Extending Drawings of Graphs to Arrangements of Pseudolines

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Abstract

In the recent study of crossing numbers, drawings of graphs that can be extended to an arrangement of pseudolines (pseudolinear drawings) have played an important role as they are a natural combinatorial extension of rectilinear (or straight-line) drawings. A characterization of the pseudolinear drawings of K_n was found recently. We extend this characterization to all graphs, by describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the pseudolines when it is possible.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms; Mathematics of computing → Graphs and surfaces

Keywords and phrases graphs, graph drawings, geometric graph drawings, arrangements of pseudolines, crossing numbers, stretchability.

Funding *Alan Arroyo*: Supported by CONACYT. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754411.
Julien Bensmail: ERC Advanced Grant GRACOL, project no. 320812.
R. Bruce Richter: Supported by NSERC.

1 Introduction

Since 2004, geometric methods have been used to make impressive progress for determining the crossing number of (certain classes of drawings of) the complete graph K_n. In particular, drawings that extend to straight lines, or, more generally, arrangements of pseudolines, have been central to this work, spurring interest in such drawings for arbitrary graphs, not just complete graphs [2, 4, 5, 6, 12].

In particular, for pseudolinear drawings, it is now known that, for $n \geq 10$, a pseudolinear drawing of K_n has more than

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

crossings [1, 14]. The number $H(n)$ is conjectured by Harary and Hill to be the smallest number of crossings over all topological drawings of K_n; that is, the crossing number $\operatorname{cr}(K_n)$ is conjectured to be $H(n)$.

A *pseudoline* is the image ℓ of a continuous injection from the real numbers $\mathbb{R}$ to the plane $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \ell$ is not connected. An *arrangement of pseudolines* is a set Σ of pseudolines such that, if ℓ, ℓ' are distinct elements of Σ, then $|\ell \cap \ell'| = 1$ and the intersection is a crossing point. Informally, a *crossing point* or *crossing* is an intersection point between two pseudolines that locally looks like a crossing point between two non parallel lines (a formal definition of crossing will be given when we introduce the notion of string). More on pseudolines and their importance for studying geometric drawings of graphs can be found in [10, 11].

A drawing D of a graph G is *pseudolinear* if there is an arrangment of pseudolines consisting of a different pseudoline ℓ_e for each edge e of G and such that $D[e] \subseteq \ell_e$.

In the study of crossing numbers, restricting the drawing to either straight lines or pseudolines yields the rectilinear crossing number $\overline{\text{cr}}(K_n)$ or the pseudolinear crossing number $\widetilde{\text{cr}}(K_n)$, respectively. Clearly $\overline{\text{cr}}(K_n) \geq \widetilde{\text{cr}}(K_n)$ and the geometric methods prove that $\widetilde{\text{cr}}(K_n) > H(n)$, for $n \geq 10$.

A *good drawing* is one where no edge self-intersects and any two edges share at most one point—either a crossing or a common end point— and no three edges share a common crossing. One somewhat surprising result is from Aichholzer et al.: *a good drawing of K_n in the plane is homeomorphic to a pseudolinear drawing if and only if it does not contain a non-planar drawing of K_4 whose crossing is incident with the unbounded face of the K_4* [2] (see Figure 1). By ignoring the grey edges from Figure 1, we see that any such drawing of K_4 contains a *B-configuration*, depicted as the third drawing of the first row of Figure 2. Based on our Theorem 2, Theorem 2.5.1 from [3] shows that any non-pseudolinear drawing contains a B-configuration. Thus, either Fig. 1 or the B-configuration can be used to characterize pseudolinear drawings of K_n. In [4] pseudolinear drawings of K_n are characterized as *f-convex*, and in [5] are characterized as monotone and free of a specific drawing of K_4.

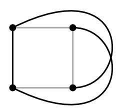


Figure 1 Non-pseudolinear K_4 with its crossing incident with the outer face.

Twenty-five years earlier, Thomassen [19] proved a similar theorem for a 1-planar drawing (that is, a drawing in which each edge is crossed at most once). The B- and W-configurations are shown as the third and fourth drawings in the first row of Figure 2. Thomassen's theorem is: *if D is a 1-planar drawing of graph G, then D is homeomorphic to a rectilinear drawing of G if and only if D contains no B- or W-configuration.*

Thomassen presented in [19] the *clouds* (first column in Figure 2) as an infinite family of drawings that are minimally non-pseudolinear.

Shortly after Thomassen's paper, Bienstock and Dean proved that if $\text{cr}(G) \leq 3$, then $\overline{\text{cr}}(G) = \text{cr}(G)$ [7]. They also exhibited examples based on overlapping W-configurations to show the result fails for $\text{cr}(G) = 4$; such graphs can have arbitrarily large rectilinear crossing number.

Despite the existence of infinitely many obstructions to pseudolinearity, we characterize them all.

▶ **Theorem 1.** A good drawing of a graph G is pseudolinear if and only if it does not contain one of the infinitely many obstructions shown in Figure 1.

The drawings in Figure 2 are obtained from the *clouds* (first column) by replacing at most two crossings by vertices. The formal statement of Theorem 1 is Theorem 15 in Section 6; also a more general version of this statement, Theorem 2, is discussed below. Our result draws a

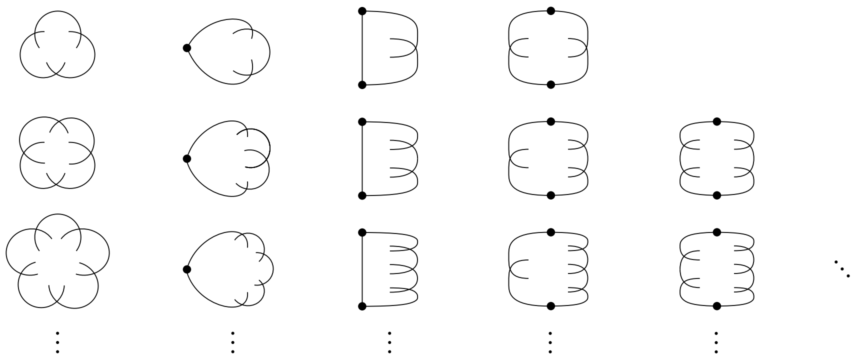


Figure 2 Obstructions to pseudolinearity.

line between the class of pseudolinear drawings and the class of rectilinear drawings: Our result shows that recognizing pseudolinear drawings is a combinatorial/topological problem and implies a polynomial-time algorithm to detect pseudolinear drawings (Theorem 14). This contrast with the rather real algebraic geometry problem of deciding the *stretchability* of a drawing, defined as the problem of deciding whether a given drawing is homeomorphic to a rectilinear drawing. Mnëv [16, 17] showed that deciding the stretchability of an arrangement of pseudolines is $\exists\mathbb{R}$-hard, implying the $\exists\mathbb{R}$-hardness for the problem of deciding the stretchability of a graph drawing. Since $\mathsf{NP} \subseteq \exists\mathbb{R}$ [15, 18, 8], this in particular shows that the stretchability problem is NP-hard. We refer to Matoušek's survey [15] for an approachable introduction to the complexity class $\exists\mathbb{R}$.

The natural setting for our characterization is strings embedded in the plane. An *arc* σ is the image $f([0,1])$ of the compact interval $[0,1]$ under a continuous map $f : [0,1] \to \mathbb{R}^2$. Let $S(\sigma) = \{p \in \sigma \ : \ |f^{-1}(p)| \geq 2\}$ be the set of self-intersections of σ. A *string* is an arc σ for which $S(\sigma)$ is finite. If $S(\sigma) = \emptyset$, then σ is *simple*. If σ' is a string and $\sigma' \subseteq \sigma$, then σ' is a substring of σ.

Suppose that σ and σ' whose intersection $\sigma \cap \sigma'$ is a finite set and let $p \in \sigma \cap \sigma$. The *rotation at* p is a cyclic sequence of substrings determined by a small neighbourhood homeomorphic to the plane in which p is origin and the substrings incident with p are rays emanating from p [13, Thm. 3.1]. The strings σ_1, σ_2 cross at p if they each have two substrings that alternate $\sigma_1 - \sigma_2 - \sigma_1 - \sigma_2$ in the rotation at p.

An intersection point between of two strings σ and σ' is *ordinary* if it is either an endpoint of σ or σ', or is a *crossing*. A set Σ of strings is *ordinary* if Σ is finite and any two strings in Σ have only finitely many intersections, all of which are ordinary. All the sets of strings considered in this paper are ordinary.

If Σ is an ordinary set of strings, then its *planarization* $G(\Sigma)$ is the plane graph obtained from Σ by inserting vertices at each crossing between strings and also at the endpoints of every string in Σ. To keep track of the information given by the strings, we will always assume that each string Σ has a different color and that each edge in $G(\Sigma)$ inherits the color of the string including it.

If Σ is an ordinary set of strings, then, for a cycle C in $G(\Sigma)$ (which is a simple closed curve in $\mathbb{R}^2$) the *edges inside* C are those drawn in the closed disk bounded by C (this includes the edges of C). A vertex $v \in V(C)$ is a *rainbow* for C if all the edges incident with v and drawn inside C have different colours. The reader can verify that, for each drawing in Figure 2, if we let Σ be the edges of the drawing, then the unique cycle in $G(\Sigma)$ has at most two rainbows. Our main result characterizes these cycles as the only possible obstructions:

▶ **Theorem 2.** *An ordinary set of strings Σ can be extended to an arrangement of pseudolines if and only if every cycle C of $G(\Sigma)$ has at least three rainbows.*

Henceforth, we define any cycle C in $G(\Sigma)$ with at most two rainbows as an *obstruction*. A set of strings is *pseudolinear* if it has an extension to an arrangement of pseudolines.

Theorem 2 is our main contribution. In the next section, we show that the presence of an obstruction implies the set of ordinary strings is not pseudolinear. The converse is proved in Section 4 by extending, one small step at a time, the strings in Σ to get closer to an arrangement of pseudolines. After each extension, we must show that no obstruction has been introduced. This involves dealing with cycles in $G(\Sigma)$ that have precisely three rainbows (that we refer as *near-obstructions*). In Section 3 we show the key lemma that if G has two such near-obstructions that intersect nicely at a vertex v, then G has an obstruction. In Section 5 we present a polynomial-time algorithm for detecting obstructions and we argue why the proof of Theorem 2 implies a polynomial-time algorithm for extending a pseudolinear set of strings. Finally, in Section 6, we show how Theorem 1 follows from Theorem 2 and we present some concluding remarks.

2 A set of strings with an obstruction is not extendible

Let us start by showing the easy direction of Theorem 2:

▶ **Lemma 3.** *If the underlying graph $G(\Sigma)$ of a set Σ of strings has an obstruction, then Σ is not pseudolinear.*

Suppose that C is a cycle of $G(\Sigma)$ for some set of strings Σ. We define $\delta(C)$ as the set of vertices of C for which their two incident edges in C have different colours. In a set Σ of simple strings where no two intersect twice, $|\delta(C)| \geq 3$ for every cycle C of $G(\Sigma)$.

▶ **Lemma 4.** *Let Σ be a set of simple strings where every pair intersect at most once. Suppose that C is an obstruction with $|\delta(C)|$ as small as possible. Let $S = x_0, x_1, \dots, x_\ell$ be a path of $G(\Sigma)$ representing a substring of some string $\sigma \in \Sigma$ such that $x_0x_1 \in E(C)$, $x_1 \in \delta(C)$ and x_1 is not a rainbow of C. Then $V(C) \cap V(S) = \{x_0, x_1\}$.*

Proof. By way of contradiction, suppose that there is a vertex $x_r \in V(C) \cap V(S)$ with $r \geq 3$. Assume that $r \geq 3$ is as small as possible. Let P be the subpath of S connecting x_1 to x_r. The facts $x_0x_1 \in E(C)$, $x_1 \in \delta(C)$, and $P \subseteq \sigma$ imply that $x_1x_2 \neq E(C)$. Because x_1 is not a rainbow for C and no two strings tangentially intersect at x_1, the edge x_1x_2 is drawn in the closed disk bounded by C. By choice of r, P is an arc connecting x_1 to x_r in the interior of C.

Let C_1 and C_2 be the two cycles of $C \cup P$ containing P, labelled so that $x_0x_1 \in E(C_1)$. We shall use the minimality of $|\delta(C)|$ to show that C_1 and C_2 are not obstructions. Then, we will count rainbows in C_1 and C_2 to obtain the contradiction that C is not an obstruction.

For a cycle X, let $\rho(X)$ be the set of rainbows of X. For $i = 1, 2$, let $Q_i = V(C_i) \setminus V(P)$. As the edges of S are included in the same string, we see that $\rho(C_1) \setminus Q_1 \subseteq \{x_r\}$ and $\rho(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$. Likewise, $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$ and $\delta(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$.

Let us show that C_1 and C_2 are not obstructions. Because $|\delta(C_2)| \geq 3$ and $\delta(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$, $|\delta(C) \cap Q_2| \geq 1$. Since $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$ and $x_1 \in \delta(C)$, $|\delta(C_1)| \leq |\delta(C_1) \cap Q_1| + |\{x_r\}| \leq |\delta(C)| - 2 + |\{x_r\}| < |\delta(C)|$. Because $|\delta(C_1)| \geq 3$ and $|\delta(C_1) \setminus Q_1| \leq 1$, $|\delta(C) \cap Q_1| \geq 2$. Since $x_1 \in \delta(C) \cap \delta(C_2)$, $|\delta(C_2)| \leq |\delta(C) \cap Q_2| + |\{x_1, x_r\}| \leq |\delta(C)| - 3 + |\{x_1, x_r\}| < |\delta(C)|$. Thus, neither C_1 nor C_2 is an obstruction.

Finally, as $|\rho(C_1)| \geq 3$ and $|\rho(C_1) \setminus Q_1| \leq 1$, $|\rho(C) \cap Q_1| = |\rho(C_1) \cap Q_1| \geq 2$. Because $|\rho(C_2)| \geq 3$ and $|\rho(C_2) \setminus Q_2| \leq 2$, $|\rho(C) \cap Q_2| = |\rho(C_2) \cap Q_2| \geq 1$. Thus $|\rho(C)| \geq 3$, a contradiction. ◀

Proof of Lemma 3. By way of contradiction, suppose that Σ is pseudolinear and that $G(\Sigma)$ has an obstruction C.

Consider an extension of Σ to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings Σ' extending Σ, and in which every pair of strings in Σ' cross once. In $G(\Sigma')$, there is a cycle C' that represents the same simple closed curve as C. Because every rainbow of C' is a rainbow of C, C' has fewer than three rainbows. Therefore, we may assume that $\Sigma = \Sigma'$ and $C = C'$. Now, the ends of every string in Σ are degree-1 vertices in the outer face of $G(\Sigma)$.

As every string in Σ is simple and no two strings intersect more than once, $|\delta(C)| \geq 3$. We will assume that C is chosen to minimize $|\delta(C)|$.

Since C is an obstruction, there exists $x_1 \in \delta(C)$ such that x_1 is not a rainbow in C. Consider a neighbour x_0 of x_1 in C. Let $S = x_0, x_1, \dots x_\ell$ be the path obtained by traversing the string σ extending x_0x_1, such that x_ℓ is an end of σ. By Lemma 4, $V(S) \cap V(C) = \{x_0, x_1\}$, and because x_ℓ is in the outer face of C, the segment of σ from x_1 to x_ℓ has its relative interior in the outer face of C.

However, since x_1 is not a rainbow, there exists a string $\sigma' \in \Sigma$ including two edges at x_1 drawn inside C. Thus, σ and σ' tangentially intersect at x_1, a contradiction. ◀

3 The key lemma

In this section we present the key lemma used in the proof of Theorem 2.

A plane graph G is *path-partitioned* if for $m \geq 1$, there exists a colouring $\chi : E(G) \to \{1, \dots, m\}$ such that for each $i \in \{1, \dots, m\}$, the edges in $\chi^{-1}(i)$ induce a path $P_i \subseteq G$ where any two distinct paths P_i and P_j do not tangentially intersect. Indeed, every underlying planar graph $G(\Sigma)$ of a set of simple strings Σ is path-partitioned. Moreover, every path-partitioned plane graph can be obtained by subdividing a planarization of an ordinary set of simple strings. To extend the previously introduced notation we refer to each P_i as a string. The concepts of rainbow and obstruction naturally extend to the context of path-partitioned plane graphs.

Suppose that G is a path-partitioned plane graph. Given $v \in V(G)$, a *near-obstruction at* v is a cycle C with at most three rainbows and such that v is a rainbow of C. Understanding how near-obstructions behave is the key ingredient needed in the proof of Theorem 2:

▶ **Lemma 5.** *Let G be a path-partitioned plane graph and let $v \in V(G)$. Suppose that C_1 and C_2 are two near-obstructions at v such that the union of the closed disks bounded by C_1 and C_2 contains a small open ball centered at v. Suppose that one of the following two holds:*

1. *no obstruction of G contains v; or*
2. *the two edges of C_1 incident with v are the same as the two edges of C_2 incident with v.*

Then G has an obstruction not including v.

Given a plane graph G, a cycle $C \subseteq G$ and a vertex $v \in V(C)$, *the edges at v inside C* are the edges of G incident with v drawn inside C. Consider a homeomorphism from a small disc neighbourhood of v to the plane so that each edge segment incident with v is a straight ray from the origin (which is v). Since no two strings intersect tangentially at v, we may assume that the rotation at v has substrings of the same colour making an angle of π at v.

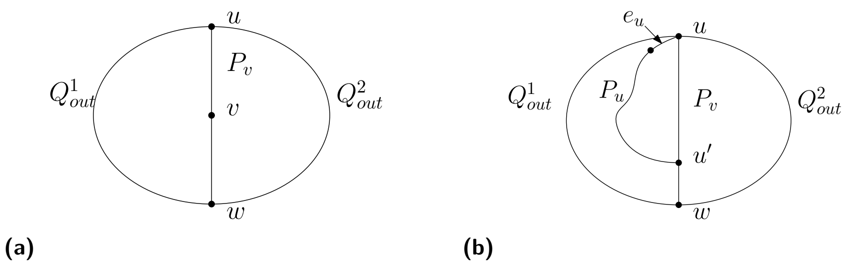


Figure 3 Auxiliary figures used in the proof of Lemma 5.

The angles between rays are the *angles at v* and we associate to them the set of edges at v drawn as rays inside them. From this geometric perspective, it is obvious that, if an angle α is less than π, then α is rainbow. This proves the second of the following facts.

▶ **Useful Facts.** *Let G be a plane path-partitioned graph and let $v \in V(G)$. Then*

1. *if α, β are two angles at v with $\alpha \subseteq \beta$ and β is rainbow, then α is rainbow; and*
2. *if α and β are two angles such $\overline{\alpha}$ is not rainbow and β is a proper subangle of the complement $\overline{\alpha}$ of α, then β is rainbow.*

Proof of Lemma 5. By way of contradiction, suppose that G has no obstruction not including v. The "small ball" hypothesis implies that v is not in the outer face of the subgraph $C_1 \cup C_2$.

We claim that $|V(C_1) \cap V(C_2)| \geq 3$. Suppose not. For $i = 1, 2$, let e_i and f_i be the edges of C_i at v and let Δ_i be the closed disk bounded by C_i. From the "small ball" hypothesis it follows that (i) Δ_1 contains the edges e_2 and f_2; and (ii) the points near v in the exterior of Δ_2 are contained in Δ_1. These two properties imply that the path $C_2 - v$ intersects C_1 at least twice, and because $v \in V(C_1) \cap V(C_2)$, $|V(C_1) \cap V(C_2)| \geq 3$.

From the last paragraph we know that $C_1 \cup C_2$ is 2-connected, and hence the outer face of $C_1 \cup C_2$ is bounded by a cycle C_{out}. We will assume that

(*) the cycles C_1 and C_2 satisfying the hypothesis of Lemma 5 are chosen so that the number of vertices of G in the disk bounded by C_{out} is minimal.

Useful Fact 1 applied to the interior angles at vertices of C_{out} shows that every vertex that is a rainbow in C_{out} is also a rainbow in each of the cycles in $\{C_1, C_2\}$ containing it. We can assume that C_{out} is not an obstruction or else we are done. We may relabel C_1 and C_2 so that two of the rainbows of C_{out}, say p and q, are also rainbows in C_1. Neither p nor q is v because $v \notin V(C_{out})$. Because C_1 is a near-obstruction, p, q and v are the only rainbows of C_1.

Since $v \notin V(C_{out})$, by following C_1 in the two directions starting at v, we find a path $P_v \subseteq C_1$ containing v in which only the ends u and w of P_v are in C_{out} (note that $u \neq w$ because $\{p, q\} \subseteq V(C_1) \cap V(C_{out})$). See Figure 3a.

As v is in the interior face of C_{out}, P_v is also in the interior of C_{out}. Let Q^1_{out}, Q^2_{out} be the uw-paths of C_{out}. One of the two closed disks bounded by $P_v \cup Q^1_{out}$ and $P_v \cup Q^2_{out}$ contains C_1. By symmetry, we may assume that C_1 is contained in the first disk. Since $C_{out} \subseteq C_1 \cup C_2$, this implies that Q^2_{out} is a subpath of C_2.

Our desired contradiction will be to find three rainbows in C_2 distinct from v. We find the first: let $C_1 - (P_v)$ be the uw-path in C_1 distinct from P_v. The disk bounded by

$(C_1 - (P_v)) \cup Q^2_{out}$ contains the one bounded by C_1. Useful Fact 1 applied to the interior angles at the vertices of $(C_1 - (P_v)) \cup Q^2_{out}$ implies that each vertex in $C_1 - (P_v)$ that is a rainbow in $(C_1 - (P_v)) \cup Q^2_{out}$ is also rainbow in C_1. Since C_1 has at most two rainbows in $C_1 - (P_v)$, namely p and q, $(C_1 - (P_v)) \cup Q^2_{out}$ has a third rainbow r_1 in the interior of Q^2_{out} (else $(C_1 - (P_v)) \cup Q^2_{out}$ is an obstruction and we are done). Note that r_1 is also a rainbow for C_2.

To find another rainbow in C_2, consider the edge e_u of C_2 incident to u and not in Q^2_{out}. We claim that either u is a rainbow in C_2 or that e_u is not included in the closed disk bounded by $P_v \cup Q^2_{out}$. Seeking a contradiction, suppose that u is not a rainbow of C_2 and that e_u is included in the disk. Then Useful Fact 2 implies that u is a rainbow in C_1. As p and q are the only rainbows of C_1 in C_{out}, u is one of p and q. Therefore u is a rainbow in C_{out}, and hence, a rainbow in C_2, a contradiction.

If u is a rainbow in C_2, then this is the desired second one. Otherwise, e_u is not in the closed disk bounded by $P_v \cup Q^2_{out}$. Let $P_u \subseteq C_2$ be the path starting at u, continuing on e_u and ending on the first vertex u' in P_v that we encounter. Let C_u be the cycle consisting of P_u and the uu'-subpath uP_vu' of P_v. See Figure 3b.

▷ Claim 6. If P_u does not have a rainbow of C_u in its interior, then either C_u is an obstruction not containing v or:

(a) C_u and C_2 are near-obstructions at v satisfying the same conditions as C_1 and C_2 in Lemma 5; and

(b) the closed disk bounded by the outer cycle of $C_u \cup C_2$ contains fewer vertices than the disk bounded by C_{out}.

Proof. Suppose that all the rainbows of C_u are located in uP_vu'. If z is a rainbow of C_u, then $z \in \{u, v, u'\}$, as otherwise z is a rainbow of C_1 distinct from p, q and v, a contradiction. Thus, if $v \notin V(C_u)$, then C_u is the desired obstruction. We may assume that $v \in V(C_u)$.

If $u' = w$, then $C_2 = P_u \cup Q^2_{out}$, violating the assumption that $v \in V(C_2)$. Thus $u' \neq w$. If $u' = v$, then the rainbows of C_u are included in $\{u, u'\}$, and hence C_u is an obstruction. However, the existence of C_u shows that both alternatives (1) and (2) in Lemma 5 fail: condition (1) fails because C_u contains v and (2) fails because the edge of P_u incident with v is in $E(C_2) \setminus E(C_1)$. Thus $u' \neq v$.

The previous two paragraphs show that C_u is a near-obstruction at v with rainbows u, v and u'. Since the interior of C_u near v is the same as the interior of C_1 near v, the pair (C_u, C_2) satisfies the "small ball" hypothesis. Thus, (a) holds.

Let C'_{out} be the outer cycle of $C_u \cup C_2$. From the fact that $C_u \cup C_2 \subseteq C_1 \cup C_2$ it follows that the disk bounded by C_{out} includes the disk bounded by C'_{out}.

Since $p, q \in V(C_{out})$, p and q are in the disk bounded by C_{out}. If both p and q are in C_2, then p, q and r_1 are rainbows in C_2, and also distinct from v, contradicting that C_2 is a near-obstruction for v. If, say $p \notin V(C_2)$, then p is not in the disk bounded by C'_{out}, which implies (b). ◀

From Claim 6(b) and assumption (*) either C_u is the desired obstruction or P_u contains a rainbow r_2 of C_2 in its interior. We assume the latter as otherwise we are done.

In the same way, the last rainbow r_3 comes by considering the edge of $C_2 - Q^2_{out}$ incident with w. It follows that v, r_1, r_2 and r_3 are four different rainbows in C_2, contradicting the fact that C_2 is a near-obstruction. ◀

4 Proof of Theorem 2

In this section we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines.

Proof of Theorem 2. It was shown in Observation 3 that the existence of obstructions implies non-extendibility. For the converse, suppose that Σ is a set of strings for which $G(\Sigma)$ has no obstructions.

We start by reducing to the case where the point set $\bigcup\Sigma$ is connected: iteratively add a new string in a face of $\bigcup\Sigma$ connecting two connected components of $\bigcup\Sigma$. No obstruction is introduced at each step (obstructions are cycles), and, eventually, the obtained set $\bigcup\Sigma$ is connected. An extension of the new set of strings contains an extension for the original set, thus we may assume that $\bigcup\Sigma$ is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$, then we slightly extend the a-end of σ into one of the faces incident with a.
- **Face-Escaping Step.** If a string $\sigma \in \Sigma$ has an end a with degree 1 in $G(\Sigma)$, and is incident with an interior face, then we extend the a-end of σ until it intersects some point in the boundary of this face.
- **Exterior-Meeting Step.** Assuming that all the strings in Σ have their two ends in the outer face and these ends have degree 1 in $G(\Sigma)$, we extend the ends of two disjoint strings so that they meet in the outer face.

Each of these three steps either increases the number of pairs of strings that intersect, or increase the number crossings (recall that a crossing between σ and σ' is a non-tangential intersection point in $\sigma \cap \sigma'$ that is not an end of σ or σ'). Moreover, these steps can be performed as long as one of the next two conditions holds: (1) at least one string does not have an end incident with the outer face; and (2) there is a pair of strings that do not cross. If none of (1) and (2) hold, then our set of strings is extendible into an arrangement of pseudolines. Henceforth, we will show that, if performed correctly, none of these steps introduces an obstruction. The proof for each step can be read independently.

▶ **Lemma 7** (Disentangling Step). *Suppose that $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$. Then we can extend the a-end of σ into one of the faces incident to a without creating an obstruction.*

Proof. A pair of different edges f and f' in $G(\Sigma)$ incident with a are *twins* if they belong to the same string in Σ. The edge $e \subseteq \sigma$ incident with a has no twin.

The fact that no pair of strings tangentially intersect at a tells us that if (f_1, f_1') and (f_2, f_2') are pairs of twins, then f_1, f_2, f_1', f_2' occur in this cyclic order for either the clockwise or counterclockwise rotation at a. Thus, we may assume that the counterclockwise rotation at a restricted to the twins and e is $e, f_1, \dots, f_t, f_1', \dots, f_t'$, where (f_i, f_i') is a twin pair for $i = 1, \dots, t$.

To avoid tangential intersections, the extension of σ at a must be in the angle between f_t and f_1' not containing e. Let $e_1, \dots, e_k$ be the counterclockwise ordered list of non-twin edges at a having an end in this angle (as depicted in Figure 4). We label $e_0 = f_t$ and $e_{k+1} = f_1'$. If there are no twins, then let $e_0 = e_{k+1} = e$.

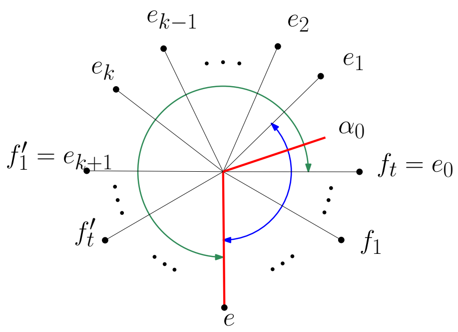


Figure 4 Substrings included in the disk bounded by C_0.

Let us consider all the possible extensions: for $i \in \{0, \ldots, k\}$, let Σ_i be the set of strings obtained from Σ by slightly extending the a-end of σ into the face containing the angle between e_i and e_{i+1}. Let α_i be the new edge at a extending σ in Σ_i (see α_0 in Figure 4).

Seeking a contradiction, suppose that, for each $i \in \{0, ..., k\}$, $G(\Sigma_i)$ contains an obstruction C_i. Since α_i contains a degree-1 vertex, α_i is not in C_i. Hence C_i is a cycle of $G(\Sigma)$. Thus, C_i is not an obstruction in $G(\Sigma)$ and becomes an obstruction in $G(\Sigma_i)$. This conversion has a simple explanation: in $G(\Sigma)$, C_i has exactly three rainbows, and one of them is a. After α_i is added, a is not a rainbow in C_i (witnessed by the edges e and α_i included in the new version of σ).

Recall from Section 3 that a *near-obstruction at a* is a cycle with exactly three rainbows, and one of them is a. Each of C_0, C_1,...,C_k is a near-obstruction at a in $G(\Sigma)$.

For a cycle $C \subseteq G$, let $\Delta(C)$ denote the closed disk bounded by C. Both e and α_0 are in $\Delta(C_0)$. Thus, either $\Delta(C_0) \supseteq \{e, f_1, f_2, \ldots, f_t, e_1\}$ (blue bidirectional arrow in Figure 4) or $\Delta(C_0) \supseteq \{f_t, e_1, \ldots, e_k, f_1', f_2', \ldots, f_t', e\}$ (green bidirectional arrow). We rule out the latter situation as the second list contains f_t and f_t', and this would imply that a is not a rainbow for C_0 in $G(\Sigma)$.

We just showed that $\{e, e_0, e_1\} \subseteq \Delta(C_0)$. By symmetry, $\{e_k, e_{k+1}, e\} \subseteq \Delta(C_k)$. Consider the largest index $i \in \{0, 1, \ldots, k-1\}$ for which $\{e, e_0, \ldots, e_{i+1}\} \subseteq \Delta(C_i)$. By the choice of i, and because $\{e, \alpha_{i+1}\} \subseteq \Delta(C_{i+1})$, $\{e, f_t', \ldots, f_1', e_k, \ldots, e_i\} \subseteq \Delta(C_{i+1})$. Apply Lemma 5 to the pair C_i and C_{i+1}, where C_i, C_{i+1} and a play the roles of C_1, C_2 and v. Condition 1 of Lemma 5 holds, and hence we obtain that $G(\Sigma)$ has an obstruction, a contradiction. ◀

▶ **Lemma 8** (Face-Escaping Step). *Suppose that there is a string σ that has an end a with degree 1 in $G(\Sigma)$, and a is incident to an interior face F. Then there is an extension σ' of σ from its a-end to a point in the boundary of F such that the set $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$ has no obstruction.*

Proof. Let W be the closed boundary walk $(x_0, e_1, x_1, e_2, \ldots, e_n, x_n)$ of F such that $x_0 = x_n = a$ and F is to the left as we traverse W (see Figure 5 for an illustration with $n = 9$). For $i = 1, \ldots, n$ we let m_i be a point in the relative interior of e_i, and let P be the list of non-necessarily distinct points m_1, x_1, m_2, $x_2 \ldots, m_n$, which are the potential ends for all the different extensions. For each $p \in P$, let Σ_p be the set of strings obtained from Σ by extending the a-end of σ by adding an arc α_p connecting a to p in F (see Figure 5). We assume that every two distinct arcs α_p and $\alpha_{p'}$ are internally disjoint.

Let f_p be the edge $e_1 \cup \alpha_p$ in $G(\Sigma_p)$; f_p has ends x_1 and p. Also, let $\sigma^p = \sigma \cup \alpha_p$. See Figure 6. Seeking a contradiction, suppose that each $G(\Sigma_p)$ has an obstruction.

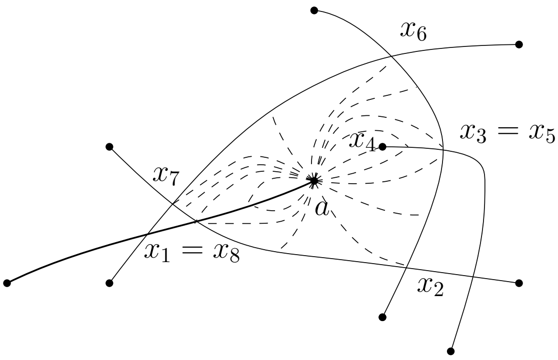


Figure 5 All possible extensions in the Face-Escaping Step.

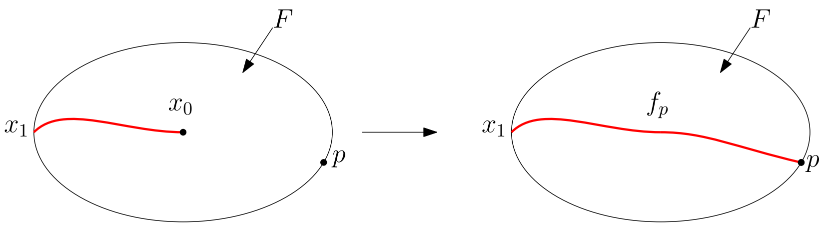


Figure 6 Transforming Σ into Σ_p.

▷ Claim 9. Let $p \in P$. Then there exists an obstruction C_p in $G(\Sigma_p)$ including f_p. Moreover,

(1) if $p \in \sigma$, then C_p can be chosen so that all its edges are included in σ^p; and
(2) if $p \notin \sigma$, then every obstruction includes f_p.

Proof. First, if $p \in \sigma$, then the string σ^p self-intersects at p; thus σ^p has a simple close curve including f_p. In this case let C_p be the cycle in $G(\Sigma_p)$ representing this simple closed curve without rainbows, and thus (1) holds.

Second, assume that $p \notin \sigma$ and let C_p be any obstruction of $G(\Sigma_p)$. For (2), we will show that $f_p \in E(C_p)$.

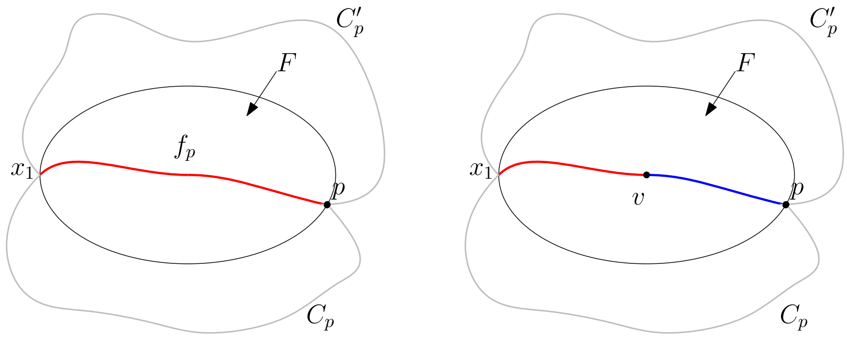
Seeking a contradiction, suppose that $f_p \notin E(C_p)$.

If $p = m_i$ for $i \in \{1, \ldots, n\}$, since m_i is the only vertex whose rotation in $G(\Sigma)$ differs from its rotation in $G(\Sigma_{m_i})$, $m_i \in V(C_p)$. Consider the cycle C of $G(\Sigma)$ obtained from C_p by replacing the subpath (x_{i-1}, m_i, x_i) by the edge $x_{i-1}x_i$. For each vertex $v \in V(C)$ the colors of the edges of $G(\Sigma)$ at v included in the disk bounded by C are the same as in $G(\Sigma_p)$ for the disk bounded by $V(C_p)$. Thus, C is an obstruction for $G(\Sigma)$, a contradiction.

Suppose now that p is one of $x_1, \ldots, x_{n-1}$. The only vertex in $G(\Sigma)$ whose rotation is different in $G(\Sigma_p)$ is p. Therefore, p is a point that is a rainbow for C_p in $G(\Sigma)$, but not a rainbow in $G(\Sigma_p)$, as witnessed by the two edges of σ^p that are incident with p and inside C_p. This contradicts the assumption that $p \notin \sigma$. Hence $f_p \in E(C_p)$. ◀

Henceforth we assume that, for $p \in P$, C_p is an obstruction in $G(\Sigma_p)$ as in Claim 9.

More can be said about the obstructions in $G(\Sigma_p)$, but for this we need some terminology. If we orient an edge e in a plane graph, then the *sides* of e are either the points near e that are to the right of e, or the points near e to the left of e. For any cycle C of G through e, exactly one side of e lies inside C. This is the side of e *covered* by C. For the next claim and in the rest of the proof we will assume that for $p \in P$, f_p is oriented from x_1 to p.



(a) χ **(b)** χ'

Figure 7 The two edge colorings χ and χ' discussed in the proof of Claim 10.

▷ Claim 10. For $p \in P$ with $p \notin \sigma$, every obstruction in $G(\Sigma_p)$ covers the same side of f_p.

Proof. Suppose that for $p \in P$ there are obstructions C_p and C'_p covering both sides of f_p. Let G' be the plane graph obtained from $G(\Sigma_p)$ by subdividing f_p, and let v be the new degree-2 vertex inside f_p.

We consider the edge-colouring χ induced by the strings in Σ_p. Let χ' be a new colouring obtained from χ by replacing the colour of the edge vp by a new colour not used in χ (see Figure 7). It is immediate that (i) χ' induces a path-partition in G'; and in the next paragraph we show that (ii) C_p and C'_p are near-obstructions for v with respect to χ'.

Consider the set of edges in the rotation at p inside the disk bounded by C_p and assume they are colored by χ. No edge from this set (except f_p) can have the same color as f_p or else $p \in \sigma$, contradicting our hypothesis. Therefore, p is a rainbow in C_p in χ if and only if p is a rainbow in C_p in χ'. Thus, when we switch from χ to χ', v is the only vertex of C_p switching identity (where the identity is to be or not to be a rainbow). As C_p is an obstruction for χ, then C'_p is a near obstruction at v for χ'. Likewise, C'_p is a near obstruction for χ'.

As Condition 2 of Lemma 5 holds for $C_1 = C_p$, $C_2 = C'_p$ and $v = v$ with respect to χ', G' has an obstruction not containing v in χ'. However, this implies the existence of an obstruction in $G(\Sigma)$ with respect to χ, a contradiction. ◀

Recall that the boundary walk of F is $W = (x_0, e_1, \ldots, e_n, x_n)$, with $x_0 = x_n = a$. Since x_1 and x_{n-1} are in σ, the extreme obstructions C_{x_1} and C_{x_2} cover the right of f_{x_1} and the left of $f_{x_{n-1}}$, respectively. Thus, there are two consecutive vertices x_{i-1}, x_i in $W - a$, such that the interior of $C_{x_{i-1}}$ covers the right of $f_{x_{i-1}}$ and the interior of C_{x_i} covers the left of f_{x_i}. Moreover, we may assume that the interior of C_{m_i} includes the left of f_{m_i} (otherwise we reflect our drawing).

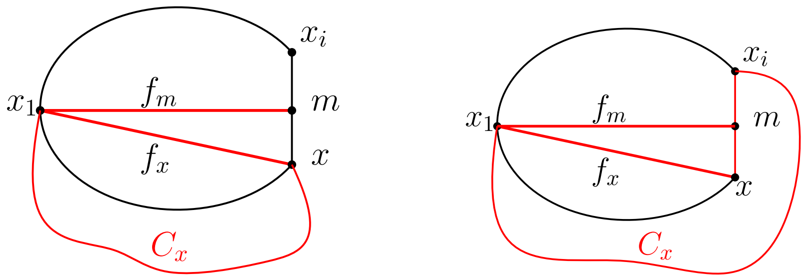
The next claim is the last ingredient to obtain a final contradiction. To make the notation simpler, we let $x = x_{i-1}$ and $m = m_i$.

▷ Claim 11. Exactly one of the following holds:

(a) $x \in \sigma$, $m \notin \sigma$ and $G(\Sigma_m)$ has an obstruction covering the side of f_m not covered by C_m; or

(b) $x \notin \sigma$ and $G(\Sigma_x)$ has an obstruction covering the side of f_x not covered by C_x.

Proof. By redrawing the arcs representing f_x and f_m, we will assume that they only intersect at x_1. In particular this redrawing creates two copies of the edge e_1.



(a) Case a.1. **(b)** Case a.2.

Figure 8 Illustrations for Claim 11.a.

First, suppose that $x \in \sigma$. For (a) we have two cases depending on whether xx_i is an edge in C_x.

Case a.1 *$xx_i \notin E(C_x)$. See Figure 8a.*

Let C'_m be the cycle obtained from C_x by replacing the edge f_x by the path $P = (x_1, f_m, m, mx, x)$. Since $x \in \sigma$, by the choice of C_x (Claim 9), all the edges in C_x are in σ^x. Therefore, by Claim 9.1, all the edges in C'_m, with the possible exception of mx, are in σ^m. Thus C'_m is an obstruction in $G(\Sigma_m)$.

Now we show that C'_m covers the right side of f_m. The disk bounded by $P \cup f_x$ is to the right of f_m as this side of $P \cup f_x$ is included in the bounded face F. Since the interior of C_x is to the right of f_x, the interior of C'_m covers the right side of f_m.

Finally, note that $m \notin \sigma$, or else, $C'_m \subseteq \sigma^m$ and hence by the choice of C_m, and Claim 10, $C'_m = C_m$. However, this contradicts that C_m covers the left side of f_m. Thus, (a) holds.

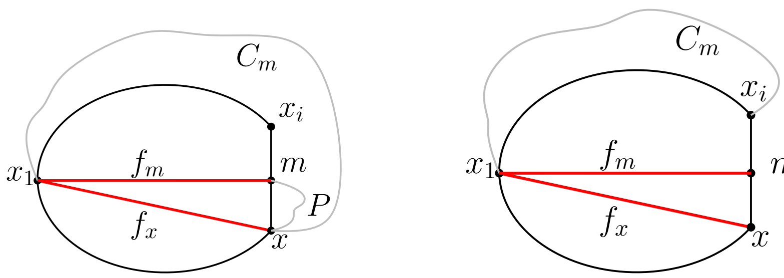
Case a.2. *$xx_i \in E(C_x)$. See Figure 8b.*

Let C'_m be the cycle obtained from C_x by replacing the path (x_1, f_x, x, xx_i, x_i) by (x_1, f_m, m, mx_i, x_i). Since $x \in \sigma$, by the choice of C_x (Claim 9), all the edges in C_x are in σ^x. Therefore all the edges in C'_m are in σ^m. Thus C'_m is an obstruction in $G(\Sigma_m)$.

Now we show that C'_m covers the right side of f_m. The disk bounded by $f_x \cup f_m \cup xm$ is to the right of f_m as this side of $f_x \cup f_m \cup xm$ is included in the bounded face F. Since the interior of C_x is to the right of f_x, the interior of C'_m covers the right side of f_m.

Finally, as $C'_m \subseteq \sigma^m$ and by the choice of C_m, $C'_m = C_m$. However, this contradicts the assumption that C_m covers the left side of f_m. Thus, (a) holds.

Turning to (b), let us suppose that $x \notin \sigma$.



(a) Case b.1. **(b)** Case b.2.

Figure 9 Illustrations for Claim 11.b.

Case b.1. $x \in V(C_m)$. *See Figure 9a.*

Let T be the triangle bounded by f_x, f_m and xm. The interior face of T is to the left of f_x and to the right of f_m. Let P be the mx-path of $C_m - f_m$ and let P' be the xx_1-path of $C_m - m$. Since the interior face of T is a subset of F, P and P' are drawn in the closure of the exterior of T (possibly $P = (m, mx, x)$).

Let C be the simple closed curve bounded by $P \cup f_x \cup f_m$ (in other words, C is obtained from T by replacing xm by P). Seeking a contradiction, suppose that xm is in the closed exterior of C. Then, P' is included inside the cycle $C' = P + xm$. Since $V(C') \subseteq V(C_m)$ and C_m is included in the disk bounded by C', the number of rainbows in C' is at most the number of rainbows in C_m. Then C' is an obstruction in $G(\Sigma_m)$ not containing f_m, contradicting Claim 9.2. Thus, xm is inside C.

Our last observation implies that P' is an arc connecting x_1 and x in the exterior of C. Since the interior of C_m covers the left of f_m, the interior of $C'_x = P' + f_x$ covers the left of f_x. The cycle C'_x is an obstruction because $V(C'_x) \subseteq V(C_m)$ and C_m is included inside C'_x.

Case b.2. $x \notin V(C_m)$. *See Figure 9b.*

In this case we let C'_x be the cycle obtained by replacing the path (x_1, f_m, m, mx_i, x_i) in C_m by the path $P = (x_1, f_x, x, xx_i, x_i)$ in $G(\Sigma_x)$. Since C_m covers the left of f_m and F is bounded, C'_x covers the left of f_x.

To show that C'_x is an obstruction, note that C_m is inside C'_x and that $V(C'_x) \setminus \{x\} \subseteq V(C_m)$. Thus, all the rainbows of C'_x in $V(C'_x) \setminus \{x\}$ are also rainbows in C_m. Since $x \notin \sigma$, we see that x is a rainbow in C'_x, but is not a vertex of C_m. To compensate, we note that m is a rainbow in C_m that is not in $V(C_x)$: if m is not rainbow, both f_m and xx_i are included in σ, implying that $x \in \sigma$. This shows that C'_x has at most as many rainbows as C_m. Therefore C'_x is the desired obstruction. ◀

Claims 10 and 11 contradict each other, so, for some $p \in P$, $G(\Sigma_p)$ has no obstructions. ◀

▶ **Lemma 12** (Exterior-Meeting Step). *If all the strings in Σ have their ends on the outer face of $G(\Sigma)$ and the ends have degree 1 in $G(\Sigma)$, then we can extend a pair disjoint strings so that they intersect without creating an obstruction.*

Proof. First, consider a simple closed curve in the outerface of $\bigcup \Sigma$ closely following the outerboundary of $\bigcup \Sigma$. Then, by slightly modifying this curve, we obtain a simple closed cuve $\mathcal{O}$ containing all the ends of the strings in Σ, but otherwise disjoint from $\bigcup \Sigma$. See Figure 10.

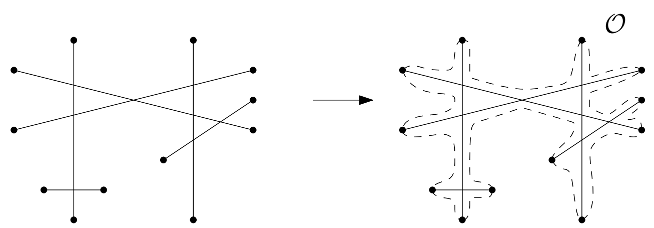


Figure 10 Construction of the curve $\mathcal{O}$.

Suppose σ_1, σ_2 are two disjoint strings in Σ. For $i = 1, 2$, let a_i, b_i be the ends of σ_i; since σ_1 and σ_2 do not cross, we may assume that these ends occur in the cyclic order a_1, b_1,

b_2, a_2. We extend the a_i-ends of σ_1 and σ_2 so that they meet in a point p in the outer face, and so that all the ends of σ_1 and σ_2 remain incident with the outer face (Figure 11). Let Σ' be the obtained set of strings.

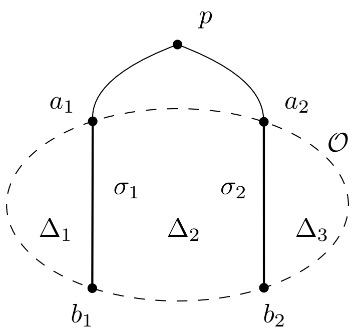


Figure 11 Exterior-Meeting Step.

Seeking a contradiction, suppose that $G(\Sigma')$ has an obstruction C. Since $G(\Sigma)$ has no obstruction, $p \in V(C)$. Our contradiction will be to find three rainbows in C. Note that p is a rainbow. To obtain a second rainbow, traverse C starting from p towards a_1. Let d_1 be the first vertex during our traversal that is not in the extended σ_1, and let c_1 be its neighbour in σ_1, one step before we reach d_1. Since b_1 has degree one, $c_1 \neq b_1$.

The strings σ_1 and σ_2 divide the disk bounded by $\mathcal{O}$ into three closed regions Δ_1, Δ_2, Δ_3 such that $\Delta_1 \cap \Delta_2 = \sigma_1$, $\Delta_2 \cap \Delta_3 = \sigma_2$ and $\Delta_1 \cap \Delta_3 = \emptyset$ (see Figure 11).

▷ Claim 13. The cycle C has a rainbow included in Δ_1.

Proof. First, suppose that $d_1 \notin \Delta_1$. In this case, c_1 is a rainbow because otherwise there would be a string σ that tangentially intersects σ_1 at c_1. Thus, if $d_1 \notin \Delta_1$, then c_1 is the desired rainbow.

Second, suppose that $d_1 \in \Delta_1$. Let P_1 be the path of C starting at c_1, continuing on the edge c_1d_1, and ending at the first vertex we encounter in σ_1. Let C' be the cycle enclosed by $P_1 \cup \sigma_1$. Since C' is not an obstruction, there is one rainbow of C' that is an interior vertex of P_1; this is the desired rainbow of C. This concludes the proof of Claim 13. ◀

Considering σ_2 instead of σ_1, Claim 13 yields a third rainbow in C inside the region Δ_3 analogous to Δ_1, contradicting that C is an obstruction. Hence Lemma 12 holds. ◀

We iteratively apply the Disentangling Step, Face-Escaping Step or Exterior-Meeting Step without creating obstructions. Each step increases the number of pairwise intersecting strings in Σ until we reach a stage where the strings are pairwise intersecting and all of them have their two ends in the unbounded face. From this we extend them into an arrangement of pseudolines. This concludes the proof of Theorem 2. ◀

5 Finding obstructions and extending strings in polynomial time

We start this section by describing an algorithm to detect obstructions. Henceforth, we assume that the input to the problem is the planarization $G(\Sigma)$ of an ordinary set of s strings Σ. For the running-time analysis, we assume that n and m are the number of vertices and edges in $G(\Sigma)$, respectively. Since $G(\Sigma)$ is planar, $m = O(n)$. Moreover, if Σ is pseudolinear, then $n \le \binom{s}{2} + 2s = \binom{s+2}{2} - 1$. At the end of this section we explain how to extend Σ (if possible) in polynomial time.

Recall that each string in Σ receives a different colour; this induces an edge-colouring on $G(\Sigma)$ where each string is a monochromatic path. An *outer-rainbow* is a vertex $x \in V(G(\Sigma))$ incident with the outer face and for which the edges incident with x have different colours. Next we describe the basic operation in our obstruction-detecting algorithm.

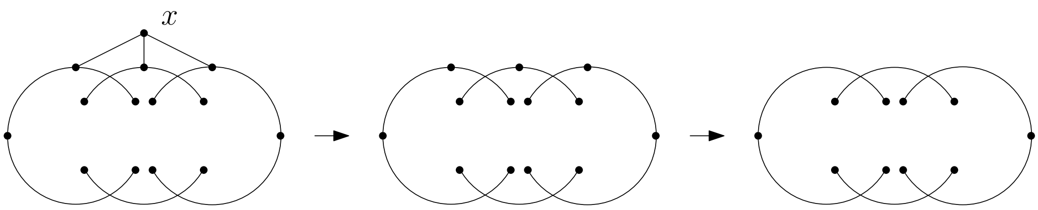


Figure 12 From Σ to $\Sigma - x$.

Outer-rainbow deletion. Given an outer-rainbow $x \in V(G(\Sigma))$, the instance $G(\Sigma - x)$ is defined by: first, removing x and the edges incident to x; second, suppressing the degree-2 vertices incident with edges of the same colour; and third, removing remaining degree-0 vertices (Figure 12 illustrates this process). Edge colours are preserved.

It is easy to verify that $G(\Sigma - x)$ is the planarization of an arrangement of strings. The colours removed by this operation are those belonging to strings that are paths of length 1 in $G(\Sigma)$ incident with x. Our obstruction-detecting algorithm relies on the following property:

(**) if x is an outer-rainbow of $G(\Sigma)$, then there is an obstruction in $G(\Sigma)$ not including x if and only if there is an obstruction in $G(\Sigma - x)$.

This property holds because cycles in $G(\Sigma) - x$ and in $G(\Sigma - x)$ are in 1-1 correspondence: two cycles correspond to each other if they are the same simple closed curve. This correspondence is obstruction-preserving.

Let us now describe the two subroutines in our algorithm. For this, we remark that an outer-rainbow of $G(\Sigma)$ is a rainbow for any cycle containing it.

Subroutine 1. *Detecting an obstruction through two outer-rainbows x and y.*

(1) Find a cycle C through x and y whose edges are incident with the outer face of $G(\Sigma)$. If C exists, then this cycle is unique and can be described as the outer boundary of the block containing x and y. If no such C exists, then output *No obstruction through x and y*. Else, go to Step 2.

(2) Find whether there is a third outer-rainbow $z \in V(C) \setminus \{x, y\}$. If such z exists, update $G(\Sigma) \longleftarrow G(\Sigma - z)$ and go to Step 1. If no such z exists, output C.

Correctness and running-time of Subroutine 1: If an obstruction through x and y exists, then x and y are in the same block (some authors use the term ‘biconnected component’). Since x and y are incident with the outer face, the outer boundary of the block containing x and y is the cycle C from Step 1. This C can be found by considering outer boundary walk W of $G(\Sigma)$ and then by finding whether x and y belong to the same non-edge block of W. Finding W is $O(m)$ and computing the blocks of W via a DFS takes $O(m)$ time.

In Step 2, if there is a third outer rainbow z in C, then no obstruction through x and y contains z. Property (**) justifies the update that takes $O(m)$ time.

A full run from Step 1 to Step 2 takes $O(m)$. Moving from Step 2 to Step 1 occurs $O(n)$ times. Thus, the total time for Subroutine 1 is $O(mn) = O(n^2)$.

Subroutine 2. *Detecting an obstruction through a single outer-rainbow x.*

(1) Find a cycle C through x whose edges are incident with the outer face of $G(\Sigma)$. If no such C exists, output *No obstruction through* x. Else, go to Step 2.
(2) Find whether there is an outer-rainbow y in $V(C) \setminus \{x\}$. If no such y exists, output C. Else, apply Subroutine 1 to x and y; if there is an obstruction C' through x and y, then output C'. Else, update $G(\Sigma) \longleftarrow G(\Sigma - y)$ and go to Step 1.

Correctness and running-time of Subroutine 2: If $G(\Sigma)$ has an obstruction through x, then there is a non-edge block in $G(\Sigma)$ containing x. The outer boundary of this block is a cycle C through x having all edges incident with the outer face. As in Subroutine 1, Step 1 takes $O(m)$ time.

Detecting the existence of y in Step 2 is $O(m)$ because to detect rainbows in C, each edge incident with a vertex in $V(C)$ is verified at most twice. The update in Step 2 is justified by Property (**). Since Step 2 may use Subroutine 1, Step 2 takes $O(n^2)$ time. As moving from Step 2 to Step 1 occurs $O(n)$ times, the total running-time for Subroutine 2 is $O(n^3)$.

We are now ready for the algorithm to detect obstructions.

Algorithm 1: *Detecting obstructions in* $G(\Sigma)$.

(1) Find a cycle C having all edges incident with the outer face. If no such C exists, output *No obstruction.* Else, go to step 2.
(2) Find whether there is an outer rainbow $x \in V(C)$. If not, output C. Else apply Subroutine 2 to x; if there is an obstruction C' through x, output C'. Else, update $G(\Sigma) \longleftarrow G(\Sigma - x)$ and go to Step 1.

Correctness and running-time of Algorithm 1: If $G(\Sigma)$ has an obstruction, then it has a non-trivial block whose outer boundary is a cycle C as in Step 1. As before, C and x as in Step 2 can be found in $O(m)$ steps. If C has not outer rainbow x, then C is an obstruction; Property (**) justifies the update in Step 2.

Since Step 2 may use Subroutine 2, a full run of Steps 1 and 2 takes $O(n^3)$ time. Since Step 2 goes to Step 1 $O(n)$ times, the running-time of Algorithm 1 is $O(n^4)$.

Algorithm 1 and the constructive proof of Theorem 2 imply the following result.

▶ **Theorem 14.** *There is a polynomial-time algorithm to recognize and extend an ordinary set of strings that are extendible to an arrangement of pseudolines.*

Proof. Let Σ be an ordinary set of s strings. First, note that if $n = |V(G(\Sigma))|$, $m = |E(G(\Sigma))|$, and Σ is extendible, then $n \leq \binom{s}{2} + 2s$. Hence $n, m = O(s^2)$.

Assume that $G(\Sigma)$ has not obstructions, by first verifying that $n \leq \binom{s}{2} + 2s$ and then running Algorithm 1. For each end in each string in Σ, we keep track of whether one of the Disentangling, Face-Escaping or Exterior-Meeting Steps apply.

The Disentangling and Face-Escaping Steps consist on extending one end a of a fixed string $\sigma \in \Sigma$ in different ways to find an obstruction-free set of strings. For the Disentangling Step, the number of possible extensions is bounded by the maximum degree of $G(\Sigma)$; for the Face-Escaping Step, the number of possible extensions is bounded by twice the length of the face containing the end that we are extending. Thus, each step lead to $O(m)$ possibilities, and testing obstructions in each of them is $O(n^4)$. Thus, the Disentangling and the Face-Escaping Steps take $O(n^5)$ time.

The Exterior-Meeting Step is $O(m^2)$ because for this step we just need to record the number of the pairwise disjoint strings in Σ and the set of strings that have ends incident with the outer face; if all the strings have their ends in the outer boundary, the extension is performed as in the proof of Lemma 12.

As there is a total of $O(s^2)$ extending steps, extending Σ is $O(s^2(n^5 + m^2)) = O(s^{12})$. ◀

6 Concluding remarks

In this work we characterized in Theorem 2 sets of strings that can be extended into arrangements of pseudolines. Moreover, we showed that the obstructions to pseudolinearity can be detected in $O(n^4)$ time, where n is the number of vertices in the planarization of the set of strings.

An easy consequence of Theorem 2 is the following (presented before as Theorem 1).

▶ **Theorem 15.** *Let D be a non-pseudolinear good drawing of a graph H. Then there is a subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma\in S} \sigma'$ is one of the drawings represented in Figure 2.*

Proof. Take C an obstruction of the planarization associated to D. Let $\delta(C) \subseteq V(C)$ be the vertices that in C are incident with two different strings in $\Sigma = \{D[e] : e \in E(H)\}$. We choose our obstruction C so that $|\delta(C)|$ is as small as possible.

Decompose C into a cyclic sequence of paths $P_0, \ldots, P_m$, where P_i connects two points in $\delta(C)$ and it is otherwise disjoint from $\delta(C)$. Using Lemma 4, one can show that $P_0, \ldots, P_m$ belong to distinct edge-arcs $\sigma_0, \ldots, \sigma_m \in \Sigma$, respectively. For each P_i, we consider the string σ'_i, obtained by slightly extending the ends of P_i that are not rainbows in C; we extend them along σ_i.

Let $x \in \delta(C)$ be an end shared by P_{i-1} and P_i. If x is not a rainbow for C, then x is a crossing between σ_{i-1} and σ_i. Moreover, the arcs added to P_{i-1} and P_i at x to obtain σ'_{i-1} and σ'_i are inside C. If x is a rainbow in C, then P_i and P_{i-1} are not extended at x, and x acts as one of the degree-2 vertices in Figure 2. The rest of the points in $\delta(C)$ are crossings in $\bigcup_{i=0}^m \sigma'_i$ facing the interior of C. Since C has at most two rainbows, $\bigcup_{i=0}^m \sigma'_i$ is one of the drawings depicted in Figure 2. ◀

Theorem 2 can also be applied to show that *a drawing of K_n is pseudolinear if and only if does not contain the B-configuration* (Theorem 2.5.1 in [3]). We sketch the proof of a specific case of this theorem in the next two paragraphs and comment on the general case afterwards.

Suppose that $G(\Sigma)$ is the planarization of a non-pseudolinear drawing D of K_n for which we would like to show that D contains a B-configuration. Consider an obstruction C of $G(\Sigma)$ minimizing $|\delta(C)|$, where $\delta(C)$ are vertices of C incident with edges in C having different colours. For illustrative purposes, let us assume that C contains two vertices from $V(K_n)$. Since C is an obstruction, u and v are the only rainbows of C.

An edge e of K_n is *involved in C* if C contains a subarc of $D[e]$ (see Figure 13). By using Lemma 4 is not hard to show that every edge involved in C is drawn inside C. Consider all the vertices incident with an edge involved in C and let D' be the drawing of the complete graph induced by these vertices. Then, D' has at most two vertices in its outer boundary, namely u and v. Thus, the outer boundary of D' is incident with at least one crossing. The K_4 containing this crossing is drawn as in Figure 1 with its crossing incident with the outer face. This K_4 contains a B-configuration.

The proof for the general case, where C does not necessarily contains two vertices of K_n, is considered in full detail in [3], and uses the complete subgraph induced by the edges involved in C combined with the fact that $|\delta(C)|$ is minimal.

A drawing is *stretchable* if it is homeomorphic to a rectilinear drawing. There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 14. Nevertheless, the following is an immediate consequence of Thomassen's main result in [19].

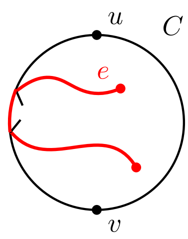


Figure 13 An edge e involved in the obstruction C.

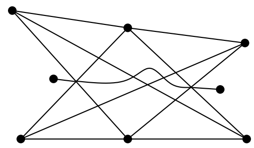


Figure 14 Non-Pappus configuration.

▶ **Corollary 16.** *A 1-planar drawing of a graph is stretchable if and only if it is pseudolinear.*

Proof. If a drawing D is stretchable then clearly it is pseudolinear. To show the converse, suppose that D is pseudolinear. Then D does not contain any obstruction, and in particular, neither of the B- and W-configurations in Figure 2 occurs in D. This condition was shown in [19] to be equivalent to being stretchable. ◀

One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-strechable arrangements of pseudolines. However, such examples seem to inevitably have some edge with multiple crossings. This leads to a natural question.

▷ Question 17. Is it true that if D is a pseudolinear drawing in which every edge is crossed at most twice, then D is stretchable?

We believe that there are other instances where pseudolinearity characterizes stretchability of drawings. A drawing is *near planar* if the removal of one edge produces a planar graph. One instance, is the following result by Eades et al. that can be translated to the language of pseudolines:

▶ **Theorem 18.** *[9] A drawing of a near-planar graph is stretchable if and only if the drawing induced by the crossed edges is pseudolinear.*

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