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A duality formula and a particle Gibbs sampler for continuous time Feynman-Kac measures on path spaces

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Abstract

Continuous time Feynman-Kac measures on path spaces are central in applied probability, partial differential equation theory, as well as in quantum physics. This article presents a new duality formula between normalized Feynman-Kac distribution and their mean field particle interpretations. Among others, this formula allows us to design a reversible particle Gibbs-Glauber sampler for continuous time Feynman-Kac integration on path spaces. We also provide new Dyson-Phillips semigroup expansions, as well as novel uniform propagation of chaos estimates for continuous time genealogical tree based particle models with respect to the time horizon and the size of the systems. Our approach is self contained and it is based on a novel stochastic perturbation analysis and backward semigroup techniques. These techniques allow to obtain sharp quantitative estimates of the convergence rate to equilibrium of particle Gibbs-Glauber samplers. To the best of our knowledge these results are the first of this kind for continuous time Feynman-Kac measures.

Keywords : Feynman-Kac formulae, interacting particle systems, genealogical trees, ancestral lines, Gibb-Glauber dynamics, propagation of chaos properties, contraction inequalities, Dyson-Phillips expansions.

Mathematics Subject Classification : 60K35, 60H35, 37L05, 47D08.

1 Introduction

Feynman-Kac measures on path spaces are central in applied probability as well as in biology and quantum physics. They also arise in a variety of application domains such as in estimation and control theory, as well as a rare event analysis. For a detailed review on Feynman-Kac measures and their application domains we refer to the books [24, 25, 37, 42], see also the more recent articles [22, 55] on branching processes and neutron transport equations and the references therein.

Their mean field type particle interpretations are defined as a system of particles jumping a given rate uniformly onto the population. From the pure numerical viewpoint, this interacting jump transition can be interpreted as an acceptance-rejection scheme with a recycling of rejected particles by duplicating the selected ones. Feynman-Kac interacting particle models encapsulate a variety of algorithms such as the diffusion Monte Carlo used to solve Schrödinger ground states, see

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for instance the series of articles [14, 16, 41, 74, 75, 58, 59] as well as section 23.5 and chapter 27 in [42] and the references therein.

Their discrete time versions are encapsulated a variety of well known algorithms such as particle filters [26] (a.k.a. sequential Monte Carlo methods in Bayesian literature [17, 24, 25, 37, 47]), the go-with the winner [1], as well as the self-avoidind random walk pruned-enrichment algorithm by Rosenbluth and Rosenbluth [76], and many others. This list is not exhaustive (see also the references therein). The research monographs [24, 25] provide a detailed discussion on these subjects with precise reference pointers.

The seminal article [2] by Andrieu, Doucet and Holenstein introduced a new way to combine Markov chain Monte Carlo methods with discrete generation particle methods. A variant of the method, where ancestors are resampled in a forward pass, was developed by Lindsten, Schön and Jordan in [60], and Lindsten and Schön [61]. In all of these studies, the validity of the particle conditional sampler is assessed by interpreting the model as a traditional Markov chain Monte Carlo sampler on an extended state space. The central idea is first to design a detailed encoding of the ancestors at each level in terms of random maps on integers, and then to extend the "target" measure on a sophisticated state space incapsulating these iterated random sequences.

In a more recent article [34], the authors provide an alternative and we believe more natural interpretation of these particle Markov chain Monte Carlo methods in terms of a duality formula extending the well known unbiasedness properties of Feynman-Kac particle measures on many-body particle measures. This article also provides sharp quantitative estimates of the convergence rate to equilibrium of the models with respect to the time horizon and the size of the systems. The analysis of these models, including backward particle Markov chain Monte Carlo samplers has been further developed in [30, 31].

The main objective of the present article is to extend these methodologies to continuous time Feynman-Kac measures on path spaces.

The first difficulty comes from the fact that the discrete time analysis [30, 31, 34] only applies to simple genetic type particle models, or equivalently to branching models with pure multinomial selection schemes. Thus, these results do not apply to discrete time approximation of continuous time models based on geometric type jumps, and any density type argument cannot be applied.

In contrast with their discrete time version, continuous time Feynman-Kac particle models are not described by conditionally independent local transitions, but in terms of interacting jump processes.

The analysis of continuous time genetic type particle models is not so developed as their discrete time versions. For instance, uniform convergence estimates are available for continuous time Feynman-Kac models with stable processes [37, 40, 41, 74]. Nevertheless, to the best of our knowledge, sharp estimates for path space models and genealogical tree based particle samplers in continuous time have never been discussed in the literature. These questions are central in the study the convergence to equilibrium of particle Gibbs-Glauber sampler on path spaces.

In the present article we provide a duality formula for continuous time Feynman-Kac measures on path-spaces (cf. theorem 1.1). This formula on generalogical tree based particle models that can be seen as an extension of well known unbiasedness properties of Feynman-Kac models to their many body version (defined in section 6.1). The second main result of the article is to design and to analyze the stability properties of a particle Gibbs-Glauber sampler of path space (cf. theorem 1.2). Our approach combines a perturbation analysis of nonlinear stochastic semigroups with propagation of chaos techniques (cf. section 5). Incidentally these techniques also provide with little efforts new uniform propagation of chaos estimates w.r.t. the time horizon (cf. corollary 5.12).

1.1 Statement of the main results

Let (X_t, V_t) be a continuous time Markov process and a bounded non negative function on some metric space (S, d_S). We denote by $D_t(S)$ the set of càdlàg paths from $[0, t]$ to S. Unless otherwise is stated, as a rule in the further development of the article

$$\widehat{X}_t := (X_s)_{s \leqslant t} \in \widehat{S} := \cup_{t \geqslant 0} D_t(S)$$

stands for the historical process of some process X_t. In this notation, we extend V_t to $D_t(S)$ by setting $\widehat{V}_t(\widehat{X}_t) = V_t(X_t)$.

The Feynman-Kac probability measures $\mathbb{Q}_t$ on the path spaces $D_t(S)$ associated with (X_t, V_t) are defined for any bounded measurable function F on $D_t(S)$ by the path integration formula

$$\mathbb{Q}_t(F) := \int F(\omega) \, \mathbb{Q}_t(d\omega) := \mathbb{Z}_t^{-1} \, \mathbb{E}\left(F(\widehat{X}_t) \, \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s) ds \right] \right) \tag{1.1}$$

where $\mathbb{Z}_t$ stands for the normalizing constant.

We also let η_t be the terminal time marginal of the measures $\mathbb{Q}_t$. In this notation, for any bounded measurable function f on S we have

$$\eta_t(f) := \int f(x) \, \eta_t(dx) := \mathbb{Z}_t^{-1} \, \mathbb{E}\left(f(X_t) \, \exp\left[-\int_0^t V_s(X_s) ds \right] \right) \tag{1.2}$$

In addition, the normalizing constant $\mathbb{Z}_t$ is given by the easily checked free energy formula

$$\mathbb{Z}_t = \mathbb{E}\left(\exp\left[-\int_0^t V_s(X_s) ds \right] \right) = \exp\left[-\int_0^t \eta_s(V_s) ds \right]$$

For a more thorough discussion on these Feynman-Kac models we refer the reader to section 3.2 and section 3.3.1. A selected list of application areas are also discussed in section 1.2.1.

The path integration formulae (1.1) can rarely be solved analytically and their numerical solving often require extensive calculations. One strategy is to interpret these probability measures on path space in terms the occupation measures of the ancestral lines of a genetic type interacting jump process [36, 37, 41, 42].

These particle interpretations are defined as follows:

Consider a system of N particles $\xi_t = (\xi_t^i)_{1 \leqslant i \leqslant N} \in S^N$ evolving independently as X_t with jump rate V_t; at each jump time the particle jumps onto a particle uniformly chosen in the pool.

This jump can be interpreted as the death of the particle and the instantaneous birth of an offspring of a particle uniformly chosen in the pool. After this birth each offspring evolves as independent copies of the process X_t.

When the i-th particle duplicates it becomes the parent of two offsprings. Running back in time we can trace the whole genealogy of each particle.

Let $A_{s,t}(i) \in [N]$ be the index of the ancestor of of the i-th particle ξ_t^i at level $s \leqslant t$. In this notation, the i-th ancestral line $\mathbb{X}_t^i := (\xi_s^{A_{s,t}(i)})_{0 \leqslant s \leqslant t} \in D_t(S)$ of the i-th particle ξ_t^i at time t is represented in a synthetic way by the following backward ancestral path

$$\xi_0^{A_{0,t}(i)} \longleftarrow \ldots \longleftarrow \xi_s^{A_{s,t}(i)} \longleftarrow \ldots \longleftarrow \xi_t^i$$

In this interpretation, the genealogical tree of the N particles $\xi_t = (\xi_t^i)_{1 \leqslant i \leqslant N}$ is defined by the N ancestral lines $\mathbb{X}_t = (\mathbb{X}_t^i)_{1 \leqslant i \leqslant N} \in D_t(S)^N$.

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We underline that the N ancestral lines $\mathbb{X}_t = (\mathbb{X}_t^i)_{1 \leqslant i \leqslant N} \in D_t(S)^N$ of length t of the N individuals $\xi_t = (\xi_t^i)_{1 \leqslant i \leqslant N}$ is also defined forward in time by a system of N path-valued particles evolving independently as the historical process $\widehat{X}_t$, with jump rate $\widehat{V}_t$ on $\widehat{S}$. At each jump time an ancestral line jumps onto another uniformly chosen ancestral line in the pool. Between jumps the ancestral lines evolves as independent copies of the historical process $\widehat{X}_t$.

Let $\mathcal{I}$ be an uniform random variable on the index set $[N] := \{1, \dots, N\}$, independent of particle model discussed above.

Given $\mathbb{X}_t$ the distribution of a randomly chosen ancestral line $\mathbb{X}_t^{\mathcal{I}}$ is given by the occupation measure of the genealogical tree. In addition, in some weak sense we have the convergence

$$m(\mathbb{X}_t) := \frac{1}{N} \sum_{i \in [N]} \delta_{\mathbb{X}_t^i} \longrightarrow_{N \to \infty} \mathbb{Q}_t \quad \text{and} \quad m(\xi_t) := \frac{1}{N} \sum_{i \in [N]} \delta_{\xi_t^i} \longrightarrow_{N \to \infty} \eta_t \qquad (1.3)$$

For a more detailed discussion and precise estimates, we refer to [37, 39, 41, 74], as well as to theorem 1.3 and section 5.3 in the present article.

The historical process $\widehat{\xi}_t = (\xi_s)_{s \leqslant t} \in D_t(S)^N$ encapsulates the ancestral lines at any time horizon, including ancestors at any time level s with no currently living descendants at a time horizon $t \geqslant s$. The process $\widehat{\xi}_t$ has the same jump rate as ξ_t. When a jump occurs, we keep and extend all historical trajectories from the randomly selected particles at that time.

For a more detailed description of the processes $(\xi_t, \widehat{\xi}_t, \mathbb{X}_t)$, we refer to section 4.2 dedicated to historical and genealogical tree evolutions, see for instance figure 8 and the complete genealogical tree presented in figure 9.

Given $\mathcal{I}$, the dual process $\widehat{\chi}_t \in D_t(S)^N$ is defined as $\widehat{\xi}_t$ but the $\mathcal{I}$-th ancestral line is a frozen path with the same law as the historical process $\widehat{X}_t$. More precisely, the jump of the $\mathcal{I}$-th particle onto the j-th one is replaced by the jump of the j-th particle onto the $\mathcal{I}$-th frozen one. In addition, at these jumps times we flip the j-th and $\mathcal{I}$-th row historical path up to the present time horizon. A schematic picture of this jump and historical permutation at time t is given below
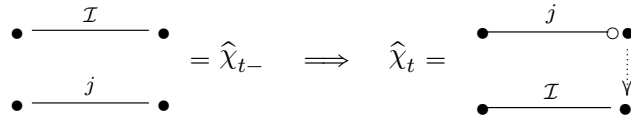
The duality terminology comes from the fact that $\widehat{\chi}_t$ coincides with the conditional stochastic process $\widehat{\xi}_t$ given a frozen ancestral line, under some many-body Feynman-Kac measure (cf. theorem 1.1 and the Bayes' formula (1.6)). We also underline that $\widehat{\chi}_t := (\widehat{\chi}_t(s))_{s \leqslant t} \in D_t(S)^N$ is not necessarily the historical path of an auxiliary Markov process.

For a more detailed description of $\widehat{\chi}_t$ in terms of generators we refer to section 4.3 dedicated to particle models with a frozen ancestral line, see for instance (4.10), the evolution diagrams given in figure 11 as well as the graphical description of the process $\widehat{\chi}_t$ provided in figure 12.

The genealogical type tree $\mathbb{Y}_t = (\mathbb{Y}_t^i)_{1 \leqslant i \leqslant N} \in D_t(S)^N$ of the dual process discussed above is also defined as $\mathbb{X}_t$. The main difference is that the $\mathcal{I}$-th ancestral line is frozen and $\mathbb{Y}_t^{\mathcal{I}} \overset{law}{=} \widehat{X}_t$.

The remaining $(N-1)$ path-valued particles $\mathbb{Y}_t^- := (\mathbb{Y}_t^j)_{j \neq \mathcal{I}}$ are defined as above with a rescaled jump rate $(1-1/N)\widehat{V}_t$, and including an auxiliary jump rate $2\widehat{V}_t/N$ at which the path-particle jump onto the first frozen ancestral line.

For a more detailed description of these dual ancestral lines, we also refer to section 4.3, see for instance the graphical description of the dual genealogical tree provided in figure 13 as well as the generator of the process in path space defined in (4.16).

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A realization of the genealogical tree associated with $N = 3$ particles with 2 interacting jumps and the first frozen ancestral line is illustrated below in figure 2 .
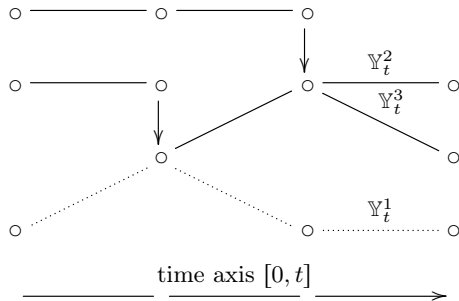

Figure 2: A genealogical tree associated with $N = 3$ particles with 2 interacting jumps. The couple of arrows stands for the interacting jumps, the dotted line represents the frozen ancestral line with $\mathcal{I} = 1$. The vertical axis stands for the state space S.

The first main result of the article is the following duality formula.

Theorem 1.1 (Duality formula)**.** *For any time horizon $t \geqslant 0$, any $N \geqslant 2$ and any bounded measurable function F on $D_t(S)^{(2N)} := (D_t(S)^N \times D_t(S)^N)$ we have the almost sure formula*

$$\mathbb{E}\left(F(\widehat{\xi}_t, \mathbb{X}_t) \, \exp\left[-\int_0^t m(\mathbb{X}_s)(\widehat{V}_s)ds\right]\right) = \mathbb{E}\left(F(\widehat{\chi}_t, \mathbb{Y}_t) \, \exp\left[-\int_0^t \widehat{V}_s(\mathbb{Y}_s^{\mathcal{I}})ds\right] \mid \mathcal{I}\right) \qquad (1.4)$$

The proof of the above theorem is provided in section 6, see for instance theorem 6.2 and corollary 6.3. We obtain the normalizing constant $\mathbb{Z}_t$ by choosing the unit function $F = 1$ in (1.4). Observe that for any F on $D_t(S)$ we have

$$(1.4) \Longrightarrow \mathbb{E}\left(F(\mathbb{X}_t^{\mathcal{I}}) \, \exp\left[-\int_0^t m(\mathbb{X}_s)(\widehat{V}_s)ds\right]\right) = \mathbb{Z}_t \times \mathbb{Q}_t(F) \qquad (1.5)$$

The last assertion yields the rather well known unbiasedness property of the occupation measure of the ancestral lines; see for instance [37] and references therein. Formula (1.4) can also be interpreted as a Bayes formula for many body probability measures on the product space $D_t(S)^{(2N)}$. Therefore, using the conditional distributions we can design a Gibbs sampler. Nevertheless to target $\mathbb{Q}_t$, we do not need to store the complete ancestral tree. A more judicious target measure is the probability measure Π_t on $D_t(S)^{N+1}$ defined by

$$\Pi_t(F) = \frac{1}{\mathbb{Z}_t} \, \mathbb{E}\left(F(\mathbb{X}_t, \mathbb{X}_t^{\mathcal{I}}) \, \exp\left[-\int_0^t m(\mathbb{X}_s)(\widehat{V}_s)ds\right]\right)$$

Let π_t be the marginal of Π_t w.r.t. the variable $\mathbb{X}_t$. Theorem 1.1 yields the Bayes formula

$$\Pi_t(d(z_1, z_2)) = \pi_t(dz_1) \, \mathbb{A}_t(z_1, dz_2) = \mathbb{Q}_t(dz_2) \, \mathbb{B}_t(z_2, dz_1) \qquad (1.6)$$

with the Markov transitions $\mathbb{A}_t$ and $\mathbb{B}_t$ defined by

$$\mathbb{A}_t(z_1, dz_2) := \mathbb{P}(\mathbb{X}_t^{\mathcal{I}} \in dz_2 \mid \mathbb{X}_t = z_1) =: m(z_1)(dz_2) \quad \text{and} \quad \mathbb{B}_t(z_2, dz_1) := \mathbb{P}(\mathbb{Y}_t \in dz_1 \mid \mathbb{Y}_t^{\mathcal{I}} = z_2)$$

The detailed proof of the above assertion is provided in section 6.2, on page 44. Equivalently, under the probability measure Π_t, any randomly chosen ancestral line is distributed with $\mathbb{Q}_t$. In addition,

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under Π_t the conditional distribution of the genealogical tree $\mathbb{X}_t$ given a selected ancestral line coincides with the one of the genealogical tree $\mathbb{Y}_t$ given the frozen selected ancestral line.

Next we provide an analytic description of the Bayes formula (1.6). Integrating (1.6) w.r.t. the first coordinate for any $z_1 \in D_t(S)^N$ we have

$$\mathbb{A}_t(z_1, dz_2) \ll (\pi_t \mathbb{A}_t)(dz_2) := \int \pi_t(dy_1) \ A_t(y_1, dz_2) = \mathbb{Q}_t(dz_2)$$

Thus, we can define the dual operator $\mathbb{A}^{\star}_{t,\pi_t}$ from $\mathbb{L}_1(\pi_t)$ into $\mathbb{L}_1(\mathbb{Q}_t)$ given for any $F \in \mathbb{L}_1(\pi_t)$ by the Radon-Nikodym formula

$$\mathbb{A}^{\star}_{t,\pi_t}(F) := \frac{d\pi_t^F \mathbb{A}_t}{d\mathbb{Q}_t} \quad \text{with} \quad \pi_t^F(dy) := \pi_t(dy) \ F(y)$$

For a more thorough discussion on dual Markov transitions, we refer the reader to [35, 72]. In addition, for any conjugate integers $1/p + 1/q = 1$ with $p, q \geqslant 1$ we have

$$(F, G) \in (\mathbb{L}_p(\pi_t) \times \mathbb{L}_q(\mathbb{Q}_t)) \qquad \mathbb{Q}_t \left(\mathbb{A}^{\star}_{t,\pi_t}(F) \ G \right) = \pi_t \left(F \ \mathbb{A}_t(G) \right)$$

We define in the same way the dual operator $\mathbb{B}^{\star}_{t,\mathbb{Q}_t}$ from $\mathbb{L}_1(\mathbb{Q}_t)$ into $\mathbb{L}_1(\pi_t)$ by

$$\mathbb{B}^{\star}_{t,\mathbb{Q}_t}(G) := \frac{d\mathbb{Q}_t^G \mathbb{B}_t}{d\pi_t} \quad \text{with} \quad \mathbb{Q}_t^G(dy) := \mathbb{Q}_t(dy) \ F(y) \Longrightarrow \pi_t \left(F \ \mathbb{B}^{\star}_{t,\mathbb{Q}_t}(G) \right) = \mathbb{Q}_t \left(\mathbb{B}_t(F) \ G \right)$$

Let $P_{s,t}$ be the Markov semigroup of the reference process X_t, that is

$$P_{s,t}(x, dy) := \mathbb{P} \left(X_t \in dy \mid X_s = x \right)$$

We consider the following regularity condition

$$(H_0) \quad \exists h > 0 \quad \text{s.t.} \quad \forall t \geqslant 0 \quad \forall x \in S \quad \rho(h) \ \mu_{t,h}(dy) \ \leqslant P_{t,t+h}(x, dy) \ \leqslant \rho(h)^{-1} \ \mu_{t,h}(dy) \qquad (1.7)$$

for some constant $\rho(h) > 0$ and some collection of probability measures $\mu_{t,h}$ on S indexed by $h > 0$ and $t \geqslant 0$ whose values do not depend on the parameters (x, y). A discussion on the above condition is provided at the end of this section.

Let $\text{osc}(F)$ and $\|\mu_1 - \mu_2\|_{\text{tv}}$ be the oscillation of a function F on $D_t(S)$ and the total variation distance between probability measures μ_1 and μ_2 (cf. section 2.1 for a more precise definition).

In this notation, the second main result of the article can be stated basically as follows.

Theorem 1.2 (Particle Gibbs-Glauber dynamics). *For any time horizon $t \geqslant 0$ the measure $\mathbb{Q}_t$ is reversible w.r.t. the Markov transition $\mathbb{K}_t := \mathbb{B}_t \mathbb{A}_t$ on $D_t(S)$ defined for any bounded measurable function F on $D_t(S)$ and any path $x \in D_t(S)$ by the formula*

$$\mathbb{K}_t(F)(x) := \mathbb{E} \left(m(\mathbb{Y}_t)(F) \mid \mathbb{Y}_t^{\mathcal{I}} = x \right)$$

In addition, when (H_0) is satisfied, for $n \geqslant 1$, any probability measure μ on $D_t(S)$ and any bounded function F on $D_t(S)$ we have the contraction inequality

$$\|\mu \mathbb{K}_t^n - \mathbb{Q}_t\|_{\text{tv}} \leqslant (c \ (t \vee 1)/N)^n \ \|\mu - \mathbb{Q}_t\|_{\text{tv}} \qquad (1.8)$$

with the n-th iterate $\mathbb{K}_t^n$ of the Markov transition $\mathbb{K}_t$ defined sequentially by the integral formula

$$\mathbb{K}_t^n(x_1, dx_3) := \int \ \mathbb{K}_t^{n-1}(x_1, dx_2) \ \mathbb{K}_t(x_2, dx_3)$$

In the above display, c stands for some explicit finite constant whose value doesn't depend on the parameters (F, t, n, N).

The proof of the above theorem is provided in section 6.2.

For any given time horizon $t \geqslant 0$, the integral operator $\mathbb{K}_t$ is the probability transition of a discrete generation Markov chain $\mathbb{X}_t^{(n)}$ taking values in the path space $D_t(S)$ and indexed by the integer parameter $n \in \mathbb{N}$; that is we have

$$\mathbb{P}\left(\mathbb{X}_t^{(n+1)} \in dz \mid \mathbb{X}_t^{(n)} = x\right) = \mathbb{K}_t(x, dz) := \int \mathbb{B}_t(x, dy) \, \mathbb{A}_t(y, dz)$$

Initially, we can sample a genealogical tree $\mathbb{X}_t = (\mathbb{X}_t^i)_{i \in [N]}$, then we pick randomly an ancestral line $\mathbb{X}_t^{\mathcal{I}}$ and set $\mathbb{X}_t^{(0)} = \mathbb{X}_t^{\mathcal{I}}$.

For any given $x \in D_t(S)$ and $y \in D_t(S)^N$, we summarize the overlapping transition of the particle Gibbs sampler graphically as follows:

$$\left\{\begin{array}{ccc} \mathbb{Y}_t^{(n)} & = & y \\ \mathbb{X}_t^{(n)} & = & x \end{array}\right\} \xrightarrow{\mathbb{B}_t} \left\{\begin{array}{ccc} \mathbb{Y}_t^{(n+1)} & = & \overline{y} \sim \left(\mathbb{Y}_t \mid \mathbb{Y}_t^{\mathcal{I}} = x\right) \\ \mathbb{X}_t^{(n)} & = & x \end{array}\right\} \xrightarrow{\mathbb{A}_t} \left\{\begin{array}{ccc} \mathbb{Y}_t^{(n+1)} & = & \overline{y} \\ \mathbb{X}_t^{(n+1)} & = & \overline{x} \sim m\left(\overline{y}\right) \end{array}\right\}.$$

A realization of the overlapping transition $\mathbb{X}_t^{(n)} \rightsquigarrow \mathbb{X}_t^{(n+1)}$ for a genealogical tree with $N = 3$ ancestral lines is illustrated by the following schematic diagram:
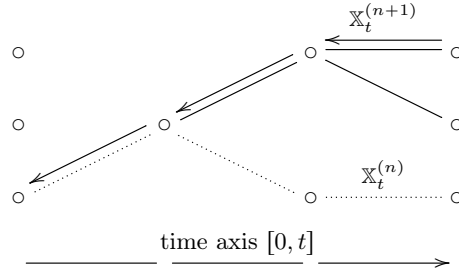

Figure 3: A realization of the transition $\mathbb{X}_t^{(n)} \rightsquigarrow \mathbb{X}_t^{(n+1)}$ of a particle Gibbs sampler on an genealogical tree with $N = 3$ ancestral lines. The dotted and plain lines account together for the three paths in the genealogical tree $\mathbb{Y}_t^{(n+1)}$; the frozen ancestral line is represented by the dotted line $\mathbb{X}_t^{(n)}$ selected at rank (n). The sequence of arrows stands for the selected ancestral line $\mathbb{X}_t^{(n+1)}$ at rank $(n+1)$. The vertical axis stands for the state space S.

As shown in (1.5), under the many-body measure Π_t a randomly selected ancestral line $\mathbb{X}_t^{\mathcal{I}}$ is unbiased but this random path is biased w.r.t. the distribution of the interacting particle system. Propagation of chaos type estimates allow to quantify this bias, see [37, 41], as well as chapter 15 in the monograph [25] for discrete time generation particle systems.

In this connexion, we underline that the Particle Gibbs-Glauber dynamics presented in theorem 1.2 allows to improve the precision of the conventional particle interpretations of Feynman-Kac measures by sampling sequentially a series of particle Gibbs samplers on path spaces with frozen trajectories. Each iteration of the sampler reduces the distance between the distribution of the ancestral path and the desired target measure. For instance, whenever (H_0) is met and $\mathbb{X}_t^{(0)} = \mathbb{X}_t^{\mathcal{I}}$, the propagation of chaos property presented in (5.10) and (1.8) yield the following theorem

Theorem 1.3. *For any $n \geqslant 1$ we have the inequalities*

$$\left\|\mathrm{Law}\left(\mathbb{X}_t^{\mathcal{I}}\right) - \mathbb{Q}_t\right\|_{\mathrm{tv}} \leqslant c \, t/N \quad \text{and} \quad \left\|\mathrm{Law}\left(\mathbb{X}_t^{(n)}\right) - \mathbb{Q}_t\right\|_{\mathrm{tv}} \leqslant (c \, t/N)^n \times \left\|\mathrm{Law}\left(\mathbb{X}_t^{\mathcal{I}}\right) - \mathbb{Q}_t\right\|_{\mathrm{tv}}$$

We end this section with some comments on our regularity assumptions.

Condition (1.7) is satisfied for jump-type elliptic diffusions on compact manifolds S with a bounded jump rate, see for instance the pioneering work of Aronson [5], Nash [69] and Varopoulos [81] on Gaussian estimates for heat kernels on manifolds. The estimates (1.8) are also met under weaker regularity assumptions such as conditions (H_1) and (H_2) stated in (3.14) and (3.15). Nevertheless these conditions depends on the stability properties of the semigroup of the unknown Feynman-Kac measures η_t; thus this type of condition is difficult to check in practice.

Also recall that Feynman-Kac semigroups for time homogeneous models equipped with a reversible reference process X_t can also be turned into conventional Markov semigroups of h-processes, see for instance [37, 41, 74] as well as chapter 27 in [42] and references therein. In this context, the long time behavior of Feynman-Kac semigroups can be discussed in terms of the spectral properties of the h-process. For quadratic potential functions and Ornstein-Uhlenbeck reference processes the Feynman-Kac model discussed above reduces to the harmonic oscillator. In some situations, the potential function can be chosen so that the exponential weight in (1.1) is an exponential change of measure, see for instance section 4.2 in [37], as well as chapter 18 in [42]. In this context the corresponding Feynman-Kac semigroup also coincides with the semigroup of a conventional Markov process. In more general cases, the spectral properties of the h-processes are unknown.

1.2 Illustrations and comments

This section gives some comments on the impact of the above results on some application domain areas. We also provide a detailed discussion on some numerical aspects of the particle Gibbs-Glauber dynamics introduced above as well as some comparisons with existing literature on interacting particle systems.

1.2.1 Some application domains

As mentioned in the introduction, the Feynman-Kac measures (1.1) and their mean field particle interpretations appear in wide variety of applications including in biology, physics, as well as in signal processing and mathematical finance.

Continuous time models arise when the process X_t is derived from physical or natural evolution principles, such as continuous time signals in target tracking filtering problems [78], stochastic population dynamics describing species competition and populations growths [56], Langevin gradient-type diffusions including their overdamped versions describing the evolution of a particle in a fluid [57], as well as Brownian fluctuations of atomic structures in molecular chemistry [54], and many others.

The potential function V_t depends on the problem at hand. In nonlinear filtering, it represents the log-likelihood of the robust optimal filter. In population dynamics, V_t can be interpreted as a killing rate of a branching process. In statistical physics and quantum mechanics, it represents the ground state energy (a.k.a. local energy) of a physical system, including molecular and atomic systems. It is clearly out of the scope of the present article to enter into the details of all of these models. For a more thorough discussion on these application domain areas, we refer to the books [24, 25, 37, 42] and the reference therein.

In most cases we are mainly interested in computing the final-time marginal of the Feynman-Kac measures (1.1). For instance, in nonlinear filtering these measures represent the robust optimal filter, while the path space measures represents the full conditional distributions of the random trajectories of the signal w.r.t. the observation process. Thus, they also solve the smoothing problem by estimating the signal states at any given time using observations from larger time intervals.

Apart from few notable exceptions such as for linear-Gaussian models in Kalman-Bucy filtering theory and for the harmonic oscillator in the spectral theory of Schrödinger operators, the flow of final-time marginal measures has no finite recursion and cannot be solved analytically.

In signal processing literature, the interacting particle system ξ_t discussed above is also known as a particle filter on path space.

In Quantum Monte Carlo literature, the particle system ξ_t discussed above is also known as the population Monte Carlo algorithm and the particles ξ_t^i are often referred as walkers or replica.

This class of processes can be interpreted as Moran type interacting particle systems [67, 68]. They can also be seen as Nanbu type interpretation of a particular spatially homogeneous generalized Boltzmann equation [36, 66].

1.2.2 Practical and numerical aspects

In some particular instances, the random paths of the process X_t can be sampled exactly on any time discretization mesh. This class of models includes linear-Gaussian and geometric-type Brownian models, as well as some piecewise deterministic processes and some classes of one-dimensional jump-diffusion processes [10, 11, 12, 13, 15]. Discretization-free simulation procedures for general diffusion processes based on sequential importance sampling techniques have also been developed in [50]. Using these discretization-free simulation procedures the interacting jump particle systems discussed in this article, including the particle Gibbs-Glauber dynamics can be sampled perfectly using conventional Poisson thinning techniques (a.k.a. Gillespie's algorithm [51]). The resulting particle sampler provides an estimate of the marginal of the Feynman-Kac measures (1.1) on the random paths w.r.t. any time discretization mesh.

More generally, the simulation of the random trajectories of X_t requires to discretize the time parameter. For a more thorough discussion on the time discretization of stochastic processes we refer to the seminal book by Kloeden and Platen [53].

This additional level of approximation may also corrupt some statistical properties of the continuous time process. For instance, the reversible properties of overdamped Langevin diffusions are lost for any Euler-Maryuama discretization of the underlying diffusion. In this context, a Metropolis-Hastings type adjustment (a.k.a. MALA) is required to recover the reversibility property w.r.t. some prescribed target invariant measure [73]. From the physical viewpoint, the random paths simulated by MALA algorithms are based *on auxiliary non physical rejection-type transitions* so that they loose their initial physical interpretation. Therefore, in physics and statistics, the unadjusted Langevin algorithm (a.k.a. ULA) is often preferred to describe the "true" random trajectories of the system. Under appropriate global Lipschitz conditions on the gradient of the confinement potential function several bias-type estimates can be found in [23, 49].

In the same vein, the sampling of the particle Gibbs-Glauber dynamics described in theorem 1.2 requires some Euler-type discretization as soon as the underlying process X_t cannot be directly sampled. In this situation, one natural strategy is to consider the discrete time version of the Feynman-Kac measures η_t defined as in (1.1) by replacing X_t by some discrete time approximation (see for instance chapter 5 in [25] and the references therein). In this context, several discrete time approximations of the particle Gibbs-Glauber dynamics discussed above can be designed using the discrete time particle Gibbs samplers discussed in [2, 34]. In contrast with MALA algorithms the reversible-type properties of the resulting Gibbs samplers in discrete time are preserved w.r.t. to the discrete-time version of the target Feynman-Kac measures. In addition, these discrete time approximations are not based on any type of auxiliary Metropolis-Hasting rejection so that they preserve their physical interpretations.

Several bias-type estimates between continuous and discrete time Feynman-Kac measures can

be found in [25, 32, 33]. Most of these estimates are concerned with the time discretization of the terminal-time marginal of the Feynman-Kac measures (1.1), including uniform estimates w.r.t. the time horizon. The extension of these results to path space models remains an important open research question.

1.2.3   Comparisons with diffusion type particle models

The interacting particle systems discussed in the present article differ from nonlinear and interacting diffusion processes arising in fluid mechanics and granular flows [7, 8, 64, 65, 79, 80]. In this context, the interaction mechanism is encapsulated in the drift of diffusion-type particles. One common feature of these interacting processes is the nonlinearity of the distribution flow associated with these stochastic processes.

One natural idea is to interpret the mean field particle systems associated with these processes as a stochastic perturbation of a nonlinear process. This interpretation allows to enter the stability properties of the nonlinear process into the convergence analysis of these particle algorithms. This technique has been developed in [28, 29, 37] for discrete time Feynman-Kac models and further extended in [74] to continuous time models. The extension to stochastic diffusion flows and McKean-Vlasov type nonlinear diffusions are developed in [4, 44].

Theorem 5.7 in the present article also provides a novel backward stochastic perturbation formula which simplifies the stability analysis of these models and provides sharp propagation of chaos estimates.

We underline that the stochastic perturbation techniques discussed above and in the present article differs from the log-Sobolev functional techniques [62, 63], entropy dissipation approaches [18, 20], as well as gradient flows in Wasserstein metric spaces, optimal transportation inequalities [9, 18, 19, 70, 71] and the more recent variational approach [3] currently used in the analysis of gradient type flow interacting diffusions.

In this connection, we mention that the backward perturbation analysis developed in the present article relies on weak Taylor expansions of the evolution semigroup of Feynman-Kac measures. We project to extend these expansions to nonlinear diffusions in a forthcoming article.

The duality formula and the particle Gibbs-Glauber dynamics introduced in this article open up a whole new avenue of research questions.

Recall that the Feynman-Kac measures (1.1) can be interpreted as the distribution of the random paths of a non absorbed particle evolving as X_t and killed at rate V_t. This class of models are often referred as particle models in absorbing medium with soft obstacles [27, 37, 41]. A natural research project is to extend this framework to absorbing medium with hard obstacles [45, 46, 82].

Another important question is to extend the Taylor expansions of the Gibbs sampler developed in [34] to continuous time models. One possible route is to combine the weak Taylor expansions developed in [43] for particle approximating measures with the backward analysis developed in the present article.

We mention that the perturbation analysis developed in [34] allows to destimate the $\mathbb{L}_p$-decays rates to equilibrium in terms of the norm of integral operators. In this connection, one important question is to quantify with more precision the exponential convergence rates to equilibrium of the Particle Gibbs-Glauber dynamics stated in theorem 1.2.

2 Some preliminary results

2.1 Basic notation

Let $\mathcal{B}(E)$ be the Banach space of bounded functions f on some measurable space $(E, \mathcal{E})$ equipped with the uniform norm $\|f\| := \sup_{x \in E} |f(x)|$. Also let $\mathrm{Osc}(E) \subset \mathcal{B}(E)$ be the subset of functions f with at most. unit oscillations; that is s.t. $\mathrm{osc}(f) := \sup_{x,y} |f(x) - f(y)| \leqslant 1$.

We also let $\mathcal{M}(E)$ be the set of finite signed measures on E, $\mathcal{M}_+(E) \subset \mathcal{M}(E)$ the subset of positive measures and $\mathcal{P}(E) \subset \mathcal{M}_+(E)$ the subset of probability measures. Given a measure μ on E we write $\mu(f)$ the Lebesgue integral given by

$$\mu(f) = \int \mu(dx)\ f(x)$$

The total variation norm on the set $\mathcal{M}(E)$ is defined by

$$\|\mu\|_{\mathrm{tv}} \quad := \quad \sup\left\{|\mu(f)| \ : \ f \in \mathrm{Osc}(E)\right\} = \frac{1}{2}\ \sup\left\{|\mu(f)| \ : \ f \in \mathcal{B}(E),\ \|f\| \leqslant 1\right\} \tag{2.1}$$

We denote by $[s, t]_n$ the collection Weyl chambers defined for any $n \geqslant 1$ by

$$[s, t]_n := \{(r_1, \ldots, r_n) \in [s, t]^n \ : \ s \leqslant r_1 \leqslant \ldots \leqslant r_n \leqslant t\} \tag{2.2}$$

We denote by $dr = dr_1 \times \ldots \times dr_n$ the Lebesgue measure on $[s, t]_n$.

For a given $N \geqslant 1$, we let $\langle N \rangle$ be the semigroup of mappings from $[N] := \{1, \ldots, N\}$ into itself equipped with the composition of mappings and the neutral element $e(i) = i$. Also let $[N]_0^2 \subset [N]^2$ be the subset of indices

$$[N]_0^2 := \{(i, j) \in [N]^2 \ : \ i \in [N] \quad \& \quad j \in [N] - \{i\}\}$$

2.2 Integral operators

We introduce some integral operator notation needed from the onset. For any bounded positive integral operator $Q(x, dy)$ and any $(\mu, f, x) \in (\mathcal{M}(E) \times \mathcal{B}(E) \times E)$ we define by $\mu Q \in \mathcal{M}(E)$ and $Q(f) \in \mathcal{B}(E)$ by the formulae

$$(\mu Q)(dy) := \int \mu(dx)Q(x, dy) \quad \text{and} \quad Q(f)(x) := \int Q(x, dy)\ f(y)$$

By Fubini theorem we have $\mu Q f := \mu(Q(f)) = (\mu Q)(f)$. Given a pair of operators Q_1 and Q_2 we denote by $Q_1 Q_2 := Q_1 \circ Q_2$ the composition of the operators defined for functions f on S by

$$(Q_1 \circ Q_2)(f) := (Q_1 Q_2)(f) = Q_1(Q_2(f)) \tag{2.3}$$

We also write Q^n the n iterate of Q defined by the recursion $Q^n(f) = Q(Q^{n-1}(f)) = Q^{n-1}(Q(f))$.

When $Q(1) > 0$ we let $\overline{Q}$ be the Markov operator

$$\overline{Q} \ : \ f \in \mathcal{B}(E) \mapsto \overline{Q}(f) := Q(f)/Q(1) \in \mathcal{B}(E)$$

We also let ϕ be the mapping from $\mathcal{P}(E)$ into itself defined by

$$\phi(\eta) = \eta Q^\eta \quad \text{with} \quad Q^\eta := \frac{Q}{\eta Q(1)} \Longrightarrow \eta Q^\eta(1) = 1 \quad \text{and} \quad \phi(\delta_x)(f) = \overline{Q}(f)(x) \tag{2.4}$$

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Notice that

$$Q^\eta(1) = \mu Q^\eta(1) \; Q^\mu(1) \Longrightarrow (\mu Q^\eta(1))^{-1} = \eta Q^\mu(1)$$

The Dobrushin ergodic coefficient $\beta_{\mathrm{dob}}(M)$ of a Markov transition $M(x, dy)$ from E into itself is defined by

$$\beta_{\mathrm{dob}}(M) := \sup_{x, y \in E} \|M(x, .) - M(y, .)\|_{\mathrm{tv}} = \sup\{\mathrm{osc}(M(f)) \; : \; f \in \mathrm{Osc}(E)\}$$

For any $\mu_1, \mu_2 \in \mathcal{P}(E)$ and any pair of Markov transition M_1, M_2 from E into itself we have

$$\beta_{\mathrm{dob}}(M_1 M_2) \leqslant \beta_{\mathrm{dob}}(M_1) \; \beta_{\mathrm{dob}}(M_2) \quad \text{and} \quad \|\mu_1 M - \mu_2 M\|_{\mathrm{tv}} \leqslant \beta_{\mathrm{dob}}(M) \; \|\mu_1 - \mu_2\|_{\mathrm{tv}} \tag{2.5}$$

2.3 Taylor expansions

The Feynman-Kac semigroups discussed in section 3.2 have the same form as the map ϕ discussed in (2.4) (see for instance (3.7)). The stochastic perturbation analysis developed in section 5 is mainly based on a second order Taylor expansion of these maps (see for instance the proof of proposition 3.5, as well as the perturbation semigroup equation presented in theorem 5.2 and the Aleeksev-Gröbner interpolation formula stated in theorem 5.7).

To describe in some details these Taylor expansions, consider the collection of first order integral operators $\partial_\eta \phi$ indexed by $\eta \in \mathcal{P}(E)$ and defined by

$$\partial_\eta \phi \; : \; f \in \mathcal{B}(E) \mapsto \partial_\eta \phi(f) = Q^\eta [f - \phi(\eta)(f)] \in \mathcal{B}(E) \Longrightarrow \eta \partial_\eta \phi = 0 = \partial_\eta \phi(1) \tag{2.6}$$

Rewritten in integral form, we have

$$\partial_\eta \phi(f)(x) = \int \partial_\eta \phi(x, dy) \; f(y) \quad \text{with} \quad \partial_\eta \phi(x, dy) := Q^\eta(x, dy) - Q^\eta(1)(x) \; \phi(\eta)(dy)$$

For any $\eta, \nu \in \mathcal{P}(E)$ we have the first order Taylor expansion

$$\phi(\nu) - \phi(\eta) \;\; = \;\; \eta Q^\nu(1) \; \times \; (\nu - \eta)\partial_\eta \phi \quad \text{with} \quad (\nu - \eta)\partial_\eta \phi(dy) := \int (\nu - \eta)(dx)\partial_\eta \phi(x, dy)$$

Also observe that

$$\partial_\eta \phi(f)(x) = Q^\eta(1)(x) \int \eta(dy) \; Q^\eta(1)(y) \; (\overline{Q}(f)(x) - \overline{Q}(f)(y))$$

$$\Longrightarrow \|\partial_\eta \phi(f)\| \leqslant \|Q^\eta(1)\| \; \mathrm{osc}(\overline{Q}(f)) \quad \text{and} \quad \|\phi(\nu) - \phi(\eta)\|_{\mathrm{tv}} \leqslant \beta_{\nu,\eta}(\phi) \; \|\nu - \eta\|_{\mathrm{tv}} \tag{2.7}$$

with

$$\beta_{\nu,\eta}(\phi) := [\|Q^\nu(1)\| \wedge \|Q^\eta(1)\|] \; \beta_{\mathrm{dob}}(\overline{Q})$$

More generally, using the identity

$$\frac{1}{x} = \sum_{0 \leqslant k < n} (1 - x)^k + \frac{(1 - x)^n}{x} \tag{2.8}$$

which is valid for any $x > 0$ and $n \geqslant 1$, we check the Taylor with remainder expansion

$$\phi(\nu) \;\; = \;\; \phi(\eta) + \sum_{1 \leqslant k \leqslant n} \frac{1}{k!} \; (\nu - \eta)^{\otimes k} \; \partial_\eta^k \phi + \frac{1}{(n + 1)!} \; (\nu - \eta)^{\otimes(n+1)} \; \overline{\partial}_{\nu,\eta}^{n+1} \phi \tag{2.9}$$

In the above display, $\partial_\eta^k \phi$ stand for the collection of integral operators

$$\partial_\eta^k \phi(f) := (-1)^{k-1} \ k! \ \left[Q^\eta(1)^{\otimes(k-1)} \otimes \partial_\eta \phi(f) \right] \quad \text{and} \quad \overline{\partial}_{\nu,\eta}^{n+1} \phi := \eta Q^\nu(1) \ \partial_\eta^{n+1} \phi$$

For any $\mu, \eta \in \mathcal{P}(E)$ we have the decomposition

$$\partial_\eta \phi(f) = Q^\eta[f - \phi(\eta)f] \ = \ \mu Q^\eta(1) \ (\partial_\mu \phi(f) + Q^\mu(1) \ [\phi(\mu) - \phi(\eta)](f))$$

2.4   Carré du champ operators

The generator L of a stochastic process X_t on some measurable space $(E, \mathcal{E})$ provides a simple and natural way to define the evolution of the random path of the process.

  These operators are also the stepping stone of stochastic calculus. They play the same role as the vector fields associated with a dynamical system and they allow to derive integration by parts formula; see for instance section 4.1 in the context of mean-field particle processes. The carré du champ operator Γ_L associated with the generator L characterizes the predictable quadratic part of these integration by parts formula (see for instance (4.2) and (4.3)).

  In this section we review some basic inequalities that allows to quantify the first order fluctuation term as well as the second order bias term in the Aleeksev-Gröbner interpolation formula stated in theorem 5.7.

  The carré du champ operator Γ_L acts on an algebra of functions $\mathcal{D}(L) \subset \mathcal{B}(E)$ and it is defined by the quadratic form

$$(f, g) \in \mathcal{D}(L)^2 \mapsto \Gamma_L(f, g) = L(fg) - fL(g) - gL(f) \in \mathcal{B}(E)$$

When $f = g$ sometimes we write $\Gamma_L(f)$ instead of $\Gamma_L(f, f)$. We also have the Cauchy-Schwartz inequality

$$|\Gamma_L(f, g)| \leqslant \sqrt{\Gamma_L(f, f)\Gamma_L(g, g)} \quad \text{and} \quad \Gamma_L(cf) = c^2 \ \Gamma_L(f) \tag{2.10}$$

The above inequality yields the estimate

$$\Gamma_L(f + g) = \Gamma_L(f) + \Gamma_L(g) + 2\Gamma_L(f, g) \leqslant \left[\sqrt{\Gamma_L(f)} + \sqrt{\Gamma_L(g)} \right]^2 \tag{2.11}$$

Let L^d be some bounded jump-type generator of the following form

$$L^d(f)(u) = \lambda(u) \int \ (f(v) - f(u)) \ J(u, dv)$$

for some bounded rate function λ and some Markov transition J on E. In this case, we have

$$\Gamma_{L^d}(f, g)(u) = \int \ L^d(u, dv) \ (\delta_v - \delta_u)^{\otimes 2} \ (f \otimes g)$$

The convergence analysis of the particle measures (1.3) developed in section 5.2 is also based on the n-th order operators given by the formula

$$\Gamma_{L^d}^{(n)}(f_1, \ldots, f_n)(u) := \int \ L^d(u, dv) \ (\delta_v - \delta_u)^{\otimes n} \ (f_1 \otimes \ldots \otimes f_n) \tag{2.12}$$

Applied to mappings of the form (2.4), for any $\mu, \eta \in \mathcal{P}(E)$ we have the carré du champ formula

$$(\eta Q^\mu(1))^2 \ \Gamma_L \ (Q^\eta(1), \partial_\eta \phi(f))$$

$$= \Gamma_L \ (Q^\mu(1), \partial_\mu \phi(f)) + \ [\phi(\mu) - \phi(\eta)](f) \ \Gamma_L \ (Q^\mu(1)) \tag{2.13}$$

for any $f \in \mathcal{D}(L)$ as soon as $Q^\eta(1), \partial_\eta \phi(f) \in \mathcal{D}(L)$.

2.5 Historical processes

The article discusses several classes of particle models evolving in path spaces, such as the complete ancestral tree models and the genealogical tree based evolutions discussed in section 1.1.

In this short section, we review some basic facts on the concatenation of paths and the description of historical semigroups.

For any $s \leqslant t$, we denote by $D_{s,t}(S)$ the set of càdlàg paths from $[s,t]$ to the metric space (S, d_S). Fo any given $x = (x_r)_{s \leqslant r \leqslant t} \in D_{s,t}(S)$ we let $x_- \in D_{s,t}(S)$ the stopped path given by

$$x_-(u) := \left\{ \begin{array}{ll} x_u & \text{if} \quad u \in [s, t[\\ x_{t-} & \text{if} \quad u = t \end{array} \right.$$

For any $r \leqslant s \leqslant t$ and any $x = (x_u)_{r \leqslant u \leqslant s} \in D_{r,s}(S)$ and $y = (y_u)_{s \leqslant u \leqslant t} \in D_{s,t}(S)$ the concatenate path $(x \vee y) \in D_{r,t}(S)$ is defined by

$$(x \vee y)(u) := \left\{ \begin{array}{ll} x_u & \text{if} \quad u \in [r, s[\\ y_u & \text{if} \quad u \in [s, t] \end{array} \right. \implies x \vee y = x_- \vee y$$

The set $D_{r,s}(S)$ can be embedded into $D_{r,t}(S)$ by considering the stopped process extension

$$x = (x_u)_{r \leqslant u \leqslant s} \in D_{r,s}(S) \mapsto x_{\wedge s} := (x_{u \wedge s})_{r \leqslant u \leqslant t} \in D_{r,s}(S) \quad \text{with} \quad u \wedge s := \min\{u, s\}$$

When $s = t$, the set $D_{s,s}(S)$ reduces to S and for any $x = (x_u)_{r \leqslant u \leqslant s} \in D_{r,s}(S)$ and $y \in S$ the concatenate path $(x \vee y) \in D_{r,s}(S)$ is given by to the càdlàg path

$$(x \vee y)(u) := \left\{ \begin{array}{ll} x_u & \text{if} \quad u \in [r, s[\\ y & \text{if} \quad u = s \end{array} \right. \implies x = x \vee x_s \quad \text{and} \quad x_- = x \vee x_{s-} \tag{2.14}$$

Let $X_{s,t}(y)$ be the stochastic semigroup of the process X_t starting at $X_s = y$ at time $s \leqslant t$. The stochastic semigroup $\widehat{X}_{s,t} : D_s(S) \mapsto D_t(S)$ of the historical process $\widehat{X}_t := (X_s)_{s \leqslant t}$ is defined for any $x \in D_s(S)$ and $s \leqslant t$ by the stop-and-go formula

$$\widehat{X}_{s,t}(x) = x \vee \widetilde{X}_{s,t}(x) = x_- \vee \widetilde{X}_{s,t}(x) \in D_t(S) \tag{2.15}$$

with the mapping

$$\widetilde{X}_{s,t} : x = (x_u)_{0 \leqslant u \leqslant s} \in D_s(S) \mapsto \widetilde{X}_{s,t}(x) = (X_{s,u}(x_s))_{s \leqslant u \leqslant t} \in D_{s,t}(S)$$

2.6 Coalescent operators

The complete ancestral as well as the genealogical tree evolutions $(\widehat{\xi}_t, \mathbb{X}_t)$ discussed in section 1.1 belong to the class of interacting jump processes in the space of càdlàg paths.

When a jump occurs in a genealogical tree evolution, a path-particle is killed and instantaneously other path-particle duplicates. When a jump occurs in a complete ancestral tree evolution, a path-particle restarts its evolution from a new selected state. In this situation, the jump of the historical particle is characterized by the concatenation of the stopped process of the historical particle with a new terminal state, from which the particle restarts its free evolution. The dual process $\widehat{\chi}_t$ of the historical process incorporates an additional permutation of the historical paths.

To describe with some precision these jumps, we need to introduce some new objects.

We denote by $\mathfrak{T}$ and $\mathfrak{C}$ the set of transpositions σ_ι and coalescent maps c_ι indexed by $\iota = (i, j) \in [N]_0^2$ and given for any $k \in [N]$ by

$$\sigma_{i,j}(k) := \left\{ \begin{array}{ll} k & \text{if} \quad k \notin \{i, j\} \\ j & \text{if} \quad k = i \\ i & \text{if} \quad k = j \end{array} \right. \quad \text{and} \quad c_{i,j}(k) := \left\{ \begin{array}{ll} k & \text{if} \quad k \neq i \\ j & \text{if} \quad k = i \end{array} \right. \tag{2.16}$$

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Definition 2.1. *For any $a, b \in \langle N \rangle$ and $x = (x_r)_{s \leqslant r \leqslant t} \in D_{s,t}(S)^N$ we set*

$$x^a := (x^{a(i)})_{i \in [N]} \in D_{s,t}(S)^N \qquad \mathbb{C}_{a,b}(x) := x^a \vee x_t^b \quad and \quad \mathbb{C}_a := \mathbb{C}_{a,a} \tag{2.17}$$

Observe that for any a_1, b_1 and any $a_2, b_2 \in \langle N \rangle$ we have

$$\mathbb{C}_{a_1,b_1} \circ \mathbb{C}_{a_2,b_2} = \mathbb{C}_{a_2 \circ a_1, b_2 \circ b_1} \quad and \quad \mathbb{C}_{a_1}(x) = x^{a_1}$$

We also mention that for any $(i,j) \in [N]_0^2$ we have

$$\mathbb{C}_{c_{i,j}} \circ \mathbb{C}_{\sigma_{j,i}} = \mathbb{C}_{\sigma_{j,i} \circ c_{i,j}} = \mathbb{C}_{c_{j,i}} \quad and \; therefore \quad \mathbb{C}_{\sigma_{i,j},c_{j,i}} = \mathbb{C}_{e,c_{i,j}} \circ \mathbb{C}_{\sigma_{i,j}} \tag{2.18}$$

Observe that for any index $k \in [N]$ we have the partition

$$[N]_0^2 := \{(i,j) \in [N]_0^2 \;:\; i \in [N] - \{k\}\} \cup \{(k,j) \in [N]_0^2 \;:\; j \in [N] - \{k\}\}$$

Definition 2.2. *For any $k \in [N]$ and any $(i,j) \in [N]_0^2$ we let*

$$\left(\mathbb{C}_{e,c_{i,j}}^k, \mathbb{C}_{c_{i,j}}^k\right) := \begin{cases} \left(\mathbb{C}_{e,c_{i,j}}, \mathbb{C}_{c_{i,j}}\right) & \text{if} \quad i \in [N] - \{k\} \\[2mm] \left(\mathbb{C}_{\sigma_{k,j},c_{j,k}}, \mathbb{C}_{c_{j,k}}\right) & \text{if} \quad i = k \end{cases} \tag{2.19}$$

A schematic picture of the jumps $x \rightsquigarrow \mathbb{C}_{e,c_{i,k}}(x) = x \vee x_t^{c_{i,k}}$ and $x \rightsquigarrow \mathbb{C}_{\sigma_{i,k},c_{i,k}}(x) = x^{\sigma_{i,k}} \vee x_t^{c_{i,k}}$ is given below

We underline that the jump in $\mathbb{C}_{e,c_{i,k}}(x)$ occurs on the k-th row trajectory while the jump in $\mathbb{C}_{\sigma_{i,k},c_{i,k}}(x)$ occurs on the i-th row trajectory.

A schematic picture of the path-valued jump $x \rightsquigarrow \mathbb{C}_{c_{i,k}}(x) = x^{c_{i,k}}$ is given below

Between jumps all the particle models discussed in this article evolve independently as independent copies of the reference process X_t introduced in section 1.1. Let $X_{s,t}^i$ with $i \in [N]$ be N independent copies of the stochastic semigroup $X_{s,t}$. We extend the stochastic semigroups introduced (2.15) to product spaces and we let $\mathcal{X}_{s,t}$, $\widehat{\mathcal{X}}_{s,t}$ and $\widetilde{\mathcal{X}}_{s,t}$ the stochastic semigroups with the i-th coordinates mappings defined for any for any $z \in S^N$ and any $x \in D_s(S)^N$ by the formulae

$$\mathcal{X}_{s,t}^i(z) := X_{s,t}^i(z^i) \qquad \widehat{\mathcal{X}}_{s,t}^i(x) := \widehat{X}_{s,t}^i(x^i) \quad and \quad \widetilde{\mathcal{X}}_{s,t}^i(x) := \widetilde{X}_{s,t}^i(x^i)$$

$$\implies \quad \widehat{\mathcal{X}}_{s,t}(x) = x \vee \widetilde{\mathcal{X}}_{s,t}(x) = x_- \vee \widetilde{\mathcal{X}}_{s,t}(x) \in D_t(S)^N \tag{2.20}$$

We also denote $\widehat{\xi}_{s,t}$ and $\widehat{\mathcal{X}}_{s,t}$ the stochastic evolution semigroup of the processes $\widehat{\xi}_t$ and $\widehat{\chi}_t$; that is, for any $s \leqslant t$ we have the stochastic evolution semigroup formulae

$$\widehat{\xi}_{s,t}(\widehat{\xi}_s) = \widehat{\xi}_t \quad and \quad \widehat{\chi}_{s,t}(\widehat{\chi}_s) = \widehat{\chi}_t \quad with \; the \; initial \; condition \quad \widehat{\xi}_0 = \xi_0 = \widehat{\chi}_0$$

We define in the same vein the stochastic evolution semigroups $\mathbb{X}_{s,t}$ and $\mathbb{Y}_{s,t}$ of the genealogical tree processes $\mathbb{X}_t$ and $\mathbb{Y}_t$.

At jump times, these path-space interacting jumps stochastic flows are described by the composition of the coalescent operators presented in (2.17) and (2.19) with the stochastic evolution semigroups (2.20).

Definition 2.3. *For any $a \in \mathfrak{C}$ and any $s \leqslant t$ and $k \in [N]$ we define*

$$
\begin{aligned}
\widehat{\mathcal{X}}_{s,t}^a &:= \mathbb{C}_a \circ \widehat{\mathcal{X}}_{s,t} & \widehat{\mathcal{X}}_{s,t}^{k,a} &:= \mathbb{C}_a^k \circ \widehat{\mathcal{X}}_{s,t} \\
\widehat{\mathcal{T}}_{s,t}^a &:= \mathbb{C}_{e,a} \circ \widehat{\mathcal{X}}_{s,t} & \widehat{\mathcal{T}}_{s,t}^{k,a} &:= \mathbb{C}_{e,a}^k \circ \widehat{\mathcal{X}}_{s,t}
\end{aligned}
\tag{2.21}
$$

Observe that

$$
\widehat{\mathcal{X}}_{s,t}^a(x) = \mathbb{C}_a(\widehat{\mathcal{X}}_{s,t}(x)) = \left(\widehat{\mathcal{X}}_{s,t}^{a(i)}(x) \right)_{i \in [N]} = \left(\widehat{X}_{s,t}^{a(i)}\left(x^{a(i)} \right) \right)_{i \in [N]}
$$

This yields for any $r \leqslant s \leqslant t$ and any $a, b \in \langle N \rangle$ and $x \in D_r(S)^N$ the composition formula

$$
\left(\widehat{\mathcal{X}}_{s,t}^a \circ \widehat{\mathcal{X}}_{r,s}^b \right)(x) = \widehat{\mathcal{X}}_{s,t}^a \left(\widehat{\mathcal{X}}_{r,s}^b(x) \right) = \left(\widehat{X}_{s,t}^{a(i)} \left(\widehat{X}_{r,s}^{(b \circ a)(i)} \left(x^{(b \circ a)(i)} \right) \right) \right)_{i \in [N]}
$$

More generally, for any sequence of mappings $a_n \in \langle N \rangle$ any $x \in D_{t_0}(S)^N$ and any non decreasing time steps t_n we have the formula

$$
\left(\widehat{\mathcal{X}}_{t_{n-1},t_n}^{a_n} \circ \ldots \circ \widehat{\mathcal{X}}_{t_0,t_1}^{a_1} \right)(x)
$$

$$
= \left(\left(\widehat{X}_{t_{n-1},t_n}^{a_n(i)} \circ \widehat{X}_{t_{n-2},t_{n-1}}^{a_{n-1,n}(i)} \circ \ldots \circ \widehat{X}_{t_0,t_1}^{a_{1,n}(i)} \right) \left(x^{a_{1,n}(i)} \right) \right)_{i \in [N]} \quad \text{with} \quad a_{p,n} := a_p \circ a_{p+1} \circ \ldots \circ a_n
$$

Finally observe that for any $a_1, \ldots, a_n \in \mathfrak{C}$ and any $k \in [N]$ and $x \in D_{t_0}(S)^N$ we have

$$
\mathbb{C}_{c_{i,j}}^k = 1_{i \neq k}\, \mathbb{C}_{c_{i,j}} + 1_{i=k}\, \mathbb{C}_{c_{j,k}}
$$

$$
\implies \quad \left(\widehat{\mathcal{X}}_{t_{n-1},t_n}^{k,a_n} \circ \ldots \circ \widehat{\mathcal{X}}_{t_0,t_1}^{k,a_1} \right)(x)^k = \widehat{X}_{t_0,t_n}^k(x^k)
\tag{2.22}
$$

The composition of the stochastic flows $\widehat{\mathcal{T}}_{s,t}^a$ is slightly more involved. For instance, for any $s \leqslant r \leqslant t$ and any $a, b \in \mathfrak{C}$ and $x \in D_s(S)^N$ we have the concatenate path formulae

$$
\widehat{\mathcal{T}}_{s,t}^a(x) = \widehat{\mathcal{X}}_{s,t}(x) \vee \mathcal{X}_{s,t}^a(x_s) \in D_t(S)^N
$$

In addition, we have the composition rule

$$
\left(\widehat{\mathcal{T}}_{r,t}^a \circ \widehat{\mathcal{T}}_{s,r}^b \right)(x) = \widehat{\mathcal{X}}_{s,r}(x) \vee \left(\widetilde{\mathcal{X}}_{r,t} \circ \widehat{\mathcal{T}}_{s,r}^b \right)(x) \vee \left(\mathcal{X}_{r,t}^a \circ \mathcal{X}_{s,r}^b \right)(x_s)
$$

The proof of the first assertion is immediate. To check the second, observe that

$$
\begin{aligned}
\widehat{\mathcal{T}}_{r,t}^a \left(\widehat{\mathcal{T}}_{s,r}^b(x) \right) &= \widehat{\mathcal{X}}_{r,t} \left(\widehat{\mathcal{X}}_{s,r}(x) \vee \mathcal{X}_{s,r}^b(x_s) \right) \vee \mathcal{X}_{r,t}^a(\mathcal{X}_{s,r}^b(x_s)) \\
&= \widehat{\mathcal{X}}_{s,r}(x) \vee \widetilde{\mathcal{X}}_{r,t} \left(\widehat{\mathcal{T}}_{s,r}^b(x) \right) \vee \mathcal{X}_{r,t}^a(\mathcal{X}_{s,r}^b(x_s)) \quad \text{(by (2.20))}
\end{aligned}
$$

This ends the proof of the lemma. ∎

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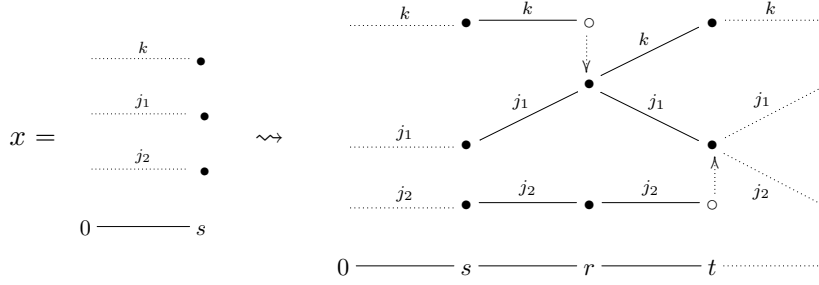A graphical description of these compositions is given below in figure 6.

Figure 6: Genealogical tree associated with the composition $\widehat{\mathcal{T}}_{r,t}^{c_{j_2},j_1} \circ \widehat{\mathcal{T}}_{s,r}^{c_k,j_1}$

The dotted lines $(\overset{j}{\dots})$ represent the paths associated with x, the plain lines $(\overset{j}{\quad})$ represent the paths associated with the stochastic semigroup X^j. A synthetic description of the composition associated with the graph in figure 6 is given below in figure 7.
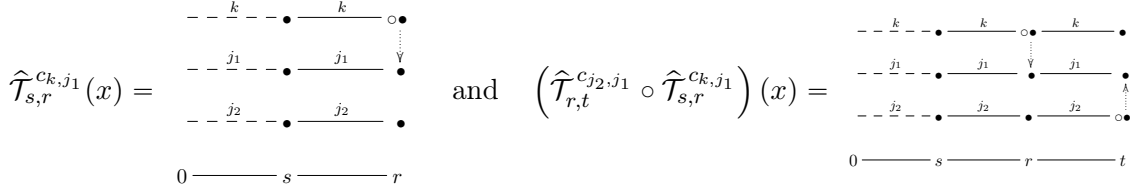

Figure 7

2.7 Empirical measures

We fix some integer $N \geqslant 2$ and for any $1 \leqslant i < j \leqslant N$ and $x = (x^i)_{1 \leqslant i \leqslant N}$ on some N-fold cartesian product E^N of some measurable space $(E, \mathcal{E})$ we set

$$
\begin{aligned}
x^{-i} &= \left(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^N\right) \in E^N \\
x^{-\{i,j\}} &= \left(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{j-1}, x^{j+1}, \dots, x^N\right) \in E^{N-2}
\end{aligned}
$$

For any $1 \leqslant i \leqslant N$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in E^N$ we consider the functions

$$
\begin{aligned}
\varphi_{x^{-i}} \; : \; u \in E \; &\mapsto \; \varphi_{x^{-i}}(u) = \left(x^1, \dots, x^{i-1}, u, x^{i+1}, \dots, x^N\right) \in E^N \\
m \; : \; x \in S^N \; &\mapsto \; m(x) = \frac{1}{N} \sum_{i \in [N]} \delta_{x^i} \in \mathcal{P}(E)
\end{aligned}
\tag{2.23}
$$

In this notation, the generator $\mathcal{L}_t$ of the stochastic flow $x \in S^N \mapsto \mathcal{X}_{s,t}(x) \in S^N$ introduced in (2.20) is given for any $F \in \mathcal{D}(\mathcal{L})$ and $x \in D_t(S)^N$ by the formula

$$
\mathcal{L}_t(F)(x) = \sum_{i \in [N]} L_t(F_{x^{-i}})(x^i)
$$

In the above display, $\mathcal{D}(\mathcal{L}) \subset \mathcal{B}(S^N)$ stands for the set of functions $F \in \mathcal{B}(S^N)$ s.t. for any $x \in S^N$ we have

$$
F_{x^{-i}} := F \circ \varphi_{x^{-i}} \in \mathcal{D}(L)
$$

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Let $X = (X^i)_{1 \leqslant i \leqslant N}$ be N independent random samples from some distribution $\eta \in \mathcal{P}(S)$. Using (2.9) we have the first order expansion

$$
\begin{aligned}
&\phi(m(X)) - \phi(\eta) \\
&= (m(X) - \eta)\partial_\eta \phi - \eta(Q^{m(X)}(1)) \times (m(X) - \eta)(Q^\eta(1)) \times (m(X) - \eta)\partial_\eta \phi
\end{aligned}
\tag{2.24}
$$

Several estimates can be derived from the above decomposition. For instance using Cauchy-Schwartz inequality we have the bias estimate

$$
\log\left(Q(1)(x)/Q(1)(y)\right) \leqslant q \Longrightarrow N \left|\mathbb{E}\left[\phi(m(X))(f)\right] - \phi(\eta)(f)\right| \leqslant e^q \operatorname{osc}(\overline{Q}(f))
$$

The stochastic perturbation analysis discussed above will be used repeatedly in section 5.2 dedicated to the convergence analysis of the empirical measures associated with the genetic particle models discussed in section 1.1. For instance, theorem 5.7 is the extended version of the second order expansion (2.24) to interacting particle occupation measures.

3 A brief review on Feynman-Kac measures

3.1 Exponential maps

For any $s \leqslant t$ and $N \geqslant 1$ we let $\mathcal{Z}_{s,t}$ be the exponential map defined by

$$
\mathcal{Z}_{s,t} \; : \; x = (x_r)_{0 \leqslant r \leqslant t} \in D_t(S)^N \mapsto \mathcal{Z}_{s,t}(x) := \exp\left(-\int_s^t m(x_r)(V_r)\, dr\right) \in [0,1]
\tag{3.1}
$$

To simplify notation, when $s = 0$ sometimes we write $\mathcal{Z}_t$ instead of $\mathcal{Z}_{0,t}$. When $N = 1$, the map $\mathcal{Z}_{s,t}$ reduces to the exponential map $Z_{s,t}$ defined by

$$
z = (z_r)_{0 \leqslant r \leqslant t} \in D_t(S) \mapsto Z_{s,t}(z) := \exp\left(-\int_s^t V_r(z_r)\, dr\right) \in [0,1]
\tag{3.2}
$$

Observe that for any $s \leqslant t$ we have

$$
Z_t := Z_{0,t} \Longrightarrow Z_t(\widehat{X}_t) = Z_s(\widehat{X}_s)\, Z_{s,t}(\widehat{X}_t) \quad \text{with the historical process } \widehat{X}_t = (X_s)_{0 \leqslant s \leqslant t}
$$

To clarify the presentation when there are no possible confusions we write $Z_t(X)$ and $Z_{s,t}(X)$ instead of $Z_t(\widehat{X}_t)$ and $Z_{s,t}(\widehat{X}_t)$.

3.2 Evolution semigroups

Consider the flow of Feynman-Kac measures $(\gamma, \eta) : t \in \mathbb{R}_+ := [0, \infty[\mapsto (\gamma_t, \eta_t) \in (\mathcal{M}_+(S) \times \mathcal{P}(S))$ defined for any $f \in \mathcal{B}(S)$ by the formulae

$$
\eta_t(f) = \gamma_t(f)/\gamma_t(1) \quad \text{with} \quad \gamma_t(f) := \mathbb{E}\left(f(X_t)\, Z_t(X)\right)
\tag{3.3}
$$

Observe that

$$
\log \mathbb{E}\left(Z_t(X)\right) = -\int_0^t \eta_s(V_s)ds
$$

This shows that

$$
\overline{Z}_t(X) := Z_t(X)/\mathbb{E}(Z_t(X)) = \exp\left[-\int_0^t \overline{V}_s(X_s)ds\right] \quad \text{with} \quad \overline{V}_t := V_t - \eta_t(V)
\tag{3.4}
$$

We also consider the Feynman-Kac semigroup

$$Q_{s,t}(f)(x) = \mathbb{E}\left(f(X_t)\, Z_{s,t}(X) \mid X_s = x\right) \quad \text{with} \quad Z_{s,t}(X) := Z_t(X)/Z_s(X) \tag{3.5}$$

When $V = 0$ the semigroup $Q_{s,t}$ resumes to the Markov semigroup $P_{s,t}$ of the reference process X_t.

Definition 3.1. *The mathematical model $(\gamma_t, \eta_t, Q_{s,t})$ defined above and the measure $\mathbb{Q}_t$ defined in (1.1) is called the Feynman-Kac model associated with the reference process and the potential function (X_t, V_t).*

We further assume that the (infinitesimal) generators L_t of X_t are well defined on some common sub-algebra $\mathcal{D}(S) \subset \mathcal{B}(S)$, and for any $s < t$ we have $Q_{s,t}(\mathcal{B}(S)) \subset \mathcal{D}(S)$.

We let $\mathcal{V}_t(f) = V_t f$ the multiplication operator on $\mathcal{B}(S)$. We also let $L_t = L_t^c + L_t^d$ be the decomposition of the generator L_t in terms of a diffusion-type operator L_t^c and a bounded jump-type generator of the following form

$$L_t^d(f)(u) = \lambda_t(u) \int \left(f(v) - f(u)\right)\, J_t(u, dv)$$

for some bounded rate function λ_t and some Markov transition J_t on S.

In this notation, for any $f \in \mathcal{D}(S)$ and $s \leqslant t$ we have

$$\partial_t \gamma_t(f) = \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L_t - \mathcal{V}_t \implies \gamma_t = \gamma_s Q_{s,t} \tag{3.6}$$

The semigroup associated with the normalized Feynman-Kac measures η_t is given for any $s \leqslant t$ by the formula

$$\eta_t = \phi_{s,t}(\eta_s) := \frac{\eta_s Q_{s,t}}{\eta_s Q_{s,t}(1)} \implies \partial_t \eta_t(f) = \Lambda_t(\eta_t)(f) := \eta_t(L_t^V(f)) + \eta_t(V_t)\, \eta_t(f) \tag{3.7}$$

with the collection of functional linear operators

$$\Lambda_t(\eta) \ : \ f \in \mathcal{D}(S) \mapsto \Lambda_t(\eta)(f) := \eta(L_t^V(f)) + \eta(V_t)\, \eta(f) \in \mathbb{R}$$

Finally we recall that $\eta_t = \mathrm{Law}(Y_t)$ can be interpreted as the law of a nonlinear Markov process Y_t associated with the collection of generators $L_{t,\eta}$ defined for any $(\eta, f, x) \in (\mathcal{P}(S), \mathcal{D}(S) \times S)$ by

$$L_{t,\eta}(f)(x) := L_t(f)(x) + V_t(x) \int \left(f(y) - f(x)\right) \eta(dy) \quad \implies \quad \Lambda_t(\eta) = \eta L_{t,\eta} \tag{3.8}$$

3.3 Path space measures

3.3.1 Historical processes

Consider a Feynman-Kac model $(\gamma_t', \eta_t', Q_{s,t}', \mathbb{Q}_t')$ associated with some auxiliary Markov process X_t' on some metric space $(S', d_{S'})$, and some bounded non negative potential functions V_t' on S'. For instance, for any function f on S' and any function F on $D_t(S')$ we have

$$\eta_t'(f) \propto \mathbb{E}\left(f(X_t')\, \exp\left[-\int_0^t V_s'(X_s')ds\right]\right) \quad \text{and} \quad \mathbb{Q}_t'(F) \propto \mathbb{E}\left(F(\widehat{X}_t')\, \exp\left[-\int_0^t \widehat{V}_s'(\widehat{X}_s')ds\right]\right)$$

with the historical process

$$\widehat{X}_t' := \left(X_s'\right)_{s \leqslant t} \quad \text{and the potential function} \quad \widehat{V}_t'\left(\widehat{X}_t'\right) := V_t'(X_t').$$

Assume that the process X_t discussed in (3.3) is given by the historical process

$$X_t := \widehat{X}'_t \in D_t(S') \quad \text{and set} \quad V_t(X_t) := V'_t(X'_t) \tag{3.9}$$

In this situation, we have $\mathbb{Q}'_t = \eta_t$, with the measure η_t defined in (3.3); that is we have

$$\mathbb{Q}'_t(F) \propto \mathbb{E}\left(F(X_t) \, \exp\left[-\int_0^t V_s(X_s) ds \right] \right) \propto \eta_t(F)$$

In the reverse angle, consider a collection of Feynman-Kac measures $(\widehat{\gamma}_t, \widehat{\eta}_t, \widehat{Q}_{s,t})$ defined as $(\gamma_t, \eta_t, Q_{s,t})$ by replacing in (3.3) and (3.5) the pair (X_t, V_t) by the historical process

$$\widehat{X}_t := (X_s)_{s \leqslant t} \in \widehat{S} := \cup_{t \geqslant 0} D_t(S) \quad \text{and the potential function} \quad \widehat{V}_t(\widehat{X}_t) := V_t(X_t) \tag{3.10}$$

For instance, replacing (X_t, V_t) by $(\widehat{X}_t, \widehat{V}_t)$ in (3.3) and (3.5) for any F on $D_t(S)$ we have the formula

$$\widehat{\eta}_t(F) \propto \mathbb{E}\left(F(\widehat{X}_t) \, \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s) ds \right] \right)$$

This shows that $\widehat{\eta}_t = \mathbb{Q}_t$, with the measure $\mathbb{Q}_t$ introduced in (1.1). We underline that $\mathbb{Q}_t$ differs from the law of the the historical process $\widehat{Y}_t := (Y_s)_{s \leqslant t}$ of the nonlinear process Y_t discussed in the end of section 3.2.

To avoid unnecessary technical discussions we distinguish these models with the following terminology.

Definition 3.2. *The Feynman-Kac model associated with the historical process $\widehat{X}_t$ and the potential function $\widehat{V}_t$ is called the path-space version of the Feynman-Kac model associated with the process X_t and the potential function V_t. In the reverse angle the Feynman-Kac model associated with the pair (X_t, V_t) is called the marginal model of the Feynman-Kac model associated with the pair $(\widehat{X}_t, \widehat{V}_t)$.*

Let L_t be the generator of X_t defined on some common sub-algebra $\mathcal{D}(L) \subset \mathcal{B}(S)$. In this situation, the generators $\widehat{L}_t$ of the historical process $\widehat{X}_t$ can be defined on some common domain $\mathcal{D}(\widehat{L}) \subset \mathcal{B}(\widehat{S})$ in two different ways:

The more conventional approach is to consider cylindrical functions

$$f(\widehat{X}_t) := \varphi(X_{s_1}, \ldots, X_{s_n}, X_t) \tag{3.11}$$

that only depend on a given finite collection of time horizons $s_i \leqslant s_{i+1} < t$, with $1 \leqslant i < n$, and some bounded functions φ from S^{n+1} into $\mathbb{R}$. The regularity of the "test" function φ depends on the process at hand. For jump-type processes, no additional regularity is required. For diffusion-type processes the function is often required to be compactly supported and twice differentiable.

Another elegant and more powerful approach is to use the functional Itô calculus theory developed by B. Dupire in an unpublished article [48] dating from 2009, and further developed in [21, 52]. This path-dependent stochastic calculus allows to consider more general functions such as running integrals or running maximum of the process X_t. It also allows to consider diffusion-type processes with a drift and a diffusion term that depends on the history of the process.

These path space models and their applications in the context of genealogical tree based particle models and historical processes were also presented in the early 2000s in previous work of one of the authors with Miclo [37, 38, 39].

The path space $\widehat{S}$ is equipped with a time-space metric $d_{\widehat{S}}$ so that $(\widehat{S}, d_{\widehat{S}})$ is a complete and separable metric space (cf. for instance proposition 1.1.13 and theorem 1.1.15 in [77]). The smoothness properties of continuous function f on S are defined in terms of time and space functional

derivatives. Thus, for diffusion-type historical processes $\widehat{X}_t$, the generator $\widehat{L}_t$ is defined on functions $F \in \mathcal{D}(\widehat{L})$ which a differentiable w.r.t. the time parameter and, as before twice differentiable with compactly supported derivatives (cf. for instance theorem 1.3.1 in [77], as well as the pathwise change of variable formula stated in theorem 5.3.6 in [6]). For instance, for the cylindrical functions discussed in (3.11) we have

$$\widehat{L}_t(f)(\widehat{X}_t) = L_t\left(\varphi(X_{s_1}, \ldots, X_{s_n}, .)\right)(X_t)$$

as soon as the section $y \mapsto \varphi(x_1, \ldots, x_n, y)$ belongs to $\mathcal{D}(L)$. In the same vein, for path-integral functionals, we have

$$f(\widehat{X}_t) := g(X_t) + \int_0^t h(X_s) \, ds \quad \Longrightarrow \quad \widehat{L}_t(f)(\widehat{X}_t) = L_t(g)(X_t) + h(X_t)$$

as soon as $g \in \mathcal{D}(L)$ and $h \in \mathcal{B}(S)$. We also have the jump formula

$$\Delta f(\widehat{X}_t) := f(\widehat{X}_t) - f(\widehat{X}_{t-}) = f(\widehat{X}_{t-} \vee X_t) - f(\widehat{X}_{t-})$$

It is clearly not the scope of the article to describe in full details the functional Itô calculus on path space. We rather refer the reader to the article [52] and the Ph.D thesis of Saporito [77]. The discrete time version of these historical processes calculus in the context of path-space interacting particle systems can be found in [38], see also [24, 25] and references therein.

In the further development of the article we shall use these ideas back and forth. We already mention that the mean field particle interpretation of the Feynman-Kac measures associated with an historical process $\widehat{X}_t$ coincides with the genealogical tree-based particle evolution of the marginal model.

Let $\mathcal{D}(\widehat{\mathcal{L}})$ be the set of functions F on $\widehat{S}^N := \cup_{t \geqslant 0} D_t(S)^N$ s.t. for any $x \in D_t(S)^N$ we have

$$F_{x^{-i}} := F \circ \varphi_{x^{-i}} \in \mathcal{D}(\widehat{L})$$

The generator $\widehat{\mathcal{L}}_t$ of the historical stochastic flow $\widehat{\mathcal{X}}_{s,t}(x)$ is given for any $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and $x \in D_t(S)^N$ by the formula

$$\widehat{\mathcal{L}}_t(F)(x) = \sum_{i \in [N]} \widehat{L}_t(F_{x^{-i}})(x^i)$$

We choose a generator $\widehat{\mathcal{L}}_t^{(2)}$ on some domain $\mathcal{D}(\widehat{\mathcal{L}}^{(2)})$ for a coupled stochastic flow $\widehat{\mathcal{X}}_{s,t}^{(2)}$ of the following form

$$\widehat{\mathcal{X}}_{s,t}^{(2)}(x,y) := (\widehat{\mathcal{X}}_{s,t}(x), \widehat{\mathcal{X}}_{s,t}(y)) \quad \text{with } (x,y) \in D_s(S)^{(2N)} := \left(D_s(S)^N \times D_s(S)^N\right).$$

For instance, we can define $\widehat{\mathcal{X}}_{s,t}^{(2)}(x,y) := (\widehat{\mathcal{X}}_{s,t}(x), \widehat{\mathcal{X}}_{s,t}(y))$ firstly when $x_s = y_s$ by

$$\widehat{\mathcal{X}}_{s,t}(x) = x \vee \widetilde{\mathcal{X}}_{s,t}(x) \qquad \widehat{\mathcal{X}}_{s,t}(y) = y \vee \widetilde{\mathcal{X}}_{s,t}(y) \quad \text{with} \quad \widetilde{\mathcal{X}}_{s,t}(x) = \widetilde{\mathcal{X}}_{s,t}(y)$$

When $x_s \neq y_s$, the processes $\widehat{\mathcal{X}}_{s,t}(x)$ and $\widehat{\mathcal{X}}_{s,t}(y)$ are chosen independent.

3.3.2 McKean measures

Definition 3.3. *The McKean measure $\mathbb{M}_t$ is the distribution on $D_t(S)$ of the historical process $\widehat{Y}_t := (Y_s)_{s \leqslant t}$ of the nonlinear process Y_t discussed in (3.8).*

Observe that the t-marginal of $\mathbb{M}_t$ coincides with η_t, that is we have

$$F(\widehat{Y}_t) = f(Y_t) \Longrightarrow \int F(y) \; \mathbb{M}_t(dy) = \mathbb{E}(F(\widehat{Y}_t)) = \mathbb{E}(f(Y_t)) = \int f(z) \; \eta_t(dz)$$

The generator $\widehat{L}_{t,\eta_t}$ of $\widehat{Y}_t$ only depends on the marginal probability η_t on S and it is given for any function $F \in \mathcal{D}(\widehat{L})$ and any $x = (x_s)_{s \leqslant t} \in D_t(S)$ by the formula

$$\widehat{L}_{t,\eta_t}(F)(x) = \widehat{L}_t(F)(x) + \widehat{V}_t(x) \int (F(x \vee z) - F(x)) \; \eta_t(dz)$$

where $\widehat{L}_t$ stands for the generator of the historical process $\widehat{X}_t := (X_s)_{s \leqslant t}$ and $\widehat{V}_t$ is the potential function on path space given in (3.10).

Following (3.10), the Feynman-Kac measures $\mathbb{Q}_t = \widehat{\eta}_t$ associated with the historical process and the potential function $(\widehat{X}_t, \widehat{V}_t)$ on path space $\widehat{S}$ satisfy for any $F \in \mathcal{D}(\widehat{L})$ the evolution equation

$$\partial_t \mathbb{Q}_t(F) = \mathbb{Q}_t(\mathbb{L}_{t,\mathbb{Q}_t}(F))$$

with the generator $\mathbb{L}_{t,\mathbb{Q}_t}$ defined for any $F \in \mathcal{D}(\widehat{L})$ and $x \in D_t(S)$ by the formula

$$\mathbb{L}_{t,\mathbb{Q}_t}(F)(x) = \widehat{L}_t(F)(x) + \widehat{V}_t(x) \int (F(y) - F(x)) \; \mathbb{Q}_t(dy) \tag{3.12}$$

Arguing as above, $\mathbb{Q}_t = \mathrm{Law}(\overline{Y}_t)$ can be interpreted as the law of a nonlinear Markov process $\overline{Y}_t \in D_t(S)$ associated with the collection of generators $\mathbb{L}_{t,\mathbb{Q}_t}$. Here again, besides the fact that $\overline{Y}_t$ is a random path, we underline that $\overline{Y}_t$ is not the historical path of an auxiliary Markov process.

The historical process $\widehat{Y}_t$ is a jump type process taking values in the path space $\widehat{S} := \cup_{t \geqslant 0} D_t(S)$. Let $(T_n)_{n \geqslant 0}$ be a collection of jump times occurring at a rate $\widehat{V}_t(\widehat{Y}_t) := V_t(Y_t)$ that only depends on the terminal value of the historical process. We use the convention $T_0 = 0$ when $n = 0$. Also let $(\widehat{Y}_t^n)_{n \geqslant 0}$ be a collection of $D_t(S)$-valued independent random variables with common law $\mathbb{Q}_t$. The terminal time variables $Y_t^n := \widehat{Y}_t^n(t)$ are independent random variables with common law η_t.

Between the jump times T_n, the process $\widehat{Y}_t$ evolves as the historical process $\widehat{X}_t$ of the reference process X_t. At each jump time T_n the predictable path $\widehat{Y}_{T_n-}$ jumps onto the càdlàg path $\widehat{Y}_{T_n}$ given by the concatenate path

$$\widehat{Y}_{T_n} := \widehat{Y}_{T_n-} \vee Y_{T_n}^n$$

In the above display, $Y_{T_n}^n$ stands for a random sample from η_{T_n} independent of $\widehat{Y}_{T_n-}$. Thus, for any $T_n \leqslant t < T_{n+1}$ we have

$$\widehat{Y}_t = \widehat{X}_{T_n,t}(\widehat{Y}_{T_n}) = \widehat{Y}_{T_n} \vee \widetilde{X}_{T_n,t}(\widehat{Y}_{T_n}^n) \quad \text{and} \quad \widehat{Y}_{T_n} = \widehat{Y}_{T_{n-1}} \vee \widetilde{X}_{T_{n-1},T_n}(\widehat{Y}_{T_{n-1}}^{n-1})$$

We summarize the above discussion with the following proposition.

Proposition 3.4. *For any time horizon $t \geqslant 0$ we have*

$$\widehat{Y}_t = \sum_{n \geqslant 0} \left(\widetilde{X}_{T_0,T_1}(\widehat{Y}_{T_0}^0) \vee \widetilde{X}_{T_1,T_2}(\widehat{Y}_{T_1}^1) \vee \ldots \vee \widetilde{X}_{T_n,t}(\widehat{Y}_{T_n}^n) \right) \; 1_{T_n \leqslant t < T_{n+1}}$$

The McKean measure $\mathbb{M}_t$ is given for any measurable function F on $D_t(S)$ by

$$\mathbb{M}_t(F) = \sum_{n \geqslant 0} \int_{[0,t]_n} \mathbb{E} \left\{ \left[\prod_{0 \leqslant p < n} Z_{r_p,r_{p+1}} \left(\widehat{X}_{r_p,r_{p+1}}(\widehat{Y}_{r_p}^p) \right) \; \widehat{V}_{r_{p+1}} \left(\widehat{X}_{r_p,r_{p+1}}(\widehat{Y}_{r_p}^p) \right) \right] \right.$$

$$\left. \times Z_{r_n,t} \left(\widehat{X}_{r_n,t}(\widehat{Y}_{r_n}^n) \right) \; F \left(\widetilde{X}_{r_0,r_1}(\widehat{Y}_{r_0}^0) \vee \ldots \vee \widetilde{X}_{r_n,t}(\widehat{Y}_{r_n}^n) \right) \right\} \; dr_1 \ldots dr_n$$

When the function $F(\widehat{X}_t) = f(X_t)$ only depends on the terminal time, we recover the fact that

$$\mathbb{M}_t(F) = \gamma_t(f) \; e^{\int_0^t \eta_s(V_s) \; ds} = \mathbb{E}\left(f(X_t) \; e^{-\int_0^t (V_s(X_s) - \eta_s(V_s))ds}\right) = \mathbb{E}(f(Y_t)) = \eta_t(f) \tag{3.13}$$

The detailed proof of the above assertion is provided in the appendix, on page 45.

3.4 Some regularity conditions

This section discusses in some details the two main regularity conditions used in the further development of the article.

Firstly, observe that the semigroup $P_{s,t}$ associated with the historical process $X_t = (X'_s)_{s \leqslant t}$ discussed in (3.9) never satisfies the regularity condition (H_0) stated in (1.7). Nevertheless it may happen that the semigroup $P'_{s,t}$ associated with X'_t satisfies condition (H_0). In this situation, to avoid repetition or unnecessary long discussions we simply say that (H'_0) is met.

We also use the following weaker conditions:

$$(H_1) \qquad \exists \alpha < \infty \quad \exists \beta > 0$$
$$\text{s.t.} \quad \forall s \leqslant t \quad \forall f \in \mathcal{B}(S) \quad \operatorname{osc}(\overline{Q}_{s,t}(f)) \leqslant \alpha \; e^{-\beta(t-s)} \; \operatorname{osc}(f) \tag{3.14}$$

$$(H_2) \qquad \exists q < \infty \quad \text{s.t.} \quad \forall s \leqslant t \quad \forall x, y \in S \qquad \log\left(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)\right) \leqslant q \tag{3.15}$$

As before when the semigroup $\overline{Q}'_{s,t}$ and $Q'_{s,t}$ of associated with a marginal Feynman-Kac model satisfy condition (H_i), to avoid repetition or unnecessary long discussions we simply say that (H'_i) is met. We mention that

$$(H_0) \Longrightarrow (H_1) \Longrightarrow (H_2)$$

The proof of the l.h.s. implication can be found in [40] (see for instance remark 2.2 and the contraction inequalities stated in proposition 2.3). In this context, the parameters (α, β) do not depends on the measure $\mu_{t,h}$ discussed in (1.7). To check the second implication observe that

$$\log\left(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)\right) = \int_s^t \left[\phi_{s,u}(\delta_y)(V_u) - \phi_{s,u}(\delta_x)(V_u)\right] \; du \tag{3.16}$$

This implies that

$$(H_1) \Longrightarrow (H_2) \quad \text{with} \quad q = \alpha\beta^{-1} \operatorname{osc}(V) \quad \text{with} \quad \operatorname{osc}(V) := \sup_{t \geqslant 0} \operatorname{osc}(V_t)$$

Using (2.7) we also have

$$(H_2) \quad \Longrightarrow \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant e^q \operatorname{osc}(f) \quad (\text{since} \quad \operatorname{osc}(\overline{Q}_{s,t}(f)) \leqslant \operatorname{osc}(f))$$
$$(H_1) \quad \Longrightarrow \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant r \; e^{-\beta(t-s)} \operatorname{osc}(f) \quad \text{with} \quad r = \alpha \; e^q \quad \text{and} \quad q = \alpha\beta^{-1} \operatorname{osc}(V) \tag{3.17}$$

We return to the historical process $X_t = (X'_s)_{s \leqslant t}$ discussed in (3.9). In this case, for any $x_s = (x'(u))_{u \leqslant s} \in D_s(S')$ we have
$$Q_{s,t}(f)(x_s) = Q'_{s,t}(f')(x'_s)$$
in the above display, f and f' stand for some bounded measurable functions on the path space $D_t(S')$ and on S' such that

$$\forall y_t = (y'_u)_{u \leqslant t} \in D_t(S') \qquad f(y_t) = f'(y'_t)$$

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This implies that

$$(H_1') \Longrightarrow (H_2) \quad \text{is met with} \quad q = \alpha\beta^{-1} \operatorname{osc}(V) \quad \text{and} \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant e^q \operatorname{osc}(f) \tag{3.18}$$

In the reverse angle, the Feynman-Kac semigroups on path space $\widehat{Q}_{s,t}$ discussed in (3.10) is defined for any $s \leqslant t$, and any $x \in D_s(S)$ and $f \in \mathcal{B}(D_t(S))$ by the formula

$$\widehat{Q}_{s,t}(f)(x) := \mathbb{E}\left(f(\widehat{X}_t) \, Z_{s,t}\left(\widehat{X}_t\right) \mid \widehat{X}_s = x\right) \tag{3.19}$$

with the exponential map $Z_{s,t}$ defined in (3.2). The above semigroup never satisfies the regularity condition (H_1) stated in (1.7). Nevertheless arguing as in (3.18) permuting $(Q'_{s,t}, Q_{s,t})$ and $(Q_{s,t}, \widehat{Q}_{s,t})$ we have

$$(H_1) \Longrightarrow (\widehat{H}_2) \tag{3.20}$$

In the above display, $(\widehat{H}_2)$ stands for the condition defined as (H_2) by replacing $Q_{s,t}$ by $\widehat{Q}_{s,t}$ and the state S by the path space $D_s(S)$.

3.5 Forward and backward equations

Proposition 3.5. *For any $s \leqslant t$ and $\eta \in \mathcal{P}(S)$ we have the Gelfand-Pettis forward and backward differential equations*

$$\partial_t \phi_{s,t}(\eta_s) = \Lambda_t(\phi_{s,t}(\eta_s)) \quad \text{and} \quad \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta)\partial_\eta \phi_{s,t} \tag{3.21}$$

In addition, for any mapping ϕ of the form (2.4) we also have

$$\partial_t \phi\left(\phi_{s,t}(\eta)\right) = \Lambda_t(\eta)\partial_{\phi_{s,t}(\eta)}\phi \quad \text{and} \quad \partial_s \phi\left(\phi_{s,t}(\eta)\right) = -\Lambda_s(\eta)\partial_{\phi_{s,t}(\eta)}\phi \tag{3.22}$$

Proof. The l.h.s. assertion in (3.21) is a direct consequence of (3.7).

The r.h.s. differential formula in (3.21) as (3.22) can be checked using brute force and lengthy calculations from the backward evolution equations associated with Feynman-Kac semigroups.

A more judicious and more direct approach is to apply a second oder perturbation approach. For instance, applying the second order Taylor with remainder expansion (2.9) to the mapping $\phi_{s,t}$ given in (3.7), for any $s + h \leqslant t$ we find that

$$\begin{aligned}
\phi_{s,t}(\eta) &= \phi_{s+h,t}\left(\eta + [\phi_{s,s+h}(\eta) - \eta]\right) \\
&= \phi_{s+h,t}(\eta) + [\phi_{s,s+h}(\eta) - \eta]\partial_\eta \phi_{s+h,t} + \frac{1}{\phi_{s,s+h}(\eta)Q^\eta(1)} \frac{1}{2} [\phi_{s,s+h}(\eta) - \eta]^{\otimes 2}\partial_\eta^2 \phi_{s+h,t}
\end{aligned}$$

On the other hand we have

$$\phi_{s,s+h}(\eta) = \eta + \Lambda_s(\eta) \, h + \mathrm{O}(h^2) \quad \text{and} \quad \phi_{s,s+h}(\eta)Q^\eta(1) = 1 + \mathrm{O}(h)$$

This yields

$$\phi_{s,t}(\eta) = \phi_{s+h,t}(\eta) + \Lambda_s(\eta)\partial_\eta \phi_{s+h,t} \ h + \mathrm{O}(h^2)$$

from which we check the backward evolution formula

$$h^{-1}\left[\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right] \longrightarrow_{h \to 0} \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta)\partial_\eta \phi_{s,t}$$

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For any mapping ϕ of the form (2.4), applying the third order Taylor with remainder expansion (2.9) we also have

$$\phi\left(\phi_{s+h,t}(\eta)\right) - \phi\left(\phi_{s,t}(\eta)\right) = (\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta))\partial_{\phi_{s,t}(\eta)}\phi$$

$$+ \frac{1}{2}\left(\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right)^{\otimes 2}\partial^2_{\phi_{s,t}(\eta)}\phi + \frac{1}{\phi_{s+h,t}(\eta)\ Q^{\phi_{s,t}(\eta)}(1)}\ \frac{1}{3}\left(\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right)^{\otimes 3}\partial^3_{\phi_{s,t}(\eta)}\phi$$

Arguing as above we check (3.22). This ends the proof of the proposition. ∎

4 Interacting genetic type particle systems

4.1 Mean field particle processes

Definition 4.1. *The N-mean field particle interpretation of the nonlinear process discussed in (3.8) is defined by the Markov process $\xi_t = \left(\xi^i_t\right)_{1\leqslant i\leqslant N} \in S^N$ with generator $\mathcal{G}_t$ given for any $F \in \mathcal{D}(\mathcal{L})$ and any $x = (x^i)_{1\leqslant i\leqslant N} \in S^N$ by*

$$\mathcal{G}_t(F)(x) := \sum_{i\in[N]} L_{t,m(x)}(F_{x^{-i}})(x^i) \tag{4.1}$$

with the collection of generators $L_{t,\mu}$ indexed by probability measures μ on S defined in (3.8).

Let $\mathcal{F} := (\mathcal{F}_t)_{t\geqslant 0}$, with $\mathcal{F}_t = \sigma(\xi_s\ :\ s\leqslant t)$ be the filtration generated by the mean field particle model defined in (4.1). Also let $\mathcal{D}([0,T], S^N)$ be the set of functions

$$F\ :\ (t,x) \in ([0,T]\times S^N) \mapsto F_t(x) \in \mathbb{R}$$

with a bounded derivative w.r.t. the first argument and s.t. $F_t \in \mathcal{D}(\mathcal{L})$, for any $t \in [0,T]$. For any $F \in \mathcal{D}([0,T], S^N)$, and any $T \geqslant 0$, applying the Itô integration by part formula (see for instance section 15.5 in [42]) we have

$$dF_t(\xi_t) = [\partial_t F_t + \mathcal{G}_t(F_t)]\,(\xi_t)\ dt + d\mathcal{M}_t(F) \tag{4.2}$$

In the above display $\mathcal{M}_t$ stands for a martingale random field on $\mathcal{D}([0,T], S^N)$ with angle bracket defined for any functions $F, G \in \mathcal{D}([0,T], S^N)$ and any time horizon $t \in [0,T]$ by the formula

$$\partial_t\langle\mathcal{M}(F), \mathcal{M}(G)\rangle_t = \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t) \tag{4.3}$$

Choosing functions of the form

$$F_t(x) = m(x)(f_t) \quad\text{and}\quad G_t(x) = m(x)(g_t) \Longrightarrow \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t) = m(\xi_t)\Gamma_{L_{t,m(\xi_t)}}(f_t, g_t) \tag{4.4}$$

for some $f, g \in \mathcal{D}([0,T], S)$, we also check that the occupation measure $m(\xi_t) \in \mathcal{P}(S)$ satisfies the stochastic equation

$$dm(\xi_t)(f_t) = [m(\xi_t)(\partial_t f_t) + \Lambda_t(m(\xi_t))(f_t)]\ dt + \frac{1}{\sqrt{N}}\ dM_t(f) \tag{4.5}$$

with a martingale random field M_t on $\mathcal{D}([0,T],S)$ with angle brackets by the formula

$$\partial_t \langle M(f), M(g) \rangle_t$$

$$= m(\xi_t)(\Gamma_{L_t}(f_t, g_t)) + \int m(\xi_t)(du) \, m(\xi_t)(dv) \, V_t(u) \, (f_t(v) - f_t(u))(g_t(v) - g_t(u))$$

With a slight abuse of notation we also write M_t the extension of the random field M_t to $\mathcal{F}$-predictable functions $\mathcal{D}([0,T],S)$.

In the further development of the article we write $(M_t^c, \mathcal{M}_t^c)$ and $(M_t^d, \mathcal{M}_t^d)$ the continuous and the discontinuous part of the martingales $(M_t, \mathcal{M}_t)$; as well as

$$L_{t,\eta} = L_{t,\eta}^c + L_{t,\eta}^d \quad \text{with} \quad L_{t,\eta}^c := L_t^c$$

The angle bracket of $\mathcal{M}_t^d$ is given for any functions $F, G \in \mathcal{D}([0,T],S^N)$ and any time horizon $t \in [0,T]$ by the formula

$$\partial_t \langle \mathcal{M}^d(F), \mathcal{M}^d(G) \rangle_t$$

$$= \sum_{i \in [N]} \int \left[F_{t,\xi_t^{-i}}(v) - F_t(\xi_t) \right] \left[G_{t,\xi_t^{-i}}(v) - G_t(\xi_t) \right] \left[V_t(\xi_t^i) \, m(\xi_t)(dv) + \lambda_t(\xi_t^i) \, J_t(\xi_t^i, dv) \right]$$

4.2 Historical and genealogical tree evolutions

The generator $\mathcal{G}_t$ of the N-mean field particle process $\xi_t = (\xi_t^i)_{1 \leqslant i \leqslant N} \in S^N$ discussed in (4.1) can be rewritten for any $F \in \mathcal{D}(\mathcal{L})$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in S^N$ in terms of the interacting jump operator

$$\mathcal{G}_t(F)(x) = \sum_{i \in [N]} \left[L_t(F_{x^{-i}})(x^i) + V_t(x^i) \int (F_{x^{-i}}(y) - F(x)) \, m(x)(dy) \right] \tag{4.6}$$

The process ξ_t is called the N-mean field particle interpretation of the Feynman-Kac measures η_t defined in (3.3).

The process $\xi_t = (\xi_t^i)_{i \in [N]} \in S^N$ can be interpreted as a genetic type particle system with mutation associated with the generator L_t and selection dictated by the potential function V_t. Between the jumps the particles ξ_t^i evolve independently as independent copies of a process with generator L_t. At rate $V_t(\xi_t^i)$ the particle is killed and instantly a particle randomly selected in the pool duplicates.

The historical process $\widehat{\xi}_t = (\xi_s)_{s \leqslant t} \in \widehat{S}^N$ coincides with the N-mean field interpretation of the historical process $\widehat{Y}_t = (Y_s)_{s \leqslant t} \in \widehat{S}$ of the nonlinear process Y_t defined in (3.8), with the path space $\widehat{S}$ defined in (3.10).

The generator $\widehat{\mathcal{G}}_t$ of the process $\widehat{\xi}_t$ given for any function $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and any $x = (x_s)_{s \leqslant t} \in D_t(S)^N$ by the operator

$$\widehat{\mathcal{G}}_t(F)(x) = \sum_{i \in [N]} \left[\widehat{L}_t(F_{x^{-i}})(x^i) + \widehat{V}_t(x^i) \int (F_{x^{-i}}(x^i \vee z) - F(x)) \, m(x_t)(dz) \right] \tag{4.7}$$

The genealogical tree evolution associated with the genetic type particle system $\xi_t \in S^N$ coincides with the N-mean field interpretation $\mathbb{X}_t = (\mathbb{X}_t^i)_{i \in [N]} \in D_t(S)^N$ of the $D_t(S)$-valued nonlinear process

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$\overline{Y}_t$ defined in (3.12). The generator $\mathbb{G}_t$ of $\mathbb{X}_t$ is given for any function $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and $x = (x_s)_{s \leqslant t} \in D_t(S)^N$ by the operator

$$\mathbb{G}_t(F)(x) := \sum_{i \in [N]} \mathbb{L}_{t,m(x)}(F_{x^{-i}})(x^i)$$

$$= \sum_{i \in [N]} \left[\widehat{L}_t(F_{x^{-i}})(x^i) + \widehat{V}_t(x^i) \int (F_{x^{-i}}(y) - F(x)) \, m(x)(dy) \right] \qquad (4.8)$$

with the collection of generators $\mathbb{L}_{t,\mu}$ indexed by probability measures μ on $D_t(S)$ defined in (3.12).

Observe that $\mathbb{G}_t$ can be deduced from the operator $\mathcal{G}_t$ by replacing in (4.1) the generator and the potential function (L_t, V_t) and the state S by $(\widehat{L}_t, \widehat{V}_t)$ and by the path space $\widehat{S}$.

The process $\mathbb{X}_t$ is called the N-mean field particle interpretation of the Feynman-Kac measures on path space $\widehat{\eta}_t = \mathbb{Q}_t$ defined in (3.12).

In contrast with $\mathbb{X}_t$, the historical process $\widehat{\xi}_t$ keeps track of all past-ancestral lines with no descendants. Thus $\widehat{\xi}_t$ can be interpreted as the complete ancestral tree associated with the genetic type particle system discussed above.

Rewritten in a slightly different form the jump generator in (4.7) is given by the formula

$$\widehat{V}_t(x^i) \int (F_{x^{-i}}(x^i \vee z) - F(x)) \, m(x_t)(dz)$$

$$= \widehat{V}_t^-(x^i) \frac{1}{N-1} \sum_{k \in [N]-\{i\}} \left(F(x \vee x_t^{c_{i,k}}) - F(x) \right) \quad \text{with} \quad \widehat{V}_t^- := \left(1 - \frac{1}{N} \right) \widehat{V}_t$$

This shows that for any $i \in [N]$ and at rate $\widehat{V}_t(\widehat{\xi}_t^i)$ the i-th particle ξ_t^i jumps onto a randomly chosen particle ξ_t^k, with $k \in [N] - \{i\}$. Also observe that the jump rate of the system $\widehat{\xi}_t$ is given by the stochastic intensity

$$\widehat{\lambda}_t(\widehat{\xi}_t) = N \, m(\widehat{\xi}_t)(\widehat{V}_t^-) = (N-1) \, m(\widehat{\xi}_t)(\widehat{V}_t) \qquad (4.9)$$

A graphical description of the historical process $\widehat{\xi}_t$ with three jump times (T_1, T_2, T_3) and the jump coalescent maps $(c_{k,j_1}, c_{j_2,j_1}, c_{k,j_2})$ is given below in figure 8.
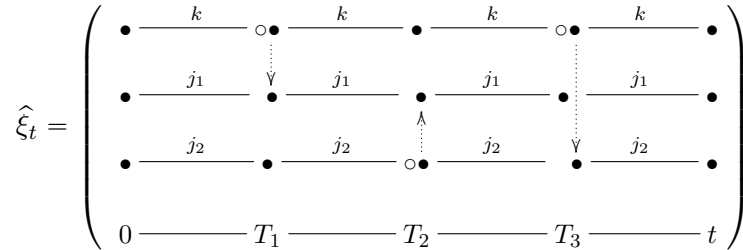

Figure 8: Graphical description of the historical process $\widehat{\xi}_t$

The plain lines ($\overset{j}{\text{———}}$) represent the path associated with N independent copies $X_{u,v}^j$ of the stochastic semigroup $X_{u,v}$ of the process X_t with generator L_t. The dotted vertical arrows represent the jumps from the i_n-th coordinate onto the j_n-th one.

The genealogical tree $\mathbb{X}_t$ associated with the historical process $\widehat{\xi}_t$ in figure 8 is given below in figure 9.
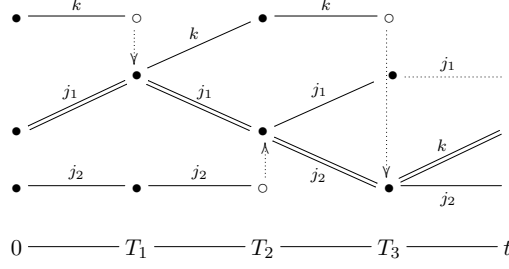
Figure 9: Description of the genealogical tree associated with the particle system ξ_t

In the above display the k-th ancestral line $\mathbb{X}_t^k$ is represented by the doubled lines ($=\!=\!=$). A graphical description of the genealogical tree $\mathbb{X}_t$ associated with the particle evolution presented in figure 9 is given below, in figure 10.
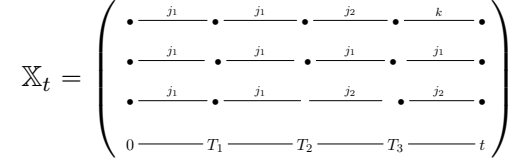

Figure 10: Graphical description of the genealogical tree $\mathbb{X}_t$

4.3 Particle models with a frozen ancestral line

Definition 4.2. *Let $\mathcal{I}$ be an uniform random sample on $[N]$. Given $\mathcal{I}$, let $\widehat{\chi}_t \in D_t(S)^N$ be the Markov process with initial condition $\widehat{\chi}_0 = \xi_0$ and generator $\widehat{\mathcal{G}}_{\mathcal{I},t}$ defined for any function $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and $x = (x_s)_{s \leqslant t} \in D_t(S)^N$, by the formula*

$$\widehat{\mathcal{G}}_{\mathcal{I},t}(F)(x) = \sum_{i \in [N]-\{\mathcal{I}\}} \left[\widehat{L}_t(F_{x^{-i}})(x^i) + \widehat{V}_t(x^i) \int (F_{x^{-i}}(x^i \vee z) - F(x)) \, m(x_t)(dz) \right]$$

$$+ \widehat{L}_t(F_{x^{-\mathcal{I}}})(x^{\mathcal{I}}) + \frac{1}{N} \sum_{i \in [N]-\{\mathcal{I}\}} \widehat{V}_t(x^i) \, (F(x^{\sigma_{i,\mathcal{I}}} \vee x_t^{c_{i,\mathcal{I}}}) - F(x))$$

(4.10)

with the concatenate operation $\vee$ defined in (2.14).

Observe that $\widehat{\chi}_t := (\widehat{\chi}_t(s))_{s \leqslant t} \in D_t(S)^N$ is not necessarily the historical path of an auxiliary Markov process.

To better connect the above generator with the operator $\widehat{\mathcal{G}}_t$ defined in (4.7) observe that

$$\widehat{\mathcal{G}}_t(F)(x) - \widehat{\mathcal{G}}_{k,t}(F)(x)$$

$$= \frac{1}{N} \, \widehat{V}_t(x^k) \sum_{i \in [N]-\{k\}} (F(x \vee x_t^{c_{k,i}}) - F(x)) - \frac{1}{N} \sum_{i \in [N]-\{k\}} \widehat{V}_t(x^i) \, (F(x^{\sigma_{i,k}} \vee x_t^{c_{i,k}}) - F(x))$$

This yields the following lemma.

Lemma 4.3. *The generators $\widehat{\mathcal{G}}_t$ and $\widehat{\mathcal{G}}_{k,t}$ defined in (4.7) and (4.10) are connected for any function $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and $x = (x_s)_{s \leqslant t} \in D_t(S)^N$ by the formula*

$$\widehat{\mathcal{G}}_t(F)(x) - m(x)(\widehat{V}_t) \ F(x) = \widehat{\mathcal{G}}_{k,t}(F)(x) - \widehat{V}_t(x^k) \ F(x) + \frac{1}{N} \sum_{i \in [N] - \{k\}} \epsilon_{k,i}(F)(x)$$

with the bounded integral operator $\epsilon_{i,k}$ defined by the anti-symmetric functions

$$\epsilon_{k,i}(F)(x) := \widehat{V}_t(x^k) \ F(x \vee x_t^{c_{k,i}}) - \widehat{V}_t(x^i) \ F(x^{\sigma_{i,k}} \vee x_t^{c_{i,k}}) = -\epsilon_{k,i}(F)(x^{\sigma_{i,k}})$$

Nevertheless, given $\mathcal{I} = k$, the end-state process ζ_t defined by

$$\zeta_t := \widehat{\mathcal{X}}_t(t) \in S^N$$

is a Markov process. Given $\mathcal{I} = k$, its generator $\mathcal{G}_{k,t}$ is given for any $F \in \mathcal{D}(\mathcal{L})$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in S^N$ by the formula

$$\mathcal{G}_{k,t}(F)(x) = \sum_{i \in [N] - \{k\}} \left[L_t(F_{x^{-i}})(x^i) + V_t(x^i) \int (F_{x^{-i}}(z) - F(x)) \ m(x)(dz) \right]$$

$$+ L_t(F_{x^{-k}})(x^k) + \frac{1}{N} \sum_{i \in [N] - \{k\}} V_t(x^i) \ (F_{x^{-i}}(x^k) - F(x)) \tag{4.11}$$

Given $\mathcal{I}$, the genealogical tree $\mathbb{Y}_t$ associated with the process $\widehat{\mathcal{X}}_t$ is the $\in D_t(S)^N$-valued process starting at $\mathbb{Y}_0 = \xi_0$ with generator $\mathbb{G}_{\mathcal{I},t}$ given for any function $F \in \mathcal{D}(\widehat{\mathcal{L}})$ and $x = (x_s)_{s \leqslant t} \in D_t(S)^N$ by the formula

$$\mathbb{G}_{\mathcal{I},t}(F)(x) = \sum_{i \in [N] - \{\mathcal{I}\}} \left[\widehat{L}_t(F_{x^{-i}})(x^i) + \widehat{V}_t(x^i) \int (F_{x^{-i}}(z) - F(x)) \ m(x)(dz) \right]$$

$$+ \widehat{L}_t(F_{x^{-\mathcal{I}}})(x^{\mathcal{I}}) + \frac{1}{N} \sum_{i \in [N] - \{\mathcal{I}\}} \widehat{V}_t(x^i) \ (F_{x^{-i}}(x^{\mathcal{I}}) - F(x)) \tag{4.12}$$

As in (4.8), $\mathbb{G}_{k,t}$ can be deduced from the operator $\mathcal{G}_{k,t}$ by replacing in (4.11) the generator and the potential function (L_t, V_t) and the state S by $(\widehat{L}_t, \widehat{V}_t)$ and by the path space $\widehat{S}$.

Observe that the historical process $\widehat{\zeta}_t^{\mathcal{I}} = (\zeta_s^k)_{s \leqslant t}$ of the $\mathcal{I}$-th particle in $\zeta_t := (\zeta_t^i)_{i \in [N]}$ evolves as the historical process $\widehat{X}_t$ and doesn't depends on the remaining $(N-1)$ interacting particles.

This yields the following proposition.

Proposition 4.4. *The $\mathcal{I}$-th ancestral line $\mathbb{Y}_t^{\mathcal{I}}$ of the genealogical tree $\mathbb{Y}_t := (\mathbb{Y}_t^i)_{i \in [N]} \in D_t(S)^N$ coincides with the historical process $\widehat{\zeta}_t^{\mathcal{I}} := (\zeta_s^{\mathcal{I}})_{s \leqslant t}$ of the $\mathcal{I}$-th particle $\zeta_t^{\mathcal{I}}$; that is, we have that*

$$\mathbb{Y}_t^{\mathcal{I}} = \widehat{\zeta}_t^{\mathcal{I}} \stackrel{law}{=} \widehat{X}_t \in D_t(S) \tag{4.13}$$

Observe that the jump rate of the system $\widehat{\mathcal{X}}_t$ is given by the stochastic intensity

$$\widehat{\lambda}_{\mathcal{I},t}(\widehat{\mathcal{X}}_t) = (N-1) \ m(\widehat{\mathcal{X}}_t^{-\mathcal{I}})(\widehat{V}_t) \tag{4.14}$$

A synthetic description of the evolution of $\widehat{\mathcal{X}}$ given $\mathcal{I} = k$ and the jumps

$$x \rightsquigarrow x^{\sigma_{j_1,k}} \vee x_t^{c_{j_1,k}} \qquad x \rightsquigarrow x \vee x_t^{c_{j_2,j_1}} \quad \text{and} \quad x \rightsquigarrow x^{\sigma_{j_2,k}} \vee x_t^{c_{j_2,k}}$$

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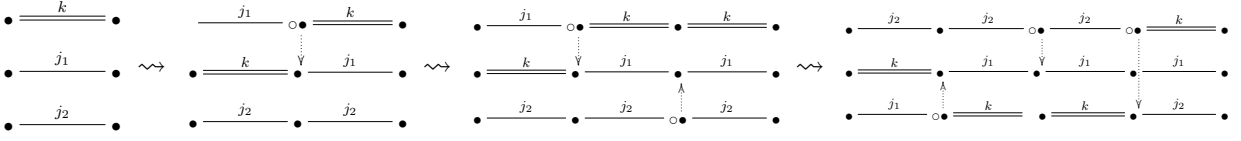on three jump times $(T_{k,1}, T_{k,2}, T_{k,3})$ is given below in figure 11.

Figure 11

In the above display the k-th ancestral line $\mathbb{Y}_t^k$ is represented by the doubled lines ($=\!=\!=$).
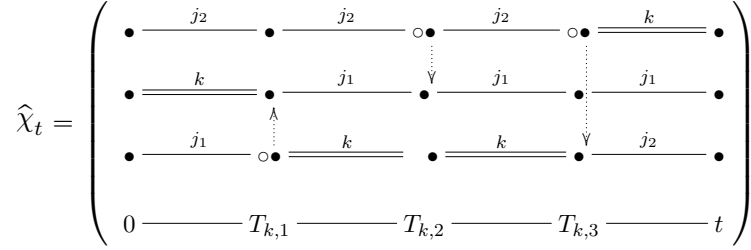A graphical description of the corresponding process $\widehat{\chi}_t$ is given below in figure 12.

Figure 12: Graphical description of the process $\widehat{\chi}_t$

The genealogical tree $\mathbb{Y}_t$ associated with the historical process $\widehat{\chi}_t$ in figure 12 is given below in figure 13. .
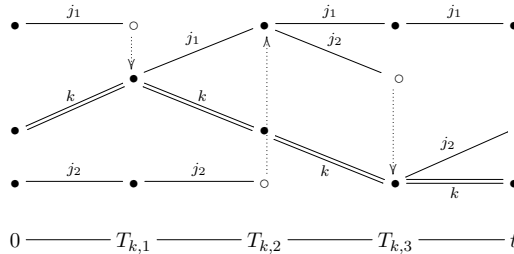

Figure 13: Genealogical tree associated with the process $\widehat{\chi}_t$ in figure 12

As in figure 8, the plain lines ($\overset{j}{-\!-\!-}$) represent the path associated with N independent copies $X_{u,v}^j$ of the stochastic semigroup $X_{u,v}$ of the process X_t with generator L_t. In this notation, we have the formula

$$\mathbb{Y}_t^k = \widehat{\zeta}_t^k = \left(X_{0,s}^k(\xi_0^k) \right)_{s \leqslant t} \in D_t(S) \tag{4.15}$$

Observe that the generator $\mathbb{G}_{\mathcal{I},t}$ of the genealogical tree $\mathbb{Y}_t$ defined in (4.12) can be rewritten for any $F \in \mathcal{D}(\widehat{\mathcal{L}})$ as follows

$$\mathbb{G}_{\mathcal{I},t}(F)(x)$$

$$= \widehat{L}_t(F_{x^{-\mathcal{I}}})(x^{\mathcal{I}}) + \sum_{i \in [N]-\{\mathcal{I}\}} \left[\widehat{L}_{x^{\mathcal{I}},t}(F_{x^{-i}})(x^i) + \widehat{V}_t^-(x^i) \int (F_{x^{-i}}(z) - F(x)) \, m(x^{-\mathcal{I}})(dz) \right] \tag{4.16}$$

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In the above display, $\widehat{L}_{z,t}$ stands for the collection of operators indexed by $z \in D_t(S)$ and defined for any function $f \in \mathcal{D}(\widehat{L})$ and any $y \in D_t(S)$ by

$$\widehat{L}_{z,t}(f)(y) := \widehat{L}_t(f)(y) + \frac{2}{N} \widehat{V}_t(y) \; (f(z) - f(y))$$

In the same vein, replacing $(\widehat{L}_t, \widehat{V}_t)$ by (L_t, V_t) in (4.16), for any function $F \in \mathcal{D}(\mathcal{L})$ and any $x = (x^i)_{1 \leqslant i \leqslant N} \in S^N$ we obtain the formula

$$\mathcal{G}_{\mathcal{I},t}(F)(x)$$

$$= L_t(F_{x^{-\mathcal{I}}})(x^{\mathcal{I}}) + \sum_{i \in [N] - \{\mathcal{I}\}} \left[L_{x^{\mathcal{I}},t}(F_{x^{-i}})(x^i) + V_t^-(x^i) \int (F_{x^{-i}}(z) - F(x)) \; m(x^{-\mathcal{I}})(dz) \right]$$

the collection of operators $L_{z,t}(f)(y)$ indexed by $z \in S$ and defined for any function $f \in \mathcal{D}(L)$ and any $y \in S$ by

$$L_{z,t}(f)(y) := L_t(f)(y) + \frac{2}{N} V_t(y) \; (f(z) - f(y)) \quad \text{and} \quad V_t^- := \left(1 - \frac{1}{N}\right) V_t$$

Definition 4.5. *For any final time horizon T and given the ancestral line $\mathbb{Y}_T^{\mathcal{I}} = \widehat{\zeta}_T^{\mathcal{I}} \in D_T(S)$, we denote by $\mathbb{Q}_t^- = \widehat{\eta}_t^-$ and η_t^- the conditional Feynman-Kac measures defined as in (3.12) and (3.3) by replacing $(\widehat{L}_t, \widehat{V}_t)$ by $(\widehat{L}_t^-, \widehat{V}_t^-)$, and respectively (L_t, V_t) by (L_t^-, V_t^-), with the quenched generators*

$$\widehat{L}_t^- := \widehat{L}_{\widehat{\zeta}_t^{\mathcal{I}},t} \quad \text{and} \quad L_t^- := L_{\zeta_t^{\mathcal{I}},t} \tag{4.17}$$

Using (4.13) we readily check the following theorem.

Theorem 4.6. *For any time horizon T the $\mathcal{I}$-ancestral line $\mathbb{Y}_T^{\mathcal{I}} \in D_T(S)$ has the same law as the historical process $\widehat{X}_t$.*

In addition, given the ancestral line $\mathbb{Y}_T^{\mathcal{I}}$, the processes defined by

$$t \in [0, T] \; \mapsto \; \mathbb{Y}_t^- = \left(\mathbb{Y}_t^i\right)_{i \in [N] - \{\mathcal{I}\}} \in D_t(S)^{N-1} \quad \text{and} \quad \zeta_t^- = \left(\zeta_t^i\right)_{i \in [N] - \{\mathcal{I}\}} \in S^{N-1}$$

coincides with the conditional $(N-1)$-mean field particle interpretation of the conditional Feynman-Kac measures $\mathbb{Q}_t^-$ and η_t^-.

4.4 Coupled historical and genealogical trees

Definition 4.7. *Let $\widehat{\Xi}_{s,t}$ be the transition semigroup of the flow $\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right)$ defined for any $F \in \mathcal{B}\left(D_t(S)^{(2N)}\right)$ and $(x, y) \in D_s(S)^{(2N)}$ by the formula*

$$\widehat{\Xi}_{s,t}(F)(x, y) := \mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right)\right) \tag{4.18}$$

Observe that the processes $\widehat{\xi}_{s,t}(x)$ and $\mathbb{X}_{s,t}(x)$ have the same terminal states $\xi_{s,t}^i(x_s^i)$.

Using (4.7) and (4.8) we check that the generator $\widehat{\mathcal{H}}_t$ of the coupled process $(\widehat{\xi}_t, \mathbb{X}_t)$ is defined by the formula

$$\widehat{\mathcal{H}}_t(F)(x, y) = \widehat{\mathcal{L}}_t^{(2)}(F)(x, y) + \widehat{\mathcal{J}}_t^{(2)}(F)(x, y) - \widehat{\lambda}_t(x) \; F(x, y) \tag{4.19}$$

with the function $\widehat{\lambda}_t$ defined in (4.9) and the jump-intensity integral operator $\widehat{\mathcal{J}}_t^{(2)}$ given by

$$\widehat{\mathcal{J}}_t^{(2)}(F)(x,y) := \sum_{\iota \in [N]_0^2} \widehat{\lambda}_t^\iota(x)\, F(\mathbb{C}_{e,c_\iota}(x), \mathbb{C}_{c_\iota}(y)) \quad \text{with} \quad \widehat{\lambda}_t^{i,j}(x) := \frac{1}{N}\, \widehat{V}_t(x^i) \tag{4.20}$$

In the above display, $c_{i,j}$ and $\mathbb{C}_{a,b}$ stands for the coalescent maps and operators defined in (2.16) and (2.17).

Definition 4.8. *Let $\widehat{\Xi}_{s,t}^{\mathcal{I}}$ be the conditional transition semigroup of the flow $\left(\widehat{\mathcal{X}}_{s,t}(x), \mathbb{Y}_{s,t}(y)\right)$ defined for any $F \in \mathcal{B}(D_t(S)^{(2N)})$ and $(x,y) \in D_t(S)^{(2N)}$ by the formula*

$$\widehat{\Xi}_{s,t}^{\mathcal{I}}(F)(x,y) := \mathbb{E}\left(F\left(\widehat{\mathcal{X}}_{s,t}(x), \mathbb{Y}_{s,t}(y)\right) \mid \mathcal{I}\right) \tag{4.21}$$

Using (4.10) and (4.12) we check that the generator $\widehat{\mathcal{H}}_{\mathcal{I},t}$ of the conditional coupled process $(\widehat{\mathcal{X}}_t, \mathbb{Y}_t)$ given $\mathcal{I}$ is defined for any $F \in \mathcal{D}(\widehat{\mathcal{L}}^{(2)})$ and $(x,y) \in D_t(S)^{(2N)}$ by the formula

$$\widehat{\mathcal{H}}_{\mathcal{I},t}(F)(x,y) = \widehat{\mathcal{L}}_t^{(2)}(F)(x,y) + \widehat{\mathcal{J}}_{\mathcal{I},t}^{(2)}(F)(x,y) - \widehat{\lambda}_{\mathcal{I},t}(x)\, F(x,y) \tag{4.22}$$

with the rate function $\widehat{\lambda}_{k,t}$ defined in (4.14) and the jump-intensity integral operator $\widehat{\mathcal{J}}_{k,t}^{(2)}$ given by

$$\widehat{\mathcal{J}}_{\mathcal{I},t}^{(2)}(F)(x,y) = \sum_{i \in [N]-\{\mathcal{I}\}} \sum_{j \in [N]-\{i\}} \widehat{\lambda}_{\mathcal{I},t}^{i,j}(x)\, F(\mathbb{C}_{e,c_{i,j}}(x), \mathbb{C}_{c_{i,j}}(y))$$

$$+ \sum_{j \in [N]-\{\mathcal{I}\}} \widehat{\lambda}_{\mathcal{I},t}^{\mathcal{I},j}(x)\, F(\mathbb{C}_{\sigma_{\mathcal{I},j},c_{j,\mathcal{I}}}(x), \mathbb{C}_{c_{j,\mathcal{I}}}(y)) \quad \text{with} \quad \widehat{\lambda}_{k,t}^{i,j} := 1_{i \neq k}\, \widehat{\lambda}_t^{i,j} + 1_{i=k}\, \widehat{\lambda}_t^{j,k}$$

In the above display, $c_{i,j}$, $\sigma_{i,j}$ and $\mathbb{C}_{a,b}$ stands for the coalescent maps and operators defined in (2.16) and (2.17). Following word-for-word the proof of lemma 4.3, we check the following lemma.

Lemma 4.9. *The jump intensities $(\widehat{\mathcal{J}}_t^{(2)}, \widehat{\mathcal{J}}_{k,t}^{(2)})$ and the generators $(\widehat{\mathcal{H}}_t, \widehat{\mathcal{H}}_{k,t})$ defined in (4.19) and (4.22) are connected for any $F \in \mathcal{D}(\widehat{\mathcal{L}}^{(2)})$ and $(x,y) \in D_t(S)^{(2N)}$ by the formulae*

$$\widehat{\mathcal{J}}_t^{(2)}(F)(x,y) = \widehat{\mathcal{J}}_{k,t}^{(2)}(F)(x,y) + \frac{1}{N} \sum_{i \in [N]-\{k\}} \epsilon'_{k,i}(F)(x,y) \tag{4.23}$$

$$\widehat{\mathcal{H}}_t(F)(x,y) - m(x)(\widehat{V}_t)\, F(x,y) = \widehat{\mathcal{H}}_{k,t}(F)(x,y) - \widehat{V}_t(x^k)\, F(x,y) + \frac{1}{N} \sum_{i \in [N]-\{k\}} \epsilon'_{k,i}(F)(x,y)$$

with the bounded integral operator $\epsilon'_{i,k}$ defined by the anti-symmetric functions

$$\epsilon'_{k,i}(F)(x,y) := \widehat{V}_t(x^k)\, F(x \vee x_t^{c_{k,i}}, y^{c_{k,i}}) - \widehat{V}_t(x^i)\, F(x^{\sigma_{i,k}} \vee x_t^{c_{i,k}}, y^{c_{i,k}}) = -\epsilon_{k,i}(F)(x^{\sigma_{i,k}}, y^{\sigma_{i,k}}) \tag{4.24}$$

Using (2.18) we check that

$$\widehat{\mathcal{J}}_{\mathcal{I},t}^{(2)}(F)(x,y) = \sum_{i \in [N]-\{\mathcal{I}\}} \sum_{j \in [N]-\{i\}} \widehat{\lambda}_{\mathcal{I},t}^{i,j}(x)\, F(\mathbb{C}_{e,c_{i,j}}(x), \mathbb{C}_{c_{i,j}}(y))$$

$$+ \sum_{j \in [N]-\{\mathcal{I}\}} \widehat{\lambda}_{\mathcal{I},t}^{\mathcal{I},j}(x)\, F(\mathbb{C}_{e,c_{\mathcal{I},j}}\left(\mathbb{C}_{\sigma_{\mathcal{I},j}}(x)\right), \mathbb{C}_{c_{\mathcal{I},j}}\left(\mathbb{C}_{\sigma_{\mathcal{I},j}}(y)\right))$$

Rewritten in a more compact form in terms of the collection of coalescent operators $\mathbb{C}_{e,a}^k$ and $\mathbb{C}_a^k$ indexed by $k \in [N]$ and $a \in \mathfrak{C}$ introduced in (2.19), we obtain for any $F \in \mathcal{D}(\widehat{\mathcal{L}}^{(2)})$ and $(x,y) \in D_t(S)^{(2N)}$ the formula

$$\widehat{\mathcal{J}}_{\mathcal{I},t}^{(2)}(F)(x,y) = \sum_{\iota \in [N]_0^2} \widehat{\lambda}_{\mathcal{I},t}^\iota(x)\, F(\mathbb{C}_{e,c_\iota}^{\mathcal{I}}(x), \mathbb{C}_{c_\iota}^{\mathcal{I}}(y)) \tag{4.25}$$

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4.5 Dyson-Phillips expansions

The main objective of this section is to express the stochastic flows $(\widehat{\xi}_{s,t}, \mathbb{X}_{s,t})$ in terms of the stochastic evolution flows discussed in (2.20) and the jump intensity (4.20).

For any given $s \geqslant 0$ and $x \in D_s(S)$ we denote by $T_n^{s,x}$ an increasing sequence of jump times with stochastic intensity $\widehat{\lambda}_t(\widehat{\xi}_{s,t}(x))$. In the further development of this section, $[s,t]_n$ stands for the Weyl chamber introduced in (2.2).

Observe that

$$\mathcal{E}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x)\right) := \exp\left(-\int_s^t \widehat{\lambda}_r\left(\widehat{\mathcal{X}}_{s,r}(x)\right) \, dr\right) = \mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x)\right)^{N-1} \tag{4.26}$$

with the exponential map $\mathcal{Z}_{s,t}$ introduced in (3.1).

Definition 4.10. *For any $n \geqslant 0$ We denote by $\widehat{\mathcal{U}}_{s,t}^{(n)}$ and $\widehat{\mathbb{U}}_{s,t}^{(n)}$ the integral semigroups defined for any $F \in \mathcal{B}(D_t(S)^{(2N)})$ and $(x,y) \in D_s(S)^{(2N)}$ by the formulae*

$$\begin{aligned}
\widehat{\mathcal{U}}_{s,t}^{(n)}(F)(x,y) &:= \mathbb{E}\left(\widehat{\mathbb{U}}_{s,t}^{(n)}(F)(x,y)\right) \\
\widehat{\mathbb{U}}_{s,t}^{(n)}(F)(x,y) &:= F\left(\widehat{\mathcal{X}}_{s,t}(x), \widehat{\mathcal{X}}_{s,t}(y)\right) \mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x)\right)^n
\end{aligned} \tag{4.27}$$

Using (4.26) we have

$$\begin{aligned}
\widehat{\mathcal{U}}_{s,t}^{(N-1)}(F)(x,y) &= \mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right) 1_{T_1^{s,x}>t}\right) \\
\widehat{\mathbb{U}}_{s,t}^{(N-1)}(F)(x,y) &= \mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right) 1_{T_1^{s,x}>t} \mid \widehat{\mathcal{X}}_{s,t}\right)
\end{aligned}$$

In addition, the semigroups $\left(\widehat{\Xi}_{s,t}, \widehat{\mathcal{U}}_{s,t}^{(N-1)}\right)$ are connected by the Gelfand-Pettis weak-sense integration formulae

$$\widehat{\Xi}_{s,t} = \widehat{\mathcal{U}}_{s,t}^{(N-1)} + \int_s^t \widehat{\mathcal{U}}_{s,r}^{(N-1)} \, \widehat{\mathcal{J}}_r^{(2)} \, \widehat{\Xi}_{r,t} \, dr$$

with the jump intensity integral operator $\widehat{\mathcal{J}}_r^{(2)}$ defined in (4.25). Iterating the above implicit formula, we obtain the following proposition.

Proposition 4.11. *For any $s \leqslant t$ and any $F \in \mathcal{B}(D_t(S)^{(2N)})$ and $(x,y) \in D_s(S)^{(2N)}$ we have the Dyson-Phillips expansion*

$$\widehat{\Xi}_{s,t}(F)(x,y) = \sum_{n \geqslant 0} \mathbb{E}\left(F(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)) \, 1_{T_n^{s,x}<t<T_{n+1}^{s,x}}\right) \tag{4.28}$$

with the iterated integrals

$$\mathbb{E}\left(F(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)) \, 1_{T_n^{s,x}<t<T_{n+1}^{s,x}}\right)$$

$$= \int_{[s,t]_n} \left(\left(\widehat{\mathcal{U}}_{r_0,r_1}^{(N-1)} \widehat{\mathcal{J}}_{r_1}^{(2)}\right) \cdots \left(\widehat{\mathcal{U}}_{r_{n-1},r_n}^{(N-1)} \widehat{\mathcal{J}}_{r_n}^{(2)}\right) \left(\widehat{\mathcal{U}}_{r_n,t}^{(N-1)}(F)\right)\right)(x) \, dr$$

We end this section with a probabilistic interpretation of (4.28) in terms of the collection of maps $\widehat{\mathcal{T}}_{s,t}^a$ and $\widehat{\mathcal{X}}_{s,t}^a$, indexed by $a \in \mathfrak{C}$ and $s \leqslant t$ defined in (2.21).

By (4.19) and (4.20), given the stochastic flows $\widehat{\mathcal{X}}_{s,t}$, the jump times $T_n^{s,x} = r_n$ and the randomly selected coalescent maps $a_n = c_{\iota_n}$ for some index $\iota_n \in [N]_0^2$ we have the stochastic evolution semigroup formulae

$$\widehat{\xi}_{r_{n-1},r_n} = \widehat{\mathcal{T}}_{r_{n-1},r_n}^{c_{\iota_n}} \quad \text{and} \quad \mathbb{X}_{r_{n-1},r_n} = \widehat{\mathcal{X}}_{r_{n-1},r_n}^{c_{\iota_n}}$$

In addition, whenever $r_n \leqslant t < r_{n+1}$ we have

$$\widehat{\xi}_{s,t} = \widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{T}}_{r,r_n}^{\iota,(n)} \quad \text{and} \quad \mathbb{X}_{s,t} = \widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{X}}_{r,r_n}^{\iota,(n)} \tag{4.29}$$

with the convention $r_0 = s$, and the compositions

$$\widehat{\mathcal{T}}_{r,r_n}^{\iota,(n)} := \widehat{\mathcal{T}}_{r_{n-1},r_n}^{c_{\iota_n}} \circ \ldots \circ \widehat{\mathcal{T}}_{r_0,r_1}^{c_{\iota_1}} \quad \text{and} \quad \widehat{\mathcal{X}}_{r,r_n}^{\iota,(n)} := \widehat{\mathcal{X}}_{r_{n-1},r_n}^{c_{\iota_n}} \circ \ldots \circ \widehat{\mathcal{X}}_{r_0,r_1}^{c_{\iota_1}} \tag{4.30}$$

The conditional probability $\mathfrak{p}_n(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$ on $\left([N]_0^{(2n)} \times [s,t]_n\right)$ of the n first jump times and the selected coalescent indices is given by the formula

$$\mathfrak{p}_n^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$$

$$:= \mathbb{P}\left((T_1^{s,x}, \ldots, T_n^{s,x}) \in d(r_1, \ldots, r_n), \quad (a_1, \ldots, a_n) = (c_{\iota_1}, \ldots, c_{\iota_n}), \quad 1_{T_{n+1}^{s,x} > t} \mid \widehat{\mathcal{X}}_{s,t}\right)$$

$$= \mathcal{E}_{r_0,r_1}\left(\widehat{\mathcal{X}}_{r_0,r_1}(x)\right) \widehat{\lambda}_{r_1}^{\iota_1}\left(\widehat{\mathcal{X}}_{r_0,r_1}(x)\right) \mathcal{E}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_0,r_1}^{c_{\iota_1}}(x)\right)\right) \widehat{\lambda}_{r_2}^{\iota_2}\left(\widehat{\mathcal{X}}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_0,r_1}^{c_{\iota_1}}(x)\right)\right) \times \ldots \times$$

$$\times \mathcal{E}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r,r_{n-1}}^{\iota,(n-1)}(x)\right)\right) \widehat{\lambda}_{r_n}^{\iota_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r,r_{n-1}}^{\iota,(n-1)}(x)\right)\right) \mathcal{E}_{r_n,t}\left(\widehat{\mathcal{X}}_{r_n,t}\left(\widehat{\mathcal{X}}_{r,r_n}^{\iota,(n)}(x)\right)\right) dr \tag{4.31}$$

This construction yields a probabilistic interpretation of the weak-sense Dyson-Phillips expansion stated in (4.28).

Theorem 4.12. *For any $s \leqslant t$, any $n \geqslant 1$ and any $F \in \mathcal{B}(D_t(S)^{(2N)})$ and $(x,y) \in D_s(S)^{(2N)}$ we have the conditional almost sure Dyson-Phillips expansion formulae*

$$\mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right) \; 1_{T_n^{s,x} < t < T_{n+1}^{s,x}} \mid \widehat{\mathcal{X}}_{s,t}\right)$$

$$= \int_{[s,t]_n} \left(\left(\widehat{\mathbb{U}}_{r_0,r_1}^{(N-1)} \widehat{\mathcal{J}}_1^{(2)}\right) \cdots \left(\widehat{\mathbb{U}}_{r_{n-1},r_n}^{(N-1)} \widehat{\mathcal{J}}_n^{(2)}\right) \left(\widehat{\mathbb{U}}_{r_n,t}^{(N-1)}(F)\right)\right)(x) \; dr$$

$$= \sum_{\iota \in [N]_0^{(2n)}} \int_{[s,t]_n} F\left(\widehat{\mathcal{T}}_{r,t}^{\iota,n}(x), \widehat{\mathcal{X}}_{r,t}^{\iota,n}(y)\right) \mathfrak{p}_n^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$$

Corollary 4.13. *For any $s \leqslant t$, and any $F \in \mathcal{B}(D_t(S)^{(2N)})$, $(x,y) \in D_s(S)^{(2N)}$ and $n \geqslant 1$ we have the almost sure formula*

$$\mathbb{E}\left(F(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)) \, \mathcal{Z}_{s,t}\left(\mathbb{X}_{s,t}(x)\right) \; 1_{T_n^{s,x} < t < T_{n+1}^{s,x}} \mid \widehat{\mathcal{X}}_{s,t}\right)$$

$$= \int_{[s,t]_n} \left(\left(\widehat{\mathbb{U}}_{r_0,r_1}^{(N)} \widehat{\mathcal{J}}_1^{(2)}\right) \cdots \left(\widehat{\mathbb{U}}_{r_{n-1},r_n}^{(N)} \widehat{\mathcal{J}}_n^{(2)}\right) \left(\widehat{\mathbb{U}}_{r_n,t}^{(N)}(F)\right)\right)(x) \; dr$$

with the semigroup $\widehat{\mathbb{U}}_{s,t}^{(N)}$ defined in (4.27), and the convention $r_0 = s$.

Proof. By theorem 4.12 we have

$$\mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right) \; \mathcal{Z}_{s,t}\left(\mathbb{X}_{s,t}(x)\right) \; 1_{T_n^{s,x} < t < T_{n+1}^{s,x}} \mid \widehat{\mathcal{X}}_{s,t}\right)$$

$$= \sum_{\iota \in [N]_0^{(2n)}} \int_{[s,t]_n} F\left(\widehat{\mathcal{T}}_{r,t}^{\iota,n}(x), \widehat{\mathcal{X}}_{r,t}^{\iota,n}(y)\right) \; \mathfrak{q}_n^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota, r))$$

with the measure

$$\mathfrak{q}_n^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota, r)) = \mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{r,t}^{\iota,n}(x)\right) \; \mathfrak{p}_n^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota, r))$$

Also observe that

$$\mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{r,t}^{\iota,n}(x)\right) = \mathcal{Z}_{s,r_1}\left(\widehat{\mathcal{X}}_{s,r_1}(x)\right) \; \mathcal{Z}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_0,r_1}^{c_{\iota_1}}(x)\right)\right)$$

$$\times \ldots \times \; \mathcal{Z}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r,r_{n-1}}^{\iota,(n-1)}(x)\right)\right) \; \mathcal{Z}_{r_n,t}\left(\widehat{\mathcal{X}}_{r,t}^{\iota,n}(x)\right)$$

In addition, for any $y \in D_{r_n}(S)^N$ and $n \geqslant 1$ we have

$$\mathcal{E}_{r_{n-1},r_n}(y) \times \mathcal{Z}_{r_{n-1},r_n}(y) = \mathcal{Z}_{r_{n-1},r_n}(y)^N$$

The end of the proof of the corollary is now easily completed. ∎

5 Perturbation analysis

5.1 Semigroup estimates

We consider a collection of generators L_t^ϵ and potential functions V_t^δ of the form

$$L_t^\epsilon = L_t + \epsilon \, \overline{L}_t \quad \text{and} \quad V_t^\delta = V_t + \delta \overline{V}_t \quad \text{with} \quad \epsilon, |\delta| \in [0,1]$$

In the above display, $\overline{V}_t$ stands for some uniformly bounded function and $\overline{L}_t$ a bounded generator of an auxiliary jump type Markov process of the form

$$\overline{L}_t(f)(x) = \lambda(x) \int \left(f(y) - f(x)\right) K_t(x, dy)$$

for some jump rate function function $\lambda(x)$ and some Markov transitions $K_t(x, dy)$ such that

$$\lambda_1 \leqslant \lambda(x) \leqslant \lambda_2 \quad \text{and} \quad \varpi_1 \, \kappa_t(dy) \leqslant K_t(x, dy) \leqslant \varpi_2 \, \kappa_t(dy)$$

In the above display, λ_i, ϖ_i stands for some positive parameters and κ_t some probability measures.

We let $P_{s,t}^\epsilon$ be the transition semigroup of the process with generator L_t^ϵ. In this notation, we have the following technical lemma.

Lemma 5.1. *Assume that $P_{s,t}$ satisfies (H_0) for some parameters h and $\rho(h) > 0$ and some probability measures $\mu_{t,h}$. In this situation, for any $\epsilon \in [0,1]$ and $t \geqslant 0$ there exists some probability measures $\mu_{t,h}^\epsilon$ such that*

$$\rho_\epsilon(h) \, \mu_{t,h}^\epsilon(dy) \leqslant P_{t,t+h}^\epsilon(x, dy) \leqslant \rho_\epsilon(h)^{-1} \, \mu_{t,h}^\epsilon(dy) \tag{5.1}$$

with the parameters

$$\rho_\epsilon(h) \quad := \quad \rho(h) \left(e^{-\epsilon\lambda_2 h} + (1 - e^{-\epsilon\lambda_2 h})\varpi_2\right) \min\left((\lambda_1/\lambda_2)(\varpi_1/\varpi_2), e^{-\epsilon(\lambda_2-\lambda_1)h}\right)$$

$$\geqslant \quad \rho(h) \ \min\left((\lambda_1/\lambda_2)(\varpi_1/\varpi_2), e^{-(\lambda_2-\lambda_1)h}\right)$$

The proof of the above lemma is provided in the appendix on page 45.

We consider the Feynman-Kac semigroup $Q_{s,t}^{\delta,\epsilon}$ defined as $Q_{s,t}$ by replacing V_t by V_t^δ and X_t by a Markov process with generator L_t^ϵ.

Also let $\phi_{s,t}^{(\delta,\epsilon)}$ be defined as $\phi_{s,t}$ by replacing $Q_{s,t}$ by $Q_{s,t}^{\delta,\epsilon}$, and set

$$L_t^{\delta,\epsilon} = \epsilon\overline{L}_t - \delta\ \overline{\mathcal{V}}_t \quad \text{and} \quad L_{t,\eta}^{\delta,\epsilon} = \epsilon\overline{L}_t - \delta(\overline{\mathcal{V}}_t - \eta(\overline{\mathcal{V}}_t)) \quad \text{with} \quad \overline{\mathcal{V}}_t(f) := \overline{V}_t\ f$$

Theorem 5.2. *For any $|\epsilon|, |\delta| \in [0,1]$ and any $s \leqslant t$ we have the perturbation semigroup Gelfand-Pettis weak-sense integration formulae*

$$Q_{s,t}^{\delta,\epsilon} - Q_{s,t} = \int_s^t Q_{s,u}^{\delta,\epsilon}\ L_u^{\delta,\epsilon}\ Q_{u,t}\ du = \int_s^t Q_{s,u}\ L_u^{\delta,\epsilon}\ Q_{u,t}^{\delta,\epsilon}\ du \tag{5.2}$$

In addition, for any $\eta \in \mathcal{P}(S)$ we have

$$\phi_{s,t}^{\delta,\epsilon}(\eta) - \phi_{s,t}(\eta) = \int_s^t \phi_{s,u}^{\delta,\epsilon}(\eta)\ L_{u,\phi_{s,u}^{\delta,\epsilon}(\eta)}^{\delta,\epsilon}\ \partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)}\phi_{u,t}\ du = \int_s^t \phi_{s,u}(\eta)\ L_{u,\phi_{s,u}(\eta)}^{\delta,\epsilon}\ \partial_{\phi_{s,u}(\eta)}\phi_{u,t}^{\delta,\epsilon}\ du$$

Proof. We check (5.2) using the fact that

$$\partial_u(Q_{s,u}^{\delta,\epsilon}Q_{u,t}) = Q_{s,u}^{\delta,\epsilon}\ (L_u^\epsilon - L_u - \delta\ \overline{\mathcal{V}}_u)\ Q_{u,t} = \epsilon\ Q_{s,u}^{\delta,\epsilon}\ \overline{L}_u\ Q_{u,t} - \delta\ Q_{s,u}^{\delta,\epsilon}\ \overline{\mathcal{V}}_u\ Q_{u,t}$$

and

$$\partial_u(Q_{s,u}Q_{u,t}^{\delta,\epsilon}) = -\epsilon\ Q_{s,u}\ \overline{L}_u\ Q_{u,t}^{\delta,\epsilon} + \delta\ Q_{s,u}\ \overline{\mathcal{V}}_u\ Q_{u,t}^{\delta,\epsilon}$$

The perturbation analysis of the normalized semigroups $\phi_{s,t}^{\delta,\epsilon}$ is slightly more involved.

Let $\Lambda_t^{\delta,\epsilon}$ be defined as Λ_t by replacing (L_t, V_t) by $(L_t^\epsilon, V_t^\delta)$. Notice that

$$h^{-1}\left[\phi_{t,t+h}^{\delta,\epsilon}(\eta) - \eta\right] = \Lambda_t^{\delta,\epsilon}(\eta) + \mathrm{O}(h)$$

For any given $s \leqslant t$, we consider the interpolating maps $u \in [s,t] \mapsto \Delta_{s,u,t}^{\delta,\epsilon}$ defined by

$$\Delta_{s,u,t}^{\delta,\epsilon} := \phi_{u,t} \circ \phi_{s,u}^{\delta,\epsilon}$$

On the other hand, for any $s \leqslant u \leqslant u + h \leqslant t$ we have the decomposition

$$\Delta_{s,u+h,t}^{\delta,\epsilon}(\eta) - \Delta_{s,u,t}^{\delta,\epsilon}(\eta)$$

$$= \phi_{u+h,t}\left(\phi_{s,u+h}^{\delta,\epsilon}(\eta)\right) - \phi_{u,t}\left(\phi_{s,u+h}^{\delta,\epsilon}(\eta)\right) + \phi_{u,t}\left(\phi_{s,u+h}^{\delta,\epsilon}(\eta)\right) - \phi_{u,t}\left(\phi_{s,u}^{\delta,\epsilon}(\eta)\right)$$

$$= -\Lambda_u(\phi_{s,u}^{\delta,\epsilon}(\eta))\left(\partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)}\phi_{u,t}\right)h$$

$$+\phi_{u,t}\left(\phi_{s,u}^{\delta,\epsilon}(\eta) + \left[\phi_{s,u+h}^{\delta,\epsilon}(\eta) - \phi_{s,u}^{\delta,\epsilon}(\eta)\right]\right) - \phi_{u,t}\left(\phi_{s,u}^{\delta,\epsilon}(\eta)\right) + \mathrm{O}(h^2)$$

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This implies that

$$h^{-1}\left[\Delta^{\delta,\epsilon}_{s,u+h,t}(\eta) - \Delta^{\delta,\epsilon}_{s,u,t}(\eta)\right]$$

$$= -\Lambda_u(\phi^{\delta,\epsilon}_{s,u}(\eta))\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t} + h^{-1}\left[\phi^{\delta,\epsilon}_{s,u+h}(\eta) - \phi^{\delta,\epsilon}_{s,u}(\eta)\right]\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t} + \mathrm{O}(h)$$

We conclude that

$$\partial_u\Delta^{\delta,\epsilon}_{s,u,t}(\eta) = \left[\Lambda^{\delta,\epsilon}_u(\phi^{\delta,\epsilon}_{s,u}(\eta)) - \Lambda_u(\phi^{\delta,\epsilon}_{s,u}(\eta))\right]\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t}$$

On the other hand, we have

$$\left[\Lambda^{\delta,\epsilon}_t(\eta) - \Lambda_t(\eta)\right](f) = \epsilon\,\eta(\overline{L}_t(f)) - \delta\,\eta(f\,(\overline{V}_t - \eta(\overline{V}_t)))$$

By symmetry arguments, this ends the proof of the theorem. ∎

Corollary 5.3. *For any $s \leqslant t$ and any $\eta \in \mathcal{P}(S)$ we have the estimates*

$$(H_1) \implies \|\phi^{\delta,\epsilon}_{s,t}(\eta) - \phi_{s,t}(\eta)\|_{\mathrm{tv}} \leqslant c\,(\epsilon + \delta)$$

$$(H_2) \implies \|\phi^{\delta,\epsilon}_{s,t}(\eta) - \phi_{s,t}(\eta)\|_{\mathrm{tv}} \leqslant c\,(\epsilon + \delta)\,(t - s)$$

for some finite constant c whose value doesn't depend on the parameters (s,t,η), nor on (ϵ,δ).

5.2 Particle stochastic flows

Given a random measure μ on some measurable state space $(E,\mathcal{E})$ and some σ-field $\mathcal{F} \subset \mathcal{E}$ we write $\mathbb{E}(\mu \mid \mathcal{F})$ the first conditional moment measure given by

$$\mathbb{E}(\mu \mid \mathcal{F}) \;:\; f \in \mathcal{B}(S) \mapsto \mathbb{E}(\mu \mid \mathcal{F})(f) := \mathbb{E}(\mu(f) \mid \mathcal{F})$$

For any $t \geqslant 0$, we let $\Delta m(\xi_t)$ be the random jump occupation measure

$$\Delta m(\xi_t) := m(\xi_t) - m(\xi_{t-}) = \Delta M_t = M_t - M_{t-}$$

with the martingale random field M_t defined in (4.5). In this notation, we have

$$N^{n-1}\,\partial_t\mathbb{E}\left[(\Delta m(\xi_t))^{\otimes n}(f^{(1)}_t \otimes \ldots \otimes f^{(n)}_t) \mid \mathcal{F}_{t-}\right] = m(\xi_{t-})\Gamma^{(n)}_{L^d_{m(\xi_{t-})}}(f^{(1)}_t, \ldots, f^{(n)}_t) \tag{5.3}$$

with the operators $\Gamma^{(n)}_{L^d_{m(\xi_{t-})}}$ defined in (2.12). When $n = 2$ the above formula resumes to

$$\partial_t\mathbb{E}\left[\Delta m(\xi_t)(f_t)\,\Delta m(\xi_t)(g_t) \mid \mathcal{F}_{t-}\right] = \frac{1}{N}\,m(\xi_{t-})\Gamma_{L^d_{m(\xi_{t-})}}(f_t, g_t)$$

$$= \partial_t\langle M^d_t(f), M^d(g)\rangle_t = \partial_t\langle \mathcal{M}^d_t(F), \mathcal{M}^d(G)\rangle_t$$

with the functions (F, G) defined in (4.4)

Definition 5.4. *For any $t \geqslant s$ and $n \geqslant 1$, we consider the integral random operators*

$$\Delta^n \phi_{s,t}(m(\xi_s)) \quad := \quad N^{n-1} \frac{1}{n!} \, (\Delta m(\xi_s))^{\otimes n} \, \overline{\partial}^n_{m(\xi_{s-})+\Delta m(\xi_s), m(\xi_{s-})} \phi_{s,t}$$

and their first variational measure

$$\Upsilon^n_{m(\xi_{s-})} \phi_{s,t} := \partial_s \mathbb{E}\left[\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right]$$

Choosing $n = 1$ we have

$$\Delta \phi_{s,t}(m(\xi_s)) \quad := \quad \Delta^1 \phi_{s,t}(m(\xi_s)) = \phi_{s,t}(m(\xi_s)) - \phi_{s,t}(m(\xi_{s-}))$$

Arguing as in the proof of (3.17) and using (5.3), for any collection of functions $f^{(n)} \in \mathrm{Osc}(S)$ we have the estimate

$$N^{n-1} \, \partial_s \, \mathbb{E}\left[\Delta \phi_{s,t}(m(\xi_s))^{\otimes n}(f_t^{(1)} \otimes \ldots \otimes f_t^{(n)}) \mid \mathcal{F}_{s-}\right] \leqslant e^{nq} \, \|\lambda + V\| \tag{5.4}$$

Proposition 5.5. *For any $t \geqslant s$ and $n \geqslant 1$, we have*

$$\Delta^n \phi_{s,t}(m(\xi_s)) = \frac{N^{n-1}}{n!} \, (\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} + \frac{1}{N} \, \Delta^{n+1} \phi_{s,t}(m(\xi_s)) \tag{5.5}$$

In addition, for any $f \in \mathcal{B}(S)$ we have

$$\Upsilon^n_{m(\xi_{s-})} \phi_{s,t}(f)$$

$$= (-1)^{n-1} \, m(\xi_{s-}) \Gamma^{(n)}_{L^d_{m(\xi_{s-})}} \left(Q^{m(\xi_{s-1})}_{s,t}(1), \ldots, Q^{m(\xi_{s-1})}_{s,t}(1), \partial_{m(\xi_{s-})} \phi_{s,t}(f)\right) + \frac{1}{N} \, \Upsilon^{n+1}_{m(\xi_{s-})} \phi_{s,t}(f) \tag{5.6}$$

Proof. We have

$$\begin{aligned}
\Delta^{n+1} \phi_{s,t}(m(\xi_s)) &= \quad N^n \left[\Delta \phi_{s,t}(m(\xi_s)) - \sum_{1 \leqslant k \leqslant n} \frac{1}{k!} \, (\Delta m(\xi_s))^{\otimes k} \partial^k_{m(\xi_{s-})} \phi_{s,t}\right] \\
&= \quad N \, \Delta^n \phi_{s,t}(m(\xi_s)) - \frac{N^n}{n!} \, (\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} \quad \Longleftrightarrow \quad (5.5)
\end{aligned}$$

This implies that

$$\partial_s \mathbb{E}\left[\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right] := \Upsilon^n_{m(\xi_{s-})} \phi_{s,t}$$

$$= \frac{N^{n-1}}{n!} \, \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} \mid \mathcal{F}_{s-}\right] + \frac{1}{N} \, \partial_s \mathbb{E}\left[\Delta^{n+1} \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right]$$

This ends the proof of the proposition. ∎

Lemma 5.6. *For any $n \geqslant 1$ and $s \leqslant t$ we have the almost sure uniform estimates*

$$(H_2) \Longrightarrow \|\Upsilon^n_{m(\xi_{s-})} \phi_{s,t}\|_{\mathrm{tv}} \leqslant 2^{n-1} e^{(n+1)q} \, \|\lambda + V\| \tag{5.7}$$

The detailed proof of the above estimate is provided in the appendix, on page 46.

In the further development of this section, for any given time horizon t and any $f \in \mathcal{B}(S)$ we let

$$s \in [0, t] \mapsto \mathcal{M}_s^d(\phi_{\cdot,t}(m(.)))(f)$$

be the martingale $s \in [0, t] \mapsto \mathcal{M}_s^d(F)$ associated with the function

$$(s, x) \in [0, t] \times S^N \mapsto F(s, x) = \phi_{s,t}(m(x))(f)$$

We also denote by

$$s \in [0, t] \mapsto M_s^c \left(\partial_{m(\xi_\bullet)} \phi_{\cdot,t}(f) \right), \quad \text{resp.} \quad M_s^c \left(Q_{\cdot,t}^{m(\xi_\bullet)}(1) \right)$$

the martingale $M_s^c(f)$ associated with the $\mathcal{F}$-predictable bounded function

$$(s, x) \in [0, t] \times S \mapsto f_s(x) = \partial_{m(\xi_{s-})} \phi_{s,t}(f)(x), \quad \text{resp.} \quad f_s(x) = Q_{s,t}^{m(\xi_{s-})}(1)(x)$$

We are now in position to state and to prove the main result of this section.

Theorem 5.7. *For any time horizon $t \geqslant 0$ and any $f \in \mathcal{B}(S)$ the interpolating function*

$$s \in [0, t] \mapsto \phi_{s,t}(m(\xi_s))(f) \in \mathbb{R}$$

satisfies the stochastic differential equation

$$
\begin{aligned}
d\phi_{s,t}(m(\xi_s))(f) \quad = \quad & \frac{1}{\sqrt{N}} \, dM_s^c \left(\partial_{m(\xi_\bullet)} \phi_{\cdot,t}(f) \right) + d\mathcal{M}_s^d \left(\phi_{\cdot,t}(m(.))(f) \right) \\
& + \frac{1}{N} \, \Upsilon_{m(\xi_{s-})}^2 \phi_{s,t}(f) \, ds - \frac{1}{N} \, m(\xi_s) \Gamma_{L_s^c} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \, ds
\end{aligned}
$$

Proof. Observe that

$$dm(\xi_s) = \Lambda_s(m(\xi_s)) \, ds + \frac{1}{\sqrt{N}} \, dM_s^c + \Delta m(\xi_s) - \underbrace{\mathbb{E}(\Delta m(\xi_s) \mid \mathcal{F}_{s-})}_{\ll ds}$$

Using Itô formula and the backward formula (3.21) we have

$$d \, \phi_{s,t}(m(\xi_s))(f) = -\Lambda_s(m(\xi_s)) \left(\partial_{m(\xi_s)} \phi_{s,t}(f) \right) \, ds + \left[\phi_{s,t}(m(\xi_{s-}) + dm(\xi_s)) - \phi_{s,t}(m(\xi_{s-})) \right](f)$$

$$= \frac{1}{\sqrt{N}} \, dM_s^c \left(\partial_{m(\xi_\bullet)} \phi_{\cdot,t}(f) \right) + d\mathcal{M}_s^d(\phi_{\cdot,t}(m(.))(f))$$

$$+ \frac{1}{2N} \left(dM_s^c \otimes dM_s^c \right) \partial_{m(\xi_s)}^2 \phi_{s,t}(f) + \partial_s \mathbb{E} \left[\Delta \phi_{s,t}(m(\xi_s))(f) - \Delta m(\xi_s) \partial_{m(\xi_{s-})} \phi_{s,t}(f) \mid \mathcal{F}_{s-} \right] \, ds$$

This ends the proof of the theorem. ∎

The above theorem can be interpreted as an Aleeksev-Gröbner formula for interpolating semi-groups on the space of measures [44].

Next corollary is a direct consequence of the recursion (5.6).

Corollary 5.8. *For any $t \geqslant 0$ and any $f \in \mathcal{B}(S)$ we have the almost sure formula*

$$\phi_{s,t}(m(\xi_s))(f) - \phi_{0,t}(m(\xi_0))(f)$$

$$= \frac{1}{\sqrt{N}} \ M_s^c \left(\partial_{m(\xi_\bullet)} \phi_{\bullet,t}(f) \right) + \mathcal{M}_s^d (\phi_{\bullet,t}(m(.)))(f)$$

$$- \frac{1}{N} \ \int_0^s m(\xi_u) \Gamma_{L_{u,m(\xi_u)}} \left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)} \phi_{u,t}(f) \right) \ du + \frac{1}{N^2} \int_0^s \Upsilon_{m(\xi_{u-})}^3 \phi_{u,t}(f) \ du$$

$$\tag{5.8}$$

Choosing $s = t$ and taking the expectation in (5.8) we obtain the following result.

Corollary 5.9. *For any $t \geqslant 0$ and $f \in \mathcal{D}(S)$ we have the formula*

$$\mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f))$$

$$= -\frac{1}{N} \ \int_0^t \mathbb{E} \left[m(\xi_s) \Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \right] \ ds + \frac{1}{N^2} \int_0^t \mathbb{E} \left[\Upsilon_{m(\xi_{s-})}^3 \phi_{s,t}(f) \right] \ ds$$

5.3 Some non-asymptotic estimates

Theorem 5.10. *For any time horizon $t \geqslant 0$ and any function $f \in \mathrm{Osc}(S)$ we have*

$$\begin{array}{rcl} (H_1) & \Longrightarrow & |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leqslant c/N \\ (H_2) & \Longrightarrow & |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leqslant c \ t/N \end{array} \tag{5.9}$$

In addition, for any function $F \in \mathrm{Osc}(D_t(S))$ we have

$$(H_1) \Longrightarrow |\mathbb{E}(m(\mathbb{X}_t)(F)) - \mathbb{Q}_t(F)| \leqslant c \ t/N \tag{5.10}$$

for some finite constant c whose value doesn't depend on the parameters (t, N).

The assertion (5.10) is a direct consequence of (3.20) and (5.9). The proof of the estimates (5.9) is mainly based on the decomposition presented in corollary 5.9. The estimates rely on elementary but rather technical carré du champ inequalities, and semigroup techniques. Thus, the detail of the proof is housed in the appendix, on page 46.

The first estimate stated in the above corollary extends the bias estimate obtained in [74] to time varying Feynman-Kac models. The central difference between homogeneous and time varying models lies on the fact that we cannot use h-process techniques. The latter allows to interpret the Feynman-Kac semigroups in terms of more conventional Markov semigroups.

We end this section with a some more or less direct consequences of the above estimates in the analysis of the measures discussed in theorem 4.6.

By corollary 5.3, for any $N > 1$ we have

$$(H_1) \Longrightarrow \|\eta_t^- - \eta_t\|_{\mathrm{tv}} \leqslant c/N \quad \text{and} \quad (H_2) \Longrightarrow \|\eta_t^- - \eta_t\|_{\mathrm{tv}} \leqslant c \ t/N$$

with the Feynman-Kac measures η_t^- defined in (4.17).

Arguing as in the proof of (5.10) we also have the almost sure estimate

$$(H_1) \Longrightarrow \|\mathbb{Q}_t^- - \mathbb{Q}_t\|_{\mathrm{tv}} \leqslant c \ t/N$$

with the Feynman-Kac measures $\mathbb{Q}_t^-$ defined in (4.17).

By (5.1), when (H_0) is satisfied, the Feynman-Kac model defined in terms of the pair (L_s^-, V_s^-) given in (4.17) satisfies the stability property (H_1). Thus, using theorem 5.10 we readily deduce the following estimates.

Corollary 5.11. *Assume that (H_1) is met. In this situation, for any $f \in \mathrm{Osc}(S)$ and $F \in \mathrm{Osc}(D_t(S))$ we have almost sure and uniform estimates given by*

$$|\mathbb{E}\left(m(\zeta_t^-)(f) \mid \widehat{\zeta}_t^{\mathcal{I}}\right) - \eta_t(f)| \leqslant c/N \quad and \quad |\mathbb{E}\left(m(\mathbb{Y}_t^-)(F) \mid \mathbb{Y}_t^{\mathcal{I}}\right) - \mathbb{Q}_t(F)| \leqslant c\,t/N$$

The above results give some information on the bias of the occupation measures. We end this section with some propagation of chaos estimate. Using (5.8), for any functions $f_i \in \mathrm{Osc}(S)$ we have

$$\mathbb{E}\left(m(\xi_t)(f_1)\, m(\xi_t)(f_2)\right) - \mathbb{E}\left(\phi_{0,t}(m(\xi_0))(f_1)\, \phi_{0,t}(m(\xi_0))(f_2)\right)$$

$$= -\frac{1}{N} \sum_{(k,l)\in\{(1,2),(2,1)\}} \int_0^t \mathbb{E}\left[\phi_{s,t}(m(\xi_s))(f_k)\, m(\xi_u)\Gamma_{L_{u,m(\xi_u)}}\left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)}\phi_{u,t}(f_l)\right)\right]\, du$$

$$+ \frac{1}{N} \int_0^t \mathbb{E}\left[m(\xi_u)\Gamma_{L_{u,m(\xi_u)}}\left(\partial_{m(\xi_u)}\phi_{u,t}(f_1), \partial_{m(\xi_u)}\phi_{u,t}(f_2)\right)\right]$$

$$+ \int_0^t \partial_s\, \mathbb{E}\left[\Delta\phi_{s,t}(m(\xi_s))(f_1)\ \ \Delta\phi_{s,t}(m(\xi_s))(f_2)\right]\, ds$$

$$+ \frac{1}{N^2} \sum_{(k,l)\in\{(1,2),(2,1)\}} \int_0^t \mathbb{E}\left[\phi_{s,t}(m(\xi_s))(f_k)\, \Upsilon_{m(\xi_{s-})}^3 \phi_{s,t}(f_l)\right]\, ds$$

By (5.4) and using the same lines of arguments as in the proof of theorem 5.10 we check the following estimates.

Corollary 5.12. *For any time horizon $t \geqslant 0$, any $f, g \in \mathrm{Osc}(S)$ and $i \neq j$ we have*

$$(H_1) \quad \Longrightarrow \quad |\mathbb{E}\left(f(\xi_t^i)\, g(\xi_t^j)\right) - \eta_t(f)\, \eta_t(g)| \leqslant c/N$$

$$(H_2) \quad \Longrightarrow \quad |\mathbb{E}\left(f(\xi_t^i)\, g(\xi_t^j)\right) - \eta_t(f)\, \eta_t(g)| \leqslant c\,t/N$$

In addition, when (H_1) is met, for any $F, G \in \mathrm{Osc}(D_t(S))$ we have

$$|\mathbb{E}\left(F(\mathbb{X}_t^i)\, G(\mathbb{X}_t^j)\right) - \mathbb{Q}_t(F)\, \mathbb{Q}_t(G)| \leqslant c\,t/N$$

as well as the almost sure estimates

$$|\mathbb{E}\left(F(\mathbb{Y}_t^i)\, G(\mathbb{Y}_t^j) \mid \mathbb{Y}_t^{\mathcal{I}}\right) - \mathbb{Q}_t(F)\, \mathbb{Q}_t(G)| \leqslant c\,t/N$$

We can extend the above arguments to any finite block of particles.

6 A duality formula

6.1 Many-body Feynman-Kac semigroups

For any given $s \geqslant 0$ and $x \in D_s(S)$ we denote by $T_{k,n}^{s,x}$ an increasing sequence of jump times with intensity $\widehat{\lambda}_{k,t}(\widehat{\mathcal{X}}_{s,t}(x))$, on the time horizon $t \in [s, \infty[$. We use the convention $T_{k,0}^{s,x} = 0$ for $n = 0$.

Definition 6.1. *Let $\widehat{\Pi}_{s,t}$ be the many-body Feynman-Kac semigroup defined for any $F \in \mathcal{B}(D_t(S)^{(2N)})$ and $(x,y) \in D_t(S)^{(2N)}$ by the formula*

$$\widehat{\Pi}_{s,t}(F)(x,y) := \mathbb{E}\left(F\left(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)\right) \mathcal{Z}_{s,t}\left(\mathbb{X}_{s,t}(x)\right)\right) \tag{6.1}$$

We are now in position to state and prove the main result of this section.

Theorem 6.2. *For any $s \leqslant t$, and any $F \in \mathcal{B}(D_t(S)^{(2N)})$, any $(x,y) \in D_s(S)^{(2N)}$ and $n \geqslant 1$ we have the almost sure formula*

$$\mathbb{E}\left(F(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)) \ \mathcal{Z}_{s,t}\left(\mathbb{X}_{s,t}(x)\right) \ 1_{T_n^{s,x} < t < T_{n+1}^{s,x}}\right)$$
$$= \mathbb{E}\left(F(\widehat{\mathcal{X}}_{s,t}(x), \mathbb{Y}_{s,t}(y)) \ Z_{s,t}\left(\mathbb{Y}_{s,t}^{\mathcal{I}}(x)\right) \ 1_{T_{\mathcal{I},n}^{s,x} < t < T_{\mathcal{I},n+1}^{s,x}} \ | \ \mathcal{I}\right) \tag{6.2}$$

with the exponential map $Z_{s,t}$ on $D_t(S)$ defined in (3.2). In addition, we have the conditional transition semigroup formula

$$\widehat{\Pi}_{s,t}(F)(x,y) = \mathbb{E}\left(F(\widehat{\mathcal{X}}_{s,t}(x), \mathbb{Y}_{s,t}(y)) \ Z_{s,t}\left(\mathbb{Y}_{s,t}^{\mathcal{I}}(x)\right) \ | \ \mathcal{I}\right)$$

Proof. By lemma 4.9 we have

$$\left(\widehat{\mathcal{U}}_{s,t}^{(N)}\widehat{\mathcal{J}}_t^{(2)}(F)\right)(x,y)$$

$$= \left(\widehat{\mathcal{U}}_{s,t}^{(N)}\widehat{\mathcal{J}}_{k,t}^{(2)}(F)\right)(x,y) + \frac{1}{N}\sum_{i \in [N]-\{k\}}\mathbb{E}\left(\epsilon'_{k,i}(F)\left(\widehat{\mathcal{X}}_{s,t}(x), \widehat{\mathcal{X}}_{s,t}(y)\right) \ \mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x)\right)^N\right)$$

with the bounded integral operators $\epsilon'_{i,k}$ defined in (4.24). Replacing in the r.h.s. expectation $\widehat{\mathcal{X}}_{s,t}$ by $\widehat{\mathcal{X}}_{s,t}^{\sigma_{i,j}}$ and using the fact that

$$\mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}^{\sigma_{i,k}}(x)\right) = \mathcal{Z}_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x)\right) \quad \text{we conclude that} \quad \widehat{\mathcal{U}}_{s,t}^{(N)}\widehat{\mathcal{J}}_{k,t}^{(2)} = \widehat{\mathcal{U}}_{s,t}^{(N)}\widehat{\mathcal{J}}_t^{(2)}.$$

This implies that for any $n \geqslant 1$ and $k \in [N]$ we have

$$\mathbb{E}\left(F(\widehat{\xi}_{s,t}(x), \mathbb{X}_{s,t}(y)) \ \mathcal{Z}_{s,t}\left(\widehat{\xi}_{s,t}(x)\right) \ 1_{T_n^{s,x} < t < T_{n+1}^{s,x}}\right)$$
$$= \int_{[s,t]_n}\left(\left(\widehat{\mathcal{U}}_{r_0,r_1}^{(N)}\widehat{\mathcal{J}}_{k,r_1}^{(2)}\right)\cdots\left(\widehat{\mathcal{U}}_{r_{n-1},r_n}^{(N)}\widehat{\mathcal{J}}_{k,r_n}^{(2)}\right)\left(\widehat{\mathcal{U}}_{r_n,t}^{(N)}(F)\right)\right)(x) \ dr \tag{6.3}$$

In the above display, $\widehat{\mathcal{U}}_{s,t}^{(N)}$ stands for the Feynman-Kac semigroup defined in (4.27) and $[s,t]_n$ the Weyl chamber introduced in (2.2). The end of proof of (6.2) is rather technical but it follows the same arguments as the ones used in the proof of theorem 4.12, thus it is provided in the appendix, on page 50.

The second assertion is obtained by summing over the number of jumps in the interval $[s,t]$. Alternatively, in terms of generators, using (4.23) and recalling that

$$\widehat{V}_t(\mathbb{X}_{s,t}^k(x)) = V_t(\xi_{s,t}^k(x_s)) = \widehat{V}_t(\widehat{\xi}_{s,t}^k(x))$$

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we check that

$$\partial_t \widehat{\Pi}_{s,t}(F)(x,y)$$

$$= \mathbb{E}\left(\left(\left(\widehat{\mathcal{H}}_{k,t}(F)\left(\widehat{\xi}_{s,t}(x),\mathbb{X}_{s,t}(y)\right) - \widehat{V}_t(\widehat{\xi}^k_{s,t}(x))\ F\left(\widehat{\xi}_{s,t}(x),\mathbb{X}_{s,t}(y)\right)\right)\ \mathcal{Z}_{s,t}\left(\widehat{\xi}_{s,t}(x)\right)\right)\right)$$

for any $k \in [N]$. This implies that

$$\partial_t \widehat{\Pi}_{s,t}(F) = \widehat{\Pi}_{s,t}\left(\widehat{\mathcal{H}}^V_{k,t}(F)\right) \quad \text{with} \quad \widehat{\mathcal{H}}^V_{k,t}(F)(x,y) := \widehat{\mathcal{H}}_{k,t}(F)\,(x,y) - \widehat{V}_t(x^k)\ F\,(x,y)$$

Consider the semigroup

$$\widehat{\Pi}^{(\mathcal{I})}_{s,t}(F)(x,y) \quad := \quad \mathbb{E}\left(F(\widehat{\mathcal{X}}_{s,t}(x),\mathbb{Y}_{s,t}(y))\ Z_{s,t}\left(\mathbb{Y}^{\mathcal{I}}_{s,t}(x)\right)\ |\ \mathcal{I}\right)$$

Arguing as above and recalling that

$$\widehat{V}_t(\widehat{\mathcal{X}}^k_{s,t}(x)) = V_t\left(\zeta^k_{s,t}(x_s)\right) = \widehat{V}_t\left(\mathbb{Y}^k_{s,t}(x)\right) \quad \text{we check that} \quad \partial_t\,\widehat{\Pi}^{(\mathcal{I})}_{s,t}(F) = \widehat{\Pi}^{(\mathcal{I})}_{s,t}\left(\widehat{\mathcal{H}}^V_{\mathcal{I},t}(F)\right).$$

This ends the proof of the theorem. ∎

Next corollary is a direct consequence of the above theorem and it extends the duality formula presented in [34] to continuous time Feynman-Kac models.

Corollary 6.3. *For any $t \geqslant 0$ and $F \in \mathcal{B}(D_t(S)^{(2N)})$ we have the almost sure formula*

$$\widehat{\Pi}_t(F) := \mathbb{E}\left(F\left(\widehat{\xi}_t,\mathbb{X}_t\right)\ \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right) = \mathbb{E}\left(F(\widehat{\mathcal{X}}_t,\mathbb{Y}_t)\ \overline{Z}_t\left(\mathbb{Y}^{\mathcal{I}}_t\right)\ |\ \mathcal{I}\ \right)$$

with the function $\overline{\mathcal{Z}}_t$ defined as $\mathcal{Z}_t$ by replacing V_t by the normalized potential $\overline{V}_t$ defined in (3.4). This yields for any $F \in \mathcal{B}(D_t(S)^{N+1})$ the duality formula

$$\Pi_t(F) := \mathbb{E}\left(F\left(\mathbb{X}_t,\mathbb{X}^{\mathcal{I}}_t\right)\ \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right) = \mathbb{E}\left(F(\mathbb{Y}_t,\mathbb{Y}^{\mathcal{I}})\ \overline{Z}_t\left(\mathbb{Y}^{\mathcal{I}}_t\right)\right) \tag{6.4}$$

6.2 Particle Gibbs samplers

For any given time horizon $t \geqslant 0$, the probability measure introduced in (6.4) takes the form

$$\Pi_t(d(z_1,z_2)) \in \mathcal{P}(E_1 \times E_2) \quad \text{with} \quad E_1 = D_t(S)^N \quad \text{and} \quad E_2 := D_t(S)$$

The first marginal of the measure Π_t is given for any $F \in \mathcal{B}(D_t(S)^N)$ by the formula

$$\pi_t(F) := \mathbb{E}\left(F\left(\mathbb{X}_t\right)\ \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right)$$

On the other hand, by corollary 6.3, for any $k \in [N]$ and $F \in \mathcal{B}(\mathcal{D}_t(S))$ we have

$$\mathbb{E}\left(F\left(\mathbb{X}^k_t\right)\ \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right) = \mathbb{E}\left(F(\mathbb{Y}^k_t)\ \overline{Z}_t\left(\mathbb{Y}^k_t\right)\ |\ \mathcal{I} = k\ \right) = \mathbb{Q}_t(F)$$

In addition, for any $k,l \in [N]$ we have

$$\mathbb{Q}_t(F) = \mathbb{E}\left(F\left(\mathbb{X}^k_t\right)\ \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right) = \mathbb{E}\left(F(\mathbb{Y}^k_t)\ \overline{Z}_t\left(\mathbb{Y}^l_t\right)\ |\ \mathcal{I} = l\ \right)$$

Thus, under $\widehat{\Pi}_t$ all the ancestral lines in $\mathbb{Y}_t$ are distributed according to $\mathbb{Q}_t$. In addition, the second marginal of the measure Π_t is given for any $F \in \mathcal{B}(D_t(S))$ by the formula

$$\mathbb{Q}_t(F) = \mathbb{E}\left(F(\mathbb{Y}^{\mathcal{I}}) \, \overline{Z}_t\left(\mathbb{Y}_t^{\mathcal{I}}\right)\right) = \mathbb{E}\left(F\left(\mathbb{X}_t^{\mathcal{I}}\right) \, \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right) = \mathbb{E}\left(m\left(\mathbb{X}_t\right)(F) \, \overline{\mathcal{Z}}_t\left(\mathbb{X}_t\right)\right)$$

This yields the duality formula stated in (1.6) and in corollary 6.3.

The transition of the Gibbs-sampler with target measure Π_t on $E := (E_1 \times E_2)$ is defined by

$$\mathbb{S}_t((z_1, z_2), d(\overline{z}_1, \overline{z}_2)) := \mathbb{B}_t(z_2, d\overline{z}_1) \, \mathbb{A}_t(\overline{z}_1, d\overline{z}_2) \tag{6.5}$$

This transition is summarized in the following synthetic diagram

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{z}_1 & \sim & (\mathbb{Y}_t \mid \mathbb{Y}_t^{\mathcal{I}} = z_2) \\ z_2 & \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{z}_1 & \\ \overline{z}_2 & \sim & m(\overline{z}_1) \end{pmatrix}$$

By construction, we have the duality property

$$\Pi_t(d(z_1, z_2)) \, \mathbb{S}_t((z_1, z_2), d(\overline{z}_1, \overline{z}_2)) = \Pi_t(d(\overline{z}_1, \overline{z}_2)) \, \mathbb{S}_t^-((\overline{z}_1, \overline{z}_2), d(z_1, z_2)) \tag{6.6}$$

with the backward transition

$$\mathbb{S}_t^-((\overline{z}_1, \overline{z}_2), d(z_1, z_2)) = \mathbb{A}_t(\overline{z}_1, dz_2) \, \mathbb{B}_t(z_2, dz_1)$$

Integrating (6.6) w.r.t. $\overline{z}_1$ we also have the reversibility property

$$\mathbb{Q}_t(dz_2) \, \mathbb{K}_t(z_2, d\overline{z}_2) = \mathbb{Q}_t(d\overline{z}_2) \, \mathbb{K}_t(\overline{z}_2, dz_2)$$

with the Markov transition $\mathbb{K}_t = \mathbb{B}_t \circ \mathbb{A}_t$ from $D_t(S)$ into itself defined for any $F \in \mathcal{B}(D_t(S))$ and $z \in D_t(S)$ by

$$\mathbb{K}_t(F)(z) := \int \mathbb{K}_t(z, d\overline{z}) \, F(\overline{z}) = \mathbb{E}\left(m(\mathbb{Y}_t)(F) \mid \mathbb{Y}_t^{\mathcal{I}} = z\right)$$

We further assume that the Markov transitions of X_t satisfy condition (H_0) and thus (H_1). In this situation, using corollary 5.11, for any time horizon $t \geqslant 0$, any function $F \in \mathrm{Osc}(D_t(S))$ and and $n \geqslant 1$ we check that

$$\|\mathbb{K}_t(F) - \mathbb{Q}_t(F)\| \leqslant c \, (t \vee 1)/N$$

for some finite constant c whose value doesn't depend on the parameters (F, t, N). This implies that

$$\mathrm{osc}(\mathbb{K}_t(F)) \leqslant c \, (t \vee 1)/N \, \mathrm{osc}(F) \quad \text{and} \quad \beta_{\mathrm{dob}}(\mathbb{K}_t) \leqslant c \, (t \vee 1)/N$$

For any $\mu \in \mathcal{P}(D_t(S))$, we conclude that

$$\|\mu \mathbb{K}_t^n - \mathbb{Q}_t\|_{\mathrm{tv}} \leqslant (c \, (t \vee 1)/N)^n \times \|\mu - \mathbb{Q}_t\|_{\mathrm{tv}}.$$

The last assertion is a direct consequence of the contraction inequalities (2.5).

Acknowledgements

Appendix

Proof of (3.13)

Observe that

$$\mathbb{M}_t(F) = \eta_0 Q_{0,t}(f) + \sum_{n \geq 1} \int_{0 \leq s_1 < \ldots < s_n \leq t} \left[\prod_{0 \leq p < n} \eta_{s_p} Q_{s_p, s_{p+1}}(V_{s_{p+1}}) \right] (\eta_{s_n} Q_{s_n, t})(f) \, ds_1 \ldots ds_n$$

Recalling that

$$\eta_s Q_{s,t}(f) = \frac{\gamma_s Q_{s,t}(f)}{\gamma_s(1)} = \frac{\gamma_t(f)}{\gamma_s(1)} = \frac{\gamma_t(1)}{\gamma_s(1)} \times \eta_t(f)$$

we find that

$$\begin{aligned}
\mathbb{M}_t(F) &= \gamma_t(f) + \sum_{n \geq 1} \int_{0 \leq s_1 < \ldots < s_n \leq t} \left[\prod_{0 \leq p < n} \eta_{s_{p+1}}(V_{s_{p+1}}) \times \frac{\gamma_{s_{p+1}}(1)}{\gamma_{s_p}(1)} \right] \frac{\gamma_t(f)}{\gamma_{s_n}(1)} \, ds_1 \ldots ds_n \\
&= \gamma_t(f) \left(1 + \sum_{n \geq 1} \int_{0 \leq s_1 < \ldots < s_n \leq t} \left[\prod_{0 \leq p < n} \eta_{s_{p+1}}(V_{s_{p+1}}) \right] ds_1 \ldots ds_n \right)
\end{aligned}$$

This ends the proof of (3.13). ∎

Proof of lemma 5.1

Let $X_{s,t}(x)$, with $t \geq s$, be the stochastic flow associated with the generator L_t starting at $X_{s,s}(x) = x$ at time $t = s$. In this notation, we have the perturbation formula

$$P_{s,t}^\epsilon(f)(x)$$

$$= \mathbb{E}\left[f(X_{s,t}(x)) \, e^{-\epsilon \int_s^t \lambda(X_{s,u}(x)) du} \right] + \int_s^t \mathbb{E}\left[\epsilon \lambda(X_{s,u}(x)) \, e^{-\epsilon \int_s^u \lambda(X_{s,v}(x)) dv} K_u(P_{u,t}^\epsilon(f))(X_{s,u}(x)) \right] du$$

For non negative functions f and any $t \geq 0$ and $h > 0$ we have

$$\begin{aligned}
P_{t,t+h}^\epsilon(f) &\leq e^{-\epsilon \lambda_1 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_1 (u-t)} \, \kappa_u P_{u,t+h}^\epsilon(f) \, du \\
&\leq e^{\epsilon(\lambda_2 - \lambda_1)h} \left[e^{-\epsilon \lambda_2 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \, \kappa_u P_{u,t+h}^\epsilon(f) \, du \right]
\end{aligned}$$

In the same vein, we have

$$P_{t,t+h}^\epsilon(f) \geq (\lambda_1/\lambda_2)(\varpi_1/\varpi_2) \left[e^{-\epsilon \lambda_2 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \, \kappa_u P_{u,t+h}^\epsilon(f) \, du \right]$$

This shows that

$$\rho_\epsilon(h) \leq \frac{d(\delta_x P_{t,t+h}^\epsilon)}{d\mu_{t,h}^\epsilon}(y) \leq \rho_\epsilon(h)^{-1}$$

with the probability measure

$$\mu_{t,h}^\epsilon \propto e^{-\epsilon \lambda_2 h} \mu_{t,h} + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \, \kappa_u P_{u,t+h}^\epsilon(f) \, du$$

This ends the proof of the lemma. ∎

Proof of lemma 5.6

For any functions $f_i \in \mathrm{Osc}(S)$ and any $l \leqslant k$ we have

$$\left| \mathbb{E}\left[\prod_{1 \leqslant l \leqslant k} \Delta m(\xi_t)(f_l) \mid \mathcal{F}_{t-} \right] \right| \leqslant \frac{1}{N^k} \left[\sum_{i \in [N]} \left(V_t(\xi_t^i) + \lambda_t(\xi_t^i) \right) \right] \leqslant \frac{1}{N^{k-1}} \, \|\lambda + V\| dt$$

$$\implies N^k \, |\Upsilon_{m(\xi_{s-})}^{k+1} \phi_{s,t}(f)| = \frac{N^k}{(k+1)!} \, \left| \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes(k+1)} \, \overline{\partial}_{m(\xi_{s-})}^{(k+1)} \phi_{s,t}(f) \mid \mathcal{F}_{s-} \right] \right|$$

$$= N^k \, \left| \partial_s \mathbb{E}\left[\frac{1}{m(\xi_s)Q_{s,t}^{m(\xi_{s-})}(1)} \, \left(\Delta m(\xi) \left(Q_{s,t}^{m(\xi_{s-})}(1) \right) \right)^k \, \Delta m(\xi) \partial_{m(\xi_{s-})} \phi_{s,t}(f) \mid \mathcal{F}_{s-} \right] \right|$$

$$\leqslant \, e^{(2+k)q} \, \|\lambda + V\|$$

Proof of theorem 5.10

We use (2.13) to check that

$$m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right)$$

$$= (\eta_s Q_{s,t}^{m(\xi_s)}(1))^2 \, m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1), \partial_{\eta_s} \phi_{s,t}(f) \right)$$

$$+ (\eta_s Q_{s,t}^{m(\xi_s)}(1))^2 \, \left[\phi_{s,t}(\eta_s) - \phi_{s,t}(m(\xi_s)) \right](f) \, m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1) \right)$$

Using (2.7) we also have the estimate

$$\left| m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \right| \leqslant e^{3q} \, \mathrm{osc}(\overline{Q}_{s,t}(f)) \, m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1) \right)$$

$$+ e^{2q} \, \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1) \right)} \, \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}} \left(\partial_{\eta_s} \phi_{s,t}(f) \right)} \tag{6.7}$$

On the other hand, we have

$$\partial_{\eta_s} \phi_{s,t}(f) = Q_{s,t}^{\eta_s}[f - \eta_t(f)] \quad \text{and} \quad Q_{s,t}^{\eta_s}(f)(x) = \mathbb{E}\left(f(X_t) \, e^{-\int_s^t \overline{V}_u(X_u) \, du} \mid X_s = x \right)$$

$$\implies \partial_s Q_{s,t}^{\eta_s}(f) = -L_s(Q_{s,t}^{\eta_s}(f)) + \overline{V}_s \, Q_{s,t}^{\eta_s}(f)$$

$$\implies \partial_s(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) = -Q_{s,t}^{\eta_s}(f) \, L_s(Q_{s,t}^{\eta_s}(g)) - Q_{s,t}^{\eta_s}(g) \, L_s(Q_{s,t}^{\eta_s}(f)) + 2\overline{V}_s \, Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)$$

We also have

$$L_{t,\mu}(f) = L_t(f) + V_t \, [\mu(f) - f] \Longleftrightarrow L_t(f) - L_{t,\mu}(f) = V_t \, [f - \mu(f)]$$

This yields the formula

$$\begin{aligned} \eta\Gamma_{L_{t,\eta}}(f,g) - \eta\Gamma_{L_t}(f,g) &= \int \eta(dx) \, \eta(dy) \, V_t(y) \, [f(y) - f(x)][g(y) - g(x)] \\ &= \eta(V_t(fg)) + \eta(V_t) \, \eta(fg) - \eta(fV_t) \, \eta(g) - \eta(gV_t) \, \eta(f) \end{aligned}$$

For any given time horizon t and $s \in [0,t]$ we have

$$dm(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}} \, dM_s(Q_{\cdot,t}^{\eta}(f)Q_{\cdot,t}^{\eta}(g))$$

$$= m(\xi_s) \left[L_{s,m(\xi_s)}(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - Q_{s,t}^{\eta_s}(f) \, L_s(Q_{s,t}^{\eta_s}(g)) \right.$$

$$\left. - Q_{s,t}^{\eta_s}(g) \, L_s(Q_{s,t}^{\eta_s}(f)) + 2\overline{V}_s \, Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] \, ds$$

This yields

$$dm(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}} \, dM_s(Q_{\cdot,t}^{\eta}(f)Q_{\cdot,t}^{\eta}(g))$$

$$= m(\xi_s) \left[\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) + V_s \, Q_{s,t}^{\eta_s}(f) \ (m(\xi_s)Q_{s,t}^{\eta_s}(g) - Q_{s,t}^{\eta_s}(g)) \right.$$

$$\left. + V_t \, Q_{s,t}^{\eta_s}(g) \ (m(\xi_s)Q_{s,t}^{\eta_s}(f) - Q_{s,t}^{\eta_s}(f)) + 2\overline{V}_s \, Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] \, ds$$

from which we check that

$$m(\xi_t)(fg) - m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}} \, M_t(Q_{\cdot,t}^{\eta}(f)Q_{\cdot,t}^{\eta}(g))$$

$$= \int_0^t m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) \, ds$$

$$+ \int_0^t m(\xi_s) \left[V_s \, Q_{s,t}^{\eta_s}(f) \ (m(\xi_s)Q_{s,t}^{\eta_s}(g) - Q_{s,t}^{\eta_s}(g)) \right.$$

$$\left. + V_s \, Q_{s,t}^{\eta_s}(g) \ (m(\xi_s)Q_{s,t}^{\eta_s}(f) - Q_{s,t}^{\eta_s}(f)) + 2(V_s - \eta_s(V_s)) \, Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] \, ds$$

After some simplifications we check that

$$\int_0^t m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) \, ds$$

$$= m(\xi_t)(fg) - m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}} \, M_t(Q_{\cdot,t}^{\eta}(f)Q_{\cdot,t}^{\eta}(g))$$

$$+ \int_0^t \left[2\eta_s(V_s) \, m(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) \right.$$

$$\left. - m(\xi_s)(V_s \, Q_{s,t}^{\eta_s}(f)) \ m(\xi_s)Q_{s,t}^{\eta_s}(g) - m(\xi_s)(V_s \, Q_{s,t}^{\eta_s}(g)) \ m(\xi_s)Q_{s,t}^{\eta_s}(f) \right] \, ds$$

Choosing $f = g = 1$ and taking the expectations we find that

$$\int_0^t \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(1)) \right] \, ds$$

$$= 1 - \eta_0(Q_{0,t}^{\eta_0}(1)^2) + 2\int_0^t \mathbb{E}\left[\eta_s(V_s) \, m(\xi_s)(Q_{s,t}^{\eta_s}(1)^2) - m(\xi_s)(V_s \, Q_{s,t}^{\eta_s}(1)) \ m(\xi_s)Q_{s,t}^{\eta_s}(1) \right] \, ds$$

$$\leqslant 1 + 2e^{2q} \, \|V\| \, t$$

Choosing $f = g = h - \eta_t(h)$, with $h \in \mathrm{Osc}(S)$ and taking the expectations we find that

$$\int_0^t \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(\partial_{\eta_s}\phi_{s,t}(h)) \right] ds$$

$$= \mathbb{E}\left[m(\xi_t)([h - \eta_t(h)]^2) \right] - \eta_0([\partial_{\eta_0}\phi_{0,t}(h)]^2)$$

$$+2\int_0^t \mathbb{E}\left[\eta_s(V_s)\, m(\xi_s)([\partial_{\eta_s}\phi_{s,t}(h)]^2) - m(\xi_s)(V_s\, \partial_{\eta_s}\phi_{s,t}(h))\; m(\xi_s)(\partial_{\eta_s}\phi_{s,t}(h)) \right] ds$$

$$\leqslant 1 + 4e^{2q}\, \|V\|\, t$$

For any $f \in \mathrm{Osc}(S)$ combining (6.7) with Cauchy-Schwartz inequality we find that that

$$\int_0^t \left| m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f) \right) \right| ds \leqslant 2e^{3q}\left[1 + 4e^{2q}\,\|V\| \right]\, t$$

Combining the above estimate with (5.7) and corollary 5.9 whenever (H_2) is met we conclude that

$$|\mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f))| \leqslant 2e^{3q}\left(1 + 4e^{2q}\,\|V\| + \frac{1}{N^2}\,2e^{q}\,\|\lambda + V\| \right)\, t/N$$

We further assume that (H_1) is satisfied. In this case, using (6.7) we also have

$$\left| m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f) \right) \right| \leqslant e^{3q}\,\alpha\, e^{-\beta(t-s)}\, m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1) \right)$$

$$+ e^{2q}\, \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1) \right)}\, \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(\partial_{\eta_s}\phi_{s,t}(f) \right)}$$

For any $\widetilde{\beta} \in \mathbb{R}$ we set

$$\widetilde{Q}_{s,t}^{\eta_s}(f)(x) := e^{\widetilde{\beta}(t-s)}\, Q_{s,t}^{\eta_s}(f)(x)$$

$$= \mathbb{E}\left(f(X_t)\, e^{-\int_s^t \widetilde{V}_u(X_u)\, du} \mid X_s = x \right) \quad \text{with} \quad \widetilde{V}_t(x) = \overline{V}_t(x) - \widetilde{\beta}$$

Arguing as above, we have

$$\partial_s(\widetilde{Q}_{s,t}^{\eta_s}(f)\widetilde{Q}_{s,t}^{\eta_s}(g)) = -\widetilde{Q}_{s,t}^{\eta_s}(f)\, L_s(\widetilde{Q}_{s,t}^{\eta_s}(g)) - \widetilde{Q}_{s,t}^{\eta_s}(g)\, L_s(\widetilde{Q}_{s,t}^{\eta_s}(f)) + 2\, \widetilde{V}_s\, \widetilde{Q}_{s,t}^{\eta_s}(f)\, \widetilde{Q}_{s,t}^{\eta_s}(g)$$

and

$$dm(\xi_s)(\widetilde{Q}_{s,t}^{\eta_s}(f)\widetilde{Q}_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}}\, dM_s(\widetilde{Q}_{\cdot,t}^{\eta}(f)\widetilde{Q}_{\cdot,t}^{\eta}(g))$$

$$= m(\xi_s)\left[\Gamma_{s,L_{s,m(\xi_s)}}(\widetilde{Q}_{s,t}^{\eta_s}(f), \widetilde{Q}_{s,t}^{\eta_s}(g)) + V_s\, \widetilde{Q}_{s,t}^{\eta_s}(f)\; (m(\xi_s)\widetilde{Q}_{s,t}^{\eta_s}(g) - \widetilde{Q}_{s,t}^{\eta_s}(g)) \right.$$

$$\left. +V_t\, \widetilde{Q}_{s,t}^{\eta_s}(g)\; (m(\xi_s)\widetilde{Q}_{s,t}^{\eta_s}(f) - \widetilde{Q}_{s,t}^{\eta_s}(f)) + 2\widetilde{V}_s\, \widetilde{Q}_{s,t}^{\eta_s}(f)\, \widetilde{Q}_{s,t}^{\eta_s}(g) \right] ds$$

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This implies that

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\, m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g))\, ds$$

$$= m(\xi_t)(fg) - e^{\widetilde{\beta}t}\, m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}}\, M_t(\widetilde{Q}_{\cdot,t}^{\eta}(f)\widetilde{Q}_{\cdot,t}^{\eta}(g))$$

$$+ \int_0^t e^{2\widetilde{\beta}(t-s)}\, \Big[2(\eta_s(V_s) + \widetilde{\beta})\, m(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - m(\xi_s)(V_s\, Q_{s,t}^{\eta_s}(f))\, m(\xi_s)Q_{s,t}^{\eta_s}(g)$$

$$-m(\xi_s)(V_s\, Q_{s,t}^{\eta_s}(g))\, m(\xi_s)Q_{s,t}^{\eta_s}(f)\Big]\, ds$$

Choosing $f = g = 1$ and $\widetilde{\beta} < 0$ we have

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\, \mathbb{E}\Big[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(1))\Big]\, ds$$

$$= 1 - e^{\widetilde{\beta}t}\, \eta_0(Q_{0,t}^{\eta_0}(1)^2)$$

$$+2\int_0^t e^{2\widetilde{\beta}(t-s)}\, \mathbb{E}\Big[(\eta_s(V_s) + \widetilde{\beta})\, m(\xi_s)(Q_{s,t}^{\eta_s}(1)^2) - m(\xi_s)(V_s\, Q_{s,t}^{\eta_s}(1))\, m(\xi_s)Q_{s,t}^{\eta_s}(1)\Big]\, ds$$

$$\leqslant 1 + e^{2q}\, \left(1 + 2|\widetilde{\beta}|^{-1}\|V\|\right) = 1 + e^{2q}\, \left(1 + 4\beta^{-1}\|V\|\right) \quad \text{when} \quad \widetilde{\beta} = -\beta/2$$

Choosing $f = g = [h - \eta_t(h)]$, with $h \in \mathrm{Osc}(S)$ and $0 < \widetilde{\beta} < \beta$ we have

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\, \mathbb{E}\Big[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(\partial_{\eta_s}\phi_{s,t}(h)\Big]\, ds$$

$$\leqslant \mathbb{E}\Big[m(\xi_t)([h - \eta_t(h)]^2)\Big]$$

$$+2\int_0^t e^{2\widetilde{\beta}(t-s)}\, \Big[(\eta_s(V_s) + \widetilde{\beta})\, m(\xi_s)\left([\partial_{\eta_s}\phi_{s,t}(h)]^2\right) - m(\xi_s)(V_s\, \partial_{\eta_s}\phi_{s,t}(h))\, m(\xi_s)\partial_{\eta_s}\phi_{s,t}(h)\Big]\, ds$$

$$\leqslant 1 + 2r^2(2\|V\| + \widetilde{\beta})\int_0^t e^{-2(\beta-\widetilde{\beta})(t-s)}\, ds$$

$$\leqslant 1 + r^2(2\|V\| + \widetilde{\beta})\, (\beta - \widetilde{\beta})^{-1} = 1 + r^2(4\|V\|\beta^{-1} + 1) \quad \text{when} \quad \widetilde{\beta} = \beta/2$$

We end the proof of the theorem using the fact that

$$|m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)|$$

$$\leqslant e^{3q}(1 + \alpha)\, e^{-\beta(t-s)/2}m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right) + e^{2q}\, e^{\beta(t-s)/2}m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(\partial_{\eta_s}\phi_{s,t}(f)\right)$$

In the last assertion we have used the fact that the estimate $\sqrt{ab} \leqslant ca + b/c$, for all $a, b, c > 0$. This ends the proof of the theorem. ∎

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Proof of (6.2)

The proof follows the same lines of arguments as the ones used in the proof of corollary 4.13.

By (4.22) and (4.25), given $\mathcal{I} = k$, the historical flow $\widehat{\mathcal{X}}_{s,t}$, the jump times $T^{s,x}_{k,n} = r_n$ as well as the randomly selected coalescent maps $a_n = c_{\iota_n}$ for some $\iota_n \in [N]_0^2$, for any $r_n \leqslant t < r_{n+1}$ we have

$$\widehat{\chi}_{s,t} = \widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{T}}^{\iota,(k,n)}_{r,r_n} \quad \text{and} \quad \mathbb{Y}_{s,t} = \widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{X}}^{\iota,(k,n)}_{r,r_n} \tag{6.8}$$

with the semigroups $(\widehat{\mathcal{T}}^{\iota,(k,n)}_{r,r_n}, \widehat{\mathcal{X}}^{\iota,(k,n)}_{r,r_n})$ defined as $(\widehat{\mathcal{T}}^{\iota,(n)}_{r,r_n}, \widehat{\mathcal{X}}^{\iota,(n)}_{r,r_n})$ by replacing in (4.30) the maps $(\widehat{\mathcal{T}}^a_{s,t}, \widehat{\mathcal{X}}^a_{s,t})$ by the maps $(\widehat{\mathcal{T}}^{k,a}_{s,t}, \widehat{\mathcal{T}}^{k,a}_{s,t})$ introduced in (2.21). This yields the almost sure formula

$$\int_{[s,t]_n} \left(\left(\widehat{\mathbb{U}}^{(N)}_{r_0,r_1} \widehat{\mathcal{J}}^{(2)}_{\mathcal{I},r_1} \right) \cdots \left(\widehat{\mathbb{U}}^{(N)}_{r_{n-1},r_n} \widehat{\mathcal{J}}^{(2)}_{\mathcal{I},r_n} \right) \left(\widehat{\mathbb{U}}^{(N)}_{r_n,t}(F) \right) \right)(x) \, dr$$

$$= \sum_{\iota \in [N]_0^{(2n)}} \int_{[s,t]_n} F\left(\left(\widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{T}}^{\iota,(\mathcal{I},n)}_{r,r_n} \right)(x), \left(\widehat{\mathcal{X}}_{r_n,t} \circ \widehat{\mathcal{X}}^{\iota,(\mathcal{I},n)}_{r,r_n} \right)(y) \right) \, \mathfrak{q}^{s,x}_{\mathcal{I},n}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$$

with the random measures

$$\mathfrak{q}^{s,x}_{\mathcal{I},n}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$$

$$:= \mathcal{Z}_{r_0,r_1}\left(\widehat{\mathcal{X}}_{r_0,r_1}(x) \right)^N \, \widehat{\lambda}^{i_1,j_1}_{\mathcal{I},r_1}\left(\widehat{\mathcal{X}}_{r_0,r_1}(x) \right)$$

$$\times \mathcal{Z}_{r_1,r_2}\left(\widehat{\mathcal{X}}_{r_1,r_2}\left(\widehat{\mathcal{X}}^{\mathcal{I},c_{\iota_1}}_{r_0,r_1}(x) \right) \right)^N \, \widehat{\lambda}^{i_2,j_2}_{\mathcal{I},r_2}\left(\widehat{\mathcal{X}}_{r_1,r_2}\left(\widehat{\mathcal{X}}^{\mathcal{I},c_{\iota_1}}_{r_0,r_1}(x) \right) \right)$$

$$\times \ldots \times \mathcal{Z}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^{\iota,(\mathcal{I},n-1)}_{r,r_{n-1}}(x) \right) \right)^N \, \widehat{\lambda}^{i_n,j_n}_{\mathcal{I},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^{\iota,(\mathcal{I},n-1)}_{r,r_{n-1}}(x) \right) \right)$$

$$\times \mathcal{Z}_{r_n,t}\left(\widehat{\mathcal{X}}_{r_n,t}\left(\widehat{\mathcal{X}}^{\iota,(n)}_{\mathcal{I},(r,r_n)}(x) \right) \right)^N \, dr$$

On the other hand, using (4.14) we have

$$\mathcal{Z}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^{\iota,(\mathcal{I},n-1)}_{r,r_{n-1}}(x) \right) \right)^N$$

$$= Z_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^k_{s,r_n}(x) \right) \times \mathcal{E}^k_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^{\iota,(\mathcal{I},n-1)}_{r,r_{n-1}}(x) \right) \right)$$

with the stochastic exponential functional

$$\mathcal{E}^k_{s,t}\left(\widehat{\mathcal{X}}_{s,t}(x) \right) := \exp\left(-\int_s^t \widehat{\lambda}_{k,r}\left(\widehat{\mathcal{X}}_{s,r}(x) \right) \, dr \right) \quad \text{with} \quad \widehat{\lambda}_{\mathcal{I},t}(x) = (N-1) \, m(x^{-\mathcal{I}})(\widehat{V}_t)$$

The last assertion comes from the fact that

$$\widehat{\mathcal{X}}^k_{r_{n-1},r_n}\left(\widehat{\mathcal{X}}^{\iota,(\mathcal{I},n-1)}_{r,r_{n-1}}(x) \right) = \widehat{\mathcal{X}}^k_{s,r_n}(x) \quad \text{by (2.22)}$$

We conclude that

$$\mathfrak{q}^{s,x}_{\mathcal{I},n}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r)) = Z_{s,t}\left(\widehat{\mathcal{X}}^k_{s,t}(x) \right) \times \mathfrak{p}^{s,x}_{\mathcal{I},n}(\widehat{\mathcal{X}}_{s,t}, d(\iota,r))$$

with the conditional probability measures of the jump times and coalescent maps

$$\mathfrak{p}_{\mathcal{I},n}^{s,x}(\widehat{\mathcal{X}}_{s,t}, d(\iota, r))$$

$$:= \mathbb{P}\left((T_{\mathcal{I},1}^{s,x}, \ldots, T_{\mathcal{I},n}^{s,x}) \in d(r_1, \ldots, r_n), \quad (a_1, \ldots, a_n) = (c_{\iota_1}, \ldots, c_{\iota_n}), \quad 1_{T_{\mathcal{I},n+1}^{s,x} > t} \mid \widehat{\mathcal{X}}_{s,t}, \ \mathcal{I} \right)$$

We end the proof of (6.2) taking the expectations and using (6.3). ■

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