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On groups with biprimary subgroups of even order

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Abstract. We investigate groups in which maximal subgroups of even order are primary or biprimary. We also research soluble groups with restriction on a number of prime devisors of some proper subgroup orders. We give applications of received results to cofactors of proper subgroups.

1. Introduction

All groups in this paper are finite.

The structure of a group depends to a large extent on the properties of its maximal subgroups. This area of the group theory has a long history. We mark out some papers of the current decade. V. S. Monakhov and V. N. Tyutyanov proved that if all maximal subgroups of a group are either simple or nilpotent, then a group is soluble and a Schmidt group [4, Theorem 1]. If every maximal subgroup of a group is either simple or supersoluble, then a group can be insoluble. The chief series of such group is of the form

$$1 \subseteq K \subseteq G, \ |G:K| \leqslant 2, \ K \simeq \mathrm{PSL}(2,p)$$

for a fit prime p, [4, Theorem 2]. V. A. Belonogov [5] enumerated all simple non-abelian groups whose every maximal subgroup is π-closed for

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some fixed set of primes π. Groups in which all maximal subgroups are Hall subgroups were also investigated. V. M. Levchuk, A. G. Liharev and V. N. Tyutyanov [6, 7] ascertained that simple groups with complemented maximal subgroups are isomorphic to $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,11)$ or $\mathrm{PSL}(5,2)$. In these groups maximal subgroups are Hall subgroups. Later V. N. Tyutyanov and T. V. Tihonenko [8] proved that the converse statement is also true, i. e. simple groups with Hall maximal subgroups are confined to three mentioned groups. V. S. Monakhov [9] described π-soluble groups with Hall maximal subgroups. In [10], it was proved that the non-abelian composition factors of insoluble groups with Hall maximal subgroups are confined up to isomorphism to $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,11)$ and $\mathrm{PSL}(5,2)$. Thus the problem of V. S. Monakhov [11, 17.92] was solved. N. V. Maslova and D. O. Revin [12] received the description of groups in which every maximal subgroup is a Hall subgroup. These results were further developed by V. A. Vedernikov [13], who investigated the structure of groups with Hall insoluble maximal subgroups.

In [14], the notion "wide subgroup" was offered, see also [15]. A subgroup H of a group G is wide if $\pi(H) = \pi(G)$. We use $|\pi(G)|$ to denote a number of different prime devisors of $|G|$. A group G is said to be quasi-k-primary if G has no wide subgroups and $k = \max_{M<\cdot G} |\pi(M)|$. Here $M <\cdot G$ denotes that M is a maximal subgroup of a group G, by $\pi(G)$ we denote the set of all prime devisors of $|G|$. A quasi-1-primary group is also called quasiprimary, quasi-2-primary group is also called quasibiprimary [16]. It is clear that if every maximal subgroup of a group is a Hall subgroup, then such group has no wide subgroups. In many simple groups there are no wide subgroups. Simple groups containing wide subgroups were enumerated in [17, 3.8]. In [18, Theorem 1], the possible non-abelian composition factors of insoluble groups with wide subgroups were described. Thus, it was satisfied the question of V. S. Monakhov [11, 11.64].

S. S. Levischenko [16] studied the structure of quasibiprimary groups. A soluble quasibiprimary group can be represented as the semidirect product $P \rtimes M$ of its normal elementary abelian Sylow subgroup P by quasiprimary maximal subgroup M [16, Theorem 3.1]. In an insoluble quasibiprimary group G, the Frattini subgroup $\Phi(G)$ is primary [16, Theorem 2.2], the quotient group $G/\Phi(G)$ is simple, and all such simple groups are enumerated [16, Theorem 2.1].

In this paper we investigate groups in which maximal subgroups of even order are primary or biprimary. In particular, we prove that if in an insoluble group G with $\Phi(G) = 1$ all maximal subgroups of even order are primary or biprimary, then every proper subgroup of G is primary or biprimary. We also research soluble groups with restriction on a number of prime devisors of some proper subgroup orders. We give an application of received results to the cofactors of proper subgroups.

2. Preliminaries

We write $H \leqslant G$ if H is a subgroup of a group G, $H < G$ if H is a proper subgroup of G, and $H \lhd G$ if H is a normal subgroup of G. If $|\pi(G)| = k$, then a group G is called k-primary; if $|\pi(G)| \leqslant k$, then G is said to be no more than k-primary. A group G is primary if $|\pi(G)| = 1$ and biprimary if $|\pi(G)| = 2$. If $p \in \pi(G)$, then we say that a group G is a pd-group.

If a number n divides a number m, then we write $n \mid m$, If a number n does not divide a number m, then we write $n \nmid m$.

Let r be a prime. By G_r we denote a Sylow r-subgroup of a group G, $G_{r'}$ denotes its r'-Hall subgroup. A group G is r-closed if $G_r \lhd G$. A group G is r-nilpotent if there is a r'-Hall subgroup $G_{r'}$ such that $G_{r'} \lhd G$. A group G is r-decomposed if it is r-closed and r-nilpotent. The largest normal soluble subgroup of a group G is called the soluble radical and denoted by $R(G)$. The Fitting subgroup and the Frattini subgroup are denoted by $F(G)$ and $\Phi(G)$, respectively.

All unexplained notations and terminology are standard. The reader is referred to [2, 3] if necessary.

Lemma 1. [20] *If all proper subgroups of an insoluble group G are soluble, G is isomorphic to one of the following groups.*

(1) $\mathrm{PSL}(3,3)$.
(2) $\mathrm{Sz}(2^p)$, *where p is an odd prime.*
(3) $\mathrm{SL}(2,2^p)$, *where p is a prime.*
(4) $\mathrm{PSL}(2,3^p)$, *where p is an odd prime.*
(5) $\mathrm{PSL}(2,p)$, *where p is a prime with $p > 5$ and $p^2+1 \equiv 0 \pmod 5$.*

We also need the following results.

Lemma 2. [1, Theorem 2.54] *The group $\mathrm{PSL}(2,p^m)$ contains only the following subgroups.*

(1) *Elementary abelian p-groups of order p, p^2, ..., p^m.*
(2) *Cyclic groups of order z, where $z \mid \frac{p^m \pm 1}{d}$, $d = (2, p^m - 1)$.*
(3) *Dihedral groups of order $2z$, where $z \mid \frac{p^m \pm 1}{d}$, $d = (2, p^m - 1)$.*

(4) *The alternating group* A_4 *when* $p > 2$ *or* $p = 2$ *and* m *is an odd number.*
(5) *The symmetric group* S_4 *when* $p^{2m} \equiv 1 \pmod{16}$.
(6) *The alternating group* A_5 *when* $p = 5$ *or* $p^{2m} \equiv 1 \pmod 5$.
(7) *The semidirect product of an elementary abelian group of order* p^k *by a cyclic group of order* t, *where* $t \mid \frac{p^k-1}{d}$ *and* $t \mid p^m-1$, $d = (2, p^m-1)$.
(8) $\mathrm{PSL}(2, p^k)$ *where* $k \mid m$.
(9) $\mathrm{PGL}(2, p^k)$ *when* p *is odd and* $2k \mid m$.

Lemma 3. [21, XI.3] *The Suzuki group* $\mathrm{Sz}(2^p)$ *contains only the following maximal subgroups.*
(1) *The Frobenius group of order* $2^{2p} \cdot (2^p - 1)$.
(2) *The dihedral group of order* $2 \cdot (2^p - 1)$.
(3) *The Frobenius group* $\langle a \rangle \leftthreetimes \langle b \rangle$, *where* $|a| = 2^p + 2^{\frac{p+1}{2}} + 1$, $|b| = 4$.
(4) *The Frobenius group* $\langle a \rangle \leftthreetimes \langle b \rangle$, *where* $|a| = 2^p - 2^{\frac{p+1}{2}} + 1$, $|b| = 4$.

Lemma 4. [16, Lemma 1.3] *If each of three consecutive positive integers is primary, then these are one of the following.*
(1) *1, 2, 3.*
(2) *2, 3, 4.*
(3) *3, 4, 5.*
(4) *7, 8, 9.*

Lemma 5. *Fix a positive integer* k *and prime* p. *Let* N *be a normal subgroup of a pd-group* G. *If every pd-subgroup of* G *is no more than* k*-primary, then every pd-subgroup of* G/N *is also no more than* k*-primary.*

Proof. Let every *pd*-subgroup of a *pd*-group G be no more than k-primary. If N contains a Sylow p-subgroup of G for some $p \in \pi(G)$, then G/N does not contain *pd*-subgroups. Let $p \mid |G/N|$ and let H/N be a *pd*-subgroup of G/N. Then H is a *pd*-subgroup of G and $|\pi(H)| \leqslant k$ by the choice of G. Consequently, $|\pi(H/N)| \leqslant |\pi(H)| \leqslant k$. □

3. Insoluble groups with biprimary subgroups of even order

Theorem 1. *Let* G *be an insoluble group such that every maximal subgroup of even order in* G *is no more than biprimary. Then* $\Phi(G)$ *is primary and* $G/\Phi(G)$ *is isomorphic to one of the following groups.*
(1) $\mathrm{PSL}(3, 3)$.
(2) $\mathrm{Sz}(2^p)$, *where* $p = 3$ *or* $p = 5$.

(3) $\mathrm{SL}(2, 2^p)$, *where* $p = 2$ *or* $p = 3$.
(4) $\mathrm{PSL}(2, 3^p)$, *where odd prime* p *is such that* $3^p - 1 = 2 \cdot q^a$ *for integer* $a \geqslant 1$ *and prime* $q > 3$, *and* $3^p + 1 = 4 \cdot r^b$ *for integer* $b \geqslant 1$ *and prime* $r > 3$, $r \neq q$.
(5) $\mathrm{PSL}(2, p)$, *where prime* p *is such that* $p > 5$, $p^2 \equiv 1 \pmod 5$ *and one of the following statements holds.*
   (5.1) $p - 1 = 2 \cdot 3^\beta$ *for integer* $\beta \geqslant 1$, *and* $p + 1 = 2^a \cdot q^b$ *for integers* $a \geqslant 2$, $b \geqslant 0$ *(if* $b = 0$, *then* $a \geqslant 3$*) and prime* $q > 3$.
   (5.2) $p - 1 = 2 \cdot q^\beta$ *for integer* $\beta \geqslant 1$ *and prime* $q > 3$, *and* $p + 1 = 2^a \cdot 3^b$ *for integers* $a \geqslant 2$ *and* $b \geqslant 1$.
   (5.3) $p - 1 = 2^\alpha$ *for integer* $\alpha > 2$, *and* $p + 1 = 2 \cdot 3^b$ *for integer* $b > 1$.

Proof. Let G be an insoluble group such that every maximal subgroup of even order in G is no more than biprimary. It is clear that $|\pi(G)| \geqslant 3$ and $2 \mid |\pi(G)|$. In addition, every proper subgroup of G is soluble. Indeed, let $H < G$. If $2 \nmid |H|$, then H is soluble as a group of odd order. If $2 \mid |H|$, then by the choice of G, H is no more than biprimary and therefore soluble.

Let G be a non-simple group.

(a) *Every proper normal subgroup of* G *is primary.*

Suppose that N is a proper normal subgroup of G such that $|\pi(N)| \geqslant 2$.

If N is of odd order, then we consider a subgroup $H = G_2 \leftthreetimes N$ in G. Assume that $G = H$. Then G is soluble, since N and G/N are soluble. But this contradicts the choice of G. Let $H < G$. Since $2 \mid |H|$, we have $|\pi(H)| \leqslant 2$ by the choice of G. But $|\pi(H)| = |\pi(G_2)| + |\pi(N)| \geqslant 3$, the contradiction.

Assume that N is of even order. Then by the choice of G, $|\pi(N)| = 2$. Since $|\pi(G)| \geqslant 3$, there is $p \in \pi(G)$ such that $p \nmid |N|$. Now, we can consider a subgroup $H = G_p \leftthreetimes N$ and prove that G is soluble similarly to that described above. This contradiction proves that every proper normal subgroup of G is primary. In particular, $\Phi(G)$ is primary.

(b) $\Phi(G)$ *is a maximal normal subgroup of* G.

Let N be a maximal normal subgroup of G containing $\Phi(G)$. By (a), N is primary. Let M be a maximal subgroup of G. If $G = MN$, then G is soluble. This contradiction implies that $N \leqslant M$ Hence $N \leqslant \Phi(G)$ and $N = \Phi(G)$. Thus, $\Phi(G)$ is a maximal normal subgroup of G and $G/\Phi(G)$ is a simple group.

In view of Lemma 5, every subgroup of even order in $G/\Phi(G)$ is no more than biprimary. Therefore we can assume without loss of generality that G is a simple group. Since every proper subgroup of G is soluble, it implies that G is isomorphic to one of five groups from Lemma 1.

(1) $G \simeq PSL(3,3)$.

It is known [3, 16] that $PSL(3,3)$ has no 3-primary subgroups.

(2) $G \simeq Sz(2^p)$, where p is an odd prime.

By Lemma 3, in G there are proper subgroups A and B such that $|A| = 2^2 \cdot a$ and $|B| = 2^2 \cdot b$, where

$$a = 2^p + 2^{\frac{p+1}{2}} + 1, \quad b = 2^p - 2^{\frac{p+1}{2}} + 1.$$

Since subgroups A and B are of even orders, it follows that they are no more than biprimary by the choice of G. Hence a and b are primary as they are odd numbers. The product of these numbers

$$a \cdot b = (2^p + 1)^2 - 2^{p+1} = 4^p + 1 = 5(4^{p-1} - 4^{p-2} + \ldots - 4 + 1)$$

is divisible by 5. Therefore a or b is a power of 5. Then $2^{2p} + 1 = 5^\alpha \cdot q^\beta$ with integer $\alpha \geqslant 1$ and prime q.

Suppose that $q = 5$. Then both a and b is divisible by 5. Consequently, $a - b = 2 \cdot 2^{\frac{p+1}{2}}$ is divisible by 5, but this is impossible. Thus, $q \neq 5$.

If $\alpha = 1$, then $2^{2p} + 1 = 5 \cdot q^\beta$. Since $q \neq 5$, we conclude that only one of a or b is equal to 5. Hence $p = 3$, since $a \geqslant 41$ and $b \geqslant 25$ when $p > 3$.

Let $\alpha > 1$. Then $k = 4^{p-1} - 4^{p-2} + \ldots - 4 + 1$ is divisible by 5. Note that

$$k = (1 - 4 + 4^2 - 4^3 + 4^4) + (-4^5 + 4^6 - 4^7 + 4^8 - 4^9) + \ldots + l.$$

It is clear that either l equals 0 or a number of summands in l is divisible by 5. On the other hand, k contains exactly p summands. Consequently, $5 \mid p$ and $p = 5$.

Now, we show that in the group $\mathrm{Sz}(2^p)$ every maximal subgroup of even order is no more than biprimary when $p = 3$ or $p = 5$. If $p = 3$, then in view of Lemma 3, maximal subgroups of $\mathrm{Sz}(2^3)$ can have the following orders.

$|M_1| = 2^6(2^3 - 1) = 2^6 \cdot 7$; $|M_2| = 2(2^3 - 1) = 2 \cdot 7$;

$|M_3| = (2^3 + 2^2 + 1) \cdot 2^2 = 2^2 \cdot 13$; $|M_4| = (2^3 - 2^2 + 1) \cdot 2^2 = 2^2 \cdot 5$.

Consequently, every maximal subgroup of $\mathrm{Sz}(2^3)$ is of even order and biprimary. Similarly, all maximal subgroups of $\mathrm{Sz}(2^5)$ is also of even order and biprimary.

(3) $G \simeq \mathrm{SL}(2, 2^p)$ with prime p.

In view of Lemma 2(3), G contains dihedral subgroups

$$D_1 = A_1 \leftthreetimes B, \quad D_2 = A_2 \leftthreetimes B,$$

where $|A_1| = 2^p - 1$, $|A_2| = 2^p + 1$, $|B| = 2$. By the choice of G, D_1 and D_2 are both no more than biprimary. Hence odd numbers $2^p - 1$ and $2^p + 1$ are prime powers. Thus, each of three consecutive numbers $2^p - 1$, 2^p and $2^p + 1$ is primary. Hence $p = 2$ or $p = 3$ by Lemma 4.

Now, we show that in the group $\mathrm{SL}(2, 2^p)$ every maximal subgroup of even order is no more than biprimary when $p = 2$ or $p = 3$. By Lemma 2, in $PSL(2, 2^3)$ there are following subgroups.

(3.1) Elementary abelian 2-groups of orders 2, 2^2, 2^3.

(3.2) Cyclic groups of orders z, where $z \mid 7$ or $z \mid 9$.

(3.3) Dihedral groups of orders $2z$, where $z \mid 7$ or $z \mid 9$.

(3.4) The semidirect product of an elementary group of order 2^k by a cyclic group of order t, where $t \mid (2^k - 1)$ and $t \mid 7$.

Elementary abelian subgroups are always primary. Since $p = 3$, it follows that $z \mid 7$ or $z \mid 9$, and cyclic subgroups (3.2) are primary, dihedral subgroups (3.3) are biprimary, subgroups (3.4) are no more than biprimary. Thus, every proper subgroup of even order in $\mathrm{PSL}(2, 2^3)$ is no more than biprimary. Similarly, we can prove that in $\mathrm{SL}(2, 2^2)$ all maximal subgroups of even order are also no more than biprimary.

(4) $G \simeq \mathrm{PSL}(2, 3^p)$ with odd prime p.

By Lemma 2(3), G contains dihedral subgroups

$$D_1 = A_1 \leftthreetimes B, \quad D_2 = A_2 \leftthreetimes B,$$

where $|A_1| = \frac{3^p-1}{2}$, $|A_2| = \frac{3^p+1}{2}$, $|B| = 2$. By the choice of G, D_1 and D_2 are both no more than biprimary. Note that

$$|D_1| = 2 \cdot \frac{3^p - 1}{2} = 3^p - 1 = 2(3^{p-1} + 3^{p-2} + \ldots + 3 + 1) = 2n.$$

A number of summands in n is odd. It follows that n is also odd and so n is primary. Hence

$$3^p - 1 = 2 \cdot n = 2 \cdot q^a$$

for prime q and integer $a \geqslant 0$. Since n is odd, we conclude that q is also odd. In addition, $3 \nmid q^a$ and $q > 3$ because $3 \nmid (3^p - 1)$.

In a similar way, using D_2, we deduce that

$$3^p + 1 = 4 \cdot r^b$$

for prime $r > 3$ and integer $b \geqslant 1$.

Note that

$$4 \cdot r^b = 3^p + 1 = (3^p - 1) + 2 = 2 \cdot q^a + 2 = 2(q^a + 1),$$

$$2 \cdot r^b - q^a = 1.$$

Consequently, $r \neq q$.

Now, we can prove that every maximal subgroup of even order in $\mathrm{PSL}(2, 3^p)$ is no more than biprimary when odd prime p satisfies the obtained relations similarly to the proof in (3).

(5) $G \simeq \mathrm{PSL}(2, p)$ with prime p such that $p > 5$ and $p^2 \equiv 1 \pmod 5$.

In view of Lemma 2(3), G contains dihedral subgroups

$$D_1 = A_1 \leftthreetimes B, \quad D_2 = A_2 \leftthreetimes B,$$

where $|A_1| = \frac{p-1}{2}$, $|A_2| = \frac{p+1}{2}$, $|B| = 2$. By the choice of G, D_1 and D_2 are both no more than biprimary. Therefore both $p-1$ and $p+1$ is also no more than biprimary. Note that $p-1$, p and $p+1$ are three consecutive integers, and $p > 5$. Hence only one of $p-1$ and $p+1$ is divisible by 3, and only one of them is divisible by 4 as they are two two consecutive even integers.

Suppose that $4 \mid (p-1)$. By Lemma 2(6), G contains a subgroup H such that $|H| = p \cdot \frac{p-1}{2}$ is an even positive integer. By the choice of G, H is no more than biprimary. Since $\frac{p-1}{2}$ is even with odd prime p, we obtain that $p - 1 = 2^\alpha$ for integer $\alpha > 2$. Further, $4 \nmid (p+1)$ and $3 \mid (p+1)$, since $4 \mid (p-1)$ and $3 \nmid (p-1)$. Hence $p + 1 = 2 \cdot 3^b$ for integer $b > 1$.

In a similar way, assuming $4 \nmid (p-1)$, we get relations (5.1) and (5.2).

Now, we can prove similarly to the proof in (3) that every maximal subgroup of even order in $\mathrm{PSL}(2, p)$ is no more than biprimary when odd prime p satisfies the obtained relations.

Theorem 1 is proved. $\square$

Corollary 1. *If in an insoluble group G every proper subgroup of even order is no more than biprimary, then the quotient group $G/\Phi(G)$ is quasibiprimary.*

Proof. The group $G/\Phi(G)$ is isomorphic to one of the groups from Theorem 1, each of which is quasibiprimary [16, Theorem 2.1]. $\square$

Corollary 2. *Let G be an insoluble group with $\Phi(G) = 1$. If every maximal subgroup of even order in G is primary or biprimary, then all proper subgroups of G are primary or biprimary.*

Proof. In view of Corollary 1, G is quasibiprimary, therefore every proper subgroup of G is no more than biprimary. □

4. Soluble groups with a limited number of divisors of the subgroup orders

In the previous paragraph, we investigate insoluble groups in which every maximal $2d$-subgroup is primary or biprimary. Now, we study soluble pd-groups all of whose maximal pd-subgroups are no more than k-primary for a fixed positive integer k. In particular, when $p = k = 2$, we get the full description of soluble groups with primary of biprimary maximal subgroups of even order.

Lemma 6. *If a group G has no wide pd-subgroups for some $p \in \pi(G)$, then every proper subgroup of G is not wide.*

Proof. Let M be a proper subgroup of G. If $p \in \pi(M)$, then $|\pi(M)| < |\pi(G)|$ by the choice of G. Suppose that $p \notin \pi(M)$. Then

$$\pi(M) \subseteq \pi(G) \setminus \{p\}, \ |\pi(M)| \leqslant |\pi(G)| - 1.$$

Thus, G has no wide maximal subgroups and so every proper subgroup of G is not wide. □

A group is called a Schmidt group if it is non-nilpotent but all its proper subgroups are nilpotent. In [19], the reader can find properties of Schmidt groups and their applications in the theory of groups and their classes. Taking into account the behaviour of Schmidt groups, it can be easily deduced

Lemma 7. *Quasiprimary groups are exhausted by the following groups.*

(1) *Cyclic groups of order pq, where p and q are different primes.*

(2) *Non-nilpotent groups of order $p^a q$ with a minimal normal subgroup of order p^a, where p and q are different primes and a is the least positive integer such that q divides $p^a - 1$.*

Let us note that further k is a fixed positive integer and p is a prime.

Theorem 2. *In a soluble pd-group G every maximal pd-subgroup is no more than k-primary if and only if either G is no more than k-primary or $G = N \rtimes M$, where N is a minimal normal and Sylow q-subgroup for some $q \in \pi(G)$, M is a quasi-$(k-1)$-primary maximal subgroup.*

Proof. Suppose that every maximal pd-subgroup of a soluble pd-group G is no more than k-primary for some $p \in \pi(G)$ and $|\pi(G)| > k$.

(1) $|\pi(G)| = k + 1$.

Since G is soluble, its maximal subgroups have primary indices and for every $r \in \pi(G)$ there is a subgroup H such that $|G : H| = r^a$, $a \in \mathbb{N}$. In particular, there is a maximal pd-subgroup M such that $|G : M| = r^a$, $r \neq p$, $a \in \mathbb{N}$. By the hypothesis, $|\pi(M)| \leqslant k$. Since $|G| = |M| \cdot |G : M|$, we get $|\pi(G)| = k + 1$.

(2) *G has no wide subgroups.*

If H is a proper pd-subgroup of G, then by the choice of G, we have

$$|\pi(H)| \leqslant k < k + 1 = |\pi(G)|,$$

that is, every pd-subgroup of G is not wide. In view of Lemma 6, G has no wide subgroups.

(3) *Every normal subgroup of G is a Hall subgroup.*

Let K be a proper normal subgroup of G, $\tau = \pi(G) \setminus \pi(K)$. Assume that K is not a Hall subgroup of G. By (2), we obtain that $|\pi(K)| < |\pi(G)|$ and $\tau \neq \varnothing$. As G is a soluble group, there is a τ-Hall subgroup M in G. Now,

$$(|M|, |K|) = 1, \ M \cap K = 1, \ KM = K \leftthreetimes M < G.$$

Since

$$\pi(K \leftthreetimes M) = \pi(K) \cup \pi(M) = \pi(K) \cup \tau = \pi(G),$$

it follows that $K \leftthreetimes M$ is a wide subgroup of G. This contradicts (2). Therefore the assumption is false and K is a Hall subgroup of G.

(4) *The completion of the proof of the necessity.*

Let N be a minimal normal subgroup of G. In view of (3), N is a Sylow q-subgroup of G for some $q \in \pi(G)$. As G is soluble, G contains a maximal subgroup M such that $|G : M| = q^a$, $a \in \mathbb{N}$ and $G = N \leftthreetimes M$. By (1), M is k-primary. Suppose that M contains a wide subgroup M_1. Then in G there is a wide subgroup $H = N \leftthreetimes M_1$, since

$$|\pi(H)| = |\pi(N)| + |\pi(M_1)| = 1 + |\pi(M)| = 1 + k = |\pi(G)|.$$

But this contradicts (2). Consequently, every proper subgroup of M is no more than $(k-1)$-primary, and so M is a quasi-$(k-1)$-primary.

Thus, we have finished the proof of the necessity.

Now, we prove the sufficiency. If a group G is no more than k-primary, then its every subgroup is no more than k-primary. Suppose that a soluble group G is of the form $G = N \leftthreetimes M$, where N is a minimal normal and Sylow q-subgroup for some $q \in \pi(G)$, M is a quasi-$(k-1)$-primary maximal subgroup. Then

$$|\pi(M)| = k,\ |\pi(G)| = |\pi(N \rtimes M)| = |\pi(N)| + |\pi(M)| = 1 + k.$$

Show that every maximal pd-subgroup H of G is no more than k-primary. If $N \nsubseteq H$, then

$$G = NH,\ N \cap H = 1,\ H \simeq M,\ |\pi(H)| = |\pi(M)| = k,$$

and H is k-primary. Let $N \subseteq H$. Then $H = N \rtimes (H \cap M)$ by Dedekind identity. Since every proper subgroup of M is no more than $(k-1)$-primary, it implies $|\pi(H \cap M)| \leqslant k - 1$. Hence $H = N \rtimes (H \cap M)$ is no more than k-primary.

Theorem 2 is proved. □

Assuming $k = 2$ in Theorem 2, we obtain

Corollary 3. *Every maximal pd-subgroup of a soluble pd-group G is no more than biprimary if and only if either G is no more than biprimary or $G = N \rtimes M$, where N is a minimal normal and Sylow q-subgroup for some $q \in \pi(G)$, M is a quasiprimary maximal subgroup.*

Note that the structure of quasiprimary groups is given in Lemma 7.

Applying Theorem 1 and Corollary 3 with $p = 2$, we get

Corollary 4. *Let every maximal subgroup of even order in a group G be primary or biprimary. If G is insoluble, then $\Phi(G)$ is primary and $G/\Phi(G) \in \Omega$. If G is soluble, then G is of odd order, or G is no more than biprimary, or $G = N \rtimes M$, where N is a minimal normal and Sylow q-subgroup for some $q \in \pi(G)$, M is a quasiprimary maximal subgroup.*

In what follows Ω denotes the set of simple groups from Theorem 1.

Corollary 5. [14, Theorem 1] *A soluble group G is quasi-k-primary if and only if $G = N \rtimes M$, where N is a minimal normal and Sylow p-subgroup for some $p \in \pi(G)$, M is a quasi-$(k-1)$-primary maximal subgroup.*

Proof. Let G be a soluble quasi-k-primary group. Then G has no wide subgroups and $|\pi(H)| \leqslant k$ for every maximal subgroup H. Hence $|\pi(G)| \geqslant k+1$. By Theorem 2, G is of the form $G = N \rtimes M$, where N is a minimal normal and Sylow p-subgroup for some $p \in \pi(G)$, M is a quasi-$(k-1)$-primary maximal subgroup.

Conversely, let a soluble group G be of the form $G = N \rtimes M$, where N is a minimal normal and Sylow p-subgroup for some $p \in \pi(G)$, M is a quasi-$(k-1)$-primary maximal subgroup. Then every proper subgroup in M is no more than $(k-1)$-primary. Since M is soluble, it implies that every maximal subgroup of M is of prime index. Hence M is k-primary and $|\pi(G)| = |\pi(N)| + |\pi(M)| = k+1$. In view of Theorem 2, every qd-subgroup H of G is no more than k-primary for some $q \in \pi(G)$, and we have

$$|\pi(H)| \leqslant k < k+1 = |\pi(G)|,$$

that is, H is not wide. By Lemma 6, G has no wide subgroups, and so G is quasi-k-primary. □

Assuming $k = 2$ in Corollary 5, we obtain the result of S. S. Lewischenko.

Corollary 6. ([16, Theorem 3.1]) *A soluble quasibiprimary group G can be represented as the semidirect product $P \rtimes M$ of its elementary abelian Sylow subgroup P and quasiprimary maximal subgroup M.*

5. On applications to cofactors

The cofactor $\mathrm{cof}_G H$ of a subgroup H in a group G is the quotient group H/H_G [22]. Here $H_G = \bigcap_{x \in G} x^{-1}Hx$ is the core of H in G, that is, the largest normal subgroup of G contained in H.

The structure of a group essentially depends on the properties of the subgroup cofactors. Y. G. Berkovich [23] studied groups in which the non-nilpotent cofactors of maximal subgroups are soluble and have nilpotent proper normal subgroups. It was proved either such groups are soluble or the quotient group by the soluble radical is the direct product of the simple groups $\mathrm{SL}(2, 2^{p_i})$, where $2^{p_i} - 1$ are pairwise different simple Mersenne primes.

J. Dixon, J. Poland and A. Remtula [24] ascertained that groups with the p-nilpotent cofactors of maximal subgroups are p-soluble when $p > 2$. If $p = 2$, then the similar statement is not true and the insoluble group $\mathrm{PGL}(2, 7)$ is a counterexample.

S. M. Evtuhova and V. S. Monakhov [25] investigated groups with the supersoluble cofactors of maximal subgroups. In such groups, the non-abelian composition factors are isomorphic to $\mathrm{PSL}(2, p)$ with prime p such that $p \equiv \pm 1 \pmod 8$. If a group with mentioned cofactor properties is soluble, then its nilpotent length is no more than 3 and the p-length is no more than 2 for all primes p. In other paper [26], these authors researched the structure of groups in which the cofactors of maximal subgroups are of squarefree orders.

I. V. Lemeshev and V. S. Monakhov [27] studied groups with π-decomposable cofactors of maximal subgroups. In particular, if such group has a nilpotent π-Hall subgroup, then the quotient group by the Fitting subgroup is π-decomposable. This implies groups with the nilpotent cofactors of maximal subgroups are metanilpotent.

L. P. Avdashkoba and S. F. Kamornikov [28] investigated a class of a soluble groups in which the cofactors of all maximal subgroups belongs to some class of groups $\mathfrak{X}$. They considered the cases when $\mathfrak{X}$ is a homomorph, a Schunck class or a formation. The proof methods of these authors are based on the construction of a local Schunck class defined by a constant group function.

In a number of papers [29]–[30], groups with restrictions on the cofactors of non-maximal subgroups were investigated.

Some authors studied the inverse problem: under what conditions on a subgroup H of a group G its cofactor belongs to the given class of groups $\mathfrak{F}$. It is known, for example, that if either H permutes with all Sylow subgroups of G [31] or H is a modular element (in the sense of Kurosh) of the lattice of all subgroups of G [32, Theorem 5.2.3], then $\mathrm{cof}_G H$ is nilpotent, and it is not necessity abelian even if G is a p-group for a prime p and H permutes with all subgroups of G [33]. In connection with these results, see also the paper of A. N. Skiba [34]).

We use the following cofactor properties.

Lemma 8. [27, Lemma 1] (1) *If $K \leqslant H \leqslant G$, then $K_G \leqslant K_H$.*

(2) *If $N \leqslant H \leqslant G$ and $N \lhd G$, then $N \leqslant H_G$ and $(H/N)_{G/N} = H_G/N$.*

(3) *If $N \lhd G$ and $H \leqslant G$, then $(H_G)N \leqslant (HN)_G$.*

(4) *If $N \leqslant H \leqslant G$ and $N \lhd G$, then $\mathrm{cof}_{G/N} H/N \simeq \mathrm{cof}_G H$.*

We also need the following results.

Lemma 9. [35, Theorem 2] *Let G be a non-p-nilpotent group. If G contains a p-decomposed maximal subgroup M, then either M_p or $M_{p'}$ is normal in G.*

Lemma 10. *A nilpotent maximal subgroup of an insoluble group with the trivial soluble radical is a Sylow 2-subgroup.*

Proof. Let G be an insoluble group, $R(G) = 1$, and M be a nilpotent maximal subgroup of G. If M is of odd order, then G is soluble [3, IV.7.4], the contradiction. Let $2 \mid |M|$. Then $M = M_2 \times M_{2'}$. Suppose that $M_2 < G_2$. Then $M_2 < N_{G_2}(M_2) \leqslant N_G(M_2)$. On the other hand, $M = N_M(M_2) \leqslant N_G(M_2) \leqslant G$. Consequently, $N_G(M_2) = G$ and $M_2 \leqslant R(G) = 1$, the contradiction. Hence M_2 is a Sylow 2-subgroup of G and M_2 is not normal in G. By Lemma 9, $M_{2'}$ is normal in G and $M_{2'} = 1$, since $R(G) = 1$. Thus, $M = M_2$ is a Sylow 2-subgroup of G. □

Lemma 11. [19, Theorem 2.4] *In a non-2-closed group there is a 2-nilpotent Schmidt 2d-subgroup.*

Lemma 12. *If G is a group, then $F(R(G)) = F(G)$.*

Proof. Since $F(R(G))$ is a characteristic subgroup in $R(G)$, then $F(R(G)) \lhd G$ and $F(R(G)) \subseteq F(G)$. On the other hand, $F(G) \subseteq R(G)$ and so $F(G) \subseteq F(R(G))$. Thus, $F(R(G)) = F(G)$. □

Theorem 3. *Let $\mathfrak{F}$ be a formation. If the cofactor of every proper pd-subgroup of a soluble pd-group G belongs to $\mathfrak{F}$ or is no more than k-primary, then $G/F(G) \in \mathfrak{F}$, or $|\pi(G/F(G))| \leqslant k$, or G is p-closed.*

Proof. Let N be a normal subgroup of a soluble *pd*-group G and H/N be a proper *pd*-subgroup of G/N. Then H is a proper *pd*-subgroup of G and, by the hypothesis, $\mathrm{cof}_G H \in \mathfrak{F}$ or $|\pi(\mathrm{cof}_G H)| \leqslant k$. In view of Lemma 8, we have $\mathrm{cof}_G H \simeq \mathrm{cof}_{G/N} H/N$, and so $\mathrm{cof}_{G/N} H/N \in \mathfrak{F}$ or

$$|\pi(\mathrm{cof}_{G/N} H/N)| = |\pi(\mathrm{cof}_G H)| \leqslant k.$$

Consequently, the hypotheses of the theorem are inherited by all quotients of G.

Suppose that $\Phi(G) = 1$. Then $G = F(G) \rtimes H$ [1, 4.23]. Since G is soluble, we have

$$H_G \leqslant C_G(F(G)) \leqslant F(G),\ H_G = 1.$$

If H is a *pd*-subgroup, then $H \in \mathfrak{F}$ or $|\pi(H)| \leqslant k$ by the hypotheses. Let $H \in \mathfrak{F}$. Then $G/F(G) \in \mathfrak{F}$. If $|\pi(H)| \leqslant k$, then

$$|\pi(G/F(G))| = |\pi(H)| \leqslant k.$$

If $p \nmid |H|$, then $F(G)$ contains a Sylow p-subgroup of G and G is p-closed. Thus, the theorem is true when $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. We apply induction on the order of G. Since G is soluble, we have $F(G/\Phi(G)) = F(G)/\Phi(G)$. Consequently,

$$(G/\Phi(G))/F(G/\Phi(G)) = (G/\Phi(G))/(F(G)/\Phi(G)) \simeq G/F(G). \quad (1)$$

By induction $G/\Phi(G)$ is p-closed, or $(G/\Phi(G))/F(G/\Phi(G)) \in \mathfrak{F}$, or

$$|\pi((G/\Phi(G))/(F(G/\Phi(G)))| \leqslant k.$$

If $G/\Phi(G)$ is p-closed, then G is also p-closed. In two other cases, (1) implies that either $G/F(G) \in \mathfrak{F}$ or $|\pi(G/F(G)| \leqslant k$.

Theorem 3 is proved. □

The following corollaries can be easily obtained.

Corollary 7. *Let $\mathfrak{F}$ be a formation. If in a soluble non-p-closed group G the cofactor of every proper pd-subgroup belongs to $\mathfrak{F}$ or is no more than k-primary, then $G/F(G) \in \mathfrak{F}$ or $|\pi(G/F(G))| \leqslant k$.*

Corollary 8. *Let $\mathfrak{F}$ be a formation. If in a soluble group G the cofactor of every proper subgroup belongs to $\mathfrak{F}$ or is no more than k-primary, then $G/F(G) \in \mathfrak{F}$ or $|\pi(G/F(G))| \leqslant k$.*

Corollary 9. [24, Proposition 7] *Let $\mathfrak{F}$ be a formation. If in a soluble group G the cofactor of every proper subgroup belongs to $\mathfrak{F}$, then $G/F(G) \in \mathfrak{F}$.*

Assuming $\mathfrak{F} = \mathfrak{N}$ in Theorem 3, we obtain

Corollary 10. *If every proper subgroup of a soluble group G either is nilpotent or has the no more than k-primary cofactor, then either G is metanilpotent or $|\pi(G/F(G))| \leqslant k$.*

Corollary 11. *If the cofactor of every proper subgroup of a soluble group G is nilpotent, then G is metanilpotent.*

Corollary 12. *If the cofactor of every proper subgroup of a soluble group G is no more than k-primary, then $|\pi(G/F(G))| \leqslant k$.*

Theorem 4. *If in a group G the cofactors of non-nilpotent subgroups of even order are primary or biprimary, then $R(G)$ is 2-closed, or $R(G)$ is metanilpotent, or $|\pi(R(G)/F(G))| \leqslant 2$, and $G/R(G)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H/R(G))$, where $H/R(G)$ is a normal subgroup of $G/R(G)$ and $H/R(G) \in \Omega$.*

Proof. If G is a soluble group, then applying Corollary 7 in assuming $p = 2$ and $\mathfrak{F} = \mathfrak{N}$, we have $G = R(G)$ is 2-closed, in particular, G is of odd order, or G is metanilpotent, or $|\pi(R(G)/F(G))| \leqslant 2$, and the theorem is true. Further, we can assume that G is insoluble.

Suppose that G is a simple group and M is a maximal subgroup of even order in G. If M is nilpotent, then by Lemma 10, M is primary. If M is not nilpotent, then by the hypothesis, M is primary or biprimary. Thus, in G every maximal subgroup of even order is no more than biprimary. In view of Theorem 1, $G \in \Omega$.

Let G be a non-simple group and $R(G) = 1$. Choose a minimal normal subgroup N in G. It is clear that N is a proper subgroup of G and can be represented as the direct product of simple non-abelian isomorphic subgroups

$$N = N_1 \times N_2 \times \ldots \times N_t, \quad t \geqslant 1.$$

Since N_1 is a simple non-abelian subgroup, then $|\pi(N_1)| > 2$ and N_1 is a non-nilpotent subgroup of even order. By the hypothesis, $|\pi(\mathrm{cof}_G N_1)| \leqslant 2$. Hence $N = N_1 = (N_1)_G$ is a simple non-abelian subgroup of G. Let H be a maximal subgroup of even order in N. If H is nilpotent, then by Lemma 10, H is primary. Suppose that H is non-nilpotent and $|\pi(H)| > 2$. Then $|\pi(\mathrm{cof}_G H)| \leqslant 2$, $H_G \neq 1$ and $H_G \subseteq N$. But N is a simple non-abelian subgroup of G, the contradiction. Thus, in N all maximal subgroup of even order are primary or biprimary. By Theorem 1, $N \in \Omega$.

Assume that $C_G(N) \neq 1$. Then $C_G(N)N = C_G(N) \times N$ and $C_G(N)$ is insoluble. Choose in $C_G(N)$ a minimal normal subgroup K of G. As we prove above, K is a simple non-abelian group. Consider the subgroup $KN = K \times N$. By Lemma 11, K contains a 2-nipotent Schmidt $2d$-subgroup S. Since $|\pi(S)| = 2$ and $|\pi(N)| > 2$, there is a prime $q \in \pi(N) \setminus \pi(S)$. Let N_q be a Sylow q-subgroup of N. Then in $K \times N$ there is a soluble 3-primary subgroup $A = S \times N_q$ of even order. By the hypothesis, $|\pi(\mathrm{cof}_G A)| \leqslant 2$. Hence

$$1 \neq (A)_G \subseteq R(G) = 1,$$

the contradiction. Consequently, the assumption is false, $C_G(N) = 1$ and G is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(N)$.

Let now $R(G) \neq 1$. Since $R(G/R(G)) = 1$, then as we prove above, $G/R(G)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H/R(G))$, where $H/R(G)$ is a normal subgroup of $G/R(G)$ and $H/R(G) \in \Omega$. In addition, by Theorem 3, $R(G)$ is 2-closed, or $R(G)$ is metanilpotent, or $|\pi(R(G)/F(R(G)))| \leqslant 2$. By Lemma 12, $|\pi(R(G)/F(G))| = |\pi(R(G)/F(R(G)))| \leqslant 2$.

Theorem 4 is proved. □

Corollary 13. *If all proper subgroups of an insoluble group G have primary or biprimary cofactors, then ~~either $R(G)$ is metanilpotent or~~ $|\pi(R(G)/F(G))| \leqslant 2$ and $G/R(G)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H/R(G))$*, where $H/R(G)$ is a subgroup of $G/R(G)$ and $H/R(G) \in \Omega$.*

Proof. Suppose that all proper subgroups of an insoluble group G have primary or biprimary cofactors. Then by Theorem 4, $G/R(G)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H/R(G))$, where $H/R(G)$ is a subgroup of $G/R(G)$ and $H/R(G) \in \Omega$. In view of Corollary 12 and Lemma 12, we have $|\pi(R(G)/F(\cancel{R(}G\cancel{)}))| \leqslant 2$. □

Corollary 13 replenishes the list of groups from Theorem 2 [30].

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