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Abstract

The aim of this work is to study the asymptotic stability of the nonlinear Korteweg-de Vries equation in the presence of a delayed term. We first consider the case where the weight of the term with delay is smaller than the weight of the term without delay and we prove a semiglobal stability result for any lengths. Secondly we study the case where the support of the term without delay is not included in the support of the term with delay. In that case, we give a local exponential stability result if the weight of the delayed term is small enough. We illustrate these results by some numerical simulations.

Keyword: KdV equation, asymptotic stability, delay

1 Introduction and main results

The Korteweg-de Vries (KdV) equation is the nonlinear dispersive partial differential equation $y_t + y_x + y_{xxx} + yy_x = 0$, which models the (unidirectional) propagation of a water wave of small amplitude in an uniform bounded channel. The KdV equation has been the subject of intensive research (see for instance [1], [26], etc).

Without delay, the exponential stability of the nonlinear KdV equation was first studied in [32], with a boundary damping and where the length of the spatial domain is $L = 1$. However, it is well-known that the length of the domain plays an important role in the controllability or the stability questions of the KdV equation. For instance, if $L = 2\pi$, there exists a solution ($y(x,t) = 1 - \cos x$) of the linearized system around 0 which has a constant L^2-norm. More generally, for a length belonging to the set of critical lengths

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},\, k, l \in \mathbb{N}^* \right\},$$

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we can construct an initial data whose corresponding solution of the linear KdV equation has a constant L^2-norm. Nevertheless, if the length is non critical (i.e. $L \notin \mathcal{N}$), the nonlinear KdV equation is locally exponentially stable, even without adding an internal or a boundary feedback as in [32] (see [24]). Moreover, for any critical length, the nonlinear KdV equation is locally exponentially stable adding a localized damping, and even a semi-global stability result holds by working directly with the nonlinear system (see [24] and [21]). We mention [6] and [30] in which the asymptotic stability for the nonlinear KdV equation for particular critical lengths has been proven without any feedback law.

The main goal of this paper is to study the asymptotic stability of the following nonlinear KdV equation with an internal delayed feedback:

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) + a(x)y(x,t) + b(x)y(x,t-h) = 0, \\ \qquad x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\ t \in (-h,0), \end{cases} \tag{1.1}$$

where $h > 0$ is the (constant) delay, $L > 0$ is the length of the spacial domain, $y(x,t)$ is the amplitude of the water wave at position x at time t, and $a = a(x)$ and $b = b(x)$ are nonnegative functions belonging to $L^\infty(0,L)$. We will also assume that $\operatorname{supp} b = \omega$, where $\operatorname{supp} b$ is the support of the function b, and

$$b(x) \geq b_0 > 0 \quad \text{a.e. in } \omega \tag{1.2}$$

where ω is an open, nonempty subset of $(0,L)$.

In the case without delay (i.e. $b = 0$) it is well-known (see for instance [24]) that for every $T > 0$, $L > 0$ and $y_0 \in L^2(0,L)$, the system (1.1) is locally well-posed in

$$\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H_0^1(0,L)).$$

We will give in Section 2 the proof of the well-posedness of (1.1) with delay (i.e. $b \neq 0$).

There are two main difficulties to study (1.1): the nonlinear character of this system (due to the presence of yy_x) and the delay in the internal feedback. In particular we have to prove that the delay in the feedback will not destabilize the system, which can be the case for other delayed systems, see for instance [11]. Very recently, the robustness with respect to the delay of the boundary stability of the nonlinear KdV equation has been study in [2], where the boundary condition is $y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0,t-h)$. The authors obtain, under an appropriate condition on the feedback gains with and without delay (i.e. $|\alpha| + |\beta| < 1$), the locally exponentially stability result for non critical length. Note that no condition about the size of the delayed feedback gain β with respect to the non-delayed feedback gain α is required. The aim of this present work is to extend these results to internal damping with delay for any length, and to study if we need a restrictive assumption on the weights a and b.

We first assume that the coefficients a and b satisfy to the following assumption:

$$\exists c_0 > 0, \quad b(x) + c_0 \leq a(x) \quad \text{ a.e. in } \omega. \tag{1.3}$$

Note that (1.2) and (1.3) imply that $\omega = \operatorname{supp} b \subset \operatorname{supp} a$ and

$$a(x) \geq b_0 + c_0 > 0 \quad \text{ a.e. in } \omega. \tag{1.4}$$

We define the Hilbert space of the initial and delayed data $\mathcal{H} := L^2(0, L) \times L^2((0, L) \times (-h, 0))$, endowed with the norm defined for all $(y, z) \in \mathcal{H}$ by

$$\|(y, z)\|_{\mathcal{H}}^2 = \int_0^L y^2(x)dx + \int_0^L \int_{-h}^0 \xi(x)z^2(x, s)dxds,$$

where ξ is a nonnegative function in $L^\infty(0, L)$ chosen such that $\operatorname{supp} \xi = \operatorname{supp} b = \omega$ and

$$b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0 \quad \text{ a.e. in } \omega. \tag{1.5}$$

Note that this choice of ξ is possible due to (1.3).

Let us now give the following definition of the energy of system (1.1), corresponding to the norm of $(y(\cdot, t), y(\cdot, t + \ \cdot))$ on $\mathcal{H}$:

$$E(t) = \int_0^L y^2(x, t)dx + h \int_\omega \int_0^1 \xi(x)y^2(x, t - h\rho)d\rho dx, \tag{1.6}$$

where $\xi \in L^\infty(0, L)$ satisfies (1.5). The first part of the energy E corresponds to the natural energy of the KdV equation, and the second part is classical when considering internal delayed terms, as in [16] for the wave equation. Formally the energy satisfies the following dissipation law

$$\frac{d}{dt} E(t) \leq -y_x^2(0, t) + \int_\omega \left(-2a(x) + b(x) + \xi(x)\right) y^2(x, t)dx - 2 \int_{(0,L)\setminus\omega} a(x)y^2(x)dx \\ + \int_\omega \left(b(x) - \xi(x)\right) y^2(x, t - h)dx, \tag{1.7}$$

see Proposition 5, which is non positive due to (1.5).

The assumption (1.3), i.e. the fact that the weight of the term with delay is smaller than the weight of the term without delay, can be found in [16], for the asymptotic stability of the wave equation with delayed feedback. This restriction about the weights of the feedbacks is also used for hyperbolic and parabolic partial differential equations in [19, 20] and even for the Schrödinger equation (which is a dispersive equation, like KdV equation) in [18]. This restrictive assumption is necessary in these cases and if they are not satisfied, it can be shown that instabilities may appear (see for instance [11], [12] with $a = 0$, or [16] in the more general case for the wave equation). However it is not the case for the delayed boundary stability of the nonlinear KdV equation (see [2]). The necessity of (1.3) here is confirmed by numerical simulations in Section 5.

We refer to [33] for the existence of the Korteweg-de Vries-type equation with delay and to the recent work [13] for the distributed stabilization of the Korteweg-de Vries-Burgers equation in the presence of input delay. Concerning the stabilization of the KdV equation by the construction of time varying feedback laws, without delay, we can cite the recent papers [10] and [31]. The rapid stabilization (or how to construct a feedback law which stabilizes the system at a prescribed decay rate) has been studied in [5], in [14] and [4] by the backstepping method, and in [9] by an integral transform. Finally a related interesting question is the global stabilization of a nonlinear KdV equation with a saturating distributed control studied recently in [15].

The first main result of this paper is a semi-global stability result for any lengths, when (1.3) holds, working directly with the nonlinear system (1.1).

Theorem 1. *Assume that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying* (1.2) *and* (1.3). *Let $L > 0$ and $R > 0$. There exists $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that*

$$E(t) \leq CE(0)e^{-\mu t}, \qquad t > 0,$$

for any solution of (1.1) *with $\|(y_0, z_0)\|_{\mathcal{H}} \leq R$.*

The semi-global character of this result comes from the fact that even if we are able to chose any radius R for the initial data, the decay rate μ depends on R.

The second main result is a local stability result in the case where $\operatorname{supp} b \not\subset \operatorname{supp} a$ (and so (1.3) does not hold), with a restrictive assumption on the length L of the domain and with the weight of the delayed feedback small enough.

Theorem 2. *Assume that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying* (1.2) *and assume that the length L fulfills* (3.30). *Let $\xi > 1$. Then there exist $\delta > 0$ (depending on ξ, L, h) and $r > 0$ sufficiently small such that if*

$$\|b\|_{L^\infty(0,L)} \leq \delta,$$

for every $(y_0, z_0) \in \mathcal{H}$ satisfying

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

the energy of system (1.1), *denoted E and defined by* (1.6) *with $\xi(x) = \xi b(x)$, decays exponentially.*

To prove this theorem, we consider a "close" auxiliary problem for which the energy is decreasing and we use a classical perturbation result of Pazy [23], inspired by Nicaise and Pignotti [17]. It is very interesting to note that we can take $a = 0$ in Theorem 2. This kind of result seems new for the asymptotic stability of the nonlinear KdV equation with delay.

The paper is organized as follows. We prove the well-posedness result and regularity of the solutions of (1.1) in Section 2. Section 3 is devoted to the case where (1.3) holds, with the proofs of a local exponential stability result by a Lyapunov functional in Subsection 3.1 (which holds for restrictive lengths of the domain but gives an estimation of the decay rate) and of the semi-global stability result, stated in Theorem 1, in Section 3.2. The study of the case $\operatorname{supp} b \not\subset \operatorname{supp} a$ of Theorem 2 is done in Section 4. Some remarks and numerical simulations are presented in Section 5.

2 Well-posedness and regularity results

2.1 Study of the linear equation

This subsection is devoted to the proof of the well-posedness of the KdV equation linearized around 0, that writes

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) + b(x)y(x,t-h) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\ t \in (-h,0). \end{cases} \tag{2.8}$$

Classically, when dealing with delayed equations and following Nicaise and Pignotti [16], we set $z(x,\rho,t) = y_{|\omega}(x,t-\rho h)$ for any $x \in \omega$, $\rho \in (0,1)$ and $t > 0$. It is easy to show that z satisfies the transport equation

$$\begin{cases} hz_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \omega,\ \rho \in (0,1),\ t > 0, \\ z(x,0,t) = y_{|\omega}(x,t), & x \in \omega,\ t > 0, \\ z(x,\rho,0) = z_{0|\omega}(x,-\rho h), & x \in \omega,\ \rho \in (0,1). \end{cases} \tag{2.9}$$

We equipped the Hilbert space $H = L^2(0,L) \times L^2(\omega \times (0,1))$ with the inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y\tilde{y}\,dx + h\int_\omega \int_0^1 \xi(x) z\tilde{z}\,d\rho dx,$$

for any $(y,z), (\tilde{y},\tilde{z}) \in H$, where ξ is a nonnegative function in $L^\infty(0,L)$ such that $\operatorname{supp}\xi = \operatorname{supp} b = \omega$ and (1.5) holds. We denote by $\|\cdot\|_H$ the associated norm and this new norm is clearly equivalent to the usual norm on H since $\xi(x) > b(x) \geq b_0 > 0$ on ω (see (1.5)).

We then rewrite (2.8) and (2.9) as a first order system:

$$\begin{cases} U_t(t) = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0 \in H, \end{cases} \tag{2.10}$$

where $U = \begin{pmatrix} y \\ z \end{pmatrix}$, $U_0 = \begin{pmatrix} y_0 \\ z_{0|\omega}(\cdot, -h\ \cdot) \end{pmatrix}$, and where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -y_{xxx} - y_x - ay - b\tilde{z}(\cdot, 1) \\ -\frac{1}{h}\, z_\rho \end{pmatrix},$$

where $\tilde{z}(\cdot, 1) \in L^2(0, L)$ is the extension of $z(\cdot, 1)$ by zero outside ω, with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{(y, z) \in H^3(0, L)\times L^2(\omega, H^1(0, 1)) \,\big|\, y(0) = y(L) = y_x(L) = 0,\ z(x, 0) = y_{|\omega}(x) \text{ in } \omega\Big\}.$$

Theorem 3. *Assume that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying (1.2) and (1.3), and that $U_0 \in H$. Then there exists a unique mild solution $U \in C([0, +\infty), H)$ for system (2.10). Moreover if $U_0 \in \mathcal{D}(\mathcal{A})$, then the solution is classical and satisfies*

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), H).$$

Proof. We first prove that the operator $\mathcal{A}$ is dissipative. Let $U = (y, z) \in \mathcal{D}(\mathcal{A})$. Then we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= -\int_0^L y_{xxx}y\,dx - \int_0^L y_x y\,dx - \int_0^L a(x)y^2dx - \int_\omega b(x)z(x,1)y(x)dx \\
&\quad - \int_\omega\int_0^1 \xi(x)z_\rho(x,\rho)z(x,\rho)\,dxd\rho \\
&= \int_0^L y_{xx}y_x\,dx - [y_{xx}y]_0^L - \frac{1}{2}[y^2]_0^L - \int_0^L a(x)y^2dx - \int_\omega b(x)z(x,1)y(x)dx - \frac{1}{2}\int_\omega \xi(x)[z^2]_0^1dx \\
&= \frac{1}{2}[y_x^2]_0^L - \int_0^L a(x)y^2dx - \int_\omega b(x)z(x,1)y(x)dx - \frac{1}{2}\int_\omega \xi(x)z^2(x,1)dx + \frac{1}{2}\int_\omega \xi(x)y^2(x)dx \\
&\le -\frac{1}{2}y_x^2(0) + \int_\omega \left(-a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2}\right) y^2(x)dx - \int_{(0,L)\setminus\omega} a(x)y^2(x)dx \\
&\quad + \int_\omega \left(\frac{b(x)}{2} - \frac{\xi(x)}{2}\right) z^2(x,1)dx.
\end{aligned}$$

If we take ξ such that (1.5) holds (which is possible due to (1.3)), then $-a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} < 0$ and $\frac{b(x)}{2} - \frac{\xi(x)}{2} < 0$ in ω, and $-a(x) \le 0$ in $(0, L) \setminus \omega$. Consequently $\langle \mathcal{A}U, U\rangle \le 0$, which means that the operator $\mathcal{A}$ is dissipative.

Secondly we show that the adjoint of $\mathcal{A}$, denoted by $\mathcal{A}^*$, is also dissipative. It is not difficult to prove that the adjoint is defined by

$$\mathcal{A}^*U = \begin{pmatrix} y_{xxx} + y_x - ay + \xi(x)\tilde{z}(\cdot, 0) \\ \frac{1}{h}\, z_\rho \end{pmatrix}, \qquad U = \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*),$$

with domain

$$\mathcal{D}(\mathcal{A}^*) = \Big\{(y, z) \in H^3(0, L)\times L^2(\omega, H^1(0, 1)) \big| y(0) = y(L) = y_x(0) = 0, z(x, 1) = -\frac{1}{\xi(x)}\, b(x)y_{|\omega}(x) \text{ in } \omega\Big\}.$$

Then for all $U = (y, z) \in \mathcal{D}(\mathcal{A}^*)$, we have

$$\langle \mathcal{A}^* U, U \rangle = \int_0^L y_{xxx} y \, dx + \int_0^L y_x y \, dx - \int_0^L a(x) y^2 dx + \int_\omega \xi(x) z(x,0) y(x) dx + \int_0^1 \int_\omega \xi(x) z_\rho z \, dx d\rho$$
$$= \; -\frac{1}{2}[y_x^2]_0^L - \int_0^L a(x) y^2 dx + \int_\omega \xi(x) z(x,0) y(x) dx + \frac{1}{2}\int_\omega \frac{1}{\xi(x)} b^2(x) y^2(x) dx - \frac{1}{2}\int_\omega \xi(x) z^2(x,0) dx$$
$$\leq -\frac{1}{2} y_x^2(L) + \int_\omega \left(-a(x) + \frac{1}{2\xi(x)} b^2(x) + \frac{\xi(x)}{2} \right) y^2(x) dx - \int_{(0,L)\setminus\omega} a(x) y^2(x) dx \leq 0,$$

since, due to (1.5), we have in ω

$$-a(x) + \frac{1}{2\xi(x)} b^2(x) + \frac{\xi(x)}{2} < -a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} \leq 0.$$

Finally, since $\mathcal{A}$ is a densely defined closed linear operator, and both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, then $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on H (see for instance Corollary 4.4 of [23]), which finishes the proof. $\square$

We denote by $\{S(t), t \geq 0\}$ the semigroup of contractions associated with $\mathcal{A}$. In the following, by abusing the notation, we identify $z_{0|\omega}$ and z_0, and the real C is a positive constant that can depend on T, h, $\|a\|_{L^\infty(0,L)}$ and $\|b\|_{L^\infty(0,L)}$. In the following proposition, we detail a few *a priori* and regularity estimates of the solutions of systems (2.8) and (2.9).

Proposition 1. *Assume that* (1.2) *and* (1.3) *are satisfied. Then, the map*

$$(y_0, z_0(\cdot, -h\ \cdot)) \quad \mapsto \quad S(\cdot)(y_0, z_0(\cdot, -h\ \cdot)) \tag{2.11}$$

is continuous from H *to* $\mathcal{B} \times C([0,T], L^2(\omega \times (0,1)))$*, and for* $(y_0, z_0(\cdot, -h\ \cdot)) \in H$*, the following estimates hold*

$$\int_0^T \int_0^L a(x) y^2(x,t) dx dt + \int_0^T \int_\omega z^2(x,1,t) dx dt$$
$$\leq C \left(\|y_0\|^2_{L^2(0,L)} + \|z_0(\cdot, -h\ \cdot)\|^2_{L^2(\omega\times(0,1))} \right), \tag{2.12}$$

$$\|y_0\|^2_{L^2(0,L)} \leq C \left(\|y\|^2_{L^2(0,T,L^2(0,L))} + \|y_x(0,\cdot)\|^2_{L^2(0,T)} + \|z_0(\cdot, -h\ \cdot)\|^2_{L^2(\omega\times(0,1))} \right), \tag{2.13}$$

$$\|z_0(\cdot, -h\,\cdot)\|^2_{L^2(\omega\times(0,1))} \leq \|z(\cdot,\cdot,T)\|^2_{L^2(\omega\times(0,1))} + \frac{1}{h} \|z(\cdot,1,\cdot)\|^2_{L^2(\omega\times(0,T))}. \tag{2.14}$$

Proof. • First of all, for any $(y_0, z_0(\cdot, -h\ \cdot)) \in H$, Theorem 3 yields $S(.)(y_0, z_0(\cdot, -h\ \cdot)) = (y, z) \in C([0,T], H)$. As the operator $\mathcal{A}$ generates a C_0 semigroup of contractions we get for all $t \in [0,T]$,

$$\int_0^L y^2(x,t) dx + h \int_\omega \int_0^1 \xi(x) z^2(x,\rho,t) dx d\rho \leq \int_0^L y_0^2(x) dx + h \int_\omega \int_0^1 \xi(x) z_0^2(x, -\rho h) dx d\rho. \tag{2.15}$$

Let now $p \in C^\infty([0,1]\times[0,T])$, $q \in C^\infty([0,L]\times[0,T])$ and $(y_0, z_0(\cdot,-h\ \cdot)) \in \mathcal{D}(\mathcal{A})$. Then multiplying (2.9) by pz and (2.8) by qy, and using some integrations by parts we get

$$\int_0^1\int_\omega \big(p(\rho,T)z^2(x,\rho,T) - p(\rho,0)z_0^2(x,-\rho h)\big)\,dxd\rho - \frac{1}{h}\int_0^T\int_0^1\int_\omega (hp_t + p_\rho)z^2 dxd\rho dt \\ + \frac{1}{h}\int_0^T\int_\omega \big(p(1,t)z^2(x,1,t) - p(0,t)y^2(x,t)\big)\,dxdt = 0 \quad (2.16)$$

$$\int_0^L \big(q(x,T)y^2(x,T) - q(x,0)y_0^2(x)\big)\,dx - \int_0^T\int_0^L (q_t + q_x + q_{xxx})y^2 dxdt + 3\int_0^T\int_0^L q_x y_x^2 dxdt \\ + \int_0^T q(0,t)y_x^2(0,t)dt + 2\int_0^T\int_0^L a(x)qy^2 dxdt + 2\int_0^T\int_0^L b(x)qy(x,t)y(x,t-h)dxdt = 0. \quad (2.17)$$

• Let us first choose $p(\rho,t) \equiv \rho$ in (2.16). Then we obtain

$$\int_0^1\int_\omega \rho\left(z^2(x,\rho,T) - z_0^2(x,-\rho h)\right)dxd\rho - \frac{1}{h}\int_0^T\int_\omega\int_0^1 z^2 d\rho dxdt + \frac{1}{h}\int_0^T\int_\omega z^2(x,1,t)dxdt = 0$$

and thanks to (2.15) we have

$$\int_0^T\int_\omega z^2(x,1,t)dxdt \le C\big(\,\|y_0\|^2_{L^2(0,L)} + \|z_0(\cdot,-h\ \cdot)\|^2_{L^2(\omega\times(0,1))}\,\big). \quad (2.18)$$

Secondly, if we choose $q(x,t) \equiv 1$ in (2.17), then we get,

$$\int_0^L \big(y^2(x,T) - y_0^2(x)\big)\,dx + \int_0^T y_x^2(0,t)dt + 2\int_0^T\int_0^L a(x)y^2 dxdt + 2\int_0^T\int_0^L b(x)y(x,t)y(x,t-h)dxdt = 0,$$

which implies

$$2\int_0^T\int_0^L a(x)y^2 dxdt \le \|y_0\|^2_{L^2(0,L)} + 2\int_0^T\int_0^L b(x)\,|y(x,t)|\,|y(x,t-h)|\,dxdt.$$

Therefore, since

$$\begin{aligned}\int_0^T\int_0^L b(x)\,|y(x,t)|\,|y(x,t-h)|\,dxdt &\le \frac{1}{2}\int_0^T\int_0^L b(x)y^2(x,t)dxdt + \frac{1}{2}\int_0^T\int_0^L b(x)y^2(x,t-h)dxdt \\ &\le \int_0^T\int_0^L b(x)y^2(x,t)dxdt + \frac{1}{2}\int_0^L\int_{-h}^0 b(x)y^2(x,t)dxdt \\ &= \int_0^T\int_0^L b(x)y^2(x,t)dxdt + \frac{1}{2}\int_\omega\int_{-h}^0 b(x)z_0^2(x,\rho)d\rho dx\end{aligned}$$

we get, using (2.15), that

$$\int_0^T\int_0^L a(x)y^2 dxdt \le C\left(\|y_0\|^2_{L^2(0,L)} + \|z_0(\cdot,-h\ \cdot)\|^2_{L^2(\omega\times(0,1))}\right),$$

that concludes the proof of (2.12).

• Taking now $q(x,t) \equiv x$ in (2.17), we can write

$$\int_0^L x\left(y^2(x,T) - y_0^2(x)\right)dx - \int_0^T\int_0^L y^2 dxdt + 3\int_0^T\int_0^L y_x^2 dxdt$$
$$+ 2\int_0^T\int_0^L xa(x)y^2 dxdt + 2\int_0^T\int_0^L xb(x)y(x,t)y(x,t-h)dxdt = 0$$

and we have

$$3\int_0^T\int_0^L y_x^2(x,t)dxdt \leq L\|y_0\|^2_{L^2(0,L)} + (1 + 2L\|b\|_{L^\infty(0,L)})\int_0^T\int_0^L y^2 dxdt$$
$$+ L\|b\|_{L^\infty(0,L)}\int_{-h}^0\int_\omega z_0^2(x,t)dxdt.$$

Using (2.15), we obtain that there exists $C > 0$ such that

$$\|y_x\|^2_{L^2(0,T,L^2(0,L))} \leq C\left(\|y_0\|^2_{L^2(0,L)} + \|z_0(\cdot,-h\ \cdot)\|^2_{L^2(\omega\times(0,1))}\right)$$

that implies, together with (2.15), the continuity of the map (2.11).

• Choosing $q(x,t) \equiv T - t$ in (2.17) implies easily inequality (2.13) since it writes

$$-\int_0^L Ty_0^2(x)dx + \int_0^T\int_0^L y^2 dxdt + \int_0^T (T-t)y_x^2(0,t)dt$$
$$+ 2\int_0^T\int_0^L (T-t)a(x)y^2 dxdt + 2\int_0^T\int_0^L (T-t)b(x)y(x,t)y(x,t-h)dxdt = 0,$$

and since we use the fact that

$$\begin{aligned}
2\int_0^T\int_0^L (T-t)b(x)y(x,t)y(x,t-h)dxdt \quad &\leq \quad T\int_0^T\int_0^L b(x)y^2(x,t)dxdt \\
&\quad +T\int_0^T\int_0^L b(x)y^2(x,t-h)dxdt \\
&\leq \quad 2T\int_0^T\int_0^L b(x)y^2(x,t)dxdt \\
&\quad +T\|b\|_{L^\infty(0,L)}\int_{-h}^0\int_\omega z_0^2(x,t)dxdt.
\end{aligned}$$

• Finally, taking $p(\rho,t) = 1$ in (2.16) yields inequality (2.14) since it writes

$$\int_0^1\int_\omega \left(z^2(x,\rho,T) - z_0^2(x,-\rho h)\right)dxd\rho + \frac{1}{h}\int_0^T\int_\omega \left(z^2(x,1,t) - y^2(x,t)\right)dxdt = 0.$$

By density of $\mathcal{D}(\mathcal{A})$ in H, the results extend to arbitrary $(y_0, z_0(\cdot,-h\ \cdot)) \in H$. □

2.2 KdV linear equation with a source term

We now consider the KdV linear equation with a right hand side:

$$\begin{cases}
y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) + b(x)y(x,t-h) = f(x,t), & x \in (0,L),\ t > 0, \\
y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\
y(x,0) = y_0(x), & x \in (0,L), \\
y(x,t) = z_0(x,t), & x \in (0,L), t \in (-h,0).
\end{cases} \tag{2.19}$$

Proposition 2. *Assume that* (1.2) *and* (1.3) *hold. For any* $(y_0, z_0(\cdot, -h\ \cdot)) \in H$ *and* $f \in L^1(0, T, L^2(0, L))$, *there exists a unique mild solution* $(y, y(\cdot, t-h\ \cdot)) \in \mathcal{B}\times C([0, T], L^2(\omega\times(0, 1)))$ *to* (2.19). *Moreover, there exists* $C > 0$ *independent of* T *such that*

$$\|(y, z)\|^2_{C([0,T],H)} \leq C \left(\|(y_0, z_0(\cdot, -h\ \cdot))\|^2_H + \|f\|^2_{L^1(0,T,L^2(0,L))} \right), \tag{2.20}$$

$$\|y_x\|^2_{L^2(0,T,L^2(0,L))} \leq C(1 + T) \left(\|(y_0, z_0(\cdot, -h\ \cdot))\|^2_H + \|f\|^2_{L^1(0,T,L^2(0,L))} \right). \tag{2.21}$$

Proof. The well-posedness of system (2.19) in $C([0, T], H)$, when we rewrite it as a first order system (see (2.10)) with source term $(f(\cdot, t), 0)$, and the proof of (2.20), stem from $\mathcal{A}$ being the infinitesimal generator of a C_0-semigroup of contractions on H (see [23]).

The proof of (2.21) follows exactly the steps of the proof of Proposition 1 (see the third step). We have to be careful to the fact that the right hand side terms are not homogeneous anymore (but involve the source f) and to note that

$$\left|\int_0^T \int_0^L fydxdt\right| \leq \int_0^T \|f\|_{L^2(0,L)} \|y\|_{L^2(0,L)}\, dt \leq \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} \int_0^T \|f\|_{L^2(0,L)}\, dt$$
$$\leq \frac{1}{2} \max_{t\in[0,T]} \|y(t)\|^2_{L^2(0,L)} + \frac{1}{2} \|f\|^2_{L^1(0,T,L^2(0,L))}.$$

□

2.3 Global existence of the solution of the nonlinear system

We endow the space $\mathcal{B}$ with the norm

$$\|y\|_{\mathcal{B}} = \max_{t\in[0,T]} \|y(.,t)\|_{L^2(0,L)} + \left(\int_0^T \|y(.,t)\|^2_{H^1_0(0,L)}\, dt\right)^{1/2}.$$

To prove the well-posedness result of the nonlinear system (1.1), we follow [24] (see also [3], [7]).

The first step is to show that the nonlinear term yy_x can be considered as a source term of the linear equation (2.19):

Proposition 3. *Let* $y \in \mathcal{B}$. *Then* $yy_x \in L^1(0, T, L^2(0, L))$ *and the map*

$$y \in \mathcal{B} \mapsto yy_x \in L^1(0, T, L^2(0, L))$$

is continuous. In particular, there exists $K > 0$ *such that, for any* $y, \tilde{y} \in \mathcal{B}$, *we have*

$$\int_0^T \|yy_x - \tilde{y}\tilde{y}_x\|_{L^2(0,L)} \leq KT^{1/4} \left(\|y\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}}\right) \|y - \tilde{y}\|_{\mathcal{B}}.$$

Proof. The proof can be found in [24], which is a variant of Proposition 4.1 of [25]. □

We are now in position to prove the global existence of solutions of (1.1):

Proposition 4. *Let $L > 0$ and assume that (1.2) and (1.3) hold. Then for every $(y_0, z_0(\cdot, -h\cdot)) \in H$, there exists a unique $y \in \mathcal{B}$ solution of system (1.1). Moreover there exists $C > 0$ such that*

$$\|y_x\|^2_{L^2(0,T,L^2(0,L))} \leq C \left(\|(y_0, z_0(\cdot, -h\cdot))\|^2_H + \|(y_0, z_0(\cdot, -h\cdot))\|^4_H \right). \tag{2.22}$$

Proof. We closely follow [24] (see also [21]): we can obtain the global existence of solution by proving the local (in time) existence and using the decay of the energy to obtain the global existence of solution. Indeed, if we prove the local (in time) existence and uniqueness of solution of (1.1), global existence will then be an immediate consequence of the *a priori* estimate

$$E(t_2) \leq E(t_1) \leq E(0), \qquad \forall 0 < t_1 < t_2, \tag{2.23}$$

provided by (1.7).

We are then reduced to prove the local (in time) existence and uniqueness of solution of (1.1). Let $(y_0, z_0(\cdot, -h\cdot)) \in H$. Given $y \in \mathcal{B}$, we consider the map $\Phi : \mathcal{B} \to \mathcal{B}$ defined by $\Phi(y) = \tilde{y}$ where $\tilde{y}$ is solution of

$$\begin{cases} \tilde{y}_t(x,t) + \tilde{y}_{xxx}(x,t) + \tilde{y}_x(x,t) + a(x)\tilde{y}(x,t) + b(x)\tilde{y}(x,t-h) = -y(x,t)y_x(x,t), \quad x \in (0,L),\, t > 0, \\ \tilde{y}(0,t) = \tilde{y}(L,t) = \tilde{y}_x(L,t) = 0, \quad t > 0, \\ \tilde{y}(x,0) = y_0(x), \quad x \in (0,L), \\ \tilde{y}(x,t) = z_0(x,t), \quad x \in (0,L),\, t \in (-h,0). \end{cases}$$

Clearly $y \in \mathcal{B}$ is a solution of (1.1) if and only if y is a fixed point of the map Φ.

From (2.20), (2.21) and Proposition 3, we get

$$\begin{aligned} \|\Phi(y)\|_{\mathcal{B}} = \|\tilde{y}\|_{\mathcal{B}} &\leq C(1+\sqrt{T}) \left(\|(y_0, z_0(\cdot, -h\cdot))\|_H + \int_0^T \|yy_x(t)\|_{L^2(0,L)} dt \right) \\ &\leq C(1+\sqrt{T}) \left(\|(y_0, z_0(\cdot, -h\cdot))\|_H + T^{\frac{1}{4}} \|y\|^2_{\mathcal{B}} \right) \\ &\leq C(1+\sqrt{T}) \|(y_0, z_0(\cdot, -h\cdot))\|_H + 2CT^{\frac{1}{4}} \|y\|^2_{\mathcal{B}}, \end{aligned}$$

with $T < 1$. Moreover, for the same reasons, we have

$$\begin{aligned} \|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} &\leq C(1+\sqrt{T}) \int_0^T \|-y_1 y_{1,x} + y_2 y_{2,x}\|_{L^2(0,L)} dt \\ &\leq C(1+\sqrt{T}) T^{\frac{1}{4}} \left(\|y_1\|_{\mathcal{B}} + \|y_2\|_{\mathcal{B}} \right) \|y_1 - y_2\|_{\mathcal{B}}. \end{aligned}$$

We consider Φ restricted to the closed ball $\{y \in \mathcal{B}, \|y\|_{\mathcal{B}} \leq R\}$ with $R > 0$ to be chosen later. Then

$$\|\Phi(y)\|_{\mathcal{B}} \leq C(1+\sqrt{T}) \|(y_0, z_0(\cdot, -h\cdot))\|_H + 2CT^{\frac{1}{4}} R^2 \text{ and } \|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq 2C(1+\sqrt{T}) T^{\frac{1}{4}} R \|y_1 - y_2\|_{\mathcal{B}}.$$

So if we take $R = 2C \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H$ and $T > 0$ satisfying

$$\sqrt{T} + 8C^2 \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H T^{\frac{1}{4}} < 1 \qquad \text{and} \qquad T < \min\left\{1, \frac{1}{(4CR)^4}\right\},$$

then $\|\Phi(y)\|_{\mathcal{B}} < R$ and $\|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq C_1 \|y_1 - y_2\|_{\mathcal{B}}$, with $C_1 < 1$. Consequently, we can apply the Banach fixed point theorem and the map Φ has a unique fixed point.

To prove (2.22), we multiply the first equation of (1.1) by xy and integrate to obtain

$$\begin{aligned} &3\int_0^T \int_0^L y_x^2(x,t)dxdt + \int_0^L xy^2(x,T)dx + 2\int_0^T \int_0^L a(x)xy^2(x,t)dxdt \\ &= \int_0^T \int_0^L y^2(x,t)dxdt + \int_0^L xy_0^2(x)dx - 2\int_0^T \int_0^L b(x)xy(x,t)y(x,t-h)dt + \frac{2}{3}\int_0^T \int_0^L y^3(x,t)dxdt. \end{aligned}$$

Then, (2.23) yields to

$$\begin{aligned} \int_0^T \int_0^L y_x^2(x,t)dxdt \leq{}& \frac{T+L}{3} \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H^2 \\ &+ \frac{2L}{3}\int_0^T \int_\omega b(x) \left|y(x,t)y(x,t-h)\right| dxdt + \frac{2}{9}\int_0^T \int_0^L |y(x,t)|^3 dxdt. \end{aligned}$$

As $H^1(0,L)$ embeds into $C([0,L])$ and using Cauchy-Schwarz inequality and (2.23), we get

$$\begin{aligned} &\int_0^T \int_0^L |y|^3(x,t)dxdt \leq \int_0^T \|y\|_{L^\infty(0,L)} \int_0^L y^2(x,t)dxdt \\ &\leq \sqrt{L}\int_0^T \|y\|_{H^1(0,L)} \int_0^L y^2(x,t)dxdt \\ &\leq \sqrt{L}\, \|y\|^2_{L^\infty(0,T,L^2(0,L))} \int_0^T \|y\|_{H^1(0,L)}\, dt \\ &\leq \sqrt{LT} \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H^2 \|y\|_{L^2(0,T,H^1(0,L))}. \end{aligned}$$

Moreover, using again (2.23), we have

$$\begin{aligned} \int_0^T \int_\omega b(x) \left|y(x,t)y(x,t-h)\right| dxdt &\leq \frac{\|b\|_{L^\infty(0,L)}}{2} \left(\int_0^T \int_\omega y^2(x,t)dxdt + \int_0^T \int_\omega y^2(x,t-h)dxdt\right) \\ &\leq \|b\|_{L^\infty(0,L)} \int_0^T \int_0^L y^2(x,t)dxdt \\ &\quad + \frac{\|b\|_{L^\infty(0,L)}}{2} \int_{-h}^0 \int_\omega z_0^2(x,t)dxdt \\ &\leq \|b\|_{L^\infty(0,L)} \left(T + \frac{1}{2}\right) \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H^2. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \int_0^T \int_0^L y_x^2(x,t)dxdt \leq{}& \left(\frac{T+L}{3} + \frac{L\,\|b\|_{L^\infty(0,L)}\,(2T+1)}{3}\right) \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H^2 \\ &+ \frac{2\sqrt{LT}}{9} \left\|(y_0, z_0(\cdot, -h\cdot))\right\|_H^2 \|y\|_{L^2(0,T,H^1(0,L))}. \end{aligned}$$

Using Young's inequality, there exists $C > 0$ such that (2.22) holds. □

3 Asymptotic stability results when (1.3) holds

3.1 Local stability result by a Lyapunov functional

The goal of this subsection is to prove a local stability result which is based on the appropriate choice of a Lyapunov functional. We start by proving the decay of the energy of (1.1).

Proposition 5. *Let (1.2) and (1.3) be satisfied. Then, for any regular solution of (1.1) the energy E defined by (1.6) is non-increasing and there exists a positive constant C_1 such that*

$$\frac{d}{dt}E(t) \leq -C_1\left[y_x^2(0,t) + \int_0^L a(x)y^2(x,t)dx + \int_\omega y^2(x,t-h)dx\right] \leq 0. \tag{3.24}$$

Proof. Differentiating (1.6) and using (1.1), we obtain,

$$\begin{aligned}
\frac{d}{dt}E(t) &= -2\int_0^L y(x,t)(y_{xxx} + y_x + yy_x + ay)(x,t)dx - 2\int_0^L b(x)y(x,t)y(x,t-h)dx \\
&\quad -2\int_0^1\int_\omega \xi(x)y(x,t-h\rho)\partial_\rho y(x,t-h\rho)d\rho \\
= &\quad -y_x^2(0,t) - 2\int_0^L a(x)y^2(x,t)dx - 2\int_0^L b(x)y(x,t)y(x,t-h)dx + \int_\omega \xi(x)y^2(x,t)dx \\
&\quad -\int_\omega \xi(x)y^2(x,t-h)dx.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{d}{dt}E(t) \leq -y_x^2(0,t) + \int_\omega \left(-2a(x) + b(x) + \xi(x)\right)y^2(x,t)dx - 2\int_{(0,L)\setminus\omega} a(x)y^2(x)dx \\
+ \int_\omega \left(b(x) - \xi(x)\right)y^2(x,t-h)dx.
\end{aligned} \tag{3.25}$$

Assumptions (1.4) and (1.5) (due to (1.2)-(1.3)) end the proof. □

This result is not sufficient to prove the exponential stability. Therefore, we choose now the following Lyapunov functional (similar to the one in [2]):

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \tag{3.26}$$

where μ_1 and μ_2 are positive constants that will be fixed small enough later on, E is the energy defined by (1.6), V_1 is defined by

$$V_1(t) = \int_0^L xy^2(x,t)dx, \tag{3.27}$$

and V_2 is defined by

$$V_2(t) = h\int_\omega\int_0^1 (1-\rho)y^2(x,t-h\rho)dxd\rho. \tag{3.28}$$

It is clear that the two energies E and V are equivalent, in the sense that

$$E(t) \leq V(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right) E(t) \tag{3.29}$$

(see (1.2) and (1.5)).

Proposition 6. *Assume that a and b are nonnegative functions belonging to $L^\infty(0, L)$ satisfying* (1.2) *and* (1.3)*, and assume that the length L fulfills*

$$L < \pi\sqrt{3}. \tag{3.30}$$

Then, there exists $r > 0$ sufficiently small, such that for every $(y_0, z_0) \in \mathcal{H}$ satisfying

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

the energy of system (1.1)*, denoted E and defined by* (1.6)*, decays exponentially. More precisely, there exist two positive constants γ and κ such that*

$$E(t) \leq \kappa E(0) e^{-2\gamma t}, \qquad t > 0,$$

where for μ_1, μ_2 sufficiently small

$$\gamma \leq \min\left\{\frac{(9\pi^2 - 3L^2 - 2L^{3/2} r \pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2h(\mu_2 + \|\xi\|_{L^\infty(0,L)})}\right\}, \tag{3.31}$$

$$\kappa \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right).$$

Proof. Let y be a solution of (1.1) with $(y_0, z_0(\cdot, -h\cdot)) \in \mathcal{D}(\mathcal{A})$ satisfying $\|(y_0, z_0(\cdot, -h\cdot))\|_H \leq r$. Differentiating (3.27) and using (1.1), we obtain by using several integrations by parts

$$\begin{aligned}
\frac{d}{dt} V_1(t) &= -2\int_0^L xy(x,t)(y_{xxx} + y_x + yy_x + ay)(x,t)dx - 2\int_0^L xb(x)y(x,t)y(x,t-h)dx \\
&= -2\int_0^L y_x^2(x,t)dx + 2\left[y(x,t)y_x(x,t)\right]_0^L - \int_0^L y_x^2(x,t)dx + \left[xy_x^2(x,t)\right]_0^L \\
&\quad + \int_0^L y^2(x,t)dx + \frac{2}{3}\int_0^L y^3(x,t)dx - 2\int_0^L xa(x)y^2(x,t)dx - 2\int_0^L xb(x)y(x,t)y(x,t-h)dx \\
&= -3\int_0^L y_x^2(x,t)dx + \int_0^L y^2(x,t)dx + \frac{2}{3}\int_0^L y^3(x,t)dx \\
&\quad - 2\int_0^L xa(x)y^2(x,t)dx - 2\int_\omega xb(x)y(x,t)y(x,t-h)dx.
\end{aligned} \tag{3.32}$$

Moreover, differentiating (3.28), using an integration by parts, we have

$$\begin{aligned}
\frac{d}{dt}V_2(t) &= 2h\int_\omega\int_0^1(1-\rho)y(x,t-h\rho)\partial_t y(x,t-h\rho)d\rho dx\\
&= -2\int_\omega\int_0^1(1-\rho)y(x,t-h\rho)\partial_\rho\left(y(x,t-h\rho)\right)d\rho dx\\
&= -\int_\omega\left[(1-\rho)y^2(x,t-h\rho)\right]_0^1 dx-\int_\omega\int_0^1 y^2(x,t-h\rho)d\rho dx\\
&= \int_\omega y^2(x,t)dx-\int_\omega\int_0^1 y^2(x,t-h\rho)d\rho dx.
\end{aligned}\tag{3.33}$$

Consequently, with (3.25), (3.26), (3.32), (3.33) and the Cauchy-Schwarz inequality, we have, for any $\gamma>0$,

$$\begin{aligned}
&\frac{d}{dt}V(t)+2\gamma V(t)\le\int_\omega\left(-2a(x)+b(x)+\xi(x)+\mu_1 Lb(x)+\mu_2\right)y^2(x,t)dx\\
&-2\int_{(0,L)\setminus\omega}a(x)y^2(x,t)dx+\int_\omega\left(b(x)-\xi(x)+\mu_1 Lb(x)\right)y^2(x,t-h)dx\\
&+(\mu_1+2\gamma+2\gamma\mu_1 L)\int_0^L y^2(x,t)dx-3\mu_1\int_0^L y_x^2(x,t)dx\\
&+\frac{2}{3}\mu_1\int_0^L y^3(x,t)dx+\int_\omega\int_0^1\left(2\gamma\xi(x)h+2\gamma\mu_2 h-\mu_2\right)y^2(x,t-h\rho)d\rho dx.
\end{aligned}$$

Poincaré inequality ($\|y\|_{L^2(0,L)}\le\frac{L}{\pi}\|y_x\|_{L^2(0,L)}$ for $y\in H_0^1(0,L)$) implies that

$$\begin{aligned}
&\frac{d}{dt}V(t)+2\gamma V(t)\le\int_\omega\left(-2a(x)+b(x)+\xi(x)+\mu_1 Lb(x)+\mu_2\right)y^2(x,t)dx\\
&+\int_\omega\left(b(x)-\xi(x)+\mu_1 Lb(x)\right)y^2(x,t-h)dx+\left(\frac{L^2\left(\mu_1+2\gamma+2\gamma\mu_1 L\right)}{\pi^2}-3\mu_1\right)\int_0^L y_x^2(x,t)dx\\
&+\frac{2}{3}\mu_1\int_0^L y^3(x,t)dx+\int_\omega\int_0^1\left(2\gamma\xi(x)h+2\gamma\mu_2 h-\mu_2\right)y^2(x,t-h\rho)d\rho dx.
\end{aligned}$$

Using (1.5), it is sufficient to take μ_1 and μ_2 sufficiently small to have $-2a(x)+b(x)+\xi(x)+\mu_1 Lb(x)+\mu_2\le 0$ and $b(x)-\xi(x)+\mu_1 Lb(x)\le 0$ for $x\in\omega$. More precisely, we can take

$$\mu_1\le\inf_{x\in\omega}\left\{\frac{2a(x)-b(x)-\xi(x)}{Lb(x)},\frac{\xi(x)-b(x)}{Lb(x)}\right\},$$
$$\mu_2\le\inf_{x\in\omega}\left\{2a(x)-b(x)-\xi(x)-\mu_1 Lb(x)\right\}.$$

For instance, by (1.5), we can take

$$0<\mu_1<\frac{c_0}{L\left\|b\right\|_{L^\infty(0,L)}},\qquad 0<\mu_2<c_0-L\mu_1\left\|b\right\|_{L^\infty(0,L)}.$$

Moreover, using Cauchy-Schwarz inequality, Proposition 5 and since $H_0^1(0,L)\subset L^\infty(0,L)$, we obtain:

$$\begin{aligned}
\int_0^L y^3(x,t)dx &\le \|y(.,t)\|^2_{L^\infty(0,L)}\int_0^L|y(x,t)|dx\\
&\le L\sqrt{L}\left\|y_x(.,t)\right\|^2_{L^2(0,L)}\left\|y(.,t)\right\|_{L^2(0,L)}\\
&\le L^{3/2}\left\|(y_0,z_0(\cdot,-h\cdot))\right\|_H\left\|y_x(.,t)\right\|^2_{L^2(0,L)}\\
&\le L^{3/2}\,r\left\|y_x(.,t)\right\|^2_{L^2(0,L)}.
\end{aligned}$$

Consequently, we have

$$\frac{d}{dt}V(t) \;+\; 2\gamma V(t) \;\;\leq\;\; \Upsilon\|y_x(t)\|^2_{L^2(0,L)} \;+\; \int_\omega\int_0^1 \left(2h\gamma(\mu_2+\xi(x))-\mu_2\right)y^2(x,t\;-\;h\rho)dxd\rho$$

where $\Upsilon = \dfrac{L^2\left(2\gamma\left(1+L\mu_1\right)+\mu_1\right)}{\pi^2} - 3\mu_1 + \dfrac{2L^{3/2}r\mu_1}{3}$.

Since L satisfies the constraint (3.30), it is possible to choose r small enough to have $r < \dfrac{3(3\pi^2-L^2)}{2L^{3/2}\pi^2}$. Then one can choose $\gamma > 0$ such that (3.31) holds in order to obtain

$$\frac{d}{dt}V(t) + 2\gamma V(t) \leq 0, \quad \forall t > 0.$$

Integrating over $(0,t)$ and using (3.29), we finally obtain that

$$E(t) \leq \left(1+\max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right)E(0)e^{-2\gamma t}, \qquad \forall t > 0.$$

By density of $\mathcal{D}(\mathcal{A})$ in H, the results extend to arbitrary $(y_0, z_0(\cdot,-h\cdot)) \in H$. □

Remark 1. *This result gives an estimation of the decay rate* (3.31)*. In particular, we can note that when the delay h increases, the decay rate γ decreases. The condition $L < \sqrt{3}\pi$ is a technical assumption and comes from the choice of the multiplier x in the expression of V_1. To find a better multiplier is an open problem as far as we know.*

3.2 Semi-global stability result

The goal of this subsection is to prove Theorem 1 using directly the nonlinear system (1.1). The proof of this theorem is based on an observability inequality for the nonlinear delayed KdV equation and the use of a contradiction argument. Consequently, the value of the decay rate can not be estimated in this approach. The two main difficulties to the semi-global stability result are the pass to the limit in the nonlinear term and the fact that this nonlinear term do not allow to use Holmgrem's theorem. Instead we will use the following unique continuation property for the nonlinear system due to Saut and Scheurer [28]:

Theorem 4. *Let $u \in L^2(0,T,H^3(0,L))$ be a solution of*

$$u_t + u_x + u_{xxx} + uu_x = 0$$

such that

$$u(x,t) = 0, \qquad \forall t \in (t_1,t_2), \, \forall x \in \omega,$$

where ω is an open nonempty subset of $(0,L)$. Then

$$u(x,t) = 0, \qquad \forall t \in (t_1,t_2), \, \forall x \in (0,L).$$

To prove Theorem 1, in order to use Theorem 4, we have to show that the limit solution in the contradiction argument is in $L^2(0,T,H^3(0,L))$.

Proof of Theorem 1. We follow [21] (see also [7]). Let y be the solution of (1.1) with $(y_0, z_0(\cdot,-h\,\cdot)) \in \mathcal{D}(\mathcal{A})$. Integrating (3.24) between 0 and $T>h$, we have

$$E(T)-E(0) \leq -C_1\left(\int_0^T y_x^2(0,t)dt + \int_0^T\int_0^L a(x)y^2(x,t)dxdt + \int_0^T\int_\omega y^2(x,t-h)dxdt\right),$$

which is equivalent to

$$\int_0^T y_x^2(0,t)dt + \int_0^T\int_0^L a(x)y^2(x,t)dxdt + \int_0^T\int_\omega y^2(x,t-h)dxdt \leq \frac{1}{C_1}\left(E(0)-E(T)\right). \quad (3.34)$$

Consequently, if we succeed to prove the observability inequality

$$\int_0^L y_0^2(x)dx + h\int_\omega\int_0^1 \xi(x)z_0^2(x,-h\rho)dxd\rho$$
$$\leq C\left(\int_0^T y_x^2(0,t)dt + \int_0^T\int_0^L a(x)y^2(x,t)dxdt + \int_0^T\int_\omega z^2(x,1,t)dxdt\right) \quad (3.35)$$

for the nonlinear system (1.1), as the energy is non-increasing, we have, using (3.34),

$$E(T) \leq E(0) \leq C\left(\int_0^T y_x^2(0,t)dt + \int_0^T\int_0^L a(x)y^2(x,t)dxdt + \int_0^T\int_\omega y^2(x,t-h)dxdt\right)$$
$$\leq \frac{C}{C_1}\left(E(0)-E(T)\right),$$

which implies that

$$E(T) \leq \gamma E(0), \text{ with } \gamma = \frac{\frac{C}{C_1}}{1+\frac{C}{C_1}} < 1. \quad (3.36)$$

Using this argument on $[(m-1)T, mT]$ for $m=1,2,...$ (which is valid because the system is invariant by translation in time), we will get

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0).$$

Therefore, we have $E(mT) \leq e^{-\nu mT}E(0)$ with $\nu = \frac{1}{T}\ln\frac{1}{\gamma} = \frac{1}{T}\ln\left(1+\frac{C_1}{C}\right) > 0$. For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T}E(0) \leq \frac{1}{\gamma}e^{-\nu t}E(0).$$

By density of $\mathcal{D}(\mathcal{A})$ in H, we deduce that the exponential decay of the energy E holds for any initial data in H.

We are then reduced to prove the observability inequality (3.35) for the nonlinear system (1.1). First, since $\int_0^L y^2 y_x dx = \frac{1}{3}[y^3]_0^L = 0$, we can obtain, similarly to (2.17) with $q \equiv 1$,

$$\int_0^L y^2(x,t)dx - \int_0^L y_0^2(x)dx + \int_0^t y_x^2(0,s)ds + 2\int_0^t \int_0^L a(x)y^2(x,s)dxds + 2\int_0^t \int_0^L b(x)y(x,s)y(x,s-h)dxds = 0.$$

By integrating this last equation between 0 and T, we have

$$T\int_0^L y_0^2(x)dx \leq \int_0^T \int_0^L y^2(x,t)dxdt + T\int_0^T y_x^2(0,t)dt + 2T\int_0^T \int_0^L a(x)y^2(x,t)dxdt + 2T\int_0^T \int_0^L b(x)y(x,t)y(x,t-h)dxdt.$$

Using the fact that

$$\int_0^T \int_0^L b(x)y(x,t)y(x,t-h)dxdt \leq \frac{\|b\|_{L^\infty(0,L)}}{2}\int_0^T \int_\omega y^2(x,t)dxdt + \frac{1}{2}\int_0^T \int_\omega b(x)y^2(x,t-h)dxdt,$$

and (1.4), there exists $C > 0$ such that

$$T\int_0^L y_0^2(x)dx \leq \|y\|^2_{L^2((0,L)\times(0,T))} + T\int_0^T y_x^2(0,t)dt + C\int_0^T \int_0^L a(x)y^2(x,t)dxdt + C\int_0^T \int_\omega b(x)y^2(x,t-h)dxdt. \quad (3.37)$$

Moreover, integrating (2.14) between 0 and T, we have,

$$T\int_0^1 \int_\omega z_0^2(x,-h\rho)d\rho dx \leq \int_0^T \int_0^1 \int_\omega z^2(x,\rho,t)dxd\rho dt + \frac{T}{h}\int_0^T \int_\omega y^2(x,t-h)dxdt,$$

with, if $T > h$, using (1.2) and (1.4),

$$\begin{aligned}
\int_0^T \int_0^1 \int_\omega z^2(x,\rho,t)dxd\rho dt &= \int_0^T \int_0^1 \int_\omega y^2(x,t-\rho h)dxd\rho dt = \frac{1}{h}\int_0^T \int_{t-h}^t \int_\omega y^2(x,u)dxdudt \\
&\leq \frac{T}{h}\int_{-h}^T \int_\omega y^2(x,u)dxdu \\
&= \frac{T}{h}\int_{-h}^{T-h} \int_\omega y^2(x,u)dxdu + \frac{T}{h}\int_{T-h}^T \int_\omega y^2(x,u)dxdu \\
&\leq \frac{T}{h}\int_0^T \int_\omega \left(y^2(x,t) + y^2(x,t-h)\right)dxdt \\
&\leq C\int_0^T \int_0^L a(x)y^2(x,t)dxdt + C\int_0^T \int_\omega b(x)y^2(x,t-h)dxdt.
\end{aligned}$$

Gathering these estimates with (3.37), we see that it suffices, in order to prove the observability inequality (3.35) for the nonlinear system (1.1), to prove that for any $T, R > 0$ there exists $K > 0$ (which depends on R and T) such that

$$K\|y\|^2_{L^2((0,L)\times(0,T))} \leq \int_0^T y_x^2(0,t)dt + \int_0^T \int_0^L a(x)y^2(x,t)dxdt + \int_0^T \int_\omega b(x)y^2(x,t-h)dxdt \quad (3.38)$$

for the solutions of the nonlinear system (1.1) with $\|(y_0, z_0(\cdot, -h\cdot))\|_H \leq R$. We do that by a contradiction argument. We assume that (3.38) does not hold and we built a sequence $(y^n)_n \subset \mathcal{B}$ solution of (1.1) with $\|(y_0^n, z_0^n(\cdot, -h\cdot))\|_H \leq R$ such that

$$\lim_{n\to+\infty} \frac{\|y^n\|^2_{L^2((0,L)\times(0,T))}}{\|y^n_x(0,\cdot)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a\,|y^n|^2\,dxdt + \int_0^T\int_\omega b\,|y^n(x,t-h)|^2\,dxdt} = +\infty.$$

We define $\lambda_n = \|y^n\|_{L^2((0,L)\times(0,T))}$ and $v_n = \frac{y^n}{\lambda_n}$. Then, v^n satisfies

$$\begin{cases} v^n_t(x,t) + v^n_{xxx}(x,t) + v^n_x(x,t) + \lambda_n v^n(x,t) v^n_x(x,t) + a(x)v^n(x,t) \\ \qquad + b(x)v^n(x,t-h) = 0, \\ v^n(0,t) = v^n(L,t) = v^n_x(L,t) = 0, \end{cases} \tag{3.39}$$

$$\|v^n\|_{L^2((0,L)\times(0,T))} = 1 \tag{3.40}$$

and

$$\|v^n_x(0,\cdot)\|^2_{L^2(0,L)} + \int_0^T\int_0^L a\,|v^n|^2\,dxdt + \int_0^T\int_\omega b(x)\,|v^n(x,t-h)|^2\,dxdt \longrightarrow_{n\to+\infty} 0. \tag{3.41}$$

Using the fact that

$$\int_0^T\int_0^L (T-t)(v^n)^2 v^n_x\,dxdt = 0,$$

we have, as for the linear case (see (2.13)) that

$$\|v^n(\cdot,0)\|^2_{L^2(0,L)} \leq C\left(\|v^n\|^2_{L^2(0,T,L^2(0,L))} + \|v^n_x(0,\cdot)\|^2_{L^2(0,T)} + \|v^n(\cdot,-h\cdot)\|^2_{L^2(\omega\times(0,1))}\right),$$

with, if $T > h$ and since (1.2) holds,

$$\begin{aligned} \|v^n(\cdot,-h\cdot)\|^2_{L^2(\omega\times(0,1))} &= h\int_{-h}^0\int_\omega |v^n(x,t)|^2\,dxdt \\ &\leq h\int_{-h}^{T-h}\int_\omega |v^n(x,t)|^2\,dxdt \leq h\int_0^T\int_\omega b(x)\,|v^n(x,t-h)|^2\,dxdt. \end{aligned}$$

Gathering this with (3.40) and (3.41), we see that $(v^n(\cdot,0))_n$ is bounded in $L^2(0,L)$. Moreover, we note that $(\lambda_n)_n$ is bounded in $\mathbb{R}$ since, due to (3.24),

$$\lambda_n = \|y^n\|_{L^2((0,L)\times(0,T))} \leq T\,\|(y^n_0, z^n_0(\cdot,-h\cdot))\|_H \leq TR.$$

Consequently, using the same inequality as (2.22) for (3.39), $(v^n)_n$ is bounded in $L^2(0,T,H^1(0,L))$. We also notice that $(v^n v^n_x)_n$ is a subset of $L^2(0,T,L^1(0,L))$, since by Cauchy-Schwarz inequality, we have

$$\|v^n v^n_x\|_{L^2(0,T,L^1(0,L))} \leq \|v^n\|_{C([0,T],L^2(0,L))}\,\|v^n\|_{L^2(0,T,H^1(0,L))}.$$

All these are used to show that $v^n_t = -(v^n_{xxx} + v^n_x + \lambda_n v^n v^n_x + av^n + bv^n(t-h))$ is bounded in $L^2(0,T,H^{-2}(0,L))$ and consequently using a result of Simon [29], the set $\{v^n\}_n$ is relatively compact in $L^2(0,T,L^2(0,L))$ and a subsequence of $(v^n)_n$, also denoted by $(v^n)_n$, converges strongly in $L^2(0,T,L^2(0,L))$ to a limit v verifying $\|v\|_{L^2(0,T,L^2(0,L))} = 1$. Furthermore, by weak lower semicontinuity, we have

$$\|v_x(0,\cdot)\|^2_{L^2(0,L)} + \int_0^T \int_0^L a\,|v|^2\,dxdt + \int_0^T \int_\omega b(x)\,|v(x,t-h)|^2\,dxdt$$
$$\leq \liminf_{n\to\infty} \left(\|v_x^n(0,\cdot)\|^2_{L^2(0,L)} + \int_0^T \int_0^L a\,|v^n|^2\,dxdt + \int_0^T \int_\omega b(x)\,|v^n(x,t-h)|^2\,dxdt \right) = 0$$

and therefore

$$v(x,t) = 0 \quad \text{in } \omega \times (-h,T) \qquad \text{and} \qquad v_x(0,t) = 0 \quad \text{in } (0,T).$$

Since $(\lambda_n)_n$ is bounded, we can also extract a subsequence, still denoted by $(\lambda_n)_n$, which converges to $\lambda \geq 0$. Consequently, the limit v satisfies

$$\begin{cases} v_t(x,t) + v_{xxx}(x,t) + v_x(x,t) + \lambda v(x,t)v_x(x,t) = 0, \quad x \in (0,L),\, t \in (0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0, \qquad t \in (0,T), \\ v(x,t) = 0, \qquad x \in \omega,\, t \in (0,T), \\ v_x(0,t) = 0, \qquad t \in (0,T), \\ \|v\|_{L^2(0,T,L^2(0,L))} = 1. \end{cases}$$

We then distinguish two cases:

- First case: $\lambda = 0$. Then the system satisfied by v is linear and we can apply Holmgrem's theorem to get that the solution v is trivial, which contradicts $\|v\|_{L^2(0,T,L^2(0,L))} = 1$.

- Second case: $\lambda > 0$. We then prove that in fact $v \in L^2(0,T,H^3(0,L))$. For that, we consider $u = v_t$. Then u satisfies

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x(x,t) + \lambda u(x,t)v_x(x,t) + \lambda v(x,t)u_x(x,t) = 0, \; x \in (0,L),\, t \in (0,T), \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \qquad t \in (0,T), \\ u(x,t) = 0, \qquad x \in \omega,\, t \in (0,T), \\ u_x(0,t) = 0, \qquad t \in (0,T), \\ u(x,0) = -v'(x,0) - v'''(x,0) - \lambda v(x,0)v'(x,0) \in H^{-3}(0,L). \end{cases}$$

  Using Lemma A.2 of [7], we get that the initial data $u(\cdot,0) \in L^2(0,L)$ and so $u = v_t \in \mathcal{B}$. We deduce that $v_{xxx} = -v_t - v_x - \lambda v v_x \in L^2(0,T,L^2(0,L))$ and then that $v \in L^2(0,T,H^3(0,L))$. Applying Theorem 4, we obtain that the solution v is trivial, which contradicts $\|v\|_{L^2(0,T,L^2(0,L))} = 1$ and ends the proof.

□

4 Study of the case where $\operatorname{supp} b \not\subset \operatorname{supp} a$

In this section, we prove the exponential stability of (1.1) in the case where $\omega = \operatorname{supp} b \not\subset \operatorname{supp} a$ (and so (1.3) does not hold). In this case, the derivative of the energy E, defined by (1.6) for classical solution of (1.1), satisfies

$$\begin{aligned}\frac{d}{dt}E(t) &= -y_x^2(0,t) - 2\int_{\text{supp}\,a} a(x)y^2(x,t)dx - 2\int_\omega b(x)y(x,t)y(x,t-h)dx + \int_\omega \xi(x)y^2(x,t)dx \\ &\quad - \int_\omega \xi(x)y^2(x,t-h)dx \\ &\leq -y_x^2(0,t) - 2\int_{\text{supp}\,a} a(x)y^2(x,t)dx + \int_\omega (b(x)+\xi(x))y^2(x,t)dx + \int_\omega (b(x)-\xi(x))y^2(x,t-h)dx,\end{aligned}$$

and so the energy is not decreasing in general due to the term $b(x) + \xi(x) > 0$ on ω.

Inspired by [17], we consider the following auxiliary problem, which is "close" to (1.1) but with a decreasing energy:

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) + a(x)y(x,t) & \\ \qquad +b(x)y(x,t-h) + \xi b(x)y(x,t) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\ t \in (-h,0), \end{cases} \tag{4.42}$$

where ξ is a positive constant. Then we consider the energy defined by (1.6) with $\xi(x) = \xi b(x)$, i.e.

$$E(t) = \int_0^L y^2(x,t)dx + h\xi \int_\omega \int_0^1 b(x)y^2(x,t-h\rho)d\rho dx. \tag{4.43}$$

Then the derivative of this energy E, for classical solution of (4.42), satisfies

$$\begin{aligned}\frac{d}{dt}E(t) &= -y_x^2(0,t) - 2\int_{\text{supp}\,a} a(x)y^2(x,t)dx - 2\int_\omega b(x)y(x,t)y(x,t-h)dx - 2\xi\int_\omega b(x)y^2(x,t)dx \\ &\quad +\xi\int_\omega b(x)y^2(x,t)dx - \xi\int_\omega b(x)y^2(x,t-h)dx \\ &\leq -y_x^2(0,t) - 2\int_{\text{supp}\,a} a(x)y^2(x,t)dx + \int_\omega (b(x)-\xi b(x))y^2(x,t)dx \\ &\quad + \int_\omega (b(x)-\xi b(x))y^2(x,t-h)dx \leq 0\end{aligned}$$

taking $\xi > 1$.

In order to prove the exponential stability of system (1.1) from the exponential stability of (4.42), we would like to use the classical perturbation result of Pazy [23]:

Theorem 5. *Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X, then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X satisfying $\|S(t)\| \leq Me^{(\omega+M\|B\|)t}$.*

The strategy to treat the case $\text{supp}\, b \not\subset \text{supp}\, a$ is then the following: we first prove the exponential stability for (4.42) linearized around 0 by the Lyapunov approach for all $L < \sqrt{3}\pi$ (allowing to have an estimation of the decay rate), then we show the exponential stability of the linear system (2.8) using Theorem 5 for all $L < \sqrt{3}\pi$ and for $\|b\|_{L^\infty(0,L)}$ small enough. Finally we obtain the local exponential stability of the nonlinear system (1.1) for all $L < \sqrt{3}\pi$ and for $\|b\|_{L^\infty(0,L)}$ small enough.

4.1 Exponential stability for a linear auxiliary system by the Lyapunov approach

We first consider the system (4.42) linearized around 0:

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) + b(x)y(x,t-h) + \xi b(x)y(x,t) = 0, \\ \qquad x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\ t \in (-h,0). \end{cases} \tag{4.44}$$

We start by showing that this system (4.44) is well-posed. As in Section 2, setting $z(x,\rho,t) = y_{|\omega}(x,t-\rho h)$ for any $x \in \omega$, $\rho \in (0,1)$ and $t > 0$, z satisfies the transport equation (2.9). We equipped the Hilbert space $H = L^2(0,L) \times L^2(\omega \times (0,1))$ with the inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y\tilde{y}\,dx + h\xi \int_\omega \int_0^1 b(x) z\tilde{z}\,d\rho dx,$$

for any $(y,z), (\tilde{y},\tilde{z}) \in H$ and where $\xi > 1$.

We then rewrite (4.44) as a first order system:

$$\begin{cases} U_t(t) = \mathcal{A}_0 U(t), \quad t > 0, \\ U(0) = U_0 \in H, \end{cases} \tag{4.45}$$

where $U = \begin{pmatrix} y \\ z \end{pmatrix}$, $U_0 = \begin{pmatrix} y_0 \\ z_{0|\omega}(\cdot, -h\ \cdot) \end{pmatrix}$, and where the operator $\mathcal{A}_0$ is defined by

$$\mathcal{A}_0 U = \begin{pmatrix} -y_{xxx} - y_x - ay - \xi by - b\tilde{z}(\cdot,1) \\ -\frac{1}{h} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \Big\{ (y,z) \in H^3(0,L) \times L^2(\omega, H^1(0,1)) \,\big|\, y(0) = y(L) = y_x(L) = 0,\ z(x,0) = y_{|\omega}(x) \text{ in } \omega \Big\}.$$

Theorem 6. *Assume that a and b are nonnegative functions in $L^\infty(0,L)$ satisfying* (1.2) *and that $U_0 \in H$. Let $\xi > 1$. Then there exists a unique mild solution $U \in C([0,+\infty), H)$ for system* (4.45)*. Moreover if $U_0 \in \mathcal{D}(\mathcal{A}_0)$, then the solution is classical and satisfies*

$$U \in C([0,+\infty), D(\mathcal{A}_0)) \cap C^1([0,+\infty), H).$$

Proof. We follow the proof of Theorem 3. We first prove that the operator $\mathcal{A}_0$ is dissipative.

Let $U=(y,z)\in\mathcal{D}(\mathcal{A}_0)$. Then we have

$$\begin{aligned}\langle\mathcal{A}_0U,U\rangle&=\frac{1}{2}[y_x^2]_0^L-\int_0^L a(x)y^2dx-\int_\omega \xi b(x)y^2dx-\int_\omega b(x)z(x,1)y(x)dx-\frac{\xi}{2}\int_\omega b(x)z^2(x,1)dx\\&\quad+\frac{\xi}{2}\int_\omega b(x)y^2(x)dx\\&\leq-\frac{1}{2}y_x^2(0)-\int_{\operatorname{supp}a}a(x)y^2(x)dx+\frac{1}{2}\int_\omega b(x)\left(1-\xi\right)y^2(x)dx+\frac{1}{2}\int_\omega b(x)\left(1-\xi\right)z^2(x,1)dx.\end{aligned}$$

If we take $\xi>1$, then $\langle\mathcal{A}_0U,U\rangle\leq 0$, which means that the operator $\mathcal{A}_0$ is dissipative.

Secondly we show that the adjoint of $\mathcal{A}_0$, denoted by $\mathcal{A}_0^*$, is also dissipative. It is not difficult to prove that the adjoint is defined by

$$\mathcal{A}_0^*U=\begin{pmatrix} y_{xxx}+y_x-ay-\xi by+\xi b\tilde{z}(\cdot,0)\\ \frac{1}{h}\,z_\rho\end{pmatrix},\qquad U=\begin{pmatrix}y\\ z\end{pmatrix}\in\mathcal{D}(\mathcal{A}_0^*),$$

with domain

$$\mathcal{D}(\mathcal{A}_0^*)=\left\{(y,z)\in H^3(0,L)\times L^2(\omega,H^1(0,1))\big|y(0)=y(L)=y_x(0)=0,z(x,1)=-\frac{1}{\xi}\,y_{|\omega}\,(x)\text{ in }\omega\right\}.$$

Then for all $U=(y,z)\in\mathcal{D}(\mathcal{A}_0^*)$,

$$\begin{aligned}\langle\mathcal{A}_0^*U,U\rangle&=-\frac{1}{2}[y_x^2]_0^L-\int_0^L a(x)y^2dx-\int_\omega\xi b(x)y^2dx+\int_\omega\xi b(x)z(x,0)y(x)dx\\&\quad+\frac{1}{2\xi}\int_\omega b(x)y^2(x)dx-\frac{\xi}{2}\int_\omega b(x)z^2(x,0)dx\\&\leq-\frac{1}{2}y_x^2(L)-\int_{\operatorname{supp}a}a(x)y^2(x)dx+\int_\omega b(x)\left(-\frac{\xi}{2}+\frac{1}{2\xi}\right)y^2(x)dx,\end{aligned}$$

and, since $\xi>1$, we have $-\frac{\xi}{2}+\frac{1}{2\xi}<0$.

Finally, the facts that $\mathcal{A}_0$ is a densely defined closed linear operator, and both $\mathcal{A}_0$ and $\mathcal{A}_0^*$ are dissipative, imply that $\mathcal{A}_0$ is the infinitesimal generator of a C_0 semigroup of contractions on H, which finishes the proof. □

We denote by $\{T(t),t\geq 0\}$ the semigroup of contractions associated with $\mathcal{A}_0$.

To prove the exponential stability of (4.44), we closely follow Subsection 3.1. More precisely, we choose the following candidate Lyapunov functional:

$$V(t)=E(t)+\mu_1V_1(t)+\mu_2V_2(t),\tag{4.46}$$

where μ_1 and μ_2 are positive constants that will be fixed small enough later on, E is the energy defined by (4.43), V_1 is defined by (3.27) and V_2 is defined by

$$V_2(t)=h\int_\omega\int_0^1(1-\rho)b(x)y^2(x,t-h\rho)dxd\rho.\tag{4.47}$$

It is clear that the two energies E and V are equivalent, in the sense that

$$E(t)\leq V(t)\leq\left(1+\max\left\{L\mu_1,\frac{\mu_2}{\xi}\right\}\right)E(t).\tag{4.48}$$

Proposition 7. *Assume that a and b are nonnegative functions in $L^\infty(0,L)$ satisfying* (1.2) *and that the length L fulfills* (3.30)*. Let $\xi > 1$. Then, for every $(y_0, z_0(\cdot, -h\cdot)) \in H$, the energy of system* (4.44)*, denoted E and defined by* (4.43)*, decays exponentially. More precisely, there exist two positive constants α and β such that*

$$E(t) \leq \beta E(0)e^{-2\alpha t}, \qquad t > 0,$$

where, for μ_1, μ_2 sufficiently small,

$$\alpha \leq \min\left\{ \frac{\mu_2}{2h(\xi+\mu_2)}, \frac{\mu_1(3\pi^2 - L^2)}{2L^2(1+\mu_1 L)} \right\}, \tag{4.49}$$

$$\beta = \left(1 + \max\left\{ L\mu_1, \frac{\mu_2}{\xi} \right\}\right).$$

Proof. Let y be a solution of (4.44) with $(y_0, z_0(\cdot, -h\cdot)) \in \mathcal{D}(\mathcal{A}_0)$. Differentiating (3.27), with (4.44), we obtain by using several integrations by parts and as in the proof of Proposition 6

$$\frac{d}{dt} V_1(t) = -3\int_0^L y_x^2(x,t)dx + \int_0^L y^2(x,t)dx - 2\int_{\operatorname{supp} a} xa(x)y^2(x,t)dx$$
$$- 2\int_\omega \xi x b(x) y^2(x,t)dx - 2\int_\omega xb(x)y(x,t)y(x,t-h)dx.$$

Moreover, differentiating (4.47), using an integration by parts, we obtain

$$\frac{d}{dt} V_2(t) = \int_\omega b(x)y^2(x,t)dx - \int_\omega\int_0^1 b(x)y^2(x,t-h\rho)d\rho dx.$$

Consequently, for any $\alpha > 0$, we get

$$\frac{d}{dt} V(t) + 2\alpha V(t) \leq -2\int_0^L a(x)(1+\mu_1 x)y^2(x,t)dx + \int_\omega b(x)\left(1 - \xi + \mu_2 + \mu_1 L\right) y^2(x,t)dx$$
$$+ \int_\omega b(x)\left(1 - \xi + \mu_1 L\right) y^2(x,t-h)dx + (\mu_1 + 2\alpha + 2\alpha\mu_1 L)\int_0^L y^2(x,t)dx$$
$$- 3\mu_1 \int_0^L y_x^2(x,t)dx + \int_\omega\int_0^1 b(x)\left(2\alpha\xi h + 2\alpha\mu_2 h - \mu_2\right) y^2(x,t-h\rho)d\rho dx.$$

Using Poincaré inequality, we obtain that

$$\frac{d}{dt} V(t) + 2\alpha V(t) \leq \int_\omega b(x)\left(1 - \xi + \mu_2 + \mu_1 L\right) y^2(x,t)dx$$
$$+ \int_\omega b(x)\left(1 - \xi + \mu_1 L\right) y^2(x,t-h)dx + \left(\frac{L^2}{\pi^2}\left(\mu_1 + 2\alpha + 2\alpha\mu_1 L\right) - 3\mu_1\right)\int_0^L y_x^2(x,t)dx$$
$$+ \int_\omega\int_0^1 b(x)\left(2\alpha\xi h + 2\alpha\mu_2 h - \mu_2\right) y^2(x,t-h\rho)d\rho dx.$$

Then, we choose $\mu_1 > 0$, $\mu_2 > 0$ and $\alpha > 0$ such that

$$\mu_1 < \frac{\xi - 1}{L}, \quad \mu_2 < \xi - 1 - L\mu_1 \quad \text{and} \quad \alpha < \min\left\{ \frac{\mu_2}{2h(\xi+\mu_2)}, \frac{\mu_1(3\pi^2 - L^2)}{2L^2(1+\mu_1 L)} \right\}.$$

This is possible by (3.30). Therefore we obtain

$$\frac{d}{dt}V(t) + 2\alpha V(t) \leq 0, \quad \forall t > 0.$$

Integrating over $(0,t)$ and using (4.48), we finally obtain that

$$E(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{\xi}\right\}\right) E(0)e^{-2\alpha t}, \qquad \forall t > 0.$$

By density of $\mathcal{D}(\mathcal{A}_0)$ in H, the results extend to arbitrary $(y_0, z_0(\cdot, -h\cdot)) \in H$. $\square$

4.2 Exponential stability for the linear KdV equation using a perturbation argument

We study now the asymptotic stability of the linear system (2.8) in the case where $\operatorname{supp} b = \omega \not\subset \operatorname{supp} a$, that we can rewrite as the first order system (2.10). It is clear that the corresponding operator $\mathcal{A}$ satisfy

$$\mathcal{A} = \mathcal{A}_0 + B,$$

with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0)$ and where the bounded operator B is defined by

$$BU = \begin{pmatrix} \xi by \\ 0 \end{pmatrix}, \qquad \forall U = \begin{pmatrix} y \\ z \end{pmatrix} \in H.$$

Proposition 8. *Assume that a and b are nonnegative functions in $L^\infty(0,L)$ satisfying* (1.2) *and assume that* (3.30) *holds. Let $\xi > 1$. Then for every $U_0 \in H$, there exists a unique mild solution $U \in C([0,+\infty), H)$ for system* (2.8)*, and for every $U_0 \in \mathcal{D}(\mathcal{A})$, the solution is classical and satisfies $U \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty), H)$. Moreover there exists $\delta > 0$ (depending on ξ, L, h) such that if*

$$\|b\|_{L^\infty(0,L)} \leq \delta,$$

then, for every $(y_0, z_0(\cdot, -h\cdot)) \in H$ the energy of system (2.8)*, denoted E and defined by* (4.43)*, decays exponentially. More precisely, there exist two positive constants ν and β (defined in Proposition 7) such that*

$$E(t) \leq \beta E(0)e^{-2\nu t}, \qquad t > 0.$$

Proof. It suffices to apply Theorem 5 and to note that, using Proposition 7 and the fact that $\|B\| = \xi \|b\|_{L^\infty(0,L)}$, we have

$$-\alpha + \sqrt{\beta}\xi \|b\|_{L^\infty(0,L)} < 0 \Leftrightarrow \|b\|_{L^\infty(0,L)} \leq \frac{\alpha}{\xi\sqrt{\beta}}.$$

$\square$

Remark 2. *Note that if h is large, the choice of b is such that $\|b\|_{L^\infty(0,L)}$ is small, due to* (4.49).

4.3 Local exponential stability for the nonlinear system (1.1)

We finally obtain the local exponential stability result enounced in Theorem 2 by considering the nonlinear KdV equation (1.1) in the case where $\operatorname{supp} b = \omega \not\subset \operatorname{supp} a$. We emphasize that we can take $a = 0$ in this theorem.

Proof of Theorem 2. It suffices to adapt Section 2.3, and more precisely to follow Proposition 5 of [7], to prove the local (for small initial data) existence of solution y of (1.1). Moreover y satisfies

$$\|y\|_{\mathcal{B}} \leq C \|(y_0, z_0(\cdot, -h\cdot))\|_H . \tag{4.50}$$

The proof of the exponential stability follows [7] for the asymptotic stability of the nonlinear KdV equation with internal feedback without delay. Consider initial data $\|(y_0, z_0(\cdot, -h\cdot))\|_H \leq r$ with r chosen later. The solution y of (1.1) can be written as $y = y^1 + y^2$ where y^1 is solution of

$$\begin{cases} y^1_t(x,t) + y^1_{xxx}(x,t) + y^1_x(x,t) + a(x)y^1(x,t) + b(x)y^1(x,t-h) = 0, & x \in (0,L),\ t > 0, \\ y^1(0,t) = y^1(L,t) = y^1_x(L,t) = 0, & t > 0, \\ y^1(x,0) = y_0(x), & x \in (0,L), \\ y^1(x,t) = z_0(x,t), & x \in (0,L),\ t \in (-h,0), \end{cases}$$

and y^2 is solution of

$$\begin{cases} y^2_t(x,t) + y^2_{xxx}(x,t) + y^2_x(x,t) + a(x)y^2(x,t) + b(x)y^2(x,t-h) = -y(x,t)y_x(x,t), & x \in (0,L),\ t > 0, \\ y^2(0,t) = y^2(L,t) = y^2_x(L,t) = 0, & t > 0, \\ y^2(x,0) = 0, & x \in (0,L), \\ y^2(x,t) = 0, & x \in (0,L),\ t \in (-h,0). \end{cases}$$

More precisely, y^1 is solution of (2.8) with initial data $(y_0, z_0(\cdot, -h\ \cdot)) \in H$ and y^2 is solution of (2.19) with initial data $(0,0)$ and right-hand side $f = -yy_x \in L^1(0,T,L^2(0,L))$. Using the fact that there exist $\gamma \in (0,1)$ and $T > 0$ such that $\left\|(y^1(T), z^1(T))\right\|_H \leq \gamma \|(y_0, z_0(\cdot, -h\ \cdot))\|_H$ (due to Proposition 8), Propositions 2 and 3, we have

$$\begin{aligned} \|(y(T), z(T))\|_H &\leq \left\|(y^1(T), z^1(T))\right\|_H + \left\|(y^2(T), z^2(T))\right\|_H \\ &\leq \gamma \|(y_0, z_0(\cdot, -h\ \cdot))\|_H + C \|yy_x\|_{L^1(0,T,L^2(0,L))} \\ &\leq \gamma \|(y_0, z_0(\cdot, -h\ \cdot))\|_H + C \|y\|^2_{\mathcal{B}}, \end{aligned} \tag{4.51}$$

with $0 < \gamma < 1$.

Therefore, gathering (4.51) and (4.50), there exists $C > 0$ so that

$$\|(y(T), z(T))\|_H \leq \|(y_0, z_0(\cdot, -h\cdot))\|_H (\gamma + C \|(y_0, z_0(\cdot, -h\cdot))\|_H)$$

which implies

$$\|(y(T), z(T))\|_H \leq \|(y_0, z_0(\cdot, -h\cdot))\|_H (\gamma + Cr) .$$

Given $\epsilon > 0$ small enough such that $\gamma + \epsilon < 1$, we can take r small enough such that $r < \frac{\epsilon}{C}$, in order to have

$$\|(y(T), z(T))\|_H \leq (\gamma + \epsilon) \|(y_0, z_0(\cdot, -h\cdot))\|_H ,$$

with $\gamma + \epsilon < 1$. The proof ends similarly to the proof of Theorem 1 (see after (3.36), using also (4.50)). □

Remark 3. *In order to apply Theorem 5, we need to have an estimation of the decay rate α of the linear auxiliary system* (4.44)*. That is why we use a Lyapunov method and we assume that* (3.30) *holds. If we prove an observability inequality as* (3.35) *(which holds without restriction on the length of the domain) for the linear or the nonlinear system, we do not have an estimation of the observability constant C in* (3.35) *since we use a contradiction argument. The decay rate of the linear auxiliary system is then given by $\alpha = \frac{1}{T}\ln\left(1 + \frac{C_1}{C}\right)$ (see the proof of Theorem 1) and we should verify that $-\alpha + \sqrt{\beta}\xi \|b\|_{L^\infty(0,L)} < 0$. But the observability constant C may depend on $\|b\|_{L^\infty(0,L)}$ and so this assumption is difficult to verify. To remove the assumption* (3.30) *in the case where* $\operatorname{supp} b = \omega \not\subset \operatorname{supp} a$ *is, to our knowledge, an interesting open question.*

5 Numerical simulations and other conclusions

In this paper, we studied the robustness with respect to the delay of the asymptotic stability of the nonlinear KdV equation with internal feedbacks. We first considered the case where the support of the weight of the internal feedback with delay b is included in the support of the weight of the internal feedback without delay a and where b is strictly smaller than a. We proved the local exponential stability result by two methods: the first one by a Lyapunov approach giving an estimation of the decay rate but with a technical limitation on the length of the domain (i.e. $L < \sqrt{3}\pi$) and a second one (a semiglobal stability result) by an observability inequality which holds for any lengths (but without information on the exponential decay rate). Secondly we considered the case where the support of the weight of the internal feedback with delay b is not included in the support of the weight of the internal feedback without delay a. In this case, if b is small enough (and even if $a = 0$), we showed that the nonlinear system is locally exponentially stable when $L < \sqrt{3}\pi$.

To illustrate these results, we present now some numerical simulations. Adapting the numerical scheme of [8] for internal feedback with delay (see also [2] and [22] for the critical generalized KdV equation), and using the parameters $T = 10$, $L = 3$, $h = 2$ and initial conditions $y_0(x) = 1 - \cos(2\pi x)$ and $z_0(x, \rho) = (1 - \cos(2\pi x))\cos(2\pi\rho h)$ with $\operatorname{supp} a = \operatorname{supp} b = (0, L/5)$ and where a and b are constant on their support, we obtain the following figure, that represents $t \mapsto \ln(E(t))$ for different values of a and b. We can see that when there is no feedback ($a = b = 0$), the energy is exponentially decreasing, and if the feedback without delay increases ($a = 1$ and $b = 0$), the energy is quickly exponentially decreasing. Moreover if the coefficient of delay b increases, then the energy is not exponentially decreasing, except if b is very small (for instance $b = 0.1$, $a = 0$) or if b is smaller than a ($a = 4$, $b = 1$). More precisely, with $a = 0$, $b = 10$ or $a = b = 10$, before that the delay acts ($t < 2$), the energy decays exponentially, which is not longer the case when b is effective ($t > 2$). Consequently Figure 1 illustrates Proposition 6 and Theorem 2.

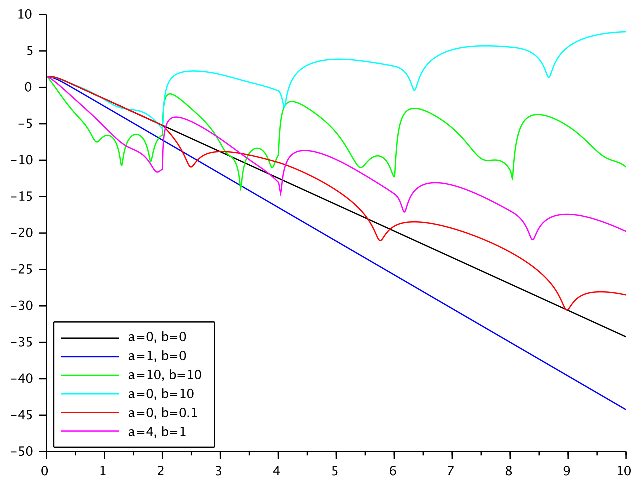

Figure 1: Representation of $t \mapsto \ln(E(t))$ for different values of a and b.

We finish this paper by considered the cases of mixed internal and boundary dampings with delay.

The most simple case is the case where we have an internal feedback without delay and a boundary feedback with delay, i.e.

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) + a(x)y(x,t) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = \beta y_x(0,t-h), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y_x(0,t) = z_0(t), & t \in (-h,0), \end{cases}$$

where $|\beta| < 1$ (see [2] for an explanation of this assumption), and where a is a nonnegative function in $L^\infty(0,L)$ such that $a(x) \geq a_0 > 0$ a.e. in ω, an open nonempty subset of $(0,L)$. In this case, it is sufficient to combine [21] and [2] to obtain the local exponential stability result for every non critical length (i.e. $L \notin \mathcal{L} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k,l \in \mathbb{N}^* \right\}$).

If we consider now the case of an internal feedback with delay and a boundary feedback without delay, i.e.

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) + b(x)y(x,t-h) = 0, \\ \qquad x \in (0,L),\, t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = \alpha y_x(0,t), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\, t \in (-h,0), \end{cases} \tag{5.52}$$

where $|\alpha| < 1$ (see [32]), and where b is a nonnegative function in $L^\infty(0,L)$ such that $b(x) \geq b_0 > 0$ a.e. in $\operatorname{supp} b = \omega$ an open nonempty subset of $(0,L)$ and where $\|b\|_{L^\infty(0,L)}$ is small enough. Then we can follow Section 4 to obtain the local exponential stability result for every $L < \sqrt{3}\pi$. For that, we introduce the following auxiliary exponentially stable system

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) + b(x)y(x,t-h) + \xi b(x)y(x,t) = 0, \\ \qquad x \in (0,L),\, t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = \alpha y_x(0,t), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \\ y(x,t) = z_0(x,t), & x \in (0,L),\, t \in (-h,0), \end{cases}$$

with the energy defined by (4.43) and $\xi > 1$. Note that we can take $\alpha = 0$ here.

An interesting question to investigate is to remove the technical assumption (3.30) in Theorem 2 or for (5.52). An other subject of future research could be the study of the asymptotic stability of the KdV equation with a delay in the nonlinear term.

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