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Analytical properties of the Lambert W function

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ANALYTICAL PROPERTIES OF THE LAMBERT W FUNCTION

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Abstract

This research studies analytical properties of one of the special functions, the Lambert W function. W function was re-discovered and included into the library of the computer-algebra system MAPLE in 1980's. Interest to the function nowadays is due to the fact that it has many applications in a wide variety of fields of science and engineering.

The project can be broken into four parts. In the first part we scrutinize a convergence of some previously known asymptotic series for the Lambert W function using an experimental approach followed by analytic investigation. Particularly, we have established the domain of convergence in real and complex cases, given a comparative analysis of the series properties and found asymptotic estimates for the expansion coefficients. The main analytical tools used herein are Implicit Function Theorem, Lagrange Inversion Theorem and Darboux's Theorem.

In the second part we consider an opportunity to improve convergence properties of the series under study in terms of the domain of convergence and rate of convergence. For this purpose we have studied a new invariant transformation defined by parameter p, which retains the basic series structure. An effect of parameter p on a size of the domain of convergence and rate of convergence of the series has been studied theoretically and numerically using MAPLE. We have found that an increase in parameter p results in an extension of the domain of convergence while the rate of convergence can be either raised or lowered.

We also considered an expansion of $W(x)$ in powers of $\ln x$. For this series we found three new forms for a representation of the expansion coefficients in terms of different special numbers and accordingly have obtained different ways

to compute the expansion coefficients. As an extra consequence we have obtained some combinatorial relations including the Carlitz-Riordan identities.

In the third part we study the properties of the polynomials appearing in the expressions for the higher derivatives of the Lambert W function. It is shown that the polynomial coefficients form a positive sequence that is log-concave and unimodal, which implies that the positive real branch of the Lambert W function is Bernstein and its derivative is a Stieltjes function.

In the fourth part we show that many functions containing W are Stieltjes functions. In terms of the result obtained in the third part, we, in fact, obtain one more way to establish that the derivative of W function is a Stieltjes function. We have extended the properties of the set of Stieltjes functions and also proved a generalization of a conjecture of Jackson, Procacci & Sokal. In addition, we have considered a relation of W to the class of completely monotonic functions and shown that W is a complete Bernstein function.

We give explicit Stieltjes representations of functions of W. We also present integral representations of W which are associated with the properties of its being a Bernstein and Pick function. Representations based on Poisson and Burniston-Siewert integrals are given as well. The results are obtained relying on the fact that the all of the above mentioned classes are characterized by their own integral forms and using Cauchy Integral Formula, Stieltjes-Perron Inversion Formula and properties of W itself.

Keywords: Lambert W function; asymptotic series; domain of convergence; special numbers; unimodal sequences; log-concave sequences; integral representations; Stieltjes functions; completely monotonic functions; Bernstein functions

Statement of Co-Authorship

Chapters 2 - 5 of this thesis consist of the following papers:

Chapter 2: G.A. Kalugin and D.J. Jeffrey, Convergence of asymptotic series for the Lambert W function, manuscript of the paper, 35 pp. (2011).

Chapter 3: G.A. Kalugin and D.J. Jeffrey, Series transformations to improve and extend convergence, CASC 2010, LNCS 6244, Eds: V.P. Gerdt et al., Springer-Verlag, 134–147 (2010).

Chapter 4: G.A. Kalugin and D.J. Jeffrey, Unimodal sequences show that Lambert W is Bernstein, C.R. Math. Rep. Acad. Sci. Canada, 33, 50–56 (2011).

Chapter 5: G.A. Kalugin, D.J. Jeffrey, R.M. Corless, and P.M. Borwein, Stieltjes and other integral representations for functions of Lambert W, 13 pp., Integral Transforms And Special Functions (2011) *accepted.*, arXiv: 1103.5640.

The original draft for each of the above articles was prepared by the author. Subsequent revisions were performed by the author and Dr. David J. Jeffrey. Development of software, analytical and numerical work using the computer algebra system MAPLE was performed by the author under supervision of Dr. David J. Jeffrey.

To my wife Olga and daughter Irina

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List of Symbols [1]

$\begin{bmatrix} n \\ m \end{bmatrix}$	-	unsigned Stirling numbers of the first kind (Stirling cycle numbers).
$\begin{Bmatrix} n \\ m \end{Bmatrix}$	-	Stirling numbers of the second kind (Stirling subset numbers).
$d(n,m)$	-	unsigned associated Stirling numbers of the first kind.
$\begin{Bmatrix} n \\ m \end{Bmatrix}_{\geq 2}$	-	2-associated Stirling numbers of the second kind (2-associated Stirling subset numbers).
$\begin{Bmatrix} n+r \\ m+r \end{Bmatrix}_r$	-	Shifted r-Stirling numbers of the second kind (non-central Stirling numbers of the second kind).
$\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle$	-	Second-order Eulerian numbers.
B_n	-	Bernoulli numbers.
$B_n^{(r)}(\lambda)$	-	Bernoulli polynomials of higher order (generalized Bernoulli polynomials).
ϖ_n	-	Bell numbers.

[1] See also Appendix B for more details.

Chapter 1

Introduction

"All truths are easy to understand once they are discovered; the point is to discover them" – Galileo Galilei

1.1 Historical remarks

The Lambert function $W(z)$ is defined as the root of the transcendental equation

$$W(z)\exp(W(z)) = z\ . \tag{1.1}$$

According to (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996), (The poster 'The Lambert W Function') the mathematical history of W function goes back to the 18th century and is associated with the names of such two great mathematicians as Johann Lambert (1728-1777) and Leonhard Euler (1707-1783). In 1758 Lambert published a series solution of the trinomial equation $x = q + x^m$ (Lambert, 1758). In 1779, stimulated by Lambert's work, Euler transformed the trinomial equation into symmetric form

$$\frac{x^\alpha - x^\beta}{\alpha - \beta} = vx^{\alpha+\beta}\ . \tag{1.2}$$

In a special case $\beta \to \alpha$, the left hand side tends to $x^\alpha \ln x$ and equation (1.2) becomes (Euler, 1779)

$$\ln x = v x^\alpha \tag{1.3}$$

or

$$y = v e^{\alpha y} \ , \tag{1.4}$$

where $y = \ln x$. Euler noticed that it would be enough to solve the equation (1.3) (or (1.4)) for $\alpha = 1$ because then it can be solved for any $\alpha \neq 0$. Euler found a solution of (1.4) for $\alpha = 1$ as a series

$$y = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} v^n \tag{1.5}$$

with the convergence radius of $1/e$.

Both Lambert and Euler left the found solution unnamed. In the modern terminology, the series (1.5) defines the *(Cayley) tree function* $T(v)$ (Flajolet & Sedgewick, 2009, p. 127-128) that thus satisfies the functional equation

$$T(z) \exp(-T(z)) = z \ .$$

This discloses a relation between the tree function and the Lambert W function (cf.(1.1))

$$W(z) = -T(-z) \ . \tag{1.6}$$

The Lambert W function was christened two hundred years later after Lambert's and Euler's works appeared. Specifically, in 1980's the function was included into the library of the computer algebra system MAPLE and as of MAPLE V RELEASE 4 it is named as **LambertW**. The letter W to designate the function was chosen more or less accidentally (Corless, Jeffrey, & Knuth, 1997) but it certainly has some significance because of a significant contribution to the study of the properties of W by E.M. Wright (Wright, 1949, 1955, 1959). It is worth noting that MAPLE provides evaluation of W with an arbitrary precision, which is implemented on basis of the asymptotic expansions studied by de Bruijn (1961) and Comtet (1970). The MAPLE implementation of W together with the publication of the fundamental paper (Corless et al., 1996) 'On the Lambert W function' by R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth opened doors for a wide use of the function in absolutely different areas. Interest to the W function is due to the fact that it has rich beautiful and useful properties, has many interesting applications and plays a significant role in various research fields, see Section 1.4.

1.2 Definition and properties of W

The Lambert W function is the multivalued inverse of the mapping $W \mapsto We^W$. The branches, denoted by W_k ($k \in \mathbb{Z}$), are defined through the equations (Corless et al., 1996)

$$\forall z \in \mathbb{C}, \qquad W_k(z)\exp(W_k(z)) = z , \tag{1.7}$$

$$W_k(z) \sim \ln_k z \text{ as } \Re z \to \infty , \tag{1.8}$$

where $\ln_k z = \ln z + 2\pi i k$, and $\ln z$ is the principal branch of natural logarithm (Jeffrey, Hare, & Corless, 1996). We will often consider the principal branch $k = 0$, therefore we shall usually abbreviate W_0 as W herein.

For convenience, we recall from (Corless et al., 1996) some properties of the principal branch that are used below. The function is continuous from above on its branch cut $\mathbb{B} \subset \mathbb{R}$, defined to be the interval $\mathbb{B} = (-\infty, -1/e]$. On the cut plane $\mathbb{C}\backslash\mathbb{B}$, the function is holomorphic. Its real values obey $-1 \leq W(x) < 0$ for $x \in [-1/e, 0)$, $W(0) = 0$ and $W(x) > 0$ for $x > 0$. The imaginary part of $W(t)$

has the following range of values for real t

$$\Im W(t) \in (0,\pi) \text{ for } t \in (-\infty,-1/e) \text{ and } \Im W(t) = 0 \text{ otherwise.} \tag{1.9}$$

$\Im W(t) \to \pi$ as $t \to -\infty$. Also, $\Im W(t)$ is continuously differentiable for $t \neq -1/e$. $\Im W(z)$ and $\Im z$ have the same sign in the cut plane $\mathbb{C}\backslash\mathbb{R}$, or equivalently

$$\Im W(z) \Im z > 0\ . \tag{1.10}$$

W has near conjugate symmetry, meaning $W(\overline{z}) = \overline{W(z)}$, except on the branch cut $\mathbb{B}$. We also note

$$W(z) = \ln z - \ln W(z) \tag{1.11}$$

in the cut plane $\mathbb{C}\backslash(-\infty,0]$. The Taylor series near $z=0$ is

$$W(z) = \sum_{n=1}^{\infty}(-n)^{n-1}\frac{z^n}{n!} \tag{1.12}$$

with the radius of convergence $1/e$ (see Appendix A.1), while the asymptotic behaviour of $W(z)$ near its branch point is given by

$$W(z) \sim -1 + \sqrt{2(ez+1)} \quad z \to -1/e\ . \tag{1.13}$$

It follows from (1.12) and (1.8) that

$$W(z)/z \to 1 \quad \text{as} \quad z \to 0\ . \tag{1.14}$$

$$W(z)/z \to 0 \quad \text{as} \quad z \to \infty\ . \tag{1.15}$$

If $z = t + is$ and $W(z) = u + iv$, then

$$e^u(u\cos v - v\sin v) = t, \quad e^u(u\sin v + v\cos v) = s\ . \tag{1.16}$$

For the case of real z, i.e. $s = 0$, the functions $u = u(t)$ and $v = v(t)$ are defined by

$$u = -v \cot v, \tag{1.17}$$

$$t = t(v) = -v \csc(v) e^{-v \cot v} . \tag{1.18}$$

For the case of pure imaginary z, i.e. $t = 0$, the functions $u = u(s)$ and $v = v(s)$ obey

$$u = v \tan v, \tag{1.19}$$

$$s = s(v) = v \sec(v) e^{v \tan v} . \tag{1.20}$$

The derivative of $W(z)$ is given by

$$W'(z) = \frac{W(z)}{z(1 + W(z))} . \tag{1.21}$$

For further we also need the derivative of function $v(t)$, defined in (1.18); it can be conveniently found by taking the imaginary part of (1.21) and using (1.17)

$$v'(t) = \frac{v}{t[v^2 + (1+u)^2]} = \frac{v}{t[v^2 + (1 - v \cot v)^2]} . \tag{1.22}$$

With (1.14) and (1.15), it follows from (1.21)

$$W'(z) \to 1 \quad \text{as} \quad z \to 0 , \tag{1.23}$$

$$W'(z) \to 0 \quad \text{as} \quad z \to \infty . \tag{1.24}$$

Near conjugate symmetry implies

$$dW(\bar{z})/d\bar{z} = \overline{dW(z)/dz} \tag{1.25}$$

For the case of real z, i.e. $s = 0$, the functions $u = u(t)$ and $v = v(t)$ are defined by

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Near conjugate symmetry implies

$$dW(\bar{z})/d\bar{z} = \overline{dW(z)/dz} \tag{1.25}$$

for $z \notin (-\infty, 0]$.

In addition, we prove the following lemma.

Lemma 1.2.1. *Function $\Im W(-t)$ is nonnegative and bounded on the real line and continuously differentiable for $t \neq 1/e$. Specifically, it is zero for $t \in (-\infty, 1/e]$ and increases from 0 to π while t changes from $1/e$ to ∞. Correspondingly, the derivative $d\Im W(-t)/dt$ is zero for $t < 1/e$ and positive for $t > 1/e$. In addition, $d\Im W(-t)/dt = o(1/t)$ as $t \to \infty$.*

Proof. Due to the above properties of function $\Im W(t)$ (see (1.9)), the function $\Im W(-t)$ is nonnegative and bounded for real t and $\Im W(-t) \to \pi$ as $t \to \infty$. The function is also continuously differentiable everywhere except $t = 1/e$. We set $v(t) = \Im W(t)$, then by (1.22)

$$v'(t) = \frac{A(v(t))}{t}, \quad A(v) = \frac{v}{v^2 + (1 - v\cot v)^2}.$$

Hence, the derivative $d\Im W(-t)/dt = A(v(-t))/t$, which implies that it is zero for $t < 1/e$ and positive for $t > 1/e$ as $v(t) = 0$ for $t > -1/e$ and $v(t) > 0$ for $t < -1/e$. It remains to assertain the estimation of the derivative $d\Im W(-t)/dt$ at large t but it immediately follows from two facts that $v(-t) \to \pi$ as $t \to \infty$ and that $A(v) \to 0$ as $v \to \pi$. □

Finally we briefly give the properties of the Lambert W function in the complex plane; a detailed discussion can be found in (Corless et al., 1996). As mentioned previously, the principal branch $W_0(z)$ has a branch point at $z = -1/e$. At this point the branches $W_0(z)$, $W_{-1}(z)$ and $W_1(z)$ have the common value -1 and therefore all of them have the above mentioned branch cut $\mathbb{B}$. In addition, $W_{-1}(z)$ and $W_1(z)$ have one more branch cut along the negative real axis $\mathbb{S} = \{z : -\infty < z \leq 0\}$ because of the extra branch point at $z = 0$. The branch cut $\mathbb{S}$, as $\mathbb{B}$, is closed on the top for the counter-clockwise continuity (Corless et al., 1996; Jeffrey et al., 1996). All other branches of $W(z)$ have only the branch cut $\mathbb{S}$.

Figure 1.1 shows the complex ranges of the branches of W (Corless et al., 1996). The principal branch W_0 is separated from the branches W_1 and W_{-1} by the curve $\{-\eta \cot \eta + \eta i : -\pi < \eta < \pi\}$ where $\eta = \Im W$ (cf. (1.17)). The boundary between W_1 and W_{-1} is just $(-\infty, -1]$. The curves separating the rest of the branches are the inverse images of the negative real axis under the map $\omega \mapsto \omega e^{\omega}$ and described by $\{-\eta \cot \eta + i\eta : 2k\pi < \pm\eta < (2k+1)\pi\}$ for natural k. In accordance with the counter-clockwise continuous convention the points forming a boundary between two branches belong to the branch below them (Jeffrey et al., 1996). A similar partition of the complex plane by the branches of the tree function $T(z) = -W(-z)$ (cf. (1.6)) was considered in (Lauwerier, 1963). The Riemann surface of W is given in Figure 1.2.

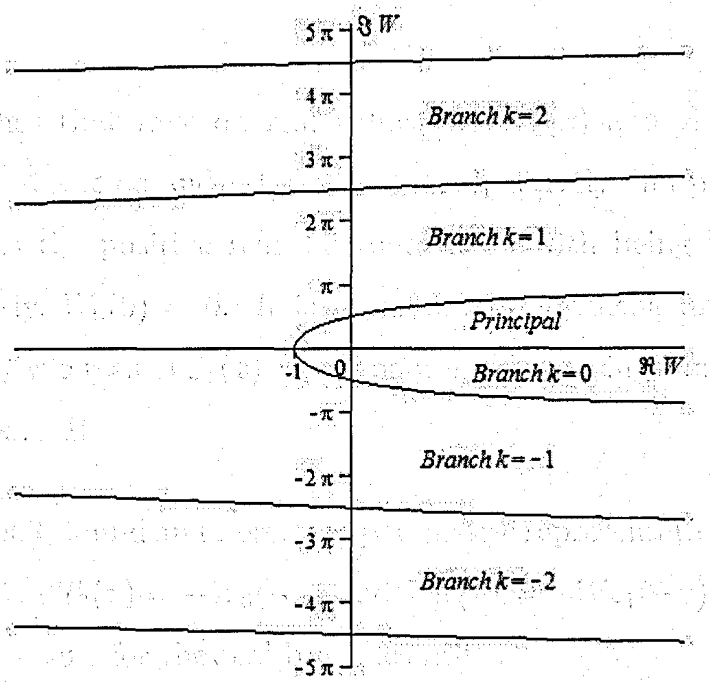

Figure 1.1: Ranges of the branches of W.

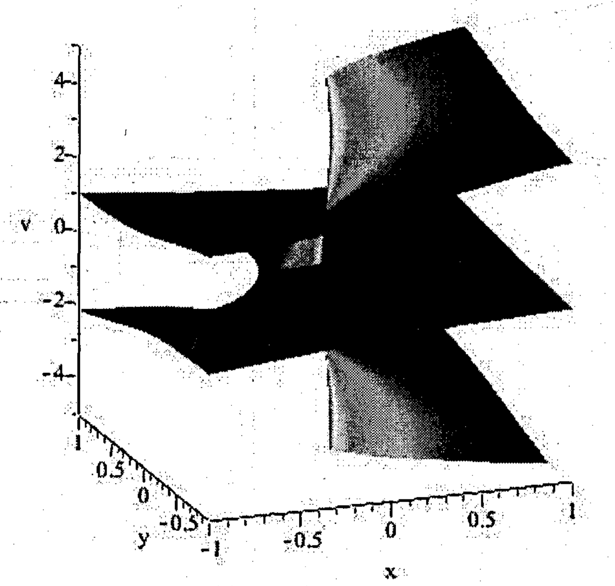

Figure 1.2: The Riemann surface of $W(z)$.

1.3 Real branches of W

The only branches that take on real values are $W_0(x)$ and $W_{-1}(x)$; they are defined for $-1/e \leq x < \infty$ and $-1/e \leq x < 0$ respectively and plotted in Figure 1.3. $W_0(x)$ maps the positive real axis onto itself with being bounded at the origin, particularly, $W_0(0) = 0$. It is a monotone increasing function with the range in $[-1, \infty)$ whereas $W_{-1}(x)$ is a monotone decreasing function and takes on values in $(-\infty, -1]$.

Lauwerier (1963) found an interesting parametric representation of the branches of the tree function $T_0(x) = -W_0(-x)$ and $T_{-1}(x) = -W_{-1}(-x)$ (cf. (1.6)). We present a similar result for the real branches of W.

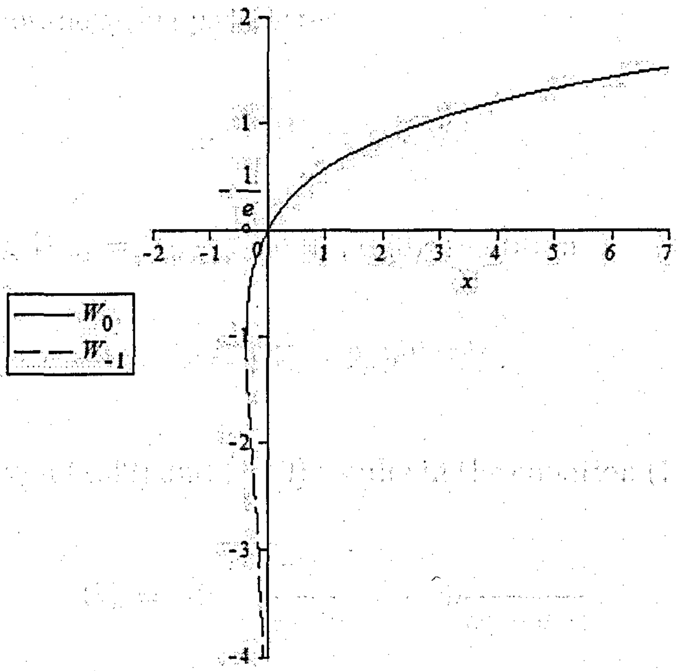

Figure 1.3: Real branches of the W function.

Theorem 1.3.1. *The branches $W_0(x)$ and $W_{-1}(x)$ admit the following parametric representation for $-1/e \leq x < 0$ (Corless et al., 1997)*

$$x = -\frac{p}{\sinh p} e^{-p \coth p} , \tag{1.26}$$

$$W_0 = -\frac{p}{\sinh p} e^{-p} , \tag{1.27}$$

$$W_{-1} = -\frac{p}{\sinh p} e^{p} , \tag{1.28}$$

where $p \geq 0$.

Proof. By definition (1.7)

$$x = W_0(x) \exp(W_0(x)) = W_0 e^{W_0} \tag{1.29}$$

and at the same time

$$x = W_{-1}(x) \exp(W_{-1}(x)) = W_{-1} e^{W_{-1}} . \tag{1.30}$$

We introduce a non-negative parameter

$$p = (W_0 - W_{-1})/2 \, . \tag{1.31}$$

Then substituting $W_{-1} = W_0 - 2p$ into (1.30) we obtain

$$x = (W_0 - 2p)e^{W_0 - 2p} \, . \tag{1.32}$$

Comparison between (1.32) and (1.29) results in the equation $(W_0 - 2p)e^{-2p} = W_0$ with a solution

$$W_0 = -2p\frac{e^{-2p}}{1 - e^{-2p}} = -2p\frac{e^{-p}}{e^p - e^{-p}} \, , \tag{1.33}$$

which is equivalent to (1.27).
Similarly, substituting $W_0 = 2p + W_{-1}$ into (1.29) we obtain

$$x = (2p + W_{-1})e^{2p + W_{-1}} \, . \tag{1.34}$$

Comparison between (1.34) and (1.30) results in $(2p + W_{-1})e^{2p} = W_{-1}$; this gives

$$W_{-1} = -2p\frac{e^{2p}}{e^{2p} - 1} = -2p\frac{e^{p}}{e^p - e^{-p}} \, , \tag{1.35}$$

which is equivalent to (1.28).
Finally, formula (1.26) can be obtained by substituting (1.33) into (1.29) or (1.35) into (1.30). □

Remark 1.3.2. The main result of Theorem 1.3.1 is that the real branches of W can be parameterized through parameter (1.31). This result, in fact, was discovered in (Barton, David, & Merrington, 1960, 1963) where studied are solutions of equation

$$e^{-a} + ka = 1 \quad (k > 0) \, , \tag{1.36}$$

where k is known (cases $k < 1$ and $k > 1$ were considered separately in the former and the latter paper respectively). In particular, in (Barton et al., 1960) the authors noted that if, in our notations, W_0 and W_{-1} are two roots of equation $\omega e^{\omega} = x$ then $W_0 - W_{-1} = a$, where a satisfies $e^{-a} - a/W_{-1} = 1$. It is easy to see that the last two relations are the same as (1.31) and (1.28) respectively in terms of $a = 2p$. The authors used this fact and the results of computations obtained for $k < 1$ in (Barton et al., 1960) to find a numerical solution for $k > 1$ in (Barton et al., 1963).

The equation (1.36) appeared in (David & Johnson, 1952), where the authors studied the truncated Poisson distribution, and described the maximum likelihood $\hat{\lambda} = a$ in terms of the truncated sample mean $\bar{x} = 1/k$. The authors rightly noted that '...it does not seem possible to obtain an explicit expression for $\hat{\lambda}$'. Today, fifty nine years later, due to the Lambert W function a solution of equation (1.36) can be written in an explicit form

$$a = \begin{cases} \dfrac{1}{k} + W_0\left(-\dfrac{1}{k}e^{-1/k}\right), & \text{if} \quad 0 < k \le 1 \\[2ex] \dfrac{1}{k} + W_{-1}\left(-\dfrac{1}{k}e^{-1/k}\right), & \text{if} \quad k > 1 \end{cases}$$

or

$$a = \frac{1}{k} + W_m\left(-\frac{1}{k}e^{-1/k}\right), \tag{1.37}$$

where $m = \lceil(\text{sgn}(1-k) - 1)/2\rceil$.

Also, an equation similar to (1.36) appears in (Valluri, Gil, Jeffrey, & Basu, 2009) where there are considered some applications of W function to quantum statistics. The equation defines the extrema in energy of the distribution function for a system comprised by a large number of non-interacting equivalent particles (cf. (Valluri et al., 2009, Eq. (18))) and has a solution of the form similar to

(1.37) (cf. (Valluri et al., 2009, Eq. (19))). In addition, it is shown in (Valluri et al., 2009) that the critical point of the integrand in the Lambert transform

$$\int_0^\infty \frac{t^{f(r)}}{e^{xt}-1}dt$$

is also defined by an equation similar to (1.36); a solution of this equation for $f(r) = 1/r$ and $x = 1$ is exactly the right-hand side of (1.37) (cf. (Valluri et al., 2009, Eq. (42))).

It is also interesting that the results of Theorem 1.3.1 can be expanded in series containing Bernoulli numbers B_n (see Appendix B) (Corless et al., 1997).

Theorem 1.3.3. *The following expansions hold (Lauwerier, 1963)*

$$-\ln(-ex) = \sum_{n=1}^{\infty} \frac{(2n+1)B_{2n}}{2n(2n)!}(2p)^{2n} \tag{1.38}$$

$$W_0 = -1 + p - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2p)^{2n} \tag{1.39}$$

$$W_{-1} = -1 - p - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2p)^{2n} \tag{1.40}$$

$$\ln(-W_0) = -p - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n)!}(2p)^{2n} \tag{1.41}$$

$$\ln(-W_{-1}) = p - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n)!}(2p)^{2n} \tag{1.42}$$

$$\frac{d}{dp}\ln(-W_0) = -1 - 2\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2p)^{2n-1} \tag{1.43}$$

$$\frac{d}{dp}\ln(-W_{-1}) = 1 - 2\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2p)^{2n-1} \tag{1.44}$$

Note. Lauwerier (1963) gave only two expansions that are similar to formulas (1.38) and (1.44); at that the factor 2 in front of the sum sign in the formula similar to (1.44) is missed in the text of his paper.

Proof. Substituting expansion (B.14)

$$\frac{2p}{e^{2p}-1} = 1 - p + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2p)^{2n} \tag{1.45}$$

into equation (1.33)

$$W_0 = -2p\frac{e^{-p}}{e^{p}-e^{-p}} = -\frac{2p}{e^{2p}-1} ,$$

we obtain (1.39) and then (1.40) using connection $W_{-1} = W_0 - 2p$.

Since $\ln(-W_0) = \ln p - p - \ln(\sinh p)$ by (1.27), we have

$$\begin{aligned}\frac{d}{dp}\ln(-W_0) &= \frac{1}{p} - 1 - \frac{\cosh p}{\sinh p} = \frac{1}{p} - 1 - \frac{e^{p}+e^{-p}}{e^{p}-e^{-p}} = \frac{1}{p} - 1 - \frac{e^{2p}+1}{e^{2p}-1} \\ &= \frac{1}{p} - 1 - \frac{(e^{2p}-1)+2}{e^{2p}-1} = \frac{1}{p} - 2 - \frac{2}{e^{2p}-1} \\ &= \frac{1}{p} - 2 - 2\left(\frac{1}{2p} - \frac{1}{2} + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(2p)^{2n-1}\right) ,\end{aligned}$$

where in the last line we substituted expansion (1.45) divided by $2p$. After eliminating brackets and rearrangements we obtain (1.43).

Further, it follows from (1.29) and (1.30) that $-W_{-1}e^{W_{-1}} = -W_0 e^{W_0}$, therefore $\ln(-W_{-1}) + W_{-1} = \ln(-W_0) + W_0$, which means $\ln(-W_{-1}) = \ln(-W_0) + W_0 - W_{-1}$, i.e.

$$\ln(-W_{-1}) = \ln(-W_0) + 2p . \tag{1.46}$$

Thus

$$\frac{d}{dp}\ln(-W_{-1}) = 2 + \frac{d}{dp}\ln(-W_0) ,$$

which together with (1.43) gives (1.44).

Now we integrate (1.43) in p to find

$$\ln(-W_0) = c - p - 2\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}2^{2n-1}\frac{p^{2n}}{2n} .$$

Since $W_0 = -1$ for $p = 0$ and so the constant $c = 0$, we obtain (1.41). Substituting (1.41) into (1.46) gives (1.42). To derive formula (1.38) we note that $-\ln(-ex) = -1 - \ln(-x) = -1 - (\ln(-W_0) + W_0)$ and substitute (1.41) and (1.39) here. □

Remark 1.3.4. The series (1.39) and (1.40) represent expansions of the real branches near the branch point $x = -1/e$ (where $W_0 = W_{-1} = -1$, i.e. $p = 0$) and together with (1.38) (or (1.26)) can be used instead of the expansions in terms of $\sqrt{2(1+ex)}$ (Corless et al., 1996).

Remark 1.3.5. By definition (1.31) the double parameter p shows the difference between values of the real branches and therefore has an obvious geometric interpretation as a distance between two points on the graph in Figure 1.3 taken at the same $x \in [-1/e, 0)$. Interestingly, by (1.46) this distance would be the same in the graph for the logarithm of the absolute values of these branches. The graphs depicted in figures 1.4 and 1.5 demonstrate the same distances between points at $x = -0.15$.

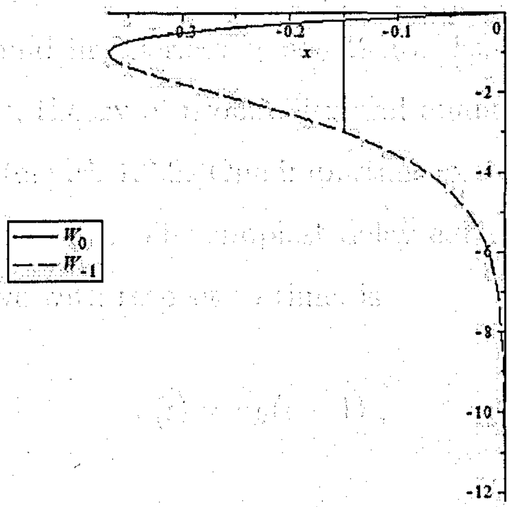

Figure 1.4: Distance between the real branches of W.

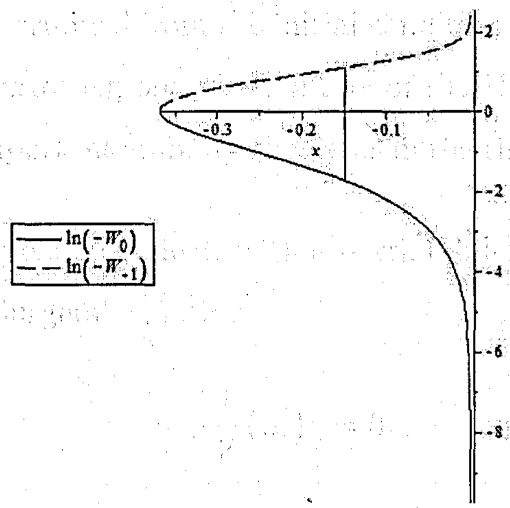

Figure 1.5: Distance between the logarithms of modulus of the real branches.

1.4 Applications

The Lambert W function has many applications in many areas of science such as combinatorics, applied mathematics, statistical mechanics, biology and others; it also gives useful analytical tools in solving engineering problems. Some of the applications can be found in (Corless et al., 1996). Examples of application of W to solving equations, theory of probability and quantum statistics have been already mentioned in Remark 1.3.2. One important example relates to the study of delay-differential equations. The simplest delay equation, using the notation $\dot{y} = \frac{dy}{dt}$ for the derivative with respect to time, is

$$\dot{y}(t) = ay(t-1) ,$$

subject to the condition on $-1 \leq t \leq 0$ that $y(t) = f(t)$, a known function. A general solution is expressed in terms of branches W_k (Heffernan & Corless, 2006)

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(W_k(a)t\right) ,$$

where the c_k can be determined from the initial conditions. One sees immediately that the solution will grow exponentially if any of the $W_k(a)$ has a positive real part, which leads to important stability theorems in the theory of delay equations.

Another interesting example deals with a partial differential equation. Let us consider the inviscid Burgers' equation

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0 \tag{1.47}$$

with the special initial condition $u(x,0) = e^{ix}$ (Weideman, 2003). A solution of (1.47) in the implicit form is given by $u = f(x - ut)$, where f is an arbitrary differentiable function. Plugging in the initial condition we find $u = e^{i(x-ut)}$. Using the Lambert W function we are able to write the solution in an explicit form

$$u = e^{ix}e^{-iut} \Rightarrow ue^{iut} = e^{ix} \Rightarrow itue^{iut} = ite^{ix} ,$$

i.e. $iut = W\left(ite^{ix}\right)$ and finally (Weideman, 2003)

$$u = \frac{W_0\left(ite^{ix}\right)}{it} .$$

A shock forms at a singular point when $ite^{ix} = -e^{-1}$, i.e $x = \pi/2 + i(1 + \ln t)$. Taking real x, we find the coordinates of the critical point in the (x,t)-plane (Weideman, 2003)

$$(x_*, t_*) = \left(\frac{\pi}{2}, \frac{1}{e}\right) .$$

One of the classical examples of application of W is connected with the problem of iterated exponentiation (Corless et al., 1996), where a function

$$h(z) = z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}}$$

is to be evaluated. Since $h(z)$ satisfies equation $h(z) = z^{h(z)}$, it can be found in closed form. Indeed, the last equation is equivalent to $h(z) = \exp(h(z)\ln z)$ or $(-h(z)\ln z)\exp(-h(z)\ln z) = -\ln z$, hence (Corless et al., 1996), (The poster 'The Lambert W Function')

$$h(z) = -\frac{W(-\ln z)}{\ln z} .$$

Many combinatorial applications of W are due to a simple connection (1.6) with the tree function. Specifically, W has applications in the enumeration of trees (Janson, Knuth, Luczak, & Pittel, 1993) and in graph theory (Flajolet, Salvy, & Schaeffer, 2004). W also participates in asymptotic estimations of Bell numbers ϖ_n (Appendix B), for example, according to (Lovász, 1993, Ex.9(b), p.17)

$$\varpi_n \sim \frac{\lambda_n^{n+1/2}}{n^{1/2}} e^{\lambda_n - n - 1} ,$$

where $\lambda_n = n/W(n)$.

It is also worthwhile noting that a generating function for the second-order Eulerian numbers is expressed in terms of the Lambert W function (cf. (B.11), (B.12)).

The engineering applications of W can be encountered in such problems as modeling of non-Gaussian noises in signal processing (Chapeau-Blondeau & Monir, 2002), combustion modeling (O'Malley, Jr., 1991; Corless et al., 1996), jet fuel problem (Anderson, 1989). We give some details for the last one following (Corless et al., 1996).

Let w_0 and w_1 be the initial and final weights of a jet airplane respectively, C_L and C_D the lift and drug coefficients, S the area of the horizontal projection of the plane's wings, ρ the ambient air density. We wish to find the thrust specific fuel consumption c_t and w_1 (to compute the weight of the fuel $w_0 - w_1$) from the equations for the endurance E_t and range R which are (Anderson, 1989,

p. 312-323)

$$E_t = \frac{C_L}{c_t C_D} \ln \frac{w_0}{w_1} , \tag{1.48}$$

$$R = \frac{2}{c_t C_D} \left(\frac{2C_L}{\rho S} \right)^{1/2} \left(w_0^{1/2} - w_1^{1/2} \right) . \tag{1.49}$$

For convenience, we introduce a negative parameter

$$A = -\frac{\sqrt{2}E_t}{R} \left(\frac{w_0}{\rho S C_L} \right)^{1/2}$$

and change variables

$$y = \sqrt{\frac{w_1}{w_0}} \quad \text{and} \quad c = \frac{C_D}{C_L} c_t E_t$$

Then equations (1.48) and (1.49) are equivalent to $c = -2 \ln y$ and

$$\frac{\ln y}{1-y} = A . \tag{1.50}$$

Clearly, it remains to solve equation (1.50). Since its left-hand side is a monotone increasing function of y with range in $(-\infty, 0)$ and its the right-hand side is a negative constant, the equation has the unique solution. We can rewrite (1.50) as $(Ay)e^{Ay} = Ae^A$ to get finally

$$y = \begin{cases} W_0(Ae^A)/A, & \text{if } A \le -1 , \\ W_{-1}(Ae^A)/A, & \text{if } -1 < A < 0 . \end{cases}$$

References

Anderson, J. D. (1989). *Introduction to Flight* (3rd ed.). New York: McGraw-Hill.

Barton, D. E., David, F. N., & Merrington, M. (1960). Tables for the solution of the exponential equation, $\exp(-a)+ka = 1$. *Biometrica*, *47*(3/4), 439–445.

Barton, D. E., David, F. N., & Merrington, M. (1963). Table for the solution of the exponential equation $\exp(b) - b/(1-p) = 1$. *Biometrica*, *50*(1/2), 169–176.

Bruijn, N. G. de. (1961). *Asymptotic Methods in Analysis* (Second ed.). Amsterdam: North-Holland.

Chapeau-Blondeau, F., & Monir, A. (2002). Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. *IEEE Transactions on Signal Processing*, *50*(9), 2160–2165.

Comtet, L. (1970). Inversion de $y^\alpha e^y$ et $y \log^\alpha y$ au moyen des nbombres de Stirling. *C. R. Acad. Sci., Paris, Sér. A-B*, *270*, A1085–A1088.

Corless, R. M., Gonnet, G., Hare, D. E., Jeffrey, D. J., & Knuth, D. E. (1996). On the Lambert W function. *Adv. Comp. Math.*, *5*(4), 329–359.

Corless, R. M., Jeffrey, D. J., & Knuth, D. E. (1997). A sequence of series for the Lambert W function. In *Issac '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (pp. 197–204). New York, NY, USA: ACM press.

David, F. N., & Johnson, N. L. (1952). The Truncated Poisson. *Biometrics*, *8*(4), 275–285.

Euler, L. (1779). De serie Lambertina plurimisque eius insignibus proprietatibus. *Leonhardi Euleri Opera Omnia, Ser. 1, Opera Mathematica*, *6*, 350–369.

Flajolet, P., Salvy, B., & Schaeffer, G. (2004). Airy phenomena and analytic combinatorics of connected graphs. *The Electronic Journal of Combinatorics*, *11*(1), R34, 30 p.

Flajolet, P., & Sedgewick, R. (2009). *Analytic Combinatorics.* Cambridge University Press.

Heffernan, J. M., & Corless, R. M. (2006). Solving some delay differential equations with computer algebra. *Mathematical Scientist*, *31*(1), 21–34.

Janson, S., Knuth, D. E., Luczak, T., & Pittel, B. (1993). The birth of the giant component. In *Random structures and algorithms 4* (pp. 233–358). John Wiley & Sons, Inc.

Jeffrey, D. J., Hare, D. E., & Corless, R. M. (1996). Unwinding the branches of the Lambert W function. *Mathematical Scientist*, *21*, 1–7.

Lambert, J. H. (1758). Observations variae in mathesin puram. *Acta Helvetica, physico-mathematico-anatomico-botanico-medica*, *3*, 128–168.

Lauwerier, H. A. (1963). The Asymptotic Expansion of the Statistical Distribution of N.V. Smirnov. *Z. Wahrscheinlichkeitstheorie*, *2*, 61–68.

Lovász, L. (1993). *Combinatorial Problems and Excercises* (2nd ed.). Amsterdam, Netherlands: North-Holland.

O'Malley, Jr., R. E. (1991). Singular Perturbation Methods for Ordinary Differential Equations. In *Applied Mathematical Sciences 89.* Springer-Verlag.

The poster 'The Lambert W Function'. (2004). http://www.orcca.on.ca/LambertW/. UWO.

Valluri, S. R., Gil, M., Jeffrey, D. J., & Basu, S. (2009). The Lambert W Function and Quantum Statistics. *J. Math. Phys.*, *50*(10), 11 p.

Weideman, J. A. C. (2003). Computing the Dynamics of Complex Singularities of nonlinear PDEs. *SIAM J. Applied Dynamical Systems*, *2*(2), 171–186.

Wright, E. M. (1949). The linear difference-differential equation with constant coefficients. *Proc. Royal Soc. Edinburgh*, *A62*, 387–393.

Wright, E. M. (1955). A non-linear difference-differential equation. *J. für reine und angewandte Mathematik*, *194*, 66–87.

Wright, E. M. (1959). Solution of the equation ze^z=a. *Bull. Amer. Math. Soc.*, *65*, 89–93.

CHAPTER 2

Convergence of asymptotic series for the Lambert W function

"No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it also gives the impression of being beautiful." – George Boole

2.1 Introduction

In this chapter we study some previously known series for the Lambert W function to specify old results, to establish the domain of convergence in real and complex cases, to give a comparative analysis of their properties and to find asymptotic expressions for the expansion coefficients. We also obtain different forms of representation of the expansion coefficients and present some combinatorial consequences including the Carlitz-Riordan identities resulting from that.

The equation $y^\alpha e^y = x$ was solved by Comtet (1970), following de Bruijn

(1961), as

$$y = \Phi_\alpha(x) = \ln x - \alpha \ln\ln x + \alpha u = \alpha\left(\frac{1-\tau}{\sigma} + u\right), \tag{2.1}$$

where

$$\sigma = \frac{\alpha}{\ln x}, \; \tau = \alpha\frac{\ln\ln x}{\ln x}, \tag{2.2}$$

and function u obeys the fundamental relation (Corless, Jeffrey, & Knuth, 1997)

$$1 - e^{-u} + \sigma u - \tau = 0. \tag{2.3}$$

Comtet (1970) further showed that u has the series development

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\begin{bmatrix} n \\ n-m+1 \end{bmatrix}\frac{\sigma^{n-m}\tau^m}{m!}, \tag{2.4}$$

where $\left[{n \atop n-m+1}\right]$ denotes Stirling cycle numbers, also called the unsigned Stirling numbers of the first kind (Graham, Knuth, & Patashnik, 1989; Corless et al., 1997). This series was further developed and rearranged in (Jeffrey, Corless, Hare, & Knuth, 1995) in terms of the 2-associated Stirling numbers of the second kind (Graham et al., 1989; Corless et al., 1997)

$$u = \sum_{m=1}^{\infty}\frac{\tau^m}{m!}\sum_{p=0}^{m-1}\left\{ {p+m-1 \atop p} \right\}_{\geq 2}\frac{(-1)^{p+m-1}}{(1+\sigma)^{p+m}}. \tag{2.5}$$

In a particular case $\alpha = 1$ the function defined by (2.1) is the Lambert W function (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996), i.e. $\Phi_1(x) = W(x)$. In the chapter, the series (2.4) and (2.5) are considered for $\alpha > 0$ in a real case and for $\alpha = 1$ in a complex case. Specifically, in the former $\sigma > 0$, i.e. in the expressions (2.2) $x \in (1, \infty)$, and in the latter real x is changed to complex z.

The Lambert W function is multivalued, its branches W_k are defined by (1.7).

With this definition, the first terms of the asymptotic series for W_k are

$$W_k(z) = \ln_k z - \ln(\ln_k z) + \frac{\ln(\ln_k z)}{\ln_k z} + O\left(\left(\frac{\ln \ln_k z}{\ln_k z}\right)^2\right) . \tag{2.6}$$

We shall mostly be concerned with the principal branch $k = 0$, which is the only branch that is finite at the origin and takes on real values on the positive real line. We shall abbreviate W_0 to W for the rest of the chapter.

The fundamental relation (2.3) possesses a remarkable property: it can itself be solved in terms of the Lambert W function (Corless et al., 1997)

$$u = W(e^s) - \frac{1-\tau}{\sigma} , \tag{2.7}$$

where

$$s = s(\sigma, \tau) = \frac{1-\tau}{\sigma} - \ln \sigma . \tag{2.8}$$

This gives a useful representation of the Lambert W function (Corless et al., 1997)

$$W(z) = W(e^s) \tag{2.9}$$

and allows to get properties of u from those of $W(z)$ and vice versa. For example, it follows from (2.7) that in the real case

$$-\pi < \Im u < \pi \tag{2.10}$$

because the same is true for W (cf. Figure 1.1).

The asymptotic series (2.4) and (2.5) were studied in the real case in (Jeffrey et al., 1995). In this chapter we specify and establish the domain of convergence of the series in both real and complex case, analyse a difference in their properties and find asymptotic expressions for the expansion coefficients in (2.5).

For $\alpha = 1$, along with the expansions (2.4) and (2.5) we study a series in terms of the second-order Eulerian numbers (Corless et al., 1997)

$$W(x) = \omega_0 + \sum_{m=1}^{\infty} \frac{1}{m!\sigma^m(1+\omega_0)^{2m-1}} \sum_{k=0}^{m-1} \left\langle\!\!\left\langle {m-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k \omega_0^{k+1} , \qquad (2.11)$$

where $\sigma = 1/\ln x$ and ω_0 denotes the Omega constant $W(1) = 0.56714329...$ A definition of the second-order Eulerian numbers is given in Appendix B.

The series (2.11) was obtained and studied in (Corless et al., 1997) and in fact represents a series of the Wright w function (Corless & Jeffrey, 2002). We give three new forms of representation of the expansion coefficients of this series as well as their asymptotic estimates.

In the chapter, it is also shown that the series (2.5) can be represented in terms of the second-order Eulerian numbers. Some combinatorial consequences following from different forms for representation of the expansion coefficients in (2.5) and (2.11) are presented, including the Carlitz-Riordan identities.

2.2 Series (2.4)

2.2.1 Convergence in real case

It is shown in (Jeffrey et al., 1995) that the series (2.4) is convergent for

$$x > x_\alpha = \begin{cases} e, & 0 < \alpha \le 1 \\ (\alpha e^\alpha)^\alpha, & \alpha > 1 \end{cases} \qquad (2.12)$$

Below we are going to confirm and specify this result. We first prove a statement in terms of variable σ and τ.

Theorem 2.2.1. *The domain of convergence of the series* (2.4) *in real case is defined by inequality*

$$\ln\sigma < 1 - \frac{\tau}{\sigma} + \Re W_{-1}\left(-e^{\frac{\tau}{\sigma}-1}\right). \tag{2.13}$$

Proof. We consider the series (2.4) in the real case and start with the fundamental relation (2.3) to write it in the form

$$G_\lambda(\sigma, u) = 0, \tag{2.14a}$$

where we introduced a variable $\lambda = \tau/\sigma$ playing a role of parameter and set

$$G_\lambda(\sigma, u) = 1 - e^{-u} + \sigma u - \sigma\lambda. \tag{2.14b}$$

By Implicit Function Theorem (Markushevich, 1965), for fixed $\lambda \in \mathbb{R}$ equations (2.14) determine a function

$$u_\lambda(\sigma) = \sum_m c_m(\lambda)\sigma^m \tag{2.15}$$

with initial condition $u_\lambda(0) = 0$ in a domain where $\partial G_\lambda(\sigma, u)/\partial u = e^{-u} + \sigma \neq 0$. Since $\partial G_\lambda(0, 0)/\partial u = 1 \neq 0$, the mentioned initial condition is justified.

To find the critical points in the complex σ-plane we first solve the equation $e^{-u} + \sigma = 0$; its roots are $u = u_*^{(k)}$, where

$$u_*^{(k)} = -\ln\sigma + i\pi(2k-1), \quad k \in \mathbb{Z}\,. \tag{2.16}$$

Substituting (2.16) into (2.14) we obtain the equation

$$\lambda - 1 + \ln\sigma - i\pi(2k-1) = 1/\sigma \tag{2.17}$$

which, after exponentiating

$$\frac{1}{\sigma}e^{1/\sigma} = -e^{\lambda-1} ,$$

can be solved for σ in terms of the Lambert W function

$$\sigma_m = 1/W_m(-e^{\lambda-1}) , \tag{2.18}$$

where the m-th root is defined by the m-th branch of W.
Comparison between equations (2.16) and (2.17) shows that at the critical points we have a relation $\lambda - 1 - u = 1/\sigma$ and hence $\Im(1/\sigma) = \Im(-u)$. Therefore, due to (2.10) a root σ_m defined by equation (2.18) is a singular point for the principal branch if its reciprocal has imaginary part in $(-\pi, \pi)$. On the other hand, in the right-hand side of equation (2.18) only branches W_0 and W_{-1} have the imaginary part in this range (Corless et al., 1996) (and thereby can provide not ~~only~~ complex but also real roots σ_m unlike the other branches). Thus we conclude that there are only two acceptable values for m, i.e. $m = -1, 0$.

Due to identity $W_m(-e^{\lambda-1}) \exp\left\{W_m(-e^{\lambda-1})\right\} = -e^{\lambda-1}$ equation (2.18) can be written as

$$\sigma_m = -\exp\left\{1 - \lambda + W_m(-e^{\lambda-1})\right\} \quad (m = -1, 0) . \tag{2.19}$$

The key point for further considerations is that the radius of convergence of the power series (2.15) is equal to the distance from the origin in the complex σ-plane to the closest singular point (Titchmarsh, 1939),(Antimirov, Kolyshkin, & Vaillancourt, 1998, p.175, Theorem 4.3.2). In other words, due to (2.19) the domain of convergence is defined by inequality

$$|\sigma| < \min_{m \in \{-1,0\}} \left| -\exp\left\{1 - \lambda + W_m(-e^{\lambda-1})\right\}\right| .$$

or

$$\ln|\sigma| < 1 - \Re\lambda + \min_{m\in\{-1,0\}} \Re W_m(-e^{\lambda-1}) . \tag{2.20}$$

Since $\Re W_{-1}(x) \leq \Re W_0(x)$ for all $x \in \mathbb{R}$ (Corless et al., 1996), after substituting $\lambda = \tau/\sigma$ the condition (2.13) follows. □

To express the condition (2.13) of convergence of the series (2.4) in terms of independent variable x in (2.2) and compare the result with (2.12) it is convenient to prove the following lemma.

Lemma 2.2.2. *Solution of inequality $\Re W_{-1}(x) > a$ for $x < 0$, where a is constant, is given by*

$$x < x_0 = \begin{cases} ae^a, & a \leq -1 \\ -e^a \eta_0 \csc\eta_0, & a > -1 \end{cases} \tag{2.21}$$

where $\eta_0 \in (0, \pi)$ is the root of equation $\eta_0 \cot\eta_0 = -a$.

Proof. We set $W_{-1}(x) = \xi + i\eta$ for real negative x where $\xi \leq -1, \eta = 0$ for $-1/e \leq x < 0$ and $\xi > -1, -\pi < \eta < 0$ for $x < -1/e$ (Corless et al., 1996). Then separating the real and imaginary parts in the defining equation (1.7) we obtain

$$x = e^{\xi}(\xi\cos\eta - \eta\sin\eta), \quad 0 = e^{\xi}(\eta\cos\eta + \xi\sin\eta) .$$

From these equations, one can find a dependence of ξ on x explicitly for $-1/e \leq x < 0$

$$\xi = W_{-1}(x) \tag{2.22}$$

and parametrically for $x < -1/e$

$$x = -\eta\csc(\eta)e^{-\eta\cot\eta} , \tag{2.23}$$

$$\xi = -\eta\cot\eta , \tag{2.24}$$

where $-\pi < \eta < 0$.

Now we consider inequality $\xi > a$ in two cases comparing a with value -1. When $a \leq -1$ the inequality $\xi > a$ holds for all $x < -1/e$ because in this case $\xi > -1$ by (2.24). For $-1/e \leq x < 0$ we solve inequality $W_{-1}(x) > a$ due to (2.22) with the result $-1/e \leq x < ae^a$. Thus $\xi > a$ for $x < ae^a$.
When $a > -1$ the inequality $\xi > a$ can have a solution only for $x < -1/e$ because $\xi \leq -1$ for the rest x. According to (2.23) and (2.24) the solution is given by $x < -\eta_0 \csc(\eta_0) \exp(-\eta_0 \cot \eta_0)$ where $\eta_0 \in (-\pi, 0)$ satisfies the equation $-\eta_0 \cot \eta_0 = a$ due to which the solution can also be written as $x < -e^a \eta_0 \csc \eta_0$ and $\eta_0 \in (0, \pi)$. Joining both cases, the lemma follows. □

Note that in the formula (2.21) when $a > -1$ but $a \neq 0$ we can also write $x_0 = ae^a / \cos \eta_0$.

Theorem 2.2.3. *The series* (2.4) *is convergent when*

$$x > x_\alpha = \begin{cases} (e/\alpha)^\alpha, & 0 < \alpha \leq 1 \\ e^{\alpha \eta_0 \csc \eta_0}, & \alpha > 1 \end{cases} \tag{2.25}$$

where η_0 *satisfies equation* $\eta_0 \cot \eta_0 = 1 - \ln \alpha$ $(0 < \eta_0 < \pi)$, *and divergent when* $x < x_\alpha$.

Proof. We consider the condition of convergence of the series (2.4) established by Theorem 2.2.1 in the real case, i.e. when $\alpha > 0$ and $x > 1$. Substituting the expressions (2.2) in (2.13) we obtain

$$\Re W_{-1}\left(-\frac{\ln x}{e}\right) > \ln \alpha - 1 \,. \tag{2.26}$$

Applying Lemma 2.2.2 to the inequality (2.26) we come to (2.25), where $x_\alpha > 1$, which justifies the assumption $x > 1$. Thus the theorem is completely proved. □

Note. The statement of Theorem 2.2.3 was independently reported by A.J.E.M. Janssen and J.S.H. van Leeuwaarden (November, 2007).

Remark 2.2.4. In the formula (2.25) when $\alpha > 1$ but $\alpha \neq e$ we can also write $x_\alpha = (e/\alpha)^{\alpha \sec \eta_0}$.

Remark 2.2.5. Due to (2.26) the condition of convergence of the series (2.4) for $\alpha = 1$ can be written as

$$\Re W_{-1}\left(-\frac{\ln x}{e}\right) > -1 \; . \tag{2.27}$$

Corollary 2.2.6. *The series* (2.4) *for* $\alpha = 1$, *i.e. for* W *function, is convergent for* $x > e$ *and divergent for* $x < e$.

Proof. Follows immediately from (2.25) for $\alpha = 1$. □

Since the statement of Corollary 2.2.6 is very important, we give one more proof of it following (Jeffrey et al., 1995).

Theorem 2.2.7. *The series (2.4) converges for all* $x > e$.

Proof. We write the equation (2.3) in the form

$$g(u) + f(u;\sigma,\tau) = 0 \; , \tag{2.28}$$

where

$$g(u) = 1 - e^{-u} \quad \text{and} \quad f(u;\sigma,\tau) = \sigma u - \tau \; .$$

We now consider the equation (2.28) with respect to u for fixed real σ and τ specified below. For any analytic function $F(\zeta)$ with a single isolated zero at $\zeta = u$ inside a contour C in the complex ζ-plane, we can use Cauchy's integral formula to write

$$u = \frac{1}{2\pi i}\int_C \frac{F'(\zeta)}{F(\zeta)}\,\zeta\, d\zeta \; . \tag{2.29}$$

Setting $F(\zeta) = g(\zeta) + f(\zeta;\sigma,\tau)$ we can write a solution of the equation (2.28) in the form

$$u = \frac{1}{2\pi i}\int_C \frac{e^{-\zeta}+\sigma}{g(\zeta)+f(\zeta;\sigma,\tau)}\,\zeta\,d\zeta\,, \tag{2.30}$$

provided a proper contour C exists.

Let us fix an arbitrary $x \in (e,\infty)$, then $0 < \sigma < 1$ as $\sigma = 1/\ln x$. Taking an arbitrary $\delta \in (0, 1-\sigma)$ we consider the following rectangular contour

$$\zeta = \begin{cases} \delta + it\,, & -2\delta^{1/2} \le t \le 2\delta^{1/2}\,, \\ t + i2\delta^{1/2}\,, & -2 \le t \le \delta\,, \\ -2 + it\,, & -2\delta^{1/2} \le t \le 2\delta^{1/2}\,, \\ t - i2\delta^{1/2}\,, & -2 \le t \le \delta\,. \end{cases} \tag{2.31}$$

It is straightforward to show that on this contour $|g| > |f|$. Hence, by Rouché's theorem g and $f+g$ have the same number of zeros within the contour. But equation $g(u) = 0$ has the unique root $u = 0$. Therefore, the function $f+g$ has a single isolated zero and the contour can be taken for the integration contour C.

In addition to satisfying the conditions of the integration, the contour allows us to evaluate the integral by expanding the denominator of the integrand in (2.30) as an absolutely and uniformly convergent power series in f/g.

$$\begin{aligned} \frac{1}{g+f} &= \frac{1}{g}\cdot\frac{1}{1+\dfrac{f}{g}} = \sum_{k=0}^{\infty}(-1)^k\frac{f^k}{g^{k+1}} = \sum_{k=0}^{\infty}(-1)^k(1-e^{-\zeta})^{-k-1}(\sigma\zeta-\tau)^k \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^k\binom{m+k}{m}(1-e^{-\zeta})^{-k-m-1}\zeta^k\sigma^k\tau^m\,. \end{aligned} \tag{2.32}$$

Substituting this expansion into (2.30) and integrating term by term, we obtain u as the sum of an absolutely convergent double power series in σ and τ, provided $x > e$.

The domain of convergence cannot be extended to $x < e$, because the series for du/dz diverges at $x = e$. This can be seen by noting that $\tau = 0$ at $x = e$. All terms reduce to zero except $m = 1$ which gives the sum

$$\frac{1}{e}\sum_{k=0}^{\infty}(-1)^k ,$$

which is divergent. □ □

Remark 2.2.8. The radius of convergence of the series (2.4) in terms of variable $\sigma = 1/\ln x$ equals unit.

We now prove a statement, relating to divergence of the series (2.4), which was found by us earlier than the conditions (2.25) but unlike Theorem 2.2.3 concerning positive α it deals with any $\alpha \neq 0$. In addition, the statement demonstrates an interesting application of the ratio test to the series (2.4).

Theorem 2.2.9. *The series* (2.4) *is divergent at least for*

$$e^{-|\alpha|} < x < e^{b|\alpha|} \tag{2.33}$$

where

$$b = \begin{cases} W(1/|\alpha|) & \textit{when } |\alpha| < 1/e\ , \\ 1 & \textit{when } |\alpha| \geq 1/e\ . \end{cases}$$

Proof. Changing indices for summing the expansion (2.4) can be written through a double series (Jeffrey et al., 1995)

$$u = \sum_{m=1}^{\infty}\sum_{l=0}^{\infty} c_{m,l}\ , \tag{2.34}$$

where

$$c_{m,l} = c_{m,l}(\sigma, \tau) = \frac{(-1)^l}{m!}\cdot\begin{bmatrix} l+m \\ l+1 \end{bmatrix}\sigma^l\tau^m$$

For the column-series $\sum_m c_{m,l}$ the ratio test gives

$$\lim_{m\to\infty}\left|\frac{c_{m+1,l}}{c_{m,l}}\right| = |\tau| \lim_{m\to\infty} \frac{\begin{bmatrix} l+m+1 \\ l+1 \end{bmatrix}}{(m+1)\begin{bmatrix} l+m \\ l+1 \end{bmatrix}} = |\tau|$$

as according to (Abramowitz & Stegan, 1970)

$$\lim_{p\to\infty} \frac{\begin{bmatrix} p+1 \\ l+1 \end{bmatrix}}{p\begin{bmatrix} p \\ l+1 \end{bmatrix}} = 1 \qquad \text{for fixed } l$$

in our notations.

For the row-series $\sum_l c_{m,l}$ we have

$$\lim_{l\to\infty}\left|\frac{c_{m,l+1}}{c_{m,l}}\right| = |\sigma| \lim_{l\to\infty} \frac{\begin{bmatrix} l+m+1 \\ l+2 \end{bmatrix}}{\begin{bmatrix} l+m \\ l+1 \end{bmatrix}} = |\sigma|$$

because by (Abramowitz & Stegan, 1970)

$$\lim_{l\to\infty} \frac{\begin{bmatrix} l+m \\ l+1 \end{bmatrix}}{(l+1)^{2m-2}} = \frac{1}{2^{m-1}(m-1)!} \qquad \text{for fixed } m.$$

According to (Limaye & Zeltser, 2009, Theorem 2.7) the series (2.34) (and therefore (2.4)) is divergent when $|\sigma| > 1$ or $|\tau| > 1$. Expressing these inequalities in terms of x by (2.2) and uniting the obtained sets we obtain the stated interval (2.33) where the series (2.4) is divergent. □

By Theorem 2.2.9 for $\alpha = 1$ the series (2.4) is divergent at least for $e^{-1} < x < e$, which is consistent with Corollary 2.2.6.

For comparison, the curves described by equations (2.25), (2.12) and (2.33) are depicted in Figure 2.1 by solid, dash and dashdot lines respectively. Thus the solid line shows the exact (lower) boundary of domain of convergence of series (2.4).

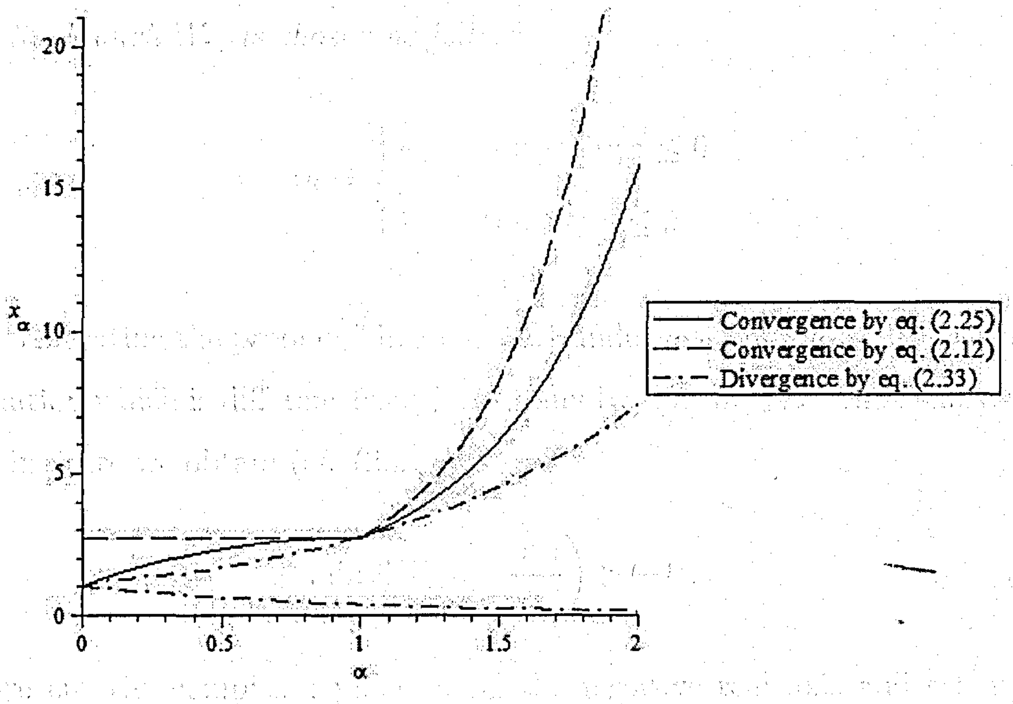

Figure 2.1: Boundary of domain of convergence of series (2.4)

2.2.2 Convergence in complex case

From now on we consider equations (2.1), (2.2) only for $\alpha = 1$ (under the same relation (2.3)) and in this subsection derive the convergence conditions for the series (2.4) in the complex case using the results obtained in Section 2.2.1 in the real case. To do that we set

$$\sigma = 1/\ln z, \tau = \ln\ln z/\ln z\,, \tag{2.35}$$

where $z = x + iy$ is a complex variable and $\ln z$ denotes the principal branch of the natural logarithm. Then the right-hand side of the series (2.4) represents a function of the complex variable z and the following theorem holds.

Theorem 2.2.10. *The domain of convergence of the series (2.4) in the complex z-plane is defined by*

$$\Re W_m\left(-\frac{\ln z}{e}\right) > -1, \tag{2.36}$$

where the branch W_m *is chosen as follows*

$$m = \begin{cases} -1, & -\pi < \arg z \le 0 \\ 1, & 0 < \arg z \le \pi \end{cases}$$

Proof. Repeating the proof of Theorem 2.2.1 under assumption $\lambda \in \mathbb{C}$ we come to an equation which is different from (2.20) only by that $m \in \mathbb{Z}$. Then substituting (2.35) in there we obtain (cf. (2.27))

$$\min_{m\in\mathbb{Z}} \Re W_m\left(-\frac{\ln z}{e}\right) > -1\,. \tag{2.37}$$

Now we cut the complex z-plane along the negative real axis and set $\arg z \in (-\pi, \pi]$. We consider inequality (2.37) in domain $D = \{z \in \mathbb{C} | -\pi < \arg z \le 0\}$ and assume that there exists some value $m = q$ such that the domain of convergence in D is defined by equation

$$\Re W_q\left(-\ln z/e\right) > -1 \tag{2.38}$$

and its continuous boundary $\mathscr{L}$ is given by

$$\Re W_q\left(-\ln z/e\right) = -1\,. \tag{2.39}$$

The domain of convergence found in real case is defined in a similar way. Specifically, in domain $\{z \in \mathbb{R} | z > 0\}$ we have $q = -1$ by (2.27) and the boundary $z = e$ by Corollary 2.2.6. We require that in the limiting case $\arg z \to 0-$ equation (2.38) become equation (2.27) and show that there is an unique value $q = -1$ satisfying this requirement. (If there were several such values of q, it would mean that the boundary $\mathscr{L}$ is composed of several pieces of different curves, and to identify them one should reduce domain D, i.e. consider its subdomains.) Substituting $W_q(-\ln z/e) = -1 + i\eta$ in the defining equation (1.7) and separating the real and imaginary parts we find

$$\sin\eta - \eta\cos\eta = \arg z \tag{2.40a}$$

$$\cos\eta + \eta\sin\eta = \ln|z| \tag{2.40b}$$

These equations describe a set of the boundary points which can be found in the following way. Given a value for $\arg z$ one can find η from (2.40a) which being substituted in (2.40b) yields the corresponding value of $\ln|z|$. However, for fixed $\arg z \in (-\pi, 0]$ the equation (2.40a) has an infinite number of solutions. We select a solution to provide a continuous transition to the real case when $\arg z \to 0-$ and when the boundary of the domain of convergence is defined by $\Re W_{-1}(-\ln z/e) = -1$ (cf. (2.27)). An elementary analysis of the equation (2.40a) shows that to meet these requirements one needs to choose a solution of this equation from the interval $\eta \in (-\pi, 0]$ and set $q = -1$ in (2.39). Since by (2.40a) such solution exists if and only if $z \in D$, the above assumption is approved and the domain of convergence in D is described by (2.38) with $q = -1$, i.e.

$$\Re W_{-1}(-\ln z/e) > -1 \; .$$

Due to the near conjugate symmetry property of W function (Corless et al., 1996), i.e. $W_k(z) = \overline{W_{-k}(\bar{z})}$ when z is not on the branch cut, we obtain the convergence condition $\Re W_1(-\ln z/e) > -1$ in the domain $\{z \in \mathbb{C} \,|\, 0 < \arg z \leq \pi\}$. Thus the theorem is completely proved. □

Remark 2.2.11. The 'branch splitting' in the proved formula (2.36) is due to the branch choices for the Lambert W function and similar to the effect that occurs in the series for W about the branch point (Corless et al., 1997, Sec. 3).

Remark 2.2.12. The inequality opposite to (2.36) defines the domain where the series (2.4) is divergent. This domain is finite (it encloses the origin $z = 0$) and contains a subdomain defined by inequality $|\sigma| > 1$. Therefore, unlike the real case (see Corollary 2.2.6) in the complex case the condition $|\sigma| < 1$ is only necessary but not sufficient for convergence of the series (2.4).

2.3 Series (2.5)

2.3.1 Convergence in real case

We regard the expansion (2.5) as a power series around $\tau = 0$ where variable σ plays a role of a parameter.

Theorem 2.3.1. *For $\alpha > 0$ and $\sigma > 0$, the radius of convergence of the series (2.5) is exactly*

$$\tau_*(\sigma) = |1 + \sigma - \sigma \ln \sigma \pm i\pi\sigma| \ , \tag{2.41}$$

which is equiavalent to the condition of convergence of the series (2.5) *as*

$$|\sigma(\ln \alpha - \ln \sigma)| < \sqrt{(1 + \sigma - \sigma \ln \sigma)^2 + \pi^2\sigma^2} \ . \tag{2.42}$$

Proof. We rewrite the fundamental relation (2.3) in the form of equation

$$F_\sigma(\tau, u) = 0, \tag{2.43}$$

where

$$F_\sigma(\tau, u) = 1 - e^{-u} + \sigma u - \tau \ , \tag{2.44}$$

and analyse this equation similarly to that in the proof of Theorem 2.2.1. By Implicit Function Theorem (Markushevich, 1965) the equation (2.43) determines a function $u_\sigma(\tau) = \sum_m c_m(\sigma)\tau^m$ with initial condition $u_\sigma(0) = 0$ in a domain where $\partial F_\sigma(\tau, u)/\partial u = e^{-u} + \sigma \neq 0$. The initial condition is justified by $\partial F_\sigma(0,0)/\partial u = 1 + \sigma \neq 0$. Since the critical points are defined by the same equation as in Theorem 2.2.1, they are given by (2.16) and the corresponding values of τ are

$$\tau_*^{(k)} = 1 - e^{-u_*^{(k)}} + \sigma u_*^{(k)} = 1 + \sigma - \sigma \ln \sigma + i\pi\sigma(2k-1), k \in \mathbb{Z} \tag{2.45}$$

The radius of convergence is equal to the distance from the origin in the complex τ-plane to the closest singular point (Antimirov et al., 1998, Theorem 4.3.2). Among the critical points (2.45) there are two the nearest to the origin equidistant points which correspond to $k = 0$ and $k = 1$:

$$\tau_*^{(0)} = 1 + \sigma - \sigma \ln \sigma - i\pi\sigma, \tag{2.46a}$$

$$\tau_*^{(1)} = 1 + \sigma - \sigma \ln \sigma + i\pi\sigma. \tag{2.46b}$$

The corresponding values of $u_*^{(k)}$ are

$$u_*^{(0)} = -\ln \sigma - i\pi, \tag{2.47a}$$

$$u_*^{(1)} = -\ln \sigma + i\pi. \tag{2.47b}$$

Since the expansion coefficients of the series (2.5) are real, the closest singularities can appear as a conjugate pair only (Hunter & Guerrieri, 1980). Based on the Weierstrass's preparation theorem (Markushevich, 1965; Adachi, 2007) we will show that the points (2.46) are singular, each corresponding to a square-root branch point of function $u = u_\sigma(\tau)$ in the complex τ-plane. We will also find a behavior of function $u = u_\sigma(\tau)$ near the points (2.46) used then for a study of an asymptotic behaviour of the expansion coefficients of the series (2.5).

Let us consider, for example, point $\tau = \tau_*^{(0)}$. Expanding the left-hand side of equation (2.43) into a Taylor series near the point $S\left(\tau_*^{(0)}, u_*^{(0)}\right)$ we obtain

$$F_\sigma(S) + \frac{\partial F_\sigma(S)}{\partial \tau}\left(\tau - \tau_*^{(0)}\right) + \frac{\partial F_\sigma(S)}{\partial u}\left(u - u_*^{(0)}\right) + \frac{\partial^2 F_\sigma(S)}{\partial \tau^2}\frac{\left(\tau - \tau_*^{(0)}\right)^2}{2}$$

$$+\frac{\partial^2 F_\sigma(S)}{\partial \tau \partial u}\left(\tau - \tau_*^{(0)}\right)\left(u - u_*^{(0)}\right) + \frac{\partial^2 F_\sigma(S)}{\partial u^2}\frac{\left(u - u_*^{(0)}\right)^2}{2} + \cdots = 0\,,$$

where dots denote the skipped terms of the higher order. Since

$$F_\sigma(S) = 0, \frac{\partial F_\sigma(S)}{\partial u} = 0, \frac{\partial^2 F_\sigma(S)}{\partial u^2} = -\exp\left(-u_*^{(0)}\right), \text{ and } \frac{\partial F_\sigma}{\partial \tau} \equiv -1$$

the last equation becomes

$$-\left(\tau - \tau_*^{(0)}\right) - \exp\left(-u_*^{(0)}\right)\frac{\left(u - u_*^{(0)}\right)^2}{2} + \cdots = 0.$$

Thus, in accordance with the Weierstrass's preparation theorem (Markushevich, 1965, p.111), equation (2.43) is locally equivalent to the equation

$$\left(\tau_*^{(0)} - \tau\right) \sim \exp\left(-u_*^{(0)}\right)\frac{\left(u - u_*^{(0)}\right)^2}{2}.$$

It follows that at $\tau = \tau_*^{(0)}$ function $u = u_\sigma(\tau)$ has a singularity corresponding to a square-root branch point as near this point

$$u \sim u_*^{(0)} \pm \exp\left(\frac{u_*^{(0)}}{2}\right)\sqrt{2\left(\tau_*^{(0)} - \tau\right)}$$

or substituting (2.47a)

$$u \sim -\ln\sigma - i\pi \pm i\sqrt{\frac{2\tau_*^{(0)}}{\sigma}}\left(1-\frac{\tau}{\tau_*^{(0)}}\right)^{\frac{1}{2}} . \tag{2.48}$$

It is not difficult to show that if we consider the values of the function (2.48) in the interior of the circle of radius (2.41) remaining in the vicinity of $\tau = \tau_*^{(0)}$ then the function (2.48) taken with the plus sign only satisfies the condition $-\pi < \Im u < 0$, which corresponds to $\Im u_*^{(0)} = -\pi < 0$ at point $\tau = \tau_*^{(0)}$ itself by (2.47a). Moreover, since in the mentioned vicinity $-\pi < \Im\tau/\sigma < 0$, we have $-\pi < \Im W < \pi$ by (2.7), which corresponds to the principal branch of W function (Corless et al., 1996). Thus we come to conclusion that the function $u = u_\sigma(\tau)$ behaves near the singularity (2.46a) like

$$u \sim -\ln\sigma - i\pi + i\sqrt{\frac{2\tau_*^{(0)}}{\sigma}}\left(1-\frac{\tau}{\tau_*^{(0)}}\right)^{\frac{1}{2}} \text{ as } \tau \to \tau_*^{(0)}. \tag{2.49a}$$

One can show in a similar way that near the singularity (2.46b) the function $u = u_\sigma(\tau)$ behaves like

$$u \sim -\ln\sigma + i\pi - i\sqrt{\frac{2\tau_*^{(1)}}{\sigma}}\left(1-\frac{\tau}{\tau_*^{(1)}}\right)^{\frac{1}{2}} \text{ as } \tau \to \tau_*^{(1)}. \tag{2.49b}$$

Thus the points (2.46) are singular and we immediately obtain expression (2.41); the inequality (2.42) follows from (2.41) as $\tau = -\sigma(\ln\sigma - \ln\alpha)$ due to (2.2). The theorem is completely proved. □

Corollary 2.3.2. *For* $\alpha = 1$, *the series* (2.5) *is convergent for* $0 < \sigma < \sigma_0$ *and divergent for* $\sigma > \sigma_0$ *where* $\sigma_0 = 224.790951...$ *is the only root of the equation*

$$|-\sigma\ln\sigma| = \sqrt{(1+\sigma-\sigma\ln\sigma)^2 + \pi^2\sigma^2} . \tag{2.50}$$

Proof. Follows immediately from (2.42). □

Thus, in terms of the variable x, the series (2.5) for the Lambert W function ($\alpha = 1$) is convergent for $x > x_0 = e^{1/\sigma_0} = 1.004458...$, which confirms and specifies the result obtained in (Jeffrey et al., 1995). It is also worth to emphasize that the domain of convergence is described by $x > x_0 > 1$ rather than $x > 1$ though x_0 is very close to unit.

Remark 2.3.3. Substituting values (2.46) and (2.47) into (2.7) we find $W(x) = -1$ for both $k = 0$ and $k = 1$. Although it is well-known that this value of the Lambert W function corresponds to its branch point and asymptotics (2.49) can be obtained immediately from the results in (Corless et al., 1996, 1997), we derived these asymptotic formulae to demonstrate a method based on the Weierstrass's preparation theorem.

Remark 2.3.4. The results of Theorem 2.3.1 correspond to the properties of the Wright ω function (see subsection 2.4.2). In particular, due to (2.8) for fixed $\sigma > 0$, the singular points of function $u_\sigma(\tau)$ can be found through those of function $\omega(s)$, $s_* = -\xi \pm i\pi$ ($\xi \leq -1$), by transformation $\tau_* = 1 - \sigma \ln \sigma - \xi\sigma \mp i\pi\sigma$. Since $\Re\tau_*$ has the minimum at $\xi = -1$, the closest singular points are defined by (2.46), which corresponds to the results of the theorem.

Remark 2.3.5. The solution $\sigma = \sigma_0$ of the equation (2.50) is much more than unit and can be found approximately with a good precision. Specifically, taking square of the both sides of (2.50) and leaving the main terms we obtain $\sigma^2 - 2\sigma^2 \ln \sigma + \pi^2\sigma^2 - 2\sigma \ln \sigma \approx 0$. Searching for a solution of the approximate equation in the form $\sigma = \exp(\frac{1+\pi^2}{2})(1 + \delta)$, where the exponential factor is an exact solution of the approximate equation with neglected last term and a correction term δ is to be determined, we obtain an approximate value in deficit $\sigma_0 \approx \exp(\frac{1+\pi^2}{2}) - \frac{1+\pi^2}{2} = 223.8126969...$. Taking into consideration of the terms of higher powers in δ in a similar way, one can obtain a more accurate value.

Remark 2.3.6. The convergence condition (2.42) has a clear geometrical interpretation in (σ, τ)-plane. For example, for $\alpha = 1$, one can show that in accordance with the inequality (2.42), when $\sigma < \sigma_0$ the curve L described by $\tau = -\sigma \ln \sigma$ is located inside the region S bounded by curves $\tau = \pm\sqrt{(1+\sigma-\sigma\ln\sigma)^2+\pi^2\sigma^2}$, which expresses the condition of convergence of the series (2.5). However, at point $\sigma = \sigma_0$ the curve L leaves the region S through the lower boundary curve that can be described for large σ by the asymptotic expression

$$\tau(\sigma) = -\sqrt{(1+\sigma-\sigma\ln\sigma)^2+\pi^2\sigma^2} = -\sigma\ln\sigma+\sigma-\frac{1+\pi^2}{2}\frac{\sigma}{\ln\sigma}+1+O\left(\frac{1}{\ln\sigma}\right).$$

It follows that afterwards the curve L remains below the lower boundary of S, which corresponds to the divergence of the series (2.5) for $\sigma > \sigma_0$.

Now we consider case $\sigma < 0$, which should be done carefully as by Implicit Function Theorem it should be $\partial F_\sigma(0,0)/\partial u \neq 0$ due to the initial condition $u_\sigma(0) = 0$ and therefore the value $\sigma = -1$ should be excluded. It follows from (2.45) that when $\sigma < 0$ and $\sigma \neq -1$, i.e. $\sigma = |\sigma|\, e^{i\pi}$ and $|\sigma| \neq 1$ there is only one the nearest to the origin singularity given by (2.46b)

$$\tau_*^{(1)} = 1+\sigma-\sigma\ln|\sigma| \tag{2.51}$$

that lies on the positive real axis. Correspondingly the radius of convergence instead of (2.41) is the modulus of the right-hand side of (2.51).

Finally, when $\sigma = 0$ the series following from (2.3)

$$u = -\ln(1-\tau) = \sum_{m=1}^{\infty}\frac{\tau^m}{m} \tag{2.52}$$

is convergent for $|\tau| < 1$.

Note. When $\sigma = -1$, $\tau_*^{(1)} = 0$ by (2.51), i.e. the series diverges everywhere. We also note that in all cases considered above the condition of convergence of the series (2.5) is described in an unique manner, particularly, the radius of convergence is given by (2.41).

2.3.2 Comparison with series (2.4)

Let us compare the domain of convergence for the series (2.4) and (2.5). Both can be represented in the form

$$u = \sum_{m=1}^{\infty} c_m(\sigma)\tau^m \tag{2.53}$$

(see (2.34) for the series (2.4)). However, by Corollary 2.3.2 and Corollary 2.2.6 the series (2.5) has a much wider domain of convergence than the series (2.4) (not only in the real case but also in the complex case, see Figure 2.2 below). To undestand this phenomenon we note that the domains D_4 and D_5 of definition of the function $c_m(\sigma)$ in the series (2.4) and (2.5) respectively are different. Specifically, the domain D_4 contains point $\sigma = -1$ where the conditions of the Implicit Function Theorem are violate, which results in restriction $|\sigma| < 1$ that appears as a necessary condition for convergence of the series (2.4). However, in the series (2.5) $c_m(\sigma) = c_m(\zeta(\sigma))$ where $\zeta(\sigma) = 1/(1+\sigma)$, i.e. the domain D_5 does not contain point $\sigma = -1$. Therefore the mentioned restriction does not appear and the domain of convergence is extending. This correpsonds to the fact that the function $\zeta = \zeta(\sigma)$ maps the interior of the unit circle $|\sigma| = 1$ into an unbounded domain which is the right half-plane $\mathbb{R}\zeta > 1/2$. Since the series (2.4) and (2.5) have common values in the domain where they are both convergent, the series (2.5) is the analytic continuation of the series (2.4).

In terms of variable ζ the series (2.5) becomes (Jeffrey et al., 1995)

$$u = \sum_{m=1}^{\infty} \frac{\tau^m}{m!} \sum_{p=0}^{m-1} \left\{ \begin{matrix} p+m-1 \\ p \end{matrix} \right\}_{\geq 2} (-1)^{p+m-1} \zeta^{p+m} \tag{2.54}$$

and can be regarded as a result of applying the Euler's transformation for improvement of convergence of series (Hardy, 1949). Indeed, the standard Euler's transformation associated with changing variable to extend a domain of convergence of the series (2.4) is $\rho = \sigma/(1+\sigma)$ (Morse & Feshbach, 1953). Since in terms of a new varibale the fundamental relation (2.3) is written as

$$1-\rho = \frac{u}{e^{-u}+u+\tau-1},$$

it would be natural to introduce variable $\zeta = 1-\rho = 1/(1+\sigma)$ rather than ρ. The series (2.5),(2.54) were first found in (Jeffrey et al., 1995).

One can also show that a representation of the Lambert W function through the function $u_\tau(\sigma) = \sum_{m=1}^{\infty} c_m(\tau)\sigma^m$, where τ plays a role of parameter, can not extend the domain of convergence established for series (2.5). Indeed, in this case equation (2.43) changes to $F_\tau(\sigma, u) = 0$ where $F_\tau(\sigma, u)$ is still defined by the right-hand side of (2.44) but with initial condition $u_\tau(0) = -\ln(1-\tau)$. By the Implicit Function Theorem it should be $\partial F_\tau(0, -\ln(1-\tau))/\partial u \neq 0$, which gives $\tau \neq 1$, i.e. $|\tau| < 1$, and substituting $\tau = -\sigma \ln \sigma$ yields $0 < \sigma < 1/\omega_0$ as a necessary condition for convergence (cf. $0 < \sigma < \sigma_0$ in Corollary 2.3.2).

Thus among the series with the considered structures the series (2.5) has as wide as possible domain of convergence.

2.3.3 Asymptotics of expansion coefficients

Once the behavior of function $u = u_\sigma(\tau)$ near the nearest to the origin singularities has been established one can find an asymptotic formula for the expansion coefficients of the series (2.5) using the Darboux's theorem about expansions at algebraic singularities (Comtet, 1970; Bender, 1974). The similar approach, based on the Weierstrass's preparation theorem and the Darboux's theorem, was applied to asymptotic enumeration of trees in (Savický & Woods, 1998).

According to the Darboux's theorem (see Appendix A.2) and found estimates (2.49) for $\sigma > 0$ the expansion coefficients in the series (2.53) have an asymptotic formula for large m as

$$c_m(\sigma) = \frac{1}{m}\left(i\sqrt{\frac{2\tau_*^{(0)}}{\sigma}}\frac{1}{\Gamma\left(-\frac{1}{2}\right)\left(\tau_*^{(0)}\right)^m m^{\frac{1}{2}}} - i\sqrt{\frac{2\tau_*^{(1)}}{\sigma}}\frac{1}{\Gamma\left(-\frac{1}{2}\right)\left(\tau_*^{(1)}\right)^m m^{\frac{1}{2}}}\right)$$
$$+ o\left(\frac{1}{m^{\frac{3}{2}}\left|\tau_*^{(1)}\right|^m}\right)$$

or

$$c_m(\sigma) \sim \frac{i}{\sqrt{2\pi\sigma}m^{\frac{3}{2}}}\left(\frac{1}{\left(\tau_*^{(1)}\right)^{m-\frac{1}{2}}} - \frac{1}{\left(\tau_*^{(0)}\right)^{m-\frac{1}{2}}}\right), \qquad (2.55)$$

as $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$. Setting $\tau_*^{(1)} = \left|\tau_*^{(1)}\right| e^{i\theta_1}$ we find

$$c_m(\sigma) \sim \sqrt{\frac{2}{\pi\sigma}}\frac{\sin\left(m-\frac{1}{2}\right)\theta_1}{\tau_*^{m-\frac{1}{2}} m^{\frac{3}{2}}}, \qquad \text{as} \quad m \to \infty \qquad (2.56)$$

where $\tau_* = \tau_*(\sigma)$ is defined by (2.41) and $\theta_1 = \arg(1 + \sigma - \sigma\ln\sigma + i\pi\sigma)$. Specifically, for $\sigma > 0$

$$\theta_1 = \begin{cases} \arctan\dfrac{\pi}{1-\ln\sigma+1/\sigma}, & \text{if } 0 < \sigma < \dfrac{1}{W(1/e)}, \\ \pi + \arctan\dfrac{\pi}{1-\ln\sigma+1/\sigma}, & \text{if } \sigma > \dfrac{1}{W(1/e)}. \end{cases}$$

It follows from (2.56) that for large m the expansion coefficients in the series (2.5) disclose their oscillatory behavior due to *sin* function though the amplitude decays as $\tau_*(\sigma) > 1$ for any $\sigma > 0$. Since the series (2.5) can be interpreted as a result of applying the Euler's transformation to the series (2.4) (see (2.54)), we note that some cases of oscillatory coefficients resulting from the Euler's transformation are studied in (Hunter, 1987).

In order to find an asymptotic formula in case when $\sigma < 0$, suffice it to take in (2.55) only the first term with (2.51)

$$c_m(\sigma) \sim -\frac{1}{\sqrt{2\pi\,|\sigma|}m^{\frac{3}{2}}\,(1-|\sigma|+|\sigma|\ln|\sigma|)^{m-\frac{1}{2}}} \quad \text{as} \quad m \to \infty \tag{2.57}$$

Finally, for case $\sigma = 0$, it follows from (2.52) that for any $m \in \mathbb{N}$

$$c_m(0) = \frac{1}{m}. \tag{2.58}$$

2.3.4 Convergence in complex case

Theorem 2.3.1 is extended to the complex case.

Theorem 2.3.7. *For complex σ, the radius of convergence of the series* (2.5) *is*

$$\tau_*(\sigma) = |1+\sigma-\sigma\ln\sigma - i\pi\sigma| \;\; when\ \Im\sigma < 0\,, \tag{2.59a}$$

$$\tau_*(\sigma) = |1+\sigma-\sigma\ln\sigma + i\pi\sigma| \;\; when\ \Im\sigma > 0\,. \tag{2.59b}$$

In the complex z-plane this is equiavalent to that the series (2.5) *is convergent everywhere in the exterior of the boundary line defined by equation*

$$|-\sigma\ln\sigma| = |1+\sigma-\sigma\ln\sigma \pm i\pi\sigma|\ , \tag{2.60}$$

where $\sigma = 1/\ln z$ and sign minus or plus is taken respectively in the upper or lower half-plane.

Proof. Repeating the proof of Theorem 2.3.1 under assumption $\sigma \in \mathbb{C}$ we obtain the same equations (2.16) and (2.45) for singular points $u_*^{(k)}$ and $\tau_*^{(k)}$ respectively, where $k \in \mathbb{Z}$. However, many of the singular points do not correspond to the principal branch of W function and relate to the other branches. We are going to find acceptable values of k for which singular points relate to the principal branch of W. Formally these values of k are not obvious because the logarithms (and their branches) in the equations (2.6) and (2.16) are different, more precisely, they are taken of different variables, z in the former and $\sigma = 1/\ln z$ in the latter.

To find acceptable values of k we substitute into equation (2.7) a relation $\tau = -\sigma \ln \sigma$ following from (2.2) to obtain

$$u = W(e^s) - 1/\sigma - \ln \sigma \,, \tag{2.61}$$

where $s = s(\sigma, \tau)$ is defined by (2.8). Let us consider values of u in the ϵ-vicinity of the point $u_*^{(k)}$. Comparing between (2.61) and (2.16) gives

$$i\pi(2k-1) + \epsilon e^{i\varphi} = W(e^s) - 1/\sigma \,,$$

where $-\pi < \varphi \leq \pi$. Setting $z = |z|\, e^{i\theta}$ $(-\pi < \theta < \pi)$ in $\sigma = 1/\ln z$ and separating the imaginary part in the last equation we obtain

$$\pi(2k-1) + \epsilon \sin \varphi = \Im W - \theta \,. \tag{2.62}$$

Since for the principal branch $-\pi < \Im W < \pi$, we find $-1/2 - \epsilon \sin \varphi/(2\pi) < k < 3/2 - \epsilon \sin \varphi/(2\pi)$, i.e. acceptable values are $k = 0$ and $k = 1$.

Now we note that both points $\tau_*^{(0)}$ and $\tau_*^{(1)}$ are singular, particularly, they correspond to a square-root branch point of function $u = u_\sigma(\tau)$ for the same reason as in the real case (see proof of Theorem 2.3.1). Taking into account this result we consider equation (2.62) for $\epsilon = 0$ in two cases $k = 0$ and $k = 1$. When $k = 0$, we have $\Im W = \theta - \pi$. Since $-\pi < \Im W < \pi$, only positive θ satisfy this equation, i.e. $0 < \theta < \pi$. Similarly when $k = 1$ we have $\Im W = \theta + \pi$ which holds for $-\pi < \theta < 0$. Thus we conclude that the curve $|-\sigma \ln \sigma| = \left|\tau_*^{(0)}(\sigma)\right|$ is located in the upper z-half-plane and the curve $|-\sigma \ln \sigma| = \left|\tau_*^{(1)}(\sigma)\right|$ is located in the lower z-half-plane, these curves being symmetric with respect to the real axis. Hence, the equation (2.60) describes the boundary of domain of convergence of the series (2.5) in the complex case. In addition, since $\sigma = (\ln|z| - i\theta)/\left|\ln z\right|^2$, θ and $\Im\sigma$ are of opposite signs and the equations (2.59) follow. The theorem is completely proved. □

Remark 2.3.8. We note that the case $|\sigma| < 1$ reveals a connection between the series (2.5) and (2.4). In particular, the case permits to expand $1/(1+\sigma)$ in powers of σ in the former that after some rearrangments can be reduced to the latter (Jeffrey et al., 1995). In accordance with Theorem 2.2.10 the series (2.4) is convergent in a subdomain V in the complex σ-plane which is defined by (2.36) (written in terms of σ) and contained in the unit disc $U = \{\sigma \in \mathbb{C} \mid |\sigma| < 1\}$ (cf. Remark 2.2.12); more precisely the boundaries of V and U have one common point $\sigma = 1$ (where both series are convergent). The series (2.5) is also convergent in V but has a wider domain of convergence being convergent in $U \cap H$ where the domain H bounded by (2.60).

The curve defined by equation (2.60) in the complex z-plane is depicted in Figure 2.2 by solid line in the upper half-plane only (corresponding to the negative sign) as it is symmetric with respect to the real axis. The exterior of this boundary line can be regarded as the domain of analytic continuation of the series (2.5) from the part of the real axis $x > x_0$ (see Corollary 2.3.2) to the complex z-plane. For comparison, in the same figure it is shown (by dash line) the boundary line of the domain of convergence of the series (2.4) defined by equation (2.36). Since the domain of convergence of the series (2.5) is located in the exterior of the curve depicted by solid line, it is wider than one for the series (2.4).

In the end of this subsection we give asymptotics for the expansion coefficients of the series (2.5) as $m \to \infty$ when $\Im\sigma \neq 0$. It follows from the proof of Theorem

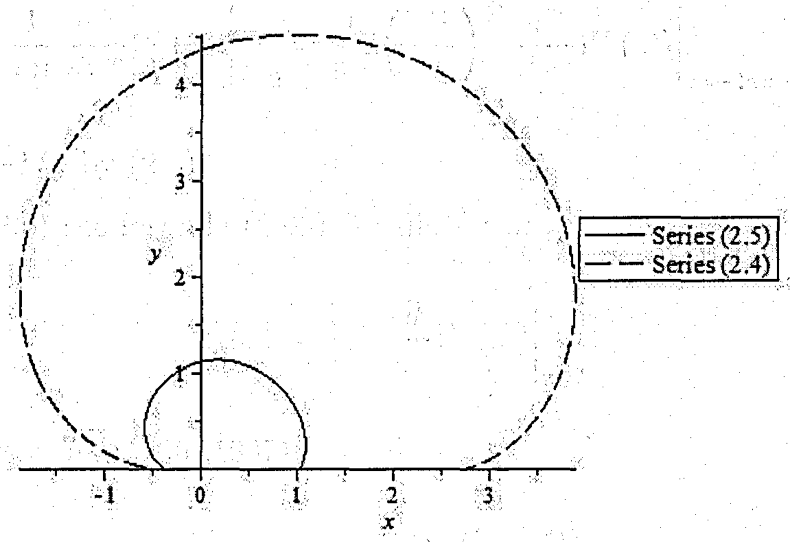

Figure 2.2: The domain of convergence of the series (2.5) in the complex z-plane.

2.3.7 that in this case there is only one singularity $\tau = \tau_*^{(0)}$ when $\Im\sigma < 0$ and $\tau = \tau_*^{(1)}$ when $\Im\sigma > 0$. Therefore, one can use formula (2.55) keeping only one corresponding term (unlike case of real σ when there occur two singularities and both terms constitute the asymptotic formula (2.56)). Thus, taking (2.46) we find

$$c_m(\sigma) \sim \frac{1}{\sqrt{2\pi\sigma}m^{3/2}} \frac{\pm i}{(1+\sigma-\sigma\ln\sigma \pm i\pi\sigma)^{m-1/2}} \quad \text{as} \quad m \to \infty,$$

where sign "+" ("−") is taken in case of positive (negative) $\Im\sigma$.

2.3.5 Representation in terms of Eulerian numbers

The expansion coefficients of the series (2.5) can be expressed in terms of the second-order Eulerian numbers (see Appendix B). To show that we combine equations (2.7) and (2.53), then the coefficients $c_m(\sigma)$ in the right-hand side of

(2.53) are

$$c_m(\sigma) = \frac{1}{m!}\,\frac{d^m}{d\tau^m}W\left(e^s\right)\bigg|_{\tau=0} = \frac{1}{m!}\left(-\frac{1}{\sigma}\right)^m \frac{d^m}{ds^m}W\left(e^s\right)\bigg|_{s=-\ln\sigma+1/\sigma} \tag{2.63}$$

as $\partial s/\partial\tau = -1/\sigma$ by (2.8).

Because of (2.7) the formula (2.63) is valid for $m \geq 2$, for $m = 1$ we have

$$c_1(\sigma) = \frac{1}{\sigma} + \frac{d}{d\tau}W\left(e^s\right)\bigg|_{\tau=0}. \tag{2.64}$$

Since (Corless et al., 1996, 1997)

$$\frac{d^m}{ds^m}W\left(e^s\right) = \frac{q_m\left(W\left(e^s\right)\right)}{\left(1+W\left(e^s\right)\right)^{2m-1}},$$

where the polynomials $q_n(r)$ can be expressed in terms of the second-order Eulerian numbers (Graham et al., 1989; Corless et al., 1997)

$$q_m(r) = \sum_{k=0}^{m-1}\left\langle\!\!\left\langle {m-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k r^{k+1},$$

and

$$W\left(e^s\right)\big|_{s=-\ln\sigma+1/\sigma} = W\left(\frac{1}{\sigma}e^{1/\sigma}\right) = \frac{1}{\sigma},$$

we finally obtain

$$c_1(\sigma) = \frac{1}{1+\sigma},\ c_m(\sigma) = \frac{(-1)^m\sigma^{m-1}}{m!(1+\sigma)^{2m-1}}\sum_{k=0}^{m-1}\left\langle\!\!\left\langle {m-1 \atop k} \right\rangle\!\!\right\rangle \frac{(-1)^k}{\sigma^{k+1}},\ m \geq 2. \tag{2.65}$$

Substituting (2.65) into the right-hand side of (2.53) results in a desireable formula

$$u = \frac{\tau}{1+\sigma} + \sum_{m=2}^{\infty}\frac{\tau^m}{m!(1+\sigma)^{2m-1}}\sum_{k=0}^{m-1}\left\langle\!\!\left\langle {m-1 \atop k} \right\rangle\!\!\right\rangle (-1)^{m-k}\sigma^{m-k-2}. \tag{2.66}$$

By introducing the variable $\zeta = 1/(1+\sigma)$ the series (2.66) can also be written as

$$u = \tau\zeta + \sum_{m=2}^{\infty} \frac{\tau^m}{m!} \sum_{k=0}^{m-1} \left\langle\!\!\left\langle {m-1 \atop k} \right\rangle\!\!\right\rangle (-1)^{m+k} \zeta^{m+k+1} (1-\zeta)^{m-k-2}. \tag{2.67}$$

We note that the expansion (2.67) does not contain terms of the second order in ζ.

The series (2.5),(2.54),(2.66), and (2.67) have the same properties including the domain of convergence and the asymtotic estimates for the expansion coefficients studied in Section 2.3.3. This fact leads to some combinatorial consequences considered in Section 2.5.

2.4 Series (2.11)

2.4.1 Different representations

The series development (2.11) was obtained in (Corless et al., 1997; Corless & Jeffrey, 2002) and represents an expansion of $W(x)$ in powers of $\sigma^{-1} = \ln x$

$$W(x) = \omega_0 + \sum_{n=1}^{\infty} a_n (\ln x)^n \tag{2.68}$$

or

$$W(e^t) = \omega_0 + \sum_{n=1}^{\infty} a_n t^n, \tag{2.69}$$

where $t = \ln x$ and (Corless & Jeffrey, 2002)

$$a_n = \frac{1}{n!(1+\omega_0)^{2n-1}} \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k \omega_0^{k+1}. \tag{2.70}$$

The formula (2.70) expresses the expansion coefficients a_n in terms of the second-order Eulerian numbers. We now show that these coefficients can also be represented through the unsigned associated Stirling numbers of the first kind $d(m,k)$ given by (Comtet, 1970)

$$[\ln(1+v)-v]^k = k!\sum_{m=2k}^{\infty}(-1)^{m+k}\,d(m,k)\frac{v^m}{m!} \tag{2.71}$$

and the 2-associated Stirling numbers of the second kind used in the series (2.5) (see also Appendix B).

Both representations can be obtained on the basis of a relation (Jeffrey, Hare, & Corless, 1996)

$$W(e^t)+\ln W(e^t)=t \tag{2.72}$$

and the Lagrange Inversion Theorem (Carathéodory, 1958). To apply this theorem it is convenient to introduce a function that is zero at $t=0$. We consider function

$$v=v(t)=W(e^t)/\omega_0-1 \tag{2.73}$$

and write (2.72) as

$$t=\omega_0\,v+\ln(1+v).$$

Then by the Lagrange Inversion Theorem we obtain

$$v=\sum_{n=1}^{\infty}\frac{t^n}{n}[v^{n-1}]\left(\omega_0+\frac{\ln(1+v)}{v}\right)^{-n} \tag{2.74}$$

where the operator $[v^p]$ represents the coefficient of v^p in a series expansion in v. Comparing (2.73),(2.69) and (2.74) leads to a formula for the coefficients a_n, which after applying the binomial theorem becomes

$$a_n=\frac{\omega_0}{n(1+\omega_0)^n}[v^{n-1}]\sum_{k=0}^{\infty}(-1)^k\binom{n-1+k}{n-1}\frac{[\ln(1+v)-v]^k}{v^k(1+\omega_0)^k}$$

or by (B.5)

$$a_n = \frac{\omega_0}{n!} \sum_{k=0}^{n-1} \frac{(-1)^{n+k-1}\, d(n+k-1,k)}{(1+\omega_0)^{n+k}}. \tag{2.75}$$

If instead of function (2.73) to take

$$h = h(t) = W(e^t) - \omega_0 - t \tag{2.76}$$

and apply the Lagrange Inversion Theorem to invert a relation

$$t = \omega_0(e^{-h} - 1) - h$$

coming from (2.72), then we find in a similar way

$$a_n = \frac{1}{n!} \sum_{k=0}^{n-1} \left\{ \begin{matrix} n+k-1 \\ k \end{matrix} \right\}_{\geq 2} \frac{(-1)^{k+1}\omega_0^k}{(1+\omega_0)^{n+k}}. \tag{2.77}$$

Finally, one more representation for the coefficients a_n can be found in the following way. Let us consider a function

$$\psi = \psi(t) = W(e^t) - t \tag{2.78}$$

which is a simplified version of functions (2.73) and (2.76): now one does not need to provide the zero function value at $t = 0$ and here $\psi(0) = \omega_0$. Then it follows from (2.72) that

$$t = e^{-\psi} - \psi. \tag{2.79}$$

This equation can also be obtained from the fundamental relation (2.3) by transformation $u = \psi + \ln t, \sigma = 1/t, \tau = \ln t/t$ which follow from (2.2).

Differentiating (2.79) in t and excluding the term $e^{-\psi}$ from the result again using (2.79) result in an initial value problem for ordinary differential equation

$$\frac{d\psi}{dt} = -\frac{1}{1+t+\psi}.$$

Searching a solution in the form of series

$$\psi(t) = \omega_0 + \sum_{n=1}^{\infty} c_n t^n \tag{2.80}$$

by substituting it into the differential equation and equating coefficients of the same power in t one can finally find

$$c_1 = -\frac{1}{1+\omega_0}, \quad c_n = -\frac{1}{n(1+\omega_0)}\left((n-1)c_{n-1} + \sum_{k=1}^{n-1} k c_k c_{n-k}\right), n = 2, 3, 4, \ldots \tag{2.81a}$$

At length combining (2.80),(2.78) and (2.69) gives

$$a_1 = 1 + c_1, \quad a_n = c_n \text{ for } n \geq 2. \tag{2.81b}$$

In practice computing the expansion coefficients in (2.68) using recurrence (2.81a) is faster and takes less digits to obtain a required level of accuracy than using either of (2.70), (2.75) or (2.77) which, however, being different representations of the same expansion coefficients, lead to some combinatorial relations considered in Section 2.5.

2.4.2 Convergence properties

The expansion (2.11) in fact represents a series of the Wright ω function (Corless et al., 1997; Corless & Jeffrey, 2002) $\omega(z) = W_{\mathcal{K}(z)}(e^z)$, where $\mathcal{K}(z) = \lceil(\Im z - \pi)/(2\pi)\rceil$ is the unwinding number of z. The Wright ω function was introduced by Corless and Jeffrey (Corless & Jeffrey, 2002) and studied in (Wright, 1959; Corless & Jeffrey, 2002). When $z \neq \xi \pm i\pi$ for $\xi \leq -1$, $\omega = \omega(z)$ satisfies equation $f(z, \omega) = 0$ where $f(z, \omega) = \omega + \ln \omega - z$ (cf. (2.72)). Applying the same approach as in Section 2.3.1 to this equation one can obtain the same results as in (Wright, 1959; Corless et al., 1997; Corless & Jeffrey, 2002). Specifically, the nearest to the origin singularities are (Corless et al., 1997)

$$z_1 = -1 - i\pi \text{ and } z_2 = -1 + i\pi. \tag{2.82}$$

Note that they are connected with the singularities (2.46) of function $u = u_\sigma(\tau)$, defined by (2.3) or (2.7), through function (2.8)

$$z_1 = s(\sigma, \tau_*^{(1)}) \text{ and } z_2 = s(\sigma, \tau_*^{(0)}).$$

Thus the radius of convergence is $\sqrt{1+\pi^2}$ (Corless et al., 1997) and the domain of convergence is given by

$$|\sigma| > \frac{1}{\sqrt{1+\pi^2}}. \tag{2.83}$$

The estimation of ω in the vicinity of the singularities (2.82) is (Wright, 1959; Corless & Jeffrey, 2002)

$$\omega \sim -1 - \operatorname{sgn}(\Im z_k)\sqrt{2z_k}\left(1 - \frac{z}{z_k}\right)^{\frac{1}{2}} \quad \text{as } z \to z_k, \qquad (k = 1, 2)$$

As in Section 2.3.3, using the Darboux's theorem one can find the asymptotic expression for the expansion coefficients in (2.68)

$$a_n \sim \sqrt{\frac{2}{\pi}} \frac{(-1)^n \sin\left(\frac{2n-1}{2} \arctan \pi\right)}{n^{\frac{3}{2}}(1+\pi^2)^{\frac{2n-1}{4}}} \text{ as } n \to \infty. \tag{2.84}$$

Thus, as in case of the series (2.5) for positive σ (see (2.56)), the expansion coefficients in the series (2.11) disclose decaying oscillations in their behavior for large n.

In real case inequality (2.83) read as $\exp(-\sqrt{1+\pi^2}) < x < \exp(\sqrt{1+\pi^2})$. Thus from the point of view of the domain of convergence the series (2.11) takes an intermediate place between the expansion of $W(x)$ at the origin (Corless et al., 1996) $W(x) = \sum_{n=1}^{\infty}(-n)^{n-1}x^n/n!$, which is valid for $-e^{-1} < x < e^{-1}$, and the series (2.5) which is valid for $x > x_0 = 1.004458...$ (see Corollary 2.3.2). These three expansions put together cover the entire region of definition of $W(x)$.

2.5 Combinatorial consequences

In this section we collect some combinatorial consequences resulting from the above obtained expressions for the expansion coefficients.

Equating the right-hand sides of (2.54) and (2.67) one can find

$$\sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle (1+\lambda)^{n-k-1}\lambda^k = \sum_{k=1}^{n} \left\{ {n+k \atop k} \right\}_{\geq 2} \lambda^{k-1}, \tag{2.85a}$$

where summation in the right-hand side starts with one as $\left\{ {n \atop 0} \right\}_{\geq 2} = 0$ (Graham et al., 1989). Setting $\mu = \lambda/(1-\lambda)$ in (2.85a) we also find

$$\sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \mu^k = \sum_{k=1}^{n} \left\{ {n+k \atop k} \right\}_{\geq 2} \mu^{k-1}(1-\mu)^{m-k}. \tag{2.85b}$$

The identities (2.85) were obtained by L. M. Smiley (2000) in a different way, where notation $\{\{\}\}$ was used instead of $\{\}_{\geq 2}$, and referred to as the Carlitz-Riordan identities (Smiley, 2001). Applying the binomial theorem to (2.85) leads to a pair of identities expressing the 2-associated Stirling numbers of the second kind through the second-order Eulerian numbers and inversely (Smiley, 2000)

$$\left\{ {n+q \atop q} \right\}_{\geq 2} = \sum_{p=0}^{n} \binom{n-p-1}{q-p-1} \left\langle\!\!\left\langle {n \atop p} \right\rangle\!\!\right\rangle$$

$$\left\langle\!\!\left\langle {n \atop q} \right\rangle\!\!\right\rangle = \sum_{p=0}^{n} (-1)^{q-p} \binom{n-p-1}{q-p} \left\{ {n+p+1 \atop p+1} \right\}_{\geq 2}$$

Some estimates can also be obtained by comparing the found asymptotic formulas (2.56) and (2.57) with the explicit expressions for the expansion coefficients in (2.5). For example, taking estimate (2.57) and the expansion coefficients in (2.5) at $\sigma = -2$ we obtain

$$\sum_{p=1}^{m-1} \left\{ {p+m-1 \atop p} \right\}_{\geq 2} \sim \frac{(m-1)!}{2\sqrt{\pi m}\,(2\ln 2 - 1)^{m-\frac{1}{2}}} \quad \text{as} \quad m \to \infty \tag{2.86}$$

where the term with $p = 0$ is skipped (cf. (2.85a). This result is consistent with the formula given in (Comtet, 1970, Ex.10(7), p.224).

Another consequence is obtained by taking the expansion coefficients in (2.5) at $\sigma = 0$ together with (2.58)

$$\sum_{p=0}^{m-1} (-1)^{p+m-1} \left\{ {p+m-1 \atop p} \right\}_{\geq 2} = (m-1)! \tag{2.87}$$

Further, comparing (2.70), (2.75) and (2.77) between one another we come to the following three identities

$$\frac{1}{(1+\omega_0)^{n-1}} \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k \omega_0^k = \sum_{k=0}^{n-1} \frac{(-1)^{n+k-1}\, d(n+k-1, k)}{(1+\omega_0)^k} \tag{2.88a}$$

$$\frac{1}{(1+\omega_0)^{n-1}} \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^{k+1} \omega_0^{k+1} = \sum_{k=0}^{n-1} \left\{ {n+k-1 \atop k} \right\}_{\geq 2} \frac{(-1)^k \omega_0^k}{(1+\omega_0)^k} \tag{2.88b}$$

$$\omega_0 \sum_{k=0}^{n-1} \frac{(-1)^{n+k}\, d(n+k-1,k)}{(1+\omega_0)^k} = \sum_{k=0}^{n-1} \left\{ {n+k-1 \atop k} \right\}_{\geq 2} \frac{(-1)^k \omega_0^k}{(1+\omega_0)^k} \tag{2.88c}$$

Finally, combining either of (2.70), (2.75) or (2.77) with (2.84) gives an asymptotic expression for the sum involved there.

Thus, in studying the expansion series for the Lambert W function, we, on the way, derived the Carlitz-Riordan identities (2.85) as well as found a formula for an alternating sum of 2-associated Stirling numbers of the second kind (2.87) and confirmed the asymptotic formula (2.86) for summation of the same numbers without the alternating factor. We also found formulas (2.88) where the Omega constant ω_0 plays a role of a magic number which connects sums involving the second-order Eulerian numbers, the associated Stirling numbers of the first kind and the 2-associated Stirling numbers of the second kind.

2.6 Concluding remarks

We ascertained the domain of convergence of the series (2.4) and (2.5) in real and complex cases and found that the series (2.5) has a much wider domain of convergence than that of the series (2.4) in both cases and provided an analysis of this fact in real case. We found asymptotic expressions for the expansion coefficients and obtained a representation of the series (2.5) in terms of the second-order Eulerian numbers.

We also considered the well-known expansion of $W(x)$ in powers of $\ln x$ and gave an asymptotic estimate for the expansion coefficients. We found three more forms for a representation of the expansion coefficients of the series in terms of the associated Stirling numbers of the first kind (2.75), the 2-associated Stirling subset numbers (2.77) and iterative formulas (2.81). Finally we presented some combinatorial consequences, including the Carlitz-Riordan identities, which result from the found different forms of the expansion coefficients of the above series.

References

Abramowitz, M., & Stegan, I. A. (1970). Handbook of mathematical functions with formulas, graphs amd mathematical tables. In *Applied mathematics 55, part 24.1.3 (III)* (p. 824). Mineola, NY: U.S. Department of Commerce, National Bureau of Standards. (9th printing)

Adachi, K. (2007). *Several Complex Variables and Integral Formulas.* Singapore: World Scientific.

Antimirov, M. Y., Kolyshkin, A. A., & Vaillancourt, R. (1998). *Complex Variables.* San Diego: Academic Press.

Bender, E. A. (1974). Asymptotic methods in enumeration. *SIAM Review*, *16*(4), 485–515.

Bruijn, N. G. de. (1961). *Asymptotic Methods in Analysis.* Amsterdam: North-Holland.

Carathéodory, C. (1958). *Theory of Functions of a Complex Variable, vol. 1.* New York: Chelsea.

Comtet, L. (1970). Inversion de $y^{\alpha}e^{y}$ et $y\log^{\alpha} y$ au moyen des nbombres de Stirling. *C. R. Acad. Sci., Paris, Sér. A-B*, *270*, A1085–A1088.

Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., & Knuth, D. E. (1996). On the Lambert W function. *Adv. Comp. Math.*, *5*(4), 329–359.

Corless, R. M., & Jeffrey, D. J. (2002). The Wright w function. In J. Calmet, B. Benhamou, O. Caprotti, L. Henocque, & V. Sorge (Eds.), *AISC-Calculemus 2002, LNAI 2385* (pp. 76–89). Springer-Verlag.

Corless, R. M., Jeffrey, D. J., & Knuth, D. E. (1997). A sequence of series for the

Lambert W function. In *Issac '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (pp. 197–204). New York, NY, USA: ACM press.

Graham, R. L., Knuth, D. E., & Patashnik, O. (1989). *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley.

Hardy, G. H. (1949). *Divergent series.* London: Oxford University Press.

Hunter, C. (1987). Oscillations in the Coefficients of Power Series. *SIAM Journal on Applied Mathematics*, *47*(3), 483–497.

Hunter, C., & Guerrieri, B. (1980). Deducing the Properties of Singularities of Functions From Their Taylor Series Coefficients. *SIAM J. Appl. Math.*, *39*(2), 248–263. Available from `http://www.jstor.org/stable/2101048`

Janssen, A. J. E. M., & Leeuwaarden, J. S. H. van. (November, 2007). *Some remarks on $\phi_\alpha(x)$.* Private communication.

Jeffrey, D. J., Corless, R. M., Hare, D. E., & Knuth, D. E. (1995). On the inversion of $y^\alpha e^y$ in terms of associated Stirling numbers. *C. R. Acad. Sc.*, *320*, 1449–1452.

Jeffrey, D. J., Hare, D. E., & Corless, R. M. (1996). Unwinding the branches of the Lambert W function. *Mathematical Scientist*, *21*, 1–7.

Limaye, B. V., & Zeltser, M. (2009). On the Pringsheim convergence of double series. *Proceedings of the Estonian Academy of Sciences*, *58*(2), 108–121.

Markushevich, A. I. (1965). *Theory of functions of a complex variable, II.* Prentice-Hall.

Morse, P. M., & Feshbach, H. (1953). *Methods of theoretical physics I.* New York: McGraw-Hill Book Company, Inc.

Savický, P., & Woods, A. R. (1998). The number of Boolean functions computed by formulas of a given size. In *Proceedings of the 8th Int. Conf., Random Structures and Algorithms 13* (pp. 349–382). John Wiley & Sons, Inc.

Smiley, L. M. (2000). *Completion of a Rational Function Sequence of Carlitz.* arXiv:math/0006106, 8 p. Available from `http://www.math.uaa.alaska`

`.edu/$\sim$smiley/cscarx.pdf`

Smiley, L. M. (2001). *Carlitz-Riordan Identities.* Available from `http://www.math.uaa.alaska.edu/$\sim$smiley/stir2eul2.html` (Wed Nov 21 03:23:23 AKST)

Titchmarsh, E. C. (1939). *The Theory of Functions.* Oxford Unversity Press.

Wright, E. M. (1959). Solution of the equation ze^z=a. *Bull. Amer. Math. Soc.*, *65*, 89–93.

Chapter 3

One-parameter asymptotic series of W

"It is the mark of an educated mind to rest satisfied with the degree of precision which the nature of the subject admits and not to seek exactness where only an approximation is possible." – Aristotle

3.1 Introduction

There are several series expansions for the principal branch W_0; one of them is a Taylor series expansion around $z = 0$ and the others are asymptotic expansions for large z although these expansions are also valid for non-principal branches around $z = 0$. One practical application of the series is to provide initial estimates for the numerical evaluation of W; these estimates can then be refined using iterative schemes. The series also have intrinsic interest. For example, the definition above of the branches W_k is based on partitioning the plane using the asymptotic series. Another interest is the fact that the asymptotic series are also convergent, and the nature of the convergence is one particular interest of this chapter.

The asymptotic series for W is defined by equation (2.1) taken with $\alpha = 1$

$$W(z) = \ln z - \ln\ln z + u \ , \tag{3.1}$$

where u has the series development in the form of either (2.4)

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{n} (-1)^{n-m} \left[\begin{matrix} n \\ n-m+1 \end{matrix} \right] \frac{\sigma^{n-m} \tau^m}{m!},$$

or (2.5)

$$u = \sum_{m=1}^{\infty} \frac{\tau^m}{m!} \sum_{k=0}^{m-1} \left\{ \begin{matrix} k+m-1 \\ k \end{matrix} \right\}_{\geq 2} (-1)^{k+m-1} \zeta^{k+m},$$

in terms of Stirling cycle numbers and the 2-associated Stirling subset numbers respectively. The used herein variables are $\sigma = 1/\ln z$, $\tau = \ln\ln z/\ln z$ and $\zeta = 1/(1+\sigma)$.

Two further expansions introduce the variables $L_\tau = \ln(1-\tau)$ and $\eta = \sigma/(1-\tau)$.

$$u = -L_\tau + \sum_{n=1}^{\infty} (-\eta)^n \sum_{m=1}^{n} (-1)^m \left[\begin{matrix} n \\ n-m+1 \end{matrix} \right] \frac{L_\tau^m}{m!}, \tag{3.2}$$

$$u = -L_\tau + \sum_{m=1}^{\infty} \frac{1}{m!} L_\tau^m \eta^m \sum_{k=0}^{m-1} \left\{ \begin{matrix} k+m-1 \\ k \end{matrix} \right\}_{\geq 2} \frac{(-1)^{k+m-1}}{(1+\eta)^{k+m}}. \tag{3.3}$$

All of these expansions are limited in their domain of applicability by the fact that σ and τ are each singular at $z = 1$, restricting their utility to $z > 1$. In addition to the domain of validity of the variables, there is the question of the domain of convergence of the series. For example, we show below that for $z \in \mathbb{R}$, series (2.4) is convergent only for $z > e$.

In the chapter we consider transformations of the above series and concentrate on their properties for $z \in \mathbb{R}$. The transformations contain a parameter p which can be varied, while retaining the basic series structure. Therefore we refer to them as *one-parameter family of invariant transformations.* The parameter effects on the domain of convergence of the series as well as their rate of convergence that is the accuracy for a given number of terms. Our goal is to improve such convergence properties of the series by varying p. Using theoretical and experimental methods with the help of the computer-algebra system MAPLE we will show that the parameter can be used to expand the domain of convergence of the series while the rate of convergence can increase or decrease with p.

3.2 Computer algebra tools

We shall be using a number of tools from MAPLE in the work below. The coefficients appearing in the expansions (2.4) and (2.5) can be computed from their generating functions as follows. The 2-associated Stirling subset numbers are defined by the generating function

$$(e^z - 1 - z)^m = m! \sum_{n \geq 0} \frac{z^n}{n!} \left\{ {n \atop m} \right\}_{\geq 2}.$$

Given numerical values for n and m, we expand the left-hand side symbolically up to the term of nth order and then extract the appropriate numerical coefficient. The next lines show an implementation of this procedure with examples in MAPLE.

```
> StirlingSubset2:=proc(n::integer, m::integer)
                   option remember;
                   local f,z;
                   f:=series( (exp(z)-1-z)^m , z , n+1);
                   if n<2*m then
                       0
                     else
                       coeff(f,z,n)*n!/m!;
                     end if;
                   end proc;
```

```
> StirlingSubset2(6,3),StirlingSubset2(9,4),StirlingSubset2(12,5);
                    15,  1260,  190575
```

It can be noted that a similar method to this is used in the standard MAPLE library for Stirling Cycle numbers, which are used in (2.4). In practice, it is more efficient to store all of the coefficients from any series expansion, but this level of detail is not shown here. Similar techniques can be used for the Eulerian numbers used below in (2.70).

Another important tool from MAPLE for this paper is computation to arbitrary precision. It is a standard topic in numerical analysis that summing series requires a close watch on the effects of working precision, otherwise one runs the risk of generating 'numerical monsters' which are completely artificial effects of the computation and do not reflect any actual mathematical properties (Essex, Davison, & Schulzky, 2000). In all of the calculations below, the MAPLE environment variable `Digits` was set and monitored to ensure that the results were reliable.

3.3 An invariant transformation

We reconsider the derivation of (2.4), trying the *ansatz*

$$W = \ln z - \ln(p + \ln z) + u \ . \tag{3.4}$$

Substituting into the defining equation $We^W = z$, we obtain

$$\left(\ln z - \ln(p + \ln z) + u \right) \frac{ze^u}{p + \ln z} = z$$

From this, it is clear that if we define

$$\sigma = \frac{1}{p + \ln z} \text{ and } \tau = \frac{p + \ln(p + \ln z)}{p + \ln z}, \tag{3.5}$$

then we recover the equation (2.3) originally given by de Bruijn for u and leading to the series (2.4). Thus the fundamental relation (2.3) is invariant with respect to p, with only the definitions of σ and τ being changed.

This remarkable property is due to the fact that the solution (2.1) of the original transcendental equation $y^\alpha e^y = x$ possesses a similarity property with respect to parameter $\alpha > 0$ in the following sense (Jeffrey, Corless, Hare, & Knuth, 1995)

$$\Phi_\alpha(x) = \alpha \Phi_1\left(\frac{x^{1/\alpha}}{\alpha}\right) = \alpha W\left(\frac{x^{1/\alpha}}{\alpha}\right). \tag{3.6}$$

Indeed, it follows from (2.1) and (3.6) that

$$W\left(\frac{x^{1/\alpha}}{\alpha}\right) = \frac{1-\tau}{\sigma} + u\,, \tag{3.7}$$

where σ and τ are defined by (2.2). The right-hand side of (3.7) does not include α explicitly. On the other hand, α is included in the left-hand side through a combination $z = x^{1/\alpha}/\alpha$. Therefore, the fundamental relation (2.3) will retain if we change variable $x = (\alpha z)^\alpha$. Substituting this formula into (2.2) and introducing parameter $p = \ln \alpha$ we obtain exactly equations (3.5).

Thus introducing the invariant parameter p generates an infinite one-parameter family of series formed by replacement of variables τ and σ in the original series with expressions (3.5). Similar series for W are associated with the invariance observed in (Jeffrey et al., 1995) and studied in (Corless, Jeffrey, & Knuth, 1997).

We now consider the properties of the transformations for $z \in \mathbb{R}$. We shall start with $p \in \mathbb{R}$ and later consider briefly one complex value of p. Both σ and

τ are singular at $z_s = e^{-p}$, with the special case $p = 0$ recovering the previous observations regarding the singularities at $z = 1$. We note that σ is monotonically decreasing on $z > z_s$. For τ, we have $\tau(z_0) = 0$ at $z_0 = \exp(z_s - p)$, with τ positive for larger z and negative for smaller. Also we note that τ has a maximum at $z = \exp(ez_s - p)$. In Figure 3.1-3.2, we plot σ and τ, defined by (3.5), for different values of p. We see that for all $z > z_s$, σ decreases with increasing p, but τ increases. In view of the form of the double sums above it is not obvious whether convergence is increased or decreased as a result of these opposed changes. This is what we wish to investigate here.

3.4 Domain of convergence

We wish to investigate first the domains of $z \in \mathbb{R}$ for which the series (2.4) and (2.5) converge, and how the domains vary with p. We begin with theoretical results. For $p = 0$ the domains of convergence are known from theorems 2.2.3 and 2.3.1. Specifically, the series (2.4) converges for $z > e$ and the series (2.5) converges for $z > z_0 = 1.004458...$ (see Corollary 5.1), under otherwise conditions the series are divergent. For arbitrary real p the following statement can be proved for the series (2.4).

Theorem 3.4.1. *The domain of convergence of the transformed series* (2.4) *is defined by equations*

$$\Re W_{-1}\left(-e^{p-1}(p + \ln z)\right) > p - 1 \text{ and } z > e^{-p} , \tag{3.8}$$

which is equivalent to

$$z > z_p = \begin{cases} e^{1-2p}, & p \leq 0 \\ e^{-p+\eta_0 \csc \eta_0}, & p > 0 \end{cases} \tag{3.9}$$

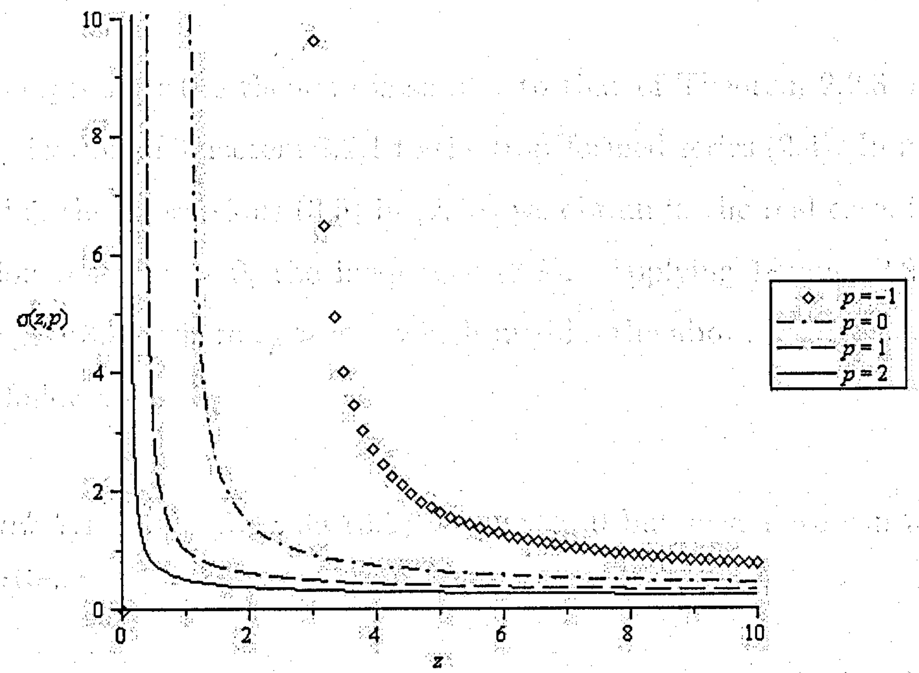

Figure 3.1: Dependence σ on z for different values of parameter p.

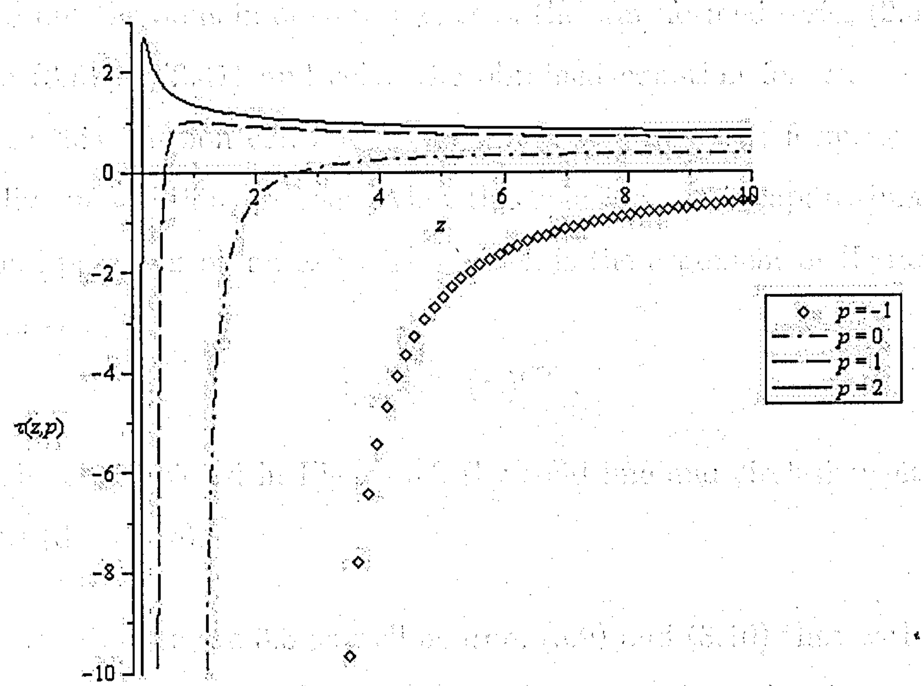

Figure 3.2: Dependence τ on z for different values of parameter p.

where $\eta_0 \in (0, \pi)$ *is the root of equation* $\eta_0 \cot \eta_0 = 1 - p$.

Proof. The proof of the theorem is similar to that of Theorem 2.2.3 and based on an application of Theorem 2.2.1 to the transformed series (2.4). In particular, substituting the expressions (3.5) in (2.13) we obtain in the real case, i.e. under assumption $p + \ln z > 0$, the inequality (3.8). Applying Lemma 2.2.2 to the latter we get (3.9), where $z_p > e^{-p}$, which justifies the above assumption and the theorem follows. □

Remark 7.1 In the formula (3.9) when $p > 0$ but $p \neq 1$ we can also write $z_p = e^{-p+(1-p)\sec \eta_0}$.

Remark 7.2 The convergence condition (3.8) can be extended to the case of complex z similar to the extension of the condition (2.27) for the untransformed series (2.4) by Theorem 2.2.10.

To find out the domain of convergence of the transformed series (2.5) we can substitute (3.5) in (2.41) and solve the obtained equation for z as a function of p. Since this solution can not be presented in an explicit form, we found it numerically. In addition, we found that this solution can be approximated with a very good precision by an expression which is the argument of W function in (3.7) taken at z_0

$$z_p = e^{-p} \left(z_0\right)^{e^{-p}} . \tag{3.10}$$

Both results are depicted in Figure 3.3 (by solid line and circles) together with curve (3.9) (dash line).

It follows from Figure 3.3 as well as from (3.9) and (3.10) that with increase of parameter p the domain of convergence of the transformed series monotonely extends. To illustrate and qualitatively verify this result we design an appropriate numerical procedure. The method is simply to compute the partial sum of a series

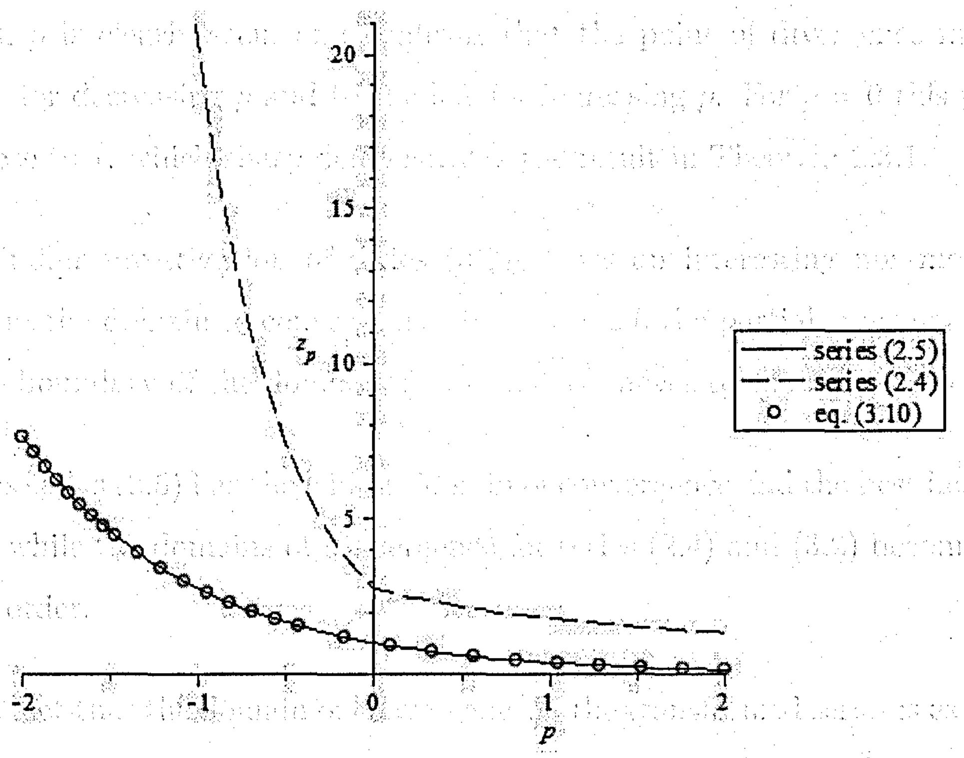

Figure 3.3: Behavior of boundary of convergence domain as a function of p for series (2.4) (dash line) and (2.5) (solid line) in real case.

to a high number of terms, using extended floating-point precision as necessary, and then to plot the ratio of the partial sum to the exact value (the exact value is obtained using a built-in MAPLE function LambertW(k,x), where a method different from series summation is used). The edge of the domain of convergence is then signaled by rapid oscillations and by marked deviations from the desired ratio of 1. (To make a graph be readable we depict only the relevant part of each curve.)

For the series (2.4) we have plotted in Figure 3.4 the partial sum to 40 terms for different values of p. For $p = 0$, we see a nice illustration of Theorem 2.2.3, with the partial sum becoming unstable in the vicinity of $z = e$. For positive p, we see the domain of convergence increased and for negative p it is decreased, in accordance with Theorem 3.4.1. Similar effects can be seen for (2.5), we plot in Figure 3.5 the partial sums for 40 terms as p varies. The domain of convergence for each p is clearly seen, and confirms that the point of divergence moves to larger z for decreasing p and to the left for increasing p. For $p = 0$ this point is very close to 1, which sharp demonstrates the result in Theorem 2.3.1.

A similar investigation of series (3.2) shows an interesting non-monotonic change in the domain of convergence. In Figure 3.6 the partial sums are plotted and the boundary of the domain of convergence moves to the right for $p \neq 0$.

Thus series (2.5) has the widest domain of convergence and the best behaviour with p, while the domains of convergence for series (2.4) and (3.2) become worse in that order.

The fact that the domain of convergence of the transformed series is extending while the parameter p is increasing can also be found in the complex case based on the results of theorems 2.3.7 and 3.4.1. To make certain of this it is sufficient for the series (2.5), to substitute expressions (3.5) (with $z \in \mathbb{C}$) into equation (2.41) and for the series (2.4), to consult Remark 7.2. The results are presented for $p = -1, -1/2, 0, 1/2$ and 1 in Figure 3.7 and Figure 3.8 for the series (2.4) and (2.5) respectively where the curves for $p = 0$ are the same as in Figure 2.2 and the points of intersection of the curves with the positive real axis correspond to the points on the curves depicted in Figure 3.3.

3.5 Rate of convergence

By rate of convergence, we are referring to the accuracy obtained by partial sums of a series. Given two series, each summed to N terms, the series giving on average a closer approximation to the converged value is said to converge more quickly. The qualification 'on average' is needed because it will be seen in the plots below that the error regarded as a function of z can show fine structure

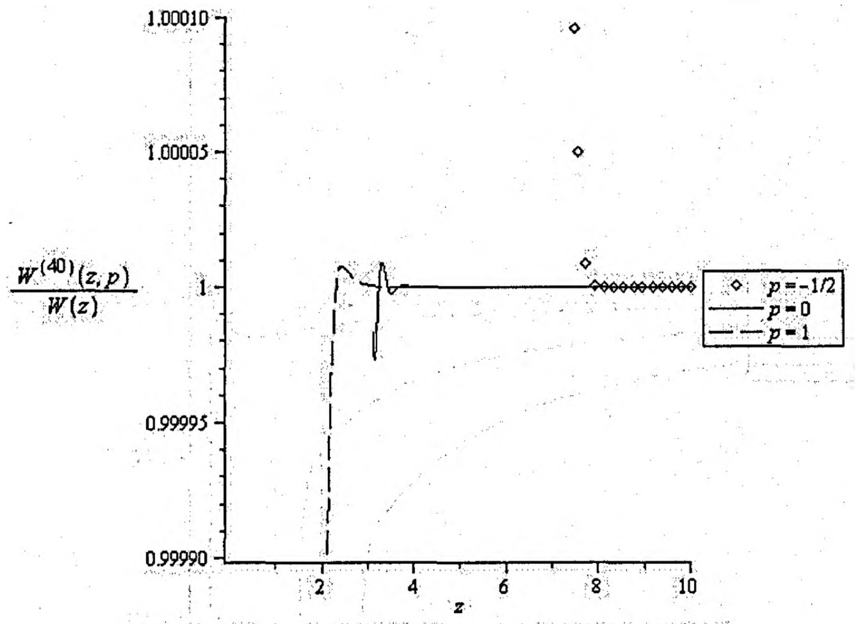

Figure 3.4: For series (2.4), the ratio $W^{(40)}(z,p)/W(z)$ as functions of z for $p = -1/2, 0, 1$.

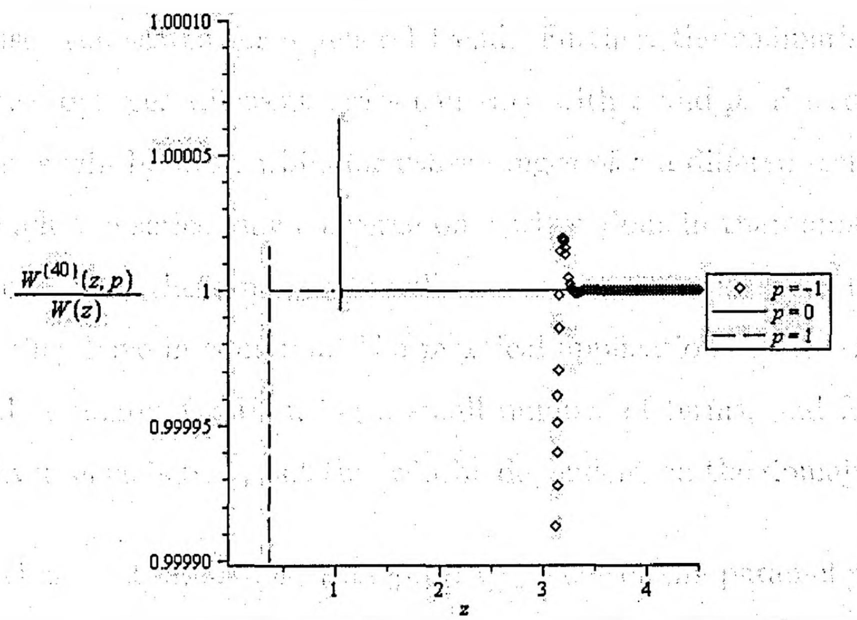

Figure 3.5: For series (2.5), the ratio $W^{(40)}(z,p)/W(z)$ as functions of z for $p = -1, 0, 1$. Compared with Figure 3.4, this shows convergence down to smaller z.

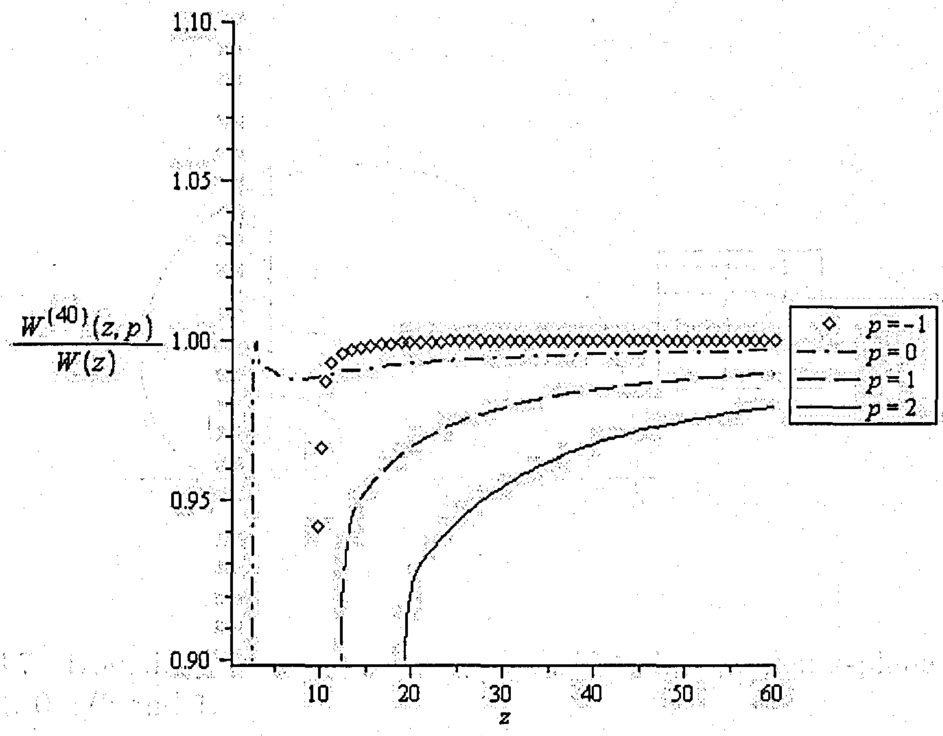

Figure 3.6: For series (3.2), the ratio $W^{(40)}(z,p)/W(z)$ against z. Compared with figures 3.4 and 3.5, the changes in convergence are no longer monotonic in p.

which confuses the search for a general trend. Further, the comparison of rate of convergence between different series can vary with z and p. For some ranges of z, one series will be best, while for other ranges of z a different series will be best. Although one series may converge on a wider domain than another, there is no guarantee that the same series will converge more quickly on the part of the domain they have in common. The practical application of these series is to obtain rapid estimates for W using a small number of terms, and for this the quickest convergence is best, but this will be dependent on the domain of z.

The previous section showed that positive values of the parameter p extend the domain of convergence of the series, but its effect on rate of convergence is different. Figures 3.9, 3.10 and 3.11 show the dependence on z of the accuracy of computations of the series (2.4),(2.5) and (3.3) respectively with $N = 10$ for $p = -1, -1/2, 0$ and 1. One can see that the behaviour of the accuracy is non-

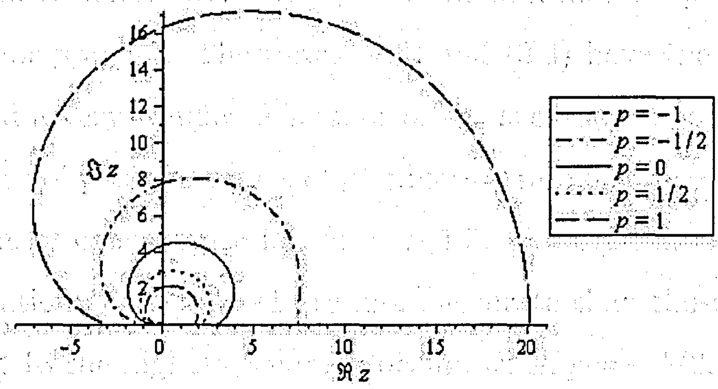

Figure 3.7: Domains of convergence of series (2.4) in complex z-plane for $p = -1, -1/2, 0, 1/2$ and 1.

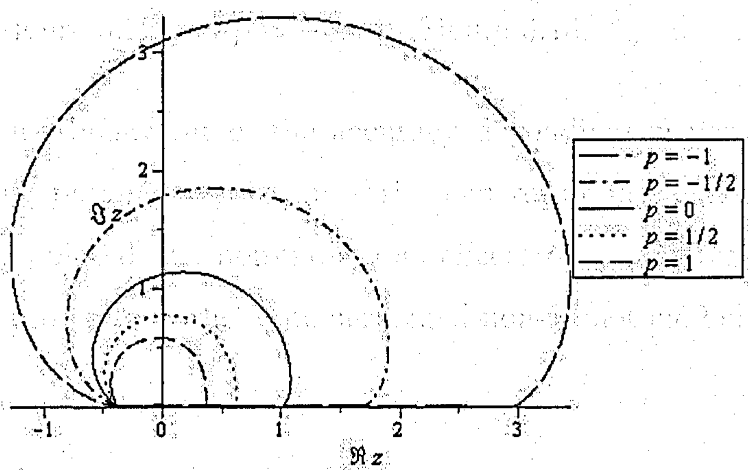

Figure 3.8: Domains of convergence of series (2.5) in complex z-plane for $p = -1, -1/2, 0, 1/2$ and 1.

monotone with respect to both z and p although some particular conclusions can be made. For example, one can observe that for the series (2.4) at least for $z < 30$ within the common domain of convergence the accuracy for $p = -1/2, 0$ and 1 is higher than for $p = -1$. The series (2.5) and (3.3) have the same domain of convergence and a very similar behaviour of the accuracy. Specifically, for these series an increase of positive values of p reduces a rate of convergence within the common domain of convergence i.e. for $z > 1.5$. However, at the same time for $z > 11$ computations with $p = -1$ are more accurate than those with positive p and for $5 < z < 18$ the highest accuracy occurs when $p = -1/2$.

The next two figures 3.12 and 3.13 display the dependence of convergence properties of the series (2.4) and (2.5) respectively on parameter p for different numbers of terms $N = 10, 20$ and 40. Again, the curves in these figures confirm that the accuracy strongly depends on parameter p and is non-monotone and show that on the whole an increase of the number of terms improves the accuracy. It is also interesting that there exists a value of p for which the accuracy at the given point is maximum; this value depends very slightly on N and approximately is $p \approx -0.75$ in Figure 3.12 and $p \approx -0.5$ in Figure 3.13.

The explained behaviour of the accuracy depending on parameter p shows that introducing parameter p in the series can result in significant changes in accuracy. The pointed out non-monotone effects of parameter p on a rate of convergence can be due to the aforementioned non-monotone behaviour of τ.

3.6 Branch −1 and complex p

The above discussion has considered only real values for the parameter p. We briefly shift our consideration to complex p and to branch -1. For z in the domain $-1/e < z < 0$, we have that $W_{-1}(z)$ takes real values in the range $[-1, -\infty)$.

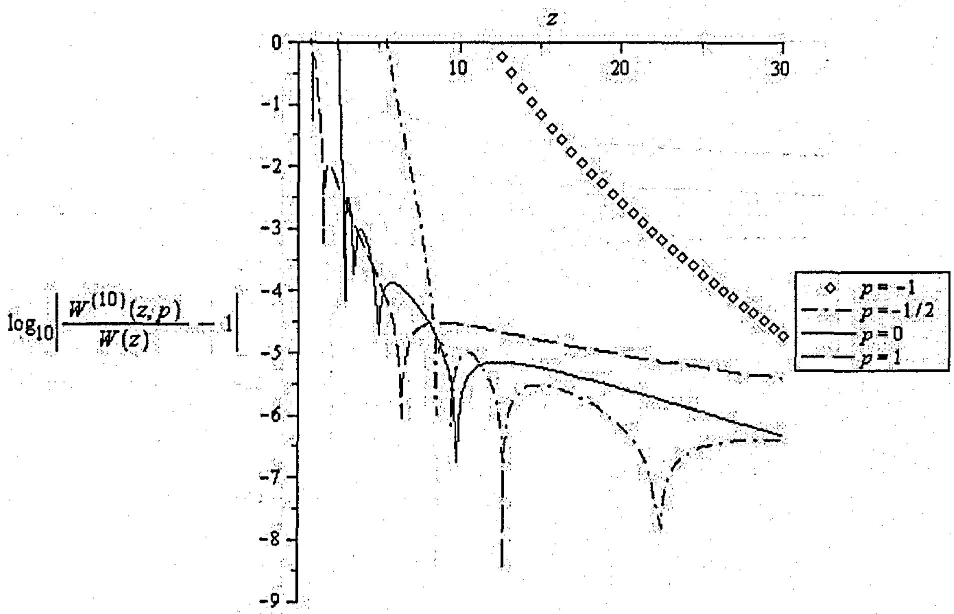

Figure 3.9: For series (2.4) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

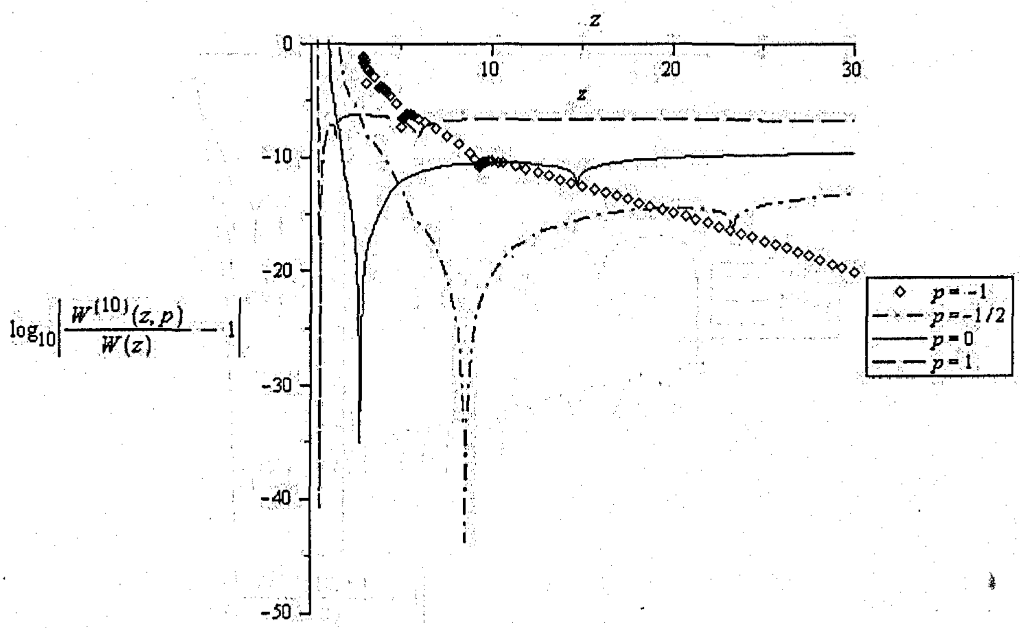

Figure 3.10: For series (2.5) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

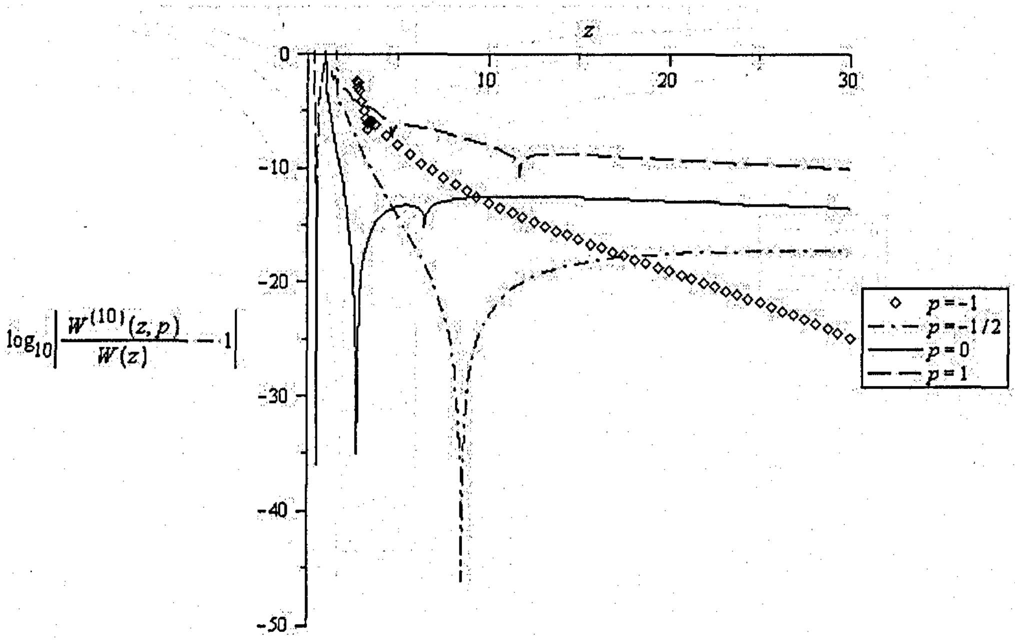

Figure 3.11: For series (3.3) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

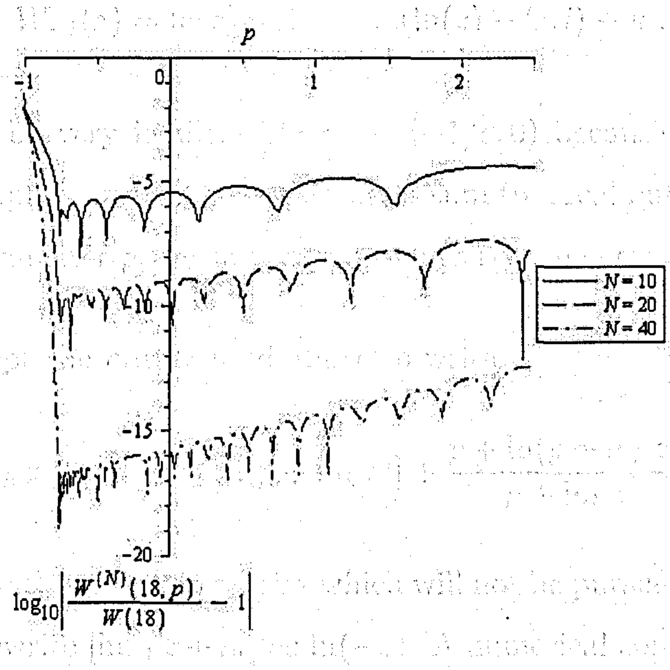

Figure 3.12: For series (2.4), the accuracy as a function of p at fixed point $z = 18$ for $N = 10, 20$ and 40.

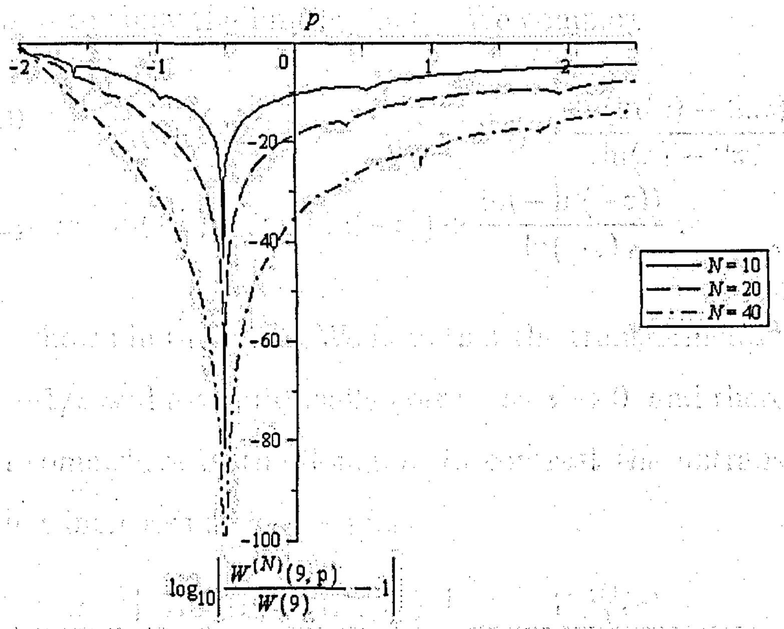

Figure 3.13: For series (2.5), the accuracy as a function of p at fixed point $z = 9$ for $N = 10, 20$ and 40.

The general asymptotic expansion (2.6) takes the form

$$W_{-1}(z) = \ln(z) - 2\pi i - \ln(\ln(z) - 2\pi i) + u \ . \tag{3.11}$$

This will clearly be very inefficient for $z \in [-1/e, 0)$ because each term in the series will be complex, and yet the series must sum to a real number. If, however, we utilize the parameter p, we can improve convergence enormously.

We again adopt the *ansatz* used above to write

$$W_k(z) = [\ln_k z + p] - [p + \ln(p + \ln_k z)] + \frac{p + \ln(p + \ln_k z)}{p + \ln_k z} + v \ , \tag{3.12}$$

where v stands for the remaining series which will not be pursued here. By setting $p = i\pi$, we can rewrite $[\ln_{-1} z + i\pi]$ as $\ln(-z)$. A numerical comparison of partial sums can be used to show the improvement. We compare

$$W_{-1}^{(1)} = \ln(z) - 2\pi i - \ln(\ln(z) - 2\pi i) + \frac{\ln(\ln(z) - 2\pi i)}{\ln(z) - 2\pi i}, \tag{3.13}$$

$$\hat{W}_{-1} = \ln(-z) - \ln(-\ln(-z)) + \frac{\ln(-\ln(-z))}{\ln(-z)}. \tag{3.14}$$

The results are shown in table 3.1. We note that the transformed series is exactly correct at $z = -1/e$ and asymptotically correct as $z \to 0$, and therefore the error is a maximum somewhere in the domain. In contrast the untransformed series has an error that increases as $z \to -1/e$.

z	$W_{-1}(z)$	$\hat{W}_{-1}(z)$	$W_{-1}^{(1)}(z)$
−0.01	−6.4728	−6.4640	−6.3210 − 0.04815i
−0.1	−3.5772	−3.4988	−3.4124 − 0.3223i
−0.2	−2.5426	−2.3810	−2.5182 − 0.5153i
−0.3	−1.7813	−1.5438	−2.0087 − 0.6621i
$-1/e$	−1	−1	−1.7597 − 0.7450i

Table 3.1: Numerical comparison of series transformation with $p = i\pi$.

The accuracy is also shown graphically in figure 3.14. Notice that although the approximation $\hat{W}_{-1}$ given in (3.14) is exactly equal to W_{-1} at $z = -e^{-1}$, the local behaviour is different. We know that W_{-1} has a square-root singularity, while $\hat{W}_{-1}$ is regular there. This is why the maximum error occurs at $z = -e^{-1}$.

3.7 Concluding remarks

We considered an invariant transformation defined by the parameter p and applied it to the series for the Lambert W function to obtain an infinite one-parameter family of series. We studied an effect of parameter p on convergence properties of the transformed series of this class. It is shown that an increase of p results in an extension of the domain of convergence of the series and thus the series

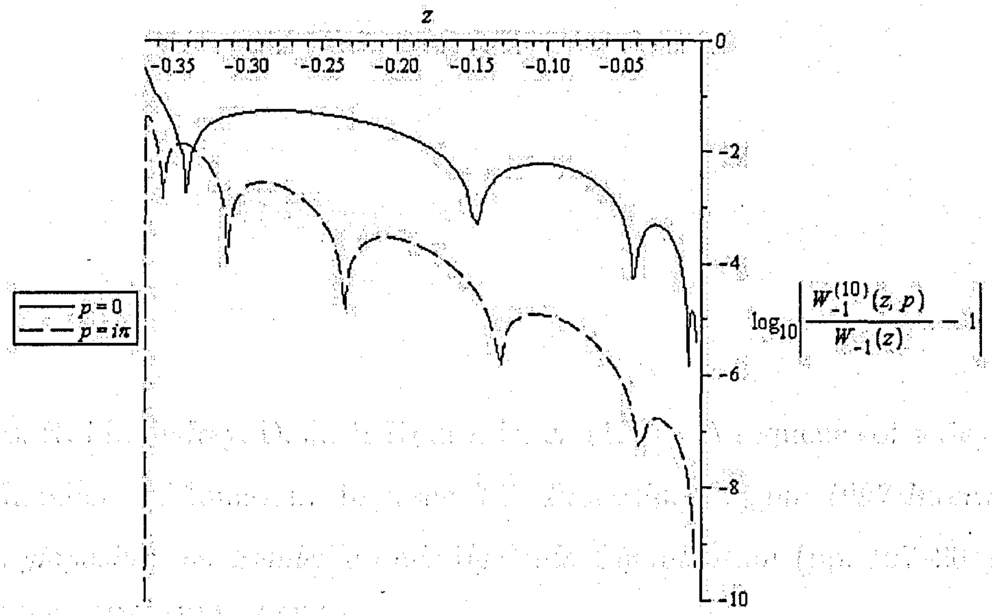

Figure 3.14: Errors in approximations (3.13) and (3.14) for W_{-1}.

obtained under the transformation with positive values of p have a wider domain of convergence than the original series does. However, at the same time a rate of convergence can be found to be reduced when the parameter p increases. Therefore in such a case within the common domain of convergence of the series with different positive values of p the series with the minimum value of p would be the most effective.

References

Corless, R. M., Jeffrey, D. J., & Knuth, D. E. (1997). A sequence of series for the Lambert W function. In *Issac '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (pp. 197–204). New York, NY, USA: ACM press.

Essex, C., Davison, M., & Schulzky, C. (2000). Numerical monsters. *SIGSAM Bulletin*, *34*(4), 16–32.

Jeffrey, D. J., Corless, R. M., Hare, D. E., & Knuth, D. E. (1995). On the inversion of $y^{\alpha}e^{y}$ in terms of associated Stirling numbers. *C. R. Acad. Sc.*, *320*, 1449–1452.

CHAPTER 4

Unimodal sequences show that Lambert W is Bernstein

"Wherever there is number, there is beauty." – Diadochus Proclus

4.1 Introduction

In this chapter we study the properties of the polynomials (their coefficients) arising in the expressions for the higher derivatives of the principal branch of the Lambert W function. We consider such properties as positiveness, unimodality and log-concavity. The most important consequence coming from the properties of the polynomial coefficients is that the derivative $dW(x)/dx$ is a completely monotonic function (Sokal, 2008). By (Berg, 2008, Definition 5.1), an infinitely differentiable function is called Bernstein function if its derivative is completely monotonic. Thus W is a Bernstein function (see also Section 5.5). Below we consider three forms of the higher derivatives of W.

4.2 First form

The nth derivative of the principal branch W is given implicitly by

$$\frac{d^n W(x)}{dx^n} = \frac{\exp(-nW(x))p_n(W(x))}{(1+W(x))^{2n-1}} \qquad \text{for } n \geq 1 , \tag{4.1}$$

where the polynomials $p_n(w)$ satisfy $p_1(w) = 1$, and the recurrence relation

$$p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p_n'(w) \qquad \text{for } n \geq 1 . \tag{4.2}$$

In (Corless, Jeffrey, & Knuth, 1997), the first five polynomials were printed explicitly:

$$\begin{aligned} &p_1(w) = 1 , \; p_2(w) = -2 - w , \; p_3(w) = 9 + 8w + 2w^2 , \\ &p_4(w) = -64 - 79w - 36w^2 - 6w^3 , \\ &p_5(w) = 625 + 974w + 622w^2 + 192w^3 + 24w^4 . \end{aligned}$$

The coefficients were also listed in (Sloane, 2008, A042977). These initial cases suggest the conjecture that each polynomial $(-1)^{n-1}p_n(w)$ has all positive coefficients, and if this is true, then $dW(x)/dx$ is a completely monotonic function (Sokal, 2008). We prove the conjecture and prove in addition that the coefficients are unimodal and log-concave.

4.2.1 Formulae for the coefficients

In view of the conjecture, we write

$$p_n(w) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k} w^k . \tag{4.3}$$

We now give several theorems regarding the coefficients.

Theorem 4.2.1. *The coefficients* $\beta_{n,k}$ *defined in* (4.3) *obey the recurrence relations*

$$\beta_{n,0} = n^{n-1} \ , \quad \beta_{n,1} = 3n^n - (n+1)^n - n^{n-1} \ , \tag{4.4}$$

$$\beta_{n,n-1} = (n-1)! \ , \quad \beta_{n,n-2} = (2n-2)(n-1)! \ , \tag{4.5}$$

$$\beta_{n+1,k} = (3n-k-1)\beta_{n,k} + n\beta_{n,k-1} - (k+1)\beta_{n,k+1} \ , \quad 2 \le k \le n-3 \ . \tag{4.6}$$

Proof. By substituting (4.3) into (4.2) and equating coefficients. □

Theorem 4.2.2. *An explicit expression for the coefficients* $\beta_{n,k}$ *is*

$$\beta_{n,k} = \sum_{m=0}^{k} \frac{1}{m!} \binom{2n-1}{k-m} \sum_{q=0}^{m} \binom{m}{q} (-1)^q (q+n)^{m+n-1} \ . \tag{4.7}$$

Proof. We rewrite (4.1) in the form

$$p_n(W(x)) = (1 + W(x))^{2n-1} e^{nW(x)} \frac{d^n W(x)}{dx^n} \ .$$

From the Taylor series of $W(x)$ around $x = 0$, given in (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996), we obtain

$$\frac{d^n W(x)}{dx^n} = \sum_{m=n}^{\infty} \frac{(-m)^{m-1}}{(m-n)!} x^{m-n} \ .$$

Substituting this into the expression of p_n, using $x = We^W$ and changing the index of summation, we obtain the equation

$$p_n(w) = (1+w)^{2n-1} \sum_{s=0}^{\infty} (-1)^{n+s-1} (n+s)^{n+s-1} \frac{w^s}{s!} e^{(n+s)w} \ . \tag{4.8}$$

We expand the right side around $w = 0$ and equate coefficients of w. □

Remark 4.2.3. The polynomials $p_n(w)$ can be expressed in terms of the diagonal Poisson transform $\mathbf{D}_n[f_s; z]$ defined in (Poblete, Viola, & Munro, 1997), namely, by (4.8)

$$p_n(w) = (-1)^{n-1}(1+w)^{2(n-1)}\mathbf{D}_n[(n+s)^{n-1}; -w] \,. \tag{4.9}$$

Theorem 4.2.4. *The coefficients can equivalently be expressed either in terms of shifted r-Stirling numbers of the second kind* $\left\{ {n+r \atop m+r} \right\}_r$ *defined in (Broder, 1984),*

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \left\{ {2n-1+m \atop n+m} \right\}_n \,, \tag{4.10}$$

or in terms of Bernoulli polynomials of higher order $B_n^{(z)}(\lambda)$ *defined in (Norlund, 1924),*

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \binom{m+n-1}{n-1} B_{n-1}^{(-m)}(n) \,, \tag{4.11}$$

or in terms of the forward difference operator Δ *(Graham, Knuth, & Patashnik, 1989, p. 188),*

$$\beta_{n,k} = \sum_{m=0}^{k} \binom{2n-1}{k-m} \frac{(-1)^m}{m!} \Delta^m n^{m+n-1} \,.$$

Proof. We convert (4.7) using identities found in (Broder, 1984) and (Lopez & Temme, 2010) respectively.

$$\left\{ {n+r \atop m+r} \right\}_r = \frac{1}{m!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^n$$

and

$$B_n^{(-m)}(r) = \frac{n!}{(m+n)!} \sum_{q=0}^{m} (-1)^{m-q} \binom{m}{q} (q+r)^{m+n} \,.$$

□

4.2.2 Expressions for the coefficients in terms of Stirling polynomials

We express the coefficients $\beta_{n,k}$ in terms of Stirling polynomials $\sigma_n(x)$ and generalized Stirling polynomials $\sigma_n^\alpha(x)$ introduced in (Graham et al., 1989, § 6.2) and (Schmidt, 2010, Sect. 5.2.1) respectively. For this purpose we apply by a straightforward way the definitions of the mentioned polynomials through the generating functions which are

$$\left(\frac{z}{e^z-1}\right)^x e^{\nu z} = \sum_{n\geq 0} \frac{B_n^{(x)}(\nu)}{n!} z^n , \tag{4.12}$$

$$\left(\frac{ze^z}{e^z-1}\right)^x = \sum_{n\geq 0} x\sigma_n(x) z^n , \tag{4.13}$$

$$\left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)^x e^{(1-\alpha)z} = \sum_{n\geq 0} \sigma_n^\alpha(x) z^n . \tag{4.14}$$

Specifically, to use polynomials $\sigma_n(x)$ we note that the left-hand side of equation (4.12) is a product of the left-hand side of (4.13) and exponential function

$$e^{(\nu-x)z} = \sum_{n\geq 0} \frac{(\nu-x)^n}{n!} z^n . \tag{4.15}$$

Then, taking the right-hand sides of these equations we obtain

$$\sum_{n\geq 0} \frac{B_n^{(x)}(\nu)}{n!} z^n = \sum_{n\geq 0} x\sigma_n(x) z^n \sum_{n\geq 0} \frac{(\nu-x)^n}{n!} z^n , \tag{4.16}$$

which gives relation

$$B_n^{(x)}(\nu) = xn! \sum_{q=0}^{n} \sigma_{n-q}(x) \frac{(\nu-x)^q}{q!} . \tag{4.17}$$

With using this relation the formula (4.11) can be written in the form

$$\beta_{n,k} = \sum_{q=0}^{n-1} \frac{1}{q!} \sum_{m=0}^{k} (-1)^{m+1} \binom{2n-1}{k-m} \frac{(m+n-1)!}{(m-1)!} (m+n)^q \sigma_{n-1-q}(-m) \,. \quad (4.18)$$

To use polynomials $\sigma_n^{\alpha}(x)$ we note that by (4.12)

$$\left(\frac{\alpha z}{e^{\alpha z}-1}\right)^x e^{\nu\alpha z} = \sum_{n\geq 0} \frac{B_n^{(x)}(\nu)}{n!} \alpha^n z^n \,. \quad (4.19)$$

The left-hand sides of equations (4.14) and (4.19) are the same when $\alpha = 1/(\nu - x + 1)$. Taking this α and equating the right-hand sides of these equations we find relation

$$B_n^{(x)}(\nu) = n!(\nu - x + 1)^n \sigma_n^{1/(\nu-x+1)}(x) \,. \quad (4.20)$$

With using this relation the formula (4.11) can be written in the form

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{2n-1}{k-m} \frac{(m+n-1)!}{m!} (m+n+1)^{n-1} \sigma_{n-1}^{1/(m+n+1)}(-m) \,. \quad (4.21)$$

4.2.3 Properties of the coefficients

We now give theorems regarding the properties of the $\beta_{n,k}$. We recall the following definitions (Stanley, 1989). A sequence $c_0, c_1 \ldots c_n$ of real numbers is said to be *unimodal* if for some $0 \leq j \leq n$ we have $c_0 \leq c_1 \leq \ldots \leq c_j \geq c_{j+1} \geq \ldots \geq c_n$, and it is said to be *logarithmically concave* (or log-concave for short) if $c_{k-1}c_{k+1} \leq c_k^2$ for all $1 \leq k \leq n-1$. We prove that for each fixed n, the $\beta_{n,k}$ are unimodal and log-concave with respect to k. Since a log-concave sequence of positive terms is unimodal (Wilf, 2005), it is convenient to start with the log-concavity property.

Theorem 4.2.5. *For fixed $n \geq 3$ the sequence $\{k!\beta_{n,k}\}_{k=0}^{n-1}$ is log-concave.*

Proof. Using (4.7) we can write

$$k!\beta_{n,k} = (2n-1)! \sum_{m=0}^{k} \binom{k}{m} x_m y_{k-m} ,$$

where

$$x_m = \sum_{j=0}^{j} \binom{m}{j} a_j , \quad a_j = (-1)^j (n+j)^{m+n-1} , \qquad (4.22)$$

and $y_m = 1/(2n-1-m)!$. Since the binomial convolution preserves the log-concavity property (Walkup, 1976; Wang & Yeh, 2007), it is sufficient to show that the sequences $\{x_m\}$ and $\{y_m\}$ are log-concave. We have

$$\begin{aligned} a_{j-1}a_{j+1} &= (-1)^{j-1}(n+j-1)^{m+n-1}(-1)^{j+1}(n+j+1)^{m+n-1} \\ &= (-1)^{2j}\left((n+j)^2-1\right)^{m+n-1} < (-1)^{2j}(n+j)^{2(m+n-1)} = a_j^2 . \end{aligned}$$

Thus the sequence $\{a_j\}$ is log-concave and so is $\{x_m\}$ due to (4.22) and the afore-mentioned property of the binomial convolution. The sequence $\{y_m\}$ is log-concave because

$$\begin{aligned} y_{m-1}y_{m+1} &= \frac{1}{(2n-1-m+1)!}\frac{1}{(2n-1-m-1)!} \\ &= \frac{2n-1-m}{2n-1-m+1}\frac{1}{(2n-1-m)!}\frac{1}{(2n-1-m)!} < y_m^2 . \end{aligned}$$

□

Now we prove that the coefficients $\beta_{n,k}$ are positive. The following two lemmas are useful.

Lemma 4.2.6. *If a positive sequence* $\{k!c_k\}_{k\geq 0}$ *is log-concave, then*

(i) $\{(k+1)c_{k+1}/c_k\}$ *is non-increasing;*

(ii) $\{c_k\}$ *is log-concave;*

(iii) *the terms c_k satisfy*

$$c_k c_m \geq \binom{k+m}{k} c_0 c_{k+m} \quad (0 \leq m \leq k+1) \; . \tag{4.23}$$

Proof. The statements (i) and (ii) are obvious. To prove (iii) we apply a method used in (Asai, Kubo, & Kuo, 2000). Specifically, by (i) we have for $0 \leq p \leq k$

$$\frac{c_{p+1}}{c_p} \geq \frac{k+p+1}{p+1} \frac{c_{k+p+1}}{c_k} \; .$$

Apply the last inequality for $p = 0, 1, 2, ..., m$ with $m \leq k+1$, and form the products of all left-hand and right-hand sides. As a result, after the cancellation we obtain

$$\frac{c_m}{c_0} \geq \frac{k+1}{1}\frac{k+2}{2} \cdots \frac{k+m}{m}\frac{c_{k+m}}{c_k} \; ,$$

which is equivalent to (4.23). □

Lemma 4.2.7. *If the coefficients $\beta_{n,k}$ are positive, then for fixed $n \geq 3$ they satisfy*

$$\frac{(k+1)\beta_{n,k+1}}{\beta_{n,k}} < n-1 \; . \tag{4.24}$$

Proof. By Theorem 4.2.5 and under the assumption of lemma, for fixed $n \geq 3$ the sequence $\{k!\beta_{n,k}\}_{k=0}^{n-1}$ meet the conditions of Lemma 4.2.6. Applying the inequality (4.23) with $m = 1$ to this sequence gives $(k+1)\beta_{n,k+1}/\beta_{n,k} \leq \beta_{n,1}/\beta_{n,0}$. Then the lemma follows as due to (4.4)

$$\frac{\beta_{n,1}}{\beta_{n,0}} = \frac{3n^n - (n+1)^n - n^{n-1}}{n^{n-1}} = 3n - n\left(1+\frac{1}{n}\right)^n - 1 < 3n - 2n - 1 = n - 1 \; .$$

□

Theorem 4.2.8. *The coefficients $\beta_{n,k}$ are positive.*

Proof. We prove the statement by induction on n. It is true for $n \leq 5$ (see §1). Assume that for some fixed n all the members of the sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ are positive. Since $\beta_{n+1,0} = (n+1)^n > 0$ and $\beta_{n+1,n} = n! > 0$ by (4.4) and (4.5), we only need to consider $k = 1, 2, ..., n-1$.

Substituting inequalities $\beta_{n,k+1} < (n-1)\beta_{n,k}/(k+1)$ and $\beta_{n,k-1} > k\beta_{n,k}/(n-1)$, which follow from (4.24), in the recurrence (4.6) immediately gives the result

$$\beta_{n+1,k} > (3n-k-1)\beta_{n,k}+n\frac{k}{n-1}\beta_{n,k}-(k+1)\frac{n-1}{k+1}\beta_{n,k} = \left(2n+\frac{k}{n-1}\right)\beta_{n,k} > 0\,.$$

Thus the proof by induction is complete. □

Corollary 4.2.9. *The sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ is log-concave and unimodal for $n \geq 3$.*

Proof. By Theorem 4.2.8 the sequence $\{\beta_{n,k}\}_{k=0}^{n-1}$ is positive, therefore by Theorem 4.2.5 and Lemma 4.2.6(ii) it is log-concave and unimodal. □

4.2.4 Relation to Carlitz's numbers

There is a relation between the coefficients $\beta_{n,k}$ and numbers $B(\kappa, j, \lambda)$ introduced by Carlitz (1980). Comparing the formula (4.10) with (Carlitz, 1980, eq.(6.3)) and taking into account that he uses the notation $R(n, m, r) = \left\{ {n+r \atop m+r} \right\}_r$, we find

$$\beta_{n,k} = (-1)^k B(n-1, n-1-k, n)\,. \tag{4.25}$$

It follows that for $n \geq 3$, the sequence $\{B(n-1, k, n)\}_{k=0}^{n-1}$ is log-concave together with $\{\beta_{n,k}\}_{k=0}^{n-1}$.

Using the property (Carlitz, 1980, eq.(2.7)) that $\sum_{j=0}^{\kappa} B(\kappa, j, \lambda) = (2\kappa-1)!!$, we can compute $p_n(w)$ at the singular point where $W = -1$ (cf. (4.1)). Thus, substituting $w = -1$ in (4.3) gives $p_n(-1) = (-1)^{n-1}(2n-3)!!$. Thus $w = -1$ is not a zero of $p_n(w)$.

We also note that the numbers $B(\kappa, j, \lambda)$ are polynomials of λ and satisfy a three-term recurrence (Carlitz, 1980, eq. (2.4))

$$B(\kappa, j, \lambda) = (\kappa + j - \lambda)B(\kappa - 1, j, \lambda) + (\kappa - j + \lambda)B(\kappa - 1, j - 1, \lambda) \quad (4.26)$$

with $B(\kappa, 0, \lambda) = (1 - \lambda)^{\bar{\kappa}}$, $B(0, j, \lambda) = \delta_{j,o}$. This gives one more way to compute the coefficients $\beta_{n,k}$, specifically, for given n and k we find a polynomial $B(n-1, n-1-k, \lambda)$ using (4.26) and then set $\lambda = n$ to use (4.25).

4.2.5 Consequences

It has been established that the coefficients of the polynomials $(-1)^{n-1}p_n(w)$ are positive, unimodal and log-concave. These properties imply an important property of W. In particular, it follows from formula (4.1) and Theorem 4.2.8 that $(-1)^{n-1}(dW/dx)^{(n-1)} > 0$ for $n \geq 1$. Since $W(x)$ is positive for all positive x (Corless et al., 1996), this means that the derivative W' is completely monotonic and W itself is a Bernstein function (Berg, 2008).

Some additional identities can be obtained from the results above. For example, computing $\beta_{n,n-1}$ by (4.10) and comparing with (4.5) gives

$$\sum_{m=0}^{n-1} (-1)^m \binom{2n-1}{n-m-1} \left\{ {2n-1+m \atop n+m} \right\}_n = (n-1)!\,.$$

A relation between $\left\{ {2n-1+m \atop n+m} \right\}_n$ and $B_{n-1}^{(-m)}(n)$ can be obtained from (4.10) and (4.11), but this is a special case of (Carlitz, 1980, eq. (7.5)). It is also interesting to note that (4.10) and (4.11) can be inverted. Indeed, in these formulae for fixed n, the sequence $(-1)^k\beta_{n,k}$ is a convolution of two sequences, therefore its generating function $G(w)$ is a product of the generating functions of these two sequences and we can write $G(w) = (1-w)^{2n-1}F(w)$, where $F(w)$ represents a generating function of the sequence $\left\{{2n-1+m \atop n+m}\right\}_n$ in case of formula (4.10) or of the sequence $\binom{m+n-1}{n-1}B_{n-1}^{(-m)}(n)$ in case of (4.11). Now, since $F(w) = G(w)(1-w)^{-(2n-1)} = G(w)\sum_{k\geq 0}\binom{2n-2+k}{2n-2}w^k$, the inverse of, for example, (4.10) is

$$\left\{{2n-1+m \atop n+m}\right\}_n = \sum_{k=0}^{n-1}(-1)^k\beta_{n,k}\binom{2n-2+m-k}{2n-2}. \tag{4.27}$$

With connection (4.25) this equation is a special case of (Carlitz, 1980, eq.(2.9)).

4.3 Second form

Compared to (4.1), there are two more forms to represent the derivatives of W. One of them is linked to the results obtained in (Dumont & Ramamonjisoa, 1996), they show that

$$\frac{d^nW(x)}{dx^n} = \frac{(-1)^{n-1}\exp(-nW(x))}{(1+W(x))^n}Q_n\left(\frac{1}{1+W(x)}\right), \tag{4.28}$$

where

$$Q_n(y) = \sum_{k=0}^{n-1} b_{n,k}y^k\,. \tag{4.29}$$

Coefficients $b_{n,k}$ satisfy the recurrence relation

$$b_{n,k} = (n-1)b_{n-1,k} + (n+k-2)b_{n-1,k-1}, \quad b_{1,0} = 1\,. \tag{4.30}$$

They are related to the Ramanujan sequence (Berndt, 1985) $\psi_k(r,s)$ and the sequence $Q_{n,k}(x)$, introduced by Shor (1995) $b_{n,k} = \psi_{k+1}(n-1,n) = Q_{n,k}(0)$. These sequences arise in the study of Cayley's formula of the number of trees, their combinatorial interpretations are given in (Shor, 1995; Zeng, 1999; Chen & Guo, 2001). The values for coefficients $b_{n,k}$ are listed in (Sloane, 2008, A054589).

Comparing the aforementioned representations for the higher derivatives gives

$$\sum_{k=0}^{n-1} \beta_{n,k} w^k = \sum_{k=0}^{n-1} b_{n,n-1-k}(1+w)^k \,. \tag{4.31}$$

It follows that the relations between coefficients $b_{n,k}$ and $\beta_{n,k}$ are

$$\beta_{n,k} = \sum_{m=k}^{n-1} \binom{m}{k} b_{n,n-1-m} \tag{4.32}$$

and

$$b_{n,n-1-k} = \sum_{m=k}^{n-1} (-1)^{m-k} \binom{m}{k} \beta_{n,m} \,. \tag{4.33}$$

4.3.1 Positiveness

In (Zeng, 1999) there is a combinatorial proof that $Q_{n,k}(x)$ is a polynomial of $(x+1)$ with non-negative integer coefficients. So $b_{n,n-1-m} = Q_{n,n-1-m}(0) > 0$.

Also, positiveness immediately follows from the recurrence (4.30) by induction on n.

4.3.2 Log-concavity

Coefficients $b_{n,k}$ are log-concave because being defined by the recurrence (4.30) they relate to triangular arrays which are necessarily log-concave (Kurtz, 1972).

Note that coefficients $\beta_{n,k}$ have a property which is stronger than log-concavity. The sequence $\{k!\beta_{n,k}\}$ is also log-concave. However, the coefficients $b_{n,k}$ do not have this property. For example, a difference $kb_{n,k}^2 - (k+1)b_{n,k-1}b_{n,k+1}$ computed with values taken from table (Sloane, 2008, A054589) for $n = 6$ and $k = 1$ is -6240, i.e. negative, therefore sequence $\{k!b_{n,k}\}$ is not log-concave.

4.3.3 Unimodality

Since a positive and log-concave sequence is unimodal, properties 1 and 2 imply unimodality of $b_{n,k}$.

Remark 4.3.1. It follows from (4.32) that the sequence $\beta_{n,k}$ is positive because so is the sequence $b_{n,k}$. In addition, since the sequence $b_{n,k}$ is log-concave, so is $\beta_{n,k}$ due to (4.31) and Brenti's criterion (Brenti, 1994). This way to ascertain the properties of the coefficients $\beta_{n,k}$ through the ones of the coefficients $b_{n,k}$ was first pointed out by Chapoton (2010) and was found independently by Pakes (2011). The author became familiar with the form (4.28) from (Chapoton, 2010).

Remark 4.3.2. Since the coefficients $\beta_{n,k}$ are related to numbers $B(\kappa, j, \lambda)$ (cf. (4.25)), the relation (4.33) will connect the $b_{n,k}$ as well to the same numbers.

4.4 Third form

One more form to represent the higher derivatives of W immediately follows from that mentioned in (Knuth, 2005, Ex. 50, p. 84, 136-137) in terms of the tree function $T(x) = -W(-x)$

$$\frac{d^n W(x)}{dx^n} = \frac{\exp(-nW(x))}{(1+W(x))^n} P_n\left(\frac{W(x)}{1+W(x)}\right), \tag{4.34}$$

where polynomials

$$P_n(x) = (-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k a_{n,k} x^k, \tag{4.35}$$

with the coefficients $a_{n,k}$ satisfying the recurrences

$$P_{n+1}(x) = n(x-2)P_n(x) + (x-1)^2 P_n'(x), \quad P_1(x) = 1$$

and

$$a_{n,0} = n^{n-1}, \quad a_{n,n-1} = (2n-3)!! \,, \\ a_{n+1,k} = 2(n+k)a_{n,k} + (n+k-1)a_{n,k-1} + (k+1)a_{n,k+1} \,. \tag{4.36}$$

The numbers $a_{n,k}$ and row sums

$$S_n = \sum_{k=0}^{n-1} a_{n,k}$$

are given in (Sloane, 2008, A048160, A005264). The exponential generating function of the sequence S_n satisfies

$$(1+x)\exp(A(x)) = 1 + 2A(x) \,.$$

This equation can be solved in terms of W function as

$$A(x) = -\frac{1}{2} - W\left(-\frac{1+x}{2\sqrt{e}}\right) \,.$$

Then, using the well-known expansion of W near the origin (Corless et al., 1996) and taking small x (more precisely, satisfying inequality $|x+1| < 2/\sqrt{e}$) one can find eventually

$$\sum_{k=0}^{n-1} a_{n,k} = \sum_{m=n}^{\infty} \frac{m^{m-1}}{(m-n)!\,2^m e^{m/2}} \,, \tag{4.37}$$

where the infinite sum on the right-hand side is thus integer.

Comparing representations (4.1) and (4.34) gives the relation between coefficients $\beta_{n,k}$ and $a_{n,k}$

$$a_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{n-1-m}{n-1-k} \beta_{n,m} \,, \tag{4.38}$$

$$\beta_{n,k} = \sum_{m=0}^{k} (-1)^m \binom{n-1-m}{n-1-k} a_{n,m} \,. \tag{4.39}$$

It follows from (4.36) by induction on n that all $a_{n,k}$ are positive. In addition, based on the relation (4.38) one can show that for fixed $n \geq 3$ the sequence $\{a_{n,k}\}_{k=0}^{n-1}$ is log-concave. Indeed, the relation can be written in the form

$$a_{n,k} = \sum_{m=0}^{k} x_m y_{k-m} \,,$$

where $x_m = (-1)^m \beta_{n,m}$ and $y_m = \binom{p+m}{p}$, $p = n-1-k$. The former is log-concave because so is $\beta_{n,m}$ by Corollary 4.2.9 and the latter is log-concave because

$$y_{m-1}y_{m+1} = \frac{m^2+(p+1)m}{m^2+(p+1)m+p} y_m^2 \leq y_m^2 \,.$$

Thus, $a_{n,k}$ is convolution of two log-concave sequences and therefore it is so as well.

At last, being positive and log-concave the sequence $\{a_{n,k}\}_{k=0}^{n-1}$ is unimodal.

4.5 Concluding remarks

In fact, the above considered polynomials (4.3), (4.29) and (4.35) arise in formulae for the following expression with the higher derivatives of $W = W(x)$

$$(1+W)^n e^{nW} \frac{d^n W}{dx^n} \,, \tag{4.40}$$

when we want to write it in terms of polynomials with respect to different combinations of W, namely, $W, 1/(1+W)$ and $W/(1+W)$ respectively. The expression

(4.40) can be written in different forms, particularly,

$$(1+W)^n e^{nW} \frac{d^n W}{dx^n} = \frac{(1+W)^n}{W^n} x^n \frac{d^n W}{dx^n} = \frac{1}{(W')^n} \frac{d^{n-1} W'}{dx^{n-1}} ,$$

where $W' = dW/dx$.

It is worth noting that Knuth (2005) considers polynomials

$$\sum_{k=0}^{n-1} a_{n,k} x^k \tag{4.41}$$

rather than (4.35). Note that the alternating factor $(-1)^k$ does not effect on log-concavity.

Finally, it follows from the above consideration that the polynomials

$$\sum_{k=0}^{n-1} \beta_{n,k} x^k , \quad \sum_{k=0}^{n-1} b_{n,k} x^k \quad \text{and} \quad \sum_{k=0}^{n-1} a_{n,k} x^k \tag{4.42}$$

have the same properties, in particular, all of them are positive, log-concave and unimodal. Positiveness of the polynomial coefficients means that the derivative W' is completely monotonic and W itself is a Bernstein function.

In addition, numerical experiments show that the polynomials share one more common property that is associated with their roots. Specifically, for $n > 2$ the even polynomials, which are of odd order, have one real root and $(n-2)/2$ pairs of complex conjugate roots while the odd polynomials, which are of even order, have only complex roots (in the form of $(n-1)/2$ complex conjugate pairs). The roots of polynomials (4.41) and (4.35) differ in the sign of their real parts only, therefore the latter have the same property of the roots.

References

Asai, N., Kubo, I., & Kuo, H.-H. (2000). Bell numbers, Log-Concavity, and Log-Convexity. *Acta Appl. Math.*, *63*, 79–87.

Berg, C. (2008). Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity. In S. Mateu & E. Porcu (Eds.), *Positive Definite Functions: From Schoenberg to Space-Time Challenges* (pp. 15–45). Castellon, Spain: Dept. of Mathematics, University Jaume I.

Berndt, B. C. (1985). *Ramanujan's Notebooks. Part 1.* New York, Berlin: Springer-Verlag.

Brenti, F. (1994). Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry: an update. *Contemporary Mathematics*, *178*, 71–84.

Broder, A. Z. (1984). The *r*-Stirling numbers. *Discrete Math.*, *49*(3), 241–259.

Carlitz, L. (1980). Weighted Stirling numbers of the first and second kind II. *Fibonacci Quart.*, *18*, 242–257.

Chapoton, F. (2010, November). *Short comment on your arxiv preprint.* Private email.

Chen, W. Y., & Guo, V. J. (2001, 14 December). *Bijections behind the Ramanujan Polynomials.* arXiv:math/0107024v2 [math.CO].

Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., & Knuth, D. E. (1996). On the Lambert W function. *Adv. Comp. Math.*, *5*(4), 329–359.

Corless, R. M., Jeffrey, D. J., & Knuth, D. E. (1997). A sequence of series for the Lambert W function. In *Issac '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (pp. 197–204). New

York, NY, USA: ACM press.

Dumont, D., & Ramamonjisoa, A. (1996). Grammaire de Ramanujan et Arbres de Cayley. *The Electronic J. Combinatorics, 3, R17.*

Graham, R. L., Knuth, D. E., & Patashnik, O. (1989). *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley.

Knuth, D. E. (2005). *The Art of Computer Programming, Vol. 4, Fascicle 3: Generating All Combinations and Partitions, Sec. 7.2.1.5* (Third ed.). Reading, Massachusetts: Addison-Wesley.

Kurtz, D. C. (1972). A note on a concavity properties of triangular arrays of numbers. *J. Combin. Theory, Ser. A*, *13*, 135–139.

Lopez, J. L., & Temme, N. M. (2010). Large degree asymptotics of generalized Bernoulli and Euler polynomials. *J. Math. Anal. Appl.*, *363*(1), 197–208.

Norlund, N. E. (1924). *Vorlesungen uber Differenzenrechnungy.* Berlin: Springer-Verlag.

Pakes, A. G. (2011). Lambert's W, infinite divisibility and Poisson mixtures. *J. Math. Anal. Appl.*, *378*(2), 480–492.

Poblete, P. V., Viola, A., & Munro, J. I. (1997). The diagonal Poisson transform and its application to the analysis of a hashing scheme. *Random Struct. Algorithms*, *10*, 221–255.

Schmidt, M. D. (2010). Generalized j-Factorial Functions, Polynomials, and Applications. *J. Integer Sequences*, *13*(6), 54 pp.

Shor, P. (1995). A new proof of Cayley's formula for counting labeled trees. *J. Combin. Theory, Ser. A*, *71*, 154–158.

Sloane, N. J. (2008). Sequences A042977, A054589, A048160 and A005264. In *The on-line encyclopedia of integer sequences.* Available from `http://www.research.att.com/~njas/sequences/`

Sokal, A. D. (2008, October). *Another question about the Lambert W.* Private email.

Stanley, R. P. (1989). Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry. *Ann. New York Acad. Sci.*, *576*, 500–535.

Walkup, D. W. (1976). Polya sequences, binomial convolution, and the union of random sets. *J. Appl. Prob.*, *13*(1), 76–85.

Wang, Y., & Yeh, Y.-N. (2007). Log-concavity and LC-positivity. *J. Combin. Theory*, *114*, 195–210.

Wilf, H. S. (2005). *Generatingfunctionology* (3rd ed.). A. K. Peters.

Zeng, J. (1999). A Ramanujan Sequence that Refines the Cayley Formula for Trees. *Ramanujan*, *3*, 45–54.

Chapter 5

Stieltjes, Poisson and similar representations of functions of W

"Integral transforms are like opera glasses: The knowledge you gain from them depends on which end you look through." – Micha Hofri

5.1 Introduction

In this chapter we show that many functions of W are members of a number of function classes, namely, the classes of Stieltjes functions, Pick functions and Bernstein functions, including subclasses Thorin-Bernstein functions and complete Bernstein functions. This is mainly due to the fact that W is a real symmetric function, in the terminology of (Baker & Graves-Morris, 1981, p. 160) (see also (Titchmarsh, 1939, p. 155)), with positive values on the positive real line. A description of the mentioned classes can be found in a review paper (Berg, 2008) and a recently published book (Schilling, Song, & Vondraček, 2010).

The above mentioned classes are of particular interest because they are characterized by their own integral forms. As a consequence, the W function is rich in integral representations and we give explicit integral representations of functions of W. In the chapter we also extend the properties of the set of Stieltjes functions in Sections 5.2 and 5.6. In addition, we give one more proof of the fact established in the previous chapter that W function is Bernstein. Moreover, we show that W is a complete Bernstein function.

The classes of Stieltjes functions and Bernstein functions are intimately connected with the class of completely monotonic functions that have many applications in different fields of science; the list of appropriate references is given in (Alzer & Berg, 2002). Therefore we shall also study the complete monotonicity of some functions containing W.

The properties and integral representations mentioned above have interesting computational implications. For example, that $W(z)/z$ is a Stieltjes function means that the poles of successive Padé approximants interlace and all lie on the negative real axis (Baker & Graves-Morris, 1981, p. 186) (here in the interval $-\infty < z < -1/e$). In addition, some of the integral representations permit spectrally convergent quadratures for numerical evaluation.

5.2 Stieltjes functions

We now review the properties of Stieljes functions, again concentrating on results that will be used in this paper. We must note at once that there exist several different definitions of Stieltjes functions in the literature, and here we follow the definition of Berg (Berg, 2008).

Definition 5.2.1. A function $f : (0, \infty) \to \mathbb{R}$ is called a *Stieltjes function* if it admits a representation

$$f(x) = a + \int_0^\infty \frac{d\sigma(t)}{x+t} \quad (x > 0), \tag{5.1}$$

where a is a non-negative constant and σ is a positive measure on $[0,\infty)$ such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$.

A Stieltjes function is also called a *Stieltjes transform* (Berg & Forst, 1975, p. 127). Except in Section 5.3.3 below, the term Stieltjes function will here always refer to definition (5.1).

Theorem 5.2.2. *The set $\mathcal{S}$ of all Stieltjes functions forms a convex cone (Berg & Forst, 1975, p. 127) and possesses the following properties.*

(i) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f(1/x)} \in \mathcal{S}$

(ii) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{xf(x)} \in \mathcal{S}$

(iii) $f \in \mathcal{S} \Rightarrow \frac{f}{cf+1} \in \mathcal{S} \quad (c \geq 0)$

(iv) $f, g \in \mathcal{S} \setminus \{0\} \Rightarrow f \circ \frac{1}{g} \in \mathcal{S}$

(v) $f, g \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f \circ g} \in \mathcal{S}$

(vi) $f, g \in \mathcal{S} \Rightarrow f^\alpha g^{1-\alpha} \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$

(vii) $f \in \mathcal{S} \Rightarrow f^\alpha \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$

(viii) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{x}\left(\frac{f(0)}{f(x)} - 1\right) \in \mathcal{S}$

(ix) $f \in \mathcal{S} \setminus \{0\}, \lim_{x \to 0+} xf(x) = c \geq 0 \Rightarrow f(x) - c/x \in \mathcal{S}$

(x) $f \in \mathcal{S} \Rightarrow f^\alpha(0) - f^\alpha(1/x) \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$

(xi) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{x}\left(1 - \frac{f(x)}{f(0)}\right) \in \mathcal{S}$

(xii) $f \in \mathcal{S}, \lim_{x \to \infty} f(x) = c > 0 \Rightarrow (c^\beta - f^\beta) \in \mathcal{S} \quad (-1 \leq \beta \leq 0)$

In the above statements constants c and $f(0) = \lim_{x \to 0+} f(x)$ are assumed to be finite.

Proof. Properties (i)-(vii) are listed in (Berg, 2008); property (vi) is due to the fact that the Stieltjes cone is logarithmically convex (Berg, 1979) and property (vii) is its immediate consequence. Property (viii) is taken from (Bender & Orszag, 1999, p. 406). Property (ix) follows from properties (ii) and (viii) in the following way: $f \in \mathcal{S} \setminus \{0\} \Rightarrow g(x) = 1/(xf(x)) \in \mathcal{S} \Rightarrow (g(0)/g(x) - 1)/x = (xf(x)/c - 1)/x \in \mathcal{S} \Rightarrow f(x) - c/x \in \mathcal{S}$. The last three properties (x)–(xii) will be proved in Section 5.6. □

A Stieltjes function f has a holomorphic extension to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying $f(\bar{z}) = \overline{f(z)}$ (see (Berg, 1979), (Alzer & Berg, 2006) and (Schilling et al., 2010, p. 11-12))

$$f(z) = a + \int_0^\infty \frac{d\sigma(t)}{z+t} \quad (|\arg(z)| < \pi). \tag{5.2}$$

In addition, a Stieltjes function $f(z)$ in the cut plane $\mathbb{C}\backslash(-\infty, 0]$ can be represented in the integral form (Baker & Graves-Morris, 1981, p.158)

$$f(z) = \int_0^\infty \frac{d\Phi(u)}{1+uz} \quad (|\arg(z)| < \pi) \ , \tag{5.3}$$

where $\Phi(u)$ is a bounded and non-decreasing function with finite real-valued moments $\int_0^\infty t^n \, d\Phi(t)$ $(n = 0, 1, 2, \dots)$. The integral (5.3) is used in (Baker & Graves-Morris, 1981, Ch. 5) for a study of Padé approximants to the Stieltjes functions; it is equivalent to the representation (5.2) by virtue of the following observation. According to properties (i) and (ii), if a function $f \in \mathcal{S}$ then $f(1/x)/x \in \mathcal{S}$ as well and hence the latter admits representation (5.1)

$$\frac{1}{x} f\left(\frac{1}{x}\right) = a + \int_0^\infty \frac{d\sigma(t)}{x+t} \ ,$$

which after replacing x with $1/x$ gives

$$f(x) = \frac{a}{x} + \int_0^\infty \frac{d\sigma(t)}{1+xt} ,$$

where the first term can be included into the integral since $a \geq 0$ and $1/x$ is a Stieltjes function (see e.g. (Berg, 2008)). Finally, one considers the holomorphic extension of the last integral to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ similar to obtaining (5.2). Conversely, starting with formula (5.3) and taking the same operatons in reverse order we will come to (5.2).

There are various kinds of necessary and sufficient conditions implying that a function f is a Stieltjes function. Some of them are based on the classical results established by R. Nevanlinna, F. Riesz, and Herglotz. Here we quote two such theorems taken from (Akhiezer, 1965, p. 93) and (Berg, 2008, Theorem 3.2).

Theorem 5.2.3. *A function $g(z)$ admits an integral representation in the upper half-plane in the form*

$$g(z) = \int_{\mathbb{R}} \frac{d\Phi(u)}{u-z} \quad (\Im z > 0) , \tag{5.4}$$

with a non-decreasing function $\Phi(u)$ of bounded variation on $\mathbb{R}$ (i.e. $\int_{\mathbb{R}} d\Phi(u) < \infty$ for smooth $\Phi(u)$), if and only if $g(z)$ is holomorphic in the upper half-plane and

$$\Im g(z) \geq 0 \quad \text{and} \quad \sup_{1<y<\infty} |yg(iy)| < \infty . \tag{5.5}$$

Note that the function $g(z)$ is in the class of Pick functions defined in Section 5.8.

To apply Theorem 5.2.3 to the integral (5.3) one should set $g(z) = -f(-1/z)/z$ (cf. (Baker & Graves-Morris, 1981, (6.12) on p. 215)), then conditions (5.5) read as

$$\Im f(-1/z)/z \leq 0 \quad \text{and} \quad \sup_{1<y<\infty} |f(i/y)| < \infty\,. \tag{5.6}$$

Theorem 5.2.4. *A function $f:(0,\infty)\to\mathbb{R}$ is a Stieltjes function if and only if $f(x)\geq 0$ for $x>0$ and there is a holomorphic extension $f(z)$, $z=x+iy$, to the cut plane $\mathbb{C}\backslash(-\infty,0]$ satisfying*

$$\Im f(z) \leq 0 \text{ for } \Im z > 0. \tag{5.7}$$

Remark 5.2.5. The inequalities (5.7) being a part of a characterization of the Stieltjes functions express a necessary condition for f to be a Stieltjes function. In the terminology of (Bender & Orszag, 1999, p. 358), a holomorphic function $f(z)$ is called a Herglotz function if $\Im f>0$ when $\Im z>0$, $\Im f=0$ when $\Im z=0$ and $\Im f<0$ when $\Im z<0$. Thus, for f to be a Stieltjes function it is necessary that f be an anti-Herglotz function (cf. (Bender & Orszag, 1999, p. 406)).

5.3 Stieltjes functions containing $W(z)$

In this section we consider a number of functions containing $W(z)$ and prove that they are Stieltjes functions. We begin with the function $W(z)/z$.

5.3.1 The function $W(z)/z$

The fact that $W(z)/z$ is a Stieltjes function could be established conveniently by applying one of the criteria stated in Section 5.2. However, we first present a direct proof that is of great importance for further investigations. Moreover, compared with using the criteria above, the present way allows us to make useful observations which are given in the remarks following the proof and used in further evidence.

Theorem 5.3.1. *$W(z)/z$ is a Stieltjes function.*

Proof. From (1.14), the function $W(z)/z$ is single-valued and holomorphic in the same domain as $W(z)$, namely $D = \{z \in \mathbb{C} \,|\, z \notin \mathbb{B}\}$, and can be represented by the Cauchy's integral formula

$$\frac{W(z)}{z} = \frac{1}{2\pi i} \int_C \frac{W(t)}{t(t-z)} dt \;, \tag{5.8}$$

where C is the standard 'keyhole' contour which consists of a small circle around the branch point $t = -1/e$ of radius, say r, and a large circle around the origin of radius, say R; the circles being connected through the upper and lower edges of the cut along the negative real axis. Then for sufficiently small r and large R the interior of the contour C encloses any point in D.

Let us consider the integral (5.8) in the limit in which $r \to 0$ and $R \to \infty$. Using asymptotic estimations (1.13) and (1.8), it is easily seen that the contributions of each circle to the integral (5.8) go to zero. As a result, in accordance with the assignment of values of W function on the branch cut, the integral becomes

$$\frac{W(z)}{z} = \frac{1}{2\pi i} \int_{-\infty}^{-1/e} \frac{W(t)}{t(t-z)} dt + \frac{1}{2\pi i} \int_{-1/e}^{-\infty} \frac{\overline{W(t)}}{t(t-z)} dt \;,$$

which reduces to

$$\frac{W(z)}{z} = \frac{1}{\pi} \int_{-\infty}^{-1/e} \frac{\Im W(t)}{t(t-z)} \, dt \;, \tag{5.9}$$

where $|\arg(z)| < \pi$. Changing t to $-t$ transforms the integral (5.9) to the form (5.2)

$$\frac{W(z)}{z} = \int_{1/e}^{\infty} \frac{1}{z+t} \frac{\mu(t)}{t} dt \;, \tag{5.10}$$

where

$$\mu(t) = \frac{1}{\pi} \Im W(-t) \;. \tag{5.11}$$

According to Lemma 1.2.1, $\mu(t) \in (0, 1)$ for $t \in (1/e, \infty)$, therefore $\int_{1/e}^{\infty} \mu(t) dt / [t(1+$

$t)] < \infty$ and the conditions in Definition 5.2.1 are satisfied. Thus the integral (5.10) is a Stieltjes function. □

Remark 5.3.2. The function $W(z)/z$ is a real symmetric function as is any Stieltjes function (this immediately follows from Definition 5.2.1), which just corresponds to the near conjugate symmetry property.

Remark 5.3.3. The representation of $W(z)/z$ in the form (5.3) equivalent to (5.10) is

$$\frac{W(z)}{z} = \int_0^e \frac{d\Phi(t)}{1+tz}\,, \tag{5.12}$$

where $d\Phi(t) = \mu(1/t)dt$. Since $\mu(1/t) \in (0,1)$ for $t \in (0,e)$ by Lemma 1.2.1, $\Phi'(t) \geq 0$ and thus $\Phi(t)$ is a bounded and non-decreasing function. In addition, all the moment integrals $\int_0^e t^n \, d\Phi(t)$ $(n = 0, 1, 2, \dots)$ exist. This remark is useful for justifying the use of Padé approximants for the evaluation of $W(z)$ based on the theory in (Baker & Graves-Morris, 1981, Ch. 5) (see Appendix C).

Remark 5.3.4. An existence of representation (5.12) also follows from Theorem 5.2.3. Indeed, for function $f(z) = W(z)/z$ conditions (5.6) read as

$$\Im W(-1/z) \geq 0 \quad \text{and} \quad \sup_{1<y<\infty} |yW(i/y)| < \infty\,.$$

The first condition is satisfied by (1.10) because $\Im(-1/z)$ and $\Im z$ are of the same sign. To verify the second condition we set $W(i/y) = u + iv$ and put $s = 1/y$ in (1.19) and (1.20). As a result, since $0 < v < \pi/2$ for $y > 0$, we obtain

$$|yW(i/y)|^2 = y^2(u^2+v^2) = y^2v^2(1+\tan^2 v) = y^2v^2/\cos^2 v = e^{-2v\tan v} \leq 1.$$

To extend the result to the lower half-plane $\Im z < 0$ it is enough to take the complex conjugate of both sides of the representation (5.3) and use the near conjugate symmetry of W. Thus Theorem 5.2.3 gives us one more way to prove that $W(z)/z$ is a Stieltjes function.

5.3.2 Other functions

By Theorem 5.3.1 $W(x)/x \in \mathcal{S}$. Using this result and the properties of the set $\mathcal{S}$ listed in Section 5.2 we now give some classes of functions that are members of $\mathcal{S}$.

Theorem 5.3.5. *The following functions belong to the set $\mathcal{S}$, for $x > 0$.*

(a) $1/(c + W(x))$, $c \geq 0$

(b) $W^\alpha(1/x), 0 \leq \alpha \leq 1$

(c) $x^\beta W^\beta(1/x), -1 \leq \beta \leq 0$

(d) $W(x)/[x(c + W(x))]$, $c \geq 0$

(e) $1/W(x) - 1/x$

(f) $c + W(x^\beta)$, $c \geq 0$, $-1 \leq \beta \leq 0$

(g) $1/(c + W(x^\alpha))$, $c \geq 0$, $0 \leq \alpha \leq 1$

(h) $x^{\alpha\beta\gamma} W^{-\alpha\gamma}(x^\beta)[1 + W(x^\beta)]^{1-\gamma}$, $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 0$, $0 \leq \gamma \leq 1$

(i) $1 - x^\alpha W^\alpha(1/x)$, $0 \leq \alpha \leq 1$

(j) $1 - x^{-\alpha\beta} W^\alpha(x^\beta)[1 + W(x^\beta)]^{-\alpha}$, $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 0$

Proof. We use the properties listed in Theorem 5.2.2.

(a) We apply property (ii) to $W(x)/x$ to find that $1/W(x) \in \mathcal{S}$ and then apply (iii) to $1/W(x)$.

(b) We first apply (i) to $f(x) = 1/W(x)$ that is in $\mathcal{S}$ by statement (a) and find $W(1/x) \in \mathcal{S}$. Then we apply (vii) to $W(1/x)$.

(c) Apply (i) to $W(x)/x$ and apply then (vii) to the result.

(d) Apply (xi) to the function in the statement (a) using $W(0) = 0$.

(e) Apply (viii) to $W(x)/x$ using (1.14) or apply (ix) to the function in the statement (a) with $c = 0$.

(f) Apply (v) to the function in the statement (a) and $g(x) = x^\beta$ $(-1 \leq \beta \leq 0)$ that is in $\mathcal{S}$ (Berg & Forst, 1975; Berg, 2008).

(g) Apply (iv) to the function in the statement (a) and $g(x) = x^{-\alpha} \in \mathcal{S}$ for $0 \leq \alpha \leq 1$.

(h) Apply (v) to functions $f(x) = W(x)/x$ and $g(x) = x^\beta$ $(-1 \leq \beta \leq 0)$ and find $x^\beta W^{-1}(x^\beta) \in \mathcal{S}$. Hence by (vii) $a(x) = x^{\alpha\beta}W^{-\alpha}(x^\beta) \in \mathcal{S}$ for $0 \leq \alpha \leq 1$. Then apply (v) to the function in the statement (a) with $c = 1$ and $g(x) = x^\beta$ to get $b(x) = 1 + W(x^\beta) \in \mathcal{S}$. Finally apply (vi) to $a(x)$ and $b(x)$.

(i) Apply (xii) to the function in the statement (c) with $\beta = -1$ using (1.14) (or apply (x) to $W(x)/x$).

(j) Apply (x) (or (xii)) to the result of application of (iv) (respectively (v)) to the function in the statement (d) with $c = 1$ and $g(x) = x^\beta$ $(-1 \leq \beta \leq 0)$.

□

Corollary 5.3.6. *The derivative $dW(x)/dx$ is a Stieltjes function.*

Proof. Follows from Theorem 5.3.5 (d) with $c = 1$ and formula (1.21). □

The next theorem proves and generalizes a conjecture in (Jackson, Procacci, & Sokal, 2009).

Theorem 5.3.7. *The following functions are Stieltjes functions for fixed real* $a \in (0, e]$

$$F_0(z) = \frac{z}{1+z} W(a(1+z)) / \left[W(a(1+z)) - W(a)\right]^2 \ , \tag{5.13}$$

$$F_1(z) = zW\left(\frac{a}{1+z}\right) \Big/ \left[W(a) - W\left(\frac{a}{1+z}\right)\right]^2 . \tag{5.14}$$

Proof. We first apply Theorem 5.2.4 to the function $F_0(z)$. To do so we note that $F_0(z) \geq 0$ for real $z > 0$ ($a \in (0, e]$) and $F_0(z)$ is a holomorphic function in the cut plane $\mathbb{C}\backslash(-\infty, 0]$ (cf. the branch cut $\mathbb{B}$). For convenience, we define a function $V(z) = \Im F_0(z)$, then it remains to show that $V(z) \leq 0$ in the upper half-plane. Since $V(z)$ is a harmonic function in the domain $\Im z > 0$, it is subharmonic there. Thus we can apply either the maximum principle for harmonic functions in the form of (Axler, Bourdon, & Ramey, 2001, Corollary 1.10) or the maximum principle for subharmonic functions (Doob, 1984, p. 19–20). In both cases, to get the desired result it is sufficient to ascertain that the superior limit of $V(z)$ at all boundary points including infinity is less than or equal to 0 (Alzer & Berg, 2002). In other words, $V(z) \leq 0$ for $\Im z > 0$ if (cf. (Koosis, 1988, p. 27))

$$\lim_{|z|\to\infty} V(z) \leq 0 \quad (\Im z > 0)$$

and

$$\limsup_{y\to 0+} V(x+iy) \leq 0 \text{ for all } x \in \mathbb{R} \ . \tag{5.15}$$

Since $F_0(z) \sim 1/\ln z$ for large z due to (1.8), $V(z) \to 0$ as $|z| \to \infty$ and the first condition is satisfied.

To verify the second condition we introduce variables $t = a(1+x)$ and $s = ay$ and set $W(t+is) = u + iv$ where $u = u(t,s), v = v(t,s)$. We also introduce a constant $b = W(a) \in (0, 1]$. Then the condition (5.15) becomes $H(t) \geq 0$ for all

$t \in \mathbb{R}$, where

$$H(t) = \limsup_{s\to 0+} \left\{ v\frac{[t(t-a)+s^2](u^2+v^2-b^2)}{(t^2+s^2)[v^2+(b-u)^2]} + as\frac{(u^2+v^2)(2b-u)-b^2u}{(t^2+s^2)[v^2+(b-u)^2]} \right\}. \tag{5.16}$$

For analysis of function $H(t)$, it is convenient to consider the following five cases: (i) $-\infty < t < -1/e$, (ii) $-1/e \leq t < 0$, (iii) $t = 0$, (iv) $(0 < t < a) \cup (a < t < \infty)$, and (v) $t = a$. We start with the case (i). Since $V(z)$ is continuous (from above) on the real line $z = x \in \mathbb{R}$, the expression under the limit sign in (5.16) is continuous in domain $\{(t,s)|t \in \mathbb{R}, s > 0\}$. Then using relation (1.17) we obtain

$$H(t) = \frac{v}{[(b+v\cot v)^2+v^2]^2}\left(\frac{v^2}{\sin^2 v} - b^2\right)\left(1-\frac{a}{t}\right).$$

We have $v \in (0,\pi)$ for $t \in (-\infty, -1/e)$, hence $v^2/\sin^2 v > 1$. Since $0 < b \leq 1$, we conclude that in case (i) $H(t) > 0$. Taking into account that $v = 0$ in cases (ii), (iv) and (v) and relations (1.19) and (1.20) in case (iii) it is not difficult to show that in all of these cases $H(t) = 0$. Thus $H(t) \geq 0$ for all real t, i.e. the condition (5.15) is satisfied and $F_0(z)$ is a Stieltjes function.

The theorem for the function $F_1(z)$ follows from the relation

$$F_1(z) = -F_0\left(-\frac{z}{1+z}\right) \tag{5.17}$$

because in terms of the conditions of Theorem 5.2.4 the transformation in the right hand-side of (5.17) retains the properties of $F_0(z)$. In particular, $\Im F_1(z) \leq 0$ for $\Im z > 0$ because, first, $\Im z$ and $\Im(-z/(1+z))$ are of the opposite signs and secondly, $\Im F_0(z) \geq 0$ for $\Im z < 0$ which follows from $F_0(\bar{z}) = \overline{F_0(z)}$ due to near conjugate symmetry and the established above non-positivity of $\Im F_0(z)$ in the upper half-plane. Thus $F_1(z)$ is also a Stieltjes function. □

Remark 5.3.8. We make a note about a behavior of functions (5.13) and (5.14) for large and small z. Specifically, using (1.8) and (1.14) one can obtain respectively $F_0(z) \to 0$ and $F_1(z) \to a/W^2(a)$ as $z \to \infty$. Using (1.21) we find $F_{0,1} \sim c/z$ as $z \to 0$, where $c = (1+W(a))^2/W(a)$.

We now have even a stronger than Theorem 5.3.7 result in the following corollary.

Corollary 5.3.9. *With the constant c defined in Remark 5.3.8 the differences* $F_{0,1} - c/z$ *are Stieltjes functions for fixed* $a \in (0, e]$.

Proof. Follows from Remark 5.3.8 and the property (ix) given in Theorem 5.2.2. □

5.3.3 Is W a Stieltjes function?

The principal branch of the Lambert W function itself is not a Stieltjes function in the sense of Definition 5.2.1. It can be shown in different ways. For example, one can apply Theorem 5.2.3 to $W(z)$ to see that the second condition (5.6) fails. Indeed, when $z = is$ we have by (1.19) and (1.20)

$$|sW(is)| = s\sqrt{u^2+v^2} = v^2 \sec^2(v) e^{v \tan v} \to \infty \quad \text{as} \quad v \to \pi/2.$$

The same conclusion can be reached using Theorem 5.2.4 because (1.10) contradicts (5.7). Finally, W is not a Stieltjes function because it is not an anti-Herglotz function (cf. Remark 5.2.5).

Note, however, that W function can be regarded as a Stieltjes function in the sense of a definition given in (Tokarzewski, 1996) and (Brodsky, Ellis, Gardi, Karliner, & Samue, 1997) or used in (Tokarzewski & Telega, 1998) and different from (5.3) by the factor z in the right hand-side. W function can also be considered as a generalized Stieltjes transform by the definition in (Saxena & Gupta, 1964) (which is different from that of the generalized Stieltjes transform defined in (Widder, 1938, p. 30) and studied, for example, in (Schwarz, Art. No. 013501) and (Sokal, 2010)). Finally, in (Schilling et al., 2010), the terms Stieltjes function and Stieltjes representation are not treated as equivalent (compare definitions (Schilling et al., 2010, p. 11) and (Schilling et al., 2010, p. 55)). By these definitions $W(z)$ has a Stieltjes representation (which is the result of multiplication of the representation (5.10) by z) though it is not a Stieltjes function.

5.4 Explicit Stieltjes representations

The Stieltjes representation for $W(z)/z$ given in (5.10) and (5.11) itself contains W, which can be regarded as self-referential. Here we give representations containing only elementary functions for this and other functions related to W.

Theorem 5.4.1. *The following representation of function $W(z)/z$ holds (The poster 'The Lambert W Function')*

$$\frac{W(z)}{z} = \frac{1}{\pi}\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{z + v\csc(v)e^{-v\cot v}}dv \quad (|\arg z| < \pi)\ . \tag{5.18}$$

Proof. We start with (5.9) and, noting (1.9), change to the variable $v = \Im W(t)$. The integral becomes

$$\frac{W(z)}{z} = \frac{1}{\pi}\int_0^{\pi} \frac{v}{t(z-t)}\frac{dv}{v'(t)}\ , \tag{5.19}$$

where the variables t and v are related by (1.18) and the derivative $v'(t)$ is defined by (1.22). After substitutions the result follows. □

Remark 5.4.2. Since the integrand in (5.18) is an even function (with respect to

v), the integral admits the symmetric form

$$\frac{W(z)}{z} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v^2 + (1 - v \cot v)^2}{z + v \csc(v) e^{-v \cot v}} dv \quad (|\arg z| < \pi) \, .$$

This integral has a C^∞ periodic extension and thus the midpoint rule is spectrally convergent for its quadrature (see e.g. (Weideman, 2002)).

We now take advantage of Corollary 5.3.6 and derive an integral representation of $W'(z)$.

Theorem 5.4.3. *The derivative of W function has the following Stieltjes integral representation*

$$W'(z) = \frac{W(z)}{z(1 + W(z))} = \frac{1}{\pi} \int_0^{\pi} \frac{dv}{z + v \csc(v) e^{-v \cot v}} \quad (|\arg z| < \pi) \, . \tag{5.20}$$

Proof. We take the formula (5.2) with $a = 0$ due to (1.24)

$$W'(z) = \int_0^{\infty} \frac{d\mu(t)}{z + t} \, , \tag{5.21}$$

where the unknown function $\mu(t)$ can be determined using the Stieltjes-Perron inversion formula (Henrici, 1977, p. 591) (see also Section A.3)

$$\mu(t) = \frac{1}{\pi} \lim_{s \to 0+} \Im \int_{-\infty}^{-t} W'(\tau + is) d\tau$$

for all continuity points on the t-axis. Since $\mu(t)$ is defined to arbitrary constant, after integrating one can set

$$\mu(t) = \frac{1}{\pi} \lim_{s \to 0+} \Im W(-t + is) = \frac{1}{\pi} \Im W_0(-t) \, , \tag{5.22}$$

where the limit uses the continuity from above of W on its branch cut. The same result can be obtained using one of Sokhotskyi's formulas (Henrici, 1986, p. 138).

Check that function (5.22) satisfies the conditions in Definition 5.2.1. By Lemma 1.2.1 the domain of integration in (5.21) is defined by $1/e < t < \infty$. In addition, the function $\mu(t)$ can be regarded as a positive measure such that $d\mu(t)/dt = o(1/t)$ at large t. Therefore $\int_{1/e}^{\infty}(1+t)^{-1}d\mu(t) < \infty$ and the conditions in Definition 5.2.1 are satisfied. Thus (5.21) takes the form

$$W'(z) = \frac{1}{\pi}\int_{1/e}^{\infty}\frac{1}{z+t}\frac{d\Im W_0(-t)}{dt}dt\,. \tag{5.23}$$

Changing to variable $v = \Im W_0(-t)$ in the integral (5.23) with using (1.18) we obtain (5.20). □

Remark 5.4.4. The formula (5.23) can also be found by considerations similar to those used in the proof of Theorem 5.3.1. In addition, (5.23) is a result of differentiating (5.10) with subsequent integration by parts. Finally, comparing formulae (5.10) and (5.23) shows that the latter is obtained from the former when we formally replace the ratios $W(z)/z$ and $\mu(t)/t$ respectively with the derivatives $dW(z)/dz$ and $d\mu(t)/dt$ at the same time.

Remark 5.4.5. The formulae (5.10) and (5.23) were also found in (Pakes, 2011).

Corollary 5.4.6.

$$\int_0^{\pi}\left\{\frac{\sin v}{v}e^{v\cot v}\right\}^p dv = \frac{\pi p^p}{p!}, \quad p \in \mathbb{N}\,. \tag{5.24}$$

Proof. The integral (5.20) can be written as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^n}{n!}z^{n-1} = \frac{1}{\pi}\int_0^{\pi}\frac{dv}{z-t}\,, \tag{5.25}$$

where t is defined by (1.18) and the left-hand side is obtained by differentiation of the series (1.12) that is convergent for $|z| < 1/e$. Since $|t| > 1/e$ and therefore

$|z| < |t|$, we can expand $(z-t)^{-1}$ in the non-negative powers of z. Equating the coefficients of the same power of z in (5.25) we obtain an equality

$$(-1)^{n-1}\frac{n^n}{n!} = -\frac{1}{\pi}\int_0^\pi \frac{dv}{t^n}$$

which after substituting (1.18) results in (5.24). □

It is obvious that if the integral (5.24) is known then going back from it to (5.25) we find (5.20). The integral (5.24) was conjectured by Nuttall for real $p \geq 0$ (Nuttall, 1985); Bouwkamp found a more general integral (Bouwkamp, 1986), for which Nuttall's conjecture is a special case, using a representation of $\pi p^p/\Gamma(p+1)$ via a Hankel-type integral. Thus the Stieltjes representation of the derivative of W function (5.20) allows one to compute the integral (5.24) and conversely, starting with the integral of Nuttall-Bouwkamp one can obtain formula (5.20) in a way completely different from that used in the proof of Theorem 5.4.3. It is interesting to note that the connection between (5.24) and Lambert W was noted by W.E. Hornor and C.C. Rousseau before W was named (see editorial remarks in (Nuttall, 1985)).

Coming back to the results of Theorem 5.3.5 we consider the assertion (a) with $c = 1$ and assertion (e) by which $1/(1+W(z)) \in \mathcal{S}$ and $1/W(z) - 1/z \in \mathcal{S}$. We can derive integral representations of these functions in the same manner as it was done for $W'(z)$ in the proof of Theorem 5.4.3. The result is in the following theorem.

Theorem 5.4.7. *The following Stieltjes integral representations hold*

$$\frac{1}{1+W(z)} = \frac{1}{\pi}\int_0^\pi \frac{dv}{1+ze^{v\cot v}\sin v/v} \quad (|\arg z| < \pi)\ , \tag{5.26}$$

$$\frac{1}{W(z)} = \frac{1}{z} + \frac{1}{\pi}\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{v\csc(v)\,(v\csc(v) + ze^{v\cot v})}dv \quad (|\arg z| < \pi)\,. \tag{5.27}$$

Corollary 5.4.8.

$$W(z) = \ln\left[1 + \frac{z}{\pi}\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{v\csc(v)\,(v\csc(v) + ze^{v\cot v})}dv\right]\,. \tag{5.28}$$

Proof. By substituting (5.27) in $W(z) = \ln(z/W(z))$ (cf. (1.11)). □

Remark 5.4.9. The formulae (5.20) and (5.26) were first found by A. Sokal (Sokal, 2008) where it was also pointed out that the found explicit Stieltjes representations can be used to obtain those for functions containing $W(1/z)$ by just replacing z with $1/z$. For example, formula (5.20) yields

$$\frac{W(1/z)}{1 + W(1/z)} = \frac{1}{\pi}\int_0^{\pi} \frac{dv}{1 + zv\csc(v)e^{-v\cot v}} \quad (|\arg z| < \pi)\,.$$

5.5 Completely monotonic functions

We denote by $\mathcal{CM}$ the set of all completely monotonic functions, which are defined as follows (Alzer & Berg, 2006).

Definition 5.5.1. A function $f : (0, \infty) \to \mathbb{R}$ is called a *completely monotonic* function if f has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x > 0$, $n = 0, 1, 2, ...$

The set of Stieltjes functions is contained in the set of completely monotonic functions, and thus all of the functions listed in Theorem 5.3.5 are completely monotone. The set $\mathcal{CM}$ is a convex cone containing the positive constant functions; a product of completely monotonic functions is again completely monotone

(Berg & Forst, 1975, p. 61). By Bernstein's theorem (Berg & Forst, 1975, Theorem 9.3), a function $f \in \mathcal{CM}$ if and only if it is of the form

$$f(x) = \int_0^\infty e^{-x\xi} d\tilde{\nu}(\xi) \quad (x > 0), \tag{5.29}$$

where $\tilde{\nu}$ is an uniquely determined positive measure on $[0, \infty)$. Completely monotonic functions are in turn connected with the set of Bernstein functions denoted by $\mathcal{B}$.

Definition 5.5.2. (Berg, 2008, Definition 5.1) A function $f : (0, \infty) \to [0, \infty)$ is called a *Bernstein function* if it is C^∞ and f' is completely monotonic.

Since $W' \in \mathcal{S} \subset \mathcal{CM}$, W is a Bernstein function. The same fact has been established in Section 4.2.5 in a different way based on the properties of the polynomials appearing in the higher derivatives of W.

A Bernstein function $f(x)$ admits the Lévy-Khintchine representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-x\xi}\right) d\nu(\xi) , \tag{5.30}$$

where $a, b \geq 0$ and ν is a positive measure on $(0, \infty)$ satisfying $\int_0^\infty \xi(1+\xi)^{-1} d\nu(\xi) < \infty$. It is called the Lévy measure. The equation (5.30) is obtained by integrating (5.29) written for f' (Berg, 2008).

An important relation between the classes $\mathcal{S}$ and $\mathcal{B}$ is given by the assertion (Berg, 2008, Theorem 5.4)

$$g \in \mathcal{S} \setminus \{0\} \Rightarrow 1/g \in \mathcal{B}. \tag{5.31}$$

Combining this with the function composition result (Berg, 2008, Corollary 5.3) that $f \in \mathcal{CM}$ and $g \in \mathcal{B}$ implies $f \circ g \in \mathcal{B}$ we obtain the following lemma.

Lemma 5.5.3. *If $f \in \mathcal{CM}$ and $g \in \mathcal{S} \setminus \{0\}$ then $f(1/g) \in \mathcal{CM}$.*

This lemma extends the list of completely monotonic functions containing W.

Theorem 5.5.4. *The following functions are completely monotonic*

(a) $x^{\lambda}W(x)$ $(x > 0,\ \lambda \le -1)$.

(b) $x^{\lambda}W^{\alpha}(x^{\beta})\left[1 + W(x^{\beta})\right]^{\gamma}$ $(x > 0,\ \alpha, \gamma \ge 0,\ -1 \le \beta \le 0,\ \lambda \le 0)$.

(c) $x^{\lambda}W^{\alpha}(x^{-\beta})\left[1 + W(x^{-\beta})\right]^{\gamma}$ $(x > 0,\ \alpha, \gamma \le 0,\ -1 \le \beta \le 0,\ \lambda \le 0)$.

(d) $1 - x^{-\alpha\beta\gamma}W^{\alpha\gamma}(x^{\beta})[1 + W(x^{\beta})]^{\gamma-1}$ $(x > 0,\ 0 \le \alpha \le 1,\ -1 \le \beta \le 0,\ 0 \le \gamma \le 1)$.

Proof. (a) Since $W(x)/x \in \mathcal{S} \subset \mathcal{CM}$ and $x^{\alpha} \in \mathcal{CM}$ for $\alpha \le 0$, the function $x^{\lambda}W(x)$ $(\lambda \le -1)$ is a product of two completely monotonic functions and the statement (a) follows.

(b) Take function $f_{\alpha}(x) = x^{-\alpha} \in \mathcal{CM}$ $(x > 0,\ \alpha \ge 0)$ and functions $g(x) = 1/W(x^{\beta})$ and $h(x) = 1/(1 + W(x^{\beta}))$ where $-1 \le \beta \le 0$. Since $1/g \in \mathcal{S}$ and $1/h \in \mathcal{S}$ by Theorem 5.3.5 (f) with $c = 0$ and $c = 1$ respectively, by Lemma 5.5.3 we have $f_{\alpha}(g(x)) = g^{-\alpha}(x) \in \mathcal{CM}$ and $f_{\gamma}(h(x)) = h^{-\gamma}(x) \in \mathcal{CM}$ $(\gamma \ge 0)$. Substituting functions $g(x)$ and $h(x)$ in the power functions and taking a product of obtained completely monotonic functions with $x^{\lambda} \in \mathcal{CM}$ $(x > 0,\ \lambda \le 0)$, the statement (b) follows.

(c) Consider function $f_{\lambda}(x) = x^{\lambda} \in \mathcal{CM}$ $(x > 0,\ \lambda \le 0)$ and functions $g(x) = W(x^{-\beta})$ and $h(x) = 1 + W(x^{-\beta})$ where $-1 \le \beta \le 0$. Since $1/g \in \mathcal{S}$ and $1/h \in \mathcal{S}$ by Theorem 5.3.5 (g) with $c = 0$ and $c = 1$ respectively, by Lemma 5.5.3 we have $f_{\alpha}(g(x)) = g^{\alpha}(x) \in \mathcal{CM}$ and $f_{\gamma}(h(x)) = h^{\gamma}(x) \in \mathcal{CM}$ for $\alpha \le 0$ and $\gamma \le 0$. Substituting functions $g(x)$ and $h(x)$ and taking a product of obtained functions with $f_{\lambda}(x)$, the statement (c) follows.

(d) By Theorem 5.3.5 (h) and the assertion (5.31), for $x > 0$, $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 0$, $0 \leq \gamma \leq 1$ we have $f(x) = g^{\alpha\gamma}(x)[1 + W(x^{\beta})]^{\gamma-1} \in \mathcal{B}$, where $g(x) = x^{-\beta}W(x^{\beta})$. In addition, the function $f(x)$ is bounded, particularly, $0 < f(x) < 1$ because $0 < [1 + W(x^{\beta})]^{\gamma-1} < 1$ and $0 < g(x) < 1$ (the latter follows from the fact that $g(x)$ goes to 0 and 1 as x tends to 0 and ∞ respectively and $g'(x) > 0$, which can be established using (1.14), (1.15) and (1.21)). Then by (Berg, 2008, Remark 5.5) the assertion (d) follows.

□

We considered only sufficient conditions for a function to be a completely monotonic. To find the necessary and sufficient conditions is a much more complicated problem so that in some cases it requires (at least as the first step) using the methods of experimental mathematics (Shemyakova, Khashin, & Jeffrey, 2010).

5.6 Complete Bernstein functions

A very important subclass in $\mathcal{B}$ is the class of complete Bernstein functions denoted by $\mathcal{CB}$.

Definition 5.6.1. (Schilling et al., 2010, Definition 6.1) A Bernstein function f is called a complete Bernstein function if the Lévy measure in (5.30) is such that $d\nu(t)/dt$ is a completely monotonic function.

We point out four connections between classes $\mathcal{CB}$ and $\mathcal{S}$ used in this paper (for additional relations between these classes see (Schilling et al., 2010, Chapter 7)). By Proposition 7.7 in (Schilling et al., 2010),

$$f \in \mathcal{S} \Rightarrow f(0) - f(x) \in \mathcal{CB}\,, \tag{5.32}$$

where the limit of $f(x)$ at $x = 0$ (from the right) is assumed to be finite. Also if f is bounded and $f \in \mathcal{CB}$, there exists a bounded $g \in \mathcal{S}$ with $\lim_{x\to\infty} g(x) = 0$ such that

$$f(x) = f(0) + g(0) - g(x)\,. \tag{5.33}$$

In addition, (Schilling et al., 2010, Theorem 7.3) and (Schilling et al., 2010, Theorem 6.2(i),(ii)) establish

$$f \in \mathcal{CB} \Leftrightarrow 1/f \in \mathcal{S}\setminus\{0\}\ , \tag{5.34}$$

$$f \in \mathcal{CB} \Leftrightarrow f(x)/x \in \mathcal{S}\ . \tag{5.35}$$

Now we go back to the properties of the set $\mathcal{S}$ listed in Section 5.2 to prove the last three properties therein. Let $f \in \mathcal{S}\setminus\{0\}$.

(x) Apply sequentially (vii), (5.32), (5.34), (i), to obtain $f^\alpha \in \mathcal{S}$ $(0 \le \alpha \le 1) \Rightarrow f^\alpha(0) - f^\alpha(x) \in \mathcal{CB} \Rightarrow g(x) = [f^\alpha(0) - f^\alpha(x)]^{-1} \in \mathcal{S} \Rightarrow 1/g(1/x) = f^\alpha(0) - f^\alpha(1/x) \in \mathcal{S}$;

(xi) Apply sequentially (5.32), (5.34), (ii), to obtain $f(0) - f(x) \in \mathcal{CB} \Rightarrow g(x) = [f(0) - f(x)]^{-1} \in \mathcal{S} \Rightarrow 1/(xg(x)) = (f(0) - f(x))/x \in \mathcal{S} \Rightarrow (1 - f(x)/f(0))/x \in \mathcal{S}$;

(xii) By (vii), $f^\alpha \in \mathcal{S}$ $(0 \le \alpha \le 1)$. Suppose that $\lim_{x\to 0} f(x) = b \le \infty$ and $\lim_{x\to\infty} f(x) = c$ where $0 < c < \infty$. Then $b^{-\alpha} \le f^{-\alpha} \le c^{-\alpha}$, i.e. $f^{-\alpha}$ is bounded. In addition, $f^{-\alpha} \in \mathcal{CB}$ by (5.34). Therefore the statement (5.33) can be applied, i.e. there exists a bounded function $g \in \mathcal{S}, \lim_{x\to\infty} g(x) = 0$ such that we can write $g(x) = g(0) + b^{-\alpha} - f^{-\alpha}(x)$. Taking the last equation in the limit $x \to \infty$ we obtain $g(0) + b^{-\alpha} = c^{-\alpha}$, hence $g = c^{-\alpha} - f^{-\alpha}$ and the assertion follows.

In closing this section we note that the statement (5.34) with $1/W \in \mathcal{S}$ (by Theorem 5.3.5(a) with $c = 0$) immediately results in $W \in \mathcal{CB}$. Being a complete Bernstein function W has an integral representation that is the result of multiplication of (5.18) by z (Schilling et al., 2010, Remark 6.4), which reflects the relation (5.35). In addition, the complete Bernstein functions are closely connected to the Pick functions considered in Section 5.8.

5.7 Bernstein representations

Not only does $W \in \mathcal{CB}$ as shown, it also belongs to another subset of Bernstein functions.

Definition 5.7.1. (Schilling et al., 2010, Definition 8.1) A Bernstein function f is called a *Thorin-Bernstein function* if the Lévy measure in (5.30) is such that $t\,d\nu(t)/dt$ is completely monotonic function.

To find out whether W is a Thorin-Bernstein function we apply Theorem 8.2 in (Schilling et al., 2010), which establishes five equivalent assertions (i)-(v) which we refer to below. In particular, in accordance with assertions (i) and (ii), $W(x)$ is a Thorin-Bernstein function because $W(x)$ maps $(0, \infty)$ to itself, $W(0) = 0$ and $W'(x) \in \mathcal{S}$. Then $W(x)$ admits two integral representations stated in the assertions (v) and (iii). The former has been already obtained; it is given by (5.10). The latter can be derived from the former, which is shown in the following theorem.

Theorem 5.7.2. *The principal branch of the W function can be represented as the integral*

$$W(z) = \frac{1}{\pi} \int_0^\pi \ln\left(1 + z\frac{\sin v}{v} e^{v \cot v}\right) dv \quad (|\arg z| < \pi) \ . \tag{5.36}$$

Proof. Integration (5.10) by parts with accounting for (1.9) gives

$$W(x) = \frac{1}{\pi}\int_{1/e}^{\infty} \ln\left(1+\frac{x}{t}\right)\frac{d}{dt}\Im W(-t)dt\,. \tag{5.37}$$

By Lemma 1.2.1 a measure $\Im W(-t)$ satisfies the requirements in the assertion (iii). Changing to the variable $v = \Im W(-t)$ with the help of (1.18) and taking a holomorphic extension of the result to the cut z-plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying near conjugate symmetry, we obtain (5.36). □

Remark 5.7.3. In the terminology of (Schilling et al., 2010, p. 75), the integral form (5.37) is the Thorin representation of W function and $\mu(t) = \Im W(-t)/\pi$ is the Thorin measure of W.

Remark 5.7.4. Differentiating the representation (5.36) for $W(z)$ gives formula (5.20) for $W'(z)$.

Remark 5.7.5. The representation (5.37) (up to changing t to $-t$) was obtained in (Caillol, 2003) as a dispersion relation for the principal branch of W function using the Cauchy's integral formula in a manner similar to the method applied for the proof of Theorem 5.3.1. The same formula was also found in (Pakes, 2011).

As a Bernstein function, W can be written in the form (5.30) with $a = 0$ and $b = 0$ due to $W(0) = 0$ and (1.15). It allows us to establish one more representation of W.

Theorem 5.7.6. *For the principal branch of W function the following formula holds*

$$W(z) = \int_0^{\infty} \frac{1-e^{-z\xi}}{\xi}\varphi(\xi)d\xi \quad (\Re z \geq 0), \tag{5.38}$$

where

$$\varphi(\xi) = \frac{1}{\pi}\int_0^{\pi} \exp\left(-\xi v \csc(v) e^{-v\cot v}\right) dv. \tag{5.39}$$

Proof. We consider the Stieltjes integral form (5.1) for the derivative

$$W'(x) = \int_0^\infty \frac{d\mu(\theta)}{x+\theta}$$

and use representation $(x+\theta)^{-1} = \int_0^\infty e^{-(x+\theta)\xi} d\xi$ to write it in the form

$$W'(x) = \int_0^\infty \left\{ \int_0^\infty e^{-\xi\theta} d\mu(\theta) \right\} e^{-x\xi} d\xi \,. \tag{5.40}$$

Comparing (5.40) and the result of differentiating (5.30) we find the relation between measures μ and ν (Berg, 2005)

$$\frac{d\nu}{d\xi} = \frac{1}{\xi} \int_0^\infty e^{-\xi\theta} d\mu(\theta) \,.$$

Using formula (5.22) and changing the variable $v = \Im W(-\theta)$ (~~see~~ (1.18)) we obtain

$$d\nu = \frac{\varphi(\xi)}{\xi} d\xi, \tag{5.41}$$

where $\varphi(\xi)$ is defined by (5.39). We collect the intermediate results and take a holomorphic continuation of (5.30) to the right half-plane $\Re z \geq 0$ where the integral (5.38) is convergent, in accordance with near conjugate symmetry (cf. Proposition 3.5 in (Schilling et al., 2010)). □

Note that by (5.39) function $\varphi(\xi) \in \mathcal{CM}$, as should be, because W is still a Thorin-Bernstein function (cf. Definition 5.7.1)

Remark 5.7.7. Formulae (5.38)–(5.39) were also obtained in (Pakes, 2011).

5.8 Pick representations

Definition 5.8.1. (Berg, 2008, Definition 4.1) A function $f(z)$ is called a *Pick function* (or *Nevanlinna function*) if it is holomorphic in the upper half-plane

$\Im z > 0$ and $\Im f \geq 0$ there.

A Pick function $f(z)$ admits an integral representation (Berg, 2008, Theorem 4.4)

$$f(z) = \alpha_0 + b_0 z + \int_{-\infty}^{\infty} \frac{1+tz}{(t-z)(1+t^2)} d\sigma(t) \quad (\Im z > 0) \,, \tag{5.42}$$

where

$$\alpha_0 = \Re f(i), \quad b_0 = \lim_{y\to\infty} \frac{f(iy)}{iy} \,, \tag{5.43}$$

and a positive measure σ satisfies

$$\lim_{s\to 0+} \frac{1}{\pi} \int_{\mathbb{R}} \Im f(t+is)\varphi(t)dt = \int_{\mathbb{R}} \varphi(t) d\sigma(t) \tag{5.44}$$

for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support. The formula (5.42) with the integral written in terms of a measure $d\tilde{\sigma}(t) = \pi(1+t^2)^{-1} d\sigma(t)$ is called a *Nevanlinna formula* (Levin, 1996, p. 100).

Since $W(z)$ is a holomorphic function in the upper half-plane $\Im z > 0$ with the property (1.10), $W(z)$ is a Pick function. It also follows from the two facts that $W \in \mathcal{CB}$ (see Section 5.6) and that the complete Bernstein functions are exactly those Pick functions which are non-negative on the positive real line (Schilling et al., 2010, Theorem 6.7). Thus W admits a representation (5.42) and in view of that the following theorem holds.

Theorem 5.8.2. *The principal branch of W function can be represented in the form*

$$W(z) = \alpha_0 + \frac{1}{\pi} \int_0^{\pi} K(z,v) t(v) dv \quad (|\arg z| < \pi) \,, \tag{5.45}$$

where $\alpha_0 = \Re W(i) = 0.3746990..$,

$$K(z,v) = \frac{(1+zt(v))\,(v^2 + (1 - v\cot v)^2)}{(z - t(v))\,(1+t^2(v))} \,, \tag{5.46}$$

and $t(v)$ is defined by (1.18).

Proof. Apply formulas (5.43)-(5.44) to function $f(z) = W(z)$

$$\alpha_0 = \Re W(i), \quad b_0 = \lim_{y\to\infty} \frac{W(iy)}{iy}, \quad d\sigma(t) = \frac{1}{\pi}\Im W(t)dt.$$

Using (1.15), we see $b_0 = 0$. Since $\Im W(t) = 0$ for $t \geq -1/e$ (cf. (1.9)), we obtain

$$W(z) = \alpha_0 + \frac{1}{\pi}\int_{-\infty}^{-1/e} \frac{1+tz}{(t-z)(1+t^2)}\Im W(t)dt \quad (\Im z > 0)\ . \tag{5.47}$$

By the change of variable $v = \Im W(t)$ in the integral (5.47) (see (1.18)) we obtain formula (5.45) that is also valid in the lower half-plane $\Im z < 0$ in accordance with near conjugate symmetry of W. □

Corollary 5.8.3.

$$\frac{W(z)}{z} = \gamma_0 \exp\left\{-\frac{1}{\pi}\int_0^{\pi} K(z,v)t(v)dv\right\} \quad (|\arg z| < \pi)\ , \tag{5.48}$$

where $\gamma_0 = e^{-\Re W(i)} = 0.6874961...$

Proof. It immediately follows from (5.45) owing to the identity $W(z)/z = e^{-W(z)}$. □

Now we take advantage of the fact that if function $f \in \mathcal{S}$ then $-f$ and $1/f$ are Pick functions (Berg, 2008). Therefore, since $W(x)/x \in \mathcal{S}$, $-W(x)/x$ and $x/W(x)$ are Pick functions that admit a representation (5.42). We can obtain a representation (5.42) for functions $-W(x)/x$ and $x/W(x)$ similar to the derivation of formula (5.45), and the result is in the following theorem.

Theorem 5.8.4. *For the principal branch of the W function the following formulas hold*

$$\frac{W(z)}{z} = \beta_0 + \frac{1}{\pi}\int_0^{\pi} K(z,v)dv \quad (|\arg z| < \pi) \ , \tag{5.49}$$

$$\frac{z}{W(z)} = \eta_0 - \frac{1}{\pi}\int_0^{\pi} K(z,v)e^{-2v\cot v}dv \quad (|\arg z| < \pi) \ , \tag{5.50}$$

where $K(z,v)$ *is defined by* (5.46), $\beta_0 = \Re\left[W(i)/i\right] = \Im W(i) = 0.5764127..$, $\eta_0 = \Re[i/W(i)] = 1.2195314..$.

The constants in (5.45) and (5.48)-(5.50) obey the relations $\alpha_0 + i\beta_0 = W(i)$, $\gamma_0 = e^{-\alpha_0} = \beta_0/\cos\beta_0$, $\eta_0 = \beta_0/(\alpha_0^2+\beta_0^2)$.

We add in one more integral representation associated with the Nevanlinna formula which follows from the result obtained by Cauer (Cauer, 1932). Specifically, based on the Riesz-Herglotz formula (Levin, 1996, p. 99) Cauer proved that if a real symmetric function $f(z)$ with non-negative real part is holomorphic in the right z-half-plane, it can be represented as

$$f(z) = z\left[b + \int_0^{\infty} \frac{dh(r)}{z^2+r}\right] , \tag{5.51}$$

where constant $b \geq 0$ and

$$h(r) = \frac{2}{\pi}\lim_{x\to 0}\Re\int_0^{\sqrt{r}} f(x+iy)dy \ . \tag{5.52}$$

In fact, the formula (5.51) follows from the Nevanlinna formula (or (5.42)) after changing the variable $z \to -iz$, which transforms the upper half-plane onto the right half-plane, and taking into account $f(\bar{z}) = \overline{f(z)}$.

Theorem 5.8.5. *The following representation of function* $W(z)/z$ *holds*

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^{\pi/2} \frac{[v^2+(1+v\tan v)^2]\, v\sec(v)e^{v\tan v}}{z^2+v^2\sec^2(v)e^{2v\tan v}}\tan v\, dv \quad (\Re z > 0) \ . \tag{5.53}$$

Proof. Since W function meets the above requirements, the formulas (5.51) and (5.52) can be applied with the result

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^\infty \frac{\Re W(is)}{z^2+s^2}ds \quad (\Re z > 0)\ ,$$

where we set $b = 0$ due to (1.15) and $r = s^2$.

Changing the variables defined by (1.19)-(1.20) we obtain

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^{\pi/2} \frac{v\tan v}{z^2+s^2(v)}\frac{ds}{dv}dv\ . \tag{5.54}$$

Similar to (1.22) one can find

$$\frac{dv}{ds} = \frac{v}{s(v)\left[v^2+(1+v\tan v)^2\right]}\ . \tag{5.55}$$

Substituting (5.55) and (1.20) into (5.54), the theorem follows. □

Remark 5.8.6. Comparison of the formula (5.53) with the representation (5.18) (taken in the right z-half-plane) shows that the integrand in the former contains z^2 rather than z, which can be profitable in using the integral representations for numerical evaluation of $W(z)$ at large z.

5.9 Poisson's integrals

Theorem 5.9.1. *The following two formulae of Poisson* [1] (Poisson, 1823) *hold for* $x \in (-1/e, e)$

$$W(x) = \frac{2}{\pi} \int_0^\pi \frac{\cos\frac{3}{2}\theta - xe^{-\cos\theta}\cos\left(\frac{5}{2}\theta + \sin\theta\right)}{1 - 2xe^{-\cos\theta}\cos(\theta + \sin\theta) + x^2 e^{-2\cos\theta}} \cos\tfrac{1}{2}\theta \; \mathrm{d}\theta \tag{5.56}$$

$$W(x) = -\frac{2}{\pi} \int_0^\pi \frac{\sin\frac{3}{2}\theta + xe^{\cos\theta}\sin\left(\frac{5}{2}\theta - \sin\theta\right)}{1 + 2xe^{\cos\theta}\cos(\theta - \sin\theta) + x^2 e^{2\cos\theta}} \sin\tfrac{1}{2}\theta \; \mathrm{d}\theta \tag{5.57}$$

Proof. We consider the defining equation (1.7) for given real $z = x$

$$We^W = x \tag{5.58}$$

and interpret it as an equation with respect to W. Then we can write the equation in the form $F(W) = 0$ where

$$F(\zeta) = \zeta - xe^{-\zeta} \, . \tag{5.59}$$

Let Γ be the positively-oriented circumference of the unit circle $|\zeta| = 1$ in the complex ζ-plane and domain G be the interior of Γ. The function $F(\zeta)$ is holomorphic in G and by Rouché's theorem it has a single isolated zero there when $|x| < 1/e$ because in this case $\left|-xe^{-\zeta}\right| < |\zeta|$ on Γ. Therefore, using Cauchy's

[1] The second formula is explicitly given in (Poisson, 1823, sec. 80, p. 501) in terms of the tree function $T(x)$ (see (5.62) and (5.63) below) and proved using the Lagrange Inversion Theorem (Whittaker & Watson, 1927, p. 133) and a series expansion of the logarithmic function $-\ln(1 - e^{ix}\phi)$ in powers of e^{ix} where the expansion coefficients ϕ^n/n are exactly the coefficients of the complex exponential Fourier series for the same function. On the other hand, today it is well known (Carathéodory, 1958, p. 143-145) that there is a tight connection between the classical Poisson Formula and the Cauchy Integral Formula. Our proof is based on the latter and thereby differs from that given in the original.

integral formula with taking Γ for the integration contour we can write

$$W = \frac{1}{2\pi i}\int_{\Gamma}\frac{F'(\zeta)}{F(\zeta)}\zeta d\zeta\,. \tag{5.60}$$

for $|x| < 1/e$.

Since $F'(\zeta) = 1 + xe^{-\zeta} = 1 + \zeta$ by (5.59) and (5.58), we obtain

$$W = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}(1+e^{i\theta})}{1 - xe^{-\cos\theta - i(\theta+\sin\theta)}}d\theta\,, \tag{5.61}$$

where we set $\zeta = e^{i\theta}, -\pi \leq \theta \leq \pi$. Separating the real and imaginary parts of the integrand in (5.61) we find that the former is an even function of θ whereas the latter is an odd one. Thus, the integral of the imaginary part vanishes, as should be, and the integral of the real part gives double the value of the integral on $[0, \pi]$. As a result, after some arrangements, we come to integral (5.56).

If, instead of (5.58), we consider the equation defining the (Cayley) 'tree function' $T(x)$ (Flajolet & Sedgewick, 2009, p.127-128)

$$Te^{-T} = x \tag{5.62}$$

and introduce function $H(\zeta) = \zeta - xe^{\zeta}$ in a similar way as function (5.59) then after analogous considerations and taking into account a relation

$$W(x) = -T(-x) \tag{5.63}$$

in the final result we obtain formula (5.57).

Now we discuss the domain of validity of the integrals (5.56) and (5.57) which is actually wider than the interval $-1/e < x < 1/e$ arisen above in applying Rouché's theorem. It immediately follows from the fact that W is a single valued function and therefore $F(\zeta)$, the denominator in (5.60), has a single zero in G

for each such x that $|\zeta| < 1$, i.e. for $-1/e < x < e$. Since Rouché's theorem is a consequence of the argument principle (see e.g. (Markushevich, 1965)), it would be instructive to obtain this result using the latter. To do this, say for integral (5.56), we apply the argument principle to function (5.59) in case when $x > 0$. It is easy to see that function $\eta = F(\zeta)$ performs a conformal mapping of the strip $\{-\infty < \Re\zeta < \infty, -\pi < \Im\zeta < \pi\}$, containing entire the domain G, to the complex η-plane cut along two semi-infinite lines on which $\eta = \xi \pm i\pi, \xi \geq 1 + \ln x$. We also cut the η-plane along the negative real axis to take $|\arg \eta| \leq \pi$ in the cut plane and consider an image of Γ which is defined by equations

$$\rho \cos \varphi = \cos \theta - xe^{-\cos\theta} \cos(\sin \theta) \; , \tag{5.64a}$$

$$\rho \sin \varphi = \sin \theta + xe^{-\cos\theta} \sin(\sin \theta) \; , \tag{5.64b}$$

where $\rho = |\eta|$ and $\varphi = \arg \eta$.

The equations (5.64) are invariant under transformation $\theta \to -\theta, \varphi \to -\varphi$ and describe a closed curve $\tilde{\Gamma}$ that is symmetric with respect to the real axis in the η-plane. Suppose that while a variable point ζ moves along Γ once in the ζ-plane, the image point $\eta = F(\zeta)$ moves on $\tilde{\Gamma}$ once in the η-plane, making one cycle about the origin. Then the change in argument of η is 2π and therefore, by the argument principle the function $F(\zeta)$ has a single zero in G (Markushevich, 1965, p. 48). For this it is necessary that two points on $\tilde{\Gamma}$ corresponding to $\varphi = 0$ and $\varphi = \pi$ are located on the real axis on the opposite sides of the origin, i.e. with positive ρ to be measured on the opposite rays. Substituting $\theta = \pi$ in (5.64) gives $\rho \cos \varphi = -1 - xe$ and $\rho \sin \varphi = 0$. It can be $\rho > 0$ only when $\varphi = \pi$; then $\rho = 1 + xe$ is positive for any $x > 0$. When $\theta = 0$, we have $\rho \cos \varphi = 1 - x/e$ and $\rho \sin \varphi = 0$. Now $\varphi = 0$ and $\rho = 1 - x/e > 0$ when $x < e$. Thus for $0 < x < e$ the curve $\tilde{\Gamma}$ encloses the origin. Since for these x the right-hand side of equation (5.64b) vanishes, i.e. $\Im\eta = 0$ sequentially at $\theta = -\pi, \theta = 0$ and $\theta = \pi$ as θ continuously changes from $-\pi$ to π, the curve $\tilde{\Gamma}$ is traversed once with exactly one cycle about the origin being made. This corresponds to the fact that the inverse of the mapping $\eta = F(\zeta)$ is continuous in the domain bounded by the curve $\tilde{\Gamma}$ and on $\tilde{\Gamma}$ itself and hence $\tilde{\Gamma}$ consists only of simple points (Markushevich, 1967, Theorem 2.22). Thus, by the argument principle the function $F(\zeta)$ has a single zero in G. Summarizing up the obtained results we conclude that the integral (5.56) is valid for $x \in (-1/e, e)$. The integral (5.57) can be considered in a similar manner. □

Remark 5.9.2. The integral representations (5.56) and (5.57) can be immediately applied to the tree function using relation (5.63).

Remark 5.9.3. We can apply the above approach to the equation (1.11). To eliminate a singularity at the origin we compose the integration contour of a small circle of radius, say r, and the unit circle, both centered at the origin and connected through the cut along the negative real axis. Then, making r go to zero, we find for $0 < x < e$

$$W(x) = \psi(x) + \frac{2}{\pi} \int_0^{\pi} \frac{\cos\frac{\theta}{2} + \theta \sin\frac{3}{2}\theta - \cos\frac{3}{2}\theta \ln x}{1 + 2\theta \sin\theta + \theta^2 - 2\cos\theta \ln x + \ln^2 x} \cos\frac{\theta}{2} d\theta \,,$$

where

$$\psi(x) = \int_0^1 \frac{t-1}{\pi^2 + (\ln x + t - \ln t)^2} dt \,.$$

5.10 Burniston-Siewert representations

One of the analytic methods for solving transcendental equations is based on a canonical solution of the suitably posed Riemann-Hilbert boundary-value problem (Henrici, 1986, p. 183-193). This method was found and developed by Burniston and Siewert (Burniston & Siewert, 1973), its versions, variations and applications were also considered by other authors. By the method, a solution of a transcendental equation is represented as a closed-form integral formula that can be regarded as an integral representation of the unknown variable. Below we consider such integrals for W function which are based on the results of application of the Burniston-Siewert method to solving equation (5.58) obtained in paper (Anastasselou & Ioakimidis, 1984a) and the classical work (Siewert & Burniston, 1973).

We start with two formulas derived in (Anastasselou & Ioakimidis, 1984a) and apply them to function (5.59)

$$W(x) = -F(0)\exp\left\{-\frac{1}{2\pi i}\int_\Gamma \frac{\ln\left(F(\zeta)/\zeta\right)}{\zeta}d\zeta\right\}\ , \tag{5.65}$$

$$W(x) = -\frac{1}{2\pi i}\int_\Gamma \ln\left(\frac{F(\zeta)}{\zeta}\right)d\zeta, \tag{5.66}$$

where the integration contour Γ is the unit circle $|\zeta| = 1$ and $x \in (-1/e, e)$. Since $F(0) = -x$ and $W(x)/x = e^{-W(x)}$, formula (5.65) is simplified

$$W(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln\left(F(\zeta)/\zeta\right)}{\zeta}d\zeta\ . \tag{5.67}$$

We set $\zeta = e^{i\theta}, -\pi \leq \theta \leq \pi$. Then, as $F(\zeta)/\zeta = F(e^{i\theta})e^{-i\theta} = R(\theta) + iI(\theta)$, where

$$\begin{aligned} R(\theta) =&\ \ 1 - xe^{-\cos\theta}\cos(\theta + \sin\theta), \\ I(\theta) =&\ \ \ \ \ \ \ xe^{-\cos\theta}\sin(\theta + \sin\theta), \end{aligned}$$

and $d\zeta/\zeta = id\theta$, the integral (5.67) is reduced to

$$W(x) = \frac{1}{2\pi}\int_0^\pi \ln\left(R^2(\theta) + I^2(\theta)\right)d\theta\ . \tag{5.68}$$

Similarly, the integral (5.66) can be represented in the form

$$W(x) = \frac{1}{2\pi} \int_0^{\pi} \left\{ 2\arctan(I(\theta)/R(\theta)) \sin\theta - \ln\left(R^2(\theta) + I^2(\theta)\right) \cos\theta \right\} d\theta \,, \tag{5.69}$$

where we have taken into account that $\arg(R(\theta) + iI(\theta)) = \arctan(I(\theta)/R(\theta))$ as $R(\theta) > 0$ for $0 < \theta < \pi$ and $-1/e < x < e$. We note that the integral (5.68) has a simpler form than (5.69). Integrals similar to the above with using a function $\Phi(\zeta) = \zeta e^{\zeta} - x$ in our notations instead of (5.59) in formulas (5.65) and (5.66) (without simplification (5.67)) are given in (Anastasselou & Ioakimidis, 1984a).

Thus the integrals (5.68) and (5.69) representing the principal branch of the Lambert W function are valid in the domain that contains interval $(-1/e, 0)$. However, there is one more branch that is also a real-valued function on this interval, this is the branch -1 with the range $(-\infty, -1)$ (recall $W_0 > -1$ and $W_0(-1/e) = W_{-1}(-1/e) = -1$) (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996). A representation of this branch can be obtained on the basis of a simple interpretation of formula (5.66) given in (Anastasselou & Ioakimidis, 1984b)

$$W_{-1}(x) = 1 - 2c - \frac{1}{2\pi i} \int_C \ln\left(\frac{F(\zeta)}{\zeta}\right) d\zeta, \tag{5.70}$$

where the circle C is defined by equation $|\zeta + c| = c - 1$ with arbitrary constant $c > 1$ and $-1/e < x < -(2c-1)e^{1-2c}$. Transformations of (5.70) leading to a definite integral are similar to those used above to obtain the integrals (5.68) and (5.69) and skipped here together with a bulky result.

We return to the principal branch and use the result in (Siewert & Burniston, 1973, formula(13)) to write (Wolfram Research, Inc.)

$$W(z) = 1 + (\ln z - 1) \exp\left(\frac{i}{2\pi} \int_0^{\infty} \ln\left(\frac{\ln z + t - \ln t + i\pi}{\ln z + t - \ln t - i\pi} \right) \frac{dt}{1+t} \right) \tag{5.71}$$

or

$$W(z) = 1 + (\ln z - 1)\exp\left\{-\frac{1}{\pi}\int_0^\infty \frac{\arg(\ln z + t - \ln t + i\pi)}{1+t}dt\right\}, \qquad (5.72)$$

where $z \notin [-1/e, 0]$. In case of real $z = x > 1/e$, when the expression $\ln z + t - \ln t$ is real and positive (for $t \in (0, \infty)$), the formula (5.72) is simplified and reduced to

$$W(x) = 1 + (\ln x - 1)\exp\left\{-\frac{1}{\pi}\int_0^\infty \arctan\left(\frac{\pi}{\ln x + t - \ln t}\right)\frac{dt}{1+t}\right\} \qquad (5.73)$$

or, after integrating by parts

$$W(x) = 1 + (\ln x - 1)\exp\left\{-\int_0^\infty \frac{t-1}{\pi^2 + (\ln x + t - \ln t)^2}\frac{\ln(1+t)}{t}dt\right\}. \qquad (5.74)$$

We emphasize that the domain $x > 1/e$ of validity of the formulae (5.73) and (5.74) is different from that of (5.68) and (5.69).

For the case $x \in (-1/e, 0)$, we refer the reader to (Siewert & Burniston, 1973, formulae (32)) where the principal branch W_0 and the branch W_{-1} are represented in the form of a combination of two expressions similar to the right-hand side of (5.72).

Remark 5.10.1. We can regard the integral in the formula (5.71) as an improper integral depending on parameter $p = \ln z$ and consider it in the limit $p \to \infty$ (when $z \to \infty$). Since the integrand is a continuous function of two variables t and p in the domain under consideration and the integral is uniformly convergent with respect to p, we can take the limit under the integral sign and find that the integral vanishes as the integrand goes to zero. Then the formula (5.71) reproduces the asymptotic result (1.8).

Finally we note that by use of elementary complex analysis in (Kheyfits, 2004) there is obtained a common closed form representation for all the branches $W_k(z)$ in the complex z-plane through simple quadratures.

5.11 Concluding remarks

In this chapter we derived various integral representations of the principal branch of the Lambert W function using different approaches. The most part of them is associated with functions of W which belong to various classes of functions admitting certain integral representations. Among other classes we considered in detail the classes of Stieltjes functions and complete monotonic functions and by the example of functions containing W in fact demonstrated different ways to establish belonging of a function to these classes.

Besides their own importance the derived integral representations have some applications. One of them has been mentioned in connection with finding Nuttall-Bouwkamp integral (5.24). Other definite integrals appear in taking the obtained integrals with a particular value of z. For example, integrals (5.18), (5.26), (5.27), (5.53) taken at $z = e$ yield respectively

$$\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{1 + v\csc(v)e^{-(1+v\cot v)}} dv = \pi \, ,$$

$$\int_0^{\pi} \frac{v^2 + (1 - v\cot v)^2}{v\csc(v)\,(v\csc(v) + e^{1+v\cot v})} dv = \frac{e-1}{e}\pi \, ,$$

$$\int_0^{\pi} \frac{dv}{1 + e^{1+v\cot v}\sin v/v} = \frac{1}{2}\pi \, ,$$

$$\int_0^{\pi/2} \frac{[v^2 + (1 + v\tan v)^2]\, v\sec(v)e^{v\tan v - 1}}{1 + v^2\sec^2(v)e^{2(v\tan v-1)}} \tan v \, dv = \frac{1}{2}\pi \, .$$

Another advantage that can be taken of the obtained results is based on a comparison between different representations of the same function. This reveals equivalent forms of the involved integrals. In addition, since some of the integrals are simpler than others, such equations can be regarded as a simplification of the latter. For example, equating integrals (5.49) and (5.18) shows that the former can be simplified and reduced to the latter.

At last we mention that the Pick representations (5.45), (5.48), (5.49), and (5.50) can be considered as integrals expressing properties of the kernel $K(z,v)$ defined by (5.46).

References

Akhiezer, N. I. (1965). *The classical moment problem and some related questions in analysis.* New-York: Hafner Publishing Company.

Alzer, H., & Berg, C. (2002). Some classes of completely monotonic functions. *Ann. Acad. Sci. Fenn., Math., 27*, 445–460.

Alzer, H., & Berg, C. (2006). Some classes of completely monotonic functions, II. *Ramanujan J., 11*(2), 225–248.

Anastasselou, E. G., & Ioakimidis, N. I. (1984a). A generalization of the Siewert-Burniston method for the determination of zeros of analytic functions. *J. Math. Phys., 25*, 2422–2425.

Anastasselou, E. G., & Ioakimidis, N. I. (1984b). A new method for obtaining exact analytical formulae for the roots of transcendental functions. *Lett. Math. Phys., 8*(2), 135–143.

Axler, S., Bourdon, P., & Ramey, W. (2001). *Harmonic Function Theory.* New-York: Springer-Verlag.

Baker, G. A., & Graves-Morris, P. R. (1981). Padé Approximants. Part I: Basic Theory. In G.-C. Rota (Ed.), *Encyclopedia of mathematics and its applications 13.* Reading, Massachusetts: Addison-Wesley.

Bender, C. M., & Orszag, S. A. (1999). *Advanced Mathematical Methods for Scientists and Engineers.* New-York: Springer-Verlag.

Berg, C. (1979). The Stieltjes cone is logarithmically convex. In I. Laine, O. Lehto, & T. Sorvali (Eds.), *Complex Analysis, Joensuu 1978, Lecture Notes in Math. 747* (pp. 46–54). Springer.

Berg, C. (2005). On a generalized Gamma convolution related to the q-calculus. In M. Ismail & E. Koelink (Eds.), *Theory and Applications of Special Functions, Developments of Mathematics 13* (pp. 61–76). Berlin: Springer.

Berg, C. (2008). Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity. In S. Mateu & E. Porcu (Eds.), *Positive Definite Functions: From Schoenberg to Space-Time Challenges* (pp. 15–45). Castellon, Spain: Dept. of Mathematics, University Jaume I.

Berg, C., & Forst, G. (1975). Potential Theory on Locally Compact Abelian Groups. In *Ergeb. math. grenzgeb. 87*. Berlin: Springer-Verlag.

Bouwkamp, C. J. (1986). A conjectured Definite Integral. Problem 85-16. *SIAM Review*, *28*(4), 568–569.

Brodsky, S. J., Ellis, J., Gardi, E., Karliner, M., & Samue, M. A. (1997). Padé Approximants, Optimal Renormalization Scales, and Momentum Flow in Feynman Diagrams. *Phys. Rev. D*, *56*, 6980–6992.

Burniston, E. E., & Siewert, C. E. (1973). The use of Riemann problems in solving a class of transcendental equations. *Proc. Camb. Phil. Soc.*, *73*, 111–118.

Caillol, J.-M. (2003). Applications of the Lambert W function to classical statistical mechanics. *Journal of Physics A: Math. Gen.*, *36*, 10431–10442.

Carathéodory, C. (1958). *Theory of Functions of a Complex Variable, Vol. 1.* New York: Chelsea.

Cauer, W. (1932). The poisson integral for functions with positive real part. *Bull. Amer. Math. Soc.*, *38*(10), 713–717.

Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., & Knuth, D. E. (1996). On the Lambert W function. *Adv. Comp. Math.*, *5*(4), 329–359.

Doob, J. L. (1984). *Classical Potential Theory and its Probabilistic Counterpart.* New-York: Springer-Verlag.

Flajolet, P., & Sedgewick, R. (2009). *Analytic Combinatorics.* Cambridge University Press.

Henrici, P. (1977). *Applied and Computational Complex Analysis 2.* John Willey & Sons.

Henrici, P. (1986). *Applied and Computational Complex Analysis 3.* John Willey & Sons.

Jackson, B., Procacci, A., & Sokal, A. D. (2009). *Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights.* arXiv:0810.4703v2 [math.CO].

Kheyfits, A. I. (2004). Closed-form representations of the Lambert W function. *Fractional Calculus and Applied Analysis*, *7*(2), 177–190.

Koosis, P. (1988). *The Logarithmic Integral I.* Cambridge University Press.

Levin, B. Y. (1996). Lectures on Entire Functions. In *Translations of mathematical monographs 150.* Providence, Rhode Island: AMS.

Markushevich, A. I. (1965). *Theory of functions of a complex variable, II.* Prentice-Hall.

Markushevich, A. I. (1967). *Theory of functions of a complex variable, III.* Prentice-Hall.

Nuttall, A. H. (1985). A conjectured Definite Integral. Problem 85-16. *SIAM Review*, *27*(4), 573.

Pakes, A. G. (2011). Lambert's W, infinite divisibility and Poisson mixtures. *J. Math. Anal. Appl.*, *378*(2), 480–492.

Poisson, S.-D. (1823). Suite du mémoire sur les intégrales définies et sur la sommation des séries. *Journal de l'École Royale Polytechnique*, *12*, 404–509.

Saxena, R. K., & Gupta, K. C. (1964). Certain properties of generalized Stieltjes transform involving Meijer's G-function. *Proceedings of the National Institute of Sciences of India*, *30*(6), 707–714.

Schilling, R. L., Song, R., & Vondraček, Z. (2010). *Bernstein functions. Theory and Applications.* Berlin: De Gruyter.

Schwarz, J. H. (Art. No. 013501). The generalized Stieltjes transform and its inverse. *J. Math. Phys.*, *46*(1). (ISSN 0022–2488, MR2113770, 8 pp., 2005)

Shemyakova, E., Khashin, S. I., & Jeffrey, D. J. (2010). A conjecture concerning a completely monotonic function. *Computers and Mathematics with Applications*, *60*, 1360–1363.

Siewert, C. E., & Burniston, E. E. (1973). Exact analytical solutions of $ze^z = a$. *J. Math. Anal. Appl.*, *43*, 626–632.

Sokal, A. D. (2008, October). *Another question about the Lambert W.* Private email.

Sokal, A. D. (2010). Real-variables characterization of generalized Stieltjes functions. *Expositiones Math.*, *28*(2), 179–185.

The poster 'The Lambert W Function'. (2004). http://www.orcca.on.ca/LambertW/. UWO.

Titchmarsh, E. C. (1939). *The Theory of Functions.* Oxford Uhversity Press.

Tokarzewski, S. (1996). Two-point padé approximants for the expansions of Stieltjes functions in real domain. *J. Comput. Appl. Math.*, *67*, 59–72.

Tokarzewski, S., & Telega, J. J. (1998). Bounds on effective moduli by analytical continuation of the Stieltjes function expanded at zero and infinity. *Z. angew. Math. Phys.*, *49*, 137–155.

Weideman, J. A. C. (2002). Numerical Integration of Periodic Functions: A Few Examples. *The American Mathematical Monthly*, *109*(1), 21–36.

Whittaker, E. T., & Watson, G. N. (1927). *A Course of Modern Analysis.* Cambridge University Press.

Widder, D. V. (1938). The Stieltjes Transform. *Trans. Am. Math. Soc.*, *43*, 7–60.

Wolfram Research, Inc. (1998-2010). Wolfram Research, Inc. Available from `http://functions.wolfram.com/ElementaryFunctions/ProductLog/07/01/01/0001/`

Chapter 6

Conclusion

"The whole is more than the sum of its parts." – Aristotle

We studied different analytical properties of the Lambert W-function. A part of them relates to the convergence of the asymptotic series of W. In particular, we ascertained the domain of convergence of the series in terms of Stirling cycle numbers and the series in terms of the 2-associated Stirling subset numbers in real and complex cases. We found that the latter has a much wider domain of convergence than the former in both cases and we provided an analysis of this fact in the real case. We also found asymptotic expressions for the expansion coefficients and obtained a representation of the series with a wider domain of convergence in terms of the second-order Eulerian numbers.

We applied an invariant transformation defined by the parameter p to the above series to obtain one-parameter families of series. We found that an increase of p results in an extension of the domain of convergence of the series. Thus the series obtained under the transformation with positive values of p have a wider domain of convergence than the original series does. However, at the same time a rate of convergence can be found to be reduced when the parameter p increases. Therefore in such a case within the common domain of convergence of the series with different positive values of p the series with the minimum value of p would be the most effective. In practice, the obtained results can be applied to compute rapid estimates for W using a small number of terms in the series at the expense of an appropriate choice of a particular value of parameter p, for example, in evaluating of the Lambert W function in computer-algebra systems.

We also considered the well-known expansion of $W(x)$ in powers of $\ln x$ and gave an asymptotic estimate for the expansion coefficients. We found three more forms for a representation of the expansion coefficients of the series in terms of the associated Stirling numbers of the first kind, the 2-associated Stirling subset numbers and iterative formulas. This allows us to compute the expansion coefficients in different ways to meet requirements in accordance with available computer resources. Finally we presented some combinatorial consequences, including the Carlitz-Riordan identities, which result from the found different forms of the expansion coefficients of the above series.

We studied three forms for the higher derivatives of the Lambert W function. Each form contains its own sequence of polynomials. It is shown that all of these polynomials have similar properties. Specifically, their coefficients form positive sequences that are log-concave and unimodal. This property implies that the principal branch of W function is Bernstein and its derivative is a Stieltjes function. Relations of the polynomial coefficients to the shifted r-Stirling numbers of the second kind, the Bernoulli polynomials of higher order as well as Carlitz numbers are found as well.

We derived various integral representations of the principal branch of the Lambert W function using different approaches. The most part of them is associated with functions of W belonging to various classes of functions admitting certain integral representations. Among other classes we considered in detail the classes of Stieltjes functions and complete monotonic functions and by the example of functions based on W in fact demonstrated different ways to establish belonging of a function to these classes.

Besides their own importance the derived integral representations have some applications. One of them is computing values of some particular definite integrals as well as more complicated consequences such as the mentioned Nuttall-Bouwkamp integral. Another one is a proof of convergence of successive Padé approximants for numerical evaluation of W function. In addition, some of the found integral representations permit spectrally convergent quadratures.

We also note that some advantages can be taken of the comparison between different representations of the same function. This reveals equivalent forms of the involved integrals. Besides, since some of the integrals are simpler than others, such equations can be regarded as a simplification of the latter.

Thus, in the accomplished work we found a number of new beautiful properties of the Lambert W function which are also useful for practical needs.

Appendix A

Analytical Tools

A.1 Lagrange Inversion Theorem

Theorem A.1.1. *Let function $\psi(\omega)$ be analytic at $\omega = 0$ and $\psi(0) \neq 0$. Then a solution of equation $z = \omega\psi(\omega)$ is given by series (Goursat, 1904, § 190)*

$$\omega = \sum_{n=1}^{\infty} \left[\frac{d^{n-1}}{d\omega^{n-1}} \left(\frac{1}{\psi(\omega)} \right)^{n} \right]_{\omega=0} \frac{z^n}{n!} . \tag{A.1}$$

Note. Theorem A.1.1 and formula (A.1) are called *Lagrange Inversion Theorem* and *Lagrange Inversion Formula* respectively. There are other forms of formula (A.1) for equations of more general form (see, e.g. (Goursat, 1904, § 189)). We give an example of application of formula (A.1).

Example. Let us consider equation $z = \omega e^{\omega}$ in the vicinity of $z = 0$ (cf. (1.7)). To apply formula (A.1) we set $\psi(\omega) = e^{\omega}$. The function $\psi(\omega)$ satisfies all the requirements of Theorem A.1.1, therefore we can write

$$\omega = \sum_{n=1}^{\infty} \left[\frac{d^{n-1}}{d\omega^{n-1}} e^{-n\omega} \right]_{\omega=0} \frac{z^n}{n!} .$$

Since

$$\frac{d^k}{d\omega^k}e^{-n\omega} = (-n)^k e^{-n\omega} ,$$

we have

$$\left[\frac{d^{n-1}}{d\omega^{n-1}}e^{-n\omega}\right]_{\omega=0} = (-n)^{n-1}$$

and finally obtain

$$\omega = \sum_{n=1}^{\infty}(-n)^{n-1}\frac{z^n}{n!} . \tag{A.2}$$

The radius of convergence of the series (A.2) is $1/e$, which can be easily seen using the ratio test

$$\left|\frac{(-n)^{n-1}/n!}{(-n-1)^n/(n+1)!}\right| = \left(\frac{n}{n-1}\right)^n \frac{n+1}{n} = \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{-n} \to e^{-1} \quad \text{as} \quad n \to \infty .$$

A.2 Darboux's Theorem

Definition A.2.1. (Bender, 1974) A function $f(z)$ is said to have an algebraic singularity at $z = a$ if $f(z)$ can be written as a function analytic at $z = a$ plus a finite sum of terms of the form

$$\frac{g(z)}{\left(1-\frac{z}{a}\right)^{\theta}} , \tag{A.3}$$

where $g(z)$ is analytic at $z = a$, $g(a) \neq 0$ and θ is a real or complex number such that $-\theta \notin \mathbb{Z}$. The real part of θ is called the weight of the singularity.

Theorem A.2.2. *(Darboux's Theorem)(Bender, 1974) Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic at $z = 0$ and have only algebraic singularities at $z = a_k$, $k = 1, 2, ..., K$ on its circle of convergence $|z| = R$. Let the leading behavior of $f(z)$ near the* *singularity* $z = a_k$ *is of the form* (A.3)

$$\frac{g_k(z)}{\left(1 - \frac{z}{a_k}\right)^{\theta_k}}$$

for each k . *Then*

$$c_n = \frac{1}{n} \sum_{k=1}^{K} \frac{g_k(a_k) n^{\theta_k}}{a_k^n \Gamma(\theta_k)} + o\left(\frac{1}{R^n n^{\vartheta - 1}}\right) ,$$

where $\vartheta = max_k \Re(\theta_k)$ *and* $\Gamma(s)$ *is the gamma function.*

A.3 Stieltjes-Perron Inversion Formula

Theorem A.3.1. *(Stieltjes-Perron Inversion Formula) (Henrici, 1977, p. 591) Let* ψ *be a bounded, nondecreasing real function defined on* $(-\infty, \infty)$, *and let* f *be defined by*

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{z + \tau} d\psi(\tau) \quad (\Im z < 0) .$$

Then for arbitrary real σ *and* τ

$$\frac{1}{2}\left[\psi(\tau^+) + \psi(\tau^-)\right] - \frac{1}{2}\left[\psi(\sigma^+) + \psi(\sigma^-)\right] = \frac{1}{\pi} \lim_{\eta \to 0+} \Im \int_{-\tau}^{-\sigma} f(\lambda - i\eta) d\lambda .$$

Appendix B

Special Numbers

Unsigned Stirling numbers of the first kind (Stirling cycle numbers) (Graham, Knuth, & Patashnik, 1989; Corless, Jeffrey, & Knuth, 1997).

Notation:

$$\begin{bmatrix} n \\ m \end{bmatrix}$$

Generating function:

$$\ln^m(1+z) = m! \sum_{n=0}^{\infty} (-1)^{m+n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} \tag{B.1}$$

Recurrence relation:

$$\begin{bmatrix} n \\ m \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0} \tag{B.2}$$

Stirling numbers of the second kind (Stirling subset numbers) (Graham et al., 1989; Corless et al., 1997).

Notation:

$$\begin{Bmatrix} n \\ m \end{Bmatrix}$$

Generating function:

$$(e^z - 1)^m = m! \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{z^n}{n!} \tag{B.3}$$

Recurrence relation:

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}, \quad \left\{ {n \atop 0} \right\} = \delta_{n,0} \tag{B.4}$$

Unsigned associated Stirling numbers of the first kind (Comtet, 1974).

Notation:

$$d(n, m)$$

Generating function:

$$[\ln(1+z) - z]^m = m! \sum_{n=2m}^{\infty} (-1)^{n+m} d(n, m) \frac{z^n}{n!} \tag{B.5}$$

Recurrence relation:

$$d(n, m) = (n-1)[d(n-1, m) + d(n-2, m-1)] , \quad d(0, 0) = 1 \tag{B.6}$$

2-associated Stirling numbers of the second kind (2-associated Stirling subset numbers) (Graham et al., 1989; Corless et al., 1997).

Notation:

$$\left\{ {n \atop m} \right\}_{\geq 2}$$

Generating function:

$$(e^z - 1 - z)^m = m! \sum_{n \geq 0} \left\{ {n \atop m} \right\}_{\geq 2} \frac{z^n}{n!} \tag{B.7}$$

Recurrence relation:

$$\left\{ {n \atop m} \right\}_{\geq 2} = m \left\{ {n-1 \atop m} \right\}_{\geq 2} + (n-1) \left\{ {n-2 \atop m-1} \right\}_{\geq 2}, \quad \left\{ {n \atop n} \right\}_{\geq 2} = \delta_{n,0} \tag{B.8}$$

Shifted r-Stirling numbers of the second kind (non-central Stirling numbers of the second kind) (Broder, 1984; Koutras, 1982).

Notation:

$$\left\{ {n+r \atop m+r} \right\}_r$$

Generating function:

$$\frac{1}{m!} e^{rz} \left(e^z - 1\right)^m = \sum_{n=0}^{\infty} \left\{ {n+r \atop m+r} \right\}_r \frac{z^n}{n!} \tag{B.9}$$

Recurrence relation:

$$\left\{ {n+r \atop m+r} \right\}_r = (m+1) \left\{ {n+r-1 \atop m+r} \right\}_{r-1} + \left\{ {n+r-1 \atop m+r-1} \right\}_{r-1}, \quad \left\{ {r \atop m+r} \right\}_r = \delta_{m,0} \tag{B.10}$$

Second-order Eulerian numbers (Graham et al., 1989; Corless et al., 1997).

Notation:

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle$$

Generating functions (Bergeron, Flajolet, & Salvy, 1992; Gosper, Jr., 1998):

$$\frac{t + W(-t\exp(z(t-1)^2 - t))}{t-1} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle t^k \frac{z^n}{n!} \tag{B.11}$$

$$\frac{1 - 1/t}{1 + 1/W\left(-t\exp\left(z(1-t)^2 - t\right)\right)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle t^k \frac{z^n}{n!} \tag{B.12}$$

Recurrence relation:

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (k+1) \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-1-k) \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle, \quad \left\langle\!\!\left\langle {n \atop n} \right\rangle\!\!\right\rangle = \delta_{n,0}, \quad \left\langle\!\!\left\langle {n \atop 0} \right\rangle\!\!\right\rangle = 1 \tag{B.13}$$

Bernoulli numbers (Graham et al., 1989).

Notation:

$$B_n$$

Generating function:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{1}{2}z+\sum_{n=1}^{\infty}B_{2n}\frac{z^{2n}}{(2n)!}\,, \tag{B.14}$$

because $B_0=1, B_1=-1/2$ and $B_{2n+1}=0$ for all natural n.

Recurrence relation (Namias, 1986):

$$B_0=1, \quad B_n=\frac{1}{2(1-2^n)}\sum_{k=0}^{n-1}\binom{n}{k}2^kB_k \tag{B.15}$$

Bernoulli polynomials of higher order (Norlund, 1924):

Notation:

$$B_n^{(r)}(\lambda)$$

Generating function:

$$\left(\frac{z}{e^z-1}\right)^r e^{\lambda z}=\sum_{n=0}^{\infty}B_n^{(r)}(\lambda)\frac{z^n}{n!} \tag{B.16}$$

Recurrence relation:

$$rB_n^{(r+1)}(\lambda)=(r-n)B_n^{(r)}(\lambda)+(\lambda-r)\frac{d}{d\lambda}B_n^{(r)}(\lambda) \tag{B.17}$$

Bell numbers (Graham et al., 1989).

Notation:

$$\varpi_n$$

Generating function:

$$\exp\left(e^{z}-1\right)=\sum_{n=0}^{\infty}\varpi_{n}\frac{z^{n}}{n!} \tag{B.18}$$

Recurrence relation:

$$\varpi_{0}=1,\quad \varpi_{n+1}=\sum_{k=0}^{n}\binom{n}{k}\varpi_{k} \tag{B.19}$$

Appendix C

Padé Approximants for The Lambert W Function

Definition C.0.2. (Baker & Graves-Morris, 1981) A Padé approximant of function $f(z)$ is a rational function

$$[L/M] = \frac{a_0 + a_1 z + \ldots + a_L z^L}{1 + b_1 z + \ldots + b_M z^M}$$

that has a Maclaurin expansion which is consistent with a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n z^n .$$

C.1 [3/2] Padé approximant to function $W(z)/z$

Take Maple commands

```
> alias(W = LambertW):
> Order := 24:
> S := series( W(z)/z, z ):
> convert(S, ratpoly, 3, 2);
```

$$\frac{1 + \frac{1159}{505} z + \frac{1193}{2020} z^2 - \frac{133}{1212} z^3}{1 + \frac{1664}{505} z + \frac{4819}{2020} z^2}$$

This rational approximation is used in Maple to evaluate $W(z)$ near $z = 0$.

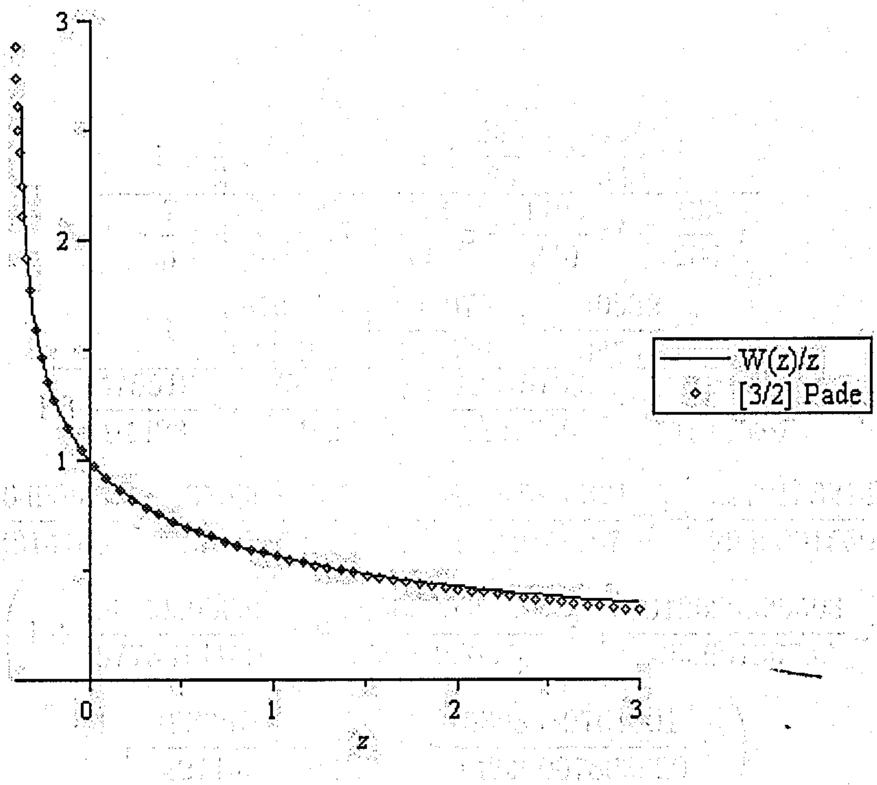

Figure C.1: [3/2] Padé approximant for W(z)/z.

C.2 Other Padé approximants to function $W(z)/z$

Create Padé table

```
> PadeTable:=seq( convert( S, ratpoly, k,k+1 ), k=1..5 );
```

PadeTable :=

$$\frac{1+\frac{4}{3}z}{1+\frac{7}{3}z+\frac{5}{6}z^2},\ \frac{1+\frac{228}{85}z+\frac{451}{340}z^2}{1+\frac{313}{85}z+\frac{1193}{340}z^2+\frac{133}{204}z^3},$$

$$\frac{1+\frac{381096}{94423}z+\frac{848073}{188846}z^2+\frac{40532}{34545}z^3}{1+\frac{475519}{94423}z+\frac{757921}{94423}z^2+\frac{12216739}{2832690}z^3+\frac{798983}{1618680}z^4},$$

$$\left(1+\frac{47306490920}{8773814169}z+\frac{37036845053}{3899472964}z^2+\frac{41047808321}{6824077687}z^3+\frac{2872158214405}{2948001560784}z^4\right)\Big/$$

$$\left(1+\frac{56080305089}{8773814169}z+\frac{505009940819}{35095256676}z^2+\frac{3312529329503}{245666796732}z^3\right.$$

$$\left.+\frac{1983576598463}{421143080112}z^4+\frac{5398089761801}{14740007803920}z^5\right),$$

$$\left(1+\frac{785811326134885740}{116440941682504219}z+\frac{1903782046557342357}{116440941682504219}z^2+\frac{1784723875258709620}{104796847514253797 1}z^3\right.$$

$$\left.+\frac{3942118362962089425 1}{558916520076020251 2}z^4+\frac{11411641023324141929 9}{146715586519955315940}z^5\right)\Big/$$

$$\left(1+\frac{902252267817389959}{116440941682504219}z+\frac{5262745803701951975}{2328818833650084 38}z^2\right.$$

$$+\frac{3214377185409113031 7}{1047968475142537971}z^3+\frac{3233215183345095345 31}{16767495602280607536}z^4$$

$$\left.+\frac{279159353353695041635 9}{586862346079821263760}z^5+\frac{8578702363330882299 1}{320106734225357052960}z^6\right)$$

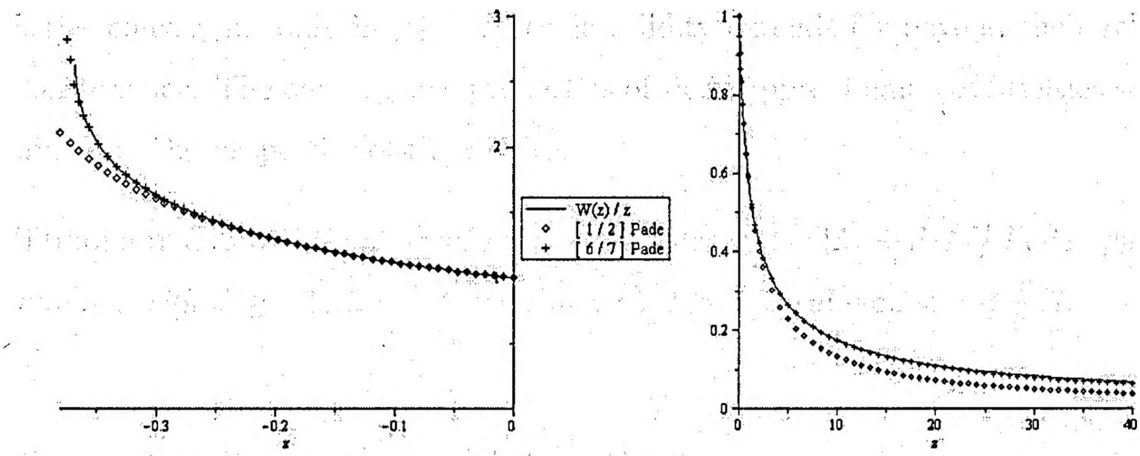

Figure C.2: [1/2] and [6/7] Padé approximants for $W(z)/z$.

C.3 Theory for Stieltjes series

A Stieltjes function is defined by the Stieltjes-integral representation

$$f(z) = \int_0^\infty \frac{d\Phi(u)}{1+zu}$$

(see ... for more details).

A formal expansion of the integral is called *a Stieltjes series*

$$f(z) = \sum_{k=0}^{\infty} f_k(-z)^k .$$

Theorem C.3.1. *(Existence) All $[M+J/M]$ Padé approximants, with $J \geq -1$, to Stieltjes series exist and are nondegenerate.*

Theorem C.3.2. *(Convergence) Let $f(z)$ be a Stieltjes series with radius of convergence $R > 0$ and for given arbitrary numbers $\Lambda > R$ and $0 < \delta < R$, a domain $D(\Lambda, \delta)$ be the set of all points in $|z| < \Lambda$ that are at least a distance δ from the cut $(-\infty, -R]$. Then any sequence of $[M_k + J_k/M_k]$ Padé approximants, with $J_k \geq -1$, of $f(z)$ converges uniformly to $f(z)$ in $D(\Lambda, \delta)$.*

Thus even though the Padé approximants are constructed from the Stieltjes series convergent only in $|z| < R$, their validity extends far beyond the circle of convergence. The convergence properties of Padé approximants of Stieltjes series hinge on the properties of their poles.

Theorem C.3.3. *(Property of poles) The poles of the $[M + J/M]$ Padé approximants, with $J \geq -1$, lie on the real axis in the interval $-\infty < z < -R$.*

C.4 Applications to W function

$W(z)/z$ is a Stieltjes function, therefore Theorems C.3.1 and C.3.2 ensure the existence and convergence of Padé approximants to this function. Moreover, since $R = 1/e$, by Theorem C.3.3 the poles of Padé approximants should lie in the interval $-\infty < z < -1/e$. What does Maple tell us about that?

```
> dens:= map( denom, PadeTable );
```

$$\begin{aligned} dens := [&6 + 14\,z + 5\,z^2, 1020 + 3756\,z + 3579\,z^2 + 665\,z^3, \\ &11330760 + 57062280\,z + 90950520\,z^2 + 48866956\,z^3 + 5592881\,z^4, \\ &14740007803920 + 94214912549520\,z + 212104175143980\,z^2 + 198751759770180\,z^3 \\ &+ 69425180946205\,z^4 + 5398089761801\,z^5, \\ &3521174076478927582560 + 27284108578797872360160\,z + 79572716551973513862000\,z^2 \\ &+ 108003073429746197865120\,z^3 + 67897518850247002251510\,z^4 \\ &+ 16749561201221702498154\,z^5 + 943657259966397052901\,z^6] \end{aligned}$$

```
> rts:=map( t->[fsolve(t,z,complex)], dens );
```

$$rts := [[-2.271779789, -0.5282202113],$$
$$[-4.093800259, -0.8444844878, -0.4436701401],$$
$$[-6.455610949, -1.270425123, -0.5987475960, -0.4125658307],$$
$$[-9.352503587, -1.799352788, -0.8039719015, -0.5077480857, -0.3974882010],$$
$$[-12.78157444, -2.429207992, -1.053237111, -0.6337948309, -0.4628223076,$$
$$-0.3889869449]]$$

```
> [min(rts), max(rts)];
```

$$[-12.78157444, -0.3889869449]$$

Compare this interval with

$$(-\infty, -0.3678794412)$$

Appendix D

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References

Baker, G. A., & Graves-Morris, P. R. (1981). Padé Approximants. Part I: Basic Theory. In G.-C. Rota (Ed.), *Encyclopedia of mathematics and its applications 13.* Reading, Massachusetts: Addison-Wesley Publishing Company, Inc.

Bender, E. A. (1974). Asymptotic Methods in Enumeration. *SIAM Review, 16*(4), 485–515.

Bergeron, F., Flajolet, P., & Salvy, B. (1992, February). Varieties of increasing trees. In J.-C. Raoult (Ed.), *CAAP'92: Proceedings of the 17th Colloquium on Trees in Algebra and Programming, LNCS 581* (pp. 24–48). Rennes, France: Springer.

Broder, A. Z. (1984). The r-Stirling numbers. *Discrete Math., 49*(3), 241–259.

Comtet, L. (1974). *Advanced combinatorics.* Dordrecht, Holland: Reidel.

Corless, R. M., Jeffrey, D. J., & Knuth, D. E. (1997). A sequence of series for the Lambert W function. In *Issac '97: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (pp. 197–204). New York, NY, USA: ACM press.

Gosper, Jr., R. W. (1998). The solutions of $ye^{y^2} = x$ and $ye^y = x$. *SIGSAM Bulletin, Communications in Computer Algebra, 32*(1), 8–10.

Goursat, E. (1904). *A course in mathematical analysis, Vol. 1.* Ginn & Company.

Graham, R. L., Knuth, D. E., & Patashnik, O. (1989). *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley.

Henrici, P. (1977). *Applied and Computational Complex Analysis 2.* John Willey

& Sons.

Kalugin, G. A., & Jeffrey, D. J. (2010). Series Transformations to Improve and Extend Convergence. In V. P. Gerdt, W. Koepf, E. W. Mayr, & E. V. Vorozhtsov (Eds.), *CASC 2010, LNCS 6244* (pp. 134–147). Springer-Verlag.

Kalugin, G. A., & Jeffrey, D. J. (2011). Unimodal sequences show that Lambert W is Bernstein. *C. R. Math. Rep. Acad. Sci. Canada*, *33*(2), 50–56.

Kalugin, G. A., Jeffrey, D. J., Corless, R. M., & Borwein, P. M. (2011). Stieltjes, Poisson and other integral representations for functions of Lambert W. *Adv. Comp. Math.*, *arXiv: 1103.5640*, 20 p.

Koutras, M. (1982). Non-central Stirling numbers and some applications. *Discrete Math.*, *42*(1), 73–89.

Namias, V. (1986). A simple derivation of Stirling's asymptotic series. *Amer. Math. Monthly*, *93*, 25–29.

Norlund, N. E. (1924). *Vorlesungen uber Differenzenrechnungy.* Berlin: Springer-Verlag.