B \to C \to A \to 0. \tag{4.1.10}$$

It is equivalently given by its *Busby invariant* $\tau : A \to \mathcal{Q}(B)$, where $\mathcal{Q}(B)$ denotes the Calkin algebra $\mathcal{M}(B)/B$, see [14, Section 15]. Let $\pi : \mathcal{M}(B) \to \mathcal{Q}(B)$ denote the quotient map. Here we assume that all Busby invariants are injective which is equivalent with B being an essential ideal of C. Also, the terms extension and Busby invariant will be used interchangeably, depending on the context.

An extension $\widetilde{\tau} : A \to \mathcal{Q}(B)$ is *(strongly) unitarily equivalent* to τ if there is a unitary $u \in \mathcal{M}(B)$ such that $\widetilde{\tau}(a) = \pi(u)\tau(a)\pi(u^*)$, for all $a \in A$. The extension τ is *trivial* if it lifts to a $*$-homomorphism into the multiplier algebra $\mathcal{M}(B)$. Since B is C^*-stable, there are standard isomorphisms $\mathcal{M}(B) \cong M_2(\mathcal{M}(B))$ which are uniquely determined up to unitary equivalence, and yield isomorphisms $\mathcal{Q}(B) \cong M_2(\mathcal{Q}(B))$. Then the addition of extensions is well-defined (up to unitary equivalence) and we say that τ is *invertible* if there is some $\tau^{-1} : A \to \mathcal{Q}(B)$ such that $\tau \oplus \tau^{-1} : A \to M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$ is trivial.

Definition 4.1.5 ([14, Def. 15.6.3])**.** By $\mathrm{Ext}(A, B)$ we denote the abelian semigroup of unitary equivalence classes of extensions of A by B, modulo the unitary equivalence classes of trivial extensions, with addition of extensions as the operation. The group of invertibles is denoted by $\mathrm{Ext}^{-1}(A, B)$.

The classification of extensions by abelian groups was initiated by Brown, Douglas and Fillmore [23] in the case $A = C(X)$, where X is a compact metrisable space, and B is the compact operators $\mathcal{K}(H)$ on a separable Hilbert space. The notation was simply $\mathrm{Ext}(C(X))$ instead of $\mathrm{Ext}(C(X),\mathcal{K}(H))$, and they proved that $\mathrm{Ext}(C(X))$ is a group. Moreover, they showed that the Ext-functor satisfies Bott periodicity and among other properties it yields a homology theory on compact metrisable spaces. However, passing to noncommutative C^*-algebras requires heavy C^*-algebraic machinery, like the Absorption Theorem of Voiculescu, the Choi-Effros Lifting Theorem and Stinespring Dilation, see [67, Section 3]. In the class of C^*-algebras A for which $\mathrm{Ext}(A)$ is a group, the Ext-functor generates a cohomology theory, which due to the commutative case is conventionally known as Ext-homology. A bit later, generalisations of the aforementioned theorems led Kasparov in [77] to define the bi-functor Ext.

Every class in $\mathrm{KK}_1(A,B)$ can be realised as a class in $\mathrm{Ext}^{-1}(A,B)$. To see this, first note that $\mathcal{B}_B(\mathbb{H}_B) \cong \mathcal{M}(B)$ and $\mathcal{K}_B(\mathbb{H}_B) \cong B$. Then each class $[\mathbb{H}_B,\rho,F] \in \mathrm{KK}_1(A,B)$ corresponds to the class of the extension $\tau : A \to \mathcal{Q}(B)$ given by $\tau(a) = \pi(P\rho(a)P)$, where $P = (F+1)/2$.

Moreover, the converse is true. If $\tau : A \to \mathcal{Q}(B)$ is an invertible extension, then $\tau \oplus \tau^{-1} : A \to M_2(\mathcal{Q}(B))$ lifts to a $*$-homomorphism into $M_2(\mathcal{M}(B))$, whose compression provides a completely positive contractive (cpc) lift of τ into $\mathcal{M}(B)$. This procedure gives an inverse to the map $\mathrm{KK}_1(A,B) \to \mathrm{Ext}^{-1}(A,B)$ described above and it holds that $\mathrm{KK}_1(A,B)$ is naturally isomorphic to $\mathrm{Ext}^{-1}(A,B)$.

If A is nuclear, the Choi-Effros Lifting Theorem [24] implies that every extension $\tau : A \to \mathcal{Q}(B)$ has a cpc lift $\sigma : A \to \mathcal{M}(B)$. Then using Kasparov's Stinespring Dilation [78] we can write σ as the compression of a $*$-homomorphism. This yields a $\mathrm{KK}_1(A,B)$ class and the inverse of τ. Consequently, $\mathrm{Ext}(A,B) = \mathrm{Ext}^{-1}(A,B)$. Although this result is very satisfying, it is not useful for explicit index theoretic calculations. The reason is that both aforementioned theorems provide abstract constructions and hence only the existence of such a map $\mathrm{Ext}(A,B) \to \mathrm{KK}_1(A,B)$. Computing this map for a specific case occupies a large portion of this thesis.

Definition 4.1.6. Let $[\tau] \in \mathrm{Ext}^{-1}(A,B)$. Any Kasparov $(A \otimes \mathbb{C}_1, B)$-bimodule whose class in $\mathrm{KK}_1(A,B)$ realises $[\tau]$, will be called a KK_1-*lift* of $[\tau]$.

Even in the case where $B = \mathcal{K}(H)$, for H being a separable Hilbert space, the above procedure does not produce explicit KK_1-lifts. The only way to construct such lifts, for extensions that we do not know their inverse, is by using brute-force. In this thesis we construct a KK_1-lift for a class in $\mathrm{Ext}(A,\mathcal{K}(H))$, for a specific separable nuclear C^*-algebra A, without using the Choi-Effros Lifting Theorem and (Kasparov's) Stinespring Dilation. Also, we work in the context of K-homology, and hence we do not have to worry about gradings. From Proposition 4.1.4 we can describe the corresponding KK_1-class exactly.

4.1.3 Spanier-Whitehead K-duality

Following [12] we give a short exposition of how the notion of Spanier-Whitehead K-duality came into existence. Let X be a polyhedron embedded in S^{n+1}. An *n-dual of X*, denoted by $\mathcal{D}_n(X)$, is a polyhedron in $S^{n+1} \setminus X$ for which some suspension is a deformation retract of the corresponding suspension of $S^{n+1} \setminus X$.

Let now X^* be a polyhedron that is a deformation retract of $S^{n+1} \setminus X$. This means X^* is an n-dual of X. By removing a point $p \in S^{n+1}$ that is neither in X nor in X^*, we can identify $S^{n+1} \setminus \{p\}$ with $\mathbb{R}^{n+1}$. Under this identification, X and X^* are subsets of $\mathbb{R}^{n+1}$.

By considering the map $\mu : X \times X^* \to S^n$ given by

$$\mu(x, y) = (x - y)/\|x - y\|_{\mathbb{R}^{n+1}}, \tag{4.1.11}$$

which is null-homotopic on $X \vee X^*$, we obtain the map $\mu : X \wedge X^* \to S^n$. The pull-back of the generator $[S^n] \in H^n(S^n)$ along μ, together with the slant product, induce an isomorphism

$$\backslash\mu^*[S^n] : H_k(X) \to H^{n-k}(X^*) \tag{4.1.12}$$

between the reduced homology and cohomology groups. The important feature of X^* is that, for large n, its stable homotopy type does not depend, up to suspension, on the embedding and the deformation retraction. Therefore, for sufficiently large n, there is the *Spanier-Whitehead dual* $\mathcal{D}(X)$ for which the isomorphism (4.1.12) holds in place of X^*. This construction is known as Spanier-Whitedead Duality. It also makes sense for finite complexes and generalises Alexander Duality.

In the noncommutative setting one tries to build a KK-theoretic Spanier-Whitehead duality between C^*-algebras. Specifically, let A be a separable C^*-algebra and similarly we search for a dual algebra $\mathcal{D}(A)$ such that, $\mathrm{K}^*(A) \cong \mathrm{K}_*(\mathcal{D}(A))$ or $\mathrm{K}^*(A) \cong \mathrm{K}_{*+1}(\mathcal{D}(A))$. A natural candidate is the Paschke dual of A, see [76] and [67, Section 5]. Given an ample representation $\rho : A \to \mathcal{B}(H)$; that is, $\rho(A) \cap \mathcal{K}(H) = \{0\}$, the *Paschke dual* is

$$\mathcal{D}_\rho(A) = \{T \in \mathcal{B}(H) : [T, \rho(A)] \subset \mathcal{K}(H)\}. \tag{4.1.13}$$

Since the representation ρ is large, from Voiculescu's Absorption Theorem we have that $\mathcal{D}_\rho(A)$ is independent of the ample ρ. In addition, if A is nuclear and unital, there is a canonical isomorphism $\mathrm{K}_*(\mathcal{D}_\rho(A)) \cong \mathrm{K}^{*+1}(A)$, similar to the classical case [76, Section 9]. However, Paschke duals do not fit nicely into the framework of KK-theory, since they are usually non-nuclear and non-separable, and hence the double Paschke dual of A will not necessarily be independent of the choice of an ample representation. However, alternatives are studied in [76].

There is a rich literature on KK-theoretic Spanier-Whitehead Duality. It first appeared in the work of Kasparov [80] where he constructed isomorphisms between the K-theory and K-homology of a compact Riemannian manifold and its cotangent bundle. Kasparov referred to this duality as KK-theoretic Poincaré duality. Later, this notion was used by Connes [28, Chapter VI] who showed self-duality for the irrational rotation algebras. Then, Kaminker and Putnam [73] studied Kasparov's K-theoretic duality for the Cuntz-Krieger algebras. In their work, they referred to this duality as Spanier-Whitehead K-duality, and they argued that it is a more appropriate name than Poincaré duality, since classical Poincaré duality relates the homology and cohomology of the same manifold. A basic difference between classical Spanier-Whitehead duality and Poincaré duality is that, for the first duality one does not have to assume orientability of the space, while in the latter, orientability is necessary. For the relation between the two dualities we refer the reader to [73, Section 2]. The following definition is based on the definitions given in [46,73,74].

Definition 4.1.7. Let A and B be two separable C^*-algebras. We say that A and B are *Spanier-Whitehead* K-*dual*, or just *dual*, if there is a K-homology class $\Delta \in \mathrm{KK}_i(A \otimes B, \mathbb{C})$ and a K-theory class $\widehat{\Delta} \in \mathrm{KK}_i(\mathbb{C}, A \otimes B)$ such that

$$\begin{aligned}\widehat{\Delta} \otimes_B \Delta &= 1_A\\ \widehat{\Delta} \otimes_A \Delta &= (-1)^i 1_B.\end{aligned}$$

Such a pair $(\widehat{\Delta}, \Delta)$ will be called a *duality pair*. In particular, if B is the opposite algebra A^{op}, we will say that A is a *Poincaré duality algebra*, and we will refer to Δ as the *fundamental class* of A.

Remark 4.1.8. To be precise, by $\widehat{\Delta} \otimes_B \Delta$ we mean the product $\widehat{\Delta} \otimes_B (\sigma_{12})^*(\Delta)$. Similarly, by $\widehat{\Delta} \otimes_A \Delta$ we mean $(\sigma_{12})_*(\widehat{\Delta}) \otimes_A \Delta$.

The following was first proved by Connes in the case $i = 0$. The general proof can be found in [46, Lemma 9]. Given a duality pair $(\widehat{\Delta}, \Delta)$ between A and B, as in Definition 4.1.7, we obtain isomorphisms

$$\widehat{\Delta} \otimes_B : \mathrm{KK}_j(B, \mathbb{C}) \to \mathrm{KK}_{j+i}(\mathbb{C}, A) \tag{4.1.14}$$

$$\otimes_A \Delta : \mathrm{KK}_j(\mathbb{C}, A) \to \mathrm{KK}_{j+i}(B, \mathbb{C}). \tag{4.1.15}$$

In fact, we obtain various isomorphisms with coefficients, see [20]. One of them is $\mathrm{KK}_i(\mathbb{C}, B \otimes A) \to \mathrm{KK}_0(B, B)$ given by $x \mapsto \tau_B(x) \otimes_{B\otimes A\otimes B} \tau^B(\Delta)$. In particular, due to duality, $(\sigma_{12})_*(\widehat{\Delta}) \in \mathrm{KK}_i(\mathbb{C}, B \otimes A)$ gets mapped uniquely to $(-1)^i 1_B$. This means that if $\widehat{\Delta}'$ was another class which produced duality with Δ, then $\widehat{\Delta}' = \widehat{\Delta}$. A similar statement can be made for $\widehat{\Delta}$. Moreover, a very interesting property is that for every invertible element ℓ in the ring $\mathrm{KK}_0(A, A)$, the classes $\ell \otimes_A \Delta$ and $\widehat{\Delta} \otimes_B \ell^{-1}$ form another duality pair for

A, B. Finally, if A, B satisfy the Universal Coefficient Theorem (UCT) with K-homology in the middle, the groups $\mathrm{K}_*(A)$ and $\mathrm{K}^*(A)$ are finitely generated, see [73]. The question whether a separable C^*-algebra has a Spanier-Whitehead dual is being addressed in [76].

In [104] Popescu and Zacharias showed Poincaré duality for higher rank graph algebras. In [46] Emerson showed that for a large class of hyperbolic groups Γ, the C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ satisfies Poincaré duality. Echterhoff, Emerson and Kim [42] showed Spanier-Whitehead K-duality for a certain class of orbifold C^*-algebras. Kaminker, Putnam and Whittaker in [74] studied the Poincaré duality for the stable and unstable Ruelle algebras $R^s = S \rtimes \mathbb{Z}$ and $R^u = U \rtimes \mathbb{Z}$. More precisely, first they proved that R^s and R^u are Spanier-Whitehead K-dual, and using the Kirchberg-Phillips classification [103] for purely infinite C^*-algebras they deduced that, whenever the K-theory groups of S and U have finite rank, each R^s and R^u is a Poincaré duality algebra.

4.2 K-duality and uniformly smooth C*-algebras

In this section we investigate how the notion of summable Fredholm modules and that of smooth extensions, over dense $*$-subalgebras of C^*-algebras, are related. Also, we compute slant products in KK-theory over separable, simple, purely infinite C^*-algebras which are not necessarily unital. Similar computations, but for unital C^*-algebras, have been carried out in [58]. Finally, we develop tools from holomorphic functional calculus that allow us to use these slant products, and deduce summability and smoothness conditions on the K-homology and Ext-groups of the C^*-algebras.

4.2.1 Smooth extensions and summable Fredholm modules

Let H be a separable Hilbert space. We now present some classes of two-sided ideals of $\mathcal{B}(H)$ that consist of compact operators and that will be used in the sequel.

For a compact operator $T \in \mathcal{K}(H)$, let $(s_n(T))_{n\in\mathbb{N}}$ be the sequence of its singular values in decreasing order, counting their multiplicities. The *Schatten p-ideal* on H, where $p > 0$, is defined as

$$\mathcal{L}^p(H) = \{T \in \mathcal{K}(H) : (s_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})\}, \tag{4.2.1}$$

and is equipped with the *Schatten p-norm*

$$\|T\|_p = \|(s_n(T))_{n\in\mathbb{N}}\|_{\ell^p(\mathbb{N})}. \tag{4.2.2}$$

The *Logarithmic integral p-ideal* on H, where $p \geq 1$, is defined as

$$\mathrm{Li}^{1/p}(H) = \{T \in \mathcal{K}(H) : s_n(T) = O((\log n)^{-1/p})\} \tag{4.2.3}$$

and has a canonical choice for a norm, see [28, p. 391]. For simplicity, denote $\mathrm{Li}^1(H)$ by $\mathrm{Li}(H)$. The following concept is important for our purpose.

Definition 4.2.1 ([60])**.** A *symmetrically normed ideal* is a two-sided ideal $\mathcal{J}$ of $\mathcal{B}(H)$ with a norm $\|\cdot\|_{\mathcal{J}}$ such that

(1) $\|SRT\|_{\mathcal{J}} \leq \|S\|\|R\|_{\mathcal{J}}\|T\|$, for all $S, T \in \mathcal{B}(H)$ and $R \in \mathcal{J}$;

(2) $\mathcal{J}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{J}}$.

If $p \geq 1$, the ideals $\mathcal{L}^p(H)$ and $\mathrm{Li}^{1/p}(H)$ are symmetrically normed, and in fact Banach $*$-ideals. However, if $p \in (0,1)$, the ideal $\mathcal{L}^p(H)$ is only a quasi-Banach space, but satisfies all the basic properties of symmetrically normed ideals [60]. The fact that it is quasi-normed will create some technical difficulties in Subsection 4.2.3.

In what follows, all the aforementioned symmetrically (quasi-)normed ideals will be denoted by $\mathcal{J}$, and will have (quasi-)norm $\|\cdot\|_{\mathcal{J}}$. Occasionally though, $\mathcal{J}$ will be the ideal of compact operators $\mathcal{K}(H)$. Moreover, by the square root of $\mathcal{J}$ we will mean the symmetrically (quasi-)normed ideal

$$\mathcal{J}^{1/2} = \mathrm{span}\{T \in \mathcal{K}(H) : T^*T,\, TT^* \in \mathcal{J}\}, \tag{4.2.4}$$

with the (quasi-)norm $\|T\|_{\mathcal{J}^{1/2}} = \|T^*T\|_{\mathcal{J}}^{1/2}$. Before presenting some background on the utility of such ideals in the study of extensions, we give the following definition. Let A be a separable C^*-algebra.

Definition 4.2.2. Let $\mathcal{A} \subset A$ be a dense $*$-subalgebra. An extension $\tau : A \to \mathcal{Q}(H)$ will be called *$\mathcal{J}$-smooth on $\mathcal{A}$* if there is a linear map $\eta : \mathcal{A} \to \mathcal{B}(H)$ such that

$$\eta(ab) - \eta(a)\eta(b),\, \eta(a^*) - \eta(a)^* \in \mathcal{J}$$

and $\tau(a) = \eta(a) + \mathcal{K}(H)$, for all $a, b \in \mathcal{A}$. If $\mathcal{J} = \mathcal{L}^p(H)$, the extension will be called *p-smooth* (or *finitely smooth*), while if $\mathcal{J} = \mathrm{Li}^{1/2}(H)$, it will be called *$\theta$-smooth*.

The notion of smooth extensions was introduced by Douglas [40] who studied 1-smooth extensions of finite complexes. Shortly after, Douglas and Voiculescu in [41] studied the p-smoothness of sphere extensions, an essential step in understanding the smoothness of extensions of C^∞-manifolds. They obtained that, for every $n \geq 2$, every $(n-1)$-smooth extension of the sphere $S^{2n-1} \subset \mathbb{C}^n$ should be trivial, and that p-smooth non-trivial extensions exist if $p > n$. This satisfying result indicates that smoothness is related with dimension. Their work was later extended by Gong [61] in the framework of finite complexes.

The basic idea in the commutative case is as follows. Consider a compact metrisable space X and an extension $\tau : C(X) \to \mathcal{Q}(H)$. If X embeds in $\mathbb{C}^n$, the canonical coordinates $\{z_i\}_{i=1}^n$ of $\mathbb{C}^n$, restricted on X, yield canonical elements $\{\tau(z_i)\}_{i=1}^n \subset \mathcal{Q}(H)$. The extension τ will be $\mathcal{J}$-smooth if there are $\{T_i\}_{i=1}^n \subset \mathcal{B}(H)$ such that, $\tau(z_i) = T_i + \mathcal{K}(H)$ and the commutators $[T_i, T_j]$ and $[T_i, T_j^*]$ lie in $\mathcal{J}$.

In the noncommutative setting, examples of smooth extensions are usually derived from the work of Connes [26] who defined smoothness in the framework of Fredholm modules. This opened the window to use the theory of smooth extensions in the computation of index pairings between K-theory and K-homology.

Definition 4.2.3 ([26,58])**.** Let $\mathcal{A} \subset A$ be a dense $*$-subalgebra. An even or odd Fredholm module (H, ρ, F) over A is $\mathcal{J}^{1/2}$*-summable on* $\mathcal{A}$ if, for every $a \in \mathcal{A}$, it holds that

$$\rho(a)(F^* - F) \in \mathcal{J},\ \rho(a)(F^2 - 1) \in \mathcal{J},\ [\rho(a), F] \in \mathcal{J}^{1/2}.$$

If $\mathcal{J}^{1/2} = \mathcal{L}^p(H)$, the Fredholm module will be called *p-summable* (or *finitely summable*), while if $\mathcal{J}^{1/2} = \mathrm{Li}^{1/2}(H)$, it will be called *θ-summable*.

The largest dense $*$-subalgebra of A on which a Fredholm module (H, ρ, F) can be $\mathcal{J}^{1/2}$-summable is called the *Hölder algebra*, which we denote by $\mathrm{Höl}_{\mathcal{J}^{1/2}}(H, \rho, F)$ (the notation is borrowed from [58]). Whenever $\mathcal{J}$ and $\mathcal{J}^{1/2}$ are Banach $*$-ideals, the Hölder algebra admits the structure of a Banach $*$-algebra and is holomorphically stable in A, as we see in [26, Appendix 3]. One can also check Subsection 4.2.3. Therefore, the inclusion $i : \mathrm{Höl}_{\mathcal{J}^{1/2}}(H, \rho, F) \hookrightarrow A$ yields an isomorphism

$$i_* : \mathrm{K}_*(\mathrm{Höl}_{\mathcal{J}^{1/2}}(H, \rho, F)) \to \mathrm{K}_*(A). \tag{4.2.5}$$

A finitely summable Fredholm module (H, ρ, F) corresponds to a cocycle $\mathrm{Ch}^*(H, \rho, F)$ in the periodic cyclic cohomology group $\mathrm{H}^*(\mathcal{A})$, for $\mathcal{A} = \mathrm{Höl}_{\mathcal{J}^{1/2}}(H, \rho, F)$. The cocycle $\mathrm{Ch}^*(H, \rho, F)$ is called the *Connes-Chern character* of (H, ρ, F), see [26]. The important feature of Connes' cyclic theory is that $\mathrm{Ch}^*(H, \rho, F)$ pairs with every $x \in \mathrm{K}_*(\mathcal{A})$ in a way that

$$\langle x, \mathrm{Ch}^*(H, \rho, F) \rangle = \langle i_*(x), [H, \rho, F] \rangle \in \mathbb{Z}, \tag{4.2.6}$$

and $\langle x, \mathrm{Ch}^*(H, \rho, F) \rangle$ is simply given by a trace formula. The Connes-Chern character can be extended to the context of θ-summable Fredholm modules, but there one has to map into the entire cyclic cohomology groups, see [28, Chapter IV]. Consequently, C^*-algebras with the following strong K-homological summability conditions are of particular interest.

Definition 4.2.4 ([48])**.** The K-homology of A is *uniformly* $\mathcal{J}$*-summable* if there is a dense $*$-subalgebra $\mathcal{A} \subset A$ such that, every $x \in \mathrm{K}^*(A)$ can be represented by a Fredholm module which is $\mathcal{J}$-summable on $\mathcal{A}$.

For a closed manifold M of dimension m, the K-homology of $C(M)$ is uniformly $\mathcal{L}^p$-summable over $C^\infty(M)$, for $p > m$. This is because every K-homology class over $C(M)$ can be represented by some pseudodifferential operator F of order 0 such that, the singular values $s_n([F,f])$ are $O(n^{-1/m})$, for every $f \in C^\infty(M)$. A detailed proof of this fact can be found in [110, Chapter 6]. In the noncommutative setting, Goffeng and Mesland [58] proved that the odd K-homology of Cuntz-Krieger algebras is uniformly $\mathcal{L}^p$-summable, for every $p > 0$, and the even K-homology is uniformly $\mathrm{Li}^{1/2}$-summable. In [48], Emerson and Nica showed that for a large class of hyperbolic groups Γ, the K-homology of $C(\partial\Gamma) \rtimes \Gamma$ is uniformly $\mathcal{L}^p$-summable, where p depends on the Hausdorff dimensions of the Gromov boundary $\partial\Gamma$, when equipped with specific metrics. In particular, they obtained that the K-homology of the reduced group C^*-algebra $C_r^*(\Gamma)$, for a finitely generated free group Γ, is uniformly $\mathcal{L}^p$-summable over the group ring $\mathbb{C}\Gamma$, whenever $p > 2$. In all the examples presented so far, the main tool for proving the existence of non-trivial K-homology classes over the C^*-algebras is the KK-theoretic Spanier-Whitehead duality. This is discussed in Subsection 4.1.3. A different approach was pursued by Rave in [110, Chapter 4] where he proved that every AF-algebra has uniformly $\mathcal{L}^1$-summable K-homology.

So far in the literature, there is no known obstruction that prevents the K-homology of C^*-algebras to be uniformly finitely summable. But, it is also not true that this finiteness condition is universal, see [58, Lemma 6] and [105] for some special counterexamples. In the unbounded picture of K-homology, though, the situation is different. Pure infiniteness is a C^*-algebraic condition that prevents the existence of finitely summable unbounded Fredholm modules. This is because purely infinite C^*-algebras are traceless, and any such module would yield a tracial state on the C^*-algebra [27]. What makes the study of such an obstruction in K-homology even more difficult, but certainly more interesting, is that classes of unbounded Fredholm modules which are at best θ-summable, may get mapped via bounded transform, to classes of finitely summable Fredholm modules. For example, this happens for the purely infinite C^*-algebras studied in [48,58].

In contrast to finite summability, the notion of θ-summability is better behaved. Specifically, any θ-summable Fredholm module lifts to some θ-summable unbounded Fredholm module [28, p. 392], and vice versa. At least, for purely infinite C^*-algebras, θ-summability in K-homology is still very interesting, since any finitely summable Fredholm module would still lift to a strictly θ-summable unbounded Fredholm module. At the end, what one hopes for, is to use tools like Getzler's spectral flow approach and obtain an index theorem [56].

Following the definition of uniform summability in K-homology we are led to consider its analogue for extensions. However, in order to present a well-defined notion, suppose that A is nuclear, and denote the group $\mathrm{Ext}(A, \mathcal{K}(H))$ by $\mathrm{Ext}(A)$.

Definition 4.2.5. We say that A is *uniformly $\mathcal{J}$-smooth* if there is a dense $*$-subalgebra $\mathcal{A} \subset A$ such that, every $x \in \mathrm{Ext}(A)$ can be represented by an extension which is $\mathcal{J}$-smooth on $\mathcal{A}$.

Let (H, ρ, F) be an odd Fredholm module over A that is $\mathcal{J}^{1/2}$-summable on a dense $*$-subalgebra $\mathcal{A}$. Then, it is not hard to show that the extension $\tau : A \to \mathcal{Q}(H)$ given by $\tau(a) = P\rho(a)P + \mathcal{K}(H)$, where $P = (F+1)/2$, is $\mathcal{J}$-smooth on $\mathcal{A}$. To see this, assume for simplicity that (H, ρ, F) is normalised so that $F^2 = 1$, $F = F^*$. This is a particularly nice instance because P is a projection and the map $\eta : A \to B(H)$, given by $\eta(a) = P\rho(a)P$, is already $*$-linear and cpc. Moreover, $\eta(ab) - \eta(a)\eta(b) \in \mathcal{J}$ for $a, b \in \mathcal{A}$, since

$$P\rho(ab)P - P\rho(a)P\rho(b)P = -P[P, \rho(a)][P, \rho(b)] \in \mathcal{J}^{1/2}\mathcal{J}^{1/2} \subset \mathcal{J}. \tag{4.2.7}$$

Hence, C^*-algebras with uniformly $\mathcal{J}^{1/2}$-summable K-homology are uniformly $\mathcal{J}$-smooth.

Remark 4.2.6. The *degree of irregularity* of A, $\inf\{p > 0 : A \text{ is uniformly } \mathcal{L}^p\text{-smooth}\}$, is invariant under isomorphisms. It would be interesting to know if it is a homotopy invariant. This is similar to a question raised in [58] about the degree of summability of A. However, as we see from the following discussion, these two notions might not be the same.

The converse direction, whether an extension $\tau : A \to \mathcal{Q}(H)$ which is $\mathcal{J}$-smooth on $\mathcal{A}$, lifts to an odd Fredholm module that is $\mathcal{J}^{1/2}$-summable on $\mathcal{A}$, is a very delicate matter and seems to be a difficult problem to solve. For example, the case where $\mathcal{J} = \mathcal{K}(H)$ requires the Choi-Effros Lifting Theorem. Nevertheless, if $\mathcal{J} \neq \mathcal{K}(H)$, it is still possible to find a reasonable characterisation for this problem. Of course, such τ should be invertible, and hence, we know that there is an isometry $V : H \to H'$ and a representation $\rho : A \to \mathcal{B}(H')$ such that $\tau(a) = V^*\rho(a)V + \mathcal{K}(H)$. Since the projection VV^* commutes with $\rho(a)$ modulo $\mathcal{K}(H')$, we obtain the odd Fredholm module

$$(H', \rho, 2VV^* - 1) \tag{4.2.8}$$

that represents $[\tau]$. Moreover, since τ is $\mathcal{J}$-smooth on $\mathcal{A}$, there exists a linear map $\eta : \mathcal{A} \to \mathcal{B}(H)$ such that $\eta(ab) - \eta(a)\eta(b) \in \mathcal{J}$, $\eta(a^*) - \eta(a)^* \in \mathcal{J}$ and $\tau(a) = \eta(a) + \mathcal{K}(H)$, for all $a, b \in \mathcal{A}$. As a result, we have $\eta(a) - V^*\rho(a)V \in \mathcal{K}(H)$, for every $a \in \mathcal{A}$. Let $\sim_{\mathcal{J}}$ and $\sim_{\mathcal{K}}$ stand for perturbations modulo $\mathcal{J}$ and $\mathcal{K}(H)$, respectively. For $a, b \in \mathcal{A}$, it holds that

$$V^*\rho(ab)V - V^*\rho(a)VV^*\rho(b)V \sim_{\mathcal{K}} \eta(ab) - \eta(a)\eta(b) \sim_{\mathcal{J}} 0. \tag{4.2.9}$$

This does not give any refined information on (4.2.8). However, if for every $a \in \mathcal{A}$ we had $\eta(a) - V^*\rho(a)V \in \mathcal{J}$, the Fredholm module (4.2.8) would be $\mathcal{J}^{1/2}$-summable over $\mathcal{A}$. To summarise, an extension $\tau : A \to \mathcal{Q}(H)$ which is $\mathcal{J}$-smooth on $\mathcal{A}$, lifts to an odd Fredholm module that is $\mathcal{J}^{1/2}$-summable on $\mathcal{A}$, if and only if, the linear map η that corresponds to τ (see Definition 4.2.2), can be chosen so that it extends to a cpc map $\widetilde{\eta} : A \to \mathcal{B}(H)$ such that $\tau(a) = \widetilde{\eta}(a) + \mathcal{K}(H)$, for all $a \in A$. This characterisation has been studied in more generality by Goffeng in [57].

Remark 4.2.7. In this thesis we try to lift a $\mathcal{J}$-smooth extension $\tau : A \otimes B \to \mathcal{Q}(H)$, where A, B are nuclear and $\mathcal{J}$ is some Schatten ideal. The extension τ is the product (in the Calkin algebra) of two representations $\rho_A : A \to \mathcal{B}(H)$ and $\rho_B : B \to \mathcal{B}(H)$ which commute modulo compacts, meaning that $\tau(x) = (\rho_A \cdot \rho_B)(x) + \mathcal{K}(H)$, for $x \in A \otimes_{\text{alg}} B$. The map $\rho_A \cdot \rho_B$ is linear, and for x, y in a $*$-subalgebra of $A \otimes_{\text{alg}} B$ that is dense in $A \otimes B$, it holds that $(\rho_A \cdot \rho_B)(xy) - (\rho_A \cdot \rho_B)(x)(\rho_A \cdot \rho_B)(y) \in \mathcal{J}$ and $(\rho_A \cdot \rho_B)(x^*) - (\rho_A \cdot \rho_B)(x)^* \in \mathcal{J}$. If $\rho_A \cdot \rho_B$ could extend to a cpc map on $A \otimes B$ then, from the discussion above, there would exist an abstract $\mathcal{J}^{1/2}$-summable Fredholm module lifting τ, since we do not know a priori the inverse of τ. However, $\rho_A \cdot \rho_B$ is not even $*$-linear. In fact, since $\rho_A(A)$ and $\rho_B(B)$ do not commute, we can find self-adjoint elements $a \in A$ and $b \in B$ that $\rho_A(a)$ and $\rho_B(b)$ do not commute. This means that $(\rho_A \cdot \rho_B)(a^* \otimes b^*) \neq (\rho_A \cdot \rho_B)(a \otimes b)^*$.

Question 1. *Suppose A is nuclear and $\mathcal{J}$ is a Schatten ideal. Does every $\mathcal{J}$-smooth extension on a dense $*$-subalgebra $\mathcal{A} \subset A$ lift to a $\mathcal{J}^{1/2}$-summable Fredholm module on $\mathcal{A}$?*

A refined version of the Choi-Effros Lifting Theorem might be the right tool to answer the above question. However, this would require a novel approach since this theorem relies heavily on C^*-algebraic methods. One tricky part of such endeavour would be to work with approximate units in $\mathcal{J}$ which are quasicentral relative to A, but in a refined way. However, this could lead to the construction of unbounded Fredholm modules over A which are $\mathcal{J}$-summable, see [129]. But for example, if A is purely infinite this is impossible. The method of constructing a lift is summarised in the following theorem, which is altered to meet our notation.

Theorem 4.2.8 ([58, Theorem 2.2.1])**.** *Suppose $\tau : A \to \mathcal{Q}(H)$ is an extension which is $\mathcal{J}$-smooth on a dense $*$-subalgebra $\mathcal{A} \subset A$, and $\eta : \mathcal{A} \to \mathcal{B}(H)$ is a linear map described in Definition 4.2.2. Assume also that there is an isometry $V : H \to H'$, and a representation $\rho : A \to \mathcal{B}(H')$ such that $\eta(a) - V^*\rho(a)V \in \mathcal{J}$, for all $a \in \mathcal{A}$. Then the odd Fredholm module $(H', \rho, 2VV^* - 1)$ over A is $\mathcal{J}^{1/2}$-summable on $\mathcal{A}$ and represents the class $[\tau] \in \mathrm{Ext}(A)$.*

4.2.2 Slant products for simple, purely infinite C^*-algebras

Our aim is to explicitly compute slant products of the form (4.1.15) for $\Delta \in \mathrm{KK}_1(A \otimes B, \mathbb{C})$. In the literature we were able to find this computation only in the case where both algebras are unital, see [58, Prop. 2.1.3, Prop. 2.1.6]. However, if A or B are not unital, certain technical difficulties arise which are described below. Nevertheless, if A, B are separable and A is also simple and purely infinite, we manage to circumvent these difficulties. This suffices for the purpose of this thesis.

Suppose that A, B are separable C^*-algebras and let (H, ρ, F) be an odd Fredholm module over $A \otimes B$, and $e \in \mathcal{M}_n(A) = \mathcal{M}_n(\mathbb{C}) \otimes A$ be a projection. In addition, let $(u_n)_{n \in \mathbb{N}}$

be an approximate identity for B such that

$$u_{n+1}u_n = u_n. \tag{4.2.10}$$

It holds that the sequence of positive elements $(\mathrm{id}_{\mathcal{M}_n(\mathbb{C})} \otimes \rho)(e \otimes u_n)$ is bounded and increasing. Therefore, it converges strongly to a positive operator $P_e \in B(\mathbb{C}^n \otimes H)$. Using (4.2.10) it is straightforward to show that P_e is a projection.

Let H_e be the Hilbert space $P_e(H)$ and $\rho_e : B \to \mathcal{B}(H_e)$ be the representation given by $\rho_e(b) = (\mathrm{id}_{\mathcal{M}_n(\mathbb{C})} \otimes \rho)(e \otimes b)$. This representation is well-defined since for every $b \in B$ we have that

$$\rho_e(b)P_e = \rho_e(b) = P_e\rho_e(b). \tag{4.2.11}$$

Finally, we consider the operator

$$F_e = P_e(\mathrm{id}_{\mathbb{C}^n} \otimes F)P_e. \tag{4.2.12}$$

Lemma 4.2.9. *The triple (H_e, ρ_e, F_e) is an odd Fredholm module over B.*

Proof. Since the triple $(\mathbb{C}^n \otimes H, \mathrm{id}_{\mathcal{M}_n(\mathbb{C})} \otimes \rho, \mathrm{id}_{\mathbb{C}^n} \otimes F)$ is an odd Fredholm module over $\mathcal{M}_n(\mathbb{C}) \otimes A \otimes B$, it suffices to prove the statement only for $n = 1$. By $\sim_{\mathcal{K}}$ we will denote a compact perturbation in $\mathcal{K}(H)$. Also, all the operators in $\mathcal{B}(H_e)$ can be thought to be in $\mathcal{B}(H)$ by letting them to be zero on $H_e^{\perp}$. Let $b \in B$, then we have that

$$\begin{aligned}
\rho_e(b)(F_e^* - F_e) &= \rho_e(b)(P_eF^*P_e - P_eFP_e) \\
&= \rho_e(b)F^*P_e - \rho_e(b)FP_e \\
&\sim_{\mathcal{K}} (F^* - F)\rho_e(b) \\
&\sim_{\mathcal{K}} 0.
\end{aligned}$$

Since $\rho_e(b)(F_e^* - F_e)$ is zero on $H_e^{\perp}$ and maps in H_e, we have $\rho_e(b)(F_e^* - F_e) \in \mathcal{K}(H_e)$. Similarly, it holds that $\rho_e(b)(F_e^2 - P_e) \in \mathcal{K}(H_e)$. Finally,

$$\begin{aligned}
[F_e, \rho_e(b)] &= P_eFP_e\rho_e(b) - \rho_e(b)P_eFP_e \\
&= P_eF\rho_e(b)P_e - P_e\rho_e(b)FP_e \\
&= P_e[F, \rho_e(b)]P_e \\
&\sim_{\mathcal{K}} 0.
\end{aligned}$$

As a result, $[F_e, \rho_e(b)] \in \mathcal{K}(H_e)$. □

In order to compute the slant product we need a good description of the K-theory classes over A. If A is unital, it is well-known that $\mathrm{K}_0(A)$ can be alternatively given by formal differences of isomorphism classes of finitely generated projective modules over A.

More precisely, if $e \in \mathcal{M}_n(A)$ is a projection, its Murray-von-Neumann equivalence class $[e]$ corresponds to the isomorphism class $[eA^n]$. Following [79, Section 6], the class $[eA^n]$ corresponds to the equivalence class of the Kasparov $(\mathbb{C}, A)$-bimodule $(eA^n, j, 0)$, where $j : \mathbb{C} \to \mathcal{B}_A(pA^n)$ is given by $j(\lambda) = \lambda \cdot \mathrm{id}$. Passing to formal differences gives a complete description of the classes in $\mathrm{KK}_0(\mathbb{C}, A)$. If A is not unital, the above description does not hold. Instead, one has to pass through the relative group $\mathrm{K}_0(\widetilde{A}, \widetilde{A}/A)$, where $\widetilde{A}$ is the unitisation of A, see [49, Section 6]. To be precise, given projections $e_0, e_1 \in \mathcal{M}_n(\widetilde{A})$, the formal difference $[e_1] - [e_0]$ corresponds to the equivalence class of the relative triple $[e_1, e_0, u] \in \mathrm{K}_0(\widetilde{A}, \widetilde{A}/A)$, where $u \in \mathcal{M}_n(\widetilde{A})$ is such that $u^*u - 1$, $uu^* - 1$, $ue_1u^* - e_0 \in \mathcal{M}_n(A)$. Then, the relative triple $[e_1, e_0, u]$ corresponds to the class $[e_1A^n \oplus e_0A^n, j_1 \oplus j_0, W]$ in $\mathrm{KK}_0(\mathbb{C}, A)$, where for $w = e_0ue_1$,

$$W = \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix}. \tag{4.2.13}$$

This is a very explicit description. However, if A is not unital, the operator W cannot be taken to be zero in general, and this makes the computation of the Kasparov product (4.1.15) more difficult.

Suppose now that A is simple and purely infinite; that is, A contains no non-trivial closed two-sided ideals and every non-zero hereditary C^*-subalgebra of A contains an infinite projection. In particular, there is a non-zero projection $p \in A$ and since A is simple, p is full, meaning that ApA is dense in A, see [22, Lemma 1.1]. Passing to hereditary C^*-subalgebras preserves pure infiniteness and simplicity. Therefore, the C^*-subalgebra pAp is simple and purely infinite. It is also unital and C^*-stably isomorphic to A. In addition, the inclusion map $\psi : pAp \to A$ induces an isomorphism $\psi_* : \mathrm{KK}_*(\mathbb{C}, pAp) \to \mathrm{KK}_*(\mathbb{C}, A)$, see [98, Prop. 1.2]. Moreover, from [30], the K-theory of pAp is given by

$$\mathrm{K}_0(pAp) = \{[e] : e \text{ is a non-zero projection in } pAp\}. \tag{4.2.14}$$

In addition, the class $[e]$ of a projection $e \in pAp$, corresponds to the isomorphism class of finitely generated projective modules $[e(pAp)]$. This is because pAp is unital. In turn, this corresponds to the equivalence class of Kasparov $(\mathbb{C}, pAp)$-bimodules $[e(pAp), j, 0]$. Composing with ψ_* we obtain the isomorphism $\mathrm{K}_0(pAp) \to \mathrm{KK}_0(\mathbb{C}, A)$ given by

$$[e] \mapsto [e(pAp) \otimes_\psi A, j \otimes_\psi \mathrm{id}, 0]. \tag{4.2.15}$$

Lemma 4.2.10. *Let A be a C^*-algebra with a full projection $p \in A$, and $e \in pAp$ be any projection. The map $U : e(pAp) \otimes_\psi A \to eA$ given on simple tensors by $ea' \otimes_\psi a \mapsto ea'a$, for $a \in A$ and $a' \in pAp$, extends to an isomorphism of Hilbert A-modules.*

Proof. Clearly, the map U extends by linearity to an A-module map on the algebraic tensor product $e(pAp) \otimes_{pAp} A$. Similarly, it holds that U preserves the inner products and thus, it extends to an isometry on the complete tensor product $e(pAp) \otimes_\psi A$. Moreover, we have that $epApA = eApA$. Since p is full, for every $c \in A$ there is $(c_n)_{n\in\mathbb{N}} \subset ApA$ such that $\lim_n c_n = c$. Consequently, $\lim_n epc_n = \lim_n ec_n = ec \in eA$, meaning that U has dense range. □

The slant products will be computed in the context of K-theory and K-homology. Then, from Proposition 4.1.4, they can be explicitly described as KK-classes. First, recall the notation of Lemma 4.2.9, the isomorphism $\psi_* : \mathrm{K}_0(pAp) \to \mathrm{K}_0(A)$, in the context of K-theory, and the description (4.2.14) of $\mathrm{K}_0(pAp)$.

Proposition 4.2.11. *Let A be a separable, simple, purely infinite C^*-algebra, and B be a separable C^*-algebra. Let $p \in A$ be a non-zero projection and (H, ρ, F) be an odd Fredholm module over $A \otimes B$, where ρ is non-degenerate. The slant product $\otimes_A[H, \rho, F] : \mathrm{K}_0(A) \to \mathrm{K}^1(B)$ is given by*

$$\psi_*([e]) \mapsto [H_e, \rho_e, F_e], \quad \textit{for } e \in pAp.$$

Proof. Using the map in (4.2.15) we see that the class $\psi_*([e])$ is represented by the Kasparov $(\mathbb{C}, A)$-bimodule

$$(e(pAp) \otimes_\psi A, j \otimes_\psi \mathrm{id}, 0).$$

The unitary $U : e(pAp) \otimes_\psi A \to eA$ of Lemma 4.2.10, makes this bimodule unitarily equivalent to $(eA, j', 0)$, where the map $j' : \mathbb{C} \to \mathcal{B}_A(eA)$ is given by $j'(\lambda)(ea) = \lambda ea$, for $a \in A$. Therefore, the Kasparov product $\psi_*([e]) \otimes_A [H, \rho, F]$ is equal to the product $\tau_B([eA, j', 0]) \otimes_{A\otimes B} [H, \rho, F]$ which, since the operator on the left KK-class is zero, it is given by

$$[(eA \otimes B) \otimes_\rho H, (j' \otimes \mathrm{id}) \otimes_\rho \mathrm{id}, G],$$

where G is any F-connection for $eA \otimes B$, see [69, Def. 2.2.4] and [69, Def. 2.2.7].

Before constructing the operator G, let us briefly mention the definition of a connection in our context. For every $x \in eA\otimes B$, let $T_x \in \mathcal{B}(H, (eA\otimes B)\otimes_\rho H)$ be given by $T_x(\xi) = x\otimes_\rho \xi$, and then $T_x^*(y \otimes_\rho \xi) = \rho(\langle x, y\rangle_{A\otimes B})\xi$. For G to be an F-connection for $eA \otimes B$ we should have that, for every $x \in eA \otimes B$,

$$T_x F - G T_x \in \mathcal{K}(H, (eA \otimes B) \otimes_\rho H),$$
$$F T_x^* - T_x^* G \in \mathcal{K}((eA \otimes B) \otimes_\rho H, H).$$

Let us first find a nicer description of the Hilbert space $(eA \otimes B) \otimes_\rho H)$. The map $W : (eA \otimes B) \otimes_\rho H \to H_e$ given on simple tensors by $x \otimes_\rho \xi \mapsto \rho(x)\xi$ is unitary. Indeed, it is straightforward to see that W preserves the inner products. Also, it has a dense range since $\rho(eA \otimes B) = P_e\rho(A \otimes B)$, and hence $\overline{\rho(eA \otimes B)H} = \overline{P_e\rho(A \otimes B)H} = P_e(\overline{\rho(A \otimes B)H}) = H_e$. The claim now is that $W^{-1}F_eW$ is an F-connection for $eA\otimes B$. Before proving the claim we should note that for $x \in eA \otimes B$ it holds that $P_e\rho(x) = \rho(x)$ and $\rho(x^*)P_e = \rho(x^*)$. We have that $T_xF - W^{-1}F_eWT_x \in \mathcal{K}(H, (eA \otimes B) \otimes_\rho H)$ if and only if $WT_xF - F_eWT_x \in \mathcal{K}(H, H_e)$. To prove the latter, let $\xi \in H$ and then we have,

$$\begin{aligned}(WT_xF - F_eWT_x)\xi &= W(x \otimes_\rho F\xi) - F_eW(x \otimes_\rho \xi)\\ &= \rho(x)F\xi - F_e\rho(x)\xi\\ &= P_e\rho(x)F\xi - P_eF\rho(x)\xi\\ &= P_e[\rho(x), F]\xi.\end{aligned}$$

Similarly, it holds that $FT_x^* - T_x^*W^{-1}F_eW \in \mathcal{K}((eA \otimes B) \otimes_\rho H, H)$ if and only if the operator $FT_x^*W^{-1} - T_x^*W^{-1}F_e \in \mathcal{K}(H_e, H)$. Recall that P_e is the strong operator limit of $(\rho(e \otimes u_n))_{n\in\mathbb{N}}$, where $(u_n)_{n\in\mathbb{N}}$ is an approximate identity for B satisfying $u_{n+1}u_n = u_n$. For $\xi \in H_e$ we have,

$$\begin{aligned}(FT_x^*W^{-1} - T_x^*W^{-1}F_e)\xi &= FT_x^*W^{-1}P_e\xi - T_x^*W^{-1}F_e\xi\\ &= \lim_{n\to\infty}(FT_x^*((e \otimes u_n) \otimes_\rho \xi) - T_x^*((e \otimes u_n) \otimes_\rho F\xi))\\ &= \lim_{n\to\infty}(F\rho(\langle x, e \otimes u_n\rangle_{A\otimes B})\xi - \rho(\langle x, e \otimes u_n\rangle_{A\otimes B})F\xi)\\ &= [F, \rho(x^*)]\xi.\end{aligned}$$

This proves the claim, that $W^{-1}F_eW$ is an F-connection for $eA \otimes B$. The proof of the proposition finishes by observing that the representation $j'' : \mathbb{C} \otimes B \to \mathcal{B}(H_e)$ given by conjugation with W, that is,

$$j''(\lambda \otimes b) = W(j' \otimes \mathrm{id}) \otimes_\rho \mathrm{id})(\lambda \otimes b)W^{-1},$$

is the same with the representation $\rho_e : B \to \mathcal{B}(H_e)$ given by $\rho_e(b) \to \rho(e \otimes b)$. This can be proved in a similar fashion as above, by evaluating $j''(\lambda\otimes b)$ on some $\xi \in H_e$ which is written as $P_e\xi$. Then, the result follows from the strong limit description of P_e. Consequently, the desired KK-class $[(eA \otimes B) \otimes_\rho H, (j' \otimes \mathrm{id}) \otimes_\rho \mathrm{id}, W^{-1}F_eW]$ is equal to $[H_e, \rho_e, F_e]$. □

Using the notation of Proposition 4.2.11, with a simple calculation, we obtain the following result.

Corollary 4.2.12. *Let A be a separable, simple, purely infinite C^*-algebra, and B be a separable C^*-algebra. Let $p \in A$ be a non-zero projection and $\tau : A \otimes B \to \mathcal{Q}(H)$ be an invertible extension. The slant product $\otimes_A[\tau] : \mathrm{K}_0(A) \to \mathrm{Ext}^{-1}(B)$ is given by*

$$\psi_*([e]) \mapsto [\tau_e], \quad \textit{for } e \in pAp,$$

where $\tau_e(b) = \tau(e \otimes b)$, for $b \in B$.

The aforementioned corollary allows us to deduce smoothness results on C^*-algebras. Specifically, if the map $\otimes_A[\tau]$ is an isomorphism and τ is $\mathcal{J}$-smooth, satisfying some additional properties, then B is uniformly $\mathcal{J}$-smooth. For proving this we need to use holomorphic functional calculus.

4.2.3 Holomorphic functional calculus and K-duality

In this subsection we aim to develop tools from holomorphic functional calculus that will allow us to prove Proposition 4.2.21. First we extend a lemma of Connes, regarding the ideals $\mathcal{L}^p(H)$ for $p \geq 1$, to general symmetrically normed ideals. Also, we extend this lemma to include the case $p \in (0,1)$. However, the latter endeavour is more elaborate since the corresponding ideals are quasi-normed. Further, we show that holomorphicity passes to corner subalgebras of C^*-algebras. First, we give the following definition which is a combination of Definitions 1 of [26, p.92] and [28, p.285].

Definition 4.2.13. Let D be a Banach algebra and $\mathcal{D}$ a subalgebra with unitisation $\widetilde{\mathcal{D}} \subset \widetilde{D}$. We say that $\mathcal{D}$ is *holomorphically closed* in D if, for every $d \in \widetilde{\mathcal{D}}$ we have

$$f(d) \in \widetilde{\mathcal{D}},$$

for any function f that is holomorphic in an open neighbourhood of the spectrum of d in $\widetilde{D}$. Moreover, we say that $\mathcal{D}$ is *holomorphically stable* in D if, for every $n \in \mathbb{N}$, the subalgebra $\mathcal{M}_n(\widetilde{\mathcal{D}})$ is *holomorphically closed* in $\mathcal{M}_n(\widetilde{D})$.

The following theorem summarises a result of Schweitzer and the Density Theorem of Karoubi, see [124], [28, Prop.2, p.285] and [26, p.92].

Theorem 4.2.14 (Density Theorem)**.** *Suppose that $\mathcal{D}$ is a holomorphically closed dense subalgebra of a Banach algebra D. Then, $\mathcal{D}$ is holomorphically stable in D. Moreover, the inclusion $i : \mathcal{D} \to D$ induces the isomorphism*

$$i_* : \mathrm{K}_*(\mathcal{D}) \to \mathrm{K}_*(D).$$

Let H be a separable Hilbert space and $\mathcal{I} \subset \mathcal{B}(H)$ be a symmetrically normed ideal. We note that, for all $R \in \mathcal{I}$, it holds

$$\|R\|_{\mathcal{B}(H)} \leq \|R\|_{\mathcal{I}}, \tag{4.2.16}$$

see [60, Chapter III]. The next lemma is basically Proposition 5a of [26, p.87], but stated more generally. We present its proof for completeness.

Lemma 4.2.15. *For every $S, T \in \mathcal{B}(H)$ such that $[S,T] \in \mathcal{I}$, and every function f that is holomorphic in a neighbourhood of the spectrum $\sigma(S)$, we have*

$$[f(S), T] \in \mathcal{I}.$$

Proof. Let S, T as in the statement and consider a function f which is holomorphic on an open neighbourhood $U \supset \sigma(S)$. Also, let $\gamma : [0,1] \to U$ be a simple closed C^1-curve around $\sigma(U)$, oriented counter-clockwise. Then, the operator

$$f(S) = \frac{1}{2\pi i} \int_\gamma^{\mathcal{B}(H)} f(z)(z-S)^{-1} dz$$

is by definition the $\mathcal{B}(H)$-valued Riemann integral

$$\frac{1}{2\pi i} \int_{[0,1]}^{\mathcal{B}(H)} f(\gamma(t))\gamma'(t)(\gamma(t)-S)^{-1} dt.$$

This integral converges since $\mathcal{B}(H)$ is a Banach space and the integrated function is (uniformly) continuous. Using the continuity and linearity of the commutator with T, we then have that

$$[f(S), T] = \frac{1}{2\pi i} \int_\gamma^{\mathcal{B}(H)} f(z)[(z-S)^{-1}, T] dz.$$

Note that the integrated function $F : \gamma([0,1]) \to (\mathcal{B}(H), \|\cdot\|)$ given by

$$F(z) = f(z)[(z-S)^{-1}, T]$$

is continuous.

A closer look now yields that, for every $z \in \mathbb{C} \setminus \sigma(S)$, one has

$$[(z-S)^{-1}, T] = (z-S)^{-1}[S,T](z-S)^{-1} \in \mathcal{I}. \tag{4.2.17}$$

As a result, the function F takes values in $\mathcal{I}$. In fact, using the symmetric property of $\mathcal{I}$, it is straightforward to prove that F is continuous also with respect to $\|\cdot\|_{\mathcal{I}}$. Now, since $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach space, the $\mathcal{I}$-valued Riemann integral

$$\frac{1}{2\pi i} \int_\gamma^{\mathcal{I}} f(z)[(z-S)^{-1}, T] dz$$

is also well-defined. Finally, observe that the inequality (4.2.16) forces the latter integral to be equal to $[f(S), T]$. □

The situation for the symmetrically quasi-normed ideals $\mathcal{L}^p(H)$, for $p \in (0,1)$, is a bit different. Specifically, it is no longer true that every continuous function $g : \gamma([0,1]) \to \mathcal{L}^p(H)$, for γ being a simple closed C^1-curve, is Riemann integrable. An example with values in $\ell^p(\mathbb{N})$ can be found in [62, Section 3]. Even assuming that g is holomorphic (differentiable) in a neighbourhood of γ does not suffice. One has to assume that g is analytic, meaning that it has local power series expansions. We note that in the context of quasi-Banach spaces, analyticity is stronger than holomorphicity because tools like the Hahn-Banach Theorem do not necessarily hold. Quasi-Banach spaces are usually studied in the context of r-Banach spaces, where $r \in (0,1]$. Let us pause for a moment and give precise definitions.

A *quasi-normed space* is a complex vector space V that is equipped with a *quasi-norm* $\|\cdot\| : V \to [0,\infty)$ which, for all $v, w \in V$, satisfies

(i) $\|v\| = 0$ only if $v = 0$;

(ii) $\|zv\| = |z|\|v\|$, for $z \in \mathbb{C}$;

(iii) $\|v + w\| \leq K(\|v\| + \|w\|)$, for some $K \geq 1$.

For $r \in (0,1]$, an *r-normed space* is a complex vector space V equipped with an *r-norm* $\|\cdot\|_r : V \to [0,\infty)$ that satisfies (i), (ii) and for every $v, w \in V$ one has

(iv) $\|v + w\|_r^r \leq \|v\|_r^r + \|w\|_r^r$.

A *quasi-Banach space* is a complete quasi-normed space, and an *r-Banach space* is a complete r-normed space. If V is r-normed with $\|\cdot\|_r$, then $\|\cdot\|_r$ is a quasi-norm with constant $K = 2^{1/r-1}$. Conversely, if $\|\cdot\|$ is a quasi-norm on V with constant K, the Aoki-Rolewicz Renorming Theorem [113] yields an equivalent r-norm $\|\cdot\|_r$ on V, where also $K = 2^{1/r-1}$, that for every $v \in V$ it holds

$$\|v\|_r \leq \|v\| \leq 2^{1/r}\|v\|_r. \tag{4.2.18}$$

Returning to our problem of integrating on simple closed C^1-curves, we mention that Riemann integration for quasi-Banach (or r-Banach) valued functions is defined in exactly the same way as for Banach-valued functions. Let V be an r-normed space with r-norm $\|\cdot\|_r$, and Γ be a simple closed C^1-curve in $\mathbb{C}$. The linear space of Riemann integrable V-valued functions on Γ is denoted by $\mathscr{R}(\Gamma, V)$. Moreover, a function $g : \Gamma \to V$ has *r-expansion on* Γ if, for every $k \geq 0$, there are $a_k \in V$ and Riemann integrable functions $f_k : \Gamma \to \mathbb{C}$ such that,

$$g(z) = \sum_{k=0}^{\infty} a_k f_k(z) \quad \text{and} \quad \sum_{k=0}^{\infty} \|a_k\|_r^r \|f_k\|_\infty^r < \infty.$$

Let us denote by $\mathscr{E}(\Gamma, V)$ the linear space of functions $\Gamma \to V$ that have r-expansions.

Proposition 4.2.16 ([62, Proposition 3.7]). *The linear space $\mathscr{E}(\Gamma, V)$ is contained in $\mathscr{R}(\Gamma, V)$.*

For the next two results, let $p \in (0,1)$ and consider the ideal $\mathcal{L}^p(H)$ equipped with the Schatten quasi-norm $\|\cdot\|_p$. Its constant is $K = 2^{1/p}$, and the renorming process gives an equivalent complete r-norm $|\cdot|_r$, where $r = p/(p+1)$.

Proposition 4.2.17. *Let $S, T \in \mathcal{B}(H)$ such that $[S,T] \in \mathcal{L}^p(H)$, and let Γ be a simple closed C^1-curve in the resolvent $\mathbb{C} \setminus \sigma(S)$. Then, the map $F : \Gamma \to \mathcal{L}^p(H)$ given by*

$$F(z) = [(z-S)^{-1}, T],$$

is Riemann integrable.

Proof. Instead of showing directly that F is Riemann integrable with respect to the quasi-norm $\|\cdot\|_p$, we will prove that it is Riemann integrable with respect to $|\cdot|_r$ and then use inequalities (4.2.18). Specifically, we will show that F is a finite sum of functions with r-expansions on Γ, and hence from Proposition 4.2.16 we obtain that F is Riemann integrable.

To begin, let us also denote by F the extension of the map on $\mathbb{C} \setminus \sigma(S)$. Observe that the map $z \mapsto (z-S)^{-1}$ defined on $\mathbb{C} \setminus \sigma(S)$ is analytic in $\mathcal{B}(H)$. Indeed, let $w \in \mathbb{C} \setminus \sigma(S)$ and for every $z \in \mathbb{C}$ with $|z-w| < 1/\|(w-S)^{-1}\|$ we have that

$$(z-S)^{-1} = \sum_{k=0}^{\infty} s_k(w)(w-z)^k,$$

where $s_k(w) = ((w-S)^{-1})^{k+1}$. Moreover, recall that, for every $z \in \mathbb{C} \setminus \sigma(S)$, it holds

$$[(z-S)^{-1}, T] = (z-S)^{-1}[S,T](z-S)^{-1}.$$

As a result, for every $w \in \mathbb{C} \setminus \sigma(S)$ and every $z \in \mathbb{C}$ with $|z-w| < 1/\|(w-S)^{-1}\|$ we have,

$$F(z) = \sum_{n=0}^{\infty} c_n(w)(w-z)^n, \tag{4.2.19}$$

where

$$c_n(w) = \sum_{j=0}^{n} s_j(w)[S,T]s_{n-j}(w).$$

The expansion (4.2.19) converges in $\mathcal{B}(H)$ and hence F is analytic in $\mathcal{B}(H)$. Clearly, the expansion (4.2.19) around $w \in \mathbb{C} \setminus \sigma(S)$ holds for all $z \in \mathbb{C}$ such that

$$|z-w| < \frac{1}{2K\|(w-S)^{-1}\|}. \tag{4.2.20}$$

This open ball will be denoted by $B(w)$.

Let now $w \in \Gamma$ and consider the open segment $\Gamma(w) = \Gamma \cap B(w)$. We claim that the expansion (4.2.19), restricted on $\Gamma(w)$, converges (absolutely) also in $(\mathcal{L}^p(H), |\cdot|_r)$. In other words, it is a local r-expansion, in the sense that, if $f_{n,w} : \Gamma(w) \to \mathbb{C}$ are given by $f_{n,w}(z) = (w - z)^n$, then

$$\sum_{n=0}^{\infty} |c_n(w)|_r^r \|f_{n,w}\|_\infty^r < \infty. \tag{4.2.21}$$

Then, since $|\cdot|_r$ is equivalent to $\|\cdot\|_p$ which majorises the operator norm $\|\cdot\|$ on $\mathcal{B}(H)$, we get that this local r-expansion agrees with F on $\Gamma(w)$. Of course, each $c_n(w) \in \mathcal{L}^p(H)$ and the functions $f_{n,w}$ are Riemann integrable. We note that for (4.2.21) to be true, it suffices to show that

$$\sum_{n=0}^{\infty} \|c_n(w)\|_p^r \|f_{n,w}\|_\infty^r < \infty. \tag{4.2.22}$$

We have that

$$\|f_{n,w}\|_\infty = \sup_{z \in \Gamma(w)} |w - z|^n \leq \frac{1}{(2K)^n \|(w - S)^{-1}\|^n}. \tag{4.2.23}$$

Moreover,

$$\begin{aligned} \|c_n(w)\|_p &\leq K^n \sum_{j=0}^{n} \|s_j(w)[S,T]s_{n-j}(w)\|_p \\ &\leq \|[S,T]\|_p K^n \sum_{j=0}^{n} \|s_j(w)\| \|s_{n-j}(w)\|, \end{aligned}$$

and since $r \in (0,1]$, we also have that

$$\|c_n(w)\|_p^r \leq (\|[S,T]\|_p K^n)^r \sum_{j=0}^{n} \|s_j(w)\|^r \|s_{n-j}(w)\|^r. \tag{4.2.24}$$

In addition, the series

$$\sum_{n=0}^{\infty} K^{rn} \|s_n(w)\|^r \|f_{n,w}\|_\infty^r < \infty$$

because from (4.2.23) we obtain that

$$\begin{aligned} K^{rn} \|s_n(w)\|^r \|f_{n,w}\|_\infty^r &= K^{rn} \|((w - S)^{-1})^{n+1}\|^r \|f_{n,w}\|_\infty^r \\ &\leq \|(w - S)^{-1}\|^r \frac{K^{rn} \|(w - S)^{-1}\|^{rn}}{2^{rn} K^{rn} \|(w - S)^{-1}\|^{rn}} \\ &= \|(w - S)^{-1}\|^r (\frac{1}{2^r})^n. \end{aligned}$$

Therefore, by noticing that $\|f_{n,w}\|_\infty^r = \|f_{1,w}\|_\infty^{rn}$, we have

$$\sum_{n=0}^{\infty} (\sum_{j=0}^{n} \|s_j(w)\|^r \|s_{n-j}(w)\|^r) K^{rn} \|f_{n,w}\|_\infty^r = (\sum_{n=0}^{\infty} K^{rn} \|s_n(w)\|^r \|f_{n,w}\|_\infty^r)^2 < \infty,$$

and then from (4.2.24) the series (4.2.22) converges.

From the compactness of Γ, there are $w_1, \ldots, w_m \in \Gamma$ that determine open segments $\Gamma(w_1), \ldots, \Gamma(w_m)$ that cover Γ, on which F has local r-expansions. Let $\{h_{w_1}, \ldots, h_{w_m}\}$ be a partition of unity subordinated to the cover $\{\Gamma(w_1), \ldots, \Gamma(w_m)\}$. Then, the function $F : \Gamma \to \mathcal{L}^p(H)$ can be written as

$$F = \sum_{j=1}^{m} F h_{w_j}.$$

Now, each $F h_{w_j}$ has an r-expansion on Γ since, if we let $\widetilde{f}_{n,w_j}$ denote the extension of $f_{n,w_j} : \Gamma(w_j) \to \mathbb{C}$ to Γ, then for every $z \in \Gamma$ one has

$$(F h_{w_j})(z) = \sum_{n=0}^{\infty} c_n(w_j)(\widetilde{f}_{n,w_j} h_{w_j})(z).$$

We still have that $\widetilde{f}_{n,w_j} h_{w_j}$ is Riemann integrable and

$$\sum_{n=0}^{\infty} |c_n(w_j)|_r^r \|\widetilde{f}_{n,w_j} h_{w_j}\|_\infty^r \leq \sum_{n=0}^{\infty} |c_n(w_j)|_r^r \|f_{n,w_j}\|_\infty^r \|h_{w_j}\|_\infty^r < \infty.$$

Consequently, the function $F : \Gamma \to (\mathcal{L}^p(H), |\cdot|_r)$ lies in $\mathscr{E}(\Gamma, \mathcal{L}^p(H))$ and hence it is Riemann integrable. □

Recall that inequality (4.2.16) holds also for the ideals $\mathcal{L}^p(H)$, for $p \in (0,1)$. Working as in Lemma 4.2.15 but applying Proposition 4.2.17 we obtain the following result.

Lemma 4.2.18. *Let $p \in (0,1)$. For every $S, T \in \mathcal{B}(H)$ such that $[S,T] \in \mathcal{L}^p(H)$ and every function f that is holomorphic in a neighbourhood of the spectrum $\sigma(S)$, we have*

$$[f(S), T] \in \mathcal{L}^p(H).$$

Let A, B be arbitrary C^*-algebras in $\mathcal{B}(H)$ and $\mathcal{I} \subset \mathcal{B}(H)$ be a symmetrically normed ideal, or a Schatten p-ideal for $p \in (0,1)$.

Lemma 4.2.19. *Let $\mathcal{A} \subset A$, $\mathcal{B} \subset B$ be dense $*$-subalgebras that commute modulo the ideal $\mathcal{I}$, meaning that $[a,b] \in \mathcal{I}$, for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then, there are dense $*$-subalgebras $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{A}}$ that*

(1) satisfy $\mathcal{A} \subset \mathcal{A}_{\mathcal{B}} \subset A$ and $\mathcal{B} \subset \mathcal{B}_{\mathcal{A}} \subset B$;

(2) are holomorphically stable;

(3) commute modulo $\mathcal{I}$.

Proof. Let $\mathcal{A}_{\mathcal{B}} = \{a \in A : [a,b] \in \mathcal{I},\ \text{for all } b \in \mathcal{B}\}$. It is straightforward to see that $\mathcal{A}_{\mathcal{B}}$ is a $*$-subalgebra of A, and since it contains $\mathcal{A}$, it is also dense in A. Moreover, one can see that

$$\widetilde{\mathcal{A}_{\mathcal{B}}} = \{a \in \widetilde{A} : [a,b] \in \mathcal{I},\ \text{for all } b \in \mathcal{B}\}.$$

Let $a \in \widetilde{\mathcal{A}_{\mathcal{B}}}$ and consider an arbitrary $b \in \mathcal{B}$. From Lemmas 4.2.15 and 4.2.18 we see that, if f is holomorphic in an open neighbourhood of the spectrum of a in $\widetilde{A}$ (note that it agrees with the spectrum in $\mathcal{B}(H)$), then it still holds that $[f(a), b] \in \mathcal{I}$. Therefore, the algebra $\mathcal{A}_{\mathcal{B}}$ is holomorphically closed in A, and from Theorem 4.2.14 we obtain that it is actually holomorphically stable in A.

Consider now $\mathcal{B}_{\mathcal{A}} = \{b \in B : [a, b] \in \mathcal{I}, \text{ for all } a \in \mathcal{A}_{\mathcal{B}}\}$. By definition $\mathcal{B}_{\mathcal{A}}$ contains $\mathcal{B}$ and therefore, is a dense $*$-subalgebra of B. Working as before we obtain that $\mathcal{B}_{\mathcal{A}}$ is holomorphically stable in B. □

Since we work with hereditary C^*-subalgebras we need the following fact.

Lemma 4.2.20. *Let A be a C^*-algebra and $\mathcal{A} \subset A$ be a holomorphically stable $*$-subalgebra. Then, for every projection $p \in \mathcal{A}$, the $*$-algebra $p\mathcal{A}p$ is holomorphically stable in pAp.*

Proof. Let $p \in \mathcal{A}$ be a projection. We claim that $p\mathcal{A}p$ is holomorphically closed in pAp. Note that pAp is unital with unit p, and hence we do not need to consider its unitisation. In order to avoid trivialities assume that $p \neq 0$. If A is not unital, we consider its unitisation $\widetilde{A}$ as well as the unitisation $\widetilde{\mathcal{A}} \subset \widetilde{A}$. This makes sense because $p\widetilde{A}p = pAp$ (similarly for $\mathcal{A}$), and therefore, we can simply assume that A and $\mathcal{A}$ are unital, with unit 1. Again, to avoid trivialities assume that $p \neq 1$.

Let $b \in p\mathcal{A}p$ and denote by $\sigma_p(b)$ the spectrum relative to pAp, and by $\sigma(b)$ the spectrum relative to A. It holds that

$$\sigma(b) = \sigma_p(b) \cup \{0\}.$$

Indeed, the element b cannot be invertible in A and hence $0 \in \sigma(b)$. Also, for $z \in \mathbb{C}$, if $z1 - b$ is invertible in A with inverse $(z1 - b)^{-1}$, then $zp - b$ is invertible in pAp with inverse

$$(zp - b)^{-1} = p(z1 - b)^{-1}p. \tag{4.2.25}$$

Consequently, $\sigma_p(b) \subset \sigma(b)$. Finally, by representing A and pAp faithfully on a Hilbert space (note that $*$-isomorphisms preserve the spectrum of elements), it is not hard to check that $\sigma(b) \setminus \{0\} \subset \sigma_p(b)$.

Consider now a function f which is holomorphic in an open set $U \supset \sigma_p(b)$. By a slight abuse of notation, let us denote by $f_p(b)$ the holomorphic functional calculus relative to pAp, and in case it can be defined, lets us denote by $f(b)$ the holomorphic functional calculus relative to A. First, assume that $0 \in U$ and hence $\sigma(b) \subset U$. This means $f(b)$ is defined. Let γ be a simple closed C^1-curve in U that encloses counter-clockwise the spectrum $\sigma(b)$ and then, using (4.2.25), we have

$$\begin{aligned} f_p(b) &= \frac{1}{2\pi i}\int_\gamma f(z)(zp-b)^{-1}dz \\ &= p(\frac{1}{2\pi i}\int_\gamma f(z)(z1-b)^{-1}dz)p \\ &= pf(b)p. \end{aligned}$$

As a result, $f_p(b) \in p\mathcal{A}p$ because $f(b) \in \mathcal{A}$.

Assume now that $0 \notin U$. Then, since $\{0\}$ and $\sigma_p(b)$ are disjoint, we can find an open ball $W \subset \mathbb{C}$ around 0, and an open set V around $\sigma_p(b)$ such that $W \cap V = \varnothing$. Consider the holomorphic function

$$g(z) = \begin{cases} f(z), & \text{if } z \in U \cap V \\ 0, & \text{if } z \in W, \end{cases}$$

and note that its domain contains $\sigma(b)$. Let γ_1, γ_2 be simple closed C^1-curves such that, γ_1 is in $U \cap V$ encircling $\sigma_p(b)$, and γ_2 is in W around 0, both with counter-clockwise orientation. Then, we have that

$$\begin{aligned} g(b) &= \frac{1}{2\pi i}\int_{\gamma_1} g(z)(z1-b)^{-1}dz + \frac{1}{2\pi i}\int_{\gamma_2} g(z)(z1-b)^{-1}dz \\ &= \frac{1}{2\pi i}\int_{\gamma_1} g(z)(z1-b)^{-1}dz. \end{aligned}$$

Therefore, using (4.2.25) we obtain that

$$\begin{aligned} pg(b)p &= \frac{1}{2\pi i}\int_{\gamma_1} f(z)(zp-b)^{-1}dz \\ &= f_p(b). \end{aligned}$$

Since $g(b) \in \mathcal{A}$ it follows that $f_p(b) \in p\mathcal{A}p$.

Finally, let $n \in \mathbb{N}$ and we claim that $\mathcal{M}_n(p\mathcal{A}p)$ is holomorphically closed in $\mathcal{M}_n(pAp)$. First, observe that $\mathcal{M}_n(pAp) = p'\mathcal{M}_n(A)p'$ and $\mathcal{M}_n(p\mathcal{A}p) = p'\mathcal{M}_n(\mathcal{A})p'$, where p' is $\operatorname{diag}(p, \dots, p) \in \mathcal{M}_n(\mathcal{A})$. Also, recall that A and $\mathcal{A}$ can be assumed to be unital, and from the hypothesis we have that $\mathcal{M}_n(\mathcal{A})$ is holomorphically closed in $\mathcal{M}_n(A)$. Then, from the proof so far, the same holds for $\mathcal{M}_n(p\mathcal{A}p) \subset \mathcal{M}_n(pAp)$. □

Now we state one of the main propositions that will be used in Subsection 6.2.3 to show that every Ruelle algebra is uniformly smooth relatively to a Schatten ideal. For the next result let $\mathcal{I} \subset \mathcal{B}(H)$ be a symmetrically normed ideal or a Schatten p-ideal, for $p \in (0, 1)$. We note that the same result can be obtained in the case where A, B are separable (with at least one of them being nuclear) and A is also unital. This follows from the index computations [58, Prop. 2.1.3, Prop. 2.1.6] for unital A, B, and our computations in Subsection 4.2.2 which show that it is not necessary for B to have a unit.

Proposition 4.2.21. *Let A be a separable simple, purely infinite C^*-algebra and B be a separable nuclear C^*-algebra. Suppose that A, B are Spanier-Whitehead* K*-dual, with the duality being implemented by $\Delta \in \mathrm{KK}_1(A \otimes B, \mathbb{C})$. Also, assume that*

(1) there are faithful representations $\rho_A : A \to \mathcal{B}(H)$, $\rho_B : B \to \mathcal{B}(H)$ so that $\rho_A(A)$ and $\rho_B(B)$ commute modulo $\mathcal{K}(H)$;

(2) Δ is represented by the invertible extension $\tau : A \otimes B \to \mathcal{Q}(H)$ given on elementary tensors by

$$\tau(a \otimes b) = \rho_A(a)\rho_B(b) + \mathcal{K}(H);$$

(3) there are dense $*$-subalgebras $\mathcal{A} \subset A$, $\mathcal{B} \subset B$ such that $\rho_A(\mathcal{A})$ and $\rho_B(\mathcal{B})$ commute modulo $\mathcal{I}$.

Then, there are holomorphically stable dense $*$-subalgebras $\mathcal{A}_{\mathcal{B}} \subset A$, $\mathcal{B}_{\mathcal{A}} \subset B$ such that τ is $\mathcal{I}$-smooth on $\mathcal{A}_{\mathcal{B}} \otimes_{alg} \mathcal{B}_{\mathcal{A}}$. Consequently, the C^*-algebra B is uniformly $\mathcal{I}$-smooth on $\mathcal{B}_{\mathcal{A}}$.

Proof. Nuclearity of B and condition (1) imply that the map $\tau : A \otimes B \to \mathcal{Q}(H)$ is a $*$-homomorphism. Moreover, from condition (3) we obtain that τ is $\mathcal{I}$-smooth on $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}$. Note that the $*$-algebra $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}$ is indeed dense in the (minimal) tensor product $A \otimes B$. In addition, from Lemma (4.2.19) we can find holomorphically stable dense $*$-subalgebras

$$\rho_A(\mathcal{A})_{\mathcal{B}} \subset \rho_A(A)$$
$$\rho_B(\mathcal{B})_{\mathcal{A}} \subset \rho_B(B),$$

that still commute modulo $\mathcal{I}$. Let $\mathcal{A}_{\mathcal{B}} \subset A$, $\mathcal{B}_{\mathcal{A}} \subset B$ be the inverse images of $\rho_A(\mathcal{A})_{\mathcal{B}}$ and $\rho_B(\mathcal{B})_{\mathcal{A}}$ and we have that τ is $\mathcal{I}$-smooth on $\mathcal{A}_{\mathcal{B}} \otimes_{\mathrm{alg}} \mathcal{B}_{\mathcal{A}}$. Also, since $*$-isomorphisms commute with holomorphic functional calculus, we have that the dense $*$-subalgebras $\mathcal{A}_{\mathcal{B}}$, $\mathcal{B}_{\mathcal{A}}$ are holomorphically stable in A, B, respectively.

Using Corollary 4.2.12 we can now deduce that B is uniformly $\mathcal{I}$-smooth on $\mathcal{B}_{\mathcal{A}}$. Indeed, for every non-zero projection $p \in A$, the map $\otimes_A[\tau] : \mathrm{K}_0(A) \to \mathrm{Ext}(B)$ is computed on K_0-classes of the form $\psi_*([e])$ which exhaust $\mathrm{K}_0(A)$. Here, the map $\psi_* : \mathrm{K}_0(pAp) \to \mathrm{K}_0(A)$ is the inclusion isomorphism and $e \in pAp$ is a projection. More precisely, we obtain that

$$\otimes_A[\tau](\psi_*([e])) = [\tau_e],$$

where the extension $\tau_e : B \to \mathcal{Q}(H)$ is given by $\tau_e(b) = \tau(e \otimes b)$, for $b \in B$.

Let $x \in \mathrm{Ext}(B)$ and we claim that x can be represented by an extension which is $\mathcal{I}$-smooth on $\mathcal{B}_{\mathcal{A}}$. First note that, since $\mathcal{A}_{\mathcal{B}}$ is dense and holomorphically stable, and A contains non-zero projections, there is a non-zero projection $q \in \mathcal{A}_{\mathcal{B}}$. Moreover, the map $\otimes_A[\tau]$ is an isomorphism because $[\tau]$ corresponds to the K-homology duality class

$\Delta \in \mathrm{KK}_1(A \otimes B, \mathbb{C})$ via a natural map. As a result, we can find a projection $e \in qAq$ such that $[\tau_e] = x$. To finalise the proof, we use Lemma 4.2.20 to show that $q\mathcal{A}_{\mathcal{B}}q$ is dense and holomorphically stable in qAq. Therefore, there is a projection $e' \in q\mathcal{A}_{\mathcal{B}}q$ such that $[e'] = [e]$, and hence $x = [\tau_e] = [\tau_{e'}]$. Since τ is $\mathcal{I}$-smooth on $\mathcal{A}_{\mathcal{B}} \otimes_{\mathrm{alg}} \mathcal{B}_{\mathcal{A}}$, the extension $\tau_{e'} : B \to \mathcal{Q}(H)$ is $\mathcal{I}$-smooth on $\mathcal{B}_{\mathcal{A}}$. $\square$

Chapter 5

Smale space C*-algebras

The purpose of this chapter is to introduce the necessary background on Smale space groupoids and their C^*-algebras. Although this chapter does not contain any new results, it provides a concise introduction to the theory.

5.1 An introduction to Smale space C*-algebras

In this section we present the basic theory on C^*-algebras built from the stable, unstable and homoclinic equivalence relations on non-wandering Smale spaces. These are groupoid C^*-algebras in the sense of Renault [111]. Our focus will be on the stable and unstable Ruelle algebras, which for a given Smale space, are Spanier-Whitehead K-dual, as was proved by Kaminker, Putnam and Whittaker in [74]. A large portion of the first two subsections can be found in the work of Ruelle [117], Putnam [108], and finally, Putnam and Spielberg [109].

Let us fix the notation and recall some notions from Section 3.1. A Smale space is denoted by (X,φ) and is assumed to be irreducible. Occasionally though, it will be mixing. We can always make these reductions due to Smale's Decomposition Theorem, see Theorem 3.1.2. The expansivity constant is $\varepsilon_X > 0$, the contraction constant is $\lambda_X > 1$, the locally defined bracket map is $[\cdot,\cdot]$, and the metric is d. For $0 < \varepsilon \leq \varepsilon_X$, the local stable and unstable sets around $x \in X$ are denoted by $X^s(x,\varepsilon)$ and $X^u(x,\varepsilon)$, and the global stable and unstable sets at $x \in X$ are

$$\begin{aligned} X^s(x) &= \{y \in X : \lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y)) = 0\} \\ X^u(x) &= \{y \in X : \lim_{n\to-\infty} d(\varphi^n(x),\varphi^n(y)) = 0\}. \end{aligned} \tag{5.1.1}$$

For a mixing Smale space the global stable and unstable sets are dense in X, since in particular, for every $x,y \in X$, the intersection $X^s(x) \cap X^u(y)$ is countable and dense in X, see [116, Proposition 7.16]. For many cases though, irreducibility suffices, because if

(X,φ) is irreducible, it holds that $X^s(P)\cap X^u(Q)$ is countable and dense, for any periodic orbits P and Q, where $X^s(P)=\bigcup_{p\in P}X^s(p)$ and $X^u(Q)=\bigcup_{q\in Q}X^u(q)$. This immediately follows from Smale's Decomposition Theorem.

5.1.1 Equivalence groupoids

For the theory of topological and étale groupoids we refer the reader to [111,127]. The stable, unstable and homoclinic equivalence relations on X are respectively defined as

$$\begin{aligned} G^s &= \{(x,y) : y\in X^s(x)\} \\ G^u &= \{(x,y) : y\in X^u(x)\} \\ G^h &= \{(x,y) : y\in X^s(x)\cap X^u(x)\}. \end{aligned} \tag{5.1.2}$$

They will be regarded as groupoids and the goal is to define some natural topologies on them. The fact that

$$\begin{aligned} X^s(x) &= \bigcup_{n\geq 0}\varphi^{-n}(X^s(\varphi^n(x),\varepsilon_X)) \\ X^u(x) &= \bigcup_{n\geq 0}\varphi^{n}(X^u(\varphi^{-n}(x),\varepsilon_X)), \end{aligned} \tag{5.1.3}$$

(see [106, Proposition 4.2.3]) suggests that we put the inductive limit topology on G^s and G^u. First one has to consider the sets

$$\begin{aligned} G^s_0 &= \{(x,y)\in X\times X : y\in X^s(x,\varepsilon_X)\} \\ G^u_0 &= \{(x,y)\in X\times X : y\in X^u(x,\varepsilon_X)\}. \end{aligned} \tag{5.1.4}$$

Then, for every $n\in\mathbb{N}$ the iterations $G^s_n=(\varphi\times\varphi)^{-n}(G^s_0)$ and $G^u_n=(\varphi\times\varphi)^{n}(G^u_0)$ can be equipped with the relative topology of $X\times X$. Since $G^s=\bigcup_{n\geq 0}G^s_n$ and $G^u=\bigcup_{n\geq 0}G^u_n$, as sets, the groupoids G^s,G^u can indeed be given the inductive limit topology. Similarly, the groupoid G^h can be given the inductive limit topology by writing $G^h=\bigcup_{n\geq 0}G^h_n$, where each $G^h_n=G^s_n\cap G^u_n$ is given the relative topology of $X\times X$. It is worth mentioning that, if (X,φ) is mixing, then every G^h-orbit is countable and dense. We will refer to these groupoids as the *stable, unstable* and *homoclinic groupoids*, respectively.

The *stable, unstable* and *homoclinic Ruelle groupoids* are the semi-direct product groupoids,

$$\begin{aligned} G^s\rtimes_\varphi\mathbb{Z} &= \{(x,n,y)\in X\times\mathbb{Z}\times X : (\varphi^n(x),y)\in G^s\} \\ G^u\rtimes_\varphi\mathbb{Z} &= \{(x,n,y)\in X\times\mathbb{Z}\times X : (\varphi^n(x),y)\in G^u\} \\ G^h\rtimes_\varphi\mathbb{Z} &= \{(x,n,y)\in X\times\mathbb{Z}\times X : (\varphi^n(x),y)\in G^h\}, \end{aligned} \tag{5.1.5}$$

with partial multiplication given by

$$(x, n, y) \cdot (y, m, z) = (x, n + m, z) \tag{5.1.6}$$

in all cases.

All these groupoids are second countable, locally compact, Hausdorff and have Haar systems. In fact, the groupoids G^h and $G^h \rtimes_\varphi \mathbb{Z}$ admit étale topologies and therefore, the counting measure induces a Haar system on them. To obtain Haar systems on the other groupoids one uses the Bowen measure μ_{B} on X, see Section 3.1. More precisely, for every $x \in X$, the restriction of μ_{B} on $[X^u(x, \varepsilon_X/2), X^s(x, \varepsilon_X/2)]$ is a product measure $\mu_x^u \times \mu_x^s$, where μ_x^u and μ_x^s are non-finite regular Borel measures defined on $X^u(x)$ and $X^s(x)$, respectively, and which have the necessary invariance and compatibility properties, see [108, Section 2]. Then, the families

$$\{\mu_x^s : x \in X\} \text{ and } \{\mu_x^u : x \in X\} \tag{5.1.7}$$

define Haar systems for G^s and G^u, respectively. Moreover, these Haar systems, combined with the counting measure on $\mathbb{Z}$, yield Haar systems on $G^s \rtimes_\varphi \mathbb{Z}$ and $G^u \rtimes_\varphi \mathbb{Z}$. Finally, all the aforementioned groupoids are amenable, see [109].

A refined construction of the stable and unstable (Ruelle) groupoids is given by Putnam and Spielberg in [109]. For such a groupoid, say G, they build an étale version G' whose C^*-algebra $C^*(G')$ is strongly Morita equivalent to $C^*(G)$. The main idea is to obtain G' by restricting G on an abstract transversal, for instance if $G = G^s$, such a transversal would be $X^u(Q)$, where Q is a periodic orbit in (X, φ). This concept is along the lines of the abstract transversals of Muhly, Renault and Williams, which happen to be closed in the unit spaces of the groupoids, see [92]. However, in our case, $X^u(Q)$ is far from being closed in the unit space X since it is a proper dense subset, and hence, the relative topology on it behaves quite badly. Therefore, one has to work as follows. Let P, Q be two periodic orbits in (X, φ). The restricted stable and unstable groupoids are

$$\begin{aligned} G^s(Q) &= \{(v, w) \in G^s : v, w \in X^u(Q)\} \\ G^u(P) &= \{(v, w) \in G^u : v, w \in X^s(P)\}. \end{aligned} \tag{5.1.8}$$

Recall that if (X, φ) is mixing, the set $X^s(x) \cap X^u(y)$ is countable and dense in X, for all $x, y \in X$. This means that $G^s(Q)$ and $G^u(P)$ meet every stable and unstable equivalence class, respectively, at countably many points. In order to construct étale topologies on $G^s(Q)$ and $G^u(P)$, one first needs to equip the unit spaces $X^u(Q)$ and $X^s(P)$ with the inductive limit topologies, using the description (5.1.3). Then, the topology on the groupoids is given by a family of local homeomorphisms in the following way.

Recall the definition of the constant $0 < \varepsilon'_X \leq \varepsilon_X/2$ from (3.1.5), which is small enough so that, for all $x, y \in X$ with $d(x,y) \leq \varepsilon'_X$, we have that both $d(x,[x,y])$ and $d(y,[x,y])$ are less than $\varepsilon_X/2$. Let $(v,w) \in G^s(Q)$, and for some $N \in \mathbb{N}$, one has $\varphi^N(w) \in X^s(\varphi^N(v), \varepsilon'_X/2)$. Due to the continuity of φ^N, we can find some $\eta > 0$ such that $\varphi^N(X^u(w,\eta)) \subset X^u(\varphi^N(w), \varepsilon'_X/2)$. The map $h^s : X^u(w,\eta) \to X^u(v, \varepsilon_X/2)$ defined by

$$h^s(z) = \varphi^{-N}[\varphi^N(z), \varphi^N(v)], \tag{5.1.9}$$

is called a *stable holonomy map* and is a homeomorphism onto its image. Also, it holds that $h^s(w) = v$. Similarly, we have *unstable holonomy maps* h^u. More precisely, for every $(v,w) \in G^u(P)$ there are $N \in \mathbb{N}$ and $\eta > 0$ so that, the map $h^u : X^s(w,\eta) \to X^s(v, \varepsilon_X/2)$ given by

$$h^u(z) = \varphi^N[\varphi^{-N}(v), \varphi^{-N}(z)], \tag{5.1.10}$$

is a homeomorphism onto its image and $h^u(w) = v$. The topologies are generated as follows.

Theorem 5.1.1 ([106, Theorem 8.3.5])**.** *The collections of sets*

$$V^s(v,w,h^s,\eta,N) = \{(h^s(z), z) : z \in X^u(w,\eta)\}$$
$$V^u(v,w,h^u,\eta,N) = \{(h^u(z), z) : z \in X^s(w,\eta)\}$$

generate second countable, locally compact and Hausdorff topologies on $G^s(Q)$ and $G^u(P)$, respectively, for which $G^s(Q)$ and $G^u(P)$ are étale groupoids.

Combing the stable and unstable holonomy maps gives the étale topology on G^h. Given $(v,w) \in G^h$, we can find big enough $N \in \mathbb{N}$ and small enough $\eta' > 0$ for which both holonomy maps h^s and h^u are defined, and then some $0 < \eta \leq \eta'$ so that, the map $h : B(w,\eta) \to B(v, \varepsilon_X/2)$ given by

$$h(z) = [h^s([z,w]), h^u([w,z])], \tag{5.1.11}$$

is a well-defined local homeomorphism sending w to v. Similarly, the following holds.

Theorem 5.1.2 ([108])**.** *The collection of sets*

$$V^h(v,w,h,\eta,N) = \{(h(z), z) : z \in B(w,\eta)\}$$

generates a second countable, locally compact and Hausdorff topology on G^h for which G^h is an étale groupoid.

It is not hard to see that all these étale topologies induce étale topologies on the semi-direct products by $\mathbb{Z}$. From now on we will refer to $G^s(Q)$ and $G^u(P)$ as the stable and unstable groupoids of the Smale space (X,φ), regardless of the choice of Q and P. Our convention is well-defined up to isomorphism of the corresponding C^*-algebras. We discuss this in the next subsection. For the same reason, the groupoids $G^s(Q)\rtimes_\varphi\mathbb{Z}$ and $G^u(P)\rtimes_\varphi\mathbb{Z}$ will be called the stable and unstable Ruelle groupoids.

5.1.2 C^*-algebras

For the purpose of this thesis it suffices to consider the C^*-algebras built from the étale groupoids of Smale spaces. We will try to give a short, but self-contained, exposition of their construction and their properties. For more details we refer the reader to [74,106,109]. For brevity, we will only construct the C^*-algebras of the stable (Ruelle) groupoids since the construction of the others is similar.

Let Q be a periodic orbit in (X,φ) and consider the stable groupoid $G^s(Q)$. Let $C_c(G^s(Q))$ denote the complex vector space of continuous functions with compact support on $G^s(Q)$. We define a convolution and an involution on $C_c(G^s(Q))$ by

$$\begin{aligned}(f\cdot g)(v,w)&=\sum_{(v,z)\in G^s(Q)}f(v,z)g(z,w)\\ f^*(v,w)&=\overline{f(w,v)}\end{aligned}\tag{5.1.12}$$

for any $f,g\in C_c(G^s(Q))$. The convolution is well-defined since every function in $C_c(G^s(Q))$ can be written as a finite sum of functions, each one having support in an open set of the form $V^s(v,w,h^s,\eta,N)$, see Theorem 5.1.1. This can be easily proved using a partition of unity argument.

To complete the $*$-algebra $C_c(G^s(Q))$ into a C^*-algebra, for every $w\in X^u(Q)$, we consider the representation $\rho_w:C_c(G^s(Q))\to\mathcal{B}(\ell^2(X^s(w)\cap X^u(Q)))$ given by

$$\rho_w(f)\xi(v)=\sum_{(v,z)\in G^s(Q)}f(v,z)\xi(z).\tag{5.1.13}$$

Then, we define the reduced C^*-algebra $C^*_r(G^s(Q))$ as the completion of

$$\bigoplus_{w\in X^u(Q)}\rho_w(C_c(G^s(Q)))\subset\bigoplus_{w\in X^u(Q)}\mathcal{B}(\ell^2(X^s(w)\cap X^u(Q))).\tag{5.1.14}$$

The representation

$$\bigoplus_{w\in X^u(Q)}\rho_w:C_c(G^s(Q))\to\bigoplus_{w\in X^u(Q)}\mathcal{B}(\ell^2(X^s(w)\cap X^u(Q)))\tag{5.1.15}$$

is called the *regular representation* and it is faithful due to [127, Prop. 3.3.3]. In fact, from [127, Prop. 3.3.3] we see that, any restriction of the regular representation over a dense subset of $X^u(Q)$ is a faithful representation of $C_c(G^s(Q))$. This allows us to find a more geometric picture of the reduced C^*-algebra.

Let P be (possibly) another periodic orbit in (X,φ) and denote the countable dense subset $X^s(P)\cap X^u(Q)$ of X by $X^h(P,Q)$. The *fundamental representation* ρ_s of $C_c(G^s(Q))$ on $\mathscr{H}=\ell^2(X^h(P,Q))$ is given, for $f\in C_c(G^s(Q))$, $\xi\in\mathscr{H}$, by

$$\rho_s(f)\xi(v)=\sum_{(v,z)\in G^s(Q)} f(v,z)\xi(z), \tag{5.1.16}$$

and is essentially the faithful representation $\bigoplus_{w\in X^h(P,Q)}\rho_w$. The *stable algebra* $\mathcal{S}(Q)$ is defined as the completion of $\rho_s(C_c(G^s(Q)))$ in $\mathcal{B}(\mathscr{H})$.

Remark 5.1.3. The C^*-algebras $C^*_r(G^s(Q))$ and $\mathcal{S}(Q)$ are isomorphic. In order to see this, first assume that (X,φ) is mixing. In [109] it is proved that $G^s(Q)$ is amenable and that $C^*_r(G^s(Q))$ is simple. As a result, the full C^*-algebra $C^*(G^s(Q))$ is also simple. Let $\pi:C_c(G^s(Q))\to\mathcal{B}(H)$ be an arbitrary faithful representation and let $C^*_\pi(G^s(Q))$ be the completion of $\pi(C_c(G^s(Q)))$ in $\mathcal{B}(H)$. The C^*-algebra $C^*_\pi(G^s(Q))$ is isomorphic to a quotient of $C^*(G^s(Q))$, and from amenability and simplicity we obtain that $C^*_r(G^s(Q))\cong C^*_\pi(G^s(Q))$. This result can be extended to the case where the Smale space is irreducible, using Smale's Decomposition Theorem and [83, Section 2.5], and it should be clear that it particularly holds for the fundamental representation ρ_s.

In many instances we can consider $P=Q$. However, later on it will be important to choose $P\neq Q$, so that $X^h(P,Q)$ contains no periodic points. We will usually suppress the notation of ρ_s and instead of writing $\rho_s(a)\xi$, for $a\in C_c(G^s(Q))$, $\xi\in\mathscr{H}$, we will simply write $a\xi$.

Lemma 5.1.4 ([74, Lemma 3.3])**.** *Suppose that $V^s(v,w,h^s,\eta,N)$ is an open set as in Theorem 5.1.1, and $a\in C_c(G^s(Q))$ with supp$(a)\subset V^s(v,w,h^s,\eta,N)$. Then for every $x\in X^h(P,Q)$ we have*

$$a\delta_x=a(h^s(x),x)\delta_{h^s(x)},$$

if $x\in X^u(w,\eta)$, and $a\delta_x=0$, otherwise.

We now move on to construct the stable Ruelle algebra which is given as a crossed product of $\mathcal{S}(Q)$ by the integers. The homeomorphism φ induces the automorphism $\Phi=\varphi\times\varphi$ on $G^s(Q)$, which yields the automorphism α_s of $C_c(G^s(Q))$ given by

$$\alpha_s(f)=f\circ\Phi^{-1}. \tag{5.1.17}$$

By continuity, α_s extends on $\mathcal{S}(Q)$. For $a\in C_c(G^s(Q))$ with $\operatorname{supp}(a)\subset V^s(v,w,h^s,\eta,N)$ and $x\in X^h(P,Q)$ such that $h^s(\varphi^{-1}(x))$ is defined, we have

$$\alpha_s(a)\delta_x=a(h^s\circ\varphi^{-1}(x),\varphi^{-1}(x))\delta_{\varphi\circ h^s\circ\varphi^{-1}(x)}. \tag{5.1.18}$$

Moreover, the homeomorphism φ induces the unitary u on $\mathscr{H}$ given by $u\delta_x = \delta_{\varphi(x)}$. Then, in [133, Lemma 3.3.1] it is proved that α_s is inner, meaning that $\alpha_s(a) = uau^*$, for every $a \in \mathcal{S}(Q)$, and hence we can form the crossed product $\mathcal{S}(Q) \rtimes_{\alpha_s} \mathbb{Z}$. The Ruelle algebras for $G^u(P)$ and G^h are constructed as above. One simply has to change the notation wherever is needed. We summarise the notation in the following definition.

Definition 5.1.5. The stable, unstable and homoclinic C^*-algebras will be denoted by $\mathcal{S}(Q), \mathcal{U}(P)$ and $\mathcal{H}$, respectively. Also, the stable, unstable and homoclinic Ruelle algebras are

$$\mathcal{R}^s(Q) = \mathcal{S}(Q) \rtimes_{\alpha_s} \mathbb{Z},$$
$$\mathcal{R}^u(P) = \mathcal{U}(P) \rtimes_{\alpha_u} \mathbb{Z},$$
$$\mathcal{R}^h = \mathcal{H} \rtimes_{\alpha_h} \mathbb{Z}.$$

Remark 5.1.6. Following [135], the Ruelle algebra $\mathcal{R}^s(Q)$ encodes the dynamical system $(\mathcal{S}(Q), \mathbb{Z}, \alpha_s)$. It is defined by first considering the complex $*$-algebra $C_c(\mathbb{Z}, \mathcal{S}(Q))$ equipped with the α_s-twisted convolution and involution

$$f \cdot_{\alpha_s} g(n) = \sum_{m \in \mathbb{Z}} f(m)\alpha_s^m(g(n-m))$$
$$f^*(n) = \alpha_s(f(-n)^*).$$

The $*$-algebra $C_c(\mathbb{Z}, \mathcal{S}(Q))$ can be represented by covariant representations (π, U), that is, representations $\pi : \mathcal{S}(Q) \to \mathcal{B}(H)$ and unitary representations $U : \mathbb{Z} \to \mathcal{B}(H)$ such that $\pi(\alpha_s(a)) = U\pi(a)U^*$, for every $a \in \mathcal{S}(Q)$. More precisely, every covariant representation (π, U) yields the representation $\pi \rtimes U : C_c(\mathbb{Z}, \mathcal{S}(Q)) \to \mathcal{B}(H)$ given by

$$\pi \rtimes U(f) = \sum_{n \in \mathbb{Z}} \pi(f(n))U(n),$$

which is called the integrated form of (π, U). Then, the Ruelle algebra $\mathcal{R}^s(Q)$ is the completion of $C_c(\mathbb{Z}, \mathcal{S}(Q))$ with respect to the norm

$$\|f\| = \sup_{(\pi,U)} \|\pi \rtimes U(f)\|,$$

where the supremum is taken over all covariant representations (π, U) of $(\mathcal{S}(Q), \mathbb{Z}, \alpha_s)$. We consider $C_c(\mathbb{Z}, \mathcal{S}(Q))$ as a $*$-subalgebra of $\mathcal{R}^s(Q)$. Also, for every covariant representation of $(\mathcal{S}(Q), \mathbb{Z}, \alpha_s)$, its integrated form produces a representation of $\mathcal{R}^s(Q)$ [135, Prop. 2.39].

Moreover, an element $f \in C_c(\mathbb{Z}, \mathcal{S}(Q))$ is often written as $\sum_{n \in \mathbb{Z}} f(n)n$. Later on, it will be (notationally) convenient to abuse this notation a bit and instead write $\sum_{n \in \mathbb{Z}} f(n)u^n$, which corresponds only to the integrated form $\mathrm{id} \rtimes U$, where $U(1) = u$. Also, when proving results about $\mathcal{R}^s(Q)$ we will often prove them on generators of the form au^n, where

$a \in \mathcal{S}(Q)$, and the results will follow from linearity and continuity. Finally, if A is an α_s-invariant $*$-subalgebra of $\mathcal{S}(Q)$, that is $\alpha_s(A) = A$, then $C_c(\mathbb{Z}, A)$ is also a $*$-subalgebra of $C_c(\mathbb{Z}, \mathcal{S}(Q))$. With the aforementioned notational convention, the algebra $C_c(\mathbb{Z}, A)$ will often be denoted by $A \rtimes_{\alpha_s,\mathrm{alg}} \mathbb{Z}$, that is, finite sums of the form $\sum_{n\in\mathbb{Z}} f(n)u^n$, where $f(n) \in A$.

As we mentioned earlier, for any choice of Q and P, the stable and unstable (Ruelle) algebras are strongly Morita equivalent to the ones in [108]. Since they are also C^*-stable (see [36, Theorem A.2] and [68, Corollary 4.5]), it follows that, for any two periodic orbits Q and Q', we have $\mathcal{S}(Q) \cong \mathcal{S}(Q')$ and $\mathcal{R}^s(Q) \cong \mathcal{R}^s(Q')$. The same holds for the unstable (Ruelle) algebras. Nevertheless, we will keep the notation Q and P to indicate that these algebras are represented on $\mathscr{H} = \ell^2(X^h(P,Q))$.

It is worthwhile mentioning that all the algebras of Definition 5.1.5 possess remarkable properties that make them fit into Elliott's classification program of simple, separable, nuclear C^*-algebras, see [44,114]. Let us present a few of them, which are additional to the C^*-stability of the stable and unstable (Ruelle) algebras.

First of all, they are all separable, nuclear and satisfy the UCT [109]. The homoclinic algebra $\mathcal{H}$ is unital, quasidiagonal [34, Corollary 4.4] and has finite nuclear dimension, see [34, Corollary 3.8]. Similarly, the C^*-algebras $\mathcal{S}(Q), \mathcal{U}(P)$ have finite nuclear dimension [34, Corollary 3.8]. Recently, it was proved that they are quasidiagonal [32]. The main idea of the proof was to show that $\mathcal{S}(Q)$ and $\mathcal{U}(P)$ have non-zero projections. Then, using the following Theorem, which we also need for the sequel, and the fact that quasidiagonality is preserved under tensoring with the compacts and passing to subalgebras, the result follows.

Theorem 5.1.7 ([108, Theorem 3.1]). *The C^*-algebras $\mathcal{H}$ and $\mathcal{S}(Q) \otimes \mathcal{U}(P)$ are strongly Morita equivalent.*

If (X, φ) is mixing, the algebras $\mathcal{S}(Q), \mathcal{U}(P)$ and $\mathcal{H}$ are simple and have unique traces given by integrating against the Bowen measure, see [108, Theorem 3.3]. The traces on $\mathcal{S}(Q), \mathcal{U}(P)$ are unbounded while the trace on $\mathcal{H}$ is a state.

Moving now to Ruelle algebras, the properties of $\mathcal{R}^h$ are similar. More precisely, it is quasidiagonal, with a unique tracial state, and of finite nuclear dimension [35, Theorem 6.2]. However, the other Ruelle algebras are very different. While $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ are $\mathcal{Z}$-stable (see [35, Corollary 6.4]), they are simple and purely infinite, see [109]. As a result, they can be classified up to isomorphism by their K-theory, see [103]. Moreover, $\mathcal{R}^s(Q)$ is Spanier-Whitehead K-dual to $\mathcal{R}^u(P)$ [74]. It is interesting to note that duality does not seem to hold, in general, for $\mathcal{S}(Q)$ and $\mathcal{U}(P)$. Before we give a reason for the latter fact, let us present some prototype examples that come from zero, one and two-dimensional Smale spaces.

Example 5.1.8. *For a topological Markov chain* (Σ_A, σ_A) *(see Subsection 3.1.2), its stable, unstable and homoclinic algebras are approximately finite dimensional [106, Section 8.5]. Moreover, the stable and unstable Ruelle algebras are* C^**-stably isomorphic to the Cuntz-Krieger algebras* $\mathcal{O}_A$ *and* $\mathcal{O}_{A^t}$*, respectively, see [31, Theorem 3.8]. This is why Ruelle algebras are thought as higher dimensional analogues of Cuntz-Krieger algebras.*

Example 5.1.9. *Another example comes from the dyadic solenoid* $(\widehat{\mathbb{T}}, \widehat{g})$ *which is the stationary inverse limit associated to the map* $g: \mathbb{T} \to \mathbb{T}$ *given by* $g(z) = z^2$*, see Subsection (3.1.4). The projection on the* 0*-th coordinate* $p: \widehat{\mathbb{T}} \to \mathbb{T}$ *defines a fibre bundle, with fibres homeomorphic to the Cantor space* Σ*, and commutes with the dynamics in the sense that* $p \circ \widehat{g} = g \circ p$*. Let* $D = \{e^{2\pi i k 2^{-n}} : k \in \mathbb{Z}, n \in \mathbb{N}\} \subset \mathbb{T}$ *be the dyadic roots of unity. Then, one can immediately see that two points* $x, y \in \widehat{\mathbb{T}}$ *are stably equivalent if and only if* $p(x) = p(y)a$*, for some* $a \in D$*. Consequently, we have that*

$$C_r^*(G^s) \cong (C(\mathbb{T}) \otimes \mathcal{K}(L^2(\Sigma))) \rtimes D \cong (C(\mathbb{T}) \rtimes D) \otimes \mathcal{K}(L^2(\Sigma)).$$

The unstable algebra is more interesting. First, one observes that there is a flow F *on* $\widehat{\mathbb{T}}$ *defined by* $p \circ F_t(x) = e^{2\pi i t} p(x)$*. Then, since the orbits of the flow are the global unstable sets of* $(\widehat{\mathbb{T}}, \widehat{g})$*, it follows that* $C_r^*(G^u) \cong C(\widehat{\mathbb{T}}) \rtimes_F \mathbb{R}$*. Further, the flow has a natural cross-section* $p^{-1}(1) \cong \Sigma$*, and the first return map* F_1 *is the so-called* 2^∞*-odometer. Then,* $C(\widehat{\mathbb{T}}) \rtimes_F \mathbb{R}$ *is strongly Morita equivalent to* $C(\Sigma) \rtimes_{F_1} \mathbb{Z}$ *(see [138, Lemma 3.3]), which by Fourier transform, is isomorphic to* $C(\mathbb{T}) \rtimes D$*. To summarise, for any choice of* Q *and* P*, it holds that*

$$\mathcal{S}(Q) \cong \mathcal{U}(P) \cong C(\widehat{\mathbb{T}}) \rtimes_F \mathbb{R}. \tag{5.1.19}$$

Example 5.1.10. *The stable and unstable* C^**-algebras of hyperbolic toral automorphisms are* C^**-stably isomorphic to irrational rotation algebras [29, Section 6]. Specifically, for the hyperbolic toral automorphism on* $\mathbb{T}^2$*, induced by the matrix*

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

the unstable groupoid G^u *is exactly the Kronecker foliation of the differential equation* $dx = \theta dy$*, where* $\theta = (1+\sqrt{5})/2$*. This is because* $(\theta, 1)$ *is the eigenvector of the eigenvalue* $\theta^2 > 1$*. Consequently, for any choice of* Q *and* P*, the unstable algebra* $\mathcal{U}(P)$ *is* C^**-stably isomorphic to the irrational rotation algebra* $\mathcal{A}_\theta$*. Similarly, the stable algebra* $\mathcal{S}(Q)$ *is* C^**-stably isomorphic to* $\mathcal{A}_{-\theta^{-1}}$*. Note that here* $\mathcal{S}(Q) \cong \mathcal{U}(P)$*.*

Returning to the duality issue, a Spanier-Whitehead duality between $\mathcal{S}(Q)$ and $\mathcal{U}(P)$ would imply that the K-theory of any of the two algebras is isomorphic to the K-homology of the other, maybe with a degree shift. However, for the stable and unstable algebras of the full 2-shift this is not possible. More precisely, both algebras are C^*-stably isomorphic to the CAR-algebra A, and it is well-known that $\mathrm{K}_0(A) \cong \mathbb{Z}[1/2]$, $\mathrm{K}_1(A) = 0$, while $\mathrm{K}^0(A) = 0$ and $\mathrm{K}^1(A) = \widehat{\mathbb{Z}}_2/\mathbb{Z}$, where $\widehat{\mathbb{Z}}_2$ is the uncountable group of 2-adic numbers, see [63]. Similarly, we cannot have duality between the stable and unstable algebras of the 2^∞-solenoid. In fact, if they were dual, each one would be a Poincaré duality algebra. However, from the isomorphism (5.1.19), Connes-Thom isomorphism [14, Section 19.3], and the calculations in [138, Example 4.9] and [75, Proposition 6.6], we have that

$$\begin{aligned}
\mathrm{K}_0(S(Q)) &\cong \mathrm{K}_1(C(\widehat{\mathbb{T}})) \cong \mathbb{Z}[1/2] \\
\mathrm{K}_1(S(Q)) &\cong \mathrm{K}_0(C(\widehat{\mathbb{T}})) \cong \mathbb{Z} \\
\mathrm{K}^0(S(Q)) &\cong \mathrm{K}^1(C(\widehat{\mathbb{T}})) \cong \widehat{\mathbb{Z}}_2/\mathbb{Z} \\
\mathrm{K}^1(S(Q)) &\cong \mathrm{K}^0(C(\widehat{\mathbb{T}})) \cong 0.
\end{aligned} \tag{5.1.20}$$

Nevertheless, for hyperbolic toral automorphisms, the algebras $\mathcal{S}(Q)$ and $\mathcal{U}(P)$ are Spanier-Whitehead K-dual, and in fact, each one is a Poincaré duality algebra [28, Chapter VI]. It would be interesting to know for which Smale spaces the stable and unstable algebras are Spanier-Whitehead K-dual.

5.2 KPW-extension class and the K-duality of Ruelle algebras

We begin by stating the main result of Kaminker, Putnam and Whittaker on Spanier-Whitehead K-duality of Ruelle algebras. Then, we give a detailed exposition since it will be heavily used in the sequel. Let (X, φ) be an irreducible Smale space with periodic orbits Q and P. For a reason that will soon be clarified, we assume that $Q \cap P = \varnothing$, and hence $X^h(P, Q)$ has no periodic points.

Theorem 5.2.1 ([74]). *For every irreducible Smale space with periodic orbits Q and P, the Ruelle algebras $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ are Spanier-Whitehead* K*-dual. Moreover, if the* K*-theory groups of $\mathcal{S}(Q)$ or $\mathcal{U}(P)$ have finite rank, then $\mathcal{R}^s(Q) \cong \mathcal{R}^u(P)$ and consequently, both Ruelle algebras are Poincaré duality algebras.*

Their work, philosophically, builds on a previous result of Kaminker and Putnam [73] who proved that, the Cuntz-Krieger algebra $\mathcal{O}_A$ is Spanier-Whitehead K-dual to $\mathcal{O}_{A^t}$, if A is an irreducible matrix. However, there are considerable differences between the two approaches. The main reason is that, for the Cuntz-Krieger algebras, one can use the fact that they are quotients of Toeplitz algebras that admit representations on Fock spaces, while for general Ruelle algebras this is not true.

5.2.1 K-theory duality class

We now describe the K-theory duality class $\widehat{\Delta} \in \mathrm{KK}_1(\mathbb{C}, \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P))$. The main idea is to construct a $*$-homomorphism $\widehat{\delta} : C_0(\mathbb{R}) \to \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$ whose class $[\widehat{\delta}]$ in $\mathrm{KK}_0(C_0(\mathbb{R}), \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P))$ corresponds to the desired $\widehat{\Delta}$, under Bott periodicity. The canonical way to construct $\widehat{\delta}$ is by first defining a partial isometry $W \in \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$ such that $W^*W = WW^*$. In our case, the partial isometry W is the twist of a projection in $\mathcal{S}(Q) \otimes \mathcal{U}(P)$ whose class in $\mathrm{K}_0(\mathcal{S}(Q) \otimes \mathcal{U}(P))$ corresponds to the K-theory class of the unit in $\mathcal{H}$, via strong Morita equivalence, see Theorem 5.1.7. The Morita equivalence bimodule is explicitly described in [32, Section 2.6]. Then, forming the $*$-homomorphism $C(\mathbb{T}) \to \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$ given by $z \mapsto W$, and restricting it on $C_0(\mathbb{R}) \cong C^*(z-1)$, we obtain $\widehat{\delta}$ as the $*$-homomorphism that maps $z-1$ to $W - W^*W$.

The next concept provides the framework to construct the projection in $\mathcal{S}(Q) \otimes \mathcal{U}(P)$. First, recall the definition of $0 < \varepsilon'_X \leq \varepsilon_X/2$ from (3.1.5).

Definition 5.2.2 ([74, Def. 5.1])**.** Suppose that $\mathcal{F} = \{f_1, \ldots, f_K\}$ are continuous, non-negative functions on X and $\mathcal{G} = \{g_1, \ldots, g_K\}$ is a subset of $X^h(P, Q)$. For any $0 < \varepsilon \leq \varepsilon'_X$, we say that $(\mathcal{F}, \mathcal{G})$ is an *ε-partition of X* if

(1) the squares of the functions in $\mathcal{F}$ form a partition of unity in $C(X)$;

(2) the elements of $\mathcal{G}$ are distinct;

(3) the support of f_k is contained in $B(g_k, \varepsilon/2)$, for each $1 \leq k \leq K$.

A straightforward compactness argument gives the following lemma.

Lemma 5.2.3 ([74, Lemma 5.2])**.** *There exists an ε'_X-partition $(\mathcal{F}, \mathcal{G})$ of X such that*

$$(\mathcal{F} \circ \varphi^{-1}, \varphi(\mathcal{G})) = (\{f_k \circ \varphi^{-1} : 1 \leq k \leq K\}, \{\varphi(g_k) : 1 \leq k \leq K\})$$

is also an ε'_X-partition of X. Moreover, $\mathcal{G}$ can be chosen so that $\mathcal{G} \cap \varphi(\mathcal{G}) = \varnothing$.

For $0 < \varepsilon \leq \varepsilon'_X$, let $(\mathcal{F}, \mathcal{G})$ be an ε-partition and define the function $p_{\mathcal{G}}$ on the groupoid $G^s(Q) \times G^u(P)$ by setting, for $(x, x') \in G^s(Q)$ and $(y, y') \in G^u(P)$,

$$p_{\mathcal{G}}((x, x'), (y, y')) = f_i([x, y]) f_j([x', y']), \tag{5.2.1}$$

whenever $x \in X^u(g_i, \varepsilon)$, $y \in X^s(g_i, \varepsilon)$, $x' \in X^u(g_j, \varepsilon)$, $y' \in X^s(g_j, \varepsilon)$ and $[x, y] = [x', y']$, and to be zero otherwise. This is well-defined because, if such a pair i, j exists, for some $(x, x'), (y, y')$, then it is unique since $g_i = [y, x]$ and $g_j = [y', x']$.

The next lemma describes $p_{\mathcal{G}}$ as a projection on $\mathscr{H} \otimes \mathscr{H}$, where $\mathscr{H} = \ell^2(X^h(P,Q))$. We will make the *standard convention* that the bracket map returns the empty set for points x, y with $d(x,y) > \varepsilon_X$ and that, the Dirac delta function on the empty set returns zero. Also, any function on the empty set is zero. Recall the unitary $u \in \mathcal{B}(\mathscr{H})$ given by $u\delta_x = \delta_{\varphi(x)}$. The next lemma follows from Lemma 5.3, Lemma 5.4 and Lemma 5.5 of [74].

Lemma 5.2.4 ([74]). *For $0 < \varepsilon \leq \varepsilon'_X$, let $(\mathcal{F}, \mathcal{G})$ be an ε-partition. Then it holds that,*

(1) $p_{\mathcal{G}} \in \mathcal{S}(Q) \otimes \mathcal{U}(P)$;

(2) for $x, z \in X^h(P,Q)$,

$$p_{\mathcal{G}}(\delta_x \otimes \delta_z) = f_k([x,z]) \sum_{i=1}^{K} f_i([x,z]) \delta_{[x,g_i]} \otimes \delta_{[g_i,z]},$$

if there is some $1 \leq k \leq K$ such that, $x \in X^u(g_k, \varepsilon)$, $z \in X^s(g_k, \varepsilon)$, and is zero if there is no such k. (If there is such k then it is unique);

(3) $p_{\mathcal{G}}$ is a projection.

If ε is chosen small enough, so that $(\mathcal{F} \circ \varphi^{-1}, \varphi(\mathcal{G}))$ is an ε'_X-partition, then

(4) $(u \otimes u) p_{\mathcal{G}} (u^* \otimes u^*) = p_{\varphi(\mathcal{G})}$.

In order to create the partial isometry $W \in \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$ with $W^*W = WW^*$, we proceed as follows. Using Lemma 5.2.3 we can find an ε'_X-partition $(\mathcal{F}, \mathcal{G})$ such that, $(\mathcal{F} \circ \varphi^{-1}, \varphi(\mathcal{G}))$ is also an ε'_X-partition and $\mathcal{G} \cap \varphi(\mathcal{G}) = \varnothing$. For $0 \leq s \leq 1$, consider

$$\mathcal{F}_s = \{(1-s)^{\frac{1}{2}} f_1, \ldots, (1-s)^{\frac{1}{2}} f_K, s^{\frac{1}{2}} f_1 \circ \varphi^{-1}, \ldots, s^{\frac{1}{2}} f_K \circ \varphi^{-1}\}, \tag{5.2.2}$$

and the set of points

$$\mathcal{G}_s = \{g_1, \ldots, g_K, \varphi(g_1), \ldots, \varphi(g_K)\}. \tag{5.2.3}$$

Each pair $(\mathcal{F}_s, \mathcal{G}_s)$ is an ε'_X-partition that gives rise to the projection $p_{\mathcal{G}_s}$. Then from Lemma 5.2.4 we have that

(i) $p_{\mathcal{G}_s}$ is a path of projections in $\mathcal{S}(Q) \otimes \mathcal{U}(P)$;

(ii) $p_{\mathcal{G}_0} = p_{\mathcal{G}}$;

(iii) $p_{\mathcal{G}_1} = p_{\varphi(\mathcal{G})}$.

As a result, $p_{\mathcal{G}}$ and $p_{\varphi(\mathcal{G})}$ are homotopic, and hence, there is a partial isometry w in $\mathcal{S}(Q) \otimes \mathcal{U}(P)$ such that $w^*w = p_{\mathcal{G}}$ and $ww^* = p_{\varphi(\mathcal{G})}$, see [131, Proposition 5.2.10]. Let now $W = (u \otimes u) p_{\mathcal{G}} w^* \in \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$, and from Lemma 5.2.4 we get $W^*W = WW^* = p_{\varphi(\mathcal{G})}$.

Definition 5.2.5 ([74, Def. 5.6]). The K-*theory duality class* $\widehat{\Delta} \in \mathrm{KK}_1(\mathbb{C}, \mathcal{R}^s(Q)\otimes\mathcal{R}^u(P))$ is defined as $[\beta] \otimes_{C_0(\mathbb{R})} [\widehat{\delta}]$, where $[\beta] \in \mathrm{KK}_1(\mathbb{C}, C_0(\mathbb{R}))$ is the Bott-class and $[\widehat{\delta}]$ in $\mathrm{KK}(C_0(\mathbb{R}), \mathcal{R}^s(Q)\otimes\mathcal{R}^u(P))$ is given by the $*$-homomorphism $\widehat{\delta} : C_0(\mathbb{R}) \to \mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)$,

$$\widehat{\delta}(z-1) = W - W^*W,$$

where $W = (u\otimes u)p_{\mathcal{G}}w^*$.

5.2.2 KPW-extension class

We now present the K-homology duality class which, together with the K-theory duality class $\widehat{\Delta}$ of Subsection 5.2.1, gives Theorem 5.2.1 of Kaminker, Putnam and Whittaker. This class is actually constructed as an extension class $[\tau_\Delta] \in \mathrm{Ext}(\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P))$ which, due to nuclearity, can be realised by an abstractly defined $\Delta \in \mathrm{KK}_1(\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P), \mathbb{C})$. A central aim of this thesis is about finding a concrete Fredholm module representative of Δ. This is studied in Sections 7.1 and 7.2.

As we saw in Subsection 5.2.1, the construction of the K-theory duality class makes use of the bracket map on Smale spaces. The bracket map axiomatises transversality, and it is not enough to construct the extension τ_Δ. One also needs to use the expanding and contracting nature of the dynamics. The main strategy is to find two representations of $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ which commute modulo compacts, and then, their product will give the extension τ_Δ. This requires a series of lemmas.

Recall, the notation of the open sets of Theorem 5.1.1 that define the étale topology on $G^s(Q)$ and $G^u(P)$. A first consequence of transversality is the following lemma. Its proof is straightforward, and therefore, is omitted.

Lemma 5.2.6 ([74, Lemma 6.1]). *For every* $a \in C_c(G^s(Q))$, $b \in C_c(G^u(P))$ *with supports* $\mathrm{supp}(a) \subset V^s(v,w,h^s,\eta,N)$ *and* $\mathrm{supp}(b) \subset V^u(v',w',h^u,\eta',N')$, *it holds that* $\mathrm{rank}(ab)$ *and* $\mathrm{rank}(ba)$ *are at most one. Consequently, for every* $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$ *we have that the products* $ab, ba \in \mathcal{K}(\mathscr{H})$.

Using the expanding and contracting nature of the dynamics, together with the fact that $Q\cap P = \varnothing$, we have the following. We include the proof for completeness.

Lemma 5.2.7 ([74, Lemma 6.2]). *For every* $a \in C_c(G^s(Q))$, $b \in C_c(G^u(P))$ *with supports* $\mathrm{supp}(a) \subset V^s(v,w,h^s,\eta,N)$ *and* $\mathrm{supp}(b) \subset V^u(v',w',h^u,\eta',N')$, *there is* $M \in \mathbb{N}$ *such that,* $\alpha_s^{-n}(a)b = b\alpha_s^{-n}(a) = 0$, *for* $n \geq M$. *As a result, for every* $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$ *we have that*

$$\lim_{n\to+\infty} \alpha_s^{-n}(a)b = 0 \quad \text{and} \quad \lim_{n\to+\infty} b\alpha_s^{-n}(a) = 0.$$

Proof. Following the notation of the statement, we will show that there is some $M \in \mathbb{N}$ such that $\alpha_s^{-n}(a)b = 0$, for every $n \geq M$. For each $n \in \mathbb{N}$, we have that

$$\alpha_s^{-n}(a)b\delta_x = a(h^s \circ \varphi^n \circ h^u(x), \varphi^n \circ h^u(x))b(h^u(x), x)\delta_{\varphi^{-n} \circ h^s \circ \varphi^n \circ h^u(x)},$$

if $x \in X^s(w', \eta')$ and $h^u(x) \in \varphi^{-n}(X^u(w, \eta))$, and zero otherwise. So, in the case we have $\alpha_s^{-n}(a)b\delta_x \neq 0$, in particular, it holds that

$$h^u(x) \in X^u(\varphi^{-n}(w), \lambda_X^{-n}\eta) \cap X^s(v', \varepsilon_X/2).$$

Since $Q \cap X^s(P) = \varnothing$, $v' \in X^s(P)$, there is $\varepsilon > 0$ so that $X^u(Q, \varepsilon) \cap X^s(v', \varepsilon_X/2) = \varnothing$. Let $M \in \mathbb{N}$ so that $X^u(\varphi^{-n}(w), \lambda_X^{-n}\eta) \subset X^u(Q, \varepsilon)$, for every $n \geq M$. Then, for every $n \geq M$ we have that $\alpha_s^{-n}(a)b = 0$. Similarly, we can prove that $b\alpha_s^{-n}(a) = 0$, for n big enough. Finally, by considering linear combinations and norm limits we obtain the general result for $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$. □

The next lemma plays a prominent role in the construction of τ_Δ. In Subsection 6.2.2 we provide a refined version of it, and therefore, we postpone its proof until then.

Lemma 5.2.8 ([74, Lemma 6.3])**.** *For any $a \in \mathcal{S}(Q)$ and $b \in \mathcal{U}(P)$, we have*

$$\lim_{n \to +\infty} \|\alpha_s^n(a)b - b\alpha_s^n(a)\| = 0$$
$$\lim_{n \to +\infty} \|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| = 0.$$

Let us consider the inflated representation $\overline{\rho_s} : \mathcal{R}^s(Q) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ given by

$$\begin{aligned} a &\mapsto \bigoplus_{n \in \mathbb{Z}} \alpha_s^n(a) \\ u &\mapsto 1 \otimes B, \end{aligned} \tag{5.2.4}$$

where $a \in \mathcal{S}(Q)$, u is the unitary given by $u\delta_x = \delta_{\varphi(x)}$, and B is the left bilateral shift given by $B\delta_n = \delta_{n-1}$. Moreover, consider the representation $\overline{\rho_u} : \mathcal{R}^u(P) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ given by

$$\begin{aligned} b &\mapsto b \otimes 1 \\ u &\mapsto u \otimes B^*, \end{aligned} \tag{5.2.5}$$

for $b \in \mathcal{U}(P)$. It is straightforward to show that both representations are covariant, and from [100, Theorem 7.7.5] we have that they are faithful (but this also follows since the C^*-algebras are simple). Also, with this representation of $\mathcal{R}^u(P)$, the commutators $[\overline{\rho_s}(a), \overline{\rho_u}(u)]$, $[\overline{\rho_u}(b), \overline{\rho_s}(u)]$ and $[\overline{\rho_s}(u), \overline{\rho_u}(u)]$ are zero. Therefore, from Lemmas 5.2.6, 5.2.7 and 5.2.8, we obtain the following.

Lemma 5.2.9. *The C^*-algebras $\overline{\rho_s}(\mathcal{R}^s(Q))$ and $\overline{\rho_u}(\mathcal{R}^u(P))$ commute modulo compact operators on $\mathscr{H}\otimes\ell^2(\mathbb{Z})$.*

As a result, since the Ruelle algebras are nuclear, we can form the extension (or rather the Busby invariant) $\tau_\Delta:\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)\to\mathcal{Q}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ given on elementary tensors (and generators of each factor) by

$$\begin{aligned}\tau_\Delta(au^j\otimes bu^{j'}) &= (\overline{\rho_s}\cdot\overline{\rho_u})(au^j\otimes bu^{j'})+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\\ &= (\sum_{n\in\mathbb{Z}}\alpha_s^n(a)\otimes e_{n,n})(1\otimes B^j)(b\otimes 1)(u^{j'}\otimes B^{-j'})+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\\ &= \sum_{n\in\mathbb{Z}}\alpha_s^n(a)bu^{j'}\otimes e_{n,n}B^{j-j'}+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\\ &= \sum_{n\in\mathbb{Z}}\alpha_s^n(a)bu^{j'}\otimes e_{n,n+j-j'}+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\end{aligned}\tag{5.2.6}$$

where $a\in\mathcal{S}(Q)$, $b\in\mathcal{U}(P)$, $j,j'\in\mathbb{Z}$, and $e_{n,m}$ are matrix units. Since (X,φ) is irreducible, $\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)$ is simple. Therefore, in order to prove that τ_Δ is not the zero extension, it suffices to show that there is some $x\in\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)$ such that $\tau_\Delta(x)$ is not zero. This follows from the next lemma.

Lemma 5.2.10 ([133, Lemma 4.4.13])**.** *There are $a\in\mathcal{S}(Q)$ and $b\in\mathcal{U}(P)$ such that the operator $\overline{\rho_s}(a)\overline{\rho_u}(b)$ is not compact.*

The extension τ_Δ can also be constructed as follows. Let $\mathcal{E}$ be the C^*-algebra generated by $\overline{\rho_s}(\mathcal{R}^s(Q))$, $\overline{\rho_u}(\mathcal{R}^u(P))$ and $\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$. Note that neither $\overline{\rho_s}(\mathcal{R}^s(Q))$ or $\overline{\rho_u}(\mathcal{R}^u(P))$ contain non-zero compact operators. Consequently, from Lemma 5.2.9, it holds that

$$\mathcal{E}/\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\cong\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P).\tag{5.2.7}$$

Definition 5.2.11 ([74, Def. 6.6])**.** The *KPW-extension class* is represented by the *KPW-extension*

$$0\to\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\to\mathcal{E}\to\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)\to 0,$$

or equivalently, by its Busby invariant $\tau_\Delta:\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)\to\mathcal{Q}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ (5.2.6). The (abstract) corresponding class $\Delta\in\mathrm{KK}_1(\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P),\mathbb{C})$ will be called the K-*homology duality class.*

Chapter 6

Noncommutative dimension of Ruelle algebras

This chapter is the noncommutative analogue of Chapter 3. We begin by constructing metrics on the (étale) Smale space groupoids (Theorem 6.1.4), using the Alexandroff-Urysohn-Frink Metrisation Theorem. These groupoid metrics are compatible with the étale topologies of the groupoids and make the range and source maps locally bi-Lipschitz, and the Smale space homeomorphism (on the groupoid level) bi-Lipschitz. Then, we show how to construct Lipschitz $*$-algebras on general étale groupoids and use this (together with the aforementioned groupoid metrics) to construct dense Lipschitz $*$-subalgebras of the stable and unstable Ruelle algebras (Propositions 6.2.6 and 6.2.7). Finally, we prove that these Lipschitz subalgebras can be enlarged to holomorphically stable $*$-subalgebras on which every C^*-algebraic extension of the Ruelle algebras, by the compacts, reduces to an algebraic extension by a Schatten p-ideal (Corollaries 6.2.19 and 6.2.21). The value $p > 0$ is related with the topological entropy and the λ-number of the Smale space, a topological invariant for Smale spaces built from self-similar metrics (Definition 6.1.16). In this last step we also make use of the K-duality of Kaminker, Putnam and Whittaker.

6.1 Metrisation of Smale space groupoids

Our aim is to construct metrics of dynamic nature on the stable and unstable groupoids of Smale spaces. This is an important step in studying the smoothness of Ruelle algebras, since these groupoid metrics will be designed to yield Lipschitz algebras on which the extensions of Ruelle algebras are uniformly smooth, as we will see in Section 6.2. An essential point in our metrisation approach is that, given a Smale space, its groupoids do not actually depend on the Smale space metric. This is studied in Subsection 6.1.1. In Subsection 6.1.2, we construct families of metrics for the stable and unstable groupoids, for which the groupoid automorphisms (5.1.18) and the groupoid inversion maps become bi-Lipschitz, and the groupoid range and source maps become locally bi-Lipschitz. For these metrics, we also give concrete distance estimates, see Proposition 6.1.15. Finally, in Subsection 6.1.4 we present an alternative family of metrics specifically constructed for the stable and unstable groupoids of topological Markov chains. They have all the properties of the general metrics of Subsection 6.1.2, but they are more tractable.

The so-called Birkhoff-Kakutani Theorem (see [82, Chapter 6]) is a metrisation tool for topological groups. It requires a countable neighbourhood basis of the unit element and, of course, the group to be Hausdorff. Therefore, on the groupoid level one could try and imitate the above theorem by considering neighbourhoods of the unit space of the groupoid. However, the Smale space groupoids are by no means close to being groups. Our metrisation tool is the quite powerful Alexandroff-Urysohn-Frink Theorem 6.1.6. Let G denote the stable or unstable groupoid. The main utility of Theorem 6.1.6 lies in building a tractable uniform structure (see [82, Chapter 6]) on G by considering countably many neighbourhoods of the diagonal of G in $G\times G$. The corresponding uniform structure should also generate the étale (Hausdorff) topology on G, thus giving a compatible metric on G.

Let (X,d,φ) be an irreducible Smale space with constants $\varepsilon_X>0$ and $\lambda_X>1$. Fix two periodic orbits Q and P and recall the basis sets of Theorem 5.1.1 that generate the étale topologies on the stable and unstable groupoids $G^s(Q)$ and $G^u(P)$.

6.1.1 Dependence of groupoids on Smale space metric

First we should investigate the algebraic and topological dependence of the groupoids $G^s(Q)$ and $G^u(P)$ on the Smale space metric d. Let $\mathrm{M}_d(X)$ be the collection of all metrics d' on X inducing the same topology with d. We introduce the collection of *hyperbolic metrics*

$$\mathrm{hM}_d(X,\varphi)=\{d'\in \mathrm{M}_d(X):(X,d',\varphi)\text{ is a Smale space}\}, \tag{6.1.1}$$

and its sub-collection of *self-similar hyperbolic metrics*

$$\mathrm{sM}_d(X,\varphi)=\{d'\in \mathrm{M}_d(X):(X,d',\varphi)\text{ is a self-similar Smale space}\}. \tag{6.1.2}$$

It always holds that $\mathrm{sM}_d(X,\varphi)\neq\varnothing$ (see Lemma 3.1.16) and in fact, if $d'\in\mathrm{sM}_d(X,\varphi)$, then for every $0<\alpha\leq 1$, the metric $(d')^{\alpha}\in\mathrm{sM}_d(X,\varphi)$.

Recall now that for $x\in X$, its global stable set is

$$X^s(x)=\{y\in X:\lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y))=0\}.$$

Since X is compact, for any $d_1,d_2\in\mathrm{M}_d(X)$, the identity map $(X,d_1)\to(X,d_2)$ is uniformly continuous. Consequently, the set $X^s(x)$ can be given in terms of any metric in $\mathrm{M}_d(X)$.

Of course, the same holds for the global unstable sets, and therefore, the stable and unstable (algebraic) groupoids $G^s(Q)$ and $G^u(P)$ can equivalently be given by any metric in $\mathrm{M}_d(X)$.

On the other hand, the dependence of the (topological) groupoids $G^s(Q)$ and $G^u(P)$ on d is a bit more subtle. First of all, the bases inducing the étale topologies, formed by the basis sets of Theorem 5.1.1, depend on the metric d and the fact that d is hyperbolic. More precisely, consider any $d' \in \mathrm{hM}_d(X,\varphi)$. Since $G^s(Q)$ and $G^u(P)$ are algebraically independent of the choice of metric in $\mathrm{hM}_d(X,\varphi)$, Theorem 5.1.1 can be applied to the Smale space (X,d',φ) and yield topological bases

$$\mathcal{B}^s(Q,d') \text{ and } \mathcal{B}^u(P,d') \tag{6.1.3}$$

for $G^s(Q)$ and $G^u(P)$, respectively. Denote by $\mathcal{T}^s(Q,d')$ and $\mathcal{T}^u(P,d')$ the corresponding étale topologies. If $d_1 \neq d_2 \in \mathrm{hM}_d(X,\varphi)$, even though $\mathcal{B}^s(Q,d_1)$ and $\mathcal{B}^u(P,d_1)$ might differ from $\mathcal{B}^s(Q,d_2)$ and $\mathcal{B}^u(P,d_2)$, we have the following natural result.

Proposition 6.1.1. *The algebraic structure of $G^s(Q)$ and $G^u(P)$ can be given by any metric in $M_d(X)$. Further, their étale topologies can be given by any metric in $hM_d(X,\varphi)$. Specifically, for any two metrics $d_1,d_2 \in hM_d(X,\varphi)$, it holds that $\mathcal{T}^s(Q,d_1) = \mathcal{T}^s(Q,d_2)$ and $\mathcal{T}^u(P,d_1) = \mathcal{T}^u(P,d_2)$.*

Proof. The part about the algebraic structure of the groupoids has already been proved. For the topological part we shall only consider $G^s(Q)$, since the proof for $G^u(P)$ is similar. Let $d_1,d_2 \in \mathrm{hM}_d(X,\varphi)$. In what follows we highlight the dependence on the metrics, wherever appropriate.

It is useful to first look at the topology of the unit space $X^u(Q)$. We begin by claiming that, for every $y \in X$ and $0 < \varepsilon_1 \leq \varepsilon_{X,d_1}$, there is some $0 < \varepsilon_2 \leq \varepsilon_{X,d_2}$ such that

$$X^u_{d_2}(y,\varepsilon_2) \subset X^u_{d_1}(y,\varepsilon_1). \tag{6.1.4}$$

Note that it suffices to check it for all $0 < \varepsilon_1 \leq \varepsilon'_{X,d_1}$, see (3.1.5) for the definition of $\varepsilon'_{X,d_1} \leq \varepsilon_{X,d_1}/2$. Fix such y and ε_1, and from [106, Lemma 4.1.5] we have

$$\begin{aligned} X^u_{d_1}(y,\varepsilon_1) &= \{z \in X : d_1(\varphi^{-n}(y),\varphi^{-n}(z)) < \varepsilon_1,\ n \geq 0\} \\ &= \bigcap_{n\geq 0} \varphi^n(B_{d_1}(\varphi^{-n}(y),\varepsilon_1)). \end{aligned}$$

Similarly, for every $0 < \varepsilon_2 \leq \varepsilon'_{X,d_2}$ we have $X^u_{d_2}(y,\varepsilon_2) = \bigcap_{n\geq 0} \varphi^n(B_{d_2}(\varphi^{-n}(y),\varepsilon_2))$. The claim follows since the identity map $(X,d_2) \to (X,d_1)$ is uniformly continuous, and hence we can find $\varepsilon_2 \in (0,\varepsilon'_{X,d_2})$ such that $B_{d_2}(z,\varepsilon_2) \subset B_{d_1}(z,\varepsilon_1)$, for every $z \in X$.

Let $x \in X$, and for every $d' \in \mathrm{hM}_d(X,\varphi)$, the collection of sets

$$\{X^u_{d'}(y,\varepsilon) : y \in X^u(x),\, 0 < \varepsilon \leq \varepsilon_{X,d'}\} \tag{6.1.5}$$

forms a basis of neighbourhood systems for a topology on $X^u(x)$. Due to (6.1.4) all these topologies coincide. In particular, the topology on $X^u(Q)$ does not depend on the choice of metric in $\mathrm{hM}_d(X,\varphi)$.

We now pass on the groupoid level. Let $(v,w) \in G^s(Q)$ and we aim to show that, for every $B_1 \in \mathcal{B}^s(Q,d_1)$ containing (v,w), there is $B_2 \in \mathcal{B}^s(Q,d_2)$ such that $(v,w) \in B_2 \subset B_1$. It is not hard to see that there is a basis set

$$V^s_{d_1}(v,w,h^s_{d_1},\eta_{d_1},N_{d_1}) \subset B_1,$$

which by definition contains (v,w). The goal is to find a smaller basis set

$$V^s_{d_2}(v,w,h^s_{d_2},\eta_{d_2},N_{d_2}) \subset V^s_{d_1}(v,w,h^s_{d_1},\eta_{d_1},N_{d_1}).$$

Since $w \in X^s(v)$, there is $N_{d_2} \in \mathbb{N}$ such that

$$\varphi^{N_{d_2}}(w) \in X^s_{d_2}(\varphi^{N_{d_2}}(v), \varepsilon'_{X,d_2}/2).$$

Then, we can find $\eta_{d_2} > 0$ and build the holonomy map $h^s_{d_2} : X^u_{d_2}(w,\eta_{d_2}) \to X^u_{d_2}(v,\varepsilon_{X,d_2}/2)$, thus obtaining the basis set $V^s_{d_2}(v,w,h^s_{d_2},\eta_{d_2},N_{d_2})$. From the inclusion (6.1.4), the fact that $h^s_{d_1}(X^u_{d_1}(w,\eta_{d_1}))$ is an open unstable neighbourhood of v, and the continuity of $h^s_{d_2}$ which maps w to v, we can choose $\eta_{d_2} > 0$ even smaller so that

$$X^u_{d_2}(w,\eta_{d_2}) \subset X^u_{d_1}(w,\eta_{d_1}) \;\text{ and }\; h^s_{d_2}(X^u_{d_2}(w,\eta_{d_2})) \subset h^s_{d_1}(X^u_{d_1}(w,\eta_{d_1})).$$

This completes the proof. □

Corollary 6.1.2. *The algebraic structure of $G^s(Q) \rtimes_\varphi \mathbb{Z}$, $G^u(P) \rtimes_\varphi \mathbb{Z}$, G^h and $G^h \rtimes_\varphi \mathbb{Z}$ can be given by any metric in $M_d(X)$. Moreover, their étale topologies can be given by any metric in $hM_d(X,\varphi)$.*

Proof. The étale topology on G^h is given by the basis sets of Theorem 5.1.2, which are built from stable and unstable holonomy maps. Therefore, following the proof of Proposition 6.1.1, we can see that its algebraic structure can be given by any metric in $M_d(X)$, and similarly, its topology by any metric in $\mathrm{hM}_d(X,\varphi)$. Let now G be any of the $G^s(Q)$, $G^u(P)$ and G^h, and consider the algebraic semi-direct product $G \rtimes_\varphi \mathbb{Z}$. Again, the algebraic structure of $G \rtimes_\varphi \mathbb{Z}$ is independent of the choice of metric in $\mathrm{M}_d(X)$. Equip now $\mathbb{Z}$ with the discrete topology and $G \times \mathbb{Z}$ with the product topology. The topology on $G \rtimes_\varphi \mathbb{Z}$ is the one transferred via the bijection $G \times \mathbb{Z} \to G \rtimes_\varphi \mathbb{Z}$, given by $(x,y,n) \mapsto (x,n,\varphi^n(y))$. The result follows. □

To summarise, the Smale space groupoids are independent of the choice of hyperbolic metric in $\mathrm{hM}_d(X,\varphi)$. As a result, one can freely choose a suitable hyperbolic metric to work with. The only subtle point is that such metric will yield specific bases for the étale topologies on the groupoids, and these bases are used to represent the Smale space C^*-algebras, see Subsection 5.1.2.

In Subsection 6.1.2 we show that, for every $d' \in \mathrm{hM}_d(X,\varphi)$, the corresponding basis $\mathcal{B}^s(Q,d')$ for $G^s(Q)$ yields a compatible metric $D_{s,d'}$ on $G^s(Q)$. These metrics are of dynamic nature and in fact, if d' is chosen to be self-similar, we can form the Lipschitz subalgebra of compactly supported functions

$$\mathrm{Lip}_c(G^s(Q),D_{s,d'}) \subset \mathcal{S}(Q).$$

Moreover, in this case, the automorphism $\varphi\times\varphi$ of $G^s(Q)$ becomes bi-Lipschitz, leading to a well-defined algebraic crossed product

$$\mathrm{Lip}_c(G^s(Q),D_{s,d'}) \rtimes_{\alpha_s,\mathrm{alg}} \mathbb{Z} \subset \mathcal{R}^s(Q),$$

see (5.1.18) for the automorphism α_s of $\mathcal{S}(Q)$. Similarly, for the unstable groupoid. All these facts are studied in Subsection 6.2.2. Consequently, it should be clear that self-similar hyperbolic metrics are the best behaved hyperbolic metrics.

Definition 6.1.3. Let $d' \in \mathrm{hM}_d(X,\varphi)$ and G be any of the aforementioned étale groupoids. The d'-*model* of G is the construction of G from the Smale space (X,d',φ).

6.1.2 Metrics and Lipschitz dynamics on Smale space groupoids

We are mainly interested in metrising $G^s(Q)$ and $G^u(P)$. Nonetheless, in a straightforward manner, these metrics induce metrics on $G^s(Q)\rtimes_\varphi\mathbb{Z}$, $G^u(P)\rtimes_\varphi\mathbb{Z}$, since the topologies on the latter groupoids are given by the product topologies on $G^s(Q)\times\mathbb{Z}$, $G^u(P)\times\mathbb{Z}$. Using analogous methods, one can construct compatible metrics on G^h and $G^h\rtimes_\varphi\mathbb{Z}$ that behave similarly with the metrics of Theorem 6.1.4. However, what follows is notationally demanding, and therefore, we avoid performing similar computations for G^h and $G^h\rtimes_\varphi\mathbb{Z}$, whose metric structure investigation is not needed in this thesis.

The groupoids $G^s(Q)$, $G^u(P)$ are second countable, locally compact and Hausdorff, and therefore, metrisable. The usual way to prove that such topological spaces are metrisable is by showing that their one-point compactifications are metrisable, as we see in [82, Chapter 4]. Unfortunately, the groupoid metrics constructed in this way are too abstract for our purpose. Instead, we use the powerful metrisation tool of Alexandroff, Urysohn and Frink [54]. The basic idea is to realise the (Hausdorff) topologies on $G^s(Q)$ and $G^u(P)$ as uniform topologies of uniform structures with countable bases, and use the fact that such uniform topologies are generated by metrics.

More precisely, for every $d' \in \mathrm{hM}_d(X,\varphi)$, using the étale topological bases $\mathcal{B}^s(Q,d')$ and $\mathcal{B}^u(P,d')$, one can build compatible metrics $D_{s,d'}$ and $D_{u,d'}$ for $G^s(Q)$ and $G^u(P)$. In particular, if $d' \in \mathrm{sM}_d(X,\varphi)$, we obtain the following nice result (Theorem 6.1.4), whose proof is achieved as a combination of Lemmas 6.1.10, 6.1.11, 6.1.12 and 6.1.14. Before stating the theorem, we should highlight a few basic facts and introduce some notation. For simplicity, we consider only $G^s(Q)$.

By $i : G^s(Q) \to G^s(Q)$ we denote the *inversion map*

$$i(x,y) = (y,x), \tag{6.1.6}$$

and by $r,s : G^s(Q) \to X^u(Q)$ we denote the *range* and *source maps*, defined as

$$r(x,y) = x, \quad s(x,y) = y. \tag{6.1.7}$$

In the sequel we will consider compatible metrics D, $\widetilde{D}$ on $G^s(Q)$, $X^u(Q)$, respectively (see (6.1.29), (6.1.33) and Lemma 6.1.14), for which r becomes locally bi-Lipschitz, that is, for every $a \in G^s(Q)$ there is $\ell_a > 0$ and $\Lambda_a \geq 1$ such that,

$$\Lambda_a^{-1} D(a,b) \leq \widetilde{D}(r(a),r(b)) \leq \Lambda_a D(a,b), \tag{6.1.8}$$

for every $b \in B_D(a,\ell_a)$. Similarly, for the map s. This should be no surprise, because for every $V \in \mathcal{B}(G^s(Q))$, where

$$\mathcal{B}(G^s(Q)) = \{W \subset G^s(Q) : W \in \mathcal{B}^s(Q,d'),\, d' \in \mathrm{hM}_{d'}(X,\varphi)\}, \tag{6.1.9}$$

it is not hard to show that the restrictions $r_V : V \to r(V)$ and $s_V : V \to s(V)$ are homeomorphisms, and therefore, the maps r,s are local homeomorphisms (note that $j(X^u(Q))$ is open is $G^s(Q)$ since $G^s(Q)$ is étale).

Theorem 6.1.4. *For every metric $d' \in sM_d(X,\varphi)$ there are compatible metrics $D_{s,d'}$ on $G^s(Q)$ and $\widetilde{D_{s,d'}}$ on $X^u(Q)$ so that, with respect to these metrics,*

(1) the groupoid automorphism $\Phi = \varphi \times \varphi : G^s(Q) \to G^s(Q)$ is bi-Lipschitz with

$$4^{-1} D_{s,d'}(a,b) \leq D_{s,d'}(\Phi(a),\Phi(b)) \leq 8 D_{s,d'}(a,b),$$

for every $a,b \in G^s(Q)$;

(2) the inversion map i is bi-Lipschitz, and for every basis set $V \in \mathcal{B}(G^s(Q))$, the restrictions r_V and s_V of the range and source maps are bi-Lipschitz. Specifically, the maps r,s are locally bi-Lipschitz.

Similarly, for every $d' \in sM_d(X,\varphi)$, there are compatible metrics $D_{u,d'}$ on $G^u(P)$ and $\widetilde{D_{u,d'}}$ on $X^s(P)$ for which, the automorphism $\Phi = \varphi \times \varphi : G^u(P) \to G^u(P)$ satisfies

$$4^{-1} D_{u,d'}(a,b) \leq D_{u,d'}(\Phi^{-1}(a), \Phi^{-1}(b)) \leq 8 D_{u,d'}(a,b),$$

for every $a,b \in G^u(P)$, the inversion map is bi-Lipschitz, and the restrictions of the range and source maps on basis sets in $\mathcal{B}(G^u(P))$ are also bi-Lipschitz.

Remark 6.1.5. Following Proposition 6.1.1, for every $d_1, d_2 \in \mathrm{hM}_d(X,\varphi)$, the d_1-model of $G^s(Q)$ is exactly the same with the d_2-model of $G^s(Q)$. They are just different ways to construct $G^s(Q)$. Nevertheless, for $d' \in \mathrm{hM}_d(X,\varphi)$, the metric $D_{s,d'}$ is better adapted to the d'-model of $G^s(Q)$.

There are several formulations of the Alexandroff-Urysohn-Frink Metrisation Theorem [82, Chapter 6]. In our case it is best to use the one in [120, Theorem 2.4.1]. Before stating it, if Z is a set, $\mathcal{U}$ is a cover of Z and $z \in Z$, let us denote the $\mathcal{U}$-*star* of z by

$$\mathrm{st}(z,\mathcal{U}) = \bigcup \{U \in \mathcal{U} : z \in U\}. \tag{6.1.10}$$

Theorem 6.1.6 (Alexandroff-Urysohn-Frink Metrisation Theorem)**.** *For a Hausdorff space Z the following are equivalent.*

(1) Z is metrisable.

(2) Z has a sequence of open covers $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that,

(a) for each $z \in Z$, the sequence $(\mathrm{st}(z,\mathcal{U}_n))_{n\in\mathbb{N}}$ is a neighbourhood basis of z;

(b) for every $U, U' \in \mathcal{U}_{n+1}$, if $U \cap U' \neq \varnothing$ then there is some $U'' \in \mathcal{U}_n$ such that $U \cup U' \subset U''$.

(3) Each $z \in Z$ has a neighbourhood basis $(V_n(z))_{n\in\mathbb{N}}$ of open sets satisfying the condition that, for every $z \in Z$ and $n \in \mathbb{N}$, there is some $j(z,n) \geq n$ so that, if $y \in Z$ with $V_{j(z,n)}(z) \cap V_{j(z,n)}(y) \neq \varnothing$, then $V_{j(z,n)}(y) \subset V_n(z)$.

Sketch of proof. The direction (1)⇒(3) is straightforward. Assume that (3) holds. For every $z \in Z$, set $k(z,1) = 1$, and for $n \geq 2$ inductively define

$$k(z,n) = \max\{n, j(z,i) : i = 1, \ldots, k(z,n-1)\}. \tag{6.1.11}$$

Then, for every $n \in \mathbb{N}$, let

$$U_n(z) = \bigcap_{i=1}^{k(z,n)} V_i(z), \tag{6.1.12}$$

and it can be shown that the covers $\mathcal{U}_n = \{U_n(z) : z \in Z\}$ form a sequence satisfying the properties of part (2).

Assume now that $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of open covers of Z satisfying the conditions of part (2). Then, it is possible to construct a metric that generates the topology on Z. Let $\mathcal{U}_0 = \{Z\}$ and for every $y, z \in Z$ define

$$\rho(y,z) = \inf\{2^{-n} : \text{there is } U \in \mathcal{U}_n \text{ such that } y, z \in U\}. \tag{6.1.13}$$

It is straightforward to show that ρ is a 2-quasimetric, that is, for every $x, y, z \in Z$ it holds

(i) $\rho(y,z) = 0$ if and only if $y = z$;

(ii) $\rho(y,z) = \rho(z,y)$;

(iii) $\rho(x,y) \leq 2\max\{\rho(x,z), \rho(z,y)\}$.

An important property which can be proved by induction is that

(iv) for $n \geq 3$ and for every $z_1, \dots, z_n \in Z$,

$$\rho(z_1, z_n) \leq 2(\rho(z_1,z_2) + \rho(z_{n-1},z_n)) + 4\sum_{i=2}^{n-2}\rho(z_i, z_{i+1}).$$

Using the quasimetric ρ we can define the metric $d_\rho : Z \times Z \to [0,1]$ given by

$$d_\rho(y,z) = \inf\{\sum_{i=1}^{n-1}\rho(z_i, z_{i+1}) : n \in \mathbb{N},\, z_i \in Z,\, z_1 = y,\, z_n = z\}. \tag{6.1.14}$$

Immediately, one can observe that d_ρ is symmetric and satisfies the triangle inequality. From (iv) we have that $\rho(y,z) \leq 4d_\rho(y,z)$, and hence, if $d_\rho(y,z) = 0$, then $y = z$. For every $z \in Z$, the metric d_ρ satisfies

$$\operatorname{st}(z, \mathcal{U}_{n+2}) \subset \overline{B}_{d_\rho}(z, 2^{-n-2}) \subset \operatorname{st}(z, \mathcal{U}_n). \tag{6.1.15}$$

Since for every $z \in Z$, the sequence $(\operatorname{st}(z,\mathcal{U}_n))_{n\in\mathbb{N}}$ is a neighbourhood basis of z, it follows that d_ρ generates the topology on Z. □

We aim to construct metrics on $G^s(Q)$ and $G^u(P)$ by building neighbourhood bases that satisfy condition (3) of Theorem 6.1.6. We consider only the stable groupoid $G^s(Q)$, since the exactly same method will work for the unstable one. To simplify the notation we will denote the elements of $G^s(Q)$ by $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ or $e = (e_1, e_2)$.

Let $d' \in \mathrm{hM}_d(X, \varphi)$ be an arbitrary metric and consider the d'-model of $G^s(Q)$. This means all topological basis sets are in $\mathcal{B}^s(Q, d')$, and all local stable and unstable sets, and the homeomorphism φ, are defined and behave with respect to d'. However, in order to keep the notation reasonably simple, we do not highlight the dependence on the metric d'. In particular, the Smale space constants will be again denoted by $\varepsilon_X > 0$ and $\lambda_X > 1$, instead of $\varepsilon_{X,d'}$ and $\lambda_{X,d'}$.

First of all, since φ is uniformly continuous on X, for every $n \geq 0$, we can consider $\eta_n \in (0, \varepsilon_X/2]$ to be the supremum amongst all $\eta \in (0, \varepsilon_X/2]$ such that $\varphi^n(X^u(z, 2\eta)) \subset \mathrm{cl}(X^u(\varphi^n(z), \varepsilon'_X/2))$, for every $z \in X$, where the closure is taken in $X^u(\varphi^n(z), \varepsilon_X)$. In particular,

$$\varphi^n(X^u(z, 2\eta_n)) \subset \mathrm{cl}(X^u(\varphi^n(z), \varepsilon'_X/2)) \tag{6.1.16}$$

for every $z \in X$. The sequence $(\eta_n)_{n\geq 0}$ is decreasing since, for every $z \in X$, we have $\varphi^{n+1}(X^u(z, 2\eta_{n+1})) \subset \mathrm{cl}(X^u(\varphi^{n+1}(z), \varepsilon'_X/2))$ and hence

$$\varphi^n(X^u(z, 2\eta_{n+1})) \subset \mathrm{cl}(X^u(\varphi^n(z), \lambda_X^{-1}\varepsilon'_X/2)),$$

meaning that $\eta_{n+1} \leq \eta_n$. Also, one can observe that $(\eta_n)_{n\geq 0}$ converges to zero.

Recall that the basis sets of the topology on $G^s(Q)$ are denoted by $V^s(a, h^s, \eta, N)$, where

(i) $N \in \mathbb{N}$ is big enough so that

$$\varphi^N(a_2) \in X^s(\varphi^N(a_1), \varepsilon'_X/2);$$

(ii) $\eta \in (0, \varepsilon_X/2]$ is small enough so that

$$\varphi^N(X^u(a_2, \eta)) \subset X^u(\varphi^N(a_2), \varepsilon'_X/2);$$

(iii) $h^s : X^u(a_2, \eta) \to X^u(a_1, \varepsilon_X/2)$ is given by

$$h^s(z) = \varphi^{-N}[\varphi^N(z), \varphi^N(a_1)].$$

For every $a \in G^s(Q)$ we define $N_a \geq 0$ to be the *first time* that

$$\varphi^{N_a}(a_2) \in X^s(\varphi^{N_a}(a_1), \varepsilon'_X/2), \tag{6.1.17}$$

and consider the sequences $(N_{a,n})_{n\geq 0}$ and $(\eta_{a,n})_{n\geq 0}$ given by

$$N_{a,n} = \max\{N_a, n\} \text{ and } \eta_{a,n} = \eta_{N_{a,n}} \tag{6.1.18}$$

Then, define the holonomy map $h^s_{a,n} : X^u(a_2, \eta_{a,n}) \to X^u(a_1, \varepsilon_X/2)$ given by

$$h^s_{a,n}(z) = \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(z), \varphi^{N_{a,n}}(a_1)], \tag{6.1.19}$$

and consider the basis set

$$V_n(a) = V^s(a, h^s_{a,n}, \eta_{a,n}, N_{a,n}). \tag{6.1.20}$$

The above condition (ii) can be relaxed by considering the closure on the right-hand side of the inclusion. Then, due to (6.1.16), each holonomy map $h^s_{a,n}$ can be extended on $X^u(a_2, 2\eta_{a,n})$.

Lemma 6.1.7. *For $a \in G^s(Q)$, the sequence of open sets $(V_n(a))_{n\geq 0}$ forms a decreasing neighbourhood basis of a.*

Proof. Let $a \in G^s(Q)$, we will show that if $b \in V_{n+1}(a)$, then $b \in V_n(a)$. Since $N_{a,n+1} \geq N_{a,n}$ it holds that $\eta_{a,n+1} \leq \eta_{a,n}$, hence $b_2 \in X^u(a_2, \eta_{a,n+1}) \subset X^u(a_2, \eta_{a,n})$, meaning that b_2 lies in the domain of the holonomy map $h^s_{a,n}$. The claim follows because $h^s_{a,n}(b_2) = b_1$, where $b_1 = h^s_{a,n+1}(b_2) = \varphi^{-N_{a,n+1}}[\varphi^{N_{a,n+1}}(b_2), \varphi^{N_{a,n+1}}(a_1)]$.

Let $V^s(c, h^s, \eta, N)$ be a basis set containing $a \in G^s(Q)$. Choose n big enough such that $N_{a,n} \geq N$ and $\eta_{a,n}$ is even smaller than η, so that $X^u(a_2, \eta_{a,n}) \subset X^u(c_2, \eta)$. Then, it holds that $V_n(a) \subset V^s(c, h^s, \eta, N)$. Indeed, let $b \in V_n(a)$ and we have that $b_2 \in X^u(a_2, \eta_{a,n})$, meaning that b_2 lies in the domain of h^s. It remains to show that $h^s(b_2) = b_1$, where $b_1 = h^s_{a,n}(b_2) = \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(b_2), \varphi^{N_{a,n}}(a_1)]$. For this we use the fact that

(i) $\varphi^N(a_1) \in X^u(\varphi^N(c_1), \varepsilon_X/2)$;

(ii) $d'(\varphi^{N+k}(a_1), \varphi^{N+k}(b_2)) < \varepsilon_X$, for all $k \in \{0, \dots, N_{a,n} - N\}$.

For part (i), observe that $\varphi^N(a_1) = [\varphi^N(a_2), \varphi^N(c_1)]$. Since $d'(\varphi^N(a_2), \varphi^N(c_2)) < \varepsilon'_X/2$ and $d'(\varphi^N(c_2), \varphi^N(c_1)) < \varepsilon'_X/2$, we have that

$$[\varphi^N(a_2), \varphi^N(c_1)] \in X^u(\varphi^N(c_1), \varepsilon_X/2).$$

For part (ii), note that $b_2 \in X^u(a_2, \eta_{a,n})$ and hence $\varphi^{N_{a,n}}(b_2) \in X^u(\varphi^{N_{a,n}}(a_2), \varepsilon'_X/2)$. Therefore, $\varphi^{N+k}(b_2) \in X^u(\varphi^{N+k}(a_2), \varepsilon'_X/2)$, for all $k \in \{0, \dots, N_{a,n} - N\}$. In addition, $\varphi^N(a_1) \in X^s(\varphi^N(a_2), \varepsilon_X/2)$, and as a result $\varphi^{N+k}(a_1) \in X^s(\varphi^{N+k}(a_2), \varepsilon_X/2)$, for every $k \in \{0, \dots, N_{a,n} - N\}$.

With conditions (i) and (ii) the following computations with the bracket map are well-defined, that is,

$$\begin{aligned} h^s(b_2) &= \varphi^{-N}[\varphi^N(b_2), \varphi^N(c_1)] \\ &= \varphi^{-N}[\varphi^N(b_2), [\varphi^N(c_1), \varphi^N(a_1)]] \\ &= \varphi^{-N}[\varphi^N(b_2), \varphi^N(a_1)] \\ &= \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(b_2), \varphi^{N_{a,n}}(a_1)] \\ &= b_1. \end{aligned}$$

This completes the proof. □

We now mimic the notation of part (3) of Theorem 6.1.6. For every $a \in G^s(Q)$ and $n \geq 0$ define

$$j(a,n) = N_{a,n} + \lceil \log_{\lambda_X} 3 \rceil, \tag{6.1.21}$$

and observe that $j(a,n) \geq \max\{N_a, n\} + 1$.

Lemma 6.1.8. *For every $a \in G^s(Q)$ and $n \geq 0$, if $b \in G^s(Q)$ with $V_{j(a,n)}(a) \cap V_{j(a,n)}(b) \neq \varnothing$, then $V_{j(a,n)}(b) \subset V_n(a)$.*

Proof. Let $n \geq 0$ and $a, b \in G^s(Q)$ such that $V_{j(a,n)}(a) \cap V_{j(a,n)}(b) \neq \varnothing$. We have

$$\begin{aligned} V_{j(a,n)}(a) &= V^s(a, h^s_{a,j(a,n)}, \eta_{a,j(a,n)}, N_{a,j(a,n)}) \\ V_{j(a,n)}(b) &= V^s(b, h^s_{b,j(a,n)}, \eta_{b,j(a,n)}, N_{b,j(a,n)}). \end{aligned}$$

Let $c \in V_{j(a,n)}(a) \cap V_{j(a,n)}(b)$ and $e \in V_{j(a,n)}(b)$. We aim to show that

$$e \in V_n(a) = V^s(a, h^s_{a,n}, \eta_{a,n}, N_{a,n}).$$

We have that

$$e_2 \in X^u(b_2, \eta_{b,j(a,n)}),\ c_2 \in X^u(b_2, \eta_{b,j(a,n)}),\ c_2 \in X^u(a_2, \eta_{a,j(a,n)}). \tag{6.1.22}$$

In any case, it holds that

$$\eta_{b,j(a,n)} \leq \eta_{a,j(a,n)}. \tag{6.1.23}$$

To check that (6.1.23) is true, first recall the decreasing sequence $(\eta_n)_{n\geq 0}$ from (6.1.16). Then, note that $N_{a,j(a,n)} = j(a,n)$ because $j(a,n) > N_a$, hence $\eta_{a,j(a,n)} = \eta_{N_{a,j(a,n)}} = \eta_{j(a,n)}$, where $\eta_{j(a,n)} \geq \eta_{N_{b,j(a,n)}} = \eta_{b,j(a,n)}$.

From (6.1.22) and (6.1.23) we obtain that $e_2 \in X^u(a_2, 3\eta_{a,j(a,n)})$. We aim to show that $3\eta_{a,j(a,n)} \leq \eta_{a,n}$, and hence $e_2 \in X^u(a_2, \eta_{a,n})$, meaning that e_2 lies in the domain of $h^s_{a,n}$. Note that $\lambda_X^{-\lceil \log_{\lambda_X} 3 \rceil} \leq 3^{-1}$, and by using (6.1.16) for $\eta_{a,j(a,n)} = \eta_{j(a,n)}$, for every $z \in X$ we have

$$\begin{aligned} \varphi^{N_{a,n}}(X^u(z, 6\eta_{a,j(a,n)})) &= \varphi^{j(a,n) - \lceil \log_{\lambda_X} 3 \rceil}(X^u(z, 6\eta_{a,j(a,n)})) \\ &\subset \varphi^{j(a,n)}(X^u(\varphi^{-\lceil \log_{\lambda_X} 3 \rceil}(z), \lambda_X^{-\lceil \log_{\lambda_X} 3 \rceil} 6\eta_{a,j(a,n)})) \\ &\subset \varphi^{j(a,n)}(X^u(\varphi^{-\lceil \log_{\lambda_X} 3 \rceil}(z), 2\eta_{a,j(a,n)})) \\ &= \varphi^{j(a,n)}(X^u(\varphi^{-\lceil \log_{\lambda_X} 3 \rceil}(z), 2\eta_{j(a,n)})) \\ &\subset \mathrm{cl}(X^u(\varphi^{N_{a,n}}(z), \varepsilon'_X/2)). \end{aligned}$$

From the choice of $\eta_{a,n}$ we obtain that $3\eta_{a,j(a,n)} \leq \eta_{a,n}$. It remains to show that $h^s_{a,n}(e_2) = e_1$, where

$$e_1 = h^s_{b,j(a,n)}(e_2) = \begin{cases} \varphi^{-N_b}[\varphi^{N_b}(e_2), \varphi^{N_b}(b_1)], & \text{if } j(a,n) \leq N_b \\ \varphi^{-j(a,n)}[\varphi^{j(a,n)}(e_2), \varphi^{j(a,n)}(b_1)], & \text{if } j(a,n) > N_b. \end{cases}$$

Assume that $j(a,n) \leq N_b$, and thus $N_{a,n} < N_b$. We will prove and use the following three facts:

(i) $\varphi^{N_{a,n}}(c_1) \in X^u(\varphi^{N_{a,n}}(a_1), \varepsilon_X/2)$;

(ii) $d'(\varphi^{N_{a,n}+k}(e_2), \varphi^{N_{a,n}+k}(c_1)) < \varepsilon_X$, for all $k \in \{0, \dots, N_b - N_{a,n}\}$;

(iii) $\varphi^{N_b}(c_1) \in X^u(\varphi^{N_b}(b_1), \varepsilon_X/2)$.

Part (i) follows since $c \in V_{j(a,n)}(a)$, meaning that $\varphi^{j(a,n)}(c_1) \in X^u(\varphi^{j(a,n)}(a_1), \varepsilon_X/2)$, and hence $\varphi^{N_{a,n}}(c_1) \in X^u(\varphi^{N_{a,n}}(a_1), \varepsilon_X/2)$. For part (ii), observe that $c \in V_n(a)$. Then, we have that $\varphi^{N_{a,n}}(c_1) \in X^s(\varphi^{N_{a,n}}(c_2), \varepsilon_X/2)$ and thus for all $k \in \{0, \dots, N_b - N_{a,n}\}$,

$$\varphi^{N_{a,n}+k}(c_1) \in X^s(\varphi^{N_{a,n}+k}(c_2), \varepsilon_X/2).$$

Also, since $e_2 \in X^u(c_2, 2\eta_{b,j(a,n)})$ we have $\varphi^{N_b}(e_2) \in \mathrm{cl}(X^u(\varphi^{N_b}(c_2), \varepsilon'_X/2))$, hence

$$\varphi^{N_{a,n}+k}(e_2) \in \mathrm{cl}(X^u(\varphi^{N_{a,n}+k}(c_2), \varepsilon'_X/2)),$$

for all $k \in \{0, \dots, N_b - N_{a,n}\}$. The claim for (ii) follows, and part (iii) is clear.

With conditions (i), (ii) and (iii) the following computations are well-defined,

$$\begin{aligned} h^s_{a,n}(e_2) &= \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(e_2), \varphi^{N_{a,n}}(a_1)] \\ &= \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(e_2), [\varphi^{N_{a,n}}(a_1), \varphi^{N_{a,n}}(c_1)]] \\ &= \varphi^{-N_{a,n}}[\varphi^{N_{a,n}}(e_2), \varphi^{N_{a,n}}(c_1)] \\ &= \varphi^{-N_b}[\varphi^{N_b}(e_2), \varphi^{N_b}(c_1)] \\ &= \varphi^{-N_b}[\varphi^{N_b}(e_2), [\varphi^{N_b}(c_1), \varphi^{N_b}(b_1)]] \\ &= \varphi^{-N_b}[\varphi^{N_b}(e_2), \varphi^{N_b}(b_1)]. \end{aligned}$$

The exactly same computations work for the case $j(a,n) > N_b$, where in the place of N_b one has to put $j(a,n)$. In any case, we obtain that $e \in V_n(a)$ and this completes the proof of the lemma. □

Lemmas 6.1.7 and 6.1.8 imply that $G^s(Q)$ satisfies condition (3) of Theorem 6.1.6. Now, following the sketched proof of Theorem 6.1.6, we aim to construct a sequence of open covers $(\mathcal{U}^s_n)_{n\in\mathbb{N}}$ satisfying condition (2). First of all, in order to be consistent with Theorem 6.1.6, for every $a \in G^s(Q)$ we consider the neighbourhood basis $(V_n(a))_{n\in\mathbb{N}}$ instead of $(V_n(a))_{n\geq 0}$.

For every $a \in G^s(Q)$, set $k(a,1) = 1$, and inductively for $n \geq 2$ define

$$k(a,n) = \max\{n, j(a,i) : i = 1,\dots,k(a,n-1)\}. \tag{6.1.24}$$

Then, for every $n \in \mathbb{N}$, let

$$U_n(a) = \bigcap_{i=1}^{k(a,n)} V_i(a), \tag{6.1.25}$$

and since $(V_n(a))_{n\in\mathbb{N}}$ is decreasing we have that $U_n(a) = V_{k(a,n)}(a)$. Then, the open covers

$$\mathcal{U}_n^s = \{U_n(a) : a \in G^s(Q)\} \tag{6.1.26}$$

form the desired sequence. The next lemma gives a precise definition of the covers $\mathcal{U}_n^s$.

Lemma 6.1.9. *For every $a \in G^s(Q)$ and $n \in \mathbb{N}$ we have that*

$$k(a,n+1) = N_{a,1} + n\lceil \log_{\lambda_X} 3\rceil.$$

Proof. By definition we have that $k(a,n+1) = \max\{n+1, j(a,i) : i = 1,\dots,k(a,n)\}$. The sequence $(j(a,m))_{m\in\mathbb{N}}$ is increasing, hence $k(a,n+1) = \max\{n+1, j(a,k(a,n))\}$. In fact, it holds that

$$k(a,n+1) = j(a,k(a,n)) \tag{6.1.27}$$

because $j(a,k(a,n)) \geq j(a,n) = N_{a,n} + \lceil \log_{\lambda_X} 3\rceil \geq n+1$. To prove the lemma we will use (6.1.27) and induction on n. For $n=1$ we have

$$k(a,2) = j(a,k(a,1)) = j(a,1) = N_{a,1} + \lceil \log_{\lambda_X} 3\rceil.$$

Assuming that $k(a,n+1) = N_{a,1} + n\lceil \log_{\lambda_X} 3\rceil$, one has that

$$\begin{aligned} k(a,n+2) &= j(a,k(a,n+1)) \\ &= N_{a,k(a,n+1)} + \lceil \log_{\lambda_X} 3\rceil \\ &= k(a,n+1) + \lceil \log_{\lambda_X} 3\rceil \\ &= N_{a,1} + (n+1)\lceil \log_{\lambda_X} 3\rceil, \end{aligned}$$

thus completing the induction. □

As we mentioned before, in order not to overload, we simplified our notation and did not highlight the dependence on the metric $d' \in \mathrm{hM}_d(X,\varphi)$. For this reason, the sequence $(\mathcal{U}_n^s)_{n\in\mathbb{N}}$ should more appropriately be denoted as $(\mathcal{U}_n^{s,d'})_{n\in\mathbb{N}}$. Now, let $\mathcal{U}_0^{s,d'} = \{G^s(Q)\}$ and according to the proof of Theorem 6.1.6, the sequence $(\mathcal{U}_n^{s,d'})_{n\geq 0}$ yields a 2-quasimetric on

$G^s(Q)$ given by

$$\rho_{s,d'}(a,b) = \inf\{2^{-n} : \text{there is } U \in \mathcal{U}_n^{s,d'} \text{ such that } a,b \in U\}. \tag{6.1.28}$$

This in turn gives the chain-metric $D_{s,d'}$ defined as

$$D_{s,d'}(a,b) = \inf\{\sum_{i=1}^{n-1} \rho_{s,d'}(c_i,c_{i+1}) : n \in \mathbb{N},\, c_i \in G^s(Q),\, c_1 = a,\, c_n = b\}, \tag{6.1.29}$$

which satisfies

$$4^{-1}\rho_{s,d'}(a,b) \leq D_{s,d'}(a,b) \leq \rho_{s,d'}(a,b) \tag{6.1.30}$$

for every $a,b \in G^s(Q)$. In fact, for every $a \in G^s(Q)$ it holds that

$$\operatorname{st}(a,\mathcal{U}_{n+2}^{s,d'}) \subset \overline{B}_{D_{s,d'}}(a,2^{-n-2}) \subset \operatorname{st}(a,\mathcal{U}_n^{s,d'}), \tag{6.1.31}$$

and since the sequence $(\operatorname{st}(a,\mathcal{U}_n^{s,d'}))_{n\geq 0}$ is a neighbourhood basis, the metric $D_{s,d'}$ generates the étale topology on $G^s(Q)$. Let now $j : X^u(Q) \to G^s(Q)$ be the *units embedding* given by

$$j(x) = (x,x). \tag{6.1.32}$$

The restriction of $D_{s,d'}$ on $j(X^u(Q))$ generates the relative topology on $j(X^u(Q))$, and hence the pull-back metric $\widetilde{D_{s,d'}} : X^u(Q) \times X^u(Q) \to [0,1]$ given by

$$\widetilde{D_{s,d'}}(x,y) = D_{s,d'}(j(x),j(y)), \tag{6.1.33}$$

generates the topology on $X^u(Q)$. As a result, we have proved the following.

Lemma 6.1.10. *For every $d' \in hM_d(X,\varphi)$, the metrics $D_{s,d'}$ and $D_{u,d'}$ induce the étale topologies on $G^s(Q)$ and $G^u(P)$. Moreover, the pull-back metrics $\widetilde{D_{s,d'}}$ and $\widetilde{D_{u,d'}}$ induce the topologies of the units spaces $X^u(Q)$ and $X^s(P)$, respectively.*

We continue by showing that for self-similar hyperbolic metrics the groupoids admit bi-Lipschitz dynamics. Again, we keep the notation to the minimum and when we work with the d'-model of $G^s(Q)$, where $d' \in \mathrm{hM}_d(X,\varphi)$, we do not indicate the dependence of the Smale space constants and the local stable and unstable sets on d'. Also, the homeomorphism φ behaves with respect to d', and the basis sets of $G^s(Q)$ are in $\mathcal{B}^s(Q,d')$. Nevertheless, we do highlight the dependence of the covers $\mathcal{U}_n^{s,d'}$, the quasimetric $\rho_{s,d'}$ and the metric $D_{s,d'}$ on d'. Before proceeding to the next lemma we should make the following observation.

Let $d' \in \mathrm{sM}_d(X,\varphi)$ and consider, for a moment, the d'-model of $G^s(Q)$. Then, for every $0 < \varepsilon \leq \lambda_X^{-1}\varepsilon_X$ and $z \in X$, we have

$$\varphi(X^u(z,\varepsilon)) = X^u(\varphi(z),\varepsilon\lambda_X). \tag{6.1.34}$$

Consequently, for the sequence $(\eta_n)_{n\geq 0}$ in (6.1.16) it holds that $\eta_n = \lambda_X^{-n}\varepsilon'_X/4$. As a result, for every $a \in G^s(Q)$, the neighbourhood basis sets (6.1.20) corresponding to d' are

$$V_n(a) = V^s(a, h^s_{a,n}, \lambda_X^{-N_{a,n}}\varepsilon'_X/4, N_{a,n}). \tag{6.1.35}$$

Lemma 6.1.11. *Let $d' \in sM_d(X,\varphi)$ and consider the metric $D_{s,d'}$ on $G^s(Q)$. Then, the automorphism $\Phi = \varphi\times\varphi : (G^s(Q), D_{s,d'}) \to (G^s(Q), D_{s,d'})$ is bi-Lipschitz with*

$$4^{-1}D_{s,d'}(a,b) \leq D_{s,d'}(\Phi(a),\Phi(b)) \leq 8D_{s,d'}(a,b),$$

for every $a,b \in G^s(Q)$. Similarly, for the metric $D_{u,d'}$ on $G^u(P)$, the automorphism $\Phi = \varphi\times\varphi : (G^u(P), D_{u,d'}) \to (G^u(P), D_{u,d'})$ satisfies

$$4^{-1}D_{u,d'}(a,b) \leq D_{u,d'}(\Phi^{-1}(a),\Phi^{-1}(b)) \leq 8D_{u,d'}(a,b),$$

for every $a,b \in G^u(P)$.

Proof. We will prove it only for the stable groupoid. Let $d' \in \mathrm{sM}_d(X,\varphi)$ and consider the d'-model of $G^s(Q)$ so that φ satisfies (6.1.34). Recall the quasimetric $\rho_{s,d'}$ in (6.1.28) and we aim to show that, for every $a,b \in G^s(Q)$, it holds

(i) $\rho_{s,d'}(\Phi^{-1}(a),\Phi^{-1}(b)) \leq \rho_{s,d'}(a,b)$;

(ii) $\rho_{s,d'}(\Phi^{-1}(a),\Phi^{-1}(b)) \geq 2^{-1}\rho_{s,d'}(a,b)$.

Then, using the inequalities (6.1.30) we obtain the result. First we prove part (i). If $a = b$, the statement is trivial. Also, since $\rho_{s,d'}$ is bounded by 1, if $\rho_{s,d'}(a,b) = 1$ then $\rho_{s,d'}(\Phi^{-1}(a),\Phi^{-1}(b)) \leq \rho_{s,d'}(a,b)$.

Assume now that $a \neq b$ and $\rho_{s,d'}(a,b) = 2^{-n}$, for some $n \geq 2$ (the case $n = 1$ is treated in the exactly same way, however we avoid to show it in an attempt to reduce notation). There is $c \in G^s(Q)$ such that $a,b \in U_n(c) \in \mathcal{U}_n^{s,d'}$. From (6.1.35) and since $k(c,n) \geq N_c$ (note that $n \geq 2$), we have that

$$U_n(c) = V_{k(c,n)}(c) = V^s(c, h^s_{c,k(c,n)}, \lambda_X^{-k(c,n)}\varepsilon'_X/4, k(c,n)).$$

For simplicity, let us drop the notation of $h^s_{c,k(c,n)}$, since it is completely determined by c, $k(c,n)$ and $\lambda_X^{-k(c,n)}\varepsilon'_X/4$. It is straightforward to see that

$$\Phi^{-1}(a), \Phi^{-1}(b) \in V^s(\Phi^{-1}(c), \lambda_X^{-k(c,n)-1}\varepsilon'_X/4, k(c,n)+1).$$

One can observe that $N_{\Phi^{-1}(c),1} \leq N_{c,1}+1$ and hence,

$$k(\Phi^{-1}(c), n) \leq k(c,n)+1.$$

Note that $k(\Phi^{-1}(c), n) \geq N_{\Phi^{-1}(c)}$, and we have

$$\begin{aligned} V^s(\Phi^{-1}(c), \lambda_X^{-k(c,n)-1}\varepsilon'_X/4, k(c,n)+1) &= V^s(\Phi^{-1}(c), \lambda_X^{-k(c,n)-1}\varepsilon'_X/4, k(\Phi^{-1}(c),n)) \\ &\subset V^s(\Phi^{-1}(c), \lambda_X^{-k(\Phi^{-1}(c),n)}\varepsilon'_X/4, k(\Phi^{-1}(c),n)) \\ &= V_{k(\Phi^{-1}(c),n)}(\Phi^{-1}(c)) \\ &= U_n(\Phi^{-1}(c)). \end{aligned}$$

Consequently, it holds that $\rho_{s,d'}(\Phi^{-1}(a), \Phi^{-1}(b)) \leq 2^{-n} = \rho_{s,d'}(a,b)$, and this completes the proof of part (i).

To prove part (ii) suppose that $\rho_{s,d'}(a,b) = 2^{-n}$, for some $n \geq 0$. The claim is that $\rho_{s,d'}(\Phi^{-1}(a), \Phi^{-1}(b)) \geq 2^{-n-1}$. Assume to the contrary that

$$\rho_{s,d'}(\Phi^{-1}(a), \Phi^{-1}(b)) \leq 2^{-n-2}.$$

Then, there is some $c \in G^s(Q)$ such that $\Phi^{-1}(a), \Phi^{-1}(b) \in U_{n+2}(c) = V_{k(c,n+2)}(c)$, where

$$V_{k(c,n+2)}(c) = V^s(c, \lambda_X^{-k(c,n+2)}\varepsilon'_X/4, k(c,n+2)).$$

One can immediately see that

$$a, b \in V^s(\Phi(c), \lambda_X^{-k(c,n+2)+1}\varepsilon'_X/4, k(c,n+2)-1).$$

Note that the latter basis set is well-defined since $k(c,n+2)-1 \geq 0$. Also, observe that $N_{\Phi(c),1} \leq N_{c,1}$, and hence

$$\begin{aligned} k(\Phi(c), n+1) &= N_{\Phi(c),1} + n\lceil \log_{\lambda_X} 3 \rceil \\ &\leq N_{c,1} + n\lceil \log_{\lambda_X} 3 \rceil \\ &\leq N_{c,1} + n\lceil \log_{\lambda_X} 3 \rceil + \lceil \log_{\lambda_X} 3 \rceil - 1 \\ &= k(c, n+2) - 1. \end{aligned}$$

As a result,

$$\begin{aligned} V^s(\Phi(c), \lambda_X^{-k(c,n+2)+1}\varepsilon'_X/4, k(c,n+2)-1) &= V^s(\Phi(c), \lambda_X^{-k(c,n+2)+1}\varepsilon'_X/4, k(\Phi(c),n+1)) \\ &\subset V^s(\Phi(c), \lambda_X^{-k(\Phi(c),n+1)}\varepsilon'_X/4, k(\Phi(c),n+1)) \\ &= V_{k(\Phi(c),n+1)}(\Phi(c)) \\ &= U_{n+1}(\Phi(c)). \end{aligned}$$

This means that $\rho_{s,d'}(a,b) \leq 2^{-n-1}$, leading to a contradiction. This completes the proof of part (ii). $\square$

We continue by looking closer at the Lipschitz structure of $G^s(Q)$ and $G^u(P)$. Again, given the groupoid metrics that correspond to $d' \in \mathrm{sM}_d(X,\varphi)$, we prefer to work with the d'-model of the groupoids, and the only part where we highlight the dependence on d' is in the covers $\mathcal{U}_n^{s,d'}$, the quasimetric $\rho_{s,d'}$ and the metric $D_{s,d'}$.

Lemma 6.1.12. *For every $d' \in sM_d(X,\varphi)$, the inversion maps on $G^s(Q)$ and $G^u(P)$ are bi-Lipschitz, with respect to the metrics $D_{s,d'}$ and $D_{u,d'}$.*

Proof. We only prove it for $G^s(Q)$ for which we consider its d'-model. Also, note that since $i^2 = \mathrm{id}$, we only have to show that the inversion map i is Λ-Lipschitz, for some $\Lambda \geq 1$ which depends on d', and then

$$\Lambda^{-1}D_{s,d'}(a,b) \leq D_{s,d'}(i(a),i(b)) \leq \Lambda D_{s,d'}(a,b),$$

for every $a,b \in G^s(Q)$. As usual, instead of working directly with $D_{s,d'}$, we work with the quasimetric $\rho_{s,d'}$, and it suffices to find $\Lambda' \geq 1$ such that,

$$\rho_{s,d'}(i(a),i(b)) \leq \Lambda'\rho_{s,d'}(a,b), \tag{6.1.36}$$

for every $a,b \in G^s(Q)$. Then, following the metric inequalities (6.1.30) we can choose $\Lambda = 4\Lambda'$. Let $M \in \mathbb{N}$ such that $\lambda_X^{-M}\varepsilon_X/2 \leq \varepsilon'_X/4$, and the claim is that we can choose

$$\Lambda' = 2^{M+1}.$$

The case where $a = b$ or $\rho_{s,d'}(a,b) = 1$ is trivial. Also, since $\rho_{s,d'}$ is bounded by 1, if $\rho_{s,d'}(a,b) = 2^{-n}$ for some $n \in \{1,\dots,M+1\}$, then $\Lambda' = 2^{M+1}$ satisfies (6.1.36). Suppose now that $\rho_{s,d'}(a,b) = 2^{-n}$ for some $n > M+1$. Then, there is $c = (c_1,c_2) \in G^s(Q)$ such that $a,b \in U_n(c) \in \mathcal{U}_n^{s,d'}$. From (6.1.35) and the fact that $n \geq 2$, and hence $k(c,n) \geq N_c$, it holds that

$$U_n(c) = V_{k(c,n)}(c) = V^s(c, h^s_{c,k(c,n)}, \lambda_X^{-k(c,n)}\varepsilon'_X/4, k(c,n)).$$

It is not hard to see that

$$h^s_{c,k(c,n)}(X^u(c_2,\lambda_X^{-k(c,n)}\varepsilon'_X/4)) \subset X^u(c_1,\lambda_X^{-k(c,n)}\varepsilon_X/2),$$

and for simplicity, we henceforth drop the notation of $h^s_{c,k(c,n)}$. Since $n-M\geq 2$, we have

$$k(c,n)\geq k(c,n)-M\geq k(c,n-M)\geq N_c=N_{i(c)},$$

and also that

$$k(c,m)=k(i(c),m),$$

for all $m\in\mathbb{N}$. As a result, the following calculations are well-defined,

$$\begin{aligned}
i(V^s(c,\lambda_X^{-k(c,n)}\varepsilon'_X/4,k(c,n))) &= i(V^s(c,\lambda_X^{-k(c,n)}\varepsilon'_X/4,k(c,n)-M))\\
&\subset V^s(i(c),\lambda_X^{-k(c,n)}\varepsilon_X/2,k(c,n)-M)\\
&\subset V^s(i(c),\lambda_X^{-k(c,n)+M}\varepsilon'_X/4,k(c,n)-M)\\
&= V^s(i(c),\lambda_X^{-k(c,n)+M}\varepsilon'_X/4,k(c,n-M))\\
&\subset V^s(i(c),\lambda_X^{-k(c,n-M)}\varepsilon'_X/4,k(c,n-M))\\
&= V_{k(i(c),n-M)}(i(c))\\
&= U_{n-M}(i(c)).
\end{aligned}$$

Consequently, $\rho_{s,d'}(i(a),i(b))\leq 2^{-n+M}=2^M\rho_{s,d'}(a,b)<2^{M+1}\rho_{s,d'}(a,b)$, and this completes the proof of the claim. □

We now prove an elementary fact about locally bi-Lipschitz maps for which we were not able to find a reference in the literature.

Lemma 6.1.13. *Suppose that $f:(Y,d_Y)\to(Z,d_Z)$ is locally bi-Lipschitz, and $K\subset Y$ is a compact set on which the restriction $f_K:K\to f(K)$ is a homeomorphism. Then, f_K is bi-Lipschitz.*

Proof. The map f is bi-Lipschitz and this means that, for every $a\in Y$ there is $\ell_a>0$ and $\Lambda_a\geq 1$ such that,

$$\Lambda_a^{-1}d_Y(a,b)\leq d_Z(f(a),f(b))\leq\Lambda_a d_Y(a,b), \tag{6.1.37}$$

for every $b\in B_{d_Y}(a,\ell_a)$. First, we show that f_K is Lipschitz. The collection of open balls $\{B_{d_Y}(a,\ell_a/2):a\in K\}$ covers the compact K, and hence there is a finite subcover

$$\{B_{d_Y}(a_i,\ell_{a_i}/2):a_i\in K,\, i=1,\ldots,m\}.$$

Consider

$$\ell_K = \min\{\ell_{a_i}/2 : i = 1, \dots, m\}$$

and

$$\Lambda_K = \max\{\operatorname{diam}(f(K))/\ell_K, \Lambda_{a_1}, \dots, \Lambda_{a_m}\}.$$

Let $a, b \in K$ and assume that $d_Y(a,b) \geq \ell_K$. Then,

$$\frac{d_Z(f(a), f(b))}{d_Y(a,b)} \leq \frac{\operatorname{diam}(f(K))}{\ell_K} \leq \Lambda_K.$$

Suppose now that $d_Y(a,b) < \ell_K$, and we know that there is some $i \in \{1, \dots, m\}$ so that $a \in B_{d_Y}(a_i, \ell_{a_i}/2)$. Therefore, it holds that $a, b \in B_{d_Y}(a_i, \ell_{a_i})$ and hence

$$d_Z(f(a), f(b)) \leq \Lambda_{a_i} d_Y(a,b) \leq \Lambda_K d_Y(a,b),$$

thus completing the proof that f_K is Lipschitz. We now show that f_K^{-1} is Lipschitz. From (6.1.37) we see that

$$\Lambda_a^{-1} d_Y(a,b) \leq d_Z(f_K(a), f_K(b)),$$

for every $a \in K$ and $b \in B_{d_Y}(a, \ell_a) \cap K$. In particular,

$$d_Y(f_K^{-1}(a'), f_K^{-1}(b')) \leq \Lambda_a d_Z(a', b'),$$

for every $a' \in f(K)$ and $b' \in f(B_{d_Y}(f_K^{-1}(a'), \ell_{f_K^{-1}(a')}) \cap K)$. As a result, since the latter sets are open neighbourhoods of each a', the map f_K^{-1} is locally Lipschitz, and a similar argument as before finishes the proof. □

For the next lemma recall that $\mathcal{B}(G^s(Q))$ is the collection of all basis set in $\mathcal{B}^s(Q, d')$, for all $d' \in \mathrm{hM}_{d'}(X, \varphi)$. Moreover, given a metric $D_{s,d'}$, recall the pull-back metric $\widetilde{D_{s,d'}}$ on the units space $X^u(Q)$ along the embedding $j : X^u(Q) \to G^s(Q)$, see (6.1.33).

Lemma 6.1.14. *For every $d' \in sM_d(X, \varphi)$ and every basis set $V \in \mathcal{B}(G^s(Q))$, the restricted range and source maps r_V and s_V are bi-Lipschitz, with respect to the metrics $D_{s,d'}$ on $G^s(Q)$ and $\widetilde{D_{s,d'}}$ on $X^u(Q)$. Similarly for $G^u(P)$.*

Proof. We will show it only for $G^s(Q)$ since the proof for $G^u(P)$ is the same. Let $d' \in \mathrm{sM}_d(X, \varphi)$ and we aim to prove that, with respect to $D_{s,d'}$ and $\widetilde{D_{s,d'}}$,

(i) s_V is bi-Lipschitz, for every $V \in \mathcal{B}^s(Q, d')$;

(ii) s_V is bi-Lipschitz, for every $V \in \mathcal{B}(G^s(Q))$;

(iii) r_V is bi-Lipschitz, for every $V \in \mathcal{B}(G^s(Q))$.

We begin with part (i). Similarly as before, it is enough to consider the quasimetric $\rho_{s,d'}$ (see the metric inequalities (6.1.30)), and we claim that, for $V \in \mathcal{B}^s(Q, d')$, there is some $M_V \in \mathbb{N}$ such that,

$$2^{-M_V} \rho_{s,d'}(a,b) \leq \rho_{s,d'}(j(s_V(a)), j(s_V(b))) \leq \rho_{s,d'}(a,b), \tag{6.1.38}$$

for every $a, b \in V$. First, we show that $\rho_{s,d'}(j(s_V(a)), j(s_V(b))) \leq \rho_{s,d'}(a,b)$. The case where $a = b$ or $\rho_{s,d'}(a,b) = 1$ is trivial. Suppose that $\rho_{s,d'}(a,b) = 2^{-n}$, for some $n \in \mathbb{N}$. Then there is $c = (c_1, c_2) \in G^s(Q)$ so that $a, b \in U_n(c) \in \mathcal{U}_n^{s,d'}$. From (6.1.35) we have that

$$U_n(c) = V_{k(c,n)}(c) = V^s(c, h^s_{c,k(c,n)}, \lambda_X^{-N_{c,k(c,n)}} \varepsilon'_X/4, N_{c,k(c,n)}),$$

and, in particular,

$$s_V(a), s_V(b) \in X^u(c_2, \lambda_X^{-N_{c,k(c,n)}} \varepsilon'_X/4).$$

As a result,

$$\begin{aligned} j(s_V(a)),\, j(s_V(b)) &\in j(X^u(c_2, \lambda_X^{-N_{c,k(c,n)}} \varepsilon'_X/4)) \\ &= V^s(j(c_2), \mathrm{id}, \lambda_X^{-N_{c,k(c,n)}} \varepsilon'_X/4, 0). \end{aligned}$$

Observe that $N_{c,k(c,n)} \geq k(j(c_2), n) = N_{j(c_2),k(j(c_2),n)}$, and hence

$$\begin{aligned} V^s(j(c_2), \mathrm{id}, \lambda_X^{-N_{c,k(c,n)}} \varepsilon'_X/4, 0) &\subset V^s(j(c_2), \mathrm{id}, \lambda_X^{-k(j(c_2),n)} \varepsilon'_X/4, 0) \\ &= V^s(j(c_2), h^s_{j(c_2),k(j(c_2),n)}, \lambda_X^{-k(j(c_2),n)} \varepsilon'_X/4, k(j(c_2),n)) \\ &= V_{k(j(c_2),n)}(j(c_2)) \\ &= U_n(j(c_2)). \end{aligned}$$

Therefore, it holds that

$$\rho_{s,d'}(j(s_V(a)), j(s_V(b))) \leq 2^{-n} = \rho_{s,d'}(a,b).$$

We now aim to show that there is $M_V \in \mathbb{N}$ such that,

$$2^{-M_V} \rho_{s,d'}(a,b) \leq \rho_{s,d'}(j(s_V(a)), j(s_V(b))),$$

for every $a, b \in V$. Let us now make the definition of V more precise and assume that $V = V^s(e, h^s, \eta, N)$, where $e \in G^s(Q)$. Also, let $N' \in \mathbb{N}$ such that $\lambda_X^{-N'} \varepsilon_X/2 \leq \varepsilon'_X/2$. Then, for $a, b \in V$ we have that

$$\varphi^N(a_2) \in X^s(\varphi^N(a_1), \varepsilon_X/2), \tag{6.1.39}$$

and in particular,

$$\varphi^{N+N'}(a_2) \in X^s(\varphi^{N+N'}(a_1), \varepsilon'_X/2). \tag{6.1.40}$$

Similarly for $b \in V$. The claim is that we can choose

$$M_V = \left\lceil \frac{\log_{\lambda_X} 2 + (N+N')(1 + \lceil \log_{\lambda_X} 3 \rceil)}{\lceil \log_{\lambda_X} 3 \rceil} \right\rceil.$$

To this purpose, suppose that $\rho_{s,d'}(a,b) = 2^{-n}$, for some $n \geq 0$, and we claim that

$$2^{-M_V - n} \leq \rho_{s,d'}(j(s_V(a)), j(s_V(b))).$$

Our approach is proof by contradiction. So, assume that

$$\rho_{s,d'}(j(s_V(a)), j(s_V(b))) \leq 2^{-M_V-n-1}.$$

Then, there is $c \in G^s(Q)$ such that

$$j(s_V(a)),\ j(s_V(b)) \in U_{M_V+n+1}(c) \in \mathcal{U}^{s,d'}_{M_V+n+1},$$

where

$$U_{M_V+n+1}(c) = V^s(c, \lambda_X^{-k(c,M_V+n+1)}\varepsilon'_X/4, k(c, M_V+n+1)).$$

In particular, it holds that

$$s_V(b) \in X^u(s_V(a), 2\lambda_X^{-k(c,M_V+n+1)}\varepsilon'_X/4). \tag{6.1.41}$$

Moreover,

$$2\lambda_X^{-k(c,M_V+n+1)}\varepsilon'_X/4 \leq \lambda_X^{-k(a,n+N+N'+1)}\varepsilon'_X/4. \tag{6.1.42}$$

Indeed, by removing $\varepsilon'_X/4$ from both sides, taking the logarithm $\log_{\lambda_X}$ and considering (6.1.40), the inequality (6.1.42) is true because

$$\begin{aligned}
\log_{\lambda_X} 2 + k(a, n+N+N'+1) &= \log_{\lambda_X} 2 + N_{a,1} + (n+N+N')\lceil \log_{\lambda_X} 3 \rceil \\
&\leq \log_{\lambda_X} 2 + (N+N') + (n+N+N')\lceil \log_{\lambda_X} 3 \rceil \\
&\leq (M_V + n)\lceil \log_{\lambda_X} 3 \rceil \\
&\leq N_{c,1} + (M_V+n)\lceil \log_{\lambda_X} 3 \rceil \\
&= k(c, M_V+n+1).
\end{aligned}$$

In addition, since $k(a, n+N+N'+1) \geq N+N' \geq N_a$, the basis set

$$V^s(a, \lambda_X^{-k(a,n+N+N'+1)}\varepsilon'_X/4, N+N')$$

is well-defined, and that is equal to

$$V^s(a, \lambda_X^{-k(a,n+N+N'+1)}\varepsilon'_X/4, k(a, n+N+N'+1)) = U_{n+N+N'+1}(a).$$

Using (6.1.39), (6.1.41) and (6.1.42), one can see that the following computations for $a = (a_1, a_2)$, $b = (b_1, b_2)$ in $V = V^s(e, h^s, \eta, N)$ are well-defined, that is,

$$\begin{aligned}
\varphi^{-N-N'}[\varphi^{N+N'}(b_2), \varphi^{N+N'}(a_1)] &= \varphi^{-N}[\varphi^N(b_2), \varphi^N(a_1)] \\
&= \varphi^{-N}[\varphi^N(b_2), [\varphi^N(a_1), \varphi^N(e_1)]] \\
&= \varphi^{-N}[\varphi^N(b_2), \varphi^N(e_1)] \\
&= b_1,
\end{aligned}$$

and hence

$$b \in V^s(a, \lambda_X^{-k(a,n+N+N'+1)}\varepsilon'_X/4, N+N').$$

Consequently,

$$\rho_{s,d'}(a,b) \leq 2^{-n-N-N'-1},$$

which leads to a contradiction since $\rho_{s,d'}(a,b) = 2^{-n}$. This completes the proof of part (i).

Part (ii) follows easily from part (i). Indeed, since $\mathcal{B}^s(Q,d')$ is a topological basis for $G^s(Q)$, part (i) implies that the source map s is locally bi-Lipschitz, with respect to $D_{s,d'}$ and $\widetilde{D_{s,d'}}$. Moreover, every $V \in \mathcal{B}(G^s(Q))$ is pre-compact and the restriction of s on $\mathrm{cl}(V)$ is again a homeomorphism. Then, the proof of part (ii) follows from Lemma 6.1.13. Finally, for part (iii) observe that the range map r is also locally bi-Lipschitz, because $r = s \circ i$ and the inversion map i is bi-Lipschitz, see Lemma 6.1.12. Similarly, the result follows from Lemma 6.1.13. □

Proposition 6.1.15. *Consider the metric $d' \in sM_d(X,\varphi)$ and the Smale space (X, d', φ) with contraction constant $\lambda_{X,d'} > 1$. Moreover, let $c \in G^s(Q)$ and consider the basis set $V^s_{d'}(c, h^s, \lambda_{X,d'}^{-N-k}\gamma, N) \in \mathcal{B}^s(Q,d')$, where $k \in \mathbb{N}$ and $\gamma > 0$ does not dependent on k. Then, we can find $\gamma' = \gamma'(\lambda_{X,d'}, \gamma) > 0$ so that, if $a, b \in V^s(c, h^s, \lambda_{X,d'}^{-N-k}\gamma, N)$, then*

$$D_{s,d'}(a,b) \leq 2^{-k/\lceil \log_{\lambda_{X,d'}} 3 \rceil}\gamma'.$$

The analogous statement holds for the metric $D_{u,d'}$ on $G^u(P)$.

Proof. Let $d' \in \mathrm{sM}_d(X,\varphi)$ and again, for the sake of simplicity, we keep the notation of the metric d' only for $\mathcal{U}_n^{s,d'}$, $\rho_{s,d'}$ and $D_{s,d'}$. Let $k' > 0$ be such that $\lambda_X^{-k'}\gamma \leq \varepsilon'_X/4$. If

$k < k' + \lceil \log_{\lambda_X} 3 \rceil$, let $\gamma'' > 0$ be big enough so that

$$2^{-k/\lceil \log_{\lambda_X} 3 \rceil}\gamma'' \geq 1 \geq D_{s,d'}(a,b).$$

If $k \geq k' + \lceil \log_{\lambda_X} 3 \rceil$, then

$$\lambda_X^{-N-k}\gamma \leq \lambda_X^{-N-k+k'}\varepsilon'_X/4 < \lambda_X^{-N}\varepsilon'_X/4.$$

By dropping the notation of the holonomy maps, for brevity, we have that

$$\begin{aligned} V^s(c, \lambda_X^{-N-k}\gamma, N) &\subset V^s(c, \lambda_X^{-N-k+k'}\varepsilon'_X/4, N) \\ &= V^s(c, \lambda_X^{-N-k+k'}\varepsilon'_X/4, N+k-k') \\ &= V_{N+k-k'}(c). \end{aligned}$$

Ideally, the goal is to find the largest $n \geq 2$ such that $k(c,n) \leq N + k - k'$, and then $V_{N+k-k'}(c) \subset V_{k(c,n)}(c) = U_n(c) \in \mathcal{U}_n^{s,d'}$, leading to

$$D_{s,d'}(a,b) \leq \rho_{s,d'}(a,b) \leq 2^{-n}.$$

For $n \geq 2$, we have

$$\begin{aligned} k(c,n) &= N_{c,1} + (n-1)\lceil \log_{\lambda_X} 3 \rceil \\ &\leq N + (n-1)\lceil \log_{\lambda_X} 3 \rceil. \end{aligned}$$

Hence we can find the largest n so that $N + (n-1)\lceil \log_{\lambda_X} 3 \rceil \leq N + k - k'$, which is

$$n = \lfloor (k-k')/\lceil \log_{\lambda_X} 3 \rceil \rfloor + 1 \geq 2.$$

It holds that $n > (k-k')/\lceil \log_{\lambda_X} 3 \rceil$ and hence

$$2^{-n} < 2^{-k/\lceil \log_{\lambda_X} 3 \rceil} 2^{k'/\lceil \log_{\lambda_X} 3 \rceil}.$$

By choosing $\gamma' = \max\{\gamma'', 2^{k'/\lceil \log_{\lambda_X} 3 \rceil}\}$ we complete the proof. □

6.1.3 Optimisation of uniform convergence rates

All the following facts about the stable groupoid and stable C^*-algebras are also true in the unstable cases. As we mentioned earlier, in Section 6.2 we intend to build a family of Lipschitz algebras of compactly supported functions, namely

$$\{\mathrm{Lip}_c(G^s(Q), D_{s,d'}) : d' \in \mathrm{sM}_d(X,\varphi)\},$$

which are dense $*$-subalgebras of $\mathcal{S}(Q)$. All these Lipschitz algebras are α_s-invariant and therefore, for every $d' \in \mathrm{sM}(X,\varphi)$, one can form the dense $*$-subalgebra

$$\Lambda_{s,d'} = \mathrm{Lip}_c(G^s(Q), D_{s,d'}) \rtimes_{\alpha_s,\mathrm{alg}} \mathbb{Z} \tag{6.1.43}$$

of the Ruelle algebra $\mathcal{R}^s(Q)$. One of our results will be that, for any $d' \in \mathrm{sM}(X,\varphi)$, the algebra $\mathcal{R}^s(Q)$ is uniformly $\mathcal{L}^p$-smooth on $\Lambda_{s,d'}$, for every $p > p(d')$, where $p(d')$ is a constant that depends on the topological entropy $\mathrm{h}(\varphi)$ and the metric d'.

It turns out that, if $d_1, d_2 \in \mathrm{sM}_d(X,\varphi)$ and the corresponding contraction constants satisfy $\lambda_{X,d_1} \geq \lambda_{X,d_2}$, we have

$$p(d_1) \leq p(d_2), \tag{6.1.44}$$

meaning that Λ_{s,d_1} is more smooth than Λ_{s,d_2}. This fact is related to Proposition 6.1.15 which offers a variety of exponential uniform convergence rates, each one depending on a choice of metric in $\mathrm{sM}_d(X,\varphi)$, that can be used to estimate the convergence rates of the limits in Lemma 5.2.8, over the aforementioned Lipschitz subalgebras. Therefore, the question is, *to what extent can we find such smooth Lipschitz subalgebras?*

Definition 6.1.16. For every $d' \in \mathrm{sM}_d(X,\varphi)$ consider the Smale space (X, d', φ) with contraction constant $\lambda_{X,d'} > 1$. The quantity

$$\lambda(X,\varphi) = \sup\{\lambda_{X,d'} : d' \in \mathrm{sM}_d(X,\varphi)\}$$

is called the λ-*number* of the Smale space (X,φ).

It turns out that the λ-numbers (also considered in [7]) are topological invariants of Smale spaces. Before proving this, we present the following lemma whose proof involves a straightforward manipulation of the Smale space axioms, and therefore is omitted.

Lemma 6.1.17. *Let (Y,ψ) be a topological dynamical system and $(Z, d_Z, \zeta, \varepsilon_Z, \lambda_Z)$ be a Smale space with locally defined bracket map $[\cdot,\cdot]_Z$. Suppose that these systems are topologically conjugate via the conjugating homeomorphism $f : (Y,\psi) \to (Z,\zeta)$. Then (Y,ψ) admits a Smale space structure with*

(1) $d_Y(y_1,y_2) = d_Z(f(y_1), f(y_2))$, *for any* $y_1, y_2 \in Y$;

(2) $\varepsilon_Y = \varepsilon_Z$ *and*

$$[y_1,y_2]_Z = f^{-1}[f(y_1), f(y_2)]_Z,$$

whenever $d_Y(y_1,y_2) \leq \varepsilon_Y$;

(3) $\lambda_Y = \lambda_Z$.

Moreover, if $(Z, d_Z, \zeta, \varepsilon_Z, \lambda_Z)$ is self-similar, the Smale space $(Y, d_Y, \psi, \varepsilon_Y, \lambda_Y)$ is too.

Proposition 6.1.18. *If the Smale space* (Y, ψ) *is topologically conjugate to the Smale space* (Z, ζ)*, then* $\lambda(Y, \psi) = \lambda(Z, \zeta)$.

Proof. Let us denote the metrics on Y and Z by d_Y and d_Z. The result follows immediately from Lemma 6.1.17, which shows that

$$\{\lambda_{X,d'} : d' \in \mathrm{sM}_{d_Y}(X, \varphi)\} = \{\lambda_{X,d'} : d' \in \mathrm{sM}_{d_Z}(X, \varphi)\}. \qquad \square$$

Interestingly enough, the λ-numbers seem to capture some topological information about the Smale space. The next result is a consequence of Theorem 3.4.6. First, note that the topological dimension of X is always finite, since X admits an expansive homeomorphism [91] (it also follows from Theorem 3.4.6).

Proposition 6.1.19. *It holds that* $\lambda(X, \varphi) < \infty$ *if and only if* $\dim X > 0$*. Specifically, if* $\dim X > 0$ *then,*

$$\lambda(X, \varphi) \leq e^{2\,\mathrm{h}(\varphi)/\dim X}.$$

Proof. First assume that $\dim X > 0$. Then from Theorem 3.4.6 we obtain that, for every $d' \in \mathrm{sM}_d(X, \varphi)$, the Hausdorff dimension satisfies

$$\dim_H(X, d') = \frac{2\,\mathrm{h}(\varphi)}{\log \lambda_{X,d'}}.$$

From Proposition 2.1.11 we have that $\dim X \leq \dim_H(X, d')$ and therefore, by taking the supremum over $d' \in \mathrm{sM}_d(X, \varphi)$, we obtain that $\lambda(X, \varphi) \leq e^{2\,\mathrm{h}(\varphi)/\dim X}$.

We now prove the converse by showing that, if $\dim X = 0$, then $\lambda(X, \varphi) = \infty$. First, we have that (X, φ) is topologically conjugate to a topological Markov chain (Y, ψ), see [107, Theorem 2.2.8]. As a result, from Proposition 6.1.18 we get that $\lambda(X, \varphi) = \lambda(Y, \psi)$. Now, it is straightforward to see that the ultrametric (3.1.8), which induces the cylinder topology on Y, can have any factor $\lambda > 1$ instead of just $\lambda = 2$. All of these ultrametrics are self-similar for ψ. As a result, $\lambda(Y, \psi) = \infty$. $\square$

Consequently, our quest for finding the smoothest Lipschitz subalgebra (6.1.43) of $\mathcal{R}^s(Q)$, by varying the metric $d' \in \mathrm{sM}_d(X, \varphi)$, has a topological limitation. Another way to optimise even further our approach of using the Alexandroff-Urysohn-Frink Metrisation Theorem, would be to find sharper estimates in Proposition 6.1.15. More precisely, the diameter of the sets $V^s_{d'}(c, h^s, \lambda_{X,d'}^{-N-k}\gamma, N)$ goes exponentially fast to zero, as k goes to infinity. However, the base of the exponent, namely

$$2^{-1/\lceil \log_{\lambda_{X,d'}} 3 \rceil},$$

is by no means random. In fact, it is the best possible (in this generality of stable and unstable groupoids) achieved by this metrisation method. Specifically, the number 3 is related to the following fact: in a metric space (Z,d), given any two sufficiently small intersecting balls $B_d(z_1,r_1)$ and $B_d(z_2,r_2)$, it holds $B_d(z_1,r_1) \subset B_d(z_2,3r)$, where $r = \max\{r_1,r_2\}$. The number 2 is related to the fact that $\rho_{s,d'}$ is a 2-quasimetric, and somehow seems to be important. In [123], the author constructs a K-quasimetric space, with $K > 2$, for which Frink's chain-metric approach does not work.

6.1.4 Groupoid ultrametrics for topological Markov chains

Again, all the following facts are also true for the unstable groupoid. Another natural question to ask is, *why attempt this complicated metrisation method in the first place?* The answer to this is still elusive, but what is certainly true is that the topological dimension of the Smale space being positive complicates the metrisation process.

Indeed, suppose that (X,d_λ,φ) is an irreducible topological Markov chain and d_λ is the usual self-similar ultrametric with expanding factor $\lambda > 1$. Then, for every ultrametric $d' \in \mathrm{sM}_{d_\lambda}(X,\varphi)$, the construction of the groupoid metric $D_{s,d'}$ could be simplified, and the exponential uniform convergence rates of Proposition 6.1.15 could become sharper. More precisely, these exponents have as bases the quantities

$$2^{-1/\lceil \log_{\lambda_{X,d'}} 3\rceil},$$

and since the ceiling function is the result of a triangle inequality (see Lemma 6.1.8), it would no longer be necessary to consider it. In fact, one would expect that, in the zero-dimensional case, the stable groupoid $G^s(Q)$ admits metrics of arbitrarily fast uniform convergence rates, since the λ-number $\lambda(X,\varphi)$ is infinite. However, if we keep the ceiling function, since it is not continuous at zero, the smallest base we can achieve is 2^{-1}. In this subsection, instead of adjusting the demanding computations of Subsection 6.1.2 to the zero-dimensional setting, we present a better suited, but different, family of groupoid ultrametrics for topological Markov chains.

Let us fix an irreducible topological Markov chain (X,d_λ,φ), for some $\lambda > 1$, and the goal is to metrise its stable groupoid $G^s(Q)$. We note that for every $\lambda_1,\lambda_2 > 1$, the metrics $d_{\lambda_1}, d_{\lambda_2}$ on X are equivalent, and therefore, the algebraic and topological structures of $G^s(Q)$ do not depend on a particular choice of metric, as we see in Subsection 6.1.1. Therefore, all the following results hold for every $\lambda > 1$, and the reason we highlight λ is because the family of these new groupoid ultrametrics for $G^s(Q)$ will be indexed by $\lambda \in (1,\infty)$. For simplicity, the elements of $G^s(Q)$ will be denoted by $a = (a_1,a_2)$, $b = (b_1,b_2)$ and $c = (c_1,c_2)$.

We now mention a few basic facts about (X,d_λ,φ). First, the expansivity constant $\varepsilon_X = \lambda^{-1}$. Also, in our calculations there is no need to consider $\varepsilon_X/2$ since d_λ is an ultrametric. Similarly, the constant $\varepsilon'_X \le \varepsilon_X/2$ (see (3.1.5)) is also not needed because, whenever $d_\lambda(x,y) \leq \lambda^{-1}$, then

$$d_\lambda(x,[x,y]),\, d_\lambda(y,[x,y]) \leq d_\lambda(x,y). \tag{6.1.45}$$

As a result, for every $c \in G^s(Q)$ it is enough to consider only the holonomy maps $h^s : X^u(c_2,\eta) \to X^u(c_1,\lambda^{-N-1})$ given by

$$h^s(z) = \varphi^{-N}[\varphi^N(z),\varphi^N(c_1)],$$

where $N \geq 0$ is such that

$$\varphi^N(c_2) \in X^s(\varphi^N(c_1),\lambda^{-1}) \tag{6.1.46}$$

and $\eta \leq \lambda^{-N-1}$. With that said, it can be assumed that the holonomy maps of all basis sets $V^s(c,h^s,\eta,N) \subset G^s(Q)$ are as above. Note that these basis sets will depend on the metric d_λ. It is important to mention that, for every $b \in V^s(c,h^s,\eta,N)$, we also have that $\varphi^N(b_2) \in X^s(\varphi^N(b_1),\lambda^{-1})$. To see this, note that $\varphi^N(b_1) = [\varphi^N(b_2),\varphi^N(c_1)]$, and also that

$$\begin{aligned} d_\lambda(\varphi^N(b_2),[\varphi^N(b_2),\varphi^N(c_1)]) &\leq d_\lambda(\varphi^N(b_2),\varphi^N(c_1)) \\ &\leq \max\{d_\lambda(\varphi^N(b_2),\varphi^N(c_2)), d_\lambda(\varphi^N(c_2),\varphi^N(c_1))\} \\ &\leq \lambda^{-1}. \end{aligned}$$

Moreover, every such h^s is actually an isometry, and hence

$$h^s : X^u(c_2,\eta) \to X^u(c_1,\eta).$$

To see this, first observe that, for every $x,y \in X$ with $d_\lambda(x,y) \leq \lambda^{-1}$, we have

$$d_\lambda(x,[y,x]) = d_\lambda(y,[x,y]). \tag{6.1.47}$$

Let now $a,b \in V^s(c,h^s,\eta,N)$, then $a_2,b_2 \in X^u(c_2,\eta)$ and $a_1,b_1 \in X^u(c_1,\lambda^{-N-1})$. From the previous discussion we obtain that $\varphi^N(b_2) \in X^s(\varphi^N(b_1),\lambda^{-1})$, and hence

$$\begin{aligned} a_1 &= h^s(a_2) \\ &= \varphi^{-N}[\varphi^N(a_2),\varphi^N(c_1)] \\ &= \varphi^{-N}[\varphi^N(a_2),[\varphi^N(c_1),\varphi^N(b_1)]] \\ &= \varphi^{-N}[\varphi^N(a_2),\varphi^N(b_1)]. \end{aligned}$$

Then, since $d_\lambda(\varphi^N(b_2),\varphi^N(a_1)) \leq \lambda^{-1}$, from (6.1.47) it holds that

$$d_\lambda(\varphi^N(b_2),\varphi^N(a_2)) = d_\lambda(\varphi^N(b_1),\varphi^N(a_1)).$$

Finally, since φ^{-1} is the λ^{-1}-multiple of an isometry on local unstable sets, it follows that

$$d_\lambda(b_2, a_2) = d_\lambda(b_1, a_1) = d_\lambda(h^s(b_2), h^s(a_2)).$$

In an attempt to metrise $G^s(Q)$, define $d_{s,\lambda} : G^s(Q) \times G^s(Q) \to [0,1]$ by

$$d_{s,\lambda}(a,b) = \begin{cases} \max\{d_\lambda(a_1,b_1), d_\lambda(a_2,b_2)\}, & \text{if } b_2 \in X^u(a_2,\lambda^{-1}),\, b_1 \in X^u(a_1,\lambda^{-1}) \\ 1, & \text{if else.} \end{cases} \tag{6.1.48}$$

Using the fact that d_λ is an ultrametric, it is not hard to show that $d_{s,\lambda}$ is also an ultrametric. However, it generates a strictly weaker topology than that of $G^s(Q)$ because there is no open ball, with respect to $d_{s,\lambda}$, that fits inside a basis set of the form $V^s(a, h^s, \eta, N)$. The reason is that every global stable set wraps densely around every local unstable set of the same mixing component.

To remedy this, consider the map $c_s : G^s(Q) \to \mathbb{N}$ that takes each $a \in G^s(Q)$ to the first time $N_a \geq 0$; that is, the minimum positive integer for which (6.1.46) holds. One should also compare this definition with the one in (6.1.17). The map c_s depends on d_λ, and the next lemma shows that it is continuous. Therefore, $G^s(Q)$ decomposes into the clopen subsets

$$G^s(Q) = \bigsqcup_{N\geq 0} c_s^{-1}(N). \tag{6.1.49}$$

In what follows, for simplicity, we omit to denote the holonomy maps, unless it is necessary.

Lemma 6.1.20. *If $a \in c_s^{-1}(N)$ and $n \geq N$, then $V^s(a, \lambda^{-n-1}, n) \subset c_s^{-1}(N)$.*

Proof. Due to self-similarity it holds that

$$V^s(a, \lambda^{-n-1}, n) = V^s(a, \lambda^{-n-1}, N).$$

Let $b \in V^s(a, \lambda^{-n-1}, N)$ and we claim that $c_s(b) = N$. We have $\varphi^N(b_2) \in X^s(\varphi^N(b_1), \lambda^{-1})$ and so $c_s(b) \leq N$. Therefore, the basis set $V^s(b, \lambda^{-n-1}, N)$ is well-defined, and one can easily show that it contains a. Now, due to self-similarity it holds that

$$V^s(b, \lambda^{-n-1}, N) = V^s(b, \lambda^{-n-1}, c_s(b)),$$

and similarly one can show that $N = c_s(a) \leq c_s(b)$. This completes the proof. □

Let now $D_{s,\lambda} : G^s(Q) \times G^s(Q) \to [0,1]$ be defined by

$$D_{s,\lambda}(a,b) = \begin{cases} d_{s,\lambda}(a,b), & \text{if } c_s(a) = c_s(b) \\ 1, & \text{otherwise.} \end{cases} \tag{6.1.50}$$

Similarly, it is straightforward to show that $D_{s,\lambda}$ is an ultrametric.

Lemma 6.1.21. *The ultrametric $D_{s,\lambda}$ generates the topology of $G^s(Q)$.*

Proof. Let $a \in G^s(Q)$ and consider the basis set $V^s(a, \lambda^{-n-1}, N)$, where $n \geq N$. Due to self-similarity the basis set is equal to $V^s(a, \lambda^{-n-1}, c_s(a))$. We claim that

$$B_{D_{s,\lambda}}(a, \lambda^{-n-1}) \subset V^s(a, \lambda^{-n-1}, c_s(a)).$$

Indeed, pick b in the open ball. We have that $c_s(b) = c_s(a) = M$ and $b_2 \in X^u(a_2, \lambda^{-n-1})$, $b_1 \in X^u(a_1, \lambda^{-n-1})$. Moreover, $b \in V^s(a, \lambda^{-n-1}, c_s(a))$ because

$$\begin{aligned}\varphi^{-M}[\varphi^M(b_2), \varphi^M(a_1)] &= \varphi^{-M}[\varphi^M(b_2), [\varphi^M(a_1), \varphi^M(b_1)]]\\ &= \varphi^{-M}[\varphi^M(b_2), \varphi^M(b_1)]\\ &= b_1.\end{aligned}$$

For the reverse inclusion, consider the open ball $B_{D_{s,\lambda}}(a, \eta)$, where $\eta \leq \lambda^{-1}$. Then,

$$B_{D_{s,\lambda}}(a, \eta) = \{b \in c_s^{-1}(c_s(a)) : b_2 \in X^u(a_2, \eta),\ b_1 \in X^u(a_1, \eta)\}.$$

Since c_s is continuous, there is $\delta > 0, N \geq 0$ such that $V^s(a, \delta, N) \subset c_s^{-1}(c_s(a))$. Choosing δ small enough gives that $V^s(a, \delta, N) \subset B_{D_{s,\lambda}}(a, \eta)$. As a result, the topology generated by $D_{s,\lambda}$ agrees with the topology of $G^s(Q)$. □

The following statement is the zero-dimensional analogue of Proposition 6.1.15.

Proposition 6.1.22. *Let $c \in G^s(Q)$ and $n \geq c_s(c)$. For every a, b in $V^s(c, \lambda^{-n-1}, n)$ it holds that*

$$D_{s,\lambda}(a, b) \leq \lambda^{-n-1}.$$

Proof. From Lemma 6.1.20 we have that $c_s(a) = c_s(b)$. Moreover, $b_2 \in X^u(a_2, \lambda^{-n-1})$ and since the holonomy map is isometric we also have that $b_1 \in X^u(a_1, \lambda^{-n-1})$. The result follows. □

The ultrametric $D_{s,\lambda}$ is very dynamic in nature. Before proving this, we should note that the automorphism $\Phi = \varphi \times \varphi : G^s(Q) \to G^s(Q)$ shifts the decomposition (6.1.49). More precisely, for every $N \in \mathbb{N}$ it holds that

$$\Phi^{-1}(c_s^{-1}(N)) = c_s^{-1}(N+1), \tag{6.1.51}$$

while the situation for $N = 0$ is different since $c_s^{-1}(1)$ is strictly contained in $\Phi^{-1}(c_s^{-1}(0))$.

Moreover, let $\widetilde{D_{s,\lambda}}$ denote the induced ultrametric on the space of units $X^u(Q)$, just like in (6.1.33). In this case, however, since $X^u(Q)$ embeds in $c_s^{-1}(0)$, it is straightforward to see that

$$\widetilde{D_{s,\lambda}}(x,y) = \begin{cases} d_\lambda(x,y), & \text{if } y \in X^u(x,\lambda^{-1}) \\ 1, & \text{if else.} \end{cases} \tag{6.1.52}$$

The next proposition is the zero-dimensional counterpart of Theorem 6.1.4. We prefer to work with Φ^{-1}.

Proposition 6.1.23. *Let (X, d_λ, φ) be an irreducible topological Markov chain with stable groupoid $G^s(Q)$. Also, for each $\kappa > 1$ consider the equivalent ultrametric d_κ. Then, there is a family of compatible ultrametrics $\{D_{s,\kappa} : \kappa > 1\}$ on $G^s(Q)$ so that, with respect to each of these ultrametrics,*

(1) the groupoid automorphism $\Phi^{-1} : G^s(Q) \to G^s(Q)$ is bi-Lipschitz with

$$\kappa^{-1} D_{s,\kappa}(a,b) \le D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) \le D_{s,\kappa}(a,b),$$

for every $a, b \in G^s(Q)$;

(2) the map Φ^{-1} is locally contracting so that, if $D_{s,\kappa}(a,b) \le \kappa^{-1}$, then

$$D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) = \kappa^{-1} D_{s,\kappa}(a,b);$$

(3) the range and source maps onto $(X^u(Q), \widetilde{D_{s,\kappa}})$ are locally isometric. Moreover, the inversion map is isometric.

Proof. Let $\kappa > 1$ and from Lemma 6.1.21 we know that $D_{s,\kappa}$ generates the topology of $G^s(Q)$. Moreover, assume that $G^s(Q)$ is built from the Smale space (X, d_κ, φ). This assumption can be made since the algebraic and topological structure of $G^s(Q)$ is given also by d_κ (Proposition 6.1.1). This means that all the following notation is relative to d_κ.

To prove parts (1) and (2), let $a, b \in G^s(Q)$ and first assume that $D_{s,\kappa}(a,b) = 1$. Then, it clearly holds that

$$D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) \le D_{s,\kappa}(a,b).$$

We claim that $D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) \ge \kappa^{-1}$. To prove this, assume to the contrary that

$$D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) \le \kappa^{-2}. \tag{6.1.53}$$

Then, $c_s(\Phi^{-1}(a)) = c_s(\Phi^{-1}(b))$

$$\varphi^{-1}(b_2) \in X^u(\varphi^{-1}(a_2), \kappa^{-2}),\ \varphi^{-1}(b_1) \in X^u(\varphi^{-1}(a_1), \kappa^{-2}).$$

Due to self-similarity we get that

$$b_2 \in X^u(a_2, \kappa^{-1}),\ b_1 \in X^u(a_1, \kappa^{-1}).$$

In addition, we have that $c_s(a) = c_s(b)$. Indeed, if $c_s(\Phi^{-1}(a)) = c_s(\Phi^{-1}(b)) \geq 1$, then $c_s(a) = c_s(\Phi^{-1}(a)) - 1$ and $c_s(b) = c_s(\Phi^{-1}(b)) - 1$. If $c_s(\Phi^{-1}(a)) = c_s(\Phi^{-1}(b)) = 0$, we have that $\Phi^{-1}(b) \in V^s(\Phi^{-1}(a), \kappa^{-2}, 0)$, and hence

$$b \in V^s(a, \kappa^{-1}, 0).$$

From Lemma 6.1.20 we get that $c_s(a) = c_s(b)$. To summarise, the assumption (6.1.53) leads to $D_{s,\kappa}(a,b) \leq \kappa^{-1}$, which is a contradiction.

Assume now that $D_{s,\kappa}(a,b) \leq \kappa^{-1}$. Then, $c_s(a) = c_s(b)$ and

$$b_2 \in X^u(a_2, \kappa^{-1}), b_1 \in X^u(a_1, \kappa^{-1}).$$

As a result,

$$\varphi^{-1}(b_2) \in X^u(\varphi^{-1}(a_2), \kappa^{-2}), \varphi^{-1}(b_1) \in X^u(\varphi^{-1}(a_1), \kappa^{-2}),$$

and $c_s(\Phi^{-1}(a)) = c_s(\Phi^{-1}(b))$. For the latter equality note that, if $c_s(a) = c_s(b) \geq 1$, then $c_s(\Phi^{-1}(a)) = c_s(a) + 1$ and $c_s(\Phi^{-1}(b)) = c_s(b) + 1$. Now, if $c_s(a) = c_s(b) = 0$, then $b \in V^s(a, \kappa^{-1}, 0)$ and hence

$$\Phi^{-1}(b) \in V^s(\Phi^{-1}(a), \kappa^{-2}, 1),$$

meaning that $c_s(\Phi^{-1}(a)) = c_s(\Phi^{-1}(b))$. Note that $c_s(\Phi^{-1}(a)) \leq 1$. In general,

$$\begin{aligned} D_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) &= d_{s,\kappa}(\Phi^{-1}(a), \Phi^{-1}(b)) \\ &= \max\{d_\kappa(\varphi^{-1}(a_1), \varphi^{-1}(b_1)), d_\kappa(\varphi^{-1}(a_2), \varphi^{-1}(b_2))\} \\ &= \max\{\kappa^{-1} d_\kappa(a_1, b_1), \kappa^{-1} d_\kappa(a_2, b_2)\} \\ &= \kappa^{-1} d_{s,\kappa}(a,b) \\ &= \kappa^{-1} D_{s,\kappa}(a,b). \end{aligned}$$

For part (3) it suffices to show that, for an arbitrary basis set $V = V^s(c, \kappa^{-n-1}, n)$, the restriction of the source map $s_V : V \to s(V)$ is an isometry. This is because every basis set is open in the topology generated by $D_{s,\kappa}$. Also, recall that s_V is bijective. Let $a, b \in V$, then $c_s(a) = c_s(b)$ and

$$b_2 \in X^u(a_2, \kappa^{-n-1}), b_1 \in X^u(a_1, \kappa^{-n-1}),$$

since $d_\kappa(a_2, b_2) = d_\kappa(a_1, b_1)$ (recall that all holonomy maps are isometric). Therefore,

$$\begin{aligned} D_{s,\kappa}(a,b) &= \max\{d_\kappa(a_1,b_1), d_\kappa(a_2,b_2)\} \\ &= d_\kappa(a_2,b_2) \\ &= \widetilde{D_{s,\kappa}}(s(a),s(b)). \end{aligned}$$

Similarly, one can show that the range map is locally isometric. The fact that the inversion is an isometry is straightforward. □

At this stage, it is still unclear if a similar approach could work for general Smale spaces, and this is the main reason for considering Theorem 6.1.4. But it should be mentioned that, if such metrisation approach is possible, the metrics might be very different from the ones obtained in Theorem 6.1.4. In fact, for the metrics $D_{s,\kappa}$ and D_{s,d_κ}, where $\kappa > 1$, even though they generate the same topology and have similar dynamical behaviour, one can see that they are not uniformly equivalent.

Specifically, there exist sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \subset G^s(Q)$ and some $t > 0$ such that $\lim_n D_{s,\kappa}(a_n,b_n) = 0$ while $D_{s,d_\kappa}(a_n,b_n) > t$, for all $n \geq 0$. The only reason this is happening is because $\lceil \log_{\lambda_{X,d_\kappa}} 3 \rceil$ is used in the definition of the sequences $j(a,n)$ in (6.1.21), which is related to a triangle inequality, see Lemma 6.1.8. However, if we remove $\lceil \log_{\lambda_{X,d_\kappa}} 3 \rceil$ from the definition of D_{s,d_κ} (in the case of SFT is no longer needed) the two metrics become equivalent.

In any case, Proposition 6.1.23 (just like Theorem 6.1.4) is exactly what we need to construct dense $\star$-subalgebras of $\mathcal{S}(Q)$, which are α_s-invariant and hence induce dense $\star$-subalgebras of $\mathcal{R}^s(Q)$, see Subsection 6.2.2.

6.2 Smoothness of Ruelle algebras

In this section we prove one of our main results; the stable and unstable Ruelle algebras are uniformly $\mathcal{L}^p$-smooth on holomorphically stable dense $\star$-subalgebras containing compactly supported Lipschitz functions. The number p depends on the topological entropy and the λ-number of the underlying Smale space, which is related to its topological dimension. This is made precise in Corollary 6.2.19. Also, in Corollary 6.2.21 we obtain the exact degree of irregularity (see Remark 4.2.6) of stabilised Cuntz-Krieger algebras. We refer the reader to Subsection 4.2.1 for details on smooth C^*-algebras. To achieve all these, we first prove that the KPW-extension (see Definition 5.2.11) is finitely-smooth. This is done in Theorems 6.2.16 and 6.2.20. An important tool in this endeavour is the K-theoretic Proposition 4.2.21.

We begin with Subsection 6.2.1 where we construct dense Lipschitz $\star$-subalgebras of C^*-algebras of étale groupoids equipped with metrics. Then, using the Smale space groupoid metrics of Theorem 6.1.4 and Proposition 6.1.23, we construct dense Lipschitz $\star$-subalgebras on which the Ruelle algebras are uniformly finitely-smooth. In the case of stabilised Cuntz-Krieger algebras, which correspond to topological Markov chains, we obtain that the degree of irregularity is zero, in other words, it agrees with the topological dimension of the underlying Smale space. This is related to the result of Goffeng and Mesland [58, Theorem 7] for non-stabilised Cuntz-Krieger algebras. However, our approach differs from theirs and provides a dynamical explanation of why the degree of irregularity is zero. Finally, all these results are essential for building a KK_1-lift of the KPW-extension class in Section 7.2.

6.2.1 Lipschitz algebras on general étale groupoids

We begin by briefly mentioning a few basic facts about étale groupoids and then we show that on étale groupoids equipped with a nice metric, one can construct dense Lipschitz $\star$-subalgebras of the corresponding groupoid C^*-algebras. We should note that all the following results seem to be classical. However, we were not able to find a reference in the literature, hence we present their proofs. For details about étale groupoids we refer the reader to [127].

Let G be a second countable, locally compact, étale groupoid whose topology is induced by a metric d_G, and let $G^{(0)}$ be the *space of units*. For simplicity assume that $G^{(0)} \subset G$. By $r, s : G \to G^{(0)}$ we denote the *range* and *source maps* defined as

$$r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) = \gamma^{-1}\gamma, \tag{6.2.1}$$

and by $G^{(2)}$ the *set of composable pairs*

$$G^{(2)} = \{(\alpha, \beta) \in G \times G : s(\alpha) = r(\beta)\}, \tag{6.2.2}$$

which is closed in $G \times G$ since $G^{(0)}$ is Hausdorff. On $G^{(2)}$ we define the *multiplication map* $m : G^{(2)} \to G$, denoted by $(\alpha, \beta) \mapsto \alpha\beta$, and for any $A, B \subset G$ we define $AB \subset G$ as

$$AB = m((A \times B) \cap G^{(2)}). \tag{6.2.3}$$

One can observe that, if $A, B \subset G$ are (pre-)compact, then AB is also (pre-)compact. Indeed, since $A \times B$ is (pre-)compact in $G \times G$ and $G^{(2)}$ is closed, then $(A \times B) \cap G^{(2)}$ is (pre-)compact in $G^{(2)}$ and hence AB is also (pre-)compact in G.

The groupoid being étale means that r, s are local homeomorphisms. In fact, there is an abundance of sets, called bisections, on which r, s are simultaneously local homeomorphisms. More precisely, a subset $B \subset G$ is a *bisection* if there is an open $V \subset G$ containing B such that, the restrictions $r_V : V \to r(V)$ and $s_V : V \to s(V)$ are homeomorphisms, with respect to the topology of G. This means that $G^{(0)}$ is open in G and hence the r, s-fibres are discrete.

Since G is second countable and Hausdorff, its topology has a countable basis $\mathcal{B}$ of open bisections. In addition, since G is locally compact, the basis can be chosen to consist of open, pre-compact bisections. Moreover, since G is étale we get that the *multiplication map* $m : G^{(2)} \to G$ is open. In particular, if $A, B \subset G$ are open, then AB is also open. An important fact to notice is that, if A, B are open, pre-compact bisections, then AB is an open, pre-compact bisection.

Let us now consider the complex vector space $C_c(G)$ which becomes a $*$-algebra with *convolution product* given by

$$(f \cdot g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta), \tag{6.2.4}$$

and *involution* given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}. \tag{6.2.5}$$

Moreover, let π be a faithful representation of $C_c(G)$ on a Hilbert space H and $C^*_\pi(G)$ to denote the completion of $\pi(C_c(G))$ in $\mathcal{B}(H)$. Since π is faithful we will often consider $C_c(G) \subset C^*_\pi(G)$ by suppressing the notation of π. The main result in this subsection is the following.

Proposition 6.2.1. *Suppose that $r, s : G \to G^{(0)}$ are locally bi-Lipschitz with respect to the metric d_G. Then, the complex vector space of compactly supported Lipschitz functions* $\mathrm{Lip}_c(G, d_G)$ *is a dense $*$-subalgebra of $C^*_\pi(G)$.*

The proof of Proposition 6.2.1 is achieved by establishing all the following lemmas.

Lemma 6.2.2. *Let (Z, d_Z) be a locally compact metric space. Then, the $*$-subalgebra of compactly supported complex valued Lipschitz functions* $\mathrm{Lip}_c(Z, d_Z)$ *is dense in $C_0(Z)$, with respect to the sup-norm.*

Proof. We use the Stone-Weierstrass Theorem for locally compact Hausdorff spaces, see [37, Theorem A.10.1]. Therefore, we have to prove that

(i) $\mathrm{Lip}_c(Z, d_Z)$ is closed under complex conjugation;

(ii) for every $x \in Z$, there is $f \in \mathrm{Lip}_c(Z, d_Z)$ such that $f(x) \neq 0$;

(iii) for every $x \neq y \in Z$, there is $f \in \mathrm{Lip}_c(Z, d_Z)$ such that $f(x) \neq f(y)$.

Part (i) clearly holds. For part (ii), let $x \in Z$ and choose $\ell_x > 0$ so small that $\overline{B}(x, \ell_x)$ is compact. Cover Z by

$$\mathcal{U} = \{B(z, \ell_x/4) : z \in Z\},$$

and from [88, Theorem 5.3] there is a partition of unity $\mathcal{F}_{\mathcal{U}}$, consisting of locally Lipschitz functions, subordinated to $\mathcal{U}$. Then, there is $f \in \mathcal{F}_{\mathcal{U}}$ such that $f(x) \neq 0$, meaning that

$x \in \operatorname{supp}(f)$, and hence

$$\operatorname{supp}(f) \subset \operatorname{st}(x, \mathcal{U}) \subset B(x, \ell_x/2).$$

From this we get that f is compactly supported. Let now $\varepsilon > 0$ be so small that the ε-neighbourhood of $\operatorname{supp}(f)$, that is,

$$N_\varepsilon(f) = \bigcup_{z \in \operatorname{supp}(f)} B(z, \varepsilon),$$

is contained in $B(x, \ell_x)$ and hence $\operatorname{cl}(N_\varepsilon(f))$ is compact. From the proof of Lemma 6.1.13 we see that the restriction of f on $\operatorname{cl}(N_\varepsilon(f))$ is Lipschitz. In fact, f is globally Lipschitz because, if $M > 0$ is such that $|f(z)| \leq M$, for all $z \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(f)$ and $w \in Z \setminus \operatorname{cl}(N_\varepsilon(f))$, then

$$\frac{|f(y) - f(w)|}{d_Z(y, w)} \leq \frac{M}{\varepsilon}.$$

Consequently, $f \in \operatorname{Lip}_c(Z, d_Z)$ and $f(x) \neq 0$.

We work similarly for part (iii). Suppose $x \neq y \in Z$. Then we can find $\ell_x > 0$ and $\ell_y > 0$ such that,

$$B(x, \ell_x) \cap B(y, \ell_y) = \varnothing, \tag{6.2.6}$$

and $\overline{B}(x, \ell_x)$, $\overline{B}(y, \ell_y)$ are compact. Let $\ell = 4^{-1} \min\{\ell_x, \ell_y\}$ and consider the cover

$$\mathcal{V} = \{B(z, \ell) : z \in Z\}.$$

There exists a partition of unity $\mathcal{F}_\mathcal{V}$ which consists of locally Lipschitz functions and is subordinated to $\mathcal{V}$. Let $f \in \mathcal{F}_\mathcal{V}$ such that $f(x) \neq 0$. Then

$$\operatorname{supp}(f) \subset \operatorname{st}(x, \mathcal{V}) \subset B(x, \ell_x/2),$$

hence $f \in \operatorname{Lip}_c(Z, d_Z)$ and from (6.2.6) we obtain that $f(y) = 0$. □

Recall that G is a second countable, locally compact, étale groupoid equipped with a metric d_G. Also, the collection $\mathcal{B}$ is a countable basis of open, pre-compact bisections.

Lemma 6.2.3. *It holds that*

$$\operatorname{Lip}_c(G, d_G) = \operatorname{span}\{f \in \operatorname{Lip}_c(G, d_G) : \textit{there is } B \in \mathcal{B} \textit{ such that } \operatorname{supp}(f) \subset B\}.$$

Proof. Let $f \in \operatorname{Lip}_c(G, d_G)$ and since $\operatorname{supp}(f)$ is compact there is a finite collection $\{B_i\}_{i=1}^n \subset \mathcal{B}$ such that $\operatorname{supp}(f) \subset \bigcup_{i=1}^n B_i$. Consider the open cover of Z

$$\mathcal{U} = \{B_i\}_{i=1}^n \cup \{Z \setminus \operatorname{supp}(f)\},$$

and let $\mathcal{F} = \{f_i\}_{i=1}^{n+1}$ be a partition of unity subordinated to $\mathcal{U}$ in an index-wise manner, which consists of locally Lipschitz functions. Then, we claim that, for every $i \in \{1, \dots, n\}$, the function f_i is globally Lipschitz with compact support. Indeed, we have $\operatorname{supp}(f_i) \subset B_i$ and since B_i is pre-compact, then $\operatorname{supp}(f_i)$ is compact. For this reason,

$$\operatorname{dist}(\operatorname{supp}(f_i), Z \setminus B_i) = \varepsilon_i > 0,$$

and for the $\varepsilon_i/2$-neighbourhood $N_{\varepsilon_i/2}(\operatorname{supp}(f_i))$ of $\operatorname{supp}(f_i)$ we have that

$$N_{\varepsilon_i/2}(\operatorname{supp}(f_i)) \subset B_i.$$

Therefore, $\operatorname{cl}(N_{\varepsilon_i/2}(\operatorname{supp}(f_i)))$ is compact, and working as in Lemma 6.2.2 we obtain that f_i is Lipschitz.

Since $\mathcal{F}$ is a partition of unity it holds that $f = \sum_{i=1}^{n+1} f f_i$. In fact, from the choice of the cover $\mathcal{U}$, it holds that

$$f = \sum_{i=1}^{n} f f_i. \tag{6.2.7}$$

By observing that $f f_i \in \operatorname{Lip}_c(G, d_G)$ and $\operatorname{supp}(f f_i) \subset B_i$ we conclude the proof. The other inclusion is straightforward. □

Lemma 6.2.4. *It holds that* $\operatorname{Lip}_c(G, d_G)$ *is a dense subspace of* $C_\pi^*(G)$.

Proof. It suffices to prove that, for $f \in C_c(G)$ and $\varepsilon > 0$, there is $g \in \operatorname{Lip}_c(G, d_G)$ such that $\|\pi(f) - \pi(g)\| < \varepsilon$. From [127, Prop. 3.2.1], for every $h \in C_c(G)$, there is a constant $K_h \geq 0$ such that, for every representation ρ of $C_c(G)$ on a Hilbert space, it holds that $\|\rho(h)\| \leq K_h$. Using a partition of unity argument like in Lemma 6.2.3 (where the partition functions are not required to be Lipschitz or locally Lipschitz), we can write h as a finite sum $\sum_i h_i$, with each h_i being compactly supported on a bisection in $\mathcal{B}$. Then, according to [127] we can choose

$$K_h = \sum_i \|h_i\|_\infty.$$

In particular, for the representation π, we get that

$$\|\pi(h)\| \leq \sum_i \|h_i\|_\infty. \tag{6.2.8}$$

In our case, working similarly as in Lemma 6.2.3 (see also (6.2.7)), we can write

$$f = \sum_{i=1}^{n} f f_i,$$

where f_i are Lipschitz, compactly supported on bisections in $\mathcal{B}$. Let $M = \max_{1 \leq i \leq n} \|f_i\|_\infty$, and from Lemma 6.2.2 there is $\widetilde{g} \in \mathrm{Lip}_c(G, d_G)$ so that $\|f - \widetilde{g}\|_\infty < \varepsilon(nM)^{-1}$. Let

$$g = \sum_{i=1}^{n} \widetilde{g} f_i,$$

and note that $g \in \mathrm{Lip}_c(G, d_G)$ and that each $(f - \widetilde{g}) f_i$ is compactly supported on a bisection in $\mathcal{B}$. Then, from (6.2.8) we get that $\|\pi(f) - \pi(g)\| < \varepsilon$ and the proof is complete. □

Lemma 6.2.5. *Supppose that the maps r, s are locally bi-Lipschitz with respect to d_G. Then, for every $f, g \in \mathrm{Lip}_c(G, d_G)$, the convolution $f \cdot g \in \mathrm{Lip}_c(G, d_G)$. Consequently, $\mathrm{Lip}_c(G, d_G)$ is a $*$-subalgebra of $C^*_\pi(G)$.*

Proof. Let $f, g \in \mathrm{Lip}_c(G, d_G)$, and by Lemma 6.2.3 it suffices to assume that $\mathrm{supp}(f) \subset A$, $\mathrm{supp}(g) \subset B$, where $A, B \in \mathcal{B}$. First we note that, for all $\gamma = \alpha\beta \in AB$ it holds that, $r_{AB}(\gamma) = r_A(\alpha)$ and $s_{AB}(\gamma) = s_B(\beta)$ and as a result,

$$\alpha = r_A^{-1}(r_{AB}(\gamma)), \;\; \beta = s_B^{-1}(s_{AB}(\gamma)). \tag{6.2.9}$$

From the definition of the convolution (6.2.4) we see that, if $(f \cdot g)(\gamma) \neq 0$, then there is a pair $(\alpha, \beta) \in G^{(2)}$ such that $\gamma = \alpha\beta$ and $f(\alpha)g(\beta) \neq 0$. Therefore, $\alpha \in A$, $\beta \in B$ and hence α, β are uniquely determined by γ as in (6.2.9).

So, if $\gamma \in AB$, then

$$(f \cdot g)(\gamma) = (f \circ r_A^{-1} \circ r_{AB})(\gamma)(g \circ s_B^{-1} \circ s_{AB})(\gamma), \tag{6.2.10}$$

and if $\gamma \notin AB$, then $(f \cdot g)(\gamma) = 0$. Moreover, $\mathrm{supp}(f)\,\mathrm{supp}(g)$ is a compact subset of AB and since

$$\mathrm{supp}(f \cdot g) \subset \mathrm{supp}(f)\,\mathrm{supp}(g),$$

we obtain that $\mathrm{supp}(f \cdot g)$ is also compact.

Assume now that $f \cdot g \neq 0$ and hence $AB \neq \varnothing$. Then, since r, s are locally bi-Lipschitz and A, B, AB are open, the restricted maps r_A, s_B, r_{AB}, s_{AB} are also locally bi-Lipschitz with respect to d_G. In particular, the inverses r_A^{-1}, s_B^{-1} are locally bi-Lipschitz. Also, since $\mathrm{supp}(f \cdot g)$ is compact in the open AB, we can find an ε-neighbourhood N_ε of $\mathrm{supp}(f \cdot g)$ such that $\mathrm{cl}(N_\varepsilon) \subset AB$. From the discussion so far we note that $r_A^{-1} \circ r_{AB}$ and $s_B^{-1} \circ s_{AB}$ are locally bi-Lipschitz homeomorphisms. Then, from Lemma 6.1.13 we obtain that $r_A^{-1} \circ r_{AB}$ and $s_B^{-1} \circ s_{AB}$ are bi-Lipschitz on the compact $\mathrm{cl}(N_\varepsilon)$, and since f, g are Lipschitz with compact support we get that $f \cdot g$ is Lipschitz on $\mathrm{cl}(N_\varepsilon)$. Since $f \cdot g$ vanishes outside of $\mathrm{cl}(N_\varepsilon)$, working as in Lemma 6.2.2 gives that $f \cdot g$ is globally Lipschitz. □

6.2.2 Lipschitz subalgebras of Ruelle algebras

We begin by proving that the groupoid metrics of Theorem 6.1.4 and Proposition 6.1.23 yield dense Lipschitz $\star$-subalgebras of the stable and unstable Ruelle algebras. Then, we study commutation relations between stable and unstable Lipschitz algebras. Eventually, this study leads to Theorems 6.2.16 and 6.2.20 which are about smooth extensions of Ruelle algebras.

Let (X,d,φ) be an irreducible Smale space and fix two periodic orbits Q,P such that $Q\cap P=\varnothing$. The étale topology of $G^s(Q)$ is generated by the basis of bisections of Theorem 5.1.1. In fact, any two hyperbolic metrics in $\mathrm{hM}_d(X,\varphi)$ produce bases of bisections that generate the exact same topology. For $d'\in \mathrm{hM}_d(X,\varphi)$, this basis is denoted by $\mathcal{B}^s(Q,d')$. Of course, the same hold for $G^u(P)$, and for $d'\in \mathrm{hM}_d(X,\varphi)$ the corresponding basis is denoted by $\mathcal{B}^u(P,d')$. For all this see Section 6.1.

Every basis $\mathcal{B}^s(Q,d')$, for $d'\in \mathrm{hM}_d(X,\varphi)$, can be used to represent the continuous compactly supported functions in $\mathcal{S}(Q)$ and $\mathcal{R}^s(Q)$. Similarly for $\mathcal{U}(P)$ and $\mathcal{R}^u(P)$. For these facts we refer to Subsection 5.1.2. Moreover, if in particular $d'\in \mathrm{sM}_d(X,\varphi)$, the basis $\mathcal{B}^s(Q,d')$ yields a compatible dynamical metric $D_{s,d'}$ on $G^s(Q)$, as we see in Theorem 6.1.4. Similarly, one has a metric $D_{u,d'}$ on $G^u(P)$. If (X,d,φ) is a topological Markov chain, a different construction (Proposition 6.1.23) produces ultrametrics $D_{s,\kappa}$ and $D_{u,\kappa}$ on $G^s(Q)$ and $G^u(P)$, respectively, for every expanding factor $\kappa>1$.

All these aforementioned facts combined with the results of Subsection 6.2.1 allow us to deduce the following.

Proposition 6.2.6. *Let $d'\in sM_d(X,\varphi)$. The complex vector space of compactly supported Lipschitz functions $\mathrm{Lip}_c(G^s(Q),D_{s,d'})$ forms a dense $\star$-subalgebra of $\mathcal{S}(Q)$. Moreover, it is α_s-invariant, and therefore, the algebraic crossed product*

$$\Lambda_{s,d'}(Q,\alpha_s)=\mathrm{Lip}_c(G^s(Q),D_{s,d'})\rtimes_{\alpha_s,alg}\mathbb{Z}$$

is a well-defined dense $\star$-subalgebra of $\mathcal{R}^s(Q)$. Similarly, in the unstable case we obtain the dense $\star$-subalgebra

$$\Lambda_{u,d'}(P,\alpha_u)=\mathrm{Lip}_c(G^u(P),D_{u,d'})\rtimes_{\alpha_u,alg}\mathbb{Z}$$

of the Ruelle algebra $\mathcal{R}^u(P)$.

Proof. We only prove it for the stable case. From Theorem 6.1.4 we have that the range and source maps of $G^s(Q)$ are locally bi-Lipschitz with respect to $D_{s,d'}$. Therefore, from Proposition 6.2.1 we obtain that $\mathrm{Lip}_c(G^s(Q),D_{s,d'})$ is a convolution $\star$-algebra whose image in $\mathcal{S}(Q)$ is dense.

Moreover, the groupoid automorphism $\Phi = \varphi \times \varphi : G^s(Q) \to G^s(Q)$ is bi-Lipschitz with respect to $D_{s,d'}$ (Theorem 6.1.4), and since for $a \in C_c(G^s(Q))$ we have $\alpha_s(a) = a \circ \Phi^{-1}$, then

$$\alpha_s(\mathrm{Lip}_c(G^s(Q), D_{s,d'})) = \mathrm{Lip}_c(G^s(Q), D_{s,d'}).$$

As a result, we can form the algebraic crossed product $\Lambda_{s,d'}(Q,\alpha_s)$ which is a $\ast$-subalgebra of $\mathcal{R}^s(Q)$, see Remark 5.1.6. Finally, $\Lambda_{s,d'}(Q,\alpha_s)$ is dense in $\mathcal{R}^s(Q)$ since $\mathrm{Lip}_c(G^s(Q), D_{s,d'})$ is dense in $\mathcal{S}(Q)$, see [135, Remark 2.30]. □

In exactly the same way, the ultrametrics of Proposition 6.1.23 allow us to build (roughly speaking) dense Lipschitz $\ast$-subalgebras of stabilised Cuntz-Krieger algebras.

Proposition 6.2.7. *Suppose that (X, d, φ) is an irreducible topological Markov chain. For all $\kappa > 1$, the vector space $\mathrm{Lip}_c(G^s(Q), D_{s,\kappa})$ forms a dense $\ast$-subalgebra of $\mathcal{S}(Q)$. Moreover, it is α_s-invariant, and therefore, the algebraic crossed product*

$$\Lambda_{s,\kappa}(Q,\alpha_s) = \mathrm{Lip}_c(G^s(Q), D_{s,\kappa}) \rtimes_{\alpha_s, alg} \mathbb{Z}$$

is a well-defined dense $\ast$-subalgebra of $\mathcal{R}^s(Q)$. Similarly, in the unstable case we obtain the dense $\ast$-subalgebra

$$\Lambda_{u,\kappa}(P,\alpha_u) = \mathrm{Lip}_c(G^u(P), D_{u,\kappa}) \rtimes_{\alpha_u, alg} \mathbb{Z}$$

of the Ruelle algebra $\mathcal{R}^u(P)$.

Lets us briefly recall the definition of the Spanier-Whitehead K-homology duality class Δ in $\mathrm{KK}_1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P), \mathbb{C})$ in terms of the KPW-extension τ_Δ. For more details see Subsection 5.2.2.

The extension τ_Δ is given by the product (in the Calkin algebra) of two faithful representations $\overline{\rho_s} : \mathcal{R}^s(Q) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ and $\overline{\rho_u} : \mathcal{R}^u(P) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ which commute modulo $\mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$. Here $\mathscr{H} = \ell^2(X^h(P,Q))$. On elementary tensors (and generators of each factor), the extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ is given by

$$\begin{aligned} \tau_\Delta(au^j \otimes bu^{j'}) &= \overline{\rho_s}(au^j)\overline{\rho_u}(bu^{j'}) + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})) \\ &= \sum_{n\in\mathbb{Z}} \alpha_s^n(a) bu^{j'} \otimes e_{n,n+j-j'} + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})), \end{aligned} \tag{6.2.11}$$

where $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$, $j, j' \in \mathbb{Z}$, and $e_{n,m}$ are matrix units.

Remark 6.2.8. Let $d' \in \mathrm{sM}_d(X,\varphi)$, and our goal is to study τ_Δ on the dense $\ast$-subalgebra $\Lambda_{s,d'}(Q,\alpha_s) \otimes_{\mathrm{alg}} \Lambda_{u,d'}(P,\alpha_u)$. Due to the linearity of $\overline{\rho_s} \cdot \overline{\rho_u}$, our arguments can be simplified. First, since the tensor product is algebraic, we can simply work on elementary tensors $x \otimes y$, for $x \in \Lambda_{s,d'}(Q,\alpha_s)$, $y \in \Lambda_{u,d'}(P,\alpha_u)$. Further, since both factors of the tensor algebra are algebraic crossed products, it suffices to work on generators of the form $au^j \otimes bu^{j'}$, where $a \in \mathrm{Lip}_c(G^s(Q), D_{s,d'})$, $b \in \mathrm{Lip}_c(G^u(P), D_{u,d'})$ and $j, j' \in \mathbb{Z}$. Finally, since the groupoids are étale, we can also assume that a and b are supported on bisections. Of course, the same reductions hold for all $d' \in \mathrm{sM}_d(X, \varphi)$, and the algebras $\Lambda_{s,\kappa}(Q, \alpha_s) \otimes_{\mathrm{alg}} \Lambda_{u,\kappa}(P, \alpha_u)$, where $\kappa > 1$.

At this point our intentions should be clear. All these Lipschitz algebras will be shown to be "smooth" subalgebras of Ruelle algebras, which can also be extended to holomorphically stable ones. In order to do this, we have to study the smoothness of the KPW-extension on these Lipschitz algebras. In fact, we aim to investigate how much the algebras $\overline{\rho_s}(\Lambda_{s,d'}(Q, \alpha_s))$ and $\overline{\rho_u}(\Lambda_{u,d'}(P, \alpha_u))$ commute, for $d' \in \mathrm{sM}_d(X, \varphi)$. This will yield results for general Smale spaces. Now, if (X, d, φ) is a topological Markov chain, we will be able to obtain sharp estimates by studying how much the algebras $\overline{\rho_s}(\Lambda_{s,\kappa}(Q, \alpha_s))$ and $\overline{\rho_u}(\Lambda_{u,\kappa}(P, \alpha_u))$ commute, for $\kappa > 1$. Since τ_Δ in (6.2.11) is given by off-diagonal matrices, the next lemma about singular values will be useful.

Lemma 6.2.9. *Consider a compact operator $T = \bigoplus_{n \in \mathbb{N}} T_n \in \mathcal{B}(\bigoplus_{n \in \mathbb{N}} H_n)$. Assume that there are constants $C_1, C_2 > 0$ so that, for every $\varepsilon > 0$ there are $n_0 \in \mathbb{N}$ and $\alpha_\varepsilon, \beta_\varepsilon > 1$ such that, for all $n \geq n_0$, it holds*

(1) $\mathrm{rank}(T_n) \leq C_1 \alpha_\varepsilon^n$;

(2) $\|T_n\| \leq C_2 \beta_\varepsilon^{-n}$.

Then, for every $\varepsilon > 0$ it holds $s_n(T) = O(n^{-\log_{\alpha_\varepsilon} \beta_\varepsilon})$. Consequently, if we also assume that $\alpha_\varepsilon, \beta_\varepsilon$ converge to $\alpha, \beta > 1$ as ε approaches zero, then $(s_n(T))_{n \in \mathbb{N}} \in \ell^p(\mathbb{N})$, for every $p > \log_\beta \alpha$. Moreover, if (2) is replaced by $\|T_n\| \leq C_2 n^{-\gamma_\varepsilon}$ for some $\gamma_\varepsilon > 0$ then, for every $\varepsilon > 0$ we have $s_n(T) = O((\log n)^{-\gamma_\varepsilon})$.

Proof. Let $\varepsilon > 0$ and consider $n_0 \in \mathbb{N}$ and $\alpha_\varepsilon, \beta_\varepsilon > 1$ as in the statement. For every $n \in \mathbb{N}$, define $R_n = \sum_{i=1}^n \mathrm{rank}(T_i) + 1$. The claim is that, for every $n \geq n_0$, we have

$$s_{R_n}(T) \leq C_2 \beta_\varepsilon^{-n-1}.$$

Indeed, from [60, Chapter II] we know that, for any two compact operators W_1, W_2 acting on the same Hilbert space, the singular values satisfy

$$s_{n+m-1}(W_1 + W_2) \leq s_n(W_1) + s_m(W_2),$$

for all $n, m \in \mathbb{N}$. So, if $W_1 = \bigoplus_{i=1}^n T_i$ and $W_2 = \bigoplus_{i=n+1}^\infty T_i$ are seen as operators in $\mathcal{B}(\bigoplus_{i \in \mathbb{N}} H_i)$, then $W_1 + W_2 = T$ and

$$s_{R_n}(W_1) = 0,\ s_1(W_2) \leq C_2 \beta_\varepsilon^{-n-1}.$$

Also, for $n \geq n_0$, we have that

$$R_n \leq R_{n_0-1} + C_1 \sum_{j=n_0}^{n} \alpha_\varepsilon^j.$$

For simplicity, the right hand side can be written as $R'_n = Q_{n_0} + C_1\alpha_\varepsilon^{n+1}/(\alpha_\varepsilon - 1)$, where $Q_{n_0} = R_{n_0-1} - C_1\alpha_\varepsilon^{n_0}/(\alpha_\varepsilon - 1)$. Therefore, for every $n \geq n_0$ we have that

$$s_{R'_n}(T) \leq s_{R_n}(T) \leq C_2\beta_\varepsilon^{-n-1}. \tag{6.2.12}$$

Since $(R'_n)_{n\geq n_0}$ is increasing to infinity, for every $m \geq R'_{n_0}$ we can find $n \geq n_0$ such that $R'_n \leq m < R'_{n+1}$. Note that $m > Q_{n_0}$, and since $m < R'_{n+1}$ we get that

$$n > \log_{\alpha_\varepsilon}(m - Q_{n_0}) + \log_{\alpha_\varepsilon}((\alpha_\varepsilon - 1)/C_1) - 2.$$

Finally, from $R'_n \leq m$ and (6.2.12) we obtain that

$$\begin{aligned} s_m(T) &\leq C_2\beta_\varepsilon^{-n-1} \\ &\leq C_2 C_{\alpha_\varepsilon,\beta_\varepsilon}\beta_\varepsilon^{-\log_{\alpha_\varepsilon}(m-Q_{n_0})} \\ &= C_2 C_{\alpha_\varepsilon,\beta_\varepsilon}(m - Q_{n_0})^{-\log_{\alpha_\varepsilon}\beta_\varepsilon}, \end{aligned}$$

where $C_{\alpha_\varepsilon,\beta_\varepsilon} = \beta_\varepsilon^{-\log_{\alpha_\varepsilon}((\alpha_\varepsilon-1)/C_1)+1}$.

To summarise, for every $\varepsilon > 0$ there are $n_\varepsilon \in \mathbb{N}$ and $C_\varepsilon > 0$ such that, for all $n \geq n_\varepsilon$, we have

$$s_n(T) \leq C_\varepsilon n^{-\log_{\alpha_\varepsilon}\beta_\varepsilon}.$$

This means, for all $\varepsilon > 0$ and $p > \log_{\beta_\varepsilon}\alpha_\varepsilon$ one has $(s_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. As a result, if $\alpha_\varepsilon, \beta_\varepsilon$ converge to $\alpha, \beta > 1$ as ε approaches zero, then for every $p > \log_\beta \alpha$ we have $(s_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Finally, the case where $(\|T_n\|)_{n\in\mathbb{N}}$ goes polynomially fast to zero is dealt in exactly the same way. Just note that $(\log_{a_\varepsilon} n)^{-\gamma_\varepsilon} = O((\log n)^{-\gamma_\varepsilon})$. □

We are now in position to study commutation relations between the aforementioned stable and unstable Lipschitz algebras. Lemma 6.2.9 suggests that the Hilbert space $\mathscr{H} \otimes \ell^2(\mathbb{Z})$ should also be seen as $\bigoplus_{n\in\mathbb{Z}} \mathscr{H}$, and that it suffices to estimate the ranks and norms of the off-diagonal entries of the operators $\overline{\rho_s}(au^j)\overline{\rho_u}(bu^{j'}) - \overline{\rho_u}(bu^{j'})\overline{\rho_s}(au^j)$ in $\mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ that appear in (6.2.11). We begin with the following lemma that is derived from [84, Theorem 2.3].

Lemma 6.2.10. *Let (Y, ψ) be a mixing Smale space and consider $x, y \in Y$. Let $B \subset X^u(x)$ and $C \subset X^s(y)$ be open with compact closure. Then, for every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have*

$$\#\psi^k(B) \cap C < e^{(\mathrm{h}(\psi)+\varepsilon)k}.$$

Proof. Let $\varepsilon > 0$ and define $h^k_{B,C} = \psi^k(B) \cap \psi^{-k}(C)$, for $k \geq 0$. In [84, Theorem 2.3] it is proved that

$$\lim_{k\to\infty} \frac{\log(\# h^k_{B,C})}{2k} = \mathrm{h}(\psi),$$

and hence there is some $j_0 \in \mathbb{N}$ so that, for every $k \geq j_0$, one has

$$\#\psi^{2k}(B) \cap C < e^{(\mathrm{h}(\psi)+\varepsilon)2k}.$$

Doing the same for $\psi(B)$ in place of B, we obtain some $j_0' \in \mathbb{N}$ such that, for every $k \geq j_0'$, we have

$$\#\psi^{2k+1}(B) \cap C < e^{(\mathrm{h}(\psi)+\varepsilon)2k}.$$

Let $k_0 = 2\max\{j_0, j_0'\}$ and the proof is complete. □

Recall Smale's Decomposition Theorem 3.1.2. Here we will use a different notation. Let $X_1, \ldots, X_M$ be the decomposition of the irreducible (X, φ) into disjoint clopen sets which are cyclically permuted by φ, and where $\varphi^M|_{X_i}$ is mixing, for all $1 \leq i \leq M$. The next result does not dependent on any choice of hyperbolic metric in $\mathrm{hM}_d(X, \varphi)$, and this is why we do not highlight any metric on (X, φ).

Lemma 6.2.11. *Let $a \in C_c(G^s(Q))$ and $b \in C_c(G^u(P))$. Then, for every $n \in \mathbb{Z}$ the operator $\alpha_s^n(a)b$ has finite rank. Moreover,*

(i) there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(\alpha_s^n(a)b) = 0$, for every $n \leq -n_0$;

(ii) there exists a constant $C_1 > 0$ so that, for every $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, we have

$$\mathrm{rank}(\alpha_s^n(a)b) < C_1 e^{(\mathrm{h}(\varphi)+\varepsilon)n}.$$

Similarly for the operators $b\alpha_s^n(a)$, where $n \in \mathbb{Z}$.

Proof. Using a partition of unity argument (see Lemma 6.2.3) we can decompose a, b into finite sums of compactly supported functions

$$a = \sum_{m=1}^{m'} a_m,\ b = \sum_{\ell=1}^{\ell'} b_\ell,$$

with $\mathrm{supp}(a_m) \subset V^s(v_m, w_m, h^s_m, \eta_m, N_m)$ and $\mathrm{supp}(b_\ell) \subset V^u(v_\ell', w_\ell', h^u_\ell, \eta_\ell', N_\ell')$. Also, for every $n \in \mathbb{Z}$ we have $\alpha_s^n(a) \in C_c(G^s(Q))$. Therefore, we can use the same partition of unity argument on $\alpha_s^n(a)$, and from Lemma 5.2.6 we obtain that $\alpha_s^n(a)b$ has finite rank. Moreover, using the above decomposition of a, b and Lemma 5.2.7 we obtain part (i).

Let $\varepsilon > 0$ and to prove part (ii) it suffices to show that, for every m, ℓ there is some $n(a_m, b_\ell, \varepsilon)$ so that, for all $n \geq n(a_m, b_\ell, \varepsilon)$, we have

$$\operatorname{rank}(\alpha_s^n(a_m)b_\ell) < e^{(\mathrm{h}(\varphi)+\varepsilon)n}.$$

The result follows with $C_1 = m'\ell'$ and n_1 being the maximum among all $n(a_m, b_\ell, \varepsilon)$. Indeed, fix some $n \geq 0$ and m, ℓ, and if $x \in X^s(w'_\ell, \eta'_\ell)$, $\varphi^{-n} \circ h^u_\ell(x) \in X^u(w_m, \eta_m)$ we have that

$$\alpha_s^n(a_m)b_\ell\delta_x = a_m(h^s_m \circ \varphi^{-n} \circ h^u_\ell(x), \varphi^{-n} \circ h^u_\ell(x))b_\ell(h^u_\ell(x), x)\delta_{\varphi^n \circ h^s_m \circ \varphi^{-n} \circ h^u_\ell(x)},$$

otherwise $\alpha_s^n(a_m)b_\ell\delta_x = 0$. For brevity let $B_m = X^u(w_m, \eta_m), C_\ell = X^s(v'_\ell, \varepsilon_X/2)$ and so, if $\alpha_s^n(a_m)b_\ell\delta_x \neq 0$, then

$$h^u_\ell(x) \in \varphi^n(B_m) \cap C_\ell.$$

Since $\varphi^n(B_m) \cap C_\ell$ is finite and h^u_ℓ is injective it holds that

$$\operatorname{rank}(\alpha_s^n(a_m)b_\ell) \leq \#\varphi^n(B_m) \cap C_\ell.$$

Let $j_m \geq 0$ be the first time that $\varphi^{j_m}(w_m)$ lies in the same mixing component, say X_j, with v'_ℓ. Then, for every $k \geq 0$ the points

$$\varphi^{j_m + kM}(w_m) \in X_j.$$

If $n \not\equiv j_m \mod M$, then $\varphi^n(B_m) \cap C_\ell = \varnothing$ and hence $\alpha_s^n(a_m)b_\ell = 0$. Now, if $n = j_m + kM$, for some $k \geq 0$, we can write

$$\varphi^n(B_m) \cap C_\ell = (\varphi^M|_{X_j})^k(\varphi^{j_m}(B_m)) \cap C_\ell.$$

Using Lemma 6.2.10 for $\psi = \varphi^M|_{X_j}, B = \varphi^{j_m}(B_m)$ and $C = C_\ell$ we obtain some $k_0 \in \mathbb{N}$ so that, if $k \geq k_0$, then

$$\begin{aligned}
\#\varphi^n(B_m) \cap C_\ell &= \#(\varphi^M|_{X_j})^k(\varphi^{j_m}(B_m)) \cap C_\ell \\
&< e^{(\mathrm{h}(\varphi^M|_{X_j})+\varepsilon)k} \\
&\leq e^{(\mathrm{h}(\varphi^M|_{X_j})+\varepsilon)(n/M)}.
\end{aligned}$$

Since $\mathrm{h}(\varphi^M|_{X_j}) = M\,\mathrm{h}(\varphi)$, we can define $n(a_m, b_\ell, \varepsilon) = j_m + k_0 M$ and the proof of part (ii) is complete. ◻

Lemma 6.2.12. *Consider a metric* $d' \in sM_d(X,\varphi)$ *and let* $a \in \operatorname{Lip}_c(G^s(Q), D_{s,d'})$ *and* $b \in \operatorname{Lip}_c(G^u(P), D_{u,d'})$. *Then, there exist* $C_2 > 0$ *and* $n_2 \in \mathbb{N}$ *such that, for all* $n \geq n_2$, *it holds*

$$\|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| \leq C_2 2^{-n/\lceil \log_{\lambda_{X,d'}} 3\rceil}.$$

As a result, the constants C_2 *and* n_2 *can be chosen bigger so that*

$$\|\alpha_s^n(a)b - b\alpha_s^n(a)\| \leq C_2 2^{-(n/2)/\lceil \log_{\lambda_{X,d'}} 3\rceil},$$

for all $n \geq n_2$.

Proof. All notation is relative to d' and for brevity we omit to indicate it, except for the case of the contraction constant $\lambda_{X,d'} > 1$. First assume that $\operatorname{supp}(a) \subset V^s(v,w,h^s,\eta,N)$ and $\operatorname{supp}(b) \subset V^u(v',w',h^u,\eta',N')$. As in Lemma 6.2.11 we see that, if $\alpha_s^n(a)\alpha_u^{-n}(b) \neq 0$, then $X^s(\varphi^{-n}(v')) \cap X^u(\varphi^n(w)) \neq \varnothing$. In particular, $\varphi^{-n}(v')$ and $\varphi^n(w)$ are in the same mixing component, otherwise $\alpha_s^n(a)\alpha_u^{-n}(b) = 0$. So, the possibility that $\alpha_s^n(a)\alpha_u^{-n}(b) \neq 0$ occurs periodically.

Similarly, if $\alpha_u^{-n}(b)\alpha_s^n(a) \neq 0$, then $X^s(\varphi^{-n}(w')) \cap X^u(\varphi^n(v)) \neq \varnothing$, hence $\varphi^{-n}(w')$, $\varphi^n(v)$ lie in the same mixing component. But this can also happen if $\alpha_s^n(a)\alpha_u^{-n}(b) \neq 0$, since the points $\varphi^{-n}(w')$, $\varphi^n(v)$ are always in the same mixing component with $\varphi^{-n}(v')$, $\varphi^n(w)$, respectively. This last argument works symmetrically, and to summarise, the only chance for the operator $\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)$ to be non-zero is when n lies in a certain arithmetic-like strictly increasing sequence. Therefore, for the sake of brevity, we can simply assume that (X,φ) is mixing.

We have that, if

$$\varphi^n(x) \in X^s(w',\eta'),\ \varphi^{-2n} \circ h^u \circ \varphi^n(x) \in X^u(w,\eta), \tag{6.2.13}$$

then $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x$ equals

$$a(h^s \circ \varphi^{-2n} \circ h^u \circ \varphi^n(x), \varphi^{-2n} \circ h^u \circ \varphi^n(x))b(h^u \circ \varphi^n(x), \varphi^n(x))\delta_{\varphi^n \circ h^s \circ \varphi^{-2n} \circ h^u \circ \varphi^n(x)},$$

otherwise $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x = 0$. Similarly, if

$$\varphi^{-n}(x) \in X^u(w,\eta),\ \varphi^{2n} \circ h^s \circ \varphi^{-n}(x) \in X^s(w',\eta'), \tag{6.2.14}$$

then $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x$ equals

$$b(h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n}(x), \varphi^{2n} \circ h^s \circ \varphi^{-n}(x))a(h^s \circ \varphi^{-n}(x), \varphi^{-n}(x))\delta_{\varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n}(x)},$$

and $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x = 0$ otherwise.

Let now source(a) and source(b) denote the images of supp(a), supp(b) via the groupoid source map. Then, source(a) is a compact subset of $X^u(w,\eta)$ and source(b) is a compact subset of $X^s(w',\eta')$. From compactness, there is an $\varepsilon>0$ such that the ε-neighbourhoods $N_\varepsilon(\text{source}(a))\subset X^u(w,\eta)$ and $N_\varepsilon(\text{source}(b))\subset X^s(w',\eta')$. We claim that there is $n_2\in\mathbb{N}$ such that, for every $n\geq n_2$, if $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x\neq 0$, then (6.2.14) holds and hence the above formula for $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x$ is valid. However, this does not necessarily mean that $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x\neq 0$. The same can be said for $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x$, if $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x\neq 0$. Indeed, let $n_2\geq N,N'$ and such that $\lambda_{X,d'}^{-2n_2+N}\varepsilon_X/2$, $\lambda_{X,d'}^{-2n_2+N'}\varepsilon_X/2<\varepsilon$, and for $n\geq n_2$ suppose that $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x\neq 0$. Then,

$$\varphi^n(x)\in\text{source}(b),\ \varphi^{-2n}\circ h^u\circ\varphi^n(x)\in\text{source}(a),$$

and since

$$\varphi^{-2n}\circ h^u\circ\varphi^n(x)\in X^u(\varphi^{-n}(x),\lambda_{X,d'}^{-2n+N'}\varepsilon_X/2), \tag{6.2.15}$$

we obtain that $\varphi^{-n}(x)\in N_\varepsilon(\text{source}(a))\subset X^u(w,\eta)$. Hence, the expression $\varphi^{2n}\circ h^s\circ\varphi^{-n}(x)$ is well-defined and also

$$\varphi^{2n}\circ h^s\circ\varphi^{-n}(x)\in X^s(\varphi^n(x),\lambda_{X,d'}^{-2n+N}\varepsilon_X/2), \tag{6.2.16}$$

meaning that $\varphi^{2n}\circ h^s\circ\varphi^{-n}(x)\in N_\varepsilon(\text{source}(b))\subset X^s(w',\eta')$. This proves the claim. Moreover, from [108, Lemma 2.2], we can actually choose n_2 big enough so that, for every $n\geq n_2$, if $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x\neq 0$ or $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x\neq 0$ then,

$$\varphi^n\circ h^s\circ\varphi^{-2n}\circ h^u\circ\varphi^n(x)=\varphi^{-n}\circ h^u\circ\varphi^{2n}\circ h^s\circ\varphi^{-n}(x).$$

For $n\geq n_2$ and $x\in X^h(P,Q)$ such that $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x\neq 0$ or $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x\neq 0$, define

$$\begin{aligned}
x_1&=x\\
x_2&=\varphi^n\circ h^s\circ\varphi^{-n}(x)\\
x_3&=\varphi^n\circ h^s\circ\varphi^{-2n}\circ h^u\circ\varphi^n(x)=\varphi^{-n}\circ h^u\circ\varphi^{2n}\circ h^s\circ\varphi^{-n}(x)\\
x_4&=\varphi^{-n}\circ h^u\circ\varphi^n(x).
\end{aligned}$$

For an even bigger n_2 we can guarantee that

$$\begin{aligned}
x_1&\in X^s(x_2,\varepsilon_X'/2)\\
x_3&\in X^s(x_4,\varepsilon_X'/2)\\
x_1&\in X^u(x_4,\varepsilon_X'/2)\\
x_3&\in X^u(x_2,\varepsilon_X'/2).
\end{aligned}$$

From the discussion so far, and since $\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)$ takes basis vectors to basis vectors, it holds that, for $n \geq n_2$,

$$\|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\|$$

$$= \sup_x |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^n(x_4), \varphi^n(x_1)) - b(\varphi^n(x_3), \varphi^n(x_2))a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))|,$$

where the supremum is taken over all $x \in X^h(P,Q)$ such that $\alpha_s^n(a)\alpha_u^{-n}(b)\delta_x \neq 0$ or $\alpha_u^{-n}(b)\alpha_s^n(a)\delta_x \neq 0$. In order to estimate the norm, let us denote $e = (\varphi^{-n}(x_3), \varphi^{-n}(x_4))$, $f = (\varphi^{-n}(x_2), \varphi^{-n}(x_1))$, $e' = (\varphi^n(x_4), \varphi^n(x_1))$ and $f' = (\varphi^n(x_3), \varphi^n(x_2))$. Then, since a, b are Lipschitz, we have that

$$\begin{aligned} |a(e) - a(f)| &\leq \mathrm{Lip}(a) D_{s,d'}(e,f) \\ |b(e') - b(f')| &\leq \mathrm{Lip}(b) D_{u,d'}(e',f'). \end{aligned}$$

Again, if we consider a bigger n_2 so that $\lambda_{X,d'}^{-n_2+N}\varepsilon_X/2$, $\lambda_{X,d'}^{-n_2+N'}\varepsilon_X/2 \leq \varepsilon'_X/2$ then, for every $n \geq n_2$, the basis sets $V^u(f', \lambda_{X,d'}^{-2n+N}\varepsilon_X/2, n)$, $V^s(f, \lambda_{X,d'}^{-2n+N'}\varepsilon_X/2, n)$ are well-defined, and from (6.2.15), (6.2.16) they contain e' and e, respectively. Now, from Proposition 6.1.15 we can find $\gamma, \gamma' > 0$ (independent of e, f and e', f') such that

$$\begin{aligned} D_{s,d'}(e,f) &\leq 2^{-n/\lceil \log_{\lambda_{X,d'}} 3 \rceil}\gamma \\ D_{u,d'}(e',f') &\leq 2^{-n/\lceil \log_{\lambda_{X,d'}} 3 \rceil}\gamma'. \end{aligned}$$

Using the fact that a, b are compactly supported, we obtain $C_1 > 0$ such that, for every $n \geq n_2$, it holds

$$\|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| \leq C_1 2^{-n/\lceil \log_{\lambda_{X,d'}} 3 \rceil}. \tag{6.2.17}$$

More generally, if a, b are not necessarily supported on bisections, then we can use Lemma 6.2.3 and obtain a constant $C_2 > 0$ such that (6.2.17) holds with C_2 instead of C_1.

Finally, it is not hard to see that

$$\|\alpha_s^{2n}(a)b - b\alpha_s^{2n}(a)\| = \|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\|,$$

and hence

$$\|\alpha_s^{2n}(a)b - b\alpha_s^{2n}(a)\| \leq C_2 2^{-n/\lceil \log_{\lambda_{X,d'}} 3 \rceil}, \tag{6.2.18}$$

for all $n \geq n_2$. To complete the proof, observe that (6.2.18) can be also applied to $\alpha_s(a)$ in place of a. □

In exactly the same way, using Proposition 6.1.22, one can prove the following.

Lemma 6.2.13. *Suppose that (X, d, φ) is an irreducible topological Markov chain, and let $\kappa > 1$. Then, for every $a \in \operatorname{Lip}_c(G^s(Q), D_{s,\kappa})$ and $b \in \operatorname{Lip}_c(G^u(P), D_{u,\kappa})$, there exist $C_2 > 0$ and $n_2 \in \mathbb{N}$ such that, for all $n \geq n_2$, it holds*

$$\|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| \leq C_2\kappa^{-2n}.$$

As a result, the constants C_2 and n_2 can be chosen bigger so that

$$\|\alpha_s^n(a)b - b\alpha_s^n(a)\| \leq C_2\kappa^{-n},$$

for all $n \geq n_2$.

We now state the main results of this subsection.

Proposition 6.2.14. *Let $d' \in sM_d(X, \varphi)$. The algebras $\overline{\rho_s}(\Lambda_{s,d'}(Q, \alpha_s))$, $\overline{\rho_u}(\Lambda_{u,d'}(P, \alpha_u))$ commute modulo the Schatten ideal $\mathcal{L}^p(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$, for every*

$$p > \frac{2\operatorname{h}(\varphi)\lceil \log_{\lambda_{X,d'}} 3\rceil}{\log 2}.$$

Proof. To simplify the notation, let $p(d') = 2\operatorname{h}(\varphi)\lceil \log_{\lambda_{X,d'}} 3\rceil / \log 2$. It suffices to show that, for every $a \in \operatorname{Lip}_c(G^s(Q), D_{s,d'})$, $b \in \operatorname{Lip}_c(G^u(P), D_{u,d'})$ and $j, j' \in \mathbb{Z}$, it holds

$$[\overline{\rho_s}(au^j), \overline{\rho_u}(bu^{j'})] \in \mathcal{L}^p(\mathscr{H} \otimes \ell^2(\mathbb{Z})),$$

for every $p > p(d')$. By the construction of $\overline{\rho_s}$ and $\overline{\rho_u}$ we have that

$$[\overline{\rho_s}(au^j), \overline{\rho_u}(bu^{j'})] = [\overline{\rho_s}(a), \overline{\rho_u}(b)]\overline{\rho_s}(u^j)\overline{\rho_u}(u^{j'}),$$

and therefore it suffices to show that the singular values of the compact operator

$$R = [\overline{\rho_s}(a), \overline{\rho_u}(b)]$$

satisfy $(s_n(R))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, for every $p > p(d')$.

We have that $R = \bigoplus_{n\in\mathbb{Z}} R_n$, where $R_n = \alpha_s^n(a)b - b\alpha_s^n(a)$. From Lemma 6.2.11 we obtain that each R_n has finite rank and that there is some $n_0 \in \mathbb{N}$ such that, for all $n \leq -n_0$, it holds $R_n = 0$. For this reason, consider the compact operator $T = \bigoplus_{n\in\mathbb{N}} T_n \in \mathcal{B}(\bigoplus_{n\in\mathbb{N}} \mathscr{H})$, where each $T_n = R_{n-n_0}$, and using Lemma 6.2.9 it suffices to show that $(s_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, for every $p > p(d')$.

From Lemma 6.2.11 we can find a constant $C_1 > 0$ so that, for every $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, it holds

$$\operatorname{rank}(T_n) \leq C_1 e^{(\operatorname{h}(\varphi)+\varepsilon)n}. \tag{6.2.19}$$

Also, from Lemma 6.2.12 we can find $C_2 > 0$ and $n_2 \in \mathbb{N}$ such that, for all $n \geq n_2$, one has

$$\|T_n\| \leq C_2 2^{-(n/2)/\lceil \log_{\lambda_{X,d'}} 3 \rceil}. \tag{6.2.20}$$

Both (6.2.19) and (6.2.20) have been simplified to not include the integer n_0. Following Lemma 6.2.9, we have that $\alpha_\varepsilon = e^{\mathrm{h}(\varphi)+\varepsilon}$, $\alpha = e^{\mathrm{h}(\varphi)} > 1$, $\beta_\varepsilon = \beta = 2^{(1/2)/\lceil \log_{\lambda_{X,d'}} 3 \rceil}$, and also that, for every $p > \log_\beta \alpha$, the sequence $(s_n(T))_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. In fact, we have that

$$\log_\beta \alpha = \frac{2\,\mathrm{h}(\varphi)\lceil \log_{\lambda_{X,d'}} 3 \rceil}{\log 2},$$

and the proof is complete. □

Similarly, one can show the following.

Proposition 6.2.15. *Suppose that (X, d, φ) is an irreducible topological Markov chain, and let $\kappa > 1$. The algebras $\overline{\rho_s}(\Lambda_{s,\kappa}(Q, \alpha_s))$ and $\overline{\rho_u}(\Lambda_{u,\kappa}(P, \alpha_u))$ commute modulo the Schatten ideal $\mathcal{L}^p(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$, for every*

$$p > \frac{\mathrm{h}(\varphi)}{\log \kappa}.$$

6.2.3 Smooth extensions of Ruelle algebras

We now present the main results regarding smooth extensions of Ruelle algebras. Let (X, d, φ) be an irreducible Smale space with periodic orbits Q, P and consider the stable, unstable groupoids $G^s(Q), G^u(P)$. Recall (from Section 6.1) that, for every self-similar hyperbolic metric $d' \in \mathrm{sM}_d(X, \varphi)$, we have the dynamical groupoid metrics $D_{s,d'}, D_{u,d'}$ of $G^s(Q), G^u(P)$, respectively. Moreover, in the case where (X, d, φ) is zero-dimensional, we also have the dynamical groupoid ultrametrics $D_{s,\kappa}, D_{u,\kappa}$ of $G^s(Q), G^u(P)$, which are defined for every expanding factor $\kappa > 1$. In what follows, we will also use the λ-number $\lambda(X, \varphi)$ of (X, d, φ), which is a topological invariant related to the family of metrics $\mathrm{sM}_d(X, \varphi)$, see Subsection 6.1.3. Finally, for every $d' \in \mathrm{sM}_d(X, \varphi)$ and $\kappa > 1$, recall (from Subsection 6.2.2) the dense (Lipschitz) $\star$-subalgebras $\Lambda_{s,d'}(Q, \alpha_s)$, $\Lambda_{u,d'}(P, \alpha_u)$ and $\Lambda_{s,\kappa}(Q, \alpha_s)$, $\Lambda_{u,\kappa}(P, \alpha_u)$ (the zero-dimensional case) of the Ruelle algebras $\mathcal{R}^s(Q), \mathcal{R}^u(P)$, respectively. Recall that the KPW-extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ is the product (in the Calkin algebra) of two faithful representations $\overline{\rho_s} : \mathcal{R}^s(Q) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ and $\overline{\rho_u} : \mathcal{R}^u(P) \to \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ which commute modulo $\mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$. The next result is one of our main theorems that will allow us to deduce the uniform finite smoothness of Ruelle algebras. Its proof is omitted since it is a straightforward application of Propositions 6.2.6 and 6.2.14, Lemma 4.2.19 (see also the first part of the proof of Proposition 4.2.21) and Remark 6.2.8.

Theorem 6.2.16. *For every* $d' \in sM_d(X,\varphi)$, *there exist holomorphically stable dense* $*$*-subalgebras* $\mathrm{H}_{s,u,d'}(Q,\alpha_s) \subset \mathcal{R}^s(Q)$ *and* $\mathrm{H}_{u,s,d'}(P,\alpha_u) \subset \mathcal{R}^u(P)$ *that*

(1) contain $\Lambda_{s,d'}(Q,\alpha_s)$ *and* $\Lambda_{u,d'}(P,\alpha_u)$, *respectively;*

(2) the KPW-extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ *is p-smooth on the algebra* $\mathrm{H}_{s,u,d'}(Q,\alpha_s) \otimes_{alg} \mathrm{H}_{u,s,d'}(P,\alpha_u)$, *for every*

$$p > \frac{2\,\mathrm{h}(\varphi)\lceil \log_{\lambda_{X,d'}} 3\rceil}{\log 2}. \tag{6.2.21}$$

As a result, from a topological perspective, for every

$$p > \frac{2\,\mathrm{h}(\varphi)}{\log 2} + \frac{2\,\mathrm{h}(\varphi)\log_{\lambda(X,\varphi)} 3}{\log 2}, \tag{6.2.22}$$

there is $d' \in sM_d(X,\varphi)$ *such that* τ_Δ *is p-smooth on* $\mathrm{H}_{s,u,d'}(Q,\alpha_s) \otimes_{alg} \mathrm{H}_{u,s,d'}(P,\alpha_u)$.

Remark 6.2.17. One of the indices of $\mathrm{H}_{s,u,d'}(Q,\alpha_s) \subset \mathcal{R}^s(Q)$ is the letter "u" which stands for "unstable". Even though this subalgebra lies only in the stable Ruelle algebra $\mathcal{R}^s(Q)$, its construction depends on the Lipschitz algebra $\overline{\rho_u}(\Lambda_{u,d'}(P,\alpha_u))$ in $\mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$. For more details see the proof of Lemma 4.2.19. Similarly for the subalgebra $\mathrm{H}_{u,s,d'}(P,\alpha_u) \subset \mathcal{R}^u(P)$.

Remark 6.2.18. Following the discussion in Subsection 6.1.3, in this generality of Smale spaces, the lower bound (6.2.21), for given $d' \in \mathrm{sM}_d(X,\varphi)$, is the best possible. But, there is a subtlety. The ceiling function is not continuous at the integers, and hence we cannot derive, in general, that τ_Δ is p-smooth for every

$$p > \frac{2\,\mathrm{h}(\varphi)\lceil \log_{\lambda(X,\varphi)} 3\rceil}{\log 2}, \tag{6.2.23}$$

where $\lambda(X,\varphi) = \sup\{\lambda_{X,d'} : d' \in \mathrm{sM}_d(X,\varphi)\}$. This can easily be seen if (X,d,φ) is zero-dimensional. In this case $\lambda(X,\varphi) = \infty$ and the smallest lower bound of (6.2.21) is $2\,\mathrm{h}(\varphi)/\log 2$, while the one of (6.2.23) is zero. Equation (6.2.22) comes from the fact that $\lceil x\rceil < x+1$, for all $x \in \mathbb{R}$.

From the Spanier-Whitehead K-duality of Kaminker, Putnam and Whittaker (for this see Theorem 5.2.1 and also Subsection 5.2.2), Theorem 6.2.16 and Proposition 4.2.21 we obtain the following important corollary.

Corollary 6.2.19. *For all* $d' \in sM_d(X,\varphi)$, *the Ruelle algebras* $\mathcal{R}^s(Q)$ *and* $\mathcal{R}^u(P)$ *are uniformly* $\mathcal{L}^p$*-smooth on the holomorphically stable dense* $*$*-subalgebras* $\mathrm{H}_{s,u,d'}(Q,\alpha_s)$ *and* $\mathrm{H}_{u,s,d'}(P,\alpha_u)$, *respectively, for every*

$$p > \frac{2\,\mathrm{h}(\varphi)\lceil \log_{\lambda_{X,d'}} 3\rceil}{\log 2}.$$

In particular, for every

$$p > \frac{2\,\mathrm{h}(\varphi)}{\log 2} + \frac{2\,\mathrm{h}(\varphi)\log_{\lambda(X,\varphi)} 3}{\log 2},$$

there is a metric $d' \in sM_d(X,\varphi)$ *such that* $\mathcal{R}^s(Q)$ *and* $\mathcal{R}^u(P)$ *are uniformly* $\mathcal{L}^p$*-smooth on* $\mathrm{H}_{s,u,d'}(Q,\alpha_s)$ *and* $\mathrm{H}_{u,s,d'}(P,\alpha_u)$*, respectively.*

We now move on to sharpen the above results in the case (X,d,φ) is a topological Markov chain. This is done using the ultrametrics $D_{s,\kappa}$, $D_{u,\kappa}$ of Proposition 6.1.23, which are indexed over every expanding factor $\kappa > 1$. The next two results are strongly related to the fact that, in this case, the λ-number $\lambda(X,\varphi)$ is infinite. The proof of Theorem 6.2.20 follows from Propositions 6.2.6 and 6.2.15, Lemma 4.2.19 and Remark 6.2.8.

Theorem 6.2.20. *Suppose that* (X,d,φ) *is an irreducible topological Markov chain, and let* $\kappa > 1$*. There exist holomorphically stable dense* $*$*-subalgebras* $\mathrm{H}_{s,u,\kappa}(Q,\alpha_s) \subset \mathcal{R}^s(Q)$ *and* $\mathrm{H}_{u,s,\kappa}(P,\alpha_u) \subset \mathcal{R}^u(P)$ *that*

(1) contain $\Lambda_{s,\kappa}(Q,\alpha_s)$ *and* $\Lambda_{u,\kappa}(P,\alpha_u)$*, respectively;*

(2) the KPW-extension $\tau_\Delta : \mathcal{R}^s(Q)\otimes\mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ *is p-smooth on the algebra* $\mathrm{H}_{s,u,\kappa}(Q,\alpha_s)\otimes_{alg}\mathrm{H}_{u,s,\kappa}(P,\alpha_u)$*, for every*

$$p > \frac{\mathrm{h}(\varphi)}{\log \kappa}. \tag{6.2.24}$$

As a result, the extension τ_Δ *is optimally smooth. More precisely, for every* $p > 0$ *there is* $\kappa > 1$ *such that* τ_Δ *is p-smooth on* $\mathrm{H}_{s,u,\kappa}(Q,\alpha_s)\otimes_{alg}\mathrm{H}_{u,s,\kappa}(P,\alpha_u)$*.*

Similarly, as before, we obtain the following.

Corollary 6.2.21. *Suppose that* (X,d,φ) *is an irreducible topological Markov chain. Then, for every* $\kappa > 1$*, the Ruelle algebras* $\mathcal{R}^s(Q)$ *and* $\mathcal{R}^u(P)$ *are uniformly* $\mathcal{L}^p$*-smooth on the holomorphically stable dense* $*$*-subalgebras* $\mathrm{H}_{s,u,\kappa}(Q,\alpha_s)$ *and* $\mathrm{H}_{u,s,\kappa}(P,\alpha_u)$*, for any*

$$p > \frac{\mathrm{h}(\varphi)}{\log \kappa}.$$

Therefore, both Ruelle algebras have zero degree of irregularity. Specifically, for every $p > 0$ *there is* $\kappa > 1$ *such that* $\mathcal{R}^s(Q)$ *and* $\mathcal{R}^u(P)$ *are uniformly* $\mathcal{L}^p$*-smooth on* $\mathrm{H}_{s,u,\kappa}(Q,\alpha_s)$ *and* $\mathrm{H}_{u,s,\kappa}(P,\alpha_u)$*, respectively.*

Remark 6.2.22. Corollary 6.2.21 says that the degree of irregularity (see Remark 4.2.6) agrees with the topological dimension of the Markov chain. Moreover, it implies that every stabilised Cuntz-Krieger algebra, of an irreducible topological Markov chain, has zero degree of irregularity. It would be very interesting to know if the converse is also true. More precisely, *is it true that a stable (unstable) Ruelle algebra, with zero degree of irregularity, is the stable (unstable) Ruelle algebra a topological Markov chain?* To answer this affirmatively one could try and show that, if $\dim X > 0$, then the degree of irregularity is strictly positive. This does not seem absurd since, if $\dim X > 0$, then

$$\lambda(X, \varphi) \leq e^{2\,\mathrm{h}(\varphi)/\dim X},$$

as we see in Proposition 6.1.19, and both lower bounds (6.2.21) and (6.2.24) depend on $1/\log(\nu)$, where $\nu > 1$ is some contraction/expansion constant (or expanding factor). It would be a surprise to us if the above question had a negative answer. In any case, to tackle it requires a thorough investigation of the geometry of Smale spaces.

Chapter 7

Geometric K-duality for Ruelle algebras

In this final chapter we construct θ-summable Fredholm module representatives for the KPW-extension class of the K-duality between the stable and unstable Ruelle algebras (Theorem 7.2.1). Our method uses almost all the machinery that we have developed in the previous chapters. The heart of our construction lies in Theorem 3.3.2, which generalises Bowen's Markov partitions, and is about refining sequences of δ-fat Markov partitions with controlled multiplicities, diameters and Lebesgue covering numbers. This is the point where, to each such refining sequence, we assign a sequence of Lipschitz partitions of unity with controlled Lipschitz constants.

7.1 Building a KK_1-lift of the KPW-extension class

Let (X, d, φ) be an irreducible Smale space and fix two periodic orbits Q and P such that $Q \cap P = \varnothing$. Observe that the set $X^h(P, Q)$ does not contain periodic points. Due to nuclearity, the KPW-extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ is invertible. Therefore, its class $[\tau_\Delta]$ is realised by an abstract $\Delta \in \mathrm{KK}_1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P), \mathbb{C})$. For simplicity, consider

$$\Delta \in \mathrm{K}^1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)).$$

Here we develop tools for building a geometric Fredholm module representative of Δ. More precisely, we aim to find an explicit

(i) representation $\rho : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$;

(ii) isometry $V : \mathscr{H} \otimes \ell^2(\mathbb{Z}) \to \mathcal{H}$;

(iii) unitary $U \in \mathcal{B}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$, such that (here $\mathrm{Ad}_\pi(U)$ is conjugation by $\pi(U) \in \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$)

$$(\mathrm{Ad}_\pi(U) \circ \tau_\Delta)(x) = V^* \rho(x) V + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})), \tag{7.1.1}$$

for all $x \in \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$, and hence deduce that Δ is represented by the odd Fredholm module $(\mathcal{H}, \rho, 2VV^* - 1)$ over $\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$. In this section we construct (i), (ii), (iii), and in Section 7.2 we prove (7.1.1) and investigate summability properties of the Fredholm module $(\mathcal{H}, \rho, 2VV^* - 1)$.

To be clear, Fredholm modules of the form $(\mathcal{H}, \rho, 2VV^* - 1)$ exist due to the Choi-Effros Lifting Theorem and Stinespring's Dilation. However, they are too abstract for index computations, let alone being summable. Consequently, finding a concrete geometric Fredholm module representative requires brute force.

In our case, it is straightforward to construct ρ, since it should be an untwisted version of the extension τ_Δ. It turns out that the Hilbert space $\mathcal{H} = \mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z})$ is sufficiently large for this purpose. On the other hand, constructing a suitable isometry V is hard. One of the main reasons that causes this difficulty is the simultaneous actions of two copies of $\mathbb{Z}$ taking place in

$$\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) = (\mathcal{S}(Q) \rtimes_{\alpha_s} \mathbb{Z}) \otimes (\mathcal{U}(P) \rtimes_{\alpha_u} \mathbb{Z}).$$

In other words, finding a suitable V would be a lot easier if we only had to consider the C^*-algebras $(\mathcal{S}(Q) \rtimes_{\alpha_s} \mathbb{Z}) \otimes \mathcal{U}(P)$ or $\mathcal{S}(Q) \otimes (\mathcal{U}(P) \rtimes_{\alpha_u} \mathbb{Z})$. Our method is to construct V through an averaging process that asymptotically commutes with the $\mathbb{Z}^2$-action, and so that VV^* commutes modulo compacts with the coefficient algebra $\mathcal{S}(Q) \otimes \mathcal{U}(P)$. The main ingredient are our generalised Markov partitions, as we see in the sequel.

A main result in lifting K-homological duality classes was achieved by Goffeng and Mesland [58] in the case of Cuntz-Krieger algebras. The authors used an earlier result of Kaminker and Putnam [73] about the Spanier-Whitehead K-duality between the Cuntz-Krieger algebras $\mathcal{O}_M$ and $\mathcal{O}_{M^t}$, for an irreducible matrix M with 0 and 1 entries. In that paper, Kaminker and Putnam considered the representations of the Toeplitz algebras $\mathcal{T}_M$ and $\mathcal{T}_{M^t}$ on the corresponding Fock space $\mathcal{F}_M$, which yield commuting extensions $\tau_M : \mathcal{O}_M \to \mathcal{Q}(\mathcal{F}_M)$ and $\tau_{M^t} : \mathcal{O}_{M^t} \to \mathcal{Q}(\mathcal{F}_M)$. The K-homology duality class was then defined to be the Ext-class of $\tau_M \cdot \tau_{M^t} : \mathcal{O}_M \otimes \mathcal{O}_{M^t} \to \mathcal{Q}(\mathcal{F}_M)$. Now, since τ_M and τ_{M^t} do not lift to representations in $\mathcal{B}(\mathcal{F}_M)$, Goffeng and Mesland used KMS-states to construct representations $\rho_M : \mathcal{O}_M \to \mathcal{B}(\mathcal{H}_M)$ and $\rho_{M^t} : \mathcal{O}_{M^t} \to \mathcal{B}(\mathcal{H}_{M^t})$, and also an isometry $W : \mathcal{F}_M \to \mathcal{H}_M \otimes \mathcal{H}_{M^t}$ so that the $\mathrm{KK}_1(\mathcal{O}_M \otimes \mathcal{O}_{M^t}, \mathbb{C})$-class $[\mathcal{H}_M \otimes \mathcal{H}_{M^t}, \rho_M \otimes \rho_{M^t}, 2WW^* - 1]$ lifts $[\tau_M \cdot \tau_{M^t}]$. The aforementioned Fredholm module was shown to be θ-summable.

The situation for the Ruelle algebras differs considerably, mostly because there is no Fock space representation. Also, the KPW-extension τ_Δ is, in fact, the product of two commuting representations in the Calkin algebra $\mathcal{Q}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ that lift to non-commuting representations in $\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$, while this is not happening in the Cuntz-Krieger algebra picture. But this is to our favour. Moreover, we do not consider the double Hilbert space $\mathscr{H}\otimes\ell^2(\mathbb{Z})\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z})$, but rather the Hilbert space $\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z})=\bigoplus_{n\in\mathbb{Z}}\mathscr{H}\otimes\mathscr{H}$. This allows us to create the averaging isometry V and use reasonably easy compactness and singular-values arguments.

7.1.1 The symmetric representation

We now present the untwisted version of the KPW-extension τ_Δ. In what follows, we consider operators $a\in\mathcal{S}(Q)$, $b\in\mathcal{U}(P)$ and $j,j'\in\mathbb{Z}$, as well as, matrix units $e_{n,m}$. First, we recall the representations $\overline{\rho_s}:\mathcal{R}^s(Q)\to\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ and $\overline{\rho_u}:\mathcal{R}^u(P)\to\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ which are respectively given on generators by

$$a\mapsto\bigoplus_{n\in\mathbb{Z}}\alpha_s^n(a)\ \text{ and }\ u\to 1\otimes B,$$

as well as,

$$b\mapsto b\otimes 1\ \text{ and }\ u\mapsto u\otimes B^*.$$

Also, recall that $\overline{\rho_s}(\mathcal{R}^s(Q)),\overline{\rho_u}(\mathcal{R}^u(P))$ commute modulo $\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ and that on elementary tensors and generators one has that $\tau_\Delta:\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)\to\mathcal{Q}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ is given by

$$\begin{aligned}\tau_\Delta(au^j\otimes bu^{j'})&=(\overline{\rho_s}\cdot\overline{\rho_u})(au^j\otimes bu^{j'})+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\\&=\overline{\rho_u}(bu^{j'})\overline{\rho_s}(au^j)+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))\\&=\sum_{n\in\mathbb{Z}}b\alpha_s^n(a)u^{j'}\otimes e_{n,n+j-j'}+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z})).\end{aligned}\tag{7.1.2}$$

Since the only non-commuting parts of $\overline{\rho_s}(\mathcal{R}^s(Q))$ and $\overline{\rho_u}(\mathcal{R}^u(P))$ are the coefficient algebras $\overline{\rho_s}(\mathcal{S}(Q))$ and $\overline{\rho_u}(\mathcal{U}(P))$, the Hilbert space $\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z})$ is spacious enough to accommodate a representation of $\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)$. Indeed, consider first the (covariant) representations of $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ on $\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z})$ which are respectively given on generators by

$$\begin{gathered}a\mapsto 1\otimes\bigoplus_{n\in\mathbb{Z}}\alpha_s^n(a)\ \text{ and }\ u\mapsto 1\otimes 1\otimes B,\\ b\mapsto b\otimes 1\otimes 1\ \text{ and }\ u\mapsto u\otimes u\otimes B^*.\end{gathered}$$

Note that we put $u \mapsto u \otimes u \otimes B^*$ instead of $u \mapsto u \otimes 1 \otimes B^*$ so that the two representations commute. Their product is the representation $\widetilde{\rho} : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z}))$ given by

$$\widetilde{\rho}(au^j \otimes bu^{j'}) = \sum_{n\in\mathbb{Z}} bu^{j'} \otimes \alpha_s^n(a)u^{j'} \otimes e_{n,n+j-j'}. \tag{7.1.3}$$

It should be clear now that the representation (7.1.3) is the untwisted version of (7.1.2). But there is an additional step we need to take. First, observe that in both (7.1.2) and (7.1.3), the dynamics are basically applied only on $a \in \mathcal{S}(Q)$ and not on $b \in \mathcal{U}(P)$. There is nothing wrong about it. However, if we do not alter the representation now, so that b has dynamics, we will have to include this kind of dynamics later in the isometry V. So, lets do it now. The idea is to take half of the dynamics of a and put them on b.

Let $U = \bigoplus_{n\in\mathbb{Z}} u^{-\lfloor n/2 \rfloor}$ and consider the symmetrised KPW-extension $\mathrm{Ad}_\pi(U) \circ \tau_\Delta$, which of course yields the same Ext-class with τ_Δ and is given by

$$\begin{aligned}(\mathrm{Ad}_\pi(U) \circ \tau_\Delta)(au^j \otimes bu^{j'}) &= (\mathrm{Ad}(U) \circ (\overline{\rho_s} \cdot \overline{\rho_u}))(au^j \otimes bu^{j'}) + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})) \\ &= \sum_{n\in\mathbb{Z}} u^{-\lfloor \frac{n}{2} \rfloor} b\alpha_s^n(a) u^{j'+\lfloor \frac{n+j-j'}{2} \rfloor} \otimes e_{n,n+j-j'} + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})).\end{aligned} \tag{7.1.4}$$

Each of these off-diagonal matrices is clearly the sum of the even-coordinates matrix

$$\sum_{n\in\mathbb{Z}} \alpha_u^{-n}(b)\alpha_s^n(a)u^{j'+\lfloor \frac{j-j'}{2} \rfloor} \otimes e_{2n,2n+j-j'} + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})),$$

and the odd-coordinates matrix

$$\sum_{n\in\mathbb{Z}} \alpha_u^{-n}(b)\alpha_s^{n+1}(a)u^{j'+\lfloor \frac{j-j'+1}{2} \rfloor} \otimes e_{2n+1,2n+1+j-j'} + \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})).$$

Similarly, we let $W = \bigoplus_{n\in\mathbb{Z}} u^{-\lfloor n/2 \rfloor} \otimes u^{-\lfloor n/2 \rfloor}$ and define the *symmetric representation* $\rho = \mathrm{Ad}(W) \circ \widetilde{\rho}$. It is given by

$$\rho(au^j \otimes bu^{j'}) = \sum_{n\in\mathbb{Z}} u^{-\lfloor \frac{n}{2} \rfloor} bu^{j'+\lfloor \frac{n+j-j'}{2} \rfloor} \otimes u^{-\lfloor \frac{n}{2} \rfloor}\alpha_s^n(a)u^{j'+\lfloor \frac{n+j-j'}{2} \rfloor} \otimes e_{n,n+j-j'}. \tag{7.1.5}$$

For given $au^j \otimes bu^{j'}$, the corresponding even-coordinates matrix is

$$\sum_{n\in\mathbb{Z}} \alpha_u^{-n}(b)u^{j'+\lfloor \frac{j-j'}{2} \rfloor} \otimes \alpha_s^n(a)u^{j'+\lfloor \frac{j-j'}{2} \rfloor} \otimes e_{2n,2n+j-j'},$$

and the odd-coordinates matrix is

$$\sum_{n\in\mathbb{Z}} \alpha_u^{-n}(b)u^{j'+\lfloor \frac{j-j'+1}{2} \rfloor} \otimes \alpha_s^{n+1}(a)u^{j'+\lfloor \frac{j-j'+1}{2} \rfloor} \otimes e_{2n+1,2n+1+j-j'}.$$

As before, the representation (7.1.5) is the untwisted version of (7.1.4).

7.1.2 Intuition for the isometry

Finding a suitable isometry V is difficult, and without doubt it requires intuition. This subsection contains an exposition of ideas that eventually led us to find V. Not all of them are mathematically rigorous, however we believe their presentation will add to the understanding of the reader.

We want to find V that satisfies (7.1.1), for the symmetric representation ρ in (7.1.5). Using linearity, and the continuity of $V^*\rho(\cdot)V$, this can be reduced into finding V such that, for all $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$ and $j, j' \in \mathbb{Z}$, it holds

$$V^*\rho(au^j \otimes bu^{j'})V - (\mathrm{Ad}(U) \circ (\overline{\rho_s} \cdot \overline{\rho_u}))(au^j \otimes bu^{j'}) \in \mathcal{K}(\mathscr{H} \otimes \ell^2(\mathbb{Z})). \tag{7.1.6}$$

Since the operators $\rho(au^j \otimes bu^{j'})$ and $(\mathrm{Ad}(U)\circ(\overline{\rho_s}\cdot\overline{\rho_u}))(au^j \otimes bu^{j'})$ are off-diagonal matrices (and actually in the same off-diagonal), it seems natural to expect V to be diagonal; that is, $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where each $V_n : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is an isometry. With this in mind, calculating (7.1.6) for a given $au^j \otimes bu^{j'}$, we obtain a rather long expression which can be written as the sum of the even-coordinates matrix

$$\sum_{n\in\mathbb{Z}} (V_{2n}^*(\alpha_u^{-n}(b)u^{j'+\lfloor\frac{j-j'}{2}\rfloor} \otimes \alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor})V_{2n+j-j'} - \alpha_u^{-n}(b)\alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor}) \otimes e_{2n,2n+j-j'} \tag{7.1.7}$$

and the odd-coordinates matrix, involving the terms $\alpha_s^{n+1}(a), \alpha_u^{-n}(b)$, $u^{j'+\lfloor\frac{j-j'+1}{2}\rfloor}$ and V_{2n+1}^*, $V_{2n+1+j-j'}$ at the matrix unit $e_{2n+1,2n+1+j-j'}$.

Therefore, to show that for every $au^j \otimes bu^{j'}$ the operator (7.1.6) is compact, we simply have to show that, for every $n \in \mathbb{Z}$,

$$V_{2n}^*(\alpha_u^{-n}(b)u^{j'+\lfloor\frac{j-j'}{2}\rfloor} \otimes \alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor})V_{2n+j-j'} - \alpha_u^{-n}(b)\alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor} \in \mathcal{K}(\mathscr{H}), \tag{7.1.8}$$

and also that

$$\lim_{n\to\pm\infty} \|V_{2n}^*(\alpha_u^{-n}(b)u^{j'+\lfloor\frac{j-j'}{2}\rfloor} \otimes \alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor})V_{2n+j-j'} - \alpha_u^{-n}(b)\alpha_s^n(a)u^{j'+\lfloor\frac{j-j'}{2}\rfloor}\| = 0, \tag{7.1.9}$$

and similarly for the odd-coordinates matrix. To simplify the notation, it suffices to prove that, for every $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$ and $i, k, l \in \mathbb{Z}$, we have

$$V_{2n+l}^*(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i \in \mathcal{K}(\mathscr{H}), \tag{7.1.10}$$

and also that

$$\lim_{n\to\pm\infty} \|V_{2n+l}^*(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i\| = 0. \tag{7.1.11}$$

Then, the result about the odd-coordinates matrices follows by applying (7.1.10) and (7.1.11) to $\alpha_s(a)$ in place of a. The sequel will be about finding an isometry V that satisfies (7.1.10) and (7.1.11).

As we see in (7.1.10) and (7.1.11), the $\mathbb{Z}$-actions are not negligible. Consequently, such an isometry $V = \bigoplus_{n\in\mathbb{Z}} V_n$ has to satisfy the following quasi-invariance properties,

$$\lim_{n\to\pm\infty} \|V_{n+1} - V_n\| = 0, \tag{7.1.12}$$

and in addition,

$$\lim_{n\to\pm\infty} \|(u\otimes u)V_n - V_n u\| = 0. \tag{7.1.13}$$

Then, condition (7.1.11) holds if, for every $a\in\mathcal{S}(Q)$, $b\in\mathcal{U}(P)$, we have

$$\lim_{n\to\pm\infty} \|V_{2n}^*(\alpha_u^{-n}(b)\otimes\alpha_s^n(a))V_{2n} - \alpha_u^{-n}(b)\alpha_s^n(a)\| = 0. \tag{7.1.14}$$

To summarise, we search for V that satisfies (7.1.10), (7.1.12), (7.1.13) and (7.1.14). Then, the triple $(\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z}),\rho,2VV^*-1)$ is an odd Fredholm module representative of the K-homology duality class $\Delta\in \mathrm{K}^1(\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P))$.

Remark 7.1.1. It is conditions (7.1.12), (7.1.13) and (7.1.14) that allow us to intuitively find the isometry V. However, studying the summability of $(\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z}),\rho,2VV^*-1)$ requires an exact estimation of the convergence rates in (7.1.11), without making any simplifications using triangle inequalities. Let us look ahead to Section 7.2 for a brief moment. For compactly supported a and b, the (finite) rank of the operators in (7.1.10) increases exponentially fast as n approaches $+\infty$, and is eventually zero as n approaches $-\infty$. In particular, for a,b in the dense Lipschitz $*$-subalgebras of Proposition 6.2.6, the sequence (7.1.14) decreases exponentially fast as n approaches $+\infty$, as it happens in Lemma 6.2.12. Finally, the sequences in (7.1.12) and (7.1.13) are $O(n^{-1/2})$, and hence the sequence in (7.1.11) is $O(n^{-1/2})$. Consequently, from Theorems 4.2.8 and 6.2.16, and Lemma 6.2.9, we obtain that the Fredholm module is $\mathrm{Li}^{1/4}$-summable; almost θ-summable but not exactly. However, a closer look at the sequence in (7.1.11) shows that it is $O(n^{-1})$ and hence $(\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z}),\rho,2VV^*-1)$ is θ-summable. This is related to the fact that for the final constructed isometry, the sequences $\|V_n^*V_{n+1}-1\|$ and $\|V_n^*(u\otimes u)V_n - u\|$ are $O(n^{-1})$, but it is not clear how to use them to simplify (7.1.11).

In order to find such V we need to understand the dynamics of the KPW-extension. The $2n$-th coordinate of $(\mathrm{Ad}(U)\circ(\overline{\rho_s}\cdot\overline{\rho_u}))(a\otimes b)$ (see also (7.1.14)) is $\alpha_u^{-n}(b)\alpha_s^n(a)$. If a,b are compactly supported, there is $n_0\in\mathbb{N}$ so that $\alpha_u^{-n}(b)\alpha_s^n(a)=0$, when $n\leq -n_0$. Therefore, the interesting part is when n approaches $+\infty$. In this case, whenever $\alpha_u^{-n}(b)\alpha_s^n(a)\neq 0$, the dynamics are complicated as the sources and ranges of $\alpha_u^{-n}(b)$ and $\alpha_s^n(a)$, that stretch along large segments of global stable and unstable sets, intersect in many places, thus creating small rectangular neighbourhoods, like a grid. In fact, the larger the n, the more and smaller neighbourhoods we get. The symmetric representation ρ untwists this picture by separating stable and unstable sets, and the isometry V has to compress it back to the twisted version, with as small an error as possible. Since the aforementioned rectangular neighbourhoods are small, the compression should be related (locally) to the bracket map of the Smale space. Note that this would not be the case if we were working with the non-symmetrised extension τ_Δ and representation $\widetilde{\rho}$, because the rectangular neighbourhoods created by $b\alpha_s^n(a) \neq 0$ would typically be large in the unstable direction, and therefore, the compression through the bracket map would not be applicable.

Now that we have built some intuition about V, let us be even less mathematically rigorous, but we assure the reader it will be rewarding at the end. We search for isometries $\mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$. Roughly speaking, since $\mathscr{H} = \ell^2(X^h(P,Q))$, these correspond to inclusions $X^h(P,Q) \hookrightarrow X^h(P,Q) \times X^h(P,Q)$. But since $X^h(P,Q)$ is just a countable dense subset of X, we should look for topological embeddings $X \hookrightarrow X \times X$. The first embedding that comes to mind is the diagonal embedding, however it seems to be too "small" for our purposes. Inspired by the easy part of Whitney's Embedding Theorem [132] we proceed as follows.

Let $\mathcal{U} = \{U_1, \ldots, U_\ell\}$ be a cover of X by open rectangles (see Definition 3.2.1), and choose a point $g_r \in U_r$, for all $1 \leq r \leq \ell$. It holds that $U_r = [X^u(g_r, U_r), X^s(g_r, U_r)]$. Moreover, we have the homeomorphisms $\psi_r : U_r \to X^u(g_r, U_r) \times X^s(g_r, U_r)$ given by

$$\psi_r(x) = ([x, g_r], [g_r, x]). \tag{7.1.15}$$

Each ψ_r is the inverse of the bracket map at g_r, and the family $\{U_r, \psi_r\}_{r=1}^{\ell}$ can be considered as an *abstract foliated atlas* (note that a non-empty intersection of rectangles is again a rectangle). Indeed, the notions of transversality and holonomy maps are well-defined in this setting. Also, due to irreducibility of the Smale space (X, φ), the topological dimensions $\dim X^u(y)$, $\dim X^s(y)$ are independent of $y \in X$ and $\dim X^u(y) + \dim X^s(y) = \dim X$, see [33, Prop. 5.29]. However, the local leaves $X^u(y, \varepsilon_X)$, $X^s(y, \varepsilon_X)$ are in general non-Euclidean.

At this point let us make the following (rather vague) assumptions. First, that X is a topological manifold, and let $m = \dim X$, $m_u = \dim X^u(y)$, $m_s = \dim X^s(y)$, for some $y \in X$. Second, that the each homeomorphism ψ_r yields a homeomorphism $\widetilde{\psi_r} : U_r \to \mathbb{R}^m$ (onto its image) that has the form

$$\widetilde{\psi_r}(x) = (\psi_r^u([x, g_r]), \psi_r^s([g_r, x])), \tag{7.1.16}$$

where $\psi_r^u : X^u(g_r, U_r) \to \mathbb{R}^{m_u}$ and $\psi_r^s : X^s(g_r, U_r) \to \mathbb{R}^{m_s}$.

Let now $\{F_r\}_{r=1}^{\ell}$ be a partition of unity subordinated to the cover $\{U_r\}_{r=1}^{\ell}$. Then, if we extend every $\widetilde{\psi}_r$ to be zero outside of U_r, the map $\Psi : X \to \mathbb{R}^{\ell(m+1)}$ defined as

$$\Psi = (F_1\widetilde{\psi}_1, \dots, F_\ell\widetilde{\psi}_\ell, F_1, \dots, F_\ell), \tag{7.1.17}$$

is continuous. Also, the coordinates $(F_1, \dots, F_\ell)$ are included to guarantee that Ψ is injective.

With this in mind, and following the discussion so far, we can make a good guess of what the desired isometries $\mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ should look like. Consider again the cover $\{U_r\}_{r=1}^{\ell}$ of open rectangles, choose the points $\{g_r\}_{r=1}^{\ell}$ to be in $X^h(P,Q)$, and let $\{F_r\}_{r=1}^{\ell}$ be a partition of unity subordinated to this cover. Our model isometry is defined on basis vectors as

$$\delta_x \mapsto \sum_{r=1}^{\ell} F_r(x)^{1/2}\delta_{[x,g_r]} \otimes \delta_{[g_r,x]}. \tag{7.1.18}$$

It is well-defined with respect to the *standard convention*; the bracket map returns the empty set for points x, y with $d(x,y) > \varepsilon_X$ and that, the Dirac delta function on the empty set returns zero. The fact that it is actually an isometry is proved in Subsection 7.1.4. Now, it is hard to miss that this isometry is the topological embedding Ψ (7.1.17) in disguise. One difference is that in the isometry we consider the square roots of the partition functions, but this is only because $\sum_r F_r(x) = 1$. Moreover, we do not have something that corresponds to the coordinates $(F_1, \dots, F_\ell)$ of Ψ. However, there is no need because (7.1.18) maps onto orthogonal vectors, and using the bracket axioms, injectivity is guaranteed (after all it is an isometry).

Remark 7.1.2. It is important to mention that the isometry (7.1.18) is strongly related to the ε-partitions needed to define the K-theory duality class of Kaminker, Putnam and Whittaker, see Subsection 5.2.1. In fact, its range projection can be written in the form of Lemma 5.2.4, if the rectangles have sufficiently small diameter. So here is the intriguing part. The same ingredient that was needed to define the K-theory duality class appears to be also needed for defining the K-homology duality class.

Finally, the quasi-invariance properties (7.1.12), (7.1.13) can be achieved by defining each isometry V_n as the average of "basic" isometries $\mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ of the type (7.1.18), with mutually orthogonal ranges. For this to happen, one has to carefully choose the covers $\{U_r\}_{r=1}^{\ell}$, the set of points $\{g_r\}_{r=1}^{\ell} \subset X^h(P,Q)$ and the partitions of unity $\{F_r\}_{r=1}^{\ell}$. The key tool is Theorem 3.3.2 which provides refining sequences of δ-fat Markov partitions. This is studied in Subsections 7.1.3 and 7.1.4.

7.1.3 Partitions of unity, aperiodic samples of Markov partitions

We develop tools that will lead to the construction of the isometry V of equation (7.1.1). The main ingredient will be a refining sequence of δ-fat Markov partitions $(\mathcal{R}_n^{\delta})_{n\geq 0}$ obtained by Theorem 3.3.2. This is a special refining sequence of (X,d) where each cover consists of open rectangles. It will be useful to recall that this refining sequence is built from a refining sequence of Markov partitions $(\mathcal{R}_n)_{n\geq 0}$ with $\#\mathcal{R}_n^{\delta} = \#\mathcal{R}_n$, as this will simplify the notation. The sequence $(\mathcal{R}_n^{\delta})_{n\geq 0}$ has many important properties, but here we will only use that, for every $n\geq 0$, the multiplicity $\mathrm{m}(\mathcal{R}_n^{\delta}) \leq (\#\mathcal{R}_1)^2$.

The first tool will be a sequence of Lipschitz partitions of unity $(\mathcal{F}_n)_{n\geq 0}$ on X associated to the sequence $(\mathcal{R}_n^{\delta})_{n\geq 0}$. The second tool of our construction will be a set $\mathcal{G}\subset X^h(P,Q)$ of dynamically independent points that are chosen from each rectangle in every open cover $\mathcal{R}_n^{\delta}$. Before constructing these tools recall that $Q\cap P=\varnothing$ and hence the set $X^h(P,Q)$ has no periodic points. For the sequel, it will be convenient to choose a random ordering in every $\mathcal{R}_n^{\delta}$ and write

$$\mathcal{R}_n^{\delta} = \{R_{n,k}^{\delta} : 1\leq k\leq \#\mathcal{R}_n\}. \tag{7.1.19}$$

For every $n\in\mathbb{N}$ and $1\leq k\leq \#\mathcal{R}_n$ consider the function $h_{n,k}: X\to[0,\infty)$ given by

$$h_{n,k}(x) = d(x, X\smallsetminus R_{n,k}^{\delta}). \tag{7.1.20}$$

Every such function is 1-Lipschitz by the triangle inequality. For completeness, since $\mathcal{R}_0^{\delta}=\{X\}$, we can define $h_{0,1}(x)=1$, for all $x\in X$. However, this is not important. Then, for every $n\geq 0$, the family of functions $\mathcal{F}_n=\{F_{n,k}:1\leq k\leq\#\mathcal{R}_n\}$ on X, defined as

$$F_{n,k}(x) = \frac{h_{n,k}(x)}{\sum_j h_{n,j}(x)}, \tag{7.1.21}$$

forms a partition of unity on X. Note that $\mathcal{F}_n$ is not subordinated to $\mathcal{R}_n^{\delta}$, unless (X,d) is zero-dimensional, because $F_{n,k}(x)>0$ if and only if $x\in R_{n,k}^{\delta}$. From a simple (but interesting) calculation found in [13, Proposition 1] and Theorem 3.3.2 we obtain the following.

Proposition 7.1.3. *For every $n\geq 0$ and $1\leq k\leq\#\mathcal{R}_n$, the Lipschitz constant of $F_{n,k}$ satisfies*

$$\mathrm{Lip}(F_{n,k}) \leq \frac{2(\#\mathcal{R}_1)^2+1}{\mathrm{Leb}(\mathcal{R}_n^{\delta})}.$$

We now move on to the construction of the second key tool. For this we require the next definition. Recall that by $\mathcal{R}^{\delta}$ we denote the set of vertices $\coprod_{n\geq 0}\mathcal{R}_n^{\delta}$ of the approximation graph corresponding to $(\mathcal{R}_n^{\delta})_{n\geq 0}$, see Theorem 3.3.2.

Definition 7.1.4. An $\mathcal{R}^{\delta}$*-sampling function* is a function $c:\mathcal{R}^{\delta}\to X$ such that $c(R^{\delta})\in R^{\delta}$, for all $R^{\delta}\in\mathcal{R}^{\delta}$. The image $c(\mathcal{R}^{\delta})$ will be called an $\mathcal{R}^{\delta}$*-sample* of the Smale space (X,φ).

Remark 7.1.5. The existence of an $\mathcal{R}^\delta$-sampling function follows from the Axiom of Choice. Also, every $\mathcal{R}^\delta$-sample is a dense subset of X.

The following definition will have a crucial role in constructing the isometry V.

Definition 7.1.6. An *aperiodic* $\mathcal{R}^\delta$*-sample* of the Smale space (X,φ) is the image $c(\mathcal{R}^\delta)$ of an injective $\mathcal{R}^\delta$-sampling function $c:\mathcal{R}^\delta\to X$ where $\varphi^j(c(\mathcal{R}^\delta))\cap c(\mathcal{R}^\delta)=\varnothing$, for all $j\in\mathbb{Z}\smallsetminus\{0\}$.

Remark 7.1.7. The ordering in (7.1.19) yields the bijection

$$\{(n,k): n\geq 0,\ 1\leq k\leq \#\mathcal{R}_n\}\to\mathcal{R}^\delta$$

that maps (n,k) to the rectangle $R^\delta_{n,k}$. Therefore, given an injective $\mathcal{R}^\delta$-sampling function c, one has that $c(R^\delta_{n,k})\neq c(R^\delta_{m,\ell})$, whenever $(n,k)\neq(m,\ell)$. In Proposition 7.1.10 we construct a particular injective $\mathcal{R}^\delta$-sampling function c' whose image is aperiodic, and from that point on, every $c'(R^\delta_{n,k})$ will be denoted by $g_{n,k}$ and the image $c'(\mathcal{R}^\delta)$ will be $\mathcal{G}=\{g_{n,k}: n\geq 0,\ 1\leq k\leq\#\mathcal{R}_n\}$.

We now aim to construct an aperiodic $\mathcal{R}^\delta$-sample of (X,φ) inside $X^h(P,Q)$. First, we need the next lemma which is a consequence of irreducibility.

Lemma 7.1.8. *For every $x\in X^h(P,Q)$, the full orbit $O(x)=\{\varphi^n(x):n\in\mathbb{Z}\}$ is not dense in X.*

Proof. Let $z\in X\smallsetminus(P\cup Q)$ and $\eta=d(z,P\cup Q)>0$, and assume to the contrary that there is some $x\in X^h(P,Q)$ for which $O(x)$ is dense in X. Since $x\in X^h(P,Q)$, there is some $N\in\mathbb{N}$ so that for every $n>N$ we have $d(\varphi^n(x),P)<\eta/2$ and $d(\varphi^{-n}(x),Q)<\eta/2$. Due to irreducibility (see Proposition 2.1.3), the space X has no isolated points and hence the set $T(x)=O(x)\smallsetminus\{\varphi^{-N}(x),\dots,x,\dots,\varphi^N(x)\}$ is still dense in X. However, $T(x)\cap B(z,\eta/3)=\varnothing$, contradicting the fact that $T(x)$ is dense in X. □

Further, we require the next elementary lemma which holds for dynamical systems (Z,ψ), where Z is an infinite topological space and $\psi:Z\to Z$ is a continuous map.

Lemma 7.1.9. *For a dynamical system (Z,ψ) the following are equivalent:*

(1) (Z,ψ) is irreducible;

(2) every closed, ψ-invariant, proper subset of Z has empty interior.

Proof. Suppose that (Z,ψ) is irreducible. Then, for every ordered pair of non-empty open sets $U,V\subset Z$, there is some $m\in\mathbb{N}$ such that $\psi^m(U)\cap V\neq\varnothing$. Now, assume to the contrary that there is a closed, ψ-invariant, proper subset $K\subset Z$ with non-empty interior U. Then we have $\psi^n(U)\subset K$, for all $n\in\mathbb{N}$. Taking $V=Z\smallsetminus K$ gives a contradiction to (Z,ψ) being irreducible.

Conversely, assume condition (2) and that there are non-empty open sets U, V in Z such that for every $n \in \mathbb{N}$ it holds $\psi^n(U) \cap V = \varnothing$. Then $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} \psi^n(U))$ is a closed, ψ-invariant, proper subset of Z with non-empty interior, contradicting condition (2). □

For the next proposition we will use the ordering (7.1.19), see also Remark 7.1.7.

Proposition 7.1.10. *The Smale space (X, φ) has an aperiodic $\mathcal{R}^\delta$-sample that is denoted by $\mathcal{G}$ and is a subset of $X^h(P, Q)$.*

Proof. The sampling function $c : \mathcal{R}^\delta \to X$ will be constructed inductively. First, note that from Lemmas 7.1.8 and 7.1.9, for every $g \in X^h(P, Q)$, the set $\mathrm{cl}(O(g))$ is a closed, φ-invariant, proper subset of X with empty interior. Therefore, every $X \setminus \mathrm{cl}(O(g))$ is open and dense in X. Also, note that the intersection of an open dense set with a dense set is a dense set.

Order the set $B = \{(n, k) : n \geq 0,\ 1 \leq k \leq \#\mathcal{R}_n\}$, that is in bijection with $\mathcal{R}^\delta$, using the lexicographic order $(n, k) \leq (m, \ell)$, if $n < m$ or ($n = m$ and $k \leq \ell$). It will be convenient to write $B = \{e_i : i \in \mathbb{N}\}$, where $e_1 = (0, 1)$ and $e_i < e_{i+1}$, for all $n \in \mathbb{N}$. We will construct a set $\mathcal{G} = \{g_{e_i} : i \in \mathbb{N}\}$ in $X^h(P, Q)$ as follows:

(i) Choose $g_{e_1} \in X^h(P, Q)$;

(ii) Having chosen $g_{e_1}, \dots, g_{e_n}$, choose

$$g_{e_{n+1}} \in R^\delta_{e_{n+1}} \cap \big(X^h(P, Q) \setminus \bigcup_{i=1}^{n} \mathrm{cl}(O(g_{e_i}))\big).$$

The sampling function $c : \mathcal{R}^\delta \to X$ is given by $c(R^\delta_{e_i}) = g_{e_i} \in R^\delta_{e_i}$, where $i \in \mathbb{N}$, and the $\mathcal{R}^\delta$-sample is $c(\mathcal{R}^\delta) = \mathcal{G}$. To see that it is injective, let $e_m \neq e_n$ and assume that $e_m \leq e_n$. Then,

$$g_{e_n} \in X^h(P, Q) \setminus \bigcup_{i=1}^{n-1} \mathrm{cl}(O(g_{e_i})),$$

while $g_{e_m} \in \bigcup_{i=1}^{n-1} \mathrm{cl}(O(g_{e_i}))$, and hence $g_{e_n} \neq g_{e_m}$.

Moreover, $\varphi^j(\mathcal{G}) \cap \mathcal{G} = \varnothing$, for all $j \in \mathbb{Z} \setminus \{0\}$. For this, assume to the contrary that there are $g_{e_m}, g_{e_n} \in \mathcal{G}$ and $j \in \mathbb{Z} \setminus \{0\}$ such $\varphi^j(g_{e_n}) = g_{e_m}$. If $n = m$ we get a contradiction because g_{e_n} is not periodic. Assume that $m < n$. Since $X^h(P, Q)$ and every orbit is φ-invariant, we have that

$$\varphi^j(g_{e_n}) \in X^h(P, Q) \setminus \bigcup_{i=1}^{n-1} \mathrm{cl}(O(g_{e_i})),$$

while $g_{e_m} \in \bigcup_{i=1}^{n-1} \mathrm{cl}(O(g_{e_i}))$, and hence $\varphi^j(g_{e_n}) \neq g_{e_m}$, leading to a contradiction. Similarly if $n < m$. Finally, we can drop the notation of the e_i's and get the desired aperiodic $\mathcal{R}^\delta$-sample

$$\mathcal{G} = \{g_{n,k} : n \geq 0,\ 1 \leq k \leq \#\mathcal{R}_n\}$$

of the Smale space (X, φ). □

7.1.4 The averaging isometries

In this subsection we construct the isometry V of equation (7.1.1). Then, in Section 7.2 we prove that V, together with the symmetric representation ρ in (7.1.5), yields a θ-summable Fredholm module representative of the KPW-extension class. Following the discussion in Subsection 7.1.2, the isometry V should have the form $\bigoplus_{n\in\mathbb{Z}} V_n$, for isometries $V_n : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$. Now, each V_n should be an average of isometries with mutually orthogonal ranges (of the form (7.1.18)) so that the quasi-invariance properties (7.1.12) and (7.1.13) are met, and also the property (7.1.14) holds.

Let $(\mathcal{R}_n^\delta)_{n\geq 0}$ be a refining sequence of δ-fat Markov partitions of the irreducible Smale space (X, d, φ), obtained by Theorem 3.3.2. Also, recall that the set of vertices of the corresponding approximation graph is $\mathcal{R}^\delta = \coprod_{n\geq 0} \mathcal{R}_n^\delta$. Similarly as in Subsection 7.1.3, choose a random ordering in every $\mathcal{R}_n^\delta$ and write

$$\mathcal{R}_n^\delta = \{R_{n,k}^\delta : 1 \leq k \leq \#\mathcal{R}_n\}. \tag{7.1.22}$$

For our purposes here, it is important to note that, for every $n \geq 2$ and $1 \leq k \leq \#\mathcal{R}_n$, if $x, y \in R_{n,k}^\delta$, then

$$d(\varphi^r(x), \varphi^r(y)) \leq \varepsilon'_X, \tag{7.1.23}$$

for all $|r| \leq n-1$. Our goal is to show that the refining sequence $(\mathcal{R}_n^\delta)_{n\geq 0}$ produces a family

$$\mathcal{T} = \{\iota_{n,r} : n \in \mathbb{N},\, |r| \leq n-1\} \tag{7.1.24}$$

of *basic isometries* $\iota_{n,r} : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ with mutually orthogonal ranges, which we can average in a certain way. From the results of Subsection 7.1.3 (specifically see (7.1.21) and Proposition 7.1.10)

(i) for every $n \geq 0$, there is a Lipschitz partition of unity $\mathcal{F}_n = \{F_{n,k} : 1 \leq k \leq \#\mathcal{R}_n\}$ on X such that $F_{n,k}(x) > 0$ if and only if $x \in R_{n,k}^\delta$. We will be interested in the square roots of these functions and so we consider the family

$$\mathcal{F}_n^{1/2} = \{f_{n,k} : 1 \leq k \leq \#\mathcal{R}_n\},$$

where $f_{n,k} = F_{n,k}^{1/2}$;

(ii) there is an aperiodic $\mathcal{R}^\delta$-sample

$$\mathcal{G} = \{g_{n,k} \in R_{n,k}^\delta : n \geq 0,\, 1 \leq k \leq \#\mathcal{R}_n\} \subset X^h(P, Q),$$

meaning that $g_{n,k} \neq g_{m,\ell}$, if $(n,k) \neq (m,\ell)$, and $\varphi^r(\mathcal{G}) \cap \mathcal{G} = \varnothing$, for all $r \in \mathbb{Z} \setminus \{0\}$.

Given (i) and (ii) we now construct the family $\mathcal{T}$. For every $n \in \mathbb{N}$ we define the operator $\iota_{n,0} : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ that on basis vectors is given by

$$\iota_{n,0}(\delta_y) = \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(y)\delta_{[y,g_{n,k}]} \otimes \delta_{[g_{n,k},y]}. \tag{7.1.25}$$

It is well-defined with respect to the *standard convention*; the bracket map returns the empty set for points x, y with $d(x,y) > \varepsilon_X$ and that, the Dirac delta function on the empty set returns zero. Recall that each $R^{\delta}_{n,k}$ is a rectangle with $\operatorname{diam}(R^{\delta}_{n,k}) \leq \varepsilon'_X$ and therefore we can use the notation $X^u(g_{n,k}, R^{\delta}_{n,k}) = X^u(g_{n,k}, 2\varepsilon'_X) \cap R^{\delta}_{n,k}$. Similarly for the stable case.

The adjoint is given on basis vectors by

$$\iota^*_{n,0}(\delta_x \otimes \delta_z) = f_{n,k}([x,z])\delta_{[x,z]}, \tag{7.1.26}$$

if $x \in X^u(g_{n,k}, R^{\delta}_{n,k})$, $z \in X^s(g_{n,k}, R^{\delta}_{n,k})$ (for a unique k), and is zero otherwise. Indeed, we have that

$$< \iota_{n,0}(\delta_y), \delta_x \otimes \delta_z >= \sum_k f_{n,k}(y),$$

where the sum is taken over all k such that $f_{n,k}(y) \neq 0$ and $x = [y, g_{n,k}]$, $z = [g_{n,k}, y]$. Let k_1 be one of these k's and since $f_{n,k_1}(y) \neq 0$ it holds $y \in R^{\delta}_{n,k_1}$. Consequently, we have $x \in X^u(g_{n,k_1}, R^{\delta}_{n,k_1})$, $z \in X^s(g_{n,k_1}, R^{\delta}_{n,k_1})$. Now, if k_2 is one of these k's then,

$$g_{n,k_2} \in X^s(g_{n,k_1}, \varepsilon_X) \cap X^u(g_{n,k_1}, \varepsilon_X) = \{g_{n,k_1}\},$$

and hence $k_2 = k_1$. Finally, note that $[x,z] = y$.

Lemma 7.1.11. *For every $n \in \mathbb{N}$ the operator $\iota_{n,0}$ is an isometry.*

Proof. For $\delta_y \in \mathscr{H}$ we have

$$\begin{aligned}
\iota^*_{n,0}\iota_{n,0}(\delta_y) &= \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(y)\iota^*_{n,0}(\delta_{[y,g_{n,k}]} \otimes \delta_{[g_{n,k},y]}) \\
&= \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(y)^2 \delta_y \\
&= \delta_y.
\end{aligned}$$

This completes the proof. ☐

Let $n \in \mathbb{N}$ and for every $r \in \mathbb{Z}$ we define the isometry $\iota_{n,r} = (u \otimes u)^r \iota_{n,0} u^{-r}$, where u is the unitary on $\mathscr{H}$ given by $u(\delta_x) = \delta_{\varphi(x)}$. The sequence $(\iota_{n,r})_{r \in \mathbb{Z}}$ can be considered as the $\mathbb{Z}$-orbit of $\iota_{n,0}$, and its interesting part lies in the central terms for $|r| \leq n-1$. Indeed, if

$|r| \leq n-1$ then from (7.1.23) we have that

$$\iota_{n,r}(\delta_y) = \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(\varphi^{-r}(y))\delta_{[y,\varphi^r(g_{n,k})]} \otimes \delta_{[\varphi^r(g_{n,k}),y]}. \tag{7.1.27}$$

Then, it is straightforward to see that

$$\iota_{n,r}^*(\delta_x \otimes \delta_z) = f_{n,k}(\varphi^{-r}[x,z])\delta_{[x,z]}, \tag{7.1.28}$$

if $\varphi^{-r}(x) \in X^u(g_{n,k}, R_{n,k}^\delta)$, $\varphi^{-r}(z) \in X^s(g_{n,k}, R_{n,k}^\delta)$ (for a unique k), and is zero otherwise. Since $\mathcal{G}$ is an aperiodic $\mathcal{R}^\delta$-sample, we obtain the following orthogonality result.

Lemma 7.1.12. *For every $m,n \in \mathbb{N}$ and $r,s \in \mathbb{Z}$ with $|r| \leq n-1$, $|s| \leq m-1$ so that $(n,r) \neq (m,s)$, we have that $\iota_{m,s}^*\iota_{n,r} = 0$.*

Proof. Let m,n,r,s as in the statement and assume to the contrary that $\iota_{m,s}^*\iota_{n,r} \neq 0$. Then there is a basis vector $\delta_y \in \mathscr{H}$ such that $\iota_{m,s}^*\iota_{n,r}(\delta_y) \neq 0$. We have that

$$\iota_{m,s}^*\iota_{n,r}(\delta_y) = \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(\varphi^{-r}(y))\iota_{m,s}^*(\delta_{[y,\varphi^r(g_{n,k})]} \otimes \delta_{[\varphi^r(g_{n,k}),y]}),$$

and hence there is k such that $f_{n,k}(\varphi^{-r}(y))\iota_{m,s}^*(\delta_{[y,\varphi^r(g_{n,k})]} \otimes \delta_{[\varphi^r(g_{n,k}),y]}) \neq 0$. Now, since $f_{n,k}(\varphi^{-r}(y)) \neq 0$ and $|r| \leq n-1$ we have that $\varphi^{-r}(y) \in R_{n,k}^\delta$ and so $d(y,\varphi^r(g_{n,k})) \leq \varepsilon_X'$. Therefore,

$$[y,\varphi^r(g_{n,k})] \in X^u(\varphi^r(g_{n,k}),\varepsilon_X/2), \;\; [\varphi^r(g_{n,k}),y] \in X^s(\varphi^r(g_{n,k}),\varepsilon_X/2). \tag{7.1.29}$$

Moreover, since $\iota_{m,s}^*(\delta_{[y,\varphi^r(g_{n,k})]} \otimes \delta_{[\varphi^r(g_{n,k}),y]}) \neq 0$, there exists some $1 \leq l \leq \mathcal{R}_m$ so that

$$\varphi^{-s}[y,\varphi^r(g_{n,k})] \in X^u(g_{m,l}, R_{m,l}^\delta), \;\; \varphi^{-s}[\varphi^r(g_{n,k}),y] \in X^s(g_{m,l}, R_{m,l}^\delta).$$

The fact that $|s| \leq m-1$ implies that

$$[y,\varphi^r(g_{n,k})] \in X^u(\varphi^s(g_{m,l}),\varepsilon_X'), \;\; [\varphi^r(g_{n,k}),y] \in X^s(\varphi^s(g_{m,l}),\varepsilon_X'),$$

and using (7.1.29) we get that

$$\varphi^r(g_{n,k}) \in X^s(\varphi^s(g_{m,l}),\varepsilon_X) \cap X^u(\varphi^s(g_{m,l}),\varepsilon_X). \tag{7.1.30}$$

Consequently, $\varphi^r(g_{n,k}) = \varphi^s(g_{m,l})$. If $r = s$ then $g_{n,k} = g_{m,l}$. Since the $\mathcal{R}^\delta$-sample is aperiodic we get that $n = m$, obtaining a contradiction. If $r \neq s$, then $\varphi^{r-s}(g_{n,k}) = g_{m,l}$ and hence $\varphi^{r-s}(\mathcal{G}) \cap \mathcal{G} \neq \varnothing$, leading again to a contradiction since $\mathcal{G}$ is an aperiodic $\mathcal{R}^\delta$-sample. To summarise, we showed that $\iota_{m,s}^*\iota_{n,r} = 0$. □

Remark 7.1.13. We follow the notation of (7.1.27). Restricting to isometries $\iota_{n,r}$ with $|r| \leq n-1$ allows us to control the coefficient functions $f_{n,k} \circ \varphi^{-r}$. More precisely, in Section 7.2 we will consider sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in X that $d(x_n, y_n) \leq \lambda_X^{-n}$, eventually for $n \in \mathbb{N}$. For these sequences we want to have that

$$\lim_{n\to\infty} |f_{n,k} \circ \varphi^{-r}(x_n) - f_{n,k} \circ \varphi^{-r}(y_n)| = 0, \tag{7.1.31}$$

for every k, r, or even better that the limit is converging exponentially fast to zero. From Proposition 7.1.3 we have that for $c = (2(\#\mathcal{R}_1)^2 + 1)^{1/2}$, each $f_{n,k}$ is 1/2-Hölder with coefficient

$$\text{Höl}(f_{n,k}) \leq \frac{c}{\text{Leb}(\mathcal{R}_n^\delta)^{1/2}}.$$

If the metric d is self-similar then φ, φ^{-1} are λ_X-Lipschitz, and from Theorem 3.3.2 part (7), there is a constant $c' > 0$ such that for every n, k it holds

$$\text{Höl}(f_{n,k}) \leq c'\lambda_X^{n/2}.$$

Hence, each $f_{n,k} \circ \varphi^{-r}$ is 1/2-Hölder with coefficient $\text{Höl}(f_{n,k} \circ \varphi^{-r}) \leq c'\lambda_X^{(n+|r|)/2} \leq c'\lambda_X^n$. Therefore, the limit (7.1.32) might not be zero. This can be fixed if we slow down (linearly) the coefficient functions; for every $n \in \mathbb{N}$, $k \in \{1, \dots, \mathcal{R}_{\lceil n/4\rceil}\}$ and $|r| \leq \lceil n/4\rceil - 1$ it holds that

$$|f_{\lceil n/4\rceil,k} \circ \varphi^{-r}(x_n) - f_{\lceil n/4\rceil,k} \circ \varphi^{-r}(y_n)| \leq c'\lambda_X^{1-n/4}. \tag{7.1.32}$$

This slow down trick works also for a non-self-similar metric d, for which the map φ is bi-Lipschitz. However, the slow down might not be linear, and this will affect negatively the summability properties of the Fredholm module representative of the KPW-extension class.

In order to visualise the basic isometries we consider their range projections. Let $n \in \mathbb{N}$, and for every $r \in \mathbb{Z}$ consider the projections $p_{n,r} = \iota_{n,r}\iota_{n,r}^* = (u \otimes u)^r p_{n,0}(u \otimes u)^{-r}$. For $|r| \leq n-1$ they are given on basis vectors by

$$p_{n,r}(\delta_x \otimes \delta_z) = f_{n,l}(\varphi^{-r}[x,z]) \sum_{k=1}^{\#\mathcal{R}_n} f_{n,k}(\varphi^{-r}[x,z])\delta_{[x,\varphi^r(g_{n,k})]} \otimes \delta_{[\varphi^r(g_{n,k}),z]}, \tag{7.1.33}$$

if $\varphi^{-r}(x) \in X^u(g_{n,l}, R_{n,l}^\delta)$, $\varphi^{-r}(z) \in X^s(g_{n,l}, R_{n,l}^\delta)$ (for a unique l), and is zero otherwise. Moreover, due to Lemma 7.1.12, for every $m, n \in \mathbb{N}$ and $r, s \in \mathbb{Z}$ with $|r| \leq n-1$, $|s| \leq m-1$ and $(n,r) \neq (m,s)$, we have that

$$p_{m,s}p_{n,r} = 0. \tag{7.1.34}$$

It should be clear that these projections are related to the projections that produce the K-theory duality class, see Subsection 5.2.1. Indeed, since the diameters of $\mathcal{R}_n^\delta$ converge to zero, there is some $n_0 \in \mathbb{N}$ such that $\overline{\text{diam}}(\mathcal{R}_n^\delta) \leq \varepsilon'_X/4$, for all $n \geq n_0$. Letting $\mathcal{G}_n = \{g_{n,k} \in R_{n,k}^\delta : 1 \leq k \leq \#\mathcal{R}_n\}$, the pair $(\mathcal{F}_n^{1/2}, \mathcal{G}_n)$ is an ε'_X-partition and hence, for every $n \geq n_0$, the projection $p_{n,0} \in \mathcal{S}(Q) \otimes \mathcal{U}(P) \subset \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. Since conjugation by u yields automorphisms of $\mathcal{S}(Q)$ and $\mathcal{U}(P)$, we also have that $p_{n,r} \in \mathcal{S}(Q) \otimes \mathcal{U}(P)$, for all $r \in \mathbb{Z}$. In fact, for every $n \geq n_0$ and $|r|, |s| \leq n-1$ there is a a homotopy in $\mathcal{S}(Q) \otimes \mathcal{U}(P)$ between $p_{n,r}$ and $p_{n,s}$.

Remark 7.1.14. It is interesting to note that for every $n \geq n_0$ and $|r| \leq n-1$, the projection $p_{n,r}$ yields an odd K-homology class for the homoclinic algebra $\mathcal{H}$. To see this, consider $\mathcal{S}(Q) \otimes \mathcal{U}(P) \subset \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ and let $\sigma_{12} : \mathcal{S}(Q) \otimes \mathcal{U}(P) \to \mathcal{U}(P) \otimes \mathcal{S}(Q)$ be the flip isomorphism. From Lemma 5.2.6 we have that for every $a \in \mathcal{S}(Q)$, $b \in \mathcal{U}(P)$ the products $ab, ba \in \mathcal{K}(\mathscr{H})$. Then, since $p_{n,r} \in \mathcal{S}(Q) \otimes \mathcal{U}(P)$, the triple

$$P_{n,r} = (\mathscr{H} \otimes \mathscr{H}, \sigma_{12}, 2p_{n,r} - 1)$$

is an odd Fredholm module over $\mathcal{S}(Q) \otimes \mathcal{U}(P)$. Now, Theorem 5.1.7 gives a Morita equivalence bimodule E between $\mathcal{H}$ and $\mathcal{S}(Q) \otimes \mathcal{U}(P)$, and hence the class

$$[E] \otimes_{\mathcal{S}(Q) \otimes \mathcal{U}(P)} [P_{n,r}] \in \mathrm{KK}_1(\mathcal{H}, \mathbb{C}).$$

We now construct the isometry $V = \bigoplus_{n \in \mathbb{Z}} V_n$, from isometries $V_n : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$. It will be useful to depict the family $\mathcal{T} = \{\iota_{n,r} : n \in \mathbb{N}, |r| \leq n-1\}$ as a triangle

$$\begin{array}{ccccccccc}
 & & & & \iota_{1,0} & & & & \\
 & & & \iota_{2,-1} & \iota_{2,0} & \iota_{2,1} & & & \\
 & & \iota_{3,-2} & \iota_{3,-1} & \iota_{3,0} & \iota_{3,1} & \iota_{3,2} & & \\
 & \iota_{4,-3} & \iota_{4,-2} & \iota_{4,-1} & \iota_{4,0} & \iota_{4,1} & \iota_{4,2} & \iota_{4,3} & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The gist is to find an appropriate way to average in $\mathcal{T}$ and construct each V_n. Recall that we want to achieve

$$\lim_{n \to \pm\infty} \|V_{n+1} - V_n\| = 0, \tag{7.1.35}$$

and

$$\lim_{n \to \pm\infty} \|(u \otimes u)V_n - V_n u\| = 0. \tag{7.1.36}$$

Every V_n will be the average of finitely many isometries. Also, the bigger the n, the more isometries will be averaged. The first quasi-invariance condition indicates that, the isometries whose average is V_n, should coincide more and more with the isometries that produce V_{n+1}. The second quasi-invariance condition suggests that the chosen isometries for every V_n, should be further and further away (horizontally) from the boundaries of the triangle; the sets $\{\iota_{n,-n+1} : n \in \mathbb{N}\}$ and $\{\iota_{n,n-1} : n \in \mathbb{N}\}$. In this way, every V_n, eventually, will not be shifted outside of the triangle by powers of u and $u \otimes u$. Of course, to achieve (7.1.36) for any fixed power of u and $u \otimes u$, it suffices to chose isometries from $\mathcal{T}$ that are simply not in these two boundaries. However, we are interested in achieving condition (7.1.11) which requires not moving (eventually) outside of the triangle, at all. This also forces to construct each V_n as a moving average directed to the bottom of the triangle.

More precisely, let $\iota_{0,0} : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ be the diagonal embedding, and for every $m \geq 0$ define

$$\theta_m = \sum_{r=-m}^{m} \iota_{2m,r}. \tag{7.1.37}$$

Due to Lemma 7.1.12, for every $m, m' \in \mathbb{N}$ with $m \neq m'$, it holds that $\theta_m^* \theta_m = (2m+1)I$ and $\theta_m^* \theta_{m'} = 0$. Therefore, for every $n \geq 0$, the operator

$$W_n = c_n^{-1} \sum_{m=n}^{2n} \theta_m, \tag{7.1.38}$$

where $c_n = ((n+1)(3n+1))^{1/2}$, is an isometry in $\mathcal{B}(\mathscr{H}, \mathscr{H} \otimes \mathscr{H})$. Remark 7.1.13 suggests to slow down the sequence $(W_n)_{n\geq 0}$ by considering the sequence $(\gamma_n)_{n\geq 0}$ with $\gamma_n = \lceil n/16 \rceil$, and define

$$V_n = W_{\gamma_n}. \tag{7.1.39}$$

Moreover, for $n < 0$, we define $V_n = V_{-n}$ and obtain the desired isometry

$$V = \bigoplus_{n\in\mathbb{Z}} V_n. \tag{7.1.40}$$

Remark 7.1.15. Let $m \in \mathbb{N}$ and $|r| \leq m-1$. It is straightforward to see that the open cover $\varphi^r(\mathcal{R}_m^\delta)$, corresponding to the basic isometry $\iota_{m,r}$, refines the δ-fat Markov partition $\mathcal{R}_{m-|r|}^\delta$. This means that the open covers associated with the basic isometries constituting each W_n (7.1.38), refine the δ-fat Markov partition $\mathcal{R}_n^\delta$. Consequently, as n goes to $+\infty$, the diameter of the supports of the coefficient functions (square roots of partitions of unity) of the basic isometries in W_n tends to zero. The same holds for the isometry $V_n = W_{\gamma_{|n|}}$, as $|n|$ tends to infinity.

Proposition 7.1.16. *For every $j \in \mathbb{Z}$, the isometry $V : \mathscr{H} \otimes \ell^2(\mathbb{Z}) \to \mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z})$ satisfies*

$$\lim_{n\to\pm\infty} \|V_{n+j} - V_n\| = 0$$
$$\lim_{n\to\pm\infty} \|(u \otimes u)^j V_n - V_n u^j\| = 0.$$

In particular, both sequences are $O(|n|^{-1/2})$.

Proof. It suffices to consider the limits to $+\infty$ and the isometries W_n instead of V_n. Also, we can assume that $j \geq 0$. For the first limit we estimate $\|W_{n+j} - W_n\|^2$. Specifically, we will show that for every $\eta \in \mathscr{H}$ with $\|\eta\| = 1$, the inner products $\langle W_{n+j}(\eta), W_n(\eta)\rangle$ and $\langle W_n(\eta), W_{n+j}(\eta)\rangle$ are independent of η and converge to one. Indeed, for $n \geq j$ we have that

$$
\begin{aligned}
W_{n+j}^* W_n &= (c_{n+j}c_n)^{-1}\Big(\sum_{m=n+j}^{2(n+j)} \theta_m^*\Big)\Big(\sum_{m'=n}^{2n} \theta_{m'}\Big) \\
&= (c_{n+j}c_n)^{-1} \sum_{m=n+j}^{2n} (2m+1)I \\
&= \frac{(n+1-j)(3n+1+j)}{((n+1+j)(3n+1+3j)(n+1)(3n+1))^{1/2}} I.
\end{aligned}
$$

Consequently, it holds that $\|W_{n+j} - W_n\| = O(n^{-1/2})$.

Similarly, we estimate $\|(u\otimes u)^j W_n - W_n u^j\|^2$ by showing that $W_n^*(u\otimes u)^j W_n$ is a multiple of u^j. For $n \geq j$ we have that

$$
\begin{aligned}
(u \otimes u)^j W_n &= c_n^{-1} \sum_{m=n}^{2n} (u \otimes u)^j \sum_{r=-m}^{m} \iota_{2m,r} \\
&= c_n^{-1} \sum_{m=n}^{2n} \sum_{r=-m+j}^{m+j} \iota_{2m,r} u^j.
\end{aligned}
$$

Since the basic isometries have mutually orthogonal ranges we have

$$
\begin{aligned}
W_n^*(u \otimes u)^j W_n &= c_n^{-2} \sum_{m=n}^{2n} \sum_{r=-m+j}^{m} \iota_{2m,r}^* \iota_{2m,r} u^j \\
&= \frac{3n+1-j}{3n+1} u^j.
\end{aligned}
$$

As a result, we have $\|(u \otimes u)^j W_n - W_n u^j\| = O(n^{-1/2})$. □

7.2 Fredholm modules for the KPW-extension class

We now prove that the constructions of Section 7.1 yield θ-summable Fredholm module representatives for the KPW-extension class of a Smale space. This section consists of four elaborate lemmas whose proof requires most of the tools and techniques developed in this thesis. For intuition and a better understanding we advise the reader to look at Section 7.1 first, especially Subsection 7.1.2.

Let (X, d, φ) be an irreducible Smale space and fix two periodic orbits Q, P such that $Q \cap P = \varnothing$. Consider the stable and unstable groupoids $G^s(Q)$ and $G^u(P)$. The algebraic and topological structures of the groupoids do not depend on the choice of hyperbolic metric (compatible with d), and this allows some flexibility in their study. In Theorem 6.1.4 we showed that every self-similar hyperbolic metric $d' \in \mathrm{sM}_d(X,\varphi)$ yields nicely behaved compatible groupoid metrics $D_{s,d'}$ and $D_{u,d'}$ on $G^s(Q)$ and $G^u(P)$. In Proposition 6.2.6 we proved that these metrics produce the dense (Lipschitz) $*$-subalgebras $\Lambda_{s,d'}(Q,\alpha_s)$, $\Lambda_{u,d'}(P,\alpha_u)$ of the Ruelle algebras $\mathcal{R}^s(Q)$, $\mathcal{R}^u(P)$. Then, in Theorem 6.2.16 it was proved that the KPW-extension $\tau_\Delta : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{Q}(\mathscr{H} \otimes \ell^2(\mathbb{Z}))$ (5.2.6) is finitely-smooth on the algebra $\Lambda_{s,d'}(Q,\alpha_s) \otimes_{\mathrm{alg}} \Lambda_{u,d'}(P,\alpha_u)$, for every $d' \in \mathrm{sM}_d(X,\varphi)$.

One of the technical parts of Section 7.1 was constructing the symmetric representation (7.1.5)

$$\rho : \mathcal{R}^s(Q) \otimes \mathcal{R}^u(P) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z})).$$

With a lot more effort, using a refining sequence of δ-fat Markov partitions $(\mathcal{R}_n^\delta)_{n\geq 0}$, we then constructed the isometry (7.1.40)

$$V = \bigoplus_{n\in\mathbb{Z}} V_n : \mathscr{H} \otimes \ell^2(\mathbb{Z}) \to \mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z}).$$

Each isometry V_n is the average of basic isometries of the form $\iota_{m,s} : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ (7.1.27). All basic isometries depend on the metric d (while $\rho, \mathscr{H}$ do not), hence the same is true for V. Let us denote V by V_d and consider the triple

$$F_d = (\mathscr{H} \otimes \mathscr{H} \otimes \ell^2(\mathbb{Z}), \rho, 2V_d V_d{}^* - 1). \tag{7.2.1}$$

Similarly, for each $d' \in \mathrm{sM}_d(X,\varphi)$ there is a triple $F_{d'}$ for the Smale space (X,d',φ). The main result of this section is about θ-summable Fredholm module representatives of the abstract K-homology class of τ_Δ, namely $\Delta \in \mathrm{K}^1(\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P))$.

Theorem 7.2.1. *For every metric $d' \in sM_d(X,\varphi)$, the triple $F_{d'}$ is an odd Fredholm module over $\mathcal{R}^s(Q) \otimes \mathcal{R}^u(P)$ that represents the duality class Δ, and is θ-summable on $\Lambda_{s,d'}(Q,\alpha_s) \otimes_{alg} \Lambda_{u,d'}(P,\alpha_u)$.*

Remark 7.2.2. The index-theoretic calculations of Proposition 4.2.11 give an exhaustive description of the K_1-homology classes of $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ in terms of Fredholm modules constructed by Markov partitions. Using similar techniques, the same can be done for the K_0-homology classes. Now, if $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ were unital, the θ-summability of any $F_{d'}$ would (in a straightforward manner) imply the uniform $\mathrm{Li}^{1/2}$-summability (see Definition 4.2.4) of the K-homology of both $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$. However, these Ruelle algebras are never unital. Nevertheless, it seems still possible to transfer the θ-summability of each $F_{d'}$. We will deal with this in a future project where we also intend to lift the θ-summable Fredholm modules over $\mathcal{R}^s(Q)$ and $\mathcal{R}^u(P)$ (obtained from any $F_{d'}$) to θ-summable spectral triples. Moreover, we believe that every $F_{d'}$ corresponds to interesting Fredholm modules over $\mathcal{S}(Q) \otimes \mathcal{U}(P)$, $\mathcal{S}(Q)$ and $\mathcal{U}(P)$ using the pull-back.

In this section we focus only on θ-summable Fredholm module representatives of Δ, and we do not aim to obtain the fine dimension estimates of Section 6.2. Hence, there is no need to keep track of which self-similar metric we are using. For the same reason, there is no point in considering the more tractable metrics and Lipschitz algebras of Propositions 6.1.23 and 6.2.7, in the case (X,d,φ) is a topological Markov chain. Henceforth, we assume that the Smale space (X,d,φ) is self-similar, we consider only the metric d, we denote V_d simply by V, and we prove Theorem 7.2.1 by showing that the triple

$$F=(\mathscr{H}\otimes\mathscr{H}\otimes\ell^2(\mathbb{Z}),\rho,2VV^*-1) \tag{7.2.2}$$

represents Δ.

Recall the representations $\overline{\rho_s}:\mathcal{R}^s(Q)\to\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$, $\overline{\rho_u}:\mathcal{R}^u(P)\to\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$ which commute modulo compacts and whose product in the Calkin algebra gives τ_Δ. Also, recall the unitary $U=\bigoplus_{n\in\mathbb{Z}}u^{-\lfloor n/2\rfloor}$ and the symmetrised KPW-extension (7.1.4) $\mathrm{Ad}_\pi(U)\circ\tau_\Delta$ which on $x\in\mathcal{R}^s(Q)\otimes_{\mathrm{alg}}\mathcal{R}^u(P)$ is given by

$$(\mathrm{Ad}_\pi(U)\circ\tau_\Delta)(x)=(\mathrm{Ad}(U)\circ(\overline{\rho_s}\cdot\overline{\rho_u}))(x)+\mathcal{K}(\mathscr{H}\otimes\ell^2(\mathbb{Z})). \tag{7.2.3}$$

In Proposition 6.2.14 we proved that $\overline{\rho_s}(\Lambda_{s,d}(Q,\alpha_s))$ and $\overline{\rho_u}(\Lambda_{u,d}(P,\alpha_u))$ commute modulo a Schatten ideal $\mathcal{J}$. Therefore, for every $x,y\in\Lambda_{s,d}(Q,\alpha_s)\otimes_{\mathrm{alg}}\Lambda_{u,d}(P,\alpha_u)$, we have that

$$\begin{gathered}(\overline{\rho_s}\cdot\overline{\rho_u})(xy)-(\overline{\rho_s}\cdot\overline{\rho_u})(x)(\overline{\rho_s}\cdot\overline{\rho_u})(y)\in\mathcal{J}\\ \text{and}\ \ (\overline{\rho_s}\cdot\overline{\rho_u})(x^*)-(\overline{\rho_s}\cdot\overline{\rho_u})(x)^*\in\mathcal{J}.\end{gathered} \tag{7.2.4}$$

Since $U\in\mathcal{B}(\mathscr{H}\otimes\ell^2(\mathbb{Z}))$, the same conditions hold for the linear map $\mathrm{Ad}(U)\circ(\overline{\rho_s}\cdot\overline{\rho_u})$. Here we aim to show that, for all $x\in\Lambda_{s,d}(Q,\alpha_s)\otimes_{\mathrm{alg}}\Lambda_{u,d}(P,\alpha_u)$, it holds

$$V^*\rho(x)V-(\mathrm{Ad}(U)\circ(\overline{\rho_s}\cdot\overline{\rho_u}))(x)\in\mathrm{Li}(\mathscr{H}\otimes\ell^2(\mathbb{Z})). \tag{7.2.5}$$

Then, using Theorem 4.2.8 we obtain that the triple F from (7.2.2) is an odd Fredholm module over $\mathcal{R}^s(Q)\otimes\mathcal{R}^u(P)$ that is θ-summable on $\Lambda_{s,d}(Q,\alpha_s)\otimes_{\mathrm{alg}}\Lambda_{u,d}(P,\alpha_u)$ and whose K-homology class lifts the extension class $[\mathrm{Ad}_\pi(U)\circ\tau_\Delta]=[\tau_\Delta]$, hence represents Δ. In this way, we complete the proof of Theorem 7.2.1.

From Subsection 7.1.2, in particular conditions (7.1.10) and (7.1.11), the proof of (7.2.5) is an immediate application of the following Lemmas 7.2.3 and 7.2.7, as well as, of Lemma 6.2.9 which is about singular values. From (7.1.38), recall that for every $n\in\mathbb{N}$ we have the isometry

$$W_n=c_n^{-1}\sum_{m=n}^{2n}\sum_{r=-m}^{m}\iota_{2m,r}, \tag{7.2.6}$$

where $c_n=((n+1)(3n+1))^{1/2}$, and for $n\in\mathbb{Z}$ we have $V_n=W_{\gamma_{|n|}}$, where $\gamma_{|n|}=\lceil |n|/16\rceil$.

In the next result, the fact that the metric d is self-similar is not important.

Lemma 7.2.3. *Let $a \in C_c(G^s(Q))$, $b \in C_c(G^u(P))$ and $i,k,l \in \mathbb{Z}$. Then, for every $n \in \mathbb{Z}$, the operator*

$$T_n = V^*_{2n+l}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i$$

has finite rank. In particular,

(i) there is $n_0 \in \mathbb{N}$ such that $\operatorname{rank}(T_n) = 0$, for all $n \leq -n_0$;

(ii) there is a constant $C_1 > 0$ so that, for every $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$, we have $\operatorname{rank}(T_n) \leq C_1 e^{2(\mathrm{h}(\varphi)+\varepsilon)n}$.

Proof. Assume that $\operatorname{supp}(a) \subset V^s(v,w,h^s,\eta,N)$ and $\operatorname{supp}(b) \subset V^u(v',w',h^u,\eta',N')$ and denote each $V^*_{2n+l}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k}$ by S_n. Fix some $n \in \mathbb{Z}$. The operator $\alpha_u^{-n}(b)\alpha_s^n(a)u^i$ has finite rank (see Lemma 6.2.11) and also

$$\operatorname{rank}(\alpha_u^{-n}(b)\alpha_s^n(a)u^i) = \operatorname{rank}(\alpha_u^{-n}(b)\alpha_s^n(a)) = \operatorname{rank}(b\alpha_s^{2n}(a)). \tag{7.2.7}$$

We claim that the operator S_n has finite rank too. If $y \in X^h(P,Q)$ and $S_n(\delta_y) \neq 0$, then there is some basic isometry $\iota_{2m,r}$, for $m \in \{\gamma_{|2n+k|},\dots,2\gamma_{|2n+k|}\}$ and $|r| \leq m$, such that $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y) \neq 0$. We have

$$(u^i \otimes u^i)\iota_{2m,r}(\delta_y) = \sum_{j=1}^{\#\mathcal{R}_{2m}} f_{2m,j}(\varphi^{-r}(y))\delta_{\varphi^i[y,\varphi^r(g_{2m,j})]} \otimes \delta_{\varphi^i[\varphi^r(g_{2m,j}),y]},$$

and hence there is some $j \in \{1,\dots,\#\mathcal{R}_{2m}\}$ such that $f_{2m,j}(\varphi^{-r}(y)) \neq 0$, meaning that $\varphi^i[y,\varphi^r(g_{2m,j})]$, $\varphi^i[\varphi^r(g_{2m,j}),y]$ are well-defined, and

$$\alpha_u^{-n}(b)\delta_{\varphi^i[y,\varphi^r(g_{2m,j})]} \neq 0,$$
$$\alpha_s^n(a)\delta_{\varphi^i[\varphi^r(g_{2m,j}),y]} \neq 0.$$

Consequently, we have that

$$\begin{aligned}\varphi^{n+i}[y,\varphi^r(g_{2m,j})] &\in X^s(w',\eta'),\\ \varphi^{i-n}[\varphi^r(g_{2m,j}),y] &\in X^u(w,\eta).\end{aligned} \tag{7.2.8}$$

Let $\ell \in \mathbb{N}$ be large enough so that $\ell+n+i > 0$ and $i-n-\ell < 0$. Then,

$$\varphi^{\ell+n+i}(y) \in X^s(\varphi^{\ell+n+i}[y,\varphi^r(g_{2m,j})], \lambda_X^{-(\ell+n+i)}\varepsilon_X/2)$$

and

$$\varphi^{\ell+n+i}[y,\varphi^r(g_{2m,j})] \in X^s(\varphi^\ell(w'),\lambda_X^{-\ell}\eta').$$

Therefore, $\varphi^{\ell+n+i}(y) \in X^s(\varphi^\ell(w'), \varepsilon_X)$, or equivalently

$$y \in \varphi^{-(\ell+n+i)}(X^s(\varphi^\ell(w'), \varepsilon_X)). \tag{7.2.9}$$

Similarly, we have that

$$y \in \varphi^{-(i-n-\ell)}(X^u(\varphi^{-\ell}(w), \varepsilon_X)). \tag{7.2.10}$$

The intersection of the sets in (7.2.9) and (7.2.10) is finite since the first is a compact segment of the global stable set $X^s(\varphi^{-(n+i)}(w'))$, and the second is a compact segment of the global unstable set $X^u(\varphi^{-(i-n)}(w))$. As a result, we obtain that the operator S_n has finite rank.

We now prove part (i). From (7.2.7) and Lemma 6.2.11 we can find $n_0 \in \mathbb{N}$ so that $\operatorname{rank}(\alpha_u^{-n}(b)\alpha_s^n(a)u^i) = 0$, if $n \leq -n_0$. We aim to choose a (possibly) larger n_0 so that we also have $\operatorname{rank}(S_n) = 0$, if $n \leq -n_0$. First note that since $Q \cap P = \varnothing$, there is some $\varepsilon_0 > 0$ so that $B(Q, \varepsilon_0) \cap B(P, \varepsilon_0) = \varnothing$. Also, note that $w \in X^u(Q)$, $w' \in X^s(P)$, and for every $m \in \{\gamma_{|2n+k|}, \dots, 2\gamma_{|2n+k|}\}$ and $|r| \leq m$ it holds that

$$\overline{\operatorname{diam}}(\varphi^r(\mathcal{R}^\delta_{2m})) \leq \lambda_X^{1-\gamma_{|2n+k|}} \varepsilon_X,$$

see Remark 7.1.15 and Theorem 3.3.2. It suffices to consider $n_0 \in \mathbb{N}$ large enough so that, if $n \leq -n_0$, we also have that

(i) $n + i < 0$, $i - n > 0$;

(ii) $\varphi^{-(n+i)}(w') \in B(P, \varepsilon_0/4)$, $\lambda_X^{n+i}\eta' < \varepsilon_0/4$;

(iii) $\varphi^{-(i-n)}(w) \in B(Q, \varepsilon_0/4)$, $\lambda_X^{-(i-n)}\eta < \varepsilon_0/4$;

(iv) $\lambda_X^{1-\gamma_{|2n+k|}} \varepsilon_X < \varepsilon_0/2$.

Suppose that $n \leq -n_0$ and assume to the contrary that there is $y \in X^h(P, Q)$ with $S_n(\delta_y) \neq 0$. Following the proof so far, we can find $m \in \{\gamma_{|2n+k|}, \dots, 2\gamma_{|2n+k|}\}$, $|r| \leq m$ and $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ so that $f_{2m,j}(\varphi^{-r}(y)) \neq 0$ and (7.2.8) holds. We have that

$$\begin{aligned}
&[y, \varphi^r(g_{2m,j})] \in X^s(\varphi^{-(n+i)}(w'), \lambda_X^{n+i}\eta'), \\
&[\varphi^r(g_{2m,j}), y] \in X^u(\varphi^{-(i-n)}(w), \lambda_X^{-(i-n)}\eta).
\end{aligned}$$

As a result, we get that $[y, \varphi^r(g_{2m,j})] \in B(P, \varepsilon_0/2)$ and $[\varphi^r(g_{2m,j}), y] \in B(Q, \varepsilon_0/2)$. Also, since the coefficient function $f_{2m,j} \circ \varphi^{-r}$ is non-zero exactly on the rectangle $\varphi^r(R^\delta_{2m,j})$, we have that y, $[y, \varphi^r(g_{2m,j})]$, $[\varphi^r(g_{2m,j}), y] \in \varphi^r(R^\delta_{2m,j})$ and hence their mutual distances are less than $\varepsilon_0/2$. Therefore, we obtain that $y \in B(Q, \varepsilon_0) \cap B(P, \varepsilon_0)$, which is a contradiction.

Let $\varepsilon > 0$ and we prove part (ii). Again, from (7.2.7) and Lemma 6.2.11, there is some $n_1 \in \mathbb{N}$ such that, if $n \geq n_1$, then it holds that

$$\operatorname{rank}(\alpha_u^{-n}(b)\alpha_s^n(a)u^i) < e^{2(\mathrm{h}(\varphi)+\varepsilon)n}.$$

We now prove that a similar statement holds for S_n, given that $n \in \mathbb{N}$ is sufficiently large. First, consider a (possibly) larger n_1 so that $n_1 \geq |i|$ and assume that $n \geq n_1$. If $y \in X^h(P,Q)$ and $S_n(\delta_y) \neq 0$, similarly as before, we can find $m \in \{\gamma_{|2n+k|}, \dots, 2\gamma_{|2n+k|}\}$, $|r| \leq m$ and $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ so that $f_{2m,j}(\varphi^{-r}(y)) \neq 0$ and (7.2.8) holds. However, this time we have that

$$\begin{aligned}
\varphi^{n+i}(y) &\in X^s(\varphi^{n+i}[y, \varphi^r(g_{2m,j})], \lambda_X^{-(n+i)}\varepsilon_X/2),\\
\varphi^{i-n}(y) &\in X^u(\varphi^{i-n}[\varphi^r(g_{2m,j}), y], \lambda_X^{i-n}\varepsilon_X/2),
\end{aligned}$$

and hence it holds that

$$y \in \varphi^{-(n+i)}(X^s(w', \varepsilon_X)) \cap \varphi^{n-i}(X^u(w, \varepsilon_X)).$$

Working as in Lemma 6.2.11, we can choose n_1 even larger so that, if $n \geq n_1$, it holds

$$\begin{aligned}
\operatorname{rank}(S_n) &\leq \#\varphi^{-(n+i)}(X^s(w', \varepsilon_X)) \cap \varphi^{n-i}(X^u(w, \varepsilon_X))\\
&= \#X^s(w', \varepsilon_X) \cap \varphi^{2n}(X^u(w, \varepsilon_X))\\
&< e^{2(\mathrm{h}(\varphi)+\varepsilon)n}.
\end{aligned}$$

This completes the proof of the lemma. □

The next "orthogonality" lemma holds since $\mathcal{G} = \{g_{n,k} \in R^\delta_{n,k} : n \geq 0,\, 1 \leq k \leq \#\mathcal{R}_n\}$ is an aperiodic $\mathcal{R}^\delta$-sample, see Definition 7.1.6. Also, although it was not needed so far, for the next lemma it will be convenient to highlight that the diameters in $\mathcal{R}^\delta_1$ are strictly less than ε'_X. In fact, we have that $\overline{\operatorname{diam}}(\mathcal{R}^\delta_1) < 7\varepsilon'_X/12$, see (3.3.4).

Lemma 7.2.4. *Let $a \in C_c(G^s(Q))$, $b \in C_c(G^u(P))$ and $i \in \mathbb{Z}$. Also, let $m, t \in \mathbb{N}$ and $r, s \in \mathbb{Z}$ with $|r+i| \leq m-1$, $|s| \leq t-1$ and $(t,s) \neq (m, r+i)$. Then, there is $n_2 = n_2(a,b) \in \mathbb{N}$ so that, for all $n \geq n_2$, it holds*

$$\iota^*_{t,s}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{m,r} = 0.$$

Proof. Assume that $\operatorname{supp}(a) \subset V^s(v, w, h^s, \eta, N)$ and $\operatorname{supp}(b) \subset V^u(v', w', h^u, \eta', N')$. First, we should note that if $n \geq N, N'$ then, for every $x \in \varphi^n(X^u(w, \eta))$ it holds that

$$\varphi^n \circ h^s \circ \varphi^{-n}(x) \in X^s(x, \lambda_X^{-n+N}\varepsilon_X/2),$$

and similarly, for every $x \in \varphi^{-n}(X^s(w', \eta'))$ we have that

$$\varphi^{-n} \circ h^u \circ \varphi^n(x) \in X^u(x, \lambda_X^{-n+N'} \varepsilon_X/2).$$

We define $n_2 \in \mathbb{N}$ to be slightly larger than N, N' so that $\lambda_X^{-n_2+N} \varepsilon_X/2$, $\lambda_X^{-n_2+N'} \varepsilon_X/2 \leq \varepsilon'_X/6$.

Consider now some $n \geq n_2$ and assume to the contrary that there is $y \in X^h(P, Q)$ such that

$$\iota_{t,s}^*(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{m,r}(\delta_y) \neq 0. \tag{7.2.11}$$

Since $|r+i| \leq m-1$, we have that

$$(u^i \otimes u^i)\iota_{m,r}(\delta_y) = \sum_{j=1}^{\#\mathcal{R}_m} f_{m,j}(\varphi^{-r}(y))\delta_{[\varphi^i(y), \varphi^{r+i}(g_{m,j})]} \otimes \delta_{[\varphi^{r+i}(g_{m,j}), \varphi^i(y)]}.$$

Then, from (7.2.11) there is some $j \in \{1, \ldots, \#\mathcal{R}_m\}$ so that

$$f_{m,j}(\varphi^{-r}(y))\iota_{t,s}^*(\alpha_u^{-n}(b)\delta_{[\varphi^i(y), \varphi^{r+i}(g_{m,j})]} \otimes \alpha_s^n(a)\delta_{[\varphi^{r+i}(g_{m,j}), \varphi^i(y)]}) \neq 0. \tag{7.2.12}$$

In particular, $\varphi^{-n} \circ h^u \circ \varphi^n[\varphi^i(y), \varphi^{r+i}(g_{m,j})]$ and $\varphi^n \circ h^s \circ \varphi^{-n}[\varphi^{r+i}(g_{m,j}), \varphi^i(y)]$ are well-defined, and there is a (unique) $k \in \{1, \ldots, \#\mathcal{R}_t\}$ so that

$$\begin{aligned}
\varphi^{-s-n} \circ h^u \circ \varphi^n[\varphi^i(y), \varphi^{r+i}(g_{m,j})] &\in X^u(g_{t,k}, R_{t,k}^\delta), \\
\varphi^{n-s} \circ h^s \circ \varphi^{-n}[\varphi^{r+i}(g_{m,j}), \varphi^i(y)] &\in X^s(g_{t,k}, R_{t,k}^\delta).
\end{aligned}$$

Since $|s| \leq t-1$, it follows that

$$\begin{aligned}
\varphi^{-n} \circ h^u \circ \varphi^n[\varphi^i(y), \varphi^{r+i}(g_{m,j})] &\in X^u(\varphi^s(g_{t,k}), 7\varepsilon'_X/12), \\
\varphi^n \circ h^s \circ \varphi^{-n}[\varphi^{r+i}(g_{m,j}), \varphi^i(y)] &\in X^s(\varphi^s(g_{t,k}), 7\varepsilon'_X/12).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\varphi^{-n} \circ h^u \circ \varphi^n[\varphi^i(y), \varphi^{r+i}(g_{m,j})] &\in X^u([\varphi^i(y), \varphi^{r+i}(g_{m,j})], \varepsilon'_X/6), \\
\varphi^n \circ h^s \circ \varphi^{-n}[\varphi^{r+i}(g_{m,j}), \varphi^i(y)] &\in X^s([\varphi^{r+i}(g_{m,j}), \varphi^i(y)], \varepsilon'_X/6),
\end{aligned}$$

and since $\varphi^i(y), \varphi^{r+i}(g_{m,j}) \in \varphi^{r+i}(R_{m,j}^\delta)$ and $\varphi^{r+i}(R_{m,j}^\delta)$ is a rectangle with diameter less than $7\varepsilon'_X/12$, we also have that

$$\begin{aligned}
[\varphi^i(y), \varphi^{r+i}(g_{m,j})] &\in X^u(\varphi^{r+i}(g_{m,j}), 7\varepsilon'_X/12), \\
[\varphi^{r+i}(g_{m,j}), \varphi^i(y)] &\in X^s(\varphi^{r+i}(g_{m,j}), 7\varepsilon'_X/12).
\end{aligned}$$

Hence, we have $\varphi^{r+i}(g_{m,j}) \in X^s(\varphi^s(g_{t,k}), \varepsilon_X) \cap X^u(\varphi^s(g_{t,k}), \varepsilon_X) = \{\varphi^s(g_{t,k})\}$. If $s \neq r+i$ then $\varphi^{r+i-s}(g_{m,j}) = g_{t,k}$, which is a contradiction since $\mathcal{G}$ is aperiodic. Finally, if $s = r+i$ then $g_{m,j} = g_{t,k}$, and from aperiodicity it holds that $m = t$. However, this is again a contradiction since $(t,s) \neq (m, r+i)$. This completes the proof. $\square$

For the next lemma, the fact that the metric d is self-similar plays an important role. Also, the proof is quite elaborate and we advise the reader to try and understand the proof of Lemma 6.2.12 first. Further, recall the sequence $\gamma_{|n|} = \lceil |n|/16 \rceil$, for $n \in \mathbb{Z}$, which we use to define the isometry V_n (7.2.6).

Lemma 7.2.5. *Let $a \in \mathrm{Lip}_c(G^s(Q), D_{s,d})$, $b \in \mathrm{Lip}_c(G^u(P), D_{u,d})$ and $i,k \in \mathbb{Z}$. Then, there is a constant $C_2 > 0$ and some $n_3 \in \mathbb{N}$ so that $2n_3 + k > 0$ and $\gamma_{2n_3+k} \geq |i| + 1$ and, for all $n \geq n_3$, if $m \in \{\gamma_{2n+k}, \dots, 2\gamma_{2n+k}\}$, $|r| \leq m$, it holds that*

$$\|\iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i\| \leq C_2(\lambda_X^{-n/8} + 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}).$$

Proof. Assume that $\mathrm{supp}(a) \subset V^s(v, w, h^s, \eta, N)$ and $\mathrm{supp}(b) \subset V^u(v', w', h^u, \eta', N')$. First, choose $n_3 \in \mathbb{N}$ so that $2n_3 + k > 0$ and $\gamma_{2n_3+k} \geq |i| + 1$, and since $(\gamma_n)_{n\in\mathbb{N}}$ is increasing we have that $\gamma_{2n+k} \geq |i| + 1$, for all $n \geq n_3$. Therefore, for all m, r as in the statement it holds that $|r+i| \leq 2m - 1$. Along the proof, the integer n_3 will be chosen sufficiently large for our arguments to work. Also, the constant C_2 will be described at the end of the proof. Moreover, for a large portion of the proof, the fact that we consider $m \in \mathbb{N}$ that increases with respect to $n \in \mathbb{N}$; meaning $m \geq \gamma_{2n+k}$, is not important, and what one should keep in mind is that m, r are such that $|r+i| \leq 2m-1$. When it becomes important though, it will be highlighted.

Let $n \geq n_3$ and m, r as in the statement. First note that if $\alpha_u^{-n}(b)\alpha_s^n(a)u^i \neq 0$, then $X^s(\varphi^{-n}(w')) \cap X^u(\varphi^n(v)) \neq \varnothing$ and hence $\varphi^{-n}(w')$, $\varphi^n(v)$ must lie in the same mixing component. Therefore, in this case, all points $\varphi^{-n}(w')$, $\varphi^{-n}(v')$, $\varphi^n(w)$, $\varphi^n(v)$ lie in the same mixing component. Similarly, if $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r} \neq 0$, then from (7.2.9) and (7.2.10) of Lemma 7.2.3 we see that $X^s(\varphi^{-n}(w')) \cap X^u(\varphi^n(w)) \neq \varnothing$, and hence $\varphi^{-n}(w')$, $\varphi^{-n}(v')$, $\varphi^n(w)$, $\varphi^n(v)$ are in the same mixing component. As a result, the only possibility for the operators $\iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}$ and $\alpha_u^{-n}(b)\alpha_s^n(a)u^i$ to be non-zero is when n takes values in a certain strictly increasing arithmetic-like sequence. Therefore, we can assume that (X, d, φ) is mixing.

The proof starts with a similar compactness argument as in Lemma 6.2.12. Recall that $\mathrm{source}(a)$ and $\mathrm{source}(b)$ denote the images of $\mathrm{supp}(a)$, $\mathrm{supp}(b)$ via the groupoid source map. Both $\mathrm{source}(a)$ and $\mathrm{source}(b)$ are compact subsets of $X^u(w,\eta)$ and $X^s(w',\eta')$, respectively. From compactness, we can find an $\varepsilon > 0$ such that the ε-neighbourhoods $N_\varepsilon(\mathrm{source}(a)) \subset X^u(w,\eta)$ and $N_\varepsilon(\mathrm{source}(b)) \subset X^s(w',\eta')$. Let us now choose $n_3 \in \mathbb{N}$ to be sufficiently large so that

$$\lambda^{i-n_3}\varepsilon_X,\, \lambda^{-i-n_3}\varepsilon_X,\, \lambda^{-n_3+N}\varepsilon_X < \varepsilon.$$

In particular, this also means that $n_3 \geq N, |i|$. Consider $y \in X^h(P,Q)$ and $n \geq n_3$ with $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y \neq 0$. Then

$$\varphi^{-n+i}(y) \in \mathrm{source}(a),\, \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y) \in \mathrm{source}(b),$$

and $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y$ equals

$$b(h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y), \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y))a(h^s \circ \varphi^{-n+i}(y), \varphi^{-n+i}(y))\delta_{\varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)}.$$

The goal is to find a similar description for $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y)$ which is written as

$$\sum_{j=1}^{\#\mathcal{R}_{2m}} f_{2m,j}(\varphi^{-r}(y))\alpha_u^{-n}(b)\delta_{\varphi^i[y,\varphi^r(g_{2m,j})]} \otimes \alpha_s^n(a)\delta_{\varphi^i[\varphi^r(g_{2m,j}),y]}.$$

Indeed, for every $j \in \{1,\ldots,\#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$ we have that

$$\begin{aligned}\varphi^{-n+i}[\varphi^r(g_{2m,j}),y] &\in X^u(\varphi^{-n+i}(y), \lambda_X^{-n+i}\varepsilon_X/2)\\ &\subset N_{\varepsilon/2}(\mathrm{source}(a))\\ &\subset X^u(w,\eta).\end{aligned}$$

As a result, $\alpha_s^n(a)\delta_{\varphi^i[\varphi^r(g_{2m,j}),y]}$ equals

$$a(h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}),y], \varphi^{-n+i}[\varphi^r(g_{2m,j}),y])\delta_{\varphi^n \circ h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}),y]}.$$

Moreover, for such j it holds that

$$\begin{aligned}\varphi^{n+i}[y,\varphi^r(g_{2m,j})] &\in X^s(\varphi^{n+i}(y), \lambda_X^{-(n+i)}\varepsilon_X/2)\\ &\subset X^s(\varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y), \lambda_X^{-(n+i)}\varepsilon_X/2 + \lambda_X^{-2n+N}\varepsilon_X/2)\\ &\subset N_\varepsilon(\mathrm{source}(b))\\ &\subset X^u(w',\eta').\end{aligned}$$

Therefore, $\alpha_u^{-n}(b)\delta_{\varphi^i[y,\varphi^r(g_{2m,j})]}$ equals

$$b(h^u \circ \varphi^{n+i}[y,\varphi^r(g_{2m,j})], \varphi^{n+i}[y,\varphi^r(g_{2m,j})])\delta_{\varphi^{-n} \circ h^u \circ \varphi^{n+i}[y,\varphi^r(g_{2m,j})]}.$$

Conversely, let $y \in X^h(P,Q)$ and $n \geq n_3$. If $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y) \neq 0$, then there is some $j_0 \in \{1,\dots,\#\mathcal{R}_{2m}\}$ so that

$$f_{2m,j_0}(\varphi^{-r}(y))\alpha_u^{-n}(b)\delta_{\varphi^i[y,\varphi^r(g_{2m,j_0})]} \otimes \alpha_s^n(a)\delta_{\varphi^i[\varphi^r(g_{2m,j_0}),y]} \neq 0.$$

Specifically, the expressions $\varphi^i[y,\varphi^r(g_{2m,j_0})]$, $\varphi^i[\varphi^r(g_{2m,j_0}),y]$ are well-defined and

$$\varphi^{-n+i}[\varphi^r(g_{2m,j_0}),y] \in \operatorname{source}(a),\ \varphi^{n+i}[y,\varphi^r(g_{2m,j_0})] \in \operatorname{source}(b).$$

Working as before we obtain that $\varphi^{-n+i}(y) \in N_{\varepsilon/2}(\operatorname{source}(a)) \subset X^u(w,\eta)$ and also that $\varphi^{n+i}(y) \in N_{\varepsilon/2}(\operatorname{source}(b)) \subset X^u(w',\eta')$. Therefore, the point $\varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)$ is well-defined and lies so close to $\varphi^{n+i}(y)$ (in the stable direction) so that it is actually in $N_\varepsilon(\operatorname{source}(b)) \subset X^u(w',\eta')$. As a result, $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y$ is given by

$$b(h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y), \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y))a(h^s \circ \varphi^{-n+i}(y), \varphi^{-n+i}(y))\delta_{\varphi^{-n}\circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)}.$$

Moreover, for every $j \in \{1,\dots,\#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$, by comparing with $\varphi^{-n+i}(y)$, $\varphi^{n+i}(y)$, we have that

$$\begin{aligned}\varphi^{-n+i}[\varphi^r(g_{2m,j}),y] &\in N_\varepsilon(\operatorname{source}(a)),\\ \varphi^{n+i}[y,\varphi^r(g_{2m,j})] &\in N_\varepsilon(\operatorname{source}(b)),\end{aligned}$$

and hence lie in $X^u(w,\eta)$ and $X^u(w',\eta')$, respectively.

To summarise, let $n \geq n_3$ and consider $y \in X^h(P,Q)$ so that $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y \neq 0$ or $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y) \neq 0$. Then, the expression $\varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)$ is well-defined, and the same holds for $\varphi^n \circ h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}),y]$ and $\varphi^{-n} \circ h^u \circ \varphi^{n+i}[y,\varphi^r(g_{2m,j})]$, for all $j \in \{1,\dots,\#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$. Our goal now is to evaluate the operator $\iota_{2m,r+i}^*(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}$ on δ_y.

At this point we shall consider a (possibly) larger n_3 so that $n_3 \geq N'$. Then for every y as above we have that

$$d(\varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y), \varphi^i(y)) \leq K_0\lambda_X^{-n}, \tag{7.2.13}$$

where $K_0 = \lambda_X^{N'}\varepsilon_X/2 + \lambda_X^N\varepsilon_X/2$. Moreover, from [108, Lemma 2.2], for a (possibly) larger n_3 we can guarantee that

$$\varphi^n \circ h^s \circ \varphi^{-2n} \circ h^u \circ \varphi^n(\varphi^i(y)) = \varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n}(\varphi^i(y)). \tag{7.2.14}$$

Consequently, we obtain the important fact that

$$\varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y) = [\varphi^{-n} \circ h^u \circ \varphi^{n+i}(y), \varphi^n \circ h^s \circ \varphi^{-n+i}(y)]. \tag{7.2.15}$$

In addition, for every $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$, we claim that

$$\begin{aligned} \varphi^n \circ h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}), y] &= [\varphi^{r+i}(g_{2m,j}), \varphi^n \circ h^s \circ \varphi^{-n+i}(y)], \\ \varphi^{-n} \circ h^u \circ \varphi^{n+i}[y, \varphi^r(g_{2m,j})] &= [\varphi^{-n} \circ h^u \circ \varphi^{n+i}(y), \varphi^{r+i}(g_{2m,j})]. \end{aligned} \tag{7.2.16}$$

We now prove the stable case, and the unstable case is similar. We have that both $\varphi^{-r}(y)$, $g_{2m,j}$ lie in $R^{\delta}_{2m,j}$, and since $|r+i| \leq 2m-1$ it holds that

$$\varphi^i[\varphi^r(g_{2m,j}), y] = [\varphi^{r+i}(g_{2m,j}), \varphi^i(y)].$$

For brevity denote $x = \varphi^{r+i}(g_{2m,j})$, $z = \varphi^i(y)$. For every $0 \leq \ell \leq n-N$ we have that

$$\begin{aligned} d(\varphi^{\ell+N-n}(z), \varphi^{\ell+N} \circ h^s \circ \varphi^{-n}(z)) &\leq \lambda^{-\ell} d(\varphi^{N-n}(z), \varphi^N \circ h^s \circ \varphi^{-n}(z)) \\ &= \lambda^{-\ell} d(\varphi^{N-n}(z), [\varphi^{N-n}(z), \varphi^N(v)]) \\ &\leq \lambda^{-\ell}\varepsilon_X/2, \end{aligned}$$

and also that $d(\varphi^{\ell+N-n}(z), \varphi^{\ell+N-n}[x,z]) \leq \lambda_X^{\ell+N-n}\varepsilon_X/2$. Therefore, for all $0 \leq \ell \leq n-N$ it holds that

$$d(\varphi^{\ell+N-n}[x,z], \varphi^{\ell+N} \circ h^s \circ \varphi^{-n}(z)) \leq \varepsilon_X,$$

and in addition

$$\begin{aligned} [\varphi^{\ell+N-n}[x,z], \varphi^{\ell+N} \circ h^s \circ \varphi^{-n}(z)] &= \varphi^{\ell}[\varphi^{N-n}[x,z], [\varphi^{N-n}(z), \varphi^N(v)]] \\ &= \varphi^{\ell}[\varphi^{N-n}[x,z], \varphi^N(v)]. \end{aligned}$$

For $\ell = n-N$ we obtain that

$$\begin{aligned} [x, \varphi^n \circ h^s \circ \varphi^{-n}(z)] &= [[x,z], \varphi^n \circ h^s \circ \varphi^{-n}(z)] \\ &= \varphi^{n-N}[\varphi^{N-n}[x,z], \varphi^N(v)] \\ &= \varphi^n \circ h^s \circ \varphi^{-n}[x,z], \end{aligned}$$

and this completes the proof of the claim. In fact, we can assume that n_3 is slightly larger (and still independent of y) so that,

$$\begin{aligned} [\varphi^{-n} \circ h^u \circ \varphi^{n+i}(y), \varphi^{r+i}(g_{2m,j})] &\in X^u(\varphi^{r+i}(g_{2m,j}), \varepsilon_X/2), \\ [\varphi^{r+i}(g_{2m,j}), \varphi^n \circ h^s \circ \varphi^{-n+i}(y)] &\in X^s(\varphi^{r+i}(g_{2m,j}), \varepsilon_X/2). \end{aligned} \tag{7.2.17}$$

This last condition allows us to use the adjoint $\iota^*_{2m,r+i}$ on $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y)$ which (from the proof so far) is a linear combination of the basis vectors

$$\delta_{[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^{r+i}(g_{2m,j})]} \otimes \delta_{[\varphi^{r+i}(g_{2m,j}),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)]},$$

where $j \in \{1,\dots,\#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$. For these j we have that the expression

$$\iota^*_{2m,r+i}(\delta_{[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^{r+i}(g_{2m,j})]} \otimes \delta_{[\varphi^{r+i}(g_{2m,j}),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)]})$$

is equal to

$$f_{2m,j}(\varphi^{-(r+i)}[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)])\delta_{[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)]},$$

which is equivalently written as

$$f_{2m,j}(\varphi^{-(r+i)}\circ\varphi^{-n}\circ h^u\circ\varphi^{2n}\circ h^s\circ\varphi^{-n+i}(y))\delta_{\varphi^{-n}\circ h^u\circ\varphi^{2n}\circ h^s\circ\varphi^{-n+i}(y)}.$$

Remark 7.2.6. In general, the points $\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y)$, $\varphi^n\circ h^s\circ\varphi^{-n+i}(y)$ might not be in the same rectangle with $\varphi^{r+i}(g_{2m,j})$, namely the rectangle $\varphi^{r+i}(R^\delta_{2m,j})$. In turn, this might lead to

$$[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)] \notin \varphi^{r+i}(R^\delta_{2m,j}),$$

and since $f_{2m,j}(\varphi^{-(r+i)}(x)) \neq 0$ if and only if $x \in \varphi^{r+i}(R^\delta_{2m,j})$, we will have that

$$\iota^*_{2m,r+i}(\delta_{[\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y),\varphi^{r+i}(g_{2m,j})]} \otimes \delta_{[\varphi^{r+i}(g_{2m,j}),\varphi^n\circ h^s\circ\varphi^{-n+i}(y)]}) = 0.$$

However, this is exactly what the slow-down sequence $(\gamma_n)_{n\in\mathbb{N}}$ fixes; it forces the index $2m$ to go slower to infinity than n. In this way, the refining process of the δ-fat Markov partitions $(\mathcal{R}^\delta_n)_{n\in\mathbb{N}}$ has a lag so that both $\varphi^{-n}\circ h^u\circ\varphi^{n+i}(y)$, $\varphi^n\circ h^s\circ\varphi^{-n+i}(y)$ manage to converge to $\varphi^i(y)$ fast enough so that they tend to lie in the same rectangle(s) with $\varphi^i(y)$, and hence (eventually) with $\varphi^{r+i}(g_{2m,j})$. This can be achieved by using the Lebesgue covering numbers of the δ-fat Markov partitions which can be explicitly described using the self-similarity of the metric d, see Theorem 3.3.2.

At this point let us summarise what we have proved so far. Let $n \geq n_3$ and $y \in X^h(P,Q)$ so that $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y \neq 0$ or $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y) \neq 0$. Then, the following expressions are well-defined

$$\begin{aligned}
y_1^s &= \varphi^{-n+i}(y)\\
y_2^s &= h^s \circ \varphi^{-n+i}(y)\\
y_1^u &= \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)\\
y_2^u &= h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y)\\
y^{s,u} &= \varphi^{-n} \circ h^u \circ \varphi^{2n} \circ h^s \circ \varphi^{-n+i}(y),
\end{aligned}$$

and for every $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$, the same holds for the expressions

$$\begin{aligned}
y_{1,j}^s &= \varphi^{-n+i}[\varphi^r(g_{2m,j}), y]\\
y_{2,j}^s &= h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}), y]\\
y_{1,j}^u &= \varphi^{n+i}[y, \varphi^r(g_{2m,j})]\\
y_{2,j}^u &= h^u \circ \varphi^{n+i}[y, \varphi^r(g_{2m,j})].
\end{aligned}$$

Moreover, the operator $\iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i$ evaluated on the basis vector δ_y is given by

$$\Big(\sum_j f_{2m,j}(\varphi^{-r}(y))f_{2m,j}(\varphi^{-r-i}(y^{s,u}))b(y_{2,j}^u, y_{1,j}^u)a(y_{2,j}^s, y_{1,j}^s) - b(y_2^u, y_1^u)a(y_2^s, y_1^s)\Big)\delta_{y^{s,u}},$$

where the sum is taken over the $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$. As a result, for $n \geq n_3$, we have that

$$\|\iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i\|$$

$$= \sup_y \Big|\sum_j f_{2m,j}(\varphi^{-r}(y))f_{2m,j}(\varphi^{-r-i}(y^{s,u}))b(y_{2,j}^u, y_{1,j}^u)a(y_{2,j}^s, y_{1,j}^s) - b(y_2^u, y_1^u)a(y_2^s, y_1^s)\Big|,$$

where the supremum is taken over the $y \in X^h(P,Q)$ such that $\alpha_u^{-n}(b)\alpha_s^n(a)u^i\delta_y \neq 0$ or $(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}(\delta_y) \neq 0$.

By writing $f_{2m,j}(\varphi^{-r-i}(y^{s,u}))$ as $f_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f_{2m,j}(\varphi^{-r}(y)) + f_{2m,j}(\varphi^{-r}(y))$, we first consider

$$\Big|\sum_j f_{2m,j}(\varphi^{-r}(y))\big(f_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f_{2m,j}(\varphi^{-r}(y))\big)b(y_{2,j}^u, y_{1,j}^u)a(y_{2,j}^s, y_{1,j}^s)\Big|$$

$$\leq \|b\|_\infty \|a\|_\infty \sum_j |f_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f_{2m,j}(\varphi^{-r}(y))|.$$

Following Theorem 3.3.2, the number of $j \in \{1, \dots, \#\mathcal{R}_{2m}\}$ with $f_{2m,j}(\varphi^{-r}(y)) \neq 0$ is at most $(\#\mathcal{R}_1)^2$. Also, for such j it holds that

$$\begin{aligned}|f_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f_{2m,j}(\varphi^{-r}(y))| &\leq |f^2_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f^2_{2m,j}(\varphi^{-r}(y))|^{1/2} \\ &\leq \mathrm{Lip}(f^2_{2m,j} \circ \varphi^{-r})^{1/2} d(\varphi^{-i}(y^{s,u}), y)^{1/2}.\end{aligned}$$

Again from Theorem 3.3.2, since the metric d is self-similar, there is a constant $0 < \eta \leq \varepsilon_X$ so that $\mathrm{Leb}(\mathcal{R}^{\delta}_{2m}) \geq \lambda_X^{-2m+1}\eta$. Then, using the fact that φ is λ_X-bi-Lipschitz and that $|r| \leq m \leq 2\gamma_{2n+k} < n/4 + k/8 + 2$, from Proposition 7.1.3 we obtain a constant $K_1 > 0$ that

$$\mathrm{Lip}(f^2_{2m,j} \circ \varphi^{-r}) < K_1 \lambda_X^{3n/4}.$$

Moreover, from (7.2.13) we have that $d(y^{s,u}, \varphi^i(y)) \leq K_0 \lambda_X^{-n}$ and hence

$$d(\varphi^{-i}(y^{s,u}), y) \leq K_0 \lambda_X^{|i|} \lambda_X^{-n}.$$

Consequently, it holds that

$$|\sum_j f_{2m,j}(\varphi^{-r}(y))(f_{2m,j}(\varphi^{-r-i}(y^{s,u})) - f_{2m,j}(\varphi^{-r}(y)))b(y^u_{2,j}, y^u_{1,j})a(y^s_{2,j}, y^s_{1,j})| < K_2 \lambda_X^{-n/8}, \tag{7.2.18}$$

where $K_2 = \|b\|_\infty \|a\|_\infty (\#\mathcal{R}_1)^2 (K_0 K_1 \lambda_X^{|i|})^{1/2}$.

For the other term of the triangle inequality we use the fact that

$$\sum_j f_{2m,j}(\varphi^{-r}(y))^2 = 1,$$

and consider

$$\begin{aligned}&|\sum_j f_{2m,j}(\varphi^{-r}(y))^2 (b(y^u_{2,j}, y^u_{1,j})a(y^s_{2,j}, y^s_{1,j}) - b(y^u_2, y^u_1)a(y^s_2, y^s_1))| \\ &\leq \sum_j f_{2m,j}(\varphi^{-r}(y))^2 |b(y^u_{2,j}, y^u_{1,j})a(y^s_{2,j}, y^s_{1,j}) - b(y^u_2, y^u_1)a(y^s_2, y^s_1)|.\end{aligned}$$

Since a, b are Lipschitz, for every j we have that

$$\begin{aligned}|a(y^s_{2,j}, y^s_{1,j}) - a(y^s_2, y^s_1)| &\leq \mathrm{Lip}(a) D_{s,d}((y^s_{2,j}, y^s_{1,j}), (y^s_2, y^s_1)) \\ |b(y^u_{2,j}, y^u_{1,j}) - b(y^u_2, y^u_1)| &\leq \mathrm{Lip}(b) D_{u,d}((y^u_{2,j}, y^u_{1,j}), (y^u_2, y^u_1)).\end{aligned}$$

To estimate $D_{s,d}((y^s_{2,j}, y^s_{1,j}), (y^s_2, y^s_1))$ recall that $y^s_1 = \varphi^{-n+i}(y)$, $y^s_2 = h^s \circ \varphi^{-n+i}(y)$ and also $y^s_{1,j} = \varphi^{-n+i}[\varphi^r(g_{2m,j}), y]$, $y^s_{2,j} = h^s \circ \varphi^{-n+i}[\varphi^r(g_{2m,j}), y]$. Moreover, note that n is large enough so that

$$\varphi^n(y^s_1) \in X^s(\varphi^n(y^s_2), \varepsilon'_X/2), \ \varphi^n(y^s_{1,j}) \in X^s(\varphi^n(y^s_{2,j}), \varepsilon'_X/2).$$

We aim to show that $(y^s_{2,j}, y^s_{1,j})$ lies in a sufficiently small bisection around (y^s_2, y^s_1), and this requires to be precise about the distance of y^s_1 with $y^s_{1,j}$. We have that both y, $\varphi^r(g_{2m,j})$ lie in the rectangle $\varphi^r(R^\delta_{2m,j}) \in \varphi^r(\mathcal{R}^\delta_{2m})$. From Remark 7.1.15 we see that the cover $\varphi^r(\mathcal{R}^\delta_{2m})$ refines $\mathcal{R}^\delta_{\gamma_{2n+k}}$ and hence from Theorem 3.3.2, using the fact that $\gamma_{2n+k} \geq n/8 + k/16$, it holds that

$$\operatorname{diam}(\varphi^r(R^\delta_{2m,j})) \leq K_3 \lambda_X^{-n/8},$$

where $K_3 = \lambda_X^{1-k/16}\varepsilon_X$. Assuming that n is sufficiently large so that $K_3\lambda_X^{-n/8} < \varepsilon_X$, one has that $[\varphi^r(g_{2m,j}), y] \in X^u(y, K_3\lambda_X^{-n/8})$ and hence

$$\varphi^{-n+i}[\varphi^r(g_{2m,j}), y] \in X^u(\varphi^{-n+i}(y), \lambda^i K_3 \lambda_X^{-n-n/8}).$$

Now, assuming that n is a bit larger so that $\lambda^i K_3 \lambda_X^{-n/8} < \varepsilon'_X/2$, we can consider the bisection $V^s(y^s_2, y^s_1, \lambda^i K_3 \lambda_X^{-n-n/8}, n)$ and it is straightforward to show that

$$(y^s_{2,j}, y^s_{1,j}) \in V^s(y^s_2, y^s_1, \lambda^i K_3 \lambda_X^{-n-n/8}, n).$$

Then, using Proposition 6.1.15 we can find a constant $K_4 > 0$ (independent of the points $(y^s_{2,j}, y^s_{1,j}), (y^s_2, y^s_1)$ and n) so that

$$D_{s,d}((y^s_{2,j}, y^s_{1,j}), (y^s_2, y^s_1)) \leq K_4 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}.$$

Working in exactly the same way, but assuming n is slightly larger than before, we can find a constant $K_5 > 0$ so that

$$D_{u,d}((y^u_{2,j}, y^u_{1,j}), (y^u_2, y^u_1)) \leq K_5 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}.$$

To conclude, there is a constant $K_6 > 0$ (independent of y and n) so that

$$|\sum_j f_{2m,j}(\varphi^{-r}(y))^2 (b(y^u_{2,j}, y^u_{1,j}) a(y^s_{2,j}, y^s_{1,j}) - b(y^u_2, y^u_1) a(y^s_2, y^s_1))| \leq K_6 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}. \quad (7.2.19)$$

As a result, from (7.2.18) and (7.2.19), if n_3 is sufficiently large, for every $n \geq n_3$ we have that

$$\|\iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i\| \leq C_2(\lambda_X^{-n/8} + 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}),$$

where $C_2 = \max\{K_2, K_6\}$. If either a or b is not supported on a bisection, then the same result holds for large enough C_2, n_3. This completes the proof. $\square$

The next result is an immediate application of Lemmas 7.2.4 and 7.2.5, and together with Lemmas 7.2.3 and 6.2.9 (which is about singular values), it completes the proof of (7.2.5) and hence of Theorem 7.2.1.

Lemma 7.2.7. *Let $a \in \mathrm{Lip}_c(G^s(Q), D_{s,d})$, $b \in \mathrm{Lip}_c(G^u(P), D_{u,d})$ and $i,k,l \in \mathbb{Z}$. Then, there is a constant $C_3 > 0$ and $n_4 \in \mathbb{N}$ so that, for every $n \geq n_4$, it holds that*

$$\|V^*_{2n+l}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k} - \alpha_u^{-n}(b)\alpha_s^n(a)u^i\| \leq \frac{C_3}{n}.$$

Proof. Without loss of generality we can assume that $l \leq k$ and $i \geq 0$. Let us consider $n \in \mathbb{N}$ large enough so that $2n+k$, $2n+l > 0$, $\gamma_{2n+k} \geq i+1$. Then, it holds that

$$V^*_{2n+l}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k}$$

$$= c^{-1}_{\gamma_{2n+l}} c^{-1}_{\gamma_{2n+k}} \sum_{m'=\gamma_{2n+l}}^{2\gamma_{2n+l}} \sum_{r'=-m'}^{m'} \sum_{m=\gamma_{2n+k}}^{2\gamma_{2n+k}} \sum_{r=-m}^{m} \iota^*_{2m',r'}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}.$$

From the "orthogonality" Lemma 7.2.4 we can find $n_2 \in \mathbb{N}$ so that, for every $n \geq n_2$, we have $\gamma_{2n+k} \leq 2\gamma_{2n+l}$ and the last expression is equal to

$$c^{-1}_{\gamma_{2n+l}} c^{-1}_{\gamma_{2n+k}} \sum_{m=\gamma_{2n+k}}^{2\gamma_{2n+l}} \sum_{r=-m}^{m-i} \iota^*_{2m,r+i}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)\iota_{2m,r}.$$

At this point, let us consider the numbers $\zeta_n \in (0,1]$ given by

$$\zeta_n = c^{-1}_{\gamma_{2n+l}} c^{-1}_{\gamma_{2n+k}} \sum_{m=\gamma_{2n+k}}^{2\gamma_{2n+l}} \sum_{r=-m}^{m-i} 1,$$

and with elementary computations one can see that there is a constant $K > 0$ such that $1 - \zeta_n \leq K/n$. Then, we have that

$$\|(1-\zeta_n)\alpha_u^{-n}(b)\alpha_s^n(a)u^i\| \leq \frac{K\|b\|\|a\|}{n}. \tag{7.2.20}$$

From Lemma 7.2.5, there is a constant $C_2 > 0$ and $n_3 \in \mathbb{N}$ so that, if in addition $n \geq n_3$ then

$$\|V^*_{2n+l}(\alpha_u^{-n}(b)u^i \otimes \alpha_s^n(a)u^i)V_{2n+k} - \zeta_n \alpha_u^{-n}(b)\alpha_s^n(a)u^i\| \leq C_2(\lambda_X^{-n/8} + 2^{-n/(8\lceil \log_{\lambda_X} 3 \rceil)}). \tag{7.2.21}$$

The proof follows from (7.2.20) and (7.2.21). $\square$

Conclusion

In this work we developed the notion of approximation graphs to study the fractal geometry and noncommutative geometry of Smale spaces. One of our main insights was to generalise Bowen's Markov partitions to δ-fat Markov partitions and use them to construct geometric approximation graphs encoding fine topological and metric properties of Smale spaces.

Using approximation graphs of δ-fat Markov partitions we were able to transfer, up to topological conjugacy, the Ahlfors regularity of the Parry measure on topological Markov chains down to the Bowen measure on Smale spaces. As a consequence we obtained new estimates of fractal dimensions of Smale spaces and an abundance of Smale spaces on which Bowen's measure is Ahlfors regular.

In the noncommutative setting, to each approximation graph of δ-fat Markov partitions we associated a sequence of Lipschitz partitions of unity with controlled Lipschitz constants and a sampling function satisfying certain aperiodicity conditions. With these tools, given a Smale space, we constructed the first explicit Fredholm module representatives for the KPW-extension class of the K-duality between the stable and unstable Ruelle algebras $\mathcal{R}^s$ and $\mathcal{R}^u$. Further, by taking a novel approach we constructed dynamical metrics on the Smale space groupoids and obtained dense $\star$-subalgebras $\Lambda_s \subset \mathcal{R}^s$ and $\Lambda_u \subset \mathcal{R}^u$ so that the KPW-extension is p-smooth on $\Lambda_s \otimes_{\text{alg}} \Lambda_u$, where p is related to dimensional data of the Smale space. Then, by calculating slant products in KK-theory for simple, purely infinite C^*-algebras and by developing tools from holomorphic functional calculus, we were able to transfer the smoothness of the KPW-extension on $\mathcal{R}^s$ and $\mathcal{R}^u$. In this way, we enlarged the class of uniformly finitely smooth C^*-algebras built from hyperbolic dynamical systems by adding the broad class of the stable and unstable Ruelle algebras. Finally, using the aforementioned dense $\star$-subalgebras we obtained the θ-summability of our Fredholm module representatives of the KPW-extension class.

The applications of our work are far reaching and exciting. In future projects we intend to construct several interesting spectral triples on Smale space C^*-algebras, obtain sharper estimates for the degree of irregularity of Ruelle algebras, obtain Lefschetz fixed point formulas for endomorphisms of Ruelle algebras [47] and try connecting our geometric K-duality of Ruelle algebras with the Poincaré duality of crossed products formed by certain hyperbolic groups acting on their Gromov boundary [46] (see [74] for a relevant discussion).

Bibliography

[1] R. L. Adler; Symbolic dynamics and Markov partitions, *Bull. Amer. Math. Soc.* **35** (1998), 1–56.

[2] R. L. Adler, A. G. Konheim, M. H. McAndrew; Topological entropy, *Trans. Amer. Math. Soc.* **114** (1965), 309–319.

[3] L. Ahlfors; Zur Theorie der Uberlagerungsflachen, *Acta Math.* **65** (1935), 157–194.

[4] L. Ambrosio, P. Tilli; *Topics on analysis in metric spaces*, Oxford Univ. Press, Oxford, 2004.

[5] M. Amini, G. A. Elliott, N. Golestani; The category of Bratteli diagrams, *Canad. J. Math.* **67** (2015), 990–1023.

[6] J. E. Anderson, I. F. Putnam; Topological invariants for substitution tilings and their associated C^*-algebras, *Erg. Th. Dyn. Sys.* **18** (1998), 509–537.

[7] A. Artigue; Self-similar hyperbolicity, *Erg. Th. Dyn. Sys.* **38** (2018), 2422–2446.

[8] P. Assouad; Étude d'une dimension métrique liée àl la possibilité de plongements dans $\mathbb{R}^n$, *C. R. Acad. Sci. Paris* **288** (1979), 731–734.

[9] P. Assouad; Pseudodistances, facteurs et dimension métrique, Séminaire d'Analyse Harmonique 1979–1980, *Publ. Math. Orsay* **80** (1980), 1–33.

[10] P. Assouad; Plongements Lipschitziens dans $\mathbb{R}^n$, *Bull. Soc. Math. France* **111** (1983), 429–448.

[11] L. Barreira; A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems, *Erg. Th. Dyn. Sys.* **16** (1996), 871–927.

[12] J. C. Becker, D. H. Gottlieb; *A History of Duality in Algebraic Topology*, North–Holland, 1999.

[13] G. C. Bell; Property A for groups acting on metric spaces, *Topol. Appl.* **130** (2003), 239–251.

[14] B. Blackadar; K-*Theory for Operator Algebras*, Cambridge Univ. Press, Cambridge, 1998.

[15] R. Bowen; *Equilibrium States and the Ergodic Theory of Anosov diffeomorphisms*, Springer, Berlin, 2008.

[16] R. Bowen; Markov partitions for Axiom A diffeomorphisms, *Amer. J. Math.* **92** (1970), 725–747.

[17] R. Bowen; Markov partitions and minimal sets for Axiom A diffeomorphisms, *Amer. J. Math.* **92** (1970), 907–918.

[18] R. Bowen; Periodic points and measures for Axiom A diffeomorphisms, *Trans. Amer. Math. Soc.* **154** (1971), 377–397.

[19] M. Brin, G. Stuck; *Introduction to Dynamical Systems*, Cambridge Univ. Press, Cambridge, 2002.

[20] J. Brodzki, V. Mathai, J. Rosenberg, R. J. Szabo; D-branes, RR-fields and duality on noncommutative manifolds., *Comm. Math. Phys.* **77** (2008), 643–706.

[21] N. Brodskiy, J. Dydak, J. Higes, A. Mitra; Assouad-Nagata dimension via Lipschitz extensions, *Israel J. Math.* **171** (2009), 405–423.

[22] L. G. Brown; Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pac. J. Math.* **71** (1977), 335–348.

[23] L. G. Brown, R. G. Douglas, P. A. Fillmore; Extensions of C^*-algebras and K-homology, *Ann. of Math.* **105** (1977), 265–324.

[24] M. D. Choi, E. G. Effros; The completely positive lifting problem for C^*-algebras, *Ann. of Math.* **104** (1976), 585–609.

[25] E. Christensen, C. Ivan; Sums of two dimensional spectral triples, *Math. Scand.* **100** (2007), 35–60.

[26] A. Connes; Noncommutative differential geometry. I, II, *Inst. Hautes Études Sci. Publ. Math.* **62** (1986), 257–360.

[27] A. Connes; Compact metric spaces, Fredholm modules and hyperfiniteness, *Erg. Th. Dyn. Sys.* **9** (1989), 207–230.

[28] A. Connes; *Noncommutative Geometry*, Academic Press Inc., London and San Diego, 1994.

[29] A. Connes, M. Marcolli; *A walk in the noncommutative garden, An invitation to noncommutative geometry*, World Sci. Publ., Hackensack, NJ, 2008.

[30] J. Cuntz; K-theory for certain C^*-algebras, *Ann. of Math.* **113** (1981), 181–197.

[31] J. Cuntz, W. Krieger; A Class of C^*-algebras and Topological Markov Chains, *Invent. Math.* **56** (1980), 251–268.

[32] R. Deeley, M. Goffeng, A. Yashinski; Smale space C^*-algebras have nonzero projections, *Proc. Amer. Math. Soc.* **148** (2020), 1625–1639.

[33] R. Deeley, B. R. Killough, M. F. Whittaker; Dynamical correspondences for Smale spaces, *New York J. Math.* **22** (2016), 943–988.

[34] R. Deeley, K. Strung; Nuclear dimension and classification of C^*-algebras associated to Smale spaces, *Trans. Amer. Math. Soc.* **370** (2018), 3467–3485.

[35] R. Deeley, K. Strung; Group actions on Smale space C^*-algebras, *Erg. Th. Dyn. Sys.* **40** (2019), 2368–2398.

[36] R. Deeley, A. Yashinski; The stable algebra of a Wieler Solenoid: Inductive limits and K-theory, *Erg. Th. Dyn. Sys.* (2019), 1–35.

[37] A. Deitmar, S. Echterhoff; *Principles of harmonic analysis*, Springer, New York, 2009.

[38] R. Diestel; *Graph Theory*, Springer, New York, 2005.

[39] E. Le Donne, T. Rajala; Assouad dimension, Nagata dimension, and uniformly close metric tangents, *Indiana Univ. Math. J.* **64** (2015), 21–54.

[40] R. G. Douglas; On the smoothness of elements of Ext, *Topics in modern operator theory*, Birkhäuser Verlag (1981), 63–69.

[41] R. G. Douglas, D. Voiculescu; On the smoothness of sphere extensions, *J. Operator Th.* **6** (1981), 103–111.

[42] S. Echterhoff, H. Emerson, H. J. Kim; KK-theoretic duality for proper twisted actions, *Math. Ann.* **340** (2008), 839–873.

[43] S. Echterhoff, H. Emerson, H. J. Kim; A Lefschetz fixed-point formula for certain orbifold C^*-algebras, *J. Noncomm. Geom.* **4** (2010), 125–155.

[44] G. A. Elliott; On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443** (1993), 179–219.

[45] H. Emerson; *An Example of Non-commutative Poincaré Duality Arising from Hyperbolic Dynamics*, Ph.D. Thesis, Pennsylvania State Univ., 2001.

[46] H. Emerson; Noncommutative Poincaré duality for boundary actions of hyperbolic groups, *J. Reine Angew. Math.* **564** (2003), 1–33.

[47] H. Emerson; Lefschetz numbers for C^*-algebras, *Canad. Math. Bull.* **54** (2011), 82–99.

[48] H. Emerson, B. Nica; K-homological finiteness and hyperbolic groups, *J. Reine Angew. Math.* **745** (2018), 189–229.

[49] H. Emerson; *An introduction to C^*-algebras and Noncommutative Geometry.* Book in preparation

[50] K. J. Falconer; *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, New Jersey, 2014.

[51] A. Fathi; Expansiveness, hyperbolicity and Hausdorff dimension, *Comm. Math. Phys.* **126** (1989), 249–262.

[52] J. M. Fraser; Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.* **366** (2014), 6687–6733.

[53] D. Fried; Finitely presented dynamical systems, *Erg. Th. Dyn. Sys.* **7** (1987), 489–507.

[54] A. H. Frink; Distance functions and the metrization problem, *Bull. Amer. Math. Soc.* **43** (1937), 133–142.

[55] D. M. Gerontogiannis; Ahlfors regularity and fractal dimension of Smale spaces, *Erg. Th. Dyn. Sys.* (to appear), arXiv:2004.07367 (2020).

[56] E. Getzler; The odd Chern character in cyclic homology and spectral flow, *Topol.* **32** (1993), 489–507.

[57] M. Goffeng; Equivariant extensions of $*$-algebras, *New York J. Math.* **16** (2010), 369–385.

[58] M. Goffeng, B. Mesland; Spectral triples and finite summability on Cuntz-Krieger algebras, *Doc. Math.* **20** (2015), 89–170.

[59] M. Goffeng, B. Mesland; *Spectral triples on $\mathcal{O}_N$*, Conference proceedings, MATRIX-program "Refining C^*-algebraic invariants for dynamics using KK-theory" in Creswick, Australia, 2016.

[60] I. C. Gohberg, M. G. Krein; *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc., Providence, 1969.

[61] G. Gong; Smooth extensions for finite CW complexes, *Trans. Amer. Math. Soc.* **342** (1994), 343–358.

[62] B. Gramsch; Integration und holomorphe Funktionen in lokalbeschränkten Räumen, *Math. Ann.* **162** (1966), 190–210.

[63] T. Hadfield; K-*homology of the CAR-algebra*, arXiv:math/0112236 (2001), preprint.

[64] J. Heinonen; *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

[65] R. H. Herman, I. F. Putnam, C. F. Skau; Ordered Bratteli diagrams, dimension groups and topological dynamics, *Internat. J. Math.* **3** (1992), 827–864.

[66] N. Higson; A characterization of KK-theory, *Pac. J. Math.* **126** (1987), 253–276.

[67] N. Higson, J. Roe; *Analytic* K-*homology*, Oxford Univ. Press, Oxford, 2000.

[68] J. Hjelmborg, M. Rørdam, On stability of C^*-algebras, *J. Funct. Anal.* **155** (1998), 153–170.

[69] K. K. Jensen, K. Thomsen; *Elements of* KK-*theory*, Birkhauser, Basel-Boston-Berlin, 1991.

[70] A. Julien, J. Kellendonk, J. Savinien; *On the noncommutative geometry of tilings*, in Mathematics of Aperiodic Order, Birkhäuser/Springer, Basel, 2015.

[71] A. Käenmäki, E. Rossi; Weak separation condition, Assouad dimension, and Furstenberg homogeneity, *Ann. Acad. Sci. Fenn. Math.* **41** (2016), 465–490.

[72] A. Käenmäki, M. Vilppolainen; Separation conditions on controlled Moran constructions, *Fund. Math.* **200** (2008), 69–100.

[73] J. Kaminker, I. F. Putnam; K-theoretic duality of shifts of finite type, *Comm. Math. Phys.* **187** (1997), 509–522.

[74] J. Kaminker, I. F. Putnam, M. F. Whittaker; K-theoretic duality for hyperbolic dynamical systems, *J. Reine Angew. Math.* **730** (2017), 263–299.

[75] J. Kaminker, C. L. Schochet; K-theory and Steenrod homology: applications to the Brown-Douglas-Fillmore theory of operator algebras, *Trans. Amer. Math. Soc.* **227** (1977), 63–107.

[76] J. Kaminker, C. L. Schochet; Spanier-Whitehead K-duality for C^*-algebras, *J. Topol. Anal.* **11** (2019), 21–52.

[77] G. G. Kasparov; K-functors in the extension theory of C^*-algebras, Funkcional. Anal, i Prilozen. **13** (1979), 73–74. (Russian)

[78] G. G. Kasparov; Hilbert C^*-modules: theorems of Stinespring and Voiculescu, *J. Operator Th.*, **4** (1980), 133–150.

[79] G. G. Kasparov; The operator K-functor and extensions of C^*-algebras., *Math. USSR, Izv.* **16** (1981), 513–572.

[80] G. G. Kasparov; Equivariant KK-theory and the Novikov Conjecture, *Invent. Math.* **91** (1988), 147–201.

[81] A. Katok, B. Hasselblatt; *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, Cambridge, 1995.

[82] J. L. Kelley; *General Topology*, D. Van Nostrand Company, Princeton, 1955.

[83] B. Killough; *Ring Structures on the K-Theory of C^*-Algebras Associated to Smale Spaces*, Ph.D. Thesis, Univ. of Victoria, 2009.

[84] B. Killough, I. F. Putnam; Bowen measure from heteroclinic points, *Canad. J. Math.* **64** (2012), 1341–1358.

[85] E. C. Lance; *Hilbert C^*-modules – a toolkit for operator algebraists*, Cambridge Univ. Press, Cambridge, 1995.

[86] U. Lang, T. Schlichenmaier; Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions, *Internat. Math. Res. Notic.* **58** (2005), 3625–3655.

[87] J. Luukkainen; Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.* **35** (1998), 23–76.

[88] J. Luukainnen, J. Vaisala; Elements of Lipschitz Topology, *Ann. Acad. Sci. Fenn. Math.* **3** (1977), 85–122.

[89] J. M. Mackay; Assouad dimension of self-affine carpets, *Conform. Geom. Dyn.* **15** (2011), 177–187.

[90] J. M. Mackay, J. T. Tyson; *Conformal dimension. Theory and application*, Amer. Math. Soc., Providence, RI, 2010.

[91] R. Mañé; Expansive homeomorphisms and topological dimension, *Trans. Amer. Math. Soc.* **252** (1979), 313–319.

[92] P. S. Muhly, J. N. Renault, D. P. Williams; Equivalence and isomorphism for groupoid C^*-algebras, *J. Operator Th.* **17** (1987), 3–22.

[93] J. Nagata; *Modern dimension theory*, Heldermann Verlag, Berlin, 1983.

[94] A. Naor, O. Neiman; Assouad's theorem with dimension independent of the snowflaking, *Rev. Mat. Iberoamer.* **28** (2012), 1123–1142.

[95] L. Olsen; On the Assouad dimension of graph directed Moran fractals, *Fractals* **19** (2011), 221–226.

[96] I. Palmer; *Riemannian Geometry of Compact Metric Spaces*, Ph.D. Thesis, Georgia Inst. Tech., 2010.

[97] W. Parry; Intrinsic Markov chains, *Trans. Amer. Math. Soc.* **112** (1964), 55–66.

[98] W. L. Paschke; K-Theory for actions of the circle group on C^*-algebras, *J. Operator Th.* **6** (1981), 125–133.

[99] J. Pearson, J. Bellissard; Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets, *J. Noncomm. Geom.* **3** (2009), 447–480.

[100] G. K. Pedersen; *C^*-Algebras and their Automorphism Groups*, Academic Press, London, New York, San Francisco, 1979.

[101] Y. Pesin; *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Univ. Press, Chicago, 1997.

[102] M. Pollicott, H. Weiss; The dimensions of some self–affine limit sets in the plane and hyperbolic sets, *J. Stat. Phys.* **77** (1994), 841–866.

[103] N. C. Phillips; A Classification Theorem for Nuclear Purely Infinite Simple C^*-algebras, *Doc. Math.* **5** (2000), 49–114.

[104] I. Popescu, J. Zacharias; E-theoretic duality for higher rank graph algebras, *K-Theory* **34** (2005), 265–282.

[105] M. Puschnigg; Finitely summable Fredholm modules over higher rank groups and lattices, *J. K-Theory* **8** (2011), 223–239.

[106] I. F. Putnam; *Smale Spaces and C^*-Algebras*, Lecture Notes, Univ. of Victoria, 2006.

[107] I. F. Putnam; A Homology Theory for Smale Spaces, *Mem. Amer. Math. Soc.* **232** (2014), viii+122.

[108] I. F. Putnam; C^*-algebras from Smale spaces, *Canad. J. Math.* **48** (1996), 175–195.

[109] I. F. Putnam, J. Spielberg; The Structure of C^*-Algebras Associated with Hyperbolic Dynamical Systems, *J. Func. Analysis* **163** (1999), 279–299.

[110] S. Rave; *On finitely summable* K*-homology*, Ph.D. Thesis, Univ. of Münster, 2012.

[111] J. Renault; *A groupoid approach to C^*-algebras*, Springer-Verlag, New York, 1980.

[112] J. Robinson; *Dimensions, Embeddings, and Attractors*, Cambridge Univ. Press, Cambridge, 2010.

[113] S. Rolewicz; On a certain class of linear metric spaces, *Bull. Acad. Pol. Sci.* **5** (1957), 471–473.

[114] M. Rørdam, Classification of nuclear, simple C^*-algebras, *Encyclopaedia Math. Sci.*, Springer, Berlin, 2002.

[115] D. Ruelle, D. Sullivan; Currents, flows, and diffeomorphisms, *Topol.* **14** (1975), 319–327.

[116] D. Ruelle; *Thermodynamic Formalism*, Cambridge Univ. Press, Cambridge, 2004.

[117] D. Ruelle; Noncommutative algebras for hyperbolic diffeomorphisms, *Invent. Math.* **93** (1988), 1–13.

[118] E. Ruiz, A. Sims, A. P. W. Sørensen; UCT-Kirchberg algebras have nuclear dimension one, *Adv. Math.* **279** (2015), 1–28.

[119] K. Sakai; Shadowing properties of L-hyperbolic homeomorphisms, *Topol. Appl.* **112** (2001), 229–243.

[120] K. Sakai; *Geometric Aspects of General Topology*, Springer, Berlin, 2013.

[121] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Sci. Publ., New York and London, 1993.

[122] A. H. Schoenfeld; Continuous surjections from Cantor sets to compact metric spaces, *Proc. Amer. Math. Soc.* **46** (1974), 141–142.

[123] V. Schroeder; Quasi-metric and metric spaces, *Conform. Geom. Dyn.* **10** (2006), 355–360.

[124] L. B. Schweitzer; A short proof that $\mathcal{M}_n(A)$ is local if A is local and Fréchet, *Internat. J. Math.* **3** (1992), 581–589.

[125] S. Semmes; On the nonexistence of bilipschitz parametrizations and geometric problems about A^∞-weights, *Rev. Mat. Iberoamer.* **12** (1996), 337–410.

[126] K. Simon, B. Solomyak; Hausdorff dimension for horseshoes in $\mathbb{R}^3$, *Erg. Th. Dyn. Sys.* **19** (1999), 1343–1363.

[127] A. Sims; Étale groupoids and their C^*-algebras, arXiv:1710.10897v1 (2017), preprint.

[128] S. Smale; Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[129] D. V. Voiculescu; On the existence of quasicentral approximate units relative to normed ideals. Part I, *J. Func. Analysis* **91** (1990), 1–36.

[130] P. Walters; *An Introduction to Ergodic Theory*, Springer, New York, 1982.

[131] N. E. Wegge-Olsen; *K-Theory and C^*-algebas*, Oxford Univ. Press, Oxford, New York, Tokyo, 1993.

[132] H. Whitney; The self-intersections of a smooth n-manifolds in $2n$-space, *Ann. of Math.* **45** (1944), 220–246.

[133] M. F. Whittaker; *Poincaré Duality and Spectral Triples for Hyperbolic Dynamical Systems*, Ph.D. Thesis, Univ. of Victoria, 2010.

[134] S. Wieler; *Smale spaces with totally disconnected local stable sets*, Ph.D. Thesis, Univ. of Victoria, 2012.

[135] D. P. Williams; *Crossed products of C^*-algebras*, Amer. Math. Soc., Providence, RI, 2007.

[136] R. F. Williams; Expanding attractors, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 169–203.

[137] W. Winter, J. Zacharias; The nuclear dimension of C^*-algebras, *Adv. Math.* **224** (2010), 461–498.

[138] I. Yi; K-theory of C^*-algebras from one-dimensional generalized solenoids, *J. Operator Th.* **50** (2003), 283–295.

[139] M. Zähle; Harmonic calculus on fractals, a measure geometric approach. II, *Trans. Amer. Math. Soc.* **357** (2005), 3407–3423.