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Abstract

Single and double inverted pendulum systems subjected to delayed state feedback are analyzed in terms of stabilizability. The maximum (critical) delay that allows a stable closed-loop system is determined via the Multiplicity-Induced-Dominancy (MID) property of the characteristic roots, i.e., the dominant (rightmost) roots are associated with higher multiplicity under certain conditions of the system parameters. Other methods, such as tracking the changes of the D-curves with increasing delay and the Walton-Marshall method are also demonstrated for the example of the single pendulum. For the double inverted pendulum subjected to full state feedback, the number of control gains is four, and application of numerical methods requires therefore high computational effort (i.e., optimization in a four-dimensional space). It is shown that, with the MID-

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based approach, the critical delay and the associated control gains can be determined directly using the characteristic equation and its derivatives.

Keywords: Delayed state feedback, stability, stabilizability, critical delay

1. Introduction

Stabilization of time delay systems is an important task in engineering control technology. Time delays typically arise in feedback systems and are often associated with unstable behaviour. Selection of the control parameters in order to stabilize the closed-loop system is therefore a crucial task. Stabilizability is however limited by the size of the feedback delay [17, 34]. The *critical delay*, τ_{crit}, is the maximum delay in the feedback loop, for which stabilization is still possible. If the delay is less than τ_{crit} then there exists a collection of control gains for which the closed-loop system is stable. If the delay is larger than τ_{crit} then the closed-loop system is unstable for any control gain combinations.

It is known that in some simple case studies, the limit of stabilizability and the critical delay are associated with multiple rightmost roots [13, 27, 4]. The so-called *Multiplicity-Induced-Dominancy* (MID) has proved to be a useful property in determining the critical delay for various classes of delay systems. Application of the MID property to scalar and to second-order equations subjected to delayed proportional feedback was conducted in [5] and in [6, 8], respectively. Extension to the delayed proportional-derivative (PD) feedback was studied in [4, 7] where dominancy is proven using the argument principle. Motivated by the above examples, critical delay of feedback systems may be determined by the analysis of the multiplicity of the rightmost characteristic roots.

In this paper, the MID-based method is applied to a two-degree-of-freedom undamped unstable system, namely, the double inverted pendulum, subjected to delayed PD feedback. First, the method is demonstrated for the benchmark problem of the inverted pendulum with delayed PD feedback, where a closed-form expression of the critical delay is established. Then, the critical delay is determined for the double inverted pendulum with two system parameters (length of the pendulums) and four control gains (associated with the angular position and angular velocity of both pendulum segments).

2. Stabilizability as function of feedback delay

There are several analytical and numerical methods that analyze the stability and stabilizability of time delay systems. These allow to determine the regions of control gains and feedback delay for which the system may be stabilized. In the following subsections, two particular methods are briefly presented in the framework of the characteristic equation

$$D(s,\tau) = A(s) + B(s)\mathrm{e}^{-s\tau} = 0, \tag{1}$$

where $\deg(A) > \deg(B)$. In the case of a mechanical system subjected to delayed feedback control, polynomial A expresses the behavior of the plant, while B corresponds to the controller. Indeed, the coefficients of A typically represent mechanical properties of the system (mass, mass moment of inertia, center of gravity), and, therefore are considered to be fixed, while the coefficients of B are associated with the control parameters to be tuned (control gains). The goal is to determine the values of the delay τ for which (1) has all its roots in the left-half plane.

2.1. Walton-Marshall method

The Walton-Marshall (WM) method [32] is a numerical tool that approximates the critical delay for fixed control and system parameters. The main steps of the method are the following. First, stability is determined for $\tau = 0$. Then, the domain of the delay for which the system remains stable is established. If a root $s = \mathrm{i}\omega$ satisfies $D(s,\tau) = 0$ for a certain τ, then $D(\bar{s},\tau) = 0$ is also satisfied ($\bar{s} = -\mathrm{i}\omega$ is the complex conjugate of s). Consequently, the problem of finding roots on the imaginary axis reduces to solving the system of equations

$$\begin{cases} A(s) + B(s)\mathrm{e}^{-s\tau} = 0 \\ A(-s) + B(-s)\mathrm{e}^{+s\tau} = 0 \end{cases}. \tag{2}$$

Eliminating the exponential term yields

$$A(s)A(-s) - B(s)B(-s) = 0. \tag{3}$$

Substituting $s = \mathrm{i}\omega$ into (3), one obtains a polynomial in ω^2:

$$W(\omega^2) = A(\mathrm{i}\omega)A(-\mathrm{i}\omega) - B(\mathrm{i}\omega)B(-\mathrm{i}\omega) = 0. \tag{4}$$

Imaginary roots correspond to positive ω^2 solutions of (4). If there are no such roots, then the system is stable/unstable independently of the delay. Otherwise, candidates for the critical delay are obtained after substituting the corresponding root $s = \mathrm{i}\omega$ into (1), namely

$$\tan \omega\tau = \frac{\mathrm{Im}\left\{\frac{A(\mathrm{i}\omega)}{B(\mathrm{i}\omega)}\right\}}{\mathrm{Re}\left\{\frac{-A(\mathrm{i}\omega)}{B(\mathrm{i}\omega)}\right\}}. \tag{5}$$

The next move consists in investigating whether the corresponding root $s = \mathrm{i}\omega$ crosses the imaginary axis from left to right as τ increases. Indeed, in [32], the root crossing direction is determined by the properties of W. If a root ω crosses from left to right, then W changes sign from positive to negative, i.e., $W'(\omega^2) < 0$. The maximum delay is determined by investigating all the crossing roots and their corresponding crossing directions.

2.2. Multiplicity-Induced-Dominancy

It is shown in [8, 7], that the admissible multiplicity of the zero spectral value for (1) is bounded by the *Polya and Szegő bound* denoted by PB_S, which is the degree of the quasipolynomial (i.e., the sum of the polynomials' degrees plus the number of delays). The critical delay and the corresponding critical control gains are associated with the rightmost root $s = 0$ achieving the maximal admissible multiplicity. While multiplicity can be investigated by the vanishing of the successive derivatives of (1) with respect to s, the dominancy of the root $s = 0$ requires further inquiry in most cases. This latter property is demonstrated in the next sections for both the single and the double inverted pendulum.

3. Motivation: single inverted pendulum

Stabilization of an inverted pendulum with delayed feedback is a benchmark problem in control engineering literature [2, 28, 26, 33] as well as in understanding human balancing and human motor control [18, 21, 20, 23]. As a matter of fact, many control concepts are often implemented in simple inverted pendulum systems, see, e.g., [22, 12, 26, 11]. As mentioned in the Introduction, stabilization by a delayed state feedback is not possible if the delay is larger than a critical value [30]. In this section, the stabilization of the single inverted pendulum is presented briefly, and the relation between the multiplicity of the rightmost characteristic roots and the critical delay is demonstrated. The critical delay is determined by three different techniques: 1) by analytical derivations; 2) by the MID approach; and 3) by the WM method.

3.1. Mechanical model

The mechanical model of the pendulum-cart system is shown in Fig. 1. The mass of the cart is assumed to be negligible compared to the mass of the pendulum. The linearized equation governing the motion of the stick is

$$\ddot{\varphi}(t) - \frac{mgl}{2I}\varphi(t) = \frac{l}{2I}F(t), \tag{6}$$

where φ is the angular position of the stick, m, l and $I = 1/12ml^2$ are the mass, length and the mass moment of inertia of the stick, respectively, g is the gravitational acceleration and F is the control force. In case of delayed PD feedback of the angular position, the control force may be modeled as

$$F(t) = k_{\mathrm{p}}\varphi(t-\tau) + k_{\mathrm{d}}\dot{\varphi}(t-\tau), \tag{7}$$

where k_{p} and k_{d} are the proportional and the derivative control gains, respectively, and τ is the feedback delay.

Hence, the characteristic function generated by (6) and (7) reads

$$D(s) = s^2 + a_0 + (b_0 + b_1 s)\mathrm{e}^{-s\tau}, \tag{8}$$

where

$$a_0 = -\frac{mgl}{2I}, \quad b_0 = \frac{k_{\mathrm{p}}l}{2I}, \quad b_1 = \frac{k_{\mathrm{d}}l}{2I}. \tag{9}$$

It can be seen that the coefficient a_0 involves only the plant parameters (m, I, l), while b_0 and b_1 depend on the control parameters k_{p} and k_{d}, too.

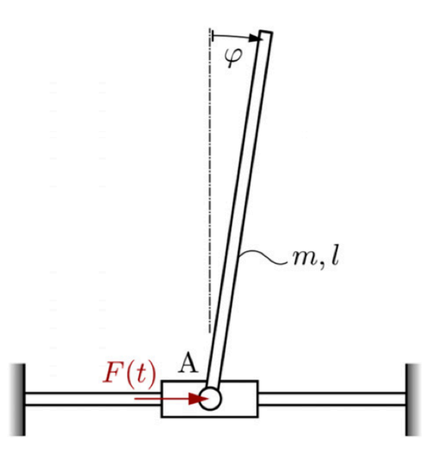

Figure 1: Mechanical model of the pendulum-cart system.

3.2. Analytical stabilizability analysis

Substituting $s = \pm \mathrm{i}\omega (\omega \geq 0)$ into the characteristic equation (8) and decomposing into real and imaginary parts yields the D-curves in the implicit form

$$a_0 - \omega^2 + b_0 \cos(\tau\omega) + b_1 \omega \sin(\tau\omega) = 0, \tag{10}$$

$$b_1 \omega \cos(\tau\omega) - b_0 \sin(\tau\omega) = 0. \tag{11}$$

If $\omega = 0$ then (10) gives $b_0 = -a_0$, which is a so-called real root boundary [1]. If $\omega > 0$, then (10) and (11) give the parametric curve

$$b_0 = (\omega^2 - a_0) \cos(\omega\tau), \tag{12}$$

$$b_1 = \frac{\omega^2 - a_0}{\omega} \sin(\omega\tau), \tag{13}$$

which is called complex root boundary [1]. The D-curves split the plane (b_0, b_1) into an infinite number of domains, where each domain exhibits a constant number of unstable characteristic roots. Stability can be attained by computing the number of unstable

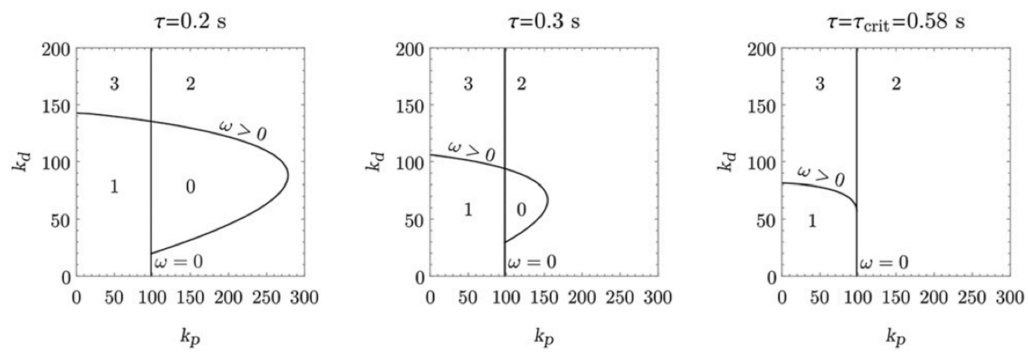

Figure 2: D-curves and the number of unstable characteristic roots of (8) for different delays. The stable region disappears when $\tau = \tau_{\rm crit}$ and stabilization is not possible if $\tau > \tau_{\rm crit}$.

characteristic roots in each individual domain and/or by checking the root tendency (root-crossing direction) along the D-curves [29, 24, 19]. Some sample stability diagrams for different feedback delays are depicted in Fig. 2 for the set of parameters in Table 1.

As observed in Fig. 2, the stable domain shrinks with increasing feedback delay, and completely vanished at a critical value $\tau_{\rm crit}$, when the tangent of the parametric curve (37)-(38) is vertical. This happens at the limit

$$\lim_{\omega\to 0}\frac{{\rm d}k_{\rm p}}{{\rm d}k_{\rm d}} = \lim_{\omega\to 0}\frac{\frac{{\rm d}k_{\rm p}}{{\rm d}\omega}}{\frac{{\rm d}k_{\rm d}}{{\rm d}\omega}} = \frac{3a_0\tau^2+6}{a_0\tau^3+6\tau} = 0, \tag{14}$$

from which we infer, the critical delay $\tau_{\rm crit} = \sqrt{-2/a_0}$, i.e., $\tau_{\rm crit} = 0.58$ s for the set of parameters in Table 1.

Table 1: Mechanical parameters of the inverted single pendulum

Parameter	Notation	Value
Mass	m	10 kg
Length	l	10 m
Mass moment of inertia	I	$83.33\ \mathrm{kgm}^2$

3.3. MID-based stabilizing controller design

The critical delay may also be recovered using the MID property. The following result is a direct consequence of Theorem 4.2 from [9]. It provides a bound for the quasipolynomial roots' multiplicity. In addition, it explicitly computes the stabilizing MID-based controller's gains and delay.

Proposition 1 *Considering equation* (8)*, the following assertions hold.*

i) The multiplicity of any given root of the quasipolynomial function (8) *is bounded by* 3*.*

ii) For an arbitrary positive delay τ, the quasipolynomial (8) *admits a real spectral value at $s = s_\pm$ with algebraic multiplicity* 3 *if and only if,*

$$s_\pm = \frac{-2 \pm \sqrt{2 - \tau^2 a_0}}{\tau}, \tag{15}$$

and the system parameters satisfy:

$$\begin{cases} b_0 = \left(2\,a_0 + \dfrac{10\,s_\pm}{\tau} + \dfrac{6}{\tau^2}\right) \mathrm{e}^{s_\pm \tau}, \\ b_1 = \left(2\,s_\pm + \dfrac{2}{\tau}\right) \mathrm{e}^{s_\pm \tau}. \end{cases} \tag{$\star_\pm$}$$

iii) If $(\star_+)$ (respectively $(\star_-)$) is satisfied then $s = s_+$ is the spectral abscissa corresponding to (8) *(respectively s_- cannot be the spectral abscissa corresponding to* (8)*). Furthermore, for an arbitrary delay τ the multiple spectral value at s_- is always dominated by a single real root s_0.*

iv) If $(\star_+)$ is satisfied then the trivial solution is asymptotically stable if, and only if, $\tau \in \left]0,\ \sqrt{-\frac{2}{a_0}}\right[$.

Thanks to the above result which is proved using the principle argument in [9], one exploits $(\star_+)$ to assert that the maximal admissible multiplicity of the rightmost root at $s = 0$ of (8) is 3. Next, (9) provides the critical feedback delay and the associated control gains

$$\tau_{\text{crit}} = \sqrt{-\frac{2}{a_0}}, \tag{16}$$

$$k_{\text{p,crit}} = -\frac{2I}{l} a_0, \tag{17}$$

$$k_{\text{d,crit}} = -\frac{2I}{l} a_0 \tau_{\text{crit}}. \tag{18}$$

The corresponding numerical results are $\tau_{\text{crit}} = 0.58$ s, $k_{\text{p,crit}} = 98.1$ N/rad, $k_{\text{d,crit}} = 57.2$ Ns/rad. An alternative proof of the dominancy of the triple root at $s = 0$ may be deduced following [5]. As a matter of fact, characteristic function may be rewritten as follows

$$\begin{aligned} D(s) = s^2 + a_0 + e^{-s\tau_1}(-a_0 - a_0\tau_1 s) \\ = s^2 + a_0 - a_0 e^{-s\tau_1}(1 + \tau_1 s) \\ = s^2 \left(1 + \int_0^1 a_0 \tau_1^2 t e^{-s\tau_1 t} \mathrm{d}t\right) \\ = s^2 \left(1 - \int_0^1 2t e^{-s\tau_1 t} \mathrm{d}t\right). \end{aligned} \tag{19}$$

In order to prove that there exists no root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ of (19) such that $\gamma_1 > 0$, substitute s_1 into (19) which leads to

$$\begin{aligned} 1 = \left|\int_0^1 2t e^{-s_1\tau_1 t} \mathrm{d}t\right| \leq \int_0^1 \left|2t e^{-s_1\tau_1 t}\right| \mathrm{d}t \\ = \int_0^1 2t e^{-\gamma_1\tau_1 t} \mathrm{d}t =: f(\gamma_1). \end{aligned} \tag{20}$$

Observing that

$$f(\gamma_1 = 0) = \int_0^1 2t \, \mathrm{d}t = 1, \tag{21}$$

it follows that for $\gamma_1 > 0$ the value of the integral is $f(\gamma_1) < 1$ which proves the inconsistency of the hypothesis that the characteristic function (19) has an unstable root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > 0$. Hence, no characteristic roots exist with positive real part, thus, the triple root $s = 0$ is indeed the dominant root.

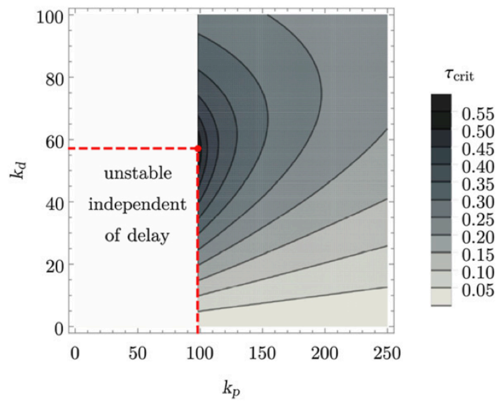

Figure 3: Critical delays for the single inverted pendulum.

3.4. *Critical delay by the WM method*

The WM method can also be applied to determine the critical delay (delay margin) for a fixed pair of control gains $(k_\mathrm{p}, k_\mathrm{d})$. In this case (4) reads

$$W(\omega^2) = \omega^4 + \left(\frac{mgl}{I} - \frac{k_\mathrm{d}^2 l^2}{4I^2}\right)\omega^2 + \left(\frac{m^2 g^2 l^2}{4I^2} - \frac{k_\mathrm{p}^2 l^2}{4I^2}\right). \tag{22}$$

Only one positive root ω_0 exists for any combination of the gains k_p and k_d and the critical delay is provided by (5). The critical delays over the plane $(k_\mathrm{p}, k_\mathrm{d})$ are shown in Fig. 3 where the maximal critical delay is attained by the critical gains $k_{\mathrm{p,crit}} = 98.1$ N/rad, $k_{\mathrm{d,crit}} = 57.2$ Ns/rad.

4. The double inverted pendulum

The double inverted pendulum is often referred to as the most simple nonlinear multi-body system featuring all the properties of higher-degree-of-freedom nonlinear systems, such as complexity or chaos. As such, it is often used to demonstrate control concepts of complex systems [10, 16, 14]. The corresponding full state feedback involves the angular position and angular velocity of both pendulum segments, which implies that the number of control gains is four. Hence optimization and stabilization shall be performed in the four dimensional space of the control gains. Note that the double inverted pendulum is also an often used important model in human balancing research [25, 31, 23].

4.1. Mechanical model

The mechanical model of the double inverted pendulum is shown in Fig 4. The parameters of the model are $L_1 = 5\ \mathrm{m}, L_2 = 5\ \mathrm{m}, m_1 = 5\ \mathrm{kg}, m_2 = 5\ \mathrm{kg}$. The control force may be modeled as

$$F(t) = k_{\mathrm{p}1}\varphi_1(t-\tau) + k_{\mathrm{d}1}\dot{\varphi}_1(t-\tau) + k_{\mathrm{p}2}\varphi_2(t-\tau) + k_{\mathrm{d}2}\dot{\varphi}_2(t-\tau), \tag{23}$$

where subscript 1 and 2 refer to the lower and upper pendulum, respectively. The linearized equation of motion reads

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{Q}(t), \tag{24}$$

where the mass and the stiffness matrix are

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

$$\mathbf{S} = \begin{pmatrix} -22.0725 & 4.4145 \\ 13.2435 & -7.3575 \end{pmatrix} \tag{26}$$

and

$$\mathbf{Q}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}. \tag{27}$$

4.2. Stabilizability analysis

Stability and stabilizability analysis of the double inverted pendulum is less straightforward compared to that of the single pendulum, since the orders of the plant and the controller are higher. Although D-curves can be generated in a similar way, the stable domains should be represented in the four-dimensional space $(k_{\mathrm{p}1}, k_{\mathrm{p}2}, k_{\mathrm{d}1}, k_{\mathrm{d}2})$. In this section, stabilizability in terms of the critical delay is determined using the MID property and the results are confirmed with the numerical semidiscretization method.

First, the characteristic polynomial (1) with

$$A(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0, \tag{28}$$

$$B(s) = b_3 s^3 + b_2 s^2 + b_1 s + b_0 \tag{29}$$

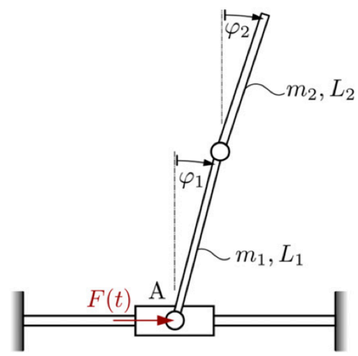

Figure 4: Double inverted pendulum

is investigated. The coefficients a_0, a_1, a_2 and a_3 are determined solely by the physical parameters L_1, L_2, m_1, m_2. The coefficients b_0, b_1, b_2 and b_3 of the delayed part of the quasipolynomial depend also on the control gains.

Now, we aim to find the control gains $k_{\mathrm{p1}}, k_{\mathrm{p2}}, k_{\mathrm{d1}}$ and k_{d2} (or equivalently, the parameters b_0, b_1, b_2 and b_3) that ensure the maximum feedback delay τ. The maximal admissible multiplicity of the zero spectral value in this case is $2+4+3-1=8$. However, the number of unknown parameters being five (b_0, b_1, b_2, b_3 and τ), we cannot expect a multiplicity higher than five. As a matter of fact, zeroing the characteristic function and the first four derivatives yields

$$D(0) = a_0 + b_0 = 0, \tag{30}$$

$$D'(0) = a_1 + b_1 - b_0\tau = 0, \tag{31}$$

$$D^{(\mathrm{II})}(0) = 2a_2 + 2b_2 - 2b_1\tau + b_0\tau^2 = 0, \tag{32}$$

$$D^{(\mathrm{III})}(0) = 6a_3 + 6b_3 - 6b_2\tau + 3b_1\tau^2 - b_0\tau^3 = 0, \tag{33}$$

$$D^{(\mathrm{IV})}(0) = 24 - 24b_3\tau + 12b_2\tau^2 - 4b_1\tau^3 + b_0\tau^4 = 0, \tag{34}$$

so that zeroing the fifth derivative

$$D^{(\mathrm{V})}(0) = -60a_3\tau^2 - 40a_2\tau^3 - 15a_1\tau^4 - 4a_0\tau^5 = 0 \tag{35}$$

is in contradiction with (34), which suggests that the maximal admissible multiplicity is 5. Solving b_0, b_1, b_2 and b_3 from (30)-(33) and substituting into (34) yields a polynomial in τ. As in the previous section, the coefficients of the polynomial may be expressed in terms of the parameters of the plant as

$$P(\tau) = 24 + 24a_3\tau + 12a_2\tau^2 + 4a_1\tau^3 + a_0\tau^4, \tag{36}$$

where

$$b_0 = -a_0, \tag{37}$$

$$b_1 = -a_1 - a_0\tau, \tag{38}$$

$$b_2 = -\frac{1}{2}\left(2a_2 + 2a_1\tau + a_0\tau^2\right), \tag{39}$$

$$b_3 = -\frac{1}{6}\left(6a_3 + 6a_2\tau + 3a_1\tau^2 + a_0\tau^3\right). \tag{40}$$

The critical delay is the smallest positive τ solution of (36).

In the particular case of the double undamped inverted pendulum, the coefficients a_1 and a_3 are zero. For the sake of simplicity, we assume that the two pendulums are made of the same material, so their length is proportional to their mass. Without loss of generality, we consider the case where $m_1 = \rho L_1$ and $m_2 = \rho L_2$ with $\rho = 1$ kg/m. Consequently, the mass moment of inertia is $I_1 = L_1^3/12$ and $I_2 = L_2^3/12$ for the center of gravity of each pendulum.

The non-zero coefficients of the characteristic equation are

$$a_0 = \frac{9g^2(L_1 + 2L_2)}{L_1^2 L_2}, \tag{41}$$

$$a_2 = -\frac{3g(L_1^2 + 7L_1L_2 + 2L_2^2)}{2L_1^2 L_2}, \tag{42}$$

$$b_0 = -\frac{9g(L_1 + 2L_2)(k_{\mathrm{p1}} + k_{\mathrm{p2}})}{L_1^2 L_2(L_1 + L_2)}, \tag{43}$$

$$b_1 = -\frac{9(k_{\mathrm{d1}} + k_{\mathrm{d2}})g(L_1 + 2L_2)}{L_1^2 L_2(L_1 + L_2)}, \tag{44}$$

$$b_2 = \frac{6L_1L_2k_{\mathrm{p1}} + 3L_2^2k_{\mathrm{p1}} - 3L_1^2k_{\mathrm{p2}}}{L_1^2 L_2(L_1 + L_2)}, \tag{45}$$

$$b_3 = \frac{-3k_{\mathrm{d2}}L_1^2 + 3k_{\mathrm{d1}}l_2(2L_1 + L_2)}{L_1^2 L_2(L_1 + L_2)}. \tag{46}$$

Note that for a physically realistic case $a_0 > 0$ and $a_2 < 0$. This property will be exploited later. The system is investigated with respect to the lengths L_1 and L_2 of the pendulum segments.

If $a_1 = 0$ and $a_3 = 0$ then the polynomial (36) reduces to an incomplete quadratic equation of the form

$$P(\tau) = 24 + 12a_2\tau^2 + a_0\tau^4 = 0. \tag{47}$$

Descartes' rule of signs indicates that the number of positive roots of (47) is two, hence, two positive and two negative solutions are obtained for τ. The smaller positive value is the critical delay.

The two positive solutions for τ are

$$\tau_1 = \sqrt{\frac{2\left(-3a_2 - \sqrt{3}\sqrt{3a_2^2 - 2a_0}\right)}{a_0}}, \tag{48}$$

$$\tau_2 = \sqrt{\frac{2\left(-3a_2 + \sqrt{3}\sqrt{3a_2^2 - 2a_0}\right)}{a_0}}. \tag{49}$$

After substituting (41) and (42), one can see that $3a_2^2 - 2a_0 > 0$, hence τ_1 and τ_2 are positive real roots indeed. The dominance of the quintuple root $s = 0$ for the case $\tau = \tau_1 < \tau_2$ is proved following the argument in [5].

If $\tau = \tau_1$ and the coefficients b_0, b_1, b_2 and b_3 satisfy equations (37), (38), (39) and (40), respectively, then the characteristic function can be written in the following form:

$$\begin{aligned} D(s) =& s^4 + a_2 s^2 + a_0 \\ & - \frac{1}{6} e^{-s\tau_1} \Big(6a_0 + 6a_0\tau_1 s + (3a_0\tau_1^2 + 6a_2)s^2 \\ & + \tau_1(a_0\tau_1^2 + 6a_2)s^3 \Big) \\ =& s^4 \left(1 - \int_0^1 e^{-s\tau_1 t} \tau_1^2 t \left(-a_2 - \frac{1}{6} a_0 \tau_1^2 t^2 \right) \mathrm{d}t \right). \end{aligned} \tag{50}$$

To prove that there exists no root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ of (50) such that $\gamma_1 > 0$, substitute s_1 into (50), one gets

$$\begin{aligned} 1 =& \left| \int_0^1 e^{-s_1\tau_1 t} \tau_1^2 t \left(-a_2 - \frac{1}{6} a_0 \tau_1^2 t^2 \right) \mathrm{d}t \right| \\ \le & \int_0^1 e^{-\gamma_1\tau_1 t} \tau_1^2 t \left| a_2 + \frac{1}{6} a_0 \tau_1^2 t^2 \right| \mathrm{d}t \\ =& - \int_0^1 e^{-\gamma_1\tau_1 t} \tau_1^2 t \left(a_2 + \frac{1}{6} a_0 \tau_1^2 t^2 \right) \mathrm{d}t =: f(\gamma_1), \end{aligned} \tag{51}$$

since

$$a_2 + \frac{1}{6} a_0 \tau_1^2 t^2 = a_2(1 - t^2) - \frac{1}{\sqrt{3}} t^2 \sqrt{3a_2^2 - 2a_0} < 0. \tag{52}$$

For $\gamma_1 = 0$

$$f(\gamma_1 = 0) = - \int_0^1 \tau_1^2 t \left(a_2 + \frac{1}{6} a_0 \tau_1^2 t^2 \right) \mathrm{d}t = 1, \tag{53}$$

so that for $\gamma_1 > 0$ the value of the integral is $f(\gamma_1) < 1$ which proves the inconsistency of the hypothesis that the characteristic function (50) has an unstable root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > 0$.

The critical delay obtained by the MID-based method over the plane (L_1, L_2) is shown in Fig. 5. In order to verify the results numerically, the critical delay was determined by the semidiscretization method [15] combined with an interval halving technique [3] over a fixed grid of control parameters for $L_1 = 5$ m and $L_2 = 5$ m. In this

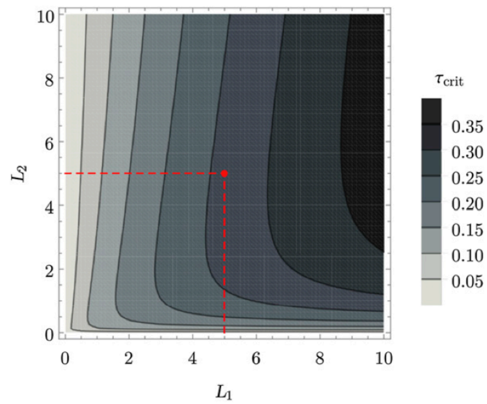

Figure 5: Critical delay for the double inverted pendulum deremined by the MID-based concept.

case, $\tau_{\text{crit}} = 0.26$ s. A section of the stability regions in the four-dimensional space of the control gains is represented in Fig. 6 for $\tau = 0.25$ s. As can be seen, the stable region disappears indeed at the critical parameters.

5. Conclusion

The MID property is employed to determine the critical feedback delay for the stabilization of a double inverted pendulum with full delayed state feedback. The method was shown to determine the critical delay with significantly smaller computational effort compared to other numerical methods, e.g., the Walton-Marschall or the semidiscretization method for a series of control gains combination. The main benefit of applying the MID-based approach is that the control space does not have to be swept, since the control gains associated with the critical delay are derived in closed form. Furthermore, in order to get the critical delay only the zeros of a quasipolynomial should be determined, while the corresponding control gains can be calculated by solving a finite set of linear equations. This is a useful feature especially for higher order plants or controllers design.

To the best knowledge of the authors, similar analytical method to determine the critical delay for higher-order system such as the double inverted pendulum is not available in the literature yet. The method applied in this paper to the double inverted

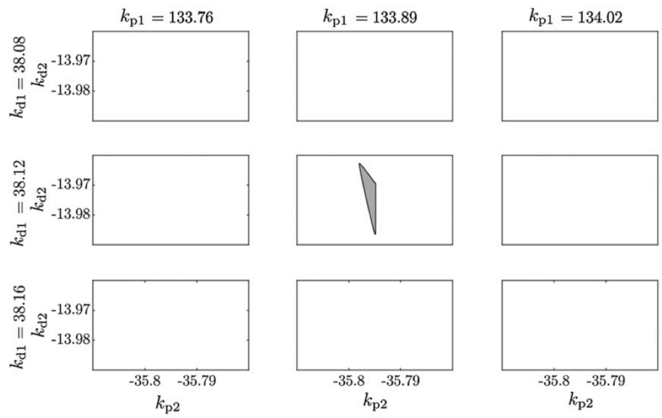

Figure 6: Stability diagrams for the double inverted pendulum with $L_1 = L_2 = 5$ m in the neighbourhood of the critical point when $\tau = 0.25$ s.

pendulum can be adopted to more general systems under certain conditions on the characteristic function, see, e.g. [9].

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